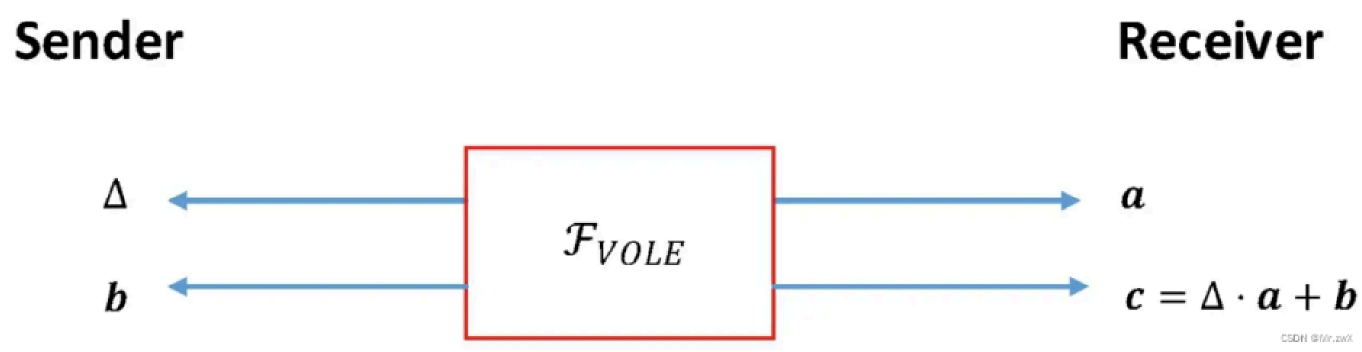
# VOLE (Vector Oblivious Linear Evaluation)

## Part 1 VOLE's General Idea

VOLE sends $\Delta$ and b to the sender, a and c to the receiver with linearity of c,b,a:
$c = \Delta \cdot a + b$

Nowadays, the mainstream VOLE is based on LPN assumption.

At abstract level:




## Part 2 VOLE's construction based on LPN assumption

## 2.1 Pre-Knoeledge

1. LPN assumption: go to file: LPN.md
2. FSS (Functional Secret Sharing): go to file: FSS.md
3. VOLE generator
   - VOLE defines 2 algorithm, i.e. VOLE = (Setup,Expand)
     - $Setup(1^{\lambda}, \mathrm{F}, n, x)$ returns a pair of seeds $(seed_0, seed_1)$, where $seed_1$ includes x
     - $Expand(\sigma, seed_{\sigma})$, if $\sigma = 0$, it returns $(u, v)$; otherwise, it returns $w$.
   - Correctness:
     - $(u, v) \leftarrow Expand(0, seed_0)$;
     - $w \leftarrow Expand(1, seed_1)$, where $w = u \cdot x + v$.
   - Saftey
     - for different input x', we have computationally indistinguishable $(seed_0, seed_1)$:
     - The vector $(u, v)$ returned by $Expand(0, seed_0)$ and random vector $(u^{'}, v^{'})$ are computationally indistinguishable.

## 2.2 VOLE's construction

1. First attempt
   - for setup, we have: (a and b are random vectors, hence c is also random)
     - $seed_0 \leftarrow (a, b) \in_R \mathrm{F}^k \times \mathrm{F}^k$;
     - $seed_1 \leftarrow (c = a \cdot x + b, x) \in \mathrm{F}^k \times \mathrm{F}^k$;
   - for expand, we have: ($C_{k,n} \in \mathrm{F}^{k \times n} (k < n)$, is a broadcast parameter matrix)
     - $Expand(0, seed_0) = (a \cdot C_{k,n}, b \cdot C_{k,n})$;
     - $Expand(1, seed_1) = c \cdot C_{k,n}$
2. Second attempt
   - In the above attempt, Expand maintains a linear relationship, but the resulting strings are not (pseudo-) random. Try to solve this problem with the help of the LPN hypothesis, defining the new Expand algorithm as follows:
   - randomly generate a matrix $C_{k,n} \in \mathrm{F}^{k \times n} (k < n)$ and broadcast them, then use FSS to generate a set of random vectors $\mu, v_b, v_c$ ($v_b + v_c =$

$x \cdot \mu$)

- $Expand(0, seed_0) = (a \cdot C_{k,n} + \mu, b \cdot C_{k,n} - v_b)$;
- $Expand(1, seed_1) = c \cdot C_{k,n} + v_c$

- It's easy to verify that the output of Expand is (pseudo-) random. the Linearity is guaranteed by ‘$v_b + v_c = x \cdot \mu$‘.

