99 國立交通大學

所別：資訊聯招　　　　科目：線性代數與離散數學

1. (a) (4%) $A=\begin{bmatrix}1&0&0&0\\2&2&0&0\\3&5&3&0\\4&6&7&4\end{bmatrix}\begin{bmatrix}1&1&-1&1\\0&1&-1&1\\0&0&2&6\\0&0&1&5\end{bmatrix}$

Find a row echelon form of A and find $\det(A^{-1})$.

1. (b) (8%) $A=\begin{bmatrix}a_1&a_2&a_3&a_4&a_5\\b_1&b_2&b_3&b_4&b_5\\c_1&c_2&c_3&c_4&c_5\\d_1&d_2&d_3&d_4&d_5\end{bmatrix}$. Applying Gaussian elimination, we obtain a row echelon form of A as follows: $U=\begin{bmatrix}1&1&2&2&1\\0&1&1&3&1\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$. Suppose $\begin{bmatrix}a_1\\b_1\\c_1\\d_1\end{bmatrix}=\begin{bmatrix}-3\\5\\2\\1\end{bmatrix}$, $\begin{bmatrix}a_2\\b_2\\c_2\\d_2\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}a_5\\b_5\\c_5\\d_5\end{bmatrix}=\begin{bmatrix}4\\-3\\7\\-1\end{bmatrix}$. Find the third and fourth column of A, find a basis for the row space of A and find a basis for the null space $N(A)$ of A.

1. (c) Let A be a 4×5 matrix and let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ be the column vectors of A. If columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4$ are linearly independent and $\boldsymbol{a}_3=\boldsymbol{a}_1+2\boldsymbol{a}_2$, $\boldsymbol{a}_5=2\boldsymbol{a}_1-\boldsymbol{a}_2+3\boldsymbol{a}_4$,

(i) (2%) What is the dimension of the null space $N(A)$?

(ii) (2%) Determine the reduced row echelon form of A.

2. (a) (5%) If A is an $n \times n$ matrix and diagonalizable, and λ is an eigenvalue of multiplicity n, then what is the dimension of the null space of $A - \lambda I$ and what is A?

(b) Let A be an $n \times n$ matrix, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent vectors in $\mathbf{R}^n$.

(i) (2%) What must be true about A for $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ to be linearly independent?

(ii) (3%) Justify your answer for (i) using a formal proof that is based on the definition of linear independence for $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

3. Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation defined by

$$L((x_1, x_2)^T) = (x_1 + x_2, 2x_1 - x_2)^T,$$

and let $E = [(1, 2)^T, (-1, 1)^T]$ and $F = [(1, 0)^T, (0, 1)^T]$ be bases for $\mathbf{R}^2$.

(a) (2%) Find the matrix A representing L with respect to the ordered bases E and F.

(b) (2%) Find the transition matrix S representing the change of bases from F to E.

(c) (3%) Find the matrix B representing L with respect to the ordered bases F and E using the matrices A and S.

4. (7%) True or false? (答對每小題得 1 分，答錯每小題倒扣 1 分，未作答以 0 分計；本題合計得分為負時以 0 分計)

(a) If A is an $m \times n$ matrix, then the null space of A^T and the column space of A are orthogonal complements.

(b) If A is an $m \times n$ matrix, then the nullity of A^T plus the rank of A equals m.

(c) If A is an $n \times n$ matrix and has fewer than n distinct eigenvalues, then A is defective.

(d) If A is an $n \times n$ matrix, then the rank of A is equal to its nonzero eigenvalues.

(e) If A is an $m \times n$ matrix, then there is always an orthogonal matrix that can diagonalize A^TA.

(f) If A is a nonsquare $m \times n$ matrix and $A^TA = I$, then the rows of A form an orthonormal set.

(g) If A is an $m \times n$ matrix and $A^TA = I$, then $\mathbf{x} - A^TA\mathbf{x}$ is orthogonal to the column space of A for any vector $\mathbf{x} \in \mathbf{R}^m$.

5. The Fibonacci sequence can be recursively defined by $x_1 = 1, x_2 = 1, x_n = x_{n-1} + x_{n-2}, n \geq 3$.

(a) (1%) Determine the matrix A that can recursively generate the Fibonacci sequence by $\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}$.

(b) (2%) Staring with $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show that $\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = A^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) (3%) Find a matrix P that diagonalizes A.

