Chapter 01

(1)a matrix $A \in F^{m \times n}$ may

左反未必存在,右反未必存在,或皆不純在➔zero matrix

存在左反矩陣 ➔ $n \le m$

存在右反矩陣 ➔ $m \le n$

(2) $(R_{ij})^{-1} = R_{ij}$, $(R_i^{\,k})^{-1} = R_i^{\frac{1}{k}}$, $(R_{ij}^{\,(k)})^{-1} = R_{ij}^{\,(-k)}$

(3) 對 A 做列運算⇔把 A 乘上一個列基本矩陣

If $A \in F^{m \times n}$,對 A 做某一型列運算 r 得到 B ,則 $B = EA$, $E = r(I_m)$ 爲列基本矩陣

(4) $A \in F^{m \times n}, F = R(\ \ or\ \ C)$

$Ax = 0$ 有非 0 解 ⇔ $Ax = 0$ 有無限多解

(5) **Row echelon form**

(a) 全部為零的列在矩陣最底下

(b) 不全為零的列，其第一個**非零元素(pivot)**為 1

(c) 對兩相鄰的非零列而言，較高列之領先 1 出現在較 低列之領先 1 的左邊

Reduced Row echelon form

A matrix be a Row echelon form and

在領先 1 的那一行除了領先 1 以外的位置全部為零

(6) make row echelon form be $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & M & N \end{array}\right]$

$$(a) no\ \ solution \Rightarrow \begin{cases} M = 0 \\ N \neq 0 \end{cases}$$

$$(b) unique\ \ solution \Rightarrow M \neq 0$$

$$(c) infinite\ \ solution \Rightarrow \begin{cases} M = 0 \\ N = 0 \end{cases}$$

(7) $A \in F^{n \times n}$ following are equivalent statement

(1)A:可逆

(2) $\forall b \in F^{n \times 1},\ Ax = b$ 具唯一解

(3) $\forall b \in F^{n \times 1},\ Ax = b$ 有解

Ex 1.1　10 間學校

$A\in F^{m\times n},B\in F^{n\times m}$ prove $tr(AB)=tr(BA)$
$suppose \quad C=AB\in F^{m\times m},D=BA\in F^{n\times n}$ $tr(AB)=tr(C)=\sum_{i=1}^{m}c_{ii}=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}b_{ji}=\sum_{j=1}^{n}\sum_{i=1}^{m}b_{ji}a_{ij}=\sum_{j=1}^{n}d_{jj}=tr(D)=tr(BA)$

Ex 1.2　8 間學校

$prove \quad \nexists \quad A,B\in F^{m\times n} \ni AB-BA=I$
$suppose \quad exist \quad A,B\in F^{m\times n}\ni AB-BA=I$ $\Rightarrow n=tr(AB-BA)=tr(AB)-tr(BA)=0 \quad \rightarrow\leftarrow$

Ex 1.4

Prove A,B: upper triangular matrix ➔ C=AB : upper triangular matrix (A,B: lower triangular matrix ➔ C=AB : low triangular matrix)
$\because A,B: upper \quad triangular \Rightarrow a_{ij}=b_{ij}=0,\forall i>j$ $claim: c_{ij}=0,\forall i>j$ $given \quad i>j, c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+...+a_{i(i-1)}b_{(i-1)j}+a_{ii}b_{ij}+...+a_{in}b_{nj}$ $\because a_{i1}=a_{i2}=...=a_{i(i-1)}=0\Rightarrow c_{ij}=a_{ii}b_{ij}+...+a_{in}b_{nj}$ $and \quad b_{ij}=0,\forall i>j \quad ,\therefore c_{ij}=a_{ii}b_{ij}+...+a_{in}b_{nj}=0$

Ex 1.4

(1) A 可逆 (2) $A\vec{x}=\vec{0}$ 只有零解 (3) $A\sim I$ (4)A 能寫成若干個列基本矩陣
$(1)\Rightarrow(2)$ $A\vec{x}=\vec{0}$,因爲 A 可逆,$\therefore A^{-1}A\vec{x}=A^{-1}\vec{0}\Rightarrow\vec{x}=\vec{0}$ $(2)\Rightarrow(3)$ $A\vec{x}=\vec{0}$ 只有零解$\Rightarrow$ $A\vec{x}=\vec{0}$ 有唯一解$\Rightarrow rank(A)=n$ A 可化簡到列梯形矩陣 R,且 R 有 n 個非零列,因爲 $R:n*n$,所以 R 不具零列, $R=I_n$ $(3)\Rightarrow(4)$ $A\sim I$ 存在基本矩陣 $E_1,E_2,....,E_k$, $E_1,E_2,....,E_k \ni E_k...E_2E_1A=I$ $A=E_1^{-1}E_2^{-1}...E_k^{-1}$ $(4)\Rightarrow(1)$ 基本矩陣之 inverse 可逆,且 A 爲數個列基本矩陣相乘,所以 A 可逆

Ex 1.5

$A \in F^{m\times m}, C \in F^{n\times n}$ prove $X = \begin{bmatrix} A & B \\ O & C \end{bmatrix}, B \in F^{m\times n}, O \in F^{n\times m}$ invertable,and $X^{-1} = ?$
$let\ \ Y = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}, P \in F^{m\times m}, Q \in F^{m\times n}, R \in F^{n\times m}, S \in F^{n\times n}$ $suppose\ \ \begin{bmatrix} A & B \\ O & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_n \end{bmatrix}$ $\Rightarrow \begin{bmatrix} AP+BR & AQ+BS \\ CR & CS \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_n \end{bmatrix}, \ldots\ldots\ ,\ \ pick\ \ Y = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ O & C^{-1} \end{bmatrix} = X^{-1}$

95 輔大電子

$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$ (a) find the reduced echelon form of A (b) find the reduced row echelon form of A
$(a)\begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix} r_{12}^{-2} r_{13}^{-3} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 & -3 \end{bmatrix}$ $r_{23}^{-1} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} r_3^{-\frac{1}{4}} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $(b)\begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} r_{21}^{-2} r_{31}^{-2} r_{32}^{-1} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Ex 1.12

solve linear system by Ax=b $A=\begin{bmatrix}1 & -1 & -2 & -8\\ -2 & 1 & 2 & 9\\ 3 & 0 & 2 & 1\end{bmatrix}, b=\begin{bmatrix}-3\\ 5\\ -8\end{bmatrix}$

$$LU=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 3 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -2 & -8\\ 0 & -1 & -2 & -7\\ 0 & 0 & 2 & 4\end{bmatrix}$$

$Ax=b \Rightarrow LUx=b$, $let\ \ Ux=y \Rightarrow Ly=b\ \ got\ \ y$, $Ux=y \Rightarrow\ \ got\ \ x$

$$\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 3 & -3 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}-3\\ 5\\ -8\end{bmatrix} \Rightarrow \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}-3\\ -1\\ -2\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -2 & -8\\ 0 & -1 & -2 & -7\\ 0 & 0 & 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-3\\ -1\\ -2\end{bmatrix} \Rightarrow \begin{cases}x_1 = x_4 - 2\\ x_2 = -3x_4 + 3\\ x_3 = -2x_4 - 1\end{cases}$$

Ex 1.9

$\begin{cases}kx+y+z=1\\ x+ky+z=1\\ x+y+kz=1\end{cases}$ k= ? when $(a)no\ \ solution$ $(b)unique\ \ solution$ $(c)infinite\ \ solution$

$$(a)no\ \ solution \Rightarrow \begin{cases}-(k-1)(k+2)=0\\ 1-k\neq 0\end{cases} \Rightarrow k=-2$$

$$(b)unique\ \ solution \Rightarrow -(k-1)(k+2)\neq 0 \Rightarrow k \notin \{1,-2\}$$

$$(c)infinite\ \ solution \Rightarrow \begin{cases}-(k-1)(k+2)=0\\ 1-k=0\end{cases} \Rightarrow k=1$$

96 輔大資工

Use Gaussian elimination to solve the following system of linear equation $\begin{cases} 2x_1 - x_2 + 3x_3 + 4x_4 = 9 \\ x_1 - 2x_2 + 7x_4 = 11 \\ 3x_1 - 3x_2 + x_3 + 5x_4 = 8 \\ 2x_1 + x_2 + 4x_3 + 4x_4 = 10 \end{cases}$
$\left[\begin{array}{cccc\|c} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{array}\right] \to r_{21}^{-2}, r_{23}^{-3}, r_{24}^{-2} \left[\begin{array}{cccc\|c} 0 & -1 & 7 & -10 & -13 \\ 1 & 0 & -2 & 7 & 11 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{array}\right]$ $\to r_{41}^{1}, r_{43}^{3} \left[\begin{array}{cccc\|c} 0 & 0 & 15 & -20 & -25 \\ 1 & 0 & -2 & 7 & 11 \\ 0 & 0 & 31 & -46 & -61 \\ 0 & 1 & 8 & -10 & -12 \end{array}\right] \to \left[\begin{array}{cccc\|c} 0 & 0 & 3 & -4 & -5 \\ 1 & 0 & -2 & 7 & 11 \\ 0 & 0 & 0 & -14 & -28 \\ 0 & 1 & 8 & -10 & -12 \end{array}\right]$ 然後硬ㄠ成 **Row echelon form** $\left[\begin{array}{cccc\|c} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & 8 & 10 & -12 \\ 0 & 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \\ x_4 = 2 \end{cases}$

96 中華應數

$\begin{cases} 2x_1 + 5x_2 + (2\alpha+1)x_3 = 4 \\ x_1 + 3x_2 + \alpha x_3 = -1 \\ -3x_1 - 5x_2 - x_3 = \beta \end{cases}$ what condition of α, β (a)unique solution (b)infinite solution (c)no solution
$\left[\begin{array}{ccc\|c} 1 & 3 & \alpha & -1 \\ 2 & 5 & 2\alpha+1 & 4 \\ -3 & -5 & -1 & \beta \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & \alpha & -1 \\ 0 & -1 & 1 & 6 \\ 0 & 4 & 3\alpha-1 & \beta-3 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & \alpha & -1 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & 3\alpha+3 & \beta+21 \end{array}\right]$ $(a)\alpha \neq -1 \quad (b)\alpha = -1, \beta = -21 \quad (c)\alpha = -1, \beta \neq -21$

Example : 1-3 例 19

$A=\begin{bmatrix}0 & a\\ b & c\end{bmatrix}$, $abc\neq 0$,find elementary matrics E_1,E_2,E_3,E_4 such that $A=E_4E_3E_2E_1$

$$\begin{bmatrix}0 & a\\ b & c\end{bmatrix}\rightarrow r_{12}\begin{bmatrix}b & c\\ 0 & a\end{bmatrix}\rightarrow r_{21}^{-\frac{c}{a}}\begin{bmatrix}b & 0\\ 0 & a\end{bmatrix}\rightarrow r_1^{\frac{1}{b}}\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}\rightarrow r_2^{\frac{1}{a}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$$R_2^{\frac{1}{a}}R_1^{\frac{1}{b}}R_{21}^{-\frac{c}{a}}R_{12}A=I\Rightarrow (R_2^{\frac{1}{a}}R_1^{\frac{1}{b}}R_{21}^{-\frac{c}{a}}R_{12})=A^{-1}$$

$$\Rightarrow (R_2^{\frac{1}{a}}R_1^{\frac{1}{b}}R_{21}^{-\frac{c}{a}}R_{12})^{-1}=A\Rightarrow A=R_{12}R_{21}^{\frac{c}{a}}R_1^{b}R_2^{a}$$

$$=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{c}{a}\\ 0 & 1\end{bmatrix}\begin{bmatrix}b & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}=E_4E_3E_2E_1$$

Example : 1-2-6

$A=\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 3 & 0 & 0\\ 0 & -4 & 5 & 0\\ 0 & 0 & -6 & 7\end{bmatrix}$, $B=(I+A)^{-1}(I-A)$,find $(I+B)^{-1}$

$$I+B=I+(I+A)^{-1}(I-A)=(I+A)^{-1}(I+A)+(I+A)^{-1}(I-A)$$

$$=(I+A)^{-1}(I+A+I-A)=2I(I+A)^{-1}=2(I+A)^{-1}$$

$$(I+B)^{-1}=\left(2(I+A)^{-1}\right)^{-1}=\frac{1}{2}(I+A)=\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 2 & 0 & 0\\ 0 & -2 & 3 & 0\\ 0 & 0 & -3 & 4\end{bmatrix}$$

96 台大資工

$A=\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 3 & 0 & 0\\ 0 & -4 & 5 & 0\\ 0 & 0 & -6 & 7\end{bmatrix}$, $I=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$, $B=(I+A)^{-1}(I-A)$,calculate $(I+B)^{-1}$

先算 $I+B=I+(I+A)^{-1}(I-A)=(I+A)^{-1}(I+A)+(I+A)^{-1}(I-A)$

$$=(I+A)^{-1}(2I+A-A)=2I(I+A)^{-1}=2(I+A)^{-1}$$

$$(I+B)^{-1}=(2(I+A)^{-1})^{-1}=\frac{1}{2}(I+A)=\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 2 & 0 & 0\\ 0 & -2 & 3 & 0\\ 0 & 0 & -3 & 4\end{bmatrix}$$

Example : 1-7-2

Find elementary matrices $E_1, E_2, \ldots, E_k$ such that $A = BE_1E_2 \ldots E_k$ where $A = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to c_{12}{}^{3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to c_{13} \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to c_1{}^{4} \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ $E_1 = C_{12}{}^{3} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = C_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = C_1{}^{4} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = BC_{12}{}^{3}C_{13}C_1{}^{4}$

96 中原應數

Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no LU decomposition
假設 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 之 LU 存在 $L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}, U \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ $A = LU \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \Rightarrow \begin{cases} ad = 0, ae = 1 \\ bd = 1, be + cf = 0 \end{cases}$ $\begin{cases} ae = 1 \Rightarrow a \neq 0 \\ ad = 0 \Rightarrow d = 0 \\ bd = 1 \Rightarrow d \neq 0 \end{cases}$ 矛盾

95 政大資科

Find the LU factorization of $A=\begin{bmatrix}2&4&-1&5&-2\\-4&-5&3&-8&1\\2&-5&-4&1&8\\-6&0&7&-3&1\end{bmatrix}$
$A=\begin{bmatrix}2&4&-1&5&-2\\-4&-5&3&-8&1\\2&-5&-4&1&8\\-6&0&7&-3&1\end{bmatrix}r_{12}^{\ 2},r_{13}^{\ -1},r_{14}^{\ 3}\to\begin{bmatrix}2&4&-1&5&-2\\0&3&1&2&-3\\0&-9&-3&-4&10\\0&12&4&12&-5\end{bmatrix}$ $r_{23}^{\ 3},r_{24}^{\ -4}\begin{bmatrix}2&4&-1&5&-2\\0&3&1&2&-3\\0&0&0&2&1\\0&0&0&4&7\end{bmatrix},r_{34}^{\ -2}\begin{bmatrix}2&4&-1&5&-2\\0&3&1&2&-3\\0&0&0&2&1\\0&0&0&0&5\end{bmatrix}=U$ $R_{34}^{\ -2}R_{24}^{\ -4}R_{23}^{\ 3}R_{14}^{\ 3}R_{13}^{\ -1}R_{12}^{\ 2}A=U\Rightarrow L=(R_{34}^{\ -2}R_{24}^{\ -4}R_{23}^{\ 3}R_{14}^{\ 3}R_{13}^{\ -1}R_{12}^{\ 2})^{-1}$ $L=R_{12}^{\ -2}R_{13}^{\ 1}R_{14}^{\ -3}R_{23}^{\ -3}R_{24}^{\ 4}R_{34}^{\ 2}=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\1&-3&1&0\\-3&4&2&1\end{bmatrix}\Rightarrow LU=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\1&-3&1&0\\-3&4&2&1\end{bmatrix}\begin{bmatrix}2&4&-1&5&-2\\0&3&1&2&-3\\0&0&0&2&1\\0&0&0&0&5\end{bmatrix}$

96 中原應數　95 彰師資工

LL^T ,LDU decomposition for $A=\begin{bmatrix}4&-4&8\\-4&13&1\\8&1&26\end{bmatrix}$
$A=\begin{bmatrix}4&-4&8\\-4&13&1\\8&1&26\end{bmatrix}r_{12}^{\ 1},r_{13}^{\ -2}\to\begin{bmatrix}4&-4&8\\0&9&9\\0&9&10\end{bmatrix}r_{23}^{\ -1}\to\begin{bmatrix}4&-4&8\\0&9&9\\0&0&1\end{bmatrix}=U$ $LU=\begin{bmatrix}1&0&0\\-1&1&0\\2&1&1\end{bmatrix}\begin{bmatrix}4&-4&8\\0&9&9\\0&0&1\end{bmatrix}\Rightarrow LDU=\begin{bmatrix}1&0&0\\-1&1&0\\2&1&1\end{bmatrix}\begin{bmatrix}4&0&0\\0&9&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&1\\0&0&1\end{bmatrix}$ $\Rightarrow L\sqrt{D}\sqrt{D}U=\begin{bmatrix}1&0&0\\-1&1&0\\2&1&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&1\\0&0&1\end{bmatrix}$ $\Rightarrow LL^T=\begin{bmatrix}-2&0&0\\-2&3&0\\4&3&1\end{bmatrix}\begin{bmatrix}2&-2&4\\0&3&3\\0&0&1\end{bmatrix}$

Chapter 02

(1) $A = [a_{ij}] \in F^{n \times n}$

$$(1)\det(r_{ij}(A)) = \det(c_{ij}(A)) = -\det(A)$$
$$(2)\det(r_i^{(k)}(A)) = \det(c_i^{(k)}(A)) = k\det(A)$$
$$(3)\det(r_{ij}^{(k)}(A)) = \det(c_{ij}^{(k)}(A)) = \det(A)$$

$$\det(\alpha A) = (\det(r_i^{(\alpha)}(A)))^n = \alpha^n \det(A)$$

(2) $E = R_{ij} = r_{ij}(I) \qquad \det(E) \in \{-1, k, 1\}, \therefore \det(E) \neq 0$

(3) $A \in F^{n \times n} \qquad A \times adj(A) = adj(A) \times A = \det(A) \times I_n$

Ex 2.5

$A \in F^{n \times n}$ Prove $A : invertable \Leftrightarrow \det(A) \neq 0$
$(\Rightarrow) \because A : invertable \Rightarrow A = E_1E_2....E_k(row \quad elementary \quad matrices)$ $\det(A) = \det(E_1E_2....E_k) = \det(E_1)\det(E_2....E_k) = ...$ $= \det(E_1)\det(E_2)\det(E_3....E_k) = \det(E_1)\det(E_2)....\det(E_k)$ $\because \det(E) \in \{-1, 1, k\} \quad , \quad \therefore \det(A) \neq 0$ $(\Leftarrow) if \quad A : noninvertable \Rightarrow rank(A) < n$ $A \sim_r R : row \quad echelon \quad form \Rightarrow \exists E_1, E_2, ..., E_k \ni E_k...E_2E_1A = R$ $\because rank(A) < n \quad , \quad \therefore R \quad contains \quad unless \quad 1 \quad zero-row \Rightarrow \det(R) = 0$ $0 = \det(R) = \det(E_k...E_2E_1A) = \det(E_k)...\det(E_1)\det(A)$ $\because \det(E) \in \{-1, 1, k\} \quad , \quad \therefore \det(A) = 0$

Ex 2.6

Prove $A \in F^{n \times n} \quad , \quad \det(AB) = \det(A)\det(B)$
$(1) B : noninvertable \Rightarrow \exists x \neq 0 \ni Bx = 0$ $\Rightarrow ABx = A0 = 0 \Rightarrow AB : noninvertable$ $\Rightarrow \det(AB) = 0$ $\because B : noninvertable \Rightarrow \det(B) = 0 \quad \therefore \det(AB) = 0 = \det(A)\det(B)$ $(2) B : invertable \Rightarrow B = E_1E_2...E_k$ $\Rightarrow \det(AB) = \det(AE_1E_2...E_k) = \det(AE_1E_2...E_{k-1})\det(E_k)$ $= \det(AE_1E_2...E_{k-2})\det(E_{k-1})\det(E_k) = ... = \det(A)\det(E_1)...\det(E_{k-1})\det(E_k)$ $= \det(A)\det(E_1E_{k-1})...\det(E_{k-1})\det(E_k) = \det(A)\det(E_1E_{k-1}.....E_k)$ $= \det(A)\det(B)$

96 成大電信

Find the determinant of $A=\begin{bmatrix}1&a&a^2&a^3\\1&b&b^2&b^3\\1&c&c^2&c^3\\1&x&x^2&x^3\end{bmatrix}$

$$\begin{vmatrix}1&a&a^2&a^3\\1&b&b^2&b^3\\1&c&c^2&c^3\\1&x&x^2&x^3\end{vmatrix} c_{34}{}^{-a}\rightarrow \begin{vmatrix}1&a&a^2&0\\1&b&b^2&b^2(b-a)\\1&c&c^2&c^2(c-a)\\1&x&x^2&x^2(x-a)\end{vmatrix} c_{23}{}^{-a}\rightarrow \begin{vmatrix}1&a&0&0\\1&b&b(b-a)&b^2(b-a)\\1&c&c(c-a)&c^2(c-a)\\1&x&x(x-a)&x^2(x-a)\end{vmatrix}$$

$$c_{12}{}^{-a}\rightarrow \begin{vmatrix}1&0&0&0\\1&b-a&b(b-a)&b^2(b-a)\\1&c-a&c(c-a)&c^2(c-a)\\1&x-a&x(x-a)&x^2(x-a)\end{vmatrix}=\begin{vmatrix}b-a&b(b-a)&b^2(b-a)\\c-a&c(c-a)&c^2(c-a)\\x-a&x(x-a)&x^2(x-a)\end{vmatrix}$$

$$=(b-a)(c-a)(x-a)\begin{vmatrix}1&b&b^2\\1&c&c^2\\1&x&x^2\end{vmatrix} c_{23}{}^{-b},c_{12}{}^{-b}\rightarrow (b-a)(c-a)(x-a)\begin{vmatrix}1&0&0\\1&c-b&c(c-b)\\1&x-b&x(x-b)\end{vmatrix}$$

$$=(b-a)(c-a)(x-a)\begin{vmatrix}c-b&c(c-b)\\x-b&x(x-b)\end{vmatrix}=(b-a)(c-a)(x-a)(c-b)(x-b)\begin{vmatrix}1&c\\1&x\end{vmatrix}$$

$$=(b-a)(c-a)(x-a)(c-b)(x-b)(x-c)$$

Ex 2.12

$(a)\det(adj(A))=?\qquad (b)adj(adj(A))=?$

$(a)\because A\times adj(A)=\det(A)\times I$

$\Rightarrow \det(A\times adj(A))=\det(\det(A)\times I)$

$\Rightarrow \det(A)\times\det(adj(A))=\det(\det(A))\times\det(I)$

$$\Rightarrow \det(A)\times\det(adj(A))=\det(\det(A))=\begin{bmatrix}\det(A)&&\\&\ddots&\\&&\det(A)\end{bmatrix}=\det(A)^n$$

$$\Rightarrow \det(adj(A))=\frac{\det(A)^n}{\det(A)}=\det(A)^{n-1}$$

(b)

