FACULTY OF IT-HCMUS

Mathematical Method in Visual Data Analysis

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Mathematical Method in Visual Data Analysis

Lecture 2:
Metric space

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2.2. The basic concepts in metric space

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2.2.1. Metric space

Definition 1. (*Metric space*)

Metric space (X, d) is space X together with a real-valued function d, $d: XxX \to R$, which measures the distance between pairs of points x and y in X, d obeys the following axioms:

$$(i)\ 0 < d(x, y) < \infty\ \forall x, y \in X, x \neq y$$

$$(ii)\ d(x, x) = 0\ \forall x \in X$$

$$(iii)\ d(x, y) = d(y, x)\ \forall x, y \in X$$

$$(iv)\ d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$$

Such a function d is called a metric.

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2.2.2. Cauchy sequences

Definition 2. (*Cauchy Sequences*)

A sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is called a Cauchy sequence if :

$$\forall \varepsilon > 0, \exists N > 0 \text{ sao cho}$$
$$d(x_n, x_m) < \varepsilon \quad \forall n, m > N$$

.

2.2.3. Convergent sequence

Definition 3. (Convergent sequence)

A sequence of points $\{x_n\}_{n=1}^{\infty}$ in metric space (X, d) is said to converge to a point $x \in X$ if:

$$\forall \varepsilon > 0, \exists N > 0 \text{ so that}$$
$$d(x_n, x) < \varepsilon \ \forall n > N$$

$x \in X$, to which the sequence converges, is called the limit of the sequence, and we use the notation

$$x = \lim_{n \to \infty} x_n$$

2.2.4. Complete Metric Space

Theorem 1. (*Convergent sequence & Cauchy sequence*)

A sequence of points $\{x_n\}_{n=1}^{\infty}$ in metric space (X, d) converges to a point $x \in X$, then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Definition 4. (Complete metric space)

A metric space (X, d) is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit $x \in X$.

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2.2.5. Metric space Hausdorff

Definition 5. (Metric *space Hausdorff*)

Let (X, d) be complete metric space. Then $\mathcal{H}(X)$ denotes the space whose points are the compact subsets of X, other than the empty set.

Definition 6. (*Metric in space Hausdorff*)

Let (X, d) be complete space metric. Let $X \in \mathcal{H}(X)$.

Hausdorff distance between points A and B in $\mathcal{H}(X)$ is defined by

$$h(A, B) = Max\{d(A, B), d(B, A)\}$$

$$where$$

$$d(A, B) = Max\{d(x, B): x \in A\}$$

$$d(x, B) = Min\{d(x, y): y \in B\}$$

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2.2.5. Metric space Hausdorff

Định lý 2. (*The completeness of metric space Hausdorff*)

Let (X, d) be complete space metric. Then $(\mathcal{H}(X), h(d))$is a complete metric space.

Moreover, if $\{A_n \in \mathcal{H}(X)\}_{n=1}^{\infty}$ is a Cauchy sequence then

$$A = \lim_{n\to\infty} A_n \in \mathcal{H}(X)$$

$$A = \{x \in X : \exists \text{ dãy } Cauchy\ \{x_n \in A_n\} \to x\}$$

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2.2.6. Contraction mapping and fixed point

Definition 1. (Contraction mapping)

A transformation $f: X \to X$ on a metric space (X, d) is called contractive or a contraction mapping if

$$\exists s, 0 \le s < 1 \; such \; that$$
$$d\big(f(x), f(y)\big) \le s.d(x, y) \;\; \forall x, y \in X.$$

Any such number s is called a contractivity factor for f

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2.2.6. Contraction mapping and fixed point

Theorem 3. (Contraction mapping)

Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d). Then f possesses exactly one fixed point $x_f \in X$ and moreover for any point $x \in X$, the sequence $\{f^{on}(x)\} \to x_f$. That is

$$\lim_{n\to\infty} f^{on}(x) = x_f, \forall x \in X.$$

2.2.6. Contraction mapping and fixed point

Theorem 4. (Fixed point approximation)

Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d) with contractivity factor s.

The fixed point x_f is approximated by the following expression

$$d(f^{on}(x), x_f) \leq \frac{s^n}{1-s} d(x, f(x)), \forall x \in X.$$

2.2.6. Contraction mapping and fixed point

Theorem 5. (approximate x by fixed point)

Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d) with contractivity factor s, fiexed point $x_f \in X$. Then

$$d(x, x_f) \leq \frac{1}{1-s} d(x, f(x)), \forall x \in X.$$

2.2.7. Contraction mapping on metric space Hausdorff

Lemma 1. (*contraction mapping*)

Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d) with contractivity factor s. Then $w: \mathcal{H}(X) \to \mathcal{H}(X)$ define by

$$w(B) = \{w(x): x \in B\} \, \forall B \in \mathcal{H}(X)\}$$

Is a contraction mapping on $(\mathcal{H}(X), h(d))$ with contractivity factor s.

2.2.7. Contraction mapping on metric space Hausdorff

Lemma 2. (*contraction mapping sequence*)

Let (X, d) be metric space.
Let $\{w_n\}_{n=1}^{N}$ be contraction mappings on $(\mathcal{H}(X), h(d))$ with contractivity factor for w_n be denoted by s_n for each n.
Define $w: \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$W(B) = w_1(B) \cup w_2(B) \cup \ldots w_N(B)$$
$$= \bigcup_{n=1}^{N} w_n(B) \quad , \qquad for\ each\ B \in \mathcal{H}(X)$$

Then W is a contraction mapping with contractivity factor

$$s = Max\{s_n : n = 1,2, \ldots, N\}$$

2.2.7. Contraction mapping on metric space Hausdorff

Theorem 6. (*fixed set in metric space Hausdorff*)

Let $\{X; w_n, n = 1,2,\ldots,N\}$ be a iterated function system with contractivity factor s. Then the transformation $w: \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B) \quad , \qquad for\ all\ B \in \mathcal{H}(X)$$

is a contraction mapping on the complete metric space $(\mathcal{H}(X), h(d))$ with contractivity factor s.

Its unique fixed set, $A \in \mathcal{H}(X)$, obeys

$$A = W(A) = \bigcup_{n=1}^{N} w_n(A),$$

$$A = \lim_{n\to\infty} W^{on}(B), B \in \mathcal{H}(X)$$

2.2.7. Contraction mapping on metric space Hausdorff

Theorem 7. (*approximate fixed set in metric space Hausdorff*)

Let $\{X; w_n, n = 1,2,\ldots,N\}$ be a iterated function system with contractivity factor s. Then the transformation $w: \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B) \quad , \qquad for\ all\ B \in \mathcal{H}(X)$$

$A \in \mathcal{H}(X)$ is a fixed set approximated by

$$h(W^{on}(B), A) \le \frac{s^n}{1-s} h(B, W(B)), \forall B \in \mathcal{H}(X)$$

2.2.7. Contraction mapping on metric space Hausdorff

Theorem 8. (approximate O by fixed set)

Let O is subset of $\mathcal{H}(X)$

Let $\{X; w_n, n = 1,2,\dots,N\}$ be a iterated function system with contractivity factor s. Then the transformation $w: \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B) \quad , \qquad for\ all\ B \in \mathcal{H}(X)$$

$A \in \mathcal{H}(X)$ is a fixed set of W,

$$h(O, A) \le \frac{1}{1-s} h(O, W(O))$$