

# Fourier Sine-Series Decay and Accelerated Impulse-Response Inversion

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## Abstract

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We prove that if a function on  $[0, T]$  vanishes at both ends and is sufficiently smooth, then its **half-range sine Fourier coefficients decay like**  $1/n^3$ . This underpins a practical technique for reconstructing a **causal** impulse response  $K(t)$  from frequency data  $R(f)$ : remove the jump at  $t = 0$  by subtracting a simple exponential, invert a rapidly convergent sine series, then add the exponential back to recover the objective response function. We also show mathematically why a jump at  $t = 0$  induces a high-frequency tail

$$\Im R(f) \sim -\frac{K_0}{2\pi f}, \quad (|f| \rightarrow \infty),$$

and for verification purpose we will select a fully analytic example: **rectangle**  $R(f) \leftrightarrow$  **(causal) sinc**  $K(t) = \frac{\sin(\pi t)}{\pi t} u(t)$ . A short proof of the classical sine-transform identity and a simple trick for differentiating  $\log \left| \frac{x+a}{x-a} \right|$  are included.

Almost all the existing literature stress the fact that the impulse response function needs to be obtained from the damping coefficients because the added mass coefficient integration cannot give stable results. However, seldom is known that a small trick can improve the added mass coefficient integration significantly, i.e., one order higher than the damping coefficient integration.

**Originality:** the tail-removal technique or accelerated convergence was not invented by me at all, but initially motivated by Benthien's paper on acoustics<sup>1</sup>. However, due to lack of acoustic and stronger mathematic backgrounds, I could not fully prove the method myself when I first found the paper about 2010 which might also be the reason for the

unhelpful communication with Professor Benthien. So I started to dig into the Fourier theory and functional analysis.

**Analytic in RHP +  $H(s) \rightarrow 0 \implies$  causal  $h(t)$**

**Claim.** Let  $H(s)$  be analytic on  $s : \Re s > 0$  and satisfy

$$\lim_{|s| \rightarrow \infty, \Re s > 0} H(s) = 0.$$

Define  $h(t)$  by Bromwich inversion on any vertical line  $\Re s = c > 0$ :

$$h(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{c-iR}^{c+iR} H(s) e^{st} ds.$$

Then  $h(t) = 0$  for all  $t < 0$  (i.e.,  $h$  is causal).

**Note:** the extra assumption  $H(s) \rightarrow 0$  excludes impulsive terms at  $t = 0$ , e.g.,  $H(s) = 1$  would give  $\delta(t)$ .

**Proof:** uses only analyticity on  $\Re s > 0$  and  $H(s) \rightarrow 0$ .

**Cauchy's convergence test is applicable here and provides a deeper insight** into why the acceleration technique works.

The two approaches, the direct inversion and the accelerated one, are discussed.

## 1. Direct Inversion

When inverting the impulse response  $K(t)$  directly from the frequency RAO  $R(f)$  using the formula involving  $B(f)$ , we face a problem if  $K(t)$  has a jump at  $t = 0$ . As shown, this leads to:

$$B(f) \sim -\frac{K_0}{2\pi f} \quad \text{as } |f| \rightarrow \infty$$

The inversion integral is:

$$K(t) = \dots - 2 \int_0^\infty B(f) \sin(2\pi ft) df$$

- **The Problem:** The integrand  $B(f) \sin(2\pi ft)$  decays like  $1/f$  which as well-known *diverges*. This means the infinite integral defining  $K(t)$  is not absolutely convergent.
- **Cauchy's Test and Conditional Convergence:** The integral only converges because the oscillatory factor  $\sin(2\pi ft)$  causes cancellations. This is known as **conditional convergence**. Cauchy's convergence criterion for an integral  $\int_a^\infty g(f)df$  requires that for any  $\epsilon > 0$ , there exists an  $N$  such that for all  $b_2 > b_1 > N$ , the partial integral satisfies  $\left| \int_{b_1}^{b_2} g(f)df \right| < \epsilon$ . For a conditionally convergent integral, this condition is met, but the convergence is slow and sensitive to the choice of upper limit.

In practice, this means: if one tries to compute this integral numerically by truncating at a finite frequency  $\Omega$ , as in WAMIT<sup>2</sup>, the result will converge very slowly (like  $1/\Omega$ ) and will oscillate as one changes  $\Omega$ .