$A\times adj(A)=\det(A)\times I$

$\Rightarrow A\ \ replace\ \ by\ \ adj(A)\Rightarrow adj(A)\times adj(adj(A))=\det(adj(A))\times I$

$$\Rightarrow adj(adj(A))=\det(adj(A))\times adj(A)^{-1}=\det(adj(A))\times\frac{A}{\det(A)}$$

$$=\det(A)^{n-1}\times\frac{A}{\det(A)}=A\det(A)^{n-2}$$

Ex 2.7

$A: 2\times 2, \lvert A\rvert = 4$, $(a)\det(3A)=?$, $(b)\det(A^2)=?$, $(c)\det(5A^TA^{-1})=?$
$(a)3^2\det(A)=36$ $(b)\det(A^2)=\det(A)^2=16$ $(c)5^2\det(A)\det(A)^{-1}=25$

A,B: 4×4 , $\det(A)=2, \det(B)=-1$, $\det(3A^TA^{-2}BA^TB^{-1})$
$\det(3A^TA^{-2}BA^TB^{-1})=3^4\det(A^T)\det(A^{-1})\det(A^{-1})\det(B)\det(A^T)\det(B^{-1})=81$

Compute determine of $A=\begin{bmatrix} 8 & 4 & 0 & 0 & 0 & 0 \\ -6 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 8 \\ 0 & 0 & 0 & 0 & 12 & 14 \end{bmatrix}$

$$A=\begin{bmatrix} B & & O \\ & C & \\ O & & D \end{bmatrix}, \det(A)=\det(B)\det(C)\det(D)$$

$$\det(A)=\det\begin{bmatrix} 8 & 4 \\ -6 & 2 \end{bmatrix}\det\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\det\begin{bmatrix} -4 & 8 \\ 12 & 14 \end{bmatrix}=40\times 5\times(-152)$$

Example 2-2-7

$$A=\begin{bmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

Evaluate the determinant of A by expanding along the second column

$$(-1)(-1)\det\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}+\det\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}+(-1)\det\begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$

$$+3\det\begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}=7+(-2)-(-6)+3(-3)=2$$

Example 2-3-21

$$A=\begin{bmatrix}2&1&0&0\\0&2&0&0\\20&20&3&1\\20&20&1&1\end{bmatrix}, det(A)=?$$

$$\because A,B,C\in F^{n\times n}$$

$$(1)det\begin{bmatrix}A&C\\O&B\end{bmatrix}=\det(A)\det(B)\quad(2)det\begin{bmatrix}A&O\\C&B\end{bmatrix}=\det(A)\det(B)$$

$$det(A)=det\begin{bmatrix}2&1\\0&2\end{bmatrix}det\begin{bmatrix}3&1\\1&1\end{bmatrix}=4*2=8$$

Example 2-3-8

$$D_n=\begin{bmatrix}10&1&0&0&\cdots&0\\25&10&1&0&\cdots&0\\0&25&10&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&25&10&1\\0&0&0&0&25&10\end{bmatrix}, det(D_n)=?$$

令 $a_n=det(D_n)$

$$det(D_n)=10det(D_{n-1})-25det(D_{n-2})$$
$$\Rightarrow a_n=10a_{n-1}-25a_{n-2}$$

$$a_1=\det[10]=10, a_2=\det\begin{bmatrix}10&1\\25&10\end{bmatrix}=75$$

遞迴關係式 $\begin{cases}a_n=10a_{n-1}-25a_{n-2}\\a_1=10,a_2=75\end{cases}$

特徵方程式:

$let\quad a_n=(c_1n+c_2)5^n$ ， $a_1=10,a_2=75$ 代入

$$\Rightarrow a_n=(n+1)5^n$$

$$a_n=10\det\begin{bmatrix}10&1&0&\cdots&0\\25&10&1&\cdots&0\\0&25&10&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&10\end{bmatrix}-\det\begin{bmatrix}25&1&0&\cdots&0\\0&10&1&\cdots&0\\0&25&10&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&10\end{bmatrix}$$

因爲對第一列展開

$$\det\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

$$\det\begin{bmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{bmatrix} = [(2-1)(3-1)(4-1)(5-1)][(3-2)(4-2)(5-2)][(4-3)(5-3)](5-4)$$

Ex 2.11

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}, adj(A) = ?, A^{-1} = ?$$

$$cof(a_{11}) = \det\begin{bmatrix} 6 & 2 \\ 0 & -3 \end{bmatrix} = -18, cof(a_{12}) = -\det\begin{bmatrix} 5 & 2 \\ 1 & -3 \end{bmatrix} = 17$$

$$cof(a_{13}) = \det\begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = -6, cof(a_{21}) = -\det\begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = -6$$

$$cof(a_{22}) = \det\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = -10, cof(a_{23}) = -\det\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = -2$$

$$cof(a_{31}) = \det\begin{bmatrix} -2 & 1 \\ 6 & 2 \end{bmatrix} = -10, cof(a_{32}) = -\det\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = -1$$

$$cof(a_{33}) = \det\begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} = 28$$

$$adj(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}, A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{-94}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

Use Cramer-Rule to solve $\begin{cases} 4x_1 + 5x_2 = 2 \\ 11x_1 + x_2 + x_3 = 3 \\ x_1 + 5x_2 + x_3 = 1 \end{cases}$

$$\Delta = \det\begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 4 & 5 & 0 \\ 11 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 0 & -15 & -4 \\ 0 & -54 & -10 \end{bmatrix} = -66$$

$$\Delta_1 = \det\begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 0 & -5 & -2 \\ 0 & -14 & -2 \end{bmatrix} = -18$$

$$\Delta_2 = \det\begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 11 & 3 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & -8 & -10 \end{bmatrix} = -22$$

$$\Delta_3 = \det\begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 4 & 5 & 2 \\ 11 & 1 & 3 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 & 1 \\ 0 & -15 & -2 \\ 0 & -54 & -8 \end{bmatrix} = 12$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{3}{11}, x_2 = \frac{\Delta_2}{\Delta} = \frac{2}{11}, x_3 = \frac{\Delta_3}{\Delta} = \frac{-2}{11}$$

Chapter 03

(1) necessary condition for subspace (用來舉反例不是 subspace)

$$W \ \ be \ \ subspace \ \ of \ \ V$$
$$(1)\left\{\vec{0}\right\} \in W$$
$$(2) if \ \ v \in W \ \ \Rightarrow \ \ -v \in W$$

(2) $V = F^{n \times n}, W_1, W_2 \ \ be \ \ subspace \ \ of \ \ V$

$$(a) W_1 \cap W_2 \ \ be \ \ subspace \ \ of \ \ V$$
$$(b) W_1 \subseteq W_2 \ \ or \ \ W_2 \subseteq W_1 \Leftrightarrow W_1 \cup W_2 \ \ be \ \ subspace \ \ of \ \ V$$

(3) 4 subspace

$A \in F^{m \times n}$

$column \ \ space : CS(A) = \{Ax \mid x \in F^{n \times 1}\}$ $\qquad CS(A) \ \ be \ \ subspace \ \ of \ \ F^{m \times 1}$

$row \ \ space \ \ : RS(A) = \{xA \mid x \in F^{1 \times m}\}$ $\qquad RS(A) \ \ be \ \ subspace \ \ of \ \ F^{1 \times n}$

$kernel \ \ space : ker(A) = \{x \in F^{n \times 1} \mid Ax = 0\}$ $\qquad ker(A) \ \ be \ \ subspace \ \ of \ \ F^{n \times 1}$

$left \ \ kernel \ \ space : Lker(A) = \{x \in F^{1 \times m} \mid xA = 0\}$ $\qquad Lker(A) \ \ be \ \ subspace \ \ of \ \ F^{1 \times m}$

(4) $S = \{v_1, v_2, ..., v_k\}$ is finite set

$(1) S \ \ linear \ \ dependent \ \ set \Leftrightarrow \exists \alpha_1, \alpha_2, ..., \alpha_k \in F : not \ \ all \ \ zero \ni \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0$

$(2) S \ \ linear \ \ independent \ \ set \Leftrightarrow if \ \ \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_k = 0$

(5) $V : vector \ \ space \ \ over \ \ F$, $S \subseteq V$

(a)S 生成 V ($span(S) = V$)

(b)S : linear independent set 稱 S 為 V 之 basis

(a)basis 未必惟一 (b)basis 之元素個數稱 dimension , denoted dim(S)

(c)S 為 V 之 basis $\Leftrightarrow$ S 為最小生成集

(d)S 為 V 之 basis $\Leftrightarrow$ S 為最大獨立集

(6) $V : vector \ \ space \ \ over \ \ F$, $\dim(V) = n$, $S \subseteq V$

$|S| > n \Rightarrow S \ \ not \ \ LI \ \ set$, $|S| < n \Rightarrow S \ \ not \ \ span \ \ V$

n 為最大獨立集 element 個數,所以 > n 不可能為獨立集

n 為最小生成集 element 個數, 所以 < n 不可能生成 V

(7) $V : vector \ \ space \ \ over \ \ F$, $|S| = n, \dim(V) = n$

(1)若 S 生成 V,則 S 為 V 之基底

(2) 若 S 為獨立集,則 S 為 V 之基底

Ex 3.1

$V = F^{n\times n}, W_1 = \{A \in V \mid A^T = A\}$ $prove\ \ W_1\ \ be\ \ subspace\ \ of\ \ V$
$(1)\forall \alpha, \beta \in F, A, B \in W_1$ $\Rightarrow \begin{cases} A^T = A \\ B^T = B \end{cases} \Rightarrow (\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$ $\alpha A + \beta B \in W_1$, $\therefore W_1\ \ be\ \ subspace\ \ of\ \ V$

Ex 3.2

$prove\ \ following\ \ be\ \ subspace\ \ of\ \ R^3$ $W_2 = \{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$
$0 = (0,0,0)^T \notin W_2 \Rightarrow W_2\ \ not\ \ be\ \ subspace\ \ of\ \ R^3$

Ex 3.3

$prove\ \ \ W_1, W_2\ \ be\ \ subspace\ \ of\ \ V \Rightarrow W_1 \cap W_2\ \ be\ \ subspace\ \ of\ \ V$
$\because W_1 \subseteq_s V, W_2 \subseteq_s V \Rightarrow W_1 \cap W_2 \subseteq V$ $\because\ \ 0 \in W_1, 0 \in W_2\ \ \Rightarrow 0 \in W_1 \cap W_2 \Rightarrow W_1 \cap W_2 \neq \varnothing$ $\forall u, v \in W_1 \cap W_2, \alpha, \beta \in R \Rightarrow u, v \in W_1, u, v \in W_2$ $\because W_1 \subseteq_s V, W_2 \subseteq_s V$ $\Rightarrow \alpha u + \beta v \in W_1, \alpha u + \beta v \in W_2$ $\Rightarrow \alpha u + \beta v \in W_1 \cap W_2$ 所以$W_1 \cap W_2\ \ be\ \ subspace\ \ of\ \ V$

Ex 3.22

$V = F^{n\times n}, W_1 = \{A\in V \mid A^T = A\}, W_2 = \{A \in V \mid A^T = -A\}, W_1, W_2$ *be subspace of* V *prove* $V = W_1 \oplus W_2$
$(1) claim: V = W_1 + W_2$ $\forall A = V, A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = B + C, B = \frac{A+A^T}{2}, C = \frac{A-A^T}{2}$ $\because B^T = (\frac{A+A^T}{2})^T = \frac{A+A^T}{2} = B \Rightarrow B \in W_1$ $\because C^T = (\frac{A-A^T}{2})^T = \frac{-A+A^T}{2} = -C \Rightarrow C \in W_2$ $\Rightarrow A = B + C \in W_1 + W_2$ (2)欲證W_1, W_2爲獨立子空間,只需證$W_1 \cap W_2 = \{0\}$ $\forall A \in W_1 \cap W_2 \Rightarrow A \in W_1, A \in W_2 \Rightarrow A^T = A, A^T = -A$ $\Rightarrow A = -A \Rightarrow 2A = O \Rightarrow A = O \Rightarrow W_1 \cap W_2 = \{0\}$

Ex 3.4

Determine the following be vector space or not on $R^{2\times 2}$ (a) the set of all 2×2 singular matrices (b) the set of all 2×2 nonsingular matrices (c) the set of all 2×2 diagonal matrices (d) the set of all 2×2 with integer entry (e) the set of all 2×2 such that $\det(A) = 0$
$(a) A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(b) A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha = 0, \alpha A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $(c) true$ $(d) A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha = \frac{1}{2} \Rightarrow \alpha A \notin R^{2\times 2}$ $(e) A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Ex 3.6

$W=\{A:2\times 2 \mid A^T=A\}, M_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}, M_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix}, M_3=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ Show $W=spam\{M_1,M_2,M_3\}$
$\forall A=\begin{bmatrix}a&b\\b&c\end{bmatrix}\in W$,令 $\begin{bmatrix}a&b\\b&c\end{bmatrix}=x\begin{bmatrix}1&0\\0&1\end{bmatrix}+y\begin{bmatrix}1&0\\0&-1\end{bmatrix}+z\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\begin{cases}a=x+y\\b=z\\c=x-y\end{cases}\Rightarrow\begin{cases}x=\dfrac{a+c}{2}\\y=\dfrac{a-c}{2}\\z=b\end{cases}$ 所以 $W=spam\{M_1,M_2,M_3\}$

Ex 3.7

Show $\{u,v,w\}=\{(1,2,3),(0,1,2),(0,0,1)\}$ $span$ R^3
$\forall v=(a,b,c)\in R^3,(a,b,c)=\alpha(1,2,3)+\beta(0,1,2)+\gamma(0,0,1)$ $\begin{cases}a=\alpha\\b=2\alpha+\beta\\c=3\alpha+2\beta+\gamma\end{cases}\Rightarrow\begin{cases}\alpha=a\\\beta=-2a+b\\\gamma=a-2b+c\end{cases}$

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Let U be the subspace of R^3 generated by the vector $(1,2,0),(-3,1,2)$ Let V be the subspace of R^3 generated by the vector $(-1,5,2),(4,1,-2)$ Show that $U=V$
$\begin{bmatrix}1&2&0\\-3&1&2\\-1&5&2\\4&1&-2\end{bmatrix}\sim\begin{bmatrix}1&2&0\\-3&1&2\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow span(V)\subseteq_s span(U)$ $\begin{bmatrix}-1&5&2\\4&1&-2\\1&2&0\\-3&1&2\end{bmatrix}\sim\begin{bmatrix}-1&5&2\\4&1&-2\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow span(U)\subseteq_s span(V)$ $\therefore span(U)=span(V)$

Ex 3.12　　96 成大電信

$$S_1=\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\},S_2=\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix}\right\},$$

Prove S_1 and S_2 span the same subspace of R^3

$$\left[\begin{array}{ccc}1&0&-1\\1&1&0\\0&1&1\\\hline 2&1&-1\\1&2&1\end{array}\right]\to r\ldots\left[\begin{array}{ccc}1&0&-1\\0&1&1\\0&0&0\\\hline 0&0&0\\0&0&0\end{array}\right]\Rightarrow span(S_2)\subseteq span(S_1)$$

$$\left[\begin{array}{ccc}1&2&1\\2&1&-1\\\hline 1&0&-1\\1&1&0\\0&1&1\end{array}\right]\to r\ldots\left[\begin{array}{ccc}1&2&1\\0&-3&-3\\\hline 0&0&0\\0&0&0\\0&0&0\end{array}\right]\Rightarrow span(S_1)\subseteq span(S_2)$$

$\therefore span(S_1)=span(S_2)$

Ex 3.8

$u,v,w\in R_3\left[x\right]$, $\{u,v,w\}$ be LI or LD ?

$u=x^3-3x^2+5x+1, v=x^3-x^2+8x+2, w=2x^3-4x^2+9x+5$

$$\begin{bmatrix}1&-3&5&1\\1&-1&8&2\\2&-4&9&5\end{bmatrix}\to r\ldots\begin{bmatrix}1&-3&5&1\\0&2&3&1\\0&0&-4&2\end{bmatrix}$$ nonzero row ➔LI

Ex 3.9

$\{u,v,w\}:LI$　　prove $\{u,u+v,u+v+w\}:LI$

Let $\alpha u+\beta(u+v)+\gamma(u+v+w)=0$

$\Rightarrow(\alpha+\beta+\gamma)u+(\beta+\gamma)v+\gamma w=0$

$\because\{u,v,w\}:LI$

$$\therefore\begin{cases}\alpha+\beta+\gamma=0\\\beta+\gamma=0\\\gamma=0\end{cases}\Rightarrow\alpha=\beta=\gamma=0\Rightarrow\{u,u+v,u+v+w\}:LI$$

Ex 3.14 96 中原應數 台大電機

$V=R^{3\times 3},W_1=\{A\in V\mid A^T=A\},W_2=\{A\in V\mid A^T=-A\}$,find a basis and dimension of W_1,W_2 W_1:symmetric matrix W_2:skew-symmetric matrix
$(1)\forall A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\in W_1,A^T=A\Rightarrow\begin{bmatrix}a&d&g\\b&e&h\\c&f&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$ $\Rightarrow b=d,c=g,f=h\Rightarrow A=\begin{bmatrix}a&b&c\\b&e&f\\c&f&i\end{bmatrix}$ $\beta=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ $\dim(W_1)=6$ $(2)\forall A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\in W_1,A^T=-A\Rightarrow\begin{bmatrix}a&d&g\\b&e&h\\c&f&i\end{bmatrix}=-\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$ $\Rightarrow a=e=i=0,b=-d,c=-g,f=-h\Rightarrow A=\begin{bmatrix}0&b&c\\-b&0&f\\-c&-f&0\end{bmatrix}$ $\beta=\left\{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right\}$ $\dim(W_2)=3$

Ex 3.16

(1) $V=R^2$, $S=\{(1,0),(1,1),(2,3),(3,4)\}$ S 生成 V,但不 LI $S_1=S-\{(2,3)\}$,依然 S_1 生成 V,但不 LI $S_2=S_1-\{(3,4)\}=\{(1,0),(1,1)\}$, LI 所以取 $\{(1,0),(1,1)\}$ 為 V 之 1 組 basis
(2) $V=R^3$, $S=\{(1,0,0),(1,1,0)\}$ S : LI 但不生成 V $S_1=S\cup\{(0,0,1)\}=\{(1,0,0),(1,1,0),(0,0,1)\}$ 生成 V 所以取 $\{(1,0,0),(1,1,0),(0,0,1)\}$ 為 V 之 1 組 basis

Ex 3.15

Find a basis of R^4 spanned by $v_1=\begin{bmatrix}1\\-1\\2\\1\end{bmatrix}, v_2=\begin{bmatrix}1\\1\\1\\3\end{bmatrix}, v_3=\begin{bmatrix}-1\\3\\-3\\1\end{bmatrix}, v_4=\begin{bmatrix}2\\1\\2\\6\end{bmatrix}$
$let\ \ W=span\{v_1,v_2,v_3,v_4\}$ $\begin{bmatrix}1&-1&2&1\\1&1&1&3\\-1&3&-3&1\\2&1&2&6\end{bmatrix}\rightarrow r\ldots\begin{bmatrix}1&-1&2&1\\0&2&-1&2\\0&0&0&0\\0&0&\frac{-1}{2}&1\end{bmatrix}$ $let\ \ W=span\{v_1,v_2,v_3,v_4\}$ $\Rightarrow W=span\{v_1,v_2,v_4\}$ $\because \{v_1,v_2,v_4\}:LI$ $\Rightarrow pick\ \ \{v_1,v_2,v_4\}\ \ be\ \ a\ \ basis\ \ of\ \ W$

Ex 3.17

$V=R^3$ 很強那個定理的應用
$S_1=\{(1,2,3),(2,1,4),(3,1,4),(4,4,9)\}:not\ \ LI\ \ ,\because \lvert S_1\rvert>\dim(V)=3$
$S_2=\{(1,1,1),(2,1,4)\}:not\ \ span\ \ V,\because \lvert S_2\rvert<\dim(V)=3$

Ex 3.18

(a)find a basis for the subspace W_1 of the vector $\begin{bmatrix}a&b&c&d\end{bmatrix}^T$ with a+c+d=0   (b)find a basis for the subspace W_2 of the vector $\begin{bmatrix}a&b&c&d\end{bmatrix}^T$ with a+b=0,c=2d
(a) $\forall v\in\begin{bmatrix}a&b&c&d\end{bmatrix}^T\in W_1\Rightarrow a+c+d=0,d=-a-c\Rightarrow v=\begin{bmatrix}a&b&c&-a-c\end{bmatrix}^T$ $pick\ \ \left\{\begin{bmatrix}1&0&0&-1\end{bmatrix}^T,\begin{bmatrix}0&1&0&0\end{bmatrix}^T,\begin{bmatrix}0&0&1&-1\end{bmatrix}^T\right\}\ \ be\ \ a\ \ basis\ \ of\ \ W_1$ $(b)\forall v\in\begin{bmatrix}a&b&c&d\end{bmatrix}^T\in W_2\Rightarrow\begin{cases}a+b=0\\c=2d\end{cases}\Rightarrow\begin{cases}a=-b\\c=2d\end{cases}\Rightarrow v=\begin{bmatrix}-b&b&2d&d\end{bmatrix}^T$ $pick\ \ \left\{\begin{bmatrix}-1&1&0&0\end{bmatrix}^T,\begin{bmatrix}0&0&2&1\end{bmatrix}^T\right\}\ \ be\ \ a\ \ basis\ \ of\ \ W_2$

(15)

Find a basis for the plane 3x-2y+7z=0 ?
$3x-2y+7z=0\Rightarrow 7z=-3x+2y$ $\Rightarrow pick\ \ \{(7,0,-3),(0,7,2)\}\ \ be\ \ a\ \ basis$

