Notes ⋆ Sobolev Spaces and the Variational Formulation of Boundary Value Problems in One Dimension

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Abstract

I have added some personal proofs, explanations and explicit computations for some proofs, remarks in the Chapter 8, [1].

Contents

1 Motivation

Consider the following problem.

Problem 1.1. *Given $f \in C([a,b])$, find a function u satisfying*

$$\begin{cases} -u'' + u = f \text{ on } [a,b] \\ u(a) = u(b) = 0 \end{cases} \tag{1.1}$$

A *classical*- or *strong*- solution of (1.1) is a C^2 function on $[a,b]$ satisfying (1.1) in the usual sense. It is well known that (1.1) can be solved explicitly by a very simple calculation, but we ignore this feature so as to illustrate the method on this elementary example.

Multiply (1.1) by $\varphi \in C^1([a,b])$ and integrate by parts; we obtain

$$\int_a^b u'\varphi' + \int_a^b u\varphi = \int_a^b f\varphi, \ \ \forall \varphi \in C^1([a,b]), \varphi(a) = \varphi(b) = 0 \tag{1.2}$$

Note that (1.2) makes sense as soon as $u \in C^1([a,b])$ (where (1.1) requires two derivatives on u); in fact, in suffices to know that $u, u \in L^1(a,b)$,[1] whereu' has a meaning yet to be made precise. Let us say (provisionally) that a C^1 function u that satisfies (1.2) is a *weak* solution of (1.1).

The following program outlines the main steps of the *variational approach* in the theory of partial differential equations.

Step A. The notion of *weak solution* is made precise. This involves *Sobolev spaces*, which are our *basic tools.*

Step B. *Existence and uniqueness of a weak solution* is established by a variational method via the Lax-Milgram theorem.

Step C. The weak solution is proved to be of class C^2 (for example): this is a *regularity* result.

Step D. A *classical* solution is recovered by showing that any weak solution that is C^2 is a classical solution.

To carry out Step D is very simple. In fact, suppose that $u \in C^2([a,b]), u(a) = u(b) = 0$, and that u satisfies (1.2). Integrating (1.2) by parts we obtain

$$\int_a^b (-u'' + u - f)\varphi = 0, \ \ \forall \varphi \in C^1([a,b]), \varphi(a) = \varphi(b) = 0 \tag{1.3}$$

and therefore

$$\int_a^b (-u'' + u - f)\varphi = 0, \ \ \forall \varphi \in C_c^1((a,b)) \tag{1.4}$$

It follows (see Corollary 4.15, [1]) that $-u'' + u = f$ a.e. on (a,b) and thus everywhere on $[a,b]$, since $u \in C^2([a,b])$.

[1]Indeed, we need that all integrals in (1.2) are well-defined. Since $u, u' \in L^1(a,b), \varphi \in C^1([a,b]), f \in C([a,b])$, the functions $u'\varphi', u\varphi, f\varphi$ are Lebesgue measurable. Since $\|u'\varphi'\|_{L^1(a,b)} \le \max_{x\in[a,b]} |\varphi'(x)| \|u'\|_{L^1(a,b)} < +\infty$, i.e., $u'\varphi' \in L^1(a,b)$, the first integral in (1.2) is well-defined. Other integrals in (1.2) is handled similarly.

2 The Sobolev Space $W^{1,p}(I)$

Let $I = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \le p \le \infty$.

Definition 1.2. The *Sobolev space* $W^{1,p}(I)^2$ is defined to be

$$W^{1,p}(I) = \left\{ u \in L^p(I); \exists g \in L^p(I), \int_I u\varphi' = -\int_I g\varphi, \ \ \forall \varphi \in C_c^1(I) \right\} \tag{2.1}$$

We set

$$H^1(I) = W^{1,2}(I) \tag{2.2}$$

For $u \in W^{1,p}(I)$ we denote $u' = g$. Note that this makes sense: g is well defined a.e. Indeed, suppose that there exists another $\bar{g} \in L^p(I)$ satisfying

$$\int_I u\varphi' = -\int_I \bar{g}\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.3}$$

we subtract (2.3) from

$$\int_I u\varphi' = -\int_I g\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.4}$$

to obtain

$$\int_I (\bar{g} - g)\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.5}$$

In particular,

$$\int_I (\bar{g} - g)\varphi, \ \ \forall \varphi \in C_c^\infty(I) \tag{2.6}$$

By Corollary 4.24, [1], we deduce

$$\bar{g} = g \text{ a.e. on } I \tag{2.7}$$

i.e., g is well defined a.e. □

Remark 1.3. In the definition of $W^{1,p}$ we call φ a *test function*. We could equally well have used $C_c^\infty(I)$ as the class of test functions because if $\varphi \in C_c^1(I)$, then $\rho_n \star \varphi \in C_c^\infty(I)$ for n large enough and $\rho_n \star \varphi \to \varphi$ in C^1 (see [1] Section 4.4; of course, φ is extended to be 0 outside I).

Proof of Remark 1.3. We recall that a sequence of mollifiers $(\rho_n)_{n\ge 1}$ is any sequence of functions on $\mathbb{R}^N$ such that

$$\rho_n \in C_c^\infty\left(\mathbb{R}^N\right), \ \ \text{supp}\ \rho_n \subset \overline{B\left(0, \frac{1}{n}\right)}, \ \ \int \rho_n = 1, \ \ \rho_n \ge 0 \text{ on } \mathbb{R}^N \tag{2.8}$$

[2] If there is no confusion we shall write $W^{1,p}$ instead of $W^{1,p}(I)$ and H^1 instead of $H^1(I)$.

Since $\varphi \in C_c^1(I)$, we can put supp $\varphi = [c,d] \subset I = (a,b)$ where $a < c \leq d < b$. Thus,

$$\begin{aligned} \text{supp }(\rho_n \star \varphi) &\subset \overline{\text{supp }\rho_n + \text{supp }\varphi} && (2.9)\\ &\subset \overline{B\left(0,\frac{1}{n}\right)} + \text{supp }\varphi && (2.10)\\ &= \overline{B\left(0,\frac{1}{n}\right)} + [c,d] && (2.11)\end{aligned}$$

For n large enough, e.g., $n > n_0 := \dfrac{1}{\min\{c-a, b-d\}}$, $\rho_n \star \varphi \in C_c^\infty(I)$ holds.

To prove the second argument, we proceed as in the proof of Proposition 4.21, [1]. Given $0 < \varepsilon < \frac{1}{n_0}$ there exists $\delta > 0$ (depending on the compact set supp φ and ε) such that

$$|\varphi(x-y) - \varphi(x)| + |\varphi'(x-y) - \varphi'(x)| < \varepsilon \tag{2.12}$$

for $\forall x \in$ supp φ, $\forall y \in B(0,\delta)$. We have, for $x \in \mathbb{R}$,

$$\begin{aligned} (\rho_n \star \varphi)(x) - \varphi(x) &= \int (\varphi(x-y) - \varphi(x))\,\rho_n(y)\,dy && (2.13)\\ &= \int_{B\left(0,\frac{1}{n}\right)} (\varphi(x-y) - \varphi(x))\,\rho_n(y)\,dy && (2.14)\\ (\rho_n * \varphi)'(x) - \varphi'(x) &= \frac{d}{dx}\int \varphi(x-y)\,\rho_n(y)\,dy - \int \varphi'(x)\,\rho_n(y)\,dy && (2.15)\\ &= \int \frac{d}{dx}(\varphi(x-y)\,\rho_n(y))\,dy - \int \varphi'(x)\,\rho_n(y)\,dy && (2.16)\\ &= \int (\varphi'(x-y) - \varphi'(x))\,\rho_n(y)\,dy && (2.17)\\ &= \int_{B\left(0,\frac{1}{n}\right)} (\varphi'(x-y) - \varphi'(x))\,\rho_n(y)\,dy && (2.18)\end{aligned}$$

For $n \geq \frac{1}{\delta}$ and $x \in$ supp φ we obtain

$$\begin{aligned} &|(\rho_n \star \varphi)(x) - \varphi(x)| + \left|(\rho_n \star \varphi)'(x) - \varphi'(x)\right| && (2.19)\\ &\leq \int_{B\left(0,\frac{1}{n}\right)} (|\varphi(x-y) - \varphi(x)| + |\varphi'(x-y) - \varphi'(x)|)\,\rho_n(y)\,dy && (2.20)\\ &\leq \varepsilon \int_{B\left(0,\frac{1}{n}\right)} \rho_n(y)\,dy, \text{ by (2.12)} && (2.21)\\ &= \varepsilon && (2.22)\end{aligned}$$

Since x is taken arbitrarily, we deduce from (2.19)-(2.22) that

$$\begin{aligned} \|\rho_n \star \varphi - \varphi\|_{C^1(\mathbb{R})} &= \sup_{x\in\mathbb{R}} |(\rho_n \star \varphi)(x) - \varphi(x)| && (2.23)\\ &\quad + \sup_{x\in\mathbb{R}} \left|(\rho_n \star \varphi)'(x) - \varphi'(x)\right| \leq \varepsilon && (2.24)\end{aligned}$$

i.e., $\rho \star \varphi \to \varphi$ in C^1. $\square$

Remark 1.4. It is clear that if $u \in C^1(I) \cap L^p(I)$ and if $u' \in L^p(I)$ (here u' is the usual derivative of u) then $u \in W^{1,p}(I)$[3]. Moreover, the usual derivative of u coincides with its derivative in the $W^{1,p}$ sense - so that notation is consistent! In particular, if I is bounded, $C^1(\bar{I}) \subset W^{1,p}(I)$ for all $1 \leq p \leq \infty$.

Example 1.5. Let $I = (-1, 1)$. As an exercise show the following

1. The function $u(x) = |x|$ belongs to $W^{1,p}(I)$ for every $1 \leq p \leq \infty$ and $u' = g$, where

$$g(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ -1, & \text{if } -1 < x < 0 \end{cases} \tag{2.25}$$

   More generally, a continuous function of $\bar{I}$ that is piecewise C^1 on $\bar{I}$ belongs to $W^{1,p}(I)$ for all $1 \leq p \leq \infty$.

2. The function g above does *not* belong to $W^{1,p}(I)$ for any $1 \leq p \leq \infty$.

PROOF OF EXAMPLE 1.15.

1. It is easily to check that u, g belong to $L^p(I)$ for every $1 \leq p \leq \infty$. It remains to check (2.1)

$$\int_I u\varphi' = \int_{-1}^{1} |x| \varphi'(x)\, dx \tag{2.26}$$

$$= -\int_{-1}^{0} x\varphi'(x)\, dx + \int_0^1 x\varphi'(x)\, dx \tag{2.27}$$

$$= -x\varphi(x)|_{-1}^{0} + \int_{-1}^{0} \varphi(x)\, dx + x\varphi(x)|_0^1 - \int_0^1 \varphi(x)\, dx \tag{2.28}$$

$$= -\int_{-1}^{1} \operatorname{sgn}(x)\varphi(x)\, dx \tag{2.29}$$

$$= -\int_{-1}^{1} g(x)\varphi(x)\, dx, \quad \forall \varphi \in C_c^1(I) \tag{2.30}$$

   Hence, $u \in W^{1,p}(I)$ for every $1 \leq p \leq \infty$ and $u' = g$.

