Active Recall Questions

Mathematics Study Notes

March 1, 2026

Introduction

Mathematics is not a subject that can be mastered through passive reading alone. True understanding requires the ability to retrieve concepts, definitions, and problem-solving strategies from memory. One of the most effective techniques for achieving this is *active recall*, especially when implemented through a structured list of self-check questions.

What is Active Recall?

Active recall is the process of deliberately retrieving information from memory rather than re-exposing oneself to it. Instead of rereading notes or textbooks, the learner attempts to answer questions, reconstruct proofs, or solve problems without looking at the solution.

In mathematics, this might include:

- Stating definitions (e.g., convergence of a series),
- Recalling theorems (e.g., the Integral Test),
- Re-deriving formulas,
- Solving representative problems from memory.

Role of Self-Check Questions

A list of self-check questions provides structure to active recall. These questions act as prompts that guide the learner through the essential components of a topic. For example, in studying infinite series, one might ask:

- What is the definition of convergence?
- When can the Integral Test be applied?
- Why does removing finitely many terms not affect convergence?

Such questions ensure that the learner engages with both conceptual understanding and procedural knowledge.

Why It Works

Active recall strengthens memory through effortful retrieval. Each time information is successfully recalled, the neural pathways associated with that knowledge become more robust. This leads to:

- Improved long-term retention,
- Faster recall during problem solving,
- Better ability to connect different concepts.

Moreover, attempting to answer questions reveals gaps in understanding. These gaps are often invisible during passive review but become immediately apparent during recall attempts.

2. Applications of Integration

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2.4 Arc Length of a Curve and Surface Area

https://openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area

1. How can the length of a curve be approximated using line segments?
2. How does partitioning an interval lead to a Riemann sum for arc length?
3. What is the formula for the arc length of a curve $y = f(x)$ on $[a, b]$?
4. What assumptions must be satisfied by $f(x)$ for the arc length formula to apply?
5. Why does the arc length formula involve $\sqrt{1 + [f'(x)]^2}$?
6. How is the Pythagorean theorem used in deriving the arc length formula?
7. What difficulties may arise when evaluating arc length integrals?
8. How do you compute arc length when the curve is given as $x = g(y)$?
9. What is the formula for arc length in terms of y?
10. How does the derivation of arc length for $x = g(y)$ differ from that for $y = f(x)$?
11. What role do smoothness and differentiability play in arc length problems?
12. How can you decide whether to integrate with respect to x or y?
13. What are common substitutions used when evaluating arc length integrals?
14. How can symmetry simplify arc length computations?
15. What is a surface of revolution?
16. How can a surface of revolution be approximated using frustums of cones?
17. What is a frustum, and why is it useful in this context?
18. How does approximating a surface with frustums lead to a Riemann sum?
19. How is the surface area formula derived from arc length ideas?
20. What is the formula for the surface area when rotating $y = f(x)$ about the x-axis?
21. What is the formula for the surface area when rotating $x = f(y)$ about the y-axis?
22. Why does the surface area formula include a factor of 2π?
23. What geometric quantity does the factor $2\pi f(x)$ or $2\pi x$ represent?
24. How does arc length appear inside the surface area integral?
25. How do you modify the surface area formula when the curve is given as $x = g(y)$?
26. What similarities exist between arc length and surface area formulas?

2.5 Physical Applications

https://openstax.org/books/calculus-volume-2/pages/2-5-physical-applications

1. What is a density function, and how can it vary along a rod or region?
2. How do you compute the mass of a thin rod with variable linear density?
3. What is the formula for mass in terms of an integral over an interval?
4. How does the concept of a Riemann sum apply to mass calculations?
5. How is area density different from linear density?
6. How do you compute the mass of a two-dimensional region with variable density?
7. What is radial density, and in what situations is it used?
8. How does symmetry simplify mass calculations in physical problems?
9. What is work in the context of physics?
10. How is work defined in terms of force and displacement?
11. How can work be approximated using sums of small contributions?
12. What is the general formula for work done by a variable force?
13. How do you compute work when the force depends on position?
14. What is Hooke's Law, and how does it relate force to displacement?
15. How is work computed when stretching or compressing a spring?
16. Why does the integral for spring work involve the force function $F(x) = kx$?
17. How do you set up limits of integration in spring problems?
18. What is a pumping problem in calculus?
19. How is work computed when lifting a fluid (e.g., water) from a tank?
20. How do you model a fluid as a collection of thin horizontal slices?
21. What quantities must be determined for each slice in a pumping problem?
22. How do you determine the distance each slice must be lifted?
23. How does density of a fluid affect the work calculation?
24. What is the role of weight density in fluid problems?
25. How do you express the work required to pump fluid as an integral?
26. What is hydrostatic pressure?
27. How is pressure related to depth in a fluid?
28. What units are used for pressure?
29. How do you compute the force exerted by a fluid on a submerged surface?
30. How is force related to pressure and area?
31. How do you model a submerged surface using horizontal strips?
32. How do you compute the force on each strip?
33. How is the total force found using an integral?
34. What assumptions are made about fluid density in hydrostatic problems?
35. How can changing the shape of a container affect work or force calculations?

2.6 Moments and Centers of Mass

https://openstax.org/books/calculus-volume-2/pages/2-6-moments-and-centers-of-mass

1. What is the definition of the center of mass of a system?
2. How can the center of mass be interpreted physically as a "balancing point"?
3. What is a moment of a system of masses with respect to a point (e.g., the origin)?
4. For point masses on a line, how do you compute the total mass?
5. For point masses on a line, how do you compute the moment with respect to the origin?
6. How do you compute the center of mass $\bar{x}$ for discrete point masses on a line?
7. How does the formula for center of mass relate to a weighted average?
8. For point masses in the plane, how are the moments M_x and M_y defined?
9. How do you compute the coordinates $(\bar{x}, \bar{y})$ of the center of mass in the plane?
10. Why does the center of mass depend more heavily on regions with larger mass?
11. What changes when moving from discrete masses to a continuous mass distribution?
12. What is a lamina?
13. What assumption is often made about density when studying laminas?
14. How is mass computed for a continuous distribution along a line?
15. How is mass computed for a lamina bounded by a function $f(x)$ on $[a, b]$?
16. How do you define the moment of a continuous mass distribution using integrals?
17. How are M_x and M_y computed for a lamina using integrals?
18. How do you compute the center of mass $(\bar{x}, \bar{y})$ for a lamina?
19. What is the centroid of a region?
20. When does the centroid coincide with the center of mass?
21. Why does constant density simplify center of mass calculations?
22. How can symmetry be used to determine the center of mass without integration?
23. Where is the center of mass of a rectangular lamina located?
24. What role does partitioning (Riemann sums) play in deriving formulas for mass and moments?
25. How do integrals arise naturally from approximating a lamina by small rectangles?
26. What is the geometric meaning of M_x and M_y?
27. Why is the center of mass independent of density when density is constant?
28. How would variable density change the formulas for mass and moments?
29. How does the concept of center of mass connect to real-world applications (e.g., balance, engineering)?
30. What is the relationship between center of mass and equilibrium?
31. How can you check whether your computed center of mass is reasonable using symmetry or intuition?
32. How does the idea of "balancing point" generalize from 1D to 2D regions?
33. What are the key steps to solve a typical center of mass problem for a region?
34. What are the key differences between discrete and continuous mass systems?
35. How can you set up integrals for regions bounded by curves when finding centers of mass?

4. Introduction to Differential Equations

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4.1 Basics of Differential Equations

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1. What is a differential equation?
2. What is meant by a solution to a differential equation?
3. How can you verify that a function $y = f(x)$ is a solution to a given differential equation?
4. Give an example of a differential equation and one of its solutions.
5. Why are solutions to differential equations often not unique?
6. What property of derivatives explains the non-uniqueness of solutions?
7. What is the order of a differential equation?
8. How do you determine the order of a differential equation from its expression?
9. What is the order of the differential equation $y' - 4y = x^2 - 3x + 4$?
10. What is the order of the differential equation $x^2y''' - 3xy'' + xy' - 3y = \sin x$?
11. What is a general solution of a differential equation?
12. Why does a general solution typically include an arbitrary constant C?
13. For the differential equation $y' = 2x$, what is its general solution?
14. What does the family of functions $y = x^2 + C$ represent?
15. What is a particular solution of a differential equation?
16. How do you obtain a particular solution from a general solution?
17. What is an initial-value problem?
18. How does an initial condition help determine a unique solution?
19. Solve the initial-value problem:
$$y' = 2x, \quad y(2) = 7.$$
20. What is the difference between a general solution and a particular solution?
21. What does it mean for a function to satisfy a differential equation?
22. Verify that $y = e^{-3x} + 2x + 3$ is a solution of
$$y' + 3y = 6x + 11.$$
23. Verify that $y = 2e^{3x} - 2x - 2$ is a solution of
$$y' - 3y = 6x + 4.$$
24. When checking a solution, why must both y and its derivatives be substituted into the equation?
25. Can two different functions be solutions to the same differential equation? Explain why.
26. What role does the constant C play geometrically in the family of solutions?
27. How would the graph of $y = x^2 + C$ change as C varies?
28. Why is identifying the order of a differential equation useful?
29. What information is needed in addition to a differential equation to determine a unique solution?

4.2 Direction Fields and Numerical Methods

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1. What is a direction field (slope field) for a differential equation?
2. For what type of differential equations are direction fields typically used?
3. What is the general form of a first-order differential equation used in direction fields?
4. What does each small line segment in a direction field represent?
5. How is the slope of a line segment at a point (x, y) determined?
6. How do you construct a direction field for a differential equation $y' = f(x, y)$?
7. Why can a direction field be drawn without solving the differential equation explicitly?
8. What qualitative information can a direction field provide about solutions?
9. What is a solution curve in the context of a direction field?
10. How can you sketch a solution curve using a direction field?
11. Why do solution curves follow the line segments in a direction field?
12. What is an initial condition in the context of direction fields?
13. How do you use an initial point (x_0, y_0) to sketch a particular solution?
14. Why do solution curves corresponding to different initial conditions typically not intersect?
15. What is an equilibrium solution?
16. How can equilibrium solutions be identified from a differential equation?
17. How do equilibrium solutions appear in a direction field?
18. What does it mean for an equilibrium solution to be stable?
19. What does it mean for an equilibrium solution to be unstable?
20. How can you determine stability visually from a direction field?
21. What is the purpose of numerical methods in differential equations?
22. Why are numerical methods needed when solving differential equations?
23. What is Euler's Method?
24. What type of problems is Euler's Method used to approximate?
25. What is the idea behind Euler's Method in terms of tangent line approximation?
26. What is the step size h in Euler's Method?
27. How does the choice of step size affect the accuracy of Euler's Method?
28. What is the iterative formula for Euler's Method?
29. Starting from (x_n, y_n), how do you compute (x_{n+1}, y_{n+1})?
30. Write the update rule:
$$y_{n+1} = y_n + hf(x_n, y_n).$$
31. How is x_{n+1} computed from x_n?

