Numerical approximation of highly oscillatory integrals

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Abstract. The purpose of this essay is the investigation of efficient methods for the numerical integration of highly oscillatory functions, over both univariate and multivariate domains. Such integrals have an unwarranted reputation for being difficult to compute. We will demonstrate that high oscillation is in fact beneficial: the methods discussed in this paper improve with accuracy as the frequency of oscillation increases. The asymptotic expansion will provide a point of departure, allowing us to prove that other, convergent methods have the same asymptotic behaviour, up to arbitrarily high order. This includes Filon-type methods, which require moments and Levin-type methods, which do not require moments but are typically less accurate and are not available in certain situations. Though we focus on the exponential oscillator, we also demonstrate the effectiveness of these methods for other oscillators such as the Bessel and Airy functions. The methods are also applicable in certain cases where the integral is badly behaved; such as integrating over an infinite interval or when the integrand has an infinite number of oscillations.

Extent of original research. Section 2 is a review section: only Corollary 2.2 and the example in Figure 1 are due to me. All of the research is my own in Section 3 through Section 8. In Section 9, the paragraphs on changing the interval of integration are my own research. This starts with the sentence that begins "At first sight, ..." on the top of page 30, and ends on the middle of page 31 with the sentence "...Levin-type method, see Figure 19.". The rest of Section 9 consists of quoted results. All of my research was done on my own, except for Theorem 7.1, which is based on conversations with David Levin for the asymptotic expansion of the integral of the Airy function.

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1. Introduction.

In its most general form, a highly oscillatory integral is

$$I[f] = \int_{\Omega} f(\boldsymbol{x})\, y_{\omega}(\boldsymbol{x})\ \mathrm{d}V,$$

where f is a smooth function, y_ω is an oscillatory function with parameter ω and Ω is some domain. The parameter ω is a positive real number that represents the frequency of oscillations: large ω implies that the number of oscillations of y_ω in Ω is large. The goal of this essay is the numerical approximation of such integrals, with attention paid to the asymptotics of the errors of the approximations, as $\omega \to \infty$. Most of the existing research deals with the exponential oscillator case where $y_\omega(t) = \mathrm{e}^{\mathrm{i}\omega g(t)}$, for some function $g \in C^\infty$.

Highly oscillatory integrals play a valuable role in applications. Using the Magnus expansion [**10**], highly oscillatory differential equations of the form $y'' + g(t)y = 0$, where $g(t) \to \infty$ while the derivatives of g are moderate, can be expressed in terms of an infinite sum of highly oscillatory integrals. Differential equations of this form appear in many areas, including special functions, e.g., the Airy function. From the field of acoustics, the boundary element method requires the evaluation of highly oscillatory integrals, in order to solve integral equations with oscillatory kernels [**8**]. Other areas of application include fluid dynamics, image analysis and more.

For large values of ω, traditional quadrature techniques fail to approximate $I[f]$ efficiently. Each sample point for Gauss-Legendre quadrature is effectively a random value on the range of oscillation, unless the number of sample points is sufficiently greater than the number of oscillations. In the univariate $y_\omega = \mathrm{e}^{\mathrm{i}\omega g}$ case with no stationary points, the integral $I[f]$ is $O\big(\omega^{-1}\big)$ for increasing ω [**18**]. This compares with an error of order $O(1)$ that the traditional techniques have. This implies that it is more accurate to approximate $I[f]$ by zero than to use Gauss-Legendre quadrature! It is safe to say that any approximation that is less accurate than equating the integral to zero is fairly useless. Letting the number of sample points depend on ω, on the other hand, results in an enormous amount of computation for large ω. For the multivariate case, the number of sample points needed to effectively use repeated univariate quadrature grows exponentially with each dimension. The method of stationary phase [**14**] is also unsuitable for our needs, as it only provides an asymptotic approximation.

We will demonstrate several methods for approximating $I[f]$ such that the accuracy improves as the frequency ω increases. Until Section 7, we focus on the exponential oscillator $y_\omega = \mathrm{e}^{\mathrm{i}\omega g}$. Section 2 contains a brief overview of the asymptotic expansion and Filon-type methods. Like the asymptotic expansion, there exists Filon-type methods with arbitrarily high asymptotic order. Unlike the asymptotic expansion, the error of a Filon-type method can be made arbitrarily small. Section 3 describes a univariate Levin-type method, which has the benefits of the Filon-type methods without requiring moments. Section 4 discusses the multivariate asymptotic expansion and Filon-type methods, then Section 5 develops a Levin-type method for multidimensional domains Ω, where Ω need not be square, nor even a polytope. In Section 6, we show that by choosing a collocation basis wisely, the asymptotic order of a Levin-type method can be further increased. Section 7 contains research on oscillators besides $\mathrm{e}^{\mathrm{i}\omega g}$, where y_ω satisfies some known differential equation. A classic example is the Airy function $y_\omega(x) = \mathrm{Ai}\,(-\omega x)$, cf. [**2**].

The last two sections look at how to handle problems where the integral is badly behaved. Section 8 investigates handling integrals over unbounded domains, as well as integrals with an infinite number of oscillations within the interval of integration. In the exponential oscillator case, this corresponds to $g' \to \infty$ at one of the endpoints of the interval. Section 9 investigates stationary points, as well as critical points in higher dimensions.

2. Asymptotic expansion and Filon-type methods.

This section consists of an overview of the relevant material from [**11**]. We focus on the exponential oscillator

$$I[f] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x.$$

Until Section 9, we assume that $g' \neq 0$ in $[a, b]$, in other words there are no stationary points. The idea behind the methods presented in this essay is to derive first an asymptotic expansion for $I[f]$, which we then use to find the order of error of other, more accurate, methods. The key observation is that

$$
\begin{aligned}
I[f] &= \int_a^b f \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x = \frac{1}{\mathrm{i}\omega}\int_a^b \frac{f}{g'}\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{\mathrm{i}\omega g}\right]\,\mathrm{d}x = \frac{1}{\mathrm{i}\omega}\left[\frac{f}{g'}\mathrm{e}^{\mathrm{i}\omega g}\right]_a^b - \frac{1}{\mathrm{i}\omega}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f}{g'}\right]\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x \\
&= Q[f] - \frac{1}{\mathrm{i}\omega} I\left[\left(\frac{f}{g'}\right)'\right],
\end{aligned}
$$

where $Q[f] = \frac{1}{\mathrm{i}\omega}\left[\frac{f}{g'}\mathrm{e}^{\mathrm{i}\omega g}\right]_a^b$. Because g' is nonzero, there are no problems associated with dividing by g'. Note that the integral in the error term is $O\big(\omega^{-1}\big)$ [**18**], hence $Q[f]$ approximates $I[f]$ with an error of order $O\big(\omega^{-2}\big)$. Moreover, the error term is another highly oscillatory integral, hence we can iterate this procedure. By continuing this process, we derive the following *asymptotic expansion*:

Theorem 2.1. *Suppose that $g' \neq 0$ in $[a, b]$. Then*

$$
I[f] \sim -\sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k}\left(\sigma_k(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k(a)\mathrm{e}^{\mathrm{i}\omega g(a)}\right),
$$

where

$$
\sigma_1 = \frac{f}{g'}, \qquad \sigma_{k+1} = \frac{\sigma_k'}{g'}, \qquad k \geq 1.
$$

The error term for approximating $I[f]$ by the first s terms of this expansion is $\frac{1}{(\mathrm{i}\omega)^s} I[\sigma_s'] = \frac{1}{(\mathrm{i}\omega)^s} I\big[\sigma_{s+1} g'\big]$. The following corollary, from [**15**], will be used in the proof of the order of error for Filon-type and Levin-type methods.

Corollary 2.2. *Suppose that $f = \boldsymbol{\mathcal{O}}(\omega^{-n})$, where $\boldsymbol{\mathcal{O}}(\omega^{-n})$ states that the $L^\infty[a, b]$ norm of f and its derivatives are all $O(\omega^{-n})$, cf. Appendix A. Furthermore, suppose that*

$$
\begin{aligned}
0 &= f(a) = f'(a) = \cdots = f^{(s-1)}(a)\,, \\
0 &= f(b) = f'(b) = \cdots = f^{(s-1)}(b)\,.
\end{aligned}
$$

Then $I[f] \sim O\big(\omega^{-n-s-1}\big)$, for $\omega \to \infty$.

Proof: Each σ_k depends on f and its first $k-1$ derivatives, in the sense that it is a sum of terms independent of ω, each multiplied by some function in the set $\big\{f, \ldots, f^{(k-1)}\big\}$. Thus it follows that $0 = \sigma_k(a) = \sigma_k(b)$ for all $k \leq s$, and the first s terms of the asymptotic expansion are identically zero. By expanding out to the $(s+1)$-term expansion we obtain

$$
I[f] = -\frac{1}{(-\mathrm{i}\omega)^{s+1}}\left\{\sigma_{s+1}(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_{s+1}(a)\mathrm{e}^{\mathrm{i}\omega g(a)}\right\} + \frac{1}{(-\mathrm{i}\omega)^{s+1}}\int_a^b g'\sigma_{s+2}\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x.
$$

From the properties of $\boldsymbol{\mathcal{O}}(\cdot)$ in Appendix A, we know that $\sigma_{s+1} = \boldsymbol{\mathcal{O}}(\omega^{-n})$. Thence $\sigma_{s+1}(b)$ and $\sigma_{s+1}(a)$ are $O(\omega^{-n})$. Furthermore, the integral is also of order $O(\omega^{-n})$, and all three terms are $O\big(\omega^{-n-s-1}\big)$.

Q.E.D.

We could, of course, use the partial sums of the asymptotic expansion to approximate $I[f]$. This approximation would improve with accuracy as the frequency of oscillations ω increased. Unfortunately, the expansion will typically not converge for fixed ω, thus there is a limit to the accuracy of an asymptotic expansion. Hence we derive a *Filon-type method*, a method which will provide convergent approximations whilst retaining the asymptotic behaviour of an asymptotic expansion. The idea is to approximate f by a polynomial $v = \sum_{k=0}^{n} c_k x^k$ using Hermite interpolation, i.e., determine the coefficients c_k by solving the system

$$
v(x_k) = f(x_k)\,, v'(x_k) = f'(x_k)\,, \ldots, v^{(m_k-1)}(x_k) = f^{(m_k-1)}(x_k)\,, \qquad k = 0, 1, \ldots, x_\nu,
$$

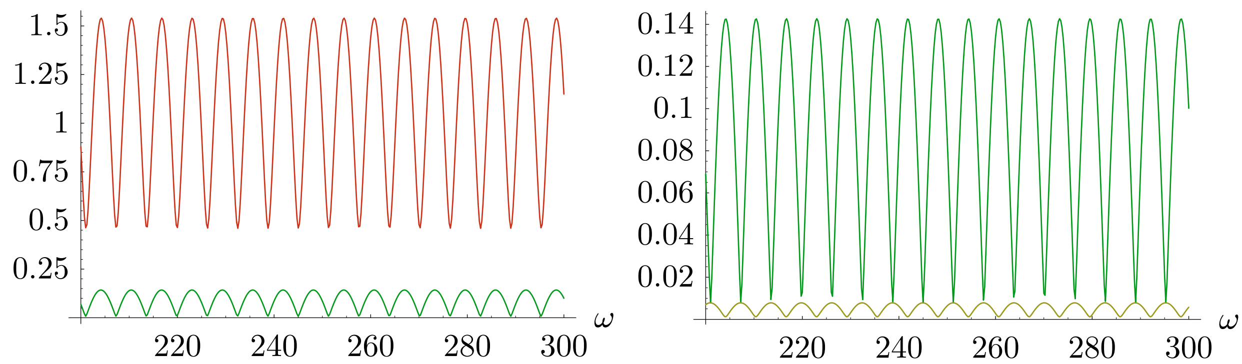


Figure 1: The error scaled by ω^3 of the asymptotic expansion (left figure, top), $Q^F[f]$ with only endpoints and multiplicities both two (left figure, bottom)/(right figure, top), and $Q^F[f]$ with nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{2,1,2\}$ (right figure, bottom) for $I[f] = \int_0^1 \cos x \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}x$.

for some set of *nodes* $\{x_0, \ldots, x_\nu\}$ and *multiplicities* $\{m_0, \ldots, m_\nu\}$. We will assume for simplicity that $x_0 = a$ and $x_\nu = b$. If the moments of $\mathrm{e}^{\mathrm{i}\omega g}$ are available, then we can calculate $I[v]$ explicitly. We define a Filon-type method as

$$Q^F[f] = I[v] = \sum_{k=0}^{n} c_k I\left[x^k\right].$$

Because the accuracy of $Q^F[f]$ depends on the accuracy of v interpolating f, adding additional sample points and multiplicities will typically decrease the error. If v converges uniformly to f, then the approximation $Q^F[f]$ converges to the solution $I[f]$. We can easily prove the asymptotic order of this method:

Theorem 2.3. *Let* $s = \min\{m_0, m_\nu\}$. *Then*

$$I[f] - Q^F[f] \sim O\left(\omega^{-s-1}\right).$$

Proof: The order of error of this method follows immediately from Corollary 2.2:

$$I[f] - Q^F[f] = I[f] - I[v] = I[f - v] \sim O\left(\omega^{-s-1}\right)$$

as $\omega \to \infty$, since $f - v$ and its first $s - 1$ derivatives are zero at the endpoints.

Q.E.D.

We will now compare Filon-type methods to the asymptotic expansion numerically to show that we can indeed decrease the error by adding interpolation points, using an example from [**15**]. Consider the Fourier oscillator $\mathrm{e}^{\mathrm{i}\omega x}$ with $f(x) = \cos x$ integrating over the interval $(0,1)$. In Figure 1 we compare several methods of order three: the two-term asymptotic expansion, $Q^F[f]$ with nodes $\{0,1\}$ and multiplicities $\{2,2\}$, and $Q^F[f]$ with nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{2,1,2\}$. Even when sampling f only at the endpoints of the interval, the Filon-type method represents a significant improvement over the asymptotic expansion, having approximately one-twelfth the error, while using exactly the same information. Adding a single interpolation point results in an error almost indistinguishable from zero in comparison to the asymptotic expansion. Adding additional node points continues to have a similar effect.

3. Univariate Levin-type method.

The major problem with using Filon-type methods is that they require explicit formulæ for the moments $I\left[x^k\right]$, which are not known for general functions g. To address this issue, we investigate another method for approximating highly oscillatory integrals, which was originally developed in [**13**]. This method uses collocation instead of interpolation, removing the requirement that moments are computable. If there exists a function F such that $\frac{\mathrm{d}}{\mathrm{d}x}\left[F\mathrm{e}^{\mathrm{i}\omega g}\right] = f\mathrm{e}^{\mathrm{i}\omega g}$, then

$$I[f] = \int_a^b f\mathrm{e}^{\mathrm{i}\omega g}\ \mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[F\mathrm{e}^{\mathrm{i}\omega g}\right]\ \mathrm{d}x = \left[F\mathrm{e}^{\mathrm{i}\omega g}\right]_a^b .$$

We can rewrite this condition as $\mathcal{L}[F] = f$ for the operator

$$\mathcal{L}[F] = F' + \mathrm{i}\omega g' F.$$

If we can approximate F, then we can approximate $I[f]$ easily. In order to do so, we use collocation with the operator $\mathcal{L}$. Let $v = \sum_{k=0}^{\nu} c_k \psi_k$ for some *basis* $\{\psi_k\}$. Given a sequence of nodes $\{x_0, \ldots, x_\nu\}$, we determine the coefficents c_k by solving the collocation system

$$\mathcal{L}[v]\,(x_0) = f(x_0), \ldots, \mathcal{L}[v]\,(x_\nu) = f(x_\nu).$$

We can then define the approximation $Q^L[f]$ to be

$$Q^L[f] = \int_a^b \mathcal{L}[v]\,\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[v\mathrm{e}^{\mathrm{i}\omega g}\right]\ \mathrm{d}x = \left[v\mathrm{e}^{\mathrm{i}\omega g}\right]_a^b .$$

It was proved in [**13**] that, whenever the endpoints of the interval are used in the collocation system, $I[f] - Q^L[f] = O\big(\omega^{-2}\big)$.

We obtain a *Levin-type method* by generalizing this method to include multiplicities, i.e. we associate a sequence of multiplicities $\{m_0, \ldots, m_\nu\}$ to the nodes $\{x_0, \ldots, x_\nu\}$. This idea was presented by the current author in [**15**]. The collocation system now has the form:

$$\mathcal{L}[v]\,(x_k) = f(x_k), \mathcal{L}[v]'\,(x_k) = f'(x_k), \ldots, \mathcal{L}[v]^{(m_k-1)}\,(x_k) = f^{(m_k-1)}(x_k), \qquad k = 0, 1, \ldots, \nu. \tag{3.1}$$

If every multiplicity m_k is one, then this is equivalent to the original Levin method. We will prove that, as in a Filon-type method, if the multiplicities at the endpoint are greater than or equal to s, then $I[f] - Q^L[f] \sim O\big(\omega^{-s-1}\big)$. Thus we obtain the same asymptotic and convergent behaviour as a Filon-type method without requiring moments, and using exactly the same information about f and g. In order to prove the order of error, we require that the *regularity condition* is satisfied, which states that the set of functions $\{g'\psi_k\}$ can interpolate any function at the given nodes and multiplicities.

Theorem 3.1. *Suppose that the regularity condition is satisfied. Then*

$$I[f] - Q^L[f] \sim O\big(\omega^{-s-1}\big)\,,$$

where $s = \min\{m_0, m_\nu\}$ *and*

$$Q^L[f] = v(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

Proof: The error term of the approximation is $I[f] - Q^L[f] = I[f - \mathcal{L}[v]]$. In order to use Corollary 2.2 we need to show that $f - \mathcal{L}[v] = \boldsymbol{\mathcal{O}}(1)$. Since f is independent of ω, we need only worry about $\mathcal{L}[v]$. Using Cramer's rule, we will show that each c_k is of order $O\big(\omega^{-1}\big)$. Define the operator $\mathcal{P}[f]$, written in partitioned form, as

$$\mathcal{P}[f] = \begin{pmatrix} \rho_0[f] \\ \vdots \\ \rho_\nu[f] \end{pmatrix}, \qquad \text{where} \qquad \rho_k[f] = \begin{pmatrix} f(x_k) \\ \vdots \\ f^{(m_k-1)}(x_k) \end{pmatrix}.$$

Basically, $\mathcal{P}[f]$ maps f to its values at every node in $\{x_0, \ldots, x_\nu\}$ with multiplicities $\{m_0, \ldots, m_\nu\}$. Note that the system (3.1) can be written as $A\boldsymbol{c} = \boldsymbol{f}$, for $\boldsymbol{c} = [c_0, \cdots, c_n]^\top$ and

$$A = [\mathcal{P}[\mathcal{L}[\psi_0]]\,, \cdots, \mathcal{P}[\mathcal{L}[\psi_n]]] = [\mathcal{P}[\psi_0'] + \mathrm{i}\omega\mathcal{P}[g'\psi_0]\,, \cdots, \mathcal{P}[\psi_n'] + \mathrm{i}\omega\mathcal{P}[g'\psi_n]] = P + \mathrm{i}\omega G,$$

where

$$P = [\mathcal{P}[\psi_0']\,, \cdots, \mathcal{P}[\psi_n']]\,, \qquad G = [\mathcal{P}[g'\psi_0]\,, \cdots, \mathcal{P}[g'\psi_n]]\,, \qquad \boldsymbol{f} = \mathcal{P}[f]\,.$$

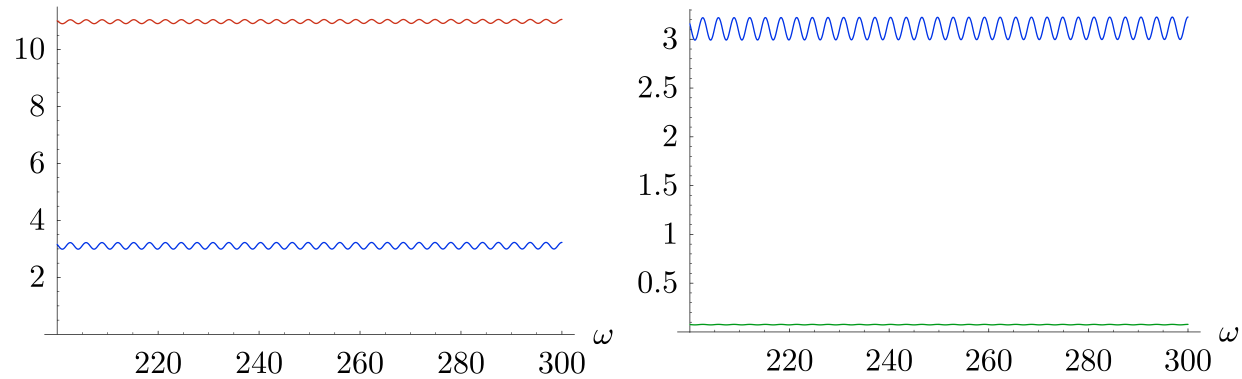


Figure 2: The error scaled by ω^3 of the asymptotic expansion (left figure, top), $Q^L[f]$ (left figure, bottom)/(right figure, top) and $Q^F[f]$ (right figure, bottom) both with only endpoints and multiplicities two for $I[f] = \int_0^1 \cos(x) \mathrm{e}^{\mathrm{i}\omega(x^2+x)}\,\mathrm{d}x$.

