Mathematical Analysis & Numerical Analysis
Giải Tích Toán Học & Giải Tích Số

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Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
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1. *Mathematical Analysis – Giải Tích Toán Học*.
   PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.pdf.
   TEX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.tex.

Mục lục

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    Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

1 Basic

Resources – Tài nguyên.

1. [LL01]. Elliott Lieb, Michael Loss. *Analysis.*

2. [Rud76]. Walter Rudin. *Principles Principles of Mathematical Analysis.*

3. [Rud73; Rud87]. Walter Rudin. *Real & Complex Analysis.*

4. [Tao22a]. Terence Tao. *Analysis I.*

5. [Tao22b]. Terence Tao. *Analysis II.*

"Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones." – [LL01, Chap. 1, p. 1]

2 C_0 Semigroup – Nửa Nhóm C_0

Resources – Tài nguyên.

1. [AK16]. Cung Thế Anh, Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng.*

"In mathematical analysis, a C_0-*semigroup*, also known as a *strongly continuous 1-parameter semigroup*, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant ODEs, strongly continuous semigroups provide solutions of linear constant coefficient ODEs in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations & PDEs. Formally, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+, +)$ on some Banach space X that is continuous in the strong toperator topology." -Wikipedia/C_0-semigroup

3 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

1. [Car16]. Manfredo P. do Carmo. *Differential Geometry of Curves & Surfaces.*

2. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries.*

3. [Küh15]. Wolfgang Kühnel. *Differential Geometry.*

4. [Wal15]. Shawn W. Walker. *The Shapes of Things.*

    "Differential geometry is the detailed study of the *shape* of a surface (manifold), including *local* & *global* properties. A plane in $\mathbb{R}^3$ is a very simple surface & does not require many tools to characterize. An "arbitrarily" shaped surface, e.g., hood of a car, has many distinguished geometric features (e.g., highly curved regions, regions of near flatness, etc.). Characterizing these features quantitatively & qualitatively requires the tools of differential geometry. Geometric details are important in many physical & biological processes, e.g., surface tension, biomembranes.

    The framework of differential geometry is built by 1st defining a local map (i.e., surface parameterization) which defines the surface. Then, a calculus framework is built up on the surface analogous to the standard "Euclidean calculus". Other approaches are also possible, e.g., those with implicit surfaces defined by level sets & distance functions. But parameterizations, though

arbitrary, are quite useful in a variety of settings $\Rightarrow$ stick mostly with those. Emphasize: The geometry of a surface does not depend on a particular parameterization. Otherwise, we will emphasize the distinction between `object 1` & `object 2`.

We will use this "abuse" of notation when there is no possibility of ambiguity.

Open set. The concept of open set is critical in multivariate calculus to properly define differentiability. The notation for referencing boundaries of sets, as well as the closure of sets, is practical for referencing geometric details of solid objects & their surfaces.

Compactness. Compact support is useful for ignoring boundary effects. This concept is needed to keep the "action of a function" away from the boundary of a set, or to localize the function in a region of interest. 1 reason is to avoid potential difficulties with differentiating a function at its boundary of definition. Or, more commonly, we wish to ignore a quantity depending on the value of a function at a boundary point, e.g., $\int_{\partial S} f = 0$ if f has compact support in S.

Topological mapping/homeomorphism. A bijective, continuous mapping $\mathbf{\Phi}$ whose inverse $\mathbf{\Phi}^{-1}$ is also continuous is called a *topological mapping* or *homeomorphism*. Point sets that can be topologically mapped onto each other are said to be *homeomorphic*. Sets that are homeomorphic have the "same topology", i.e., their connectedness is the same; they have the same kinds of "holes". See [Wal15, Sect. 2.3.1] for what can happen when a mapping is not a homeomorphism.

Rigid motion mapping. A mapping $\mathbf{\Phi}$ is called a *rigid motion* if any pair of points $\mathbf{a}, \mathbf{b}$ are the same distance apart as the corresponding pair $\mathbf{\Phi}(\mathbf{a}), \mathbf{\Phi}(\mathbf{b})$.

Orthogonal Transformations. Define the (affine) linear map $\mathbf{\Phi}$ (transformation)

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{1}$$

If A satisfies the properties $A^{-1} = A^\top$, $\det A = 1$ then $\mathbf{\Phi}$ represents a rigid motion. Basically, $\mathbf{\Phi}$ consists of a rotation represented by A followed by a translation represented by $\mathbf{b}$. A rigid motion can be used to transition from 1 Cartesian coordinate system to another. If $\mathbf{b} = \mathbf{0}$ & $A^{-1} = A^\top$, $\det A = 1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is a linear map known as a *direct orthogonal transformation*, which is nothing more than a rotation of the coordinate system with the origin as the center. If $A^{-1} = A^\top$, $\det A = 1$ is replaced by $A^{-1} = A^\top$, $\det A = -1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is called an *opposite orthogonal transformation*, which consists of a rotation about the origin & a reflection in a plane. Both $A^{-1} = A^\top$, $\det A = \pm 1$ are examples of *orthogonal matrices*.

