Precalculus 4.2 Key Points

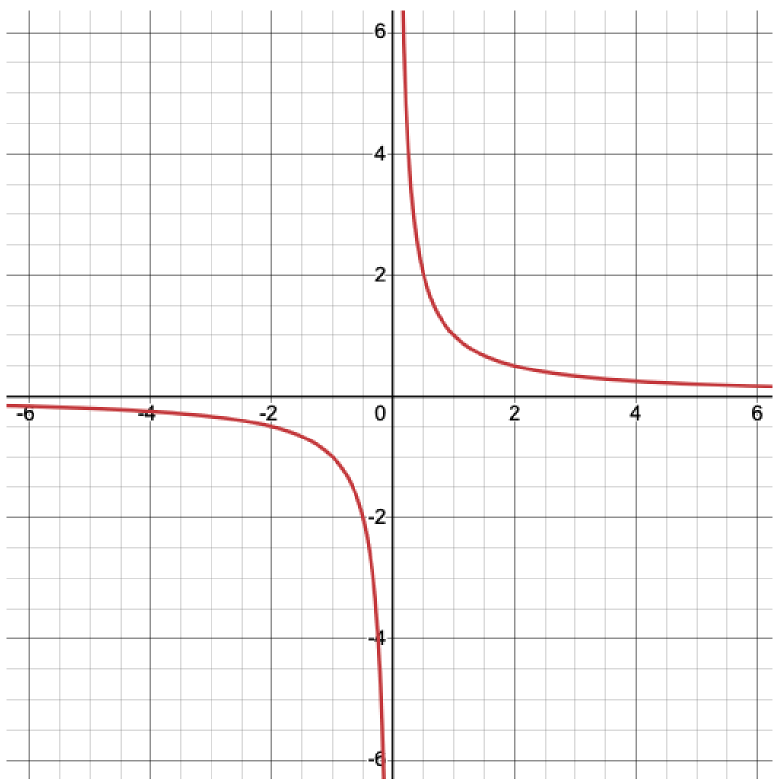
Transformations of Rational Functions:

Rational Functions can be written as a fraction of polynomials, such as $\frac{x-1}{x+3}$ or $\frac{1}{x^2+3x}$

A general form of rational functions is $y = \frac{a}{x-h} + k$, where

- a is the vertical stretch/compression
- h is the horizontal shift
- k is the vertical shift

The parent graph (the most basic graph) is $y = \frac{1}{x}$, where $a = 1$, $h = 0$, and $k = 0$



Notice that this function has a vertical and horizontal asymptote. Using the general format of $y = \frac{a}{x-h} + k$, the vertical asymptote will be at $x = h$ and the horizontal asymptote will be at $y = k$

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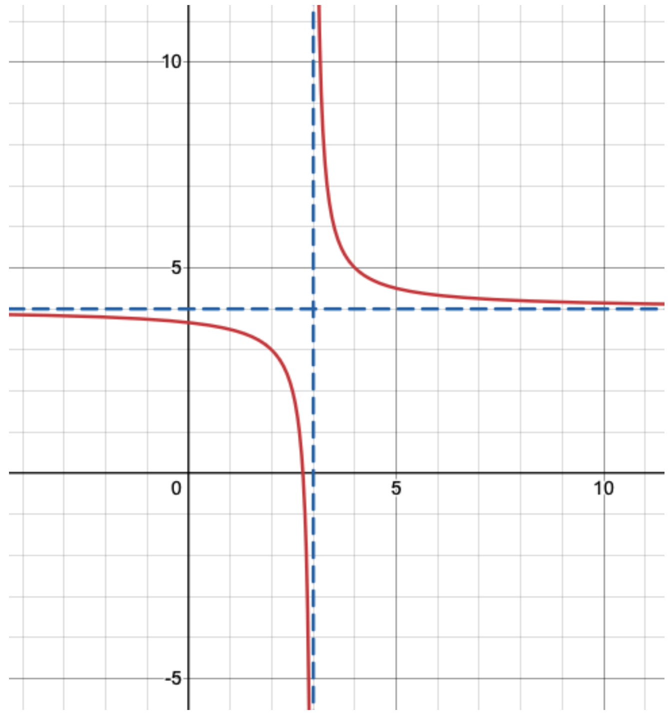
Here are a couple examples to give you a sense of how to graph them:

