Duong Minh Duc, Real Analysis

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Abstract

I retype and correct some errors in [1].

*Typer.

Contents

1 L^p Spaces

Definition 1.1. Let X be a nonempty set. Let $\mathfrak{M}$ be a nonempty family of subsets in X satisfying the following properties.

$$\Omega \in \mathfrak{M} \tag{1.1}$$

$$\Omega \backslash A \in \mathfrak{M}, \ \ \forall A \in \mathfrak{M} \tag{1.2}$$

$$\bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}, \ \ \forall \{A_n\}_{n=1}^{\infty} \subset \mathfrak{M} \tag{1.3}$$

then we call $\mathfrak{M}$ a *σ-algebra* in X.

Definition 1.2. Let $\mu : \mathfrak{M} \to [0, \infty]$ be a mapping satisfying the following properties

1. *Countably Additive.* If $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathfrak{M}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.4}$$

2. There exists B in $\mathfrak{M}$ such that $\mu(B) < \infty$.

Then we call μ a *positive measure* in X.

Proposition 1.3. *There exists a σ-algebra and a positive measure μ in space $\mathbb{R}^n$ satisfying the following properties*

1. *All open sets and closed sets in $\mathbb{R}^n$ belong to $\mathfrak{M}$.*
2. *For every cell $[a_1, b_1] \times \cdots \times [a_n, b_n]$,*

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i) \tag{1.5}$$

$$\mu((a_1, b_1) \times \cdots \times (a_n, b_n)) = \prod_{i=1}^{n} (b_i - a_i) \tag{1.6}$$

3. *μ is preserved through a translation transformation*

$$\mu(E + a) = \mu(E), \ \ \forall E \in \mathfrak{M}, a \in \mathbb{R}^n \tag{1.7}$$

4. *Through a homothetic transformation,*

$$\mu(cE) = |c|^n \mu(E), \ \ \forall E \in \mathfrak{M}, c \in \mathbb{R}^n \backslash \{0\} \tag{1.8}$$

Definition 1.4. We call $\mathfrak{M}$ and μ *Lebesgue σ-algebra* and *Lebesgue measure* in $\mathbb{R}^n$, respectively.

From now on, we use $\mathfrak{M}$ and μ to denote *Lebesgue σ-algebra* and *Lebesgue measure* in $\mathbb{R}^n$, respectively.

Theorem 1.5. *Let E be a measurable set in $\mathbb{R}^n$ for which $\mu(E) < \infty$, and ε be a positive real number. Then there exist a compact set K and an open set V such that*

$$K \subset E \subset V \tag{1.9}$$

$$\mu(V \backslash K) < \varepsilon \tag{1.10}$$

Theorem 1.6. *Let K be a compact set and an open set V in $\mathbb{R}^n$ such that $K \subset V$. Then there exists a continuous function $\varphi : \mathbb{R}^n \to [0, 1]$ such that*

$$\varphi(x) = \begin{cases} 1, & \forall x \in K \\ 0, & \forall x \in \mathbb{R}^n \backslash V \end{cases} \tag{1.11}$$

Problem 1.7. *Let E be a measurable set in $\mathbb{R}^n$ for which $\mu(E) < \infty$, and ε be a positive real number. Then there exists a continuous function $\varphi : \mathbb{R}^n \to [0, 1]$ such that*

$$\mu(\{x \in R^n : \chi_E(x) \neq \varphi(x)\}) < \varepsilon \tag{1.12}$$

Hint. Use Theorem 1.5 and Theorem 1.6. □

Definition 1.8. Let Ω be a measurable set in $\mathbb{R}^n$, $c_1, \ldots, c_m$ be m real numbers, and $A_1, \ldots, A_m$ be m measurable sets contained in Ω. Define

$$s(x) = \sum_{i=1}^{m} c_i \chi_{A_i}(x), \quad \forall x \in \Omega \tag{1.13}$$

Then we call f a *simple function* in Ω.

Definition 1.9. Let Ω be a measurable set in $\mathbb{R}^n$. Let f be a mapping from Ω into $[-\infty, \infty]$. We call f a *measurable mapping* in Ω if $f^{-1}((a, \infty]) \in \mathfrak{M}$ for all real number a.

Theorem 1.10. *Let f be a measurable function in a measurable set Ω. Then there exists a sequence of simple function $\{t_n\}_{n=1}^{\infty}$ in Ω such that*

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq s_n(x) \leq f(x), \quad \forall x \in \Omega, \forall n \in \mathbb{Z}_+ \tag{1.14}$$

and

$$\lim_{n \to \infty} s_n(x) = f(x), \quad \forall x \in \Omega \tag{1.15}$$

Definition 1.11. Let $E \in \mathfrak{M}$. Define

$$\int_E s d\mu = \sum_{k=1}^{m} c_k \mu(A_k \cap E) \tag{1.16}$$

and call $\int_E s d\mu$ the *integral* of s in E. This integral can be ∞.

Definition 1.22. Let Ω be a measurable set in $\mathbb{R}^n$, let $E \in \mathfrak{M}$, and f be a measurable function from Ω into $[0,\infty]$. Define $\mathfrak{F}(f)$ be the family of all simple functions s in Ω such that $0 \leq s \leq f$, and define

$$\int_E f d\mu = \sup_{s\in\mathfrak{F}(f)} \int_E s d\mu \tag{1.17}$$

We call $\int_E f d\mu$ *Lebesgue integral* of f in E with respect to measure μ. This integral of f can be ∞.

