CH5、遞迴關係

遞迴關係與其應用問題

目錄：

5.1 遞迴關係式

$a_n = a_{n-1} + a_{n-2} + n$

$a_0 = 0, a_1 = 1$ //Initial Condition/Bounded Conition

令 $n=2^k$

$a_n = 2a_n/2 + 2 = 2(2a_{n/4} + 2) + 2$

$= 2^2 a_{n/(2^2)} + 2^2 + 2$

$= 2^3 a_{n/(2^3)} + 2^3 + 2^2 + 2$

$= 2^{k-1} a_{n/[2^(k-1)]} + 2^{k-1} + \ldots + 2^2 + 2$

$= 2[(2^k-1)/(2-1)] - 2^{k-1} = 2\times2^k-2-2^{k-1} = 2n-2-n/2$

5.2 常係數線性遞迴

$c_n a_n + c_{n-1}a_{n-1} + \ldots + c_{n-k}a_{n-k}\ldots(*)$

其中 $c_n, \ldots, c_{k-1}$ 為 Constant，且 $c_n ,\ldots , c_{n-k} \neq 0$，稱為 k 階常係數線性遞迴

當 f(n)=0，稱其為 Homogeneous，反之：Non-homogenous

一、求齊次解

取 $a_n = A\alpha_n$，代入(*)

$c_n A\alpha_n + c_{n-1}A\alpha_{n-1} + \ldots + c_{n-k}A\alpha_{n-k} = 0$

$A\alpha_{n-k}(c_n\alpha_k + \ldots + c_{n-k}\alpha_0) = 0$

稱為(*)之特徵方程式，具 k 個根

Case 1：相異根

α 具有相異根 $\alpha_1, \ldots, \alpha_k$，則 $a_n=d_1\alpha_1 + \ldots + a_k\alpha_{kn}$

例(99 清大)：$a_n=2a_{n-1}+5a_{n-2}-6a_{n-3}, a_0=7, a_1=-4, a_2=8$

$x^3-2x^2-5x+6 = (x-1)(x^2-x-6) = (x-1)(x-3)(x+2)$

$a_n = -3^n + 3(-2)^n + 5$

Case 2：重根

α 具有 r 個根 $\alpha_1, \ldots, \alpha_k$，$\alpha_i$ 具有重數 m_i

例(99 中正)：$a_n=6a_{n-1}-5a_{n-2}, a_0=2, a_1=9$

$\alpha^2-6\alpha+9 =0 \Longrightarrow (\alpha-3)^2=0$

$a_n=c_0 3^n + c_1 n3^n \Longrightarrow a_n=2\times3^n + 3^n\times n, n\geq0$

Case 3：共軛複根

α 具有共軛複根：$\alpha_1=\delta + i\omega$、$\alpha_2=\delta - i\omega, \omega\neq0$

$a_n = c_1 d_1^n + c_2 d_2^n = c_1(\delta + i\omega)^n + c_2(\delta - i\omega)^n$

Euler Formula：$e^{i\theta} = \cos\theta + i\sin\theta$

$e\pi i + 1 = 0, \alpha_1=\delta + i\omega = p\cos\theta + ip\sin\theta = p(\cos\theta + i\sin\theta) = pe^{i\theta}$

$a_n = c_1(pe^{i\theta})^n + c_2(pe^{-i\theta})^n = c_1p^n e^{in\theta} + c_2p^n e^{-in\theta}$

$= c_1p^n[\cos(n\theta) + i\sin(n\theta)] + c_2p^n[\cos(n\theta) + i\sin(n\theta)]$

$= p^n[(c_1+c_2)\cos(n\theta) + i(c_1-c_2)\sin(n\theta)]$

$= p^n[b_1\cos(n\theta) + b_2\sin(n\theta)]$

例(98 逢甲)：$a_n - a_{n-1} + 4a_{n-2} = 0,\ a_1=2,\ a_2=0$

$\alpha^2 - 2\alpha + 4 = 0 \Longrightarrow \alpha = 1 \pm \sqrt{(3)}i$
$a_n = 2^n[B_1\cos(n\pi/3) + B_2\sin(n\pi/3)] \Longrightarrow B_1=1,\ B_2=[\sqrt{(3)}]/3$
$\Longrightarrow a_n = 2^n[\cos(n\pi/3) + 1/\sqrt{(3)}\sin(n\pi/3)],\ \forall n \geq 1$

