

國立交通大學 108 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(1102)

考試日期：108 年 2 月 13 日 第 2 節

系所班別：資訊聯招

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【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

1. (8 points) Let V be the vector space of 2×2 matrices with real entries, and P_3 the vector space of real polynomials of degree 3 or less. Define the linear transformation $T: V \rightarrow P_3$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + (b-d)x - (a+c)x^2 + (a+b-c-d)x^3.$$

Find the rank and nullity of T .

2. (5 points) Let A be an $n \times n$ matrix with real entries and n is odd. Show that it is not possible for $A^2 + I = O$, in which I is the identity matrix and O is the zero matrix.
3. Let $C[-1, 1]$ be the vector space over R of all continuous functions defined on the interval $[-1, 1]$. Let $V = \{f(x) \in C[-1, 1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$.
- (2 points) Prove that V is a subspace of $C[-1, 1]$.
 - (5 points) Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V .
 - (5 points) Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V .

4. Given $A = \begin{bmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 6 \end{bmatrix}$.

If the Gram-Schmidt process is applied to determine an orthonormal basis for $R(A) = \{b \in R^m | b = A_{mn}x\}$ and QR factorization of A , then, after the first one orthonormal vector q_1 and r_{11} are computed, we have

$$Q = [q_1 \quad q_2 \quad q_3] = \begin{bmatrix} 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \end{bmatrix} \text{ and } R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & - & - \\ 0 & - & - \\ 0 & - & - \end{bmatrix}.$$

- (5 points) Finish above process and determine q_2 and q_3 , and fill in the columns of Q .
 - (5 points) Finish above process and determine R .
 - (5 points) Use the QR factorization to find the least squares solution to $Ax = b$.
5. Let $C = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ is a 3×3 matrix.
- (5 points) Find the value of $\lim_{n \rightarrow \infty} C^n$.
 - (5 points) Compute the value of e^C .
6. Let $Q(x)$ be the statement " $x + 1 > 2x$." If the domain consists of all integers, what are these truth values?
- (2 points) $\exists x Q(x)$
 - (2 points) $\forall x Q(x)$
 - (2 points) $\exists x \neg Q(x)$
 - (2 points) $\forall x \neg Q(x)$
7. Determine whether each of the following conditional statements is a tautology or not. If yes, provide a proof. If no, provide a counter example.
- (3 points) $p \rightarrow (\neg q \vee r)$
 - (3 points) $\neg p \rightarrow (q \rightarrow r)$
 - (3 points) $(p \rightarrow q) \vee (\neg p \rightarrow r)$
8. Determine whether each of these functions is a bijection from R to R .
- (2 points) $f(x) = -3x + 4$
 - (2 points) $f(x) = -3x^2 + 7$
 - (2 points) $f(x) = (x + 1)/(x + 2)$
 - (2 points) $f(x) = x^5 + 1$

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9. Let R and B be sets of red and blue balls, respectively, with $m = |R|$ and $n = |B|$. Suppose that $m < n$. Let $G = (V, E)$ be a undirected graph with vertex set V and edge set E such that $V = R \cup B$ and $E = \{(u, v) \mid u \in R \text{ and } v \in B\}$. Let E' be an edge cut of G with the minimum number of edges.
- (4 points) What is the value of $|E'|$?
 - (6 points) Let $G' = G - E'$. What is the value of $\sum_{v \in G'} \deg(v)$, where $\deg(v)$ is the node degree of v ?
10. (5 points) Let S be the set of all bit strings of length n . Let \angle be a binary relation defined on S such that $a \angle b$ if and only if a and b differ in exactly k bit positions for any $a, b \in S$. Let $R = \{(a, b) \mid a \angle b, a, b \in S\}$. If we represent R using a zero-one matrix, how many 1's are there in the matrix?
11. A DNA sequence of length n is a sequence of n molecules, where each molecule is represented by either 'C', 'G', 'A', or 'T'. Consider a sequence a_1, a_2, \dots, a_n , where a_i , $1 \leq i \leq n$, denotes the number of DNA sequences of length i that contain two consecutive 'G's. We may express a_n as a recurrence relation
- $$a_n = s \times a_{n-1} + t \times a_{n-2} + f(n),$$
- where s and t are positive integers and $f(n)$ is a function of n .
- (3 points) $a_4 = ?$
 - (3 points) What is the value of $s + t$?
 - (4 points) What is the definition of $f(n)$?