3. Formal construction based on formal 2 attempts
   - Suppose under LPN assumption, we have public parameters $\mathbb{F}, k, n, t = rn, C \in \mathbb{F}^{k \times n}$, then the generator of VOLE canbe defined as:
     - $Setup(1^\lambda, x)$:
       1. randomly generate $(a, b) \in \mathbb{F}^k \times \mathbb{F}^k,\ \mu \in \mathbb{F}^n$, which satisfy $HW(\mu) = t$;
       2. calculate $c = a \cdot b$;
       3. $(K_0, K_1) \leftarrow FSS.Gen(1^\lambda, f)$, which satisfy $FSS.Eval(0, K_0) + FSS.Eval(1, K_1) = x \cdot \mu$;
       4. $seed_0 \leftarrow (K_0, \mu, a, b),\ seed_1 \leftarrow (K_1, x, c)$;
       5. output $(seed_0, seed_1)$.
     - $Expand(\sigma, seed_\sigma)$:
       1. if $\sigma = 0,\ seed_0 = (K_0, \mu, a, b)$, calculate $v_0 \leftarrow FSS.Eval(0, K_0)$, output $(u, v) \leftarrow (a \cdot C + \mu, b \cdot C - v_0)$;
       2. if $\sigma = 1,\ seed_1 = (K_1, x, C)$, calculate $v_1 \leftarrow FSS.Eval(1, K_1)$, output $w \leftarrow c \cdot C + v_1$;

# Part 3 VOLE's application in MPC, Multipication

Recall that in the multipication gate, how to calculate cross term is a difficult problem.

$$ $$uint(x) \cdot uint(y) = (x_0 + x_1 - 2^m w_x) \cdot (y_0 + y_1 - 2^n w_y)$$

$$ $$= x_0 y_0 + x_1 y_1 + x_0 y_1 + x_1 y_0 - 2^m w_x y - 2^n w_y x + 2^l w_x w_y$$

For $\Pi_{UMult}^{m,n}$

- $x_b y_b$ can be computed locally;
- $2^l w_x w_y$ is modulo-reduced when $mod\ L$;
- $w_x y$ and $w_y x$ can be computed with $F_{wrap}$ and $F_{Mux}$, using OT
- Focus on $x_b y_{1-b}$ ($F_{CrossTerm}^{m,n}$, use COT)

Main work is focused on $F_{CrossTerm}^{m,n}$, we choose party with less bit as receiver and the other as sender. Therefore, the receiver can offer less comparison than the other party if it is the receiver.

In sirnn.md, crossterm is addressed by COT. However, it can also be tackled by VOLE. Take $x_0 y_1$ as an example:

- $P_0$ has input $x_0$, $P_1$ has input $y_1$;
- Let $P_0$ calculate $v = b \cdot C - v_0$ with $Expand()$ locally;
- Let $P_1$ calculate $w = c \cdot C - v_1$ with $Expand()$ locally;
- therefore, $x_0 y_1 = w - v = v_0 + v_1 + c \cdot C - b \cdot C$.

# Part 4 Use VOLE generator to generate VOLE

The VOLE generator is essentially a pseudo-random number generator that generates two strings of pseudorandom numbers that happen to be linearly correlated.

## Pre-calculate:

1. From Tursted Third Party (TTP) draw random number $r_x \xleftarrow{R}$ F;
2. use VOLE generator to calculate the seeds: $(seed_0, seed_1) \xleftarrow{R} G.setup(1^\lambda, r_x)$;
3. output $seed_0$ to P0, $(seed_1, r_x)$ to P1.

## Offline:

1. P0 calculate $(r_u, r_v) \leftarrow G.Expand(0, seed_0)$;
2. P1 calculate $r_w \leftarrow G.Expand(1, seed_1)$;

## Online:

Now, P0 has private input (u,v), P1 has w

1. P1 sends $m_x \leftarrow x - r_x$ to P0
2. P0 sends $m_u \leftarrow u - r_u,\ m_v \leftarrow m_x \cdot r_u + v - r_v$ back to P1
3. P1 calculate $w \leftarrow m_u \cdot x + m_v + r_w$;

## Correctness:

Discuss the correctness of VOLE protocol. The random vector computed in the offline calculation stage meets $r_x r_u + r_v = r_w$, so the vector w computed in the online calculation stage P1 meets:

$$w = m_u x + m_v x + r_w = (u - r_u x) + (m_x r_u + v - r_v) + (r_x r_u + r_v) = ux + v$$

# Reference:

https://zhuanlan.zhihu.com/p/606020139?utm_id=0