CSE6367 Assignment1

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Problem 1

a. Prove that if A is an n × n (real) symmetric matrix, then there exists an n × n (real) orthogonal matrix U and n × n (real) diagonal matrix D such that A = U · D · U T.

① (a) Here

A is a matrix which is symmetrical. $n \times n$. We know for a symmetric matrix $A = A^T$; The matrix is equal to its transpose matrix.

Now $A = U.D.U^T$ [U is otthogonal matrix
D is diagonal matrix]

If we transpose $A^T = (U D . U^T)^T$

$= (U^T)^T . D^T . U^T$

$= U D^T U^T$.

Diagonal matrix is also a symmetric matrix because everything off the diagonal is a zero.

$\therefore A^T = U D U^T$.

So, A^T is also UDU^T.

Thuse if A is a symmetric matrix there is an orthogal and a diagonal matrix such the $A = UDU^T$

b. Given

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix},$$

plot the regions $\{Ax \mid x \in R^2$ 2 and $||x||_2 = 1\}$, and $\{Ax \mid x \in R^2$ and $||x||_2 \leq 1\}$. Explain your solution.

1 (b) the matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Let, Any matrix A.

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0.$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0.$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0.$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0.$$

So the roots are,

$$\left.\begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = 1 \end{array}\right\} \text{those are eigen values.}$$

Now for eigen vectors..

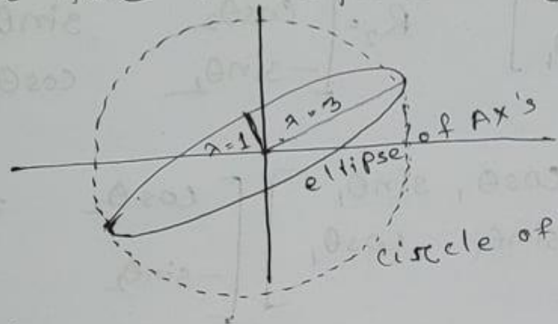
For $\lambda = 3$. eigenvector.

$$v = \begin{bmatrix} +1 \\ +1 \end{bmatrix} t$$

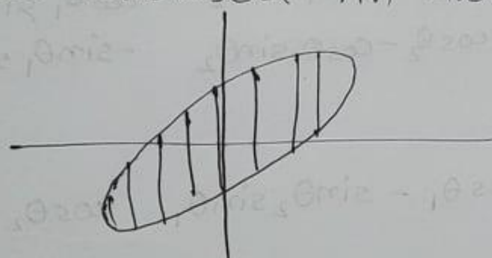
for $\lambda = 1$ eigenvector.

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t.$$

For $\{Ax \mid x \in R^2 \ \|x\|_2 = 1\}$ it will look like an ellipase. The edge will be the solution for transform on $\|x\|_2 = 1$.



For $\|x\|_2 \le 1$ the region looks like elipse will be enclosed in that.



Problem 2

Show that the determinant of a rotation matrix is ±1.

A rotation matrix in n dimension is nxn special orthogonal matrix ,whose determinant is 1.If we take a set of all rotation matrices that forms a group, it is known as the rotation group or the special orthogonal group .

Now, A is an orthogonal(rotation) matrix, We know-

A^TA =AA^T=I (Identity matrix)

If we take determinants on both sides,

Det(AA^T)=Det(I)

Det(A)Det(A^T)=Det(I)

Transpose does not change the value of the determinant. And determinant of an identity matrix is 1. So,

Det(A)Det(A^t) =1

Det(A^2) =1

Det(a)=+1 OR -1

Problem 3

a. Let R1 and R2 be two rotation matrices on the plane. Prove or disprove the following:. $R_1 R_2 = R_2 R_1$

3 (a)

R_1 and R_2 are two rotation matrix on 2D.

$$R_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

Now, $$R_1 R_2 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$R_1 R_2 = \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \\ -\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{bmatrix}$$

for, $$R_2 R_1 = \begin{bmatrix} \cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 & \cos\theta_2 \sin\theta_1 - \sin\theta_2 \cos\theta_1 \\ -\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1 & -\sin\theta_2 \sin\theta_1 + \cos\theta_2 \cos\theta_1 \end{bmatrix}$$

Here multiplication is commutative $\cos\theta_1 \cos\theta_2 = \cos\theta_2 \cos\theta_1$

Thus, $R_1 R_2 = R_2 R_1$

b. Let R1 and R2 be two rotation matrices in 3D space. Prove or disprove the following:. $R_1 R_2 = R_2 R_1$

If the matrices preserve each other's eigenspaces, they can commute or R1R2=R2R1.