Problem 1.23. *Let Ω be a measurable set in $\mathbb{R}^n$, $E \in \Omega$, and f be a measurable function from Ω into $[0,\infty]$. Suppose that $\mu(E) = 0$. Prove that*

$$\int_E f d\mu = 0 \tag{1.18}$$

HINT. Consider that f is a simple function. □

Problem 1.24. *Let Ω be a measurable set in $\mathbb{R}^n$ and f be a measurable function from Ω into $[0,\infty]$. Suppose*

$$\int_\Omega f d\mu < \infty \tag{1.19}$$

Prove that

$$\mu(\{x \in \Omega : f(x) = \infty\}) = 0 \tag{1.20}$$

HINT. Let $\alpha \in (0,\infty)$. Define

$$B = \{x \in \Omega : f(x) \geq \alpha\} \tag{1.21}$$

Prove

$$\int_\Omega f d\mu \geq \int_B f d\mu \tag{1.22}$$

$$\geq \int_B \alpha\chi_B d\mu \tag{1.23}$$

$$= \alpha\mu(B) \tag{1.24}$$

Try it. □

Definition 1.25. Let Ω be a measurable set in $\mathbb{R}^n$, $E \in \mathfrak{M}$, and f be a real measurable function in Ω. Then $|f|$ is a function from Ω into $[0,\infty)$. Suppose

$$\int_\Omega |f| d\mu < \infty \tag{1.25}$$

Define

$$f^+(x) = \max\{f(x), 0\} \tag{1.26}$$

$$f^-(x) = \max\{-f(x), 0\} \tag{1.27}$$

and

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \tag{1.28}$$

We call $\int_E f d\mu$ the *Lebesgue integral* of f in E with respect to μ. This integral of f is a real number.

Theorem 1.26 (Monotone Convergence Theorem). *Let Ω be a measurable set in $\mathbb{R}^n$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable mappings from Ω into $[0,\infty]$, and f be a mapping from X into $[0,\infty]$. Suppose that*

$$f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots, \quad \forall x \in \Omega \tag{1.29}$$

and

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in \Omega \tag{1.30}$$

Then

$$\int_X f d\mu = \int_X \lim_{n\to\infty} f_n(x)\, d\mu = \lim_{n\to\infty} \int_X f_n d\mu \tag{1.31}$$

Lemma 1.27 (Fatou's Lemma). *Let Ω be a measurable set in $\mathbb{R}^n$, $E \in \mathfrak{M}$, and $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable mappings from Ω into $[0,\infty]$. Then the following inequality holds*

$$\int_E \liminf_{n\to\infty} g_n d\mu \le \liminf_{n\to\infty} \int_E g_n d\mu \tag{1.32}$$

Theorem 1.28 (Lebesgue Dominated Convergence Theorem). *Let Ω be a measurable set in $\mathbb{R}^n$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions in Ω, and f be a function in X. Suppose that there exists a Lebesgue integrable function g in Ω such that*

$$|f_n(x)| \le g(x), \quad \forall x \in \Omega, \forall n \in Z_+ \tag{1.33}$$

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in \Omega \tag{1.34}$$

Then f in Lebesgue integrable in Ω and

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu \tag{1.35}$$

$$\lim_{n\to\infty} \int_\Omega |f_n - f|\, d\mu = 0 \tag{1.36}$$

Problem 1.29. *Let Ω be a measurable set in $\mathbb{R}^n$, f, g be two Lebesgue integrable funcitons in Ω. Define*

$$B = \{x \in \Omega : f(x) \neq g(x)\} \tag{1.37}$$

Suppose $\mu(B) = 0$. Let E be a measurable set contained in Ω. Prove that

$$\int_E f d\mu = \int_E g d\mu \tag{1.38}$$

HINT. Prove

$$\int_E f d\mu = \int_{E\cap B} f d\mu + \int_{E\setminus B} f d\mu \tag{1.39}$$

Try it. □

Problem 1.30. *Let Ω be a measurable set in $\mathbb{R}^n$, and f, g be two Lebesgue integrable functions in Ω. Define*

$$B = \{x \in \Omega : f(x) \neq g(x)\} \tag{1.40}$$

Suppose that for all measurable set E contained in Ω

$$\int_E f d\mu = \int_E g d\mu \tag{1.41}$$

Prove that $\mu(B) = 0$.

HINT. Define

$$C_n = \left\{x \in \Omega : f(x) - g(x) > \frac{1}{n}\right\} \tag{1.42}$$

$$D_n = \left\{x \in \Omega : g(x) - f(x) > \frac{1}{n}\right\} \tag{1.43}$$

for all positive integers n. Notice that

$$B = \left(\bigcup_{n=1}^{\infty} C_n\right) \cup \left(\bigcup_{n=1}^{\infty} D_n\right) \tag{1.44}$$

and

$$\int_{C_n} (f-g)\, d\mu \geq \int_{C_n} \frac{1}{n} d\mu \tag{1.45}$$

$$= \frac{1}{n}\mu(C_n) \tag{1.46}$$

Try it. □

Definition 1.31. Define $M(\Omega)$ be the set of all real measurable functions in Ω. Let f, g in $M(\Omega)$, we denote $f \sim g$ if

$$\mu(\{x \in \Omega : g(x) - f(x) \neq 0\}) = 0 \tag{1.47}$$

Problem 1.32. *Prove the relation $\sim$ is a equivalent relation in $M(\Omega)$.*

HINT. Let f, g, h in $M(\Omega)$, prove that

$$f \sim f \tag{1.48}$$

$$f \sim g \Rightarrow g \sim f \tag{1.49}$$

$$(f \sim g) \wedge (g \sim h) \Rightarrow f \sim h \tag{1.50}$$

Definition 1.33. Suppose that for all $x \in \Omega$, there exists a *property* $P(x)$. We say that P *holds almost everywhere* (abbr. *a.e.*) in Ω if

$$\mu(\{x \in \Omega : P(x) \text{ fails}\}) = 0 \tag{1.51}$$

Definition 1.34. Let $f \in M(\Omega)$. Define

$$\widetilde{f} = \{g \in M(\Omega) : g \sim f\} \tag{1.52}$$

$$N(\Omega) = \left\{\widetilde{h} : h \in M(\Omega)\right\} \tag{1.53}$$

Definition 1.35. Let α be a real number, and $f, g \in M(\Omega)$. Put $h = f + g$ and $k = \alpha f$ and

$$\widetilde{f} + \widetilde{g} = \widetilde{h} \tag{1.54}$$

$$\alpha \widetilde{f} = \widetilde{k} \tag{1.55}$$

Problem 1.36. *Prove that $N(\Omega)$ is a vector space with defined addition and multiplication operators defined by* (1.54) *and* (1.55)*.*

