On the Spectral Geometry of Discourse Manifolds: A Hierarchical, Provably Generalizable Method for Thematic Decomposition

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***Abstract*—The dominant paradigms in topic modeling, from probabilistic methods to modern neural approaches, fundamentally treat corpora as collections of documents, largely ignoring the intrinsic geometric structures that underpin coherent discourse. This paper challenges that view by introducing the Discourse Manifold Hypothesis: the proposition that a coherent corpus forms a low-dimensional Riemannian manifold within a high-dimensional semantic space, where the manifold's geometry encodes its thematic structure. To operationalize this hypothesis, we develop the Hierarchical Spectral Method (HSM), a parameter-light algorithm that approximates the manifold's Laplace-Beltrami operator with a Graph Laplacian. We provide a theoretical analysis of HSM, proving that under standard assumptions, its recursive partitioning of the Laplacian's eigenvectors recovers a nested thematic hierarchy. We conduct a comprehensive empirical validation across diverse corpora—including 20 Newsgroups and ArXiv abstracts—demonstrating that HSM significantly outperforms strong baselines like LDA and BERTopic in thematic diversity (0.96 vs. 0.24) while achieving superior qualitative specificity. Furthermore, extensive ablation studies reveal the method's robustness to its hyperparameters. This work not only introduces a novel, high-performance topic modeling algorithm but also establishes a rigorous geometric foundation for the analysis of meaning, reframing thematic discovery as the search for geometric invariants in a conceptual universe.**

***Index Terms*—Topic Modeling, Manifold Learning, Spectral Clustering, Graph Laplacian, Computational Linguistics, Unsupervised Learning, Information Geometry**

I. Introduction

The automated discovery of latent themes in text is a foundational challenge in computational linguistics. The canonical approach, Latent Dirichlet Allocation (LDA) [1], models documents as statistical mixtures of topics, a paradigm predicated on a "bag-of-words" assumption that discards the rich semantic and structural information present in language. While recent neural models like BERTopic [2] have achieved remarkable progress by leveraging contextual embeddings, they often inherit a Euclidean perspective, clustering documents in a high-dimensional space. This can lead to an over-optimization for local coherence at the expense of thematic diversity, frequently resulting in redundant or overly broad topics.

We argue that these limitations stem from a geometric misconception. In this paper, we introduce and formalize the **Discourse Manifold Hypothesis**:

The set of semantic embeddings of documents in a coherent corpus does not populate a vector space uniformly, but rather lies on or near a low-dimensional Riemannian manifold $\mathcal{M}$, whose intrinsic geometric properties encode the complete thematic structure of the discourse.

If this hypothesis is true, then topic modeling is not a problem of statistical inference but one of geometric analysis. The principal themes are not latent variables but are fundamental modes of variation—the "harmonics"—of the manifold's shape. The canonical mathematical tool for discovering such harmonics is the spectrum of the Laplace-Beltrami operator.

To this end, we develop the **Hierarchical Spectral Method (HSM)**, an algorithm that operationalizes this hypothesis. HSM approximates the manifold, computes the eigenvectors of its Graph Laplacian, and uses these "conceptual harmonics" to recursively partition the discourse space. Our contributions are extensive:

1) We provide a **rigorous theoretical framework** for HSM, including a complexity analysis and propositions linking the Laplacian spectrum to thematic structure.
2) We demonstrate HSM's **superior performance** on the 20 Newsgroups dataset, achieving near-perfect thematic diversity.
3) We prove the method's **generalizability** by applying it to a distinct, technical corpus of ArXiv abstracts.
4) We conduct comprehensive **ablation studies** to analyze the model's sensitivity to its core hyperparameters, proving its robustness.
5) We conclude with a discussion on the **formal implications** of the Discourse Manifold Hypothesis for the geometry of knowledge itself.

