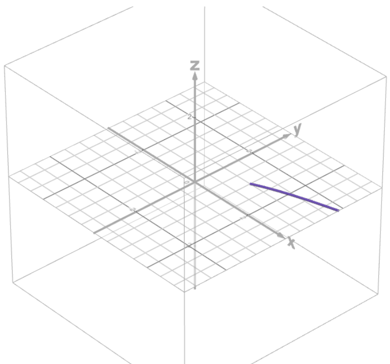
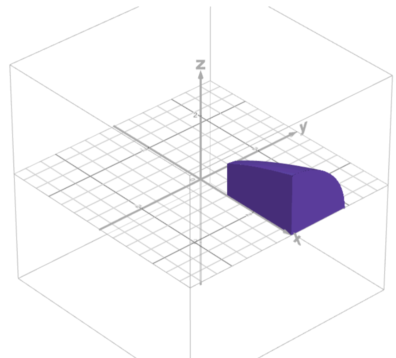
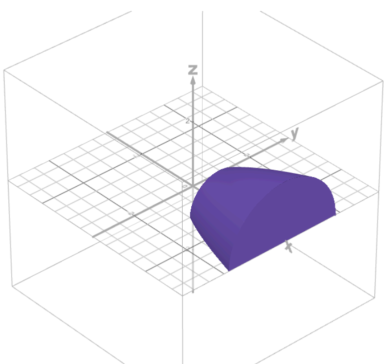
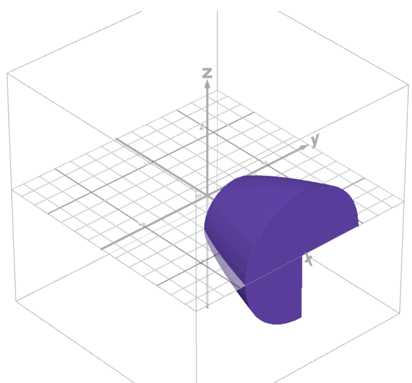
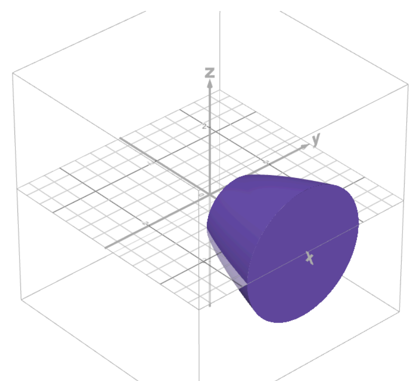
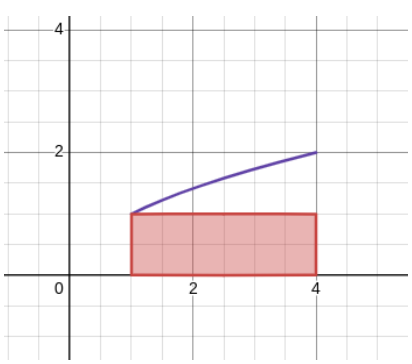
Disk Method

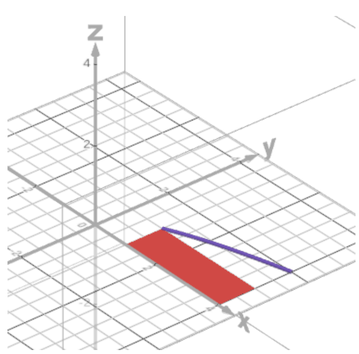
If we want to find the volume of a solid when a function is rotated about an axis, we can use the disk method

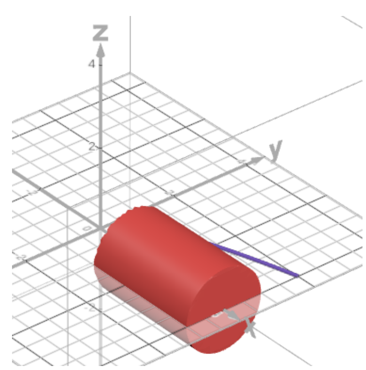
For example, let's say we wanted to rotate $f(x) = \sqrt{x}$ around the x-axis between $x = 1$ and $x = 4$, like so:

Let's begin with an approximation of the volume using a cylinder
If we take a rectangle under $f(x) = \sqrt{x}$ and rotate around the x-axis, we can create a cylinder





