Homework 6.1 & 6.2

CUNY MSDS DATA 609

Duubar Villalobos Jimenez mydvtech@gmail.com

November 18, 2018

Contents

1 Problems

The below problems are taken from the text book:

A First Course in Mathematical Modeling, 5th Edition. Frank R. Giordano, William P. Fox, Steven B. Horton. ISBN-13: 9781285050904.

1.1 Exercise #1 Page 529

In the following problem, verify that the given function pairs is a solution to the first-order system.

$x = -e^t, \quad y = e^t$

$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = -x$

1.1.1 Solution

1.1.1.1 Verifying for $x = -e^t$

$\frac{dx}{dt} = \frac{d}{dt}\left(-e^t\right)$

$\frac{dx}{dt} = -\frac{d}{dt}\left(e^t\right)$

$\frac{dx}{dt} = -[e^t]$; since $y = e^t$

$\therefore \frac{dx}{dt} = -y$

1.1.1.2 Verifying for $y = e^t$

$\frac{dy}{dt} = \frac{d}{dt}\left(e^t\right)$

$\frac{dy}{dt} = e^t$

Since $e^t = -[-e^t]$

$\frac{dy}{dt} = -[-e^t]$; since $x = -e^t$

$\therefore \frac{dy}{dt} = -x$

1.2 Exercise #6 Page 529

In the following problem, find and classify the rest points of the given autonomous system.

$\frac{dx}{dt} = -(y-1), \quad \frac{dy}{dt} = x - 2$

1.2.1 Solution

In order to find the rest points, we need to evaluate $f(x, y) = 0$ and $g(x, y) = 0$, where $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$

1.2.1.1 Evaluating $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = -y + 1$

$0 = -y + 1$

$\therefore y = 1$

1.2.1.2 Evaluating $\frac{dy}{dt} = 0$

$\frac{dy}{dt} = x - 2$

$0 = x - 2$

$\therefore x = 2$

From the above, we have determined our point to be $x_0 = 2$ and $y_0 = 1$.

