FACULTY OF IT-HCMUS

Mathematical Method in Visual Data Analysis

Lecturer: Assoc Prof. Lý Quốc Ngọc

HCMc, 10-2022

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Mathematical Method in Visual Data Analysis

Lecture 4: Vector space

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4.1. Vector space

4.1.1. Definition (Vector space)

A vector space E over the field T be a non-empty set satisfies 8 following properties:

$$i)\ x + y = y + x$$
$$ii)\ (x + y) + z = x + (y + z)$$
$$iii) \exists 0 \in E : x + 0 = x$$
$$iv) \forall x \in E, \exists(-x) \in E : x + (-x) = 0$$

4.1. Vector space

4.1.1. Definition (Vector space)

A vector space V over the field T be a non-empty set satisfies 8 following properties:

$$v)\ (\alpha+\beta).x=\alpha.x+\beta.x$$
$$vi)\ \alpha.(x+y)\ =\alpha.x+\alpha.y$$
$$vii)(\alpha.\beta).x=\alpha.(\beta.x)$$
$$viii)1.x=x$$

4.2. Linear map, Linear Transformation

4.2.1. Definition (Linear map)

Given vector space E, F over the field T.

The linear map from E to F is the mapping f from E to F obeys

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \forall x_1, x_2 \in E$$
$$f(k.x) = k.f(x) \quad \forall x \in E, \forall k \in T$$

4.2. Linear map, Linear Transformation

4.2.2. Theorem (Linear map**)**

Let $e_1, e_2, \ldots e_n$ be a basis of E,

$f_1, f_2, \ldots f_n$ be arbitrary vectors of F.

Then there exists a unique linear mapping $f: E \to F$ such that:

$$f(e_i) = f_i, i = 1..n$$

4.2.3. The matrix associated with a linear map

Let

E be a n-dimensional vector space,

F be a m-dimensional vector space,

f be the linear map from E to F.

Let $e_1, e_2, ..., e_n$ be the basis vectors in E,

$g_1, g_2, ..., g_m$ be the basis vectors in F

The mapping f maps $e_1, e_2, ..., e_n$ to $f(e_1), f(e_2), ..., f(e_n)$

4.2.3. The matrix associated with a linear map

Express $f(e_1), f(e_2),..., f(e_n)$ in terms of $g_1, g_2,..., g_m$

$$
\begin{aligned}
f(e_1) &= a_{11}g_1 + a_{12}g_2 + ... + a_{1m}g_m \\
f(e_2) &= a_{21}g_1 + a_{22}g_2 + ... + a_{2m}g_m \qquad (1) \\
&\vdots \\
f(e_n) &= a_{n1}g_1 + a_{n2}g_2 + ... + a_{nm}g_m
\end{aligned}
$$

4.2.3. The matrix associated with a linear map

The matrix created by the row coordinates of

$$f(e_1), f(e_2), \ldots, f(e_n)$$

is called the matrix associated with f in the basis

$$e_1, e_2, \ldots, e_n \text{ và } g_1, g_2, \ldots, g_m$$

(1) Có thể được viết:

$$f(e) = A.g$$

4.2.3. The matrix associated with a linear map

Calculate the coordinates of $f(x)$ in F

$$f(x) = f(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i . f(e_i)$$

$$= \sum_{i=1}^{n} x_i . \sum_{j=1}^{m} a_{ij} g_j = \sum_{j=1}^{m} (\sum_{i-1}^{n} x_i . a_{ij}).g_j$$

$$(f(x)) = (x).A$$

4.2.4. Definition (Linear Transformation)

The linear transformation A of vector space E is a linear mapping from E to E.

The linear mapping A from E to E satisfies:

$$\mathrm{A}(x_1 + x_2) = \mathrm{A}(x_1) + \mathrm{A}(x_2) \quad \forall x_1, x_2 \in E$$

$$\mathrm{A}(k.x) = k.\mathrm{A}(x) \quad \forall x \in E, \forall k \in T$$

4.2.5. Theorem (the invariant subspace of linear transformation**)**

Every linear transformation in an n-dimensional real vector space E $(n \geq 1)$ has at least one 1-dimensional or 2-dimensional invariant subspace.

