Nonlinear Programming Assignment 002

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Abstract

This assignment aims at solving some selected problems for the final exam of the course *Theory of Nonlinear Programming*.

Contents

1 Directional, Gâteaux & Fréchet Differentiable Functions

Problem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined by*

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \tag{1.1}$$

1. *Is f directional differentiable at $x_0 = (0,0)$?*
2. *Is f Gâteaux differentiable at $x_0 = (0,0)$?*

Solution.

1. Let $d \in \mathbb{R}^2$ be a vector $d = (d_1, d_2)^T$, and $x_0 = (0,0)$. If $d = (0,0)$, we have $f'(x_0; d) = 0$ from the definition of directional derivative. If $d \neq (0,0)$, we compute

$$\lim_{t\to 0} \frac{f(x_0+td) - f(x_0)}{t} = \lim_{t\to 0} \frac{f(td_1, td_2) - f(0,0)}{t} \tag{1.2}$$

$$= \lim_{t\to 0} \frac{d_1 d_2^2}{d_1^2 + t^2 d_2^4}. \tag{1.3}$$

   We consider the following two cases depending on d_1. If $d_1 = 0$ (hence $d_2 \neq 0$ due to the assumption $d \neq (0,0)$), then the term in the limit in (1.3) equals zero, so this limit also equals zero. If $d_1 \neq 0$, then

$$\lim_{t\to 0} \frac{d_1 d_2^2}{d_1^2 + t^2 d_2^4} = \frac{d_2^2}{d_1}. \tag{1.4}$$

   Combining both cases, we deduces that f is directional differentiable at x_0 and its directional derivative is given by

$$f'(x_0; d) = \begin{cases} 0, & \text{if } d_1 = 0, \\ \frac{d_2^2}{d_1} & \text{if } d_1 \neq 0. \end{cases} \tag{1.5}$$

   where $d := (d_1, d_2) \in \mathbb{R}^2$.

2. The directional derivative $f'(x_0; d)$ given by (1.5) is not linear in term of the variable d. Hence, f is not Gâteaux differentiable at x_0.

This completes our solution. □

Problem 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined by*

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \tag{1.6}$$

1. *Is f directional differentiable at $x_0 = (0,0)$?*
2. *Is f Gâteaux differentiable at $x_0 = (0,0)$?*

3. *Is f Fréchet differentiable at $x_0 = (0,0)$?*

SOLUTION.

1. Let $d \in \mathbb{R}^2$ be a vector $d = (d_1, d_2)^T$, and $x_0 = (0,0)$. If $d = (0,0)$, we have $f'(x_0; d) = 0$ by the definition of directional derivative. If $d \neq (0,0)$, we compute

$$\lim_{t \to 0} \frac{f(x_0 + td) - f(x_0)}{t} = \lim_{t \to 0} \frac{f(td_1, td_2) - f(0,0)}{t} \tag{1.7}$$

$$= \lim_{t \to 0} \frac{td_1^3 d_2}{t^2 d_1^4 + d_2^2}. \tag{1.8}$$

We consider the following two cases depending on d_2. If $d_2 = 0$ (hence $d_1 \neq 0$ due to the assumption $d \neq (0,0)$), then the term in the limit in (1.8) equals zero, so this limit also equals zero. If $d_2 \neq 0$, then

$$\lim_{t \to 0} \left| \frac{td_1^3 d_2}{t^2 d_1^4 + d_2^2} \right| \leq \lim_{t \to 0} \left| \frac{td_1^3}{d_2} \right| = 0, \tag{1.9}$$

i.e., the limit in (1.8) also equals zero in this case. Combining both cases, we deduce that f is directional differentiable at x_0 and its directional derivative is given by $f'(x_0; d) = 0$ for all $d \in \mathbb{R}^2$.

2. From the above result, we have

$$f'(x_0; d) = 0 = \begin{pmatrix} 0 & 0 \end{pmatrix} d, \quad \forall d \in \mathbb{R}^2, \tag{1.10}$$

which is linear in d. Hence, f is Gâteaux differentiable at $x_0 = (0,0)$.

3. We claim that f is not Fréchet differentiable at $x_0 = (0,0)$. To this end, we suppose for the contrary that f is Fréchet differentiable at $x_0 = (0,0)$, by definition of Fréchet differentiability, there exists a linear function $l : \mathbb{R}^2 \to \mathbb{R}$, $l(x) = \langle l, x \rangle = l_1 x_1 + l_2 x_2$ such that

$$\lim_{\|h\| \to 0} \frac{f(x_0 + h) - f(x_0) - \langle l, h \rangle}{\|h\|} = 0. \tag{1.11}$$

Denote $h = (h_1, h_2)^T \in \mathbb{R}^2$, then (1.11) becomes

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \left(\frac{h_1^3 h_2}{h_1^4 + h_2^2} - l_1 h_1 - l_2 h_2 \right) = 0. \tag{1.12}$$

In particular, if we take $h = (h_1, 0)$ for which $h_1 \neq 0$ and $h_1 \to 0$, then (1.12) gives $\lim\limits_{h_1 \to 0} \frac{l_1 h_1}{|h_1|} = 0$, thus, $l_1 = 0$. Similarly, taking $h = (0, h_2)$ for which $h_2 \neq 0$ and $h_2 \to 0$ gives $l_2 = 0$. Substituting $l_1 = l_2 = 0$ back to (1.12) gives

$$\lim_{\|h\| \to 0} \frac{h_1^3 h_2}{(h_1^4 + h_2^2)\sqrt{h_1^2 + h_2^2}} = 0. \tag{1.13}$$

But (1.13) is not true since, for instance, taking $h_2 = h_1^2$ in (1.13), i.e., $h = (h_1, h_1^2)$, for which $h_1 \neq 0$ and $h_1 \to 0$, gives

$$\lim_{h_1 \to 0} \frac{1}{2\sqrt{1 + h_1^2}} = 0, \tag{1.14}$$

which is absurd, since the limit in the left-hand side of (1.14) is $\frac{1}{2}$.

This contradiction ends our proof. □

Problem 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by*

$$f(x,y) = \left(x^3, y^2\right). \tag{1.15}$$

Consider $x_0 = (0,0), y_0 = (1,1)$.

Does there exist any $z \in [x_0, y_0] = \{tx_0 + (1-t)\, y_0 | t \in [0,1]\}$ such that

$$\|f(y_0) - f(x_0)\| = \nabla f(z)[y_0 - x_0]. \tag{1.16}$$

Solution. We compute

$$\begin{aligned} \nabla f(x,y) &= \left(3x^2, 2y\right), \quad \forall (x,y) \in \mathbb{R}^2, &(1.17)\\ [x_0, y_0] &= \{tx_0 + (1-t)\, y_0 | t \in [0,1]\} &(1.18)\\ &= \{(1-t, 1-t)\, | t \in [0,1]\} &(1.19)\\ &= \{(t,t)\, | t \in [0,1]\}, &(1.20)\\ \|f(y_0) - f(x_0)\| &= \|(1,1) - (0,0)\| = \sqrt{2}, &(1.21) \end{aligned}$$

Hence, putting $z = (t,t)$ for $t \in [0,1]$, (1.16) is equivalent to the following quadratic equation

$$3t^2 + 2t = \sqrt{2}, \tag{1.22}$$

which has a root $t_0 = \frac{1}{3}\left(\sqrt{1+3\sqrt{2}} - 1\right) \in [0,1]$. Hence, $z_0 = (t_0, t_0)$ satisfies the requirement. □

2 Tangent Cones & Asymptotic Contingent Cones

Definition 2.1 (Contingent set of first and second orders). Let X be a normed space, $M \subset X$ and $x_0 \in X$.

1. The *contingent cone* (or, *tangent cone, Bouligand cone*) of M at x_0 is determined by

$$T(M, x_0) = \left\{u \in X | \exists t_n \to 0^+, \exists u_n \to u, x_0 + t_n u_n \in M, \ \ \forall n \in \mathbb{N}\right\}. \tag{2.1}$$

2. The *second-order contingent set* of M at x_0 in the direction u is determined by

$$T^2(M, x_0, u) = \left\{ \begin{array}{l} w \in X | \exists t_n \to 0^+, \exists w_n \to w, \\ x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M, \ \ \forall n \in \mathbb{N} \end{array} \right\}. \tag{2.2}$$

3. The *asymptotic contingent cone of second order* of M at x_0 in the direction u is determined by

$$T''(M, x_0, u) = \left\{ \begin{array}{l} w \in X | \exists (t_n, r_n) \to \left(0^+, 0^+\right) : \frac{t_n}{r_n} \to 0, \\ \exists w_n \to w, x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in M, \ \ \forall n \in \mathbb{N} \end{array} \right\}. \tag{2.3}$$

Problem 4. *Let*

$$M = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2^2 \leq 0\right\}, \tag{2.4}$$

and $x_0 = (0, 0)$.

1. *Compute* $T(M, x_0)$.
2. *Consider* $u = (0, 1)$*, compute* $T^2(M, x_0, u)$ *and* $T''(M, x_0, u)$.

Solution.

1. Setting $X = \mathbb{R}^2$, we notice that $x_0 = (0, 0) \in M$. We claim that

$$T(M, x_0) = \widehat{T}(M, x_0) := \left\{(x, y) \in \mathbb{R}^2 | x \leq 0\right\}. \tag{2.5}$$

To prove (2.5), we prove the following inclusions.

(a) *Prove* $T(M, x_0) \subset \widehat{T}(M, x_0)$. Taking $u = (x, y) \in T(M, x_0)$, by (2.1), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the fact $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives

$$t_n x_n + t_n^2 y_n^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (2.6) then implies

$$x_n + t_n y_n^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

We see at a glance from (2.7) that $x_n \leq 0$ for all $n \in \mathbb{N}$. Hence $x \leq 0$ (since $x_n \to x$ as $n \to \infty$). Now let $n \to \infty$ in (2.7) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $x \leq 0$ as just mentioned. Hence, $u \in \widehat{T}(M, x_0)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}(M, x_0) \subset T(M, x_0)$. Taking $u = (x, y) \in \mathbb{R}^2$ satisfying $x \leq 0$, we claim that $u \in T(M, x_0)$. To this end, we now choose $x_n = x - \frac{1}{n} < 0, y_n = y$ for all $n \in \mathbb{N}$. This choice ensures that $u_n := (x_n, y_n) \to u := (x, y)$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. The latter gives, using (2.7) again,

$$x - \frac{1}{n} + t_n y^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

If $y = 0$, (2.8) holds obviously for all $t_n > 0$, thus, we can choose an arbitrary sequence t_n's of positive reals satisfying $t_n \to 0^+$. If $y \neq 0$, (2.8) is equivalent to

$$t_n \leq \frac{\frac{1}{n} - x}{y^2}, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

The term in the right-hand side of (2.9) is positive for all $n \in \mathbb{N}$. Hence we can choose t_n's satisfying (2.9) and $t_n \to 0^+$ as $n \to \infty$. This choice implies that $u \in T(M, x_0)$, i.e., the second inclusion is also proved.