The rest point is $(2, 1)$

1.2.1.3 Verification

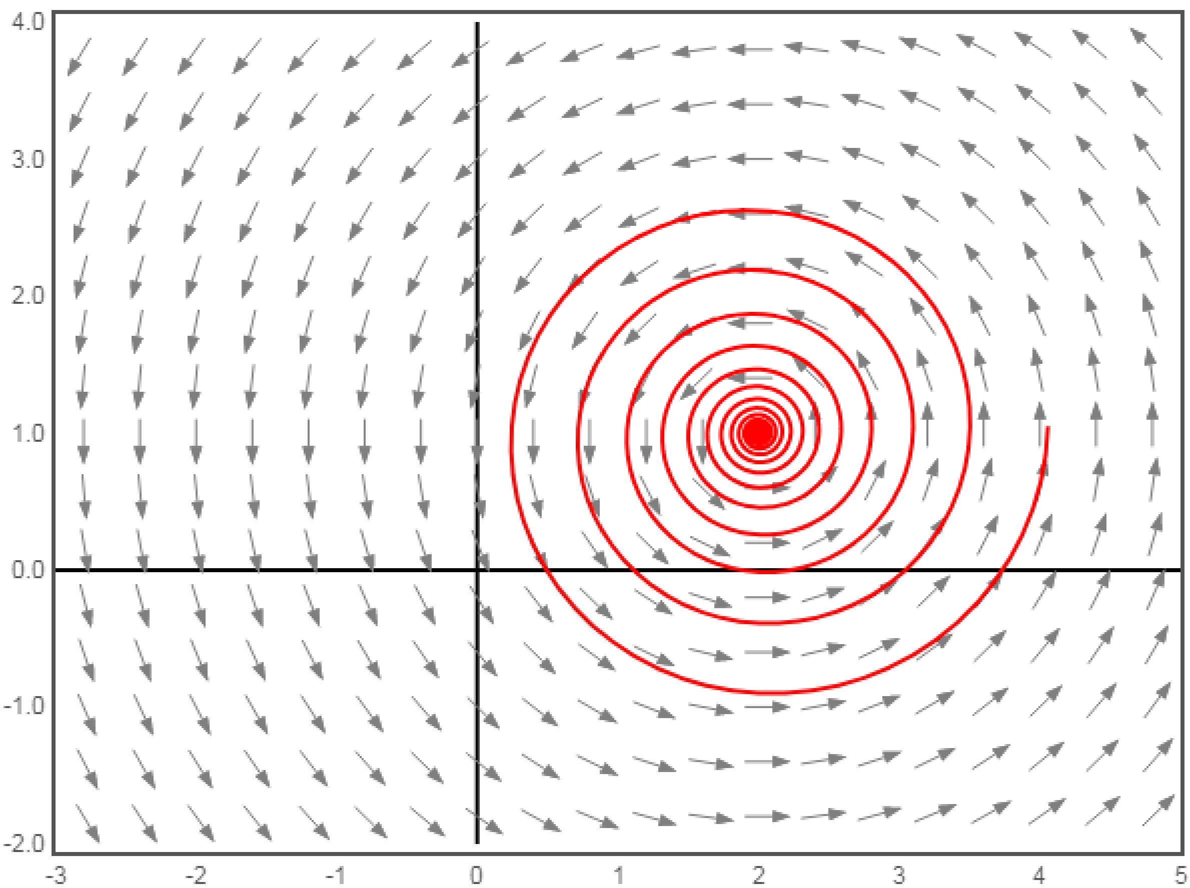
$f(x_0, y_0)$ and $g(x_0, y_0)$

$f(x_0, y_0) = -(y_0 - 1)$ and $g(x_0, y_0) = x_0 - 2$

$f(2, 1) = -(1 - 1)$ and $g(2, 1) = 2 - 2$

$f(2, 1) = 0$ and $g(2, 1) = 0$

1.2.1.4 Graphical visualization



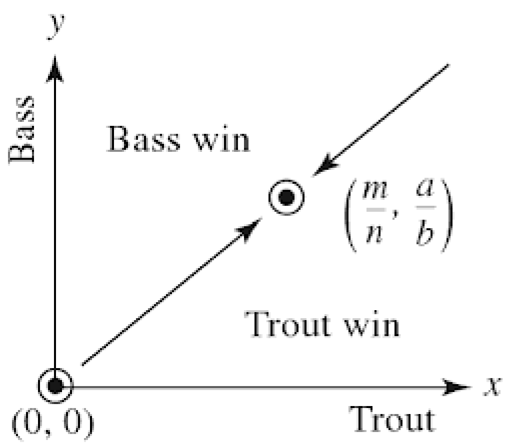
The above graph was reproduced using a slope field tool located at: https://www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html

The above graph was obtained by selecting the Euler's method, with $\Delta t = 0.1$ and a starting point $x_1 = 2.01, y_1 = 1.01$.

From the above graph, we can easily conclude that the rest point ins **unstable**; that is, due to the trajectory does not approach the the point $(x_0 = 2, y_0 = 1)$ as $t \to \infty^+$.

1.3 Exercise #7 Page 536

Show that the two trajectories loading to $(m/n, a/b)$ shown in the following Figure are unique.



1.3.1 a.

From system (12.6) derive the following equation:

$$\frac{dy}{dx} = \frac{(m - nx)y}{(a - by)x}$$

1.3.1.1 Solution

From (12.6) we have as follows:

$\frac{dx}{dt} = (a - by)x$ and $\frac{dy}{dt} = (m - nx)y$.

This implies as follows:

$dt = \frac{dx}{(a-by)x}$ and $dt = \frac{dy}{(m-nx)y}$

From there, we have as follows:

$dt = dt$

$\frac{dx}{(a-by)x} = \frac{dy}{(m-nx)y}$

$\therefore \frac{dy}{dx} = \frac{(m-nx)y}{(a-by)x}$

1.3.2 b.

