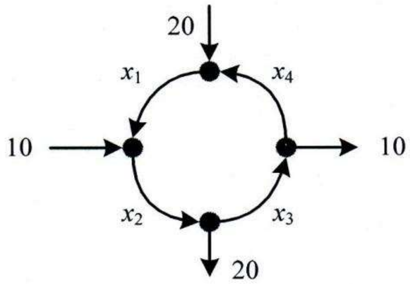
100 國立交通大學

所別：資訊聯招　　　　科目：線性代數與離散數學

1. (5%; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分為負時, 以 0 分計; 未作答亦以 0 分計)

   Consider a system of linear equations for solving the flows of traffic (in vehicles per minute) through a one-way circle shown below, where $x_i \geq 0$ denotes the traffic flow of the i-th segment of the circle, $i = 1, \ldots, 4$. Which of the followings are true?



   (a) This system of equations can be reduced to a strictly triangular form.

   (b) The row echelon form of the augmented matrix representing this system of equations is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 20 \\ 0 & 1 & 0 & -1 & 20 \\ 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$.

   (c) This linear system is consistent.

   (d) The largest traffic flow is x_1.

   (e) When solve this traffic flow problem where the flow directions in the above figure are all reversed, we can obtain a system of equations equivalent to the original one (when the flow directions are as indicated in the above figure).

2. (5%; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分為負時, 以 0 分計; 未作答亦以 0 分計)

   Which of the followings are true?

   (a) There exists a matrix A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

   (b) A system of three linear equations in two unknowns is always inconsistent.

   (c) If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular.

   (d) If A and B are nonsingular $n \times n$ matrices, then $A + B$ is also nonsingular.

   (e) If A, B, and $A + B$ are nonsingular $n \times n$ matrices, then $A^{-1} + B^{-1}$ is also nonsingular.

3. (5%; 複選，答對每個選項得 1.25 分，答錯每個選項扣 1.25 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Which of the followings are true?

(a) The adjoint of the matrix $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

(b) $\det\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix} = (a+3)(a-1)^3$.

(c) For a nonsingular matrix $A_{3\times 3} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$, $\det(A^{-1}B) = 1$ where

$$B = \begin{bmatrix} \boldsymbol{a}_2^T \\ 2\boldsymbol{a}_1^T - \boldsymbol{a}_2^T + \boldsymbol{a}_3^T \\ \boldsymbol{a}_1^T \end{bmatrix}.$$

(d) If $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 3 & 5 \end{bmatrix} = 7$ and $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 0 & 1 \end{bmatrix} = 9$, then $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 6 & 9 \end{bmatrix} = 5$.

4. (5%; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Which of the followings are true?

(a) The formula $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D) - \det(C)\det(B)$ always hold for square matrices A, B, C, and D.

(b) The formula $\det(A^k) = (\det(A))^k$ always hold for positive integer k and square matrix A.

(c) The formula $\det(A) = \det(-A)$ always hold for square matrix A.

(d) The formula $\det(A^T A) > 0$ always hold for nonsingular matrix A.

(e) The formula $\det(A) = 1$ always hold for orthogonal matrix A.

5. (5%; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分為負時, 以 0 分計; 未作答亦以 0 分計)

Which of the followings are true?

(a) Let $\boldsymbol{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by $\alpha \circ x = x^\alpha$ for each $x \in \boldsymbol{R}^+$ and for any real number α. Define the operation of addition, denote $\oplus$, by $x \oplus y = x \cdot y$ for all $x, y \in \boldsymbol{R}^+$. Then $\boldsymbol{R}^+$ is a vector space with these operations.

(b) Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by $\alpha \circ (x_1, x_2) = (\alpha x_1, \alpha x_2)$ and $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$. Then S is a vector space with these operations.

(c) If A is an $r \times s$ matrix, then the rank of A plus the nullity of A equals r.

(d) Let S_1 and S_2 be two subspaces of $\boldsymbol{R}^4$ consisting of all vectors of the form $(a + b, a - b + 2c, b, c)^T$ and $(a + b, a - b + 2c, a + b, b - c)^T$ respectively, where a, b and c are real numbers. Then the dimension of S_1 plus the dimension of S_2 equals 5.

(e) Any two finite dimensional vector spaces with the same dimension are isomorphic.

6. (5%; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分為負時, 以 0 分計; 未作答亦以 0 分計)

Let A, B and C be three matrices, and $AB = C$. Which of the followings are true?

(a) The row space of C is a subspace of the row space of A.

(b) The column space of C is a subspace of the column space of A.

(c) The row space of C is a subspace of the row space of B.

(d) If the column vectors of A are linearly dependent, then the column vectors of C are linearly dependent.

(e) If the row vectors of A are linearly dependent, then the row vectors of C are linearly dependent.

7. (5%; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Which of the followings are true?

(a) Let $L : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a linear transformation. If A is the standard matrix representation of L, then an $n \times n$ matrix B will also be a matrix representation of L if and only if B is similar to A.

(b) Let $L : V \to W$ be a linear transformation. $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are linearly dependent in V, if and only if $L(\boldsymbol{v}_1), L(\boldsymbol{v}_2), \ldots, L(\boldsymbol{v}_k)$ are linearly dependent in W.

(c) The transition matrix from one basis to another must be nonsingular, and a matrix representation of a linear transformation can be singular.

