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國立交通大學

所別：資訊聯招　科目：線性代數與離散數學

Linear Algebra with Applications

1. Let $A=\begin{bmatrix}-1 & 1\\ 1 & 2\\ 2 & 1\end{bmatrix}$ and S be the subspace of R^3 spanned by the column vector of A.

   (a) (3%) Find an orthonormal basis for $S^{\perp}$, the orthogonal complement of S.

   (b) (4%) Let $P=A(A^TA)^{-1}A^T$, if $\boldsymbol{x}\in R^3$ and $\boldsymbol{x}\notin S\cup S^{\perp}$, show that $P\boldsymbol{x}\in S$.

2. Consider the inner product space $C[0, 1]$ which is the set of all functions that have a continuous first order derivative on $[0, 1]$. The inner product of two functions $f(x)$ and $g(x)$ is defined by $<f,g>=\int_0^1 f(x)g(x)dx$.

   (a) (4%) Use the Gram-Schmidt process to find an orthonormal basis E for the subspace S spanned by the vectors 1 and x^2.

   (b) (4%) Find the best least squares approximation to the function $\sqrt{x}$ on the interval $[0, 1]$ by a function in S.

3. Define a linear transformation T from R^3 to R^3 as follows

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\1\end{bmatrix};T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\-1\end{bmatrix};T\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

   (a) (3%) Find the matrix of the linear transformation T.

   (b) (3%) Find a basis of the kernel of the transformation T.

   (c) (3%) Find the coordinate vector of $T\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$ with respect to the basis $=\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$.

4. (a) (3%) Consider a 3×2 matrix A and a 2×5 matrix B. How many possible dimensions of $\ker(AB)$ are there? What are they? You must justify your answers! (Note $\ker(AB)$ is the kernel of matrix AB)

(b) (3%) Define a linear transformation T from $\mathbf{R}^{2\times2}$ to $\mathbf{R}^{2\times2}$ by

$$T(M) = M\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}M.$$

Find a basis of the image of T with respect to the standard basis

$$U = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right).$$

5. Students A and B were asked to solve the eigenvalues of the same matrix $M = \begin{bmatrix}a & b & c\\0 & d & 1\\0 & 2 & e\end{bmatrix}$. Unfortunately, Student A mistook the value of d and obtained the eigenvalues 0, 1, and 3. Student B mistook the value of e and obtained the eigenvalues 1, 1, and -2. Assume there were no other mistakes happened when they were solving the eigenvalues.

(a) (3%) Find the value of a.

(b) (6%) Assume that the sum of correct eigenvalues of M is 1. Find the correct values of d and e.

(c) (3%) Find the correct eigenvalues of M.

6. (8%) Find a singular value decomposition for the matrix $M = \begin{bmatrix}0 & 1\\1 & 1\\1 & 0\end{bmatrix}$.

7.

(a) (3%) Consider a rectangle area cut through by several straight lines. Each straight line touches two sides of the rectangle. An example is shown in Figure 1. What is the least number of colors needed to color such a rectangle area so that neighboring regions have different colors?

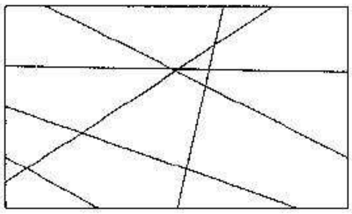


Figure 1: A 2-colorable rectangle.

(b) (4%) Consider the following decimal number:

0.01234567891011121314151617181920212 2...

The 1st digit after the decimal point is 0; the second digit is 1; the 12th digit is 0; etc. What is the 1,000th digit after the decimal point?

(c) (4%) A natural number n may be written as $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where p_i's are prime numbers and a_i's are natural numbers. Calculate the product of all the divisors of the number n.

(d) (4%) Find the coefficients of x^{58} and x^{59} in $(x^2 - \frac{1}{x})^{100}$. Please write down your answer first and then explain how you obtain the answers.

(e) (4%) We are going to prove by induction that $\sum_{i=1}^{n} Q(i) = n^2(n+1)$. For which choice of $Q(i)$ will the induction work?