(17)

Let $\{v_1,v_2\}$ be a basis for a vector space V, show that the vector $\{u_1,u_2\}$ where $u_1=v_1+v_2, u_2=v_1-v_2$ is also a basis for V
Because $\{v_1,v_2\}$ be a basis for a vector space V $\therefore \dim(V)=2$ 則 $\{u_1,u_2\}$ 只需証 LI 即爲 basis $\alpha u_1+\beta u_2=0 \Rightarrow \alpha(v_1+v_2)+\beta(v_1-v_2)=0$ $(\alpha+\beta)v_1+(\alpha-\beta)v_2=0$ $\because \{v_1,v_2\}:\text{LI} \Rightarrow \begin{cases}\alpha+\beta=0\\ \alpha-\beta=0\end{cases} \Rightarrow \alpha=0,\beta=0$ $\Rightarrow \therefore \{u_1,u_2\}:\text{LI}$

Ex 3.13

$V=R^{2\times2}, W=\left\{\begin{bmatrix} a & c \\ c & b\end{bmatrix} \middle\vert\, a,b,c\in R\right\}$,find a basis and dimension
$(a,b,c)=(1,0,0)\Rightarrow\begin{bmatrix}1&0\\0&0\end{bmatrix},(a,b,c)=(0,1,0)\Rightarrow\begin{bmatrix}0&0\\0&1\end{bmatrix},(a,b,c)=(0,0,1)\Rightarrow\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\therefore \begin{bmatrix}a&c\\c&b\end{bmatrix}=a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&0\\0&1\end{bmatrix}+c\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $pick \quad \beta=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\} \quad be \quad a \quad basis, \dim(W)=3$

(19)

$V=U+W, U=span\{(1,0,1,1),(2,1,1,2)\}\in R^4$ $,W=span\{(0,1,1,0),(2,0,1,2)\}\in R^4$ Determine the dimension of the subspace V
$\begin{bmatrix}1&0&1&1\\2&1&1&2\\0&1&1&0\\2&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&1&1\\0&1&-1&0\\0&1&1&0\\0&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&1&1\\0&1&-1&0\\0&0&2&0\\0&0&0&0\end{bmatrix}$ $pick \quad \{(1,0,1,1),(0,1,-1,0),(0,0,2,0)\} \quad be \quad a \quad basis \quad ,\dim(V)=3$

Ex 3.20

Let U *be subspace of* $R_3[x]$ *spanned by* $1+2x+x^3, 1-x-x^2$ *Let* V *be subspace of* $R_3[x]$ *spanned by* $x+x^2-3x^3, 2+2x-2x^3$ Find the dimension of U+V
$U = span\{1+2x+x^3, 1-x-x^2\}, V = span\{x+x^2-3x^3, 2+2x-2x^3\}$ $U+V = span\{1+2x+x^3, 1-x-x^2, x+x^2-3x^3, 2+2x-2x^3\}$ $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 0 & -2 \end{bmatrix} \to r\ldots \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *pcik* $\{1+2x+x^3, -3x-x^2-x^3, x^2-5x^3\}$ *be basis of* $U+V$

96 竹師應數

$W_1 = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in V, a,b,c \in F \right\}, W_2 = \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V, a,b \in F \right\}$ (a) prove W_1, W_2 are subspace of V (b) find the dimension of $W_1, W_2, W_1+W_2, W_1 \cap W_2$
(a) $\forall \alpha,\beta \in F, A = \begin{bmatrix} a_1 & b_1 \\ c_1 & a_1 \end{bmatrix}, B = \begin{bmatrix} a_2 & b_2 \\ c_2 & a_2 \end{bmatrix}, \alpha A + \beta B = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha a_1 + \beta a_2 \end{bmatrix} \in W_1$ $\because 0 \in W_1, W_1 \neq \varnothing, W_1 \subseteq V \Rightarrow W_1 \subseteq_s V$ $\forall \alpha,\beta \in F, A = \begin{bmatrix} 0 & a_1 \\ -a_1 & b_1 \end{bmatrix}, B = \begin{bmatrix} 0 & a_2 \\ -a_2 & b_2 \end{bmatrix}, \alpha A + \beta B = \begin{bmatrix} 0 & \alpha a_1 + \beta a_2 \\ -(\alpha a_1 + \beta a_2) & \alpha b_1 + \beta b_2 \end{bmatrix} \in W_2$ $\because 0 \in W_2, W_2 \neq \varnothing, W_2 \subseteq V \Rightarrow W_2 \subseteq_s V$ (b) *pick* $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *be basis of* W_1, W_2 $W_1 \cap W_2 = span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *pick* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *be basis of* $W_1 \cap W_2$ $\dim(W_1) = 3, \dim(W_2) = 2, \dim(W_1 + W_2) = 4$ $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) = 1$

Chapter 04

(1)linear transformation 必要條件 (用此來說明不是 linear transformation)

$T \in L(V, V^{/})$

$(1)T(0) = 0$

$(2)\forall v \in V, T(-v) = -T(v)$

$(3)\forall u, v \in V, T(u - v) = T(u) - T(v)$

(2)rotation axis on R^3

$$rotation \quad x: \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, rotation \quad y: \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, rotation \quad z: \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $T \in L(V, V^{/}), \beta = \{v_1, v_2, ..., v_n\}$ *be span set or basis of V*

$\mathrm{Im}(T) = span(T(\beta)) = span\{T(v_1), T(v_2), ..., T(v_n)\}$ 未必 LI

(4) $T \in L(V, V^{/}), \dim(V) = n < \infty$

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Im}(T)) = nullity(T) + rank(T)$$

(5) $T \in L(V, V^{/})$

$T: one-to-one \Leftrightarrow \ker(T) = \{0\} \Leftrightarrow nullity(T) = 0 \Leftrightarrow \dim(V) = rank(T)$

$T: onto \Leftrightarrow \mathrm{Im}(T) = V^{/} \Leftrightarrow rank(T) = \dim(V^{/}) \Leftrightarrow nullity(T) = \dim(V) - \dim(V^{/})$

(6) $T \in L(V, V^{/})$

$(1)T: one-to-one \Rightarrow \dim(V) \le \dim(V^{/})$ 左邊爲度較小,不可能 1-1

$(2)T: onto \Rightarrow \dim(V) \ge \dim(V^{/})$ 右邊爲度較小,不可能 onto

$(3)T: one-to-one, onto \Rightarrow \dim(V) = \dim(V^{/})$ 同構及同維 (限有限維)

(7) $T \in L(V, V^{/}), \dim(V) = \dim(V^{/}) = n < \infty$

$T: one-to-one \Leftrightarrow T: onto$ (若左右維度相同,則 1-1 iff onto)

(8) $T \in L(V, V^{/})$

(1)T 保相依

(2)T 保獨立 $\Leftrightarrow T \quad : one-to-one$

(3)T 保生成 $\Leftrightarrow T \quad : onto$

(9) $A,B\in F^{m\times n}$

(1) A 具左反矩陣 $\Leftrightarrow$ $n=rank(A)\le m$

$\Leftrightarrow$ $\dim(CS(A))=n$ $\Leftrightarrow$ A 行獨立於 $F^{m\times 1}$

$\Leftrightarrow$ $\dim(RS(A))=n$ $\Leftrightarrow$ A 列生成 $F^{1\times n}$

$\Leftrightarrow$ $A\vec{x}=\vec{b}$ 至多一解

(2) A 具右反矩陣 $\Leftrightarrow$ $m=rank(A)\le n$

$\Leftrightarrow$ $\dim(CS(A))=m$ $\Leftrightarrow$ A 行生成 $F^{m\times 1}$

$\Leftrightarrow$ $\dim(RS(A))=m$ $\Leftrightarrow$ A 列獨立於 $F^{1\times n}$

$\Leftrightarrow$ $A\vec{x}=\vec{b}$ 至少一解

(10) $A\vec{x}=\vec{b}$ 有解 $\Leftrightarrow$ $\vec{b}\in CS(A)$

Ex 4.1

$T:R^3\to R^2$ *define by be* $T(a_1,a_2,a_3)=(a_1-a_2,2a_3)$ *show T is a linear transformation*
$\forall\alpha\in F,u=(a_1,a_2,a_3),v=(b_1,b_2,b_3)\in R^3$ $(a)T(u+v)=T(a_1+b_1,a_2+b_2,a_3+b_3)=((a_1+b_1)-(a_2+b_2),2(a_3+b_3))$ $=(a_1-a_2,2a_3)+(b_1-b_2,2b_3)=T(u)+T(v)$ $(b)\forall\alpha\in F,T(\alpha v)=(\alpha a_1-\alpha a_2,2\alpha a_3)=\alpha(a_1-a_2,2a_3)=\alpha T(v)$

Ex 4.2

$T:F^{n\times n}\to F^{n\times n}$ *define by be* $T(A)=A^T$ *show T is a linear operator*
$\forall\alpha\in F,A,B\in F^{n\times n}$ $(a)T(A+B)=(A+B)^T=A^T+B^T=T(A)+T(B)$ $(b)T(\alpha A)=(\alpha A)^T=\alpha A^T=\alpha T(A)$

Ex 4.15

Prove $A \in F^{m\times n}, B \in F^{n\times p}$ $\quad rank(A+B) \le rank(A) + rank(B)$
$CS(A+B) \subseteq CS(A)+CS(B)$ $\Rightarrow \dim(CS(A+B)) \le \dim(CS(A)+CS(B))$ $= \dim(CS(A)) + \dim(CS(B)) - \dim(CS(A)\cap CS(B))$ $\le \dim(CS(A)) + \dim(CS(B))$ $\Rightarrow rank(A+B) \subseteq rank(A) + rank(B)$

95 台科資工

Let $T: R^n \to R^n$ be a linear transformation Prove that if T one-to-one if and only if T onto
$T \quad one-to-one \Leftrightarrow \ker(T) = \{\bar{0}\} \Leftrightarrow nullity(T) = 0$ $\Leftrightarrow rank(T) = \dim(V) \Leftrightarrow \mathrm{Im}(T) = R^n \Leftrightarrow T \quad onto$

Ex 4.14

Prove $A \in F^{m\times n}, B \in F^{n\times p}$ $\quad rank(AB) \le \min\{rank(A), rank(B)\}$
(a) $prove \quad CS(AB) \subseteq CS(A)$ $\forall \vec{y} \in CS(AB)$ $\Rightarrow \exists \vec{x} \in \vec{y} = (AB)\vec{x}$ $\Rightarrow \vec{y} = A(B\vec{x}) \in CS(A)$ $\therefore CS(AB) \subseteq CS(A)$ $\Rightarrow \dim(CS(AB)) \le \dim(CS(A))$ $\Rightarrow rank(AB) \le rank(A)$ (b) $prove \quad RS(AB) \subseteq RS(B)$ $\forall \vec{y} \in RS(AB)$ $\Rightarrow \exists \vec{x} \in \vec{y} = \vec{x}(AB)$ $\Rightarrow \vec{y} = (\vec{x}A)B \in RS(B)$ $\therefore RS(AB) \subseteq RS(B)$ $\Rightarrow \dim(RS(AB)) \le \dim(RS(B))$ $\Rightarrow rank(AB) \le rank(B)$

Let T be a linear transformation , show that if $\{T(v_1),T(v_2),....,T(v_p)\}$ is linear independent ,then $\{v_1,v_2,....,v_p\}$ is linear independent
若 $\alpha_1 v_1+\alpha_2 v_2+....+\alpha_p v_p=0$ $T(\alpha_1 v_1+\alpha_2 v_2+....+\alpha_p v_p)=T(0)=0$ $\alpha_1 T(v_1)+\alpha_2 T(v_2)+....+\alpha_p T(v_p)=0$ $\because T(v_1),T(v_2),....,T(v_p)\ \ be\ \ linear\ \ independent$ $\Rightarrow \alpha_1=\alpha_2=.....=\alpha_p=0$ $\therefore v_1,v_2,....,v_p\ \ be\ \ linear\ \ independent$

Ex 4.3

Determine following be linear transformation or not $(1)T:R^{n\times n}\to R^n\ \ defined\ \ by\ \ T(A)=\det(A)$ $(2)T:R^{n\times n}\to R^n\ \ defined\ \ by\ \ T(A)=tr(A)$ $(3)T:R^{n\times n}\to R^{n\times n}\ \ defined\ \ by\ \ T(A)=A+A^T$ $(4)T:R_2[x]\to R_3[x]\ \ defined\ \ by\ \ T(f(x))=xf(x)+f^1(x)$ $(5)T:R^2\to R^2\ \ defined\ \ by\ \ T(a,b)=(\|a\|,b)$ $(6)T:R^2\to R^2\ \ defined\ \ by\ \ T(a,b)=(a,b+3)$
$(1)T(2\begin{bmatrix}1&0\\0&1\end{bmatrix})=T(\begin{bmatrix}2&0\\0&2\end{bmatrix})=4\neq 2T(\begin{bmatrix}1&0\\0&1\end{bmatrix})=2\ \ ,not\ \ LT$ $(2)\forall \alpha,\beta\in F,A,B\in R^{n\times n}$ $T(\alpha A+\beta B)=tr(\alpha A+\beta B)=\alpha tr(A)+\beta tr(B)=T(\alpha A)+T(\beta B)$ $(3)LT$ $T(\alpha A+\beta B)=(\alpha A+\beta B)+(\alpha A+\beta B)^T=(\alpha A+\beta B)+(\alpha A^T+\beta B^T)$ $=\alpha(A+A^T)+\beta(B+B^T)=\alpha T(A)+\beta T(B)$ $(4)LT$ $T(\alpha f(x)+\beta g(x))=\alpha xf(x)+\alpha f^1(x)+\beta xg(x)+\beta g^1(x)$ $=\alpha(xf(x)+f^1(x))+\beta(xg(x)+g^1(x))=\alpha T(f(x))+\beta T(g(x))$ $(5)not\ \ LT\ \ ,\ \ T(\alpha A)=\alpha^2A^2\neq\alpha A^2=\alpha T(A)$ $(6)not\ \ LT\ \ ,\ \ T(-v)\neq -T(v)$ $(7)not\ \ LT\ \ ,\ \ T(\vec{0})\neq\vec{0}$

95 銘傳資工

Find a linear transformation $T:R^3 \to R^2$ such that

$$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix},T\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix},T\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$let\quad \begin{bmatrix}x\\y\\z\end{bmatrix}=\alpha\begin{bmatrix}1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\0\\1\end{bmatrix}+\gamma\begin{bmatrix}0\\1\\1\end{bmatrix}\Rightarrow\begin{cases}x=\alpha+\beta\\y=\alpha+\gamma\\z=\beta+\gamma\end{cases}\Rightarrow\begin{cases}\alpha=\dfrac{x+y-z}{2}\\\beta=\dfrac{x-y+z}{2}\\\gamma=\dfrac{-x+y+z}{2}\end{cases}$$

$$\because\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{x+y-z}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}+\frac{x-y+z}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{-x+y+z}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\therefore T\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{x+y-z}{2}\begin{bmatrix}2\\1\end{bmatrix}+\frac{x-y+z}{2}\begin{bmatrix}1\\-1\end{bmatrix}+\frac{-x+y+z}{2}\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\dfrac{3x+y-z}{2}\\y-z\end{bmatrix}$$

Let $T:R^3 \to R^3$ be linear transformation satisfy

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\1\end{bmatrix},T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)=\begin{bmatrix}2\\1\\2\end{bmatrix},T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)=\begin{bmatrix}2\\0\\0\end{bmatrix}$$, Find $T\left(\begin{bmatrix}3\\1\\-5\end{bmatrix}\right)$ =?

$$let\quad \begin{bmatrix}3\\1\\-5\end{bmatrix}=\alpha\begin{bmatrix}1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\2\\1\end{bmatrix}+\gamma\begin{bmatrix}0\\1\\2\end{bmatrix}\Rightarrow\begin{cases}\alpha+\beta=3\\\alpha+2\beta+\gamma=1\\\beta+2\gamma=-5\end{cases}$$

$$\Rightarrow \alpha=2,\beta=1,\gamma=-3$$

$$T\left(\begin{bmatrix}3\\1\\-5\end{bmatrix}\right)=\alpha T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)+\beta T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)+\gamma T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)$$

$$=2\begin{bmatrix}0\\1\\1\end{bmatrix}+\begin{bmatrix}2\\1\\2\end{bmatrix}-3\begin{bmatrix}2\\0\\0\end{bmatrix}=\begin{bmatrix}-4\\3\\4\end{bmatrix}$$

Let V be the vector space of all polynomials p(x) with degree at most two, and let $T: V \to V$ be the linear transformation $T(p(x)) = \frac{d}{dx} p(x)$ Suppose $p_1(x) = -x+1, p_2(x) = x+1, p_3(x) = x^2+1$ be a basis of V,find the matrix of T in the basis $\{p_1, p_2, p_3\}$ of V
$T(p_1(x)) = -1 = -\frac{1}{2}p_1(x) - \frac{1}{2}p_2(x) + 0p_3(x)$ $T(p_2(x)) = 1 = \frac{1}{2}p_1(x) + \frac{1}{2}p_2(x) + 0p_3(x)$ $T(p_3(x)) = 2x = -p_1(x) + p_2(x) + 0p_3(x)$ $[T]_\beta = \begin{bmatrix} -1/2 & 1/2 & -1 \\ -1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let the order basis for $R_3[x]$ be $B = \{x^3, x^2, x, 1\}$ and let $T: R_3[x] \to R_3[x]$ be Defined by $T(p(x)) = \frac{d}{dx} p(x)$ (a) find the matrix representation (b) Use A to find $T(4x^3 - 5x^2 + 10x - 13)$
$(a)\begin{cases} T(x^3) = 3x^2 = 0x^3 + 3x^2 + 0x + 0(1) \\ T(x^2) = 2x = 0x^3 + 0x^2 + 2x + 0(1) \\ T(x) = 1 = 0x^3 + 0x^2 + 0x + 1(1) \\ T(1) = 0 = 0x^3 + 0x^2 + 0x + 0(1) \end{cases} \Rightarrow A = [T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $(b)\left[T(4x^3 - 5x^2 + 10x - 13)\right]_B = [T]_B\left[(4x^3 - 5x^2 + 10x - 13)\right]_B$ $= A\begin{bmatrix} 4 \\ -5 \\ 10 \\ 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ 10 \\ 13 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ -10 \\ 10 \end{bmatrix}$ $\therefore T(4x^3 - 5x^2 + 10x - 13) = 12x^2 - 10x + 10$

Let $T:R^2 \to R^2$ be a linear transformation defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$ and let basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ (a) find the matrix for T with respect to the basis B (b) let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,find $[x]_B$ and $[T(x)]_B$
$(1)\begin{cases} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{cases} \Rightarrow [T]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $(2)[x]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [T(x)]_B = [T]_B [x]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

95 宜蘭電子

$T:R^2 \to R^3$ be defined by $L(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$ find a matrix A suh that $L(x) = Ax$ for all x in R^2
$\because L(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \therefore pick \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} \ni L(x) = Ax$

96 台科資工

Let $T:P_2 \to M_{2\times2}$ be a linear transformation such that $T(a+bx+cx^2)=\begin{bmatrix} a & b+c \\ a+b & c \end{bmatrix}$,find the matrix representation of T with respect to the standard bases of $P_2=\{1,x,x^2\}$ and $M_{2\times2}=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$
$T(1)=\begin{bmatrix}1&0\\1&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+1\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$' $T(x)=\begin{bmatrix}1&0\\1&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\0&0\end{bmatrix}+1\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$ $T(x^2)=\begin{bmatrix}1&0\\1&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}$ $\Rightarrow\begin{bmatrix}1&0&0\\0&1&1\\1&1&0\\0&0&1\end{bmatrix}$

96 屏教大應數

Let T be linear transformation on $M_{2\times2}$ defined by $T(A)=A^T$,,find the matrix representation of T
取 標準基底 $\beta=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ $T(\beta)=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ $[T]_\beta=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$

95 雲科資工

$T(f)=f^{/}+f^{//}$ *from* P_2 *to* P_2 ,find the matrix A for the linear transformation T
pick $\beta=\{1,x,x^2\}$ $T(1)=0,T(x)=1,T(x^2)=2+2x\Rightarrow[T]_\beta=\begin{bmatrix}0&1&2\\0&0&2\\0&0&0\end{bmatrix}$

Ex 4.10

$$let\ \ T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_2\\-5x_1+13x_2\\-7x_1+16x_2\end{bmatrix}\ \ be\ linear\ transformation\ from\ R^2\ to\ R^3$$

$$\beta=\left\{\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\right\},\gamma=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\ ,find\ the\ matrix\ for\ T\ respect\ \beta\ and\ \gamma$$

法(1)

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}1\\-2\\-5\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}-2\begin{bmatrix}0\\1\\0\end{bmatrix}-5\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\left(T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right)_\gamma=\begin{bmatrix}1\\-2\\-5\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)=\begin{bmatrix}2\\1\\-3\end{bmatrix}=2\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}-3\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\left(T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)\right)_\gamma=\begin{bmatrix}2\\1\\-3\end{bmatrix}$$

$$[T]_\beta^\gamma=\begin{bmatrix}1&2\\-2&1\\-5&-3\end{bmatrix}$$

法(2)

$$pick\ \ B=\{e_1,e_2\}\ \ in\ \ R^2,\ \ C=\{e_1,e_2,e_3\}\ \ in\ \ R^3$$

$$[T]_\beta^\gamma=[I]_C^\gamma[T]_B^C[I]_\beta^B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1\\-5&13\\-7&16\end{bmatrix}\begin{bmatrix}3&5\\1&2\end{bmatrix}=\begin{bmatrix}1&2\\-2&1\\-5&-3\end{bmatrix}$$

Ex 4.11

$$S=\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\1\end{bmatrix}\right\},T=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\ ,\ (1)\,find\left[I\right]_T^S\ ,\ (2)\left[\vec{v}\right]_T=\begin{bmatrix}2\\0\\1\end{bmatrix},\ \left[\vec{v}\right]_S=?$$

