Efficient Local Search with Conflict Minimization: A Case Study of the n-Queens Problem

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***Abstract*— Backtracking search is frequently applied to solve a constraint-based search problem, but it often suffers from exponential growth of computing time. We present an alternative to backtracking search: local search with conflict minimization. We have applied this general search framework to study a benchmark constraint-based search problem, the n-queens problem. An efficient local search algorithm for the n-queens problem was implemented. This algorithm, running in *linear time*, does not backtrack. It is capable of finding a solution for extremely large size n-queens problems. For example, on a workstation computer, it can find a solution for 3 000 000 queens in less than 55 s.**

***Index Terms*— Conflict minimization, local search, n-queens problem, nonbacktracking search**

I. Introduction

A constraint-based search problem has three components: variables, values, and constraints. The goal is to find an assignment of values to variables such that all constraints are satisfied. Backtracking search is generally applied to solve a constraint-based search problem. Many heuristics have been developed to improve the efficiency of backtracking search, for example, search rearrangement [4], [20], [21]; backmarking and backjump [7], [8]; and lookahead and forward checking [15]. Suffering from exponential growth of computing time, backtracking search techniques are not able to solve a large-scale constraint-based search problem.

We present an alternative to backtracking search: local search with conflict minimization. We describe a case study of this general framework on a benchmark constraint-based search problem, i.e., the n-queens problem. The n-queens problem is to place n indistinguishable objects on an $n \times n$ grid so that no two objects are placed on the same row, the same column, or the same diagonal.

The purpose of this study is twofold. First, the n-queens problem is a typical problem of constraint satisfaction. A general search algorithm to the n-queens problem is useful in solving other constraint satisfaction problems [14]. Second, the n-queens problem is itself a model of *the maximal coverage problem*. A solution to the n-queens problem guarantees that each object can be accessed from any one of its eight neighboring directions (two vertical, two horizontal, and four diagonal directions) without a conflict (see Fig. 1). Our general search algorithm for the n-queens problem has made many applications possible [14]. Some direct applications of this result include very large scale integration (VLSI) testing, air traffic control, modern communication systems, data/message routing in a multiprocessor computer, load balancing in a multiprocessor computer, computer task scheduling, computer resource management, optical parallel processing, and data compression.

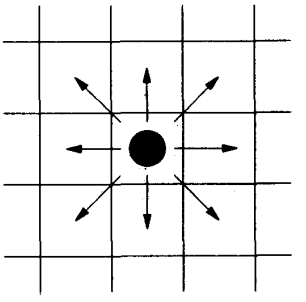

Fig. 1. Eight neighboring directions for an object.

The rest of the paper is organized as follows. In Section II, we briefly review prior work. A linear time algorithm to solve the n-queens problem is presented in Section III. A mathematical analysis of the algorithm is given in Section IV. Performance results are shown in Section V. We describe a representative application of the n-queens problem in Section VI. Section VII concludes this paper.

II. Prior Work

Solutions exist for the n-queens problem with n greater than or equal to 4 [2]. Empirical observations of smaller-size problems show that the number of solutions increases exponentially with increasing n.

Numerous closed-form analytical solutions to the n-queens problem have been published since 1918 [1]–[3], [6], [16], [22]. They basically compose a solution to the n-queens problem by providing an explicit formula for a queen placement or by patching together solutions to the smaller-size problems. As pointed out by Ahrens [2], such analytical solutions have a limitation in that they generate only a small number of solutions from a restricted subset of solutions. In addition, a problem-solving strategy derived from an analytical solution

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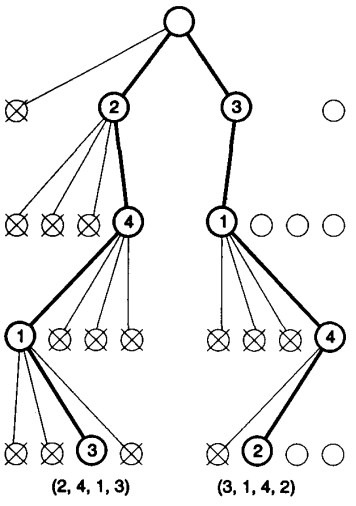


Fig. 2. A partial search tree for the 4-queens problem.

to the n-queens problem cannot be used for a general search problem [11], [17]. This is not the case with a search-based algorithm.

