國立交通大學 98 學年度碩士班考試入學試題

科目：線性代數與離散數學(1002)　　考試日期：98年3月15日 第 2 節
系所班別：資訊系所跨組聯招　　組別：資訊聯招　　第 1 頁，共 5 頁
【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

離散數學 **Discrete Mathematics** There are two problems in the discrete mathematics part. The first problem consists of 6 independent questions, as 1.1, 1.2, 1.3, 1.4, 1.5 and 1.6. The second problem consists of 8 questions. Please answer them in order. For each question, you need to write down your answer first and then explain why.

Problem 1.

1.1. (5 points) When we toss a coin, we obtain either head or tail. Now we toss a coin 5 times. There are 2^5 possible outcomes. How many of them contain no two consecutive heads?

1.2. (4 points) Let A be the set $\{1, 2, 3\}$ and B be the set $\{3.14, 2.71\}$. Let the notation 2^X denote the set of all subsets of X (assume X is a set). Let the notation $X \times Y$ denote the Cartesian product of the two sets X and Y. How many elements are there in the set $2^{2^A \times 2^B}$? You need to write down your answer first and explain why.

1.3. (4 points) How many partitions are there on a set of 4 elements?

1.4. (4 points) What is the smallest (positive) number n satisfying:

- When divided by 2, the result is a square.
- When divided by 3, the result is a cube.

1.5. (4 points) Let a, b, c, d be positive integers. Assume $a^3 = b^2$ and $c^3 = d^2$. If $c - a = 25$, what are a, b, c, d?

1.6. (4 points) Let a, and b be two symbols. The natation a^3 denotes the string aaa, that is, a string of three a's. Similarly, the notation a^4 denotes the string of four a's. Similarly, the notation a^k denotes the string of k a's. Find a 1-1 mapping from $\mathcal{N}$ to $\{a^k b^{jk} \mid j, k \in \mathcal{N}\}$.

Problem 2.

2.1 (3 points) The complement of a simple graph G is the simple graph $\overline{G}$ with the same vertices as G. An edge exists in $\overline{G}$ if and only if it does not exist in G. If G is a simple graph with 11 edges and its complementary graph $\overline{G}$ has 10 edges, then how many vertices does G have?

2.2 (3 poins) Is there a unique binary tree with 6 vertices whose preorder vertex listing is ABCEFD and whose inorder vertex listing is ACFEBD. Justify your answer.

2.3 (3 points) N_h is defined as the minimum number of vertices in a balanced binary tree of height h. Find N_2, N_3.

2.4 (3 points) Let n(T) denote the number of vertices in a full binary tree T and h(T) the height of T. Find the value range of n(T) in terms of h(T).

2.5 (3 points) Under what conditions is an edge in a connected graph G contained in every spanning tree of G.

2.6 (3 points) Let a_r denote the number of bacteria there are on the *r*th day in a controlled environment. We define the rate of growth on the *r*th day to be a_r-2a_{r-1}. If the rate of growth doubles every day, formulate the recurrence relation a_r, given that $a_0 = 1$.

2.7 (3points) F_n is the nth Fibonacci number, where n is a positive number.

Compute $F_{n+1}F_{n-1} - (F_n)^2$

2.8 (4 points) Let (A, ★) and (B, ♦) be two algebraic systems with operators ★ and ♦ defined on A and B, respectively. *f* is called a homomorphism from(A, ★) to (B, ♦) if there exists a function *f* from A onto B such that for any x and y in A *f*(x★y)=*f*(x)♦*f*(y) Justify briefly whether such a homomorphism exists.

A=

★	a	b	c	d
a	a	a	d	c
b	b	a	c	d
c	c	b	a	b
d	d	d	b	a

B=

♦	α	β
α	α	β
β	β	α

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線性代數

3. (**3 points**; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分爲負時，以 0 分計；未作答亦以 0 分計) Assume $\mathbf{A}$ is an $m \times n$ matrix with rank r and $\mathbf{b}$ is a column vector. Which statements are true?
   A. If $m > r$ and $n = r$, then $\mathbf{Ax} = \mathbf{b}$ must have no solution for some $\mathbf{b}$ and exactly one solution for other $\mathbf{b}$.
   B. If $m > r$ and $n > r$, then $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions for some $\mathbf{b}$ and exactly one solution for other $\mathbf{b}$.
   C. If $n = r$, then $\mathbf{Ax} = \mathbf{b}$ has either one solution or none.

4. (**5 points**; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分爲負時，本題以 0 分計；未作答亦以 0 分計) Suppose $\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. Let $S_{12} = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$ and $S_{23} = \text{span}(\mathbf{q}_2, \mathbf{q}_3)$. Which statements are true?
   A. The union of the two subspaces S_{12} and S_{23} forms a vector space.
   B. The intersection of the two subspaces S_{12} and S_{23} forms a vector space.
   C. The span($\mathbf{q}_1$) is an orthogonal complement of the subspace S_{23}.
   D. The rows of $\mathbf{Q}$ form a basis for the row space.
   E. The dimension of the row space of $\mathbf{Q}$ is 3.