Interpretation of transformations. Can interpret $\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in 2 different ways. Consider a point $P \in \mathbb{R}^3$ with coordinates $\mathbf{x}$:

- *Alias* (Euler perspective). Viewing (1) as a transformation of coordinates, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of the same point w.r.t. 2 different coordinate systems, equivalently, the point is referenced by 2 different "names" (sets of coordinates).

- *Alibi* (Lagrange perspective). Viewing (1) as a mapping of sets, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of 2 different points w.r.t. the same coordinate system, equivalently, the point at $\widetilde{\mathbf{x}}$ "was previously" at $\mathbf{x}$ before applying the map.

The concept of material point is directly related to the alibi viewpoint. One can think of a "particle" of material, i.e., *material point*, initially located at $\mathbf{x}$, that then moves to $\widetilde{\mathbf{x}}$ because of some physical process. The transformation (1) simply represents the kinematic outcome of that physical process, which is a standard concept in deformable continuum mechanics, especially nonlinear elasticity.

General transformations. In general, transformation may not be linear. The alias viewpoint yields a *curvilinear* coordinate system. The alibi viewpoint implies that the set S is *deformed* into the set $S' = \mathbf{\Phi}(S)$.

Parametric approach − what is a surface? A *surface* is a set of points in space that is "regular enough". A random scattering of points in space does not match our intuitive notion of what a surface is, i.e., it is not regular enough. The boundary of a sphere does match our notion of a surface, i.e., regular enough to be a surface because a sphere is "smooth". *Intuition*: Can think of creating a surface as deforming a flat rubber sheet into a curved sheet. Let $U \subset \mathbb{R}^2$ be a "flat" domain & let $\mathbf{X} : U \to \mathbb{R}^3$ be this deforming transformation, i.e., for each point $(s_1, s_2)^\top \in U$ there is a corresponding point $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ s.t. $\mathbf{x} = \mathbf{X}(s_1, s_2)$. Let $\Gamma = \mathbf{X}(U)$ denote the surface obtained from "deforming" U. Call $\mathbf{x} = \mathbf{X}(s_1, s_2)$ a *parametric representation* of the surface Γ, where s_1, s_2 are called the *parameters* of the representation. Refer to U as a *reference* domain.

Allowable parameterization/immersion. If use $\mathbf{x} = \mathbf{X}(s_1, s_2)$ to define surfaces, then we must place assumptions on $\mathbf{X}$ to guarantee that $\Gamma = \mathbf{X}(U)$ is a valid surface. At the bare minimum, $\mathbf{X}$ must be continuous to avoid "tearing" the rubber sheet. But if want to perform calculus on Γ, need more:

Assumption 1 (Regularity assumptions on $\mathbf{X}$)**.** *An allowable parameterization/immersion is a parameterization of the form* $\mathbf{x} = \mathbf{X}(s_1, s_2)$ *satisfying:*

(A1) The function $\mathbf{X}(s_1, s_2) \in C^\infty(U)$ *& each point* $\mathbf{x} = \mathbf{X}(s_1, s_2) \in \Gamma$ *corresponds to just 1 point* $(s_1, s_2) \in U$, *i.e.,* $\mathbf{X}$ *is injective.*

(A2) The Jacobian matrix $J = [\partial_{s_1}\mathbf{X}, \partial_{s_2}\mathbf{X}]$ *is of rank 2 on* U, *i.e., the columns of* J *are linearly independent.*

Regular surface. The fundamental property that makes a set of points in $\mathbb{R}^3$ a surface is that it *locally looks like a plane* at every point. If you "zoom into" a surface, it should look flat. Definition defining a surface in terms of a parameterization is inadequate. Want to define a set in $\mathbb{R}^3$ that is "intrinsically" 2D & is smooth enough so we can perform calculus on it, without regard to a specific parameterization.

Definition 1 (Regular surface)**.**

Remark 1 (Local chart)**.**

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

3.1 Calculus on Surfaces

Goal. Define & develop the fundamental tools of calculus on a regular surface. Start with the notion of differentiability of functions defined only on a surface. Define the concept of vector fields in a surface. Then proceed to develop the gradient & Laplacian operators w.r.t. a surface. These operators allow for alternative expressions of the summed & Gaussian curvatures. Derive integration by parts on surfaces, i.e., the domain of integration is a surface. Conclude with some useful identities & inequalities. Always take Γ: a regular surface, either with or without a boundary.

4 Functional Analysis – Giải Tích Hàm

Resources – Tài nguyên.

1. [Rud91]. WALTER RUDIN. *Functional Analysis.*

2. YOSIDA.

5 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

1. [ABT18]. RICHARD ASTER, BRIAN BORCHERS, CLIFFORD H. THURBER. *Parameter Estimation & Inverse Problems.*

2. [Kir21]. ANDREAS KIRSCH. *An Introduction to The Mathematical Theory of Inverse Problems.*

3. [IJ15]. KAZUFUMI ITO, BANGTI JIN. *Inverse Problems.*

6 Measure & Integration – Độ Đo & Tích Phân

Resources – Tài nguyên.

