Precalculus 3.1 Key Points

Rational Expressions:

Rational expressions are fractions that contain polynomial expressions, such as $\frac{3x+5}{2x-4}$ and $\frac{6x^2-4x+2}{7x-11}$

There are a few different ways we can combine rational expressions:

<u>Addition</u>

Example 1: $\frac{6x+5}{2x} + \frac{3x-2}{2x}$

In this problem, notice that both expressions have $2x$ as a denominator (they have a common denominator). So, we can add them just like how we would add $\frac{2}{5} + \frac{4}{5} = \frac{2+4}{5} = \frac{6}{5}$

$$\frac{6x+5}{2x} + \frac{3x-2}{2x} = \frac{(6x+5)+(3x-2)}{2x} = \frac{6x+3x+5-2}{2x} = \frac{9x+3}{2}$$

Example 2: $\frac{x^2+x-6}{(x+4)(x-2)} + \frac{4x-11}{x+4}$

If the expressions do not have a common denominator, see if you can factor and cancel out some terms to get a common denominator

$$\frac{x^2+x-6}{(x+4)(x-2)} + \frac{4x-11}{x+4} = \frac{(x+3)(x-2)}{(x+4)(x-2)} + \frac{4x-11}{x+4} = \frac{x+3}{x+4} + \frac{4x-11}{x+4} = \frac{5x-8}{x+4}$$

Example 3: $\frac{2x^2-13x+6}{x^2-5x-6} + \frac{9x^2-15x}{3x^2+3x}$

Sometimes, this requires you to cancel out factors in both expressions

$$\frac{2x^2-13x+6}{x^2-5x-6} + \frac{9x^2-15x}{3x^2+3x} = \frac{(2x-1)(x-6)}{(x+1)(x-6)} + \frac{3x(3x-5)}{3x(x+1)} = \frac{2x-1}{x+1} + \frac{3x-5}{x+1} = \frac{5x-6}{x+1}$$

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Example 4: $\frac{9x+1}{2x}+\frac{x-10}{4x}$

You might find that you cannot cancel out any factors but still do not have a common denominator. In this case, multiply the top and bottom of an expression to create a common denominator. For example, for fractions with integers, we could add them like

$\frac{5}{4}+\frac{7}{2}=\frac{5}{4}+(\frac{7}{2}\bullet\frac{2}{2})=\frac{5}{4}+\frac{14}{4}=\frac{19}{4}$ Note that the total value of the expression has not changed because we multiplied by $\frac{2}{2}$, which is the same thing as multiplying by 1

$$\frac{9x+1}{2x}+\frac{x-10}{4x}=\frac{2(9x+1)}{4x}+\frac{x-10}{4x}=\frac{18x+2+x-10}{4x}=\frac{19x-8}{4x}$$

Example 5: $\frac{7x+3}{3x-2}+\frac{9x-1}{2x+1}$

Sometimes, this requires you to multiply to change the denominators of both expressions. Hint: One method that will always create a common denominator is to multiply each expression by the denominator of the other expression.

$$\frac{7x+3}{3x-2}+\frac{9x-1}{2x+1}=(\frac{2x+1}{2x+1}\bullet\frac{7x+3}{3x-2})+(\frac{9x-1}{2x+1}\bullet\frac{3x-2}{3x-2})=\frac{(7x+3)(2x+1)}{(2x+1)(3x-2)}=$$

$$(\frac{(9x-1)(3x-2)}{(2x+1)(3x-2)})+\frac{(7x+3)(2x+1)+(9x-1)(3x-2)}{(2x+1)(3x-2)}=\frac{(14x^2+13x+3)+(27x^2-21x+2)}{(2x+1)(3x-2)}=\frac{41x^2-8x+5}{(2x+1)(3x-2)}$$

Example 6: $\frac{3+x}{5x}-\frac{7}{5x}$

For subtraction problems, you can combine expressions with common denominators in the same way as you do with addition problems (and use the same techniques to get the expressions to have a common denominator if they do not already), just subtract the numerator instead of adding.

$$\frac{3+x}{5x}-\frac{7}{5x}=\frac{3+x-7}{5x}=\frac{x-4}{5x}$$

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Example 7: $\frac{2x+5}{3x} - \frac{4x-6}{3x}$

Make sure to distribute the negative if there are multiple terms in numerator

$$\frac{2x+5}{3x} - \frac{4x-6}{3x} = \frac{(2x+5)-(4x-6)}{3x} = \frac{2x+5-4x+6}{3x} = \frac{-2x+11}{3x}$$

Multiplication

Example 1: $\frac{x+5}{2x} \bullet \frac{3x-1}{4}$

For multiplication problems, multiply the numerators together, and then multiply the denominators together

$$\frac{x+5}{2x} \bullet \frac{3x-1}{4} = \frac{(x+5)(3x-1)}{(2x)(4)} = \frac{3x^2+14x-5}{8x}$$