5. (**4 points**; 複選，答對每個選項得 1 分，答錯每個選項扣 1 分；本題合計得分爲負時，以 0 分計；未作答亦以 0 分計) Which statements are correct?
   A. Assume V and W are vector spaces and $L: V \to W$ is a linear transformation. Let $\ker(L)$ denote the kernel of L and $L(S)$ denote the image of S for any subspace S of V. If $\dim(V) = n$ and $\dim(W) = m$, then $\dim(\ker(L)) + \dim(L(V)) = m$. (Assume n and m are finite.)
   B. Using the same notations in the previous question, if $\mathbf{x} \in \ker(L)$, then $L(\mathbf{v} + \mathbf{x}) = L(\mathbf{v})$ for any $\mathbf{v} \in V$.
   C. Let P_3 be the space consisting of all polynomial of degree no more than 3, and D be the differentiation operator on P_3. Then, $\ker(D) = \{\mathbf{0}\}$.
   D. If $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, then $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{B} - \lambda\mathbf{I})$ for any scalar λ.

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6. (**6 points**; 複選, 答對每個選項得2分, 答錯每個選項扣2分; 本題合計得分爲負時, 本題以 0 分計; 未作答亦以 0 分計) Which statements are correct?

   A. Let $\mathbf{u}_1 = (-1,2,1)$, $\mathbf{u}_2 = (1,1,-2)$, $\mathbf{v} = (10,5,10)$, and $S = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$. The (shortest) distance between $\mathbf{v}$ and S is $\frac{17\sqrt{30}}{7}$.

   B. For the same setting in the previous question, the least square solution of the system $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{v}$ is $x_1 = \frac{11}{7}$ and $x_2 = -\frac{4}{7}$.

   C. Let V be an inner product space, and $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ denote the inner product of any two vectors $\mathbf{u}_1, \mathbf{u}_2 \in V$. If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an ordered basis of V, then for any vector $\mathbf{u} \in V$, the coordinate of $\mathbf{u}$ can be given by $[\mathbf{u}]_B = [\langle \mathbf{u}, \mathbf{v}_1 \rangle \quad \langle \mathbf{u}, \mathbf{v}_2 \rangle \quad \cdots \quad \langle \mathbf{u}, \mathbf{v}_n \rangle]^T$.

7. (**6 points**; 複選, 答對每個選項得2分, 答錯每個選項扣2分; 本題合計得分爲負時, 本題以 0 分計; 未作答亦以 0 分計) Which statements are correct?

   A. Assume $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{B}$ is a $m \times p$ matrix. If $\mathbf{X}$ is an $n \times p$ unknown matrix, then the system $\mathbf{A}^T\mathbf{A}\mathbf{X} = \mathbf{A}^T\mathbf{B}$ always has a solution. (Here we assume $m \geq n$.)

   B. The matrix $\begin{bmatrix} -\frac{7}{25}i & \frac{24}{25}i \\ \frac{24}{25}i & \frac{7}{25}i \end{bmatrix}$ possesses a complete orthonormal set of eigenvectors.

   C. $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not defective.

8. (**6 points**; 複選, 答對每個選項得2分, 答錯每個選項扣2分; 本題合計得分爲負時, 本題以 0 分計; 未作答亦以 0 分計) Which statements are true?

   A. Assume $\mathbf{A}_{3x3} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ and $\mathbf{B}_{3x3} = \begin{bmatrix} 2\mathbf{a}_2{}^T \\ \mathbf{a}_1{}^T + \mathbf{a}_2{}^T + \mathbf{a}_3{}^T \\ \mathbf{a}_3{}^T \end{bmatrix}$. If det $\mathbf{A}$ =2, then det($\mathbf{A}\mathbf{B}^{-1}$)= 1.

   B. If $\mathbf{P}_{3x3}$ is a projection matrix that projects any vector in $\mathbf{R}^3$ onto the vector

$\mathbf{u}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, then there must be two eigenvectors that correspond to the eigenvalue of 0.

C. If $\mathbf{A}$ is a 3×3 matrix with 3 distinct eigenvalues 0, 1, 2, then the matrix $(\mathbf{A}+\mathbf{I})$ must be invertible.

9. **(10 points)** Suppose there is an election every year in a country and the total population of this country remains fixed. If 60% of the people voted for K Party whereas 40% of the people voted for D Party in the election last time. However, 8% of K Party voters and 4% of D Party voters change their minds and vote for the rival party each year. What will the percentages of K Party and D Party voters be after n years, when n approaches infinity?

10. Consider the vector space $C[0,1]$ with inner product defined by $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, where $C[0,1]$ denotes the set of all real-valued functions that are defined and continuous on the closed interval [0,1].

    A. **(5 points)** Use the Gram-Schmidt process to find an orthonormal basis for the subspace S spanned by 1, and x.

    B. **(5 points)** Find the best least squares approximation to e^x on the interval [0,1] by a function in S.