Pick $C=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$$\begin{tikzcd} \mathbf{S} \arrow[rr, shift right, "A"'] & & \mathbf{C} \arrow[ll, shift right, "A^{-1}"'] \\ & \mathbf{T} \arrow[ul] \arrow[ur, "\mathbf{B}"'] & \end{tikzcd}$$

$$[I]_T^S=A^{-1}B=[I]_C^S[I]_T^C=\left([I]_S^C\right)^{-1}[I]_T^C$$

$$=\begin{bmatrix}-1&0&-2\\2&1&2\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}-1&0&0\\1&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&0\\-1&-1&1\\0&-1&0\end{bmatrix}$$

$$\left[\vec{v}\right]_S=[I]_T^S\left[\vec{v}\right]_T=\begin{bmatrix}1&2&0\\-1&-1&1\\0&-1&0\end{bmatrix}\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}2\\-1\\0\end{bmatrix}$$

Ex 4.12

$T:R^2\to R^3$ by $T:R^2\to R^3\ ,\ T(\vec{x})=A\vec{x}\ ,\ A=\begin{bmatrix}1&-2\\2&1\\1&1\end{bmatrix}$

$$S=\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}\ ,\ V=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}\ ,\ [T]_S^V=?$$

$let\quad B=\{e_1,e_2\},C=\{e_1,e_2,e_3\}\Rightarrow[T]_B^C=A$

$$[T]_S^V=[I]_C^V[T]_B^C[I]_S^B=\begin{bmatrix}1&0&1\\1&1&-1\\0&1&1\end{bmatrix}^{-1}\begin{bmatrix}1&-2\\2&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}$$

$B=\left\{\begin{bmatrix}7\\5\end{bmatrix},\begin{bmatrix}-3\\-1\end{bmatrix}\right\},C=\left\{\begin{bmatrix}1\\-5\end{bmatrix},\begin{bmatrix}-2\\2\end{bmatrix}\right\}$,find the change-of coordinate matrix from "B to C" and "C to B"
$let\ \ D=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}\Rightarrow[I]_B^D=\begin{bmatrix}7&-3\\5&-1\end{bmatrix},[I]_C^D=\begin{bmatrix}1&-2\\-5&2\end{bmatrix}$ $[I]_B^C=[I]_D^C[I]_B^D=\begin{bmatrix}1&-2\\-5&2\end{bmatrix}^{-1}\begin{bmatrix}7&-3\\5&-1\end{bmatrix}=\begin{bmatrix}-3&1\\-5&-2\end{bmatrix}$ $[I]_C^B=\left([I]_B^C\right)^{-1}=\begin{bmatrix}-2&1\\-5&3\end{bmatrix}$

Let $f:R^3\to R^2$ be defined by $f(x,y,z)=(2x-y,2y-z)$,Determine the matrix of f relative to the order basis $\{(1,1,1),(0,1,1),(0,0,1)\}$ and $\{(0,1),(1,1)\}$
法(1) $f(1,1,1)=(1,1)=0(0,1)+1(1,1)$ $f(0,1,1)=(-1,1)=2(0,1)+(-1)(1,1)$ ➔ $\begin{bmatrix}0&2&-1\\1&-1&0\end{bmatrix}$ $f(0,0,1)=(0,-1)=(-1)(0,1)+0(1,1)$ 法(2) $pick\ \ B=\{e_1,e_2,e_3\}\ \ in\ \ R^3,\ \ C=\{e_1,e_2\}\ \ in\ \ R^2$ $[f]_\beta^\gamma=[I]_C^\gamma[f]_B^C[I]_\beta^B=\begin{bmatrix}0&1\\1&1\end{bmatrix}^{-1}\begin{bmatrix}2&-1&0\\0&2&-1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}0&2&1\\1&-1&0\end{bmatrix}$

96 東吳數學

$T(p(x))=p(x)+(1+x)p'(x)$ (a)find the matrix A representing T with respect to $[1,x,x^2]$ (b)find the matrix B representing T with respect to $[1,1+x,1+x+x^2]$ (c)find the matrix C such that $B=C^{-1}AC$
$\beta=[1,x,x^2]$, $\gamma=[1,1+x,1+x^2]$ $(a)A=[T]_\beta=\begin{bmatrix}1&1&0\\0&2&2\\0&0&3\end{bmatrix}$ $(b)B=[T]_\gamma=\begin{bmatrix}1&0&-2\\0&2&1\\0&0&3\end{bmatrix}$ $(c)B=[T]_\gamma=[I]_\beta^\gamma[T]_\beta^\beta[I]_\gamma^\beta=C^{-1}AC\Rightarrow C=[I]_\gamma^\beta=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

Let D be differentiation operator on P_3 ,let matrix B representing D with respective to $\left[1,x,x^2\right]$ and the matrix A representing D with respective to $\left[1,2x,4x^2-2\right]$ Find a matrix S and S^{-1} $\ni A=S^{-1}BS$
$\beta=\left[1,x,x^2\right],\gamma=\left[1,2x,4x^2-2\right]\Rightarrow B=[D]_\beta, A=[D]_\gamma$ $A=[D]_\gamma=[I]_\beta^\gamma[D]_\beta[I]_\gamma^\beta=\left([I]_\gamma^\beta\right)^{-1}[D]_\beta[I]_\gamma^\beta=S^{-1}BS$ $S=[I]_\gamma^\beta=\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix},S^{-1}=\left([I]_\gamma^\beta\right)^{-1}=\left(\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}\right)^{-1}=\begin{bmatrix}1&0&\frac{1}{2}\\0&\frac{1}{2}&0\\0&0&\frac{1}{4}\end{bmatrix}$

Define $T:R^{2\times2}\to T:R_2[x]$ by $T\begin{bmatrix}a&b\\c&d\end{bmatrix}=(a+b)+2dx+bx^2$ $B=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\},B^1=\{1,x,x^2\}$ be basis Find matrix representation of T relative B and B^1
$T\begin{bmatrix}1&0\\0&0\end{bmatrix}=1=1+0x+0x^2,T\begin{bmatrix}0&1\\0&0\end{bmatrix}=1+x^2=1+0x+1x^2$ $T\begin{bmatrix}0&0\\1&0\end{bmatrix}=0=0+0x+0x^2,T\begin{bmatrix}0&0\\0&1\end{bmatrix}=2x=0+2x+0x^2$ $[T]_B^{B^1}=\begin{bmatrix}1&1&0&0\\0&0&0&2\\0&1&0&0\end{bmatrix}$

96 中華資工

Let L be the operator on P_3 defined be $L(p(x)) = xp^{/}(x) + p^{//}(x)$

(a) find the matrix A representing L with respect to $[1, x, x^2]$

(b) find the matrix B representing L with respect to $[1, 1+x, 1+x^2]$

(c) find the matrix S such that $B = S^{-1}AS$

$$L(1) = 0, L(x) = x, L(x^2) = 2 + 2x^2 \Rightarrow A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L(1) = 0, L(1+x) = x, L(1+x^2) = 2 + 2x^2 \Rightarrow B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\beta = [1, x, x^2], \gamma = [1, 1+x, 1+x^2]$$

$$B = S^{-1}AS \Leftrightarrow [L]_\gamma^\gamma = [I]_\beta^\gamma [L]_\beta^\beta [I]_\gamma^\beta, \therefore S = [I]_\gamma^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

94 台科資工

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}, basis \quad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(a) find the matrix for T with respect to the basis B

(b) $let \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, find \quad [x]_B \quad and \quad [T(x)]_B$

$$(a) [T]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(b) x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix}, a = 2, b = -1 \Rightarrow [x]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$[T(x)]_B = [T]_B [x]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

94 台科電機

L be linear transformation A represent L with respect to $\beta=\{1,x,x^2,x^3\}$ B represent L with respect to $\gamma=\{1-x,1+x,x^2+x^3,x^3\}$ Find matrix $S \ni B=SAS^{-1}$
$[L]_\gamma=[I]_\beta^\gamma[L]_\beta[I]_\gamma^\beta \Rightarrow S=[I]_\beta^\gamma=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$

94 彰師資工

Let D be differentiation operator on P_3 ,let matrix B representing D with respective to $\left[1,x,x^2\right]$ and the matrix A representing D with respective to $\left[1,2x,4x^2-2\right]$ Find a matrix S and S^{-1} $\ni A=S^{-1}BS$
$\beta=\left[1,x,x^2\right],\gamma=\left[1,2x,4x^2-2\right]\Rightarrow B=[D]_\beta,A=[D]_\gamma$ $A=[D]_\gamma=[I]_\beta^\gamma[D]_\beta[I]_\gamma^\beta=\left([I]_\gamma^\beta\right)^{-1}[D]_\beta[I]_\gamma^\beta=S^{-1}BS$ $S=[I]_\gamma^\beta=\begin{bmatrix}1 & 0 & -2\\ 0 & 2 & 0\\ 0 & 0 & 4\end{bmatrix},S^{-1}=\left([I]_\gamma^\beta\right)^{-1}=\left(\begin{bmatrix}1 & 0 & -2\\ 0 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}\right)^{-1}=\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}$

Ex 4.10

$$T: LT \;\; from \;\; R^4 \;\; to \;\; R^3, T\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 2u_1 + u_2 \\ u_1 - u_2 \\ 3u_3 + 2u_4 \end{bmatrix}$$

(a) *find the null space of T by finding basis*

(b) *find the range of T by finding basis*

(c) *determine the nullity and rank of T*

$$(a)\forall \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \ker(T) \Rightarrow T\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 2u_1 + u_2 \\ u_1 - u_2 \\ 3u_3 + 2u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow pick \;\; \left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\} \;\; be \;\; a \;\; basis \;\; of \;\; \mathrm{k}$$

$$(b) pick \;\; \beta = \{e_1, e_2, e_3, e_4\} \;\; , \mathrm{Im}(T) = span(T(\beta)) = span\{T(e_1), T(e_2), T(e_3), T(e_4)\}$$

$$\Rightarrow \mathrm{Im}(T) = span\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \to r \begin{bmatrix} 0 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$pick \left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\} be \;\; a \;\; basis \;\; of \;\; \mathrm{Im}(T)$$

$$(c) nullity(T) = \dim(\ker(T)) = 1, rank(T) = \dim(\mathrm{Im}(T)) = 3$$

Ex 4.11

$$Let \;\; T: R^{2\times2} \to R^{2\times2} \;\; be \;\; linear \;\; operator \;\; defined \;\; by \;\; T(X) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Find Image and nullity of T ?

$$let \;\; X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow T(X) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b+c & 2b+d \\ d & d \end{bmatrix}$$

$$\mathrm{Im}(T) = span\left\{ T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}), T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \right\} = span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$pick \;\; \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \;\; be \;\; basis \;\; of \;\; \mathrm{Im}(T) \Rightarrow rank(T) = 3$$

$$\because rank(T) + nullity(T) = 4 \Rightarrow nullity(T) = 1$$

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for R^4 ,if $T{:}R^4 \to R^3$ is a linear transformation for which $T(e_1) = (1,2,1), T(e_2) = (0,1,0), T(e_3) = (1,3,0), T(e_4) = (1,1,1)$ (a) find the basis for the range of T (b) find the basis for the kernel of T
$(a)\,\mathrm{Im}(T) = span\{T(e_1), T(e_2), T(e_3), T(e_3)\}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} pick \quad \{(1,2,1),(0,1,0),(0,0,-1)\} \\ be \;\; a \;\; basis \;\; of \quad \mathrm{Im}(T) \end{cases}$ $(b) \forall x = (a,b,c,d) \in \ker(T)$ $\Rightarrow (0,0,0) = T(x) = aT(e_1) + bT(e_2) + cT(e_3) + dT(e_3)$ $= a(1,2,1) + b(0,1,0) + c(1,3,0) + d(1,1,1) = (a + c + d, 2a + b + 3c + d, a + d)$ $\Rightarrow \begin{cases} a + c + d = 0 \\ 2a + b + 3c + d = 0 \\ a + d = 0 \end{cases} \Rightarrow \begin{cases} c = 0 \\ b = d \\ a = -d \end{cases} \Rightarrow \begin{cases} pick \quad \{(-1,1,0,1)\} \\ be \;\; a \;\; basis \;\; of \quad \ker(T) \end{cases}$

Let T be linear transformation from R^3 into R^2 $T(1,1,1) = (2,2), T(0,1,1) = (0,1), T(0,0,1) = (-1,1)$ Find the null space of T
$\forall (a,b,c) \in \ker(T), (a,b,c) = \alpha(1,1,1) + \beta(0,1,1) + \gamma(0,0,1)$ $\alpha = a, \beta = -a + b, \gamma = -b + c$ $T(a,b,c) = a(2,2) + (-a + b)(0,1) + (-b + c)(-1,1)$ $= (2a + b - c, a + c)$ $\begin{cases} 2a + b - c = 0 \\ a + c = 0 \end{cases} \Rightarrow \begin{cases} b = 3c \\ a = -c \end{cases} \Rightarrow \ker(T) = \{(-c, 3c, c) \mid c \in R\} = span\{(-1,3,1)\}$

Let $T: R^{2\times 2} \to R^{2\times 2}$ be the linear operator defined by $T(X)=\begin{bmatrix}1&1\\0&0\end{bmatrix}X+X\begin{bmatrix}0&0\\1&1\end{bmatrix}$

(a) find the range of T

(b) find the null space of T

(c) find the nullity(T) and dimension of range

$$(a)pick\ \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}\ be\ \ basis$$

$$T\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix},T\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&2\\0&0\end{bmatrix},T\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix},T\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\1&2&0&0\\1&0&0&0\\0&1&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&1&1\end{bmatrix}\Rightarrow \mathrm{Im}(T)=span\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&2\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$$

$$(b)\forall\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\ker(T)\Rightarrow T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a+b+c&2b+d\\d&d\end{bmatrix}\Rightarrow\begin{cases}a+b+c=0\\2b+d=0\\d=0\end{cases}\Rightarrow b=0,d=0,a=-c$$

$$\ker(T)=span\left\{\begin{bmatrix}1&0\\-1&0\end{bmatrix}\right\}$$

$$(c)\dim(\mathrm{Im}(T))=3,nullity(T)=1$$

96 成大電信

Let $T: R^4 \to R^3$ be a linear transformation defined by $T(x_1,x_2,x_3,x_4)=(x_1+x_2,x_3+x_4,x_1+x_3)$,find the kernel and range of T

$$(a)\forall\vec{x}=(x_1,x_2,x_3,x_4)\in\ker(T)$$

$$\Rightarrow T(\vec{x})=T(x_1,x_2,x_3,x_4)=\vec{0}$$

$$\Rightarrow\begin{cases}x_1+x_2=0\\x_3+x_4=0\\x_1+x_3=0\end{cases}\Rightarrow\vec{x}=(x_4,-x_4,-x_4,x_4)$$

$$\therefore\ker(T)=\{(t,-t,-t,t)\mid t\in R\}=span\{(1,-1,-1,1)\}$$

$$(b)nullity(T)=1\Rightarrow rank(T)=3\Rightarrow \mathrm{Im}(T)=R^3$$

Ex 4.13

Let $T: R^2 \to R^3$ be linear trnasformation defined by $T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a-2b \\ 3a+b \\ a+b \end{bmatrix}$ $(a)$is T one-to-one? $(b)$is T onto?
$(a) nullity(T) = \dim(V) - rank(T) = 2 = 0 \Rightarrow 1-1$ or $(a) \forall x = \begin{bmatrix} a \\ b \end{bmatrix} \in \ker(T) \Rightarrow T(x) = \begin{bmatrix} a-2b \\ 3a+b \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a-2b=0 \\ 3a+b=0 \\ a+b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$ $x = 0 \Rightarrow \ker(T) = \{0\} \Rightarrow T: one-to-one$ $(b) \operatorname{Im}(T) = span\left\{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = span\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ $rank(T) = 2$ $\because rank(T) = 2 \neq 3 = \dim(V') \Rightarrow not\ \ onto$

96 暨大資工

$L: R^3 \to R^3$ be defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ (a)Is L onto ? (b) find a basis for range L (c) find a basis for ker L
$L(\vec{x}) = A\vec{x} \Rightarrow R(L) = CS(A), \ker(L) = \ker(A)$ (a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,pick $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ be a basis of R(L)   (b) $rank(L) = 2 \neq 3 = \dim(R^3)$ not onto   (c) $solve$ $\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow \ker(L) = \left\{\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} \middle\vert\, t \in R\right\}$, 取 $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

91 暨南資工

$T:R^3\to R^3, defined\ \ by\ \ T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3x-2y\\-2x+3y\\5z\end{bmatrix}$ (a) find a basis of ker(T) (b)find a basis of R(T)
$(a)\ker(T)=0\Rightarrow pick\ \ \varnothing\ \ be\ \ a\ \ basis\ \ of\ \ \ker(T)$ $(b)pick\ \ \{e_1,e_2,e_3\}\ \ be\ \ a\ \ basis\ \ of\ \ R(T)$

96 嘉大應數

Define $T:M_{2\times2}(R)\to M_{2\times2}(R)$ by $T(A)=\begin{bmatrix}1&0\\0&0\end{bmatrix}A\begin{bmatrix}0&0\\0&1\end{bmatrix}$ (a) find the nullspace N(T) of T and the dimension of N(T) (b) find the range R(T) of T and the dimension of R(T)
$(a)\forall A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in N(T),T(A)=\begin{bmatrix}0&b\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\Rightarrow b=0$ $pick\ \ \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}\ \ be\ \ a\ \ basis\ \ of\ \ N(T)$ $(b)R(T)=span\{T(e_{11}),T(e_{12}),T(e_{21}),T(e_{22})\}$ $=span\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}=span\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}$ $pick\ \ \left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}\ \ be\ \ a\ \ basis\ \ of\ \ R(T)$ $\dim(N(T))=3,\dim(R(T))=1$

94 暨南資工

$T(X)=\begin{bmatrix}1&1\\0&0\end{bmatrix}X+X\begin{bmatrix}0&0\\1&1\end{bmatrix}$ find a basis of ker(T)
$let\ \ X=\begin{bmatrix}a&b\\c&d\end{bmatrix}\Rightarrow T(X)=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}a+b+c&2b+d\\d&d\end{bmatrix}$ $\forall x=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\ker(T)\Rightarrow T(X)=\begin{bmatrix}a+b+c&2b+d\\d&d\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ $pick\ \ \left\{\begin{bmatrix}1&0\\-1&0\end{bmatrix}\right\}\ \ be\ \ a\ \ basis\ \ of\ \ \ker(T)$

95 雲科電機

Following every transformation ,which 1-1 ,onto ?

$$(a)T:R^2 \to R^3, T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-2b \\ 3a+b \\ a+b \end{bmatrix}$$

$$(b)T:R^{2\times 2} \to R_2[x], T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+b+2dx+bx^2$$

$$(a)\forall x = \begin{bmatrix} a \\ b \end{bmatrix} \in \ker(T), T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-2b \\ 3a+b \\ a+b \end{bmatrix} \Rightarrow \begin{cases} a-2b=0 \\ 3a+b=0 \\ a+b=0 \end{cases} \Rightarrow a=0, b=0 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \ker(T) = \{0\} \Rightarrow one-to-one$$

$$\mathrm{Im}(T) = span\left\{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = span\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \end{bmatrix}$$

$$pick \quad \left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\} \quad be \quad a \quad basis \quad of \quad \mathrm{Im}(T) \Rightarrow rank(T) = 2 \neq \dim(R^3) \Rightarrow not \quad onto$$

$$(b)\,\mathrm{Im}(T) = span\left\{T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} a+b+2dx+bx^2 = span\{1, 1+x^2, 0, 2x\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow pick \quad \{1, 1+x^2, 2x\} \quad be \quad a \quad basis \quad of \quad \mathrm{Im}(T)$$

$$\Rightarrow rank(T) = 3 = \dim(R_2[x]) \Rightarrow T \quad onto$$

$$\because nullity(T) = \dim(R^{2\times 2}) - rank(T) = 4-3 = 1 \neq 0 \Rightarrow T \quad not \quad 1-1$$

Ex 4.13

$T:R^2 \to R^2 \;\; LT \;\; satisfy \;\; T(1,0)=(1,1), T(0,1)=(-1,1)$ $find \;\; direct \;\; image \;\; of \;\; S=\{(x,y)\mid ax+by=1\}$
$\forall (x,y)\in S, let \;\; T(x,y)=(x', y')$ $\Rightarrow (x', y')=T(x,y)=xT(1,0)+yT(0,1)=x(1,1)+y(-1,1)$ $\Rightarrow (x', y')=(x-y, x+y) \Rightarrow \begin{cases} x' = x-y \\ y' = x+y \end{cases} \Rightarrow \begin{cases} x=\dfrac{x'+y'}{2} \\ y=\dfrac{-x'+y'}{2} \end{cases}$ $\because ax+by=1 \Rightarrow a(\dfrac{x'+y'}{2})+b(\dfrac{-x'+y'}{2})=1 \Rightarrow (\dfrac{-a+b}{2})x' + (\dfrac{a+b}{2})y' = 1$ $T(S)=\{(x', y')\mid (\dfrac{-a+b}{2})x' + (\dfrac{a+b}{2})y' = 1\}$

Suppose that $T:R^3 \to R^3$ is linear such that $T(a_1,a_2,a_3)=\left[a_1+2a_2+a_3, -a_1+a_2+2a_3, a_1+a_3\right]$ Compute T^{-1} if it exist
$pick \;\; \beta=\{(1,0,0),(0,1,0),(0,0,1)\} \;\; be \;\; a \;\; standard \;\; basis \;\; of \;\; R^3$ $T(1,0,0)=(1,-1,1), T(0,1,0)=(2,1,0), T(0,0,1)=(1,2,1)$ $\therefore A=[T]_\beta=\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \because \det(A)=6\neq 0 \Rightarrow A \;\; be \;\; invertable \;\; matrix$ $\left[T^{-1}\right]_\beta = A^{-1} = \begin{bmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{bmatrix} \Rightarrow \begin{cases} T^{-1}([1,0,0])=(1/6, 1/2, -1/6) \\ T^{-1}([0,1,0])=(-1/3, 0, 1/3) \\ T^{-1}([0,0,1])=(1/2, -1/2, 1/2) \end{cases}$ $T^{-1}(a_1,a_2,a_3)=a_1T^{-1}(1,0,0)+a_2T^{-1}(0,1,0)+a_3T^{-1}(0,0,1)$ $=a_1(1/6, 1/2, -1/6)+a_2(-1/3, 0, 1/3)+a_3(1/2, -1/2, 1/2)$ $=\left(\dfrac{1}{6}a_1-\dfrac{1}{3}a_2+\dfrac{1}{2}a_3, \dfrac{1}{2}a_1-\dfrac{1}{2}a_3, \dfrac{-1}{6}a_1+\dfrac{1}{3}a_2+\dfrac{1}{2}a_3\right)$