Combining these, we conclude that (2.5) holds, i.e.,

$$T\left(M, x_0\right) = \left\{(x, y) \in \mathbb{R}^2 | x \leq 0\right\}. \tag{2.10}$$

2. **Compute $T^2\left(M, x_0, u\right)$.** First, we claim that

$$T^2\left(M, x_0, u\right) = \widehat{T}^2\left(M, x_0, u\right) := \left\{(x, y) \in \mathbb{R}^2 | x \leq -2\right\}. \tag{2.11}$$

To prove (2.11), we prove the following inclusions.

(a) *Prove* $T^2\left(M, x_0, u\right) \subset \widehat{T}^2\left(M, x_0, u\right)$. Taking $w = (x, y) \in T^2\left(M, x_0, u\right)$, by (2.2), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $w_n \to w$ as $n \to \infty$ and $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. Set $w_n := (x_n, y_n)$, the fact $w_n \to w$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$ gives

$$\frac{1}{2} t_n^2 x_n + \left(t_n + \frac{1}{2} t_n^2 y_n\right)^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (2.12) implies that

$$\frac{x_n}{2} + 1 + t_n y_n + \frac{1}{4} t_n^2 y_n^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Now let $n \to \infty$ in (2.13) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $x \leq -2$. Hence, $w \in \widehat{T}\left(M, x_0, u\right)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}^2\left(M, x_0, u\right) \subset T^2\left(M, x_0, u\right)$.

PROOF 1. Taking $w = (x, y)$ satisfying $x \leq -2$, we claim that $w \in T^2\left(M, x_0, u\right)$. To this end, we now choose $x_n := x - \frac{1}{n} \leq -2 - \frac{1}{n}$ and $y_n := y$ for all $n \in \mathbb{N}$. This choice ensures that $w_n := (x_n, y_n) \to w := (x, y)$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. The latter is equivalent to, using (2.13) again,

$$t_n^2 y^2 + 4 t_n y + 2\left(x - \frac{1}{n}\right) + 4 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.14}$$

We consider the following cases depending on y. If $y = 0$, then (2.14) obviously holds for all sequence t_n's since $x \leq -2$. If $y \neq 0$, consider the left-hand side of (2.14) as a quadratic equation in t_n, its discriminant is given by

$$\Delta' = 4y^2 - y^2\left[2\left(x - \frac{1}{n}\right) + 4\right] \tag{2.15}$$

$$= 2\left(\frac{1}{n} - x\right) y^2 \geq 0. \tag{2.16}$$

Thus, its two roots are given by

$$t_{n,1} = \frac{-2y - |y|\sqrt{2\left(\frac{1}{n} - x\right)}}{y^2}, \tag{2.17}$$

$$t_{n,2} = \frac{-2y + |y|\sqrt{2\left(\frac{1}{n} - x\right)}}{y^2}. \tag{2.18}$$

Since $x \leq -2$, it is easy to verify that $t_{n,1} < 0 < t_{n,2}$. If we choose a sequence t_n's such that $t_n \to 0^+$ and $0 < t_n \leq t_{n,2}$ then $x_0 + t_n u + \frac{1}{2}t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. Hence, $w \in T^2(M, x_0, u)$ and the second inclusion is also proved.

Proof 2. Use the same settings as Proof 1, we arrive at the inequality (2.14). We now define, for each $n \in \mathbb{N}$, the function

$$F_n(t) := y^2t^2 + 4yt + 2\left(x - \frac{1}{n}\right) + 4, \quad t > 0. \tag{2.19}$$

It is obvious to check $F(t)$ is continuous, and

$$F_n(0) = 2\left(x - \frac{1}{n}\right) + 4 = 2(x+2) - \frac{2}{n} < 0. \tag{2.20}$$

Thus, by continuity of F_n, we can choose $t_n > 0$ small enough such that (2.14) holds. And the chosen sequence t_n's indicates that $w \in T^2(M, x_0, u)$, i.e., the second inclusion is proved.

Combining these inclusions, we conclude that (2.11) holds, i.e.,

$$T^2(M, x_0, u) = \left\{(x, y) \in \mathbb{R}^2 | x \leq -2\right\}. \tag{2.21}$$

Compute $T''(M, x_0, u)$. We claim that

$$T''(M, x_0, u) = \widehat{T}''(M, x_0, u) := \left\{(x, y) \in \mathbb{R}^2 | x \leq 0\right\}. \tag{2.22}$$

To prove (2.22), we also prove the following two inclusions as before.

(a) *Prove* $T''(M, x_0, u) \subset \widehat{T}''(M, x_0, u)$. Taking $w = (x, y) \in T''(M, x_0, u)$, by (2.3), there exist two sequences of positive reals $\{t_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $w_n \to w$ and $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. Set $w_n := (x_n, y_n)$, the fact that $w_n \to w$ implies that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, and the fact $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$ gives

$$\frac{1}{2}t_n r_n x_n + \left(t_n + \frac{1}{2}t_n r_n y_n\right)^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.23}$$

Since $t_n, r_n > 0$ for all $n \in \mathbb{N}$, (2.23) implies that

$$\frac{x_n}{2} + \frac{t_n}{r_n} + t_n y_n + \frac{1}{4}t_n r_n y_n^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.24}$$

Now let $n \to \infty$ in (2.24) and use the given limits $x_n \to x, y_n \to y, t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0^+$ as $n \to \infty$, we obtain $x \leq 0$. Hence, $w \in \widehat{T}''(M, x_0, u)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}''(M, x_0, u) \subset T''(M, x_0, u)$.

PROOF 1. Taking $w = (x, y)$ satisfying $x \leq 0$, we claim that $w \in T''(M, x_0, u)$. To this end, we choose $x_n := x - \frac{1}{n} < 0, y_n = y$ and $t_n \leq \frac{1}{n^2}$ and $r_n = 2nt_n$ for all $n \in \mathbb{N}$, where t_n will be constrained more strictly as follows. This choice ensures that $w_n := (x_n, y_n) \to w := (x, y)$, $t_n \to 0^+$, $r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \leq \frac{1}{n^2}$ (in order for $r_n \to 0^+$) and $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. The latter is equivalent to, using (2.24) again,

$$\frac{1}{2}\left(x - \frac{1}{n}\right) + \frac{1}{2n} + t_n y + \frac{n}{2}t_n^2 y^2 \leq 0, \quad \forall n \in \mathbb{N}, \tag{2.25}$$

i.e.,

$$ny^2 t_n^2 + 2yt_n + x \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.26}$$

We consider the following cases depending on y. If $y = 0$, then (2.26) obviously holds for all sequences t_n's since $x \leq 0$. Thus we can choose a sequence of positive reals t_n's satisfying $t_n \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ arbitrarily in this case. If $y \neq 0$, consider the left-hand side of (2.26) as a quadratic equation in t_n, its discriminant is given by

$$\Delta' = (1 - nx)\, y^2 \geq 0. \tag{2.27}$$

Thus, its two roots are given by

$$t_{n,1} = \frac{-y - |y|\sqrt{1 - nx}}{ny^2}, \tag{2.28}$$

$$t_{n,2} = \frac{-y + |y|\sqrt{1 - nx}}{ny^2}. \tag{2.29}$$

Since $x \leq 0$, it is easy to verify that $t_{n,1} < 0 < t_{n,2}$. If we choose a sequence t_n's such that

$$0 < t_n \leq \min\left\{\frac{1}{n^2}, t_{n,2}\right\}, \quad \forall n \in \mathbb{N}, \tag{2.30}$$

then $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. Hence, $w \in T''(M, x_0, u)$ and the second inclusion is also proved.

PROOF 2. We choose $x_n := x - \frac{1}{n}, y_n := y, t_n = r_n^2$ for all $n \in \mathbb{N}$, where r_n's is a sequence of positive reals such that $r_n \to 0^+$ as $n \to \infty$. This choice ensures that $w_n := (x_n, y_n) \to w := (x, y)$, $t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} = r_n \to 0^+$ as $n \to \infty$. With these settings, (2.24) becomes

$$\frac{1}{2}\left(x - \frac{1}{n}\right) + r_n\left(1 + \frac{1}{2}r_n y\right)^2 \leq 0, \quad \forall n \in \mathbb{N}. \tag{2.31}$$

Define

$$F_n(r) = \frac{1}{2}\left(x - \frac{1}{n}\right) + r\left(1 + \frac{1}{2}ry\right)^2, \quad r > 0. \tag{2.32}$$

It is obvious that $F_n(r)$ is continuous and $F(0) = \frac{1}{2}\left(x - \frac{1}{n}\right) < 0$ since $x \le 0$. By continuity of F, we can choose $r_n > 0$ small enough such that (2.31) holds. And the chosen sequence r_n's indicates that $w \in T''(M, x_0, u)$, i.e., the second inclusion is also proved.

Combining these inclusions, we conclude that (2.22) holds, i.e.,

$$T''(M, x_0, u) = \left\{(x, y) \in \mathbb{R}^2 | x \le 0\right\}. \tag{2.33}$$

This completes our solution. □

Problem 5. *Let*

$$M = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1^3 - x_2^2 = 0\right\}, \tag{2.34}$$

and $x_0 = (0, 0)$.

1. *Compute* $T(M, x_0)$.
2. *Consider* $u = (1, 0)$, *compute* $T^2(M, x_0, u)$ *and* $T''(M, x_0, u)$.

Solution. Setting $X = \mathbb{R}^2$, we notice that $x_0 = (0, 0) \in M$.

1. We claim that

$$T(M, x_0) = \widehat{T}(M, x_0) := \left\{(x, 0) \in \mathbb{R}^2 | x \ge 0\right\}. \tag{2.35}$$

To prove (2.35), we prove the following inclusions.

