PDE Final Exam 2017

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Abstract

This context aims at solving the problems given in the PDE Final Exam 2017, posed by Prof. Dang Duc Trong.

Contents

1 Problems

Problem 1.1. *Let $a > 0$, $b > 0$, $\Omega = (0, a) \times (0, b)$, $S_1 = [0, a] \times \{0\}$, $S_2 = \partial\Omega \backslash S_1$, $f \in L^2(\Omega)$. Consider the following equation*

$$Lu = f \text{ in } \Omega, \tag{1.1}$$

where

$$Lu \equiv -\frac{\partial}{\partial x}\left(\left(1 + x^2\right) u_x\right) - \frac{\partial}{\partial y}\left(\left(1 + y^2\right) u_y\right), \tag{1.2}$$

with the boundary conditions $u|_{S_2} = 0$, $u_y|_{S_1} = 0$.

1. *Find the weak formulation of this boundary value problem on the solution space V needing determining.*

2. *Use the equality*

$$u^2(x, y) = 2\int_0^x u_x(s, y)\, u(s, y)\, ds, \tag{1.3}$$

   to prove that there exists $C > 0$ such that

$$\|u_x\|_2^2 + \|u_y\|_2^2 \geq C\, \|u\|_{H^1(\Omega)}^2 \text{ for all } u \in V. \tag{1.4}$$

3. *Prove that (1.1) has a weak solution in V by using Lax-Milgram theorem.*

4. *Suppose that this weak solution, say $\bar{u}$, satisfies $\bar{u} \in H^2 \cap V$, prove that this solution $\bar{u}$ satisfies the given problem.*

5. *Write the functional $J : V \to \mathbb{R}$ such that $u = \arg\min_{w \in V} J(w)$. Which boundary value problem does the minimum of J satisfy if V is replaced by H^1?*

Solution.

1. The complementary arcs S_1 and S_2 can be written as

$$S_1 = \left\{(x, y) \in \bar{\Omega}; y = 0\right\}, \tag{1.5}$$

$$S_2 = \left\{(x, y) \in \bar{\Omega}; x = 0 \text{ or } x = a \text{ or } y = a\right\} \backslash \{(0, 0) \cup (a, 0)\}. \tag{1.6}$$

   For arbitrary u and v in $C^2(\Omega)$, the integration by parts formula gives us

$$\langle Lu, v\rangle = -\int_\Omega \frac{\partial}{\partial x}\left(\left(1 + x^2\right) u_x\right) v d\Omega - \int_\Omega \frac{\partial}{\partial y}\left(\left(1 + y^2\right) u_y\right) v d\Omega \tag{1.7}$$

$$= -\int_0^b \int_0^a \frac{\partial}{\partial x}\left(\left(1 + x^2\right) u_x\right) v dx dy - \int_0^a \int_0^b \frac{\partial}{\partial y}\left(\left(1 + y^2\right) u_y\right) v dx dy \tag{1.8}$$

$$= -\int_0^b \left[\left(1+x^2\right) u_x v\big|_{x=0}^{x=a} - \int_0^a \left(1+x^2\right) u_x v_x dx \right] dy \tag{1.9}$$

$$- \int_0^a \left[\left(1+y^2\right) u_y v\big|_{y=0}^{y=b} - \int_0^b \left(1+y^2\right) u_y v_y dy \right] dx \tag{1.10}$$

$$= \int_\Omega \left[\left(1+x^2\right) u_x v_x + \left(1+y^2\right) u_y v_y \right] d\Omega \tag{1.11}$$

$$- \int_0^b \left(1+x^2\right) u_x v\big|_{x=0}^{x=a} dy - \int_0^a \left(1+y^2\right) u_y v\big|_{y=0}^{y=b} dx. \tag{1.12}$$

Now define

$$V := \left\{ v \in H^1(\Omega) ; v = 0 \text{ on } S_2 \right\}. \tag{1.13}$$

and note that for $u \in V$, $v \in V$,

$$\int_0^b \left(1+x^2\right) u_x v\big|_{x=0}^{x=a} dy + \int_0^a \left(1+y^2\right) u_y v\big|_{y=0}^{y=b} dx = \int_0^a \left(1+y^2\right) u_y(x,0)\, v(x,0)\, dx. \tag{1.14}$$