## 2. Accelerated Inversion (After Removing the Jump)

The core of the acceleration technique is to subtract the function  $K_0 e^{-\alpha t} u(t)$ , whose transform has the same problematic  $1/f$  tail. This creates a new function  $\hat{K}(t)$  that is continuous at  $t = 0$ .

- **The Result:** The new frequency function  $\hat{R}_{odd}(f)$  decays like  $O(1/f^3)$ .
- **Application of Cauchy's Test:** An integrand decaying as  $1/f^3$  leads to an *absolutely convergent* integral. The tail of the integral from  $\Omega$  to infinity is bounded by something proportional to  $\int_\Omega^\infty f^{-3} df \sim 1/\Omega^2$ .

In practice, this means: when we numerically invert the accelerated version, the result converges **rapidly and monotonically** as we increase  $\Omega$ . The partial sums of the series (or the numerical integral) form a Cauchy sequence that converges much faster.

## Summary

Feature	Standard Inversion (with jump)	Accelerated Inversion (jump removed)
Spectral Decay	$O(1/f)$	$O(1/f^3)$
Integral Convergence	Conditional	Absolute

Feature	Standard Inversion (with jump)	Accelerated Inversion (jump removed)
Cauchy Criterion	Holds, but convergence is slow and oscillatory	Holds with rapid, stable convergence
Numerical Truncation Error	$\sim O(1/\Omega)$	$\sim O(1/\Omega^2)$

**Cauchy’s convergence test explains the qualitative difference between the two methods.**

The acceleration technique improves the decay rate of the coefficients, which in turn ensures that the sequence of partial sums satisfies the Cauchy criterion in a much stronger, more useful way for numerical computation.

The statement that the sine series “converges absolutely and uniformly” for the accelerated case is a direct consequence of this sufficiently fast decay ( $O(1/n^3)$ ), which implies the sequence of partial sums is Cauchy.

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## Frequency data from radiation problem

From the radiation problem, the added mass and damping matrix can be computed from the complex potential  $\varphi_j$  with  $j$  the index for mode. WAMIT outputs:

$$R'_{ij}(\omega) = A_{ij}(\omega) - i \frac{B_{ij}(\omega)}{\omega} = \rho \iint_{S_b} n_i \varphi_j dS, \quad \omega = 2\pi f.$$

Equivalently, if we define the radiation-impedance matrix<sup>3</sup>,  $Z_{ij}(\omega) = B_{ij}(\omega) + i\omega A_{ij}(\omega)$ , then  $R'_{ij} = Z_{ij}/(i\omega)$ .

As  $\omega \rightarrow \infty$ :  $A_{ij}(\omega) \rightarrow A_{ij}(\infty)$ ,  $B_{ij}(\omega) \rightarrow 0$ , so

$$R'_{ij}(\omega) \rightarrow A_{ij}(\infty).$$

## What the time-frequency IRF relation uses

The **radiation memory kernel**  $L_{ij}(t)$  (no Dirac spikes due to physical requirement) is defined by the one-sided Fourier transform

$$R_{ij}(\omega) = A_{ij}(\omega) - A_{ij}(\infty) - i \frac{B_{ij}(\omega)}{\omega} = \int_0^\infty L_{ij}(t) e^{-i\omega t} dt.$$

Because of  $L_{ij} \in L^1$  and  $L_{ij}(0) = 0$ , we obtain  $R_{ij}(\omega) \rightarrow 0$  as  $|\omega| \rightarrow \infty$ .

- **High frequency:**  $R'(\omega) \rightarrow A(\infty)$  but  $R(\omega) \rightarrow 0$ .
- **Parity (with  $e^{-i\omega t}$ ):**  $\Re R(\omega) = A(\omega) - A(\infty)$  is even,  $\Im R(\omega) = -B(\omega)/\omega$  is odd.

## The relation between $R$ and $R'$

Just subtract the infinite-frequency added mass:

$$R_{ij}(\omega) = R'_{ij}(\omega) - A_{ij}(\infty) \quad (\text{or } R'_{ij} = R_{ij} + A_{ij}(\infty)).$$

## Time-domain interpretation

Using  $\mathcal{F}\{\delta(t)\} = 1$ ,

$$R'_{ij}(\omega) = \int_0^\infty [L_{ij}(t) + A_{ij}(\infty)\delta(t)] e^{-i\omega t} dt.$$

So  $R'$  corresponds to the kernel

$$\underbrace{L_{ij}(t)}_{\text{memory (decays for } t>0)} + \underbrace{A_{ij}(\infty)\delta(t)}_{\text{instantaneous inertia}},$$

while  $R$  is the transform of the **memory-only** part  $L_{ij}(t)$ .