This means that a vector in a matrix's eigenspace won't leave that particular eigenspace when the other is applied, so the original matrix's transformation works fine on that. In 2D, no matter what, a rotation matrix's eigenvectors are [i,1] [i,1] and [−i,1] [−i,1]. As they all such matrices have the same eigenvectors, they commute.

But in 3D, in a rotation matrix, there's always a real eigenvalue, so that it has a real eigen vector associated with it-the axis of rotation. But this vector doesn't share values with the rest of the eigenvectors for the rotation matrix. This is an eigenspace of dimension 1, rotations with different axes can't share eigenvectors, so they can't commute.

Problem 4

Let R be a 3D rotation matrix. Claim: 1 is an eigenvalue of R. Is the claim true? If so, what is the physical meaning of the corresponding eigenvector?

We know, for any matrix R ,$Rx=\lambda x$

$(R-\lambda I)\, x=0$

$R-\lambda I=0$

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \det \begin{vmatrix} \cos\theta-\lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

We will get one real and two complex eigen values. We will take the real value

$$\Rightarrow (1-\lambda)\left[(\cos\theta-\lambda)^2+\sin^2\theta\right]=0.$$

$$\Rightarrow (1-\lambda)\left(\cos^2\theta-2\cos\theta\,\lambda+\lambda^2+\sin^2\theta\right)=0$$

$$\Rightarrow (1-\lambda)=0 \qquad \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$$

$$\Rightarrow \boxed{\lambda = 1} \qquad = \cos\theta \pm \sqrt{\cos^2\theta-1}$$

$$= \cos\theta \pm i\sin\theta.$$

So one of the eigen values are 1.

Eigen vector corresponding $\lambda = 1$.

$$\begin{bmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$(\cos\theta - 1)\, x + y \sin\theta = 0$$

$$-\sin\theta\, x + (\cos\theta - 1)\, y = 0$$

$$x = 0 = z$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector correcesponding 1 is $(0, 0, 1)$. that is the axis. Every rotation in 3D amount to rotation about an axis by an angle We are ignoring the other values as they are complex.

The eigenvector gives out about what is it's state, and usually includes some probability in its coefficient since we get more than one eigenvector.

Problem 5

We denote by $\sigma_i(B)$ the ith singular value of B (sorted in descending order). Prove that if A_1 and A_2 are $m \times n$ matrices, then for all i and j in N:

$$\sigma_{i+j-1}(A_1 + A_2) \le \sigma_i(A_1) + \sigma_j(A_2).$$

(5) Let $A_1 = U_1 \Sigma_1 V_1 = \sum_{k=1}^{\min(m,n)} \sigma_k(A) u_k v_k^T$.

$$A_1^{i-1} = \sum_{k=1}^{i-1} \sigma_k(A_1) u_k v_k^T A_2^{j-1}.$$

$$\sum_{i=1}^{k} \sigma_i(A \cdots A_m) < \sum_{i=1}^{k} \sigma_i(A_1 \cdots \sigma(A_m)) \text{ for } k = 1, \dots, n.$$

we know $\prod_{i=1}^{k} \sigma_i(AB) \le \prod_{i=1}^{k} \sigma_i(A\sigma_i(B))$

By singular value decomposition

$A_1 \cdots A_m = V \Sigma W^*$ Then $V^*(A_1 \cdots A_m)W = \mathrm{diag}(\sigma_1(A_1 A$

$\cdots \sigma_n(A_1 \cdots A_m))$.