   Now, we consider an arbitrary continuous function of $\bar{I}$ that is piecewise C^1 on $\bar{I}$. Let n be a positive integer, we consider a partition of $\bar{I}$

$$-1 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1 \tag{2.31}$$

   and a function u defined by

$$u = \sum_{i=1}^{n} u_i \chi_{[a_{i-1}, a_i)} \tag{2.32}$$

   where $u_i \in C^1([a_{i-1}, a_i])$, $i = 1, \ldots, n$ such that $u_i(a_i) = u_{i+1}(a_i)$, $i = 1, \ldots, n-1$. Define

$$g = \sum_{i=1}^{n} {u_i}' \chi_{(a_{i-1}, a_i)} \tag{2.33}$$

[3]Integrating $\int_a^b u\varphi'$ by parts yields $\int_a^b u\varphi' = u(b)\varphi(b) - u(a)\varphi(a) - \int_a^b u'\varphi = -\int_a^b u'\varphi$ since $\varphi(a) = \varphi(b) = 0$.

we have $g \in L^p(I)$ for $1 \le p \le \infty$ since ${u_i}' \in C((a_{i-1}, a_i))$, $i = 1, \dots, n$. It only remains to check (2.1). For all $\varphi \in C_c^1(I)$,

$$\begin{aligned}
\int_I u\varphi' &= \int_I \left(\sum_{i=1}^n u_i \chi_{[a_{i-1},a_i)}\right)\varphi' && (2.34)\\
&= \sum_{i=1}^n \int_I u_i \chi_{[a_{i-1},a_i)}\varphi' && (2.35)\\
&= \sum_{i=1}^n \int_{a_{i-1}}^{a_i} u_i\varphi' && (2.36)\\
&= \sum_{i=1}^n \left(u_i(a_i)\varphi(a_i) - u_i(a_{i-1})\varphi(a_{i-1}) - \int_{a_{i-1}}^{a_i} {u_i}'\varphi\right) && (2.37)\\
&= \sum_{i=1}^n u_i(a_i)\varphi(a_i) - \sum_{i=1}^n u_i(a_{i-1})\varphi(a_{i-1}) - \sum_{i=1}^n \int_{a_{i-1}}^{a_i} {u_i}'\varphi && (2.38)\\
&= \sum_{i=1}^{n-1} u_i(a_i)\varphi(a_i) - \sum_{i=0}^{n-1} u_{i+1}(a_i)\varphi(a_i) - \sum_{i=1}^n \int_I {u_i}'\chi_{(a_{i-1},a_i)}\varphi && (2.39)\\
&= \sum_{i=1}^{n-1} \underbrace{(u_i(a_i) - u_{i+1}(a_i))}_{=0}\varphi(a_i) - \int_I \left(\sum_{i=1}^n {u_i}'\chi_{(a_{i-1},a_i)}\right)\varphi && (2.40)\\
&= -\int_I g\varphi && (2.41)
\end{aligned}$$

Hence, u belong to $W^{1,p}(I)$ for all $1 \le p \le \infty$, and its derivative is given by

$$u' = \sum_{i=1}^n {u_i}'\chi_{(a_{i-1},a_i)} \tag{2.42}$$

2. Fix a $p \in [0, \infty]$. Suppose for the contrary that there exists $h \in L^p(I)$ such that

$$\int_I g\varphi' = -\int_I h\varphi, \quad \forall\varphi \in C_c^1(I) \tag{2.43}$$

Then we have

$$\begin{aligned}
\int_{-1}^1 g\varphi' &= -\int_{-1}^0 \varphi' + \int_0^1 \varphi' && (2.44)\\
&= \varphi(-1) - \varphi(0) + \varphi(1) - \varphi(0) && (2.45)\\
&= -2\varphi(0), \quad \forall\varphi \in C_c^1(I) && (2.46)
\end{aligned}$$

By (2.43) and (2.44)-(2.46), we deduce that

$$\int_I h\varphi = 0, \quad \forall\varphi \in C_c^1(I\setminus\{0\}) \tag{2.47}$$

which implies $h = 0$ a.e. on $I\setminus\{0\}$. Thus $h = 0$ a.e. on I. Then

$$\int_I h\varphi = 0, \quad \forall\varphi \in C_c^1(I) \tag{2.48}$$

Combining (2.43), (2.44)-(2.46) and (2.48) yields

$$\varphi(0) = 0, \quad \forall\varphi \in C_c^1(I) \tag{2.49}$$

which is absurd. Hence $g \notin W^{1,p}(I), \forall 1 \leq p \leq \infty$. □

Remark 1.6. To define $W^{1,p}$ one can also use the language of distributions. *All* functions $u \in L^p(I)$ admit a derivative in the sense of distributions; this derivative is an element of the huge space of distributions $\mathcal{D}'(I)$. We say that $u \in W^{1,p}$ if this distributional derivative happens to lie in L^p, which is a subspace of $\mathcal{D}'(I)$. When $I = \mathbb{R}$ and $p = 2$, Sobolev spaces can also be defined using the Fourier transform.

Notation 1.7. The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p} \tag{2.50}$$

or sometimes, if $1 < p < \infty$, with the equivalent norm $\left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{\frac{1}{p}}$. The space H^1 is equipped with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + (u', v')_{L^2} = \int_a^b (uv + u'v') \tag{2.51}$$

and with the associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\right)^{\frac{1}{2}} \tag{2.52}$$

Proposition 1.8. *The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$. It is reflexive*[4] *for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The space H^1 is a separable Hilbert space.*

Proof.

1. Let (u_n) be a Cauchy sequence in $W^{1,p}$, i.e.,

$$\|u_m - u_n\|_{W^{1,p}} = \|u_m - u_n\|_{L^p} + \|u_m{}' - u_n{}'\|_{L^p} \to 0 \tag{2.53}$$

as $m, n \to +\infty$, then (u_n) and (u_n') are obviously Cauchy sequences in L^p. It follows that u_n converges to some limit u in L^p and u_n' converges to some limit g in L^p. We have

$$\int_I u_n\varphi' = -\int_I u_n{}'\varphi, \quad \forall\varphi \in C_c^1(I) \tag{2.54}$$

and in the limit

$$\int_I u\varphi' = -\int_I g\varphi, \quad \forall\varphi \in C_c^1(I) \tag{2.55}$$

Thus $u \in W^{1,p}, u' = g$, and $\|u_n - u\|_{W^{1,p}} \to 0$.

[4]This property is a *considerable* advantage of $W^{1,p}$. In the problems of the *calculus of variations*, $W^{1,p}$ is preferred over C^1, which is not reflexive. Existence of minimizers is easily established in reflexive spaces (see, e.g., Corollary 3.23).

2. $W^{1,p}$ is *reflexive* for $1 < p < \infty$. Clearly, the product space $E = L^p(I) \times L^p(I)$ is reflexive. The operator $T : W^{1,p} \to E$ defined by $Tu = [u, u']$ is an isometry from $W^{1,p}$ into E. Indeed,

$$\|Tu\|_E = \|[u, u']\|_{L^p(I)\times L^p(I)} \tag{2.56}$$
$$= \|u\|_{L^p(I)} + \|u'\|_{L^p(I)} \tag{2.57}$$
$$= \|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I) \tag{2.58}$$

   Since $W^{1,p}$ is a Banach space, $T\left(W^{1,p}\right)$ is a closed subspace of E. It follows that $T\left(W^{1,p}\right)$ is reflexive (see Prop. 3.20, [1]). Consequently, $W^{1,p}(I)$ is also reflexive.

3. $W^{1,p}$ is *separable* for $1 \leq p < \infty$. Clearly, the product space $E = L^p(I) \times L^p(I)$ is separable. Thus $T\left(W^{1,p}\right)$ which is a subset of E, is also separable (by Prop. 3.25, [1]). Consequently $W^{1,p}$ is separable. □

Remark 1.9. It is convenient to keep in mind the following fact, which we have used in the proof of Proposition 8.1: Let (u_n) be a sequence in $W^{1,p}$ such that $u_n \to u$ in L^p and (u_n') converges to some limit in L^p; then $u \in W^{1,p}$ and $\|u_n - u\|_{W^{1,p}} \to 0$. In fact, when $1 < p \leq \infty$ it suffices to know that $u_n \to u$ in L^p and $\|u_n{}'\|_{L^p}$ stays *bounded* to conclude that $u \in W^{1,p}$ (see Exercise 8.2, [1]).

The functions in $W^{1,p}$ are roughly speaking the primitives of the L^p functions. More precisely, we have the following.

Theorem 1.10. *Let $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, and I bounded or unbounded; then there exists a function $\widetilde{u} \in C\left(\bar{I}\right)$ such that*

$$u = \widetilde{u} \text{ a.e. on } I \tag{2.59}$$

and

$$\widetilde{u}(x) - \widetilde{u}(y) = \int_y^x u'(t)\,dt, \quad \forall x, y \in \bar{I} \tag{2.60}$$

Remark 1.11. Let us emphasize the content of Theorem 1.10. First, not that if one function u belongs to $W^{1,p}$ then all functions v such that $v = u$ a.e. on I also belong to $W^{1,p}$ (this follows directly from the definition of $W^{1,p}$). Theorem 1.10 asserts that every function $u \in W^{1,p}$ admits one and only one *continuous representative* on $\bar{I}$, i.e., there exists a continuous function on $\bar{I}$ that belongs to the equivalence class of u ($v \sim u$ if $v = u$ a.e.). When it is useful[5] we replace u by its continuous representative. In order to simplify the notation we also write u for its continuous representative. We finally point out that the property "u has a continuous representative" is not the same as "u is continuous a.e."

Remark 1.12. It follows from Theorem 8.2 that if $u \in W^{1,p}$ and if $u' \in C\left(\bar{I}\right)$ (i.e., u' admits a continuous representative on $\bar{I}$), then $u \in C^1\left(\bar{I}\right)$; more precisely, $\widetilde{u} \in C^1\left(\bar{I}\right)$, but as mentioned above, we dot not distinguish u and $\widetilde{u}$.

[5]For example, in other to give a meaning to $u(x)$ for every $x \in \bar{I}$.

In the proof of Theorem 8.2 we shall use the following lemmas.

Lemma 1.13. *Let $f \in L^1_{loc}(I)$ be such that*

$$\int_I f\varphi' = 0, \quad \forall \varphi \in C_c^1(I) \tag{2.61}$$

Then there exists a constant C such that $f = C$ a.e. on I.

Proof. Fix a function $\psi \in C_c(I)$ such that $\int_I \psi = 1$. For any function $w \in C_c(I)$ there exists $\varphi \in C_c^1(I)$ such that

$$\varphi' = w - \left(\int_I w\right)\psi \tag{2.62}$$

Indeed, the function $h = w - \left(\int_I w\right)\psi$ is continuous, has compact support in I, and

$$\int_I h = \int_I \left(w - \left(\int_I w\right)\psi\right) \tag{2.63}$$

$$= \int_I w - \int_I w \int_I \psi \tag{2.64}$$

$$= 0 \tag{2.65}$$

Therefore h has a (unique) primitive with compact support in I. We deduce from (2.61) that

$$\int_I f\left[w - \left(\int_I w\right)\psi\right] = 0, \quad \forall w \in C_c(I) \tag{2.66}$$

Since

$$\int_I f\left[w - \left(\int_I w\right)\psi\right] = \int_I fw - \int_I \left(\int_I w\right) f\psi \tag{2.67}$$

$$= \int_I fw - \left(\int_I w\right)\left(\int_I f\psi\right) \tag{2.68}$$

$$= \int_I fw - \int_I \left(\int_I f\psi\right) w \tag{2.69}$$

$$= \int_I \left[f - \left(\int_I f\psi\right)\right] w \tag{2.70}$$

holds for all $w \in C_c(I)$, (2.66) is equivalent to

$$\int_I \left[f - \left(\int_I f\psi\right)\right] w = 0, \quad \forall w \in C_c(I) \tag{2.71}$$

and therefore (by Corollary. 4.24, [1]) $f - \left(\int_I f\psi\right) = 0$ a.e. on I, i.e., $f = C$ a.e. on I with $C = \int_I f\psi$. ☐

Lemma 1.14. *Let $g \in L^1_{loc}(I)$; for y_0 fixed in I, set*

$$v(x) = \int_{y_0}^{x} g(t)\,dt, \quad x \in I \tag{2.72}$$

Then $v \in C(I)$ *and*

$$\int_I v\varphi' = -\int_I g\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.73}$$

Proof. We have

$$\int_I v\varphi' = \int_I \left(\int_{y_0}^x g(t)\,dt\right)\varphi'(x)\,dx \tag{2.74}$$

$$= \int_a^{y_0} \left(\int_{y_0}^x g(t)\,dt\right)\varphi'(x)\,dx + \int_{y_0}^b \left(\int_{y_0}^x g(t)\,dt\right)\varphi'(x)\,dx \tag{2.75}$$

$$= -\int_a^{y_0} dx \int_x^{y_0} g(t)\,\varphi'(x)\,dt + \int_{y_0}^b dx \int_{y_0}^x g(t)\,\varphi'(x)\,dt \tag{2.76}$$

By Fubini's theorem

$$\int_I v\varphi' = -\int_a^{y_0} g(t)\,dt \int_a^t \varphi'(x)\,dx + \int_{y_0}^b g(t)\,dt \int_t^b \varphi'(x)\,dx \tag{2.77}$$

$$= -\int_a^{y_0} g(t)\left(\varphi(t) - \varphi(a)\right)dt + \int_{y_0}^b g(t)\left(\varphi(b) - \varphi(t)\right)dt \tag{2.78}$$

$$= -\int_a^{y_0} g(t)\,\varphi(t)\,dt - \int_{y_0}^b g(t)\,\varphi(t)\,dt \tag{2.79}$$

$$= -\int_I g(t)\,\varphi(t)\,dt \tag{2.80}$$

Hence, (2.73) holds. □

Proof of Theorem 1.10. Fix $y_0 \in I$ and set $\bar{u}(x) = \int_{y_0}^x u'(t)\,dt$. By Lemma 1.14 we have

$$\int_I \bar{u}\varphi' = -\int_I u'\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.81}$$

Combining (2.81) with

$$\int_I u\varphi' = -\int_I u'\varphi, \ \ \forall \varphi \in C_c^1(I) \tag{2.82}$$

yields

$$\int_I (u - \bar{u})\,\varphi' = 0, \ \ \forall \varphi \in C_c^1(I) \tag{2.83}$$

It follows from Lemma 8.13 that $u - \bar{u} = C$ a.e. on I. The function $\tilde{u}(x) = \bar{u}(x) + C$ has the desired properties. □

Remark 1.15. Lemma 1.14 shows that the primitive v of a function $g \in L^p$ belongs to $W^{1,p}$ provided we also know that $v \in L^p$, which is always the case when I is bounded.