This is exactly the split that appears in the Cummins equation:

$$F_i^{\text{rad}}(t) = - \sum_j A_{ij}(\infty) \ddot{\xi}_j(t) - \sum_j \int_0^t L_{ij}(t - \tau) \dot{\xi}_j(\tau) d\tau.$$

## How to Invert WAMIT Radiation Data

1. From WAMIT computations:  $A_{ij}(\omega)$ ,  $B_{ij}(\omega)$ .
2. Build  $R'_{ij}(\omega) = A_{ij}(\omega) - iB_{ij}(\omega)/\omega$ .
3. Determine  $A_{ij}(\infty)$  (either average of high-frequency  $A_{ij}$ , or WAMIT's direct computation).
4. Form the memory transform  $R_{ij}(\omega) = R'_{ij}(\omega) - A_{ij}(\infty)$ .
5. Invert to time response:

$$L_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ij}(\omega) e^{i\omega t} d\omega = \int_0^{\infty} R_{ij}(f) e^{-i2\pi f t} df, \quad (t > 0).$$

Two alternatives to numerical solutions:

1.  $L(t) = \frac{2}{\pi} \int_0^{\infty} \frac{B(\omega)}{\omega} \sin \omega t d\omega,$
2.  $L(t) = \frac{2}{\pi} \int_0^{\infty} (A(\omega) - A(\infty)) \cos \omega t d\omega,$

If in nondimensional data<sup>2</sup>,

$$L(t) = 4 \int_0^{\infty} [A(2\pi f) - A(\infty)] \cos(2\pi f t) df = -4 \int_0^{\infty} B(2\pi f) \sin(2\pi f t) df.$$

## General Inversion from Frequency Data

We use the two-sided Fourier pair

$$R(f) = \int_{-\infty}^{\infty} K(t)e^{-i2\pi ft} dt, \quad K(t) = \int_{-\infty}^{\infty} R(f)e^{i2\pi ft} df.$$

Write  $R(f) = A(f) + iB(f)$ , so  $A = \Re R$ ,  $B = \Im R$ . For real  $K$ ,  $A$  is even and  $B$  is odd, i.e.,  $R(f)$  is a Hermitian function since  $R^*(f) = R(-f)$ , where the  $*$  indicates the complex conjugate.

For  $t \in (0, T)$ , a half-range sine series reads

$$g(t) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right), \quad b_n = \frac{2}{T} \int_0^T g(t) \sin\left(\frac{n\pi t}{T}\right) dt.$$

## The $1/n^3$ rule for sine coefficients

**Theorem (endpoint zeros  $\rightarrow 1/n^3$  decay).**

Let  $g \in C^2[0, T]$  with  $g(0) = g(T) = 0$  and  $g''$  absolutely continuous (equivalently  $g''' \in L^1(0, T)$ ). Then the sine coefficients  $b_n$  satisfy

$$|b_n| = O(n^{-3}).$$

Consequently the sine series converges absolutely and uniformly, and the tail after  $N$  terms is  $O(N^{-2})$ .

*Idea.* Three integrations by parts applied to  $b_n = \frac{2}{T} \int_0^T g(t) \sin(\frac{n\pi t}{T}) dt$  kill boundary terms because  $g(0) = g(T) = 0$  and  $\sin(n\pi) = 0$ . The surviving term is proportional to  $1/n^3$  times a bounded quantity involving  $g''(0)$ ,  $g''(T)$  and  $\int_0^T |g'''| dt$ . See [Appendix A](#).

**Generalization.** If  $g \in C^k$  and  $g^{(j)}(0) = g^{(j)}(T) = 0$  for  $j = 0, \dots, k-1$ , with  $g^{(k)}$  absolutely continuous, then  $b_n = O(n^{-(k+1)})$ .