(d) (2%) Derive an explicit formula for the n-th term of the Fibonacci sequence.

(e) (2%) Determine the limit of $\dfrac{x_n}{x_{n-1}}$ as n approaches infinity.

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6. For this problem, please write down your answer first and then give your explanatio
You need to answer the question one by one and in order.

Question (a). (5%) Let Y be a set. The notation 2^Y denotes the set of all subsets of Y. Fo example, $2^{\{\alpha,\beta\}} = \{\{\}, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$. Let U, V be two sets. The notation $U \times V$ denotes the set $\{(a, b) \mid a \in U, b \in V\}$. Let X be a non-empty set. A subset S of $X \times X$ i symmetric if and only if it satisfies the following condition:

$$\text{if } (a, b) \in S \text{ then } (b, a) \in S, \text{ for all elements } a, b \in X.$$

Let $P = \{1, 2, 3\}$ and $Q = \{1, 2\}$. How many subsets of $(2^P \times Q) \times (2^P \times Q)$ ar symmetric?

Question (b). (5%) Let $Fib(n)$ denote the predicate "n is a Fibonacci number" and $Prime($ denote the predicate "n is a prime". Please translate the following statement into formula in predicate logic:

"There is a Fibonacci number that is a prime."

Question (c). (5%) What is the remainder of 2^{99} divided by 33?

Question (d). (5%) Consider the arithmetic sequence: 1, 4, 7, ..., 100. Find the least k such that among any k distinct numbers randomly chosen from the above arithmetic sequence, there are two whose sum is exactly 104.

Question (e). (5%) Given 9 left parentheses and 9 right parentheses, how many properly *nested sequence of parentheses* can be formed? A sequence of left and right parentheses *is properly nested* if (1) the number of left parentheses in any prefix of the sequence is no less than the number of right parentheses and (2) the number of left parentheses is equal to the number of right parentheses in the whole sequence. For example, "[[[ ]] [[[ ]]] [[[ ]]]" is properly nested whereas "[[ ]]] [[ ]" is not. Intuitively, the usual arithmetical expressions, such as $((a + b) * (c - d))$, are all properly nested.

7. For this problem, please write down your answer first and then give your explanation. You need to answer the questions one by one and in order.

Question (a). (6%) Each correct answer gets 2 points.

An m-ary tree is a rooted tree each of whose internal vertices has no more than m children. The height of a rooted tree is the largest path length from the root to any vertex. Let h denote the height of an m-ary tree.

(a) The maximal number of leaves in an m-ary tree of height h.

(b) Let a_h denote the maximal number of vertices in an m-ary tree of height h. Give a recurrence relation of a_h and also an initial condition.

(c) If $m = 2$, give an explicit form of a_h.

Question (b). (10%) Each correct answer gets 2 points. Please highlight your answer.

(a) Let $R = \{(a, b), (b, c), (c, c)\}$ be a relation on set $\{a, b, c\}$. What is the reflexive closure of R?

(b) Let $x_1 \in \{1, 3, 5\}$, $x_2 \in \{1, 2, 3\}$, and $x_3 \in \{0, 1, 2, 3, \ldots\}$, and a_n be the number of solutions of $x_1 + x_2 + x_3 = n$. Give the generating function $g(x)$ of a_n.

(c) Let e, v, and r, respectively, denote the number of edges, vertices, and regions of a planar representation of a connected simple planar graph. Please give Euler's formula of planar graphs.

(d) Give the binary tree represent of the equation $(x + (y/z)) + 1$.

(e) Consider a connected graph with 10 vertices. If all its edges are with weight 1, what is the weight of its minimum spanning tree?

Question (e). (9%) Each correct answer gets 3 points.

Solve the linear recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2n + 1$.

(a) Let $a_n^{(h)}$ be the general solution for $a_n = 5a_{n-1} - 6a_{n-2}$. Find $a_n^{(h)}$.

(b) Let $a_n^{(p)} = sn + t$ for some s and t be a solution for $a_n = 5a_{n-1} - 6a_{n-2} + 2n + 1$.

Find $a_n^{(p)}$.

(c) If $a_0 = 5$ and $a_1 = 6$, find a_n for $a_n = 5a_{n-1} - 6a_{n-2} + 2n + 1$.

(Hint: Let $a_n = a_n^{(h)} + a_n^{(p)}$.)