$$y = \frac{1}{x-3} + 4$$

Notice the vertical asymptote at $x = 3$ and the horizontal asymptote at $y = 4$

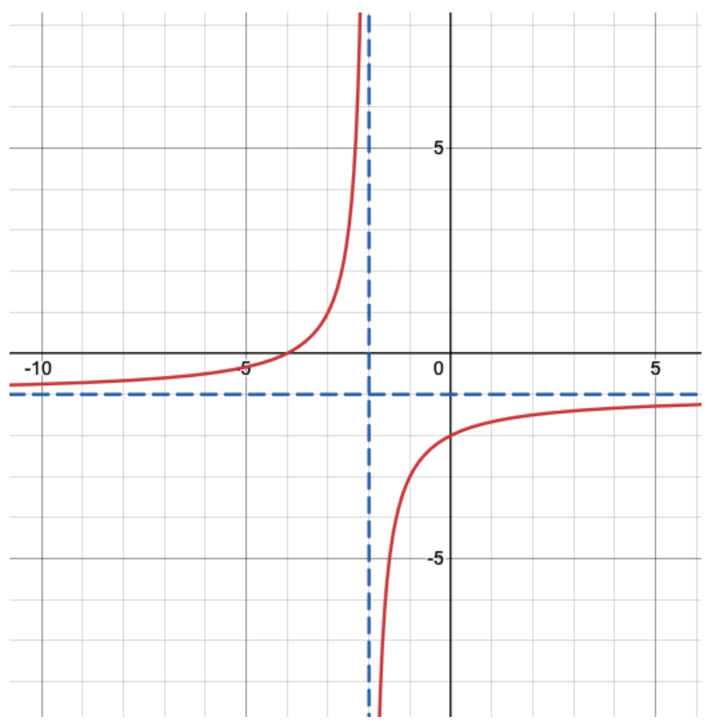


$$y = \frac{-2}{x+2} - 1$$

Notice the vertical asymptote at $x = -2$ and the horizontal asymptote at $y = -1$

Because the a value is negative, the graph is flipped vertically across the horizontal asymptote

Since the $|a|$ value is greater than 1, the graph is vertically stretched (If a was -1, the graph would have points at $(-3, 0)$ and $(-1, -2)$)



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Graphing Rational Functions:

To graph rational functions, identify the asymptotes. These will be the most important guides in your graph.

There are 3 types of asymptotes: Verical, Horizontal, Slant/Oblique

All rational functions will have a vertical asymptote.
They may have a horizontal or slant asymptote. They cannot have both a horizontal and slant asymptote.

Note: There are several steps to graph rational functions. Some steps can be done in the order of your preference.

Firstly, identify if the function has a horizontal asymptote, slant asymptote, or neither.

- If the degree of the numerator is greater than the degree of the denominator by more than 1 (e.g. $\frac{x^3+2}{x+2}$) then the graph will have neither a horizontal asymptote nor a slant asymptote
- If the degree of the numerator is greater than the degree of the denominator by exactly 1 (e.g. $\frac{x^2+2}{x+2}$) then the graph will have a slant asymptote
- If the degree of the numerator is equal to the degree of the denominator (e.g. $\frac{x^3+2}{x^3-2}$) then the graph will have a horizontal asymptote that is the ratio of the two leading coefficients ($\frac{-3x^2+3}{2x^2-5}$ will have a horizontal asymptote at $y = \frac{-3}{2}$)
- If the degree of the numerator is less than the degree of the denominator by exactly 1 (e.g. $\frac{x+2}{x^2+2}$) then the graph will have a horizontal asymptote at $y = 0$