A search-based algorithm overcomes the above difficulties and limitations. A backtracking search systematically generates all possible solutions for the n-queens problem [4], [7], [8], [20], [21]. Here we start with the first column and first row, and assign to it the first queen. Then we select the second queen and assign it to the second column such that no queens attack. We continue with the placement of the next queen to the next column until all queens are placed. If at any point we run out of row positions for a queen to be placed, we simply go back one step (backtrack), choose another conflict-free row position on the previous column, and continue the process. If we are able to assign row positions to all the queens, then we have found a solution. Otherwise, there are no solutions.

A backtracking search can be represented in a (search) tree representation. As shown in Fig. 2, for a 4-queens problem, each quadruple of queens forms a possible path from the root to a leaf node in the search tree. The search tree consists of only 2 leaf nodes (two solutions). A partial search tree in this case is shown in Fig. 2.

In the worst case, a backtracking search is exponential in time. In an empirical study, Stone and Stone [26] used various search heuristics and solved the n-queens problem for n up to 96. Recently, Kalé [18] gave a new backtracking heuristic for the n-queens problem. His heuristic starts placing queens from the edge inward for the middle-third rows, and from the middle position outward for the first-third and the last-third rows. The row placement is interleaved with column placement. The order of rows and columns is determined by the number of free positions. Measurements of this heuristic show that it is capable of finding a solution in time proportional to $O(n^2)$. Kalé's method is capable of finding all solutions to the n-queens problem. All solutions may be feasible for smaller n-queens problems, but the number of solutions becomes excessive for larger n. The capability of finding all solutions

```
1.  procedure Queen_Search (queen : array [1..n] of integer)
2.  var
3.      k : integer;
4.  begin
5.      k := Initial_Search(queen);
6.      Final_Search(k,queen);
7.  end;
```

Fig. 3. A linear time n-queens search algorithm.

thus diminishes with increasing n. In practice, backtracking approaches provide a very limited class of solutions to the large-size n-queens problem. Successive solutions produced by a backtracking search are very similar. For large n, it is difficult for a backtracking search to find solutions that are significantly distinct in the solution space.

A backtracking search is not able to solve a large-size n-queens problem. It is desirable to investigate some alternative, efficient search approaches for the n-queens problem. Recently, Sosič and Gu [23]–[25] have given several probabilistic local search algorithms that are based on a conflict minimization heuristic. We have compared our local search algorithms with a backtracking search approach and found them significantly faster [9]. The algorithms run in polynomial time or linear time, and do not backtrack at all. They are capable of solving a large-size n-queens problem. For example, on a workstation computer, our algorithm can find a solution for 3 000 000 queens in less than 55 s [25].

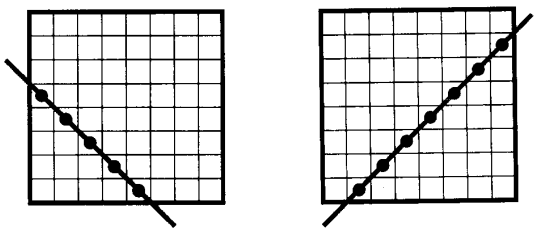
Based on the same idea of using a local search and a conflict minimization heuristic, Minton *et al.* gave a similar algorithm to solve the n-queens problem [19]. In the algorithm, queens are successively placed on columns such that each new queen is attacked by the minimum number of queens that are already placed. Once all queens are placed, conflicts are resolved by moving conflicting queens one queen at a time.

III. A Linear Time Search Algorithm for the n-Queens Problem

Two types of collisions may occur among queens, collisions on rows or columns, and collisions on diagonals. Collisions on rows and columns can simply be avoided by a permutation operation.

Let $queen_i$ denote the row number of the queen in the ith column. For the n-queens problem, there are n queens with row positions $queens_1, \cdots$ queen$_n$. Since each column has exactly one queen, no two queens will attack each other on the same column. If numbers $queen_1, \cdots, queen_n$ form a permutation of integers $1, \cdots, n$, then each row is occupied by exactly one queen. A permutation of integers $1, \cdots, n$ thus denotes a placement of queens where no two queens attack each other on the same row or the same column. The problem then remains to resolve any collisions among queens that may occur on diagonal lines.