1. [EG15]. LAWRENCE C. EVANS, RONALD F. GARIEPY. *Measure Theory & Fine Properties of Functions.*

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

"The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$-dimensional volume 'in the other direction' by 1st computing the n-dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y, which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*" – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

7 Mean-Field Game Theory – Lý Thuyết Trò Chơi Trường Trung Bình

Community – Cộng đồng. NICHOLETTA TCHOU (FRENCH), ĐÀO MẠNH KHANG (VIETNAMESE), MICHAEL HINTERMÜLLER (AUSTRIAN), STEVEN-MARIAN STENGL (GERMAN).

7.1 Wikipedia/mean-field game theory

"*Mean-field game theory* is the study of strategic decision making by small interacting agents in very large populations. It lies at the intersection of game theory with stochastic analysis & control theory. The use of the term "mean field" is inspired by mean-field theory in physics, which considers the behavior of systems of large numbers of particles where individual particles have negligible impacts upon the system. In other words, each agent acts according to his minimization or maximization problem taking into account other agents' decisions & because their population is large we can assume the number of agents goes to infinity & a representative agent exists.

In traditional game theory, the subject of study is usually a game with 2 players & discrete time space, & extends the results to more complex situations by induction. However, for games in continuous time with continuous states (differential games or stochastic differential games) this strategy cannot be used because of the complexity that the dynamic interactions generate. On the other hand with MFGs we can handle large numbers of players through the mean representative agent & at the same time describe complex state dynamics.

This class of problems was considered in the economics literature by Boyan Jovanovic & Robert W. Rosenthal, in the engineering literature by Minyi Huang, Roland Malhame, & Peter E. Caines & independently & around the same time by mathematicians Jean-Michel Lasry & Pierre-Louis Lions.

In continuous time a mean-field game is typically composed of a Hamilton-Jacobi-Bellman equation that describes the optimal control problem of an individual & a Fokker–Planck equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions it can be proved that a class of mean-field games is the limit as $N \to \infty$ of an N-player Nash equilibrium.

A related concept to that of mean-field games is "mean-field-type control". In this case, a social planner controls the distribution of states & chooses a control strategy. The solution to a mean-field-type control problem can typically be expressed as a dual adjoint Hamilton-Jacobi-Bellman equation coupled with Kolmogorov equation. Mean-field-type game theory is the multi-agent generalization of the single-agent mean-field-type control.

7.1.1 General Form of a Mean-field Game

The system of equations

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, m, Du) = 0, \\ \partial_t m - \nu \Delta m - \nabla \cdot (D_p H(x, m, Du)m) = 0, \\ \qquad\qquad\qquad\qquad m(0) = m_0, \\ \qquad\qquad\qquad\quad u(T, x) = G(x, m(T)), \end{cases}$$

can be used to model a typical Mean-field game. The basic dynamics of this set of equations can be explained by an average agent's optimal control problem. In a mean-field game, an average agent can control their movement α to influence the population's overall location by

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t,$$

where ν: a parameter, B_t: a standard Brownian motion. By controlling their movement, the agent aims to minimize their overall expected cost C throughout the time period $[0, T]$:

$$C = \mathbb{E}\left[\int_0^T L(X_s, \alpha_s, m(s)) \, \mathrm{d}s + G(X_T, m(T)) \right],$$

where $L(X_s, \alpha_s, m(s))$ is the running cost at time s & $G(X_T, m(T))$ is the terminal cost at time T. By this definition, at time t & position x, the value function $u(t, x)$ can be determined as

$$u(t, x) = \inf_\alpha \mathbb{E}\left[\int_t^T L(X_s, \alpha_s, m(s)) \, \mathrm{d}s + G(X_T, m(T)) \right].$$

Given the definition of the value function $u(t, x)$, it can be tracked by the Hamilton-Jacobi equation. The optimal action of the average players $\alpha^*(t, x)$ can be determined as $\alpha^*(t, x) = D_p H(x, m, Du)$. As all agents are relatively small & cannot single-handedly change the dynamics of the population, they will individually adapt the optimal control & the population would move in that way. This is similar to a Nash Equilibrium, in which all agents act in response to a specific set of others' strategies. The optimal control solution then leads to the Kolmogorov-Fokker-Planck equation $\partial_t m - \nu \Delta m - \nabla \cdot (D_p H(x, m, Du)m) = 0$.

7.1.2 Finite State Games

A prominent category of mean field is games with a finite number of states & a finite number of actions per player. For those games, the analog of the Hamilton-Jacobi-Bellman equation is the Bellman equation, & the discrete version of the Fokker-Planck equation is the Kolmogorov equation. Specifically, for discrete-time models, the players' strategy is the Kolmogorv equation's probability matrix. In continuous time models, players have the ability to control the transition rate matrix.