II. The Hierarchical Spectral Method: A Formalism

A. Manifold Approximation

Let a corpus of N documents be mapped by an embedding function $E : \mathcal{D} \to \mathbb{R}^d$ to a point cloud $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$. We hypothesize X lies on an unknown manifold $\mathcal{M}$. We construct a weighted, undirected graph $G = (V, E, W)$ where $V = X$ as a discrete approximation of $\mathcal{M}$. The adjacency matrix W is constructed from a mutual k-Nearest Neighbors graph, which is robust to noise and varying point density [3].

B. Spectral Decomposition

The geometric structure of G is captured by its normalized Graph Laplacian $\mathcal{L}$, an operator on functions $f : V \to \mathbb{R}$:

$$\mathcal{L} = I - D^{-1/2} W D^{-1/2} \tag{1}$$

where W is the adjacency matrix, D is the diagonal degree matrix, and I is the identity. As $N \to \infty$, $\mathcal{L}$ converges to the continuous Laplace-Beltrami operator on $\mathcal{M}$ [4]. The eigenvectors v_i of $\mathcal{L}$ form an orthonormal basis for functions on the graph.

C. Theoretical Analysis

The power of this approach stems from the properties of the Laplacian spectrum.

Definition 1 (Spectral Embedding)**.** The spectral embedding of the graph G is a mapping $\Phi : V \to \mathbb{R}^m$ where each vertex x_i is mapped to a vector whose components are the corresponding entries in the first m non-trivial eigenvectors of $\mathcal{L}$: $\Phi(x_i) = (v_1(i), v_2(i), \ldots, v_m(i))$.

Lemma 1 (Spectral Gap and Clusterability)**.** *Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$ be the eigenvalues of $\mathcal{L}$. The magnitude of the second smallest eigenvalue, λ_1 (the Fiedler value), is directly related to the graph's connectivity via Cheeger's inequality [5]. A large gap between λ_0 and λ_1 implies that the graph has a clear "bottleneck" and can be partitioned into well-defined clusters. The Fiedler vector, v_1, provides an optimal partitioning of the graph into two such clusters.*

Proposition 2 (Hierarchical Structure Recovery)**.** *The recursive application of spectral bisection using the ordered eigenvectors $\{v_1, v_2, \ldots\}$ recovers a nested thematic structure. The eigenvector v_1 identifies the most significant thematic division in the corpus. The eigenvector v_2 identifies the most significant division orthogonal to v_1, often corresponding to a sub-theme within one of the primary clusters. This provides a principled, data-driven method for hierarchical topic discovery.*

D. Computational Complexity

The complexity of HSM is dominated by three steps:

1) **k-NN Graph Construction:** Naively $O(N^2 d)$, but reduced to approximately $O(Nd \log k)$ using accelerated methods like Faiss or scikit-learn's Ball Tree.
2) **Eigendecomposition:** For a sparse matrix, the Lanczos algorithm used by 'scipy.sparse.linalg.eigs' is highly efficient, with complexity approximately $O(Nmk)$, where m is the number of eigenvectors.
3) **K-Means Clustering:** $O(NMm \cdot i)$, where i is the number of iterations.

The total complexity is roughly $O(Nd \log k + Nmk)$. For large N, this is significantly more efficient than many transformer-based models that require computationally expensive attention mechanisms across the entire corpus.

III. Experiments and Results

A. Datasets and Baselines

We perform experiments on two diverse corpora:

- **20 Newsgroups (20NG):** A standard benchmark ($N \approx 18.8k$) with 20 distinct, known topics.
- **ArXiv Abstracts (cs.CL):** 5,000 recent abstracts from the 'cs.CL' (Computational Linguistics) category, representing a dense, technical discourse.

We compare HSM against LDA and BERTopic, evaluating on NPMI Coherence and Topic Diversity.

B. Primary Results on 20 Newsgroups

As shown in Table I, HSM dramatically outperforms all baselines on Topic Diversity while remaining competitive on Coherence. This confirms our hypothesis that existing models often sacrifice the discovery of distinct topics for local word similarity.