A linear time search algorithm for the n-queens problem is shown in Fig. 3. It consists of an initial search phase that starts during problem generation and a final search phase that removes conflicts. During the initial search (function *Initial_Search*), an initial permutation of the row positions of the queens is generated. This permutation produces a small number of collisions among queens. These collisions are



(a) (b)

Fig. 4. The characterization of diagonals. (a) Constant sum of indexes on a negative slope diagonal. (b) Constant difference of indexes on positive slope diagonal.

```
1.  function Initial_Search (queen : array [1..n] of integer)
2.  var
3.      i,j,m : integer;
4.  begin
5.      for i := 1 to n do queen[i] := i;
6.      j := 1;
                    (* place queens without collisions *)
7.      for i := 1 to 3.08 * n do begin
8.          m := Random(j,n);
9.          Swap(j,m);
10.         if Partial_Collisions(j) = 0 then j := j + 1;
11.         else Swap(j,m);
12.     end;
                    (* place queens with possible collisions *)
13.     for i := j to n do begin
14.         m := Random(i,n);
15.         Swap(i,m);
16.     end;
                    (* return the number of queens with possible collisions *)
17.     Initial_Search := n - j + 1;
18. end;
```

Fig. 5. The initial search.

removed by the final search procedure. During the final search (procedure *Final_Search*), two queens are chosen for conflict minimization. If a swap of two queens reduces the number of collisions, then two queens are swapped. Otherwise, no action is taken. The final search is repeated until all collisions are eliminated, that is, a solution is found. The details of the initial and final search phases are described in the subsequent sections.

A. The Initial Search

The initial search produces a permutation of numbers $1, \cdots, n$ in variables $queen_1, \cdots queen_n$. This initial placement of row positions generally produces collisions between queens on the diagonal lines. The number of collisions on a diagonal line is one less than the number of queens on the line. Exceptions are empty diagonal lines, where there are no collisions. The sum of collisions on all diagonal lines is the total number of collisions between queens.

The number of collisions on a diagonal line can be computed in constant time using a characterization of diagonal lines. Let i be a *row index*, and let j be a *column index*. Then $i + j$ is constant on any diagonal line with a negative slope, and $i - j$ is constant on any diagonal line with a positive slope (see Fig. 4). For $queen_i$, its diagonal lines are calculated as $i+queen_i$ and $i-queen_i$. For the n-queens problem, an array of size $2n-1$ keeps tracking the number of queens on $2n-1$ negative diagonal lines. Similarly, another array of size $2n - 1$ keeps tracking the number of queens on $2n - 1$ positive diagonal lines. Using the two arrays, it takes constant time to find the number of collisions on a diagonal line.

One way to produce an initial placement is to generate a random permutation of n numbers. It was observed that a random permutation of n numbers tends to generate $0.5285n$ collisions among queens as n approaches infinity [23], [24]. A random permutation of 1 000 000 queens would generate approximately 528 500 collisions. Since this number of collisions is very high, the algorithm performance can be improved significantly if collisions in the initial permutation are minimized.

The initial minimization of collisions is done by function *Initial_Search* (see Fig. 5). There are some auxiliary functions and procedures. Function $Random(x, n)$ returns a random integer between x and n. Procedure $Swap(i, j)$ performs a swap of $queen_i$ and $queen_j$ that were generated during initialization. Function $Partial\_Collisions(i)$ returns the number of collisions on diagonals to the left that pass through position $(i, queen[i])$. This function gives the number of collisions with queens in columns smaller than i. Function $Total\_Collisions(i)$ returns the total number of collisions on diagonals that pass through position $(i, queen[i])$.

Queens are placed on successive columns from left to right. The position for each new queen is *randomly* generated from rows that are not occupied by queens in columns to the left. If the new queen is attacked by the queen in a column to the left, then a new random position is generated. New random positions are generated until a conflict-free place is found for the queen. Then the algorithm places the next queen in the next column.

After the random number generator is called a specified number of times, the remaining queens are placed randomly on empty rows in columns on the right, regardless of conflicts on diagonal lines. The number of queens on the right with possible collisions is returned as the value of function *Initial_Search*.

For a problem size n, $3.08n$ calls to the random number generator are performed, not counting the placement of queens with collisions on the right. Each call to the random number generator represents an attempt to place a queen. Number 3.08 has been chosen based on a mathematical analysis that is presented in Section IV. Experimental measurements have confirmed such a decision.

Since $3.08n$ steps are performed, the process of initial search requires a linear time. Experimental results and the analysis in Section IV show that the number of queens with a possible conflict produced by the initial search can be treated as a constant or as almost a constant for all n.