Proposition 1.16. *Let* $u \in L^p$ *with* $1 < p \leq \infty$. *The following properties are equivalent.*

1. $u \in W^{1,p}$.

2. *There is a constant C such that*

$$\left|\int_I u\varphi'\right| \leq C\|\varphi\|_{L^{p'}(I)}, \quad \forall\varphi \in C_c^1(I) \tag{2.84}$$

Furthermore, we can take $C = \|u'\|_{L^p(I)}$ in (2.84).

Proof.

1. (1) $\Rightarrow$ (2). By definition of $W^{1,p}$, we have

$$\left|\int_I u\varphi'\right| = \left|\int_I u'\varphi\right| \tag{2.85}$$

$$\leq \|u'\|_{L^p(I)}\|\varphi\|_{L^{p'}(I)}, \quad \forall\varphi \in C_c^1(I) \tag{2.86}$$

where we have used *Hölder's inequality*, see Theorem 4.6, [1].

2. The linear functional

$$\varphi \in C_c^1(I) \mapsto \int_I u\varphi' \tag{2.87}$$

is defined on a dense subspace of $L^{p'}$ (see Theorem 4.12, [1].) (since $p' < \infty$[6]) and it is continuous for the $L^{p'}$ norm. Indeed, (2.84) gives

$$\left|\int_I u\varphi' - \int_I u\psi'\right| = \left|\int_I u(\varphi-\psi)'\right| \tag{2.88}$$

$$\leq C\|\varphi-\psi\|_{L^{p'}(I)}, \quad \forall\varphi,\psi \in C_c^1(I) \tag{2.89}$$

Thus, given $\varepsilon > 0$, (2.88)-(2.89) implies

$$\|\varphi-\psi\|_{L^{p'}(I)} < \frac{\varepsilon}{C} \Rightarrow \left|\int_I u\varphi' - \int_I u\psi'\right| < \varepsilon \tag{2.90}$$

i.e., the linear functional defined by (2.87) is continuous.

Therefore it extends to a bounded linear functional F defined on all of $L^{p'}$ (applying the Hahn-Banach theorem, or simply extension by continuity). By the Riesz representation theorems (Theorem 4.11 and 4.14, [1]) there exists $g \in L^p$ such that

$$\langle F,\varphi\rangle = \int_I g\varphi, \quad \forall\varphi \in L^{p'} \tag{2.91}$$

i.e.,

$$\int_I u\varphi' = \int_I g\varphi, \quad \forall\varphi \in L^{p'} \tag{2.92}$$

In particular,

$$\int_I u\varphi' = \int_I g\varphi, \quad \forall\varphi \in C_c^1 \tag{2.93}$$

and thus $u \in W^{1,p}$. □

[6]Notice the condition $p > 1$ used in the hypothesis of Proposition 1.16.

Remark 1.17. *absolutely continuous functions and functions of bounded variation).* When $p = 1$ the implication (1) $\Rightarrow$ (2) remains true but not the converse. To illustrate this fact, suppose that I is bounded. The function u satisfying (1) with $p = 1$, i.e., the functions of $W^{1,1}(I)$, are called the *absolutely continuous* functions. They are also characterized by the property

$$(AC) \quad \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that for every finite sequence of} \\ \text{disjoint intervals } (a_k, b_k) \subset I \text{ such that } \sum |b_k - a_k| < \delta, \\ \text{we have } \sum |u(b_k) - u(a_k)| < \varepsilon \end{cases} \tag{2.94}$$

On the other hand, the functions u satisfying (2) with $p = 1$, i.e.,

$$\left| \int_I u\varphi' \right| \le \|u'\|_{L^1(I)} \|\varphi\|_{L^\infty(I)}, \quad \forall \varphi \in C_c^1(I) \tag{2.95}$$

are called functions of *bounded variation*; these functions can be characterized in many different ways.

1. They are the difference of two bounded nondecreasing functions (possibly discontinuous) on I.
2. They are the functions u satisfying the property

$$(BV) \quad \begin{cases} \text{there exists a constant } C \text{ such that} \\ \sum_{i=0}^{k-1} |u(t_{i+1}) - u(t_i)| \le C \text{ for all } t_0 < t_1 < \dots < t_k \text{ in } I \end{cases} \tag{2.96}$$

3. They are the functions $u \in L^1(I)$ that have as distributional derivative a bounded measure.

Note that functions of bounded variation need not have a continuous representative.

Proposition 1.18. *A function u in $L^\infty(I)$ belongs to $W^{1,\infty}(I)$ if and only if there exists a constant C such that*

$$|u(x) - u(y)| \le C|x - y| \text{ for a.e. } x, y \in I \tag{2.97}$$

Proof. If $u \in W^{1,\infty}(I)$ we may apply Theorem 1.10 to deduce that

$$|u(x) - u(y)| = |\tilde{u}(x) - \tilde{u}(y)| \tag{2.98}$$

$$= \left| \int_x^y u'(t)\, dt \right| \tag{2.99}$$

$$\le \|u'\|_{L^\infty(I)} |x - y| \text{ for a.e. } x, y \in I \tag{2.100}$$

Conversely, let $\varphi \in C_c^1(I)$. For $h \in \mathbb{R}$, with $|h|$ small enough, we have

$$\int_I (u(x+h) - u(x))\varphi(x)\, dx = \int_I u(x)(\varphi(x-h) - \varphi(x))\, dx \tag{2.101}$$

Indeed, since $\varphi \in C_c^1(I)$, we assume that supp $\varphi = [c_\varphi, d_\varphi] \subset (a,b)$. Choosing $h < b - d_\varphi$ gives

$$\int_a^b u(x+h)\varphi(x)\,dx = \int_{c_\varphi}^{d_\varphi} u(x+h)\varphi(x)\,dx \tag{2.102}$$

$$= \int_{c_\varphi+h}^{d_\varphi+h} u(t)\varphi(t-h)\,dt \tag{2.103}$$

$$= \int_a^b u(x)\varphi(x-h)\,dx \tag{2.104}$$

Thus

$$\int_I (u(x+h)-u(x))\varphi(x)\,dx \tag{2.105}$$

$$= \int_a^b u(x+h)\varphi(x)\,dx - \int_a^b u(x)\varphi(x)\,dx \tag{2.106}$$

$$= \int_a^b u(x)\varphi(x-h)\,dx - \int_a^b u(x)\varphi(x)\,dx \tag{2.107}$$

$$= \int_I u(x)(\varphi(x-h)-\varphi(x))\,dx \tag{2.108}$$

(these integrals make sense for h small, since φ is supported in a compact subset of I). Using the assumption on U we obtain

$$\left|\int_I u(x)(\varphi(x-h)-\varphi(x))\,dx\right| = \left|\int_I (u(x+h)-u(x))\varphi(x)\,dx\right| \tag{2.109}$$

$$\leq \int_I |u(x+h)-u(x)|\,|\varphi(x)|\,dx \tag{2.110}$$

$$\leq C\,|h|\,\|\varphi\|_{L^1(I)}, \text{ by (2.97)} \tag{2.111}$$

Dividing by $|h|$ and letting $h \to 0$, we are led to

$$\left|\int_I u\varphi'\right| \leq C\|\varphi\|_{L^1(I)}, \quad \forall \varphi \in C_c^1(I) \tag{2.112}$$

We now apply Proposition 1.16 and conclude that $u \in W^{1,\infty}$. □

The L^p-version of Proposition 1.18 reads as follows.

Proposition 1.19. *Let $u \in L^p(\mathbb{R})$ with $1 < p < \infty$. The following properties are equivalent.*

1. $u \in W^{1,p}(\mathbb{R})$.
2. *There exists a constant C such that for all $h \in \mathbb{R}$,*

$$\|\tau_h u - u\|_{L^p(\mathbb{R})} \leq C\,|h| \tag{2.113}$$

Moreover, one can choose $C = \|u'\|_{L^p(\mathbb{R})}$ in (2).

Recall that $(\tau_h u)(x) = u(x+h)$.

PROOF.

1. (1) $\Rightarrow$ (2). (This implication is also valid when $p=1$.) By Theorem 1.10 we have, for all x and h in $\mathbb{R}$,

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)\,dt \tag{2.114}$$

$$= h\int_0^1 u'(x+sh)\,ds \tag{2.115}$$

Thus

$$|u(x+h) - u(x)| \leq |h| \int_0^1 |u'(x+sh)|\,ds \tag{2.116}$$

Applying Hölder's inequality, we have

$$|u(x+h) - u(x)|^p \leq |h|^p \int_0^1 |u'(x+sh)|^p ds \tag{2.117}$$

It then follows that

$$\int_{\mathbb{R}} |u(x+h) - u(x)|^p dx \leq |h|^p \int_{\mathbb{R}} dx \int_0^1 |u'(x+sh)|^p ds \tag{2.118}$$

$$\leq |h|^p \int_0^1 ds \int_{\mathbb{R}} |u'(x+sh)|^p dx \tag{2.119}$$

But for $0 < s < 1$,

$$\int_{\mathbb{R}} |u'(x+sh)|^p dx = \int_{\mathbb{R}} |u'(y)|^p dy \tag{2.120}$$

Combining (2.118)-(2.119) with (2.120) yields

$$\|\tau_h u - u\|_{L^p(\mathbb{R})} \leq \|u'\|_{L^p(\mathbb{R})} |h| \tag{2.121}$$

2. (2) $\Rightarrow$ (1). Let $\varphi \in C_c^1(\mathbb{R})$. For all $h \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} (u(x+h) - u(x))\,\varphi(x)\,dx = \int_{\mathbb{R}} u(x)\,(\varphi(x-h) - \varphi(x))\,dx \tag{2.122}$$

Using Hölder's inequality and (2) one obtains

$$\left|\int_{\mathbb{R}} (u(x+h) - u(x))\,\varphi(x)\,dx\right| \leq C\,|h|\,\|\varphi\|_{L^{p'}(\mathbb{R})} \tag{2.123}$$

and thus

$$\left|\int_{\mathbb{R}} u(x)\,(\varphi(x-h) - \varphi(x))\,dx\right| \leq C\,|h|\,\|\varphi\|_{L^{p'}(\mathbb{R})} \tag{2.124}$$

Dividing by $|h|$ and letting $h \to 0$, we obtain

$$\left|\int_{\mathbb{R}} u\varphi'\right| \leq C\|\varphi\|_{L^{p'}(\mathbb{R})} \tag{2.125}$$

We may apply Proposition 1.16 once more and conclude that $u \in W^{1,p}(\mathbb{R})$.

This completes our proof. $\square$

Certain basic analytic operations have a meaning only for functions defined on all of $\mathbb{R}$ (for example convolution and Fourier transform). It is therefore useful to be able to extend a function $u \in W^{1,p}(I)$ to a function $\bar{u} \in W^{1,p}(\mathbb{R})$.[7] The following result addresses this point.

Theorem 1.20 (extension operator). *Let $1 \leq p \leq \infty$. There exists a bounded linear operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$, called an extension operator, satisfying the following properties.*

1. $Pu|_I = u, \quad \forall u \in W^{1,p}(I)$.
2. $\|Pu\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^p(I)}, \quad \forall u \in W^{1,p}(I)$.
3. $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I)$.

where C depends only on $|I| \leq \infty$.[8]

PROOF. Beginning with the case $I = (0, \infty)$ we show that extension by reflexion

$$(Pu)(x) = u^\star(x) = \begin{cases} u(x) & \text{if } x \geq 0 \\ u(-x) & \text{if } x < 0 \end{cases} \tag{2.126}$$

works. Clearly we have

$$\|u^\star\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)} \tag{2.127}$$

Proof of (2.127). Consider two cases.

1. *Case $p = \infty$.* (2.127) is obvious since $\|u^\star\|_{L^p(\mathbb{R})} = \|u\|_{L^p(I)}$.
2. *Case $1 \leq p < \infty$.* We have

$$\|u^\star\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |u^\star(x)|^p dx\right)^{\frac{1}{p}} \tag{2.128}$$

$$= \left(\int_{-\infty}^{0} |u(-x)|^p dx + \int_{0}^{\infty} |u(x)|^p dx\right)^{\frac{1}{p}} \tag{2.129}$$

$$= \left(\int_{0}^{\infty} |u(t)|^p dt + \int_{0}^{\infty} |u(x)|^p dx\right)^{\frac{1}{p}}, \text{ put } t = -x \tag{2.130}$$

$$= 2^{\frac{1}{p}}\|u\|_{L^p(I)} \tag{2.131}$$

$$\leq 2\|u\|_{L^p(I)} \text{ since } p \geq 1 \tag{2.132}$$

Setting

$$v(x) = \begin{cases} u'(x) & \text{if } x > 0 \\ -u'(-x) & \text{if } x < 0 \end{cases} \tag{2.133}$$

[7] If u is extended as 0 outside I then the resulting function will not, in general, be in $W^{1,p}(\mathbb{R})$ (see Remark 1.11 and Section 8.3, [1]).