Suppose that $\mathrm{T}: R^3 \to R^3$ is linear

$T(1,0,0)=(2,3,2), T(0,1,0)=(3,3,4), T(0,0,1)=(1,1,1)$

(a) find $T(1,2,3)$ (b) $T^{-1}(0,1,0)$ (c) $T^{-1}(1,2,3)$

(a) *pick* $\beta=\{(1,0,0),(0,1,0),(0,0,1)\}$ *be a standard basis of* R^3

$$\therefore A=[T]_\beta=\begin{bmatrix}2&3&1\\3&3&1\\2&4&1\end{bmatrix} \Rightarrow [T(1,2,3)]_\beta=\begin{bmatrix}2&3&1\\3&3&1\\2&4&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}11\\12\\13\end{bmatrix}$$

$\therefore T(1,2,3)=(11,12,13)$

$$(b)[T^{-1}(0,1,0)]_\beta=\begin{bmatrix}2&3&1\\3&3&1\\2&4&1\end{bmatrix}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1&1&0\\-1&0&1\\6&-2&-3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

$\therefore T^{-1}(0,1,0)=(1,0,-2)$

$$(c)[T^{-1}(1,2,3)]_\beta=\begin{bmatrix}-1&1&0\\-1&0&1\\6&-2&-3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\2\\-7\end{bmatrix}$$

$\therefore T^{-1}(1,2,3)=(1,2,-7)$

$T^{-1}(1,0,0)=(-1,-1,6), T^{-1}(0,1,0)=(1,0,-2), T^{-1}(0,0,1)=(0,1,-3)$

$T^{-1}(a,b,c)=a(-1,-1,6)+b(1,0,-2)+c(0,1,-3)=(-a+b,-a+c,6a-2b-3c)$

Let $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}$ (a) find a basis for the row space of A (b) find a basis for the column space of A
$(a)A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix},\ let\ \ U=\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}$ $\Rightarrow A=R_{12}R_{13}^{(-2)}U\Rightarrow RS(A)=RS(U)$ $\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}\sim\begin{bmatrix}2&4&6\\0&0&4\\0&0&0\end{bmatrix}\Rightarrow\begin{cases}pick\{[2\ \ 4\ \ 6],[0\ \ 0\ \ 4]\}\\be\ \ a\ \ basis\ \ of\ \ RS(A)=RS(U)\end{cases}$ $(b)A=\begin{bmatrix}0&0&4\\2&4&6\\-4&-8&-10\end{bmatrix}\sim\begin{bmatrix}0&0&4\\2&4&6\\0&0&2\end{bmatrix}$ $\Rightarrow pick\left\{\begin{bmatrix}0\\2\\-4\end{bmatrix},\begin{bmatrix}4\\6\\-10\end{bmatrix}\right\}\ be\ \ a\ \ basis\ \ of\ \ CS(A)$

94 中山電機

Find the $N(A),R(A^T),N(A^T),R(A)$ of $A=\begin{bmatrix}1&1&2\\0&1&1\\1&3&4\end{bmatrix}$
$(1)\begin{cases}x_1+x_2+2x_3=0\\x_2+x_3=0\end{cases}\Rightarrow\begin{cases}x_1=-x_3\\x_2=-x_3\end{cases}\Rightarrow N(A)=span\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$ $(2)R(A^T)=span\{[1\ \ 1\ \ 2],[0\ \ 1\ \ 1]\}$ $(3)N(A^T)=R(A)^{\perp},\forall x=\begin{bmatrix}a\\b\\c\end{bmatrix}\in R(A)^{\perp}\Rightarrow<\begin{bmatrix}a\\b\\c\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}>=0,<\begin{bmatrix}a\\b\\c\end{bmatrix},\begin{bmatrix}1\\1\\3\end{bmatrix}>=0$ $\Rightarrow\begin{cases}a+c=0\\a+b+3c=0\end{cases}\Rightarrow N(A^T)=span\left\{\begin{bmatrix}-1\\-2\\1\end{bmatrix}\right\}\qquad(4)R(A^T)=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$

Let $T: R^3 \to R^3$ be the linear transformation defined by

$T(x) = (x_1 - x_3, -2x_1 + 3x_2 - x_3, 3x_1 - 3x_2)^T$

(a) find the standard matrix representation A for the linear operator T
(b) find the LU decomposition
(c) find a basis for the column space of A
(d) find a basis for the nullspace of A

(a) Pick $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be a basis of R^3 $\Rightarrow T(\beta) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$\Rightarrow [T]_\beta = A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \end{bmatrix}$$

(b) $(b) \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \end{bmatrix} r_{12}^{\ 2}, r_{13}^{\ -3}, r_{23}^{\ 1} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} = U, L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$

$(c)\ pick\ \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \right\}\ be\ a\ basis\ of\ CS(A)$

$$(d) \forall x \in \ker(A) \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ 3x_2 - 3x_3 = 0 \end{cases} \Rightarrow x = (x_3, x_3, x_3)$$

$pick\ \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}\ be\ a\ basis\ of\ \ker(A)$

Chapter 05

(1)suppose $A,B\in F^{n\times n}$ if $A\sim B$

$(1)tr(A)=tr(B)$

$(2)det(A)=det(B)$

$(3)rank(A)=rank(B)$

$(4)nallity(A)=nallity(B)$

(2) $A\in F^{n\times n}$

$A:invertable \Leftrightarrow 0\notin\lambda(A)$

證明: $A:invertable \Leftrightarrow \det(A-0I)\neq 0 \Leftrightarrow 0\notin\lambda(A)$

(3)

(1) $T\in L(V,V)$, $\dim(V)=n$

T 可對角化 $\Leftrightarrow$ T 中存在 n 個 LI 之 eigenvectors

(2) $A:n\times n$, $\dim(A)=n$

A 可對角化 $\Leftrightarrow$ A 中存在 n 個 LI 之 eigenvectors

(4) $A\in F^{n\times n}, p_A(x)$ $split$ $over$ F and $p_A(x)=(\lambda_1-x)(\lambda_2-x)...(\lambda_n-x)$

$(1)det(A)=p_A(x)=\lambda_1\lambda_2...\lambda_n$,determinant 爲所有 eigenvalue 乘積 (含重根)

$(2)tr(A)=\lambda_1+\lambda_2+...+\lambda_n$,trace 爲所有 eigenvalue 和 (含重根)

(5) $A\in F^{n\times n}$

A 有 n 個相異 eigenvalue $\Rightarrow$ A 可對角化

但 A 可對角化未必保證有 n 個相異 eigenvalue

Ex 5.1

Prove $A\sim B\Rightarrow B\sim A$ and $A\sim B, B\sim C\Rightarrow A\sim C$
$(a)\because A\sim B,\therefore\exists\ P\in F^{n\times n}\ni P^{-1}AP=B\Rightarrow A=PBP^{-1}\Rightarrow B\sim A$ $(b)\because A\sim B, B\sim C,\therefore\exists\ P\ni P^{-1}AP=B,\exists\ Q\ni Q^{-1}BQ=C$ $\Rightarrow Q^{-1}P^{-1}APQ=C\Rightarrow(PQ)^{-1}APQ=C\Rightarrow A\sim C$

Ex 5.3

Prove $A,B \in F^{n\times n}$ if $A \sim B$ $(1) tr(A) = tr(B)$ $(2) det(A) = det(B)$ $(3) rank(A) = rank(B)$ $(4) nullity(A) = nullity(B)$
$A \sim B \Rightarrow \exists \; P \ni P^{-1}AP = B$ $(1) tr(B) = tr(P^{-1}AP) = tr(APP^{-1}) = tr(A)$ $(2) \det(B) = \det(P^{-1}AP) = \det(P)^{-1}\det(A)\det(P) = \det(A)$ $(3) rank(B) = rank(P^{-1}AP) = rank(AP) = rank(A)$ $\because \;\; matrix \;\; \times \;\; invertable \;\; matrix \;\; ,rank \;\; be \;\; same$ $(4) nullity(B) = n - rank(B) = n - rank(A) = nullity(A)$

Ex 5.17

$A,B \in F^{n\times n}, if \;\; A \sim B \;\; then$ prove $(1) p_A(x) = p_B(x)$ $(2) \lambda(A) = \lambda(B)$
(a) $\because A \sim B \Rightarrow \exists P : invertable \ni P^{-1}AP = B$ $p_B(x) = \det(B - xI) = \det(P^{-1}AP - xI) = \det(P^{-1}AP - xP^{-1}IP) = \det(P^{-1}AP - P^{-1}xIP)$ $= \det(P^{-1}(A - xI)P) = \det(P^{-1})\det(A - xI)\det(P) = \det(A - xI) = p_A(x)$ (b) by (a) 顯然成立

Ex 5.13

$A=\begin{bmatrix}5&4&2\\4&5&2\\2&2&2\end{bmatrix}$ (1)characteristic polynomial (2)eigenvalue (3)eigenvector

$p_A(A)=\det(A-xI)=-(x-1)^2(x-10)$

A 的 eigenvalue 爲 {1 , 10}

$V(1)=\ker(A-I)=\ker\begin{bmatrix}4&4&2\\4&4&2\\2&2&1\end{bmatrix}=span\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}$ 爲 A 相對於 1 之 eigenspace

$V(10)=\ker(A-10I)=\ker\begin{bmatrix}-5&4&2\\4&-5&2\\2&2&-8\end{bmatrix}=span\left\{\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$ 爲 A 相對於 10 之 eigenspace

A 相對於 1 之 eigenvector $r\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\2\end{bmatrix}$,r,s 不全爲零

A 相對於 10 之 eigenvector $t\begin{bmatrix}2\\2\\1\end{bmatrix},t\neq 0$

Ex 5.12

$find\ \ eigenvalue\ \ of\ \ T(x,y,z)=(2x+y,y-z,2y+4z)$

$pick\ \ \beta=\{e_1,e_2,e_3\}\ \ be\ \ standard\ \ basis\ \ of\ \ R^3$

$T(e_1)=(2,0,0),T(e_2)=(1,1,2),T(e_3)=(0,-1,4)$

$A=[T]_\beta=\begin{bmatrix}2&1&0\\0&1&-1\\0&2&4\end{bmatrix}$

$p_T(x)=p_A(x)=\det(A-xI)=-(x-2)^2(x-3)\Rightarrow\lambda(T)=\{2,3\}$

Ex 5.19

$A=\begin{bmatrix}-3 & 1 & 0\\ 1 & -3 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{cases}(1)\, find\ \ the\ \ eigenvalues\ \ and\ \ corresponding\ \ eigenvector\ \ of\ \ A^5\\ (2)\, find\ \ the\ \ eigenvalues\ \ and\ \ corresponding\ \ eigenvector\ \ of\ \ A^{-1}\end{cases}$
$\lambda(A)=\{-2,-4,3\}, V(-2)=span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}, V(3)=span\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, V(-4)=span\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$ $eigenvalues\ \ of\ \ A^5 \Rightarrow \lambda(A)=\{(-2)^5,(-4)^5,3^5\}, eigenvector\ \ same$ $eigenvalues\ \ of\ \ A^{-1} \Rightarrow \lambda(A)=\{-\frac{1}{2},-\frac{1}{4},\frac{1}{3}\}, eigenvector\ \ same$

Ex 5.15

$T:R^{2\times2}\to R^{2\times2}\ \ ,T\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a+b & b+c\\ c+d & d\end{bmatrix}$,Find eigenvalue and eigenvectors of T
$pick\ \ \beta=\{E_{11},E_{12},E_{21},E_{22}\}$ $T(E_{11})=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, T(E_{12})=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, T(E_{21})=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, T(E_{22})=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$ $A=[T]_\beta=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}\Rightarrow p_T(x)=p_A(x)=(x-1)^4\Rightarrow \lambda(T)=\{1\}$ $V(1)=\ker(A-I)=\ker\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}=span\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$ T 相對於 1 之 eigenvector 爲 $t\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, t\neq 0$

96 台中教大

$A=\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix}$ (a) find the eigenvalue of A (b) is $\begin{bmatrix}6\\4\end{bmatrix}$ an eigenvector of A ?
$(a)p_A(x)=\det(A-xI)=\det\begin{bmatrix}4-x & -3\\ 2 & -1-x\end{bmatrix}=(x-1)(x-2)\Rightarrow\lambda(A)=\{1,2\}$ $(b)\because\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix}=\begin{bmatrix}12\\8\end{bmatrix}=2\begin{bmatrix}6\\4\end{bmatrix}$ 所以 $\begin{bmatrix}6\\4\end{bmatrix}$ 為 A 相對於 $\lambda=2$ 的 eigenvector

Let $T:R^3\to R^3$ be given by $T(x,y,z)=(x-2z,0,-2x+4z)$ Find eigenvalue ,eigenvector, characteristic polynomial of T
$pick\ \ \beta=\{e_1,e_2,e_3\}\ \ be\ \ standard\ \ basis\ \ of\ \ R^3$ $pick\ \ A=[T]_\beta=\begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 4\end{bmatrix},p_A(x)=-x^2(x-5)$ $\ker(A-0I)=span\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\},\ker(A-5I)=span\left\{\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}$ T 對應於 0 之 eigenvector 為 $s(0,1,0)+t(2,0,1)$,s,t 不全為零 T 對應於 5 為 $r(-1,0,2),r\neq 0$

96 交大電子

$T:M_{2\times2}\to M_{2\times2}$ be a linear transformation defined by $T\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a+b&b+c\\c+d&d\end{bmatrix}$ Find the eigenvector and eigenvalues of this transformation
取 $\beta=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ 為 $M_{2\times2}$ 一組 basis $T\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix},T\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\0&0\end{bmatrix},T\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix},T\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\1&1\end{bmatrix}$ $A=[T]_\beta=\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\Rightarrow p_A(x)=(x-1)^4$ $\lambda(A)=\{1,1,1,1\}$ $V(1)=\ker(A-I)=\ker\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}=span\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$ 取 $t\begin{bmatrix}1&0\\0&0\end{bmatrix},t\neq0$ 為 T 相對於 $\lambda=1$ 的 eigenvector

96 中興應數

$T:P_2\to P_2$ be a linear transformation defined by $T(a_0+a_1x+a_2x^2)=(2a_1-2a_2)+(2a_0+3a_2)x+3a_2x^2$ Find the eigenvalue of T
取 $\beta=\{1,x,x^2\}$ 為 P_2 之一組基底 $T(1)=2x,T(x)=2,T(x^2)=-2+3x+3x^2$ $A=[T]_\beta=\begin{bmatrix}0&2&-2\\2&0&3\\0&0&3\end{bmatrix}\Rightarrow p_A(x)=\det(A-xI)=-(x+2)(x-2)(x-3)$ $\Rightarrow\lambda(T)=\{-2,2,3\}$

94 政大資科

Find eigenvalue of $A=\begin{bmatrix}3&2&1&1&2&3\\1&2&3&4&5&6\\1&1&1&7&8&9\\0&0&0&4&0&0\\0&0&0&3&3&4\\0&0&0&5&4&3\end{bmatrix}$,show the detail
$let\quad A=\begin{bmatrix}B&C\\O&D\end{bmatrix},B=\begin{bmatrix}3&2&1\\1&2&3\\1&1&1\end{bmatrix},C=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix},D=\begin{bmatrix}4&0&0\\3&3&4\\5&4&3\end{bmatrix}$ $Char_B(x)=-x(x-1)(x-5),Char_D(x)=-(x+1)(x-4)(x-7)$ $Char_A(x)=\det\begin{bmatrix}B-xI&C\\O&D-xI\end{bmatrix}=\det(B-xI)\det(D-xI)$ $=x(x-1)(x-5)(x+1)(x-4)(x-7)$ $\lambda(A)=\{0,1,5,-1,4,7\}$

Ex 5.20

Following matrix diagonalizable ? $(a)A=\begin{bmatrix}2&1&1\\0&1&0\\1&-1&2\end{bmatrix}$, $(b)B=\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$
$(a)\lambda(A)=\{1,3\},V(1)=span\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\},V(3)=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ $,A\quad has\quad no\quad 3\quad LI\quad eigenvector\Rightarrow not\quad diagonalizable$ $(b)\lambda(B)=\{-3,5\},V(1)=span\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}3\\0\\1\end{bmatrix}\right\},V(3)=span\left\{\begin{bmatrix}-1\\-2\\1\end{bmatrix}\right\},$ $B\quad has\quad no\quad 3\quad LI\quad eigenvector\Rightarrow diagonalizable$

Ex 5.23

Following matrix diagonalizable ? $(a)A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$, $(b)B=\begin{bmatrix}3&1&0\\0&3&0\\0&0&4\end{bmatrix}$
$(a)p_A(x)=-(x-1)(x-2)^2$ $am(1)=1,am(2)=2,gm(1)=1,gm(2)=2$ $\because am(1)=gm(1)\,\&\,am(2)=gm(2)\Rightarrow A:diagonalizable$ $(b)p_B(x)=-(x-3)^2(x-4)$ $am(3)=2,am(4)=1,gm(3)=1$ $\because am(3)=2\neq 1=gm(3)\Rightarrow A:not\quad diagonalizable$

Following matrix be $n\times n$ matrix,under what condition matrix are defective $A=\begin{bmatrix}1&1&1\\1&1&1\\0&0&\alpha\end{bmatrix},B=\begin{bmatrix}1&2&0\\2&1&0\\2&-1&\alpha\end{bmatrix},C=\begin{bmatrix}4&6&-2\\-1&-1&1\\0&0&\alpha\end{bmatrix}$
$p_A(x)=\det(A-xI)=-x(x-2)(x-\alpha)$ $(a)\alpha\notin\{0,2\},A\quad has\quad 3\quad distinct\quad eigenvalue\quad A:diagonalizable\Rightarrow not\quad defective$ $(b)\alpha=0,nullity(A-0I)=2\Rightarrow m(0)=gm(0)\Rightarrow A:diagonalizable\Rightarrow not\quad defective$ $(c)\alpha=2,nullity(A-2I)=1\Rightarrow m(2)\neq gm(2)\Rightarrow A:not\quad diagonalizable\Rightarrow defective$ $p_B(x)=\det(A-xI)=-(x+1)(x-3)(x-\alpha)$ $(a)\alpha\notin\{-1,3\},A\quad has\quad 3\quad distinct\quad eigenvalue\quad A:diagonalizable\Rightarrow not\quad defective$ $(b)\alpha=-1,nullity(A+I)=1\Rightarrow m(1)=gm(1)\Rightarrow A:not\quad diagonalizable\Rightarrow defective$ $(c)\alpha=3,nullity(A-3I)=1\Rightarrow m(3)\neq gm(3)\Rightarrow A:not\quad diagonalizable\Rightarrow defective$ $p_C(x)=\det(A-xI)=-(x-1)(x-2)(x-\alpha)$ $(a)\alpha\notin\{1,2\},A\quad has\quad 3\quad distinct\quad eigenvalue\quad A:diagonalizable\Rightarrow not\quad defective$ $(b)\alpha=1,nullity(A-1I)=1\Rightarrow m(1)\neq gm(1)\Rightarrow A:not\quad diagonalizable\Rightarrow defective$ $(c)\alpha=2,nullity(A-2I)=2\Rightarrow m(2)=gm(2)\Rightarrow A:diagonalizable\Rightarrow not\quad defective$

Ex 5.24

$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$,find $P \ni P^{-1}AP=D$
$p_A(x)=-(x+1)^2(x-2)\Rightarrow \lambda(A)=\{-1,2\}$ $V(1)=\ker(A+I)=\ker\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=span\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ $V(-2)=\ker(A-2I)=\ker\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}=span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ $pick\ \ P=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&2\end{bmatrix}$

Let T be the linear operator on R^3 be defined by $T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4x+z\\2x+3y+2z\\x+4z\end{bmatrix}$ Find an order basis γ such that $[T]_\gamma$ is a diagonal matrix
$pick\ \ [T]_\beta=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix},p_A(x)=\det(A-xI)=-(x-3)^2(x-5)$ $V(3)=\ker(A-3I)=span\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\},\ker(A-5I)=span\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ $pick\ \ \gamma=\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}\Rightarrow[T]_\gamma=\begin{bmatrix}3&0&0\\0&3&0\\0&0&5\end{bmatrix}$

Ex 5.25

Consider the linear transformation T taking x in R^3 to $L(x)$ in R^3 by $L(x)=(x_1+x_2,2x_2+x_3,3x_3)$ Find an ordered basis for both the domain and range of T so that the corresponding matrix representation of L is diagonal ,and find the matrix representation
取$\beta=\{e_1,e_2,e_3\}$ 爲 R^3的一組 basis $T(e_1)=(1,0,0),T(e_2)=(1,2,0),T(e_3)=(0,1,3)\Rightarrow A=[T]_\beta=\begin{bmatrix}1&1&0\\0&2&1\\0&0&3\end{bmatrix}$ $p_A(x)=\det(A-xI)=-(x-1)(x-2)(x-3)\Rightarrow\lambda(A)=\{1,2,3\}$ $V(1)=\ker(A-I)=\ker\begin{bmatrix}0&1&0\\0&1&0\\0&0&2\end{bmatrix}=span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ $V(2)=\ker(A-2I)=\ker\begin{bmatrix}-1&1&0\\0&0&1\\0&0&1\end{bmatrix}=span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ $V(3)=\ker(A-3I)=\ker\begin{bmatrix}-2&1&0\\0&-1&1\\0&0&0\end{bmatrix}=span\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ $pick\quad\gamma=\{(1,0,0),(1,1,0),(1,2,2)\}\ni[T]_\gamma=D=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

Ex 5.27

$A\begin{bmatrix}0.4&-0.3\\0.4&1.2\end{bmatrix},\lim_{k\to\infty}A^k=?$
$pick\quad P=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}1&0\\0&0.6\end{bmatrix}$ $\Rightarrow A=PDP^{-1}$ $\Rightarrow A^k=PD^kP^{-1}=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&0.6^k\end{bmatrix}\begin{bmatrix}1/2&3/4\\-1/2&-1/4\end{bmatrix}$ $\Rightarrow\lim_{k\to\infty}A^k=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1/2&3/4\\-1/2&-1/4\end{bmatrix}=\begin{bmatrix}-1/2&-3/2\\1&3/2\end{bmatrix}$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + 5x_3 - 10x_4 \\ x_1 + 5x_3 \\ x_1 + 3x_4 \end{bmatrix}$$

