Data Structure and Programming, Spring 2025
Written Homework #1

Pei-Yuan, Wu
peiyuanwu@ntu.edu.tw

Chiao-Yun, Chin b10901154@ntu.edu.tw
Yi-Chen, Wang b10901143@ntu.edu.tw
Yi- Chou, Lin b10901149@ntu.edu.tw

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Late policy: $Score = Score \times 0.7^{\lceil h \rceil/24}$, $h =$ late hours ≤ 48

If the logarithm is written without a base, it refers to the natural logarithm, i.e., $\log_e x$.

1. **(16%)** Let $f(n)$, $T(n)$ be a real function and $\phi(n)$ be a positive function.

   (a) Prove that if $\lim_{n\to\infty} f(n)$ exists, then

   $$\liminf_{n\to\infty} f(n) = \lim_{n\to\infty} f(n) = \limsup_{n\to\infty} f(n).$$

   (b) Prove that if $T(n) \in \Theta(\phi(n))$ and $\lim_{n\to\infty} \frac{|T(n)|}{\phi(n)}$ exists, then

   $$0 < \lim_{n\to\infty} \frac{|T(n)|}{\phi(n)} < \infty.$$

   (c) Prove or disprove that if $\lim_{n\to\infty} \frac{|T(n)|}{\phi(n)}$ exists and $0 < \lim_{n\to\infty} \frac{|T(n)|}{\phi(n)} < \infty$, then $T(n) \in \Theta(\phi(n))$.

   (d) Prove or disprove that if $T(n) \in \Theta(\phi(n))$, then $\lim_{n\to\infty} \frac{|T(n)|}{\phi(n)}$ is a finite positive number.

2. **(16%) Properties of Big O-Notation**

   Assume $f(n)$, $g(n)$, $h(n)$ are all positive function. Prove the following properties.

(a) If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$.

(b) If $f(n) \in O(g(n))$, then $g(n) \in \Omega(f(n))$.

(c) If $f(n) \in \omega(g(n))$, then $g(n) \in o(f(n))$.

(d) "Either $f(n) \in O(g(n))$ or $g(n) \in O(f(n))$ is true." is false.(inclusive or)

3. **(20%)Asymptotic growth comparison**

   Prove or disprove the following statements: (Define $\lfloor x \rfloor := \sup\{m \in \mathbb{Z} : m \leq x\}$, $\lceil x \rceil := \inf\{n \in \mathbb{Z} : x \leq n\}$.)

   (a) $2^{2n} \in O(2^{n+1024})$

   (b) $\log_{1024}(n^2) \in o(\log_2(n^{1024}))$

   (c) $\log(n!) \in \Theta(n \log(n))$

   (d) $\lfloor x^{2.5} \rfloor \in \Omega(x^{2.5})$

   (e) $x\lceil \frac{x}{2} \rceil \in O(x^2)$

4. **(16%)** Rank the following functions by order of growth. Here $f(n) \succ g(n)$ and $f(n) \sim g(n)$ denote $f(n) \in \omega(g(n))$ and $f(n) \in \Theta(g(n))$, respectively. Please justify your answers. (That is, justify every $\succ$ and $\sim$ relation in the order sequence.)

   (a) $f_1(n) = n(\frac{3}{2})^n$

   (b) $f_2(n) = 10^{\log n^2}$

   (c) $f_3(n) = n^6$

   (d) $f_4(n) = \sqrt{\log n}$

   (e) $f_5(n) = 3^n$

   (f) $f_6(n) = 6^{6^n}$

   (g) $f_7(n) = n^n$

   (h) $f_8(n) = n^{\log \log n}$

   (i) $f_9(n) = \log \log n$

   (j) $f_{10}(n) = 10^{10^{10}}$

   (k) $f_{11}(n) = n!$

   (l) $f_{12}(n) = (\log n)^{1.5}$

   (m) $f_{13}(n) = \sum_{k=1}^{n} k^5$

5. **(20%)** Assume all functions are positive functions. Prove or disprove the following statements.

   (a) If $f(n) \in O(h(n))$, and $g(n) \in O(h(n))$, then $f(n) + g(n) \in O(h(n))$.

   (b) If $f(n) \in O(h(n))$, and $g(n) \in O(h(n))$, then $|f(n) - g(n)| \in O(h(n))$.

(c) If $f(n) \in O(g(n))$, then $(f(n))^2 \in O((g(n))^2)$.

(d) If $f(n) \in O(g(n))$, then $2^{f(n)} \in O(2^{g(n)})$.

(e) If $f(n)$ is increasing function, and $g(n) \in o(h(n))$, then $f(g(n)) \in o(f(h(n)))$.

6. **(12%) Algorithm cost analysis**

Figure 1 presents a sample code for calculating the summation $\sum_{k=1}^{n} k^k$. Given that the time complexity of addition(x+y) and assign(x=y) operation are $\Theta(1)$.

(a) If the time complexity of multiplication(x*y) is $\Theta(1)$. What is the time complexity of the function in the sample code? Explain your answer.

(b) If the time complexity of multiplication(x*y) is $\Theta(\min(x, y))$. What is the time complexity of the function in the sample code? Explain your answer.

```
def func(n):
    sum = 0
    for k in range(1,n+1):
        x = 1
        for j in range(1,k+1):
            x = x*k
        sum = sum + x
    return sum
```

```
func(1) = 1,   1^1
func(2) = 5,   1^1+2^2
func(3) = 32,  1^1+2^2+3^3
func(4) = 288, 1^1+2^2+3^3+4^4
```

Figure 1: sample code and corresponding results