A discrete mean field game can be defined by a tuple $\mathcal{G} = (\mathcal{E}, \mathcal{A}, \{Q_a\}, \mathbf{m}_0, \{c_a\}, \beta)$ where $\mathcal{E}$ is the state space, $\mathcal{A}$ the action set, Q_a the transition rate matrices, $\mathbf{m}_0$ the initial state, $\{c_a\}$ the cost functions & $\beta \in \mathbb{R}$ a discount factor. Furthermore, a mixed strategy is a measurable function $\pi : \mathbb{R}^+ \times \mathbb{E} \to \mathcal{P}(\mathcal{A})$, that associates to each state $i \in \mathcal{E}$ & each time $t \geq 0$ a probability

measure $\pi_i(t) \in \mathcal{P}(\mathcal{A})$ on the set of possible actions. Thus $\pi_{i,a}(t)$ is the probability that, at time t a player in state i takes action a, under strategy π. Additionally, rate matrices $\{Q_a(\mathbf{m}^\pi(t))\}_{a \in \mathcal{A}}$ define the evolution over the time of population distribution, where $\mathbf{m}^\pi(t) \in \mathcal{P}(\mathcal{E})$ is the population distribution at time t.

7.1.3 Linear-quadratic Gaussian game problem

From Caines (2009), a relatively simple model of large-scale games is the linear-quadratic Gaussian model. The individual agent's dynamics are modeled as a stochastic differential equation

$$dX_i = (a_i X_i + b_i u_i)dt + \sigma_i dW_i, \ i = 1, \dots, N,$$

where X_i: the state of the ith agent, u_i: control of the ith agent, W_i: independent Wiener processes $\forall i = 1, \dots, N$. The individual agent's cost is

$$J_i(u_i, \nu) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}[(X_i - \nu)^2 + r u_i^2]\, \mathrm{d}t\right], \ \nu = \Phi\left(\frac{1}{N}\sum_{k \neq i}^N X_k + \eta\right).$$

The coupling between agents occurs in the cost function.

7.1.4 General & Applied Use

The paradigm of Mean Field Games has become a major connection between distributed decision-making & stochastic modeling. Starting out tin the stochastic control literature, it is gaining rapid adoption across a range of applications, including:

1. **Financial market.** Carmona reviews applications in financial engineering & economics that can be cast & tackled within the framework of the MFG paradigm. Carmona argues that models in macroeconomics, contract theory, finance, . . ., greatly benefit from the switch to continuous time from the more traditional discrete-time models. He considers only continuous time models in his review chapter, including systemic risk, price impact, optimal execution, models for bank runs, high-frequency trading, & cryptocurrencies.

2. **Crowd motions.** MFG assumes that individuals are smart players which try to optimize their strategy & path w.r.t. certain costs (equilibrium with rational expectations approach). MFG models are useful to describe the anticipation phenomenon: the forward part describes the crowd evolution while the backward gives the process of how the anticipations are built. Additionally, compared to multi-agent microscopic model computations, MFG only requires lower computational costs for the macroscopic simulations. Some researchers have turned to MFG in order to model the interaction between populations & study the decision-making process of intelligent agents, including aversion & congestion behavior between 2 groups of pedestrians, departure time choice of morning commuters, & decision-making processes for autonomous vehicle.

3. **Control & mitigation of Epidemics.** Since the epidemic has affected society & individuals significantly, MFG & mean-field controls (MFCs) provide a perspective to study & understand the underlying population dynamics, especially in the context of the Covid-19 pandemic response. MFG has been used to extend the SIR-type dynamics with spatial effects or allowing for individuals to choose their behaviors & control their contributions to the spread of the disease. MFC is applied to design the optimal strategy to control the virus spreading within a spatial domain, control individuals' decisions to limit their social interactions, & support the government's nonpharmaceutical interventions." – Wikipedia/mean-field game theory

8 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

1. [Bre11]. HAÏM BREZIS. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations.*

2. [Eva10]. LAWRENCE C. EVANS. *Partial Differential Equations.*

3. [GT01]. DAVID GILBARG, NEIL S. TRUDINGER. *Elliptic Partial Differential Equations of 2nd Order.*

8.1 Weak solution – Nghiệm yếu

Definition 2 (Weak solution – Nghiệm yếu)**.** *"In mathematics, a* weak solution *(also called a* generalized solution*) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of* distributions*." – Wikipedia/weak solution*

"Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough." – Wikipedia/weak solution

Example 1 (1st-order wave equation). *The 1st-order wave equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}^2$ with $u = u(t,x)$ has the weak form $\int_{\mathbb{R}^2} u\partial_t\varphi + u\partial_x\varphi \, dt \, dx = 0$ has a solution $u(t,x) = |t - x|$ which may be checked by splitting the integrals over region $\{x \geq t\}$ & $\{x \leq t\}$ where u is smooth.*

"The notion of weak solution based on distribution is sometimes inadequate. In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. Hamilton-Jacobi equation, there is a very different definition of weak solution called *viscosity solution*." – Wikipedia/weak solution

8.1.1 General idea

When solving a differential equation in u, one can rewrite it using a test function φ s.t. whatever derivatives in u show up in the equation, they are "transferred" via integration by parts to φ, resulting in an equation without derivatives of u. This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x}, \partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}),$$

where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ varies over some finite set in $\mathbb{N}^d$ & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x}, \partial)u(\mathbf{x} = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) \, d\mathbf{x} = 0,$$

where the differential operator $Q(\mathbf{x}, \partial)$ is given by the formula

$$Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) = \sum (-1)^{|\alpha|}\partial^\alpha[a_\alpha(\mathbf{x})\varphi(\mathbf{x})],$$

which is the formal adjoint of $P(\mathbf{x}, \partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$-times differentiable function u defined on the open set W s.t. $P(\mathbf{x}, \partial)u(\mathbf{x}) = 0$, $\forall \mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) \, d\mathbf{x} = 0$, $\forall \varphi \in C_c^\infty(W)$.