[8] One can take $C = 4$ in (2) and $C = 4\left(1 + \frac{1}{|I|}\right)$ in (3).

we easily check that $v \in L^p(\mathbb{R})$[9] and

$$u^\star(x) - u^\star(0) = \int_0^x v(t)\,dt, \quad \forall x \in \mathbb{R} \tag{2.134}$$

Proof of (2.134). We consider two cases.

1. *Case* $x \geq 0$. We have

$$\begin{aligned} u^\star(x) - u^\star(0) &= u(x) - u(0) && (2.135)\\ &= \int_0^x u'(t)\,dt, \text{ by Theorem 1.10} && (2.136)\\ &= \int_0^x v(t)\,dt && (2.137)\end{aligned}$$

2. *Case* $x < 0$. Similarly,

$$\begin{aligned} u^\star(x) - u^\star(0) &= u(-x) - u(0) && (2.138)\\ &= \int_0^{-x} u'(s)\,ds, \text{ by Theorem 1.10} && (2.139)\\ &= -\int_0^x u'(-t)\,dt, \text{ put } t = -s && (2.140)\\ &= \int_0^x v(t)\,dt && (2.141)\end{aligned}$$

Hence, (2.134) holds for all $x \in \mathbb{R}$.

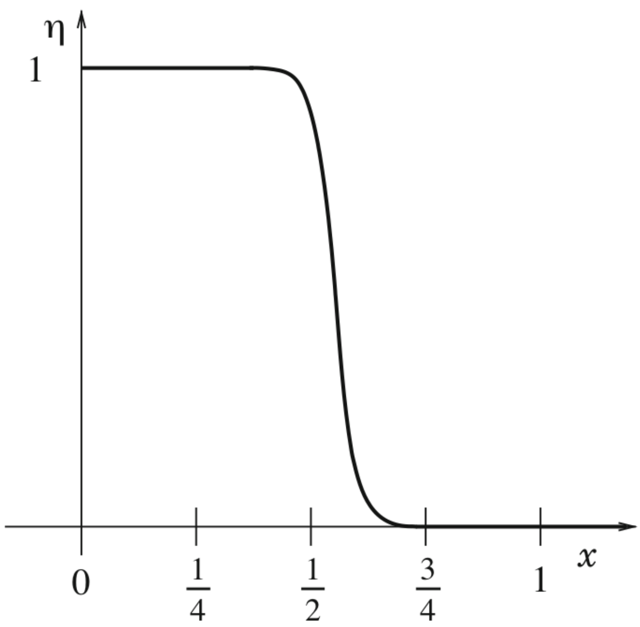

Figure 1: Function $\eta(x)$.

[9] We need to check the following two conditions: (1) v is measurable in $\mathbb{R}$, which is deduced from the fact that u' is measurable (note that $u' \in L^p(I)$). (2) $\int_{\mathbb{R}} |v|^p < \infty$, which can be easily verified by computing this integral. Explicitly, $\int_{\mathbb{R}} |v|^p = 2\int_I |u'|^p < \infty$.

It follows that $u^\star \in W^{1,p}(\mathbb{R})$ (see Remark 1.10) and $\|u^\star\|_{W^{1,p}(\mathbb{R})} \leq 2\|u\|_{W^{1,p}(I)}$, which is verified as follows.

$$\begin{aligned}
\|u^\star\|_{W^{1,p}(\mathbb{R})} &= \|u^\star\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})} && (2.142)\\
&= 2^{\frac{1}{p}}\left(\|u\|_{L^p(I)} + \|u'\|_{L^p(I)}\right), \text{ by (2.131)} && (2.143)\\
&= 2^{\frac{1}{p}}\|u\|_{W^{1,p}(I)} && (2.144)\\
&\leq 2\|u\|_{W^{1,p}(I)}, \text{ since } p \geq 1 && (2.145)
\end{aligned}$$

Now consider the case of a *bounded interval* I; without loss of generality we can take $I = (0,1)$. *Fix* a function $\eta \in C^1(\mathbb{R})$, $0 \leq \eta \leq 1$, such that

$$\eta(x) = \begin{cases} 1 \text{ if } x < \frac{1}{4} \\ 0 \text{ if } x > \frac{3}{4} \end{cases} \tag{2.146}$$

See Figure 1.1.

Given a function f on $(0,1)$ set

$$\tilde{f}(x) = \begin{cases} f(x) \text{ if } 0 < x < 1 \\ 0 \text{ if } x > 1 \end{cases} \tag{2.147}$$

We shall need the following lemma.

Lemma 1.21. *Let $u \in W^{1,p}(I)$. Then*

$$\eta\tilde{u} \in W^{1,p}(0,\infty) \tag{2.148}$$

$$(\eta\tilde{u})' = \eta'\tilde{u} + \eta\widetilde{u'} \tag{2.149}$$

Proof. Let $\varphi \in C_c^1((0,\infty))$; then

$$\begin{aligned}
\int_0^\infty \eta\tilde{u}\varphi' &= \int_0^1 \eta u\varphi' && (2.150)\\
&= \int_0^1 u\left((\eta\varphi)' - \eta'\varphi\right) && (2.151)\\
&= -\int_0^1 u'\eta\varphi - \int_0^1 u\eta'\varphi \text{ since } \eta\varphi \in C_c^1((0,1)) && (2.152)\\
&= -\int_0^\infty \left(\widetilde{u'}\eta + \tilde{u}\eta'\right)\varphi && (2.153)
\end{aligned}$$

i.e., (2.148)-(2.149) holds. □

Proof of Theorem 1.20, concluded. Given $u \in W^{1,p}(I)$, write

$$u = \eta u + (1-\eta)u \tag{2.154}$$

The function ηu is *first* extended to $(0,\infty)$ by $\eta\tilde{u}$ (in view of Lemma 1.21) and *then* to $\mathbb{R}$ by reflexion. In this way we obtain a function $v_1 \in W^{1,p}(\mathbb{R})$ that extends ηu and such that

$$\|v_1\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)} \tag{2.155}$$

$$\|v_1\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(I)} \tag{2.156}$$

(where C depends on $\|\eta'\|_{L^\infty}$).

Proof of (2.155). Consider two cases for p.

1. *Case* $p=\infty$. Since v_1 is an extension of $\eta\tilde{u}$ obtained by reflexion as $u^\star$ is u's extension in the first argument of this proof, we have

$$\begin{aligned}
\|v_1\|_{L^\infty(\mathbb{R})} &= \|\eta\tilde{u}\|_{L^\infty(0,\infty)} && (2.157)\\
&= \|\eta u\|_{L^\infty(I)} && (2.158)\\
&\leq \|u\|_{L^\infty(I)} \text{ since } 0\leq\eta\leq 1 && (2.159)
\end{aligned}$$

2. *Case* $1\leq p<\infty$. Similarly, we have

$$\begin{aligned}
\|v_1\|_{L^p(\mathbb{R})} &= 2^{\frac{1}{p}}\|\eta\tilde{u}\|_{L^p(0,\infty)} && (2.160)\\
&= 2^{\frac{1}{p}}\left(\int_0^{\frac{3}{4}}|\eta\tilde{u}|^p\right)^{\frac{1}{p}}, \text{ by (2.146)-(2.148)} && (2.161)\\
&\leq 2^{\frac{1}{p}}\left(\int_0^{\frac{3}{4}}|u|^p\right)^{\frac{1}{p}} && (2.162)\\
&\leq 2^{\frac{1}{p}}\left(\int_0^{1}|u|^p\right)^{\frac{1}{p}} && (2.163)\\
&\leq 2\|u\|_{L^p(I)} \text{ since } p\geq 1 && (2.164)
\end{aligned}$$

Proof of (2.156).[10] Similar to the first argument of this proof, we have

$$v_1{}'(x) = \begin{cases} (\eta\tilde{u})'(x) & \text{if } x>0\\ -(\eta\tilde{u})'(-x) & \text{if } x<0\end{cases} \tag{2.165}$$

We also consider two cases again.

1. *Case* $p=\infty$. As in the first argument, we have

$$\begin{aligned}
\|v_1{}'\|_{L^\infty(R)} &= \|(\eta\tilde{u})'\|_{L^\infty(0,\infty)} && (2.166)\\
&= \left\|\eta'\tilde{u}+\eta\widetilde{u'}\right\|_{L^\infty(0,\infty)} && (2.167)\\
&= \|\eta'u+\eta u'\|_{L^\infty(I)} && (2.168)\\
&\leq \max\{\|\eta'\|_{L^\infty},1\}\left(\|u\|_{L^\infty(I)}+\|u'\|_{L^\infty(I)}\right) && (2.169)\\
&= \max\{\|\eta'\|_{L^\infty},1\}\|u\|_{W^{1,\infty}(I)} && (2.170)
\end{aligned}$$

Thus

$$\begin{aligned}
\|v_1\|_{W^{1,\infty}(\mathbb{R})} &= \|v_1\|_{L^\infty(\mathbb{R})}+\|v_1{}'\|_{L^\infty(\mathbb{R})} && (2.171)\\
&\leq \|u\|_{L^\infty(I)}+\max\{\|\eta'\|_{L^\infty},1\}\|u\|_{W^{1,\infty}(I)} && (2.172)
\end{aligned}$$

[10] The main purpose of this proof is to find such a constant C explicitly.

$$\leq \|u\|_{W^{1,\infty}(I)} + \max\{\|\eta'\|_{L^\infty}, 1\} \|u\|_{W^{1,\infty}(I)} \tag{2.173}$$
$$= (1 + \max\{\|\eta'\|_{L^\infty}, 1\}) \|u\|_{W^{1,\infty}(I)} \tag{2.174}$$

Hence, we can choose $C = 1 + \max\{\|\eta'\|_{L^\infty}, 1\}$ in (2.156) when $p = \infty$.

2. *Case* $1 \leq p < \infty$. Similarly, we have

$$\|v_1{}'\|_{L^p(\mathbb{R})}^p = 2\int_0^\infty \left|\eta'\tilde{u} + \eta\widetilde{u'}\right|^p \tag{2.175}$$
$$= 2\int_0^1 |\eta' u + \eta u'|^p \tag{2.176}$$
$$\leq 2\int_0^1 (|\eta' u| + |\eta u'|)^p \tag{2.177}$$
$$\leq 2\max\left\{\|\eta'\|_{L^\infty}^p, 1\right\}\int_0^1 (|u| + |u'|)^p \tag{2.178}$$
$$\leq 2\max\left\{\|\eta'\|_{L^\infty}^p, 1\right\}\int_0^1 2^{p-1}\left(|u|^p + |u'|^p\right) \tag{2.179}$$
$$= 2^p \max\left\{\|\eta'\|_{L^\infty}^p, 1\right\}\|u\|_{W^{1,p}(I)}^p \tag{2.180}$$

where we have used the following familiar elementary inequality

$$(a+b)^p \leq 2^{p-1}(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}, \forall p \geq 1 \tag{2.181}$$

Thus

$$\|v_1\|_{W^{1,p}(\mathbb{R})} = \|v_1\|_{L^p(\mathbb{R})} + \|v_1{}'\|_{L^p(\mathbb{R})} \tag{2.182}$$
$$\leq 2\|u\|_{L^p(I)} + 2\max\{\|\eta'\|_{L^\infty}, 1\}\|u\|_{W^{1,p}(I)} \tag{2.183}$$
$$\leq 2\|u\|_{W^{1,p}(I)} + 2\max\{\|\eta'\|_{L^\infty}, 1\}\|u\|_{W^{1,p}(I)} \tag{2.184}$$
$$= 2(1 + \max\{\|\eta'\|_{L^\infty}, 1\})\|u\|_{W^{1,p}(I)} \tag{2.185}$$

Hence, we can choose $C = 2(1 + \max\{\|\eta'\|_{L^\infty}, 1\})$ in (2.156) when $1 \leq p < \infty$.

Proceed in the same way with $(1-\eta)u$, that is, *first* extend $(1-\eta)u$ to $(-\infty, 1)$ by 0 on $(-\infty, 0)$ and *then* extend to $\mathbb{R}$ by reflection (this time about the point 1, not 0). In this way we obtain a function $v_2 \in W^{1,p}(\mathbb{R})$ that extends $(1-\eta)u$ and satisfies

$$\|v_2\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)} \tag{2.186}$$
$$\|v_2\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(I)} \tag{2.187}$$

Proof of (2.186)-(2.187). Similar to (2.148), given a function f on $(0,1)$ set

$$\hat{f}(x) = \begin{cases} f(x) \text{ if } 0 < x < 1 \\ 0 \text{ if } x < 0 \end{cases} \tag{2.188}$$

We shall need the following lemma, which is very similar to Lemma 1.21.