二、非齊次, $f(n) \neq 0$
$a_n = a_n^{(h)} \quad + \quad a_n^{(p)}$
齊次解　特解，與 $f(n)$有關

Case 1：多項式
若 $f(n)=c_0+c_1n+\ldots+c_mn^m,\ c_m \neq 0$
$a_n^{(p)} = n^r(d_0+d_1n+\ldots+d_mn^m)$，其中 r 為特徵根 1 之重數

例(99 交大)：$a_n - 5a_{n-1} + 6a_{n-2} = 2n+1,\ a_0=5,\ a_1=6$

令 $a_n^{(p)}=(d_0+d_1n)$ 代入原式
$(d_0+d_1n) - 5[d_0+d_1(n-1)] + 6[d_0+d_1(n-2)] = 2n+1 \Longrightarrow d_0=4,\ d_1=1,\ a_n^{(p)} = 4+n$

$a_n=c_12^n+c_23^n+(4+n)$ 代入原式 $\Longrightarrow c_1=2,\ c_2=-1 \Longrightarrow a_n=2\times2^n - 3^n + (4+n)$

例(99 清大)：$a_{n+1} - 3a_n + 2a_{n-1} = 3,\ a_0=1,\ a_1=2$

$\alpha^2-3\alpha+2,\ \alpha=1$ or 2
$a_n^{(h)} = c_1 + c_22^n$
$a_n^{(p)} = n(d_0)$ 代入原式
$d_0(n+1) - 3d_0(n) + 2d_0(n-1) = 3 \Longrightarrow d_0=-3$

$a_n=c_1 + c_22^n + (-3)n$ 代入原式可求解

Case 2：指數
若 $f(n)=(c_0+c_1n+\ldots+c_mn^m)\alpha^n,\ c_m \neq 0$
$a_n^{(p)} = n^r(d_0+d_1n+\ldots+d_mn^m)\alpha^r$，其中 r 為特徵根之重數

例(99 海大)：$a_n - 6a_{n-1} + 9a_{n-2} = 3^n,\ a_0=1,\ a_1=2$

$\alpha^2-6\alpha+9,\ \alpha=3$
$a_n^{(h)} = c_03^n + c_1n3^n \Longrightarrow c_0=1,\ c_1=-5/6$
$a_n^{(p)} = n^2(d_0)3^n$ 代入原式
$\Longrightarrow d_0=1/2$
$\Longrightarrow a_n=(1 - 5n/6 + 1/2n^2)3^n,\ n \geq 0$

Case 3：三角函數
若 $f(n)=c_1p^n\cos(n\theta)$或 $c_2p^n\sin(n\theta)$
$a_n^{(p)} = p^n(B_1\cos(n\theta) + B_2\sin(n\theta))$

例(99 成大)：$a_{n+2}-a_n = \sin(n\pi/2)$

$\alpha^2-1=0,\ \alpha=\pm 1$

$a_n^{(h)} = c_1 + c_2(-1)^n$

$a_n^{(p)} = B_1\cos(n\pi/2) + B_2\sin(n\pi/2)$代入原式

$\Longrightarrow B_1=0,\ B_2=1/2$

$\Longrightarrow$ 再求得 $c_1, c_2 \Longrightarrow$ 求得 a_n

5.3 轉換法

例(99 政大)：$f(n) = 9f(n/3) + 2n^2$

令 $n=3^k \Longrightarrow f(3^k) = 9f(3^{k-1}) + 2\times 9^k$
令 $a_k=f(3^k) \Longrightarrow a_k = 9a_{k-1} + 2\times 9^k,\ \alpha = 9$
$a_n^{(h)} = c9^k$
$a_n^{(p)}=k(d_0)9^k \Longrightarrow d_0=2$
$\Longrightarrow a_n = (c+2\log_3 n)n^2$
$= O(n^2\log_3 n)$

例(97 清大)：$a_n=a_{n-1}*a_{n-2}$、$a_0=1,\ a_1=2$

雙邊取 $\log \Longrightarrow \log a_n = \log a_{n-1} + \log a_{n-2}$
令 $b_n = \log a_n \Longrightarrow b_n = b_{n-1} + b_{n-2}$