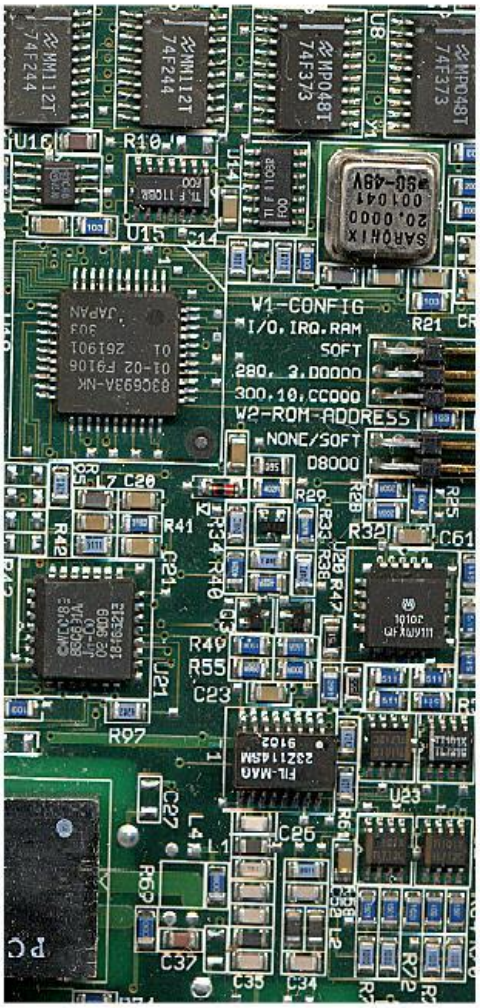
$\therefore \sigma_1(A_1 \cdots A_m) \cdots \sigma_n(A_1 \cdots A_m) = |\det V^*(A_1 \cdots A_m) W| \le (\sigma_1(A_1) \cdots \sigma_n(A_m))$

$\therefore \sigma_{i+j-1}(A_1 + A_2) \le \sigma_i(A_1) + \sigma_j(A_2)$.

Problem 6

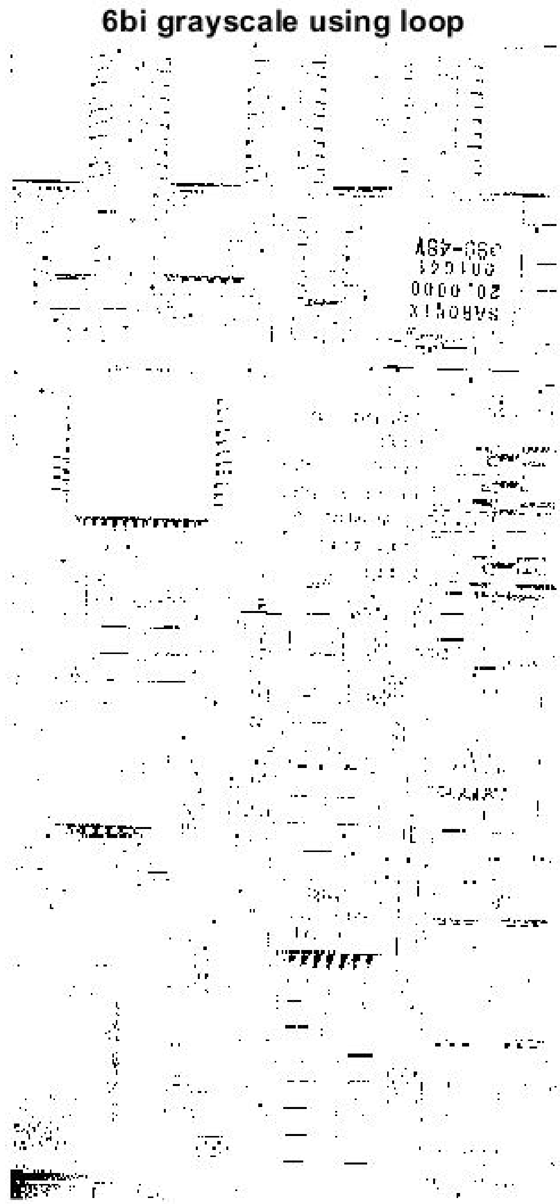
All the images and tables from the matlab programming assignment.

6a.