B. The Final Search

During the initial search, some queens produce collisions on diagonal lines. Assume there are k such queens. A simple local search technique is used during the final search to resolve the positions of these k queens.

Conflict Minimization: A conflict minimization heuristic navigates the search activity (see Fig. 6). Two queens are chosen for a possible swap. The first queen is one of the queens with collisions; the second queen is chosen at *random*. If the swap of row positions of a pair of queens produces no colli-

TABLE I
THE NUMBER OF INITIAL PERMUTATIONS TO FIND 100 SOLUTIONS

Number of queens n	200	300	400	500	600	700	800	900
Number of initial permutations	168	127	106	105	101	100	102	101

```
1.  procedure Final_Search ( k : integer, queen : array [1..n] of integer)
2.  var
3.      i,j : integer;
4.      b : boolean;
5.  begin
6.      for i := n - k + 1 to n do begin
7.          if Total_Collisions(i) > 0 then
8.              repeat
9.                  j = Random(1,n);
10.                 Swap(i,j);
11.                 b := (Total_Collisions(i) > 0) or (Total_Collisions(j) > 0);
12.                 if b then Swap(i,j);
13.             until not b;
14.     end;
15. end;
```

Fig. 6. The final search.

sions, then the swap is performed. This conflict minimization heuristic is applied until there are no collisions left, that is, a solution is found.

Termination Criteria: If no solution is found after a number of search steps, a new initial permutation is generated and a new search process is started. We have limited the number of search steps to 7000. This number has been chosen based on mathematical analysis and empirical observations (see Section IV-B). We have executed the algorithm 100 times for each n between 4 and 1000. Table I shows how many initial permutations are needed to obtain 100 solutions for a given n. For $n > 400$, the algorithm almost always finds a solutions from only one initial permutation.

For $n < 200$, the random method of choosing a pair of queens to swap does not perform very well, so a different approach has been chosen. Each of the k queens with possible collisions is systematically tested with all other queens for a possible swap. If all possible pairs are tested without a successful swap, a new initial permutation is generated, and the search is repeated. For n close to 200, this method requires approximately four initial permutations to find a solution.

The running time of the algorithm can be estimated as follows. The initial search can be done in linear time, as shown in Section III-A and Section IV. Analysis in Section IV shows that the number of steps in the final search is constant and independent of n. This has been verified by experimental results. Taking together the initial and final search phases, it follows that the algorithm has a linear running time. This is confirmed by experimental data presented in Section V. Using different random generator seeds, the algorithm produces different solutions, but its execution time remains fairly stable. Very robust behavior is another advantage of this probabilistic local search algorithm.

IV. AN ANALYSIS OF THE ALGORITHM

This section presents a mathematical analysis of the algorithm. The analysis gives an estimated number of computing steps for the initial search phase and the final search phase, which shows a linear time complexity of the algorithm.

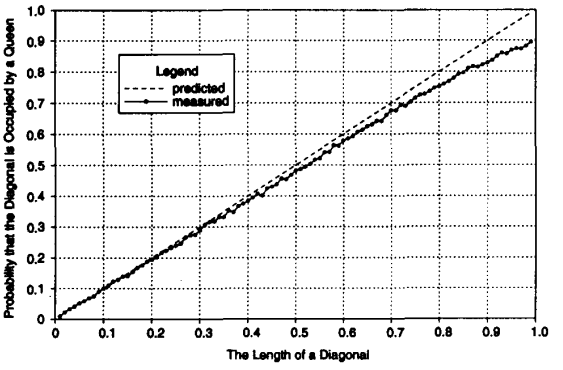


Fig. 7. Predicted and measured probabilities that there is a queen on a diagonal of length $\frac{k}{n}$. The average of 10 000 samples for $n = 10\,000$.

Following a common mathematical technique, we first transform the problem from a discrete domain to a continuous domain. The original chessboard is mapped to a unit square. A position (i, j) on the original $n \times n$ board is mapped to a point $\left(\frac{i}{n}, \frac{j}{n}\right)$ in the unit square. This simplifies the analysis of the problem as n approaches infinity without affecting the results.