Solving the system $G\boldsymbol{c} = \boldsymbol{f}$ is equivalent to interpolating f by $\{g'\psi_k\}$ at the given nodes and multiplicities. Thus the regularity condition ensures that $\det G \neq 0$. It follows that $\det A = (\mathrm{i}\omega)^{n+1} \det G + O(\omega^n)$, hence large enough ω ensures that A is nonsingular and $(\det A)^{-1} = O\big(\omega^{-n-1}\big)$. Furthermore $\det D_k = O(\omega^n)$, for D_k defined as the matrix A with the $(k+1)$th column replaced by $\boldsymbol{f}$. Hence

$$c_k = \frac{\det D_k}{\det A} = O\big(\omega^{-1}\big)\,.$$

It follows that $v = \boldsymbol{\mathcal{O}}\big(\omega^{-1}\big)$; hence $\mathcal{L}[v] = \boldsymbol{\mathcal{O}}(1)$, and the theorem follows.

Q.E.D.

Theorem 3.2 provides a simplified version of the regularity condition. It is especially helpful as it ensures that the standard polynomial basis can be used with a Levin-type method and any choice of nodes and multiplicities. Recall from [**17**] that a Chebyshev set is a basis that spans a set M that satisfies the Haar condition; in other words, that every function $u \in M$ has less than $n+1$ roots to the equations $u(x) = 0$ in the interval $[a, b]$.

Theorem 3.2. *Suppose that the basis $\{\psi_0, \ldots, \psi_n\}$ is a Chebyshev set. Then the regularity condition is satisfied for all choices of nodes and multiplicities.*

Proof: Let M be equal to the span of $\{\psi_0, \ldots, \psi_n\}$. We begin by showing that $\{g'\psi_0, \ldots, g'\psi_n\}$ is a Chebyshev set. Note that $\{g'\psi_0, \ldots, g'\psi_n\}$ is a family of linearly independent functions, since $\sum c_k g'\psi_k = g' \sum c_k \psi_k$ and $g' \neq 0$. Let $\tilde{M} = \operatorname{span}\{g'\psi_0, \ldots, g'\psi_n\}$ and $\tilde{u} \in \tilde{M}$, where $\tilde{u}$ is not identically zero. We know that $\tilde{u} = g'u$ for some $u \in M$, and u is equal to zero less than $n+1$ times. But if $u(x) \neq 0$ then $\tilde{u}(x) \neq 0$. Thus $\tilde{M}$ satisfies the Haar condition. It follows that the basis $\{g'\psi_k\}$ can interpolate at any points $\{y_0, \ldots, y_n\}$ [**17**]. Thus, by a trivial limiting argument, we know that it can interpolate at the points $\{x_0, \ldots, x_\nu\}$ with multiplicities $\{m_0, \ldots, m_\nu\}$.

Q.E.D.

The following example, taken directly from [**15**], will demonstrate the effectiveness of this method. Consider the integral $\int_0^1 \cos(x)\mathrm{e}^{\mathrm{i}\omega\left(x^2+x\right)}\,\mathrm{d}x$, in other words $f(x) = \cos x$ and $g(x) = x^2 + x$. We have no stationary points and moments are computable, hence all the methods discussed so far are applicable. We compare the asymptotic expansion with a Filon-type method and a Levin-type method, both with nodes $\{0, 1\}$ and multiplicities both two. For this choice of f and g, the Levin-type method is a significant improvement over the asymptotic expansion, whilst the Filon-type method is even more accurate.

Figure 3 compares the Levin-type method and the Filon-type method with the addition of two sample points. This graph helps emphasize the effectiveness of adding node points within the interval of integration. With just two node points, only one of which has multiplicity greater than one, the error of $Q^L[f]$ is less than

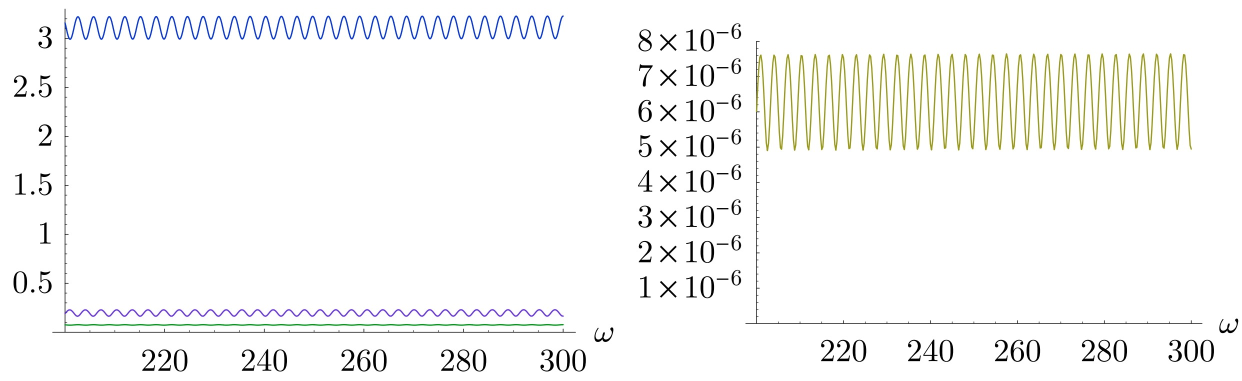


Figure 3: The error scaled by ω^3 of $Q^L[f]$ (left figure, top) and $Q^F[f]$ (left figure, bottom) both with only endpoints and multiplicities two compared to $Q^L[f]$ (left figure, middle) and $Q^F[f]$ (right figure) both with nodes $\left\{0, \frac{1}{4}, \frac{2}{3}, 1\right\}$ and multiplicities $\{2, 2, 1, 2\}$ for $I[f] = \int_0^1 \cos x \mathrm{e}^{\mathrm{i}\omega(x^2+x)}\,\mathrm{d}x$.

a sixth of what it was. In fact it is fairly close to the former $Q^F[f]$ while still not requiring the knowledge of moments. On the other hand, adding the same node points and multiplicities to $Q^F[f]$ results in an error indistinguishable from zero in comparison to the original $Q^L[f]$. It should be emphasized that even $Q^L[f]$ with only endpoints is still a very effective method, as all the values in this graph are divided by $\omega^3 \geq 200^3 = 8 \cdot 10^6$.

4. Multivariate asymptotic expansion.

With a firm concept of how to handle the univariate case, we now begin delving into the approximation of higher dimensional integrals in the form

$$I[f] = I_g[f, \Omega] = \int_\Omega f(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\,\mathrm{d}V,$$

where the domain Ω has a piecewise smooth boundary. As is the theme of this essay, we mirror the univariate methods by first deriving an asymptotic expansion, which we then use to prove the order of error for multivariate Filon-type and Levin-type methods. We begin by investigating the case where the non-resonance condition is satisfied, which is somewhat similar in spirit to the condition that g' is nonzero within the interval of integration. The *non-resonance condition* is satisfied if, for every $\boldsymbol{x}$ on the boundary of Ω, $\nabla g(\boldsymbol{x})$ is not orthogonal to the boundary of Ω at $\boldsymbol{x}$. In addition, $\nabla g \neq 0$ in the closure of Ω, i.e. there are no critical points. Note that the non-resonance condition does not hold true if g is linear and Ω has a completely smooth boundary, such as a circle, since ∇g must be orthogonal to at least one point in $\partial\Omega$.

Based on results from [**12**], we derive the following asymptotic expansion. We also use the notion of a vertex of Ω, for which the definition may not be immediately obvious. Specifically, we define the *vertices* of Ω as:

- If Ω consists of a single point in $\mathbb{R}^d$, then that point is a vertex of Ω.
- Otherwise, let $\{Z_\ell\}$ be an enumeration of the smooth components of the boundary of Ω, where each Z_ℓ is of one dimension less than Ω, and has a piecewise smooth boundary itself. Then $\boldsymbol{v} \in \partial\Omega$ is a vertex of Ω if and only if $\boldsymbol{v}$ is a vertex of some Z_ℓ.

In other words, the vertices are the endpoints of all the smooth one-dimensional edges in the boundary of Ω. In two-dimensions, these are the points where the boundary is not smooth. We denote the partial derivative operator as $\mathcal{D}^{\boldsymbol{m}}$ for $\boldsymbol{m} \in \mathbb{N}^d$ and define $|\,\boldsymbol{m}\,|$ as the sum of the entries of $\boldsymbol{m}$, cf. Appendix A.

Theorem 4.1. *Suppose that Ω has a piecewise smooth boundary, and that the non-resonance condition is satisfied. Then, for $\omega \to \infty$,*

$$I_g[f, \Omega] \sim \sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{k+d}} \Theta_k[f],$$

where $\Theta_k[f]$ depends on $\mathcal{D}^{\boldsymbol{m}} f$ for all $|\boldsymbol{m}| \le k$, evaluated at the vertices of Ω.

Proof: Fix an integer $s \ge 1$. From [**12**] we know that, if a domain S is a polytope and g has no critical points in the closure S, then

$$I_g[f,S] = Q^A_{g,s}[f,S] + \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s, S],$$

where

$$Q^A_{g,s}[f,S] = -\sum_{k=0}^{s-1} \frac{1}{(-\mathrm{i}\omega)^{k+1}} \int_{\partial S} \boldsymbol{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}S,$$

$\boldsymbol{n}$ is the outward facing unit normal and

$$\sigma_0 = f, \qquad \sigma_{k+1} = \nabla \cdot \left[\frac{\sigma_k}{\|\nabla g\|^2} \nabla g\right], \qquad k = 0, 1, \ldots.$$

Let $\{S_0, S_1, \ldots\}$ be a sequence of polytope domains such that $\lim S_j = \Omega$, where each S_j is a tessellation of Ω. Because ∇g is continuous, there is an open set U containing the closure of Ω such that $\nabla g \neq 0$ in U. Assume that each $S_j \subset U$, which is true whenever a sufficiently fine grid is used.

Note that σ_k is bounded in U for all k, because there are no critical points. Hence, due to the boundedness of each integrand and the dominating convergence theorem, it is clear that

$$\begin{aligned} I_g[f, S_j] &\to I_g[f,\Omega], \\ \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s, S_j] &\to \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s,\Omega], \\ \int_{\partial S_j} \boldsymbol{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}S &\to \int_{\partial\Omega} \boldsymbol{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}S. \end{aligned}$$

It follows that $I_g[f,\Omega] = Q^A_{g,s}[f,\Omega] + \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s,\Omega] = Q^A_{g,s}[f,\Omega] + O(\omega^{-s-d})$, using the fact that $I_g[\sigma_s,\Omega] = O(\omega^{-d})$ [**18**].

We now prove the theorem by expressing $Q^A_{g,s}[f,\Omega]$ in terms of its asymptotic expansion. Assume the theorem holds true for lower dimensions, where the univariate case follows from Theorem 2.1. For each ℓ, there exists a domain $\Omega_\ell \in \mathbb{R}^{d-1}$ and a smooth map $T_\ell : \Omega_\ell \to Z_\ell$ that parameterizes Z_ℓ by Ω_ℓ, where every vertex of Ω_ℓ corresponds to a vertex of Z_ℓ, and vice-versa. We can rewrite each surface integral in $Q^A_{g,s}[f,\Omega]$ as a sum of standard integrals:

$$\int_{\partial\Omega} \boldsymbol{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}S = \sum_\ell \int_{Z_\ell} \boldsymbol{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}S = \sum_\ell I_{g_\ell}[f_\ell, \Omega_\ell], \tag{4.1}$$

where f_ℓ is a smooth function multiplied by $\sigma_k \circ T_\ell$, and $g_\ell = g \circ T_\ell$. It follows from the definition of the non-resonance condition that the function g_ℓ satisfies the non-resonance condition in Ω_ℓ. Thus, by our assumption,

$$I_{g_\ell}[f_\ell, \Omega_\ell] \sim \sum_{i=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{i+d-1}} \Theta_i[f_\ell],$$

where $\Theta_i[f_\ell]$ depends on $\mathcal{D}^{\boldsymbol{m}} f_\ell$ for $|\boldsymbol{m}| \le i$ applied at the vertices of Ω_ℓ. But $\mathcal{D}^{\boldsymbol{m}} f_\ell$ depends on $\mathcal{D}^{\boldsymbol{m}}[\sigma_k \circ T_\ell]$ for $|\boldsymbol{m}| \le i$ applied at the vertices of Ω_ℓ, which in turn depends on $\mathcal{D}^{\boldsymbol{m}} f$ for $|\boldsymbol{m}| \le i + k$, now evaluated at the vertices of Z_ℓ, which are also vertices of Ω. The theorem follows from plugging these asymptotic expansions into the definition of $Q^A_{g,s}[f,\Omega]$.

Q.E.D.

It is not necessary to find $\Theta_k[f]$ explicitly as we only use this asymptotic expansion for error analysis, not as a means of approximation. The following corollary serves the same purpose as Corollary 2.2: it will be used to prove the order of error for a multivariate Levin-type method.

Corollary 4.2. *Let V be the set of all vertices of a domain Ω. Suppose that $f = \boldsymbol{\mathcal{O}}(\omega^{-n})$. Suppose further that*

$$0 = \mathcal{D}^{\boldsymbol{m}} f(\boldsymbol{v})$$

for all $\boldsymbol{v} \in V$ and $\boldsymbol{m} \in \mathbb{N}^d$ such that $0 \leq |\,\boldsymbol{m}\,| \leq s-1$. Then

$$I_g[f,\Omega] \sim O\big(\omega^{-n-s-d}\big)\,.$$

Proof: We prove this corollary by induction on the dimension d, with the univariate case following from Corollary 2.2. We begin by showing that $Q^A_{g,s+d}[f,\Omega] = O\big(\omega^{-n-s-d}\big)$. Since every σ_k depends on f and its partial derivatives, it follows that $\sigma_k = \boldsymbol{\mathcal{O}}(\omega^{-n})$. Furthermore, $0 = \mathcal{D}^{\boldsymbol{m}}\sigma_k(\boldsymbol{v})$ for all $\boldsymbol{v} \in V$ and every $|\,\boldsymbol{m}\,| \leq s-k-1$, where $0 \leq k \leq s-1$. Hence (4.1) is of order $O\big(\omega^{-n-(s-k)-(d-1)}\big)$ for all $0 \leq k \leq s-1$. For $k \geq s$, we know that (4.1) is at least of order $O\big(\omega^{-n-(d-1)}\big)$. Since each (4.1) is multiplied by $(-\mathrm{i}\omega)^{-k-1}$ in the construction of $Q^A_{g,s+d}[f,\Omega]$, it follows that $Q^A_{g,s+d}[f,\Omega] = O\big(\omega^{-n-s-d}\big)$. Finally,

$$\big|\, I_g[f,\Omega] - Q^A_{g,s+d}[f,\Omega]\,\big| = \left|\, \frac{1}{(-\mathrm{i}\omega)^{-s-d}} I_g\left[\sigma_{s+d},\Omega\right] \,\right| = O\big(\omega^{-s-n-d}\big)\,,$$

since $\left\|\sigma_{s+d}\right\|_\infty = O(\omega^{-n})$. Thus $I_g[f,\Omega] \sim O\big(\omega^{-s-n-d}\big)$.

Q.E.D.

We find a generalization of Filon-type methods for multivariate integrals in [**12**]. As in the univariate case, the function f is interpolated by a polynomial v, and moments are assumed to be available. Define

$$Q^F_g[f,\Omega] = I_g[v,\Omega]\,,$$

where v is the Hermite interpolation polynomial of f at a given set of nodes $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_\nu\}$ with multiplicities $\{m_0,\ldots,m_\nu\}$, obtained by solving the system

$$\mathcal{D}^{\boldsymbol{m}} v(\boldsymbol{x}_k) = \mathcal{D}^{\boldsymbol{m}} f(\boldsymbol{x}_k), \qquad 0 \leq |\,\boldsymbol{m}\,| \leq m_k - 1, \qquad k = 0,1,\ldots,\nu.$$

From Corollary 4.2, it is clear that

$$Q^F_g[f,\Omega] - I_g[f,\Omega] = O\big(\omega^{-s-d}\big)\,,$$

where s is the minimum multiplicity associated with a vertex. Note that we require explicit formulæ for the moments $I_g\Big[x_1^{k_1}\cdots x_d^{k_d},\Omega\Big]$. This is a much more stringent condition in the multivariate setting than the univariate condition: it depends not only on the oscillator g, but also on the domain of integration Ω. However, knowledge of such moments is known if Ω is a simplex and g is affine—i.e., linear plus a constant.

Remark: In this section we used a weaker definition for the non-resonance condition than that which was found in [**12**]. Also, for the cited result in Theorem 4.1, we only require that g has no critical points, whereas the original statement requires that the non-resonance condition holds. This is due to the proofs cited from that paper holding true for the weaker conditions, without any other alterations.

5. Multivariate Levin-type method.

In this section, based on [**16**], we will derive a Levin-type method for the multivariate highly oscillatory integral $I_g[f,\Omega]$. As in the univariate case, we will not require moments. This enables the approximation of highly oscillatory integrals with more complicated oscillators and over more complicated domains than was possible with a Filon-type method. We begin by demonstrating how to derive a multivariate Levin-type

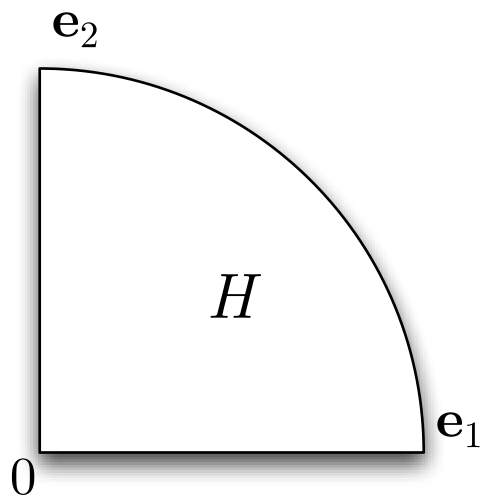


Figure 4: A unit quarter circle H, where $\boldsymbol{e}_1 = [1,0]^\top$ and $\boldsymbol{e}_2 = [0,1]^\top$.

method on a two-dimensional domain, namely a quarter unit circle H as seen in Figure 4. Afterwards, we generalize the technique to higher dimensional and more general domains.