Moreover, if $u \in V$ satisfies $u_y|_{S_1} = 0$, then

$$\int_0^a \left(1+y^2\right) u_y(x,0)\, v(x,0)\, dx = 0. \tag{1.15}$$

Thus, if u is a classical solution of the given boundary value problem, u must satisfy its weak formulation given by

$$K[u,v] = F[v] \text{ for all } v \in V, \tag{1.16}$$

where, for $u \in V$ and $v \in V$,

$$K[u,v] := \int_\Omega \left[\left(1+x^2\right) u_x v_x + \left(1+y^2\right) u_y v_y \right] d\Omega, \tag{1.17}$$

$$F[v] := \int_\Omega f v d\Omega. \tag{1.18}$$

2. Here are two solutions.

   Solution 1. Recall that $\|u\|_{H^1}^2 := \|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2$, let $u \in V$ be given. Since $u(0,y) = 0$ for all $y \in (0,b]$, we have

$$2\int_0^x u_x(s,y)\, u(s,y)\, ds = 2\int_{u(0,y)}^{u(x,y)} s ds \tag{1.19}$$

$$= u^2(x,y) - u^2(0,y) \tag{1.20}$$

$$= u^2(x,y) \text{ for all } y \in (0,b]. \tag{1.21}$$

Then using successively Cauchy-Schwarz inequality for integrals and Cauchy inequality $2ab \leq a^2 + b^2$ yields

$$u^2(x,y) = 2\int_0^x u_x(s,y)\,u(s,y)\,ds \tag{1.22}$$

$$\leq 2\left(\int_0^x \frac{u_x^2(s,y)}{M}ds\right)^{\frac{1}{2}}\left(\int_0^x Mu^2(s,y)\,ds\right)^{\frac{1}{2}} \tag{1.23}$$

$$\leq \int_0^x \frac{u_x^2(s,y)}{M}ds + \int_0^x Mu^2(s,y)\,ds, \tag{1.24}$$

for all $(x,y) \in [0,a] \times (0,b]$, where M is a positive constant depending only on a, b. Thus,

$$\|u\|_2^2 = \int_0^a \left(\int_0^b u^2(x,y)\,dy\right) dx \tag{1.25}$$

$$\leq \int_0^a \left(\int_0^b \int_0^x \frac{u_x^2(s,y)}{M}dsdy + \int_0^b \int_0^x Mu^2(s,y)\,dsdy\right) dx \tag{1.26}$$

$$\leq \int_0^a \left(\int_0^b \int_0^a \frac{u_x^2(s,y)}{M}dsdy + \int_0^b \int_0^a Mu^2(s,y)\,dsdy\right) dx \tag{1.27}$$

$$= \int_0^a \left(\frac{1}{M}\|u_x\|_2^2 + M\|u\|_2^2\right) dx \tag{1.28}$$

$$= \frac{a}{M}\|u_x\|_2^2 + aM\|u\|_2^2. \tag{1.29}$$

Now we choose M such that $aM < 1$, for instance, $M = \frac{1}{2a}$. Then the last estimate implies that

$$\|u\|_2^2 \leq 4a^2\|u_x\|_2^2. \tag{1.30}$$

Similarly, since $u(x,b) = 0$ for all $x \in [0,a]$, we have

$$2\int_y^b u_y(x,r)\,u(x,r)\,dr = 2\int_{u(x,y)}^{u(x,b)} rdr \tag{1.31}$$

$$= u^2(x,b) - u^2(x,y) \tag{1.32}$$

$$= -u^2(x,y) \ \text{ for all } x \in [0,a]. \tag{1.33}$$

Then

$$u^2(x,y) = -2\int_y^b u_y(x,r)\,u(x,r)\,dr \tag{1.34}$$

$$\leq 2\left(\int_y^b \frac{u_y^2(x,r)}{N}dr\right)^{\frac{1}{2}}\left(\int_y^b Nu^2(x,r)\,dr\right)^{\frac{1}{2}} \tag{1.35}$$

$$\leq \int_y^b \frac{u_y^2(x,r)}{N}dr + \int_y^b Nu^2(x,r)\,dr, \tag{1.36}$$

for all $(x,y) \in \overline{\Omega}$, where N is a positive constant depending only on a, b, and thus

$$\|u\|_2^2 = \int_0^b \left(\int_0^a u^2(x,y)\,dx \right) dy \tag{1.37}$$

$$\le \int_0^b \left(\int_0^a \int_y^b \frac{u_y^2(x,r)}{N} drdx + \int_0^a \int_y^b Nu^2(x,r)\,drdx \right) dy \tag{1.38}$$