We assume that the queens are uniformly distributed on the board such that no two queens attack each other. In each column, there is exactly one queen. Thus, the probability that a given position is occupied by a queen is $\frac{1}{n}$. If a diagonal spans over k positions on the chessboard, we assume that the diagonal is occupied by a queen with a probability of $\frac{k}{n}$. We have compared this predicted probability to measured probabilities for n equal to 10 000. The average values of 10 000 measurements are shown in Fig. 7. It can be seen that the predicted probabilities are close to the measured probabilities. When we map the board to a unit square, a diagonal line from point (x_0, y_0) to point (x_1, y_1) covers $|x_1 - x_0|$ positions, and it is occupied by a queen with probability $|x_1 - x_0|$.

Let a and b be two diagonal lines that cross each other. Let $p(a)$ represent the probability that there is a queen on diagonal line a, and let $p(b)$ represent the probability that there is a queen on diagonal line b. The combined probability $p(a + b)$ that a common position is attacked from diagonal a or diagonal b is computed as the probability of the sum a and b:

$$p(a + b) = p(a) + (b) - p(a)p(b). \quad (1)$$

This is a well-known formula for the probability of the sum.

A. An Analysis of the Initial Search

During the initial search, all queens are placed in successive columns, starting with the column on the left. We seek an estimated number of steps during the initial search.

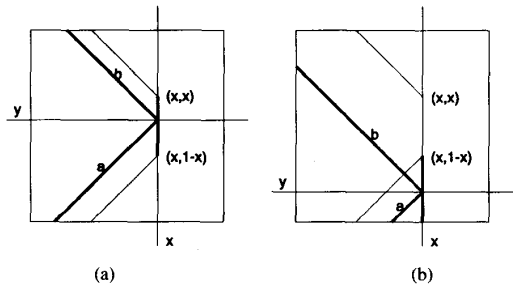


Fig. 8. Left diagonals through a column between 0 and 0.5. (a) $0 < x < 0.5, x < y < 1 - x$. (b) $0 < x < 0.5, y < x$.

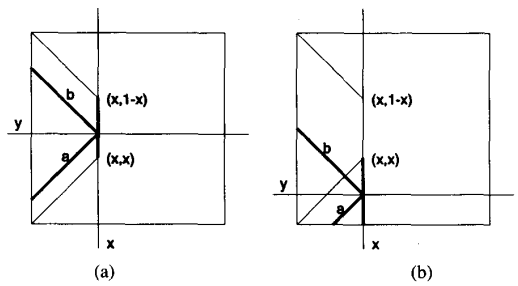


Fig. 9. Left diagonals through a column between 0.5 and 1.0. (a) $0.5 < x < 1, 1 - x < y < x$. (b) $0.5 < x < 1, y < 1 - x$.

Let $f(x)$ denote the number of steps for placing the first x queens. The value of $f(1)$ is the total number of steps during the initial search. Let $p(x)$ be the probability that a random queen on column x is attacked by some previously placed queen to the left. The value of $f(x)$ can be computed from $p(x)$ as follows:

$$f(x) = \int_0^x \frac{1}{1 - p(x)} dx. \tag{2}$$

We split the calculation of $p(x)$ into two parts: the interval with $0 < x < 0.5$ and the interval with $0.5 < x < 1$. The probabilities of a collision in these two intervals are denoted as $p_1(x) = p(x; x < 0.5)$ and $p_2(x) = p(x; x > 0.5)$, respectively. The algorithm prevents the construction of row and column collisions, so we take into account only diagonal collisions.

First, we calculate $p_1(x)$. Let x denote a column between 0 and 0.5, and let y be a point on this column. Let a and b be two left diagonal lines that pass through point (x, y). It follows from (1) that point (x, y) is occupied with the following probability:

$$p(x, y) = p(a + b) = p(a) + p(b) - p(a)p(b).$$

We separate the values of y into three intervals:

1) $x < y < 1 - x$,
2) $y < x$, and
3) $y > 1 - x$.

Case 1: If $x < y < 1 - x$, then a and b have equal length (see Fig. 9(a)). In this case, $p(a) = x$ and $p(b) = x$. It follows that point (x, y) is attacked by a queen with the following probability:

$$p(x, y) = p(a + b) = x + x - x^2 = 2x - x^2.$$

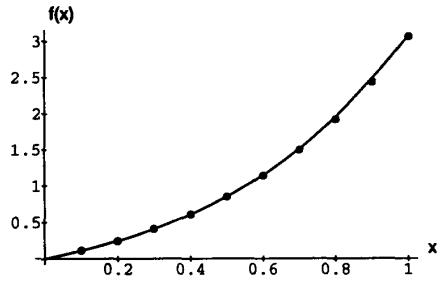


Fig. 10. The initial search: Predicted performance (curve) and measured performance (points).