(a) *Prove* $T(M, x_0) \subset \widehat{T}(M, x_0)$. Taking $u = (x, y) \in T(M, x_0)$, by (2.1), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the fact that $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives

$$t_n^3 x_n^3 - t_n^2 y_n^2 = 0, \quad \forall n \in \mathbb{N}. \tag{2.36}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (2.36) then implies

$$t_n x_n^3 = y_n^2, \quad \forall n \in \mathbb{N}. \tag{2.37}$$

We see at a glance from (2.37) that $x_n \ge 0$ for all $n \in \mathbb{N}$. Hence, $x \ge 0$ (since $x_n \to x$ as $n \to \infty$). Now let $n \to \infty$ in (2.37) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $y = 0$. Hence, $u \in \widehat{T}(M, x_0)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}(M, x_0) \subset T(M, x_0)$. Taking $u = (x, 0)$ for which $x \geq 0$, we claim that $u \in T(M, x_0)$. To this end, we now choose $x_n := x + \frac{1}{n} > 0, y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. This choice ensures that $u_n := (x_n, y_n) \to u := (x, 0)$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. The latter gives, using (2.37) again,

$$t_n \left(x + \frac{1}{n} \right)^3 = \frac{1}{n^4}, \quad \forall n \in \mathbb{N}. \tag{2.38}$$

i.e.,

$$t_n = \frac{1}{n^4 \left(x + \frac{1}{n} \right)^3}, \quad \forall n \in \mathbb{N}. \tag{2.39}$$

It is easy to check that $t_n > 0$ (since $x \geq 0$) and $t_n \to 0^+$ as $n \to \infty$.[1] Hence, $u \in T(M, x_0)$ and the second inclusion is also proved.

Combining these inclusions, we conclude that (2.35) holds, i.e.,

$$T(M, x_0) = \left\{ (x, 0) \in \mathbb{R}^2 | x \geq 0 \right\}. \tag{2.40}$$

2. **Compute $T^2(M, x_0, u)$.** First, we claim that

$$T^2(M, x_0, u) = \emptyset. \tag{2.41}$$

Indeed, suppose for the contrary that there exists $w = (x, y) \in T^2(M, x_0, u)$, by (2.2), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $w_n \to w$ as $n \to \infty$ and $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. Set $w_n := (x_n, y_n)$, the fact $w_n \to w$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$ gives

$$\left(t_n + \frac{1}{2} t_n^2 x_n \right)^3 = \frac{1}{4} t_n^4 y_n^2, \forall n \in \mathbb{N}. \tag{2.42}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (2.42) implies

$$\left(1 + \frac{1}{2} t_n x_n \right)^3 = \frac{1}{4} t_n y_n^2, \quad \forall n \in \mathbb{N}. \tag{2.43}$$

Now let $n \to \infty$ in (2.43) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $1 = 0$, which is absurd. This contradiction implies that (2.41) is true.

Compute $T''(M, x_0, u)$. We claim that $T''(M, x_0, u) = \mathbb{R}^2$. To prove this, taking $w = (x, y) \in \mathbb{R}^2$ arbitrarily, we claim that $w \in T''(M, x_0, u)$. The event $x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in M$ for all $n \in \mathbb{N}$ is equivalent to the following equality

$$\left(1 + \frac{1}{2} r_n x_n \right)^3 = \frac{r_n^2 y_n^2}{4 t_n}, \quad \forall n \in \mathbb{N}. \tag{2.44}$$

We now consider the following cases depending on y.

[1] If $x = 0$, then $t_n = \frac{1}{n} \to 0$ as $n \to \infty$. If $x < 0$, then $t_n \to \frac{1}{x^3} \lim_{n \to \infty} \frac{1}{n^4} = 0$ as $n \to \infty$.

(a) *Case* $y \neq 0$. In this case, we choose $x_n := x$ and t_n, r_n for which

$$\frac{r_n^2}{t_n} = \frac{4}{y^2}, \quad \forall n \in \mathbb{N}, \tag{2.45}$$

i.e., $t_n = \frac{r_n^2 y^2}{4}$ for all $n \in \mathbb{N}$. This choice ensures that $x_n \to x, t_n \to 0^+$ and $\frac{t_n}{r_n} = \frac{r_n y^2}{4} \to 0^+$ as $n \to \infty$ provided $r_n \to 0^+$. If now suffices to choose r_n's and y_n's in order that $r_n \to 0^+, y_n \to y$ as $n \to \infty$ and (2.44) holds. We consider the following cases depending on x.

- *Case* $x = 0$. We choose $y_n := y$ for all $n \in \mathbb{N}$ and an arbitrarily sequence of positive reals r_n's such that $r_n \to 0^+$ as $n \to \infty$. With this setting, it is easy to verify that (2.44) holds (both sides of this equality equal 1) and other conditions for $T''(M, x_0, u)$ meet. Hence, $w \in T''(M, x_0, u)$ in this case.
- *Case* $x \neq 0$. In this case, (2.44) becomes

$$\left(1 + \frac{1}{2} r_n x\right)^3 = \frac{y_n^2}{y^2}, \quad \forall n \in \mathbb{N}. \tag{2.46}$$

i.e.,

$$r_n = \frac{2}{x}\left(\sqrt[3]{\frac{y_n^2}{y^2}} - 1\right), \quad \forall n \in \mathbb{N}. \tag{2.47}$$

If $x > 0$, we choose

$$y_n = y + \frac{\operatorname{sign}(y)}{n}, \quad \forall n \in \mathbb{N}, \tag{2.48}$$

to ensure that $y_n \to y$ and

$$r_n = \frac{2}{x}\left(\sqrt[3]{\frac{\left(|y| + \frac{1}{n}\right)^2}{y^2}} - 1\right) \to 0^+, \quad \forall n \in \mathbb{N}. \tag{2.49}$$

Similarly, if $x < 0$, the choice

$$y_n = y - \frac{\operatorname{sign}(y)}{n}, \forall n \in \mathbb{N}, \tag{2.50}$$

ensures that $y_n \to y$ and

$$r_n = \frac{2}{x}\left(\sqrt[3]{\frac{\left(|y| - \frac{1}{n}\right)^2}{y^2}} - 1\right) \to 0^+, \quad \forall n \in \mathbb{N}. \tag{2.51}$$

We can write both (2.48) and (2.50) in a more compact form

$$y_n = y + \frac{\operatorname{sign}(xy)}{n}, \quad \forall n \in \mathbb{N}, \tag{2.52}$$

to ensure that $y_n \to y$ and $r_n \to 0^+$ as $n \to \infty$. Hence, we also have $w \in T''(M, x_0, u)$ in this case.

(b) *Case* $y = 0$. In this case, we choose $x_n := x, y_n := \frac{1}{n}$ for all $n \in \mathbb{N}$, (2.44) then becomes

$$\left(1 + \frac{1}{2} r_n x\right)^3 = \frac{r_n^2}{4n^2 t_n}, \quad \forall n \in \mathbb{N}. \tag{2.53}$$

We now consider the following cases depending on x.

- *Case* $x = 0$. Choose r_n's as an arbitrary sequence of positive reals satisfying $r_n \to 0^+$ as $n \to \infty$, and then choose t_n as follows,

$$t_n = \frac{r_n^2}{4n^2}, \quad \forall n \in \mathbb{N}. \tag{2.54}$$

This choice ensures that (2.53) holds and $t_n \to 0^+$ and $\frac{t_n}{r_n} = \frac{r_n}{4n^2} \to 0^+$ as $n \to \infty$. Thus, $w \in T'' (M, x_0, u)$ in this case.

- *Case* $x > 0$. We choose t_n such that

$$\frac{r_n^2}{4n^2 t_n} = 1 + \frac{1}{n}, \quad \forall n \in \mathbb{N}, \tag{2.55}$$

i.e.,

$$t_n = \frac{r_n^2}{4n\,(n+1)}, \quad \forall n \in \mathbb{N}. \tag{2.56}$$

Provided $r_n \to 0^+$ as $n \to \infty$, this choice of t_n's ensures that $t_n \to 0^+$ and $\frac{t_n}{r_n} = \frac{r_n}{4n(n+1)} \to 0^+$ as $n \to \infty$. Substituting the chosen t_n's into (2.53) yields

$$\left(1 + \frac{1}{2} r_n x\right)^3 = 1 + \frac{1}{n}, \quad \forall n \in \mathbb{N}, \tag{2.57}$$

i.e.,

$$r_n = \frac{2}{x} \left(\sqrt[3]{1 + \frac{1}{n}} - 1\right), \quad \forall n \in \mathbb{N}. \tag{2.58}$$

It is easy to verify that $r_n \to 0^+$ as $n \to \infty$. Thus, $w \in T'' (M, x_0, u)$ in this case.

- *Case* $x < 0$. This case can be handled similarly with some modifications. We choose t_n such that

$$\frac{r_n^2}{4n^2 t_n} = 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.59}$$

i.e.,

$$t_n = \frac{r_n^2}{4n\,(n-1)}, \quad \forall n \in \mathbb{N} : n > 1. \tag{2.60}$$

($t_1 > 0$ is arbitrary) Provided $r_n \to 0^+$ as $n \to \infty$, this choice of t_n's ensures that $t_n \to 0^+$ and $\frac{t_n}{r_n} = \frac{r_n}{4n(n-1)} \to 0^+$ as $n \to \infty$. Substituting the chosen t_n's into (2.53) yields

$$\left(1 + \frac{1}{2} r_n x\right)^3 = 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}, \tag{2.61}$$

i.e.,

$$r_n = \frac{2}{x}\left(\sqrt[3]{1 - \frac{1}{n}} - 1\right), \quad \forall n \in \mathbb{N}. \tag{2.62}$$

It is easy to verify that $r_n \to 0^+$ as $n \to \infty$. Thus, $w \in T''(M, x_0, u)$ in this case.

Remark 2.2. In both cases $x > 0, x < 0$, we can set in a more compact form as follows. Choose

$$t_n = \frac{r_n^2}{4n\,(n + \operatorname{sign}(x))}, \quad \forall n \in \mathbb{N}, \tag{2.63}$$

and then (2.53) gives

$$r_n = \frac{2}{x}\left(\sqrt[3]{1 + \frac{\operatorname{sign}(x)}{n}} - 1\right), \quad \forall n \in \mathbb{N}. \tag{2.64}$$

The second inclusion is also proved.

Combining these inclusions, we conclude that $T''(M, x_0, u) = \mathbb{R}^2$. This completes our solution. □

The following problem gives us some basic properties of contingent cone of first order.

Problem 6. *Let X be a normed space, $M \subset X$ and $x_0 \in X$.*

1. *If $T(M, x_0) \neq \emptyset$ then $x_0 \in \overline{M}$ (where $\overline{M}$ is the closure of the set M).*
2. *$T(M, x_0)$ is a closed cone.*
3. *$T(M, x_0) \subset \overline{\text{cone}\,(M - x_0)}$.*

   Moreover, if M is a convex set then
4. $T(M, x_0) = \overline{\text{cone}\,(M - x_0)}$, and hence, $T(M, x_0)$ is a convex set.
5. $T(M, x_0) = \{v \in X | \forall t_n \to 0^+, \forall v_n \to v, x_0 + t_n v_n \in M\}$.

