1 Designing the Kalman Filter

1.1 Problem Definition

Goal: Estimate x and v, where x is the position and v is the velocity, given measurements $z = x + \epsilon_R$, $\epsilon_R \sim \mathcal{N}(0, \sigma_R^2)$.

State vector:

$$\mathbf{x}_k = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$$

Measurement:

$$\mathbf{z}_k = \begin{bmatrix} z_k \end{bmatrix}$$

1.2 Time Evolution

$$x_{k+1} = x_k + v_k \cdot \delta t + \frac{1}{2} a \Delta t^2 \tag{1}$$

$$v_{k+1} = v_k + a\Delta t \tag{2}$$

If we did not include acceleration, then the speed would be fixed to a constant, and we would not allow our object to change speed. Let us now write the time evolution in vector form:

$$\mathbf{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} a$$

and now we define $\mathbf{F}$

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

and $\mathbf{G}$

$$\mathbf{G} = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$$

So that the equations of motion can be written as

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k \tag{3}$$

we assume that $a \sim \mathcal{N}(0, \sigma_a^2)$, i.e., that the acceleration is normally distributed. The speed can then change by some noise which corresponds to the acceleration.

1.3 Incorporate Measurements

Our measurement, $z = x + \epsilon_R$, $\epsilon_R \sim \mathcal{N}(0, \sigma_R^2)$, in vector form reads

$$\mathbf{z}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} mathbf x_k + \epsilon_k$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

1.4 Prediction Step

In the prediction step, the goal is to propagate the state $\mathbf{x}_k \to \mathbf{x}_{k+1}$.

If $\mathbf{x}_k$ is a random variable defined as follows:

$$\mathbf{x}_k \sim \mathcal{N}(\hat{\mathbf{x}}_k, \mathbf{P}_k) \tag{4}$$

Then the Kalman predict equations are

$$\hat{\mathbf{x}}_{k+1} = \mathbf{F}\hat{\mathbf{x}}_k \tag{5}$$

$$\mathbf{P}_{k+1} = \mathbf{F}\mathbf{P}_k\mathbf{F}^T + \mathbf{G}\sigma_a^2\mathbf{G}^T \tag{6}$$

1.5 Measurement Step

In the measurement step, we incorporate knowledge of z_k into our estimate of the state vector, $\mathbf{x}_k$. The update equations are as follows:

First, we compute the innovation and the innovation covariance

$$\mathbf{y} = \mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_k \tag{7}$$

$$\mathbf{S}_k = \mathbf{H}\mathbf{P}_k\mathbf{H}^T + \mathbf{R} \tag{8}$$

where $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} \sigma_R^2 \end{bmatrix}$$

We also need to compute the Kalman gain

$$\mathbf{K} = \mathbf{P}\mathbf{H}^T\mathbf{S}_k^{-1} \tag{9}$$

And finally, we update. The new mean, incorporating the measurement, is given by

$$\hat{\mathbf{x}}_{|z} = \hat{\mathbf{x}}_k + \mathbf{K}\mathbf{y} \tag{10}$$

And the covariance matrix given the measurement is

$$\mathbf{P}_{|z} = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_k \tag{11}$$