Case 2: If $y < x$, then $p(a) = y$ and $p(b) = x$ (see Fig. 9(b)). Point (x, y) is attacked by a queen with the following probability:

$$p(x, y) = p(a + b) = y + x - yx.$$

Case 3: If $y > 1 - x$, the probability is equivalent to the case of $y < x$.

To compute the probability $p_1(x)$ that a random position in column x is attacked by a previously placed queen, we need to find an average over all y between 0 and 1. This average is computed by summing probabilities over all y as follows:

$$p_1(x) = 2 \int_0^x y + x - yx \, dy + \int_x^{1-x} 2x - x^2 dy = 2x - 2x^2 + x^3. \tag{3}$$

To calculate $p_2(x)$, let x denote a column between 0.5 and 1, and let y be a point on this column. Let a and b be two left diagonal lines that pass through point (x, y). We separate the values of y in three intervals as follows:

(a) $1 - x < y < x$, (b) $y < 1 - x$ and (c) $y > x$.

Case 1: If $1 - x < y < x$, then $p(a) = y$ and $p(b) = 1 - y$ (see Fig. 8(a)). Point (x, y) is attacked by a queen with the following probability:

$$p(x, y) = p(a + b) = 1 - y + y - (1 - y)y = 1 - y + y^2.$$

Case 2: If $y < 1 - x$, then $p(a) = x$ and $p(b) = y$ (see Fig. 8(b)). The probability $p(x, y)$ is as follows:

$$p(x, y) = p(a + b) = x + y - xy.$$

Case 3: If $y > x$, the probability is equivalent to the case of $y < 1 - x$.

The probability $p_2(x)$ that a random position in column x is attacked by a queen from left is computed by the following integral:

$$p_2(x) = 2 \int_0^{1-x} x + y - xy \, dy + \int_{1-x}^x 1 - y + y^2 dy = \frac{1}{6} + x - \frac{x^3}{3}. \tag{4}$$

Using $p_1(x)$ from (3) and $p_2(x)$ from (4), we can calculate the number of steps $f(x)$ to place x queens (see (2)). If x is less than 0.5, then we have the following formula:

$$f(x) = \int_0^x \frac{1}{1 - p_1(x)} dx.$$

TABLE II
THE NUMBER OF QUEENS, PLACED DURING THE INITIAL SEARCH, USING $3.08n$ STEPS (10 runs)

Number of queens n	100	1000	10 000	100 000	1 000 000
Average	91	979	9969	99 977	999 975
Minimum	84	959	9919	99 964	999 946
Maximum	97	990	9990	99 988	999 987

If x is greater than 0.5, then we have the following formula:

$$f(x) = \int_0^{0.5} \frac{1}{1 - p_1(x)} dx + \int_{0.5}^{x} \frac{1}{1 - p_2(x)} dx.$$

Fig. 10 shows the comparison between our model and experimental measurements of the real algorithm execution. The curve in the figure represents function $f(x)$. The dots represent the measured number of steps to place x queens in 0.1 increments of x. Points are an average of 100 execution runs taken for a problem with $n = 100\,000$.

The number of steps required to place all queens without collisions is equal to $f(1) = 3.08$, which means that for problem size n, the initial search will attempt on average $3.08\,n$ positions before all queens are placed. The initial search takes time that is linear in n.

Although our model corresponds very closely to the real performance of the algorithm, the assumption at the beginning of this section that all queens can be placed randomly on each column is not accurate for the last few queens placed. Since most rows are already occupied, the number of choices for the queens on the right is very restricted. We have measured the number of queens that are placed by the initial search without collisions, using $3.08n$ steps. Table II shows the average number, the minimum number, and the maximum number of queens placed during 10 runs. These experimental results demonstrate that the initial search can place almost all queens without collisions, even for very large values of n. A similar observation was made for the method described in [19].

B. An Analysis of the Final Search

During one iteration of the final search, an attacked queen on the right is swapped with a random queen, so that neither of these queens is under attack after the swap. We want to calculate the number of attempts A until a successful swap is performed. This number multiplied by the number of queens under an attack gives an estimate of the total number of steps during the final search.

