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This document will provide some of the mathematical approaches I took to solving question 88 on projecteuler.net. It will also include some interesting mathematical insights into the question that I did not require to solve the problem, but were remarkable nevertheless.

Here is the question:

A natural number, N, that can be written as the sum and product of a given sequence of at least two natural numbers $(a_1, a_2, ..., a_k)$ is called a product-sum number:

$$N = a_1 + a_2 + ... + a_k = a_1 \times a_2 \times ... \times a_k$$

For a given sequence of size, k, we shall call the smallest N with this property a minimal product-sum number. The minimal product-sum numbers for sets of size, $k = 2, 3, 4, 5$, and 6 are as follows.

$$\begin{aligned}
&k = 2 : 4 = 2 \times 2 = 2 + 2 \\
&k = 3 : 6 = 1 \times 2 \times 3 = 1 + 2 + 3 \\
&k = 4 : 8 = 1 \times 1 \times 2 \times 4 = 1 + 1 + 2 + 4 \\
&k = 5 : 8 = 1 \times 1 \times 2 \times 2 \times 2 = 1 + 1 + 2 + 2 + 2 \\
&k = 6 : 12 = 1 \times 1 \times 1 \times 1 \times 2 \times 6 = 1 + 1 + 1 + 1 + 2 + 6
\end{aligned}$$

Hence for $2 \leq k \leq 6$, the sum of all the minimal product-sum numbers is $4 + 6 + 8 + 12 = 30$; note that 8 is only counted once in the sum. In fact, as the complete set of minimal product-sum numbers for $2 \leq k \leq 12$ is $\{4, 6, 8, 12, 15, 16\}$, the sum is 61. What is the sum of all the minimal product-sum numbers for $2 \leq k \leq 12000$?

Results

First we will show that $2k$ is a product sum number given a sequence of size k.

Lemma 1. *If we define our minimal product-sum number of all sequences of size k to be N_k, then $k \leq N_k \leq 2k$.*

Proof. Let $A = (a_1, a_2, ..., a_k)$ be our sequence. Now suppose $a_1 = 2$, $a_2 = k$ and $a_i = 1 \ \forall \ 3 \le i \le k$. Then

$$\sum_{i=1}^{k} a_i = 2 + k + (k - 2)$$
$$= 2k$$
$$= \prod_{i=1}^{k} a_i$$

This tells us that the minimal product sum-number is bounded above by $2k$. Also, given k natural numbers, their sum must clearly be greater than or equal to k. The lemma follows. $\square$

The question asks for the sum of all distinct N_k for $2 \le k \le 12000$. Lemma 1 reveals that N_k is bounded by $2k$. So for this question, the largest N_k possible is 24000. We are still concerned however, with how many elements of the sequence $(a_1, a_2, ..., a_k)$ we must pay attention to. For example consider N_{12000}, it would take a very long time to compute all possible sequences of length 12000 who's product is equivalent to its sum. This brings us to our next lemma which will greatly reduce the computations necessary to solve such a task.

Lemma 2. *Given some sequence $A = (a_1, a_2, ..., a_k)$ of natural numbers, if $\sum A = \prod A$ then there are at most $\log_2(2k)$ $a_i \in A$ such that $a_i > 1$.*

Proof. We have already shown that $N_k \le 2k$ by Lemma 1. Notice that if we were to add 1's to our sequence, it would not change $\prod A$. Then 2 is the smallest possible element to influence the product of elements in A. Consider the largest i such that $2^i \le 2k$. Then $i = \lfloor \log_2(2k) \rfloor$, implying we have at most $\lfloor \log_2(2k) \rfloor$ elements of A that are greater than 1. $\square$

This is a very strong lemma. It tells us that if k grows linearly in size, then the number of elements ≥ 2 grows logarithmicly. In other words, for $k = 12000$, N_k can be factored into at most $\lfloor \log_2(24000) \rfloor = 14$ elements.

Now we will show some very interesting patterns in product-sum numbers.

Theorem 1. *Let $A = (a_1, a_2, ..., a_k)$ be a sequence of natural numbers for which there exists a product-sum number l_k. If k is even, then l_k is even and there exists some $a_j \in A$ such that a_j is even.*

Proof. Clearly the existance of an even a_i guarantees that l_k is even. It follows that we should consider the case where a_i is odd for all $1 \leq i \leq k$, then

$$l_k = \sum_i^k a_i = k + \sum_i^k a_i - 1$$

Since $a_i - 1$ is even for all i, and k is even, then l_k must be even. The product of odd numbers is odd, but l_k is even and $l_k = \prod A$ implying there exists some even $a_j \in A$. This reveals that l_k is divisible by 2 if k is even and in particular, we know that there exists some even $a_j \in A$. $\square$

We can actually conclude an even stronger result than the one Theorem 1 gives.

Theorem 2. *Let $A = (a_1, a_2, ..., a_k)$ be a sequence of natural numbers for which there exists a product-sum number l_k. If k is even, then l_k is divisible by 4 and there exists some $a_j \in A$ such that a_j is even.*

Proof. If there exist 2 or more even $a_i \in A$, we are done and clearly 4 is a factor of l_k, so suppose there exists only one even integer $a_j \in A$ and that a_j is not divisible by 4. Note that since a_j factors into $\prod A$, then $\frac{l_k}{a_j} \in \mathbb{N}$. It follows

$$
\begin{aligned}
l_k &= \left(a_j + \sum_{i \neq j} a_i\right) \\
\implies \frac{l_k}{a_j} &= \left(1 + \frac{\sum_{i \neq j} a_i}{a_j}\right) \in \mathbb{N} \\
&\implies \frac{\sum_{i \neq j} a_i}{a_j} \in \mathbb{N} \\
&\implies \sum_{i \neq j} a_i \text{ is even}
\end{aligned}
$$

But $\sum_{i \neq j} a_i$ is a sum of $k - 1$ odd elements by hypothesis. An odd sum of odd numbers is odd, so we have arrived at a contradiction and there exists some $a_h \in A$ such that $h \neq j$ and a_h is even. Therefore if k is even, then l_k is divisible by 4. $\square$