$$\le \int_0^b \left(\int_0^a \int_0^b \frac{u_y^2(x,r)}{N} drdx + \int_0^a \int_0^b Nu^2(x,r)\,drdx \right) dy \tag{1.39}$$

$$= \int_0^b \left(\frac{1}{N} \|u_y\|_2^2 + N \|u\|_2^2 \right) dy \tag{1.40}$$

$$= \frac{b}{N} \|u_y\|_2^2 + bN \|u\|_2^2 . \tag{1.41}$$

Now we choose N such that $bN < 1$, for instance, $N = \frac{1}{2b}$. Then the last estimate implies that

$$\|u\|_2^2 \le 4b^2 \|u_y\|_2^2 . \tag{1.42}$$

Combining (1.30) and (1.42) yields

$$\|u_x\|_2^2 + \|u_y\|_2^2 \ge \frac{1}{4} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \|u\|_2^2 . \tag{1.43}$$

We now choose $C > 0$ such that

$$\frac{C}{1-C} = \frac{1}{4} \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \tag{1.44}$$

i.e.,

$$C = \frac{a^2+b^2}{a^2+b^2+4a^2b^2}, \tag{1.45}$$

then (1.4) holds.

Solution 2. We only need the inequality $\|u\|_2^2 \le 4a^2 \|u_x\|_2^2$ in the previous solution,

$$\|u_x\|_2^2 + \|u_y\|_2^2 \ge \frac{1}{2} \|u_x\|_2^2 + \frac{1}{8a^2} \|u\|_2^2 + \|u_y\|_2^2 \tag{1.46}$$

$$\ge \min\left\{ \frac{1}{2}, \frac{1}{8a^2} \right\} \left(\|u_x\|_2^2 + \|u\|_2^2 + \|u_y\|_2^2 \right) \tag{1.47}$$

$$= \min\left\{ \frac{1}{2}, \frac{1}{8a^2} \right\} \|u\|_{H^1}^2 . \tag{1.48}$$

3. Consider the defined bilinear form $K[u,v]$, it is continuous since

$$K[u,v] = \int_\Omega \left[\left(1+x^2\right) u_x v_x + \left(1+y^2\right) u_y v_y \right] d\Omega \tag{1.49}$$

$$
\begin{aligned}
&\leq \max\left\{1+a^2, 1+b^2\right\} \int_\Omega |u_x v_x + u_y v_y| \, d\Omega && (1.50)\\
&\leq \left(1+\max\{a,b\}^2\right) \int_\Omega \left(u_x^2+u_y^2\right)^{\frac{1}{2}} \left(v_x^2+v_y^2\right)^{\frac{1}{2}} d\Omega && (1.51)\\
&\leq \left(1+\max\{a,b\}^2\right) \left(\int_\Omega \left(u_x^2+u_y^2\right) d\Omega\right)^{\frac{1}{2}} \left(\int_\Omega \left(v_x^2+v_y^2\right) d\Omega\right)^{\frac{1}{2}} && (1.52)\\
&\leq \left(1+\max\{a,b\}^2\right) \left(\int_\Omega \left(u^2+u_x^2+u_y^2\right) d\Omega\right)^{\frac{1}{2}} \left(\int_\Omega \left(v^2+v_x^2+v_y^2\right) d\Omega\right)^{\frac{1}{2}} && (1.53)\\
&= \left(1+\max\{a,b\}^2\right) \|u\|_{H^1} \|v\|_{H^1}, \quad \forall u, v \in V, && (1.54)
\end{aligned}
$$

and it is coercive since

$$
\begin{aligned}
K[u,u] &= \int_\Omega \left[\left(1+x^2\right) u_x^2 + \left(1+y^2\right) u_y^2\right] d\Omega && (1.55)\\
&\geq \int_\Omega \left(u_x^2+u_y^2\right) d\Omega && (1.56)\\
&= \|u_x\|_2^2 + \|u_y\|_2^2 && (1.57)\\
&\geq C \|u\|_{H_1}^2, \quad \forall u \in V, && (1.58)
\end{aligned}
$$

where C is the constant given in Solution 1 or Solution 2 of the previous result.

