Efficient Algorithm for Inverting Impulse Response Functions from Frequency Response RAOs

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Introduction

The inversion of impulse response functions (IRFs) from frequency-domain data, such as response amplitude operators (RAOs), is a critical step in time-domain simulations for hydrodynamics and structural dynamics. Traditional methods often rely on damping coefficients due to the notorious numerical instability of added-mass-based integrals, ref. Renato Skejic (2008)[1, 2]. Many authors claimed (without any proof) that for frequencies above the highest frequency specified in the data, $B(f)$ is assumed to decay to zero with f^{-3} at infinite frequency which is however found to be incorrect. One of the advanced approaches was proposed in WAMIT[3] through high-frequency integral compensation for the truncation error. It is found that with dense frequency grid, available sufficiently high-frequency data and Filon integration, a higher accuracy can be achieved, but it becomes rarely practical in daily engineering usage due to the vast computation demanding. As far as I know, among the commercial software products, Orcaflex[2] seems to be the most reliable but with some forceful cutoff scaling and SIMA/SIMO[4] is shown with the initial response overestimated and even give some incorrect results for cross terms.

Here, we introduce a robust algorithm that leverages the rapid decay of half-range sine Fourier series or Fast Fourier Transform to achieve higher-order accuracy, particularly by enhancing the added-mass integration. This approach addresses key limitations in existing literature, such as slow convergence and sensitivity to high-frequency truncation.

> Here a dedicated literature review is prepared.

Theoretical Foundation: Rapid Decay of Sine Series Coefficients

The algorithm is grounded in the mathematical property that if a function $K(t)$ on $[0, T]$ vanishes at both endpoints ($K(0) = K(T) = 0$) and is sufficiently smooth (e.g., continuously differentiable), its half-range sine Fourier coefficients decay as $O(1/n^3)$. This rapid decay underpins the efficiency of the method, as it ensures that fewer terms are needed in the series expansion for accurate reconstruction. The half-range sine series is defined as:

$$K(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right), \quad b_n = \frac{2}{T}\int_0^T K(t) \sin\left(\frac{n\pi t}{T}\right) dt.$$

The $1/n^3$ decay rate arises from integration by parts: if $K(t)$ has continuous derivatives up to order 2, the coefficients b_n scale as $O(1/n^3)$, accelerating convergence compared to standard Fourier series.

Algorithm Steps for Causal IRF Reconstruction

The algorithm transforms frequency-domain data $R(f)$ (e.g., RAOs) into a causal IRF $K(t)$ through the following steps:

1. **Remove the Jump at $t = 0$ by Subtracting an Exponential**:
   The impulse response $K(t)$ often exhibits a discontinuity at $t = 0$ due to non-zero initial values, leading to slow decay in the frequency domain. To address this, we subtract a simple exponential function $K_e(t) = K_0 e^{-\alpha t} u(t)$ (where $u(t)$ is the unit step function) that captures the jump. The modified function $\tilde{K}(t) = K(t) - K_e(t)$ is continuous at $t = 0$, ensuring faster convergence. The parameters K_0 and α are chosen to match the high-frequency behavior of $R(f)$; specifically, K_0 is determined from the asymptotic tail $\Im R(f) \sim -K_0/(2\pi f)$ as $|f| \to \infty$.

2. **Invert the Rapidly Convergent Sine Series or FFT**:
   The function $\tilde{K}(t)$ is now smooth and vanishes at $t = 0$ and $t = T$ (for a sufficiently large T). Its half-range sine Fourier series converges rapidly with $O(1/n^3)$ decay. The inversion involves:

   - Computing the sine coefficients from the frequency data $R(f)$ using the relation between $R(f)$ and $K(t)$ via the Fourier transform. For example, if $R(f)$ is the Fourier transform of $K(t)$, the coefficients can be derived from samples of $\Im R(f)$.
   - Truncating the series to a small number of terms due to the rapid decay, significantly reducing computational cost.

3. **Add the Exponential Back**:
   After reconstructing $\tilde{K}(t)$ via the sine series, the full IRF is obtained as $K(t) = \tilde{K}(t) + K_e(t)$. This restores the causal nature of the response, with $K(t) = 0$ for $t < 0$.

Why a Jump at $t = 0$ Forces the High-Frequency Tail

The discontinuity at $t = 0$ imposes a specific high-frequency behavior on the frequency response. Mathematically, if $K(t)$ has a jump K_0 at $t = 0$, the Fourier transform $R(f) = \int_{-\infty}^{\infty} K(t) e^{-i2\pi f t} dt$ exhibits an asymptotic tail derived from integration by parts:

$$\Im R(f) \sim -\frac{K_0}{2\pi f} \quad \text{as} \quad |f| \to \infty.$$

This results from the dominant contribution of the jump to the imaginary part of $R(f)$, as the derivative of $K(t)$ includes a Dirac delta function at $t = 0$. This tail behavior is critical for accurately estimating K_0 in the exponential subtraction step.

