FEEDBACK LINEARISATION

Nonlinear system

$$\begin{cases} \dot{x}_1 = -\dfrac{B_l}{J_l} x_1 - \dfrac{k}{J_l} x_2 - \dfrac{mgl}{J_l}\cos(x_2) + \dfrac{k}{J_l} \cdot x_4 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = \dfrac{k}{J_m} x_2 - \dfrac{B_m}{J_m} x_3 - \dfrac{k}{J_m} x_4 + \dfrac{u}{J_m} \\ \dot{x}_4 = x_3 \end{cases}$$

$B_l = 0$

$$\begin{cases} \dot{x}_1 = \boxed{} - \dfrac{k}{J_l} x_2 - \dfrac{mgl}{J_l}\cos(x_2) + \dfrac{k}{J_l} \cdot x_4 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = \dfrac{k}{J_m} x_2 - \dfrac{B_m}{J_m} x_3 - \dfrac{k}{J_m} x_4 + \dfrac{u}{J_m} \\ \dot{x}_4 = x_3 \end{cases}$$

- output $\quad y = \dfrac{\pi}{4} - \vartheta_l = \dfrac{\pi}{4} - x_2$

CHECKING RELATIVE DEGREE OF THE SYSTEM

$\rightarrow \dot{y} = -\dot{x}_2 = -x_1 \qquad r > 1$

$\rightarrow \ddot{y} = -\ddot{x}_2 = -\dot{x}_1 = +\dfrac{k}{J_l} x_2 + \dfrac{mgl}{J_l}\cos(x_2) - \dfrac{k}{J_l} x_4 \qquad r > 2$

$\rightarrow \dddot{y} = -\dddot{x}_2 = -\ddot{x}_1 = \dfrac{k}{J_l}\dot{x}_2 - \dfrac{mgl}{J_l}\sin(x_2)\cdot\dot{x}_2 - \dfrac{k}{J_l}\dot{x}_4 = \dfrac{k}{J_l} x_1 - \dfrac{mgl}{J_l}\sin(x_2) x_1 - \dfrac{k}{J_l} x_3$

$\rightarrow y'''' = -x_2'''' = -\dddot{x}_1 = \dfrac{k}{J_l}\left(-\dfrac{k}{J_l} x_2 - \dfrac{mgl}{J_l}\cos(x_2) + \dfrac{k}{J_l}\cdot x_4\right) - \dfrac{mgl}{J_l}\cos(x_2) x_1^2 - \dfrac{mgl}{J_l}\sin(x_2)\left(-\dfrac{k}{J_l} x_2 - \dfrac{mgl}{J_l}\cos(x_2) + \dfrac{k}{J_l} x_4\right) - \dfrac{k}{J_l}\left(\dfrac{k}{J_m} x_2 - \dfrac{B_m}{J_m} x_3 + - k/J_m\, x_4 + u/J_m\right)$

$r = 4 \longrightarrow$ Fully linearizeable

CHANGE OF COORDINATES

$$\delta x = x - \bar{x} \rightarrow x = \delta x + \bar{x}$$

$$\begin{cases} \dot{x}_1 = -\dfrac{k}{J_l}x_2 - \dfrac{mgl}{J_l}\cos(x_2) + \dfrac{k}{J_L}\cdot x_4 \\[2mm] \dot{x}_2 = x_1 \\[2mm] \dot{x}_3 = \dfrac{k}{J_m}x_2 - \dfrac{B_m}{J_m}x_3 - \dfrac{k}{J_m}x_4 + \dfrac{u}{J_m} \\[2mm] \dot{x}_4 = x_3 \\[2mm] y = \dfrac{\pi}{4} - x_2 \end{cases}$$

$$\bar{x}_2 = \frac{\pi}{4} \qquad \bar{x}_4 = \frac{\sqrt{2}}{2}\frac{mgl}{k} + \frac{\pi}{4}$$

