Method of Subsolutions and Supersolutions for a Nonlinear Poisson Equation

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Abstract

In this context, we are interested in the method of subsolutions & supersolutions for a non-linear Poisson equation, which is presented in [**1**], p. 543. This material is used for our representation in the class *Sobolev spaces and elliptic equations* which is taught by Prof. Nicoletta Tchou in Université de Rennes 1, 2018.

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1 Main Results

Let U be a bounded open subset of $\mathbb{R}^N$ with smooth boundary.

In this context, we focus on the maximum principle, which is a basic property of elliptic PDE, and demonstrate how various resulting comparison arguments can be used to solve certain semilinear problems.

The idea is to exploit *order properties* for solutions. More precisely, we will show that if we can find a subsolution $\underline{u}$ & a supersolution $\overline{u}$ of a particular boundary-value problem and if furthermore $\underline{u} \le \overline{u}$, then there in fact exists a solution satisfying $\underline{u} \le u \le \overline{u}$.

We will investigate this boundary-value problem for the nonlinear Poisson equation

$$\begin{cases} -\Delta u = f(u), & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth, with

$$|f'(z)| \le C_f, \ \ \forall z \in \mathbb{R}, \tag{1.2}$$

for some constant C_f.

Definition 1.1. *(i) We say that $\overline{u} \in H^1(U)$ is a* weak supersolution *of problem (1.1) if*

$$\int_U D\overline{u} \cdot Dv dx \ge \int_U f(\overline{u})\, v dx, \ \ \forall v \in H_0^1(U), \ v \ge 0 \ a.e. \tag{1.3}$$

(ii) Similarly, $\underline{u} \in H^1(U)$ is a weak subsolution *of (1.1) provided*

$$\int_U D\underline{u} \cdot Dv dx \le \int_U f(\underline{u})\, v dx, \ \ \forall v \in H_0^1(U), \ v \ge 0 \ a.e. \tag{1.4}$$

(iii) We say $u \in H_0^1(U)$ is a weak solution *of (1.1) if*

$$\int_U Du \cdot Dv dx = \int_U f(u)\, v dx, \ \ \forall v \in H_0^1(U). \tag{1.5}$$

Remark 1.1. *If $\overline{u}, \underline{u} \in C^2(U)$, then from (1.3) & (1.4) it follows that*

$$-\Delta \overline{u} \ge f(\overline{u}), \ -\Delta \underline{u} \le f(\underline{u}), \ \ in\ U. \tag{1.6}$$

Proof of Remark 1.1. It suffices to prove the first inequality in (1.6), the second one is treated similarly. Since $\overline{u}$ is a weak supersolution of (1.1), the integral inequality (1.3) holds. Applying Green's formula to the LHS of (1.3) yields

$$\int_U (-\Delta \overline{u} - f(\overline{u}))\, v dx \ge 0, \ \ \forall v \in H_0^1(U), \ v \ge 0 \text{ a.e.}, \tag{1.7}$$

in particular,

$$\int_U \left(-\Delta\overline{u} - f\left(\overline{u}\right)\right) v dx \geq 0, \;\; \forall v \in C_c^{\infty}\left(U\right), \; v \geq 0. \tag{1.8}$$

We suppose for the contrary that there exists a point $x_0 \in U$ such that $-\Delta\overline{u}\left(x_0\right) < f\left(\overline{u}\left(x_0\right)\right)$. Since $\overline{u} \in C^2\left(U\right)$ and f is smooth, the last inequality implies that there exists a ball $B\left(x_0, r\right) \in U$ such that

$$-\Delta\overline{u} < f\left(\overline{u}\right), \text{ in } B\left(x_0, r\right). \tag{1.9}$$

Then plugging an arbitrary function $v \in C_c^{\infty}\left(U\right)$, $v \geq 0$ satisfying $v > 0$ in $B\left(x_0, \frac{r}{2}\right)$ into (1.8) yields a contradiction. Therefore, the desired result follows. □

Theorem 1.1 (Existence of a solution between sub- and supersolutions)**.** *Assume there exists a weak supersolution $\overline{u}$ and a weak subsolution $\underline{u}$ of (1.1) satisfying*

$$\underline{u} \leq 0, \; \overline{u} \geq 0 \;\textit{ on } \partial U \textit{ in the trace sense}, \; \underline{u} \leq \overline{u} \textit{ a.e. in } U. \tag{1.10}$$

Then there exists a weak solution u of (1.1), such that

$$\underline{u} \leq u \leq \overline{u} \textit{ a.e. in } U. \tag{1.11}$$

Proof. 1. Fix a number $\lambda > 0$ so large that

$$\text{the mapping } z \mapsto f\left(z\right) + \lambda z \text{ is nondecreasing,} \tag{1.12}$$

this is possible as a consequence of hypothesis (1.2)[1].