Lemma 1.22. *Let* $u \in W^{1,p}(I)$. *Then*

$$(1-\eta)\hat{u} \in W^{1,p}(-\infty, 1) \tag{2.189}$$

$$((1-\eta)\hat{u})' = -\eta'\hat{u} + (1-\eta)\widehat{u'} \tag{2.190}$$

Proof. Let $\varphi \in C_c^1((-\infty,1))$; then

$$\int_{-\infty}^{1} (1-\eta)\hat{u}\varphi' = \int_{0}^{1} (1-\eta) u\varphi' \tag{2.191}$$

$$= \int_{0}^{1} u\left(((1-\eta)\varphi)' - (1-\eta)'\varphi\right) \tag{2.192}$$

$$= -\int_{0}^{1} u'(1-\eta)\varphi + \int_{0}^{1} u\eta'\varphi \tag{2.193}$$

$$= -\int_{-\infty}^{1} \left(\widehat{u'}(1-\eta) - \hat{u}\eta'\right)\varphi \tag{2.194}$$

Therefore, (2.189)-(2.190) holds. □

Return the the proof of Theorem 1.20, the function $(1-\eta)u$ is *first* extended to $(-\infty,1)$ by $(1-\eta)\hat{u}$ (in view of Lemma 1.22) and *then* to $\mathbb{R}$ by reflection about the point 1, i.e.,

$$v_2(x) = \begin{cases} (1-\eta(x))\hat{u}(x) & \text{if } x<1 \\ (1-\eta(2-x))\hat{u}(2-x) & \text{if } x>1 \end{cases} \tag{2.195}$$

We now prove (2.186)-(2.187) as the previous case.

Proof of (2.186). We consider two cases as before.

1. *Case* $p=\infty$. We have

$$\|v_2\|_{L^\infty(\mathbb{R})} = \|(1-\eta)\hat{u}\|_{L^\infty(-\infty,1)} \tag{2.196}$$

$$= \|(1-\eta)u\|_{L^\infty(I)} \tag{2.197}$$

$$\leq \|u\|_{L^\infty(I)} \tag{2.198}$$

2. *Case* $1 \leq p < \infty$. Similar to the proof of (2.155), we have

$$\|v_2\|_{L^p(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |v_2|^p\right)^{\frac{1}{p}} \tag{2.199}$$

$$= \begin{pmatrix} \int_{-\infty}^{1} |(1-\eta(x))\hat{u}(x)|^p dx \\ +\int_{1}^{\infty} |(1-\eta(2-x))\hat{u}(2-x)|^p dx \end{pmatrix}^{\frac{1}{p}} \tag{2.200}$$

$$= \begin{pmatrix} \int_{0}^{1} |(1-\eta(x))\hat{u}(x)|^p dx \\ +\int_{-\infty}^{1} |(1-\eta(t))\hat{u}(t)|^p dt \end{pmatrix}^{\frac{1}{p}}, \text{ put } t=2-x \tag{2.201}$$

$$= 2^{\frac{1}{p}} \|(1-\eta)u\|_{L^p(I)} \tag{2.202}$$

$$\leq 2\|u\|_{L^p(I)} \text{ since } p \geq 1 \tag{2.203}$$

Proof of (2.187). We again consider two cases.

1. *Case* $p = \infty$. As in the first argument, with a light modification, we have

$$\begin{aligned}
\|{v_2}'\|_{L^\infty(\mathbb{R})} &= \left\|\left((1-\eta)\,\hat{u}\right)'\right\|_{L^\infty(-\infty,1)} && (2.204)\\
&= \left\|-\eta'\hat{u} + (1-\eta)\,\widehat{u'}\right\|_{L^\infty(-\infty,1)}, \text{ by (2.190)} && (2.205)\\
&= \|-\eta' u + (1-\eta)\,u'\|_{L^\infty(I)} && (2.206)\\
&\le \max\{\|\eta'\|_{L^\infty}, 1\}\left(\|u\|_{L^\infty(I)} + \|u'\|_{L^\infty(I)}\right) && (2.207)\\
&= \max\{\|\eta'\|_{L^\infty}, 1\}\,\|u\|_{W^{1,\infty}(I)} && (2.208)
\end{aligned}$$

Thus

$$\begin{aligned}
\|v_2\|_{W^{1,\infty}(\mathbb{R})} &= \|v_2\|_{L^\infty(\mathbb{R})} + \|{v_2}'\|_{L^\infty(\mathbb{R})} && (2.209)\\
&\le \|u\|_{L^\infty(I)} + \max\{\|\eta'\|_{L^\infty}, 1\}\,\|u\|_{W^{1,\infty}(I)} && (2.210)\\
&\le (1 + \max\{\|\eta'\|_{L^\infty}, 1\})\,\|u\|_{W^{1,\infty}(I)} && (2.211)
\end{aligned}$$

Hence, we can again choose $C = 1 + \max\{\|\eta'\|_{L^\infty}, 1\}$ in (2.187) when $p = \infty$.

2. *Case* $\le p < \infty$. We have

$$\begin{aligned}
\|{v_2}'\|^p_{L^p(\mathbb{R})} &= 2\int_{-\infty}^{1} \left|-\eta'\hat{u} + (1-\eta)\,\widehat{u'}\right|^p && (2.212)\\
&= 2\int_0^1 |-\eta' u + (1-\eta)\,u'|^p && (2.213)\\
&\le 2\int_0^1 (|\eta' u| + |(1-\eta)\,u'|)^p && (2.214)\\
&\le 2\max\left\{\|\eta'\|^p_{L^\infty}, 1\right\}\int_0^1 (|u| + |u'|)^p && (2.215)\\
&\le 2^p\max\left\{\|\eta'\|^p_{L^\infty}, 1\right\}\int_0^1 \left(|u|^p + |u'|^p\right) && (2.216)\\
&= 2^p\max\left\{\|\eta'\|^p_{L^\infty}, 1\right\}\|u\|^p_{W^{1,p}(I)} && (2.217)
\end{aligned}$$

where we again use (2.181). Hence,

$$\begin{aligned}
\|v_2\|_{W^{1,p}(\mathbb{R})} &= \|v_2\|_{L^p(\mathbb{R})} + \|{v_2}'\|_{L^p(\mathbb{R})} && (2.218)\\
&\le 2\|u\|_{L^p(I)} + 2\max\{\|\eta'\|_{L^\infty}, 1\}\,\|u\|_{W^{1,p}(I)} && (2.219)\\
&\le 2\|u\|_{W^{1,p}(I)} + 2\max\{\|\eta'\|_{L^\infty}, 1\}\,\|u\|_{W^{1,p}(I)} && (2.220)\\
&= 2\,(1 + \max\{\|\eta'\|_{L^\infty}, 1\})\,\|u\|_{W^{1,p}(I)} && (2.221)
\end{aligned}$$

Hence, we can choose $C = 2\,(1 + \max\{\|\eta'\|_{L^\infty}, 1\})$ in (2.187) when $1 \le p < \infty$.

Return to the proof of Theorem 1.20, then $PU = v_1 + v_2$ satisfies the condition of the theorem. More explicitly, we have

$$Pu|_I = u, \quad \forall u \in W^{1,p}(I), 1 \le p \le \infty \qquad (2.222)$$

and

1. For $p = \infty$,

$$\|Pu\|_{L^\infty(\mathbb{R})} \le 2\|u\|_{L^\infty(I)} \tag{2.223}$$
$$\|Pu\|_{W^{1,\infty}(\mathbb{R})} \le 2\left(1 + \max\left\{\|\eta'\|_{L^\infty}, 1\right\}\right)\|u\|_{W^{1,\infty}(I)} \tag{2.224}$$

   for all $u \in W^{1,\infty}(I)$.

2. For $1 \le p < \infty$,

$$\|Pu\|_{L^p(\mathbb{R})} \le 4\|u\|_{L^p(I)} \tag{2.225}$$
$$\|Pu\|_{W^{1,p}(\mathbb{R})} \le 4\left(1 + \max\left\{\|\eta'\|_{L^\infty}, 1\right\}\right)\|u\|_{W^{1,p}(I)} \tag{2.226}$$

   for all $u \in W^{1,p}(I)$.

With some suitable choices of η, (2.223)-(2.226) will be more easy to use. □

Certain properties of C^1 functions remain true for $W^{1,p}$ functions (see for example Corollaries 8.10 and 8.11, [1]). It is convenient to establish these properties by a *density* argument based on the following result.

Theorem 1.23 (density). *Le $u \in W^{1,p}(I)$ with $1 \le p < \infty$. Then there exists a sequence (u_n) in $C_c^\infty(\mathbb{R})$ such that $u_n|_I \to u$ in $W^{1,p}(I)$.*

Remark 1.24. In general, there is no sequence (u_n) in $C_c^\infty(I)$ such that $u_n \to u$ in $W^{1,p}(I)$ (See Section 1.3). This is contrast to L^p spaces: recall that for every function $u \in L^p(I)$ there s a sequence (u_n) in $C_c^\infty(I)$ such that $u_n \to u$ in $L^p(I)$ (see Corollary 4.23, [1]).

Proof. We can always suppose $I = \mathbb{R}$; otherwise, extend u to a function in $W^{1,p}(\mathbb{R})$ by Theorem 1.20. We use the *basic techniques of convolution* (which makes functions C^∞) and *cut-off* (which makes their support compact).

(a) Convolution.
We shall need the following lemma.

Lemma 1.25. *Let $\rho \in L^1(\mathbb{R})$ and $v \in W^{1,p}(\mathbb{R})$ with $1 \le p \le \infty$. Then $\rho \star v \in W^{1,p}(\mathbb{R})$ and $(p \star v)' = \rho \star v'$.*

Proof. First, suppose that ρ has compact support. We already know (Theorem 4.15, [1]) that $\rho \star v \in L^p(\mathbb{R})$. Let $\varphi \in C_c^1(\mathbb{R})$; from Propositions 4.16 and 4.20, [1], we have

$$\int (\rho \star v)\varphi' = \int v\left(\breve{\rho} \star \varphi'\right) \tag{2.227}$$
$$= \int v\left(\breve{\rho} \star \varphi\right)' \tag{2.228}$$
$$= -\int v'\left(\breve{\rho} \star \varphi\right) \tag{2.229}$$
$$= -\int (\rho \star v')\varphi \tag{2.230}$$

from which it follows that

$$\rho \star v \in W^{1,p} \tag{2.231}$$
$$(\rho \star v)' = \rho \star v' \tag{2.232}$$

If ρ does not have compact support introduce a sequence (ρ_n) from $C_c(\mathbb{R})$ such that $\rho_n \to \rho$ in $L^1(\mathbb{R})$ (see Corollary 4.23). From the above, we get

$$\rho_n \star v \in W^{1,p} \tag{2.233}$$
$$(\rho_n \star v)' = \rho_n \star v' \tag{2.234}$$

But $\rho_n \star v \to \rho \star v$ in $L^p(\mathbb{R})$ and $\rho_n \star v' \to \rho \star v'$ in $L^p(\mathbb{R})$ (by Theorem 4.15, [1]). We conclude with the help of Remark 1.9 that

$$\rho \star v \in W^{1,p} \tag{2.235}$$
$$(\rho \star v)' = \rho \star v' \tag{2.236}$$

(b) Cut-off.
Fix a function $\zeta \in C_c^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ and

$$\zeta(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| \ge 2 \end{cases} \tag{2.237}$$

Define the sequence

$$\zeta_n(x) = \zeta\left(\frac{x}{n}\right) \text{ for } n = 1, 2, \dots \tag{2.238}$$

It follows easily from the dominated convergence theorem that if a function f belongs to $L^p(\mathbb{R})$ with $1 \le p < \infty$, then $\zeta_n f \to f$ in $L^p(\mathbb{R})$.

(c) Conclusion.
Choose a sequence of mollifiers (ρ_n). We claim that the sequence $u_n = \zeta_n(\rho_n \star u)$ converges to u in $W^{1,p}(\mathbb{R})$. First, we have $\|u_n - u\|_p \to 0$. In fact, write

$$u_n - u = \zeta_n((\rho_n \star u) - u) + (\zeta_n u - u) \tag{2.239}$$

and thus

$$\|u_n - u\|_p \le \|\zeta_n((\rho_n \star u) - u)\|_p + \|\zeta_n u - u\|_p \tag{2.240}$$
$$\le \|\zeta_n\|_\infty \|(\rho_n \star u) - u\|_p + \|\zeta_n u - u\|_p \tag{2.241}$$
$$= \|(\rho_n \star u) - u\|_p + \|\zeta_n u - u\|_p \to 0 \tag{2.242}$$

where we have use the following inequality

$$\|fg\|_p \le \|f\|_\infty \|g\|_p, \ \ \forall f \in C_c^\infty(\mathbb{R}), \ \ \forall g \in L^p(\mathbb{R}) \tag{2.243}$$

Next, by Lemma 1.25, we have

$$u_n{}' = \zeta_n{}'(\rho_n \star u) + \zeta_n(\rho_n \star u') \tag{2.244}$$

Therefore

$$\|u_n{}' - u'\|_p = \left\|\zeta_n{}'(\rho_n \star u) + \zeta_n(\rho_n \star u') - u'\right\|_p \tag{2.245}$$
$$\leq \left\|\zeta_n{}'(\rho_n \star u)\right\|_p + \|\zeta_n(\rho_n \star u') - u'\|_p \tag{2.246}$$
$$\leq \left\|\zeta_n{}'\right\|_\infty \|\rho_n \star u\|_p + \|\zeta_n(\rho_n \star u' - u')\|_p + \|\zeta_n u' - u'\|_p \tag{2.247}$$
$$\leq \frac{C}{n}\|\rho_n\|_1 \|u\|_p + \|\zeta_n\|_\infty \|\rho_n \star u' - u'\|_p + \|\zeta_n u' - u'\|_p \tag{2.248}$$
$$= \frac{C}{n}\|u\|_p + \|\rho_n \star u' - u'\|_p + \|\zeta_n u' - u'\|_p \to 0 \tag{2.249}$$

where $C = \|\zeta'\|_\infty$.