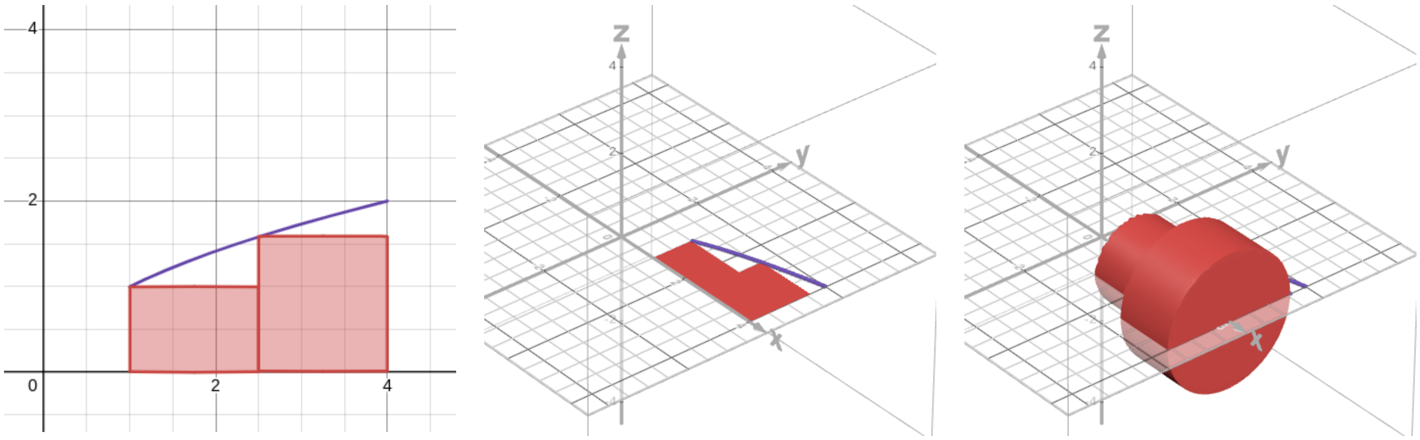


Disk Method

The volume of a cylinder is $\pi r^2 h$, where our radius is $f(x)$ at $x = 1$ and our height is $4 - 1 = 3$, so our volume approximation is $\pi(f(1))^2(4 - 1) = \pi(\sqrt{1})^2(3) = 3\pi \approx 9.425\ un^3$

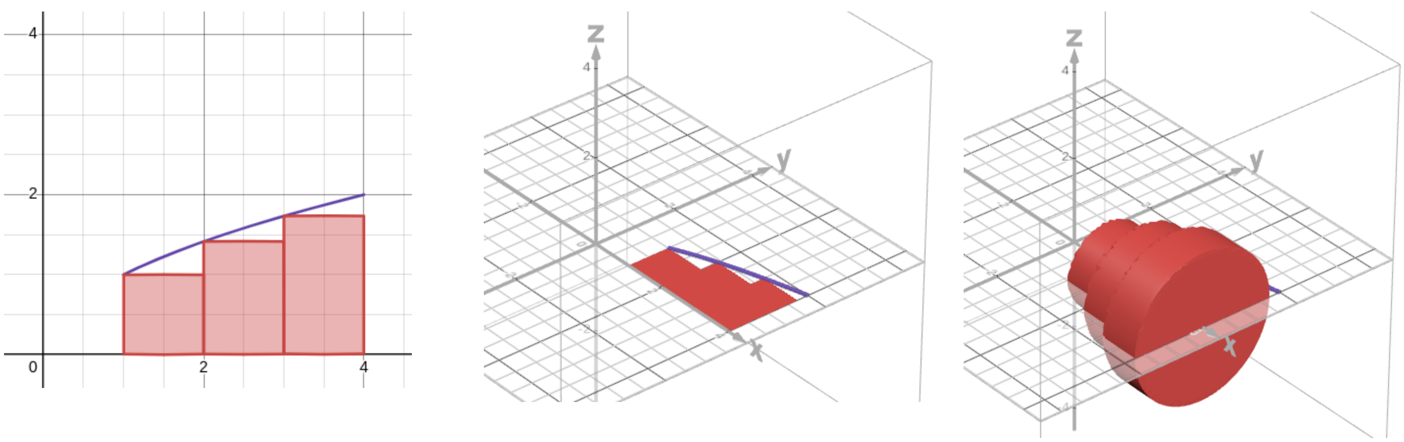
If we increase the number of rectangles we use to create cylinders to approximate the volume of the solid, we get a better approximation.

2 cylinders:



$$\pi(\sqrt{1})^2(1.5) + \pi(\sqrt{2.5})^2(1.5) = 1.5\pi + 3.75\pi = 5.25\pi \approx 16.493\ un^3$$

3 cylinders:



$$\pi(\sqrt{1})^2(1) + \pi(\sqrt{2})^2(1) + \pi(\sqrt{3})^2(1) = 6\pi \approx 18.850\ un^3$$

Disk Method

We can express the volume as a summation, where n is the number of cylinders, and Δx is the height of each cylinder:

$$V = \sum_{i=0}^{n-1} \pi r^2 h = \sum_{i=0}^{n-1} \pi(f(a + i\Delta x))^2 (\frac{b-a}{n})$$

$$V = \sum_{i=0}^{2} \pi(\sqrt{(1 + i(1))})^2 (\frac{4-1}{3}) = \sum_{i=0}^{2} \pi(1 + i)(\frac{3}{3}) = 6\pi \approx 18.850\, un^3$$

If we want to get an exact answer, we can take the limit as $n \rightarrow \infty$

$$V = \lim_{n \rightarrow \infty} \sum_{i=0}^{n} \pi(\sqrt{1 + \frac{3i}{n}})^2 (\frac{3}{n}) = \frac{15}{2}\pi \approx 23.562 un^3$$

Written as an integral:

$$V = \int_1^4 \pi(\sqrt{x})^2\, dx = \pi \int_1^4 x\, dx = 7.5\pi \approx 23.562\, un^3$$

Thus, the general equation to find the volume of a solid using the disk method is $V = \int_a^b \pi(f(x))^2\, dx$