$$\bar{u} = \frac{\sqrt{2}}{2}\frac{mgl}{k}$$

$$\begin{cases} \delta\dot{x}_1 = -\dfrac{k}{J_l}(\delta x_2 + \bar{x}_2) - \dfrac{mgl}{J_l}\cos(\delta x_2 + \bar{x}_2) + \dfrac{k}{J_l}(\delta x_4 + \bar{x}_4) \\[2mm] \delta\dot{x}_2 = \delta x_1 + \cancel{\bar{x}_1} \\[2mm] \delta\dot{x}_3 = \dfrac{k}{J_m}(\delta x_2 + \dot{x}_2) - \dfrac{B_m}{J_m}(\delta x_3 + \bar{x}_3) - \dfrac{k}{J_m}\left(\delta x_4 + \dot{x}_4\right) + \dfrac{1}{J_m}(\delta u + \bar{u}) \\[2mm] \delta\dot{x}_4 = \delta x_3 + \bar{x}_3 \\[2mm] \delta y = \dfrac{\pi}{4} - (\delta x_2 + \bar{x}_2) \end{cases}$$

$$\begin{cases} \delta\dot{x}_1 = -\dfrac{k}{J_l}\left(\delta x_2 + \dfrac{\pi}{4}\right) - \dfrac{mgl}{J_l}\left[\cos(\delta x_2)\dfrac{\sqrt{2}}{2} - \sin(\delta x_2)\dfrac{\sqrt{2}}{2}\right] \\ \qquad + \dfrac{k}{J_l}\left(\delta x_4 + \dfrac{\sqrt{2}}{2}\dfrac{mgl}{k} + \dfrac{\pi}{4}\right) \\[2mm] \delta\dot{x}_2 = \delta x_1 \\[2mm] \delta\dot{x}_3 = \dfrac{k}{J_m}\left(\delta x_2 + \dfrac{\pi}{4}\right) - \dfrac{B_m}{J_m}\cdot\delta x_3 - \dfrac{k}{J_m}\cdot\left(\delta x_4 + \dfrac{\sqrt{2}}{2}\dfrac{mgl}{k} + \dfrac{\pi}{4}\right) + \dfrac{1}{J_m}\Big(\delta u + \\ \qquad + \dfrac{\sqrt{2}}{2}mgl\Big) \\[2mm] \delta\dot{x}_4 = \delta x_3 \end{cases}$$

$$\delta y = -\delta x_2$$

DIFFEOMORPHISM

$\tilde{X}_k = y^{(k-1)} \qquad k = 1, \dots 4$

$\tilde{X}_k = \varphi_k = y^{(k-1)} \qquad k = 1, \dots 4$

$\tilde{X}_1 = -\delta X_2$

$\tilde{X}_2 = -\delta \dot{X}_2 = -\delta X_1$

$$\tilde{X}_3 = -\delta \dot{X}_1 = +\frac{k}{Jl}\left(\delta X_2 + \frac{\pi}{4}\right) + \frac{mgl}{Jl}\left[\cos(\delta X_2)\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\sin(\delta X_2)\right] - \frac{k}{Jl}\left(\delta X_4 + \frac{\sqrt{2}}{2}\frac{mgl}{k} + \frac{\pi}{4}\right)$$

$$\tilde{X}_4 = \frac{k}{Jl}\delta\dot{X}_2 + \frac{mgl}{Jl}\cdot\frac{\sqrt{2}}{2}\left(-\sin(\delta X_2)\,\delta\dot{X}_2\right) - \frac{mgl}{Jl}\cdot\frac{\sqrt{2}}{2}\cdot\cos(\delta X_2)\cdot\delta\dot{X}_2 - \frac{k}{Jl}\cdot\delta\dot{X}_4 =$$

$$= \frac{k}{Jl}\delta X_1 - \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\cdot\sin(\delta X_2)\cdot\delta X_1 - \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\cos(\delta X_2)\cdot\delta X_1 - \frac{k}{Jl}\delta X_3$$

* verificare che è un DIFF. → fare lo Jacobiano

Computation of $\varphi(x)^{-1}$

- $\delta X_2 = -\tilde{X}_1$
- $\delta X_1 = -\tilde{X}_2$
- $$\tilde{X}_3 = \frac{k}{Jl}\left(-\tilde{X}_1 + \frac{\pi}{4}\right) + \frac{mgl}{Jl}\left(\cos(\tilde{X}_1)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sin(\tilde{X}_1)\right) - \frac{k}{Jl}\cdot\delta X_4 - \frac{k}{Jl}\cdot\left(\frac{\sqrt{2}}{2}\frac{mgl}{k} + \frac{\pi}{4}\right) =$$