32. Apply Euler's Method to approximate the solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

for one step.

33. Apply Euler's Method for two steps and express the result explicitly.
34. What geometric idea explains why Euler's Method works?
35. Why does Euler's Method accumulate error over multiple steps?
36. What is the difference between the exact solution and a numerical approximation?
37. How can decreasing the step size h improve the approximation?
38. What is the trade-off when choosing a very small step size?
39. How is Euler's Method related to linear approximation (tangent lines)?
40. In what sense does Euler's Method "follow" the direction field?
41. How would you approximate a solution curve using only a direction field (without formulas)?
42. What are the limitations of direction fields?
43. What are the limitations of Euler's Method?
44. When would you prefer a graphical method over an analytical solution?
45. When would you prefer a numerical method over solving analytically?
46. How can direction fields and Euler's Method be used together?

4.3 Separable Equations

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1. What is a separable differential equation?
2. In what form must a differential equation be written to be considered separable?
3. Write the general form:
$$\frac{dy}{dx} = f(x)g(y).$$
4. Why is such an equation called "separable"?
5. What does it mean to separate variables in a differential equation?
6. How do you rewrite a separable differential equation in differential form?
7. Write the separated form:
$$\frac{dy}{g(y)} = f(x)\,dx.$$
8. What is the main idea behind the method of separation of variables?
9. What are the steps of the separation of variables method?
10. Why must you check for values where $g(y) = 0$ before separating variables?
11. What do solutions of $g(y) = 0$ represent?
12. Why can constant solutions be lost during the separation process?
13. After separating variables, what is the next step?
14. Why do we integrate both sides of the equation?
15. What form does the equation take after integration?
16. Why is an arbitrary constant C introduced after integration?
17. When solving, why might it not be possible to express y explicitly as a function of x?
18. What is an implicit solution?
19. Give an example of an implicit solution arising from separation of variables.
20. How do you apply an initial condition to a separable differential equation?
21. What is an initial-value problem in this context?
22. After integrating, how do you solve for the constant using an initial condition?
23. Solve the general structure:
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$
24. For the equation $y' = (x^2 - 4)(3y + 2)$, how do you separate variables?
25. What constant solution arises from $3y + 2 = 0$?
26. After separation, what integral must be computed:
$$\int \frac{dy}{3y + 2}?$$
27. What substitution is useful for integrating expressions like $\frac{1}{3y+2}$?
28. Why is substitution often needed when integrating the y-side?

29. How do you interpret the final solution after integrating both sides?
30. For the equation $y' = (2x + 3)(y^2 - 4)$, how do you separate variables?
31. What constant solutions arise from $y^2 - 4 = 0$?
32. Why do these constant solutions need to be included separately?
33. What integration technique is typically used for expressions like $\frac{1}{y^2-4}$?
34. How does partial fraction decomposition appear in separable equations?
35. Why are integration techniques from earlier calculus essential here?
36. What is an autonomous differential equation?
37. How can you recognize an autonomous equation from its form?
38. Why is an equation of the form $y' = g(y)$ separable?
39. What is the general solution to an equation of the form $y' = f(x)$?
40. What is the general solution to an equation of the form $y' = g(y)$?
41. How does separation of variables generalize these simpler cases?
42. In what situations is separation of variables especially useful?
43. Why are separable equations common in physics and engineering?
44. What are the limitations of the separation of variables method?
45. Can every differential equation be solved using separation of variables?
46. How can you recognize when a differential equation is not separable?
47. What is the overall strategy when solving separable differential equations?

4.4 The Logistic Equation

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1. What real-world limitation motivates the logistic model of population growth?
2. Why is exponential growth not realistic for large populations?
3. What is the carrying capacity of a population?
4. What does the symbol K represent in the logistic equation?
5. What does the symbol r represent in the logistic equation?
6. Write the logistic differential equation:
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$
7. Why is the logistic equation nonlinear?
8. What happens to $\frac{dP}{dt}$ when P is very small?
9. How does the logistic equation behave when $P \ll K$?
10. How does the logistic model approximate exponential growth for small populations?
11. What happens to $\frac{dP}{dt}$ when $P = K$?
12. What does it mean physically when $P = K$?
13. What happens to $\frac{dP}{dt}$ when $P > K$?
14. Why does the population decrease when $P > K$?
15. What are the equilibrium solutions of the logistic equation?
16. Solve:
$$rP\left(1 - \frac{P}{K}\right) = 0.$$
17. Why are $P = 0$ and $P = K$ equilibrium solutions?
18. What is meant by stability of an equilibrium solution?
19. Is $P = 0$ stable, unstable, or semi-stable? Explain.
20. Is $P = K$ stable, unstable, or semi-stable? Explain.
21. What is a phase line?
22. How do you construct a phase line for the logistic equation?
23. What does the phase line indicate about the sign of $\frac{dP}{dt}$?
24. For which values of P is the population increasing?
25. For which values of P is the population decreasing?
26. How does the phase line describe long-term behavior of solutions?
27. What happens to $P(t)$ as $t \to \infty$ when $0 < P_0 < K$?
28. What happens to $P(t)$ as $t \to \infty$ when $P_0 > K$?
29. Why does the solution approach K but never reach it exactly?
30. What is the general strategy to solve the logistic differential equation?
31. Why is the logistic equation separable?

32. Rewrite the logistic equation in separated form.
33. What algebraic manipulation is needed before integrating?
34. Why is partial fraction decomposition used in solving the logistic equation?
35. What is the general solution of the logistic differential equation?
36. Write the solution:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}.$$

37. What is the role of the initial condition $P(0) = P_0$?
38. How does the solution behave as $t \to \infty$?
39. How does the solution behave as $t \to -\infty$?
40. What is the shape of the graph of a logistic function?
41. Why is the logistic curve called sigmoidal (S-shaped)?
42. What is a point of inflection?
43. Why does the logistic solution have a point of inflection?
44. At what population value does the inflection point occur?
45. Why is the growth rate maximal at $P = \frac{K}{2}$?
46. How can you find the inflection point using the second derivative?
47. What happens to concavity before and after the inflection point?
48. How does the logistic model improve on exponential growth models?
49. In what types of real-world systems is the logistic equation used?
50. What assumptions underlie the logistic model?
51. What are the limitations of the logistic equation?

4.5 First-order Linear Equations

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1. What is a first-order linear differential equation?
2. What is the general (standard) form of a first-order linear differential equation?
3. Write the standard form:
$$y' + p(x)y = q(x).$$
4. How can any first-order linear differential equation be rewritten into standard form?
5. What roles do the functions $p(x)$ and $q(x)$ play?
6. Why is the equation called "linear"?
7. What distinguishes a linear differential equation from a separable one?
8. Can every linear differential equation be solved by separation of variables? Why or why not?
9. What is an integrating factor?
10. Why is an integrating factor useful when solving linear differential equations?
11. What property do we want the integrating factor to create?
12. What is the formula for the integrating factor $\mu(x)$?
13. Write:
$$\mu(x) = e^{\int p(x)\,dx}.$$
14. How is the integrating factor derived?
15. What differential equation does $\mu(x)$ satisfy?
16. Write:
$$\mu'(x) = \mu(x)p(x).$$
17. After multiplying by $\mu(x)$, what form does the equation take?
18. Why does the left-hand side become a derivative of a product?
19. Write the identity:
$$\frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu(x)p(x)y.$$
20. After multiplying by $\mu(x)$, what equation do we obtain?
21. Write:
$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$
22. What is the next step after obtaining this form?
23. Why can both sides now be integrated easily?
24. After integrating, what form does the solution take?
25. Write the general solution:
$$\mu(x)y = \int \mu(x)q(x)\,dx + C.$$
26. How do you solve for $y(x)$ after integration?
27. What is the full step-by-step strategy for solving first-order linear equations?
28. List the five steps of the method.

29. Why is it important to first rewrite the equation in standard form?
30. What happens if the coefficient of y' is not 1?
31. How do you handle an equation of the form $a(x)y' + b(x)y = c(x)$?
32. Why is dividing by $a(x)$ necessary?
33. What is an initial-value problem in this context?
34. How do you apply an initial condition to determine the constant C?
35. Solve symbolically:
$$y' + p(x)y = 0.$$
36. What is the general solution to the homogeneous equation?
37. How does the homogeneous solution relate to the general solution?
38. What is the difference between homogeneous and nonhomogeneous equations?
39. What happens when $q(x) = 0$?
40. What happens when $q(x) \neq 0$?
41. Why does the solution involve an integral of $\mu(x)q(x)$?
42. What integration techniques are often required in solving these equations?
43. How does the integrating factor method simplify the problem conceptually?
44. What is the geometric interpretation of multiplying by $\mu(x)$?
45. Why does the method always work for first-order linear equations?
46. What are common mistakes when applying the integrating factor method?
47. How can you verify that a solution is correct?
48. In what applications do first-order linear equations appear?
49. How are they used in modeling motion with air resistance?
50. How are they used in electrical circuits?
51. Why are first-order linear equations important in applied mathematics?