Hint. Prove that the addition and multiplication operators defined by (1.54) and (1.55) are well-defined as follows. Let $f_1, g_1 \in M(\Omega)$. Put $h_1 = f_1 + g_1$, $k_1 = \alpha f_1$. Check that $h \sim h_1, k \sim k_1$. Then check that the defined $+, \cdot$ operators satisfy all laws of a vector space. □

Problem 1.37. *Let $f, g \in M(\Omega)$ for which*

$$f(x) \le g(x) \text{ a.e. on } \Omega \tag{1.56}$$

Define

$$A = \{x \in \Omega : f(x) > g(x)\} \tag{1.57}$$

and

$$u(x) = \begin{cases} f(x), & \forall x \in \Omega \backslash A \\ 0, & \forall x \in A \end{cases} \tag{1.58}$$

$$v(x) = \begin{cases} g(x), & \forall x \in \Omega \backslash A \\ 0, & \forall x \in A \end{cases} \tag{1.59}$$

Prove that

$$u(x) = f(x) \text{ a.e. on } \Omega \tag{1.60}$$

$$v(x) = g(x) \text{ a.e. on } \Omega \tag{1.61}$$

$$v(x) \le u(x), \quad \forall x \in \Omega \tag{1.62}$$

Definition 1.38. Given $u, v \in N(\Omega)$, $f \in u$ and $g \in v$. We denote $u \le v$ if

$$\mu(\{x \in \Omega : f(x) > g(x)\}) = 0 \tag{1.63}$$

Problem 1.39. *Prove that $u \le v$ is well-defined.*

Hint. Given $h \in u$ and $k \in v$. Prove that

$$\mu(\{x \in \Omega : h(x) > k(x)\}) = 0 \tag{1.64}$$

Definition 1.40. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $N(\Omega)$ and $u \in N(\Omega)$. We say that $\{u_n\}_{n=1}^{\infty}$ *converges to* u in Ω if there exists $f \in u$ and $f_n \in u_n$ such that $\{f_n\}_{n=1}^{\infty}$ converges to f a.e. on Ω.

Problem 1.41. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $N(\Omega)$ converging to u in $N(\Omega)$. Let $g \in u$ and $g_n \in u_n$. Prove that $\{g_n\}_{n=1}^{\infty}$ converges to g a.e. on Ω.*

HINT. Define

$$B = \Omega \backslash \left\{ x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x) \right\} \tag{1.65}$$

$$A_n = \{x \in \Omega : f_n(x) \neq g_n(x)\} \tag{1.66}$$

$$A = \{x \in \Omega : f(x) \neq g(x)\} \tag{1.67}$$

$$D = A \cup B \cup \left(\bigcup_{n=1}^{\infty} A_n \right) \tag{1.68}$$

Prove $\mu(D) = 0$ and $\{g_n\}_{n=0}^{\infty}$ converges to g in $\Omega \backslash D$.

Definition 1.42. Let $p \in [1, \infty)$, $\widetilde{f} \in L^p(\Omega)$ and $g \in \widetilde{f}$. Define

$$\int_{\Omega} \widetilde{f} dx = \int_{\Omega} g dx \tag{1.69}$$

We call $\int_{\Omega} \widetilde{f} dx$ the *integral* of $\widetilde{f}$.

Problem 1.43. *Prove that $\int_{\Omega} \widetilde{f} dx$ is well-defined.*

Problem 1.44. *Suppose $\int_{\Omega} \left| \widetilde{f} \right| dx = 0$ and $\mu(\Omega) > 0$. Prove that $\widetilde{f} = \widetilde{0}$.*

HINT. Given $h \in \tilde{f}$, prove that $h \sim 0$. □

Problem 1.45. *Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in $N(\Omega)$ and u in $N(\Omega)$.Suppose*

1. $\{u_n\}_{n=0}^{\infty}$ converges to u in Ω.
2. $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots$ *in* Ω.

Prove that

$$\lim_{n\to\infty} \int_{\Omega} u_n dx = \int_{\Omega} u dx \tag{1.70}$$

HINT. Let $f_n \in u_n$. Define

$$B_n = \{x \in \Omega : f_n(x) > f_{n+1}(x)\} \tag{1.71}$$

and

$$B = \bigcup_{n=1}^{\infty} B_n \tag{1.72}$$

Prove that $\mu(B) = 0$, then apply previous problems. □

Definition 1.46. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in $N(\Omega)$ and $f_n \in u_n$. Define

1. $\liminf_{n\to\infty} u_n$ is the equivalent class of $\liminf_{n\to\infty} f_n$.

2. $\limsup_{n\to\infty} u_n$ is the equivalent class of $\limsup_{n\to\infty} f_n$.

Problem 1.47. *Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in $N(\Omega)$. Suppose $u_n \geq 0$ in Ω for all positive integers n. Prove that*

$$\int_{\Omega} \liminf_{n\to\infty} u_n dx \leq \liminf_{n\to\infty} \int_{\Omega} u_n dx \tag{1.73}$$

HINT. Proceed as Problem 1.45. □

Problem 1.48. *Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in $N(\Omega)$ and u in $N(\Omega)$. Suppose that there exists a v in $N(\Omega)$ such that*

1. $\{u_n\}_{n=0}^{\infty}$ converges to u in Ω.

2. $|u_n| \leq v$ in Ω, $\forall n \in \mathbb{Z}_+$.

3. $\int_{\Omega} v dx < \infty$.

Prove that

$$\lim_{n\to\infty} \int_{\Omega} u_n dx = \int_{\Omega} u dx \tag{1.74}$$

and

$$\lim_{n\to\infty} \int_{\Omega} |u_n - u| \, dx = 0 \tag{1.75}$$

HINT. Proceed as Problem 1.45. □

Definition 1.49. Given $p \in [1, \infty)$. We denote by $L^p(\Omega)$ the set of all class of function u in $N(\Omega)$ for which

$$\int_{\Omega} |u|^p dx < \infty \tag{1.76}$$

We define

$$\|u\|_p = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}, \quad \forall u \in L^p(\Omega) \tag{1.77}$$

Definition 1.50. We denote by $L^{\infty}(\Omega)$ the set of all class of functions u in $N(\Omega)$ such that there exist a f in u and a real number M such that

$$\mu(\{u : |f(x)| > M\}) = 0 \tag{1.78}$$

We define

$$\|u\|_{\infty} = \inf \{M \geq 0 : \mu(\{x \in \Omega : |f(x)| > M\}) = 0\} \tag{1.79}$$

for all $u \in L^{\infty}(\Omega)$ and $f \in u$.