Fully Analytic Example: Rectangle $R(f)$ ↔ Causal Sinc $K(t)$

Consider a rectangular frequency response $R(f) = \mathrm{rect}(f)$, defined as $R(f) = 1$ for $|f| \leq 1/2$ and $R(f) = 0$ otherwise. The corresponding IRF is the causal sinc function:

$$K(t) = \frac{\sin(\pi t)}{\pi t} u(t).$$

- **Verification**: The Fourier transform of $K(t)$ yields $R(f)$, demonstrating causality via the unit step $u(t)$.
- **Algorithm Application**:
  - The function $K(t)$ is continuous at $t = 0$ with $K(0) = 1$, so no jump removal is needed.
  - The sine series of $K(t)$ on $[0, T]$ decays as $O(1/n^3)$ due to smoothness, allowing efficient truncation.
    This example highlights the algorithm's accuracy for band-limited systems.

Short Proof of the Sine-Transform Identity

A key identity used in the algorithm is the Fourier sine transform relation. For a causal function $K(t)$, the imaginary part of $R(f)$ is given by:

$$\Im R(f) = -\int_0^\infty K(t) \sin(2\pi f t) dt.$$

Proof: Since $K(t)$ is real and causal, $R(f) = \int_0^\infty K(t) e^{-i2\pi f t} dt$. Expanding the exponential yields $\Im R(f) = -\int_0^\infty K(t) \sin(2\pi f t) dt$. This identity connects the frequency data to the sine series coefficients.

Importance of Initial Response

$$K_0 = h(0) = \frac{2}{\pi} \int_0^\infty c(\omega) \mathrm{d}\omega,$$

that some methods cannot obtain the correct $K(0)$ as accurate as the FFT–based ones.

The integration based on plain trapezoid even plus a tailing C_c/ω^2 correction (or **higher-order tail model**, e.g. $+D_c/\omega^4$), will not work. But this can be improved by using the **"trapezoidal rule with end correction"** from G. Benthien 2008[5].

$$\int_a^b f(x), dx \approx h\Big[\tfrac{1}{2} f(a) + f(a+h) + \cdots + f(b-h) + \tfrac{1}{2} f(b)\Big] - \frac{h^2}{12}\big(f'(b) - f'(a)\big), \quad h = \frac{b-a}{n}.$$

where the **second-order endpoint correction** requires the derivatives at the ends and the accuracy is found to match Filon integration or other methods.

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1. FFT and Hilbert-FFT can be used to calculate $K(0)$

Those two methods essentially compute

$$K(t) = \mathcal{F}^{-1}\{R(f) - K_0/(\alpha + i2\pi f)\}(t) + K_0 e^{-\alpha t}$$

on a periodic grid. The IFFT uses **all** the frequency samples, and for $t = 0$ the exponential factor is 1, so $K(0) \approx K_0$ almost by construction; small errors in the integral are partly smeared by the periodic/quadrature structure.

The Filon / real-part-only methods rely more directly on the **absolute value** of K_0. If the integral estimate is off by a percent or two, you visibly see an offset in the early-time IRF.

So improving the accuracy of $K_0 = \int_0^\infty c(\omega)\mathrm{d}\omega$ improves all the non-FFT methods much more than it helps FFT itself.

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2. Improved K_0 using end-corrected trapezoidal integral

1. Integrate $c(\omega)$ over the **measured** band $[0, \Omega]$ with **trapezoid + end correction** (eq. 2.65).
2. Add the analytic tail $\int_\Omega^\infty C_c/\omega^2, d\omega = C_c/\Omega$.
3. Multiply by $2/\pi$ to get K_0.

Assumptions:

- Assuming frequencies ω_i are **approximately uniform**.

- We approximate derivatives at the endpoints with second-order one-sided differences:

$$f'(0) \approx \frac{-3f_0 + 4f_1 - f_2}{2h}, \quad f'(\Omega) \approx \frac{3f_n - 4f_{n-1} + f_{n-2}}{2h}.$$

Conclusion

The proposed algorithm leverages the rapid decay of sine series to provide an efficient, accurate, and robust method for IRF inversion. By addressing the jump at $t = 0$ and utilizing the $1/n^3$ convergence, it outperforms traditional methods in computational efficiency. The improvement to added-mass integration further enhances its applicability, making it a versatile tool for time-domain analysis. Future work could extend this to multi-body systems and non-linear interactions.

1. Renato Skejic (2008), NTNU Doctoral these 2008:55, Maneuvering and Seakeeping of a Single Ship and of Two Ships in Interaction. ↩

2. [Vessel theory: Impulse response and convolution](#) ↩ ↩

3. WAMIT R© USER MANUAL Versions 6.4. ↩

4. SIMO 4.20.4 Theory Manual, 2021 or [SIMA Documentation]https://sima.sintef.no/doc/4.8.0/sima/index.html) ↩

5. Benthien, G.W., Acoustic Array Interactions in the Time Domain, *J. Acoust. Soc. Am.* 123, 3114 (2008) ↩