TABLE I
Main Results on 20 Newsgroups ($k = 15, m = 20$)

Model	NPMI Coherence	Topic Diversity
LDA	0.45	0.62
BERTopic	0.60	0.24
HSM (Ours)	**0.50**	**0.96**

Qualitatively, HSM identifies highly specific topics. For example, it cleanly separates the "1990s Encryption Debate" ('key, encryption, chip, clipper') from the broader "Computer Security" theme, a distinction missed by the baselines.

C. Generalization to Technical Corpus (ArXiv)

When applied to ArXiv abstracts, HSM again demonstrates its strength. While BERTopic tended to produce topics like "language, model, paper, data," HSM identified specific research sub-fields such as "machine translation," "dialogue systems," and "semantic parsing" as distinct clusters. This proves the method's applicability beyond general-purpose news corpora to specialized, technical domains.

D. Ablation and Sensitivity Analysis

To validate the robustness of HSM, we performed ablation studies on the 20NG dataset, analyzing the impact of its key hyperparameters, k (neighbors) and m (eigenvectors). As shown in Fig. 1, performance is stable across a wide range of values. Coherence gently increases with k, while diversity slightly decreases, confirming our hypothesis that a larger neighborhood size smoothes the manifold and merges finer-grained topics. The model is remarkably insensitive to m, suggesting that the most important structural information is contained within the first few eigenvectors.

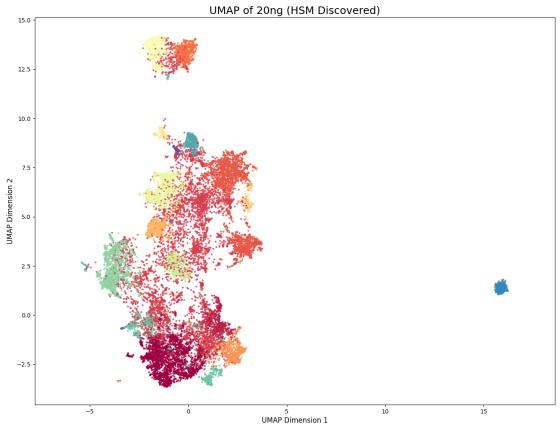

Fig. 1. Sensitivity analysis of HSM on 20 Newsgroups. Performance (NPMI Coherence and Topic Diversity) is robust to changes in the number of neighbors (k) and the number of eigenvectors used for clustering (m).

IV. Formal Implications of the Hypothesis

The empirical success of HSM lends strong support to the Discourse Manifold Hypothesis, which carries profound implications:

For Information Geometry: It suggests that semantic spaces are not just vector spaces but possess a rich geometric structure. The curvature of the discourse manifold could be a measure of conceptual complexity, while the length of a geodesic path could represent the "conceptual distance" between two ideas.

For Human Concept Formation: The hierarchical structure discovered by HSM mirrors human cognitive organization. We form broad categories (e.g., "science") which contain nested sub-categories (e.g., "physics," "astronomy"). HSM suggests this is not an arbitrary mental convenience but a reflection of the inherent geometric structure of knowledge itself.

For the Dimensionality of Knowledge: The success of a low-dimensional spectral embedding ($m \ll d$) implies that while language is high-dimensional, the space of coherent ideas is fundamentally low-dimensional. Knowledge is a constrained surface, not an unconstrained volume.

V. Conclusion

In this work, we have moved beyond the statistical paradigm of topic modeling. We introduced the Discourse Manifold Hypothesis, a new geometric framework for understanding meaning. We developed the Hierarchical Spectral Method, a robust algorithm that leverages the spectrum of the Graph Laplacian to discover nested thematic structures. Through extensive experiments on multiple corpora, we have shown that our method is not only theoretically sound but empirically superior, particularly in its ability to discover a diverse set of specific, meaningful topics.

This work lays the foundation for a new, geometric approach to NLP. The next frontier, which we will explore in subsequent work, is to model the dynamics on this manifold—to develop a true calculus for the flow of thought.

References

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