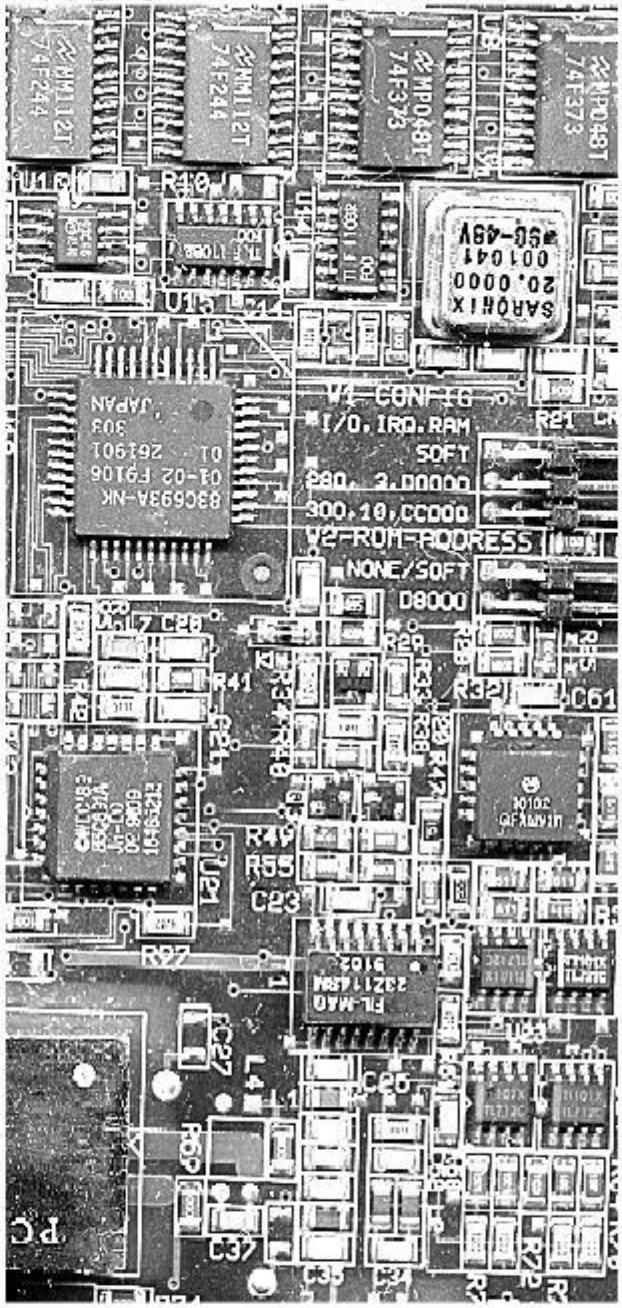
problem 6a extracted image

6b (i)



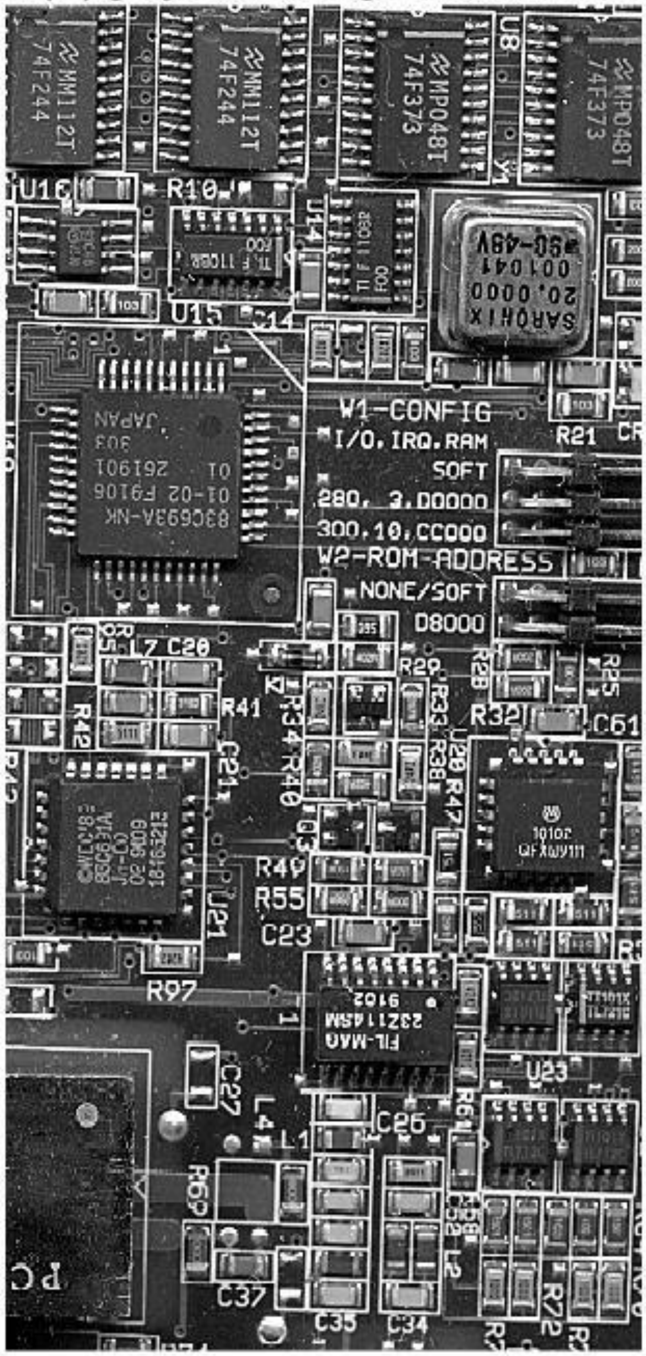
6b (ii)