5. Sequences and Series

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5.1 Sequences

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1. What is an infinite sequence? How is it typically denoted?
2. How can a sequence be interpreted as a function? What is its domain?
3. What is the index of a sequence, and what role does it play?
4. What is an explicit formula for a sequence? Give an example.
5. What is a recursive definition of a sequence? How does it differ from an explicit formula?
6. Given a recursive sequence, how can you compute its first few terms?
7. What does it mean to graph a sequence? How does it differ from graphing a function?
8. What is the limit of a sequence?
9. What does it mean for a sequence a_n to converge to a number L?
10. Write the formal ϵ–N definition of convergence of a sequence.
11. What does it mean for a sequence to diverge?
12. What are some different ways a sequence can diverge?
13. What does it mean for a sequence to diverge to infinity? To negative infinity?
14. Why does changing a finite number of terms of a sequence not affect its convergence?
15. How can limits of functions be used to determine limits of sequences?
16. State the theorem relating $\lim_{x\to\infty} f(x)$ and $\lim_{n\to\infty} a_n$ when $a_n = f(n)$.
17. Evaluate $\lim_{n\to\infty} \frac{1}{n}$ using a function-based argument.
18. What happens to r^n as $n \to \infty$ when:
    - $0 < r < 1$?
    - $r = 1$?
    - $r > 1$?
19. How can limit laws be applied to sequences?
20. How do you find the limit of a sequence that is a sum or product of simpler sequences?
21. What is a bounded sequence?
22. What does it mean for a sequence to be increasing? Decreasing?
23. What is a monotonic (monotone) sequence?
24. State the Monotone Convergence Theorem.
25. Why is the Monotone Convergence Theorem useful?
26. Give an example of a sequence that is bounded and increasing.
27. Give an example of a sequence that is unbounded.
28. Give an example of a sequence that oscillates and does not converge.

29. How can you determine convergence from the behavior of the terms as $n \to \infty$?
30. What is the difference between intuition ("approaches a value") and the formal definition of convergence?
31. How can sequences be used to model real-world processes or iterative procedures?
32. How are sequences related to infinite series?

5.2 Infinite Series

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1. What is an infinite series? How is it related to a sequence?
2. How is an infinite series written using summation notation?
3. What is the difference between a sequence a_n and the series $\sum_{n=1}^{\infty} a_n$?
4. What is a partial sum of a series? How is the k-th partial sum S_k defined?
5. How can an infinite series be defined as a limit of partial sums?
6. What does it mean for an infinite series to converge?
7. What does it mean for an infinite series to diverge?
8. What is the relationship between the convergence of a series and the convergence of its sequence of partial sums?
9. What does it mean for a series to diverge to infinity?
10. How can you determine whether a series converges by examining its partial sums?
11. What is a geometric series? Write its general form.
12. What is the common ratio of a geometric series?
13. Under what condition does a geometric series converge?
14. What is the formula for the sum of a convergent geometric series?
15. What happens if the common ratio r of a geometric series satisfies $|r| \geq 1$?
16. How can you derive the formula for the sum of a geometric series?
17. What is a telescoping series?
18. How do terms cancel in a telescoping series?
19. How can you evaluate the sum of a telescoping series?
20. Why are telescoping series often easier to evaluate than other series?
21. Give an example of a telescoping series and describe its behavior.
22. What is the harmonic series? Write its general form.
23. Does the harmonic series converge or diverge?
24. Why is it not sufficient for $a_n \to 0$ to guarantee that $\sum a_n$ converges?
25. What is the divergence test (nth-term test for divergence)?
26. If $\lim_{n\to\infty} a_n \neq 0$, what can you conclude about $\sum a_n$?
27. If $\lim_{n\to\infty} a_n = 0$, what can you conclude about $\sum a_n$?
28. How are infinite series used in applications such as approximations or modeling?
29. How does the concept of an infinite series extend the idea of a finite sum?
30. How can infinite series be used to represent functions?
31. What is the intuitive meaning of "adding infinitely many terms"?
32. How does the order of terms affect the value of a finite sum? Does this idea carry over to infinite series?

5.3 The Divergence and Integral Tests

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1. Why is it often difficult to determine convergence of a series by directly evaluating partial sums?
2. What is the divergence test (nth-term test)?
3. State the divergence test formally.
4. Why does $\sum_{n=1}^{\infty} a_n$ converging imply that $a_n \to 0$?
5. If $\lim_{n\to\infty} a_n \neq 0$, what can you conclude about the series $\sum a_n$?
6. If $\lim_{n\to\infty} a_n = 0$, what can you conclude about the series $\sum a_n$?
7. Why is the divergence test only useful for proving divergence and not convergence?
8. Give an example of a series where $a_n \to 0$ but the series diverges.
9. How do you apply the divergence test in practice to a given series?
10. What kinds of limits of a_n guarantee divergence (nonzero limit, limit does not exist)?
11. What is the idea behind comparing a series to an improper integral?
12. What is the integral test?
13. What conditions must a function $f(x)$ satisfy to apply the integral test (positivity, continuity, monotonicity)?
14. How are the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x), dx$ related in the integral test?
15. State the integral test formally.
16. If the improper integral $\int_1^{\infty} f(x), dx$ converges, what can you conclude about the series?
17. If the improper integral diverges, what can you conclude about the series?
18. How can the integral test be used to prove that the harmonic series diverges?
19. What is a p-series? Write its general form.
20. For what values of p does the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?
21. For what values of p does the p-series diverge?
22. How can the integral test be used to establish the convergence properties of p-series?
23. What is the remainder R_N of a series after N terms?
24. How can the integral test be used to estimate the remainder of a series?
25. State the inequality involving the remainder R_N and integrals.
26. How do upper and lower bounds for R_N arise from the integral test?
27. Why is it useful to estimate the error when approximating a series by partial sums?
28. How can you choose N to ensure the error is less than a desired tolerance?
29. What is the geometric interpretation of the integral test using areas under curves?
30. How does monotonicity of $f(x)$ play a role in bounding the series with integrals?
31. In what situations is the integral test particularly useful compared to other tests?

5.4 Comparison Tests

https://openstax.org/books/calculus-volume-2/pages/5-4-comparison-tests

1. What is the main idea behind comparison tests for infinite series?
2. For what type of series (in terms of sign of terms) are comparison tests typically used?
3. Why are geometric series and p-series commonly used in comparison tests?
4. State the Direct Comparison Test for series $\sum a_n$ and $\sum b_n$ with positive terms.
5. If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, what can be concluded about $\sum a_n$?
6. If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, what can be concluded about $\sum a_n$?
7. In which cases is the Direct Comparison Test inconclusive?
8. How do you choose an appropriate comparison series b_n for a given series a_n?
9. Why is it useful to compare a series to a p-series of the form $\sum \frac{1}{n^p}$?
10. What is the behavior of the p-series $\sum \frac{1}{n^p}$ for different values of p?
11. What is the behavior of a geometric series $\sum ar^n$ depending on the value of r?
12. State the Limit Comparison Test for two series $\sum a_n$ and $\sum b_n$ with positive terms.
13. What does it mean if
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$
where $0 < c < \infty$?
14. What can be concluded if the limit in the Limit Comparison Test is 0?
15. What can be concluded if the limit in the Limit Comparison Test is ∞?
16. When is the Limit Comparison Test more useful than the Direct Comparison Test?
17. Why does the Limit Comparison Test work even when inequalities between a_n and b_n are hard to establish?
18. What are common functions or expressions you simplify when applying the Limit Comparison Test?
19. How do you simplify rational expressions in n when applying comparison tests?
20. How do logarithmic terms (e.g., $\ln n$) affect comparison with p-series?
21. How do roots (e.g., $\sqrt{n}$) affect comparison with p-series?
22. Give an example of a series that can be compared to $\sum \frac{1}{n^2}$ and explain why.
23. Give an example of a series that can be compared to $\sum \frac{1}{n}$ and explain why.
24. What is the general strategy for proving convergence using comparison tests?
25. What is the general strategy for proving divergence using comparison tests?
26. Why must terms a_n and b_n be positive for these tests?
27. Can comparison tests determine the exact sum of a series? Why or why not?
28. How do comparison tests relate to the Integral Test conceptually?
29. What role do asymptotic behaviors of functions play in the Limit Comparison Test?
30. How can you justify that two sequences have the same "growth rate" for comparison?
31. What are typical mistakes when applying the Direct Comparison Test?

32. What are typical mistakes when applying the Limit Comparison Test?
33. How can you check your result after applying a comparison test?
34. When comparing $\frac{1}{n^2+1}$ to $\frac{1}{n^2}$, why does the comparison work?
35. What properties of inequalities are essential when using the Direct Comparison Test?

5.5 Alternating Series

https://openstax.org/books/calculus-volume-2/pages/5-5-alternating-series

1. What is an alternating series? Write its general form.
2. How can an alternating series be written using the factor $(-1)^n$ or $(-1)^{n+1}$?
3. In the representation $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, what conditions are imposed on b_n?
4. State the Alternating Series Test (Leibniz Test).
5. What two conditions must the sequence $\{b_n\}$ satisfy for the Alternating Series Test to apply?
6. Why is the condition $\lim_{n\to\infty} b_n = 0$ necessary for convergence?
7. Does the Alternating Series Test guarantee absolute convergence or only convergence? Explain.
8. Give an example of a series that converges by the Alternating Series Test.
9. What is meant by absolute convergence?
10. What is meant by conditional convergence?
11. What is the relationship between absolute convergence and convergence of a series?
12. Give an example of a series that converges conditionally but not absolutely.
13. How can you test whether an alternating series converges absolutely?
14. If $\sum |a_n|$ diverges but $\sum a_n$ converges, what type of convergence does the series have?
15. What is the alternating harmonic series? Does it converge absolutely or conditionally?
16. Define the nth partial sum S_N of a series.
17. What is the remainder R_N of a series?
18. For an alternating series satisfying the Alternating Series Test, what inequality bounds the remainder R_N?
19. Write the inequality relating $|R_N|$ and b_{N+1}.
20. What does the remainder estimate tell us about the error when approximating a sum by S_N?
21. How can you use the remainder estimate to determine how many terms are needed for a desired accuracy?
22. If you approximate the sum of an alternating series by S_N, what is the maximum possible error?
23. Why do partial sums of an alternating series tend to oscillate around the true sum?
24. Describe the behavior of even and odd partial sums in an alternating series that satisfies the test.
25. How can you estimate the sum of an alternating series to within a given tolerance ε?
26. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, how would you bound the error after N terms?
27. What role does monotonicity ($b_{n+1} \leq b_n$) play in the Alternating Series Test?
28. Can an alternating series converge if the sequence b_n is not decreasing? What does the test say about this?
29. Compare the convergence behavior of $\sum \frac{(-1)^{n+1}}{n}$ and $\sum \frac{1}{n}$.
30. Why is it often difficult to compute the exact sum of an alternating series?
31. What practical advantage does the Alternating Series Estimation Theorem provide?
32. How does the Alternating Series Test differ from comparison or ratio tests?

33. When analyzing a series, why is it useful to check absolute convergence first?
34. What happens if $\lim_{n\to\infty} a_n \neq 0$ for an alternating series?
35. How would you structure a complete convergence test for a series that alternates in sign?