Now write $u_0 = \underline{u}$, and then given u_k, $k = 0, 1, 2, \ldots$, inductively define $u_{k+1} \in H_0^1\left(U\right)$ to be the unique weak solution of the linear boundary-value problem[2]

$$(\mathrm{P}_{k+1}) \;\; \begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f\left(u_k\right) + \lambda u_k, & \text{in } U, \\ u_{k+1} = 0, & \text{on } \partial U. \end{cases} \tag{1.13}$$

2. We claim

$$\underline{u} = u_0 \leq u_1 \leq \ldots \leq u_k \leq \ldots \text{ a.e. in } U. \tag{1.14}$$

To confirm this, first note from (1.13) for $k = 0$, i.e., (P_1), that

$$\int_U \left(Du_1 \cdot Dv + \lambda u_1 v\right) dx = \int_U \left(f\left(u_0\right) + \lambda u_0\right) v dx, \;\; \forall v \in H_0^1\left(U\right). \tag{1.15}$$

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[1]Indeed, consider the (smooth) mappings $h_\lambda : \mathbb{R} \to \mathbb{R}$ defined by $h_\lambda\left(z\right) := f\left(z\right) + \lambda z$, $\forall z \in \mathbb{R}$. Its first derivative is given by ${h_\lambda}'\left(z\right) = f'\left(z\right) + \lambda \geq \lambda - C_f$, $\forall z \in \mathbb{R}$. Thus, if $\lambda \geq C_f$, the mapping h_λ is nondecreasing.

[2]Combining the fact that $u_0 = \underline{u} \in H^1\left(U\right)$ with Lemma 3.1 yields $f\left(u_0\right) + \lambda u_0 \in H^1\left(U\right)$. Hence, there exists a unique weak solution, say u_1, of (P_1) such that $u_1 \in H_0^1\left(U\right)$. Inductively, for (P_{k+1}), combining the fact that $u_k \in H_0^1\left(U\right)$ and Lemma 3.1 gives us $f\left(u_k\right) + \lambda u_k \in H^1\left(U\right)$. Then there exists a unique weak solution $u_{k+1} \in H_0^1\left(U\right)$ of (P_{k+1}).

Subtracting (1.15) from (1.4), recall $u_0 = \underline{u}$, yields

$$\int_U D(u_0 - u_1) \cdot Dv dx \leq \int_U \lambda (u_1 - u_0)\, v dx, \ \ \forall v \in H_0^1(U), \ v \geq 0 \text{ a.e.}. \tag{1.16}$$

Set

$$v := (u_0 - u_1)^+ \in H_0^1(U), \ \ v \geq 0 \text{ a.e.}, \tag{1.17}$$

we find

$$\int_U \left[D(u_0 - u_1) \cdot D(u_0 - u_1)^+ + \lambda (u_0 - u_1)(u_0 - u_1)^+ \right] dx \leq 0. \tag{1.18}$$

But, by Lemma 3.2,

$$D(u_0 - u_1)^+ = \begin{cases} D(u_0 - u_1) & \text{a.e. on } \{u_0 \geq u_1\}, \\ 0 & \text{a.e. on } \{u_0 \leq u_1\}. \end{cases} \tag{1.19}$$

Consequently,

$$\int_{\{u_0 \geq u_1\}} \left(|D(u_0 - u_1)|^2 + \lambda (u_0 - u_1)^2 \right) dx \leq 0, \tag{1.20}$$

so that $u_0 \leq u_1$ a.e. in U.