Separate variables,integrate and exponentiate to obtain:

$$y^a e^{-by} = K x^m e^{-nx}$$

where K is a constant of integration.

1.3.2.1 Solution

From

$$\frac{dy}{dx} = \frac{(m-nx)y}{(a-by)x}$$

We have as follows:

$(a-by)\frac{dy}{y} = (m-nx)\frac{dx}{x}$

$\int(a-by)\frac{dy}{y} = \int(m-nx)\frac{dx}{x}$

$\int \frac{a}{y}dy - \int bdy = \int \frac{m}{x}dx - \int ndx$

$a\, ln|y| + K_1 - by + K_2 = m\, ln|x| + K_3 - nx + K_4$

$e^{ln|y^a| - by + K_{12}} = e^{ln|x^m| - nx + K_{34}}$

$e^{ln|y^a|}e^{-by}e^{K_{12}} = e^{ln|x^m|}e^{-nx}e^{K_{34}}$

$y^a e^{-by} = x^m e^{-nx}\frac{e^{K_{34}}}{e^{K_{12}}}$

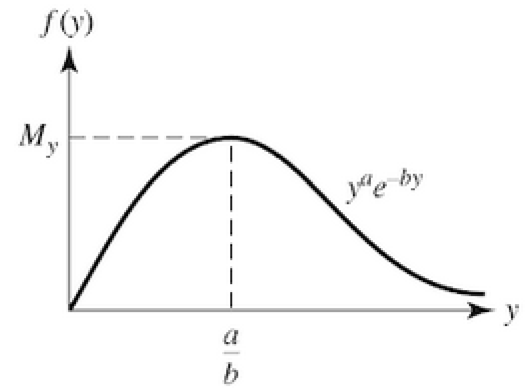
Let $K = \frac{e^{K_{34}}}{e^{K_{12}}}$

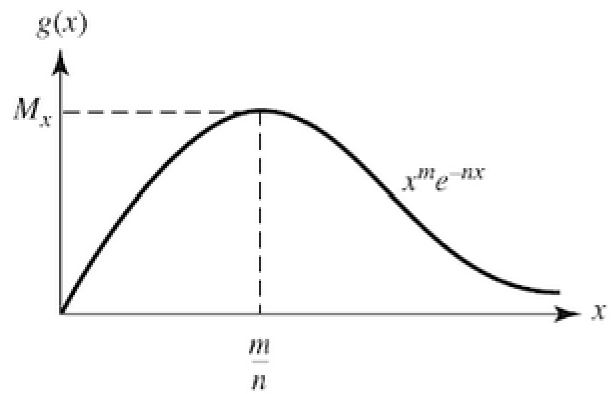
$\therefore y^a e^{-by} = K\, x^m e^{-nx}$

1.3.3 c.

Let $f(y) = y^a/e^{by}$ and $g(x) = x^m/e^{nx}$. Show that $f(y)$ has a unique maximum of $M_y = (a/eb)^a$ when $y = a/b$ as shown in the following figure. Similarly, show that $g(x)$ has unique maximum $M_x = (m/en)^m$ when $x = m/n$, also shown in the following figure.

**Please note that the text book has a typo in the problem description in which indicate "unique maximum $M_x = (x/en)^m$" but it should be "unique maximum $M_x = (m/en)^m$".





1.3.3.1 Solution

In order to find our critical points, we need to find our first derivatives.