5.6 Ratio and Root Tests

https://openstax.org/books/calculus-volume-2/pages/5-6-ratio-and-root-tests

1. Why is the condition $\lim_{n\to\infty} a_n = 0$ not sufficient to guarantee convergence of $\sum a_n$?
2. Give an example of two series where $a_n \to 0$ in both cases, but one converges and the other diverges.
3. What does it mean for the terms of a series to approach zero "fast enough"?
4. State the Ratio Test for a series $\sum a_n$.
5. Define
$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$
6. What conclusion can be drawn if $\rho < 1$ in the Ratio Test?
7. What conclusion can be drawn if $\rho > 1$ (or $\rho = \infty$) in the Ratio Test?
8. What happens if $\rho = 1$ in the Ratio Test?
9. What type of convergence does the Ratio Test establish when $\rho < 1$?
10. For what kinds of series is the Ratio Test especially useful?
11. Apply the Ratio Test conceptually: what happens to the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ for factorial-type expressions?
12. State the Root Test for a series $\sum a_n$.
13. Define
$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$
14. What conclusion can be drawn if $\rho < 1$ in the Root Test?
15. What conclusion can be drawn if $\rho > 1$ in the Root Test?
16. What happens if $\rho = 1$ in the Root Test?
17. Compare the Ratio Test and Root Test: what is similar about their conclusions?
18. In what situations is the Root Test particularly useful?
19. How does the Root Test behave for expressions involving powers like $(b_n)^n$?
20. What does it mean for a test to be "inconclusive"?
21. Give an example of a series where the Ratio Test is inconclusive.
22. Give an example of a series where the Root Test is inconclusive.
23. Why might one test succeed when another test fails?
24. What is absolute convergence, and how do these tests relate to it?
25. Why do both the Ratio and Root Tests use absolute values?
26. How are these tests useful for power series?
27. What general strategy can be used to choose an appropriate convergence test for a given series?
28. When should you consider using the Ratio Test over other tests?
29. When should you consider using the Root Test over other tests?
30. What is a good first step before applying either the Ratio or Root Test?
31. How can simplifying expressions help when applying these tests?

32. Why are these tests often easier to apply than comparison tests?
33. Can the Ratio or Root Test determine conditional convergence?
34. How do these tests relate to geometric series?
35. How can you sometimes modify a series to make the Root Test easier to apply?
36. Why is it important to recognize patterns like factorials, exponentials, or powers before choosing a test?
37. Summarize the decision process after computing ρ in either test.

6. Power Series

https://openstax.org/books/calculus-volume-2/pages/6-introduction

6.1 Power Series and Functions

https://openstax.org/books/calculus-volume-2/pages/6-1-power-series-and-functions

1. What is the general form of a power series centered at $x = 0$?
2. Write the general form of a power series centered at $x = a$.
3. Explain in your own words why a power series can be thought of as an "infinite polynomial."
4. Give two examples of power series centered at $x = 0$ with explicit formulas for their coefficients c_n.
5. What does it mean for a power series to be "centered at" a specific point?
6. State the three possible convergence behaviors of a power series centered at $x = a$.
7. Define the *interval of convergence* for a power series.
8. Define the *radius of convergence* of a power series.
9. Why does every power series always converge at its center?
10. Using the geometric series
$$\sum_{n=0}^{\infty} x^n,$$
determine its interval and radius of convergence.
11. How can the Ratio Test be used to help find the radius of convergence of a power series?
12. True/False: If a power series converges for some value x_0, then it converges for all x with $|x - a| < |x_0 - a|$. Explain your answer.
13. What is the basic idea behind representing a given function $f(x)$ with a power series?
14. Give an example of a function that can be expressed with a power series based on geometric series manipulation.
15. How would you represent the function $\frac{1}{1-x}$ as a power series?
16. Describe why it is sometimes necessary to truncate a power series when approximating a function in practice.
17. Explain the relationship between power series and functions that are infinitely differentiable near their center.
18. For a power series with finite radius of convergence R, what can be said about convergence at the endpoints $x = a \pm R$?
19. How are the endpoints of the interval of convergence typically tested if the Ratio Test is inconclusive?
20. What implications does the convergence of a power series have for the differentiability and integrability of the represented function on its interval of convergence?

6.2 Properties of Power Series

https://openstax.org/books/calculus-volume-2/pages/6-2-properties-of-power-series

1. What is the general result about combining two power series

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=0}^{\infty} d_n x^n$$

   that converge on the same interval I? State the conditions for addition, scalar multiplication, and multiplication by a power of x.

2. Suppose two power series converge to functions $f(x)$ and $g(x)$ on an interval I. What function does the series

$$\sum_{n=0}^{\infty} (c_n + d_n) x^n$$

   converge to on I?

3. For a power series $\sum_{n=0}^{\infty} c_n x^n$ that converges to $f(x)$ on an interval I, what is the power series that converges to $b\,x^m f(x)$ for a real number b and integer $m \geq 0$?

4. If $\sum_{n=0}^{\infty} c_n x^n$ converges to $f(x)$ on an interval I, what is the new series that represents $f(b\,x^m)$ when $|b\,x^m|$ is in the interval I?

5. What theorem guarantees that a power series can be differentiated and integrated term-by-term on its interval of convergence? What does this theorem say about the resulting series and their convergence?

6. Given a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

   write the term-by-term differentiated series for $f'(x)$ in summation form.

7. Given the same $f(x)$ above, write the term-by-term integrated series representing an antiderivative of $f(x)$.

8. Does term-by-term differentiation or integration change the radius of convergence of a power series? Explain and describe what can happen at the endpoints.

9. Using the known geometric series for $\frac{1}{1-x}$, what is the power series representation of $\frac{1}{(1-x)^2}$ found by term-by-term differentiation? Specify its interval of convergence and endpoint behavior.

10. What is the interval of convergence and endpoint behavior for the series representation of $\ln(1+x)$ found by term-by-term integration of $\frac{1}{1+x}$?

11. Why does term-by-term differentiation or integration *not* guarantee endpoint convergence even though it preserves radius of convergence? Describe a specific example.

12. Consider a power series that converges on $(-R, R)$. If x lies in this interval, explain whether the term-by-term derivative must converge at the same endpoints as the original series.

6.3 Taylor and Maclaurin Series

https://openstax.org/books/calculus-volume-2/pages/6-3-taylor-and-maclaurin-series

1. What is the general form of the Taylor series for a function $f(x)$ centered at $x = a$? Write it using summation notation.
2. How does the Taylor series simplify when the center $a = 0$? What name is given to this special case?
3. For a function $f(x)$ with derivatives of all orders at $x = a$, write the explicit expression for its n-th Taylor polynomial $p_n(x)$ centered at a.
4. Explain why the coefficients of a power series representation of $f(x)$ at $x = a$ must be given by derivatives of f evaluated at a.
5. What condition must hold for a function's Taylor series to equal the original function $f(x)$ on an interval? Define the remainder $R_n(x)$ in this context.
6. State the Uniqueness of Taylor Series theorem: if a power series converges to $f(x)$ on an interval containing a, what can be said about that series?
7. Define the remainder term $R_n(x)$ for a Taylor polynomial approximation and write the formula for the Lagrange form of the remainder.
8. What does Taylor's Theorem with remainder tell us about the error when approximating $f(x)$ using its n-th Taylor polynomial?
9. Given a specific function $f(x)$, describe the steps to find the Maclaurin polynomial of degree n.
10. Find the Maclaurin series for e^x by writing out several derivatives at 0 and using the definition.
11. Find the Maclaurin series for $\sin x$ and $\cos x$. Explain the pattern in the derivatives that allows you to write the general term.
12. How can the ratio test be used to determine the radius of convergence for a Taylor or Maclaurin series?
13. For a given Taylor series, how would you check whether it actually converges to $f(x)$ (not just converges in value)?
14. Why is it significant that Maclaurin polynomials are just Taylor polynomials centered at zero?
15. Suppose you are given a function $f(x)$ with known derivatives at $x = a$. Write a question asking you to compute the 2nd and 3rd Taylor polynomials explicitly and compare them to the function near a.
16. Create a question that asks you to estimate the error in using the 3rd Taylor polynomial to approximate a function value at a given point using the remainder bound.
17. Why does the remainder $R_n(x) \to 0$ as $n \to \infty$ matter for Taylor series convergence?
18. Provide the Maclaurin series expansion for $\ln(1 + x)$ and specify its interval of convergence.
19. For a given nonzero center a, how would you find the Taylor series of $\ln(x)$ about $x = a$ and determine its interval of convergence?
20. Write a practice question that asks for the Taylor series of a non-elementary function (e.g., e^{-x^2}) and discuss whether it converges to the function.
21. Formulate a question about how to bound the remainder $R_n(x)$ using information about the maximum value of a derivative on an interval.
22. Write a question that asks you to use Taylor's theorem to prove that the Maclaurin series for e^x converges for all real x.
23. Create a problem asking you to approximate a definite integral using a Taylor or Maclaurin series where the antiderivative is non-elementary.

6.4 Working with Taylor Series

https://openstax.org/books/calculus-volume-2/pages/6-4-working-with-taylor-series

1. What is the Taylor series of a function f centered at $x = a$?
2. What is the Maclaurin series? How does it relate to the Taylor series?
3. Write the Taylor series of f centered at a using sigma notation.
4. What is the nth Taylor polynomial $T_n(x)$ of a function f centered at a?
5. What is the relationship between the Taylor polynomial $T_n(x)$ and the full Taylor series?
6. Compute the Maclaurin series of e^x.
7. Compute the Maclaurin series of $\sin x$.
8. Compute the Maclaurin series of $\cos x$.
9. Compute the Maclaurin series of $\dfrac{1}{1-x}$ and state its interval of convergence.
10. How can you obtain the Maclaurin series for $\ln(1+x)$ from the geometric series?
11. How can you obtain the Maclaurin series for $\arctan x$?
12. Given a known power series, how can you:
    (a) Differentiate it term-by-term?
    (b) Integrate it term-by-term?
13. What happens to the radius of convergence when you differentiate or integrate a power series?
14. How can you construct the Taylor series for e^{2x} using the known series for e^x?
15. How can you construct the Taylor series for $\sin(3x)$?
16. How can you construct the Taylor series for xe^x?
17. How can you construct the Taylor series for $\dfrac{x}{1-x^2}$?
18. How do you find the Taylor series of a function centered at a point $a \neq 0$?
19. Find the Taylor series of $\ln x$ centered at $x = 1$.
20. What substitution allows you to rewrite $\ln x$ in terms of $\ln(1+u)$?
21. How can you approximate a function using its Taylor polynomial?
22. What is Taylor's theorem?
23. What is the remainder term $R_n(x)$ in Taylor's theorem?
24. State the Lagrange form of the remainder.
25. How can you use the remainder term to bound the error of a Taylor approximation?
26. Suppose $|f^{(n+1)}(x)| \leq M$ on an interval containing a and x. What inequality bounds $|R_n(x)|$?
27. How do you determine how large n must be to guarantee a given approximation error?
28. Use a Taylor polynomial to approximate $e^{0.1}$. How can you estimate the error?
29. Use a Taylor polynomial to approximate $\sin(0.2)$. How can you estimate the error?
30. What does it mean for a function to be equal to its Taylor series?
31. Give an example of a function whose Taylor series converges but does not equal the function.