6b (ii) grayscale using matrix operation



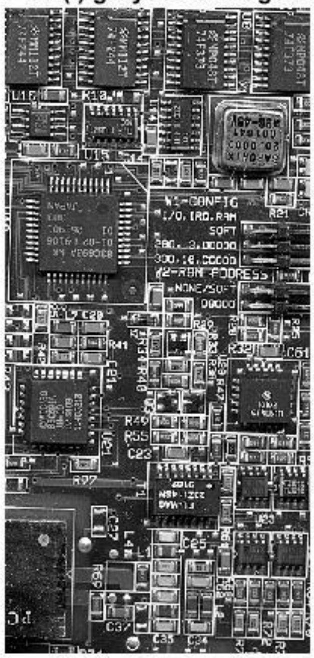
6b (iii)

6b (iii) grayscale using matlab function

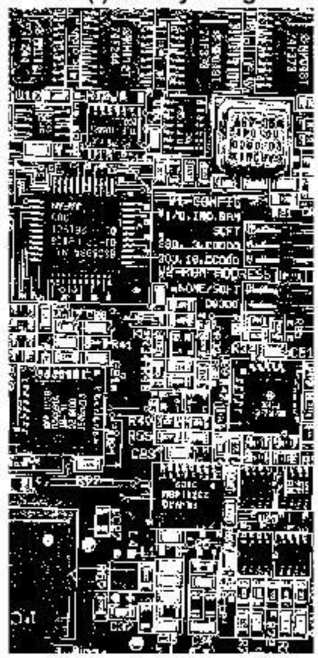


6c (i).

6c (i) grayscale image

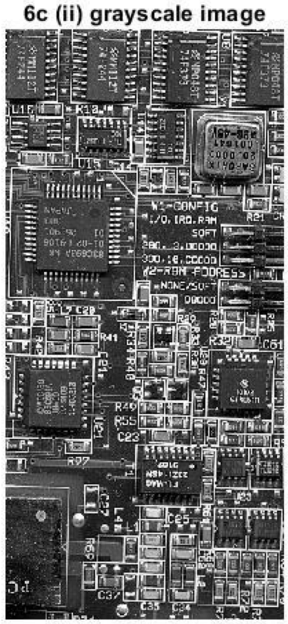


6c (i) binary image

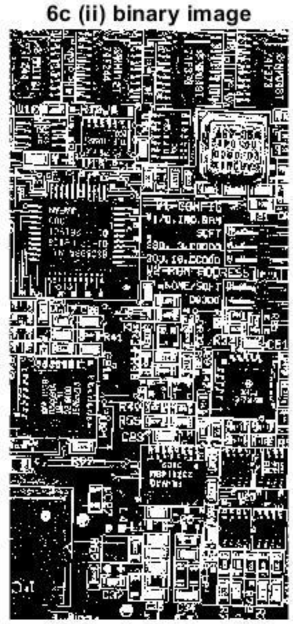


6c (ii)