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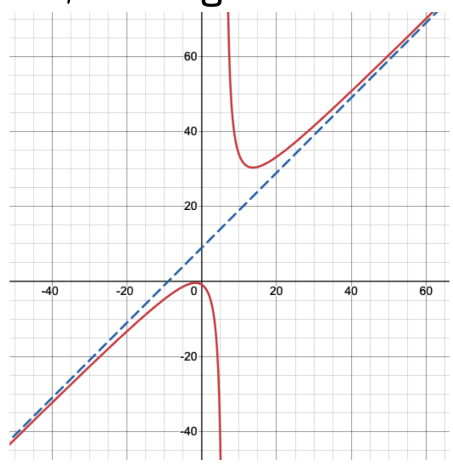
If you have determined that the graph has a slant asymptote, divide the rational function using polynomial division to determine the equation of the asymptote.

For example, take $\frac{x^2+3x+5}{x-6}$

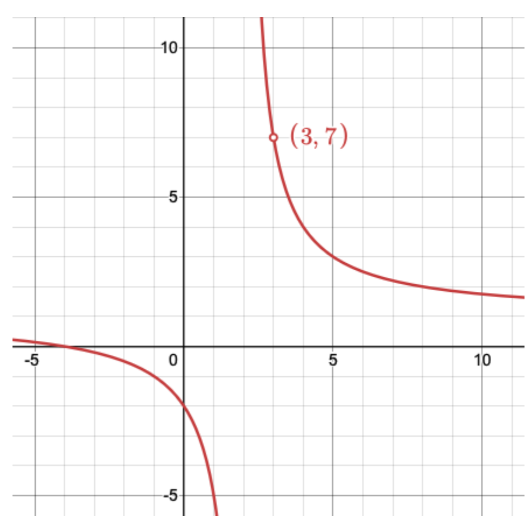
Dividing using polynomial division (either long division, an area model, or synthetic division) yields $x + 9 + \frac{59}{x-6}$

We can ignore the remainder, leaving us with the slant asymptote of $x + 9$



To determine the vertical asymptote(s), begin by factoring the numerator and denominator, if possible. If the same factor is on the top and bottom, cancel them out but make sure to keep track of them, as they will be the holes of the function. For example: $\frac{x^2+7x+12}{x^2+x-6} = \frac{(x+3)(x+4)}{(x+3)(x-2)} = \frac{x+4}{x-2}$, $x \neq 3$

Notice that plugging 3 into the original function would make the fraction equal to $\frac{0}{0}$, which is undefined. To graph this function, continue by graphing $\frac{x+4}{x-2}$, but place an open circle where the function would be at $x = 3$ to represent the discontinuity



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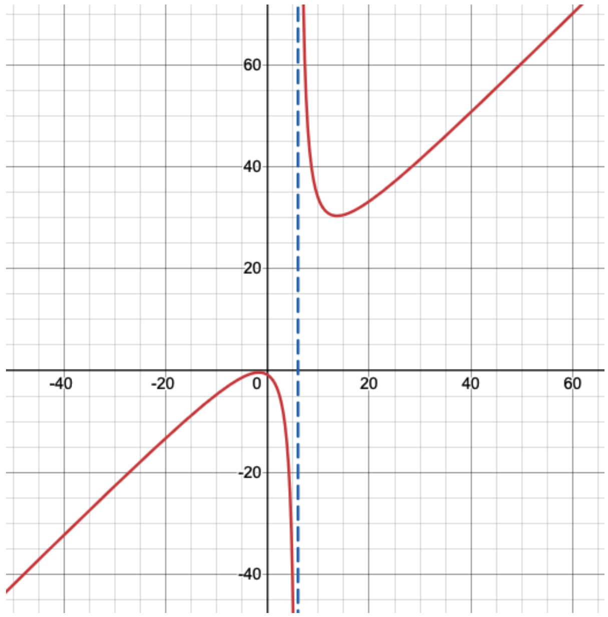
If all factors have already been cancelled out, the vertical asymptote(s) will be at the zeroes of the denominator. For example, if we set the denominator of $\frac{x^2+3x+5}{x-6}$ equal to zero, we get $x - 6 = 0$, where $x = 6$ Thus, this graph has a vertical asymptote at $x = 6$