32. Why does the Taylor series of e^x converge for all x?
33. Why does the geometric series converge only for $|x| < 1$?
34. How does the factorial in the denominator affect convergence of Taylor series?
35. When approximating near $x = a$, why is it better to center the Taylor polynomial at a rather than at 0?
36. If you need a very accurate approximation near $x = 2$, which center should you choose and why?
37. How does increasing n affect:
    (a) The accuracy of approximation?
    (b) The computational cost?
38. What is the difference between a Taylor polynomial and a Taylor series?
39. How does Taylor series provide a bridge between polynomials and general smooth functions?

Cauchy Formula for Repeated Integration

Basic understanding

1. What problem does the Cauchy formula for repeated integration solve?
2. What is meant by the "n-th repeated integral" of a function?
3. Write the definition of the n-fold repeated integral of a function f with the base point a:

$$f^{(-n)}(x) = ?$$

4. Under what conditions on f does the formula hold?
5. What is the main result of the Cauchy formula for repeated integration? State it explicitly.

Formula and structure

1. Rewrite the repeated integral as a single integral:

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt.$$

2. What is the role of the factor $\frac{1}{(n-1)!}$?
3. Why does the kernel $(x-t)^{n-1}$ appear in the formula?
4. What happens in the special case $n = 1$? Verify that the formula reduces correctly.
5. Compute explicitly $f^{(-2)}(x)$ using both:
   - the definition (nested integrals),
   - the Cauchy formula,

   and verify that they agree.

Proof understanding

1. What proof technique is used to prove the formula?
2. What is the base case of the induction?
3. Show that for $n = 1$:

$$f^{(-1)}(x) = \int_a^x f(t)\, dt.$$

4. What assumption is made in the induction step?
5. What must be proven in the induction step?
6. Where is the Leibniz integral rule used in the proof?
7. Compute:

$$\frac{d}{dx}\left[\frac{1}{n!}\int_a^x (x-t)^n f(t)\, dt\right]$$

   and explain why it gives the desired recursive relation.
8. How does the induction step transform an $(n+1)$-fold integral into the desired single-integral form?
9. Why does the induction argument prove the formula for all $n \in \mathbb{N}$?

Conceptual understanding

1. Why can repeated integration be "compressed" into a single integral?
2. How is this formula related to convolution?
3. Interpret the kernel $(x-t)^{n-1}$ geometrically or intuitively.
4. How does this formula simplify practical computations?
5. In what situations is this formula especially useful?

Generalizations

1. How can the formula be extended to non-integer orders of integration?
2. What replaces the factorial $(n-1)!$ in the generalized version?
3. State the fractional integral version:
$$(J^\alpha f)(x) = ?$$
4. What condition must α satisfy?
5. What is the name of this generalization?
6. How does the formula relate to fractional derivatives (when $\alpha < 0$)?

Connections and further ideas

1. How is this formula connected to fractional calculus?
2. What is a differintegral?
3. How can repeated integration be used to define differentiation of non-integer order?
4. How is this formula generalized to higher dimensions?
5. What is the Riesz potential?

Practice problems

1. Compute:
$$\int_0^x \int_0^{t_1} t_2 \, dt_2 \, dt_1$$
using the Cauchy formula.
2. Compute the n-fold integral of $f(t) = 1$.
3. Compute the n-fold integral of $f(t) = t^k$.
4. Show that:
$$(J^n f)(x) = (f * g_n)(x)$$
for an appropriate kernel g_n.
5. Verify the formula numerically for a simple function (e.g. $f(t) = e^t$).

Change of Variables in Multiple Integrals

Conceptual Foundations

1. What is the general idea behind a change of variables in a double integral?
2. What is meant by a transformation
$$T(u, v) = (x(u, v), y(u, v))?$$
3. What conditions must a transformation $T(u, v)$ satisfy in order for the change of variables theorem to apply?
4. What is the Jacobian matrix of a transformation $T(u, v) = (x(u, v), y(u, v))$?
5. Define the Jacobian determinant:
$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}.$$
6. Write the explicit formula for the Jacobian determinant in terms of partial derivatives.
7. Why does the absolute value of the Jacobian determinant appear in the change-of-variables formula?
8. What geometric meaning does the Jacobian determinant have?
9. What does it mean if the Jacobian determinant is zero at a point?
10. What happens to area elements under a transformation?

Change of Variables Theorem (Double Integrals)

1. State the Change of Variables Theorem for double integrals.
2. Complete the formula:
$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$
3. What is the relationship between the regions R and S?
4. Why is it often easier to integrate over S rather than R?
5. Describe the general steps required to compute a double integral using change of variables.

Computing Jacobians

1. Compute the Jacobian determinant of the transformation:
$$x = u^2 - v^2, \quad y = 2uv.$$
2. Compute the Jacobian determinant for:
$$x = au + bv, \quad y = cu + dv.$$
3. For what condition on a, b, c, d is the transformation invertible?
4. How does the Jacobian determinant behave under composition of transformations?
5. If T has Jacobian J_T and its inverse has Jacobian $J_{T^{-1}}$, what is the relationship between them?

Polar Coordinates

1. Write the transformation from polar to Cartesian coordinates.
2. Compute the Jacobian determinant for the polar transformation:
$$x = r\cos\theta, \quad y = r\sin\theta.$$
3. Why does the area element become
$$dA = r\,dr\,d\theta$$
in polar coordinates?
4. Rewrite the integral
$$\iint_R f(x, y)\,dA$$
in polar coordinates.
5. Describe a region in the plane that is particularly well suited for polar coordinates.
6. Evaluate conceptually:
$$\iint_R (x^2 + y^2)\,dA$$
over a disk using polar coordinates.

Geometric Interpretation

1. How does a small rectangle in the uv-plane transform under T?
2. How does the Jacobian determinant approximate local area scaling?
3. What is the geometric interpretation of a negative Jacobian determinant?
4. Why do we take the absolute value in the change of variables formula?

Strategy and Problem Solving

1. When facing a difficult region R, how do you decide on a good transformation?
2. Why are linear transformations often useful for parallelogram-shaped regions?
3. How can you transform an elliptical region into a circular region?
4. What is the advantage of transforming complicated boundaries into rectangular ones?
5. What common algebra mistakes occur when computing Jacobians?
6. Why is it important to rewrite the integrand entirely in terms of u and v?
7. What must always be changed: the integrand, the differential, the region?
8. How can you check whether your transformation is one-to-one?

Extension to Triple Integrals

1. State the change of variables formula for triple integrals.
2. Write the definition of the Jacobian determinant for a transformation
$$(u, v, w) \mapsto (x, y, z).$$
3. How does volume scale under a transformation in $\mathbb{R}^3$?
4. What is the Jacobian determinant for spherical coordinates?
5. What is the Jacobian determinant for cylindrical coordinates?

Deep Understanding / Proof-Level Questions

1. How can the change of variables theorem be justified using Riemann sums?
2. Why does the determinant appear naturally in linear approximations?
3. How does the change-of-variables formula relate to linear algebra?
4. In what sense is the Jacobian determinant a multidimensional generalization of dx/du?
5. Why is differentiability essential in the theorem?
6. What would fail if the transformation were not invertible?
7. How does this theorem connect to the substitution rule in single-variable calculus?

Differentiation Under the Integral Sign

1. What is the statement of Leibniz's rule for differentiating under the integral sign with constant limits?
2. What is the general form of Leibniz's rule when the limits of integration depend on the parameter?
3. What conditions must be satisfied to justify differentiation under the integral sign (continuity, uniform convergence, dominated convergence, etc.)?
4. What role does continuity of the integrand and its partial derivative play in applying differentiation under the integral sign?
5. How does the Dominated Convergence Theorem justify differentiation under the integral sign?
6. When can uniform convergence be used to justify differentiation under the integral sign?
7. What is the difference between pointwise convergence and uniform convergence in this context?
8. How do you compute $\frac{d}{d\alpha}\int_a^b f(x,\alpha), dx$?
9. How do you compute $\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha), dx$?
10. What additional terms appear when the limits of integration depend on the parameter?
11. How do you apply differentiation under the integral sign to evaluate difficult integrals?
12. What is the idea behind Feynman's technique for evaluating integrals using parameters?
13. How can introducing a parameter simplify the evaluation of an integral?
14. Give an example where differentiating under the integral sign transforms an integral into a simpler one.
15. How do you recover the original integral after differentiating with respect to a parameter?
16. What initial or boundary conditions are needed after integrating back with respect to the parameter?
17. How do you handle constants of integration when using this method?
18. When is it valid to interchange differentiation and integration?
19. What are common pitfalls when applying differentiation under the integral sign?
20. How does this technique apply to improper integrals?
21. What extra care is needed when the interval of integration is infinite?
22. How do you justify differentiation under the integral sign for $\int_0^\infty f(x,\alpha), dx$?
23. Can differentiation under the integral sign be applied multiple times? Under what conditions?
24. What is the relationship between differentiation under the integral sign and parameter-dependent integrals?
25. How does this method connect to solving differential equations?
26. How can this technique be used in probability theory (e.g., moment generating functions)?
27. How does differentiation under the integral sign relate to Fourier transforms or Laplace transforms?
28. What are some classic integrals that are evaluated using this method?
29. How would you construct your own parameterized integral to apply this technique?
30. How do you verify that your final answer is correct after applying the method?