Solution.

1. Suppose that $T(M, x_0) \neq \emptyset$, we can take, for instance, $u \in T(M, x_0)$. Then there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \to u$ and $x_0 + t_n u_n \in M$ for

all $n \in \mathbb{N}$. Set $x_n := x_0 + t_n u_n \in M$. Since $u_n \to u$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \|u_n - u\| \leq 1. \tag{2.65}$$

We now prove that $x_n \to x_0$ as $n \to \infty$. Indeed, for $n \geq N$,

$$\|x_n - x_0\| = \|t_n u_n\| = t_n \|u_n\| \leq t_n (\|u\| + 1). \tag{2.66}$$

Since $t_n \to 0^+$, (2.66) implies that $x_n \to x_0$ as $n \to \infty$, i.e., $x_0 \in \overline{M}$.

2. We first prove that $T(M, x_0)$ is a cone. Let $u \in T(M, x_0)$ arbitrarily, we need to prove that $tu \in T(M, x_0)$ for all $t > 0$. By (2.1), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \to u$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Fix $t > 0$ arbitrarily, if we set $v_n := tu_n$ and $s_n = \frac{t_n}{t}$ for all $n \in \mathbb{N}$, then $s_n \to 0^+$, $v_n \to tu$ and $x_0 + s_n v_n = x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$, i.e., $tu \in T(M, x_0)$. Since $t > 0$ and $u \in T(M, x_0)$ are chosen arbitrarily, this implies that $T(M, x_0)$ is a cone.

   To prove that $T(M, x_0)$ is closed, let $\{u_n\}_{m=0}^{\infty} \subset T(M, x_0)$ such that $u_m \to u$ as $m \to \infty$. We need to prove that $u \in T(M, x_0)$. To this end, by definition (2.1), for each $m \in \mathbb{N}$, there exist a sequence $t_{m,n} \to 0^+$ as $n \to \infty$ and a sequence $\{u_{m,n}\}_{n=0}^{\infty} \subset X$ such that $u_{m,n} \to u_m$ as $n \to \infty$ and $x_0 + t_{m,n} u_{m,n} \in M$ for all $n \in \mathbb{N}$, in addition, $\|u_{m,m} - u_m\| \leq \frac{1}{m}$ for all $m \in \mathbb{N}$[2]. We claim that

$$u_{m,m} \to u \text{ and } x_0 + t_{m,m} u_{m,m} \in M, \;\; \forall m \in \mathbb{N}. \tag{2.69}$$

   The latter is obvious since $x_0 + t_{m,n} u_{m,n} \in M$ for all $m, n \in \mathbb{N}$. We now prove the former in (2.69). With the help of triangle inequality for the norm of X,

$$\begin{aligned} \|u_{m,m} - u\| &\leq \|u_{m,m} - u_m\| + \|u_m - u\| && (2.70) \\ &\leq \frac{1}{m} + \|u_m - u\| \to 0 \text{ as } m \to \infty, && (2.71) \end{aligned}$$

   i.e., $u \in T(M, x_0)$. Hence, $T(M, x_0)$ is a closed cone.

3. The convex conical hull of $M - x_0$ is given by (see, e.g., [1], Def. 4.19, p.94)

$$\operatorname{cone}(M - x_0) := \left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in M - x_0, \lambda_i > 0, k \geq 1 \right\}. \tag{2.72}$$

[2]This is possible, since for each $m \in \mathbb{N}$, there exists a sequence $\{u_{m,n}\}_{n=0}^{\infty} \subset X$ such that $u_{m,n} \to u_m$ as $n \to \infty$. By definition of limits, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \|u_{m,n} - u_m\| \leq \frac{1}{m}. \tag{2.67}$$

Hence, we can drop all the terms $u_{m,1}, \ldots, u_{m,n-1}$ from the sequence. Re-indexing $\widehat{u}_{m,n} := u_{m,N+n-1}$ for all $n \in \mathbb{N}$, we have, in particular,

$$\|\widehat{u}_{m,m} - u_m\| = \left\|\widehat{u}_{m,N+m-1} - u_m\right\| \leq \frac{1}{m}. \tag{2.68}$$

We now ignore the old sequence $\{u_{m,n}\}_{n=0}^{\infty}$ and use the new sequence, by abuse notation, $\{u_{m,n}\}_{n=0}^{\infty}$ which is exactly $\{\widehat{u}_{m,n}\}_{n=0}^{\infty}$ just defined.

Take $u \in T(M, x_0)$ arbitrarily, we need to prove that $u \in \overline{\text{cone}(M - x_0)}$. By (2.1) again, there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \to u$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. The fact that $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives us $t_n u_n \in M - x_0$ for all $n \in \mathbb{N}$. Choosing $k = 1, \lambda_1 = \frac{1}{t_n} > 0, x_1 = t_n u_n \in M - x_0$ in (2.72) gives $u_n \in \text{cone}(M - x_0)$ for all $n \in \mathbb{N}$. Combining this with the fact that $u_n \to u$, we conclude that $u \in \overline{\text{cone}(M - x_0)}$. Therefore,

$$T(M, x_0) \subset \overline{\text{cone}(M - x_0)}. \tag{2.73}$$

4. First Proof. We now assume (until the end of the proof of this problem) that M is a convex set and $x_0 \in M$[3]. To prove $T(M, x_0) = \overline{\text{cone}(M - x_0)}$, due to (2.73), it suffices to prove that $T(M, x_0) \supset \overline{\text{cone}(M - x_0)}$. First, we need the following lemma (see, e.g., [2], Lemma 2.4.11, p.41).

Lemma 2.3. *Let M be a nonempty convex set and $x_0 \in M$. Then*

$$M - x_0 \subset T(M, x_0). \tag{2.74}$$

Proof of Lemma 2.3. Let $u \in M$. We need to show that $u - x_0 \in T(M, x_0)$. To this end, choose $\{t_n\}_{n=1}^{\infty} \subset [0, 1]$ such that $t_n \to 0^+$, and put $u_n := u - x_0$ (hence $u_n \to u - x_0$ obviously) and put

$$\begin{aligned} x_n &:= x_0 + t_n(u - x_0) && (2.75)\\ &= (1 - t_n)x_0 + t_n u \in M, \quad \forall n \in \mathbb{N}, && (2.76) \end{aligned}$$

as M is convex. By (2.1), $u - x_0 \in T(M, x_0)$. □

Return to our proof, since we have proved that $T(M, x_0)$ is a closed cone, we only need to prove prove that $T(M, x_0) \supset \text{cone}(M - x_0)$. Using the fact that the convex conical hull of an arbitrary nonempty set is the intersection of all closed convex cones that contain that sets, it suffices to prove that $T(M, x_0)$ is convex (and thus is a closed convex cone). Take $u, v \in T(M, x_0)$, we need to prove that $\lambda u + (1 - \lambda)v \in T(M, x_0)$ for all $\lambda \in [0, 1]$. But since $T(M, x_0)$ is a cone, we deduce that $\lambda u \in T(M, x_0)$ and $(1 - \lambda)v \in T(M, x_0)$. Hence, it suffices to prove the following stronger statement[4]

$$u + v \in T(M, x_0), \quad \forall u, v \in T(M, x_0). \tag{2.77}$$

By (2.1), there exists sequences of positive reals $\{t_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $s_n \to 0^+$ and sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ such that $u_n \to u, v_n \to v$ and

$$x_0 + t_n u_n \in M, \quad x_0 + s_n v_n \in M, \quad \forall n \in \mathbb{N}. \tag{2.78}$$

Since M is convex, it is deduced from (2.78) that

$$\alpha(x_0 + t_n u_n) + (1 - \alpha)(x_0 + s_n v_n) \in M, \quad \forall \alpha \in [0, 1], n \in \mathbb{N}. \tag{2.79}$$

[3] The definition of tangent cone in [1] also requires this.

[4] *A cone K is convex if and only if $K + K \subset K$.* (see, e.g., [2], Proposition 2.4.2, p.38.)

In particular, choosing $\alpha = \frac{s_n}{t_n+s_n}$ in (2.79) gives

$$x_0 + \frac{t_n s_n}{t_n + s_n}(u_n + v_n) \in M, \ \ \forall n \in \mathbb{N}. \tag{2.80}$$

Hence, if we choose $w_n := u_n + v_n \to u + v$ and $r_n := \frac{t_n s_n}{t_n+s_n} \to 0^{+}$[5]. By (2.1), $u + v \in T(M, x_0)$. This completes our proof. □

Second Proof. We have the following result (see, e.g., [2], Proposition 2.4.8, p.40)

$$\mathrm{cone}S = \mathbb{R}_+(\mathrm{conv}S) = \mathrm{conv}(\mathbb{R}_+ S), \tag{2.81}$$

for an arbitrary nonempty set S. Since M is convex, $M - x_0$ is also convex (as a Minkowski sum of convex sets), hence $\mathrm{conv}(M - x_0) = M - x_0$ (see [1], Corollary 4.12, p.91) and

$$\overline{\mathrm{cone}(M - x_0)} = \overline{\mathbb{R}_+(\mathrm{conv}(M - x_0))} = \overline{\mathbb{R}_+(M - x_0)}. \tag{2.82}$$

It suffices to prove $\overline{\mathbb{R}_+(M - x_0)} \subset T(M, x_0)$. By Lemma 2.3, we have $M - x_0 \subset T(M, x_0)$. Since $T(M, x_0)$ is a closed cone, this yields $\overline{\mathbb{R}_+(M - x_0)} \subset T(M, x_0)$. A direct consequence of this fact is that $T(M, x_0)$ is a closed convex cone.

5. *(Need correcting)* Suppose the set in the right-hand side is nonempty, i.e., there exists $v \in X$ such that

$$\forall t_n \to 0^+, \forall v_n \to v, x_0 + t_n v_n \in M, \ \ \forall n \in \mathbb{N}. \tag{2.83}$$

If we take t_1 and v_1 arbitrarily, then $x_0 + t_1 v_1$ still belongs to M. Hence, $M = X$? Should (2.83) be corrected as "$\forall t_n \to 0^+, \forall v_n \to v, x_0 + t_n v_n \in M$ *for* n *large enough*"? This problem needs correcting.