Problem 1.51. *Let $p \in [1, \infty]$, $u \in L^p(\Omega)$ and $f \in u$. Prove that*

$$\mu(\{x \in \Omega : f(x) = \infty\}) = 0 \tag{1.80}$$

Problem 1.52. *Prove that* $\|\cdot\|_\infty$ *is a norm in* $L^\infty(\Omega)$.

Theorem 1.53 (Hölder). *Let* $p, q \in (1, \infty)$ *such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.81}$$

and $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$. *Then the following inequality holds*

$$\left| \int_\Omega fg dx \right| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{1.82}$$

Theorem 1.54 (Minkowski). *Let* $p \in [1, \infty)$, $f, g \in L^p(E)$. *Then*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \tag{1.83}$$

Problem 1.55. *Given* $p \in [1, \infty)$. *Prove that* $\left(L^p(\Omega), \|\cdot\|_p\right)$ *is a normed space.*

Problem 1.56. *Given* $p \in [1, \infty)$ *and* $\{u_n\}_{n=1}^\infty$ *in* $L^p(\Omega)$, *suppose that* $\sum\limits_{n=1}^{\infty} u_n$ *converges to* u *in* $L^p(\Omega)$. *Prove that*

$$\|u\|_{L^p(\Omega)} \leq \sum_{n=1}^{\infty} \|u_n\|_{L^p(\Omega)} \tag{1.84}$$

Hint. Prove

$$\left\| \sum_{n=1}^{m} u_n \right\|_{L^p(\Omega)} \leq \sum_{n=1}^{m} \|u_n\|_{L^p(\Omega)} \tag{1.85}$$

for all positive integers m. □

Problem 1.57. *Let* $p \in [1, \infty)$ *and* $\{v_n\}_{n=1}^\infty$ *be a sequence in* $L^p(\Omega)$ *and* $v \in L^p(\Omega)$. *Suppose that there exist* $g \in L^p(\Omega)$, $f \in u$ *and* $f_n \in u_n$ *for all positive integers* n, *such that*

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \forall x \in \Omega \tag{1.86}$$

$$|f_n(x)| \leq g(x) \tag{1.87}$$

Prove that

$$\lim_{n \to \infty} \int_\Omega |u_n - u|^p dx = 0 \tag{1.88}$$

Hint. Use Lebesgue dominated convergence theorem to prove

$$\lim_{n \to \infty} \int_\Omega |f_n - f|^p dx = 0 \tag{1.89}$$

Try it. □

Problem 1.58. *Let $p \in [1, \infty)$ and $\{u_n\}_{n=1}^{\infty}$ be a sequence in $L^p(\Omega)$. Suppose that*

$$\sum_{n=1}^{\infty} \|u_n\|_{L^p(\Omega)} \leq 1 \tag{1.90}$$

Let $f_n \in u_n$ for all positive integers n. Prove that $\sum\limits_{n=1}^{\infty} f_n(x)$ converges a.e. in Ω, and there exists a $v \in L^p(\Omega)$ such that

$$\left|\sum_{n=1}^{m} u_n(x)\right| \leq v, \quad \forall m \in \mathbb{Z}_+ \tag{1.91}$$

Hint. Define

$$g_m(x) = \sum_{n=1}^{m} |f_n(x)| \tag{1.92}$$

$$g(x) = \sum_{n=1}^{\infty} |f_n(x)| \tag{1.93}$$

for all $x \in \Omega$.

Use Minkowski theorem to prove $\|g_m\|_{L^p(\Omega)} \leq 1$.

Notice that

$$g = \liminf_{m \to \infty} g_m \tag{1.94}$$

$$g^p = \liminf_{m \to \infty} g_m^p \tag{1.95}$$

Apply Fatou's lemma to prove $\|g\|_{L^p(\Omega)} < \infty$.

Prove that $g(x)$ is a real number a.e. on Ω. □

Problem 1.59. *Let $p \in [1, \infty)$ and $\{v_n\}_{n=1}^{\infty}$ be a sequence converging to v in $L^p(\Omega)$. Given $w_n \in v_n$ and $w \in v$. Prove that there exists subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{w_n\}_{n=1}^{\infty}$ and h such that*

$$w_{n_k} \to w \text{ as } k \to \infty \text{ a.e. in } \Omega \tag{1.96}$$

$$\int_{\Omega} h^p dx < \infty \tag{1.97}$$

$$|w_{n_k}(x)| \leq h(x) \text{ a.e. in } \Omega \tag{1.98}$$

Hint. Choose a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{w_n\}_{n=1}^{\infty}$ such that

$$\left\|w_{n_{k+1}} - w_{n_k}\right\|_{L^p(\Omega)} \leq \frac{1}{2^k}, \quad \forall k \in \mathbb{Z}_+ \tag{1.99}$$

Apply Problem 1.57 for

$$u_k = v_{n_{k+1}} - v_{n_k} \tag{1.100}$$

$$f_k = w_{n_{k+1}} - w_{n_k} \tag{1.101}$$

Define $h = g + |w_{n_1}|$. Notice that

$$w_{n_{k+1}} = w_{n_1} + \sum_{j=1}^{k} f_j \tag{1.102}$$

converges a.e. to w in Ω. Use Problem 1.57, prove

$$\lim_{k\to\infty} \|w_{n_k} - w\|_{L^p(\Omega)} = 0 \tag{1.103}$$

to deduce $\|w - z\|_{L^p(\Omega)} = 0$. □

Problem 1.60. *Given $p \in [1, \infty)$. Prove that $L^p(\Omega)$ is a Banach space.*

HINT. Let $\{v_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\Omega)$. Choose a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that

$$\left\|v_{n_{k+1}} - v_{n_k}\right\|_{L^p(\Omega)} \leq \frac{1}{2^k}, \forall k \in \mathbb{Z}_+ \tag{1.104}$$

HINT. Proceed as Problem 1.59. □

Problem 1.61. *Prove that $L^\infty(\Omega)$ is a Banach space.*

Problem 1.62. *Prove that $L^2(\Omega)$ is a Hilbert space with the following scalar product*

$$\langle u, v\rangle = \int_\Omega uv dx, \ \ \forall u, v \in L^2(\Omega) \tag{1.105}$$

Problem 1.63. *Given $p \in [1, \infty)$. Define*

$$S = \{u \in L^p(\Omega) : \text{ there exists a simple functions } \in u\} \tag{1.106}$$

Prove that S is dense in $L^p(\Omega)$.