Euler's Method

1. What problem does Euler's method aim to solve? State the general form of an initial value problem.
2. What is the geometric interpretation of Euler's method in terms of tangent lines?
3. Starting from the differential equation $y'(x) = f(x, y)$, derive the Euler update formula.
4. Write the recursive formula for Euler's method, defining all variables involved.
5. What role does the step size h play in Euler's method?
6. Given an initial condition $y(x_0) = y_0$, how do you compute y_1 using Euler's method?
7. How do you compute y_{n+1} from y_n in Euler's method?
8. What is meant by a numerical approximation to a solution of an ODE?
9. What is the local truncation error in Euler's method?
10. What is the global truncation error in Euler's method?
11. What is the order of accuracy of Euler's method?
12. How does decreasing the step size h affect the accuracy of Euler's method?
13. What is the trade-off between accuracy and computational cost in Euler's method?
14. Apply Euler's method with step size $h = 0.1$ to approximate the solution of $y' = y$, $y(0) = 1$ at $x = 0.1$.
15. Apply Euler's method for two steps to approximate the solution of $y' = x + y$, $y(0) = 1$, with $h = 0.5$.
16. What types of errors can accumulate in Euler's method over many steps?
17. What is meant by stability in the context of numerical methods for ODEs?
18. Why can Euler's method become unstable for certain differential equations?
19. Give an example of a differential equation where Euler's method performs poorly.
20. What is the difference between explicit Euler and implicit Euler methods?
21. Write the update formula for the implicit Euler method.
22. What advantage does the implicit Euler method have over the explicit version?
23. What is a stiff differential equation, and why is Euler's method problematic for such equations?
24. How can you modify Euler's method to improve accuracy (e.g., midpoint or improved Euler methods)?
25. What is the idea behind the Heun method (improved Euler method)?
26. Compare Euler's method with higher-order methods like Runge–Kutta methods.
27. Why is Euler's method rarely used in practice for high-precision computations?
28. How can you estimate the error in Euler's method during computation?
29. What happens if the function $f(x, y)$ is not smooth? How does this affect Euler's method?
30. In what situations might Euler's method still be useful despite its limitations?
31. How would you implement Euler's method algorithmically?
32. Write pseudocode for Euler's method.
33. How does Euler's method relate to the Taylor series expansion of the solution?

34. What term(s) of the Taylor series does Euler's method include or neglect?
35. Can Euler's method be used for systems of differential equations? How?
36. How does Euler's method behave when applied backward in time (negative step size)?
37. What are common pitfalls when using Euler's method in practice?
38. How would you visualize the steps of Euler's method on a slope field?
39. What is the difference between consistency, stability, and convergence in the context of Euler's method?
40. State the relationship between consistency, stability, and convergence (Lax equivalence theorem).

The Gamma Function

1. What is the definition of the Gamma function $\Gamma(x)$ for $x > 0$?
2. For which values of x is the integral definition of $\Gamma(x)$ valid?
3. Show that $\Gamma(1) = 1$.
4. Prove the functional equation $\Gamma(x+1) = x\Gamma(x)$.
5. How does the Gamma function extend the factorial function? State the relationship between $\Gamma(n)$ and $(n-1)!$ for $n \in \mathbb{N}$.
6. Compute $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$ using the functional equation.
7. What is the value of $\Gamma\left(\frac{1}{2}\right)$?
8. How can the integral $\int_0^\infty e^{-t^2}, dt$ be used to compute $\Gamma\left(\frac{1}{2}\right)$?
9. What substitution transforms the Gaussian integral into the Gamma function integral?
10. What is the reflection formula for the Gamma function?
11. Use the reflection formula to compute $\Gamma\left(\frac{1}{2}\right)$.
12. What is the duplication formula (Legendre's formula) for the Gamma function?
13. What is the behavior of $\Gamma(x)$ near $x = 0$?
14. Does the Gamma function have any poles? If so, where are they located?
15. Is the Gamma function defined for negative non-integer values? What happens at negative integers?
16. What is the logarithmic derivative of the Gamma function called?
17. Define the digamma function $\psi(x)$.
18. What is the integral representation of $\Gamma(x)$ involving $e^{-t}t^{x-1}$?
19. What is an alternative integral representation of $\Gamma(x)$ using a limit (Euler's limit form)?
20. What is Stirling's approximation for $\Gamma(x)$ for large x?
21. How can Stirling's formula be used to approximate $n!$?
22. What is the Beta function $B(x, y)$?
23. How is the Beta function related to the Gamma function?
24. Show that $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
25. What is the integral definition of the Beta function?
26. How can you derive the Beta-Gamma relationship using a change of variables?
27. What is the asymptotic growth rate of $\Gamma(x)$ as $x \to \infty$?
28. Is the Gamma function convex or concave on $(0, \infty)$?
29. What is the logarithmic convexity (log-convexity) property of $\Gamma(x)$?
30. State Bohr–Mollerup theorem and explain its significance.
31. How does the Gamma function behave under scaling transformations?
32. What is the value of $\Gamma\left(\frac{3}{2}\right)$?
33. Compute $\Gamma\left(\frac{5}{2}\right)$ using recursion.

34. Express $\Gamma\left(n+\frac{1}{2}\right)$ in terms of factorials and $\sqrt{\pi}$.
35. What is the relation between $\Gamma(x)$ and improper integrals in probability theory?
36. How does the Gamma function appear in the definition of the Gamma distribution?
37. What is the probability density function of the Gamma distribution?
38. What role does $\Gamma(x)$ play in normalizing probability distributions?
39. How can $\Gamma(x)$ be analytically continued to complex numbers?
40. Is $\Gamma(x)$ an entire function? Why or why not?
41. Where are the zeros of the Gamma function located?
42. What is the Weierstrass product representation of $\Gamma(x)$?
43. How does $\Gamma(x)$ relate to complex analysis and contour integrals?
44. What is Euler's constant γ and how does it relate to $\Gamma(x)$?
45. What is the series expansion of $\log \Gamma(x)$ near $x = 1$?
46. How is the Gamma function used in evaluating definite integrals?
47. Give an example of an integral that evaluates to a Gamma function.
48. How does the substitution $t = -\ln u$ relate the Gamma function to integrals over $(0, 1)$?
49. What is the connection between $\Gamma(x)$ and factorial moments?
50. Why is the Gamma function important in physics and engineering?

Hessian Matrix

Basic Definitions

1. What is the Hessian matrix of a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$?
2. Write the general form of the Hessian matrix for a function $f(x_1, x_2, \ldots, x_n)$.
3. What are the entries of the Hessian matrix in terms of partial derivatives?
4. For $f(x, y)$, write the Hessian explicitly.

Computation

1. Compute the Hessian of $f(x, y) = x^2 + y^2$.
2. Compute the Hessian of $f(x, y) = x^2y + y^3$.
3. Compute the Hessian of $f(x, y, z) = xyz$.
4. Compute the Hessian of $f(x, y) = e^{x^2+y^2}$.

Interpretation

1. What information does the Hessian matrix encode about a function?
2. How is the Hessian related to the second-order Taylor expansion?
3. What does the Hessian tell you about local curvature?

Critical Points and Optimization

1. What is a critical point of a multivariable function?
2. State the second derivative test using the Hessian for functions of two variables.
3. What are the conditions on the Hessian for:
4. a local minimum?
5. a local maximum?
6. a saddle point?
7. What does it mean for the Hessian to be positive definite?
8. What does it mean for the Hessian to be negative definite?
9. What happens if the Hessian is indefinite?
10. What happens if the determinant of the Hessian is zero?

Matrix Properties

1. When is the Hessian symmetric?
2. State the conditions under which mixed partial derivatives are equal.
3. Why is symmetry of the Hessian important?

Eigenvalues and Geometry

1. How are eigenvalues of the Hessian related to curvature?
2. What does it mean if all eigenvalues are positive?
3. What does it mean if eigenvalues have mixed signs?
4. How do eigenvectors of the Hessian relate to principal directions?

Advanced Connections

1. How does the Hessian relate to convexity of a function?
2. State the condition for a function to be convex using the Hessian.
3. What is the Hessian in the context of optimization algorithms (e.g., Newton's method)?
4. How is the Hessian used in quadratic approximation?
5. What is the bordered Hessian and when is it used?

Conceptual Checks

1. Why does the first derivative test fail for multivariable functions without second-order information?
2. Can a function have a zero gradient but not be an extremum? Explain using the Hessian.
3. How does the Hessian generalize the second derivative from single-variable calculus?

IEEE-754 Floating Point

Basic Structure

1. What are the three components of an IEEE-754 floating point number?
2. What is the purpose of the sign bit?
3. How is the exponent stored (biased or unbiased)?
4. What is the mantissa (significand), and what does it represent?
5. Write the general formula for a normalized floating point number.
6. For single precision (32-bit), how many bits are allocated to:
   - sign
   - exponent
   - fraction
7. For double precision (64-bit), how many bits are allocated to each field?

Exponent and Bias

1. What is the exponent bias in single precision?
2. What is the exponent bias in double precision?
3. Why is a bias used instead of signed exponent representation?
4. How do you recover the true exponent from the stored exponent?
5. What exponent value corresponds to zero after bias correction?

Normalization

1. What does it mean for a floating point number to be normalized?
2. Why is there an implicit leading 1 in normalized numbers?
3. Write the formula for a normalized IEEE-754 number including the implicit bit.
4. What happens when the exponent field is all zeros?
5. What happens when the exponent field is all ones?

Special Values

1. How is zero represented in IEEE-754?
2. Why are there two representations of zero?
3. How is infinity represented?
4. How is NaN (Not a Number) represented?
5. What is the difference between quiet NaN and signaling NaN?

Subnormal Numbers

1. What are subnormal (denormal) numbers?
2. Why do subnormal numbers exist?
3. How is their representation different from normalized numbers?
4. What is the smallest positive subnormal number in single precision?
5. What is the trade-off when using subnormal numbers?

Precision and Rounding

1. What is machine epsilon?
2. How many significant decimal digits does single precision provide?
3. How many significant decimal digits does double precision provide?
4. What are the common IEEE-754 rounding modes?
5. What does "round to nearest, ties to even" mean?

Range and Limits

1. What is the largest representable finite number in single precision?
2. What is the smallest positive normalized number?
3. What is the smallest positive subnormal number?
4. How does exponent size affect range?
5. How does mantissa size affect precision?