8.2 Viscosity solution – Nghiệm trơn/nhớt

Example 2 (Viscosity solution for Hamilton–Jacobi equation). *Hamilton–Jacobi equation.*

8.3 Very weak solution – Nghiệm rất yếu

Example 3 (Very weak solution of porous medium equation (PME) [Váz07]). .

Example 4 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term [DDN20]). *Given $f \in L_\delta^1(\Omega), \lambda \geq 0$, a function $u \in L_\delta^1(\Omega)$ is called a* very weak solution *of*

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & in \ \Omega, \\ |u|^{m-1}u = 1 & on \ \Gamma, \end{cases}$$

if $|u|^{m-1}u \in L^1(\Omega)$ and

$$\int_\Omega u^m\Delta\varphi + \lambda u\varphi \, d\mathbf{x} = \int_\Omega f\varphi \, d\mathbf{x} - \int_\Gamma \partial_\mathbf{n}\varphi \, d\mathbf{x}.$$

Example 5 (Very weak solution of NSEs [Tsa18]). .

8.4 Navier–Stokes Equations

Resources – Tài nguyên.

1. [Lad69]. O. A. LADYZHENSKAYA. *The Mathematical Theory of Viscous Incompressible Flow.*

2. [Soh01a; Soh01b]. HERMANN SOHR. *The NSEs: An Elementary Functional Analytic Approach.*

   Primary objective. To develop an elementary & self-contained approach to the mathematical theory of a viscous incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$, described by NSEs. Formulate the theory for a completely general domain Ω.

3. [Tem77; Tem00]. ROGER TEMAM. *NSES: Theory & Numerical Analysis.*

4. [Tsa18]. TAI-PENG TSAI. *Lectures on NSEs.*

9 Sobolev Spaces – Không Gian Sobolev

Resources – Tài nguyên.

1. [AF03]. Robert A. Adams, John J. F. Fournier. *Sobolev Spaces.*

2. [Gag57]. Emilio Gagliardo. *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.*

3. Necǎs.

4. [Tar06]. Luc Tartar. *An Introduction to Sobolev Spaces & Interpolation Spaces.*

10 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

1. [LeV07]. Randall J. LeVeque. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems.*

11 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

1. [BS08]. Susanne C. Brenner, L. Ridgway Scott. *The Mathematical Theory of FEMs.*

2. [EG04]. Alexandre Ern, Jean-Luc Guermond. *Theory & Practice of Finite Elements.*

3. [GR86]. Vivette Girault, Pierre-Arnaud Raviart. *FEMs for NSEs.*

4. [Gun89]. Max D. Gunzburger. *FEMs for Viscous Incompressible Flows.*

5. [Joh16]. Volker John. *FEMs for Incompressible Flow Problems.*

I met Volker John, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagonrinsky, k-ϵ & their simulations.

12 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

1. [EGH19]. Robert Eymard, Thierry Gallouët, Raphaèle Herbin. *Finite Volume Methods.*

2. [LeV02]. Randall J. LeVeque. *FEMs for Hyperbolic Problems.*

13 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

13.1 Wikipedia/Henri Berestycki

Henri Berestycki (born Mar 25, 1951, in Paris)[1] is a French mathematician who obtained his PhD from Université Paris VI - Université Pierre et Marie Curie in 1975.

His Dissertation was titled *Contributions à l'étude des problèmes elliptiques non linéaires*, and his doctoral advisor was Haim Brezis.[2]

He was an L.E. Dickson Instructor in Mathematics at the University of Chicago from 1975–77, after which he returned to France to continue his research.

He has made many contributions in *nonlinear analysis*, ranging from *nonlinear elliptic equations, hamiltonian systems, spectral theory of elliptic operators*, and with applications to the description of *mathematical modelling of fluid mechanics and combustion.*

His current research interests include the mathematical modelling of financial markets, mathematical models in biology and especially in ecology, and modelling in social sciences (in particular, urban planning and criminology).

For these latter topics, he obtained an ERC Advanced grant in 2012.

He worked at the French National Center of Scientific Research (CNRS), then moved to an appointment as Professor at Univ. Paris XIII (1983–1985).

He became a Professor of Mathematics in 1988 at Université Pierre et Marie Curie, Paris VI (1988–2001 of "exceptional class" since 1993), and became Professor at Ecole normale supérieure, Paris (1994–1999), and part-time professor Ecole Polytechnique (1987–1999).

He is also a visiting Professor in the Department of Mathematics at the University of Chicago, and was also co-director of the Stevanovich Center of Financial Mathematics in Chicago.

He is currently the Directeur d'études (Research Professor) at École des hautes études en sciences sociales (EHESS), since 2001.