It is easy to prove that $F \in V^\star$[1]. Indeed, F is linear since

$$
F[\alpha u + \beta v] = \int_\Omega f(\alpha u + \beta v) \, d\Omega = \alpha F[u] + \beta F[v], \tag{1.59}
$$

for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $u \in V$, $v \in V$. It is also continuous because

$$
F[v] = \int_\Omega f v d\Omega \leq \|f\|_2 \|v\|_2 \leq \|f\|_2 \|v\|_{H^1}, \quad \forall v \in V. \tag{1.60}
$$

Now applying Lax-Milgram theorem (see, e.g., [1, Corollary 5.8, p. 140]) yields that there exists a unique element $\bar{u} \in V$ such that

$$
K[\bar{u}, v] = F[v], \quad \forall v \in V. \tag{1.61}
$$

4. Suppose that $\bar{u}$ satisfies (1.61) and $\bar{u} \in H^2 \cap V$, integrating by parts (1.61) again gives us

$$
-\int_\Omega \left[\frac{\partial}{\partial x}\left(\left(1+x^2\right)\bar{u}_x\right) + \frac{\partial}{\partial y}\left(\left(1+y^2\right)\bar{u}_y\right)\right] v d\Omega - \int_0^a \bar{u}_y(x,0)\, v(x,0)\, dx = \int_\Omega f v d\Omega, \tag{1.62}
$$

for all $v \in V$, or equivalently,

$$
\langle L\bar{u}, v\rangle - \int_0^a \bar{u}_y(x,0)\, v(x,0)\, dx = F[v], \quad \forall v \in V. \tag{1.63}
$$

[1]The notation $V^\star$ denotes the the *dual space* of V, that is, the space of all *continuous linear functionals* on V, see, e.g., [1, p. 3].

We now consider the function $\bar{v} := \bar{u}_y \chi_{S_1}$, i.e.,

$$\bar{v}(x,y) := \begin{cases} 0, & \text{if } (x,y) \in \overline{\Omega} \backslash S_1, \\ \bar{u}_y(x,0), & \text{if } (x,y) \in S_1, \end{cases} \tag{1.64}$$

Since $\bar{v} = 0$ a.e. in Ω and $\bar{v}|_{S_2} = 0$, we have $\bar{v} \in V$. Plugging $v = \bar{v}$ into (1.63) yields

$$\int_0^a \bar{u}_y^2(x,0)\,dx = 0, \tag{1.65}$$

which implies $\bar{u}_y|_{S_1} = 0$ and thus $\bar{u}$ satisfies the given boundary conditions. Substituting $\bar{u}_y(x,0) = 0$ for all $x \in [0,a]$ back to (1.63) yields

$$\langle L\bar{u}, v\rangle = F[v], \ \ \forall v \in V, \tag{1.66}$$

In particular,

$$\langle L\bar{u} - f, v\rangle = 0, \ \ \forall v \in C_c^\infty(\Omega). \tag{1.67}$$

It follows (see [1, Corollary 4.24, p. 110]) that $L\bar{u} = f$ a.e. on Ω. Therefore, $\bar{u}$ satisfies the given boundary value problem almost everywhere.[2]

5. Since $K[\cdot,\cdot]$ is also symmetric, the later statement of Lax-Milgram theorem 2.1 gives us $\bar{u} = \arg\min_{w\in V} J(w)$, where the functional $J(\cdot)$ is defined by

$$\begin{aligned} J(w) :&= \frac{1}{2}K[w,w] - F[w] \qquad (1.68)\\ &= \frac{1}{2}\int_\Omega \left[\left(1+x^2\right)w_x^2 + \left(1+y^2\right)w_y^2 - 2fw\right]d\Omega. \qquad (1.69)\end{aligned}$$

If V is replaced by H^1, we denote $\widetilde{u} := \arg\min_{w\in H^1} J(w)$. Both the domain $\mathcal{A}$ of the functional J and a set $\mathcal{M}$ of *comparison functions* are set as H^1, i.e., $\mathcal{A} = \mathcal{M} = H^1(\Omega)$ (see, e.g., [2, p. 189]). Notice that $H^1(\Omega)$ is dense in $L^2(\Omega)$. For all $u \in H^1(\Omega)$, $v \in H^1(\Omega)$, we have