4.3. Eigen value – Eigen vector

4.3.1. Definition (Eigen value – Eigen vector)

Let A be the linear transformation of n-dimensional vector space E.

The subspace E_1 of E is the invariant subspace of A if A transforms every vector x of E_1 to vector of E_1

$$Ax = \lambda . x$$

Finding the eigenvector and the one-dimensional invariant subspace are equivalent problems.

$$A(k.x) = k.A(x) = k.(\lambda.x) = \lambda.(k.x)$$

4.3. Eigen value – Eigen vector

4.3.2. Calculation method (Eigen value – Eigen vector)

How to calculate eigenvalues, eigenvectors.

Let $e_1, e_2, \ldots, e_n$ be the basis of E, The matrix $A = [a_{ij}]$ associated with the linear transformation A.

Let vector x be an eigenvector of A corresponding to the eigen value λ, and its row coordinate $(x_1, x_2, \ldots, x_n)$

$$Ax = \lambda . x$$

$$(x). A = \lambda(x)$$

4.3. Eigen value – Eigen vector

4.3.2. Calculation method (Eigen value – Eigen vector)

$$\begin{cases} x_1a_{11} + x_2a_{21} + \ldots + x_na_{n1} = \lambda x_1 \\ x_1a_{12} + x_2a_{22} + \ldots + x_na_{n2} = \lambda x_2 \\ \vdots \\ x_1a_{1n} + x_2a_{2n} + \ldots + x_na_{nn} = \lambda x_n \end{cases} \Rightarrow \begin{cases} x_1(a_{11} - \lambda) + x_2a_{21} + \ldots + x_na_{n1} = 0 \\ x_1a_{12} + x_2(a_{22} - \lambda) + \ldots + x_na_{n2} = 0 \\ \vdots \\ x_1a_{1n} + x_2a_{2n} + \ldots + x_n(a_{nn} - \lambda) = 0 \end{cases}$$

$$\Rightarrow \begin{vmatrix} (a_{11} - \lambda) & a_{21} \ldots\ldots\ldots & a_{n1} \\ a_{12} & (a_{22} - \lambda) \ldots a_{n2} & \\ \vdots & & \\ a_{1n} & a_{2n} & (a_{nn} - \lambda) \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} (a_{11} - \lambda) & a_{12} \ldots\ldots\ldots & a_{1n} \\ a_{21} & (a_{22} - \lambda) \ldots a_{2n} & \\ \vdots & & \\ a_{n1} & a_{n2} & (a_{nn} - \lambda) \end{vmatrix} = 0$$

4.3. Eigen value – Eigen vector

4.3.2. Calculation method (Eigen value – Eigen vector)

Suppose that the linear transform has n independent eigen vector.

Suppose that A is the linear transform, $e_1, e_2, \ldots, e_n$ are n its independent eigen vectors.

$$\mathrm{A}e_i = \lambda_i . e_i , i = 1,2,\ldots,n$$

If n eigenvectors are chosen as the basis vectors, then the matrix of the linear transformation is diagonal :

$$A = \begin{bmatrix} \lambda_1 \, 0 \ldots 0 \\ 0 \;\; \lambda_2 \ldots 0 \\ \vdots \\ 0 \;\; 0 \ldots \lambda_n \end{bmatrix}$$

4.4. Linear forms, Bilinear forms, Quadratic forms

4.4.1. Definition (Linear forms)

Given vector space E over the field T

The linear form on E is the linear map φ from E to T obeys

$$\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2) \quad \forall x_1, x_2 \in E$$

$$\varphi(k.x) = k.\varphi(x) \quad \forall x \in E, \forall k \in T$$

4.4. Linear forms, Bilinear forms, Quadratic forms

4.4.2. Definition (Bilinear forms)

Given vector space E, F over the field T

The mapping φ from $E \times F$ to T

φ is the bilinear form on $E \times F$ if it is the linear form of x when y is fixed, the linear form of y when x is fixed.

$$1)\ \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$$

$$\varphi(k.x, y) = k.\varphi(x, y)$$

$$2)\ \varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$$

$$\varphi(x, l.y) = l.\varphi(x, y)$$

4.4. Linear forms, Bilinear forms, Quadratic forms

4.4.2. Definition (Bilinear forms)

Suppose that E is n-dimensional vector space, F is m-dimensional vector space.