We end our proof here. □

The following problem gives us some basic properties of second-order contingent set.

Problem 7. *Let X be a normed space, $M \subset X$ and $x_0, u \in X$.*

1. *If $u \notin T(M, x_0)$ then $T^2(M, x_0, u)$ and $T''(M, x_0, u)$ are empty sets.*
2. *$T^2(M, x_0, 0) = T''(M, x_0, 0) = T(M, x_0)$.*
3. *$T''(M, x_0, u)$ is a cone while $T^2(M, x_0, u)$ is not a cone in general.*
4. *If X is finite dimensional and $u \in T(M, x_0)$ then*

$$T^2(M, x_0, u) \cup T''(M, x_0, u) \neq \emptyset. \tag{2.84}$$

Solution.

[5] Indeed, $0 < r_n = t_n \underbrace{\frac{s_n}{t_n + s_n}}_{<1} < t_n \to 0^+$ as $n \to \infty$.

1. Suppose that $u \notin T(M, x_0)$, we claim that

$$T^2(M, x_0, u) = T''(M, x_0, u) = \emptyset. \tag{2.85}$$

To prove $T^2(M, x_0, u) = \emptyset$, we suppose for the contrary that there exists a $w \in T^2(M, x_0, u)$. By (2.2), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to w$ as $n \to \infty$ and $x_0 + t_n u + \frac{1}{2}t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. To obtain a contradiction, we now prove that $u \in T(M, x_0)$. Indeed, setting $u_n := u + \frac{1}{2}t_n w_n$ for all $n \in \mathbb{N}$, it is obvious to verify that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$, i.e., $u \in T(M, x_0)$. This contradiction implies that $T^2(M, x_0, u) = \emptyset$.

Similarly, to prove $T''(M, x_0, u) = \emptyset$, we suppose for the contrary that there exists a $w \in T''(M, x_0, u)$. By (2.3), there exist two sequences of positive reals $\{t_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to w$ and $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. To obtain a contradiction, we now prove that $u \in T(M, x_0)$ as above. Indeed, setting $u_n := u + \frac{1}{2}r_n w_n$ for all $n \in \mathbb{N}$, it is obvious to verify that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$, i.e., $u \in T(M, x_0)$. This contradiction implies that $T''(M, x_0, u) = \emptyset$. Hence, (2.85) holds.

2. We claim that

$$T^2(M, x_0, 0) = T''(M, x_0, 0) = T(M, x_0). \tag{2.86}$$

Prove $T^2(M, x_0, 0) = T''(M, x_0, 0)$. Taking $w \in T^2(M, x_0, 0)$, by (2.2), there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to w$ as $n \to \infty$ and

$$x_0 + \frac{1}{2}t_n^2 w_n \in M, \ \ \forall n \in \mathbb{N}. \tag{2.87}$$

Setting $\widehat{t}_n = t_n\sqrt{t_n}, r_n = \sqrt{t_n}$ for all $n \in \mathbb{N}$, this choice ensures that $\widehat{t}_n \to 0^+, r_n \to 0^+, \dfrac{\widehat{t}_n}{r_n} = t_n \to 0$ as $n \to \infty$. Moreover, (2.87) can be rewritten as

$$x_0 + \frac{1}{2}\widehat{t}_n r_n w_n \in M, \ \ \forall n \in \mathbb{N}. \tag{2.88}$$

i.e., $w \in T''(M, x_0, u)$. Notice that this argument is reversible (choose $t_n = \sqrt{\widehat{t}_n r_n}$ for all $n \in \mathbb{N}$ for the converse inclusion). Hence $T^2(M, x_0, 0) = T''(M, x_0, 0)$.

Prove $T^2(M, x_0, 0) = T(M, x_0)$. Briefly, this equality is easily deduced from

$$x_0 + t_n w_n \in M \Leftrightarrow x_0 + \frac{1}{2}\widehat{t}_n^2 w_n \in M, \tag{2.89}$$

which holds by choosing $t_n := \frac{1}{2}\widehat{t}_n^2 \to 0^+$ for the inclusion $T^2(M, x_0, 0) \subset T(M, x_0)$ and $\widehat{t}_n = \sqrt{2t_n}$ for the converse.

Prove $T''(M,x_0,0) = T(M,x_0)$. (This part is unnecessary but I also provide it here for completeness) Similarly, this equality is easily deduced from

$$x_0 + t_n w_n \in M \Leftrightarrow x_0 + \frac{1}{2}\widehat{t}_n r_n w_n \in M, \tag{2.90}$$

which holds by choosing $t_n := \frac{1}{2}\widehat{t}_n r_n$ for the inclusion $T''(M,x_0,0) \subset T(M,x_0)$ and, for instance, $\widehat{t}_n = 2t_n^{\frac{2}{3}}, r_n = t_n^{\frac{1}{3}}$ for the converse.

3. To prove that $T''(M,x_0,u)$ is a cone, taking $w \in T''(M,x_0,u)$, we will prove that $tw \in T''(M,x_0,u)$ for all $t > 0$. Fix $t > 0$ arbitrary, by (2.3), there exist two sequences of positive reals $\{t_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to w$ and $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. By setting $\widehat{r}_n := \frac{r_n}{t}$ and $\widehat{w}_n := tw_n$, we have $\widehat{w}_n \to tw$ and

$$x_0 + t_n u + \frac{1}{2}t_n \widehat{r}_n \widehat{w}_n \in M, \quad \forall n \in \mathbb{N}, \tag{2.91}$$

i.e., $tw \in T''(M,x_0,u)$. Since $t > 0$ and $w \in T''(M,x_0,u)$ are chosen arbitrarily, we conclude that $T''(M,x_0,u)$ is a cone.

To prove that $T^2(M,x_0,u)$ is not a cone in general, we go back to the setting of Problem 4. We have proved that

$$T^2(M,x_0,u) = \left\{(x,y) \in \mathbb{R}^2 | x \le -2\right\}. \tag{2.92}$$

Taking $w := (x,y) \in T^2(M,x_0,u)$, we have $x \le -2$. But this does not implies that $tw \in T^2(M,x_0,u)$ for all $t > 0$. Indeed, choosing $t := -\frac{1}{x} > 0$ yields $tx = -1 > -2$, i.e., $tw \notin T^2(M,x_0,u)$. It follows that $T^2(M,x_0,u)$ is not a cone in general.

4. Since X is finite-dimensional, we can assume that $X = \mathbb{R}^n$ without loss of generality. In [4], Remark 7, p.88, the authors have proved the following stronger results, from which (2.84) follows directly.

Theorem 2.4. *Let $M \subset \mathbb{R}^n$, $x_0 \in \overline{M}$ and $u \in \mathbb{R}^n$.*

1. $0 \in T''(M,x_0,u) \Leftrightarrow u \in T(M,x_0)$.
2. *If $T^2(M,x_0,u) = \emptyset$ and $u \in T(M,x_0)$, then there exists $w \in T''(M,x_0,u)$, $w \neq 0$, such that $w^T u = 0$.*

Proof of Theorem 2.4.

1. *Prove* $0 \in T''(M,x_0,u) \Rightarrow u \in T(M,x_0)$. Let $0 \in T''(M,x_0,u)$, then there exist two sequences of positive reals $\{t_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+, r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to 0$ and $x_0 + t_n u + \frac{1}{2}t_n r_n w_n \in M$ for all $n \in \mathbb{N}$. By setting $u_n := u + \frac{1}{2}r_n w_n$ for all $n \in \mathbb{N}$, we have $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$, i.e., $u \in T(M,x_0)$.

   Prove $u \in T(M,x_0) \Rightarrow 0 \in T''(M,x_0,u)$. If $T(M,x_0) = \{0\}$, then $u = 0$ and by the result obtained in Problem 7.2, $T''(M,x_0,u) = T(M,x_0) = \{0\}$ and the conclusion is true.

   If $T(M,x_0) \neq \{0\}$, choose $u \in T(M,x_0) \setminus \{0\}$. We have two cases depending on $T^2(M,x_0,u)$.

(a) *Case* $T^2(M, x_0, u) \neq \emptyset$. Pick $w \in T^2(M, x_0, u)$, then there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ as $n \to \infty$ and a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $w_n \to w$ as $n \to \infty$ and $x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M$ for all $n \in \mathbb{N}$. Setting $r_n := \sqrt{t_n}$, $\widehat{w}_n := \sqrt{t_n} w_n$ for all $n \in \mathbb{N}$, we have $r_n \to 0^+$, $\frac{t_n}{r_n} = \sqrt{t_n} \to 0^+$, $\widehat{w}_n \to 0$ as $n \to \infty$ and

$$x_0 + t_n u + \frac{1}{2} t_n r_n \widehat{w}_n = x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M, \quad \forall n \in \mathbb{N}. \tag{2.93}$$

i.e., $0 \in T''(M, x_0, u)$.

(b) *Case* $T^2(M, x_0, u) = \emptyset$. As $u \in T(M, x_0) \setminus \{0\}$, there exist a sequence of positive real $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \in X$ such that $u_n \to u$ as $n \to \infty$ and $x_n := x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. By Lemma 3.4, [5], p.129, there exists a subsequence, denoted again t_n's (and also x_n's), such that either

- $\dfrac{x_n - x_0 - t_n u}{\frac{1}{2} t_n^2} \to w$ for some $w \in T^2(M, x_0, u) \cap u^{\perp}$ or
- there exists a sequence $r_n \to 0^+$ such that $\frac{t_n}{r_n} \to 0$ and

$$\frac{x_n - x_0 - t_n u}{\frac{1}{2} t_n r_n} \to w, \tag{2.94}$$

  for some $w \in T''(M, x_0, u) \cap u^{\perp} \setminus \{0\}$.

As $T^2(M, x_0, u) = \emptyset$, the second condition is satisfied. Hence, as $T''(M, x_0, v) \neq \emptyset$, being this set a closed cone, we conclude that $0 \in T''\big(M, x_), u\big)$.

2. At the same time we have proved this part of Theorem 2.4. Indeed, the assumptions of part (1) imply that $u \neq 0$, as if $u = 0$, then $T^2(M, x_0, 0) = T(M, x_0) \neq \emptyset$, in contradiction with the assumption.

We end the proof of Theorem 2.4 here. □

3 Theory of Optimality Conditions

In this section, we will discuss about the theory of optimality conditions for the following problems

$$(P): \qquad \min f(x) \text{ s.t. } x \in \Omega. \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^n$.