Conversion and Interpretation

1. Convert a decimal number to IEEE-754 format step by step.
2. Convert an IEEE-754 bit pattern to decimal.
3. How do you interpret the exponent and mantissa fields together?
4. What mistakes are common when converting floating point numbers?

Arithmetic and Errors

1. Why is floating point addition not associative?
2. Give an example where $(a + b) + c \neq a + (b + c)$.
3. What is rounding error?
4. What is catastrophic cancellation?
5. Why do floating point errors accumulate in iterative computations?

Edge Cases and Pitfalls

1. Why can `0.1` not be represented exactly in binary floating point?
2. What happens when overflow occurs?
3. What happens when underflow occurs?
4. Why should you avoid direct equality comparison of floats?
5. How do you safely compare floating point numbers?

Implementation Understanding

1. How would you extract the sign, exponent, and mantissa from a 32-bit integer?
2. How can bit manipulation be used to inspect a float in memory?
3. What is the difference between float and double in terms of memory and precision?
4. When should you prefer double over float?
5. When might floating point be inappropriate (e.g., finance)?

Derivative of the Inverse Function

1. What is the definition of the inverse of a function f? Under what condition does an inverse exist?
2. What does it mean for a function to be one-to-one (injective), and why is this property necessary for the existence of an inverse?
3. State the relationship between a function and its inverse using composition.
4. If $y = f(x)$, how do you express x in terms of f^{-1}?
5. What is the geometric relationship between the graphs of f and f^{-1}?
6. How can you determine whether a function has an inverse using the horizontal line test?
7. State the formula for the derivative of an inverse function:
$$(f^{-1})'(x) = ?$$
8. Derive the formula for $\left(f^{-1}\right)'(x)$ starting from the identity $f(f^{-1}(x)) = x$.
9. If $f(a) = b$, express $\left(f^{-1}\right)'(b)$ in terms of $f'(a)$.
10. Under what conditions on f does the derivative of the inverse function exist?
11. Why must $f'(a) \neq 0$ for the inverse function to be differentiable at $b = f(a)$?
12. Compute $\left(f^{-1}\right)'(x)$ when $f(x) = x^3 + x$.
13. Let $f(x) = \sin x$ restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. What is $f^{-1}(x)$ and its derivative?
14. Find the derivative of $\arctan(x)$ using the inverse function derivative formula.
15. Show that the derivative of $\ln x$ can be derived from the inverse of e^x.
16. If $f'(x)$ is known, describe the steps to compute $\left(f^{-1}\right)'(x)$ at a specific point.
17. What happens to $\left(f^{-1}\right)'(x)$ if $f'(x)$ is very large? What is the geometric interpretation?
18. What happens to $\left(f^{-1}\right)'(x)$ if $f'(x)$ is very small (but nonzero)?
19. Explain why the slopes of inverse functions at corresponding points are reciprocals.
20. If f is decreasing and invertible, what can you say about the sign of $\left(f^{-1}\right)'(x)$?
21. Let $f(x) = x^5$. Compute $\left(f^{-1}\right)'(x)$ explicitly.
22. If $f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, compute the derivative of its inverse.
23. Verify that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ using the inverse function rule.
24. How does implicit differentiation relate to finding the derivative of an inverse function?
25. If f is differentiable and strictly monotonic on an interval, what can be said about the differentiability of f^{-1}?
26. Give an example of a function that is invertible but whose inverse is not differentiable at some point.
27. What is the domain of f^{-1} in terms of the range of f?
28. What is the range of f^{-1} in terms of the domain of f?
29. How can you use the inverse function derivative formula to avoid explicitly finding the inverse?
30. Let $f(2) = 5$ and $f'(2) = 3$. Find $\left(f^{-1}\right)'(5)$.

Jacobian Matrix and Determinant

Basic Definitions

1. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$F(x_1, \dots, x_n) = (f_1(x), \dots, f_m(x)).$$

   What is the definition of the *Jacobian matrix* $J_F(x)$?

2. Write explicitly the Jacobian matrix:

$$J_F(x) = \left(\frac{\partial f_i}{\partial x_j} \right).$$

   What are its dimensions?

3. In which case is the Jacobian matrix square?

4. When $m = n$, how is the *Jacobian determinant* defined?

5. What does the notation

$$\det J_F(x)$$

   represent?

Jacobian as Derivative

1. How is the Jacobian matrix related to the total derivative (Fréchet derivative) of a function $F : \mathbb{R}^n \to \mathbb{R}^m$?

2. State the linear approximation formula:

$$F(x + h) \approx F(x) + J_F(x)h.$$

3. What does the Jacobian matrix represent geometrically at a point?

4. What condition on $J_F(x)$ ensures that F is locally invertible near x?

Chain Rule

1. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^p$. State the multivariable chain rule for Jacobians:

$$J_{G \circ F}(x) = ?$$

2. When $n = m = p$, what is the corresponding formula for determinants?

3. Why does this imply

$$\det J_{G \circ F}(x) = \det J_G(F(x)) \cdot \det J_F(x)?$$

Geometric Interpretation

1. How does the Jacobian determinant describe local volume scaling?

2. What does it mean if $\det J_F(x) > 0$?

3. What does it mean if $\det J_F(x) < 0$?

4. What does it mean if $\det J_F(x) = 0$?

5. How does the absolute value $|\det J_F(x)|$ relate to infinitesimal volume elements?

Change of Variables in Integration

1. State the change-of-variables formula in $\mathbb{R}^n$:

$$\int_{F(U)} g(y)\, dy = \int_U g(F(x))\, |\det J_F(x)|\, dx.$$

2. Why is the absolute value necessary?
3. What conditions must F satisfy for the change-of-variables formula to hold?
4. Compute the Jacobian determinant for the transformation from Cartesian to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

5. Why does the area element become

$$dx\, dy = r\, dr\, d\theta?$$

Special Cases and Examples

1. Compute the Jacobian matrix of

$$F(x, y) = (x^2 y, e^{xy}).$$

2. Compute the Jacobian determinant of

$$F(x, y) = (u, v)$$

   where

$$u = x + y, \quad v = x - y.$$

3. For a linear transformation $F(x) = Ax$, what is $J_F(x)$?
4. How is $\det J_F(x)$ related to $\det A$ in the linear case?
5. If F is a rotation in $\mathbb{R}^2$, what is its Jacobian determinant?

Inverse Function Theorem

1. State the inverse function theorem in terms of the Jacobian determinant.
2. Why does $\det J_F(x_0) \neq 0$ imply local invertibility?
3. How is the Jacobian of the inverse function expressed in terms of J_F?

Higher Concepts

1. How does the Jacobian generalize the derivative of a single-variable function?
2. In what sense is the Jacobian matrix the best linear approximation?
3. How does the determinant relate to orientation?
4. What is the Jacobian in the context of manifolds and coordinate changes?
5. How does the Jacobian relate to differential forms?

Newton's Method

1. What is the goal of a root-finding algorithm?
2. State the Newton (Newton–Raphson) method iteration formula for finding a root of a function $f(x)$.
3. How is the Newton iteration formula derived from the tangent line approximation?
4. Write the equation of the tangent line to $f(x)$ at a point x_n.
5. How do you obtain the next iterate x_{n+1} from the tangent line?
6. What assumptions about $f(x)$ are required for Newton's method to work?
7. What is meant by an initial guess x_0, and why is it important?
8. Define what it means for Newton's method to converge.
9. What is quadratic convergence?
10. Under what conditions does Newton's method exhibit quadratic convergence?
11. What happens if $f'(x_n) = 0$ during the iteration?
12. Why can Newton's method fail to converge for some initial guesses?
13. Give an example of a function and initial guess where Newton's method diverges.
14. What is meant by the basin of attraction of a root?
15. How does the choice of initial guess affect which root Newton's method converges to?
16. What is a multiple root, and how does it affect convergence of Newton's method?
17. How can Newton's method be modified to handle multiple roots?
18. What stopping criteria can be used to terminate Newton's method?
19. Compare stopping criteria based on $|x_{n+1} - x_n|$ and $|f(x_n)|$.
20. What is the computational cost of one Newton iteration?
21. How does Newton's method compare to the bisection method in terms of speed and reliability?
22. Why is Newton's method considered a local method?
23. What is the geometric interpretation of Newton's method?
24. How can Newton's method be extended to systems of nonlinear equations?
25. Write the Newton iteration for a system of equations using the Jacobian matrix.
26. What role does the Jacobian matrix play in multidimensional Newton's method?
27. What issues arise when the Jacobian is singular or nearly singular?
28. What is a damped (or modified) Newton method?
29. Why might damping improve convergence?
30. How can Newton's method be used to find extrema of a function?
31. How is Newton's method related to solving $f'(x) = 0$?
32. What is the difference between Newton's method for optimization and for root finding?
33. What is the order of convergence of Newton's method compared to secant and bisection methods?
34. What is the secant method, and how does it approximate Newton's method?

35. When might the secant method be preferred over Newton's method?
36. What is the effect of numerical errors in evaluating $f(x)$ or $f'(x)$?
37. How can automatic differentiation help in implementing Newton's method?
38. What are practical safeguards used in implementations of Newton's method?
39. What is line search in the context of Newton's method?
40. How can Newton's method behave near inflection points?
41. What happens if the function is not differentiable at the root?
42. How can Newton's method be visualized graphically?
43. Why can Newton's method oscillate between points instead of converging?
44. How can you detect divergence in practice?
45. What are hybrid methods that combine Newton's method with bracketing methods?
46. Why are hybrid methods often preferred in practice?
47. What is the role of scaling in Newton's method for systems?
48. How does conditioning of the problem affect convergence?
49. What is meant by the error $e_n = x_n - r$, and how does it evolve in Newton's method?
50. Derive the error recurrence relation for Newton's method near a simple root.
51. Why does Newton's method converge faster near the root than far from it?