$$\begin{aligned} &\frac{J[u+\varepsilon v] - J[u]}{\varepsilon} \qquad (1.70)\\ &= \frac{1}{2\varepsilon}\begin{bmatrix} \int_\Omega \left[\left(1+x^2\right)(u_x+\varepsilon v_x)^2 + \left(1+y^2\right)(u_y+\varepsilon v_y)^2 - 2f(u+\varepsilon v)\right]d\Omega \\ -\int_\Omega \left[\left(1+x^2\right)u_x^2 + \left(1+y^2\right)u_y^2 - 2fu\right]d\Omega \end{bmatrix} \qquad (1.71)\\ &= \frac{1}{2\varepsilon}\left[\int_\Omega \left[\left(1+x^2\right)\left(2\varepsilon u_x v_x + \varepsilon^2 v_x^2\right) + \left(1+y^2\right)\left(2\varepsilon u_y v_y + \varepsilon^2 v_y^2\right) - 2\varepsilon f v\right]d\Omega\right] \qquad (1.72)\\ &= \frac{1}{2}\left[\int_\Omega \left[\left(1+x^2\right)\left(2u_x v_x + \varepsilon v_x^2\right) + \left(1+y^2\right)\left(2u_y v_y + \varepsilon v_y^2\right) - 2fv\right]d\Omega\right], \qquad (1.73)\end{aligned}$$

[2] If the stronger assumption $\bar{u} \in C^2(\Omega) \cap V$ is active, then the validity of the equality $L\bar{u} = f$ can be passed from "almost everywhere" to "everywhere" in Ω by the smoothness of $\bar{u}$.

and thus the variation of J at u in the direction v is calculated by

$$\delta J[u;v] := \lim_{\varepsilon\to 0}\frac{J[u+\varepsilon v]-F[u]}{\varepsilon} \tag{1.74}$$
$$= \frac{1}{2}\lim_{\varepsilon\to 0}\int_\Omega \left[\left(1+x^2\right)\left(2u_xv_x+\varepsilon v_x^2\right)+\left(1+y^2\right)\left(2u_yv_y+\varepsilon v_y^2\right)-2fv\right]d\Omega \tag{1.75}$$
$$= \int_\Omega \left[\left(1+x^2\right)u_xv_x+\left(1+y^2\right)u_yv_y-fv\right]d\Omega. \tag{1.76}$$

Integrating by parts, as above, the last integral yields

$$\delta J[u;v] = \langle Lu-f,v\rangle + \int_0^b \left(1+x^2\right)u_xv\Big|_{x=0}^{x=a}dy + \int_0^a \left(1+y^2\right)u_yv\Big|_{y=0}^{y=b}dx, \tag{1.77}$$

for all $v \in H^1(\Omega)$.

Now applying Theorem 2.2 to J and its minimizer $\widetilde{u} \in H^1$ yields

$$\delta J[\widetilde{u};v] = 0,\ \ \forall v \in H^1(\Omega), \tag{1.78}$$

i.e.,

$$\langle L\widetilde{u}-f,v\rangle + \int_0^b \left(1+x^2\right)\widetilde{u}_xv\Big|_{x=0}^{x=a}dy + \int_0^a \left(1+y^2\right)\widetilde{u}_yv\Big|_{y=0}^{y=b}dx = 0,\ \ \forall v\in H^1(\Omega). \tag{1.79}$$

Use the same trick as in the proof of previous statement, plugging successively $v = \widetilde{u}_x\chi_{\{0\}\times[0,b]}$, $v = \widetilde{u}_x\chi_{\{a\}\times[0,b]}$, $v = \widetilde{u}_y\chi_{[0,a]\times\{0\}}$, and $v = \widetilde{u}_y\chi_{[0,a]\times\{b\}}$ into (1.79) gives us $\widetilde{u}_x\big|_{\{0,a\}\times[0,b]} = \widetilde{u}_y\big|_{[0,a]\times\{0,b\}} = 0$ a.e. in $\bar{\Omega}$. This is equivalent to the Neumann boundary condition

$$\frac{\partial \widetilde{u}}{\partial \overrightarrow{\mathbf{n}}} = 0,\ \text{on } \partial\Omega, \tag{1.80}$$

where $\overrightarrow{\mathbf{n}}$ denotes the exterior normal to the boundary $\partial\Omega$. Substituting (1.80) back to (1.79) yields

$$\langle L\widetilde{u}-f,v\rangle = 0,\ \ \forall v\in H^1(\Omega), \tag{1.81}$$

Use the same argument before, we deduce that $\widetilde{u}$ satisfies the following Neumann boundary value problem

$$Lu = f,\ \text{in } \Omega, \tag{1.82}$$
$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}} = 0,\ \text{on } \partial\Omega. \tag{1.83}$$

This completes our solution. □

Problem 1.2. *Let L be the operator given in Problem 1.1. Consider the following problem*

$$\begin{cases} u_t + Lu = 0, & \textit{in } \Omega\times(0,+\infty), \\ u(\mathbf{x},0) = g(\mathbf{x}), & \textit{in } \Omega, \\ u(\mathbf{x},t) = 0, & \textit{on } \partial\Omega\times[0,+\infty). \end{cases} \tag{1.84}$$

1. *Determine* $D(L) \subset L^2(\Omega)$.