1.3.3.1.1 Find first derivatives

$f(y) = y^a/e^{by}$ implies that the first derivative is:

$y^l = ay^{a-1}e^{-}by - y^a e^{-by}$

$g(x) = x^m/e^{nx}$ implies that the first derivative is:

$g^l = mx^{m-1}e^{m-1} - nx^m e^{-nx}$

1.3.3.2 Find critical points

From the above, we can do that by performing:

$y^l = 0$ and $g^l = 0$

Let's resolve for $y^l = 0$

$$0 = \left[\frac{a}{y} - b\right] y^a e^{-by}$$

From the above the only part can could become zero is as follows:

$0 = \left[\frac{a}{y} - b\right]$

Resulting in

$y = \frac{a}{b}$

Let's resolve for $g^l = 0$

$$0 = \left[\frac{m}{x} - n\right] x^m e^{-nx}$$

From the above the only part can could become zero is as follows:

$0 = \left[\frac{m}{x} - n\right]$

Resulting in

$x = \frac{m}{n}$

Hence, we have our critical point $(x = \frac{m}{n}, y = \frac{a}{b})$

In order to know if this critical point is a maximum value, we need to evaluate it in the second derivative of the given functions.

$$f^{ll} = a^2 y^{a-2} e^{-by} + b^2 y^a e^{-by} - a y^{a-2} e^{-by} - 2ab y^{a-1} e^{-by}$$

By evaluating $y = \frac{a}{b}$ we obtain a negative value, indicating that this point is a maximum value. This evaluation process is rather complicated to write it down and it has been completed online by using https://www.wolframalpha.com.

Now, by evaluating $y = \frac{a}{b}$ on the original function, we have as follows:

$f(\frac{a}{b}) = e^{-a}\left(\frac{a}{b}\right)^a = \left(\frac{a}{eb}\right)^a$

$$g^{ll} = m^2 x^{m-2} e^{-nx} + n^2 x^m e^{-nx} - m x^{m-2} e^{-nx} - 2mn x^{m-1} e^{-nx}$$

By evaluating $x = \frac{m}{n}$ we obtain a negative value, indicating that this point is a maximum value. This evaluation process is rather complicated to write it down and it has been completed online by using https://www.wolframalpha.com.

Now, by evaluating $x = \frac{m}{n}$ on the original function, we have as follows:

$g(\frac{m}{n}) = e^{-m}\left(\frac{m}{n}\right)^m = \left(\frac{m}{en}\right)^m$

1.3.4 d.

Consider what happens as (x, y) approaches $(m/n, a/b)$. Take limits in part (b) as $x \to m/n$ and $y \to a/b$ to show that

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = K$$

or $M_y/M_x = K$. Thus, any solution trajectory that approaches $(m/n, a/b)$ must satisfy

$$\frac{y^a}{e^{by}} = \left(\frac{M_y}{M_x}\right)\left(\frac{x^m}{e^{nx}}\right)$$

1.3.4.1 Solution

In this case, I will proceed to calculate the limit as follows:

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \left[\left(\frac{(a/b)^a}{e^{b(a/b)}}\right)\left(\frac{e^{n(m/n)}}{(m/n)^m}\right)\right]$$

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \left(\frac{(a/b)^a}{e^a}\right)\left(\frac{e^m}{(m/n)^m}\right)$$

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \left(\frac{a/b}{e}\right)^a\left(\frac{e}{m/n}\right)^m$$

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \left(\frac{a}{eb}\right)^a\left(\frac{en}{m}\right)^m$$

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \left(\frac{a}{eb}\right)^a\left(\frac{m}{en}\right)^{-m}$$

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \frac{\left(\frac{a}{eb}\right)^a}{\left(\frac{m}{en}\right)^m}$$

Since $M_y = (a/eb)^a$ and $M_x = (m/en)^m$, we have as follows:

$$lim_{\substack{y\to a/b \\ x\to m/n}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = \frac{M_y}{M_x}$$

Hence

$$K = \frac{M_y}{M_x}$$

Now, from (b), we had

$$y^a e^{-by} = Kx^m e^{-nx}$$

or

$$\frac{y^a}{e^{by}} = K\frac{x^m}{e^{nx}}$$

We can replace K as follows:

$$\frac{y^a}{e^{by}} = \frac{M_y}{M_x} \cdot \frac{x^m}{e^{nx}}$$

1.4 Exercise #2 Page 576

Consider a company that allows back ordering. That is, the company notifies customers that a temporary stock-out exist and that their order will be filled out shortly. What conditions might argue for such a policy? What effect does such a policy have on outage costs? Should costs be assigned to stock-outs? Why? How would you make such an assignment? What assumptions are implied by the model in the figure 13.7? Suppose a "loss of goodwill cost" of w dollars per unit per day is assigned to each stock-out. Compute the optimal order quantity Q^* and interpret your model.