Note: You may be able to leave your answer in a form such as $\frac{(x+5)(3x-1)}{8x}$ depending on the problem given

Example 2: $\frac{4x+1}{6x} \bullet \frac{x^2+6x+8}{x+4}$

You may need to factor and cancel out some terms before (or after) multiplying

$$\frac{4x+1}{6x} \bullet \frac{x^2+6x+8}{x+4} = \frac{4x+1}{6x} \bullet \frac{(x+4)(x+2)}{x+4} = \frac{4x+1}{6x} \bullet \frac{x+2}{1} = \frac{(4x+1)(x+2)}{6x}$$

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Example 3: $\frac{5x+1}{9x} \div \frac{x-4}{7x}$

For division problems, combine the two expressions by multiplying by the reciprocal of the second expression

$$\frac{5x+1}{9x} \div \frac{x-4}{7x} = \frac{5x+1}{9x} \bullet \frac{7x}{x-4} = \frac{(5x+1)(7x)}{(9x)(x-4)} = \frac{x(5x+1)(7)}{x(9)(x-4)} = \frac{(5x+1)(7)}{(9)(x-4)} = \frac{35x+7}{9x-36}$$

Note: You may also see division problems written like $\frac{\frac{5x+1}{9x}}{\frac{x-4}{7x}}$, which is the same thing as $\frac{5x+1}{9x} \div \frac{x-4}{7x}$

You may also encounter <u>complex fractions</u>, which are a fraction inside of a fraction, such as $\frac{a}{1-\frac{b}{c}}$

To simply, you need to get a common denominator. In this example, begin by multiplying "1" by $\frac{c}{c}$ to get a common denominator between the two terms in the bottom of the fraction so you can combine them

$$\frac{a}{1-\frac{b}{c}} = \frac{a}{(\frac{c}{c})1-\frac{b}{c}} = \frac{a}{\frac{c}{c}-\frac{b}{c}} = \frac{a}{\frac{c-b}{c}}$$

Now, you can simplify the fraction using "keep, change flip," or multiplying by the reciprocal:

$$\frac{a}{\frac{c-b}{c}} = \frac{a}{1} \bullet \frac{c}{c-b} = \frac{ac}{c-b}$$

Now, the complex fraction has been simplified.

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U-substitution:

If you are dealing with a complicated equation, you can use U-substitution to turn it into a simpler form.

Here are a couple of examples:

1. $tan^2(x) + tan(x) - 12 = 0$

Notice that this is almost in the form of a quadratic. We can replace $cos(x)$ with u to put it into the form of a quadratic to solve it and then substitute $cos(x)$ back in afterwards.
$tan^2(x) + tan(x) - 12 = 0 \Rightarrow u^2 + u - 12 = 0$, where $u = tan(x)$

Now, we can factor and solve the quadratic using the box and diamond method or the grouping method:
$(u + 4)(u - 3) = 0$
$u = -4$ or $u = 3$

Now, we can substitute $cos(x)$ back in to solve our original equation:
$tan(x) = -4$ or $tan(x) = 3$
$x = tan^{-1}(-4) \approx -1.326$ or $x = tan^{-1}(3) \approx 1.249$

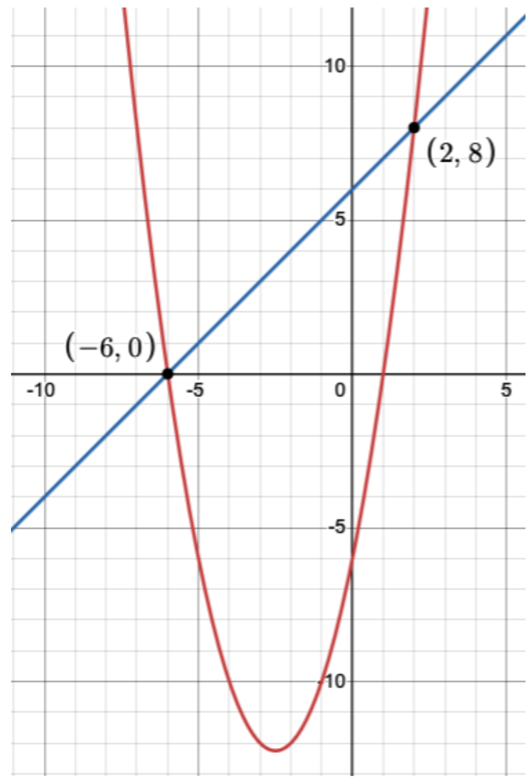
2. $x^4 - 12x^2 + 32 = 0$

In this example, we will similarly simplify this problem by substituting u into the equation.
$x^4 - 12x^2 + 32 = 0 \Rightarrow x^2 - 12x + 32 = 0$, where $u = x^2$

$(u - 4)(u - 8) = 0$
$u = 4$ or $u = 8 \Rightarrow x^2 = 4$ or $x^2 = 8$
$x = \pm 2$ or $x = \pm\sqrt{8}$

Solving Nonlinear Systems of Equations:

To solve systems equations between functions that are not just linear equations, you can graph the function and find the points of intersection or you can solve it algebraically using substitution.