2. *Do not use Poincaré inequality, prove that there exists* $C > 0$ *such that*

$$\|u_x\|_2^2 + \|u_y\|_2^2 \geq C \|u\|_{H^1}^2, \quad \forall u \in H_0^1(\Omega). \tag{1.85}$$

3. *Prove that the operator* L *is maximal monotone.*

4. *Prove that* L *is symmetric, then deduce that* L *is self-adjoint.*

5. *Use Hille-Yosida theorem, prove that for* $g \in L^2(\Omega)$ *the problem* (1.84) *has a solution*

$$u \in C\left([0,+\infty]; L^2(\Omega)\right) \cap C\left((0,+\infty); D(L)\right) \cap C^1\left((0,+\infty); L^2(\Omega)\right). \tag{1.86}$$

Proof. 1. In order that Lv makes sense for all $v \in D(L)$, it is required that $D(L) \subset H^2(\Omega)$. For $u \in H^2(\Omega)$, $v \in H^2(\Omega)$, the integration by parts formula gives us

$$\begin{aligned}
\langle Lu, v\rangle &= -\int_\Omega \left[\frac{\partial}{\partial x}\left(\left(1+x^2\right)u_x\right) + \frac{\partial}{\partial y}\left(\left(1+y^2\right)u_y\right)\right] v d\Omega & (1.87)\\
&= \int_\Omega \left[\left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right] d\Omega & (1.88)\\
&\quad - \int_0^b \left(1+x^2\right)u_x v\Big|_{x=0}^{x=a} dy - \int_0^a \left(1+y^2\right)u_y v\Big|_{y=0}^{y=b} dx. & (1.89)
\end{aligned}$$

Notice that $H^2(\Omega) \cap H_0^1(\Omega)$ is a linear subspace of $L^2(\Omega)$, we choose the domain of the operator L as $D(L) := H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\langle Lu, v\rangle = \int_\Omega \left[\left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right] d\Omega, \quad \forall u, v \in D(L). \tag{1.90}$$

2. Similar to the proof of the second statement of Problem 1.1, let $u \in H_0^1(\Omega)$ be given. Since $u(x,y) = 0$ for $(x,y) \in \partial\Omega$ we can modify the argument presented above to prove (1.85).

3. The given unbounded linear operator $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is monotone since

$$\langle Lv, v\rangle = \int_\Omega \left[\left(1+x^2\right)v_x^2 + \left(1+y^2\right)v_y^2\right] d\Omega \geq 0, \quad \forall v \in D(L). \tag{1.91}$$

To prove that L is maximal monotone, it suffices to verify that $R(I+L) = L^2(\Omega)$, i.e.,

$$\forall f \in L^2(\Omega), \; \exists u \in D(L) \text{ such that } u + Lu = f. \tag{1.92}$$

Given $f \in L^2(\Omega)$, the main idea is to modify the arguments given in the proof of Problem 1.1 for the operator $S := I + L$, instead of L, as follows.

Step 1. *Find the weak formulation of the problem*

$$\begin{cases} Su = f, & \text{in } \Omega, \\ u(x,y) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.93}$$

on the solution space $H_0^1(\Omega)$: For any $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$,

$$\begin{aligned}\langle Su, v\rangle &= \langle u, v\rangle + \langle Lu, v\rangle &(1.94)\\ &= \int_\Omega \left[u - \frac{\partial}{\partial x}\left(\left(1+x^2\right)u_x\right) - \frac{\partial}{\partial y}\left(\left(1+y^2\right)u_y\right)\right] v d\Omega &(1.95)\\ &= \int_\Omega \left[uv + \left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right] d\Omega. &(1.96)\end{aligned}$$

The weak formulation of (1.93) is then given by

$$R[u,v] = F[v], \quad \forall v \in H_0^1(\Omega), \tag{1.97}$$

where, for $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$,

$$R[u,v] := \int_\Omega \left[uv + \left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right] d\Omega. \tag{1.98}$$