In the univariate case, we determined the collocation operator $\mathcal{L}[v]$ using the fundamental theorem of calculus. We mimic this by using the generalized Stokes' theorem. Suppose we have a bivariate function $\boldsymbol{F}(x,y) = [F_1(x,y), F_2(x,y)]^\top$ such that

$$I[f] = \int_{\partial H} \mathrm{e}^{\mathrm{i}\omega g} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_{\partial H} \mathrm{e}^{\mathrm{i}\omega g} (F_1 \,\mathrm{d}y - F_2 \,\mathrm{d}x), \tag{5.1}$$

where $\mathrm{d}\boldsymbol{s} = [\,\mathrm{d}y, -\mathrm{d}x]^\top$ is the surface differential. Define the differential form $\rho = \mathrm{e}^{\mathrm{i}\omega g(x,y)} \boldsymbol{F}(x,y) \cdot \mathrm{d}\boldsymbol{s}$. Then

$$\begin{aligned} \mathrm{d}\rho &= (F_{1,x} + \mathrm{i}\omega g_x F_1) \mathrm{e}^{\mathrm{i}\omega g} \,\mathrm{d}x \wedge \mathrm{d}y - (F_{2,y} + \mathrm{i}\omega g_y F_2) \mathrm{e}^{\mathrm{i}\omega g} \,\mathrm{d}y \wedge \mathrm{d}x \\ &= (F_{1,x} + F_{2,y} + \mathrm{i}\omega (g_x F_1 + g_y F_2)) \mathrm{e}^{\mathrm{i}\omega g} \,\mathrm{d}x \wedge \mathrm{d}y \\ &= (\nabla \cdot \boldsymbol{F} + \mathrm{i}\omega \nabla g \cdot \boldsymbol{F}) \mathrm{e}^{\mathrm{i}\omega g} \,\mathrm{d}x \wedge \mathrm{d}y \\ &= \mathcal{L}[\boldsymbol{F}] \,\mathrm{e}^{\mathrm{i}\omega g} \,\mathrm{d}x \wedge \mathrm{d}y, \end{aligned} \tag{5.2}$$

where $\mathcal{L}[\boldsymbol{F}] = \nabla \cdot \boldsymbol{F} + \mathrm{i}\omega \nabla g \cdot \boldsymbol{F}$. We can rewrite the condition (5.1) as $\mathcal{L}[\boldsymbol{F}] = f$.

We now use the operator $\mathcal{L}[\boldsymbol{F}]$ to collocate f. Let $\boldsymbol{v}(x,y) = \sum_{k=0}^{n} c_k \boldsymbol{\psi}_k(x,y)$, for some basis $\{\boldsymbol{\psi}_k\}$, where $\boldsymbol{\psi}_k : \mathbb{R}^2 \to \mathbb{R}^2$. Given a sequence of nodes $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_\nu\} \subset \mathbb{R}^2$ and multiplicities $\{m_0, \ldots, m_\nu\}$, we determine the coefficients c_k by solving the system

$$\mathcal{D}^{\boldsymbol{m}} \mathcal{L}[\boldsymbol{v}]\,(\boldsymbol{x}_k) = \mathcal{D}^{\boldsymbol{m}} f(\boldsymbol{x}_k), \qquad 0 \le |\,\boldsymbol{m}\,| \le m_k - 1, \qquad k = 0, 1, \ldots, \nu,$$

where again $\boldsymbol{m} \in \mathbb{N}^d$ and $|\,\boldsymbol{m}\,|$ is the sum of the rows of the vector $\boldsymbol{m}$. We then obtain, using $T_1(t) =$

$[\cos t, \sin t]^\top$, $T_2(t) = [0, 1-t]^\top$ and $T_3(t) = [t, 0]^\top$ as the positively oriented boundary,

$$
\begin{aligned}
I_g[f, H] \approx I_g[\mathcal{L}[\boldsymbol{v}], H] &= \iint_H \mathcal{L}[\boldsymbol{v}]\, \mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x \wedge \mathrm{d}y = \iint_H \mathrm{d}\rho = \oint_{\partial H} \rho = \oint_{\partial H} \mathrm{e}^{\mathrm{i}\omega g} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} \\
&= \int_0^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}\omega g(T_1(t))} \boldsymbol{v}(T_1(t)) \cdot T_1'(t)\, dt + \int_0^1 \mathrm{e}^{\mathrm{i}\omega g(T_2(t))} \boldsymbol{v}(T_2(t)) \cdot T_2'(t)\, dt + \\
&\quad \int_0^1 \mathrm{e}^{\mathrm{i}\omega g(T_3(t))} \boldsymbol{v}(T_3(t)) \cdot T_3'(t)\, dt \\
&= \int_0^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}\omega g(\cos t, \sin t)} \left[v_2(\cos t, \sin t) \cos t - v_1(\cos t, \sin t) \sin t \right] dt - \\
&\quad \int_0^1 v_2(0, 1-t) \mathrm{e}^{\mathrm{i}\omega g(0,1-t)}\, dt + \int_0^1 v_1(t, 0) \mathrm{e}^{\mathrm{i}\omega g(t,0)}\, dt.
\end{aligned}
$$

This is a sum of three univariate highly oscillatory integrals, with oscillators $\mathrm{e}^{\mathrm{i}\omega g(\cos t, \sin t)}$, $\mathrm{e}^{\mathrm{i}\omega g(0,1-t)}$, and $\mathrm{e}^{\mathrm{i}\omega g(t,0)}$. If we assume that these three oscillators have no stationary points, which can be shown to be equivalent to the non-resonance condition, then we can approximate each of these integrals with a univariate Levin-type method, as described in Section 3. Hence we define:

$$
Q_g^L[f, H] = Q_{g_1}^L \left[f_1, \left(0, \frac{\pi}{2}\right) \right] + Q_{g_2}^L \left[f_2, (0, 1) \right] + Q_{g_3}^L \left[f_3, (0, 1) \right],
$$

for $f_1(t) = v_2(\cos t, \sin t) \cos t - v_1(\cos t, \sin t) \sin t$, $g_1(t) = g(\cos t, \sin t)$, $f_2(t) = -v_2(0, 1-t)$, $g_2(t) = g(0, 1-t)$, $f_3(t) = v_1(t, 0)$ and $g_3(t) = g(t, 0)$.

We approach the general case in a similar manner. Suppose we are given *nodes* $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_\nu\}$ in $\Omega \subset \mathbb{R}^d$, *multiplicities* $\{m_0, \ldots, m_\nu\}$ and *basis functions* $\{\boldsymbol{\psi}_k\}$, where $\boldsymbol{\psi}_k : \mathbb{R}^d \to \mathbb{R}^d$. Assume further that we are given a positive-oriented boundary of Ω defined by a set of functions $T_\ell : \Omega_\ell \to \mathbb{R}^d$, where $\Omega_\ell \subset \mathbb{R}^{d-1}$ and the ℓth boundary component Z_ℓ is the image of T_ℓ. Furthermore, assume we have the same information—nodes, multiplicities, basis and boundary parameterization—for each Ω_ℓ, recursively down to the one-dimensional edges. We define a Levin-type method $Q_g^L[f, \Omega]$ recursively as follows:

- If $\Omega = (a, b) \subset \mathbb{R}$, then $Q_g^L[f, \Omega]$ is equivalent to a univariate Levin-type method from Section 3.
- If $\Omega \subset \mathbb{R}^d$, the definition of $\mathcal{L}[\boldsymbol{v}]$ remains

  $$
  \mathcal{L}[\boldsymbol{v}] = \nabla \cdot \boldsymbol{v} + \mathrm{i}\omega \nabla g \cdot \boldsymbol{v}.
  $$

  Define $\boldsymbol{v} = \sum_{k=0}^n c_k \boldsymbol{\psi}_k$, where $n+1$ will be the number of equations in the system (5.3). We then determine the coefficients c_k by solving the collocation system

  $$
  \mathcal{D}^{\boldsymbol{m}} \mathcal{L}[\boldsymbol{v}]\,(\boldsymbol{x}_k) = \mathcal{D}^{\boldsymbol{m}} f(\boldsymbol{x}_k), \qquad 0 \le |\,\boldsymbol{m}\,| \le m_k - 1, \qquad k = 0, 1, \ldots, \nu. \tag{5.3}
  $$

  We now define

  $$
  Q_g^L[f, \Omega] = \sum Q_{g_\ell}^L \left[f_\ell, \Omega_\ell \right], \tag{5.4}
  $$

  where $g_\ell(\boldsymbol{x}) = g(T_\ell(\boldsymbol{x}))$ and $f_\ell = \boldsymbol{v}(T_\ell(\boldsymbol{x})) \cdot \boldsymbol{J}_{T_\ell}^d(\boldsymbol{x})$, cf. Appendix A for the definition of $\boldsymbol{J}_{T_\ell}^d(\boldsymbol{x})$. Assume that the nodes and multiplicities for each Levin-type method $Q_{g_\ell}^L[f_\ell, \Omega_\ell]$ contain the vertices of Ω_ℓ with the same multiplicity as the associated vertex of Ω. In other words, if $\boldsymbol{x}_j = T_\ell(\boldsymbol{u})$ is a vertex of Ω, then $\boldsymbol{u}$ has a multiplicity of m_j.

The *regularity condition* for the multivariate case is defined by the following two conditions:

- The basis $\{\nabla g \cdot \boldsymbol{\psi}_k\}$ can interpolate at the given nodes and multiplicities.
- The regularity condition is satisfied for each Levin-type method in the right-hand side of (5.4).

We thus derive the following theorem:

Theorem 5.1. *Suppose that both the non-resonance and regularity condition are satisfied. Suppose further that* $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_\nu\}$ *contains all the vertices of* Ω*, namely,* $\{\boldsymbol{x}_{i_1},\cdots,\boldsymbol{x}_{i_\nu}\}$*. Then*

$$I[f]-Q^L[f]\sim O\left(\omega^{-s-d}\right),$$

where $s=\min\{m_{i_1},\cdots,m_{i_\nu}\}$.

Proof: Assume the theorem holds for all dimensions less than d. The univariate case was proved in Theorem 3.1. We begin by showing that

$$I_g[f,\Omega]-I_g[\mathcal{L}[\boldsymbol{v}],\Omega]=I_g[f-\mathcal{L}[\boldsymbol{v}],\Omega]=O\left(\omega^{-s-d}\right).$$

This will follow if $\mathcal{L}[\boldsymbol{v}]=\boldsymbol{\mathcal{O}}(1)$. Let

$$\mathcal{P}[f]=\begin{pmatrix}\rho_0[f]\\ \vdots\\ \rho_\nu[f]\end{pmatrix},\qquad \text{for}\qquad \rho_k[f]=\begin{pmatrix}\mathcal{D}^{\boldsymbol{p}_{k,1}}f(\boldsymbol{x}_k)\\ \vdots\\ \mathcal{D}^{\boldsymbol{p}_{k,n_k}}f(\boldsymbol{x}_k)\end{pmatrix},\qquad k=0,1,\ldots,\nu,$$

where $\boldsymbol{p}_{k,1},\ldots,\boldsymbol{p}_{k,n_k}\in\mathbb{N}^d$, $n_k=\frac{1}{2}m_k(m_k+1)$, are the lexicographically ordered vectors such that $\left|\boldsymbol{p}_{k,i}\right|\le m_k-1$. As in the proof of Theorem 3.1, $\mathcal{P}[f]$ maps f to itself evaluated at the given nodes and multiplicities. Note that (5.3) has the form $A\boldsymbol{c}=\boldsymbol{f}$, where

$$A=[\mathcal{P}[\mathcal{L}[\boldsymbol{\psi}_0]],\cdots,\mathcal{P}[\mathcal{L}[\boldsymbol{\psi}_n]]]=[\mathcal{P}[\nabla\cdot\boldsymbol{\psi}_0]+\mathrm{i}\omega\mathcal{P}[\nabla g\cdot\boldsymbol{\psi}_0],\cdots,\mathcal{P}[\nabla\cdot\boldsymbol{\psi}_n]+\mathrm{i}\omega\mathcal{P}[\nabla g\cdot\boldsymbol{\psi}_n]]=P+\mathrm{i}\omega G,$$

for

$$P=[\mathcal{P}[\nabla\cdot\boldsymbol{\psi}_0],\cdots,\mathcal{P}[\nabla\cdot\boldsymbol{\psi}_n]],\qquad G=[\mathcal{P}[\nabla g\cdot\boldsymbol{\psi}_0],\cdots,\mathcal{P}[\nabla g\cdot\boldsymbol{\psi}_n]],\qquad \boldsymbol{f}=\mathcal{P}[f].$$

Note that G is the matrix associated with the system resulting from the basis $\{\nabla g\cdot\boldsymbol{\psi}_k\}$ interpolating at the given nodes and multiplicities, hence the regularity condition ensures that $\det G$ is nonsingular. By the same logic as in Theorem 3.1, it follows that the A is nonsingular for large ω and $c_k=O\left(\omega^{-1}\right)$. Thus $\mathcal{L}[\boldsymbol{v}]=\boldsymbol{\mathcal{O}}(1)$, and Corollary 4.2 states that $I_g[f,\Omega]-I_g[\mathcal{L}[\boldsymbol{v}],\Omega]=O\left(\omega^{-s-d}\right)$.

We now show that

$$Q_g^L[f,\Omega]-I_g[\mathcal{L}[\boldsymbol{v}],\Omega]=O\left(\omega^{-s-d}\right).$$

Define the differential form $\rho=\mathrm{e}^{\mathrm{i}\omega g}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}$, where $\mathrm{d}\boldsymbol{s}$ is the surface differential, cf. Appendix A. It can easily be seen that $\mathrm{d}\rho=\mathcal{L}[\boldsymbol{v}]\,\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}V$, see (5.2). Thus

$$I_g[\mathcal{L}[\boldsymbol{v}],\Omega]=\int_\Omega\mathrm{d}\rho=\int_{\partial\Omega}\rho=\sum_\ell\int_{Z_\ell}\rho,$$

where $Z_\ell=T_\ell(\Omega_\ell)$. Furthermore, using the definition of the integral of differential form, cf. Appendix A:

$$\begin{aligned}\int_{Z_\ell}\rho&=\int_{Z_\ell}\mathrm{e}^{\mathrm{i}\omega g}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}=\int_{\Omega_\ell}\mathrm{e}^{\mathrm{i}\omega g(T_\ell(\boldsymbol{x}))}\boldsymbol{v}(T_\ell(\boldsymbol{x}))\cdot\boldsymbol{J}_{T_\ell}^d(\boldsymbol{x})\,\mathrm{d}V\\&=\sum_{j=0}^n c_j\int_{\Omega_\ell}\mathrm{e}^{\mathrm{i}\omega g(T_\ell(\boldsymbol{x}))}\boldsymbol{\psi}_j(T_\ell(\boldsymbol{x}))\cdot\boldsymbol{J}_{T_\ell}^d(\boldsymbol{x})\,\mathrm{d}V\\&=\sum_{j=0}^n c_jI_{g_\ell}\left[f_{\ell,j},\Omega_\ell\right],\end{aligned}$$

for $f_{\ell,j}(\boldsymbol{x})=\boldsymbol{\psi}_j(T_\ell(\boldsymbol{x}))\cdot\boldsymbol{J}_{T_\ell}^d(\boldsymbol{x})$. By assumption, since the non-resonance and regularity conditions are satisfied, $Q_{g_\ell}^L\left[f_{\ell,j},\Omega_\ell\right]-I_{g_\ell}\left[f_{\ell,j},\Omega_\ell\right]=O\left(\omega^{-s-d+1}\right)$, where this Levin-type method has the same nodes and multiplicities as $Q_{g_\ell}^L\left[f_\ell,\Omega_\ell\right]$ in (5.4). Due to the linearity of Q^L, $Q_{g_\ell}^L\left[f_\ell,\Omega_\ell\right]=\sum_{j=0}^n c_jQ_{g_\ell}^L\left[f_{\ell,j},\Omega_\ell\right]$. Thus

$$Q_g^L[f,\Omega]-I_g[\mathcal{L}[\boldsymbol{v}],\Omega]=\sum_\ell\left(Q_{g_\ell}^L\left[f_\ell,\Omega_\ell\right]-\int_{Z_\ell}\rho\right)$$

$$
\begin{aligned}
&= \sum_{\ell} \sum_{j=0}^{n} c_j \left(Q^L_{g_\ell}\left[f_{\ell,j}, \Omega_\ell\right] - I_{g_\ell}\left[f_{\ell,j}, \Omega_\ell\right] \right) \\
&= \sum_{\ell} \sum_{j=0}^{n} O\big(\omega^{-1}\big)\, O\big(\omega^{-s-d+1}\big) = O\big(\omega^{-s-d}\big)\,.
\end{aligned} \tag{5.5}
$$

Putting both parts together we obtain that $I_g[f,\Omega] - Q^L_g[f,\Omega] = O\big(\omega^{-s-d}\big)$.

Q.E.D.

Admittedly the regularity condition seems strict, however in practice it typically holds. There is no equivalent to a Chebyshev set in higher dimensions [**3**], so we can not generalize Theorem 3.2. We can, however, under certain circumstances show that the regularity condition is satisfied whenever the standard polynomial basis can interpolate at the given nodes and multiplicities. The following corollary states, for simplicial domains and affine g, that a Levin-type method is equivalent to a Filon-type method with the standard polynomial basis. This is the main problem domain where Filon-type methods are effective, so in essence Levin-type methods are an extension to Filon-type methods.

Corollary 5.2. *If g is affine, then $I_g[\mathcal{L}[\boldsymbol{v}], \Omega] = Q^F_g[f,\Omega]$ whenever $\boldsymbol{\psi}_k = \psi_k\, \boldsymbol{t}$, where ψ_k is the standard polynomial basis and $\boldsymbol{t} \in \mathbb{R}^d$ is chosen so that $\boldsymbol{t} \cdot \nabla g \neq 0$. Furthermore, if Ω is the d-dimensional simplex S_d, then $Q^L_g[f, S_d]$ is equivalent to $Q^F_g[f, S_d]$ whenever a sufficient number of sample points are taken.*

Proof: Note that solving a Levin-type method collocation system is equivalent to interpolating with the basis $\tilde{\psi}_j = \mathcal{L}\big[\boldsymbol{\psi}_j\big] = \boldsymbol{t} \cdot \nabla \psi_j + \mathrm{i}\omega \psi_j \boldsymbol{t} \cdot \nabla g$. We begin by showing that $\tilde{\psi}_k$ and ψ_k are equivalent. Assume that $\left\{\tilde{\psi}_0, \ldots, \tilde{\psi}_{j-1}\right\}$ has equivalent span to $\left\{\psi_0, \ldots, \psi_{j-1}\right\}$. This is true for the case $\psi_0 \equiv 1$ since $\mathcal{L}[\boldsymbol{t}] = \mathrm{i}\omega \boldsymbol{t} \cdot \nabla g = C$, where $C \neq 0$ by hypothesis. Note that $\psi_j(x_1, \ldots, x_d) = x_1^{p_1} \ldots x_d^{p_d}$ for some nonnegative integers p_k. Then, for $\boldsymbol{t} = [t_1, \cdots, t_d]^\top$,

$$
\begin{aligned}
\tilde{\psi}_j =& \mathrm{i}\omega \psi_j \boldsymbol{t} \cdot \nabla g + \boldsymbol{t} \cdot \nabla \psi_j = C\psi_j + \sum_{k=1}^{d} t_k \mathcal{D}^{\boldsymbol{e}_k} \psi_j \\
=& C\psi_j + \sum_{k=1}^{d} t_k p_k x_1^{p_1} \ldots x_{k-1}^{p_{k-1}} x_k^{p_k - 1} x_{k+1}^{p_{k+1}} \ldots x_d^{p_d}.
\end{aligned}
$$

The sum is a polynomial of degree less than the degree of ψ_j, hence it lies in the span of $\left\{\psi_0, \ldots, \psi_{j-1}\right\}$. Thus ψ_j lies in the span of $\left\{\tilde{\psi}_0, \ldots, \tilde{\psi}_j\right\}$. It follows that interpolation by each of these two bases is equivalent, or in other words $I_g[\mathcal{L}[\boldsymbol{v}], \Omega] = Q^F_g[f, \Omega]$.