## Impulse responses and positive-frequency inversion

For real  $K(t)$ , we may write

$$K(t) = \int_{-\infty}^{\infty} (A(f) \cos(2\pi ft) - B(f) \sin(2\pi ft)) df. \quad (1)$$

This identity is exact (no causality assumed and  $A$  is even and  $B$  is odd only from  $K \in \mathbb{R}$ ). In particular,

$$K(t) = 2 \int_0^{\infty} A(f) \cos(2\pi ft) df - 2 \int_0^{\infty} B(f) \sin(2\pi ft) df. \quad (2)$$

In many applications, one-sided formulas will be used

$$K(t) = 4 \int_0^{\infty} A(f) \cos(2\pi ft) df = -4 \int_0^{\infty} B(f) \sin(2\pi ft) df. \quad (3)$$

These match (2) when their specific normalization for the “single-sided” spectrum is used or when causality/Hilbert-transform relations let one integral equal the other; we keep (2) as our ground truth and use (3) when stated explicitly. Due to causality, the frequency data  $A(f)$  and  $B(f)$  actually forms Hilbert tranform pair, i.e., they are not independent.

For example, in radiation response data, the  $A(f)$  represents added mass subtracted infinity value and  $B(f)$  the damping. In fact, added mass and damping are mathematically related through the Kramers-Kronig relations<sup>4,3</sup>.

## Why a jump at $t = 0$ gives $\Im R(f) \sim -K_0/(2\pi f)$

$C^1$ : function smoothness ( $f'$  exists + continuous).

$L^1 := \{f : \int |f(t)| dt < \infty\}$

Assume  $K$  is piecewise  $C^1$ ,  $K' \in L^1$ ,  $K(\pm\infty) = 0$ , and let

$$K_0 := K(0^+) - K(0^-)$$

be the jump at  $t = 0$ . Then as  $|f| \rightarrow \infty$ ,



$$\boxed{R(f) = \frac{K_0}{i2\pi f} + o\left(\frac{1}{f}\right), \quad \Im R(f) = -\frac{K_0}{2\pi f} + o\left(\frac{1}{f}\right).} \quad (4)$$

**Proof by IBP.** Split the transform at  $t = 0$  and integrate by parts on  $(-\infty, 0)$  and  $(0, \infty)$ . Boundary terms give  $K_0/(i2\pi f)$ ; the remaining integrals contain  $\widehat{K'}(f)/(i2\pi f)$ , which tends to 0 by Riemann–Lebesgue lemma. Alternatively, a simple distribution proof can use  $\mathcal{F}\{u(t)\} = \mathcal{P}\frac{1}{i2\pi f} + \frac{1}{2}\delta(f)$ .

**Implication.** Direct inversion of (2), especially the  $B$ -term, converges only conditionally (slow  $1/f$  tail) when  $K$  has a jump.

## Convergence Acceleration by removing the jump

Let  $K$  be (causal) with jump  $K_0 = K(0^+)$ . Fix  $\alpha > 0$  and define

$$\hat{K}(t) := K(t) - K_0 e^{-\alpha t} u(t), \quad t \in \mathbb{R}. \quad (5)$$

Then  $\hat{K}(0^+) = 0$  and, choosing  $T$  with  $e^{-\alpha T} \ll 1$ , also  $\hat{K}(T) \approx 0$ . Extend  $\hat{K}$  to an **odd** function on  $[-T, T]$ . By the theorem above, its sine coefficients decay as  $1/n^3$ ; numerically, its sine series (or discrete sine transform) converges very rapidly.

In frequency, (5) corresponds to

$$\hat{R}(f) = R(f) - \frac{K_0}{\alpha + i2\pi f}. \quad (6)$$

If instead we form the transform of the **odd extension** (call it  $\hat{R}_{\text{odd}}$ ), it is purely imaginary and

$$\boxed{\hat{R}_{\text{odd}}(f) = 2i\Im R(f) + i\frac{4\pi K_0 f}{\alpha^2 + (2\pi f)^2}.} \quad (7)$$

The added term cancels the  $1/f$  tail from the jump, so  $\hat{R}_{\text{odd}}(f) = O(1/f^3)$  and the sine series of  $\hat{K}$  enjoys  $1/n^3$  coefficient decay.

**Recover  $K$ .** Compute  $\hat{K}$  from  $\hat{R}_{\text{odd}}$  (fast), then add back:

$$K(t) = \hat{K}(t) + K_0 e^{-\alpha t} u(t). \quad (8)$$

## Analytic example: rect $R(f)$ and causal sinc $K(t)$

In this section, we will utilize some special analytic functions to illustrate the technique of convergence acceleration.