Definition 3.1. Consider the problem $(P), m \in \mathbb{N}^*$,

1. $x_0 \in \Omega$ is called *local minimizer* of (P) if there exists a neighborhood U of x_0 such that

$$f(x) \geq f(x_0), \quad \forall x \in U \cap \Omega. \tag{3.2}$$

2. $x_0 \in \Omega$ is called *strictly local minimizer of order* m of (P) if there exist a neighborhood U of x_0 and a positive real number α such that

$$f(x) \geq f(x_0) + \alpha \|x - x_0\|^m, \quad \forall x \in U \cap \Omega. \tag{3.3}$$

Theorem 3.2 (First-order necessary optimality condition).
If x_0 is a local minimizer of (P) then

$$\langle \nabla f(x_0), u \rangle \geq 0, \quad \forall u \in T(\Omega, x_0). \tag{3.4}$$

Theorem 3.3 (First-order sufficient optimality condition).

1. *If f is a convex function, Ω is a convex set and*

$$\langle \nabla f(x_0), u \rangle \geq 0, \quad \forall u \in T(\Omega, x_0), \tag{3.5}$$

then x_0 is a minimizer of (P).

2. *If for all $u \in T(\Omega, x_0)$ satisfying $\|u\| = 1$, $\langle \nabla f(x_0), u \rangle > 0$ holds then x_0 is a strictly local minimizer of first order of (P).*

We consider some exercises which apply the first order optimality conditions.

Problem 8 (Fermat's rule). *Use Theorem 3.2, prove that if x_0 is a local minimizer of* (P) and $x_0 \in \operatorname{int} \Omega$ *then* $\nabla f(x_0) = 0$, *then apply this result to find solutions of the following problems.*

1. (P) : $\quad \min x^2 + 3y^2 - 2xy - 4x - 8y$ s.t. $(x, y) \in \mathbb{R}^2$.
2. (P) : $\quad \min xyze^{-x-y-z}$ s.t. $(x, y, z) \in \mathbb{R}^3$.

Solution. Since x_0 is a local minimizer of (P), by Definition 3.1 and Theorem 3.2, there exists a neighborhood U of x_0 such that

$$f(x) \geq f(x_0), \quad \forall x \in U \cap \Omega, \tag{3.6}$$

and

$$\langle \nabla f(x_0), u \rangle \geq 0, \quad \forall u \in T(\Omega, x_0). \tag{3.7}$$

Since $x_0 \in \operatorname{int} \Omega$, there exists a positive real $r > 0$ such that $B_r(x_0) \subset \Omega$.

We claim that if $x_0 \in \operatorname{int} \Omega$ then $T(\Omega, x_0) = \mathbb{R}^n$. To prove this claim, for arbitrary $u \in \mathbb{R}^n$, we will prove that $u \in T(\Omega, x_0)$. Indeed, we can take an arbitrary sequence $\{u_n\}_{n=1}^{\infty} \in \mathbb{R}^n$ such that $u_n \to u$ as $n \to \infty$. We now choose a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ whose each term t_n is small enough such that

$$x_0 + t_n u_n \in B_r(x_0). \tag{3.8}$$

For instance, we can take t_n's satisfying

$$0 < t_n < \frac{r}{\|u_n\|}, \quad \forall n \in \mathbb{N}, \tag{3.9}$$

in order that (3.8) holds for all $n \in \mathbb{N}$. Thus, $u \in T(\Omega, x_0)$. And this yields $T(\Omega, x_0) = \mathbb{R}^n$ as we claimed.

Use this result, we now can proceed as the proof of Theorem 2.7, [1], p.35 as follows. If $d \in \mathbb{R}^n$, then

$$f'(x_0; d) = \lim_{t \to 0} \frac{f(x_0 + td) - f(x_0)}{t} = \langle \nabla f(x_0), d \rangle . \tag{3.10}$$

By theorem 3.2, since $T(M, x_0) = \mathbb{R}^n$, we have

$$\langle \nabla f(x_0), d \rangle \geq 0, \ \ \forall d \in T(M, x_0) = \mathbb{R}^n. \tag{3.11}$$

In particular, picking $d = -\nabla f(x_0)$ gives $-\|\nabla f(x_0)\|^2 \geq 0$, that is, $\nabla f(x_0) = 0$.

We now apply this result to find solutions of the above problems.

1. Setting $\Omega = \mathbb{R}^n$, and

$$f(x, y) = x^2 + 3y^2 - 2xy - 4x - 8y, \ \ \forall (x, y) \in \mathbb{R}^2. \tag{3.12}$$

By Fermat's rule, we consider the equation $\nabla f(x_0, y_0) = 0$, i.e.,

$$\nabla f(x_0, y_0) = (2x_0 - 2y_0 - 4, 6y_0 - 2x_0 - 8) = 0. \tag{3.13}$$

This gives a linear system of equations

$$x_0 - y_0 = 2, \tag{3.14}$$
$$x_0 - 3y_0 = -4. \tag{3.15}$$

Solving this yields $x_0 = 5, y_0 = 3$.

Check. The point $(x_0, y_0) = (5, 2)$ is a global minimizer of $f(x, y)$. Indeed, since $f(x_0, y_0) = f(5, 3) = -22$, it suffices to prove

$$f(x, y) + 22 \geq 0, \ \ \forall (x, y) \in \mathbb{R}^2. \tag{3.16}$$

Fixed $y \in \mathbb{R}$, we set

$$F_y(x) := f(x, y) + 22, \ \ \forall x \in \mathbb{R}. \tag{3.17}$$

The first derivative of $F_y(x)$ is

$$\frac{d}{dx} F_y(x) = 2(x - y - 2), \ \ \forall x \in \mathbb{R}. \tag{3.18}$$

By surveying the sign of $\frac{d}{dx} F_y(x)$, it follows from (3.18) that

$$\min_{x \in \mathbb{R}} F_y(x) = F_y(y + 2) = 2(y - 3)^2 \geq 0. \tag{3.19}$$

Hence,

$$\min(f(x, y) + 22) = \min_{y \in \mathbb{R}} \min_{x \in \mathbb{R}} F_y(x) \tag{3.20}$$
$$= \min_{y \in \mathbb{R}} 2(y - 3)^2 \tag{3.21}$$
$$= 0, \tag{3.22}$$

which is attained at $y = 3$ (and thus) $x = y + 2 = 5$.

2. Setting $\Omega = \mathbb{R}^3$, and

$$f(x,y,z) = xyze^{-x-y-z}, \quad \forall (x,y,z) \in \mathbb{R}^3. \tag{3.23}$$

We have

$$\nabla f(x,y,z) = e^{-x-y-z}\left(yz(1-x), zx(1-y), xy(1-z)\right), \tag{3.24}$$

for all $(x,y,z) \in \mathbb{R}^3$. The equation $\nabla f(x_0,y_0,z_0) = 0$ gives the following system of equations

$$y_0 z_0 (1-x_0) = 0, \tag{3.25}$$
$$z_0 x_0 (1-y_0) = 0, \tag{3.26}$$
$$x_0 y_0 (1-z_0) = 0. \tag{3.27}$$

The roots of this system are $(1,1,1)$, $(a,0,0)$ for all $a \in \mathbb{R}$ and their permutations.

If exactly two in three numbers x_0, y_0, z_0 is equal to 0 (for instance, $(x_0,y_0,z_0) = (a,0,0)$ for some nonzero $a \in \mathbb{R}$) then we can choose in a neighborhood of (x_0,y_0,z_0) a point (x,y,z) for which $xyz < 0$ [6] (for instance, choose $(x,y,z) = \left(a, \frac{\operatorname{sign}(a)}{n}, -\frac{1}{n}\right)$ with n small enough), and thus $f(x,y,z) < 0$. But $f(x_0,y_0,z_0) = 0$, we deduce that (x_0,y_0,z_0) is not a local minimizer of f. We only need to consider the remaining cases, i.e., $(x_0,y_0,z_0) = (0,0,0)$ and $(x_0,y_0,z_0) = (1,1,1)$.

For $(x_0,y_0,z_0) = (0,0,0)$, we choose $x_n = y_n = \frac{1}{n}, z_n = -\frac{2}{n}$. This choice ensures that $x_n + y_n + z_n = 0$ and (x_n,y_n,z_n) lies in any given neighborhood of $(0,0,0)$ provided n is large enough. We then have

$$f(x_n,y_n,z_n) = -\frac{2}{n^3} < 0 = f(0,0,0). \tag{3.28}$$

This implies that $(0,0,0)$ is not a local minimizer.

For $(x_0,y_0,z_0) = (1,1,1)$, we choose $x_n = 1 + \frac{2}{n}, y_n = z_n = 1 - \frac{1}{n}$. This choice ensures that $x_n + y_n + z_n = 3$ and (x_n,y_n,z_n) lies in any given neighborhood of $(1,1,1)$ if n is large enough. By Cauchy inequality, we have

$$x_n y_n z_n = \left(1+\frac{2}{n}\right)\left(1-\frac{1}{n}\right)^2 \le \left(\frac{1+\frac{2}{n}+1-\frac{1}{n}+1-\frac{1}{n}}{3}\right)^3 = 1. \tag{3.29}$$

Since $x_n \neq y_n$, the equality does not hold and thus this gives us $x_n y_n z_n < 1$. We then have

$$f(x_n,y_n,z_n) = \frac{x_n y_n z_n}{e^3} < \frac{1}{e^3} = f(1,1,1). \tag{3.30}$$

This implies that $(1,1,1)$ is not a local minimizer. In face, we can use Hessian matrix of f to prove that $(1,1,1)$ is a local maximizer. Finally, we conclude that there does not exist any local minimizers of (P).

[6] This is easily handled by considering the signs of the three coordinates.

Furthermore, the fact that f has no global minimizer can be easily demonstrated by choosing $x = -n, y = n-1, y = 2$ for $n \in \mathbb{N}$,

$$f(-n, n-1, 2) = -2n(n-1)e \to -\infty \text{ as } n \to +\infty. \tag{3.31}$$

This completes our solution. □

Problem 9. *Consider the following problem*

$$(P): \quad \min x^2 - y \text{ s.t. } (x,y) \in \Omega = \left\{(x,y) \in \mathbb{R}^2 | x + y^3 \geq 0\right\}. \tag{3.32}$$

1. *Compute the tangent cone of Ω at the point $x_0 = (0,0)$.*
2. *Apply the first-order necessary optimality condition, check whether $x_0 = (0,0)$ is a local minimizer of (P) or not.*

Solution. Setting $X = \mathbb{R}^2$, $f(x,y) = x^2 - y$ for $(x,y) \in \mathbb{R}^2$, we notice that $x_0 = (0,0) \in \Omega$.