We prove the second part of the theorem by induction, where the case of $\Omega = S_1$ holds true by the definition $Q^L_g[f, S_1] = I_g[\mathcal{L}[\boldsymbol{v}], S_1]$. Now assume it is true for each dimension less than d. Since g is affine and each boundary T_ℓ of the simplex is affine we know that each g_ℓ is affine. Furthermore we know that the Jacobian determinants of T_ℓ are constants, hence each f_ℓ is a polynomial. Thus $Q^L_{g_\ell}\left[f_\ell, S_{d-1}\right] = Q^F_{g_\ell}\left[f_\ell, S_{d-1}\right] = I_{g_\ell}\left[f_\ell, S_{d-1}\right]$, as long as enough sample points are taken so that f_ℓ lies in the span of the interpolation basis. Hence $Q^L_g[f, S_d] = I_g[\mathcal{L}[\boldsymbol{v}], S_d] = Q^F_g[f, S_d]$.

Q.E.D.

An important consequence of this corollary is that, in the two-dimensional case, a Levin-type method provides an approximation whenever the standard polynomial basis can interpolate f at the given nodes and multiplicities, assuming that g is affine and the non-resonance condition is satisfied in Ω.

We can now demonstrate the effectiveness of this method with several numerical examples. For simplicity, we take $\boldsymbol{\psi}_k = \psi_k\, \mathbf{1}$, where ψ_k is the d-dimensional polynomial basis. Note that this attaches an artificial orientation to this approximation scheme, however, this will not affect the asymptotics of the method. We begin with the case of integrating over a simplex, which Corollary 5.2 showed is equivalent to a Filon-type method. Let $f(x,y,z,t) = x^2$, $g(x,y,z,t) = x - 2y + 3z - 4t$ and approximate $I_g[f, S_4]$ by $Q^L_g[f, S_4]$ collocating only at the vertices with multiplicities all one. As expected, we obtain an error of order $O\big(\omega^{-5}\big)$, as

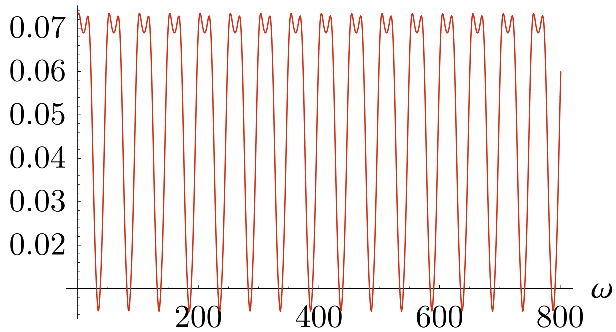
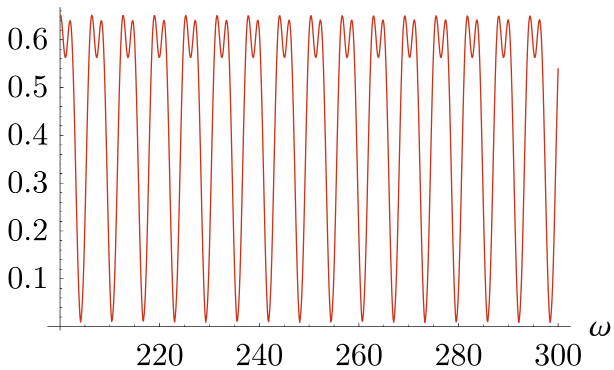


Figure 5: The error scaled by ω^5 of $Q_g^L[f, S_4]$ collocating only at the vertices with multiplicities all one, for $I_g[f, S_4] = \int_{S_4} x^2 \mathrm{e}^{\mathrm{i}\omega(x-2y+3z-4t)}\,\mathrm{d}V$.



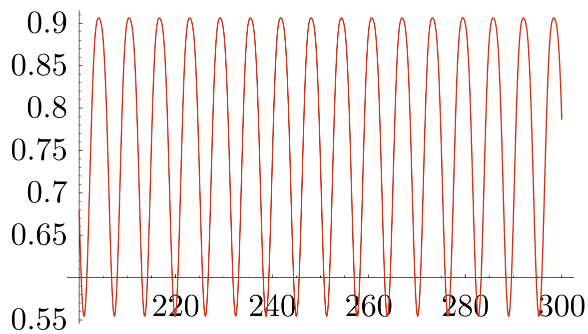


Figure 6: The error scaled by ω^3 of $Q_g^L[f, S_2]$ collocating only at the vertices with multiplicities all one (left figure), and the error scaled by ω^4 with vertex multiplicities all two and an additional point at $\left[\frac{1}{3}, \frac{1}{3}\right]^\top$ with multiplicity one (right figure), for $I_g[f, S_2] = \int_{S_2} \left(\frac{1}{x+1} + \frac{2}{y+1}\right) \mathrm{e}^{\mathrm{i}\omega(2x-y)}\,\mathrm{d}V$.

seen in Figure 5. Because this Levin-type method is equivalent to a Filon-type method, it would have solved this integral exactly had we increased the number of node points so that $\psi_k(x, y, z, t) = x^2$ was included as a basis vector.

Now consider the more complicated function $f(x, y) = \frac{1}{x+1} + \frac{2}{y+1}$ with oscillator $g(x, y) = 2x - y$, approximated by $Q_g^L[f, S_2]$, again only sampling at the vertices with multiplicities all one. As expected we obtain an order of error of $O(\omega^{-3})$. By adding an additional multiplicity to each vertex, as well as the sample point $\left[\frac{1}{3}, \frac{1}{3}\right]^\top$ with multiplicity one to ensure that we have ten equations in our system as required by polynomial interpolation, we increase the order by one to $O(\omega^{-4})$. Both of these cases can be seen in Figure 6. Note that the different scale factor means that the right-hand graph is in fact much more accurate, as it has about $1/\omega$th the error.

Because Levin-type methods do not require moments, they allow us to integrate over more complicated domains that satisfy the non-resonance condition, without resorting to tessellation. For example, we return to the case of the quarter unit circle H. Let $f(x, y) = \mathrm{e}^x \cos xy$, $g(x, y) = x^2 + x - y^2 - y$, and choose vertices for nodes with multiplicities all one. Note that g is nonlinear, in addition to the domain not being a simplex. Despite these difficulties, $Q_g^L[f, H]$ still attains an order of error $O(\omega^{-3})$, as seen in the left hand side of Figure 7. If we increase the multiplicities at the vertices to two, adding an additional node at $\left[\frac{1}{3}, \frac{1}{3}\right]^\top$ with multiplicity one, we obtain an error of order $O(\omega^{-4})$. This can be seen in the right-hand side of Figure 7. This example is significant since, due to the unavailability of moments, Filon-type methods fail to provide

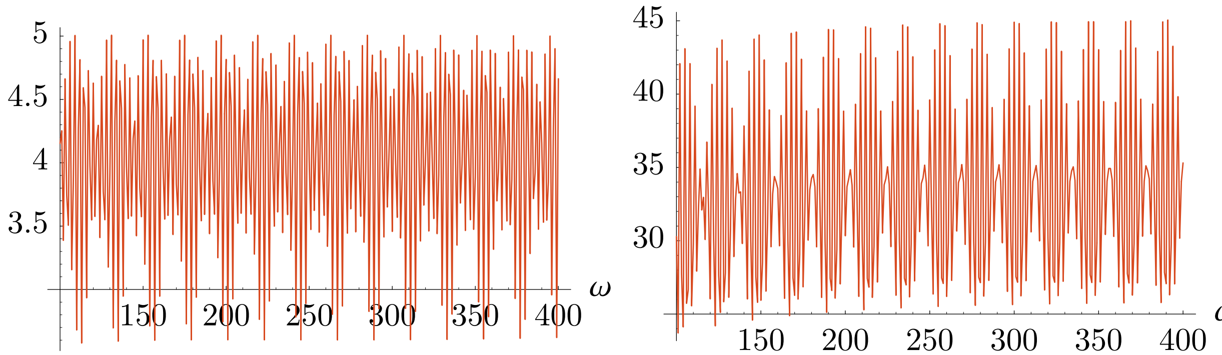


Figure 7: The error scaled by ω^3 of $Q_g^L[f,H]$ collocating only at the vertices with multiplicities all one (left figure), and the error scaled by ω^4 with vertex multiplicities all two and an additional point at $\left[\frac{1}{3},\frac{1}{3}\right]^\top$ with multiplicity one (right figure), for $I_g[f,H] = \int_H \mathrm{e}^x \cos xy \; \mathrm{e}^{\mathrm{i}\omega(x^2+x-y^2-y)} \,\mathrm{d}V$.

approximations in a quarter circle, let alone with nonlinear g. Were g linear, we could have tessellated H to obtain a polytope, but that would have resulted in an unnecessarily large number of calculations. With nonlinear g we do not even have this option, hence Filon-type methods are completely unsuitable.

6. Asymptotic basis condition.

It is important to note that, for a Levin-type method, there is no particular reason to use polynomials for $\{\boldsymbol{\psi}_k\}$. Not only can we greatly improve the accuracy of the approximation by choosing the basis wisely, but surprisingly we can even obtain higher asymptotic order. The *asymptotic basis condition* is satisfied if the basis $\{\boldsymbol{\psi}_0,\ldots,\boldsymbol{\psi}_n\}$ satisfies the following conditions:

$$\nabla g \cdot \boldsymbol{\psi}_1 = f, \qquad \nabla g \cdot \boldsymbol{\psi}_{k+1} = \nabla \cdot \boldsymbol{\psi}_k, \qquad k = 1, 2, \ldots.$$

For the univariate case, this condition becomes

$$\psi_1 = \frac{f}{g'}, \qquad , \psi_{k+1} = \frac{\psi_k'}{g'}, \qquad k = 1, 2, \ldots.$$

Note that this is equivalent to defining $\psi_k = \sigma_k$, where σ_k was defined in the asymptotic expansion, cf. Theorem 2.1, hence the term asymptotic basis condition. Surprisingly, this increases the asymptotic order to $O\big(\omega^{-\tilde{n}-s-d}\big)$, where s is again the minimum vertex multiplicity and $\tilde{n}+1$ is equal to the minimum of the number of equations in every collocation system (5.3) solved for in the definition of Q^L, recursively down to the univariate integrals. It follows that if $\Omega \subset \mathbb{R}$, then $\tilde{n} = n$. As an example, if we are collocating on a two-dimensional simplex at only the three vertices with multiplicities all one, then the initial collocation system has three equations, whilst each boundary collocation system has only two equations. Thus $\tilde{n}+1 = \min\{3,2,2,2\} = 2$, and the order is $O\big(\omega^{-2-1-2}\big) = O\big(\omega^{-5}\big)$.

The following lemma is used extensively in the proof of the asymptotic order:

Lemma 6.1. *Suppose $\{\boldsymbol{\psi}_k\}$ satisfies the asymptotic basis condition. Then, for $k \geq 1$,*

$$\det\left[\boldsymbol{g}_k, \boldsymbol{a}_k, \cdots, \boldsymbol{a}_{k+j}, B\right] = \det\left[\boldsymbol{g}_k, \boldsymbol{g}_{k+1}, \cdots, \boldsymbol{g}_{k+j+1}, B\right],$$

where B represents all remaining columns that render the matrices square and $\boldsymbol{a}_k = \boldsymbol{p}_k + \mathrm{i}\omega \boldsymbol{g}_k$, for

$$\boldsymbol{p}_k = \mathcal{P}[\nabla \cdot \boldsymbol{\psi}_k], \qquad \boldsymbol{g}_k = \mathcal{P}[\nabla g \cdot \boldsymbol{\psi}_k].$$

Proof: We know that $\boldsymbol{p}_k = \mathcal{P}[\nabla \cdot \boldsymbol{\psi}_k] = \mathcal{P}\left[\nabla g \cdot \boldsymbol{\psi}_{k+1}\right] = \boldsymbol{g}_{k+1}$. Thus we can multiply the first column by $\mathrm{i}\omega$ and subtract it from the second to obtain

$$\det\left[\boldsymbol{g}_k, \boldsymbol{p}_k + \mathrm{i}\omega \boldsymbol{g}_k, \cdots, \boldsymbol{a}_{k+j}, B\right] = \det\left[\boldsymbol{g}_k, \boldsymbol{g}_{k+1}, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_{k+j}, B\right].$$

The lemma follows by repeating this process on the remaining columns.

Q.E.D.

This lemma holds for any column interchange on both sides of the determinant. We can now prove the theorem:

Theorem 6.2. *Suppose every basis $\{\boldsymbol{\psi}_k\}$ in a Levin-typem method satisfies the asymptotic basis condition. Then*

$$Q_g^L[f,\Omega] - I_g[f,\Omega] \sim O\left(\omega^{-\tilde{n}-s-1}\right).$$

Proof: We begin by showing that $\mathcal{L}[\boldsymbol{v}] - f = \boldsymbol{\mathcal{O}}(\omega^{-n})$. Note that

$$\begin{aligned}
\mathcal{L}[\boldsymbol{v}] - f =& \sum_{k=0}^{n} c_k \mathcal{L}[\boldsymbol{\psi}_k] - f = \sum_{k=0}^{n} c_k \left(\nabla \cdot \boldsymbol{\psi}_k + \mathrm{i}\omega \nabla g \cdot \boldsymbol{\psi}_k\right) - f \\
=& c_0 \nabla \cdot \boldsymbol{\psi}_0 + \mathrm{i}\omega c_0 \nabla g \cdot \boldsymbol{\psi}_0 + \sum_{k=1}^{n} c_k \left(\nabla g \cdot \boldsymbol{\psi}_{k+1} + \mathrm{i}\omega \nabla g \cdot \boldsymbol{\psi}_k\right) - \nabla g \cdot \boldsymbol{\psi}_1 \\
=& c_0 \nabla \cdot \boldsymbol{\psi}_0 + \nabla g \cdot \left[\mathrm{i}\omega c_0 \boldsymbol{\psi}_0 + (\mathrm{i}\omega c_1 - 1)\boldsymbol{\psi}_1 + \sum_{k=2}^{n} (c_{k-1} + \mathrm{i}\omega c_k)\boldsymbol{\psi}_k + c_n \boldsymbol{\psi}_{n+1}\right] \\
=& \frac{\det D_0}{\det A} \nabla \cdot \boldsymbol{\psi}_0 + \frac{\nabla g}{\det A} \cdot \Bigg[\mathrm{i}\omega \det D_0 \boldsymbol{\psi}_0 + (\mathrm{i}\omega \det D_1 - \det A)\boldsymbol{\psi}_1 \\
& \qquad + \sum_{k=2}^{n} (\det D_{k-1} + \mathrm{i}\omega \det D_k)\boldsymbol{\psi}_k + \det D_n \boldsymbol{\psi}_{n+1}\Bigg],
\end{aligned}$$

where again D_k is the matrix A with the $(k+1)$th column replaced by $\boldsymbol{f}$. We know that $(\det A)^{-1} = O\left(\omega^{-n-1}\right)$, thus it remains to be shown that each term in the preceding equation is $O(\omega)$. This boils down to showing that each of the following terms are $O(\omega)$: $\mathrm{i}\omega \det D_0$, $\mathrm{i}\omega \det D_1 - \det A$, $\det D_{k-1} + \mathrm{i}\omega \det D_k$ for $2 \le k \le n$ and finally $\det D_n$. The first case follows directly from Lemma 6.1, since $\boldsymbol{f} = \mathcal{P}[f] = \mathcal{P}[\nabla g \cdot \boldsymbol{\psi}_1] = \boldsymbol{g}_1$, hence

$$\det D_0 = \det\left[\boldsymbol{g}_1, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\right] = \det\left[\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_{n+1}\right] = O(1).$$

The second case follows from Lemma 6.1 after rewriting the determinants as

$$\begin{aligned}
\mathrm{i}\omega \det D_1 - \det A =& \mathrm{i}\omega \det D_1 - \det\left[\boldsymbol{a}_0, \boldsymbol{p}_1 + \mathrm{i}\omega \boldsymbol{g}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\right] \\
=& \mathrm{i}\omega \det D_1 - \mathrm{i}\omega \det\left[\boldsymbol{a}_0, \boldsymbol{g}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\right] - \det\left[\boldsymbol{a}_0, \boldsymbol{p}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\right] \\
=& -\det\left[\boldsymbol{a}_0, \boldsymbol{g}_2, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\right] = O(\omega),
\end{aligned}$$

where we used the facts that $\boldsymbol{p}_1 = \boldsymbol{g}_2$. Similarly,

$$\begin{aligned}
\det D_{k-1} + \mathrm{i}\omega \det D_k =& \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_1, \boldsymbol{p}_k + \mathrm{i}\omega \boldsymbol{g}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
& + \mathrm{i}\omega \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{p}_{k-1} + \mathrm{i}\omega \boldsymbol{g}_{k-1}, \boldsymbol{g}_1, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
=& \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_1, \boldsymbol{p}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
& + \mathrm{i}\omega \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_1, \boldsymbol{g}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
& + \mathrm{i}\omega \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_k, \boldsymbol{g}_1, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
& - \omega^2 \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_{k-1}, \boldsymbol{g}_1, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
=& \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_1, \boldsymbol{p}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right] \\
& - \omega^2 \det\left[\boldsymbol{a}_0, \cdots, \boldsymbol{a}_{k-2}, \boldsymbol{g}_{k-1}, \boldsymbol{g}_1, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n\right].
\end{aligned}$$

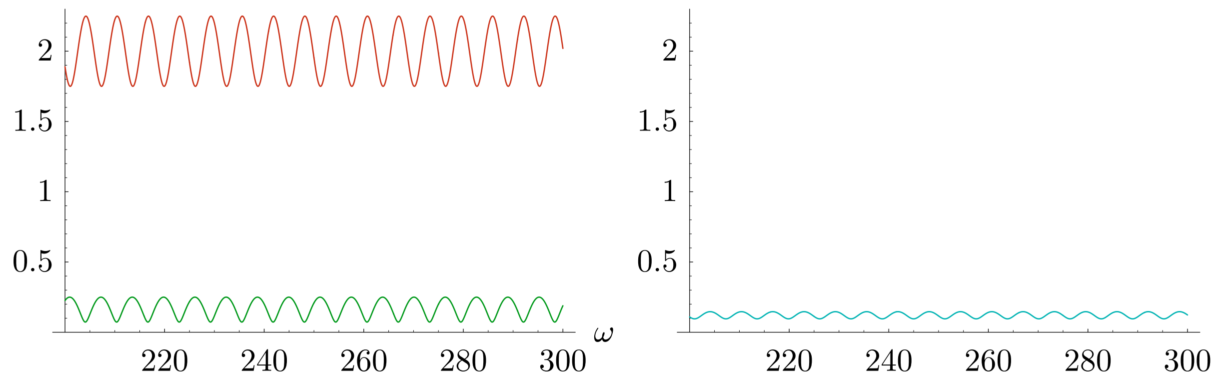


Figure 8: The error scaled by ω^4 of the asymptotic expansion (left figure, top), $Q^F[f]$ with endpoints for nodes and multiplicities two (left figure, bottom), and $Q^B[f]$ with nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities all one (right figure) for $I[f] = \int_0^1 \log(x+1)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$.