Now assume inductively that

$$u_{k-1} \leq u_k \text{ a.e. in } U. \tag{1.21}$$

From (1.13), we find, for (P_{k+1}) and (P_k), respectively,

$$\int_U (Du_{k+1} \cdot Dv + \lambda u_{k+1} v)\, dx = \int_U (f(u_k) + \lambda u_k)\, v dx, \tag{1.22}$$

$$\int_U (Du_k \cdot Dv + \lambda u_k v)\, dx = \int_U (f(u_{k-1}) + \lambda u_{k-1})\, v dx, \tag{1.23}$$

for all $v \in H_0^1(U)$.

Subtract the last two equalities, we obtain

$$\int_U \left[D(u_k - u_{k+1}) \cdot Dv + \lambda (u_k - u_{k+1})\, v \right] dx = \int_U \left[f(u_{k-1}) - f(u_k) + \lambda (u_{k-1} - u_k) \right] v dx, \tag{1.24}$$

for all $v \in H_0^1(U)$. Then set $v := (u_k - u_{k+1})^+ \in H_0^1(U)$, $v \geq 0$ a.e., we find

$$\begin{aligned} &\int_U \left[D(u_k - u_{k+1}) \cdot D(u_k - u_{k+1})^+ + \lambda (u_k - u_{k+1})(u_k - u_{k+1})^+ \right] dx \\ = &\int_U \left[f(u_{k-1}) - f(u_k) + \lambda (u_{k-1} - u_k) \right] (u_k - u_{k+1})^+ dx. \end{aligned} \tag{1.25}$$

Lemma 3.2 gives us

$$D(u_k - u_{k+1})^+ = \begin{cases} D\left(u_k - u_{k+1}\right) & \text{a.e. on } \left\{u_k \geq u_{k+1}\right\}, \\ 0 & \text{a.e. on } \left\{u_k \leq u_{k+1}\right\}. \end{cases} \tag{1.26}$$

Thus, (1.25) becomes

$$\int_{\{u_k \geq u_{k+1}\}} \left(\left|D\left(u_k - u_{k+1}\right)\right|^2 + \lambda(u_k - u_{k+1})^2\right) dx \tag{1.27}$$

$$= \int_U \left(f\left(u_{k-1}\right) + \lambda u_{k-1} - f\left(u_k\right) - \lambda u_k\right)\left(u_k - u_{k+1}\right)^+ dx \tag{1.28}$$

$$= \int_U \left(h_\lambda\left(u_{k-1}\right) - h_\lambda\left(u_k\right)\right)\left(u_k - u_{k+1}\right)^+ dx \leq 0, \tag{1.29}$$

the last inequality holding in view of (1.21) and (1.12). Therefore, $u_k \leq u_{k+1}$ a.e. in U, as asserted.

3. Next we show

$$u_k \leq \overline{u} \text{ a.e. in } U, \quad \forall k \in \mathbb{N}. \tag{1.30}$$

Statement (1.30) is valid for $k = 0$ by hypothesis (1.10). Assume now for induction that for some $k \in \mathbb{N}$,

$$u_k \leq \overline{u} \text{ a.e. in } U. \tag{1.31}$$

Then subtracting (1.3) from (1.22), we obtain

$$\int_U \left(D\left(u_{k+1} - \overline{u}\right) \cdot Dv + \lambda u_{k+1} v\right) dx \leq \int_U \left(f\left(u_k\right) + \lambda u_k - f\left(\overline{u}\right)\right) v dx, \quad \forall v \in H_0^1\left(U\right), \ v \geq 0 \text{ a.e.},$$

and thus

$$\int_U \left(D\left(u_{k+1} - \overline{u}\right) \cdot Dv + \lambda\left(u_{k+1} - \overline{u}\right) v\right) dx \tag{1.32}$$

$$\leq \int_U \left(f\left(u_k\right) + \lambda u_k - f\left(\overline{u}\right) - \lambda \overline{u}\right) v dx \tag{1.33}$$

$$= \int_U \left(h_\lambda\left(u_k\right) - h_\lambda\left(\overline{u}\right)\right) v dx \leq 0, \quad \forall v \in H_0^1\left(U\right), \ v \geq 0 \text{ a.e.}, \tag{1.34}$$

where the last inequality is deduced from (1.31), (1.12), and the positivity of v.

Taking $v := (u_{k+1} - \overline{u})^+$, we find

$$\int_{\{u_{k+1} \geq \overline{u}\}} \left(\left|D\left(u_{k+1} - \overline{u}\right)\right|^2 + \lambda(u_{k+1} - \overline{u})^2\right) dx \leq 0. \tag{1.35}$$

Thus, $u_{k+1} \leq \overline{u}$ a.e. in U. By the principle of mathematical induction, (1.30) holds.