例(99 輔大)：$a_n+na_{n-1} = n!,\ a_2=1$

同除 $n! \Longrightarrow a_n/n! + a_{n-1}/(n-1)! = 1$，*令* $b_n = a_n/n!$
$\Longrightarrow b_n + b_{n-1} = 1,\ \alpha = -1$
$b_n^{(h)} = c_1(-1)n$
$b_n^{(p)} = d_0$
求得 $d_0,\ c_1$，*可得* a_n

例(99 雲科)：$a_n-n/(n-1)a_{n-1} = n^3,\ a_1=1$

同除 n：$a_n/n - a_{n-1}/(n-1) = n^2$

例(99 台科)：$f(n) = 2f(\sqrt{(n)}) + \log_2 n,\ f(2)=1$

令 $n=2^{2\wedge k} \Longrightarrow f(2^{2\wedge k}) = 2f(2^{2\wedge(k-1)}) + 2^k$
令 $a_k = f(2^{2\wedge k}) \Longrightarrow a_k = a_{k-1} + 2^k,\ a_0 = f(2) = 1$
$a_k = (k+1)2^k \Longrightarrow f(n) = a_k = (1 + \log\log n)\log_2 n$

例(95 海大)：$a_n = 2a_{[n/2]},\ a_1=1$

令 $n=[b_{l-1}b_{l-2}\ldots b_0b_1]_2,\ b_{l-1}=1$
$a_n = a[b_{l-1}\ldots b_0]_2 = 2a[b_{l-1}\ldots b_1]_2 = 2^2a[b_{l-1}\ldots b_2]_2$
$= 2^{l-1}a_1 = 2^{l-1} = 2^{[\log 2n]}$

例(96 台科)：$a_{n+1} = -2a_n - 4b_n,\ b_{n+1} = 4a_n + 6b_n,\ a_0=1,\ b_0=0$

由 1：$b_n=(-1/4)a_{n+1} - (1/2)a_n$ *代入* 2：$((-1/4)a_{n+2} - (1/2)a_{n+1}) = 4a_n + 6((-1/4)a_{n+1} - (1/2)a_n)$
$\Longrightarrow a_{n+2} - 4a_{n+1} + 4a_n = 0$
解得 a_n *後代回，解得* b_n

5.4 生成函數法

Note：

$A(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_0 a_nx^n = \sum_1 a_{n-1}x^{n-1} = \sum_2 a_{n-2}\, x^{n-2} = \ldots$

1. a_n 型：$\sum_1 a_nx^n = A(x) - a_0 = \sum_2 a_nx^n = A(x) - a_0 - a_1x$
2. a_{n-1} 型：$\sum_1 a_{n-1}x^n = xA(x), \sum_2 \quad a_{n-1}x^n = x(A(x) - a_0)$
3. a_{n-2} 型：$\sum_2 a_{n-2}x^n = x^2A(x), \sum_3 a_{n-2}x^n = x^2(A(x) - a_0)$

例(92 師大)：$a_{n+2}-5a_{n+1}+6a_n = 2 \Longrightarrow a_n-5a_{n-1}+6a_{n-2} = 2, (n\geq2), a_0=3, a_1=7$

令 $A(x) = \sum_0 a_nx^n$

$\Longrightarrow \sum_2 a_nx^n - 5\sum_2 a_{n-1}x^n + 6\sum_2 a_{n-2}x^n = 2\sum_2 x^n$