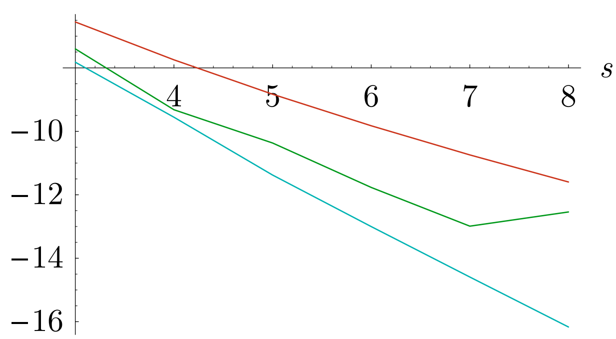


Figure 9: The base-10 logarithm of the error of the s-term asymptotic expansion (top), $Q^F[f]$ with endpoints for nodes and multiplicities s (middle), and $Q^B[f]$ with nodes $\{k/(s-1)\}_{k=0}^{s-1}$ and multiplicities all one (bottom) for $I[f] = \int_0^1 \log(x+1)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$.

Using Lemma 6.1 the first of these determinants is $O(\omega)$, whilst the second determinant has two columns equal to $\boldsymbol{g}_{k-1}$, hence is equal to zero. The last determinant $\det D_n$ is also $O(\omega)$, due to Lemma 6.1. Thus we have shown that $\mathcal{L}[\boldsymbol{v}] - f = \boldsymbol{\mathcal{O}}(\omega^{-n})$.

From Corollary 4.2, it follows that $I_g[f,\Omega] - I_g[\mathcal{L}[v],\Omega] = O\left(\omega^{-n-s-d}\right) = O\left(\omega^{-\tilde{n}-s-d}\right)$. For the univariate case the lemma has been proved, since $Q_g^L[f,(a,b)] = I_g[\mathcal{L}[v],(a,b)]$. By induction, $Q_{g_\ell}^L\left[f_{\ell,j},\Omega_\ell\right] - I_{g_\ell}\left[f_{\ell,j},\Omega_\ell\right] = O\left(\omega^{-\tilde{n}-s-(d-1)}\right)$ in (5.5). It follows that

$$
\begin{aligned}
I_g[f,\Omega] - Q_g^L[f,\Omega] &= \left(I_g[f,\Omega] - I_g[\mathcal{L}[v],\Omega]\right) - \left(Q_g^L[f,\Omega] - I_g[\mathcal{L}[v],\Omega]\right) \\
&= O\left(\omega^{-\tilde{n}-s-d}\right).
\end{aligned}
$$

Q.E.D.

We will use $Q^B[f]$ to denote a Levin-type method whose basis satisfies the asymptotic basis condition. In the univariate case, we assume that $\psi_0 \equiv 1$. Consider the integral with the Fourier oscillator and $f(x) = \log(x+1)$. We compare methods of order $O\left(\omega^{-4}\right)$. This includes the three-term asymptotic exansion, $Q^F[f]$ (which is equivalent to $Q^L[f]$) with nodes $\{0,1\}$ and multiplicities both two, and $Q^B[f]$ using nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities all one. With this set up we obtain Figure 8. The results are decent, with $Q^B[f]$ being slightly more accurate than $Q^F[f]$ on average.

The problem with the asymptotic expansion and $Q^F[f]$ with endpoints for nodes and multiplicities both s is that, in general, as $s \to \infty$ these methods diverge. Hence another worthwhile comparison is to see how

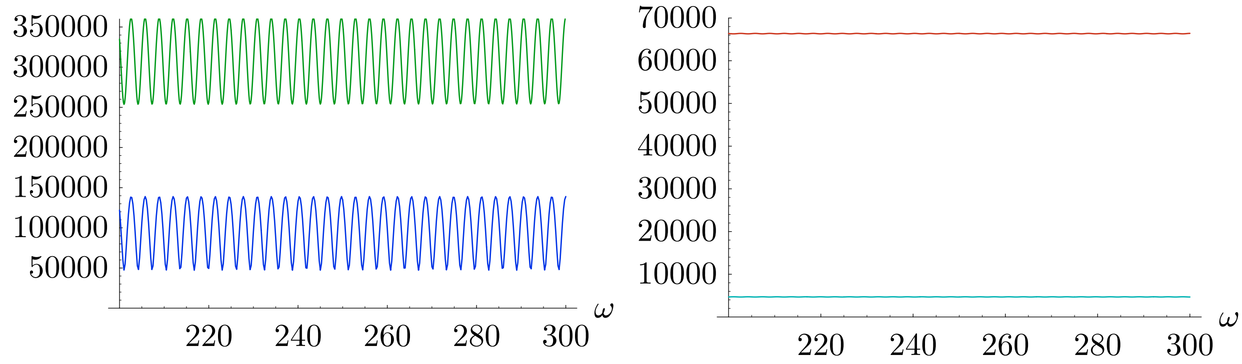


Figure 10: The error scaled by ω^3 of $Q^F[f]$ with endpoints and multiplicities both two (left figure, top), $Q^L[f]$ with endpoints and multiplicities both two (left figure, bottom), the asymptotic expansion (right figure, top), and $Q^B[f]$ with endpoints and multiplicities all one (right figure, bottom) for $I[f] = \int_0^1 \mathrm{e}^{10x}\mathrm{e}^{\mathrm{i}\omega\left(x^2+x\right)}\,\mathrm{d}x$.

s	Asym. expan.	$Q^F[f]$	$Q^L[f]$	$Q^B[f]$
2	0.0083	0.042	0.015	0.00059
3	0.00011	0.0016	0.00043	$2.8 \cdot 10^{-6}$
5	$1.7 \cdot 10^{-8}$	$1.3 \cdot 10^{-6}$	$3 \cdot 10^{-7}$	$9.9 \cdot 10^{-12}$

Table 1: The absolute value of the errors for $\omega = 200$ of the following methods of order $O\left(\omega^{-s-1}\right)$: the s-term asymptotic expansion, $Q^F[f]$ and $Q^L[f]$ with endpoints and multiplicities both s, and $Q^B[f]$ with nodes $\{k/(s-1)\}_{k=0}^{s-1}$ and multiplicities all one for $I[f] = \int_0^1 \mathrm{e}^{10x}\mathrm{e}^{200\mathrm{i}\left(x^2+x\right)}\,\mathrm{d}x$.

$Q^B[f]$ compares to these two methods for fixed ω and increasing asymptotic order. Thus fix $\omega = 50$, chosen purposely relatively small since the larger ω, the longer it takes for increasing the asymptotic order to cause the approximation to diverge. This choice results in Figure 9, where we take the base-10 logarithm of the errors. This figure clearly shows the benefit of using $Q^B[f]$ for this particular case. Though at lower orders the errors of $Q^F[f]$ and $Q^B[f]$ are very similar, at higher orders they differ by orders of magnitude.

A problem exists whenever f is not easily approximated by polynomials. In [**15**], the current author examined in detail the affect of Runge's phenomenon on Filon-type and Levin-type methods. Another similar situation is when f increases much too rapidly to be accurately approximated by polynomials. Let $f(x) = \mathrm{e}^{10x}$ and $g(x) = x^2 + x$. Note that this appears to be a ludicrously difficult example—not only do we have high oscillation but f exceeds $22,000$ in the interval of integration! Amazingly, we will see that the methods described within this paper are still very accurate, especially a Levin-type method with asymptotic basis. We compare $Q^B[f]$ which has only endpoints for nodes and multiplicities all one to the asymptotic expansion, $Q^L[f]$ and $Q^F[f]$ with only endpoints for nodes and multiplicities both two in Figure 10. In Table 1, we compare each method with different asymptotic orders. Even with only four sample points, $Q^B[f]$ has the astoundingly small error of $9.93 \cdot 10^{-12}$. This example demonstrates just how powerful these quadrature techniques are compared to Gauss-Legendre quadrature: even with $100,000$ panels Gauss-Legendre quadrature had an error of 0.11, not even close to the accuracy of the Filon-type method, to say nothing of $Q^B[f]$.

We now turn our attention to the bivariate case. For the remainder of this section, and for historical reasons, we will use the basis $\boldsymbol{\psi}_k = [\psi_k, -\psi_k]^\top$, where

$$\psi_0 \equiv 1, \qquad \psi_1 = \frac{f}{g_x - g_y}, \qquad \psi_{k+1} = \frac{\psi_{k,x} - \psi_{k,y}}{g_x - g_y}, \qquad k = 1, 2, \ldots.$$

This satisfies the asymptotic basis condition, since

$$\nabla g \cdot \boldsymbol{\psi}_1 = \frac{f}{g_x - g_y} \nabla g \cdot [1, -1]^\top = f, \qquad \nabla g \cdot \boldsymbol{\psi}_{k+1} = \frac{\psi_{k,x} - \psi_{k,y}}{g_x - g_y} \nabla g \cdot [1, -1]^\top = \psi_{k,x} - \psi_{k,y} = \nabla \cdot \boldsymbol{\psi}_k.$$

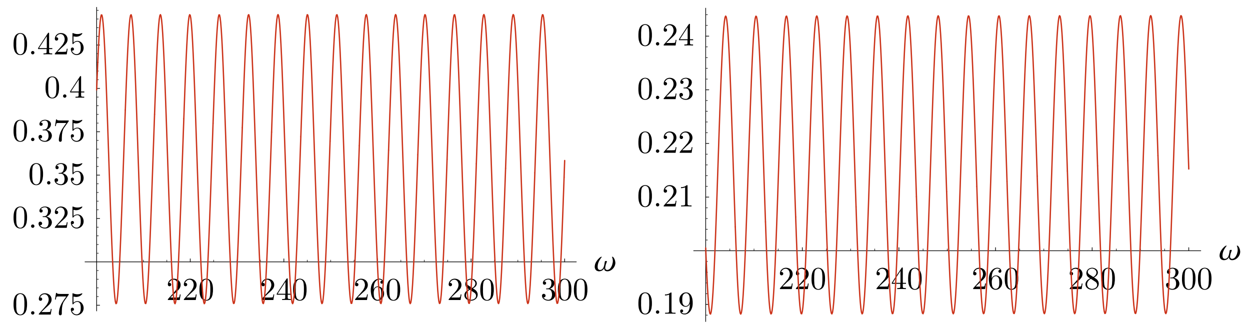


Figure 11: The error scaled by ω^4 of $Q_g^B[f, S_2]$ collocating only at the vertices with multiplicities all one (left figure), and the error scaled by ω^5 with vertices and boundary midpoints $\left\{[1/2, 0]^\top, [0, 1/2]^\top, [1/2, 1/2]^\top\right\}$ again with multiplicities all one (right figure), for $\int_{S_2} \left(\frac{1}{x+1} + \frac{2}{y+1}\right) \mathrm{e}^{\mathrm{i}\omega(2x-y)}\,\mathrm{d}V$.

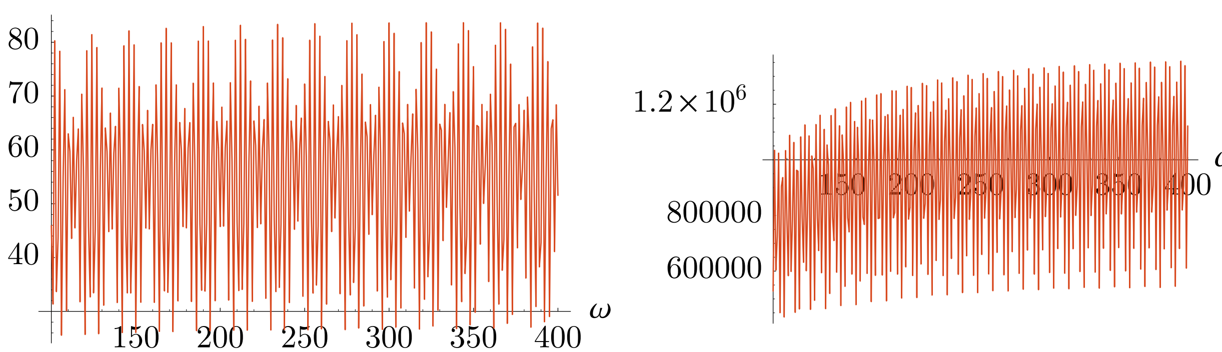


Figure 12: The error scaled by ω^4 of $Q_g^B[f, H]$ collocating only at the vertices with multiplicities all one (left figure), and the error scaled by ω^7 of $Q_g^B[f, H]$ collocating only at the vertices with multiplicities all two (left figure), for $I_g[f, H] = \int_H \mathrm{e}^x \cos xy \, \mathrm{e}^{\mathrm{i}\omega(x^2+x-y^2-y)}\,\mathrm{d}V$.

Recall the case where $f(x, y) = \frac{1}{x+1} + \frac{2}{y+1}$ with oscillator $g(x, y) = 2x - y$ over the simplex S_2. We now use $Q_g^B[f, S_2]$ in place of $Q_g^L[f, S_2]$, collocating only at the vertices. Since this results in each univariate boundary collocation having two node points, we know that $\tilde{n} = 1$. Hence we now scale the error by ω^4, i.e., we have increased the order by one, as seen in Figure 11. Since the initial two-dimensional system has three node points, adding the midpoint to the sample points of each univariate integral should increase the order again by one to $O\left(\omega^{-5}\right)$. This can be seen in the right-hand side of Figure 11.

There is nothing special about a simplex or linear g: the asymptotic basis works equally well on other domains with nonlinear g, assuming that the regularity and non-resonance conditions are satisfied. Recall the example with $f(x, y) = \mathrm{e}^x \cos xy$ and $g(x, y) = x^2 + x - y^2 - y$ on the quarter circle H. As in the simplex case, $Q_g^B[f, H]$ collocating only at vertices with multiplicities all one results in an error of $O\left(\omega^{-4}\right)$, as seen in the left-hand side of Figure 12. Note that increasing multiplicities not only increases s, but also $\tilde{n}$. If we increase the multiplicities to two, then $s = 2$ and $\tilde{n} = 3$, and the order increases to $O\left(\omega^{-7}\right)$, as seen in the right-hand side of Figure 12. It should be emphasized that, though the scale is large in the graph, the error is being divided by $\omega^7 \geq 100^7 = 10^{14}$. As a result, the errors for the right-hand graph are in fact less than the errors in the left-hand graph.

7. Higher order oscillators.

Many of the techniques discussed so far can be generalized for use with other oscillators besides the exponential oscillator. As an example, consider the integral

$$I[f] = \int_a^b f y_\omega \,\mathrm{d}x = \int_a^b f(x)\,\mathrm{Ai}\,(-\omega x)\,\mathrm{d}x,$$

where $0 < a < b$. In order to imitate the preceding sections, we need to first derive an asymptotic expansion. To accomplish this, we use an idea due to David Levin for the Airy case: replace $\mathrm{Ai}\,(-\omega x)$ by $-(\omega x)^{-1}\mathrm{Ai}''(-\omega x)$ and integrate by parts twice. We can handle other oscillators which solve differential equations in the same manner: simply write y_ω in terms of its derivatives and integrate by parts. For notational brevity, we write y in place of y_ω.

Theorem 7.1. *Assume that $f = \mathcal{O}(1)$ for increasing ω. Suppose that y satisfies a differential equation of the form*

$$py'' + qy' + \omega^\gamma r y = 0.$$

Assume that p and q are independent of ω, $r \neq 0$ in the domain of integration and $1/r = \mathcal{O}(1)$. If $\gamma > 0$, then

$$I[f] = \int_a^b f y \,\mathrm{d}x \sim \sum_{k=1}^\infty \omega^{-k\gamma} \left[\left(\frac{\sigma_k p}{r}\right)' y - \frac{\sigma_k q}{r} y - \frac{\sigma_k p}{r} y'\right]_a^b,$$

where

$$\sigma_1 = f, \qquad \sigma_{k+1} = \left(\frac{\sigma_k q}{r}\right)' - \left(\frac{\sigma_k p}{r}\right)''.$$

Proof: Let $u = 1/r$. Integrating by parts twice, we obtain

$$\begin{aligned}
I[f] &= \int_a^b f y \,\mathrm{d}x = -\int_a^b \frac{fpy'' + fqy'}{\omega^\gamma r}\,\mathrm{d}x = -\omega^{-\gamma}\int_a^b fpuy'' + fquy'\,\mathrm{d}x \\
&= \omega^{-\gamma}\left[-fpuy' - fquy\right]_a^b + \omega^{-\gamma}\int_a^b (fpu)'y' + (fqu)'y\,\mathrm{d}x \\
&= \omega^{-\gamma}\left[-fpuy' - fquy + (fpu)'y\right]_a^b + \omega^{-\gamma}\int_a^b \left[(fqu)' - (fpu)''\right] y\,\mathrm{d}x \\
&= Q[f] + \omega^{-\gamma} I[\sigma_2]\,.
\end{aligned}$$

where $Q[f] = \omega^{-\gamma}\left[-fpuy' - fquy + (fpu)'y\right]_a^b$. Since $\sigma_2 = (fqu)' - (fpu)'' = \mathcal{O}(1)$, we obtain an asymptotic expansion using induction.

Q.E.D.

We can derive a similar asymptotic expansion when y satisfies the differential equation

$$\omega^\alpha py'' + \omega^\beta qy' + \omega^\gamma ry = 0,$$

however, we will not investigate this case since all of our examples are in the form of Theorem 7.1. Corollary 7.2 follows immediately from the asymptotic expansion. It is based on Corollary 2.2, and can likewise be used to prove the order of error for Filon-type and Levin-type methods.

Corollary 7.2. *Assume that $\|y\|_\infty$ and $\|y'\|_\infty$ are $\mathrm{o}(\omega^\gamma)$, and that $\|y\|_\infty = \mathrm{o}(\|y'\|_\infty)$ as $\omega \to \infty$. Suppose that $f = \mathcal{O}(\omega^{-n})$ and $\gamma > 0$. If*

$$\begin{aligned}
0 &= f(a) = f'(a) = \cdots = f^{(s-1)}(a)\,, \\
0 &= f(b) = f'(b) = \cdots = f^{(s-1)}(b)\,,
\end{aligned}$$

then

$$I[f] \sim \begin{cases} O\left(\omega^{-n-\gamma(s+1)/2}\,\|y\|_\infty\right), & \text{if } s \text{ is odd;} \\ O\left(\omega^{-n-\gamma(s+2)/2}\,\|y'\|_\infty\right), & \text{if } s \text{ is even.} \end{cases}$$

For the case of the Airy function, we know that

$$y''(x) = \omega^2 \mathrm{Ai}''(-\omega x) = -\omega^3 x \mathrm{Ai}\,(-\omega x) = -\omega^3 x y(x)\,.$$

It follows that $p(x) = 1$, $q(x) = 0$, $r(x) = x$ and $\gamma = 3$. Thus we obtain the asymptotic expansion:

$$\begin{aligned} I[f] &\sim -\sum_{k=1}^{\infty} \frac{1}{\omega^{3k}} \left[\frac{\sigma_k(x)}{x} y'(x) + \left(\frac{\sigma_k(x)}{x} \right)' y(x) \right]_a^b \\ &= \sum_{k=1}^{\infty} \frac{1}{\omega^{3k}} \left[\omega \frac{\sigma_k(x)}{x} \mathrm{Ai}'(-\omega x) - \left(\frac{\sigma_k(x)}{x} \right)' \mathrm{Ai}\,(-\omega x) \right]_a^b , \end{aligned}$$

for $\sigma_1(x) = f(x)$ and $\sigma_{k+1}(x) = \left(\frac{\sigma_k(x)}{x} \right)''$. Using the fact that $\mathrm{Ai}'(-\omega x) = O\big(\omega^{1/4}\big)$ and $\mathrm{Ai}\,(-\omega x) = O\big(\omega^{-1/4}\big)$ [**2**], we determine that the s-step asymptotic expansion has an error of order $O\Big(\omega^{-\frac{3}{2}s - \frac{7}{4}}\Big)$. Note that we are counting the $\omega \frac{\sigma_k(x)}{x} \mathrm{Ai}'(-\omega x)$ and $\left(\frac{\sigma_k(x)}{x} \right)' \mathrm{Ai}\,(-\omega x)$ terms as separate steps in the asymptotic expansion.