(a) find the matrix A represent T

(b) find the eigenvalue and corresponding eigenspace of A

(c) find a matrix $P \ni P^{-1}AP = D$

$$(a) T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + 5x_3 - 10x_4 \\ x_1 + 5x_3 \\ x_1 + 3x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

$$(b) \lambda(A) = \{1, 2, 3\}$$

$$V(1) = \ker(A - I) = \ker\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = span\left\{\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$ 爲 A 相對於 1 的 eigenspace

$$V(2) = \ker(A - 2I) = \ker\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & -10 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = span\left\{\begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}\right\}$$ 爲 A 相對於 2 的 eigenspace

$$V(3) = \ker(A - 3I) = \ker\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -10 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = span\left\{\begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}\right\}$$ 爲 A 相對 3 的 eigenspace

$$(c) pick \quad P = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \ni P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Ex 5.26

$A=\begin{bmatrix}2&1\\1&2\end{bmatrix},(1)A^n=?,n\in N\quad(2)e^A\quad(3)\sin A=?\quad(4)A^{\frac{1}{2}}$
$(1)A^n=PD^nP^{-1}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}1^n&0\\0&3^n\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1+3^n}{2}&\frac{-1+3^n}{2}\\\frac{-1+3^n}{2}&\frac{1+3^n}{2}\end{bmatrix}$ $(2)e^A=Pe^DP^{-1}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}e^1&0\\0&e^3\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{e+e^3}{2}&\frac{-e+e^3}{2}\\\frac{-e+e^3}{2}&\frac{e+e^3}{2}\end{bmatrix}$ $(3)\sin(A)=P\sin(D)P^{-1}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}\sin 1&0\\0&\sin 3\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$ $=\begin{bmatrix}\frac{\sin 1+\sin 3}{2}&\frac{-\sin 1+\sin 3}{2}\\\frac{-\sin 1+\sin 3}{2}&\frac{\sin 1+\sin 3}{2}\end{bmatrix}$ $(4)pick\quad X=P\begin{bmatrix}\pm\sqrt{1}&0\\0&\pm\sqrt{3}\end{bmatrix}P^{-1}\Rightarrow$ $X^2=P\begin{bmatrix}\pm\sqrt{1}&0\\0&\pm\sqrt{3}\end{bmatrix}P^{-1}P\begin{bmatrix}\pm\sqrt{1}&0\\0&\pm\sqrt{3}\end{bmatrix}P^{-1}=P\begin{bmatrix}1&0\\0&3\end{bmatrix}P^{-1}=A$

$A=\begin{bmatrix}0.4&-0.3\\0.4&1.2\end{bmatrix},find\quad\lim_{k\to\infty}A^k=?$
$p_A(x)=\det(A-xI)=(x-1)(x-0.6)$ $V(1)=\ker(A-I)=span\left\{\begin{bmatrix}-1\\2\end{bmatrix}\right\},V(0.6)=\ker(A-0.6I)=span\left\{\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ $pick\quad P=\begin{bmatrix}-1&3\\2&-2\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}1&0\\0&0.6\end{bmatrix}\Rightarrow A=PDP^{-1}$ $\Rightarrow A^k=PD^kP^{-1}=\frac{1}{4}\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&0.6\end{bmatrix}^k\begin{bmatrix}2&3\\-2&-1\end{bmatrix}$ $\lim_{k\to\infty}A^k=P(\lim_{k\to\infty}D^k)P^{-1}=\frac{1}{4}\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}2&3\\2&1\end{bmatrix}$ $=\frac{1}{4}\begin{bmatrix}-1&0\\2&0\end{bmatrix}\begin{bmatrix}2&3\\2&1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2&-3\\4&6\end{bmatrix}$

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$A=\begin{bmatrix}4 & -5\\ -3 & 2\end{bmatrix}$

(a) find the characteristic equation of the matrix A

(b) find the eigenvalue of the matrix A^{10}

(c) find a basis for each eigenspace of the matrix A^{10}

(d) use diagonalization to compute A^{10}

$$(a)p_A(x)=\det(A-xI)=\det\begin{bmatrix}4-x & -5\\ -3 & 2-x\end{bmatrix}=(x+1)(x-7)$$

$$(b)\lambda(A)=\{-1,7\},\lambda(A^{10})=\{(-1)^{10},7^{10}\}$$

$$(c)\ker(A+I)=\ker\begin{bmatrix}5 & -5\\ -3 & 3\end{bmatrix}=span\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$$

取 $\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$ 為 A^{10} 相對於 $\lambda=-1$ 的 eigenspace 之 1 basis

$$\ker(A-7I)=\ker\begin{bmatrix}-3 & -5\\ -3 & -5\end{bmatrix}=span\left\{\begin{bmatrix}-5\\ 3\end{bmatrix}\right\}$$

取 $\left\{\begin{bmatrix}-5\\ 3\end{bmatrix}\right\}$ 為 A^{10} 相對於 $\lambda=7$ 的 eigenspace 之 1 basis

(d)

$$P^{-1}AP=D\Rightarrow A=PDP^{-1}\Rightarrow A^{10}=PD^{10}P^{-1}$$

$$=\begin{bmatrix}1 & -5\\ 1 & 3\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 7\end{bmatrix}^{10}\begin{bmatrix}1 & -5\\ 1 & 3\end{bmatrix}^{-1}=\begin{bmatrix}1 & -5\\ 1 & 3\end{bmatrix}\begin{bmatrix}(-1)^{10} & 0\\ 0 & 7^{10}\end{bmatrix}\begin{bmatrix}1 & -5\\ 1 & 3\end{bmatrix}^{-1}$$

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$A=\begin{bmatrix}7 & 5\\ 3 & -7\end{bmatrix}$,find a matrix $B\ni B^3=A$

$$pick\quad P=\begin{bmatrix}-1 & 5\\ 3 & 1\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}-8 & 0\\ 0 & 8\end{bmatrix}$$

$$pick\quad B=PD^{1/3}P^{-1}=P\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix}P^{-1}=\frac{1}{4}\begin{bmatrix}7 & 5\\ 3 & -7\end{bmatrix}\ni B^3=P\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix}^3P^{-1}=A$$

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Find $\cos(A)$ for $A=\begin{bmatrix}-2 & -6\\ 1 & 3\end{bmatrix}$
$p_A(x)=\det\begin{bmatrix}-2-x & -6\\ 1 & 3-x\end{bmatrix}=x(x-1)\Rightarrow \lambda(A)=\{0,1\}$ $V(0)=\ker(A)=\ker\begin{bmatrix}-2 & -6\\ 1 & 3\end{bmatrix}=span\left\{\begin{bmatrix}-3\\ 1\end{bmatrix}\right\}$ $V(1)=\ker(A-I)=\ker\begin{bmatrix}-3 & -6\\ 1 & 2\end{bmatrix}=span\left\{\begin{bmatrix}-2\\ 1\end{bmatrix}\right\}$ $pick\quad P=\begin{bmatrix}-3 & -2\\ 1 & 1\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\Rightarrow A=PDP^{-1}$ $\cos(A)=P\cos(D)P^{-1}=\begin{bmatrix}-3 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\cos(0) & 0\\ 0 & \cos(1)\end{bmatrix}\begin{bmatrix}-3 & -2\\ 1 & 1\end{bmatrix}^{-1}$

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Find a formula for A^k for $A=\begin{bmatrix}3 & 1 & 0\\ 0 & 1 & 0\\ 4 & 2 & 1\end{bmatrix}$
$pick\quad P=\begin{bmatrix}-1 & 0 & 1\\ 2 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}\Rightarrow A=PDP^{-1}$ $A^k=PD^kP^{-1}=\begin{bmatrix}-1 & 0 & 1\\ 2 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1^k & 0 & 0\\ 0 & 1^k & 0\\ 0 & 0 & 3^k\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 2 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}^{-1}=\begin{bmatrix}3^k & \frac{1}{2}(3^k-1) & 0\\ 0 & 1 & 0\\ 2(3^k)-1 & 3^k-1 & 1\end{bmatrix}$

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$A=\begin{bmatrix}2 & 5\\ 1 & -2\end{bmatrix}, A^K=?$
$A^K=PD^KP^{-1}=\frac{1}{6}\begin{bmatrix}1 & 5\\ -1 & 1\end{bmatrix}\begin{bmatrix}(-3)^k & 0\\ 0 & 3^k\end{bmatrix}\begin{bmatrix}1 & -5\\ 1 & 1\end{bmatrix}$ $=\frac{1}{6}\begin{bmatrix}(-3)^k+5\times 3^k & (-5)\times(-3)^k+5\times 3^k\\ -(-3)^k+3^k & 5(-3)^k+3^k\end{bmatrix}$

Ex 5.22

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$find \quad eigenvalue \quad, \det er\min ant \quad, trace \quad of \quad A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}$

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第 1 個 eigenvalue $\lambda_1 = 1-2=-1$,重根數爲 $n-1=4-1=3$

另 1 個 eigenvalue $\lambda_4 = tr(A)-\lambda_1 * 3 = 7$ 特徵多項式 $p_A(x) = -(-1-x)^3(7-x)$

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$\begin{bmatrix} a & b & \cdots & b \\ b & a & \vdots & b \\ \vdots & \cdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix}$	第 1 個 eigenvalue (a-b) 有 n-1 個 最後 1 個 eigenvalue $\lambda_n = tr(A)-(n-1)(a-b) = na-(n-1)(a-b) = a+(n-1)b$

$V(\lambda_1) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \right\}, V(\lambda_n) = span\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$

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Suppose A is 3×3 matrix has eigenvalue 1,2,3

(a) is a diagonalizable ?why ?

(b) determine the eigenvalue of $2A^{-1}+I$

(c) determine the determinant of $A+I$

(d) determine the determinant of $2(A^T A)$

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(a) A 爲 3×3 matrix 且具 3 相異 eigenvalue 故可對角化

$(b)\left\{2*\frac{1}{1}+1, 2*\frac{1}{2}+1, 2*\frac{1}{3}+1\right\} = \left\{3, 2, \frac{5}{3}\right\}$

$(c)(1+1)(2+1)(3+1) = 24$

$(d)2^3 \det(A)^2 = 8(1*2*3)^2 = 288$

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$A=\begin{bmatrix} 0.6 & 0.4 & 0.3 \\ 0.4 & 0.9 & 0.2 \\ 0.3 & 0.2 & 0.8 \end{bmatrix}$,sum of A`s eigenvalue ?

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$sum \quad of \quad eigenvalue = tr(A) = 0.6+0.9+0.8 = 2.3$

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矩陣解微分方程

$\vec{x}^{\,/} = A\vec{x} = PDP^{-1}\vec{x}$.......(1)

$let \quad P^{-1}\vec{x} = \vec{y} \quad ,then \quad (1) \quad be \quad \vec{x}^{\,/} = PD\vec{y}$.......(2)

$and \quad \vec{x} = P\vec{y} \quad ,differ \quad both \quad side \quad \vec{x}^{\,/} = P\vec{y}^{\,/}$..(3)

$by \quad (2)(3) \Rightarrow \vec{y}^{\,/} = D\vec{y} \quad \Rightarrow solve \quad \vec{y}$

$then \quad \vec{x} = P\vec{y}$

Solve $\begin{bmatrix} x_1^{/} \\ x_2^{/} \\ x_3^{/} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1(0) = 1, x_2(0) = 5, x_3(0) = 10$
$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \ni P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A = PDP^{-1}$ $x^{/} = Ax = PDP^{-1}x \quad , \quad let \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P^{-1}x \Rightarrow Py = x \Rightarrow (Py)^{/} = x^{/} = PDy \Rightarrow y^{/} = Dy$ $\begin{bmatrix} y_1^{/} \\ y_2^{/} \\ y_3^{/} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{cases} y_1^{/} = y_1 \\ y_2^{/} = -2y_2 \\ y_3^{/} = 3y_2 \end{cases} \Rightarrow \begin{cases} y_1 = c_1e^{t} \\ y_2 = c_2e^{-2t} \\ y_3 = c_3e^{3t} \end{cases}$ $x = Py = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} c_1e^{t} \\ c_2e^{-2t} \\ c_3e^{3t} \end{bmatrix} = \begin{bmatrix} c_1e^{t} \\ c_2e^{-2t} + c_3e^{3t} \\ 5c_3e^{3t} \end{bmatrix} \quad , put \quad initial \quad condition$ $\begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} = x(0) = \begin{bmatrix} c_1e^{t} \\ c_2 + c_3 \\ 5c_3 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 3 \Rightarrow x = \\ c_3 = 2 \end{cases} \begin{bmatrix} c_1e^{t} \\ 3e^{-2t} + 2e^{3t} \\ 10e^{3t} \end{bmatrix}$

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$$A=\begin{bmatrix}1&1&1\\a&b&c\\d&e&f\end{bmatrix}, eigenvector\ \ of\ \ A\ \ be\ \ (1,2,3),(1,0,-1),(1,-1,0)$$

$$\because (1,2,3),(1,0,-1),(1,-1,0)\ \ LI\ \ pick\ \ P=\begin{bmatrix}1&1&1\\2&0&-1\\3&-1&0\end{bmatrix}\ni P^{-1}AP=D=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$$

$$A=PDP^{-1}=\begin{bmatrix}1&1&1\\2&0&-1\\3&-1&0\end{bmatrix}\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}\frac{1}{6}&\frac{1}{6}&\frac{1}{6}\\\frac{1}{2}&\frac{1}{2}&\frac{-1}{2}\\\frac{1}{3}&\frac{-2}{3}&\frac{1}{3}\end{bmatrix}\Rightarrow\begin{cases}\frac{\lambda_1+3\lambda_2+2\lambda_3}{6}=1\\\frac{\lambda_1+3\lambda_2-4\lambda_3}{6}=1\\\frac{\lambda_1-3\lambda_2+2\lambda_3}{6}=1\end{cases}$$

$$\Rightarrow\lambda_1=6,\lambda_2=0,\lambda_3=0\Rightarrow A=\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}$$

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$$A=\begin{bmatrix}0&-1&-1\\x&y&z\\-1&-1&2\end{bmatrix}$$ we know

(i) 2 is an eigenvalue of A (ii) $[2,1,1]^T$ is an eigenvector (iii) $\det(A)=-6$

Find x,y,z

$$(a)A\begin{bmatrix}2\\1\\1\end{bmatrix}=\lambda\begin{bmatrix}2\\1\\1\end{bmatrix}\Rightarrow\begin{bmatrix}0&-1&-1\\x&y&z\\-1&-1&2\end{bmatrix}\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}-2\\2x+y+z\\-1\end{bmatrix}=\lambda\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\Rightarrow\begin{cases}\lambda=-1\\2x+y+z=-1\end{cases}\Rightarrow\det(A)=-6=2*(-1)*\lambda_3\Rightarrow\lambda_3=3$$

$$\therefore\lambda(A)=\{-1,2,3\}$$

$$(b)tr(A)=-1+2+3=4=0+y+2\Rightarrow y=2$$

$$\because 2x+y+z=-1\Rightarrow z=-2x-3$$

$$\det\begin{bmatrix}0&-1&-1\\x&2&-2x-3\\-1&-1&2\end{bmatrix}=-6\Rightarrow x=-1,z=-1$$

Chapter 07

Ex 7.4

$V=C\left[0,1\right],<f,g>,\int_0^1 f(x)g(x)dx$ $(1)f(x)=1+x,\|f\|=?$ $(2)f(x)=x,g(x)=4-3x-4x^2, prove\ \ f\perp g$
$(1)\|f\|^2=<f,f>=\int_0^1(1+x)^2dx=\frac{7}{3}\Rightarrow\|f\|=\sqrt{\frac{7}{3}}$ $(2)<f,g>=\int_0^1 x(4-3x-4x^2)dx=0$

Ex 7.6

$(a)<f,g>=\int_{-1}^1 f(x)g(x)dx\quad,\quad\|1-x\|=?$ $(b)<f,g>=\int_0^1 f(x)g(x)dx\quad,\quad f(x)=x,g(x)=x^2\quad,\quad\|g(x)\|=?,d(f,g)=?$
$(a)<v,v>=\int_{-1}^1(1-x)^2dx=\frac{1}{5}\quad,\quad\sqrt{<v,v>}=length=\sqrt{\frac{8}{3}}$ $(b)\|g(x)\|=\sqrt{<g,g>}=\sqrt{\frac{1}{5}}\quad,\quad d(f,g)=\sqrt{<f,g>}=\sqrt{\frac{1}{30}}$

Ex 7.10

$find\ \ orthonormal\ \ basis\ \ for\ \ v_1=(1,1,1,1),v_2=(1,2,3,4),v_3=(4,3,2,1)$
$u_1=v_1=(1,1,1,1),<u_1,u_1>=4$ $u_2=x_2-\frac{<x_2,u_1>}{<u_1,u_1>}u_1=(1,2,3,4)-\frac{10}{4}(1,1,1,1)=(\frac{-3}{2},\frac{-1}{2},\frac{1}{2},\frac{3}{2}),<u_2,u_2>=5$ $u_3=x_3-\frac{<x_3,u_1>}{<u_1,u_1>}u_1-\frac{<x_3,u_2>}{<u_2,u_2>}u_2=$ $(4,3,2,1)-\frac{10}{4}(1,1,1,1)-\frac{-5}{5}(\frac{-3}{2},\frac{-1}{2},\frac{1}{2},\frac{3}{2})=(0,0,0,0)$ $pick\ \ \left\{\frac{u_1}{\|u_1\|}=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),\frac{u_2}{\|u_2\|}=(\frac{-3}{2\sqrt{5}},\frac{-1}{2\sqrt{5}},\frac{1}{2\sqrt{5}},\frac{3}{2\sqrt{5}})\right\}\ \ be\ \ orthonormal\ \ basis$ 陷阱: 會把 $(0,0,0,0)$ 寫入 orthonormal set 中 ,但 orthonormal set 不包含 0

Ex 7.11

$$V = C[-1,1], < f, g > = \int_{-1}^{1} f(x)g(x)dx$$

$$W = span\{1, x, x^2, x^3\},\ find\ \ a\ \ orthonormal\ \ basis\ \ for\ \ W$$

$Sol:$

$$h_1 = 1, < h_1, h_1 > = \int_{-1}^{1} 1dx = 2$$

$$h_2 = x - \frac{< x,1 >}{2} \times 1 = x, < h_2, h_2 > = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$h_3 = x^2 - \frac{< x^2,1 >}{2} \times 1 - \frac{< x^2, x >}{\frac{2}{3}} \times x = x^2 - \frac{1}{3}, < h_3, h_3 > = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \frac{8}{45}$$

$$h_4 = x^3 - \frac{< x^3,1 >}{2} \times 1 - \frac{< x^2, x >}{\frac{2}{3}} \times x - \frac{< x^3, x^2 - \frac{1}{3} >}{\frac{8}{45}} (x^2 - \frac{1}{3}) = x^3 - \frac{3}{5} x$$

$$< h_4, h_4 > = \int_{-1}^{1} (x^3 - \frac{3}{5} x)^2 dx = \frac{8}{175} \Rightarrow Ans: \left\{ \frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}, \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}}, \frac{x^3 - \frac{3}{5} x}{\sqrt{\frac{8}{175}}} \right\}$$

$$u_1 = v_1 = 1, < u_1, u_1 > = \int_0^1 1dx = 1$$

$$u_2 = v_2 - \frac{< v_2, u_1 >}{< u_1, u_1 >} u_1 = x - \frac{\int_0^1 xdx}{1} \times 1 = x - \frac{1}{2}, < u_2, u_2 > = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$$

$$u_3 = v_3 - \frac{< v_3, u_1 >}{< u_1, u_1 >} u_1 - \frac{< v_3, u_2 >}{< u_2, u_2 >} u_2 = x^2 - \frac{\int_0^1 x^2 dx}{1} - \frac{\int_0^1 x^2 (x - \frac{1}{2}) dx}{\frac{1}{12}} (x - \frac{1}{2})$$

$$= x^2 - \frac{1}{3} - 12(\frac{1}{4} - \frac{1}{6})(x - \frac{1}{2}) = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

$$, < u_3, u_3 > = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \frac{1}{180} \quad Ans: \{1, \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}, \frac{x^2 - x + \frac{1}{6}}{\sqrt{\frac{1}{180}}}\}$$

詩歌

A 行獨立	A 列獨立
$\Leftrightarrow$ A 列生成	$\Leftrightarrow$ A 行生成
$\Leftrightarrow$ A :具左反	$\Leftrightarrow$ A :具右反
$\Leftrightarrow$ $rank(A)$ 等於行數 n	$\Leftrightarrow$ $rank(A)$ 等於列數 m
$\Leftrightarrow$ $A^H A$ 可逆	$\Leftrightarrow$ AA^H 可逆
$\Leftrightarrow$ A nonsingular	$\Leftrightarrow$ A nonsingular
$\Leftrightarrow$ $\ker(A)=\{0\}$	$\Leftrightarrow$ $Lker(A)=\{0\}$

Ex 7.13

$W = span\{(1,0,1,0),(1,1,1,0),(1,-1,0,1)\} \in R^4, v=(1,1,1,1), proj_w v = ?$

正交投影向量求法 **(1)**

對 **basis** 做 **Grad-schmidt** 代 $proj_w v = \frac{<v,u_1>}{<u_1,u_1>}u_1 + \frac{<v,u_2>}{<u_2,u_2>}u_2 + \frac{<v,u_3>}{<u_3,u_3>}u_3$

$let\ \ W = span\{v_1,v_2,v_3\}$

$Grad-schmidt\ \ for\ \ W \Rightarrow \{(1,0,1,0),(0,1,0,0),(\frac{1}{2},0,-\frac{1}{2},1)\} = \{u_1,u_2,u_3\}$

$proj_w v = \frac{<v,u_1>}{<u_1,u_1>}u_1 + \frac{<v,u_2>}{<u_2,u_2>}u_2 + \frac{<v,u_3>}{<u_3,u_3>}u_3 = (\frac{4}{3},1,\frac{2}{3},\frac{2}{3})$

Ex 7.14

$let\ \ v_1=[1,1,0]^T, v_2=[2,3,0]^T, b=[4,5,6]^T, find\ \ the\ \ projection\ \ vector$
$of\ \ b\ \ onto\ \ the\ \ plane\ \ that\ \ is\ \ spanned\ \ by\ \ the\ \ vevtor\ \ v_1,v_2$

正交投影向量求法 **(2)**

確定行獨立代 $proj_w b = A(A^H A)^{-1} A^H b$

$let\ \ W = span\{v_1,v_2\}, A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}, W = CS(A)$ ➔ $proj_w b = A(A^H A)^{-1} A^H b = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Ex 7.15　最小平方解,行相依

$$A=\begin{bmatrix}1&0&1\\0&2&2\\1&2&3\end{bmatrix},\vec{b}=\begin{bmatrix}3\\3\\3\end{bmatrix}$$