Step 2. *Prove that* (1.93) *has a weak solution in* $H_0^1(\Omega)$ *by using Lax-Milgram theorem*: Consider the defined bilinear form $R[u,v]$, it is continuous since

$$\begin{aligned}R[u,v] &= \int_\Omega \left[uv + \left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right] d\Omega &(1.99)\\ &\leq \max\left\{1+a^2, 1+b^2\right\} \int_\Omega |uv + u_x v_x + u_y v_y|\, d\Omega &(1.100)\\ &\leq \left(1+\max\{a,b\}^2\right) \int_\Omega \left(u^2+u_x^2+u_y^2\right)^{\frac{1}{2}}\left(v^2+v_x^2+v_y^2\right)^{\frac{1}{2}} d\Omega &(1.101)\\ &\leq \left(1+\max\{a,b\}^2\right)\left(\int_\Omega \left(u^2+u_x^2+u_y^2\right) d\Omega\right)^{\frac{1}{2}}\left(\int_\Omega \left(v^2+v_x^2+v_y^2\right) d\Omega\right)^{\frac{1}{2}} &(1.102)\\ &= \left(1+\max\{a,b\}^2\right)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad \forall u, v \in H_0^1(\Omega), &(1.103)\end{aligned}$$

and it is coercive since

$$\begin{aligned}R[u,u] &= \int_\Omega \left[u^2 + \left(1+x^2\right)u_x^2 + \left(1+y^2\right)u_y^2\right] d\Omega &(1.104)\\ &\geq \int_\Omega \left(u^2+u_x^2+u_y^2\right) d\Omega &(1.105)\\ &= \|u\|_{H^1(\Omega)}^2, \quad \forall u \in H_0^1(\Omega). &(1.106)\end{aligned}$$

It is easy to prove $F \in H^{-1}(\Omega)$[3]. Now applying Lax-Milgram theorem yields that there exists a unique element $\hat{u} \in H_0^1(\Omega)$ such that

$$R[\hat{u}, v] = F[v], \quad \forall v \in H_0^1(\Omega). \tag{1.107}$$

Step 3. *Suppose that this weak solution satisfies* $\hat{u} \in H^2(\Omega) \cap H_0^1(\Omega)$*, i.e.,* $\hat{u} \in D(L)$*, prove that this weak solution* $\hat{u}$ *satisfies* (1.93): Since $\hat{u} \in H_0^1(\Omega)$, $\hat{u}$ satisfies the homogeneous Dirichlet boundary conditions. It then suffices to prove that $\hat{u}$ satisfies the PDE in (1.93). To this end, integrating by parts the left-hand side of (1.107) gives us

$$\langle S\hat{u}, v\rangle = F[v], \quad \forall v \in H_0^1(\Omega), \tag{1.108}$$

[3]The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$, see [1, p. 291].

In particular, this implies

$$\langle (I+L)\,\hat{u} - f, v\rangle = 0,\ \ \forall v \in C_c^\infty(\Omega)\,. \tag{1.109}$$

It follows (see [1, Corollary 4.24, p. 110]) that $(I+L)\,\hat{u} = f$ a.e. on Ω, i.e., (1.92) holds. Therefore, L is a maximal monotone operator.

4. Note that $\overline{D(L)} = \overline{H^2(\Omega) \cap H_0^1(\Omega)} = L^2(\Omega)$[4]. The operator L is symmetric since

$$\langle u, Lv\rangle = -\int_\Omega u\left[\frac{\partial}{\partial x}\left(\left(1+x^2\right)v_x\right) + \frac{\partial}{\partial y}\left(\left(1+y^2\right)v_y\right)\right]d\Omega \tag{1.110}$$

$$= \int_\Omega \left[\left(1+x^2\right)u_x v_x + \left(1+y^2\right)u_y v_y\right]d\Omega \tag{1.111}$$

$$= \langle Lu, v\rangle\,,\ \ \forall u, v \in D(L)\,. \tag{1.112}$$

Thus, L is a maximal monotone symmetric operator. Applying [1, Proposition 7.6, p. 193] to L yields that L is self-adjoint.