$\Longrightarrow [A(x)-a_0-a_1x] - 5x[A(x)-a_0] + 6x^2A(x) = 2x^2/(1-x)$

$\Longrightarrow [1-5x+6x^2]A(x) = 2x^2/(1-x) + 3 - 8x$

$\Longrightarrow (1-2x)(1-3x)A(x) = (10x^2-11x+3)/(1-x)$

$\Longrightarrow A(x) = (10x^2-11x+3)/[(1-x)(1-2x)(1-3x)]$

$\Longrightarrow A(x) = 1/(1-x) + 0/(1-2x) + 2/(1-3x)$

$\Longrightarrow A(x) = \sum_0 x^n + 2\sum_0 (3x)^n$

$\Longrightarrow a_n = x^n$ *之係數* $= 1 + 2\times3^n, n\geq0$

5.5 應用問題

例(21 個)：和內塔

$a_n = 2a_{n-1} + 1, a_1=1$

$\Longrightarrow a_n = 2n-1, n\geq 1$

例(98 中正)：1/A 1/M 1/B，禁 A→B, B→A

$a_n = 3a_{n-1} + 2 \Longrightarrow a_n = 3^n-1$

例(24 個)：費氏數列

$a_n = [1/\sqrt{(5)}]\times\{ [(1+\sqrt{(5)}/2)]^n - (1-\sqrt{(5)}/2)]^n\}$

Note：

$\lim F_{n+1}/F_n = \lim 1/\sqrt{(5)}\alpha_{n+1} / 1/\sqrt{(5)}\alpha_n, \alpha$ 稱為 Golden Ratio

Note(變形)：

1. $a_n = a_{n-1} + a_{n-2}, a_0=0, a_1=1 \Longrightarrow a_n=F_n$
2. $a_n = a_{n-1} + a_{n-2}, a_0=1, a_1=1 \Longrightarrow a_n=F_{n+1}$
3. $a_n = a_{n-1} + a_{n-2}, a_0=1, a_1=2 \Longrightarrow a_n=F_{n+2}$

例：證：$\gcd(F_n, F_{n+1})=1$？

$F_{n+1} = F_n+F_{n-1}$

$F_n= F_{n-1} + F_{n-2}$

…

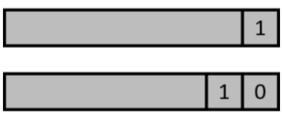
$F_2 = 1$

$\Longrightarrow \gcd(F_{n+1}, F_n) = 1$

例(99 清大)：證：$F_{n+1}\times F_{n-1} - F_n^2 = 1$？

1. *$n=1, F_2F_0 - F_1^2 = (-1) = (-1)^n$ 成立*
2. *設 $n=k$ 時成立，即 $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$*
   當 $n=k+1$ 時，$F_{k+2}F_k - F_{k+1}^2 = [(F_{k+1}F_k)\times(F_k)] - (F_k+F_{k-1})^2 = -(-1)^k = (-1)^{k+1}$
3. *由數學歸納法得證*

例(17 個)：二元 n 序列中，不含連續 0 之序列數為何？



$a_n = a_{n-1} + a_{n-2}, a_1=2, a_2=3$

$\Longrightarrow a_n = F_{n+2}$

例(99 台大)：四元 n 序列中，含偶數個 1 的序列數 a_n，寫出 a_n 之遞迴？

a_{n-1}
a_{n-1}
a_{n-1}
$4^{n-1} - a_{n-1}$
$\Longrightarrow a_n=2a_{n-1} + 4^{n-1},\ a_1=3$

例(98 中正)：n 階樓梯，每次可走 1, 2, 3 步，問有幾種走法？

$a_n = a_{n-1} + a_{n-2} + a_{n-3},\ a_1=1,\ a_2=2,\ a_3=4$

例(8 個)：切蛋糕

$a_n = a_{n-1} + n,\ a_1=2$

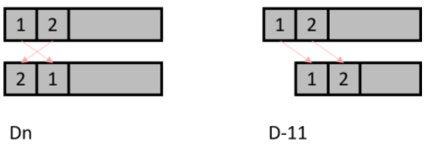
例(97 政大)：s = {1, …, n}, s 中不含連續整數之子集個數為何？

an：s 中取不連續整數之方法數
1. *n 不取*
2. *n 要取*

$a_n = a_{n-1} + a_{n-2},\ a_0=1,\ a_1=2$

例(8 個)：請導出亂序 D_n 之遞迴

$D_n = (n-1)(D_{n-1} + D_{n-2})$
$D_1=0,\ D_2=1$



例：Ackerman’s Function
A(0, n) = n+1
A(m, n) = A(m-1, 1)
A(m, n) = A(m-1, A(m, n-1))

1. $a_n = A(1, n) = A(0, A(1, n-1)) = A(0, an-1) = a_{n-1} + 1$
   $a_n = a_{n-1} + 1,\ a_0=2 \Longrightarrow a_n=n+2$
2. $b_n = A(2, n) = A(1, A(2, n-1)) = A(1, b_{n-1}) = b_{n-1} + 2$
   $b_n = b_{n-1} + 2,\ b_0 = 3 \Longrightarrow b_n = 2n+3$
3. $A(3, n) = 2n+3 - 3$