A Filon-type approximation follows immediately, since we know explicit formulæ for the moments of the Airy function in terms of the scorer function Gi and its derivative, cf. [**2**]. Clearly, if the interpolating polynomial has multiplicity at least s at each endpoint, then the error term is of order $O\Big(\omega^{-\frac{3}{2}s - \frac{7}{4}}\Big)$. However, though we have formulæ for the moments, computation of the scorer functions is difficult for large values of ω, though can be accomplished by using techniques from [**5**]. Furthermore, moments are not available for other functions we might want to integrate using these techniques: for example $y(x) = \mathrm{Ai}\,(-\omega g(x))$ for more complicated functions g.

To combat these issues, we will again derive a Levin-type method that does not require moments. The collocation in Section 3 depends on the oscillator satisfying a first order differential equation, which allows us to integrate the function explicitly. Unfortunately, second order ODEs do not lend themselves as well to collocation. If we write the integral as a system of first order differential equations, we can use a generalization of the vector-valued version of the original Levin method [**13**]. As in the exponential oscillator case, we generalize this method by adding multiplicities. Using our method of proof, we have the extra benefit of reducing the number of equations needed to obtain a given asymptotic order.

Assuming that $p \equiv 1$—which can be made true whenever $p \neq 0$—we can rewrite any function that satisfies a second order ODE as a system of first order ODEs. Our oscillator y leads to the system

$$\boldsymbol{y}'(x) = A(x)\boldsymbol{y}(x), \qquad A(x) = \begin{pmatrix} 0 & 1 \\ -\omega^\gamma r(x) & -q(x) \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}.$$

We rewrite the highly oscillatory integral as

$$I[f] = \int_a^b f y \,\mathrm{d}x = \int_a^b \boldsymbol{\varphi}\, \boldsymbol{y} \,\mathrm{d}x,$$

where $\boldsymbol{\varphi} = [f, 0]$ is a row vector-valued function. We now collocate by another row vector-valued function $\boldsymbol{v} = [v_1, v_2]$ using the operator

$$\mathcal{L}[\boldsymbol{v}] = \boldsymbol{v}' + \boldsymbol{v} A,$$

where $\boldsymbol{v} = \sum_{k=0}^{n} c_k \boldsymbol{\psi}_k$ for some set of basis functions $\{\boldsymbol{\psi}_k\}$, where $\boldsymbol{\psi}_k : \mathbb{R} \to \mathbb{R}^2$. We will not require that the multiplicity for each dimension of the operator $\mathcal{L}[\boldsymbol{v}]$ is the same. Thus assume we are given nodes $\{x_0, \ldots, x_\nu\}$ and multiplicities $\left\{m_0^{(1)}, \ldots, m_\nu^{(1)}\right\}, \left\{m_0^{(2)}, \ldots, m_\nu^{(2)}\right\}$. Again we assume that $x_0 = a$ and $x_\nu = b$. Define the operators $l_1[\boldsymbol{v}]$ and $l_2[\boldsymbol{v}]$ so that $\mathcal{L}[\boldsymbol{v}] = [l_1[\boldsymbol{v}]\,, l_2[\boldsymbol{v}]]$. In other words,

$$\begin{aligned} l_1[\boldsymbol{v}] &= v_1' - \omega^\gamma r v_2, \\ l_2[\boldsymbol{v}] &= v_2' + v_1 - q v_2. \end{aligned}$$

We determine the coeffiecents c_k by solving the collocation system

$$\begin{array}{lll} l_1[\boldsymbol{v}](x_k) = f(x_k), & \cdots, & \mathcal{D}^{m_k^{(1)}} l_1[\boldsymbol{v}](x_k) = \mathcal{D}^{m_k^{(1)}} f(x_k), \\ l_2[\boldsymbol{v}](x_k) = 0, & \cdots, & \mathcal{D}^{m_k^{(2)}} l_2[\boldsymbol{v}](x_k) = 0, \end{array} \qquad k = 0, 1, \ldots, \nu. \tag{7.1}$$

Then

$$I[f] = \int_a^b \boldsymbol{\varphi}\, \boldsymbol{y} \,\mathrm{d}x \approx \int_a^b \mathcal{L}[\boldsymbol{v}]\, \boldsymbol{y} \,\mathrm{d}x = [\boldsymbol{v}\boldsymbol{y}]_a^b = [v_1 y + v_2 y']_a^b .$$

As in the exponential oscillator case, we require a regularity condition. Define

$$G = [\mathcal{P}[\boldsymbol{\psi}_0], \cdots, \mathcal{P}[\boldsymbol{\psi}_n]], \qquad \mathcal{P}[f_1, f_2] = \begin{pmatrix} \rho_{0,1}[-\omega^\gamma r f_2] \\ \vdots \\ \rho_{\nu,1}[-\omega^\gamma r f_2] \\ \rho_{0,2}[f_2' + f_1 - q f_2] \\ \vdots \\ \rho_{\nu,2}[f_2' + f_1 - q f_2]. \end{pmatrix} \qquad \rho_{k,j}[g] = \begin{pmatrix} g(x_k) \\ g'(x_k) \\ \vdots \\ g^{\left(m_k^{(j)}-1\right)}(x_k) \end{pmatrix}.$$

The regularity condition is satisfied if G is nonsingular.

We can now prove the order of error for this method:

Theorem 7.3. *In addition to the hypotheses of Theorem 7.1, assume that $p \equiv 1$ and $r = \boldsymbol{\mathcal{O}}(1)$. If the regularity condition is satisfied, then, for $s = \min\left\{m_0^{(1)}, m_\nu^{(1)}, m_0^{(2)} + 1, m_\nu^{(2)} + 1\right\}$,*

$$I[f] - Q^L[f] \sim \begin{cases} O\left(\omega^{-\gamma(s+1)/2} \|y\|_\infty\right), & \text{if } s \text{ is odd;} \\ O\left(\omega^{-\gamma(s+2)/2} \|y'\|_\infty\right), & \text{if } s \text{ is even .} \end{cases}$$

where

$$Q^L[f] = [\boldsymbol{v}\boldsymbol{y}]_a^b = [v_1 y + v_2 y']_a^b .$$

Proof: Using Cramer's rule—in a manner similar to the proof of Theorem 3.1—we will determine that each coefficient c_k for the function $\boldsymbol{v}$ is of order $O(\omega^{-\gamma})$. The system (7.1) can be written as $A\boldsymbol{c} = \boldsymbol{f}$, for $A = P + G$, where G was defined in the regularity condition and

$$P = [\mathcal{R}[\boldsymbol{\psi}_0], \cdots, \mathcal{R}[\boldsymbol{\psi}_n]], \qquad \mathcal{R}[f_1, f_2] = \begin{pmatrix} \rho_{0,1}[f_1'] \\ \vdots \\ \rho_{\nu,1}[f_1'] \\ \rho_{0,2}[0] \\ \vdots \\ \rho_{\nu,2}[0] \end{pmatrix} = \begin{pmatrix} \rho_{0,1}[f_1'] \\ \vdots \\ \rho_{\nu,1}[f_1'] \\ \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} \rho_{0,1}[f] \\ \vdots \\ \rho_{\nu,1}[f] \\ \mathbf{0} \end{pmatrix}.$$

The regularity condition ensures that $\det G \neq 0$, hence $(\det A)^{-1} = O(\omega^{-n_1\gamma})$, where $n_1 = \sum_{k=0}^{\nu} m_k^{(1)}$. Again, let D_k be the matrix A with its $(k+1)$th column replaced by $\boldsymbol{f}$. This has one less column of order $O(\omega^\gamma)$, hence $\det D_k = O\left(\omega^{(n_1-1)\gamma}\right)$. It follows that $c_k = \det D_k (\det A)^{-1} = O(\omega^{-\gamma})$ and $\boldsymbol{v} = \boldsymbol{\mathcal{O}}(\omega^{-\gamma})$. As a result $l_1 = \boldsymbol{\mathcal{O}}(1)$ and $l_2 = \boldsymbol{\mathcal{O}}(\omega^{-\gamma})$.

Note that the function values and at least the first $s-1$ and $s-2$ derivatives of $f - l_1[\boldsymbol{v}]$ and $l_2[\boldsymbol{v}]$, respectively, are zero at the endpoints. We can write

$$I[f] - Q^L[f] = I[f] - \int_a^b l_1[\boldsymbol{v}] y + l_2[\boldsymbol{v}] y' \,\mathrm{d}x = I[f - l_1[\boldsymbol{v}]] - \int_a^b l_2[\boldsymbol{v}] y' \,\mathrm{d}x = I[f - l_1[\boldsymbol{v}]] - [l_2[\boldsymbol{v}] y]_a^b + I[l_2[\boldsymbol{v}]'] .$$

Due to Corollary 7.2, $I[f - l_1]$ has the correct order. If s is one and l_2 is not even zero at the endpoints, then $[l_2 y]_a^b = O(\omega^{-\gamma} \|y\|_\infty)$, and we obtain the correct order. For $s \geq 2$ we have $[l_2 y]_a^b = 0$, thus we can focus on the $I[l_2']$ term. Because we have already taken a derivative, we know that only the first $s-3$ derivatives are zero for l_2'. However, $l_2' = \boldsymbol{\mathcal{O}}(\omega^{-\gamma})$, hence we know from Corollary 7.2 that $I[l_2']$ also has the correct order.

Q.E.D.

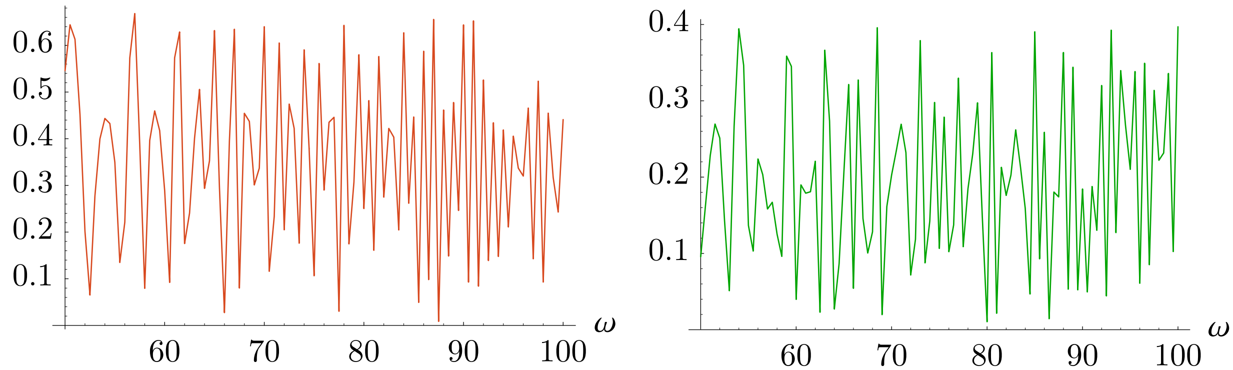


Figure 13: The error scaled by $\omega^{13/4}$ of the asymptotic expansion (left figure) and $Q^L[f]$ collocating only at the endpoints with multiplicities all one (right figure), for $I[f] = \int_1^2 \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x$.

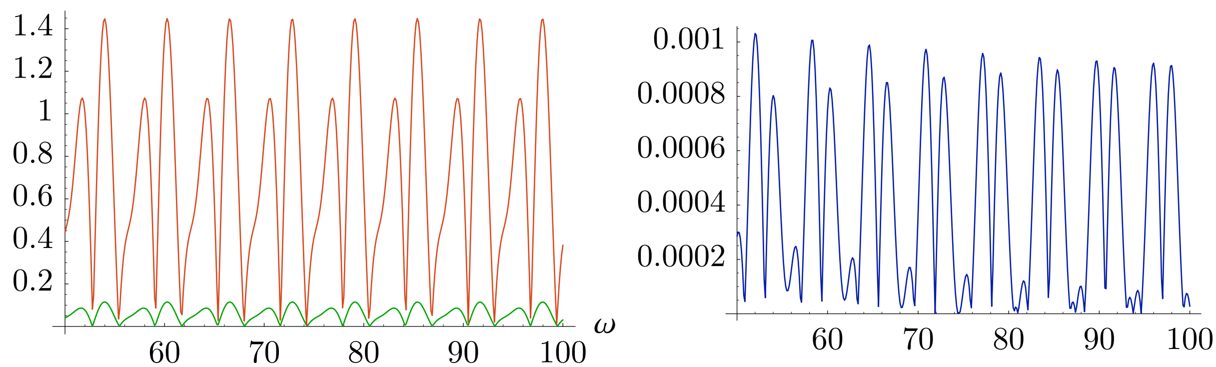


Figure 14: The error scaled by $\omega^{5/2}$ of the asymptotic expansion (left figure, top), $Q^L[f]$ collocating only at the endpoints with multiplicities both one (left figure, bottom) and $Q^L[f]$ collocating at the nodes $\{1, 5/4, 3/2, 7/4, 2\}$ with multiplicities all one (right figure), for $I[f] = \int_1^2 \cos x \, J_0(\omega x)\,\mathrm{d}x$.

For the Airy case, $Q^L[f]$ approximates $I[f]$ with an order of error $O\left(\omega^{-\frac{3}{2}s-\frac{7}{4}}\right)$. We begin with the simple example of computing the zeroth moment $\int_1^2 \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x$. In Figure 13 we compare the asymptotic expansion with order $\omega^{-13/4}$, namely $-\omega^{-2}\left[x^{-1}\mathrm{Ai}\,'(-\omega x)\right]_1^2$, to a Levin-type method, collocating only at the endpoints with multiplicities all one, using the standard polynomial basis

$$\boldsymbol{\psi}_k(x) = \begin{cases} \left[x^{k/2}, 0\right], & \text{if } k \text{ is even;} \\ \left[0, x^{(k-1)/2}\right], & \text{if } k \text{ is odd.} \end{cases}$$

This graph shows that this Levin-type method is slightly more accurate than the asymptotic expansion. Not pictured is the error when we collocate at the midpoint, in addition to the endpoints. This reduces the error of the Levin-type method further, to less than $0.13\omega^{-\frac{13}{4}}$.

Though we so far have focused on the Airy function, this technique works with other oscillators as well. Consider the Bessel function $y(x) = J_0(\omega x)$, where we know from [**2**] that

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0.$$

As a result, y satisfies the differential equation

$$x^2 y'' + x y' + \omega^2 x^2 y = 0.$$

Dividing by x^2 to ensure that $p \equiv 1$, we obtain $q(x) = x^{-1}$, $\gamma = 2$ and $r \equiv 1$. Assuming that zero is not in our interval of integration, we can use the methods developed in this section. Using the fact that

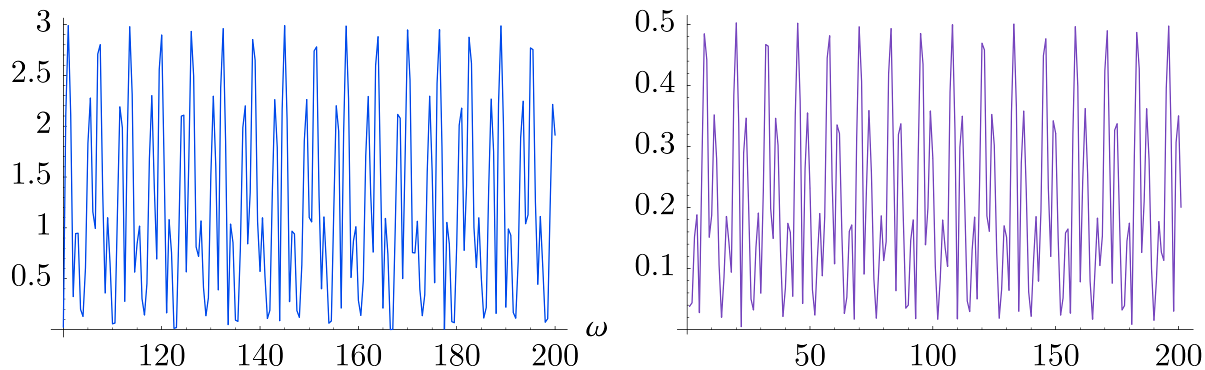


Figure 15: The error scaled by $\omega^{5/2}$ of $Q^L[f]$ collocating only at the endpoints with multiplicities all one (left figure) and the error scaled by $\omega^{7/2}$ of $Q^L[f]$ collocating only at the endpoints with multiplicities all two (right figure), for $I[f] = \int_1^2 \mathrm{e}^x\, J_2(\omega x)\,\mathrm{d}x$.

$J_0(\omega x) = O\left(\omega^{-\frac{1}{2}}\right)$ and $J_0'(\omega x) = -J_1(\omega x) = O\left(\omega^{-\frac{1}{2}}\right)$, cf. [**2**], we determine that $\|y\|_\infty = O\left(\omega^{-\frac{1}{2}}\right)$ and $\|y'\|_\infty = O\left(\omega^{\frac{1}{2}}\right)$. Thus a Levin-type method will have an error of order $O\left(\omega^{-s-\frac{3}{2}}\right)$. Consider the highly oscillatory integral $\int_1^2 \cos x\, J_0(\omega x)\,\mathrm{d}x$. In Figure 14 we compare the one-term asymptotic expansion to the Levin-type method collocating only at the endpoints with multiplicities one, and the Levin-type method collocating at $\left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}$. This figure emphasizes how much of an improvement can be made over the asymptotic expansion without significantly increasing computational costs.

As another example, consider the case where $y(x) = J_2(\omega x)$. Now y satisfies the differential equation

$$x^2 y'' + x y' + (\omega^2 x^2 - 4) y = 0.$$

We have the same parameters as J_0, except now $r(x) = 1 - 4/(\omega^2 x^2)$. Assume that $\omega > 2$, in order to ensure that r is nonzero within the interval of integration. It is not hard to see that r and $1/r$ are $\mathcal{O}(1)$, hence we can proceed without difficulty. For the integral $\int_1^2 \mathrm{e}^x J_2(\omega x)\,\mathrm{d}x$, Figure 15 compares $Q^L[f]$ collocating at the endpoints with multiplicities all one to the same with multiplicities all two. As can be seen, the order does indeed increase by one.