13.1.1 Services

- National Committee of French universities (1992–1995).

- Since 2002 director of Centre d'analyse et mathématique sociales (CAMS), CNRS -EHESS.

- Vice-president, EHESS (2004–2006).

- Member of the thesis prize committee of the universities of Paris (since 2006).

13.1.2 Awards

- Carrière Prize(1988)

- Prix Sophie Germain of the French Academy of Sciences (2004),

- Humboldt Prize in Germany (2004)

- French Legion of Honor in 2010.

- American Mathematical Society Fellowship (2012).[3]

- Foreign honorary member of the American Academy of Arts and Sciences, 2013.[4]

13.1.3 Articles

- Berestycki, Henri; Roquejoffre, Jean-Michel; Rossi, Luca; The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol.* 66 (2013), no. 4-5, 743–766.

- Barthélemy, Marc; Nadal, Jean-Pierre; Berestycki, Henri Disentangling collective trends from local dynamics. *Proc. Natl. Acad. Sci. USA* 107 (2010), no. 17, 7629–7634.

- Berestycki, Henri; Hamel, François; Nadirashvili, Nikolai Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Comm. Math. Phys.* 253 (2005), no. 2, 451–480.

- Berestycki, Henri; Hamel, François Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* 55 (2002), no. 8, 949–1032.

- Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* 81 (1996), no. 2, 467–494.

- Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S. The principal eigenvalue and maximum principle for 2nd-order elliptic operators in general domains. *Comm. Pure Appl. Math.* 47 (1994), no. 1, 47–92.

- Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345; II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 347–375.

- Bahri, Abbas; Berestycki, Henri A perturbation method in critical point theory and applications. *Trans. Amer. Math. Soc.* 267 (1981), no. 1, 1–32. □

13.2 Haïm Brezis

13.3 Lawrence Chris Evans

13.4 Peter Lax

13.5 Jacques-Louis Lions

- **Born.** May 3, 1928. Grasse, Alpes-Maritimes, France.

- **Died.** May 17, 2001 (aged 73).

- **Nationality.** French.

- **Alma mater.** University of Nancy.

- **Known for.** PDEs.

- **Awards.** Japan Prize (1991).

Scientific career.

- **Fields.** Mathematics.

- **Institutions.**
  - École Polytechnique
  - Collège de France

- **Doctoral advisor.** Laurent Schwartz.

- **Doctoral students.**
  - Alain Bensoussan
  - Jean-Michel Bismut
  - Haïm Brezis
  - Erol Gelenbe
  - Roland Glowinski
  - Roger Temam

Jacques-Louis Lions ([1] 3 May 1928 - May 17, 2001) was a French mathematician who made contributions to the theory of partial differential equations and to stochastic control, among other areas.

He received the SIAM's John von Neumann Lecture prize in 1986 and numerous other distinctions.[2][3]

Lions is listed as an ISI highly cited researcher.[4]

13.5.1 Biography

After being part of the French Résistance in 1943 and 1944, J.-L. Lions entered the École Normale Supérieure in 1947.

He was a professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, and the École polytechnique.

In 1966 he sent an invitation to Gury Marchuk, the soviet mathematician to visit Paris.

This was hand delivered by General De Gaulle during his visit to Akademgorodok in June of that year.[5]

He joined the prestigious Collège de France as well as the French Academy of Sciences in 1973.

In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught and promoted the use of numerical simulations using finite elements integration.

Throughout his career, Lions insisted on the *use of mathematics in industry*, with a particular involvement in the French space program, as well as in domains such as energy and the environment.

This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992.

Lions was elected President of the International Mathematical Union in 1991 and also received the Japan Prize and the Harvey Prize that same year.[3]

In 1992, the University of Houston awarded him an honorary doctoral degree.

He was elected president of the French Academy of Sciences in 1996 and was also a Foreign Member of the Royal Society (ForMemRS)[6] and numerous other foreign academies.[2][3]

He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English and Russian, and major contributions to several collective works, including the 4000 pages of the monumental *Mathematical analysis and numerical methods for science and technology* (in collaboration with Robert Dautray), as well as the *Handbook of numerical analysis* in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994.[7]

Both father and son have received honorary doctorates from Heriot-Watt University in 1986 and 1995 respectively.[8]

13.5.2 Books

- with Enrico Magenes: *Problèmes aux limites non homogènes et applications*. 3 vols., 1968, 1970

- *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. 1968

- with L. Cesari: *Quelques méthodes de résolution des problèmes aux limites non linéaires*. 1969

- with Roger Dautray: *Mathematical analysis and numerical methods for science and technology*. 9 vols., 1984/5

- with Philippe Ciarlet: *Handbook of numerical analysis*. 7 vols.

- with Alain Bensoussan, Papanicolaou: *Asymptotic analysis of periodic structures*. North Holland 1978

- *Controlabilité exacte, perturbations et stabilisation de systèmes distribués*[9]

- with John E. Lagnese: *Modelling Analysis and Control of Thin Plates*. □

13.6 Andrew Joseph Majda

Resources – Tài nguyên.