5.6 特殊型遞迴

Note：

$A(x) = \sum_0 a_n x^n$, $B(x) = \sum 0\ b_n x^n$, $A(x) \pm B(x) = \sum_0 (a_n \pm b_n)x^n$

$A(x) \cdot B(x) = a_0b_0 + (a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2 + \ldots$

定義：

a_n, b_n 為 2 個數列，定義 $c_n = a_n \otimes b_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0$

稱為 a_n 與 b_n 之 Convolution

Note：

1. $c_n = a_n \otimes b_n \Longrightarrow C(x) = A(x) \otimes B(x)$
2. $c_n = a_n \otimes a_n \Longrightarrow a_0a_n + a_1a_{n-1} + \ldots + a_na_0 \Longrightarrow C(x) = A(x) \cdot A(x) = A^2(x)$

例：求 n 個點的 Binary Order Tree 個數？

$k=0$, a_0a_{n-1}, $a_0=1$

$k=1$, a_1a_{n-2}

令 $A(x) = \sum_0 a_nx^n = \sum_1 (a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-1}a_2)x^n$

$= a_0a_0x + (a_0a_1+a_1a_0)x^2 + (a_0a_2+a_1a_1+a_2a_0)x^3 + \ldots$

$\Longrightarrow A(x) - a_0 = xA^2(x)$

$\Longrightarrow xA^2(x) - A(x) + 1 = 0 \Longrightarrow A(x) = [1 \pm \sqrt{(1-4x)}] / 2$

$(1-4x)^{1/2}\ \sum_0 [(1/2)(-1/2)(-3/2)\ldots((2r-3)/2)] / r!\ (-1)^r 4^r x^r$

$= - \sum_0 1/(2r-1)\ (2r)!/(r!r!2^r)\ 2^r x^r = - \sum_0 1/(2r-1)\ C_r^{2r} x^r$

$A(x) = [1-\sqrt{(1-4x)}] / 2x = 1 + \sum_0 [1/(2r-1) \times C_r^{2r} x^r] / 2x = 1/2 \sum_0 C_r^{2r}\ 1/(2r-1)\ x^{r-1}$

$\therefore a_n = 1/2\ 1/[2(n+1)-1]\ C_{n+1}^{2(n+1)} = 1/(n+1)\ C_n^{2n}$

稱為 Catalan Number

Note(變形)：

1. $a_n = a_0a_{n-1} + a_1a_{n-2} + \ldots a_{n-1}a_0$, $a_0=1$

   $\Longrightarrow a_n=c_n$
2. $a_n = a_1a_{n-1} + a_2a_{n-2} + \ldots a_{n-1}a_1$, $a_1=1$

   $\Longrightarrow a_n=c_{n-1}$
3. $a_n = a_2a_{n-1} + a_3a_{n-2} + \ldots a_{n-2}a_1$, $a_1=2$

   $\Longrightarrow a_n=c_{n-2}$

例(7 個)(99 交大)：n 個變數 $x_1, \ldots, x_n$ 之合法括號數有幾個？

$(a_k)(a_{n-k})$

$a_n = a_1a_{n-1} + a_2a_{n-2} + \ldots a_{n-1}a_1$, $a_1=1$

$\Longrightarrow a_n = c_{n-1} = 1/[(n-1)+1]\ C_{n-1}^{2\times(n-1)}$

例(99 輔大)：

R：$(x, y) \rightarrow (x+1, y)$ →

U：$(x, y) \rightarrow (x, y+1)$ ↑

$(0, 0) \rightarrow (7, 3)$且任何時間之 R 個數皆不可小於 U 之個數，有幾種方式？

全部 C_3^{10}：不合法 C_2^{10}
RUU | RURRRRR ⇔ RUU | URUUUUU
⟹ 合法：$C_3^{10} - C_2^{10}$

Note：
n 對括號，全部 C_n^{2n}，不合法：C_{n-1}^{2n} ⟹ 合法：$C_n^{2n} - C_{n-1}^{2n} = 1/(n+1)\ C_n^{2n}$