HINT. Given $v \in L^p(\Omega)$, prove that there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in S converging to v in $L^p(\Omega)$. Consider the case $v \geq 0$. Use Theorem 1.10 and Problem 1.57. □

Problem 1.64. *Given $p \in [1, \infty)$. Define*

$$C = \{u \in L^p(\Omega) : \text{ there exists a continuous function } f \in u\} \tag{1.107}$$

Prove that C is dense in $L^p(\Omega)$.

HINT. Given $v \in S$. Use Problem 1.7 to prove that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in C which converges to v in $L^p(\Omega)$.

Definition 1.65. Denote by $C_c(\mathbb{R}^n)$ the set of all real continuous function f in $\mathbb{R}^n$ such that there exists a compact set K_f containing the set

$$\{x \in \mathbb{R}^n : f(x) \neq 0\} \tag{1.108}$$

Problem 1.66. *Given $p \in [1, \infty)$. Define*

$$C_c = C \cap C_c\left(\mathbb{R}^n\right) \tag{1.109}$$

Prove that C_c is dense in $L^p(\Omega)$.

HINT. Use Problem 1.7 to prove there exists a real continuous function g_m from $\mathbb{R}^n$ into $[0, 1]$ such that

$$g_m(x) = \begin{cases} 1, & \text{if } \|x\| \leq m \\ 0, & \text{if } \|x\| > m+1 \end{cases} \tag{1.110}$$

Given $f \in C$. Define

$$f_m = g_m f \tag{1.111}$$

Apply Problem 1.57. □

Theorem 1.67. *Let $p \in [1, \infty)$, $q \in (1, \infty]$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.112}$$

and T be a continuous linear mapping from $L^p(\Omega)$ into $\mathbb{R}$. Then there exists a unique $u \in L^q(\Omega)$ for which

$$T(v) = \int_\Omega uv dx, \quad \forall v \in L^p(\Omega) \tag{1.113}$$

$$\|T\| = \|u\|_q \tag{1.114}$$

Theorem 1.68. *Let $p \in (1, \infty)$, $q \in (1, \infty)$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.115}$$

and $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $L^p(\Omega)$. Then there exist a $u \in L^p(\Omega)$ and a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ such that

$$\lim_{k \to \infty} \int_\Omega u_{n_k} v dx = \int_\Omega uv dx, \quad \forall v \in L^q(\Omega) \tag{1.116}$$

Problem 1.69. *Let $u \in L^1(\mathbb{R}^n)$ and α be a nonzero real number. Given $f \in u$, define $g(x) = f(\alpha x)$ for all $x \in \mathbb{R}^n$. Prove that g is Lebesgue integrable and*

$$\int_{\mathbb{R}^n} g d\mu = \frac{1}{|\alpha|^n} \int_{\mathbb{R}^n} f d\mu \tag{1.117}$$

HINT. Let E be a measurable set in $\mathbb{R}^n$ such that $\mu(E) < \infty$. Consider the case that f is the characteristic function of E. Prove that g is the characteristic function of $\frac{1}{\alpha}E$. □

Problem 1.70. *Let $u \in L^1(\mathbb{R}^n)$ and a be a vector in $\mathbb{R}^n$. Given $f \in u$, define $g(y) = f(a - y)$ for all $y \in \mathbb{R}^n$. Prove that g is Lebesgue integrable and*

$$\int_{\mathbb{R}^n} g d\mu = \int_{\mathbb{R}^n} f d\mu \tag{1.118}$$

Hint. Let E be a measurable set in $\mathbb{R}^n$ such that $\mu(E) < \infty$. Consider the case that f is the characteristic function of E. Prove that g is the characteristic function of $(a - E)$. $\square$

Problem 1.71. *Let $u \in L^1(\mathbb{R}^n)$ and ε be a positive real number. Prove that there exists a positive real number δ such that for all measurable set E satisfying $\mu(E) < \delta$, the following inequality holds*

$$\int_E |u|\, d\mu < \varepsilon \tag{1.119}$$

Hint. Consider the following cases: f is the characteristic function of E, f is a simple function, f is a nonnegative function. $\square$

Problem 1.72. *Let E be a measurable set in $\mathbb{R}^n$ satisfying $\mu(E) < \infty$, and $r, s \in [1, \infty)$ such that $r < s$. Prove that $L^s(E) < L^r(E)$.*

Hint. Define

$$p = \frac{s}{r} \tag{1.120}$$

$$q = \frac{1}{1 - \frac{r}{s}} \tag{1.121}$$

Let $u \in L^s(E)$, $f \in u$ and g is the characteristic function of E. Applying Hölder inequality for $|f|^r$ and g. $\square$

Problem 1.73. *Let E be a measurable set in $\mathbb{R}^n$, $p \in [1, \infty)$, $u \in L^p(E)$ and $f \in u$. Suppose*

$$\int_E f g d\mu = 0, \quad \forall g \in C_c(\mathbb{R}^n) \tag{1.122}$$

Prove that $f = 0$ a.e. in E.