5. Since L is a self-adjoint maximal monotone operator, applying Hille-Yosida theorem [1, Theorem 7.7, p. 194] yields that for every $g \in L^2(\Omega)$ there exists a unique function

$$u \in C\left([0,+\infty]\,; L^2(\Omega)\right) \cap C^1\left((0,+\infty)\,; L^2(\Omega)\right) \cap C\left((0,+\infty)\,; D(L)\right) \tag{1.113}$$

such that

$$\begin{cases} u_t + Lu = 0, & \text{in } \Omega \times (0,+\infty)\,, \\ u(x,y,0) = g(x,y)\,, & \text{in } \Omega. \end{cases} \tag{1.114}$$

Moreover (bonus), we have

$$\|u(\cdot,\cdot,t)\|_2 \le \|g\|_2, \tag{1.115}$$

$$\|u_t(\cdot,\cdot,t)\|_2 = \|Au(\cdot,\cdot,t)\|_2 \le \frac{1}{t}\|g\|_2,\ \ \forall t > 0, \tag{1.116}$$

$$u \in C^k\left((0,+\infty)\,; D\left(L^l\right)\right),\ \ \forall k, l \text{ integers}. \tag{1.117}$$

This completes our proof. □

2 Appendices

Theorem 2.1 (Lax-Milgram)**.** *Assume that $a(u,v)$ is a continuous coercive bilinear from on H. Then, given any $\varphi \in H^\star$, there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle \varphi, v\rangle\,,\ \ \forall v \in H. \tag{2.1}$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \text{ and } \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}. \tag{2.2}$$

[4] See [1, Corollary 4.23, p.109].

Theorem 2.2 ([2], Theorem 12.3, p. 189). *Let J denote a functional on domain $\mathcal{A}$ with associated set of comparison functions $\mathcal{M}$, and suppose that $u_0 \in \mathcal{A}$ is a local extreme point for J. If J has a variation at u_0, it must vanish; i.e.,*

$$\delta J\left[u_0; v\right] = 0 \text{ for all } v \in M. \tag{2.3}$$

Theorem 2.3 ([2], Theorem 12.4, p. 190). *If the subspace $\mathcal{M}$ of comparison functions is dense in $L^2(\Omega)$ and if $u_0 \in \mathcal{A}$ is a local extreme point for J, then u_0 necessarily belongs to $\mathcal{D}$ and*

$$\nabla J\left[u_0\right] = 0, \tag{2.4}$$

(the Euler equation for J).

Theorem 2.4 ([1], Corollary 4.23, p. 109). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.*

Theorem 2.5 ([1], Corollary 4.24, p. 110). *Let $\Omega \in \mathbb{R}^N$ be an open set and let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that*

$$\int uf = 0, \;\; \forall f \in C_c^\infty(\Omega). \tag{2.5}$$

Then $u = 0$ a.e. on Ω.

Theorem 2.6 ([1], Proposition 7.6, p. 193). *Let A be a maximal monotone symmetric operator. Then A is self-adjoint.*

Theorem 2.7 (Hill-Yosida, [1], Theorem 7.4, p. 185). *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1\left([0, +\infty); H\right) \cap C\left([0, +\infty); D(A)\right) \tag{2.6}$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \text{ on } [0, +\infty), \\ u(0) = u_0. \end{cases} \tag{2.7}$$

Moreover,

$$|u(t)| \leq |u_0| \text{ and } \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0|, \forall t \geq 0. \tag{2.8}$$

Theorem 2.8 ([1], Theorem 7.7, p. 194). *Let A be a self-adjoint maximal monotone operator. Then for every $u_0 \in H$ there exists a unique function*

$$u \in C\left([0, +\infty]; H\right) \cap C^1\left((0, +\infty); H\right) \cap C\left((0, +\infty); D(A)\right) \tag{2.9}$$

such that

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \text{ on } [0, +\infty), \\ u(0) = u_0. \end{cases} \tag{2.10}$$

Moreover, we have

$$|u(t)| \leq |u_0| \ \ \textit{and} \ \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq \frac{1}{t}|u_0|, \forall t \geq 0, \tag{2.11}$$

$$u \in C^k\left((0,+\infty); D\left(A^l\right)\right), \ \ \forall k, l \ \textit{integers}. \tag{2.12}$$

THE END

References

[1] Haim Brezis, *Funcitonal Analysis, Sobolev Spaces and Partial Differential Equations*, Springer.

[2] Paul DuChateau, David W. Zachmann, *Theory and Problems of Partial Differential Equations*, Schaum's outline series, McGraw-Hill.