6c (ii) grayscale image

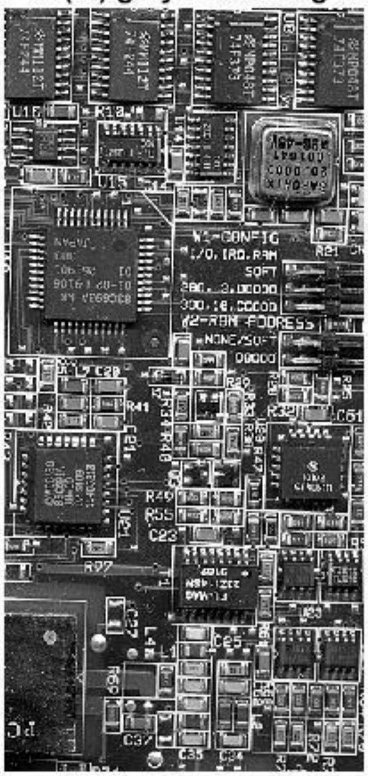


6c (ii) binary image

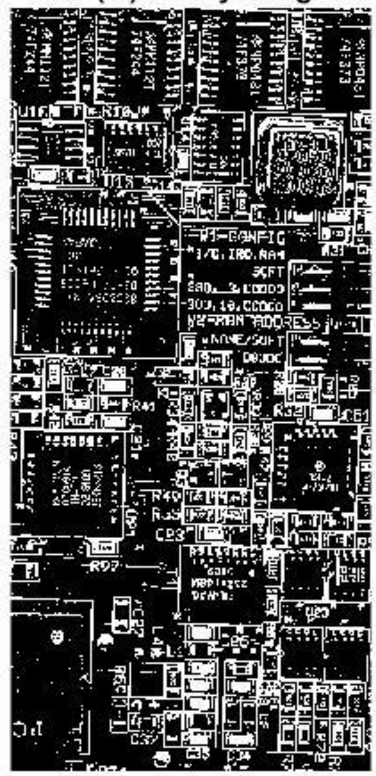


6c (iii)