Start with the **even** rectangle

$$R_{\text{even}}(f) = \mathbf{1}_{\{|f| \leq \frac{1}{2}\}}, \quad \mathcal{F}^{-1}\{R_{\text{even}}\}(t) = \frac{\sin(\pi t)}{\pi t}.$$

Make the time response **causal** by multiplying by  $u(t)$ :

$$K(t) = \frac{\sin(\pi t)}{\pi t} u(t), \quad K_0 = K(0^+) = 1. \quad (9)$$

### The transfer function of the causal half-sinc

Using  $\hat{u}(f) = \frac{1}{2}\delta(f) + \mathcal{P}\frac{1}{i2\pi f}$  and convolution in  $f$ ,

$$R(f) = (R_{\text{even}} * \hat{u})(f) = \frac{1}{2}\mathbf{1}_{\{|f| \leq \frac{1}{2}\}} - \frac{i}{2\pi} \log \left| \frac{f + \frac{1}{2}}{f - \frac{1}{2}} \right|,$$

so

$$A(f) = \frac{1}{2}\mathbf{1}_{\{|f| \leq \frac{1}{2}\}}, \quad B(f) = -\frac{1}{2\pi} \log \left| \frac{f + \frac{1}{2}}{f - \frac{1}{2}} \right|. \quad (10)$$

Expanding the log at infinity shows  $B(f) = -\frac{1}{2\pi f} + O(f^{-3})$ , in agreement with (4).

### Reconstruct $K$ from $B$ (and/or $A$ )

From (2),

$$K(t) = 2 \int_0^\infty A(f) \cos(2\pi ft) df - 2 \int_0^\infty B(f) \sin(2\pi ft) df. \quad (11)$$

A classical identity (proved in [Appendix B](#)) gives, for  $a, t > 0$ ,

$$\int_0^\infty \log \left| \frac{f+a}{f-a} \right| \sin(2\pi ft) df = \frac{\pi}{2\pi t} \sin(2\pi at) = \frac{1}{2t} \sin(2\pi at). \quad (12)$$

With  $a = \frac{1}{2}$ , the two contributions in (11) both equal  $\frac{\sin(\pi t)}{2\pi t}$ , so

$$K(t) = \frac{\sin(\pi t)}{\pi t} \quad t > 0.$$

(Using the one-sided formula  $K(t) = -4 \int_0^\infty B(f) \sin(2\pi ft) df$  yields the same result once the same normalization is adopted.)

## Acceleration

Subtract  $e^{-\alpha t}$  (since  $K_0 = 1$ ) to form  $\hat{K}$ . The odd-extension spectrum (7) becomes

$$\hat{R}_{\text{odd}}(f) = -\frac{i}{\pi} \log \left| \frac{f + \frac{1}{2}}{f - \frac{1}{2}} \right| + i \frac{4\pi f}{\alpha^2 + (2\pi f)^2},$$

whose leading  $i/(\pi f)$  terms cancel. Thus the sine coefficients of  $\hat{K}$  on  $[0, T]$  decay like  $1/n^3$ , yielding a fast, clean reconstruction; finally add back  $e^{-\alpha t}$  as in (8).

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## Numerical Implementation for Verification

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- **Choose window  $T$ :** ensure  $K(t) \approx 0$  for  $t \geq T$ . With frequency grid spacing  $\Delta f$ ,  $T \approx 1/\Delta f$ .
- **Pick  $\alpha$ :** make  $e^{-\alpha T} \ll 1$  (e.g.  $\alpha T \gtrsim 6 \Rightarrow \lesssim 10^{-3}$ ).
- **Estimate  $K_0$ :** from physics/analytic form, or from high-frequency data via  $K_0 \approx -2\pi f B(f)$  for large  $f$ .
- **Form  $\hat{R}_{\text{odd}}$ :** use (7) on the sampled frequencies.

- **Invert with a discrete sine transform on  $[0, T]$ :** get  $\hat{K}$  with  $1/n^3$  coefficient decay.
- **Recover  $K$ :**  $K(t) = \hat{K}(t) + K_0 e^{-\alpha t}$  for  $t \geq 0$ ; set  $K(t) = 0$  for  $t < 0$ .
- **Error control:** if coefficients behave like  $C/n^3$ , the partial-sum error is  $O(N^{-2})$ .