1. [Eng+23]. Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, Vlad Vicol. *In memory of Andrew J. Majda.*

   "He was hard working until the end even though he suffered from serious health issues for quite some time."

   "He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, and observed and experimental data ... Andy Majda's modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis and numerical modeling, (iii) observed phenomena and experimental data, and (iv) qualitative and/or asymptotic modeling [Maj00]."

   "Andy's legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored."

   "The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed "Majda's model" for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived "Beale-Kato-Majda criterion," which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84]."

   "At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science."

13.7 Vladimir Mazya

13.8 Jindřich Nečas

13.9 Louis Nirenberg

[Váz20]

13.10 Stanley Osher

14 Miscellaneous

Tài liệu

[ABT18]   Richard C. Aster, Brian Borchers, and Clifford H. Thurber. *Parameter estimation and inverse problems*. Third. Elsevier/Academic Press, Amsterdam, 2018, pp. xi+392. isbn: 978-0-12-804651-7. doi: 10.1016/C2015-0-02458-3. url: https://doi.org/10.1016/C2015-0-02458-3.

[AF03]   Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. isbn: 0-12-044143-8.

[AK16]   Cung Thế Anh and Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng*. Nhà Xuất Bản Đại Học Sư Phạm, 2016, p. 222.

[Bre11]   Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011, pp. xiv+599. isbn: 978-0-387-70913-0.

[BS08]   Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*. Third. Vol. 15. Texts in Applied Mathematics. Springer, New York, 2008, pp. xviii+397. isbn: 978-0-387-75933-3. doi: 10.1007/978-0-387-75934-0. url: https://doi.org/10.1007/978-0-387-75934-0.

[Car16]   Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition of [MR0394451]. Dover Publications, Inc., Mineola, NY, 2016, pp. xvi+510. isbn: 978-0-486-80699-0; 0-486-80699-5.

[DDN20]   Nguyen Anh Dao, Jesus Ildefonso Díaz, and Quan Ba Hong Nguyen. "Pointwise gradient estimates in multi-dimensional slow diffusion equations with a singular quenching term". In: *Adv. Nonlinear Stud.* 20.2 (2020), pp. 477–502. issn: 1536-1365. doi: 10.1515/ans-2020-2076. url: https://doi.org/10.1515/ans-2020-2076.

[DZ01]   M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*. Vol. 4. Advances in Design and Control. Analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, pp. xviii+482. isbn: 0-89871-489-3.

[DZ11]   M. C. Delfour and Jean-Paul Zolésio. *Shapes and geometries*. Second. Vol. 22. Advances in Design and Control. Metrics, analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, pp. xxiv+622. isbn: 978-0-898719-36-9. doi: 10.1137/1.9780898719826. url: https://doi.org/10.1137/1.9780898719826.

[EG04]   Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*. Vol. 159. Applied Mathematical Sciences. Springer-Verlag, New York, 2004, pp. xiv+524. isbn: 0-387-20574-8. doi: 10.1007/978-1-4757-4355-5. url: https://doi.org/10.1007/978-1-4757-4355-5.

[EG15]   Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299. ISBN: 978-1-4822-4238-6.

[EGH19]   Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. "Finite Volume Methods". In: *Handbook of Numerical Analysis, P.G. Ciarlet, J.L. Lions eds, 1997*. Vol. 7. 2019, pp. 713–1020. URL: https://hal.archives-ouvertes.fr/hal-02100732v2.

[Eng+23] Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, and Vlad Vicol. "In memory of Andrew J. Majda". In: *Notices Amer. Math. Soc.* 70.10 (2023), pp. 1648–1666. ISSN: 0002-9920. DOI: 10.1090/noti2810. URL: https://doi.org/10.1090/noti2810.

[Eva10]   Lawrence C. Evans. *Partial Differential Equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. URL: https://doi.org/10.1090/gsm/019.

[Gag57]   Emilio Gagliardo. "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili". In: *Rend. Sem. Mat. Univ. Padova* 27 (1957), pp. 284–305. ISSN: 0041-8994. URL: http://www.numdam.org/item?id=RSMUP_1957__27__284_0.

[GR86]   Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*. Vol. 5. Springer Series in Computational Mathematics. Theory and algorithms. Springer-Verlag, Berlin, 1986, pp. x+374. ISBN: 3-540-15796-4. DOI: 10.1007/978-3-642-61623-5. URL: https://doi.org/10.1007/978-3-642-61623-5.

[GT01]   David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001, pp. xiv+517. ISBN: 3-540-41160-7.

[Gun89]   Max D. Gunzburger. *Finite element methods for viscous incompressible flows*. Computer Science and Scientific Computing. A guide to theory, practice, and algorithms. Academic Press, Inc., Boston, MA, 1989, pp. xviii+269. ISBN: 0-12-307350-2.

[IJ15]   Kazufumi Ito and Bangti Jin. *Inverse problems*. Vol. 22. Series on Applied Mathematics. Tikhonov theory and algorithms. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015, pp. x+318. ISBN: 978-981-4596-19-0.