6c (iii) grayscale image

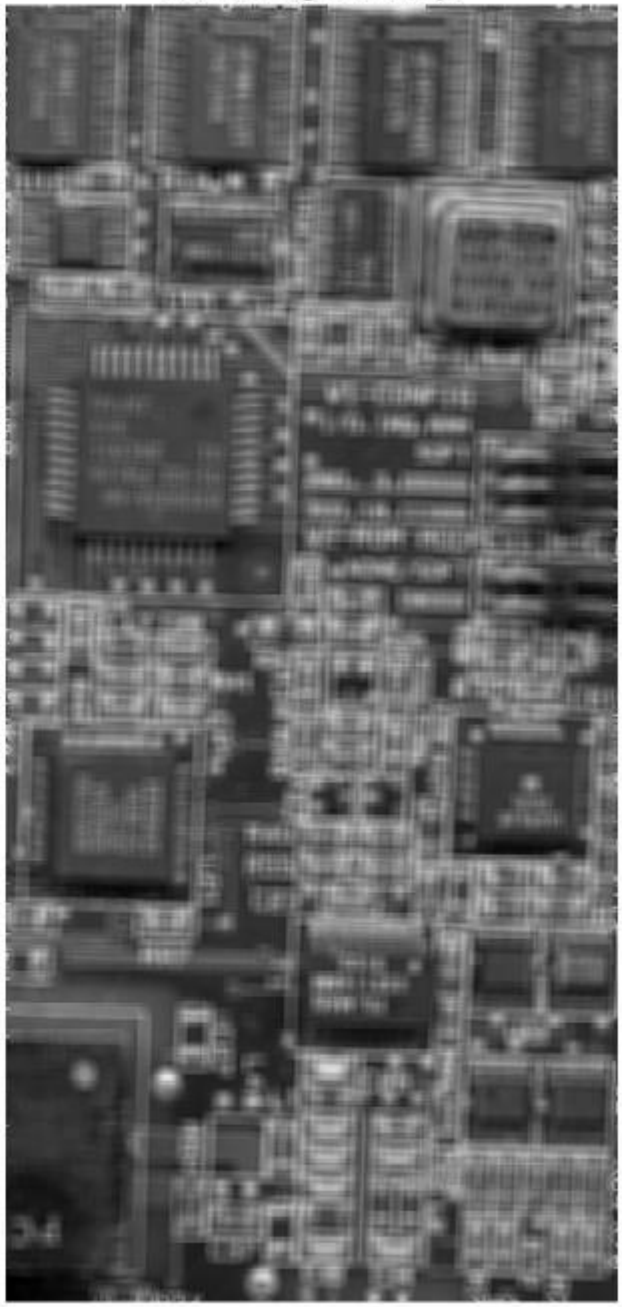


6c (iii) binary image



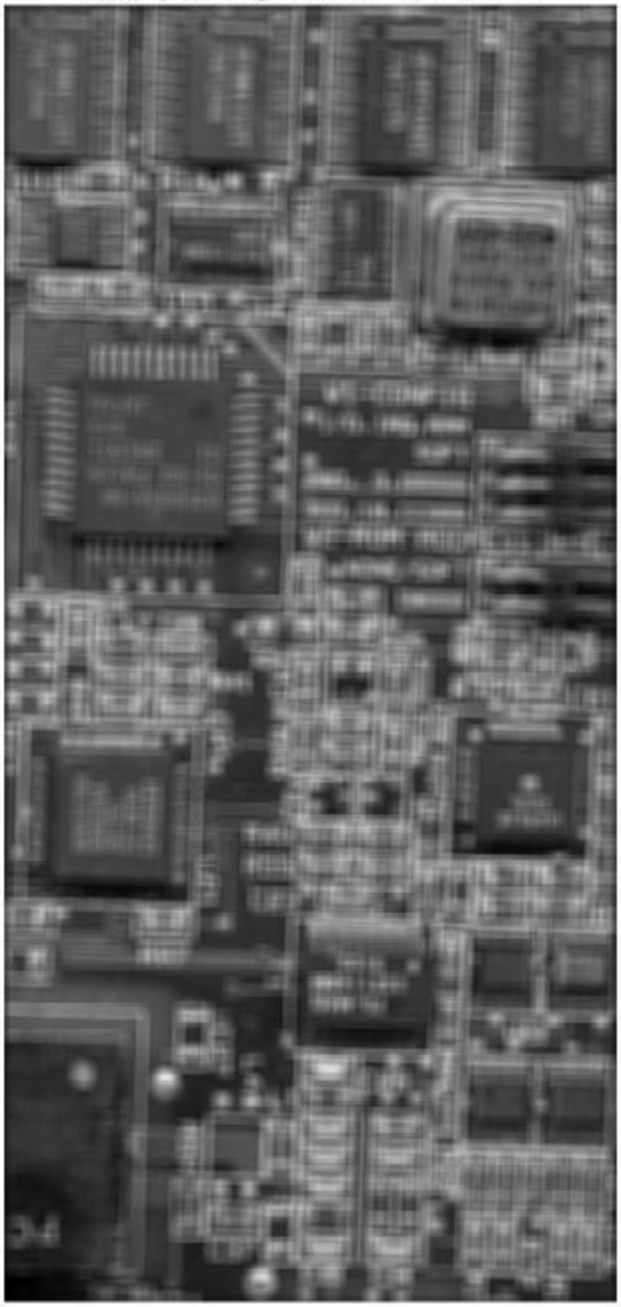
6d (i)

6di using for loop



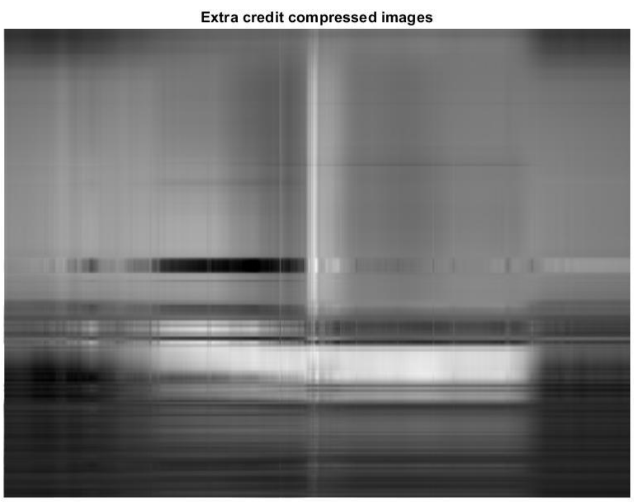
6d (ii)

6d(ii) using conv2 function



Extra credit

Original image



Extracted image with top singular value 3

Extracted image with top singular value 10

Extra credit compressed images

Extracted image with top singular value 20

Extra credit compressed images

Extracted image with top singular value 40

Table with relative error and compression ratio

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1×2 table

          relativeerror                          compressionratio
  ___________________________________    ___________________________________

  269.21    255.59    233.97    206.1    46.405    62.473    70.54    80.481

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