## Appendix A: Proof of the $1/n^3$ decay

Let  $g'(0) = g'(T) = 0$ , and

$$b_n = \frac{2}{T} \int_0^T g(t) \sin\left(\frac{n\pi t}{T}\right) dt, \quad \omega_n := \frac{n\pi}{T}.$$

Three integrations by parts, with the necessary smoothness  $g'''(t) \in L^1(0, T)$ , give

$$b_n = \frac{2T^2}{(n\pi)^3} \left[ (-1)^n g''(T) - g''(0) - \int_0^T g'''(t) \cos(\omega_n t) dt \right].$$

Thus

$$|b_n| \leq \frac{2T^2}{(n\pi)^3} \left( |g''(0)| + |g''(T)| + |g'''| \right) = O(n^{-3}).$$

Uniform absolute convergence follows by the [Weierstrass M-test](#), and  $\sum_{n>N} n^{-3} = O(N^{-2})$  gives the tail bound.

Here the necessary smoothness condition for the proof is  $g'''(t) \in L^1(0, T)$ , while  $g \in C^3[0, T]$  is unnecessarily stronger.

## Appendix B: The classical sine-transform identity

For  $a > 0, b > 0$ ,

$$\boxed{\int_0^\infty \log \left| \frac{x+a}{x-a} \right| \sin(bx) dx = \frac{\pi}{b} \sin(ab).} \quad (\text{B1})$$

**Proof.** Define  $I(a) = \int_0^\infty \log \left| \frac{x+a}{x-a} \right| \sin(bx) dx$ . Differentiate under the integral (legitimate for  $a > 0$ ):

$$I'(a) = \int_0^\infty \left( \frac{1}{x+a} + \frac{1}{x-a} \right) \sin(bx) dx = 2 \int_0^\infty \frac{x \sin(bx)}{x^2 - a^2} dx.$$

A standard integral (via contour integration, or by differentiating  $\int_0^\infty \frac{\cos(bx)}{x^2 - a^2} dx = -\frac{\pi}{2a} \sin(ab)$ ) yields

$$\int_0^\infty \frac{x \sin(bx)}{x^2 - a^2} dx = \frac{\pi}{2} \cos(ab).$$

Hence  $I'(a) = \pi \cos(ab)$ , so  $I(a) = \frac{\pi}{b} \sin(ab) + C(b)$ . Since  $I(0) = 0$ ,  $C(b) = 0$ , proving (B1).

Setting  $b = 2\pi t$  gives  $\int_0^\infty \log \left| \frac{f+a}{f-a} \right| \sin(2\pi ft) df = \frac{1}{2t} \sin(2\pi at)$ .

## Appendix C: Differentiating $\log \left| \frac{x+a}{x-a} \right|$ without case splits

Use the identity  $\log |u| = \frac{1}{2} \log(u^2)$  (valid for  $u \neq 0$ ):

$$\frac{d}{da} \log |u(a)| = \frac{u'(a)}{u(a)}.$$

With  $u_1(a) = x + a$ ,  $u_2(a) = x - a$  (for fixed  $x$ ),

$$\frac{\partial}{\partial a} \log \left| \frac{x+a}{x-a} \right| = \frac{1}{x+a} + \frac{1}{x-a} \quad (a \neq \pm x).$$

At  $a = \pm x$  the expression is singular; when it occurs inside integrals we interpret it in the **Cauchy principal-value** sense.

## Appendix D: Jumps at $t = t_0$

From time-shift theorem of Fourier transform,  $\mathcal{F}\{f(t \pm t_0)\} = e^{\pm j2\pi f t_0} \mathcal{F}\{f(t)\}$ . If  $K$  has a jump of size  $J$  at  $t = t_0$ , then the high-frequency asymptotics are

$$R(f) \sim \frac{J e^{-i2\pi f t_0}}{i2\pi f}, \quad \Im R(f) \sim -\frac{J}{2\pi f} \cos(2\pi f t_0), \quad |f| \rightarrow \infty.$$

The case  $t_0 = 0$  reduces to (4).