We assume that queens are randomly distributed with one queen per column. Let P be the probability that a random position on the board is diagonally attacked by some queen, and let F be the probability that a position in the column on the right is diagonally attacked by some queen. Given P and F, the expected number of attempts A for a successful swap is calculated as follows:

$$A = \frac{1}{(1 - F)(1 - P)}. \tag{5}$$

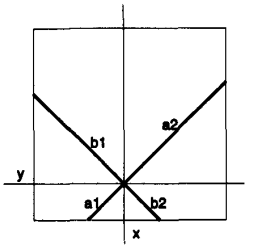


Fig. 11. Diagonals through random point (x, y).

From (4), the value of F, the probability that a position in the column on the right is diagonally attacked by some queen is equal to $p_2(1) = \frac{5}{6}$.

To calculate P, because of the eightfold symmetry of a square, we restrict ourselves to the lower-left region $0 < x < 0.5$ and $0 < y < x$. If $p(x, y)$ is the probability that a random point in this region is diagonally attacked by a queen, the value of P is equal to the following:

$$P = 8 \int_0^{0.5} \int_0^{x} p(x, y) dy dx. \tag{6}$$

Taking a point $p(x, y)$ in that region (see Fig. 11), there are four diagonal segments passing through it: a_1, a_2, b_1, and b_2. The probabilities that $p(x, y)$ is attacked from diagonal segments are as follows:

$$\begin{aligned} p(a_1) &= y, \\ p(a_2) &= 1 - x, \\ p(b_1) &= x, \\ p(b_2) &= y. \end{aligned}$$

Segments a_1 and a_2 are parts of the same diagonal a. Since each diagonal contains only one queen, (1) does not apply in this case. Probability $p(a)$ is a sum of $p(a_1)$ and $p(a_2)$:

$$p(a) = p(a_1) + p(a_2) = 1 - x + y.$$

A similar formula is valid for segments b_1 and b_2 on diagonal b:

$$p(b) = p(b_1) + p(b_2) = x + y.$$

Using (1), probability $p(x, y)$ is computed as $p(a + b)$ as follows:

$$\begin{aligned} p(x, y) = p(a + b) &= p(a) + p(b) - p(a)p(b) \\ &= 1 - x + y + x + y - (1 - x + y)(x + y) \\ &= 1 - x + x^2 + y - y^2. \end{aligned}$$

Substituting $p(x, y)$ in (6), we obtain the value of P as follows:

$$P = 8 \int_0^{0.5} \int_0^{x} 1 - x + x^2 + y - y^2 dy dx = \frac{11}{12}.$$

The number of attempts, A, to successfully swap a queen in the final search can be calculated from (5) as follows:

$$A = \frac{1}{(1 - F)(1 - P)} = \frac{1}{\frac{1}{6}\frac{1}{12}} = 72.$$

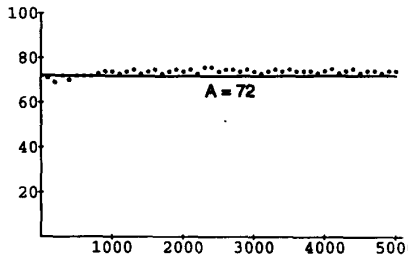


Fig. 12. The final search: Predicted performance (curve) and measured performance (points) (average of 100 runs). x-axis represents the number of queens with collisions, y-axis represents the average number of swap attempts per queen.

We have experimentally measured the number of swap attempts, A, for n equal to 100 000. A large number of experiments have been performed where the number of queens with collisions varied from 100 to 5000. Fig. 12 shows the measured number of swap attempts for each successful swap, giving the average of 100 runs for each value. The measured average value is betwen 69 and 76, which corresponds closely to the predicted value of 72.

Using a swap between a pair of queens, collisions for one queen are resolved in a constant time of approximately 72 steps, regardless of n. This constant time compares to a linear time in the method described in [19], where only one queen is moved in a single step. Assuming that the initial search produces, for any n, a constant number of queens with collisions, the final search takes constant time. In our algorithm, the limit for the number of search steps during the final search has been set to 7000. The actual average number of search steps is smaller. For example, for n equal to 1 000 000, the average number of steps in the final search in 100 runs was measured as 1448.

A mathematical analysis and measured performance of our algorithm demonstrate its linear running time. Most of the time is spent in the initial search, which executes in a linear running time. The final search requires a constant time independent of n.