1. We claim that

$$T(\Omega, x_0) = \widehat{T}(\Omega, x_0) := \left\{(x,y) \in \mathbb{R}^2 | x \geq 0\right\}. \tag{3.33}$$

To prove (3.33), we prove the following inclusions.

(a) *Prove $T(\Omega, x_0) \subset \widehat{T}(\Omega, x_0)$.* Taking $u = (x,y) \in T(M, x_0)$, there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in \Omega$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the face $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in \Omega$ for all $n \in \mathbb{N}$ gives

$$t_n x_n + t_n^3 y_n^3 \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.34}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (3.34) then implies

$$x_n + t_n^2 y_n^3 \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.35}$$

Now let $n \to \infty$ and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $x \geq 0$. Hence, $u \in \widehat{T}(\Omega, x_0)$ and our first inclusion is proved.

(b) *Prove $\widehat{T}(\Omega, x_0) \subset T(\Omega, x_0)$.* Taking $u = (x,y) \in \mathbb{R}^2$ satisfying $x \geq 0$, we claim that $u \in T(\Omega, x_0)$. To this end, we now choose $x_n = x + \frac{1}{n}, y_n = y$ for all $n \in \mathbb{N}$. This choice ensures that $u_n := (x_n, y_n) \to u := (x,y)$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $x_0 + t_n u_n \in \Omega$ for all $n \in \mathbb{N}$. The latter gives, using (3.35) again,

$$x + \frac{1}{n} + t_n^2 y^3 \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.36}$$

We consider the following cases depending on the sign of y. If $y \geq 0$, then (3.36) holds for all positive reals t_n's. Thus we can take an arbitrary sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$, this means $u \in T(\Omega, x_0)$ in this case. If $y < 0$, (3.36) gives

$$t_n \leq \sqrt{-\frac{1}{y^3}\left(x + \frac{1}{n}\right)}, \quad \forall n \in \mathbb{N}. \tag{3.37}$$

The term in the right-hand side of (3.37) is positive for all $n \in \mathbb{N}$. Hence we can choose t_n's satisfying (3.37) and $t_n \to 0^+$ as $n \to \infty$. This choice implies that $u \in T(\Omega, x_0)$, i.e., the second inclusion is also proved.

Combining these, we conclude that (3.33) holds, i.e.,

$$T(\Omega, x_0) = \left\{(x, y) \in \mathbb{R}^2 | x \geq 0\right\}. \tag{3.38}$$

2. We have $\nabla f(x, y) = (2x, -1)$ for all $(x, y) \in \mathbb{R}^2$. In particular, $\nabla f(x_0) = \nabla f(0, 0) = (0, -1)$. Consider $u := (0, 1) \in T(\Omega, x_0)$, we have

$$\langle \nabla f(x_0), u\rangle = \langle (0, -1), (0, 1)\rangle = -1 < 0. \tag{3.39}$$

By the first-order necessary optimality condition, this implies that x_0 is not a local minimizer of f.

This completes our solution. □

Problem 10. *Consider the following problem*

$$(P): \quad \min x + y^2 \text{ s.t. } (x, y) \in \Omega = \left\{(x, y) \in \mathbb{R}^2 | x - \sqrt{|y|} = 0\right\}. \tag{3.40}$$

1. *Compute the tangent cone of Ω at the point $x_0 = (0, 0)$.*
2. *Apply the first-order sufficient optimality condition, check whether x_0 is a strictly local minimizer of first order of (P) or not.*
3. *Use definition, prove that x_0 is a strictly local minimizer of first order of (P).*

Solution. Setting $X = \mathbb{R}^2$, $f(x, y) = x + y^2$ for all $(x, y) \in \mathbb{R}^2$, we notice that $x_0 = (0, 0) \in \Omega$.

1. We claim that

$$T(\Omega, x_0) = \widehat{T}(\Omega, x_0) := \left\{(x, y) \in \mathbb{R}^2 | x \geq 0, y = 0\right\}. \tag{3.41}$$

To prove (3.41), we prove the following inclusions.

(a) *Prove $T(\Omega, x_0) \subset \widehat{T}(\Omega, x_0)$.* Taking $u := (x, y) \in T(\Omega, x_0)$, there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the fact that $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives

$$t_n x_n = \sqrt{|t_n y_n|}, \quad \forall n \in \mathbb{N}. \tag{3.42}$$

We see at a glance from (3.42) that $x_n \geq 0$ for all $n \in \mathbb{N}$. Hence, $x \geq 0$ (since $x_n \to x$ as $n \to \infty$). Now squaring both sides of (3.42) yields

$$t_n^2 x_n^2 = t_n \left|y_n\right|, \quad \forall n \in \mathbb{N}. \tag{3.43}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (3.43) then implies

$$t_n x_n^2 = \left|y_n\right|, \quad \forall n \in \mathbb{N}. \tag{3.44}$$

Now let $n \to \infty$ in (3.44) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $y = 0$. Hence, $u \in \widehat{T}\left(\Omega, x_0\right)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}\left(\Omega, x_0\right) \subset T\left(\Omega, x_0\right)$. Taking $u := (x, 0)$ for which $x \geq 0$, we claim that $u \in T\left(\Omega, x_0\right)$. To this end, we now choose $x_n := x + \frac{1}{n}, y_n := \frac{1}{n^3}$ for all $n \in \mathbb{N}$. This choice ensures that $u_n := (x_n, y_n) \to u := (x, 0)$ as $n \to \infty$. It then suffices to prove that there exists a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and $x_0 + t_n u_n \in \Omega$ for all $n \in \mathbb{N}$. The latter gives, using (3.44) again,

$$t_n \left(x + \frac{1}{n}\right)^2 = \frac{1}{n^3}, \quad \forall n \in \mathbb{N}. \tag{3.45}$$

i.e.,

$$t_n = \frac{1}{n^3 \left(x + \frac{1}{n}\right)^2}, \quad \forall n \in \mathbb{N}. \tag{3.46}$$

It is easy to check that $t_n > 0$ (since $x \geq 0$) and $t_n \to 0^+$ as $n \to \infty$. Hence, $u \in T\left(\Omega, x_0\right)$ and the second inclusion is also proved.

Combining these inclusions, we conclude that (3.41) holds, i.e.,

$$T\left(\Omega, x_0\right) = \left\{(x, y) \in \mathbb{R}^2 | x \geq 0, y = 0\right\}. \tag{3.47}$$

2. Taking $u \in T\left(\Omega, x_0\right)$ satisfying $\|u\| = 1$, i.e., $u := (x, 0), x \geq 0$ for which $|x| = 1$. The only point satisfying these assumptions is $u_0 := (1, 0)$.

   We have $\nabla f(x, y) = (1, 2y)$ for all $(x, y) \in \mathbb{R}^2$. In particular, $\nabla f\left(x_0\right) = \nabla f(0, 0) = (1, 0)$. Thus,

$$\left\langle \nabla f\left(x_0\right), u_0 \right\rangle = \langle (1, 0), (1, 0) \rangle = 1 > 0. \tag{3.48}$$

   By the first-order sufficient optimality condition, (3.48) implies that x_0 is a strictly local minimizer of first order of (P).

3. By definition 3.1.2, to show that $x_0 \in \Omega$ is a strictly local minimizer of first order of (P), it suffices to prove that there exists a neighborhood U of x_0 and a positive real number α such that

$$x + y^2 \geq \alpha \sqrt{x^2 + y^2}, \quad \forall (x, y) \in U \cap \Omega. \tag{3.49}$$

We now choose

$$\alpha = \frac{1}{\sqrt{2}}, \tag{3.50}$$

$$U := B\left(x_0; 1\right) = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \le 1\right\}, \tag{3.51}$$

and then prove that (3.49) holds in this setting. Indeed, since $(x,y) \in \Omega$, $x \ge 0$ and $x^2 = |y|$. Substituting $y^2 = x^4$ into (3.49), we need to prove

$$x + x^4 \ge \frac{1}{\sqrt{2}} \sqrt{x^2 + x^4}, \quad \forall x \in [0,1]. \tag{3.52}$$

By squaring both sides,(3.52) is equivalent to

$$2\left(1 + x^3\right)^2 \ge 1 + x^2, \quad \forall x \in [0,1] \tag{3.53}$$

i.e.,

$$1 + 4x^3 + 2x^6 \ge x^2, \quad \forall x \in [0,1]. \tag{3.54}$$

The last inequality (3.54) is obviously true, since

$$\text{RHS} = x^2 \le 1 \le 1 + \underbrace{4x^3 + 2x^6}_{\ge 0} = \text{LHS}. \tag{3.55}$$

Thus, (3.49) holds for our setting.

This completes our proof. □

When we consider second-order optimality conditions (i.e., "proposed solutions" satisfy first-order optimality conditions), we continue to consider in the "critical direction" u satisfying

$$\langle \nabla f\left(x_0\right), u \rangle = 0. \tag{3.56}$$

Theorem 3.4 (Second-order necessary optimality condition).
If x_0 is a local minimizer of (P) and u satisfies $\langle \nabla f\left(x_0\right), u \rangle = 0$, then we have

1. $\langle \nabla f\left(x_0\right), w \rangle + \nabla^2 f\left(x_0\right)(u,u) > 0$ *for all* $w \in T^2\left(\Omega, x_0, u\right)$.
2. $\langle \nabla f\left(x_0\right), w \rangle \ge 0$ *for all* $w \in T''\left(\Omega, x_0, u\right)$.

Theorem 3.5 (Second-order sufficient optimality condition).
If for all $u \in T\left(\Omega, x_0\right)$ for which $\|u\| = 1$ we have

1. $\langle \nabla f\left(x_0\right), w \rangle + \nabla^2 f\left(x_0\right)(u,u) > 0$ *for all* $w \in T^2\left(\Omega, x_0, u\right)$.
2. $\langle \nabla f\left(x_0\right), w \rangle > 0$ *for all* $w \in T''\left(\Omega, x_0, u\right)$ *for which* $\|w\| = 1$.

then x_0 is a strictly local minimizer of second order of (P).

Here are some problems to apply second-order optimality conditions.

Problem 11. *Consider the following problem*

$$(P) \qquad \min x^3 - y^2 \text{ s.t. } (x,y) \in \Omega = \left\{(x,y) \in \mathbb{R}^2 | xy \ge 0\right\}. \tag{3.57}$$

1. *Compute the tangent cone of* Ω *at* $x_0 = (0, 0)$.
2. *Apply the first-order necessary optimality condition, check if* $x_0 = (0, 0)$ *is a local minimizer of* (P) *or not.*
3. *Apply the second-order necessary optimality condition, check if* $x_0 = (0, 0)$ *is a local minimizer of* (P) *or not.*

SOLUTION. Setting $Xx = \mathbb{R}^2$, $f(x, y) = x^3 - y^2$ for all $(x, y) \in \mathbb{R}^2$, we notice that $x_0 = (0, 0) \in \Omega$.

