Home Assignment - 2

1. Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the Constant c.

Sol: We have to find the positive constant c.

Thus we have $\int_{-1}^{1} f_{(X)}(x)\,dx = 1$

$$\Rightarrow \int_{-1}^{1} Cx^2\,dx = 1$$

$$\Rightarrow c\int_{-1}^{1} x^2\,dx = 1$$

$$\Rightarrow c\left[\frac{x^3}{3}\right]_{-1}^{1} = 1$$

$$\Rightarrow c\left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right] = 1$$

$$\Rightarrow c\left(\frac{2}{3}\right) = 1$$

$$\Rightarrow c = \frac{3}{2}$$

$$\therefore f_X(x) = \begin{cases} \frac{3}{2}x^2. & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad \text{for all } x \in R$$

if $Y = X^2$, find the CDF of Y.

Sol: Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad \text{for all } x \in R.$$

$R_Y = [0, \infty)$. For $y \in [0, \infty)$, we have.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx \\ &= \int_0^{\sqrt{y}} e^{-x} dx \\ &= 1 - e^{-\sqrt{y}} \end{aligned}$$

3. Let X be a continuous random with PDF given by

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X.

Sol: The expected value of the random variable is given by $E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$

According to the question,

$$E(X) = \int_0^1 x(2x) dx = \int_0^1 2x^2 dx$$

$$= \left(\frac{2}{3} x^3\right)_0^1 = \frac{2}{\,}$$

4. Let x be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$ and $Var(X)$.

Sol: We have to find the positive constant c.

Thus we have $\int_{-1}^{1} f_X(x)\, dx = 1$

$$\Rightarrow \int_{-1}^{1} cx^2\, dx = 1$$

$$\Rightarrow c\int_{-1}^{1} x^2\, dx = 1$$

$$\Rightarrow c\left[\frac{x^3}{3}\right]_{-1}^{1} = 1$$

$$\Rightarrow c\left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right] = 1 \Rightarrow c\left(\frac{2}{3}\right) = 1$$

$$\Rightarrow c = \frac{3}{2}$$

$$\therefore f_X(x) = \begin{cases} \frac{3}{2}x^2, & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now let us find $E(X)$

We know that $E(X) = \int_a^b x\, P(x)\, dx$ where $P(x)$ is probability density function.

Thus $E(X) = \int_{-1}^{1} x \cdot \frac{3}{2}x^2\, dx$

$$= \int_{-1}^{1} \frac{3}{2}x^3\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} x^3\,dx$$

$$= \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{3}{2}\left[\frac{1}{4}-\frac{1}{4}\right]$$

$$= 0$$

$\therefore E(X) = 0$

Now we have to find $Var(X)$

We know that $Var(X) = \int_a^b x^2 P(x)\,dx - \mu^2$ where $P(x)$ is probability density function and μ is mean.

Also $\mu = E(X)$

Thus $Var(X) = \int_{-1}^{1} x^2 \cdot \frac{3}{2}x^2\,dx - [E(X)]^2$

$$= \int_{-1}^{1} \frac{3}{2}x^4\,dx - [0]^2$$

$$= \frac{3}{2}\int_{-1}^{1} x^4\,dx - 0$$

$$= \frac{3}{2}\int_{-1}^{1} x^4\,dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{2}\left[\frac{1}{5}-\left(\frac{-1}{5}\right)\right]$$

$$= \frac{3}{2}\left[\frac{2}{5}\right]$$

$$\therefore Var(X) = \frac{3}{5}$$

5. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le x \le 1, 0 \le y \le \sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

Find $f_X(x)$ and $f_Y(y)$.

sol: $R_X = R_Y = [0,1]$. To find $f_X(x)$ for $0 \le x \le 1$ we can write

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{\sqrt{x}} 6xy\,dy$$

$$= 3x^2$$

Thus

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To find $f_Y(y)$ for $0 \le y \le 1$,

we can write.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{y^2}^{1} 6xy\,dx$$

$$= 3y(1-y^4)$$

Thus

$$f_Y(y) = \begin{cases} 3y(1-y^4) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

6. Suppose that there are only four possible cases:

$X = 0$ and $Y = 0$

$X = 1$ and $Y = 0$

$X = 0$ and $Y = 1$

$X = 1$ and $Y = 1$

Further assume that each of these cases has probability equal to 1/4. Compute the value of the joint distribution function: $F_{XY}(0, 1)$

Sol: The two conditions that need to be simultaneously true are:

$$X \leq 0$$

$$Y \leq 1$$

There are two cases in which they are satisfied:

$X = 0$ and $Y = 0$

$X = 0$ and $Y = 1$

Therefore, we have

$$F_{XY}(0,1) = P(X \leq 0, Y \leq 1)$$

$$= P(X = 0, Y = 0) + P(X = 0, Y = 1)$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

7. Let X and Y be two jointly continuous random variables with joint PDF.

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find ~~d~~ the constant c.

sol: We have to find the constant c.

$$\int_0^1 \int_0^1 (x + cy^2)\, dy\, dx = 1$$

$$\int_0^1 \left[yx + c\frac{y^3}{3} \right]_0^1 dx = 1$$

$$\left[\frac{x^2}{2} + c\frac{1}{3} \right]_0^1 = 1$$

$$\frac{1}{2} + c\frac{1}{3} = 1$$

$$c = \frac{3}{2}$$

8. Find $P(0 \le X \le 1/2, 0 \le Y \le 1/2)$ for the joint PDF given in Question-7.

sol:

$$P\left(0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}\right) = \int_0^{1/2} \int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dy\, dx$$

$$= \int_0^{1/2} \left[yx + \frac{3}{2}\frac{y^3}{3} \right]_0^{1/2} dx$$

$$= \cancel{6}\,\frac{1}{16} + \frac{3}{2} \times \frac{1}{3 \times 16}$$

$$= \frac{3}{32}$$

9. The number of miles driven by a truck driver falls between 300 and 700, and follows a uniform distribution.

a) Find the probability that the truck driver goes more than 650 miles a day.

b) Find the Probability that the truck driver goes between 400 and 650 miles a day.

c) At least how many miles does the truck driver travel on the furthest 10% of the days?

sol: Give that

$a = 300 \quad b = 700$

a) $P(x > c) = \frac{b - c}{b - a}$

$P(x > 650) = \frac{700 - 650}{700 - 300} = \frac{50}{400} = 0.125$

Probability $= 0.125$

b) $P(c < x < d) = \frac{d - c}{b - a}$

$P(400 < x < 650) = \frac{650 - 400}{700 - 300} = \frac{250}{400} = 0.625$

Probability $= 0.625$

c) $= a + (b - a) p$

$= 300 + (700 - 300) \times 0.1$

$= 340$

340 miles does the truck driver travel on the furthest 10% of days.

10) Suppose that the value of a stock varies each day from \$16 to \$25 with a uniform distribution

a) Find the probability that the value of the stock is more than \$19.

b) Given that the stock is greater than \$18, Find the probability that the stock is more than \$21

sol:

~~$P(X>19) =$~~

Varies each day from \$16 to \$25 with a uniform distribution. hence $a = 16$ & $b = 25$

a) $$P(x > 19) = \frac{b-x}{b-a}$$

$$= \frac{25-19}{25-16} = \frac{6}{9} = 0.667$$

0.667 = 66.7 % probability that the value of the stock is more than \$18.

b) Greater than \$18, hence $a = 18$

$$P(x > 21) = \frac{25-21}{25-18}$$

$$= \frac{4}{7} = 0.5714$$

0.5714 = 57.14 % probability that the stock is more than \$21.