$$= \delta X_4 = \frac{Jl}{k}\left[-\tilde{X}_3 + \frac{k}{Jl}\left(-\tilde{X}_1 + \frac{\pi}{4}\right) + \frac{mgl}{Jl}\left(\cos(\tilde{X}_1)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sin(\tilde{X}_1)\right) - \frac{k}{Jl}\left(\frac{\sqrt{2}}{2}\frac{mgl}{k} + \frac{\pi}{4}\right)\right] =$$

$$= -\frac{Jl}{k}\tilde{X}_3 - \tilde{X}_1 + \cancel{\frac{\pi}{4}} + \frac{mgl}{k}\cdot\frac{\sqrt{2}}{2}\left(\cos(\tilde{X}_1) + \sin(\tilde{X}_1)\right) - \frac{\sqrt{2}}{2}\frac{mgl}{k} - \cancel{\frac{\pi}{4}}$$

$$\tilde{x}_4 = \frac{k}{Jl}(-\tilde{x}_2) - \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\sin(\tilde{x}_1)\tilde{x}_2 + \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\cos(\tilde{x}_1)\tilde{x}_2 - \frac{k}{Jl}\,\delta x_3$$

$$\delta x_3 = \frac{Jl}{k}\left[-\frac{k}{Jl}\tilde{x}_2 - \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\sin(\tilde{x}_1)\cdot\tilde{x}_2 + \frac{mgl}{Jl}\frac{\sqrt{2}}{2}\cos(\tilde{x}_1)\tilde{x}_2 - \tilde{x}_4\right]$$

$$\delta x_3 = -\tilde{x}_2 - \frac{mgl}{k}\cdot\frac{\sqrt{2}}{2}\cdot\tilde{x}_2\left(\sin(\tilde{x}_1) - \cos(\tilde{x}_1)\right) - \frac{Jl}{k}\cdot\tilde{x}_4$$

$$\delta x_3 = -\left(1 + \frac{mgl}{k}\frac{\sqrt{2}}{2}(\sin(\tilde{x}_1) - \cos(\tilde{x}_1))\right)\tilde{x}_2 - \frac{Jl}{k}\tilde{x}_4$$

$$\begin{cases} \delta x_1 = -\tilde{x}_2 \\ \delta x_2 = -\tilde{x}_1 \\ \delta x_3 = -\left(1 + \frac{mgl}{k}\frac{\sqrt{2}}{2}(\sin(\tilde{x}_1) - \cos(\tilde{x}_1))\right)\tilde{x}_2 - \frac{Jl}{k}\tilde{x}_4 \\ \delta x_4 = -\frac{Jl}{k}\tilde{x}_3 - \tilde{x}_1 + \frac{mg\,l}{k}\cdot\frac{\sqrt{2}}{2}\left(\cos(\tilde{x}_1) + \sin(\tilde{x}_1)\right) - \frac{\sqrt{2}}{2}\frac{mgl}{k} \end{cases}$$

$$\varphi^{-1}(x)$$

NORMAL CANONICAL FORM

$$\ddddot{y} = -\ddddot{x}_2 = -\dddot{x}_1 = \frac{k}{Jl}\left(-\frac{k}{Jl}x_2 - \frac{mgl}{Jl}\cos(x_2) + \frac{k}{Jl}\cdot x_4\right) - \frac{mgl}{Jl}\cos(x_2)\,x_1^2$$

$$-\frac{mgl}{Jl}\sin(x_2)\left(-\frac{k}{Jl}x_2 - \frac{mgl}{Jl}\cos(x_2) + \frac{k}{Jl}x_4\right) - \frac{k}{Jl}\left(\frac{k}{Jm}x_2 - \frac{Bm}{Jm}x_3 + \right.$$

$$\left. - k/Jm\; x_4 + u/Jm\right)$$

$$\tilde{x}_k = y^{(k-1)} \qquad k = 1, \dots 4$$

$$\tilde{x}_k = \varphi_k = y^{(k-1)} \qquad k = 1, \dots 4$$

$$\tilde{x}_1 = y^{(0)} \longrightarrow \dot{\tilde{x}}_1 = \dot{y} = \tilde{x}_2$$

$$\tilde{x}_2 = y^{(1)} \longrightarrow \dot{\tilde{x}}_2 = \ddot{y} = \tilde{x}_3$$