8. Unbounded integration domains and infinite oscillations.

In the following two sections, we begin to investigate what happens when the fairly stringent conditions on g and Ω are lifted. The first question is how the methods handle the case where Ω is unbounded, for example $\Omega = (a, \infty)$. Consider the integral

$$E_1(-\mathrm{i}\omega) = \int_1^\infty \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x}\,\mathrm{d}x,$$

where E_1 is the exponential integral [**2**]. This function is important since we can derive the cosine integral Ci and sine integral Si from its real and imaginary parts. As before, we begin by deriving an asymptotic expansion, where the assumption that $a > 0$ can be weakened by reparameterizing the integral:

Theorem 8.1. *Let $\Omega = (a, \infty)$ for $a > 0$. Suppose that g and its derivatives are bounded in (a, ∞), g' does not approach zero in Ω, $f(x) \to 0$ as $x \to \infty$ and $\left(\frac{f(x)}{g'(x)}\right)' = x^\alpha u(x)$, for a smooth function u such that it and its derivatives are bounded. If $\alpha < -1$, then*

$$I[f] \sim \sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \sigma_k(a) \mathrm{e}^{\mathrm{i}\omega g(a)},$$

where, as before,

$$\sigma_1 = \frac{f}{g'}, \qquad \sigma_{k+1} = \frac{\sigma_k'}{g'}, \qquad k \geq 1.$$

Proof: Expanding out the first term of the asymptotic expansion we have

$$\int_a^M f\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x = \frac{1}{\mathrm{i}\omega}\left[\frac{f}{g'}\mathrm{e}^{\mathrm{i}\omega g}\right]_a^M - \frac{1}{\mathrm{i}\omega}\int_a^M \left(\frac{f(x)}{g'(x)}\right)' \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x.$$

We know that $\frac{f(M)}{g'(M)}\mathrm{e}^{\mathrm{i}\omega g(M)} \to 0$ as $M \to \infty$, since g' does not approach zero. Furthermore, the integral $I\left[\left(\frac{f}{g}\right)'\right]$ converges absolutely, since the integrand decays faster than x^{-1}. Finally, we obtain

$$\left(\frac{\sigma_1'(x)}{g'(x)}\right)' = \left(\frac{x^\alpha u(x)}{g'(x)}\right)' = x^\alpha\left(\frac{u'(x)}{g'(x)} + \alpha\frac{u(x)}{xg'(x)} - \frac{u(x)g''(x)}{g'(x)^2}\right).$$

It is not hard to see that $\frac{u'(x)}{g'(x)} + \alpha\frac{u(x)}{xg'(x)} - \frac{u(x)g''(x)}{g'(x)^2}$ is smooth and it and its derivatives are bounded, thus $\sigma_1'(x)$ satisfies the conditions on f, and the theorem follows by induction.

Q.E.D.

A version of Corollary 2.2 follows immediately, where now f only depends on the endpoint a. We cannot, however, use this corollary to derive a Filon-type method, since polynomials do not decay at infinity. We can show that Levin-type methods do work with any basis:

Theorem 8.2. *Suppose that f and g satisfy the requirements of Theorem 8.1. Then, using the notation of Theorem 3.1,*

$$Q^L[f] - I[f] = O\left(\omega^{-s-1}\right),$$

where $s = m_0$ and

$$Q^L[f] = -v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

Proof: Suppose each function in the set $\{\psi_0, \ldots, \psi_n\}$ satisfies the conditions on f in Theorem 8.1. Then the proof of this theorem is unaltered from Theorem 3.1, since $I[\mathcal{L}[v]] = Q^L[f]$. If $\{\psi_0, \ldots, \psi_n\}$ does not satisfy the conditions, we replace it by a basis $\left\{\tilde\psi_0, \ldots, \tilde\psi_n\right\}$ that does satisfy these properties. Define $\tilde\psi_k(x)$ so that it equals $\psi_k(x)$ for all $x_0 \le x \le x_\nu$, goes to zero smoothly in $x_\nu < x < N < \infty$ for some fixed constant $N > x_\nu$ and $\tilde\psi_k(x) \equiv 0$ for $N \le x < \infty$. The collocation system (3.1) with this new basis is unchanged from the original collocation system, hence $Q^L[f]$ is also unchanged. However, $\tilde\psi_k$ now satisfies the requisite properties, and the theorem follows.

Q.E.D.

Returning to the E_1 case, we obtain an asymptotic expansion

$$E_1(-\mathrm{i}\omega) \sim \mathrm{e}^{\mathrm{i}\omega}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!}{(-\mathrm{i}\omega)^k}.$$

It should come as no surprise that this is equivalent to the expansion in [**2**]. We can use the asymptotic basis with a Levin-type method to derive an approximation. Consider the case of arbitrarily chosen nodes $\{1,\ 5,\ 10,\ 20,\ \infty\}$ with multiplicities all one. This has an order of error $O\left(\omega^{-6}\right)$, thus we compare it to the asymptotic expansion of order $O\left(\omega^{-6}\right)$ in the left-hand side of Figure 16. Even with arbitrarily chosen nodes, $Q^B[f]$ is substantially more accurate than the asymptotic expansion; in this case it has less than a tenth of the error. We can also compare the real parts of each approximation to $-\mathrm{Ci}\,(\omega)$, where Ci is the cosine integral as defined in [**2**]. This results in the right-hand side of Figure 16.

Another potential issue is when there are an infinite number of oscillations within the interval of integration. For example, consider the integral

$$\int_0^1 \mathrm{e}^{\mathrm{i}\omega x^{-1}}\,\mathrm{d}x.$$

The convergence of such integrals follows from the definition of a Riemann integral. Assuming g' goes to infinity at a sufficiently fast rate, we can indeed derive an asymptotic expansion:

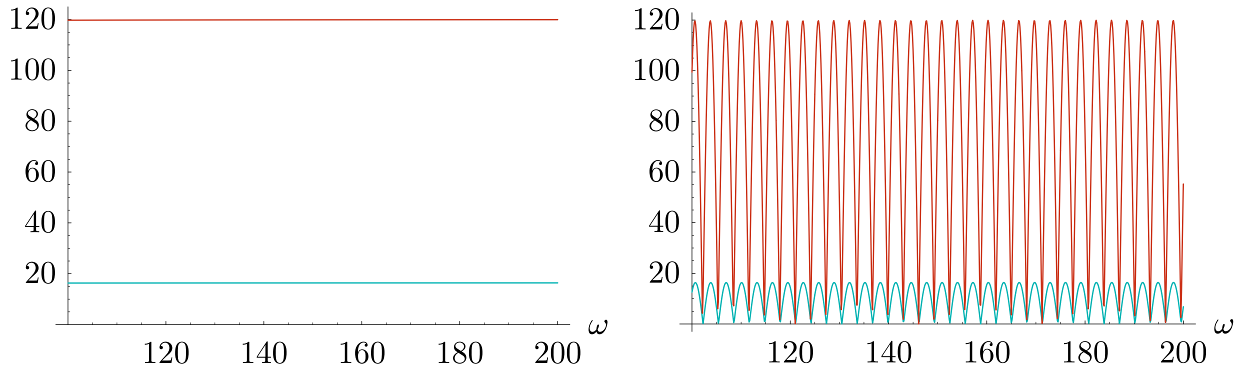


Figure 16: On the left, the error scaled by ω^6 of the asymptotic expansion (top) and $Q^B[f]$ with nodes $\{1, 5, 10, 20, \infty\}$ and multiplicities all one (bottom) for $I[f] = \int_1^\infty \frac{1}{x} \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}x$ compared to $E_1(-\mathrm{i}\omega)$. On the right, the real parts of the same approximations compared to $-\mathrm{Ci}\,(\omega)$.

Theorem 8.3. *Suppose that g is smooth, g' is nonzero in $[a, b)$, $1/g'(x) = (x-b)^\alpha u(x)$ and $f(x) = (x-b)^\beta v(x)$, where $\alpha \geq 1$. Suppose further that u, v and their derivatives are bounded. If $\alpha + \beta \geq 1$, then*

$$I[f] \sim \sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \sigma_k(a) \mathrm{e}^{\mathrm{i}\omega g(a)}.$$

Proof: Note that

$$\int_a^\epsilon f \mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x = \frac{1}{\mathrm{i}\omega}\left[\frac{f}{g'}\mathrm{e}^{\mathrm{i}\omega g}\right]_a^\epsilon - \frac{1}{\mathrm{i}\omega}\int_a^\epsilon \left(\frac{f}{g'}\right)' \mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x = \frac{1}{\mathrm{i}\omega}\left[(x-b)^{\alpha+\beta} uv \mathrm{e}^{\mathrm{i}\omega g}\right]_a^\epsilon - \frac{1}{\mathrm{i}\omega}\int_a^\epsilon (x-b)^{\alpha+\beta-1}\tilde{v}\mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x,$$

where $\tilde{v} = (\alpha+\beta)uv + (x-b)(uv)'$, which satisfies the conditions on v. Since $\alpha + \beta \geq 1 > 0$, we know that $(x-b)^{\alpha+\beta} \to 0$ as $\epsilon \to b$. Furthermore, $\tilde{\beta} = \alpha + \beta - 1 > 0$, hence the integrand is bounded. Thus we let $\epsilon \to b$ to obtain

$$I[f] = \frac{1}{\mathrm{i}\omega}\frac{f(b)}{g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{1}{\mathrm{i}\omega}\int_a^b (x-b)^{\tilde{\beta}}\tilde{v}\mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x.$$

Since $\alpha + \tilde{\beta} = 2\alpha + \beta - 1 \geq \alpha \geq 1$, we can repeat the process with $(x-b)^{\tilde{\beta}}\tilde{v}$ in place of f. The asymptotic expansion follows by induction.

Q.E.D.

An equivalent theorem holds over unbounded intervals:

Corollary 8.4. *Assume that $a > 0$. Consider the integral over (a, ∞), where $\frac{1}{g'(x)} = x^\alpha u(x)$, $f(x) = x^\beta v(x)$ and $\alpha < 0$. If $\alpha + \beta < 0$, then*

$$I[f] \sim \sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \sigma_k(a) \mathrm{e}^{\mathrm{i}\omega g(a)}.$$

Proof: The proof to this corollary is similar to Theorem 8.3. Let s be an integer large enough so that $s\alpha + \beta \leq -2$. Then the s-term expansion over (a, M) is

$$-\sum_{k=1}^{s} \frac{1}{(-\mathrm{i}\omega)^k}\left\{\sigma_k(M) - \sigma_k(a)\right\} + \frac{1}{(-\mathrm{i}\omega)^s}\int_a^M \sigma_s' \mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}x.$$

Note that $\sigma_1(x) = x^{\alpha+\beta}u(x)v(x)$, and $\sigma_1'(x) = x^{\alpha+\beta}((\alpha+\beta)x^{-1}u(x)v(x) + (u(x)v(x))'$. Hence $\sigma_k(x) = x^{k\alpha+\beta}\tilde{v}$ for some smooth function $\tilde{v}$, where $\tilde{v} = \mathcal{O}(1)$. It follows that the terms evaluated at M of the expansion vanish as $M \to \infty$. Furthermore the integral $I[\sigma_s']$ converges absolutely, since $|\,\sigma_s'(x)\,| \leq Cx^{s\alpha+\beta} \leq C'x^{-2}$.

Q.E.D.

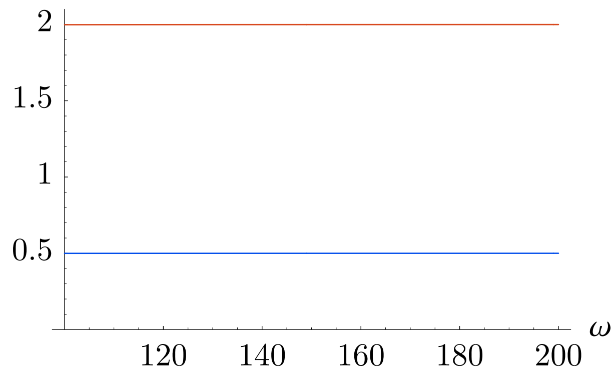


Figure 17: Errors scaled by ω^2 of the asymptotic expansion (top) compared to a Levin-type method collocating at $\{1/2, 1\}$ with multiplicities both one, for $I[f] = \int_0^1 \mathrm{e}^{\mathrm{i}\omega x^{-1}} \, \mathrm{d}x$.

A Filon-type method for the bounded interval case follows immediately, where now the order of the method depends only on the multiplicity at a. If inverse moments are available, then we can also derive a Filon-type method over an unbounded interval. Finding a Levin-type method is more difficult. We derive it for the finite interval case, though the infinite interval case can be handled in the same manner. Note that

$$\int_a^{\epsilon} \mathcal{L}[v] \, \mathrm{e}^{\mathrm{i}\omega g} \, \mathrm{d}x = v(\epsilon)\mathrm{e}^{\mathrm{i}\omega g(\epsilon)} - v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

In order for this to converge as $\epsilon \to b$, $v(\epsilon)$ must go to zero. Hence assume that the collocation basis satisfies $\psi_k(b) = 0$. In this case, we define

$$Q^L[f] = I[\mathcal{L}[v]] = -v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

The behaviour of $\mathcal{L}[v] = v' + \mathrm{i}\omega g' v$ at b depends on the order of the zeroes of ψ_k at b: if the order of the pole of g' is greater than that of the zeroes, then $L[v]$ will be unbounded at b. Thus we ensure that the order of the zeroes of each ψ_k are at least that of the order of the pole of g'. Assuming that b is not a collocation point, we can, for any basis, replace ψ_k by some smooth $\tilde{\psi}_k$ such that $\tilde{\psi}_k(x) = \psi_k(x)$ for all $a \le x \le x_\nu$, $\tilde{\psi}_k(x)$ goes to zero in $x_\nu \le x \le N < b$ and $\tilde{\psi}_k(x) \equiv 0$ for $N \le x \le b$, where N is some constant. As in Theorem 8.2, this does not effect the collocation system at all, meaning that replacing ψ_k by $\tilde{\psi}_k$ has no effect on $Q^L[f]$. Hence the requirements on the basis are effectively unchanged.

Theorem 8.5. *Suppose that f satisfies the requirements of Theorem 8.3 or Corollary 8.4. Then*

$$Q^L[f] - I[f] = O\left(\omega^{-s-1}\right),$$

where $s = m_0$ and $Q^L[f] = -v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}$.

As a numerical example, consider the integral $I[f] = \int_0^1 \mathrm{e}^{\mathrm{i}\omega x^{-1}} \, \mathrm{d}x$. Figure 17 compares a Levin-type method to the asymptotic expansion. In Figure 18, we consider the unbounded integral $\int_1^\infty \cos x \, \mathrm{e}^{\mathrm{i}\omega x^2} \, \mathrm{d}x$, and compare two Levin-type methods to the asymptotic expansion: the first Levin-type method of order $O\left(\omega^{-2}\right)$ and the second Levin-type method of order $O\left(\omega^{-3}\right)$. In all three diagrams, the Levin-type method is a clear improvement over the asymptotic expansion of the same order.

We can also generalize these techniques for higher order oscillators. For simplicity, we will focus on the case $y_\omega(x) = \mathrm{Ai}\,(-\omega x)$, over the interval (a, ∞). Assume that f and its derivatives are bounded. This integral has both an infinite domain, as well as an increasingly large frequency of oscillations at ∞. The convergence of the integral will follow from the proof of the asymptotic expansion. Recall that:

$$I[f] = \int_a^M f y_\omega \, \mathrm{d}x = -\frac{1}{\omega^3}\left[\frac{f(x)}{x} y_\omega'(x) + \left(\frac{f(x)}{x}\right)' y_\omega(x)\right]_a^M - \frac{1}{\omega^3} I\left[\left(\frac{f(x)}{x}\right)''\right]$$

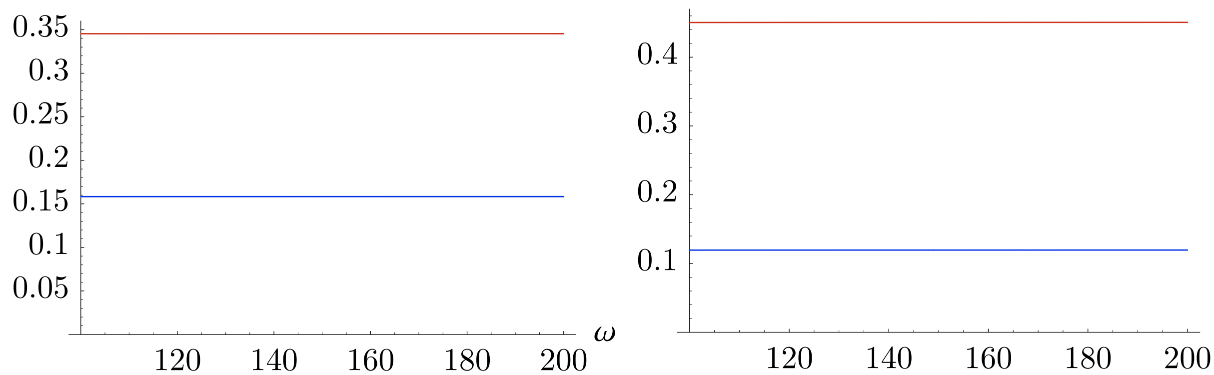


Figure 18: Errors scaled by ω^2 of the asymptotic expansion (left figure, top) compared to a Levin-type method collocating at $\{1, 2\}$ with multiplicities both one (left figure, bottom), and errors scaled by ω^3 of the asymptotic expansion (right figure, top) compared to a Levin-type method collocating at $\{1, 2\}$ with multiplicities $\{2, 1\}$ (right figure, bottom), for $I[f] = \int_1^\infty \cos x \, \mathrm{e}^{\mathrm{i}\omega x^2} \, \mathrm{d}x$.

As $M \to \infty$, the contributions from that endpoint in the first term go to zero. Moreover, note that:

$$I\left[\left(\frac{f}{\cdot}\right)''\right] = 2I\left[\frac{f}{.^3}\right] - 2I\left[\frac{f'}{.^2}\right] + I\left[\frac{f''}{\cdot}\right].$$

The first two of these integrals converge absolutely as $M \to \infty$. To prove that the last integral converges, we integrate it by parts once more. The non-integral terms evaluated at M go to zero. The remaining integral term can be written as:

$$I\left[\left(\frac{f''}{.^2}\right)''\right] = 6I\left[\frac{f''}{.^4}\right] - 4I\left[\frac{f^{(3)}}{.^3}\right] + I\left[\frac{f^{(4)}}{.^2}\right].$$

All three of these integrals converge absolutely. Thus it follows that we can let $M \to \infty$ to obtain

$$\int_a^\infty f y_\omega \, \mathrm{d}x = \frac{1}{\omega^3}\left[\frac{f(a)}{a} y_\omega'(a) + \left(\frac{f(a)}{a}\right)' y_\omega(a)\right] - \frac{1}{\omega^3} I\left[\left(\frac{f(x)}{x}\right)''\right].$$

Using induction we derive an asymptotic expansion:

Theorem 8.6. *Suppose that f and its derivatives are bounded in (a, ∞). Then*

$$\int_a^\infty f(x)\mathrm{Ai}\,(-\omega x) \, \mathrm{d}x \sim -\sum_{k=1}^\infty \frac{1}{\omega^{3k-1}}\left\{\frac{\sigma_k(a)}{a}\mathrm{Ai}\,'(-\omega a) - \frac{1}{\omega}\left(\frac{\sigma_k(a)}{a}\right)' \mathrm{Ai}\,(-\omega a)\right\},$$

for $\sigma_1(x) = f(x)$ and $\sigma_{k+1}(x) = \left(\frac{\sigma_k(x)}{x}\right)''$.

A Levin-type method can be proved using the same method as Theorem 7.3, where now

$$Q^L[f] = -\boldsymbol{v}(a)\boldsymbol{y}(a) = -v_1(a)y(a) - v_2(a)y'(a).$$

Figure 19 compares the asymptotic expansion to a Levin-type method for the first moment over the interval $(1, \infty)$. An application of this theorem will appear in the next section.