(1) $3i^2 - 2$, (2) $2i^2$, (3) $3i^3 - i$, (4) $i(3i - 1)$, (5) $3i^3 - 7i$

1[illegible]) The statement form $(p \leftrightarrow r) \rightarrow (q \leftrightarrow r)$ is equivalent to

(1) $[(\neg p \vee r) \wedge (p \vee \neg r)] \vee \neg[(\neg q \vee r) \wedge (q \vee \neg r)]$

(2) $\neg[(\neg p \vee r) \wedge (p \vee \neg r)] \wedge [(\neg q \vee r) \wedge (q \vee \neg r)]$

(3) $[(\neg p \vee r) \wedge (p \vee \neg r)] \wedge [(\neg q \vee r) \wedge (q \vee \neg r)]$

(4) $[(\neg p \vee r) \wedge (p \vee \neg r)] \vee [(\neg q \vee r) \wedge (q \vee \neg r)]$

(5) $\neg[(\neg p \vee r) \wedge (p \vee \neg r)] \vee [(\neg q \vee r) \wedge (q \vee \neg r)]$

([illegible]) (3%) How many non-negative integer solutions are there for the following equation?

$$x_1 + x_2 + x_3 + x_4 = 10$$

(a) (1%) Let $A = \{a, b, c, d, e\}$ and $R = \{(a, b), (a, d), (b, a), (b, c), (c, e), (d, d), (e, c)\}$. Draw the digraph representation of the relation R.

(b) (2%) Consider the relation R in 8(a). Give the matrix representation of R^*. Let a, b, c, d, e correspond to row numbers and also column numbers 1, 2, 3, 4, 5, respectively. Note that R^* denotes the transitive closure of R, and M_R is the matrix representation of R if

$$(M_R)_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin R \\ 1 & \text{if } (i, j) \in R \end{cases}.$$

(c) (10%) The pseudocodes below are Warshall's algorithm. Consider the relation R in 8(a). Give the matrix W at the end of the loop $k = 1$ and $k = 3$.

procedure *Warshall*(M_R : $n \times n$ zero-one matrix)

$W := M_R$

for $k := 1$ **to** n

begin

  for $i := 1$ **to** n

  begin

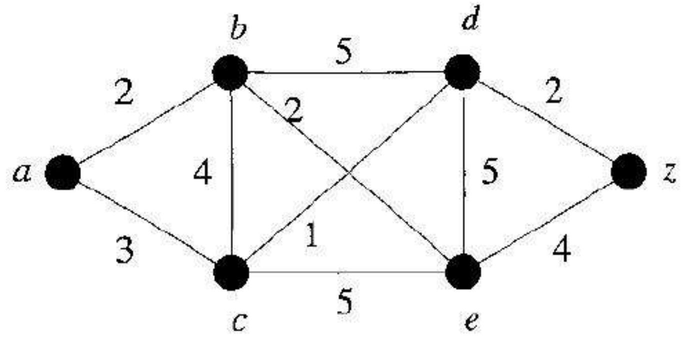
    for $j := 1$ **to** n

      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

  end

end

(d) (3%) Assume $G = (V, E, w)$ is an undirected weighted graph. For any $u \in V$, let $p(u) = \max_{\{u,v\} \in E} w(\{u, v\})$. Define $p(G) = \sum_{u \in V} p(u)$ and $c(G) = \sum_{e \in E} w(e)$. Find $p(G)$ and $c(G)$ of the following graph.



(e) (4%) Based on definitions in 8(d), prove $p(G) \le 2c(G)$.

(2%) What are directed graph in which each vertex has at most one outgoing edge? Give your answer in one sentence.

(g) (3%) Assume $G = (V, E, w)$ is a directed weighted graph. Let $p(u) = \max_{(u,v) \in E} w((u, v))$ for any $u \in V$. Define $p(G) = \sum_{u \in V} p(u)$ and $c(G) = \sum_{e \in E} w(e)$. If G is a graph as described in 8(f), is it true $p(G) = c(G)$? Please give a short explanation, no more than 5 sentences, to support your answer.