$$\tilde{x}_3 = y^{(2)} \longrightarrow \dot{\tilde{x}}_3 = \dddot{y} = \tilde{x}_4$$

$$\tilde{x}_4 = y^{(3)} \qquad \dot{\tilde{x}}_4 = \ddddot{y}$$

$$\ddddot{y} = -\frac{k^2}{Jl^2}x_2 - \frac{m.g.l.k}{Jl^2}\cdot\cos(x_2) + \frac{k^2}{Jl^2}x_4 - \frac{m.g.l}{Jl}\cos(x_2)\,x_1^2$$

$$+\frac{m.g.l.k}{Jl^2}\cdot\sin(x_2)\,x_2 + \frac{m^2.g^2.l^2}{Jl^2}\sin(x_2)\cos x_2 - \frac{m.g.l.k}{Jl^2}\sin(x_2)\,x_4$$

$$-\frac{k^2}{Jl\,Jm}x_2 + \frac{k.Bm}{Jl\,Jm}x_3 + \frac{k^2}{Jl\,Jm}x_4 - \frac{k}{Jl\,Jm}\cdot u$$

$$\dddot{y} = -\frac{k^2}{Jl^2}\left(\bar{x}_2 - \tilde{x}_1\right) - \frac{m \cdot g \cdot l \cdot k}{Jl^2} \cdot \left(\cos\left(\bar{x}_2 - \tilde{x}_1\right)\right) + \frac{k^2}{Jl^2}\left(\bar{x}_4 - \frac{Jl}{k}\tilde{x}_3 - \tilde{x}_1 + \frac{mgl}{k}\frac{\sqrt{2}}{2}\left(\cos(\tilde{x}_1)\right.\right.$$

$$\left.\left. + \sin(\tilde{x}_1)\right) - \frac{\sqrt{2}}{2}\frac{mgl}{k}\right) - \frac{mgl}{Jl}\left(\cos\left(\bar{x}_1 - \tilde{x}_2\right)\right) \cdot \left(\bar{x}_1 - \tilde{x}_2\right)^2$$

$$+ \frac{m \cdot g \cdot l \cdot k}{Jl^2} \cdot \sin\left(\bar{x}_2 - \tilde{x}_1\right) \cdot \left(\bar{x}_2 - \tilde{x}_1\right) + \frac{m^2 \cdot g^2 \cdot l^2}{Jl^2} \cdot \sin\left(\bar{x}_2 - \tilde{x}_1\right)\cos\left(\bar{x}_2 - \tilde{x}_1\right)$$

$$- \frac{m \cdot g \cdot l \cdot k}{Jl^2}\sin\left(\bar{x}_2 - \tilde{x}_1\right)\left(\bar{x}_4 - \frac{Jl}{k}\tilde{x}_3 - \tilde{x}_1 + \frac{mg l}{k}\frac{\sqrt{2}}{2}\left(\cos(\tilde{x}_1) + \sin(\tilde{x}_1)\right)\right.$$

$$\left. - \frac{\sqrt{2}}{2}\frac{mgl}{k}\right) - \frac{k^2}{Jl\,Jm}\left(\bar{x}_2 - \tilde{x}_1\right) + \frac{k \cdot Bm}{Jl\,Jm}\left(\bar{x}_3 - \left(1 + \frac{mgl}{k}\frac{\sqrt{2}}{2}\left(\sin(\tilde{x}_1) - \cos(\tilde{x}_1)\right)\right)\tilde{x}_2\right.$$

$$\left. - \frac{Jl}{k}\tilde{x}_4\right) + \frac{k^2}{Jl\,Jm}\left(\bar{x}_4 - \frac{Jl}{k}\tilde{x}_3 - \tilde{x}_1 + \frac{mgl}{k}\frac{\sqrt{2}}{2}\left(\cos(\tilde{x}_1) + \sin(\tilde{x}_1)\right) - \frac{\sqrt{2}}{2}\frac{mgl}{k}\right)$$

$$- \frac{k}{Jl\,Jm}\left(\bar{u} + d_m\right) = v \longrightarrow \text{State feedback linearizing contro law}$$

CLOSED LOOP SYSTEM S*

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{x}_3 \\ \dot{\tilde{x}}_3 = \tilde{x}_4 \\ \dot{\tilde{x}}_4 = \tilde{v} \\ y = \tilde{x}_1 \end{cases}$$