(a)Find least square solution for $Ax=b$

(b)W=CS(A),find $proj_w\vec{b}$

$$(a)solve\quad A^TA\vec{x}=A^T\vec{b}\Rightarrow\begin{bmatrix}1&0&1\\0&2&2\\1&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\12\\18\end{bmatrix}\Rightarrow\begin{cases}x_1=2-x_3\\x_2=1-x_3\end{cases}\Rightarrow Ans:\begin{bmatrix}2-x_3\\1-x_3\\x_3\end{bmatrix}$$

$$(b)proj_w\vec{b}=A\begin{bmatrix}2-x_3\\1-x_3\\x_3\end{bmatrix}=\begin{bmatrix}2\\2\\4\end{bmatrix}$$

Ex 7.17　最小平方解,行獨立

Find least square solution for $Ax=b$

$$(a)A=\begin{bmatrix}1&5\\3&1\\-2&4\end{bmatrix},b=\begin{bmatrix}4\\-2\\-3\end{bmatrix}\qquad(b)A=\begin{bmatrix}1&0&1\\0&2&2\\1&2&3\end{bmatrix},b=\begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$(a)column\quad independent\Rightarrow x=(A^TA)^{-1}A^Tb=\begin{bmatrix}\frac{2}{7}\\\frac{1}{7}\end{bmatrix}$$

$$(b)not\quad column\quad independent\Rightarrow solve\qquad A^TAx=A^Tb$$

$$A^TA=\begin{bmatrix}2&2&4\\2&8&10\\4&10&14\end{bmatrix},A^Tb=\begin{bmatrix}6\\12\\18\end{bmatrix}\Rightarrow\begin{bmatrix}2&2&4\\2&8&10\\4&10&14\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\12\\18\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2-z\\1-z\\z\end{bmatrix},z\in R$$

Ex 7.16

$(1) H = span\{(1,1,1,1),(1,0,1,1),(0,1,1,1)\}$,   *find the orthogonal projecion of the vector* $(2,3,3,1)$ *on* H (2) *find the orthogonal projecion matrix* P_W *for the subspace* W *spanned by the column space of* $A=\begin{bmatrix}1&0&5&-3\\0&1&2&4\\-1&-2&-9&-5\\1&1&7&1\end{bmatrix}$ $(3) A=\begin{bmatrix}1&1&3\\3&1&3\\0&0&0\\0&0&0\end{bmatrix}, b=\begin{bmatrix}0\\3\\1\\0\end{bmatrix}$, *find the orthogonal projecion vector of* b *on* $W=CS(A)$
$(1) let\ \ A=\begin{bmatrix}1&1&0\\1&0&1\\1&1&1\\1&1&1\end{bmatrix}, W=CS(A)\ \ \because A\ \ column\ \ independent$ $\Rightarrow proj_w v = A(A^TA)^{-1}A^Tb=(2,3,2,2)$ $(2) W=span\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix},\begin{bmatrix}0\\1\\-2\\1\end{bmatrix},\begin{bmatrix}5\\2\\-9\\7\end{bmatrix},\begin{bmatrix}-3\\4\\-5\\1\end{bmatrix}\right\}, \begin{bmatrix}1&0&-1&1\\0&1&-2&1\\5&2&-9&7\\-3&4&-5&1\end{bmatrix}\to r\begin{bmatrix}1&0&-1&1\\0&1&-2&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ $\Rightarrow W=span\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix},\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}\right\}, let\ \ B=\begin{bmatrix}1&0\\0&1\\-1&-2\\1&1\end{bmatrix}\ \ ,\ \ B(B^TB)^{-1}B^Tb=\frac{1}{3}\begin{bmatrix}2&-1&0&1\\-1&1&-1&0\\0&-1&2&-1\\1&0&-1&1\end{bmatrix}$ $(3) A^TA=\begin{bmatrix}10&4&12\\4&2&6\\12&6&18\end{bmatrix}, A^Tb=\begin{bmatrix}9\\3\\9\end{bmatrix}, A^TAx=A^Tb\Rightarrow x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\-\frac{3}{2}-3x_3\\x_3\end{bmatrix}$ $proj_w v = Ax = \begin{bmatrix}1&1&3\\3&1&3\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\frac{3}{2}\\-\frac{3}{2}-3x_3\\x_3\end{bmatrix}=\begin{bmatrix}0\\3\\0\\0\end{bmatrix}$

Ex 7.18　趨近線方程,代最小平方解

x	-2	-1	0	1
y	-3	-2	1	7

(a) Find least fit line to a linear function $y = ax + b$

(b) Find least fit line to a quadratic function $y = ax^2 + bx + c$

$$(a) let \quad A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} b \\ a \end{bmatrix}, B = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 7 \end{bmatrix}, x = (A^T A)^{-1} A^T B = \begin{bmatrix} \frac{12}{5} \\ \frac{33}{10} \end{bmatrix} \Rightarrow y = \frac{33}{10} x + \frac{12}{5}$$

$$(b) let \quad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, x = \begin{bmatrix} c \\ b \\ a \end{bmatrix}, B = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 7 \end{bmatrix}, x = (A^T A)^{-1} A^T B = \begin{bmatrix} \frac{33}{20} \\ \frac{91}{20} \\ \frac{5}{4} \end{bmatrix} \Rightarrow y = \frac{5}{4} x^2 + \frac{91}{20} x + \frac{33}{20}$$

Ex 7.20

$$let \quad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}, W = span(B), find \quad an \quad orthogonal \quad basis \quad for \quad W^{\perp}$$

$$\forall x \in \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W^{\perp} \Rightarrow < x, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} >= 0, < x, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} >= 0 \Rightarrow \begin{cases} a + c = 0 \\ a - b + c = 0 \end{cases} \Rightarrow \begin{cases} a = -c \\ b = 0 \end{cases} \Rightarrow \begin{bmatrix} -c \\ 0 \\ c \\ d \end{bmatrix}$$

$$pick \quad \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad be \quad orthogonal \quad basis \quad for \quad W^{\perp}$$

Ex 7.22

let W be a subspace of R^5 *spanned by* $w_1=(2,2,-1,0,1)$

$w_2=(-1,-1,2,-3,1), w_3=(1,1,-2,0,-1), w_4=(0,0,1,1,1)$

find the dim*ension of the orthogonal complement of W*

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to r \begin{bmatrix} 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \dim(W)=3$$

$$\because dim(V)=\dim(W)+\dim(W^{\perp}) \Rightarrow \dim(W^{\perp})=2$$

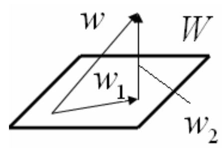
Ex 7.23

let R^4 *has the Euclidean inner product, Express* $w=\left[-1,2,6,0\right]^T$

in *the form* $w=w_1+w_2$, *where* w_1 *is in the space W spanned*

by $u_1=\left[-1,0,1,2\right]^T, u_2=\left[0,1,0,1\right]^T$, *and* w_2 *orthogonal to W*

$$let\ \ A=\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}, proj_W w = A(A^TA)^{-1}A^Tw=\begin{bmatrix} -\frac{5}{4} \\ -\frac{1}{4} \\ \frac{5}{4} \\ \frac{9}{4} \end{bmatrix}$$



$$\Rightarrow pick\ \ w_1=\left[-\frac{5}{4},-\frac{1}{4},\frac{5}{4},\frac{9}{4}\right]^T, \quad w_2=\left[\frac{1}{4},\frac{9}{4},\frac{19}{4},-\frac{9}{4}\right]^T$$

Ex 7.24

find a basis for R^3 *that include the vector* $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$$let\ \ W=span\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}, \quad \forall x=\begin{bmatrix} a \\ b \\ c \end{bmatrix}\in W^{\perp}, <\begin{bmatrix} a \\ b \\ c \end{bmatrix},\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}>=0, <\begin{bmatrix} a \\ b \\ c \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}>=0$$

$$\begin{cases} a+2c=0 \\ b+3c=0 \end{cases} \Rightarrow W^{\perp}=span\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right\}, \because R^3=W\oplus W^{\perp} \Rightarrow R^3=span\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix},\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right\}$$

Ex 7.25

find a basis for orthogonal complement of column space of matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 4 & -6 & 8\\ -2 & -3 & 2\\ -4 & 1 & -3\end{bmatrix}$
$\because R(A)^{\perp}=N(A^T)\Rightarrow\begin{bmatrix}-2 & 4 & -2 & -4\\ 2 & -6 & -3 & 1\\ -3 & 8 & 2 & -3\end{bmatrix}\to r\begin{bmatrix}-2 & 4 & -2 & -4\\ 0 & -2 & -5 & -3\\ 0 & 2 & 5 & 3\end{bmatrix}$ $\begin{cases}-2x_1+4x_2-2x_3-4x_4=0\\ 2x_2+5x_3+3x_4=0\end{cases}\Rightarrow\begin{cases}x_1=-6x_3-5x_4\\ x_2=\dfrac{-5x_3-3x_4}{2}\end{cases}\Rightarrow R(A)^{\perp}=N(A^T)=span\left\{\begin{bmatrix}-6\\ \frac{-5}{2}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-5\\ \frac{-3}{2}\\ 0\\ 1\end{bmatrix}\right\}$

Ex 7.26

let $A=\begin{bmatrix}2 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix}$*, determine the projection matrix* Q *that projects vectors in* R^3 *onto the nullspace of* A^T
let $A=\begin{bmatrix}2 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix}$*, determine the projection matrix* Q *that projects vectors in* R^3 *onto the nullspace of* A^T $\because R(A)^{\perp}=N(A^T)$ $Q=I-A(A^TA)^{-1}A=\begin{bmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$ *be rojection matrix on* $R(A)^{\perp}=N(A^T)$

Ex 7.27

let W be the subspace of R^4 *containing all vectors with* $x_1+x_2+x_3+x_4=0$ *and* $x_1+x_2-x_3-x_4=0$, *find a basis for* $W^{\perp}$
$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_1+x_2-x_3-x_4=0 \end{cases} \Rightarrow W = span\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \forall x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W^{\perp}$ $< \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} >= 0, < \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} >= 0 \Rightarrow \begin{cases} -c+d=0 \\ -a+b=0 \end{cases} \Rightarrow W = span\{(0,0,1,1),(1,1,0,0)\}$ 令解 $N(A)^{\perp} = R(A^T)$ 所以 $W = span\{(1,1,1,1),(1,1,-1,-1)\}$

Ex 7.28

find the orthogonal complement of the following subspace of R^3 $(a)\{(x,y,z) \mid x+2y+3z=0\}$ $(b)\{(x,y,z) \mid x+y+z=0, x-y+z=0\}$
(a) *let* $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, W = \{(x,y,z) \mid x+2y+3z=0\} = N(A)$ $\Rightarrow W^{\perp} = N(A)^{\perp} = R(A^T) = sapn\{(1,2,3)\}$ (b) *let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, W = \{(x,y,z) \mid x+y+z=0, x-y+z=0\} = N(A)$ $\Rightarrow W^{\perp} = N(A)^{\perp} = R(A^T) = sapn\{(1,1,1),(1,-1,1)\}$

Find the cosine of the angle between each pair of vector u and v $(a) u = (2,3,1), v = (3,-2,0)$ $(b) u = (2,0,1), v = (2,2,-1)$ $(c) u = (0,4,2,3), v = (0,-1,2,0)$
$(a) < x,y >= \|x\|\|y\| \cos\theta \Rightarrow 0 = \|x\|\|y\| \cos\theta \Rightarrow \cos\theta = 0$ $(b) < x,y >= \|x\|\|y\| \cos\theta \Rightarrow 3 = 3\sqrt{5} \cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{5}}$ $(c) < x,y >= \|x\|\|y\| \cos\theta \Rightarrow 0 = \|x\|\|y\| \cos\theta \Rightarrow \cos\theta = 0$

傳立葉係數

The vector v_1,v_2,v_3 form an orthonormal basis for R^3 where $v_1=(0,0,-1)^T, v_2=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)^T, v_3=(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)^T$,what the coordinates of vectors $w=(1,-1,1)^T$,what respect to the basis vector v_1,v_2,v_3 of R^3?
$w=<w,v_1>v_1+<w,v_2>v_2+<w,v_3>v_3=-v_1+0v_2-\frac{2}{\sqrt{2}}v_3$ 所以 w 在這組 basis 的 coordinate 爲 $\begin{bmatrix} -1 \\ 0 \\ -\frac{2}{\sqrt{2}} \end{bmatrix}$

Find an orthogonal basis for the solution set to $2w+x+3y-z=0$
$W=\{(w,x,y,z)\mid 2w+x+3y-z=0\}$ $\forall u=(w,x,y,z)\in W \Rightarrow 2w+x+3y-z=0$ $\Rightarrow z=2w+x+3y$ $W=span\{v_1=(1,0,0,2),v_2=(0,1,0,1),v_3=(0,0,1,3)\}$ $G.S.\quad for\quad v_1,v_2,v_3$ $u_1=v_1=(1,0,0,2),<u_1,u_1>=5$ $u_2=v_2-\frac{<v_2,u_1>}{<u_1,u_1>}u_1=(0,1,0,1)-\frac{2}{5}(1,0,0,2)=(-\frac{2}{5},1,0,\frac{1}{5}),<u_2,u_2>=\frac{6}{5}$ $u_3=v_3-\frac{<v_3,u_1>}{<u_1,u_1>}u_1-\frac{<v_3,u_2>}{<u_2,u_2>}u_2=(0,0,1,3)-\frac{6}{5}(1,0,0,2)-\frac{1}{2}(-\frac{2}{5},1,0,\frac{1}{5})$ $=(-1,-\frac{1}{2},1,\frac{1}{2})$ $pick\quad\{(1,0,0,2),(-\frac{2}{5},1,0,\frac{1}{5}),(-1,-\frac{1}{2},1,\frac{1}{2})\}$

Find an orthonormal basis for the null space of the matrix $A=\begin{bmatrix}1&2&1&1\\0&1&-1&2\\2&5&1&4\\1&1&2&-1\end{bmatrix}$

$$\forall x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\in W\Rightarrow\begin{cases}x_1+x_2-x_3=0\\x_2-x_4=0\end{cases}\Rightarrow\begin{cases}x_1=x_3-x_4\\x_2=x_4\end{cases}$$

$$\begin{bmatrix}1&2&1&1\\0&1&-1&2\\2&5&1&4\\1&1&2&-1\end{bmatrix}\sim\begin{bmatrix}1&2&1&1\\0&1&-1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2+x_3+x_4=0\\x_2-x_3+2x_4=0\end{cases}$$

$$\Rightarrow\begin{cases}x_1=-3x_3+3x_4\\x_2=x_3-2x_4\end{cases}\Rightarrow\ker(A)=span\left\{v_1=\begin{bmatrix}-3\\1\\1\\0\end{bmatrix},v_2=\begin{bmatrix}3\\-2\\0\\1\end{bmatrix}\right\},G.S\quad for\quad v_1,v_2$$

$$u_1=v_1=\begin{bmatrix}-3\\1\\1\\0\end{bmatrix},<u_1,u_1>=11,u_2=v_2-\frac{<v_2,u_1>}{<u_1,u_1>}u_1=\begin{bmatrix}3\\-2\\0\\1\end{bmatrix}-\frac{-11}{11}\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\1\\1\end{bmatrix},<u_2,u_2>=3$$

$$pick\quad\left\{\begin{bmatrix}\frac{-3}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\0\end{bmatrix},\begin{bmatrix}0\\\frac{-1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}\right\}\quad be\quad orthonomal\quad basis\quad for\quad\ker(A)$$

Find an orthonormal basis for the column space of $D=\begin{bmatrix}1&0&0\\1&4&0\\1&4&6\\1&4&6\end{bmatrix}$
$v_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},v_2=\begin{bmatrix}0\\4\\4\\4\end{bmatrix},v_3=\begin{bmatrix}0\\0\\6\\6\end{bmatrix},u_1=v_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},<u_1,u_1>=4$ $u_2=v_2-\frac{<v_2,u_1>}{<u_1,u_1>}u_1=\begin{bmatrix}0\\4\\4\\4\end{bmatrix}-\frac{12}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},<u_2,u_2>=12$ $u_3=v_3-\frac{<v_3,u_1>}{<u_1,u_1>}u_1-\frac{<v_3,u_2>}{<u_2,u_2>}u_1=\begin{bmatrix}0\\0\\6\\6\end{bmatrix}-\frac{12}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\frac{12}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\-4\\2\\2\end{bmatrix},<u_3,u_3>=24$ $pick\quad \left\{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{-3}{\sqrt{12}}\\\frac{1}{\sqrt{12}}\\\frac{1}{\sqrt{12}}\\\frac{1}{\sqrt{12}}\end{bmatrix},\begin{bmatrix}0\\\frac{-2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}\right\}\quad be\quad orthonomal\quad basis\quad for\quad CS(D)$

$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$,do $A=QR$ decomposition
$let\ \ v_1=\begin{bmatrix}0\\1\\1\end{bmatrix},v_2=\begin{bmatrix}1\\0\\1\end{bmatrix},v_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $u_1=v_1=\begin{bmatrix}0\\1\\1\end{bmatrix},<u_1,u_1>=2,u_2=v_2-\frac{<v_2,u_1>}{<u_1,u_1>}u_1=\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix},<u_2,u_2>=\frac{3}{2}$ $u_3=v_3-\frac{<v_3,u_1>}{<u_1,u_1>}u_1-\frac{<v_3,u_2>}{<u_2,u_2>}u_2=\begin{bmatrix}1\\1\\0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{-2}{3}\end{bmatrix},<u_3,u_3>=\frac{4}{3}$ $\begin{cases}v_1=u_1\\v_2=\frac{1}{2}u_1+u_2\\v_3=\frac{1}{2}u_1+\frac{1}{3}u_2+u_3\end{cases}\Rightarrow A=\begin{bmatrix}u_1&u_2&u_3\end{bmatrix}\begin{bmatrix}1&\frac{1}{2}&\frac{1}{2}\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&\frac{1}{3}\\1&-\frac{1}{2}&\frac{1}{3}\\1&\frac{1}{2}&-\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&\frac{1}{2}&\frac{1}{2}\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix}$ $\Rightarrow A=\begin{bmatrix}0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\sqrt{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&\frac{3}{\sqrt{6}}&\frac{1}{\sqrt{6}}\\0&0&\frac{2}{\sqrt{3}}\end{bmatrix}=QR$

Define the inner product on $R_1[x]$ as $<f,g>=\int_0^1 f(t)g(t)dt$, Find the orthogonal projection of $h(x)=4+3x-2x^2$
取 $\beta=\{1,x\}$ 爲 $R_1[x]$ 的 basis Gram-schmidt for $\{1,x\}\Rightarrow\{h_1(x),h_2(x)\}=\{1,x-\frac{1}{2}\}$ 正交投影公式 $\frac{<h_1,h_1>}{<h_1,h_1>}h_1(x)+\frac{<h_1,h_2>}{<h_2,h_2>}h_2(x)$ $=\frac{\int_0^1(4+3x-2x^2)dx}{\int_0^1 1dx}+\frac{\int_0^1(4+3x-2x^2)(x-\frac{1}{2})dx}{\int_0^1(x-\frac{1}{2})^2dx}=x+\frac{13}{3}$

Let W be the subspace of R^3 having basis $\{(1,1,2),(0,-1,3)\}$,determine the projection of the following vectors onto W $(a)(3,-1,2)$ $(b)(1,1,1)$
$let\quad A=\begin{bmatrix}1&0\\1&-1\\2&3\end{bmatrix}$ $,W=CS(A)$ $(a)\because A(A^TA)^{-1}A^T\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}5/7\\13/35\\86/35\end{bmatrix}$,所以 $(3,-1,2)$ 在 W 上的 projection 爲 $\left(\frac{5}{7},\frac{13}{35},\frac{86}{35}\right)$ $(b)\because A(A^TA)^{-1}A^T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}6/7\\38/35\\36/35\end{bmatrix}$,所以 $(1,1,1)$ 在 W 上的 projection 爲 $\left(\frac{6}{7},\frac{38}{35},\frac{36}{35}\right)$

Consider the inner product space C[0,1] with inner product defined by

$<f,g>=\int_0^1 f(x)g(x)dx$,let S be the subspace spanned by the vector 1 and x

Find the best least square approximation to $x^{\frac{1}{3}}$ on [0,1] by a function from the subspace S

Let $W=span\{1,x\}$ use Gram-schmidt for 1 and x

$\left\{u_1=1,u_2=x-\frac{1}{2}\right\}$ be an orthogonal basis for W

欲求 $x^{\frac{1}{3}}$ 在 **W** 上的 **best least square approximation** 即求 $x^{\frac{1}{3}}$ 在 **W** 上的正交投影向量

$$proj_W x^{\frac{1}{3}}=\frac{<x^{\frac{1}{3}},1>}{<1,1>}1+\frac{<x^{\frac{1}{3}},x-\frac{1}{2}>}{<x-\frac{1}{2},x-\frac{1}{2}>}(x-\frac{1}{2})$$

$$=\frac{\int_0^1 x^{\frac{1}{3}}dx}{\int_0^1 1dx}+\frac{\int_0^1 x^{\frac{1}{3}}(x-\frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2dx}(x-\frac{1}{2})=\frac{9}{14}x+\frac{3}{7}$$

Consider the inner product space C[0,1] with inner product defined by

$<f,g>=\int_0^1 f(x)g(x)dx$,let S be the subspace spanned by the vector 1 and 2x-1

Find the best least square approximation to $\sqrt{x}$

$$v_1=1,v_2=2x-1, already\ \ orthogonal\ \ so\ \ no\ \ need\ \ G.S.$$

$$<u_1,u_1>=\int_0^1 1dx=1,<u_2,u_2>=\int_0^1(2x-1)^2dx=\frac{1}{3}$$

$$\frac{<\sqrt{x},1>}{<1,1>}1+\frac{<\sqrt{x},2x-1>}{<2x-1,2x-1>}(2x-1)=\frac{2}{3}+\frac{\sqrt{4x^3}-\sqrt{x}}{\frac{1}{3}}(2x-1)$$

$$=\frac{2}{3}+\frac{\frac{4}{5}-\frac{2}{3}}{\frac{1}{3}}(2x-1)=\frac{2}{3}+\frac{2}{5}(2x-1)=\frac{4}{5}x+\frac{4}{15}$$