Roots of Unity

1. What is meant by an nth root of unity?
2. Write the equation whose solutions are the nth roots of unity.
3. How many distinct nth roots of unity are there? Why?
4. Express the nth roots of unity in exponential form using Euler's formula.
5. Write the general formula for the kth nth root of unity.
6. What is the principal (or primitive) nth root of unity?
7. How can all nth roots of unity be generated from a single primitive root?
8. What is the difference between a root of unity and a primitive root of unity?
9. State the condition under which a root of unity is primitive.
10. How many primitive nth roots of unity are there? Which function counts them?
11. What is Euler's totient function $\varphi(n)$ and how does it relate to roots of unity?
12. Represent the nth roots of unity geometrically in the complex plane.
13. What is the magnitude and argument of each nth root of unity?
14. How are the nth roots of unity distributed on the complex unit circle?
15. What is the angle between consecutive nth roots of unity?
16. Compute the sum of all nth roots of unity. What is the result?
17. Why does the sum of all nth roots of unity equal zero?
18. Compute the product of all nth roots of unity.
19. What symmetry properties do the nth roots of unity exhibit?
20. Show that the nth roots of unity form a group under multiplication.
21. What type of group is formed by the nth roots of unity?
22. What is the identity element in this group?
23. What is the inverse of a given nth root of unity?
24. Show that the set of nth roots of unity is cyclic.
25. What is a generator of the group of nth roots of unity?
26. For which values of k is ζ^k a primitive root, where ζ is a primitive nth root?
27. Solve the equation $z^n = 1$ explicitly for small values such as $n = 2, 3, 4$.
28. What are the cube roots of unity? Write them explicitly.
29. What are the fourth roots of unity? Write them explicitly.
30. How can roots of unity be used to factor the polynomial $x^n - 1$?
31. Write the factorization of $x^n - 1$ in terms of its complex roots.
32. What are cyclotomic polynomials and how are they related to roots of unity?
33. Define the nth cyclotomic polynomial $\Phi_n(x)$.
34. How does $x^n - 1$ factor into cyclotomic polynomials?
35. What is the relationship between primitive roots of unity and cyclotomic polynomials?

36. How can roots of unity be used to evaluate sums such as $\sum_{k=0}^{n-1} \zeta^k$?
37. Evaluate $\sum_{k=0}^{n-1} \zeta^{mk}$ for an integer m.
38. Under what condition does $\sum_{k=0}^{n-1} \zeta^{mk} = 0$?
39. What happens when m is a multiple of n in the above sum?
40. How are roots of unity used in discrete Fourier transforms (DFT)?
41. What is the connection between roots of unity and periodicity?
42. How can roots of unity simplify computations involving periodic sums?
43. Show that if ζ is an nth root of unity, then $\overline{\zeta} = \zeta^{-1}$.
44. What is the complex conjugate of a root of unity geometrically?
45. How do roots of unity relate to rotations in the complex plane?
46. What is the minimal polynomial of a primitive nth root of unity?
47. How does the degree of $\Phi_n(x)$ relate to $\varphi(n)$?
48. In what fields do primitive nth roots of unity lie?
49. What is the significance of roots of unity in number theory?
50. How are roots of unity used in solving polynomial equations?

Complex Numbers as Matrices / Representation Theory

https://www.youtube.com/watch?v=hsveVFoIJPM

1. What does it mean to represent an abstract algebraic object (like a group element or number) as a matrix?
2. What is the general idea of a **representation** in the context of linear algebra and group theory?
3. How can a complex number $a + bi$ be represented as a 2×2 real matrix?
4. Write explicitly the matrix corresponding to the complex number $a + bi$.
5. Show how matrix addition corresponds to addition of complex numbers under this representation.
6. Show how matrix multiplication corresponds to multiplication of complex numbers under this representation.
7. Why is this matrix representation of complex numbers structure-preserving?
8. What algebraic structure is preserved when mapping complex numbers to matrices?
9. What property must a mapping satisfy to be considered a homomorphism?
10. Verify that the mapping from complex numbers to matrices is a homomorphism.
11. What is the identity element in the complex numbers, and what matrix represents it?
12. What matrix corresponds to the imaginary unit i?
13. Compute the square of the matrix representing i and interpret the result.
14. How does this matrix representation encode the fact that $i^2 = -1$?
15. What geometric transformation in $\mathbb{R}^2$ corresponds to multiplication by i?
16. How does multiplication by a general complex number $a + bi$ act geometrically on the plane?
17. How is rotation represented in terms of matrices?
18. How does scaling appear in the matrix representation of a complex number?
19. Explain how complex multiplication combines rotation and scaling.
20. What is the determinant of the matrix corresponding to $a + bi$?
21. What is the geometric meaning of this determinant?
22. What is the trace of the matrix representation of $a + bi$?
23. How does the modulus $|a + bi|$ relate to the determinant of the matrix?
24. What condition on a and b makes the matrix invertible?
25. What is the matrix corresponding to the inverse of a complex number?
26. Show that the inverse matrix corresponds to the multiplicative inverse of the complex number.
27. How does this representation help visualize complex numbers as linear transformations?
28. What is the connection between this representation and rotations in the plane?
29. How can this idea be generalized to represent other algebraic structures using matrices?
30. What is a group representation in general?
31. Why are matrices particularly useful for representing abstract algebraic objects?
32. How does this example motivate the study of representation theory?

33. What advantages do matrix representations provide for computation?
34. How can eigenvalues of these matrices be interpreted in terms of complex numbers?
35. What happens when you diagonalize the matrix corresponding to a complex number?
36. How does this representation connect linear algebra with complex analysis?
37. In what sense are complex numbers “the same” as a subset of 2×2 real matrices?
38. What is the dimension of the vector space in which these matrices act?
39. How does this perspective change your understanding of multiplication by complex numbers?
40. Can every linear transformation of $\mathbb{R}^2$ be represented by a complex number? Why or why not?
41. What distinguishes matrices that correspond to complex numbers from arbitrary 2×2 matrices?
42. How would you test whether a given 2×2 matrix corresponds to a complex number?
43. What deeper insight does this example give about the relationship between algebra and geometry?

Geometric Meaning of the Third Derivative

https://www.youtube.com/watch?v=SovllrJUQ64

1. What is the geometric meaning of the first derivative $f'(x)$ in terms of the graph of $f(x)$?
2. How can you interpret the second derivative $f''(x)$ geometrically?
3. What does it mean for $f''(x) > 0$ in terms of the shape of the graph?
4. What does it mean for $f''(x) < 0$?
5. Define an inflection point in terms of concavity.
6. What condition on $f''(x)$ is necessary for an inflection point?
7. Why is $f''(x) = 0$ not sufficient to guarantee an inflection point?
8. What additional condition ensures that a point where $f''(x) = 0$ is actually an inflection point?
9. How can sign changes of $f''(x)$ be used to detect inflection points?
10. What is the geometric meaning of the third derivative $f'''(x)$?
11. How can $f'''(x)$ be interpreted as a rate of change of another geometric quantity?
12. If $f'''(x) > 0$, what does this say about how concavity is changing?
13. If $f'''(x) < 0$, what does this imply about the change in concavity?
14. How does $f'''(x)$ describe the "bending behavior" of a curve beyond concavity?
15. Explain how the third derivative relates to the steepness of the tangent slope.
16. How does the graph of $f'(x)$ help visualize $f''(x)$?
17. How does the graph of $f''(x)$ help visualize $f'''(x)$?
18. If $f''(x)$ has a local maximum or minimum, what can you say about $f'''(x)$ at that point?
19. What is the relationship between $f'''(x) = 0$ and extrema of $f''(x)$?
20. Can $f'''(x) = 0$ correspond to an inflection point of $f(x)$? Explain.
21. How would you distinguish between:
    - an inflection point of $f(x)$
    - an extremum of $f''(x)$
22. Describe how successive derivatives correspond to successive "rates of change" geometrically.
23. Give a physical interpretation of $f'(x)$, $f''(x)$, and $f'''(x)$ in motion (position, velocity, acceleration, ...).
24. What is the physical meaning of the third derivative in kinematics (often called jerk)?
25. How does the sign of the third derivative affect motion in a physical system?
26. Construct an example function where $f''(x) = 0$ but there is no inflection point.
27. Construct an example where $f'''(x) = 0$ but $f''(x)$ does not change sign.
28. Given a graph, how would you visually estimate where $f'''(x)$ is positive or negative?
29. How can you use higher derivatives to understand increasingly subtle features of a curve?
30. Why is the third derivative rarely discussed compared to the first and second in basic calculus?
31. In what types of problems does understanding the third derivative become important?

Second Derivative Test and Hessian

https://www.youtube.com/watch?v=z0bSdV8DA4M

Core Concept: What is the second derivative test really doing?

- What does the second derivative test attempt to determine about a function at a critical point?
- Why is checking only the first derivative insufficient for classifying critical points?
- In one variable, what does the sign of $f''(x)$ tell us geometrically?
- How can you interpret $f''(x)$ in terms of curvature?

From 1D to Multivariable Functions

- What is the analogue of the second derivative for multivariable functions?
- What is the Hessian matrix, and how is it constructed?
- Why can't we just look at "the second derivative" in multiple dimensions?
- What does the Hessian represent geometrically?

Quadratic Approximation

- What is the second-order Taylor expansion of a function near a critical point?
- Why does the linear term vanish at a critical point?
- What role does the quadratic term play in determining local behavior?
- How does the function locally resemble a quadratic form?

Quadratic Forms and Geometry

- What is a quadratic form?
- How can the expression $\mathbf{x}^T H \mathbf{x}$ describe local curvature?
- What does it mean for a quadratic form to be:
  - Positive definite?
  - Negative definite?
  - Indefinite?

Classification of Critical Points

- How does the definiteness of the Hessian classify a critical point?
- What condition corresponds to a local minimum?
- What condition corresponds to a local maximum?
- What condition corresponds to a saddle point?
- What happens if the Hessian is degenerate (determinant zero)?

Eigenvalues Interpretation

- Why are eigenvalues the key to understanding the Hessian?
- How do eigenvalues relate to curvature along different directions?
- What does it mean if all eigenvalues are positive?
- What does it mean if eigenvalues have mixed signs?
- How do eigenvectors relate to principal directions of curvature?

Geometric Intuition

- Why is a saddle point associated with "curving up in some directions and down in others"?
- How can you visualize curvature along different directions through a point?
- Why is the second derivative test really a statement about curvature in all directions?

Connection to Linear Algebra

- Why can any symmetric matrix be diagonalized?
- How does diagonalizing the Hessian simplify the quadratic form?
- What does the diagonal form reveal about the function locally?

Deeper Understanding

- Why is the second derivative test fundamentally about approximating the function by a quadratic surface?
- In what sense is the test coordinate-independent?
- How does this perspective generalize beyond $\mathbb{R}^2$ or $\mathbb{R}^3$?

Edge Cases and Limitations

- Why does the test fail when eigenvalues are zero?
- What additional analysis is needed in degenerate cases?
- Can higher-order terms change the classification?