$e_1, e_2, \ldots, e_n$ are the basis vectors of E,

$f_1, f_2, \ldots, f_n$ are the basis vectors of F.

$$\varphi(x, y) = \varphi(x_1 e_1 + x_2 e_2 + \ldots + x_n e_n, y_1 f_1 + y_2 f_2 + \ldots + y_n f_n)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \, \varphi(e_i, f_j)$$

4.4.2. Definition (Bilinear forms)

$a_{ij} = \varphi(e_i, f_j)$

Matrix $A = [a_{ij}]$ is said to be the matrix representing the bilinear form in the basis vectors $e_1, e_2, \dots, e_n$ and $f_1, f_2, \dots, f_n$

(x) is the row coordinate of x in the basis vectors $e_1, e_2, \dots, e_n$

(y) is the row coordinate of y in the basis vectors $f_1, f_2, \dots, f_n$

$$\varphi(x, y) = (x)A(y^T)$$

4.4.2. Definition (Bilinear forms)

Suppose that e, f are the bases of E, F respectively , the matrix representing bilinear form is A.

In the new base $e' \; of \; E$, $f' \; of \; F$, matrix representing bilinear form is A':

$$A' = T.A.S^T$$

T, S are the transformation matrices from the old basis to the new ones.

$$T.e = e'$$

$$S.f = f'$$

4.4.2. Definition (Bilinear forms)

If φ is the bilinear form on $E \times E$ then:

$$A' = T.A.T^T$$

T is the transition matrix from the old basis to the new one in E

$$\varphi(x,y) = (x)A(y^T) = (x')T.A.(y'T)^T = (x').(T.A.T^T).(y')^T$$

4.4. Linear forms, Bilinear forms, Quadratic forms

4.4.3. Definition (Quadratic forms)

Given vector space E over the field T

φ is the bilinear form on $E \times E$

The quadratic form w associated with φ is the mapping w from E to T

$$w(x) = \varphi(x, x) \qquad \forall x \in E, w(x) \in T$$

4.4.3. Definition (Quadratic forms)

Suppose that E is n-dimensional vector space,

$e_1, e_2, \ldots, e_n$ are the basic vectors of E,

$$w(x) = \varphi(x, x) = \varphi\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} x_j e_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \varphi(e_i, e_j)$$

$$w(x) = (x).A.(x)^T$$

$$A = [\varphi(e_i, e_j)]$$

4.4.3. Definition (Quadratic forms)

If in E, T is the transition matrix from the old basis to the new basis, then the quadratic form in the new basis is:

$$w(x) = (x).A.(x)^{\mathrm{T}} = (x')T.A.\ ((x').T)^{\mathrm{T}} = (x')T.A.T^{T}.(x')^{\mathrm{T}}$$

$$A' = T.A.T^{T}$$

4.4.4. Transform the quadratic form to the canonical form

Consider the quadratic form on n-dimensional vector space E on T.

Prove that in E, there are the basis vectors $f_1, f_2, \ldots, f_n$ such that if

$$x = \sum_{i=1}^{n} u_i . f_i$$

then

$$w(x) = k_1.u_1^2 + k_2.u_1^2 + \ldots + k_n.u_n^2$$

$k_1, k_2, \ldots k_n$ are the elements of the field T

4.5. Symmetric operators

4.5.1. Definition (Euclidean space)

A vector space E is called a Euclidean space if:

For each pair (x, y) of E corresponding to a real number called the dot product of x, y obeys:

$$\langle x, y\rangle = \langle y, x\rangle$$
$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$$
$$\langle kx, y\rangle = k\langle x, y\rangle \ \forall k$$
$$\langle x, x\rangle > 0 \ khi \ x \neq 0,$$
$$\langle x, x\rangle = 0 \ khi \ x = 0$$

4.5. Symmetric operators

4.5.2. Dot product is the bilinear form

In n-dimensional vector space E, $e_1, e_2, \ldots, e_n$ are the basic vectors.