Hint. Use Problem 1.66 and Hölder inequality, prove (1.122) holds when g is the characteristic function of a measurable set f satisfying $\mu(F) < \infty$

Problem 1.74. *Let E be a measurable set in $\mathbb{R}^n$, $p \in [1, \infty]$, $u \in L^p(E)$ and $f \in u$. Define*

$$g(x) = \begin{cases} f(x), \text{ if } x \in E \\ 0, \text{ if } x \in \mathbb{R}^n \backslash E \end{cases} \tag{1.123}$$

Prove that $|g|^p$ is Lebesgue integrable. $\square$

2 Convolution Product

We will identify $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$. Let A be a subset in $\mathbb{R}^{m+n}$ and f be a real function in A. For each $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we define

$$A_x = \{y : (x, y) \in A\} \tag{2.1}$$

$$A^y = \{x : (x,y) \in A\} \tag{2.2}$$
$$f_x(y) = f(x,y), \text{ if } y \in A_x \tag{2.3}$$
$$f^y(x) = f(x,y), \text{ if } x \in A^y \tag{2.4}$$

Theorem 2.1. *Let f be a Lebesgue measurable real function in $\mathbb{R}^{m+n}$. Then there exists measurable sets A and B, which have zero measure in $\mathbb{R}^m$ and $\mathbb{R}^n$, such that*

1. *f_x is a measurable function in $\mathbb{R}^n$ for all $x \in \mathbb{R}^m \backslash A$.*
2. *f^y is a measurable function in $\mathbb{R}^m$ for all $y \in \mathbb{R}^n \backslash B$.*

Theorem 2.2 (Fubini). *Let f be a nonnegative measurable function in $\mathbb{R}^{m+n}$. Then*

$$\int_{\mathbb{R}^{m+n}} f d\mu_{m+n} = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f_x d\mu_n \right) d\mu_m \tag{2.5}$$
$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f^y d\mu_m \right) d\mu_n \tag{2.6}$$

Theorem 2.3 (Tonelli). *Let g be a measurable real function in $\mathbb{R}^{m+n}$ such that*

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |g|_x d\mu_n \right) d\mu_m < \infty \tag{2.7}$$

Then g is Lebesgue integrable in $\mathbb{R}^{m+n}$.

The converse of Theorem 2.3 is only holds as follows.

Theorem 2.4 (Fubini). *Let g be a Lebesgue integrable function in $\mathbb{R}^{m+n}$. Then*

1. g_x is Lebesgue integrable in $\mathbb{R}^n$ for almost $x \in \mathbb{R}^n$.
2. g_y is Lebesgue integrable in $\mathbb{R}^m$ for almost $y \in \mathbb{R}^n$.
3. The following equality holds

$$\int_{\mathbb{R}^{m+n}} g d\mu_{m+n} = \int_{\mathbb{R}^m} \left(\int_{R^n} g_x d\mu_n \right) d\mu_m = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} g^y d\mu_m \right) d\mu_n \tag{2.8}$$

Problem 2.5. *Let h be a measurable function in $\mathbb{R}^n$. Define $k(x,y) = h(y)$ for all $(x,y) \in \mathbb{R}^{n+n}$. Prove that k is measurable in $\mathbb{R}^{n+n}$.*

HINT. Use definitions. □

Problem 2.6. *Let h be a measurable function in $\mathbb{R}^n$. Define $k(x,y) = h(x-y)$ for all $(x,y) \in \mathbb{R}^{n+n}$. Prove that k is measurable in $\mathbb{R}^{n+n}$.*

HINT. Consider the case that h is a continuous function. Then, use Problem 1.7. □

Problem 2.7. *Let f, g be two Lebesgue integrable functions in $\mathbb{R}^n$. Define $k(x,y) = f(y)\,g(x-y)$ for all $(x,y) \in \mathbb{R}^{n+n}$. Prove that k is Lebesgue integrable in $\mathbb{R}^{n+n}$.*

Hint. Prove that k is measurable in $\mathbb{R}^{n+n}$. Use Fubini theorem to prove

$$\int_{\mathbb{R}^{n+n}} |k(z)|\,dz = \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|\,dx \right) dy \tag{2.9}$$

$$= \left(\int_{\mathbb{R}^n} |g(t)|\,dt \right) \left(\int_{\mathbb{R}^n} |f(y)|\,dy \right) \tag{2.10}$$

Problem 2.8. *Let f, g be two Lebesgue integrable functions in $\mathbb{R}^n$. Prove that there exists a set A such that $\mu(A) = 0$ and the following integral is defined for all $x \in \mathbb{R}^n \backslash A$.*

$$\int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy, \ \ \forall x \in \mathbb{R}^n \tag{2.11}$$

Hint. Use Problem 1.24 and Fubini theorem. □

Definition 2.9. *Let f, g be two Lebesgue integrable functions in $\mathbb{R}^n$. The convolution product of f and g is a function defined by*

$$f \star g(x) = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy, \ \ \forall x \in \mathbb{R}^n \tag{2.12}$$

Then $f \star g$ is Lebesgue integrable and

$$\|f \star g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \tag{2.13}$$

Definition 2.10. Let $u, v \in L^1(\mathbb{R}^n)$. Given $f \in u$ and $g \in v$. We call the equivalent class of $f \star g$ the convolution product of u and v, which is denoted by $u \star v$. Then

$$\|u \star v\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)} \tag{2.14}$$

Problem 2.11. *Given $u, v \in L^1(\mathbb{R}^n)$. Prove that*

$$u \star v = v \star u \tag{2.15}$$

Hint. Use Problem 1.70. □

Problem 2.12. *Let $p \in (1,\infty)$, $u \in L^1(\mathbb{R}^n)$, and $v \in L^p(\mathbb{R}^n)$. Given $f \in u, g \in v$. Prove that $f \star g$ is defined a.e. and $|f \star g|^p$ is Lebesgue integrable in $\mathbb{R}^n$.*

Hint. Consider $|f| \star |g|^p$. Define $q = \frac{p}{p-1}$. Use Hölder inequality to prove