1.4.1 Solution

1.4.1.1 What conditions might argue for such a policy?

From my perspective, some conditions could be:

- Custom made or specific products in which demand is not very high.
- Products that require special handling and transportation.
- New technologies not fully developed.
- Products that have high demand with low fulfillment rate.

1.4.1.2 What effect does such a policy have on outage costs?

Definitely one of the main costs related to such a policy will be storage costs.

1.4.1.3 Should costs be assigned to stock-outs? and Why?

From my perspective, it all depends on the case. For example if it is a product that moves constant, then there should be a cost associated with it, since a stock-out will represent lost revenue. But, on the other hand, if it is a slow moving product in which not much demand is seeing; then, there should be an analysis to see if rush shipping costs could be attached.

1.4.1.4 How would you make such an assignment?

It will all depends on the product demand's over time; if it is better to have the product stored for short periods of time vs long periods of time.

1.4.1.5 What assumptions are implied by the model in Figure 13.7?

The assumptions from the model figure are:

- There are inventory cycles of an order quantity **q** consumed in **t** days that permits stock-outs.

1.4.1.6 Suppose a "loss of goodwill cost" of w dollars per unit per day is assigned to each stock-out. Compute the optimal order quantity Q^* and interpret your model.

Model Formulation

s = storage cost per day

C = cost per cycle

c = average daily cost

d = delivery cost per delivery

r = demand rate of product per day

w = loss of goodwill cost

q+ = quantity of product available

q- = quantity of product as stock-out

aq+ = average of daily inventory

Q* = optimal order quantity of product

t+ = time in days in which the product is available right away t- = time in days in order to get stock-outs

T = time in days

From above, we have as follows:

Q* = q- + q+

T = t+ + t-

aq+ = q+ / 2

C = d + s * t+ * aq+

and diving by t+, we obtain the average daily cost.

$c = \frac{d}{t+} + s \cdot aq+$

1.5 Exercise #2 Page 584

Find the local minimum value value of the function:

$$f(x, y) = 3x^2 + 6xy + 7y^2 - 2x + 4y$$

1.5.1 Solution

For this, we have to find as follows:

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Now,

$\frac{\partial f}{\partial x} = 6x + 6y - 2$

$\frac{\partial f}{\partial y} = 6x + 14y + 4$

Since we have to solve

$\begin{cases} 0 = 6x + 6y - 2 \\ 0 = 6x + 14y + 4 \end{cases}$

We have $y = -\frac{3}{4}$, $x = \frac{13}{12}$

Just to verify if the above point is truly a minimum; lets evaluate the second derivative process.

$\frac{\partial 6x+6y-2}{\partial x} = 6$

$\frac{\partial 6x+6y-2}{\partial y} = 6$

$\frac{\partial 6x+14y+4}{\partial x} = 6$

$\frac{\partial 6x+14y+4}{\partial y} = 14$

Since all resulting values are positive, we can conclude as follows:

Our minimum value will be $(x = \frac{13}{12}, y = -\frac{3}{4})$

```
plotf <- function (x, y) {
   return (3*x^2 + 6*x*y + 7*y^2 - 2*x + 4*y)
}

x <- seq(-20, 20, length= 50)
y <- x
z <- outer(x, y, plotf)
z[is.na(z)] <- 1

require(lattice)
wireframe(z, drape=T,
          col.regions=rainbow(100),
          xlab = 'x', ylab = 'y', zlab = 'f(x,y)')
```