Combining (2.240)-(2.242) and (2.245)-(2.249) yields

$$\|u_n - u\|_{W^{1,p}(\mathbb{R})} = \|u_n - u\|_{L^p(\mathbb{R})} + \|u_n{}' - u'\|_{L^p(R)} \to 0 \tag{2.250}$$

i.e., (u_n) is a desired sequence. □

The next result is an important prototype of a *Sobolev inequality* (also called a *Sobolev embedding*).

Theorem 1.26. *There exists a constant C (depending only on $|I| \leq \infty$) such that*

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I), \quad \forall 1 \leq p \leq \infty \tag{2.251}$$

In other words, $W^{1,p}(I) \subset L^\infty(I)$ with continuous injection for all $1 \leq p \leq \infty$.

*Further, if I is **bounded** then*

1. *The **injection** $W^{1,p}(I) \subset C(\bar{I})$ is **compact** for all $1 < p \leq \infty$.*
2. *The **injection** $W^{1,1}(I) \subset L^q(I)$ is **compact** for all $1 \leq q < \infty$.*

Proof. We start by proving (2.251) for $I = \mathbb{R}$; the general case then follows from this by the extension theorem (Theorem 1.20).

The case $p = \infty$ is obvious since

$$\|u\|_\infty \leq \|u\|_\infty + \|u'\|_\infty \tag{2.252}$$
$$= \|u\|_{W^{1,\infty}}, \quad \forall u \in W^{1,\infty}(R) \tag{2.253}$$

It now suffices to prove (2.251) for $1 \leq p < \infty$.

Let $v \in C_c^1(\mathbb{R})$; if $1 \leq p < \infty$ set $G(s) = |s|^{p-1}s$. The function $w = G(v)$ belongs to $C_c^1(\mathbb{R})$ and[11]

$$w' = G'(v)\, v' \tag{2.254}$$
$$= p|v|^{p-1}v' \tag{2.255}$$

Thus, for $x \in \mathbb{R}$, we have

$$G(v(x)) = \int_{-\infty}^{x} p|v(t)|^{p-1}v'(t)\, dt \tag{2.256}$$

[11] (2.255) can be easily verified by considering two cases $v \geq 0$ and $v < 0$.

and by Hölder's inequality

$$\begin{aligned}
|v(x)|^p &= |G(v(x))| && (2.257)\\
&= \left|\int_{-\infty}^{x} p|v(t)|^{p-1} v'(t)\,dt\right| && (2.258)\\
&\le p\left\|v^{p-1}v'\right\|_1 && (2.259)\\
&\le p\left\|v^{p-1}\right\|_{\frac{p}{p-1}} \|v'\|_p \text{ by Hölder's} && (2.260)\\
&= p\left(\int_{\mathbb{R}} \left|v^{p-1}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|v'\|_p && (2.261)\\
&= p\,\|v\|_p^{p-1}\,\|v'\|_p && (2.262)
\end{aligned}$$

from which we conclude that

$$\|v\|_\infty \le C\|v\|_{W^{1,p}},\quad \forall v \in C_c^1(\mathbb{R}) \tag{2.263}$$

where C is a universal constant (independent of p).[12]

Proof of (2.263). We consider three cases.

1. *Case* $p = 1$. In this case, (2.257)-(2.262) becomes

$$|v(x)| \le \|v'\|_1 \tag{2.264}$$

Thus

$$\begin{aligned}
\|v\|_\infty &\le \|v'\|_1 && (2.265)\\
&\le \|v\|_1 + \|v'\|_1 && (2.266)\\
&= \|v\|_{W^{1,p}},\quad \forall v \in C_c^1(\mathbb{R}) && (2.267)
\end{aligned}$$

Therefore, we can choose $C = 1$ in (2.263) when $p = 1$.

2. *Case* $1 < p < \infty$. To this end, we need the following inequality.

$$1 + x \ge p\left(\frac{x}{(p-1)^{p-1}}\right)^{\frac{1}{p}},\quad \forall p > 1,\ \ \forall x > 0 \tag{2.268}$$

To prove (2.268), we rewrite it as follows.

$$\frac{(p-1)^{p-1}(1+x)^p}{p^p x} \ge 1,\quad \forall p > 1,\ \ \forall x > 0 \tag{2.269}$$

Taking natural logarithm of both sides of (2.269), it suffices to prove

$$(p-1)\ln(p-1) + p\ln(1+x) \ge p\ln p + \ln x,\quad \forall p > 1,\ \ \forall x > 0 \tag{2.270}$$

Surveying the following function

$$f(p) = (p-1)\ln(p-1) + p\ln(1+x) - p\ln p - \ln x, \tag{2.271}$$

[12] Noting that $p^{\frac{1}{p}} \le e^{\frac{1}{e}},\ \ \forall p \ge 1$.

for all $p > 1$ and for all $x > 0$, yields

$$f'(p) = \ln(p-1) + \ln(1+x) - \ln p \tag{2.272}$$

$$f'(p) = 0 \Leftrightarrow p = 1 + \frac{1}{x} \tag{2.273}$$

$$f''(p) = \frac{1}{p(p-1)} > 0 \tag{2.274}$$

Hence,

$$\min_{p>1} f(p) \tag{2.275}$$

$$= f\left(1 + \frac{1}{x}\right) \tag{2.276}$$

$$= \frac{1}{x}\ln\frac{1}{x} - \left(1+\frac{1}{x}\right)\ln\left(1+\frac{1}{x}\right) - \ln x + \left(1+\frac{1}{x}\right)\ln(x+1) \tag{2.277}$$

$$= -\frac{1}{x}\ln x + \left(1+\frac{1}{x}\right)\ln\left(\frac{x}{x+1}\right) - \ln x + \left(1+\frac{1}{x}\right)\ln(x+1) \tag{2.278}$$

$$= -\left(1+\frac{1}{x}\right)\ln x + \left(1+\frac{1}{x}\right)\left(\ln\left(\frac{x}{x+1}\right) + \ln(x+1)\right) \tag{2.279}$$

$$= -\left(1+\frac{1}{x}\right)\ln x + \left(1+\frac{1}{x}\right)\ln x \tag{2.280}$$

$$= 0 \tag{2.281}$$

Thus, (2.268) holds.

Return to our proof of (2.263), we can suppose that $\|v\|_p > 0, \|v'\|_p > 0$ since there is nothing to prove on the other cases. Substituting $x = \frac{\|v'\|_p}{\|v\|_p} > 0$ into (2.268) yields

$$\|v\|_p + \|v'\|_p \geq p\left(\frac{\|v\|_p^{p-1}\|v'\|_p}{(p-1)^{p-1}}\right)^{\frac{1}{p}} \tag{2.282}$$

$$\geq p\left(\frac{|v(x)|^p}{p(p-1)^{p-1}}\right)^{\frac{1}{p}} \tag{2.283}$$

$$= \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} |v(x)| \tag{2.284}$$

$$\geq |v(x)| \tag{2.285}$$

i.e.,

$$\|v\|_\infty \leq \|v\|_{W^{1,p}}, \quad \forall v \in C_c^1(\mathbb{R}), \quad 1 < p < \infty \tag{2.286}$$

Therefore, we can also choose $C = 1$ in (2.263) when $1 < p < \infty$.

Argue now by density. Let $u \in W^{1,p}(\mathbb{R})$; there exists a sequence $(u_n) \subset C_c^1(\mathbb{R})$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R})$ (by Theorem 1.23). Applying (2.263), we see that (u_n) is a Cauchy sequence in $L^\infty(\mathbb{R})$. Indeed,

$$\|u_m - u_n\|_\infty \leq C\|u_m - u_n\|_{W^{1,p}}, \text{ by } (2.263) \tag{2.287}$$

$$\leq C\left(\|u_m - u\|_{W^{1,p}} + \|u - u_n\|_{W^{1,p}}\right) \to 0 \tag{2.288}$$

as $m, n \to \infty$.

Thus $u_n \to u$ in $L^\infty(\mathbb{R})$ and we obtain (2.251).

Proof of (1). Let $\mathcal{H}$ be the unit ball in $W^{1,p}(I)$ with $1 < p \leq \infty$. For $u \in \mathcal{H}$ we have

$$|u(x) - u(y)| = \left|\int_x^y u'(t)\,dt\right| \tag{2.289}$$

$$\leq \|u'\|_p \|1\|_{p'}, \text{ by Holder's inequality} \tag{2.290}$$

$$= \|u'\|_p |x-y|^{\frac{1}{p'}}, \text{ by } \|u\|_{W^{1,p}(I)} \leq 1 \tag{2.291}$$

$$\leq |x-y|^{\frac{1}{p'}} \tag{2.292}$$

It follows then from the Ascoli-Arzelà theorem (Theorem 4.25, [1]) that $\mathcal{H}$ has a compact closure in $C(\bar{I})$.

Indeed, applying Ascoli-Arzelà theorem to the compact metric space $(\bar{I}, |\cdot|)$, where $|\cdot|$ is usual Euclidean distance in the real number line, and the bounded subset $H = B_{W^{1,p}(I)}(0,1)$ of $C(\bar{I})$. It is deduced from (2.289)-(2.292) that $\mathcal{H}$ is uniformly equicontinuous. More explicitly, given $\varepsilon > 0$, we have

$$|x-y| < \varepsilon^{p'} \Rightarrow |u(x) - u(y)| < \varepsilon, \ \ \forall u \in \mathcal{H} \tag{2.293}$$

Then, by Ascoli-Arzelà theorem, the closure of $\mathcal{H}$ in $C(\bar{I})$ is compact. By definition of compact operator (Section 6.1, [1]), (1) holds.

Proof of (2). Let $\mathcal{H}$ be the unit ball in $W^{1,1}(I)$. Let P be the extension operator of Theorem 1.20 and set $\mathcal{F} = P(\mathcal{H})$, so that $\mathcal{H} = \mathcal{F}_{|I}$. We prove that $\mathcal{H}$ has a compact closure in $L^q(I)$ (for all $1 \leq q < \infty$) by applying Theorem 4.26 (Kolmogorov - M. Riesz - Fréchet), [1]. Clearly, $\mathcal{F}$ is bounded in $W^{1,1}(\mathbb{R})$[13]; therefore $\mathcal{F}$ is also bounded in $L^q(\mathbb{R})$, since it is bounded both in $L^1(\mathbb{R})$ and in $L^\infty(\mathbb{R})$. We now check the following condition

$$\lim_{h \to 0} \|\tau_h f - f\|_q = 0 \text{ uniformly in } f \in \mathcal{F} \tag{2.294}$$

By Proposition 1.19[14] we have, for every $f \in \mathcal{F}$,

$$\|\tau_h f - f\|_{L^1(\mathbb{R})} \leq |h|\,\|f'\|_{L^1(\mathbb{R})} \tag{2.295}$$

$$\leq C\,|h| \tag{2.296}$$

since $\mathcal{F}$ is a bounded subset of $W^{1,1}(\mathbb{R})$. Thus

$$\|\tau_h f - f\|_{L^q(\mathbb{R})}^q = \int_{\mathbb{R}} |\tau_h f - f|^q \tag{2.297}$$

$$= \int_{\mathbb{R}} |f(x+h) - f(x)|^{q-1}\,|\tau_h f - f|\,dx \tag{2.298}$$

$$\leq \left(2\|f\|_{L^\infty(\mathbb{R})}\right)^{q-1} \|\tau_h f - f\|_{L^1(\mathbb{R})} \tag{2.299}$$

[13] See the proof of Theorem 1.20.

[14] Note that the implication (1) $\Rightarrow$ (2) of Theorem 1.19 is also valid when $p = 1$.

$$\leq C\,|h| \tag{2.300}$$

and consequently

$$\|\tau_h f - f\|_{L^q(\mathbb{R})} \leq C_0 |h|^{\frac{1}{q}} \tag{2.301}$$

where C is independent of f. The desired conclusion follows since $q \neq \infty$.