9. Stationary points.

Up until this point, we have assumed that there are no stationary points in the interval of integration, i.e., $g' \neq 0$. We will now investigate relaxing this condition. The fundamental problem with stationary points is that we must divide by zero in the derivation of the asymptotic expansion. Since this creates a

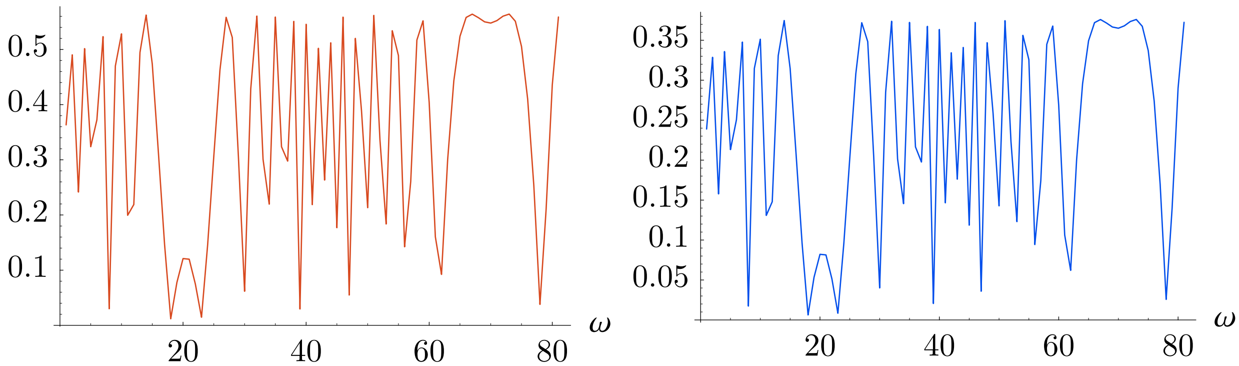


Figure 19: The error scaled by $\omega^{13/4}$ of the asymptotic expansion (left figure) compared to a Levin-type method collocating at $\{1, 3\}$ with multiplicities all one, for $I[f] = \int_1^\infty \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x$.

singularity in the integrand, we can no longer use partial integration. We obtain a solution to this problem from [**11**]. Assume that there is only one stationary point at ξ, and that $g''(\xi) \neq 0$, a condition that can similarly be weakened. Then

$$I[f] = f(\xi)\, I[1] + I[f - f(\xi)] = f(\xi)\, I[1] + \frac{1}{\mathrm{i}\omega} \int_a^b \frac{f - f(\xi)}{g'} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x.$$

The singularity in the integrand is now removable, hence we can integrate by parts:

$$I[f] = f(\xi)\, I[1] + \frac{1}{\mathrm{i}\omega} \left[\frac{f - f(\xi)}{g'} \mathrm{e}^{\mathrm{i}\omega g} \right]_a^b - \frac{1}{\mathrm{i}\omega} I\left[\left(\frac{f - f(\xi)}{g'} \right)' \right] = Q[f] - \frac{1}{\mathrm{i}\omega} I\left[\left(\frac{f - f(\xi)}{g'} \right)' \right],$$

where $Q[f] = f(\xi)\, I[1] + \frac{1}{\mathrm{i}\omega} \left[\frac{f - f(\xi)}{g'} \mathrm{e}^{\mathrm{i}\omega g} \right]_a^b$. If we assume that the first moment is available, we can compute $Q[f]$ explicitly, which approximates $I[f]$ with an error of order $O\bigl(\omega^{-3/2}\bigr)$. In other words, we have derived an approximation with an asymptotic order one more than that of the integral itself. By approximating the error term by $Q[f]$ repeatedly, we arrive at an asymptotic expansion:

Theorem 9.1. *Suppose that, for some $\xi \in (a, b)$, $g'(\xi) = 0$, $g''(\xi) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{\xi\}$. Then*

$$I[f] \sim I[1] \sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \rho_k(\xi) - \sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g}}{g'} \left\{ \rho_{m-1} - \rho_{m-1}(\xi) \right\} \right]_a^b,$$

where

$$\rho_0 = f, \qquad \rho_{k+1} = \left(\frac{\rho_k - \rho_k(\xi)}{g'} \right)', \qquad k = 0, 1, \ldots.$$

Note that the asymptotic expansion now depends on f evaluated at the endpoints of the interval and the stationary point, as well as requiring the knowledge of the first moment. In fact, in order to obtain the same asymptotic order, it requires twice as many derivatives at the stationary point, since we must use L'Hôpital's rule in order to determine $\rho_k(\xi)$. Hence the following analogue to Corollary 2.2 can be derived:

Corollary 9.2. *Suppose that*

$$0 = f(a) = f'(a) = \cdots = f^{(s-1)}(a)\,,$$
$$0 = f(\xi) = f'(\xi) = \cdots = f^{(2s-2)}(\xi)\,,$$
$$0 = f(b) = f'(b) = \cdots = f^{(s-1)}(b)\,.$$

Then $I[f] \sim O\Bigl(\omega^{-s-\frac{1}{2}}\Bigr)$.

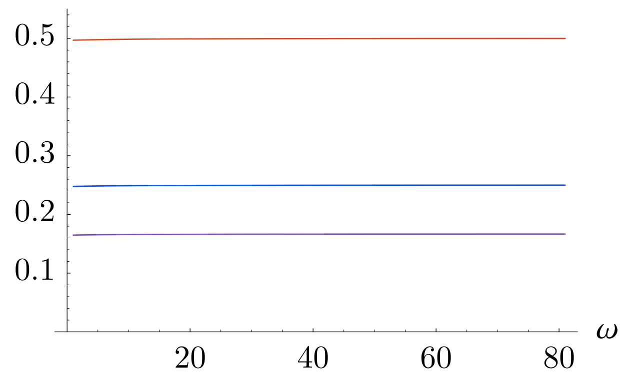


Figure 20: Error scaled by ω^2 of the asymptotic expansion (top), $Q^L[f]$ collocating at $\{\pm 1, \pm 2\}$ with multiplicities both one (middle) and $Q^L[f]$ collocating at $\{\pm 1, \pm 2, \pm 3\}$ with multiplicities all one (bottom), for $I[f] = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x$.

It follows immediately that a Filon-type method will have an order of error $O\left(\omega^{-s-\frac{1}{2}}\right)$, if $m_0, m_\nu \geq s$ and the multiplicity at the stationary point is greater than or equal to $2s-1$.

At first sight, it appears that something similar can be done for a Levin-type method; namely, ensure that the stationary point is sampled with a sufficiently large multiplicity. Unfortunately, this will not work. The regularity condition requires that the basis $\{g'\psi_k\}$ must be able to interpolate the given nodes and multiplicities. However, this basis is always identically zero at the stationary point, hence can not interpolate any nonzero function. This prevents us from using Corollary 9.2 to prove the order of error for a Levin-type method, and we are forced to look for other methods to handle this problem.

We will now present two ways of numerically approximating integrals with stationary points without using moments. For simplicity, we focus on computing the moments $I\left[x^k\right]$, since, for general f, we can always use these calculations in conjunction with a Filon-type method. The first method is to change the interval of integration so that it does not contain a stationary point. This can be accomplished if we know the value of the integral over an interval that includes (a, b) as a subset. As an example, consider the case where $g(x) = x^2$ and $a < 0 < b$, assuming initially that $f(x) = 1$. This oscillator has a single stationary point at $x = 0$. Note that $\operatorname{erf}\left(\sqrt{-\mathrm{i}\omega}\right)$ can be expressed in this form, where erf is the error function [**6**]. From the method of stationary phase [**14**], we know that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = \frac{(-1)^{(1/4)}\sqrt{\pi}}{\sqrt{\omega}}. \tag{9.1}$$

Thus, we can write $I[1]$ as

$$I[1] = \frac{(-1)^{(1/4)}\sqrt{\pi}}{\sqrt{\omega}} - \int_{-\infty}^{b} \mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x - \int_{a}^{\infty} \mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x.$$

Since g has no stationary points except at zero, we can use a Levin-type method on each integral, by using the techniques from Section 8 to handle the infinite domain integrals. We thus define

$$Q^L[f] = \frac{(-1)^{(1/4)}\sqrt{\pi}}{\sqrt{\omega}} - Q^L_g[f, (-\infty, b)] - Q^L_g[f, (a, \infty)]\,.$$

We can express the other moments in terms of elementary functions and the first moment, using the integral recurrence relationship

$$\int_a^b x\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = \frac{1}{2\mathrm{i}\omega}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = \frac{1}{2\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega b^2} - \mathrm{e}^{\mathrm{i}\omega a^2}\right),$$
$$\int_a^b x^k\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = \frac{1}{2\mathrm{i}\omega}\int_a^b x^{k-1}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x = \frac{1}{2\mathrm{i}\omega}\left(b^{k-1}\mathrm{e}^{\mathrm{i}\omega b^2} - a^{k-1}\mathrm{e}^{\mathrm{i}\omega a^2}\right) - \frac{k-1}{2\mathrm{i}\omega}\int_a^b x^{k-2}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}\omega x^2}\,\mathrm{d}x.$$

Figure 20 compares the asymptotic expansion to two Levin-type methods approximating the first moment.

Remark: It is not a coincidence that the value of the integral (9.1) is exactly the same as the contribution from the stationary point in the method of stationary phase [**14**]. Unfortunately, this cannot be generalized to other oscillators: for any other oscillator, the stationary phase contribution is asymptotic, not exact.

This technique will also work for the case of integrating the Airy function $\mathrm{Ai}\,(-\omega x)$ in a domain which contains the turning point $x = 0$. When $a < 0$, computing the integral over the interval $(a, 0)$ is numerically trivial: the integrand is non-oscillatory, and the integral itself goes to $\frac{1}{3}$ exponentially fast as $\omega \to \infty$ [**2**]. Thus assume that $a = 0$. From [**2**], we know that

$$\int_0^\infty \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x = \frac{2}{3\,\omega},$$

hence we can write

$$I[f] = \int_0^b \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x = \frac{2}{3\,\omega} - \int_b^\infty \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x.$$

From Section 8, we know how to approximate the integral $\int_b^\infty \mathrm{Ai}\,(-\omega x)\,\mathrm{d}x$, thus we have found a way of approximating $I[1]$. All other moments can be expressed explicitly in terms of Ai, Ai$'$, and the first moment, by using the recurrence relationships from [**2**]:

$$\begin{aligned}
\int x\mathrm{Ai}\,(x)\,\mathrm{d}x &= \mathrm{Ai}\,'(x)\,,\\
\int x^2\mathrm{Ai}\,(x)\,\mathrm{d}x &= x\mathrm{Ai}\,'(x) - \mathrm{Ai}\,(x)\,,\\
\int x^{k+3}\mathrm{Ai}\,(x)\,\mathrm{d}x &= x^{k+2}\mathrm{Ai}\,'(x) - x^{k+1}\mathrm{Ai}\,(x) + (n+1)(n+2)\int x^n\mathrm{Ai}\,(x)\,\mathrm{d}x.
\end{aligned}$$

For a numerical example of a Levin-type method, see Figure 19.

Returning to the original problem of stationary points with the exponential oscillator, another solution is to use analytic continuation to change the path of integration. In [**9**] we find a method based on steepest descent [**6**]. In brief, it distorts the integration interval to the path of steepest descent, and then uses generalized Gauss-Laguerre quadrature. This method has an error of order $O\left(\omega^{-2n-\frac{1}{2}}\right)$, where n is the number of quadrature points [**9**]. This suffers from two problems: the difficulty of having to compute, or at least approximate, the path of steepest descent; and handling oscillators such as $g(x) = \cos x$ which grow exponentially. Similar techniques can be used with other oscillators, for example the path of steepest descent for the Hankel function can be found in [**1**].

The multivariate case is more difficult. Resonance points, i.e., points where ∇g is orthogonal to the boundary of Ω, correspond to points in the boundary integral. This follows since, if T parameterizes the boundary and $T(\xi)$ is the resonance point, then

$$\tilde{g}(\xi) = (g(T(\xi)))' = \nabla g(T(\xi))T'(\xi) = 0.$$

Thus resonance points can be handled in the bivariate case if the existing univariate methods can handle the resulting stationary point. Research is still ongoing on how to handle critical points where $\nabla g = 0$. The most difficult situation is when there exists a curve of critical points. Suppose $\nabla g = 0$ along a curve $T \subset \Omega \subset \mathbb{R}^2$. From [**19**], we know that the asymptotic expansion of such integrals depends on

$$\int_T \frac{f}{\sqrt{g_{xx} + g_{yy}}}\,\mathrm{d}s.$$

This is a non-oscillatory integral, hence it cannot be expanded asymptotically. In order to derive a Levin-type method or Filon-type method we would need the interpolating function to be zero everywhere along the curve T. This is not in general possible, and such integrals require more research.

10. Closing remarks.

Several methods exist for approximating highly oscillatory integrals efficiently, where the accuracy improves as the frequency of oscillations increases. This is true in both the univariate and multivariate case, with different choices of oscillators, over both finite and infinite intervals and even when there are an infinite number of oscillations within the interval of integration. When moments are available, we can use a Filon-type method, whilst a Levin-type method uses collocation to provide an approximation whenever there are no stationary points. There are techniques in which we can handle the case of stationary points or turning points. In short, a large number of highly oscillatory integrals can be approximated by at least one of the methods discussed in this paper.

Many special functions have highly oscillatory integral representations. We have already shown the application of these methods to a few simple special functions, namely the exponential integral and error function. Another case is the computation of the Airy function for negative argument. Its integral representation has both an infinite number of oscillations over an unbounded interval and a stationary point, whose location depends on ω. It might be possible to combine the techniques of Section 8 and Section 9 to obtain an accurate approximation. Another possibility is to use the Magnus expansion, followed by using Levin-type methods to approximate the infinite sum of integrals. A more complicated example is the approximation of basic hypergeometric functions [**4**], which have contour integral representations that are highly oscillatory.

Another area of research is applying the techniques presented in this essay to the numerical computation of highly oscillatory differential equations. An extremely important example is the time-dependent Schrödinger equations. Magnus expansion techniques have been used recently to approximate such equations with numerical success [**7**]. Whether the integrals in such an expansion can be approximated with acceptable asymptotic behaviour remains to be seen. The applications of numerically efficient methods for approximating such equations are wide and numerous.

Acknowledgments: I would like to thank my supervisor Arieh Iserles, David Levin, Daan Huybrechs, Gates Cambridge Trust and the University of Cambridge.

Appendix A. Notation.

We define the *differential operator* $\mathcal{D}^{\boldsymbol{m}}$ as follows:

- $\mathcal{D}^{\boldsymbol{0}}$ is the identity operator.
- $\mathcal{D}^m$ for nonnegative integer $m \in \mathbb{N}$ is the mth derivative
$$\mathcal{D}^m = \frac{\mathrm{d}^m}{\mathrm{d}^m x}.$$
- $\mathcal{D}^{\boldsymbol{m}}$ for $\boldsymbol{m} = [m_1, \cdots, m_d] \in \mathbb{N}^d$ is the partial derivative
$$\mathcal{D}^{\boldsymbol{m}} = \frac{\partial^{|\,\boldsymbol{m}\,|}}{\partial x_1^{m_1} \ldots \partial x_d^{m_d}},$$
where $|\,\boldsymbol{m}\,| = \|\boldsymbol{m}\|_1 = \sum_{k=1}^{d} m_k$. Note that the absolute-value signs are not needed since each m_k is nonnegative.

The bottom two definitions are equivalent in the scalar case if we regard the scalar m as a vector in $\mathbb{N}^1$. Furthermore, it is clear that $\mathcal{D}^{\boldsymbol{m}_1}\mathcal{D}^{\boldsymbol{m}_2} = \mathcal{D}^{\boldsymbol{m}_1+\boldsymbol{m}_2}$.

Suppose f is a function from $\mathbb{R}^d$ to $\mathbb{R}$. We write $f = \boldsymbol{\mathcal{O}}(p(\omega))$ if the $L^\infty(\mathrm{cl}\,\Omega)$ norm of f and its partial derivatives are of order $O(p(\omega))$ as $\omega \to \infty$. In other words, $\|\mathcal{D}^{\boldsymbol{m}} f\|_\infty = O(p(\omega))$, for all $\boldsymbol{m} \in \mathbb{N}^d$. The most common usage is $f = \boldsymbol{\mathcal{O}}(1)$, which states that f and its derivatives are bounded in Ω for increasing ω. Note that this class of functions has the following properties, for every function $f = \boldsymbol{\mathcal{O}}(p(\omega))$, function $g = \boldsymbol{\mathcal{O}}(q(\omega))$, $c = O(r(\omega))$ and point $\boldsymbol{x} \in \mathrm{cl}\,\Omega$:

$$\begin{array}{lll} f(\boldsymbol{x}) = O(p)\,, & \mathcal{D}^{\boldsymbol{m}} f = \boldsymbol{\mathcal{O}}(p)\,, & f + g = \boldsymbol{\mathcal{O}}(\max\{p, q\})\,, \\ fg = \boldsymbol{\mathcal{O}}(pq)\,, & cf = \boldsymbol{\mathcal{O}}(pr)\,, & \int_a^b f \,\mathrm{d}V = O(p)\,. \end{array}$$

Note that if a basis ψ_k is independent of ω, hence $\boldsymbol{\mathcal{O}}(1)$, and the coefficients c_k are $O(r(\omega))$, then the linear combination $\sum c_k \psi_k$ is $\boldsymbol{\mathcal{O}}(r(\omega))$.

The definition of the *determinant matrix* of a map $T : \mathbb{R}^d \to \mathbb{R}^n$, with component functions $T_1, \ldots, T_n$, is simply the $n \times d$ matrix

$$T' = \begin{pmatrix} \mathcal{D}^{\boldsymbol{e}_1} T_1 & \cdots & \mathcal{D}^{\boldsymbol{e}_d} T_1 \\ \vdots & \ddots & \vdots \\ \mathcal{D}^{\boldsymbol{e}_1} T_n & \cdots & \mathcal{D}^{\boldsymbol{e}_d} T_n \end{pmatrix}.$$

Note that $\nabla g = g'$ when g is a scalar-valued function. The chain rule states that $(g \circ T)'(\boldsymbol{x}) = g'(T(\boldsymbol{x}))T'(\boldsymbol{x})$. The *Jacobian determinant* J_T of a function $T : \mathbb{R}^d \to \mathbb{R}^d$ is the determinant of its derivative matrix T'. For the case $T : \mathbb{R}^d \to \mathbb{R}^n$ with $n \geq d$ we define the Jacobian determinant of T for indices $i_1, \ldots, i_d$ as $J_T^{i_1,\ldots,i_d} = J_{\tilde{T}}$, where $\tilde{T} = \left[T_{i_1}, \cdots, T_{i_d}\right]^\top$.

Define the d-dimensional *surface differential* as

$$\mathrm{d}\boldsymbol{s} = \left[\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d, \ldots, (-1)^{d-1}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{d-1}\right]^\top.$$

Finally, define

$$\boldsymbol{J}_T^d(\boldsymbol{x}) = \left[J_T^{2,\ldots,d}(\boldsymbol{x}), -J_T^{1,3,\ldots,d}(\boldsymbol{x}), \cdots, (-1)^{d-1} J_T^{1,\ldots,d-1}(\boldsymbol{x})\right]^\top.$$

From the definition of the integral of a differential form, we know that if T maps $\Omega \subset \mathbb{R}^{d-1}$ onto $Z \subset \mathbb{R}^d$, then

$$\int_Z \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{s} = \int_\Omega \boldsymbol{f}(T(\boldsymbol{x})) \cdot \boldsymbol{J}_T^d(\boldsymbol{x})\,\mathrm{d}V.$$

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