[Joh16]   Volker John. *Finite element methods for incompressible flow problems*. Vol. 51. Springer Series in Computational Mathematics. Springer, Cham, 2016, pp. xiii+812. ISBN: 978-3-319-45749-9; 978-3-319-45750-5. DOI: 10.1007/978-3-319-45750-5. URL: https://doi.org/10.1007/978-3-319-45750-5.

[Kir21]   Andreas Kirsch. *An introduction to the mathematical theory of inverse problems*. Third. Vol. 120. Applied Mathematical Sciences. Springer, Cham, 2021, p. 400. ISBN: 978-3-030-63343-1; 978-3-030-63342-4. DOI: 10.1007/978-3-030-63343-1. URL: https://doi.org/10.1007/978-3-030-63343-1.

[Küh15]   Wolfgang Kühnel. *Differential geometry*. Vol. 77. Student Mathematical Library. Curves—surfaces—manifolds, Third edition [of MR1882174], Translated from the 2013 German edition by Bruce Hunt, with corrections and additions by the author. American Mathematical Society, Providence, RI, 2015, pp. xii+402. ISBN: 978-1-4704-2320-9. DOI: 10.1090/stml/077. URL: https://doi.org/10.1090/stml/077.

[Lad69]   O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969, pp. xviii+224.

[LeV02]   Randall J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002, pp. xx+558. ISBN: 0-521-81087-6; 0-521-00924-3. DOI: 10.1017/CBO9780511791253. URL: https://doi.org/10.1017/CBO9780511791253.

[LeV07]   Randall J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations*. Steady-state and time-dependent problems. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007, pp. xvi+341. ISBN: 978-0-898716-29-0. DOI: 10.1137/1.9780898717839. URL: https://doi.org/10.1137/1.9780898717839.

[LL01]   Elliott H. Lieb and Michael Loss. *Analysis*. Second. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xxii+346. ISBN: 0-8218-2783-9. DOI: 10.1090/gsm/014. URL: https://doi.org/10.1090/gsm/014.

[Rud73]   Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.

[Rud76]   Walter Rudin. *Principles of mathematical analysis*. Third. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, pp. x+342.

[Rud87]   Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

[Rud91]   Walter Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424. ISBN: 0-07-054236-8.

[Soh01a] Hermann Sohr. *The Navier-Stokes equations*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. An elementary functional analytic approach. Birkhäuser Verlag, Basel, 2001, pp. x+367. ISBN: 3-7643-6545-5. DOI: 10.1007/978-3-0348-8255-2. URL: https://doi.org/10.1007/978-3-0348-8255-2.

[Soh01b]   Hermann Sohr. *The Navier-Stokes equations*. Modern Birkhäuser Classics. An elementary functional analytic approach, [2013 reprint of the 2001 original] [MR1928881]. Birkhäuser/Springer Basel AG, Basel, 2001, pp. x+367. ISBN: 978-3-0348-0550-6; 978-3-0348-0551-3.

[Tao22a]   Terence Tao. *Analysis I*. Vol. 37. Texts and Readings in Mathematics. Fourth edition [of 2195040]. Hindustan Book Agency, New Delhi, [2022] ©2022, pp. xvi+355. ISBN: 978-81-951961-9-7.

[Tao22b]   Terence Tao. *Analysis II*. Vol. 38. Texts and Readings in Mathematics. Fourth edition [of 2195041]. Springer, Singapore; Hindustan Book Agency, New Delhi, [2022] ©2022, pp. xvii+195. ISBN: 978-9-81197-284-3. DOI: 10.1007/978-981-19-7284-3. URL: https://doi.org/10.1007/978-981-19-7284-3.

[Tar06]   Luc Tartar. *An introduction to Navier-Stokes equation and oceanography*. Vol. 1. Lecture Notes of the Unione Matematica Italiana. Springer-Verlag, Berlin; UMI, Bologna, 2006, pp. xxviii+245. ISBN: 978-3-540-35743-8; 3-540-35743-2. DOI: 10.1007/3-540-36545-1. URL: https://doi.org/10.1007/3-540-36545-1.

[Tem00]   Roger Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. AMS Chelsea Publishing, 2000, p. 408. ISBN: 0-8218-2737-5.

[Tem77]   Roger Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, pp. x+500. ISBN: 0-7204-2840-8.

[Tsa18]   Tai-Peng Tsai. *Lectures on Navier-Stokes equations*. Vol. 192. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2018, pp. xii+224. ISBN: 978-1-4704-3096-2.

[Váz07]   Juan Luis Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. Mathematical theory. The Clarendon Press, Oxford University Press, Oxford, 2007, pp. xxii+624. ISBN: 978-0-19-856903-9; 0-19-856903-3.

[Váz20]   Juan Luis Vázquez. "Remembering Louis Nirenberg and his mathematics". In: *Gac. R. Soc. Mat. Esp.* 23.2 (2020), pp. 243–261. ISSN: 1138-8927.

[Wal15]   Shawn W. Walker. *The shapes of things*. Vol. 28. Advances in Design and Control. A practical guide to differential geometry and the shape derivative. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015, pp. ix+154. ISBN: 978-1-611973-95-2. DOI: 10.1137/1.9781611973969.ch1. URL: https://doi.org/10.1137/1.9781611973969.ch1.