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|^{\frac{1}{q}} |f(y)|^{\frac{1}{p}} |g(x-y)|\,dy \right)^p dx \tag{2.16}$$

$$\leq \int_{\mathbb{R}^n} \left(\left(\int_{\mathbb{R}^n} |f(y)|\,dy \right)^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |f(y)|\,|g(x-y)|^p dy \right) \right) dx \tag{2.17}$$

$$\leq \left(\int_{\mathbb{R}^n} |f(y)|\, dy\right)^{\frac{p}{q}} \int_{\mathbb{R}^n} \left(\int_{R^n} |f(y)|\, |g(x-y)|^p dy\right) dx \tag{2.18}$$

$$\leq \left(\int_{\mathbb{R}^n} |f(y)|\, dy\right)^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |f(y)|\, dy\right) \left(\int_{\mathbb{R}^n} |g(y)|^p dy\right) \tag{2.19}$$

$$\leq \left(\int_{\mathbb{R}^n} |f(y)|\, dy\right)^{p} \left(\int_{\mathbb{R}^n} |g(y)|^p dy\right) \tag{2.20}$$

$$= \|f\|_{L^1(\mathbb{R}^n)}^p \|g\|_{L^p(\mathbb{R}^n)}^p \tag{2.21}$$

Definition 2.13. Given $p \in (1, \infty)$, $u \in L^1(\mathbb{R}^n)$, $v \in L^p(\mathbb{R}^n)$, $f \in u$ and $g \in v$. We call the equivalent class of $f \star g$ the *convolution product* of u and v, which is denoted by $u \star v$. The following inequality holds

$$\|u \star v\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^p(\mathbb{R}^n)} \tag{2.22}$$

Given $s = (s_1, \ldots, s_n)$, we define

$$|s| = \sum_{i=1}^{n} |s_i| \tag{2.23}$$

and denote partial derivative $\frac{\partial^s f}{\partial x}$ by $\frac{\partial^{|s|} f}{\partial^{s_1} x_1 \cdots \partial^{s_n} x_n}$.

Definition 2.14. Let r be a positive integer and Ω be an open set in $\mathbb{R}^n$. Define $C^r(\Omega)$ the set of real function in Ω such that all partial derivatives of order s of f exist and are continuous in Ω if $|s| \leq r$.

Definition 2.15. Let r be a positive integer and Ω be an open set in $\mathbb{R}^n$. Define $C_c^r(\Omega)$ is the set of all functions $f \in C^r(\Omega)$ such that there exists a compact set K_f for which $f(x) = 0$ for all $x \in \mathbb{R}^n \backslash K_f$. Then we define

$$C^\infty(\Omega) = \bigcap_{r=1}^{\infty} C^r(\Omega) \tag{2.24}$$

$$C_c^\infty(\Omega) = \bigcap_{r=1}^{\infty} C_c^r(\Omega) \tag{2.25}$$

Problem 2.16. *Given $p \in [1, \infty)$, $u \in L^p(\mathbb{R}^n)$, $f \in u$ and $g \in C_c^r(\mathbb{R}^n)$. Prove $f \star g \in C^r(\mathbb{R}^n)$ and*

$$\frac{\partial^s (f \star g)}{\partial x} = f \star \frac{\partial^s g}{\partial x}, \quad \forall s, |s| \leq r \tag{2.26}$$

Hint. Choose a compact set $K \subset \mathbb{R}^n$ such that $g(x) = 0$ for all $x \in \mathbb{R}^n \backslash K$. Choose a positive real number r_0 such that $K \subset B(0, r_0)$. Put $e := (1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $t \in (-1, 1) \setminus \{0\}$. Prove that

$$g(y + te) - g(y) = 0, \quad \forall y \in \mathbb{R}^n \backslash B(0, r_0 + 2) \tag{2.27}$$

Prove

$$\frac{f \star g(x + te) - f \star g(x)}{t} = \int_{B(0,r+2)} f(y) \frac{g(x + te - y) - g(x - y)}{t} dy \tag{2.28}$$

Then use Lebesgue dominated convergence theorem to prove

$$\lim_{t\to 0}\frac{f\star g(x+te)-f\star g(x)}{t}=\int_{B(0,r+2)} f(y)\frac{\partial g}{\partial x_1}(x-y)\,dy \tag{2.29}$$

$$=\int_{\mathbb{R}^n} f(y)\frac{\partial g}{\partial x_1}(x-y)\,dy \tag{2.30}$$

$$=f\star\frac{\partial g}{\partial x_1}(x) \tag{2.31}$$

Problem 2.17. *Prove that there exists a function $\rho\in C_c^\infty(\mathbb{R}^n)$ satisfying the following property*

$$\rho(x)\geq 0,\ \ \forall x\in\mathbb{R}^n \tag{2.32}$$

$$\rho(x)=0,\ \ \forall x\in\mathbb{R}^n\backslash B(0,1) \tag{2.33}$$

$$\int_{R^n}\rho d\mu=1 \tag{2.34}$$

Hint. Define

$$\phi(t)=\begin{cases} e^{\frac{1}{t^2-1}}, & \forall t\in\mathbb{R},|t|<1\\ 0, & \forall t\in\mathbb{R},|t|\geq 1\end{cases} \tag{2.35}$$

Prove that $\phi\in C^\infty(\mathbb{R})$ and

$$\phi^{(m)}(t)=\begin{cases} e^{\frac{1}{t^2-1}}\sum\limits_{\alpha,\beta}c_{m,\alpha,\beta}t^\alpha\left(t^2-1\right)^\beta, & \forall t\in\mathbb{R},|t|<1\\ 0, & \forall t\in\mathbb{R},|t|\geq 1\end{cases} \tag{2.36}$$

Define $\psi(x)=\phi\left(|x|^2\right)$ for all $x\in\mathbb{R}^n$ and

$$c=\int_{\mathbb{R}^n}\psi d\mu \tag{2.37}$$

Define

$$\rho(x)=\frac{1}{c}\psi(x),\ \ \forall x\in\mathbb{R}^n \tag{2.38}$$

Check that ρ defined by (2.38) satisfies all requirements. □

Problem 2.18. *Let ρ be defined by (2.38). Define $\rho_m(x)=m^n\rho(mx)$ for all positive integer m and for all $x\in\mathbb{R}^n$. Prove that*

$$\rho_m\in C_c^\infty(\mathbb{R}^n) \tag{2.39}$$

$$\rho_m(x)\geq 0,\ \ \forall x\in\mathbb{R}^n \tag{2.40}$$

$$\rho_m(x)=0,\ \ \forall x\in\mathbb{R}^n\backslash B\left(0,\frac{1}{m}\right) \tag{2.41}$$

$$\int_{\mathbb{R}^n}\rho_m d\mu=1 \tag{2.42}$$

Hint. Use Problem 1.69. □

Problem 2.19. *Let f be a real continuous function in $\mathbb{R}^n$. Define*

$$f_m(x) = \int_{R^n} f(y)\,\rho_m(x-y)\,dy, \quad \forall x \in \mathbb{R}^n \tag{2.43}$$