4. In light of (1.14) and (1.30), we have[3]

$$\underline{u} \leq \ldots \leq u_k \leq u_{k+1} \leq \ldots \overline{u} \text{ a.e. in } U. \tag{1.37}$$

Therefore

$$u(x) := \lim_{k \to \infty} u_k(x) \tag{1.38}$$

exists for a.e. $x \in U$. Furthermore, we have

$$u_k \to u \text{ in } L^2(U), \tag{1.39}$$

as guaranteed by the Dominated Convergence Theorem and (1.37).

Finally, we have

$$\|f(u_k)\|_{L^2(U)} \leq \|f(u_k) - f(0)\|_{L^2(U)} + \|f(0)\|_{L^2(U)} \leq C_f \|u_k\|_{L^2(U)} + |f(0)| \operatorname{vol}(U)^{\frac{1}{2}}. \tag{1.40}$$

Since we have $\|f(u_k)\|_{L^2(U)} \leq C\left(\|u_k\|_{L^2(U)} + 1\right)$ where the constant C is given by

$$C := \max\left\{C_f, |f(0)| \operatorname{vol}(U)^{\frac{1}{2}}\right\}, \tag{1.41}$$

we deduce from (1.13) that $\sup_k \|u_k\|_{H_0^1(U)} < \infty$. Indeed, substituting $v = u_{k+1} \in H_0^1(U)$ into (1.22) yields

$$\int_U \left(|Du_{k+1}|^2 + \lambda |u_{k+1}|^2\right) dx = \int_U (f(u_k) + \lambda u_k) u_{k+1} dx, \quad \forall k \in \mathbb{N}. \tag{1.42}$$

Thus,

$$\begin{aligned}
\min\{1, \lambda\} \|u_{k+1}\|_{H_0^1(U)} &\leq \int_U \left(|Du_{k+1}|^2 + \lambda |u_{k+1}|^2\right) dx && (1.43)\\
&= \int_U (f(u_k) + \lambda u_k) u_{k+1} dx && (1.44)\\
&\leq \|f(u_k)\|_{L^2(U)} \|u_{k+1}\|_{L^2(U)} + \lambda \|u_k\|_{L^2(U)} \|u_{k+1}\|_{L^2(U)} && (1.45)\\
&\leq C\left(\|u_k\|_{L^2(U)} + 1\right) \|u_{k+1}\|_{L^2(U)} + \lambda \|u_k\|_{L^2(U)} \|u_{k+1}\|_{L^2(U)} && (1.46)\\
&\leq (\lambda + C) \|\max\{|\underline{u}|, |\overline{u}|\}\|_{L^2(U)}^2 + C \|\max\{|\underline{u}|, |\overline{u}|\}\|_{L^2(U)}, && (1.47)
\end{aligned}$$

for all $k \in \mathbb{N}$. Since this bound is independent of k, we deduce that $\sup_{k \in \mathbb{N}} \|u_k\|_{H_0^1(U)} < \infty$. Hence there is a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ which converges weakly in $H_0^1(U)$ to $u \in H_0^1(U)$.

[3] As a consequence, $|u_k| \leq \max\{|\underline{u}|, |\overline{u}|\}$, $\forall k \in \mathbb{N}$, and thus

$$\|u_k\|_{L^2(U)} \leq \|\max\{|\underline{u}|, |\overline{u}|\}\|_{L^2(U)}, \quad \forall k \in \mathbb{N}. \tag{1.36}$$

5. We at last verify that u is a weak solution of problem (1.1). For this, fix $v \in H_0^1(U)$. Then from (1.13) we find

$$\int_U \left(Du_{k_{j+1}} \cdot Dv + \lambda u_{k_{j+1}} v\right) dx = \int_U \left(f\left(u_{k_j}\right) + \lambda u_{k_j}\right) v dx. \tag{1.48}$$

Let $j \to \infty$:

$$\int_U (Du \cdot Dv + \lambda uv)\, dx = \int_U (f(u) + \lambda u)\, v dx. \tag{1.49}$$

Canceling the term involving λ, we at last confirm that

$$\int_U Du \cdot Dv dx = \int_U f(u)\, v dx, \tag{1.50}$$

as desired. □

This proof illustrates the use of integration by parts, rather than the maximum principle, to establish comparisons between sub- and supersolutions.