$y = x + 6$
$y = x^2 + 5x + 6$

If we graph both functions, we can see that their two points of intersection are $(-6, 0)$ and $(2, 8)$, which are the solutions to our systems of equations.

We can also solve by substituting one side of one equation into the other equation so that we can solve for one variable

$[y] = x + 6 \Rightarrow [x^2 + 5x - 6] = x + 6$
$x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
$x = -6, 2$

Then, solve for the y values by plugging the x-values into either function

For $x = -6$

$y = (-6) + 6$
$y = 0$

For $x = 2$

$y = (2) + 6$
$y = 8$

Solutions: $(-6, 0)$ and $(2, 8)$

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Polynomial Division:

There are several different ways to divide polynomials. Three methods will be presented here, using $\frac{x^4-5x^3+7x^2-2}{x-3}$ as an example

Let's quickly review some vocabulary:
$6 \div 3 = 2$
6 is the dividend, 3 is the divisor, and 2 is the quotient

<u>Long Division</u>:

$$x-3 \;\overline{\big)\, x^4 - 5x^3 + 7x^2 + 0x - 2}$$

Make sure you write the dividend in descending order and include $0x^n$ if there is no term with the exponent of a number between the highest and lowest degree term (In simpler terms, if there is an exponent number missing, add the term $0x^{missing\ number}$)

Take the first term of the dividend (x^4) and divide it by the first term of the divisor (x). Put the result ($\frac{x^4}{x} = x^3$) on top

$$\begin{array}{r l} & x^3 \\ x-3 & \overline{\big)\, x^4 - 5x^3 + 7x^2 + 0x - 2} \end{array}$$

Then, multiply x^3 by the divisor ($x - 3$), which is $x^4 - 3x^3$, below the highest degree term in the polynomial. Flip the signs of those terms, subtract, and bring down the next term.

$$\begin{array}{r l} & x^3 \\ x-3 & \overline{\big)\, x^4 - 5x^3 + 7x^2 + 0x - 2} \\ & \underline{-x^4 + 3x^3} \;\downarrow \\ & 0 \;- 2x^3 + 7x^2 \end{array}$$

Next, take $-\ 2x^3$ and divide it by x, which results in $-\ 2x^2$. Put this number on top, and multiply it by the divisor to get $-\ 2x^3 + 6x^2$. Put this product below the highest remaining degree term in the polynomial, flip the signs, combine, and bring down the next term. Repeat this process unless you have gotten through all of the terms.

$$
\begin{array}{r|l}
 & x^3 - 2x^2 + x + 3 \\
\hline
x - 3 & x^4 - 5x^3 + 7x^2 + 0x - 2 \\
 & \underline{-x^4 + 3x^3} \\
 & 0 - 2x^3 + 7x^2 \\
 & \quad \underline{+ 2x^3 - 6x^2} \\
 & \qquad\qquad 1x^2 + 0x \\
 & \qquad\qquad \underline{-1x^2 + 3x} \\
 & \qquad\qquad\qquad 3x - 2 \\
 & \qquad\qquad\qquad \underline{-3x + 9} \\
 & \qquad\qquad\qquad\qquad 7
\end{array}
$$

If this last number is 0, then you have finished. If not, this number is your remainder. You may use this number for a special purpose for specific problems, but if you are only dividing, add to your solution a fraction with this remainder in the numerator and the divisor in the denominator. Thus, the final solution to this problem is $x^3 - 2x^2 + x + 3 + \frac{7}{x-3}$

<u>Area Model</u>:

For an area model, begin by creating a box, with the divisor on the left and the highest degree term of the dividend in the first box

	$?x^3$	$?x^2$	$?x$	$?$
x	x^4			
-3				

The first term is the coefficient of x^4 divided by x, which is x^3. This has a coefficient of 1, so we can fill that in.

	$1x^3$	$?x^2$	$?x$	$?$
x	x^4			
-3				

Now we can multiply x^3 by -3 to fill the next slot in the box

	$1x^3$	$?x^2$	$?x$	$?$
x	x^4			
-3	$-3x^3$			

Now, let's go back to the dividend, focusing on the x^3 term, $-5x^3$

We need the sum of all of the x^3 terms in the box to add to -5

We already have $-3x^3$, so we need $-2x^3$ more. Thus the next box must be $-2x^3$

	$1x^3$	$?x^2$	$?x$	$?$
x	x^4	$-2x^3$		
-3	$-3x^3$			

Now we can divide $\frac{-2x^3}{x}$, which gives us $-2x^2$, so the next question mark is -2.