(d) Any two matrices have the same trace if and only if they are similar.

(e) Let $[\boldsymbol{u}_1, \boldsymbol{u}_2]$ and $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ be ordered bases for $\boldsymbol{R}^2$, where $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$. Let L be a linear transformation from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$ whose matrix representation with respect to the order basis $[\boldsymbol{u}_1, \boldsymbol{u}_2]$ is $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Then the matrix representation of L with respect to the ordered basis $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ is $\begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix}$

8. (5%; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Consider a matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ and a system of linear equations $A\boldsymbol{x} = \boldsymbol{b} = [-1, 3, 1]^T$. Which of the following vectors $\boldsymbol{c}$ will make the two systems $A\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{c}$ and $A\boldsymbol{x} = \boldsymbol{b}$ have the same least-square error.

(a) $\boldsymbol{c} = [0, -1, -1]^T$.

(b) $\boldsymbol{c} = [3, 2, 2]^T$.

(c) $\boldsymbol{c} = [1, 0, -1]^T$.

(d) $\boldsymbol{c} = [2, 1, 2]^T$.

(e) $\boldsymbol{c} = [1, 2, 1]^T$.

9. (5%; 複選，答對每個選項得 1.25 分，答錯每個選項扣 1.25 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Suppose $A_{3\times3}$ has three distinct eigenvalues 0, 1, 3 with corresponding eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Which of the following statements are true?

(a) The rank of A must be 2.

(b) The matrix $A^2 - I$ is invertible.

(c) $\boldsymbol{v}$ and $\boldsymbol{w}$ must span the column space of A.

(d) The least-squares error of $A\boldsymbol{x} = 2\boldsymbol{v} + 3\boldsymbol{w} + \boldsymbol{u}$ must be $\|\boldsymbol{u}\|^2$.

10. (5%; 複選，答對每個選項得 1.25 分，答錯每個選項扣 1.25 分；本題合計得分為負時，以 0 分計；未作答亦以 0 分計)

Suppose $A_{3\times 3} = \boldsymbol{x}\boldsymbol{y}^T$ is a rank-1 matrix. Which of the following statements are true?

(a) The eigenvector matrix S (i.e., $AS = \Lambda S$) must be invertible.

(b) A must have one non-zero eigenvalue.

(c) When A is factorized by singular value decomposition (SVD) into $U\Sigma V^T$, the only non-zero singular value is $\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|$.

(d) When A is factorized by SVD into $U\Sigma V^T$, U must include $\dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ as one of its column vectors.

11. (a) (4%) Consider the following sequence of numbers:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …

The 1st number is 1; the 4th number is 3; etc. Define a function, for $j = 1, 2, 3, \ldots$

$g(j) =_{\text{def}}$ the k-th number in this sequence, where $k = (\lfloor (j+1)^2/2 \rfloor)$

What is $g(j)$?

11. (b) We say a binary string is *happy* if and only if it does not contain two consecutive 0's. For instance, 101010 and 011010 are happy while 000011 is not. Let $h(n) =_{\text{def}}$ the number of n-bit happy strings.

(1) (4%) What is $h(n)$ (as a recurrence relation)?

(2) (5%) Solve this recurrence relation.

11. (c) (4%) How many functions are there from a finite set A to a finite set B?

11. (d) (4%) Compute f^{1001}, where f is defined as the following permutation:

$$\begin{bmatrix} a & b & c & d & e & f & g & h \\ d & f & a & c & g & e & h & b \end{bmatrix}$$

11. (e) (4%) How many integer solutions are there for the following equation:

$$x_1 + x_2 + x_3 + 5x_4 = 18$$

Assume that $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

12. (a) (10%) Answer True/False to the following statements. Each correct answer gets 2 points, and each wrong answer deducts 2 points. If the number of points obtained in this question set is negative, it will be treated as 0.

(1) If a graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5, the graph is nonplanar.

(2) The number of regions of a general graph is given by $e - v + 2$ where e is the number of edges of the graph and v is the number of vertices of the graph.

(3) The number of nodes in a rooted tree with height h is at most $2^{h+1} - 1$.

(4) A simple graph is connected if and only if it has a spanning tree.

(5) The characteristic equation of a linear homogeneous recurrence relation of degree k with constant coefficients is a k-degree polynomial equation.

12. (b) (10%) Let $S = \{1, 2, \ldots, 40\}$. For $a, b \in S$, we write aRb if and only if $a \equiv b \bmod 7$. Please answer the following questions, and highlight your answers.

(1) (5%) Prove that R is an equivalence relation.

(2) (2%) If n is in S, let $[n]$ denote the set of elements of S that are equivalent to n. How many elements are in the equivalence class [5]?

(3) (3%) We know that $P = \{[1], [2], \ldots, [7]\}$ is a partition of S. For $[a], [b] \in P$, we write $[a] \leq [b]$ if and only if (a mod 7) $\leq$ (b mod 7). Please draw the Hasse diagram for the poset $(P, \leq)$.

12. (c) (5%) The 7-Day University needs to arrange 7 different courses, $C_1, C_2, \ldots, C_7$, in a week. One course is for one day, and all 7 courses must be arranged. $C_1, C_2, \ldots$, and C_7 can't be on Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday, respectively. How many possible arrangements does the 7-Day University have?