Find the point on the plane spanned by two vector (1,1,1) and (-1,0,2) that is closest to the point (1,4,3)
Let $W = span\{[1,1,1],[-1,0,2]\}$, $v = [1,4,3]$,   欲求 W 上與 $v = [1,4,3]$ 最靠近的點相當於求 v 在 W 上的正交投影向量 $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, A(A^T A)^{-1} A^T v = \begin{bmatrix} 2 \\ \frac{5}{2} \\ \frac{7}{2} \end{bmatrix}$ Answer: $(2, \frac{5}{2}, \frac{7}{2})$

(x,y) = (0,1),(3,4),(6,5) ，find the best squares fir by a linear function
$let \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix}, X = \begin{bmatrix} a \\ b \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ 欲求 X 最小使得 $\| AX - B \|$ 爲最小,等價於解 $A^T Ax = A^T b$,因爲 A 行獨立 $\Rightarrow A^T A$ 可逆 $X = A(A^T A)^{-1} A^T = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$, $least \quad square \quad line \quad be \quad y = \frac{2}{3} + \frac{4}{3} x$

Let $W = \{(2t, -5t, 4t \mid t \in R)\}$, find $W^{\perp}$
$pick \quad \{(2,-5,4)\} \quad be \quad a \quad basis \quad of \quad W$ $\forall v = (x,y,z) \in W^{\perp} \Rightarrow 2x - 5y + 4z = 0$ $\Rightarrow 2x = 5y - 4z$ $\Rightarrow pick \quad \{(5,2,0),(-2,0,1)\} \quad be \quad a \quad basis \quad of \quad W^{\perp}$

Let $W=\{(a,b,0,0,0,a+b):a,b\in R\}$ be subspace of R^6 ,please find $W^{\perp}$
$pick \quad \{u=(1,0,0,0,0,1),v=(0,1,0,0,0,1)\} \quad be \quad a \quad basis \quad of \quad W$ $\forall x=(x_1,x_2,x_3,x_4,x_5,x_6)\in W^{\perp}\Rightarrow<x,u>=0,<x,v>=0$ $\begin{cases}x_1+x_6=0\\x_2+x_6=0\end{cases}\Rightarrow pick \quad \{(-1,-1,0,0,0,1),(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0)\}$ $be \quad a \quad basis \quad of \quad W^{\perp}$

Find all vectors in R^4 that are perpendicular to the three vectors $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}1\\9\\9\\7\end{bmatrix}\right\}$
$\forall v=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\in W^{\perp}\Rightarrow<\begin{bmatrix}x\\y\\z\\w\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}>=0,<\begin{bmatrix}x\\y\\z\\w\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix}>=0,<\begin{bmatrix}x\\y\\z\\w\end{bmatrix},\begin{bmatrix}1\\9\\9\\7\end{bmatrix}>=0$ $\begin{cases}x+y+z+w=0\\x+2y+3z+4w=0\\x+9y+9z+7w=0\end{cases}\Rightarrow\left[\begin{array}{cccc\|c}1&1&1&1&0\\1&2&3&4&0\\1&9&9&7&0\end{array}\right]\sim\left[\begin{array}{cccc\|c}1&1&1&1&0\\0&1&2&3&0\\0&0&-8&-18&0\end{array}\right]$ $\Rightarrow\begin{cases}x+y+z+w=0\\y+2z+3w=0\\-8z-18w=0\end{cases}\Rightarrow\begin{cases}x=\frac{-1}{4}w\\y=\frac{6}{4}w\\z=\frac{-9}{4}w\end{cases}\Rightarrow pick \quad \left\{\begin{bmatrix}-1\\6\\-9\\4\end{bmatrix}\right\} \quad be \quad a \quad basis \quad of \quad W^{\perp}$

Let $V=\left\{\begin{bmatrix}a-b & b-c & 0\end{bmatrix}^T \middle\| a,b,c\in R\right\}\subseteq R^3$,find $\dim(W^{\perp})$
$pick \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \quad be \quad a \quad basis\Rightarrow dim(W)=2\Rightarrow\dim(W^{\perp})=3-dim(W)=1$

Let P be the space spanned by $u_1=(-1,0,1,2),u_2=(0,1,0,1)$,if $w=(-1,2,6,0)$ Can be expresses in the form $w=w_1+w_2$,where w_1 is in the space P and w_2 is orthogonal to P,, please find w_1,w_2
$pick\ \ A=\begin{bmatrix}-1&0\\0&1\\1&0\\2&1\end{bmatrix}\Rightarrow P=CS(A)$,w 在 P 的正交投影向量爲 $A(A^TA)^{-1}A^T\begin{bmatrix}-1\\2\\6\\0\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-5\\-1\\5\\9\end{bmatrix}\Rightarrow w_1=\frac{1}{4}(-5,-1,5,9),w_2=w-w_1=\frac{1}{4}(1,9,19,-9)$

96 暨大資工

Use Gram-schmidt process to determine orthonormal basis for $W=\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}2\\1\\-2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$
$let\ \ W=\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}2\\1\\-2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}=\{v_1,v_2,v_3\}$ $u_1=v_1=\begin{bmatrix}1\\0\\1\\0\end{bmatrix},<u_1,u_1>=2,u_2=v_2-\frac{<v_2,u_1>}{<u_1,u_1>}u_1=\begin{bmatrix}2\\1\\-2\\0\end{bmatrix}-\frac{0}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\1\\-2\\0\end{bmatrix},<u_2,u_2>=9$ $u_3=v_3-\frac{<v_3,u_1>}{<u_1,u_1>}u_1-\frac{<v_3,u_2>}{<u_2,u_2>}u_2=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}-\frac{0}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}-\frac{0}{9}\begin{bmatrix}2\\1\\-2\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},<u_3,u_3>=1$ $orthonormal\ \ basis\ \ =\{\frac{u_1}{\Vert u_1\Vert},\frac{u_2}{\Vert u_2\Vert},\frac{u_3}{\Vert u_3\Vert}\}=\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix},\begin{bmatrix}2/3\\1/3\\-2/3\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$

Let S be the subspace of R^4 spanned by $\left\{ x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(a) find $\{x_3, x_4\}$ for S such that $\{x_1, x_2, x_3, x_4\}$ is orthogonal basis for R^4

(b) express $y = (1,2,3,4)^T$ into the combination of x_1, x_2, x_3, x_4

(a) $S = \{x_1, x_2\}, \because R^4 = S \oplus S^{\perp} \Rightarrow pick \quad S^{\perp} = span\{x_3, x_4\}$

$$\forall v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in W^{\perp} \Rightarrow < \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} >= 0, < \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} >= 0$$

$$\begin{cases} y + w = 0 \\ x - z = 0 \end{cases} \Rightarrow pick \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad be \ a \ basis \ of \ W^{\perp}$$

$$pick \quad \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad be \ a \ orthogonal \ basis \ of \ R^4$$

(b)

$$y = \frac{< y, x_1 >}{< x_1, x_1 >} x_1 + \frac{< y, x_2 >}{< x_2, x_2 >} x_2 + \frac{< y, x_3 >}{< x_3, x_3 >} x_3 + \frac{< y, x_4 >}{< x_4, x_4 >} x_4$$

$$= \frac{6}{2} x_1 + \frac{-2}{2} x_2 + \frac{4}{2} x_3 + \frac{2}{2} x_4 = 3x_1 - x_2 + 2x_3 + x_4$$

96 清大資應

$V = span\{(2,0,-1,1)^T,(1,1,0,1)^T\}$ (a) Find an orthonormal basis for V (b) $b=(1,1,-3,1)^T$,use your answer (a) to find the projection p of b onto V (c) Let $A=\begin{bmatrix}2&1\\0&1\\-1&0\\1&1\end{bmatrix}$,find the least square solution
$(a)u_1=v_1=\begin{bmatrix}2\\0\\-1\\1\end{bmatrix},<u_1,u_1>=6,u_2=v_2-\frac{<v_2,v_1>}{<u_1,u_1>}v_1=\begin{bmatrix}1\\1\\0\\1\end{bmatrix}-\frac{3}{6}\begin{bmatrix}2\\0\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1/2\\1/2\end{bmatrix},<u_1,u_1>=\frac{3}{2}$ $orthonormal\ \ basis\ \ \{w_1=\frac{u_1}{\lVert u_1\rVert},w_2=\frac{u_2}{\lVert u_2\rVert}\}=\left\{\begin{bmatrix}\frac{2}{\sqrt6}\\0\\\frac{-1}{\sqrt6}\\\frac{1}{\sqrt6}\end{bmatrix},\begin{bmatrix}0\\\frac{2}{\sqrt6}\\\frac{1}{\sqrt6}\\\frac{1}{\sqrt6}\end{bmatrix}\right\}$ $(b)\overline{p}=<b,w_1>w_1+<b,w_2>w_2=\begin{bmatrix}2\\0\\-1\\1\end{bmatrix}$ $(c)solve\ \ \ A^TA\vec{x}=A^T\vec{b}\Rightarrow\vec{x}=(A^TA)^{-1}A^T\vec{b}=\begin{bmatrix}1\\0\end{bmatrix}$

96 中興資工

Find the least-square linear approximation to $f(x)=e^x$ over $[-1,1]$
$W=\{a+bx\mid a,b\in R\}=span\{1,x\}$ 定義內積$<f,g>=\int_{-1}^{1}f(x)g(x)dx$ 對 $\{1,x\}$ 做 Gram-schmidt process 得 $\{1,x\}$ $proj_we^x=\frac{<e^x,1>}{<1,1>}1+\frac{<e^x,x>}{<x,x>}x=\frac{\int_{-1}^{1}e^xdx}{\int_{-1}^{1}1dx}+\frac{\int_{-1}^{1}xe^xdx}{\int_{-1}^{1}x^2dx}x=\frac{1}{2}(e-e^{-1})+3e^{-1}x$

96 高大電機

Let W be the subspace of R^4 spanned by $A_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ Compute the projection of B onto W for $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, W = CS(A), proj_w \overline{B} = A(A^T A)^{-1} A^T \overline{B} = \begin{bmatrix} 2 \\ 16/5 \\ 2 \\ 8/5 \end{bmatrix}$

96 中興資工

Consider the vector $v = (3,2,6)$ in R^3 let $W = \{(a,b,b) \mid a,b \in R\}$ Find the projection of v onto W
法(一) Let $W = span\{(1,0,0),(0,1,1)\}$ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, W = CS(A), proj_w \vec{v} = A(A^T A)^{-1} A^T \vec{v} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$ 法(二) 此題$W = span\{(1,0,0),(0,1,1)\}$ 已正交 代 $\frac{<v,v_1>}{<v_1,v_1>} v_1 + \frac{<v,v_2>}{<v_2,v_2>} v_2 = (3,0,0) + (0,4,4) = (3,4,4)$ 較快

95 中正資工

Give a plane $x-y-2z=0$ Find the matrix P with projects any vector $b \in R^3$ onto the plane
$let\ \ W: x-y-2z=0$ $\Rightarrow W = span\{(1,1,0),(2,0,1)\}$ $let\ \ A=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}, P=A(A^TA)^{-1}A^T=\frac{1}{6}\begin{bmatrix}5&1&2\\1&5&-2\\2&-2&2\end{bmatrix}$

93 政大應數

Let S be the vector space in R^4 spanned by $\{(1,0,1,0),(0,1,0,1)\}$, $v=(1,2,3,4)$ (1) find the orthogonal projection of v onto S (2) find the minimum distance from v to S
$(a) let\ \ v_1=(1,0,1,0), v_2=(0,1,0,1)$ $\because < v_1, v_2 > = 0, \therefore pick\{v_1, v_2\}\ \ be\ \ orthogonal\ \ basis$ $proj_S v = \frac{< v, v_1 >}{< v_1, v_1 >} v_1 + \frac{< v, v_2 >}{< v_2, v_2 >} v_2 = (2,3,2,3)$ $(b) \| v - proj_S v \| = \| (-1,-1,1,1) \| = 2$

92 輔大資工

Let $S = sapn\{(3,1,-1,1),(1,-1,1,-1)\}$ be a subspace of R^4 and $b=(3,1,5,1)$ (a) find the projection of b onto S (b) compute the distance from b to S
(a)因爲 $<(3,1,-1,1),(1,-1,1,-1)> = 0$ 所以 pick $\{w_1=(3,1,-1,1), w_2=(1,-1,1,-1)\}$ be a orthogonal basis $proj_S b = \frac{< b, w_1 >}{< w_1, w_1 >} w_1 + \frac{< b, w_2 >}{< w_2, w_2 >} w_2 = \frac{6}{12}(3,1,-1,1) + \frac{6}{4}(1,-1,1,-1)$ $= (3,-1,1,-1)$ (b) distance from b to S➔ $\| b - proj_S b \| = \| (0,2,4,2) \| = \sqrt{24}$

89 交大統計

Let S span by $v_1=(1,2,1,2)^T, v_2=(2,3,1,2)^T, v_3=(3,4,-1,0)^T, v_4=(3,4,0,1)^T$ Find the orthogonal projection of $v=(1,0,0,1)^T$ onto S
$\begin{bmatrix}1&2&1&2\\2&3&1&2\\3&4&-1&0\\3&4&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$ 陷阱：**span** 不表示爲基底   ➔ pick $\{w_1=(1,0,1,0)^T, w_2=(0,1,0,1)^T, w_3=(0,0,1,1)^T\}$ be basis   Grad-schmidt for $\{w_1,w_2,w_3\}$ ➔ $\left\{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0)^T,(0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})^T,(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})^T\right\}$ $proj_S b=\frac{<b,u_1>}{<u_1,u_1>}u_1+\frac{<b,u_2>}{<u_2,u_2>}u_2+\frac{<b,u_3>}{<u_3,u_3>}u_3=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$

93 台科資工

find the projection of vector $\begin{bmatrix}2\\1\\4\end{bmatrix}$ onto the subspace spanned by $\beta=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$
$let \quad A=\begin{bmatrix}1&0\\0&2\\1&0\end{bmatrix}, b=\begin{bmatrix}2\\1\\4\end{bmatrix} \quad \Rightarrow A \quad column \quad independent$ $\Rightarrow proj_S b=A(A^TA)^{-1}A^Tb=\begin{bmatrix}3\\1\\3\end{bmatrix}$

92 台大電機

$A=\begin{bmatrix}1&1\\1&0\\1&1\\1&1\end{bmatrix},b=\begin{bmatrix}0\\0\\0\\3\end{bmatrix}$ find : (a) the normal equation (b) the least squares solution (or solutions) of the system (c) the projection of b onto the span of the columns of *A* (d) the orthogonal projection matrix for the span of the columns of *A*.
$(a)A^TAx=A^Tb,or\quad\begin{bmatrix}4&3\\3&3\end{bmatrix}x=\begin{bmatrix}3\\3\end{bmatrix}$ (b)因為 A 行獨立 ➔ $A^TAx=A^Tb\Rightarrow x=(A^TA)^{-1}A^Tb=\begin{bmatrix}0\\1\end{bmatrix}$ $(c)proj_{CS(A)}b=A(A^TA)^{-1}A^Tb=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ (d)CS(A)上的 orthogonal projection matrix $P=A(A^TA)^{-1}A^T=\frac{1}{3}\begin{bmatrix}1&0&1&1\\0&3&0&0\\1&0&1&1\\1&0&1&1\end{bmatrix}$

93 交大資科

$S=\{(1,1,0),(1,1,1)\}$,find the orthogonal projection of $(1,0,0)$ onto the subspace S
$let\quad A=\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix},W=CS(A),b=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad\Rightarrow A\ \ column\ \ independent$ $\Rightarrow\because A(A^TA)^{-1}A^Tb=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix},\therefore proj_Wb=(\frac{1}{2},\frac{1}{2},0)$

$(a) A \in C^{n \times n}$ A 可么正對角化 $\Leftrightarrow$ A :normal matrix

$(b) A \in R^{n \times n}$ **A 可正交對角化 $\Leftrightarrow$ A :symmetrix matrix**

六大算子

	T:self adjoint A:Hermitian	T: skew self adjoint A:skew Hermitian	T,A:positive definite
定義	$T^* = T$ $A^H = A$	$T^* = -T$ $A^H = -A$	$< T(\vec{x}), \vec{x} > \; > \; 0$ $\vec{x}^H A\vec{x} \; > \; 0, \forall \vec{x} \neq 0$
eigenvalue	$\forall \lambda \in R$	$\lambda = 0$ or 純虛數	$\forall \lambda > 0$
對角項	$\forall a_{ii} \in R$	$\forall a_{ii} = 0$ or 純虛數	$\forall a_{ii} > 0$
行列式	$\det(A) \in R$	$\begin{cases} \in R \text{ , } if \quad n \quad even \\ 0, \text{純虛數} \text{ , } if \quad n \quad odd \end{cases}$	$\det(A) > 0$
$\lambda_1 \neq \lambda_2$ 之 eigenvector x_1, x_2	$x_1 \perp x_2$	$x_1 \perp x_2$	$x_1 \perp x_2$

	T,A:positive semidefinite	T,A: Unitary in C	T,A: Orthogonal in R
定義	$< T(\vec{x}), \vec{x} > \; \geq \; 0$ $\vec{x}^H A\vec{x} \; \geq \; 0, \forall \vec{x} \neq 0$	$T^* T = I$ $A^H A = I$	$T^* T = I$ $A^T A = I$
eigenvalue	$\forall \lambda \geq 0$	$\forall \lvert \lambda \rvert = 1$	$\forall \lambda = \pm 1$
對角項	$\forall a_{ii} \geq 0$	?	?
行列式	$\det(A) \geq 0$	$\lvert \det(A) \rvert = 1$	$\det(A) = \pm 1$
$\lambda_1 \neq \lambda_2$ 之 eigenvector x_1, x_2	$x_1 \perp x_2$	$x_1 \perp x_2$	$x_1 \perp x_2$

Ex 8-1

$A=\begin{bmatrix}0&2&-1\\2&3&-2\\-1&-2&0\end{bmatrix}$, find a orthogonal matrix $U \ni A=UDU^T$,D:diagonal matrix
$Char_A(x)=-(x+1)^2(x-5)\Rightarrow \lambda(A)=\{-1,5\}$ $V(-1)=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\},V(5)=span\left\{\begin{bmatrix}-1\\-2\\1\end{bmatrix}\right\}$ $Gram-schmidt\quad for\quad \left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}\Rightarrow\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ $pick\quad U=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&\frac{-2}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{bmatrix},then\quad U^TAU=D=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&5\end{bmatrix}$

Ex 8-2

find a unitary matrix U that diagonalizes $A=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix}$
$V(2)=span\left\{v_1=\begin{bmatrix}-1\\1\\0\end{bmatrix},v_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\},V(8)=span\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\},G.S\quad v_1,v_2\Rightarrow\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\right\}$ $pick\quad U=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\end{bmatrix},then\quad U^TAU=D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&8\end{bmatrix}$

Ex 8-4

Determine folloing matrices are positive definite,negative definite,or indefinite $(a)\begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}$ $(b)\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ $(c)\begin{bmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{bmatrix}$
$(a)Char(x)=(x-2)(x-5)\Rightarrow all \quad eigenvalue \quad >0\Rightarrow positive \quad definite$ $(b)Char(x)=(x+3)(x+1)^2\Rightarrow all \quad eigenvalue \quad <0\Rightarrow negative \quad definite$ $(c)\det\left[6\right]=6>0,\det\begin{bmatrix} 6 & 4 \\ 4 & 5 \end{bmatrix}=14>0,\det\begin{bmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{bmatrix}=-38>0,\Rightarrow indefinite$

Ex 8-5

Fire 2 method explain following is semidefinite matrix $A=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$
$(1)\Delta_1(A)=2\geq 0,\Delta_2(A)=\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}=5\geq 0$ $\Delta_3(A)=\det\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}=0\geq 0$ $(2)p_A(x)=-x(x-3)^2,\lambda(A)=\{0,3\}\geq 0$

(1) HouseHolder matrix

假設 $w\in R^{n\times 1}, w\neq 0, H=I-\dfrac{2}{w^T w}ww^T$ 爲相對於 w 的 HouseHolder matrix

或 HouseHolder transformation 或 elementary reflextor

p.s. 當 w 爲單位向量 ➔ $H=I-2ww^T$

Ex 8-6

$w=\begin{bmatrix}1\\2\end{bmatrix}$,求相對於 w 的 HouseHolder matrix ,並說明幾何意義
$H=I-\frac{2}{w^Tw}ww^T=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\frac{2}{5}\begin{bmatrix}1&2\\2&4\end{bmatrix}=\begin{bmatrix}3/5&-4/5\\-4/5&-3/5\end{bmatrix}$ w 爲 與直線 x+2y=0 正交的向量 所以 H 爲對直線 x+2y=0 鏡射的算子

Ex 8-7

In R^3,let $W=\{(x,y,z)\mid x+3y-2z=0\}$,T is a reflection of R^3 about W,find $T(x,y,z)$
$pick\ \ (1,3,-2)\in W^{\perp},let\ \ w=\begin{bmatrix}1\\3\\-2\end{bmatrix}$ $H=I-\frac{2}{w^Tw}ww^T=\begin{bmatrix}6/7&-3/7&2/7\\-3/7&-2/7&6/7\\2/7&6/7&3/7\end{bmatrix},H\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{6x-3y+2z}{7}\\\frac{-3x-2y+6z}{7}\\\frac{2x+6y+3z}{7}\end{bmatrix}$ $T(x,y,z)=(\frac{6x-3y+2z}{7},\frac{-3x-2y+6z}{7},\frac{2x+6y+3z}{7})$

Ex 8-8

In R^3,let W be the plain x+z=0, find $T(x,y,z)$ where T is reflection of about W
$pick\ \ u=(1,0,1)\in W^{\perp},let\ \ w=\begin{bmatrix}1\\0\\1\end{bmatrix}$ $H=I-\frac{2}{w^Tw}ww^T=\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix},H\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-z\\y\\-x\end{bmatrix}\Rightarrow T(x,y,z)=(-z,y,-x)$

Ex 8-11

Find SVD of $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

(4)

$$\ker(C-16I) = span\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}, pick \quad v_1 = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\ker(C-9I) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, pick \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker(C-4I) = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, pick \quad v_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$let \quad u_1 = \frac{1}{\sigma_1} Bv_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, u_2 = \frac{1}{\sigma_2} Bv_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \frac{1}{\sigma_3} Bv_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$pick \quad U = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$B = U \sum V^T$ *be singular value decomposition of* B