2 Problems

Problem 2.1 (Exercise 6, [**1**], p. 574)**.** *Assume $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, bounded, with $f(0) = 0$ and $f'(0) \geq \lambda_1$, λ_1 denoting the principal eigenvalue for $-\Delta$ on $H_0^1(U)$. Use the method of sub- and supersolutions to show there exists a weak solution u of*

$$\begin{cases} -\Delta u = f(u), & \text{in } U, \\ u = 0, & \text{on } \partial U, \\ u > 0, & \text{in } U. \end{cases} \tag{2.1}$$

Problem 2.2 (Exercise 7, [**1**], p. 579)**.** *Assume that $\underline{u}$, $\overline{u}$ are smooth sub- and supersolutions of the boundary-value problem (1.13). Use the maximum principle to verify directly*

$$\underline{u} = u_0 \leq u_1 \leq \ldots \leq u_k \leq \ldots \overline{u}, \tag{2.2}$$

where the $\{u_k\}_{k=0}^{\infty}$ are defined in the proof of Theorem 1.1.

Solution. First of all, we need to assume in addition that U is an open set of class C^2[4] and $f \in H^m(U)$ with $m > \frac{N}{2}$. Then, using Theorem 3.5, the weak solution u_k of (P_k) in Step 1 in the above proof then satisfies $u_k \in C^2\left(\overline{U}\right)$, for all $k \in \mathbb{Z}^+$. Now subtracting the PDE in (1.13) w.r.t. (P_0) to (1.6) yields

$$\Delta(u_1 - u_0) \leq \lambda(u_1 - u_0), \text{ in } U. \tag{2.3}$$

[4]This implies the smooth sub- and supersolutions also belongs to $C\left(\overline{U}\right)$, i.e., $\underline{u}, \overline{u} \in C^2(U) \cap C\left(\overline{U}\right)$. Thus, we can apply weak maximum principle as in this proof.

Since $u_0 = \underline{u} \leq 0$ on ∂U and $u_1 = 0$ on ∂U, applying Theorem 2, ii) (Weak maximum principle for $c \geq 0$), [**1**], p. 346 gives us

$$\min_{\overline{U}} (u_1 - u_0) \geq -\max_{\partial U} (u_1 - u_0)^- = 0, \tag{2.4}$$

i.e., $u_1 \geq u_0$ in U.

At the k^{th} step, subtracting the PDEs in (P_{k+1}) and (P_k) yields

$$\begin{aligned} \Delta (u_{k+1} - u_k) + \lambda (u_k - u_{k+1}) &= f(u_{k-1}) - f(u_k) + \lambda (u_{k-1} - u_k) & (2.5)\\ &= h_\lambda (u_{k-1}) - h_\lambda (u_k) \leq 0, & (2.6) \end{aligned}$$

since h_λ is nondecreasing and $u_{k-1} \leq u_k$ obtained in the previous step. Note that $u_{k+1} - u_k = 0$ on ∂U, applying weak maximum principle similarly yields $u_{k+1} \leq u_k$ in U. The Step 3 in the above proof is handled by the maximum principle similarly. □

Instead of applying directly the weak maximum principle, the following alternative proof uses a property of subharmonicity.

Alternative proof. It suffices to prove $u_0 \leq u_1$, the rest of Step 2 and 3 of the first proof is handled similarly. We assume that U is an open *connected* set of class C^2 for simplicity. After obtaining $u_k \in C^2(\overline{U})$ for all $k \in \mathbb{Z}^+$, we assume for the contrary that $M := \max_U (u_0 - u_1) > 0$. Define

$$F := \{x \in U; u_0 - u_1 = M\}, \tag{2.7}$$

the set F is nonempty and relatively closed in U. Take $x_0 \in F$, i.e., $u_0(x_0) - u_1(x_0) = M > 0$, there exists a ball $B(x_0, r) \subset U$ such that $u_0 - u_1 > 0$ due to the smoothness of u_0 and u_1. Then (2.3) gives $\Delta (u_0 - u_1) \geq \lambda (u_0 - u_1) > 0$ in $B(x_0, r)$. Hence, $(u_0 - u_1)|_{B(x_0,r)}$ is a subharmonic function which attains a global maximum at $x_0 \in B(x_0, r)$. The maximum principle for subharmonic function implies that $u_0 - u_1 = M$ in $B(x_0, r)$. In particular, this implies that F is open, thus $F = U$. This contradicts with the smoothness of U and the fact that $u_0 - u_1 \leq 0$ in ∂U. □

3 Appendices

The following results are used in the proof of Theorem 1.1. We include them here, without proofs, for completeness.