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## Summary Notes

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- The  $1/n^3$  decay is the special case  $k = 2$  of the general " $k$  vanishing endpoint derivatives  $\Rightarrow 1/n^{k+1}$ " rule.
- The acceleration trick—subtracting  $K_0 e^{-\alpha t}$ —removes the spectral  $1/f$  tail caused by the jump, enabling  $1/n^3$  convergence of the half-range sine series on  $[0, T]$ .

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## Problem Solving Backgrounds

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Both SINTEF software SIMO and WAMIT F2T can be used to generate the Impulse response functions, but the actual calculations from project to project are neither stable nor reliable though one mathematic error in SIMO was spotted by us and was later corrected. [Orcaflex](#) was found to have robust calculations but the accuracy is not guaranteed.

Some of the correspondence logs are attached.

[EXTERNAL] Re: impulse response

Orcina [Orcina@orcina.com](mailto:Orcina@orcina.com)

Kong, Xiangjun [Xiangjun.Kong@nov.com](mailto:Xiangjun.Kong@nov.com)

Hi Kong,

When you specify frequency-dependent added mass and damping, OrcaFlex uses these data to create impulse response functions for use in the time domain model. The impulse response functions are applied using a convolution integral that takes account of the past motion of the vessel. This approach is based on work by Cummins and Wichers. Please see the OrcaFlex help page *Theory | Vessel Theory | Impulse Response and Convolution* for more details.

Ideally, OrcaFlex would evaluate the convolution integral right back to the start of the simulation at each time step. But, of course, as the simulation progresses, this will mean integrating over more and more time, which will gradually slow the simulation down more and more. We know that the impulse response should really decay to zero after a certain amount of time, so we take advantage of this and allow the user to define a “Cutoff Time” (this is defined on the Stiffness, Added Mass, Damping page of the vessel data form). OrcaFlex will then assume that the impulse response function has decayed to zero by this time and so will only evaluate the integral over the cutoff time, rather than over the whole simulation time.

Taking this approach obviously has a significant advantage in terms of processing time but, as you might guess, the cutoff time needs to be chosen with care to avoid “cutting off” the impulse response function too quickly. In addition, the cutoff introduces some approximation errors to the applied levels of damping and added mass; these errors decrease as the cut off time is increased. So, there is always a compromise to be made between using a more-accurate cutoff time, which will be slow to process, versus a less accurate shorter cutoff time, which will run faster.

The graphs presented when you click “Show Graphs...” button on the Stiffness, Added Mass, Damping page of the vessel type data form are there to help you decide whether your choice of cutoff time is adequate.

By default OrcaFlex actually allows you to choose a Cutoff Tolerance, which is defined as “a percentage error relative to the largest damping levels defined in the user data” - see the OrcaFlex help page *System Modelling: Data and Results | Vessels | Vessel Types | Stiffness, Added Mass, Damping*. Using the tolerance is often a much easier way to define your cutoff period. In fact, in many cases the default tolerance of 2% is perfectly adequate - but you can use the impulse response and damping graphs to check this for yourself.

You can find some discussion on the information presented in the Graphs in the *Stiffness, Added Mass, Damping* help page. But I've put a couple of brief notes below.

The main graph to consider when evaluating your choice of cutoff period is the damping graph. As for the impulse response function this contains two graphs; an *idealised* one (based on the damping data defined in the vessel type) and *realised* one, showing you what you will actually get using the chosen cutoff period / tolerance. Clearly, the aim is to find the shortest cutoff period that still gives a good resolution of the defined damping data. Also, remember to check the damping curves for a few row/column combinations; as suggested in the help file, as a minimum you should consider the diagonal values from the damping matrix for the degrees of freedom of interest.

Once you are happy with the damping curves, you can look at the impulse response function graphs. Again, there are *idealised* and *realised* graphs, but these are slightly different from those in the damping plot. Specifically, the idealised curve shows you the impulse response function that would be obtained if you were to use a very small time step in your simulation. The realised curve shows what you will actually get using the defined time step. Again, you simply need to check that your chosen model settings produce a response that is adequately close to the idealised response. Note also that your choice of integration scheme will also affect these graphs - this is explained in the *Stiffness, Added Mass, Damping* help page.

In addition, the x-axis scale on the impulse response curve graph matches the cut off time used, which allows you to see visually whether the responses are managing to decay to zero before the cutoff time is reached.

I hope that this helps but if you've any further questions on the impulse response functions, please do get back in touch.