When graphing rational functions, you may also be asked to determine function's end behavior and identify the x-intercept(s) and y-intercept

- The end behavior is what happens to the function as x approaches negative and positive infinity. Consider what happens when you plug in extremely small or large numbers in for x
- The x-intercepts will occur where the numerator of the function equals zero (as long as the denominator does not also equal zero)
- The y-intercept can be found by plugging in zero for x

Graphing Reciprocal Functions:

If a function can be expressed as $f(x)$, the reciprocal function is $\frac{1}{f(x)}$

You may be able to graph a reciprocal function like a rational function if the equation is $\frac{1}{x+3}$ or $\frac{1}{x-8}$

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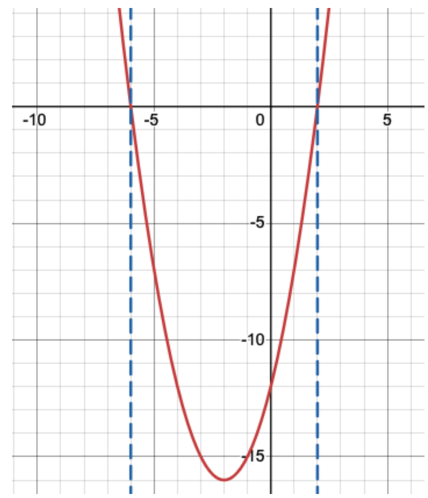
However, if the denominator has a higher degree (such as $\frac{1}{x^2+4x+4}$ or $\frac{1}{x^3-2x+x}$), then we will use a different method.

Begin by graphing the original function, $f(x)$. Wherever $f(x)$ has zeroes, the reciprocal function will have an asymptote. Once you have identified your asymptotes, consider the function's end behavior, what values the reciprocal function approaches near its asymptotes, the key points in the middle of the asymptotes, and transformations to graph it.

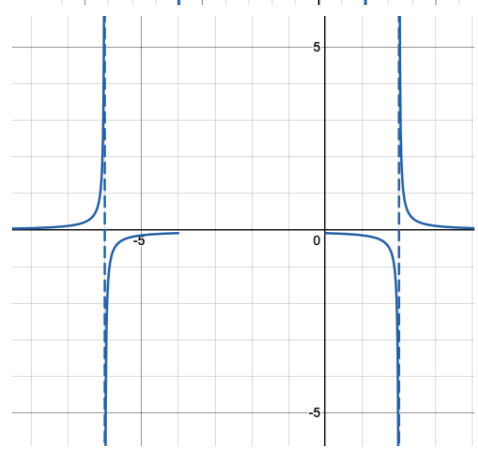
For example, let $f(x) = x^2 + 4x + 12$

The reciprocal function is $\frac{1}{x^2+4x+12}$

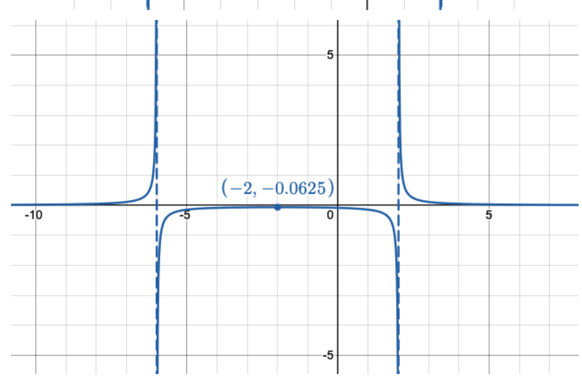


To the right is the graph of $f(x)$

Since it intersects the x-axis at $-\ 6$ and 2, the reciprocal function will have asymptotes at those locations



If we plug in values very close to the asymptotes, such as $-\ 6.0001$ or $-\ 5.9999$, along with considering end behavior, we can graph the left and right sides of the function.



To determine the point in the middle of the asymptotes, we can plug in $-\ 2$ into the reciprocal function, which gives us the point $(-\ 2,\ \ -\frac{1}{16})$ or $(-\ 2,\ \ -\ 0.0625)$