3.1 Two Properties of Weak Differentiation

The following lemmas are needed in the proof of the main theorem.

Lemma 3.1. *If $f \in L^1_{\text{loc}}(U)$ has weak partial derivative $\partial_i f \in L^1_{\text{loc}}(U)$ and $\psi \in C^\infty(U)$, then ψf is weakly differentiable with respect to x_i and*

$$\partial_i (\psi f) = \partial_i \psi f + \psi \partial_i f. \tag{3.1}$$

Proof. Let $\phi \in C_c^\infty(U)$ be any test function. Then $\psi\phi \in C_c^\infty(U)$ and the weak differentiability of f implies that

$$\int_U f\partial_i(\psi\phi)\,dx = -\int_U \partial_i f\psi\phi dx. \tag{3.2}$$

Expanding $\partial_i(\psi\phi) = \psi\partial_i\phi + \partial_i\psi\phi$ in this equation and rearranging the result, we get

$$\int_U \psi f\partial_i\phi dx = -\int_U (f\partial_i\psi + \psi\partial_i f)\,\phi dx, \quad \forall\phi \in C_c^\infty(U). \tag{3.3}$$

Thus, ψf is weakly differentiable with respect to x_i and its weak derivative is given by (3.1). ☐

Lemma 3.2. *Let $u \in H^1(U)$. Then $u^+ \in H^1(U)$ and its weak derivative is given by*

$$Du^+ := \begin{cases} Du, & a.e.\ on\ \{u > 0\}, \\ 0, & a.e.\ on\ \{u \le 0\}. \end{cases} \tag{3.4}$$

This lemma is a direct consequence of the following proposition.

Proposition 3.1. *If $u \in L^1_{\mathrm{loc}}(U)$ has the weak derivative $\partial_i u \in L^1_{\mathrm{loc}}(U)$, then $|u| \in L^1_{loc}(U)$ is weakly differentiable and*

$$\partial_i|u| = \begin{cases} \partial_i u, & if\ u > 0, \\ 0, & if\ u = 0, \\ -\partial_i u, & if\ u < 0. \end{cases} \tag{3.5}$$

Proof. Let $f^\varepsilon(t) = \sqrt{t^2 + \varepsilon^2}$. Since f^ε is C^1 and globally Lipschitz, $f^\varepsilon(u)$ is weakly differentiable, and

$$\int_U f^\varepsilon(u)\,\partial_i\phi dx = -\int_U \frac{u\partial_i u}{\sqrt{u^2+\varepsilon^2}}\phi dx, \quad \forall\phi \in C_c^\infty(U). \tag{3.6}$$

Taking the limit of this equation as $\varepsilon \to 0^+$ and using the Dominated Convergence Theorem 3.1, we conclude that

$$\int_U |u|\,\partial_i\phi dx = -\int_U \partial_i|u|\,\phi dx, \quad \forall\phi \in C_c^\infty(U), \tag{3.7}$$

where $\partial_i|u|$ is given by (3.5). ☐

Since the positive part of u is given by $u^+ := \frac{1}{2}(|u| + u)$, Proposition 3.1 implies Lemma 3.1 directly.

3.2 Dominated Convergence Theorem

Theorem 3.1 (Dominated Convergence Theorem)**.** *Assume the functions $\{f_k\}_{k=1}^\infty$ are integrable and $f_k \to f$ a.e. Suppose also $|f_k| \le g$ a.e., for some summable function g. Then*

$$\int_{\mathbb{R}^n} f_k dx \to \int_{\mathbb{R}^n} f dx. \tag{3.8}$$

3.3 Lax-Milgram Theorem

The Lax-Milgram theorem is a fairly simple abstract principle from linear functional analysis, which provides in certain circumstances the existence and uniqueness of a weak solution to some boundary-value problems.