	$1x^3$	$-2x^2$	$?x$	$?$
x	x^4	$-2x^3$		
-3	$-3x^3$			

We can continue this process by multiplying $-2x^2$ by -3 to fill in the next box, then using the x^2 term, $7x^2$, from the dividend to fill in the next box, and dividing to find the coefficient of the x term in the quotient.

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	$1x^3$	$-2x^2$	$?x$	$?$
x	x^4	$-2x^3$		
-3	$-3x^3$	$6x^2$		

$\rightarrow$

	$1x^3$	$-2x^2$	$?x$	$?$
x	x^4	$-2x^3$	x^2	
-3	$-3x^3$	$6x^2$		

$\rightarrow$

	$1x^3$	$-2x^2$	$1x$	$?$
x	x^4	$-2x^3$	x^2	
-3	$-3x^3$	$6x^2$		

Repeat this process until all of the boxes are filled

	$1x^3$	$-2x^2$	$1x$	$?$
x	x^4	$-2x^3$	x^2	
-3	$-3x^3$	$6x^2$	$-3x$	

$\rightarrow$

	$1x^3$	$-2x^2$	$1x$	$?$
x	x^4	$-2x^3$	x^2	$3x$
-3	$-3x^3$	$6x^2$	$-3x$	

$\rightarrow$

	$1x^3$	$-2x^2$	$1x$	3
x	x^4	$-2x^3$	x^2	$3x$
-3	$-3x^3$	$6x^2$	$-3x$	

$\rightarrow$

	$1x^3$	$-2x^2$	$1x$	3
x	x^4	$-2x^3$	x^2	$3x$
-3	$-3x^3$	$6x^2$	$-3x$	-9

$\rightarrow$

	$1x^3$	$-2x^2$	$1x$	3	Remainder
x	x^4	$-2x^3$	x^2	$3x$	7
-3	$-3x^3$	$6x^2$	$-3x$	-9	

The final quotient is $x^3 - 2x^2 + x + 3 + \frac{7}{x-3}$

<u>Synthetic Division</u>:

Synthetic division is a way to quickly divide a polynomial
This method only works when the divisor is x plus or minus a number, so it can be used if you are dividing by $x + 3$ or $x - 7$
If you are dividing by $x^2 + 1$ or $x^3 - 3x + 4$, then you will need to use one of the other methods.

Since we are dividing by $x - 3$, we will put 3 on the left side (Put -3 if you were diving by $x + 3$). Use the coefficients to set up the rest of the synthetic division like so:

$1x^4 - 5x^3 + 7x^2 + 0x - 2$

$$\begin{array}{c|ccccc} 3 & 1 & -5 & +7 & +0 & -2 \\ & & & & & \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{c|ccccc} 3 & 1 & -5 & +7 & +0 & -2 \\ & & & & & \\ \hline \end{array}$$

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Bring down the first coefficient. Then, multiply that by the number on the left and write it in the next spot, like so:

$$\begin{array}{r|rrrrr} 3 & 1 & -5 & +7 & +0 & -2 \\ & \downarrow & 3 & & & \\ \hline & 1 & & & & \end{array}$$

$3 \times 1 = 3$

Add $-$ 5 and 3, which is $-$ 2. Again, multiply that by the number on the left and write it in the next spot

$$\begin{array}{r|rrrrr} 3 & 1 & -5 & +7 & +0 & -2 \\ & & 3 & -6 & & \\ \hline & 1 & -2 & & & \end{array}$$

$3 \times -2 = -6$

Then, repeat until you get to the last number.

$$\begin{array}{r|rrrrr} 3 & 1 & -5 & +7 & +0 & -2 \\ & & 3 & -6 & 3 & \\ \hline & 1 & -2 & 1 & & \end{array} \quad \longrightarrow \quad \begin{array}{r|rrrrr} 3 & 1 & -5 & +7 & +0 & -2 \\ & & 3 & -6 & 3 & 9 \\ \hline & 1 & -2 & 1 & 3 & 7 \end{array}$$

$3 \times 1 = 3$

The numbers on the bottom are the coefficients for the quotient. Because the largest exponent in the dividend was x^4, we will begin by using an x^3 term. Like before, if the last number is 0, then there is no remainder, and otherwise, we use the last number for the remainder:

$$\begin{array}{r|rrrrr} 3 & 1 & -5 & +7 & +0 & -2 \\ & & 3 & -6 & 3 & 9 \\ \hline & 1 & -2 & 1 & 3 & 7 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$1x^3 - 2x^2 + 1x + 3 + \frac{7}{x-3}$$

The final quotient is $x^3 - 2x^2 + x + 3 + \frac{7}{x-3}$