Indeed, given $\varepsilon > 0$, it is deduced from (2.301) that

$$\|\tau_h f - f\|_{L^q(\mathbb{R})} < \varepsilon,\ \ \forall f \in F,\ \ \forall h \in \mathbb{R}, |h| < \frac{\varepsilon^q}{C_0^q} \tag{2.302}$$

We then deduce, by applying Kolmogorov- M. Riesz-Fréchet theorem, that the closure of $\mathcal{F}_{|I}$ in $L^q(I)$ is compact for the bounded measurable set I. Hence, by definition of compact operator again, (2) holds. □

Remark 1.27. The origin of the inequality (2.268) is given by the following idea. Look at the case $p \in \mathbb{Z}_+ \setminus \{1\}$, we can use Cauchy inequality for p nonnegative numbers

$$\begin{aligned}
\|v\|_{W^{1,p}} &= \|v\|_p + \|v'\|_p & (2.303)\\
&= \underbrace{\frac{\|v\|_p}{p-1} + \cdots + \frac{\|v\|_p}{p-1}}_{(p-1)'s} + \|v'\|_p & (2.304)\\
&\geq p\left(\frac{\|v\|_p^{p-1}\,\|v'\|_p}{(p-1)^{p-1}}\right)^{\frac{1}{p}} \text{ by Cauchy's inequality} & (2.305)\\
&\geq p\left(\frac{|v(x)|^p}{p(p-1)^{p-1}}\right)^{\frac{1}{p}} & (2.306)\\
&= \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} |v(x)| & (2.307)\\
&\geq |v(x)| & (2.308)
\end{aligned}$$

Roughly speaking, (2.268) is an extended version of this estimation for arbitrary real number $p > 1$. But proving (2.268) requires a little more calculus skills than just applying Cauchy inequality here, of course.

Remark 1.28. The injection $W^{1,1}(I) \subset C(\bar{I})$ is continuous but it is *never compact*, even if I is a bounded interval. Nevertheless, if (u_n) is a bounded sequence in $W^{1,1}(I)$ (with I bounded or unbounded) there exists a subsequence (u_{n_k}) such that $u_{n_k}(x)$ converges for *all* $x \in I$ (this is *Helly's selection theorem*). When I is *unbounded* and $1 < p \leq \infty$, we know that the injection $W^{1,p}(I) \subset L^\infty(I)$ is continuous; this injection is *never compact*. However, if (u_n) is bounded in $W^{1,p}(I)$ with $1 < p \leq \infty$ there exist a subsequence (u_{n_k}) and some $u \in W^{1,p}(I)$ such that $u_{n_k} \to u$ in $L^\infty(I)$ for every *bounded* subset J of I.

Remark 1.29. Let I be a bounded interval, let $1 \leq p \leq \infty$, and let $1 \leq q \leq \infty$. From Theorem 1.10 and (2.251) it can be shown easily that the norm

$$|||u||| = \|u'\|_p + \|u\|_q \tag{2.309}$$

is equivalent to the norm of $W^{1,p}(I)$.

PROOF OF REMARK 1.29. Consider the following cases.

1. *Case $p = q$ (including the case $p = \infty, q = \infty$).* The conclusion is obvious since $|||\cdot||| \equiv \|\cdot\|_{W^{1,p}(I)}$.

2. *Case $p \neq q$.*

   (a) *Case $1 \leq p < q = \infty$.* We have

$$\begin{aligned} |||u||| &= \|u'\|_p + \|u\|_\infty && (2.310)\\ &\leq \|u'\|_p + C\|u\|_{W^{1,p}(I)} && (2.311)\\ &\leq (1+C)\,\|u\|_{W^{1,p}(I)} && (2.312) \end{aligned}$$

   On the other hand,

$$\begin{aligned} \|u\|_{W^{1,p}(I)} &= \|u'\|_p + \|u\|_p && (2.313)\\ &\leq \|u'\|_p + |I|^{\frac{1}{p}}\|u\|_\infty && (2.314)\\ &\leq \max\left\{1, |I|^{\frac{1}{p}}\right\} |||u||| && (2.315) \end{aligned}$$

   (b) *Case $1 \leq p < q < \infty$.* We have

$$\begin{aligned} |||u||| &= \|u'\|_p + \|u\|_q && (2.316)\\ &\leq \|u'\|_p + \|u\|_{L^\infty(I)}\|1\|_q && (2.317)\\ &= \|u'\|_p + \|u\|_{L^\infty(I)}|I|^{\frac{1}{q}} && (2.318)\\ &\leq \|u'\|_p + C|I|^{\frac{1}{q}}\|u\|_{W^{1,p}(I)}, \text{ by (2.251)} && (2.319)\\ &\leq \left(1 + C|I|^{\frac{1}{q}}\right)\|u\|_{W^{1,p}(I)} && (2.320) \end{aligned}$$

   On the other hand, we have

$$\begin{aligned} \|u\|_{W^{1,p}(I)} &= \|u\|_p + \|u'\|_p && (2.321)\\ &= \|u^p\|_1^{\frac{1}{p}} + \|u'\|_p && (2.322)\\ &\leq \left(\|u^p\|_{\frac{q}{p}}\|1\|_{\frac{q}{q-p}}\right)^{\frac{1}{p}} + \|u'\|_p && (2.323)\\ &= \|u\|_q|I|^{\frac{1}{p}-\frac{1}{q}} + \|u'\|_p && (2.324)\\ &\leq \max\left\{|I|^{\frac{1}{p}-\frac{1}{q}}, 1\right\}\left(\|u\|_q + \|u'\|_p\right) && (2.325)\\ &= \max\left\{|I|^{\frac{1}{p}-\frac{1}{q}}, 1\right\} |||u||| && (2.326) \end{aligned}$$

   Reader should notice that the condition $1 \leq p < q < \infty$ guarantees $1 < \frac{q}{q-p} < \infty$.

   (c) *Case $1 \leq q < p = \infty$.*

(d) *Case* $1 \le q < p < \infty$.[15] We have

$$\|u\|_p \le \|u\|_\infty^{\frac{p-q}{p}} \|u\|_q^{\frac{q}{p}} \tag{2.327}$$

$$= k^{\frac{p-q}{p}} \left(\frac{\|u\|_\infty}{k}\right)^{\frac{p-q}{p}} \|u\|_q^{\frac{q}{p}}, \quad k > 0 \text{ is chosen later} \tag{2.328}$$

$$\le \frac{k^{\frac{p-q}{p}}}{p} \left(\frac{p-q}{k}\|u\|_\infty + q\|u\|_q\right) \tag{2.329}$$

$$\le \frac{k^{\frac{p-q}{p}}}{p} \left(\frac{C(p-q)}{k}\left(\|u'\|_p + \|u\|_p\right) + q\|u\|_q\right) \tag{2.330}$$

We now choose $k^{-\frac{q}{p}} \frac{C(p-q)}{p} < 1$, for instance, let k satisfy

$$k^{-\frac{q}{p}} \frac{C(p-q)}{p} = \frac{1}{2} \tag{2.331}$$

then (2.327)-(2.330) works.

Remark 1.30. Let I be an *unbounded* interval. If $u \in W^{1,p}(I)$, then $u \in L^q(I)$ for all $q \in [p, \infty]$, since

$$\int_I |u|^q \le \|u\|_\infty^{q-p} \|u\|_p^p \tag{2.332}$$

But in general $u \notin L^q(I)$ for $q \in [1, p)$. (Why?)

Corollary 1.31. *Suppose that I is an unbounded interval and $u \in W^{1,p}(I)$ with $1 \le p < \infty$. Then*

$$\lim_{x \in I, |x| \to \infty} u(x) = 0 \tag{2.333}$$

Proof. From Theorem 1.23 there exists a sequence (u_n) in $C_c^1(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I)$. It follows from (2.251) that

$$\|u_n - u\|_{L^\infty(I)} \le C\|u_n - u\|_{W^{1,p}(I)} \to 0 \tag{2.334}$$

as $n \to \infty$. We deduce (2.333) from this. Indeed, given $\varepsilon > 0$ we choose n large enough that $\|u_n - u\|_{L^\infty(I)} < \varepsilon$. For $|x|$ large enough, $u_n(x) = 0$ (since $u_n \in C_c^1(\mathbb{R})$) and thus $|u(x)| < \varepsilon$.

Corollary 1.32 (differentiation of a product).[16] *Let $u, v \in W^{1,p}(I)$ with $1 \le p \le \infty$. Then*

$$uv \in W^{1,p}(I) \tag{2.335}$$

and

$$(uv)' = u'v + uv' \tag{2.336}$$

[15] Hoang Cong Duc's proof.

[16] Note the *contrast* of this result with the properties of L^p functions: in general, if $u, v \in L^p$, the product uv does *not* belong to L^p. (Why?) We say that $W^{1,p}(I)$ is a *Banach algebra*.

Furthermore, the formula for integration by parts holds

$$\int_y^x u'v = u(x)\,v(x) - u(y)\,v(y) - \int_y^x uv', \;\; \forall x, y \in \bar{I} \tag{2.337}$$

PROOF. First call that $u \in L^\infty$ (by Theorem 1.26) and thus $uv \in L^p$.[17] To show that $(uv)' \in L^p$ let us begin with the case $1 \le p < \infty$. Let (u_n) and (v_n) be sequences in $C_c^1(\mathbb{R})$ such that $u_{n|I} \to u$ and $v_{n|I} \to v$ in $W^{1,p}(I)$. Thus $u_{n|I} \to u$ and $v_{n|I} \to v$ in $L^\infty(I)$ (again by Theorem 1.26). It follows that $u_n v_{n|I} \to uv$ in $L^\infty(I)$ and also in L^p.

Indeed, to prove that $u_n v_{n|I} \to uv$ in $L^\infty(I)$, using Minkowski's in equality yields

$$\left\|u_n v_{n|I} - uv\right\|_\infty \tag{2.338}$$
$$= \left\|u\left(v_{n|I} - v\right) + v_{n|I}\left(u_{n|I} - u\right)\right\|_\infty \tag{2.339}$$
$$\le \|u\|_\infty \left\|v_{n|I} - v\right\|_\infty + \left\|v_{n|I}\right\|_\infty \left\|u_{n|I} - u\right\|_\infty \tag{2.340}$$
$$\le C\left(\|u\|_\infty \left\|v_{n|I} - v\right\|_{W^{1,p}(I)} + \left\|v_{n|I}\right\|_\infty \left\|u_{n|I} - u\right\|_{W^{1,p}(I)}\right) \to 0 \tag{2.341}$$

as $n \to \infty$ since

$$\left\|v_{n|I}\right\|_\infty \le C\left\|v_{n|I}\right\|_{W^{1,p}(I)} \to C\|v\|_{W^{1,p}(I)} \text{ as } n \to \infty \tag{2.342}$$

which is bounded as $n \to \infty$.

Similarly, to prove $u_n v_{n|I} \to uv$ in $L^p(I)$, using Minkowski's in equality again yields

$$\left\|u_n v_{n|I} - uv\right\|_p \tag{2.343}$$
$$= \left\|u\left(v_{n|I} - v\right) + v_{n|I}\left(u_{n|I} - u\right)\right\|_p \tag{2.344}$$
$$\le \left\|u\left(v_{n|I} - v\right)\right\|_p + \left\|v_{n|I}\left(u_{n|I} - u\right)\right\|_p \tag{2.345}$$
$$\le \|u\|_\infty^{\frac{1}{p}} \left\|v_{n|I} - v\right\|_p + \left\|v_{n|I}\right\|_\infty^{\frac{1}{p}} \left\|u_{n|I} - u\right\|_p \tag{2.346}$$
$$\le \|u\|_\infty^{\frac{1}{p}} \left\|v_{n|I} - v\right\|_{W^{1,p}(I)} + \left\|v_{n|I}\right\|_\infty^{\frac{1}{p}} \left\|u_{n|I} - u\right\|_{W^{1,p}(I)} \to 0 \tag{2.347}$$

as $n \to \infty$, where $\left\|v_{n|I}\right\|_\infty$ is handled as (2.342). We have

$$(u_n v_n)' = {u_n}' v_n + u_n {v_n}' \to u'v + uv' \text{ in } L^p(I) \tag{2.348}$$

Applying once more Remark 1.9 to the sequence $(u_n v_n)$, we conclude that $uv \in W^{1,p}(I)$ and that (2.336) holds. Integrating (2.336), we obtain (2.337).