Then :

$$x = \sum_{i=1}^{n} x_i e_i,\ \ y = \sum_{j=1}^{n} y_j e_j$$

$$\langle x, y\rangle = \left\langle \sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \langle e_i, e_j\rangle$$

4.5. Symmetric operator

4.5.3. Definition (Symmetric operator)

The linear transformation of Euclidean space E is said to be symmetric if:

For every pair of vector x, y of E, we have:

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

4.5. Symmetric operator

4.5.4. Diagonalize the transformation matrix of the symmetric operator

A is symmetric → $|A - \lambda I| = 0$ has a real root → There exists a 1-dimensional invariant subspace → construct an orthogonal basis of eigenvectors $e_1, e_2, \dots, e_n$ → transformation matrix is diagonalized.

4.5. Symmetric operator

4.5.4. Diagonalize the transformation matrix of the symmetric operator

-The characteristic root of symmetric transformations are real.

- Every symmetric transformation has at least one 1-dimensional invariant subspace because it has a real root of the characteristic polynomial.

- A is symmetric => Find out the orthogonal matrix T such that TAT^{-1} is diagonal , diagonal elements are the characteristic root of A, each root taken a number of times equal to its multiples.

4.5. Symmetric operator

4.5.5. Reduce the quadratic form to the canonical form

Theorem:

In the n-dimensional Euclidean space E, every quadratic form $w(x)$ can be reduced to a unique canonical form

$$w(x) = \sum_{i=1}^{n} \lambda_i u_i^2$$

by the orthogonal transformation, λ_i are the eigenvalues of w

.

4.5. Symmetric operator

4.5.5. Reduce the quadratic form to the canonical form

In the n-dimensional Euclidean space E, the quadratic form $w(x)$

$\varphi(x,y)$ is the symmetric bilinear form for $w(x)$

In E, choose the orthogonal basis $e_1, e_2, \ldots, e_n$

$$\varphi(x,y)=\left\langle \sum_{i=1}^{n} x_i e_i , \sum_{i=1}^{n} y_j e_j \right\rangle = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} x_i a_{ij}\right) y_j = \sum_{j=1}^{n} z_j y_j$$

$$a_{ij}=\varphi(e_i,e_j)=\varphi(e_j,e_i)=a_{ji}$$

$$z_j=\sum_{i=1}^{n} x_i a_{ij},\ j=1,2,\ldots,n$$

4.5. Symmetric operator

4.5.5. Reduce the quadratic form to the canonical form

$$a_{ij} = \varphi(e_i, e_j) = \varphi(e_j, e_i) = a_{ji}$$

$$z_j = \sum_{i=1}^{n} x_i a_{ij}, \; j = 1,2,\ldots,n$$

$$\Rightarrow z = Ax$$

4.5. Symmetric operator

4.5.5. Reduce the quadratic form to the canonical form

The linear transformation A is symmetric because its transformation in the orthogonal basis system $e_1, e_2, \ldots, e_n$ is symmetric. According to the invariant subspace theorem of the symmetric transformation, in E there is a orthogonal basis of the eigenvectors of A: $f_1, f_2, \ldots, f_n$

In this basis: $x = \sum_{i=1}^{n} u_i f_i, \; y = \sum_{j=1}^{n} v_j f_j$

$$\varphi(x,y) = \langle Ax, y \rangle = \left\langle A\left(\sum_{i=1}^{n} u_i f_i,\right), \sum_{j=1}^{n} v_j f_j \right\rangle = \left\langle \left(\sum_{i=1}^{n} u_i (Af_i)\right), \sum_{j=1}^{n} v_j f_j \right\rangle$$

$$= \left\langle \left(\sum_{i=1}^{n} \lambda_i u_i f_i\right), \sum_{j=1}^{n} v_j f_j \right\rangle = \sum_{i=1}^{n} \lambda_i u_i v_i$$

$$w(x) = \varphi(x,x) = \sum_{i=1}^{n} \lambda_i u_i^2$$