As is common with probabilistic algorithms [5], our algorithm to solve the n-queens problem does not guarantee worst-case running time for a particular instance of the algorithm execution, but it exhibits excellent performance and a very robust behavior. The anlaysis gives a bound on the number of expected execution steps. Hence, an unsuccessful attempt may be halted, and the algorithm may be restarted from the beginning.

V. Real Execution Time of the Algorithm

The real execution time of our linear search algorithm, which was programmed in C and run on an IBM RS 6000/530 computer, is illustrated in Table III. For each n presented, we made 10 execution runs with different random numbers. These runs show that the algorithm exhibits very stable behavior for large n. Because of memory size limitation, the largest n we were able to run was 3 000 000. Since the algorithm was able to find a solution in less than 0.1 s for $n < 10\,000$, these results are not shown in the table. Approximate time to place one queen for these small values of n was around 15 μs.

TABLE III
The Execution Time of the Linear Time Search Algorithm on an IBM RS 6000/530 Computer
(average of 10 runs; time units in s)

Number of queens n	10^4	10^5	10^6	2×10^6	3×10^6
Time of the first run	0.1	1.1	16.9	35.7	54.8
Time of the second run	0.1	1.1	17.0	35.8	54.8
Time of the third run	0.1	1.1	17.1	35.9	54.6
Time of the fourth run	0.1	1.1	17.0	35.8	54.8
Time of the fifth run	0.1	1.1	17.0	35.8	54.7
Time of the sixth run	0.1	1.1	17.0	35.8	54.6
Time of the seventh run	0.1	1.1	17.0	35.8	54.7
Time of the eighth run	0.1	1.1	17.0	35.8	54.7
Time of the ninth run	0.1	1.1	17.0	35.8	54.7
Time of the tenth run	0.1	1.1	17.0	35.8	54.7
Avg. time to find a solution	0.1	1.1	17.0	35.8	54.7

VI. Applications of the n-Queens Problem

Since each solution to the n-queens problem forms a nonconflict pattern, the n-queens problem has many practical scientific and engineering applications. We give a representative example of its applications.

To achieve high communication bandwidth in a narrowband directional communication system, an array of n transmitters/receivers must be placed without any interference with each other. With n transmitters/receivers placed in a nonconflict pattern, which corresponds to a solution to the n-queens problem, each transmitter/receiver can communicate with the outside world freely in eight directions (i.e., two horizontal directions, two vertical directions, and four diagonal directions) without being obscured by other transmitters/receivers. Fig. 13 shows one such placement of 10 transmitters/receivers. This placement follows from one of the solutions to the 10-queens problem.

VII. Conclusion

A linear time local search algorithm with a conflict minimization heuristic is presented. This algorithm is significantly faster than any backtracking search algorithm and is capable of solving the n-queens problem in linear time. The performance is achieved through the application of a general conflict minimization heuristic within a general local search framework. The algorithm demonstrates that a guided, partial initial search can significantly reduce later search efforts.

With its general conflict minimization heuristic and its general local search framework, the ideas behind this algorithm have been applied to many large-scale constraint-based optimization problems. This efficient algorithm has made many difficult scientific and engineering applications possible.

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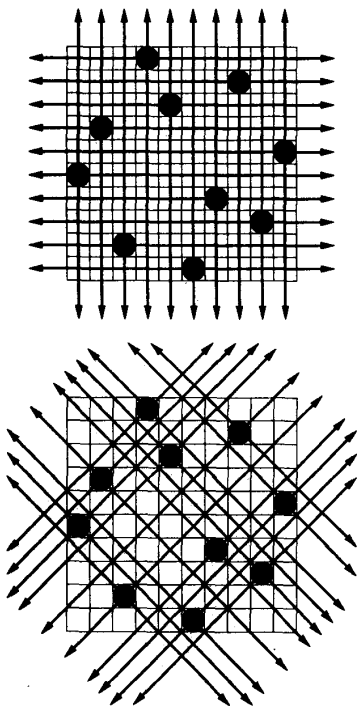

Fig. 13. A nonconflict placement of 10 transmitters/receivers, each of which can communicate in any vertical, horizontal, and diagonal direction.

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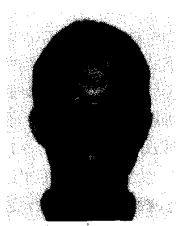

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