Let r and ϵ be two positive real numbers. Prove that there exists a positive integer N such that

$$|f(x) - f_m(x)| \leq \varepsilon, \quad \forall m \geq N, x \in B(0,r) \tag{2.44}$$

Hint. Choose N such that

$$|f(x) - f(z)| \leq \varepsilon, \quad \forall x, z \in B'(0, r+1), |x-z| \leq \frac{1}{N} \tag{2.45}$$

Prove

$$f(x) - f_m(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y))\,\rho_m(y)\,dy \tag{2.46}$$

$$= \int_{B\left(0,\frac{1}{m}\right)} (f(x) - f(x-y))\,\rho_m(y)\,dy \tag{2.47}$$

Problem 2.20. *Given $p \in [1,\infty)$, $u \in L^p(\mathbb{R}^n)$ and $f \in u$. Prove that there exists a sequence $\{f_m\}_{m=1}^{\infty}$ in $C_c^{\infty}(\mathbb{R}^n)$ such that*

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |f - f_m|^p d\mu = 0 \tag{2.48}$$

Hint. Define

$$g_k(x) = \begin{cases} f(x), & \text{if } |x| < k \\ 0, & \text{if } |x| \geq k \end{cases} \tag{2.49}$$

Use Lebesgue dominated convergence theorem to prove

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |f - g_m|^p d\mu = 0 \tag{2.50}$$

Then apply Problem 1.64, Problem 2.18 and Lebesgue dominated convergence theorem. □

3 Fourier Transform

Definition 3.1. Let f, g be two real Lebesgue integrable functions in $\mathbb{R}^n$. Define

$$\int_{\mathbb{R}^n} (f + ig)\,d\mu = \int_{\mathbb{R}^n} f d\mu + i \int_{\mathbb{R}^n} g d\mu \tag{3.1}$$

Definition 3.2. Let f be a real Lebesgue integrable function in $\mathbb{R}$. Define

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(x)\,e^{-2\pi i t x} d\mu, \quad \forall t \in \mathbb{R}^n \tag{3.2}$$

We call $\widehat{f}$ the *Fourier transform* of f.

Problem 3.3. *Prove that $\widehat{f}$ is continuous in $\mathbb{R}^n$.*

HINT. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence converging to t in $\mathbb{R}^n$. Use Lebesgue dominated convergence theorem to prove that

$$\lim_{m\to\infty} \int_{R^n} f(x) \left(e^{-2\pi i t_m x} - e^{-2\pi i t x}\right) d\mu = 0 \tag{3.3}$$

Definition 3.4. Let $u \in L^1(\mathbb{R}^n)$ and $f \in u$. We denote the equivalent class of $\widehat{f}$ by $\widehat{u}$. We call $\widehat{u}$ the Fourier transform of u.

Problem 3.5. *Let f be a real Lebesgue integrable function in $\mathbb{R}^n$ and $z \in \mathbb{R}^n$. Define*

$$f_z(x) = f(x+z), \quad \forall x \in \mathbb{R}^n \tag{3.4}$$

Prove that

$$\widehat{f_z}(t) = \widehat{f}(t)\, e^{-2\pi i t z}, \quad \forall t \in \mathbb{R}^n \tag{3.5}$$

HINT. Use definition. □

Problem 3.6. *Given $f \in C_c^{\infty}(\mathbb{R}^n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Define $g = \frac{\partial^\alpha f}{\partial x}$. Prove that*

$$\widehat{g}(t) = \widehat{f}(t) \prod_{j=1}^{n} t_j^{\alpha_j}, \quad \forall t \in \mathbb{R}^n \tag{3.6}$$

HINT. Use definition and Lebesgue dominated convergence theorem. □

Problem 3.7. *Given $f \in C_c^{\infty}(\mathbb{R}^n)$. Prove that*

$$\lim_{|t|\to\infty} \widehat{f}(t) = 0 \tag{3.7}$$

HINT. Use Problem 1.66 and Problem 1.30. □

Problem 3.8. *Denote by $C_0(\mathbb{R}^n)$ the set of all real continuous functions h in $\mathbb{R}^n$ such that $h(t) \to 0$ as $|t| \to \infty$. Prove that $C_0(\mathbb{R}^n)$ is a normed space equipped the following norm*

$$\|h\|_{\infty} = \sup\{|h(t)| : t \in \mathbb{R}^n\} \tag{3.8}$$

Problem 3.9. *Prove that the mapping $u \to \widehat{u}$ is a continuous linear mapping from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.*

HINT. Use previous problems. □

Problem 3.10. *Let f and g be two real Lebesgue integrable functions in $\mathbb{R}^n$. Define $h = f \star g$ for all $x \in \mathbb{R}^n$. Prove that*

$$\widehat{h}(t) = \widehat{f}(t)\, \widehat{g}(t), \quad \forall t \in \mathbb{R}^n \tag{3.9}$$

HINT. Prove that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \right) e^{-2\pi i t x} dx \tag{3.10}$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)\, e^{-2\pi i t y} g(x-y)\, dy \right) e^{-2\pi i t (x-y)} dx \tag{3.11}$$

Use Fubini theorem. □

Problem 3.11. *Let f, g be two real Lebesgue integrable functions in $\mathbb{R}^n$. Prove that*

$$\int_{\mathbb{R}^n} \widehat{g} f d\mu = \int_{\mathbb{R}^n} \widehat{f} g d\mu \tag{3.12}$$

HINT. Use Fubini theorem to prove

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)\, e^{-2\pi i t x} dx \right) g(t)\, dt = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(t)\, e^{-2\pi i t x} dt \right) f(x)\, dx \tag{3.13}$$

Try it. □

Theorem 3.12. *Suppose that f and $\widehat{f}$ are Lebesgue integrable functions in $\mathbb{R}^n$. Then*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(t)\, e^{2\pi i t x} dt, \text{ a.e. in } \mathbb{R}^n \tag{3.14}$$

THE END

References

[1] Duong Minh Duc, *Real Analysis*, Faculty of Math and Computer Science, Ho Chi Minh University of Science.