Best Regards,

Sarah

Sarah Ellwood

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Email: [orcina@orcina.com](mailto:orcina@orcina.com)

"Kong, Xiangjun" [Xiangjun.Kong@nov.com](mailto:Xiangjun.Kong@nov.com) 14-Apr-16 20:57 >>>

Hi,

I am one Orcaflex user in NOV-NORWAY.

Recently we are digging into the impulse response functions and we found there are two curves: "Idealized" and "At model timestep" which differ to some extent.

Is the "Idealized" based on extrapolation of the damping before integration using Filon?  
Are you able to provide the theoretical backgrounds for the two methods?

Appreciate your support,

Kong

**From:** Yang, Qinzhen

**Sent:** 05 December 2013 00:03



**To:** Hovde, Geir Olav

**Subject:** RE: Retardation function

Please see the attached spreadsheet for comparison of retardation functions.

I just compared two components: Surge-Surge and Yaw-Yaw.

For SIMO, I have used time step 0.5s and 0.05s. For Orcaflex, I have no choice but time step 0.02s.

Looks like SIMO provides some strange results at small time lag. At large time lag, the results agree well, but seems there are some high frequency oscillations from SIMO when choosing very small time steps (0.05s) .

Another strange thing for SIMO is that the results are different if you use a different time step.

But I am not sure if this is the reason for the strange yaw angle.

Regards,

Qinzheng

From: Karl Kaasen [Karl.Kaasen@marintek.sintef.no](mailto:Karl.Kaasen@marintek.sintef.no) Sent: 4. mai 2016 17:22 To: Hovde, Geir Olav [GeirOlav.Hovde@nov.com](mailto:GeirOlav.Hovde@nov.com) Cc: MAR SIMO [d-mar-simo@sintef.no](mailto:d-mar-simo@sintef.no); Kong, Xiangjun [Xiangjun.Kong@nov.com](mailto:Xiangjun.Kong@nov.com); Jan Hoff [Jan.Hoff@marintek.sintef.no](mailto:Jan.Hoff@marintek.sintef.no) Subject: [EXTERNAL] RE: Retardation functions in SIMO

Hi Geir Olav,

Thank you for sending the shrewd observations by your colleague Xiangjun Kong.

And Dr. Kong is right.

The minus sign in equation 4.18 in the SIMO manual is a typo. However, I do not think this equation is used in practice by SIMO.

Further, as far as I can judge, there is a factor of two missing in equation 4.19. This factor is present in equation 4.16, though (I am

referring to the manual of SIMO version 4.6), which ought to be the basis for eq. 4.19, rather than eq. 4.15 .

I think the missing factor is the explanation of the wrong value of the retardation function when  $t = 0$ .

We have been aware of this error for some time (although not known its cause until now). Its effect on the retardation force luckily appears to modest, as it gives the force a jagged appearance without destroying its basic character.

We will correct this error at once.

Best Regards

Karl E. Kaasen

From: Hovde, Geir Olav  
[<mailto:GeirOlav.Hovde@nov.com>]

Sent: 3. mai 2016 22:40

To: MAR SIMO <[d-mar-simo@sintef.no](mailto:d-mar-simo@sintef.no) <<mailto:d-mar-simo@sintef.no>]

Cc: Kong, Xiangjun <<mailto:Xiangjun.Kong@nov.com>

Subject: Retardation functions in SIMO

Hi,

We are puzzled with the retardation function in SIMO.

We don't get it to match with other references, and would like to get your view on it.

One of our engineers has described our questions in the attachment.

If not clear please contact Kong for a more detailed explanation (he is on copy).

Best Regards,

Geir Olav Hovde | Manager, Mooring and Riser Systems

## References

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1. Benthien, G.W., Acoustic Array Interactions in the Time Domain, *J. Acoust. Soc. Am.* 123, 3114 (2008) [↩](#)
2. WAMIT R © USER MANUAL Versions 6.4. [↩](#) [↩](#)
3. Johannes Falnes and Adi Kurniawan, Ocean Waves and Oscillating Systems, linear interactions including wave-energy extraction, 2nd Edition, 2020, Cambridge University Press. [↩](#) [↩](#)
4. Kotik J and Mangulis V, 1962. On the Kramers-Kronig relations for ship motions. *Int. Shipbuilding Progress*, 9, No. 97, 361-368. [↩](#)