1. We claim that

$$T(\Omega, x_0) = \Omega = \left\{(x, y) \in \mathbb{R}^2 | xy \geq 0\right\} \tag{3.58}$$

To prove (3.58), we prove the following inclusions.

(a) *Prove* $T(\Omega, x_0) \subset \Omega$. Taking $u := (x, y) \in T(\Omega, x_0)$, there exist a sequence of positive reals $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the fact that $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives

$$t_n^2 x_n y_n \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.59}$$

Since $t_n \geq 0$ for all $n \in \mathbb{N}$, (3.59) then implies

$$x_n y_n \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.60}$$

Now let $n \to \infty$ in (3.60) and use the given limits $x_n \to x, y_n \to y$ and $t_n \to 0^+$, we obtain $xy \geq 0$. Hence, $u \in \Omega$ and our first inclusion is proved.

(b) *Prove* $\Omega \subset T(\Omega, x_0)$. Taking $u := (x, y) \in \Omega$, i.e., $xy \geq 0$, we claim that $u \in T(\Omega, x_0)$. To this end, we consider the following cases depending on the common sign of x and y.

- *Case* $x \geq 0, y \geq 0$. We can choose $x_n := x + \frac{1}{n}, y_n := y + \frac{1}{n}$ for all $n \in \mathbb{N}$ and an arbitrary sequence of positive reals t_n's such that $t_n \to 0^+$ as $n \to \infty$. These choices will ensure that (3.59) holds. Thus, $u \in T(\Omega, x_0)$ in this case.
- *Case* $x \leq 0, y \leq 0$. Similarly, we can choose that $x_n := x - \frac{1}{n}, y_n := y - \frac{1}{n}$ for all $n \in \mathbb{N}$ and an arbitrary sequence of positive reals t_n's such that $t_n \to 0^+$ as $n \to \infty$. Hence, we also deduce that $u \in T(\Omega, x_0)$ in this case.

What we have just proved is the second inclusion.

Combining these inclusions, we conclude that (3.58) holds.

2. We have $\nabla f(x, y) = \left(3x^2, -2y\right)$ for all $(x, y) \in \mathbb{R}^2$. In particular, $\nabla f(x_0) = \nabla f(0, 0) = (0, 0)$. Thus,

$$\langle \nabla f(x_0), u \rangle = 0, \quad \forall u \in \Omega \equiv T(\Omega, x_0), \tag{3.61}$$

which satisfies the conclusion of Theorem 3.2. However, we can not deduce from (3.61) that x_0 is a local minimizer of (P).

3. We claim that x_0 is not a local minimizer of (P). Suppose for the contrary that x_0 is a local minimizer of (P), we have $\langle \nabla f(x_0), u \rangle = 0$ for all $u \in \Omega$ by above argument. Compute

$$\nabla^2 f(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}, \forall (x,y) \in \mathbb{R}^2. \tag{3.62}$$

In particular,

$$\nabla^2 f(x_0) = \nabla^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}. \tag{3.63}$$

Denote $u := (x,y) \in \Omega$, we have

$$\langle \nabla f(x_0), w \rangle + \nabla^2 f(x_0)(u,u) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.64}$$

$$= -2y^2 \le 0, \tag{3.65}$$

for all $w \in T^2(\Omega, x_0, u)$, which contradicts to the second-order necessary optimality condition. Therefore, x_0 is not a local minimizer of (P).

This completes our proof. □

Problem 12. *Consider the following problem*

$$(P) \qquad \min x^3 + y^2 \text{ s.t. } (x,y) \in \Omega = \left\{ (x,y) \in \mathbb{R}^2 | x - y^2 \ge 0 \right\}. \tag{3.66}$$

1. *Compute the tangent cone of Ω at $x_0 = (0,0)$.*
2. *Prove that x_0 satisfies the first-order necessary optimality condition.*
3. *Use definition, prove that x_0 is not a strictly local minimizer of first order of (P).*
4. *Use definition, check if x_0 is a strictly local minimizer of second order of (P).*
5. *Apply the second-order necessary optimality condition, check if $x_0 = (0,0)$ is a strictly local minimizer of second order of (P) or not.*

Solution. Setting $X = \mathbb{R}^2$, $f(x,y) = x^3 + y^2$ for all $(x,y) \in \mathbb{R}^2$. We notice that $x_0 = (0,0) \in \Omega$.

1. We claim that

$$T(\Omega, x_0) = \widehat{T}(\Omega, x_0) := \left\{ (x,y) \in \mathbb{R}^2 | x \ge 0 \right\}. \tag{3.67}$$

To prove (3.67), we prove the following inclusions.

(a) *Prove $T(\Omega, x_0) \subset \widehat{T}(\Omega, x_0)$.* Taking $u := (x,y) \in T(\Omega, x_0)$, there exist a sequence of positive reals $\{t_n\}_{n=1}^\infty$ such that $t_n \to 0^+$ and a sequence $\{u_n\}_{n=1}^\infty \subset \mathbb{R}^2$ such that $u_n \to u$ as $n \to \infty$ and $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$. Set $u_n := (x_n, y_n)$, the fact that $u_n \to u$ implies that $x_n \to x$ and $y_n \to y$, and the fact $x_0 + t_n u_n \in M$ for all $n \in \mathbb{N}$ gives

$$t_n x_n \geq t_n^2 y_n^2, \quad \forall n \in \mathbb{N}. \tag{3.68}$$

Since $t_n > 0$ for all $n \in \mathbb{N}$, (3.68) implies that

$$x_n \geq t_n y_n^2, \quad \forall n \in \mathbb{N}. \tag{3.69}$$

Let $n \to \infty$ in (3.69), we obtain $x \geq 0$. Hence, $u \in \widehat{T}(\Omega, x_0)$ and our first inclusion is proved.

(b) *Prove* $\widehat{T}(\Omega, x_0) \subset T(\Omega, x_0)$. Taking $u := (x, y)$ for which $x \geq 0$, we claim that $u \in T(\Omega, x_0)$. To this end, we choose $x_n := x + \frac{1}{n}, y_n := y$ for all $n \in \mathbb{N}$. If $y = 0$ then (3.69) obviously holds, $x + \frac{1}{n} \geq 0$ for all $n \in \mathbb{N}$, and we can choose an arbitrary sequence of positive reals t_n's for which $t_n \to 0^+$. If $y \neq 0$, (3.69) gives

$$t_n \leq \frac{1}{y^2}\left(x + \frac{1}{n}\right), \quad \forall n \in \mathbb{N}. \tag{3.70}$$

The right-hand side of (3.70) is positive. Thus, we can choose an arbitrary sequence of positive reals t_n's satisfying (3.70) and $t_n \to 0^+$ as $n \to \infty$. In both cases, we deduce that $u \in T(\Omega, x_0)$ and our inclusion is also proved.

Combining these inclusions, we conclude that (3.67) holds, i.e.,

$$T(\Omega, x_0) = \left\{(x, y) \in \mathbb{R}^2 | x \geq 0\right\}. \tag{3.71}$$

2. We have $\nabla f(x, y) = \left(3x^2, 2y\right)$ for all $(x, y) \in \mathbb{R}^2$. In particular, $\nabla f(x_0) = \nabla f(0, 0) = (0, 0)$. We then have

$$\langle \nabla f(x_0), u \rangle = \langle (0, 0), (x, y) \rangle = 0, \quad \forall u \in T(\Omega, x_0), \tag{3.72}$$

i.e., x_0 satisfies the first-order necessary optimality condition.

3. Solution 1. Suppose for the contrary that x_0 is a strictly local minimizer of first order of (P), by definition 3.1, there exist a neighborhood U of x_0 and a positive real number α such that

$$x^3 + y^2 \geq \alpha\sqrt{x^2 + y^2}, \quad \forall (x, y) \in U \cap \Omega. \tag{3.73}$$

For $x > 0$ small enough, $(x, \sqrt{x}) \in U \cap \Omega$ holds. Then (3.73) gives

$$\alpha \leq \frac{x^3 + x}{\sqrt{x^2 + x}} = \sqrt{x} \cdot \frac{x^2 + 1}{\sqrt{x + 1}} \to 0 \text{ as } x \to 0, \tag{3.74}$$

which contradicts the positivity of α. This contradiction implies that x_0 is not a strictly local minimizer of first order of (P).

Solution 2. Similarly, for $x > 0$ small enough, $(x, 0) \in U \cap \Omega$ holds. Then (3.73) gives

$$\alpha \leq x^2 \to 0 \text{ as } x \to 0, \tag{3.75}$$

which also contradicts the positivity of α. Therefore, x_0 is not a strictly local minimizer of first order of (P).

4. Similarly, suppose for the contrary that x_0 is a strictly local minimizer of second order of (P), by definition 3.1, there exists a neighborhood U of x_0 and a positive real number α such that

$$x^3 + y^2 \geq \alpha \left(x^2 + y^2\right), \quad \forall (x, y) \in U \cap \Omega. \tag{3.76}$$

For $x > 0$ small enough, $(x, 0) \in U \cap \Omega$ holds. Then (3.76) gives

$$\alpha \leq x \to 0 \text{ as } x \to 0, \tag{3.77}$$

which contradicts the positivity of α. Therefore, x_0 is not a strictly local minimizer of second order of (P).

5. We claim that x_0 is not a strictly local minimizer of second order of (P). To show this, we suppose for the contrary that x_0 is a strictly local minimizer of second order of (P), and thus, is a local minimizer of (P). We has showed that $\langle \nabla f(x_0), u \rangle = 0$ for all $u \in \Omega$ in (2). Thus,

$$\langle \nabla f(x_0), w \rangle + \nabla^2 f(x_0)(u, u) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.78}$$

$$= 2y^2, \tag{3.79}$$

for all $u := (x, y) \in \Omega$ and for all $w \in T^2(\Omega, x_0, u)$. Hence, for $u = (x, 0), x \geq 0$, (3.78)-(3.79) gives

$$\langle \nabla f(x_0), w \rangle + \nabla^2 f(x_0)(u, u) = 0, \tag{3.80}$$

which contradicts the second-order necessary condition. This contradiction illustrates that x_0 is not a strictly local minimizer of second order of (P).

This ends our proof. □

THE END

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