Assume that H is a real Hilbert space, with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, we let $\langle\cdot,\cdot\rangle$ denote the pairing of H with its dual space.

Theorem 3.2 (Lax-Milgram Theorem)**.** *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that*

$$|B[u,v]| \leq \alpha \|u\| \|v\|, \quad \forall u, v \in H, \tag{3.9}$$

and

$$\beta \|u\|^2 \leq B[u,v], \quad \forall u \in H. \tag{3.10}$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v\rangle, \quad \forall v \in H. \tag{3.11}$$

3.4 Weak Convergence

Let X denote a real Banach space.

Definition 3.1 (Weak convergence[5])**.** *We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$* converges weakly *to $u \in X$, written $u_k \rightharpoonup u$, if $\langle u^*, u_k\rangle \to \langle u^*, u\rangle$ for each bounded linear functional $u^* \in X^*$.*

Theorem 3.3 (Weak compactness)**.** *Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$.*

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, *a bounded sequence in a Hilbert space contains a weakly convergent subsequence.*

3.5 Homogeneous Dirichlet Problem for the PDE $-\Delta u + \lambda u = f$, with $\lambda > 0$

This section is an obvious modification to the homogeneous Dirichlet problem for the Laplacian $-\Delta u + u = f$ presented in [**2**], p. 291.

Let $U \subset \mathbb{R}^N$ be an open bounded set. We are looking for a function $u : \overline{U} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{3.12}$$

where $\lambda > 0$, and f is a given function on U.

[5] See [**1**], Sec. D.4, p. 723.

Definition 3.2. *A* weak solution *of* (3.12) *is a function* $u \in H_0^1(U)$ *satisfying*

$$\int_U (Du \cdot Dv + \lambda uv)\, dx = \int_U fv dx, \quad \forall v \in H_0^1(U). \tag{3.13}$$

We now focus on the existence and uniqueness of a weak solution of (3.12).

Theorem 3.4 (Dirichlet's principle)**.** *Given any* $f \in L^2(U)$*, there exists a unique weak solution* $u \in H_0^1(U)$ *of* (3.12)*. Furthermore,* u *is obtained by*

$$\min_{v \in H_0^1(U)} \left\{ \frac{1}{2} \int_U \left(|Dv|^2 + \lambda |v|^2 \right) dx - \int_U fv dx \right\}. \tag{3.14}$$

Proof. Apply Lax-Milgram in the Hilbert space $H = H_0^1(U)$ with the bilinear form

$$B[u, v] := \int_U (Du \cdot Dv + \lambda uv)\, dx, \quad \forall u, v \in H_0^1(U), \tag{3.15}$$

and the linear functional $\phi : v \mapsto \int_U fv dx$, $\forall v \in H_0^1(U)$. □

The following theorem gives more regularity on the weak solution.

Theorem 3.5 (Regularity for the Dirichlet problem, see [**2**], p. 298)**.** *Let* U *be an open set of class* C^2 *with* ∂U *bounded (or else* $U = \mathbb{R}_+^N$*). Let* $f \in L^2(U)$ *and let* $u \in H_0^1(U)$ *satisfy*

$$\int_U (Du \cdot D\varphi + \lambda u\varphi)\, dx = \int_U f\varphi dx, \quad \forall \varphi \in H_0^1(U). \tag{3.16}$$

Then $u \in H^2(U)$ *and* $\|u\|_{H^2} \leq C\|f\|_{L^2}$*, where* C *is a constant depending only on* U*. Furthermore, if* U *is of class* C^{m+2} *and* $f \in H^m(U)$*, then*

$$u \in H^{m+2}(U) \;\; \textit{and} \;\; \|u\|_{H^{m+2}} \leq C\|f\|_{H^m}. \tag{3.17}$$

In particular, if $f \in H^m(U)$ *with* $m > \frac{N}{2}$*, then* $u \in C^2\left(\overline{U}\right)$*. Finally, if* U *is of class* C^∞ *and if* $f \in C^\infty\left(\overline{U}\right)$*, then* $u \in C^\infty\left(\overline{U}\right)$*.*

References

[1] Lawrence C. Evans. *Partial Differential Equations*, 2e. Graduate Studies in Mathematics, Volume 19, AMS.

[2] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer.

[3] John K. Hunter. *Notes on Partial Differential Equations*. 2014.