We now turn to the case $p = \infty$; let $u, v \in W^{1,\infty}(I)$. Thus $uv \in L^\infty(I)$ and $u'v + uv' \in L^\infty(I)$. It remains to check that

$$\int_I uv\varphi' = -\int_I (u'v + uv')\,\varphi, \;\; \forall \varphi \in C_c^1(I) \tag{2.349}$$

For this, fix a bounded open interval $J \subset I$ such that $\operatorname{supp}\varphi \subset J$. Thus $u, v \in W^{1,p}(J)$ for all $p < \infty$ and from the above we know that

$$\int_J uv\varphi' = -\int_J (u'v + uv')\,\varphi \tag{2.350}$$

[17] Indeed, $\int_I |uv|^p \le \|u\|_\infty \int_I |v|^p < \infty$ since $u \in L^\infty$ and $v \in L^p$.

that is,

$$\int_I uv\varphi' = -\int_I (u'v + uv')\,\varphi \tag{2.351}$$

Corollary 1.33 (differentiation of a composition). *Let $G \in C^1(\mathbb{R})$ be such that*[18] $G(0) = 0$, *and let $u \in W^{1,p}(I)$ with $1 \le p \le \infty$. Then*

$$G \circ u \in W^{1,p}(I) \tag{2.352}$$

$$(G \circ u)' = (G' \circ u)\,u' \tag{2.353}$$

Proof. Let $M = \|u\|_\infty$. We have $M < \infty$ since $u \in W^{1,p}(I)$ and (2.251). Since $G(0) = 0$, there exists a constant C such that $|G(s)| \le C|s|$ for all $s \in [-M, +M]$. Thus $|G \circ u| \le C|u|$; it follows that $G \circ u \in L^p(I)$. Similarly, $(G' \circ u)\,u' \in L^p(I)$. Indeed, since $G' \in C(\mathbb{R})$, there exists a constant C_1 such that $|G'(s) - G'(0)| \le C_1|s|$ for all $s \in [-M, +M]$. Hence,

$$|G'(u(x))| \le C_1|u(x)| + |G'(0)| \tag{2.354}$$

$$\le C_1\|u\|_\infty + |G'(0)| \tag{2.355}$$

Thus

$$|(G' \circ u)\,u'| \le C_1\|u\|_\infty |u'| + |G'(0)|\,|u'| \tag{2.356}$$

which immediately implies that $(G' \circ u)\,u' \in L^p(I)$ as stated.

It remains to verify that

$$\int_I (G \circ u)\,\varphi' = -\int_I (G' \circ u)\,u'\varphi, \quad \forall \varphi \in C_c^1(I) \tag{2.357}$$

Suppose first that $1 \le p < \infty$. Then there exists a sequence (u_n) from $C_c^1(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I)$ and also in $L^\infty(I)$. Thus $(G \circ u_n)_{|I} \to G \circ u$ in $L^\infty(I)$ and $(G' \circ u)\,{u_{n|I}}' \to (G' \circ u)\,u'$ in $L^p(I)$. Clearly (by the standard rules for C^1 functions) we have

$$\int_I (G \circ u_n)\,\varphi' = -\int_I (G' \circ u_n)\,{u_n}'\varphi, \quad \forall \varphi \in C_c^1(I) \tag{2.358}$$

from which we deduce (2.357). For the case $p = \infty$ proceed in the same manner as in the proof of Corollary 1.32. More explicitly, fix a bounded open interval $J \subset I$ such that $\operatorname{supp}\varphi \subset J$. Thus $u \in W^{1,p}(J)$ for all $p < \infty$ and by (2.357) we know that

$$\int_J (G \circ u)\,\varphi' = -\int_J (G' \circ u)\,u'\varphi \tag{2.359}$$

that is,

$$\int_I (G \circ u)\,\varphi' = -\int_I (G' \circ u)\,u'\varphi \tag{2.360}$$

This completes our proof. □

[18] This restriction is unnecessary when I is bounded (or also if I is unbounded and $p = \infty$). It is essential if I is unbounded and $1 \le p < \infty$ (Why?).

The Sobolev Spaces $W^{m,p}$.

Definition 1.34. Given an integer $m \geq 2$ and a real number $1 \leq p \leq \infty$ we define by induction the space

$$W^{m,p}(I) = \left\{u \in W^{m-1,p}(I)\,; u' \in W^{m-1,p}(I)\right\} \tag{2.361}$$

We also set

$$H^m(I) = W^{m,2}(I) \tag{2.362}$$

It is easily shown (Why?)that $u \in W^{m,p}(I)$ if and only if there exist m functions $g_1, \ldots, g_m \in L^p(I)$ such that

$$\int_I u D^j \varphi = (-1)^j \int_I g_j \varphi, \quad \forall \varphi \in C_c^\infty(I), \quad \forall j = 1, 2, \ldots, m \tag{2.363}$$

where $D^j\varphi$ denotes the jth derivative of φ. When $u \in W^{m,p}(I)$ we may thus consider the successive derivatives of u: $u' = g_1, (u')' = g_2, \ldots$, up to order m. They are denoted by $Du, D^2u, \ldots, D^m u$. The space $W^{m,p}(I)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_p + \sum_{\alpha=1}^{m} \|D^\alpha u\|_p \tag{2.364}$$

and the space $H^m(I)$ is equipped with the scalar product

$$(u,v)_{H^m} = (u,v)_{L^2} + \sum_{\alpha=1}^{m} (D^\alpha u, D^\alpha v)_{L^2} \tag{2.365}$$

$$= \int_I uv + \sum_{\alpha=1}^{m} \int_I D^\alpha u D^\alpha v \tag{2.366}$$

One can show that the norm $\|\cdot\|_{W^{m,p}}$ is equivalent to the norm

$$|||u||| = \|u\|_p + \|D^m u\|_p \tag{2.367}$$

Proof of the equivalent of $\|\cdot\|_{W^{m,p}}$ and $|||\cdot|||$. (Why?)

More precisely, one proves that for every integer j, $1 \leq j \leq m-1$, and for every $\varepsilon > 0$ there exists a constant C (depending on ε and $|I| \leq \infty$) such that

$$\left\|D^j u\right\|_p \leq \varepsilon \|D^m u\|_p + C\|u\|_p, \quad \forall u \in W^{m,p}(I) \tag{2.368}$$

((Why?)see Exercise 8.6, [1] for the case $|I| < \infty$).

The reader can extend to the space $W^{m,p}$ all the properties shown for $W^{1,p}$; for example, if I is bounded, $W^{m,p}(I) \subset C^{m-1}(\bar{I})$ with continuous injection (resp. compact injection for $1 < p \leq \infty$).

3 The Space $W_0^{1,p}$

Definition 1.35. Given $1 \leq p < \infty$, denote by $W_0^{1,p}(I)$ the closure of $C_c^1(I)$ in $W^{1,p}(I)$.[19] Set

$$H_0^1(I) = W_0^{1,2}(I) \tag{3.1}$$

[19]We doe not define $W_0^{1,p}$ for $p = \infty$.

The space $W_0^{1,2}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space H_0^1 is equipped with the scalar product of H^1.[20]

The space $W_0^{1,p}$ is a separable Banach space. Moreover, it is reflexive for $p > 1$. The space H_0^1 is a separable Hilbert space.

Remark 1.36. When $I = \mathbb{R}$ we know that $C_c^1(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ (see Theorem 1.23) and therefore $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$.

Remark 1.37. Using a sequence of mollifiers (ρ_n) it is easy to check the following: (Why?)

1. C_c^∞ is dense in $W_0^{1,p}(I)$.
2. If $u \in W^{1,p}(I) \cap C_c(I)$ then $u \in W_0^{1,p}(I)$.

Our next result provides a basic characterization of functions in $W_0^{1,p}(I)$.

Theorem 1.38. *Let $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I.*

Remark 1.39. Theorem 1.38 explains the central role played by the space $W_0^{1,p}(I)$. Differential equations (or partial differential equations) are often coupled with *boundary conditions*, i.e., the value of u is prescribed on ∂I.

Proof of Theorem 1.38. If $u \in W_0^{1,p}$, there exists a sequence (u_n) in $C_c^1(I)$ such that $u_n \to u$ in $W^{1,p}(I)$ (by definition of $W_0^{1,p}$. Therefore $u_n \to u$ uniformly on $\bar{I}$ [21] and as a consequence $u = 0$ on ∂I ($u_n = 0$ on ∂I).

Conversely, let $u \in W^{1,p}(I)$ be such that $u = 0$ on ∂I. Fix any function $G \in C^1(\mathbb{R})$ such that

$$G(t) = \begin{cases} 0 \text{ if } |t| \le 1 \\ t \text{ if } |t| \ge 2 \end{cases} \tag{3.4}$$

and

$$|G(t)| \le |t|, \quad \forall t \in \mathbb{R} \tag{3.5}$$

Set $u_n = \frac{1}{n} G(nu)$, so that $u_n \in W^{1,p}(I)$ (by Corollary 1.33). On the other hand,

$$\text{supp } u_n \subset \left\{ x \in I; |u(x)| \ge \frac{1}{n} \right\} \tag{3.6}$$

and thus supp u_n is in a compact subset of I (using the fact that $u = 0$ on ∂I and $u(x) \to 0$ as $|x| \to \infty, x \in I$, see Corollary 1.31). Therefore $u_n \in W_0^{1,p}(I)$ (Why?) (see Remark 1.37). Finally, one easily checks that $u_n \to u$ in $W^{1,p}(I)$

[20] When there is no confusion we often write $W_0^{1,p}$ and H_0^1 instead of $W_0^{1,p}(I)$ and $H_0^1(I)$.

[21] Indeed, by (2.251),

$$\begin{aligned} \|u_n - u\|_{L^\infty(\bar{I})} &= \|u_n - u\|_{L^\infty(I)} \qquad (3.2) \\ &\le C\|u_n - u\|_{W^{1,p}(I)} \to 0 \qquad (3.3) \end{aligned}$$

as $n \to \infty$, where the first equality is deduced by the fact that ∂I has zero measure.

by the dominated convergence theorem. (Why?)(dominated convergence theorem for L^p) Thus $u \in W_0^{1,p}(I)$. $\square$

Remark 1.40. Let us mention two other characterizations of $W_0^{1,p}$ functions:

1. Let $1 \leq p < \infty$ and let $u \in L^p(I)$. Define $\bar{u}$ by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in I \\ 0 & \text{if } x \in \mathbb{R} \backslash I \end{cases} \tag{3.7}$$

   Then $u \in W_0^{1,p}(I)$ if and only if $\bar{u} \in W^{1,p}(\mathbb{R})$. (Why?)

2. Let $1 < p < \infty$ and let $u \in L^p(I)$. Then u belongs to $W_0^{1,p}(I)$ if and only if there exists a constant C such that (Why?)

$$\left| \int_I u\varphi' \right| \leq C\|\varphi\|_{L^{p'}(I)}, \quad \forall \varphi \in C_c^1(\mathbb{R}) \tag{3.8}$$

Proposition 1.41 (Poincaré's inequality). *Suppose I is a **bounded** interval. Then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C\|u'\|_{L^p(I)}, \quad \forall u \in W_0^{1,p}(I) \tag{3.9}$$

In other words, on $W_0^{1,p}$, the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}$ norm.

Proof. Let $u \in W_0^{1,p}(I)$ (with $I = (a,b)$). Since $u(a) = 0$, we have

$$\begin{aligned} |u(x)| &= |u(x) - u(a)| && (3.10) \\ &= \left| \int_a^x u'(t)\,dt \right| && (3.11) \\ &\leq \int_a^x |u'(t)|\,dt && (3.12) \\ &\leq \int_a^b |u'(t)|\,dt && (3.13) \\ &= \|u'\|_{L^1(I)} && (3.14) \end{aligned}$$

Thus $\|u\|_{L^\infty(I)} \leq \|u'\|_{L^1(I)}$ and (3.9) then follows by Hölder's inequality. More explicitly,

$$\begin{aligned} \|u\|_{W^{1,p}(I)} &= \|u\|_p + \|u'\|_p && (3.15) \\ &\leq |I|^{\frac{1}{p}} \|u\|_\infty + \|u'\|_p && (3.16) \\ &\leq |I|^{\frac{1}{p}} \|u'\|_{L^1(I)} + \|u'\|_p && (3.17) \\ &\leq |I|^{\frac{1}{p}} |I|^{\frac{1}{p'}} \|u'\|_p + \|u'\|_p && (3.18) \\ &= (|I| + 1)\|u'\|_p && (3.19) \end{aligned}$$

Thus, we can take $C = |I| + 1$ in (3.9). $\square$

Remark 1.42. If I is bounded, the expression $(u', v')_{L^2} = \int_I u'v'$ defines a scalar product on H_0^1 and the associated norm, i.e., $\|u'\|_{L^2}$, is equivalent to the H^1 norm.

$$\|u'\|_{L^2(I)} \leq \|u\|_{H^1(I)} \leq C\|u'\|_{L^2(I)}, \quad \forall u \in H_0^1(I) \tag{3.20}$$

Remark 1.43. Given an integer $m \geq 2$ and a real number $1 \leq p < \infty$, the space $W_0^{m,p}(I)$ is defined as the closure of $C_c^m(I)$ in $W^{m,p}(I)$. Once shows (see Exercise 8.9, [1]) that

$$W_0^{m,p}(I) = \left\{u \in W^{m,p}(I)\,; u = Du = \cdots = D^{m-1}u = 0 \text{ on } \partial I\right\} \tag{3.21}$$

It is essential to notice the *distinction* between

$$W_0^{2,p}(I) = \left\{u \in W^{2,p}(I)\,; u = Du = 0 \text{ on } \partial I\right\} \tag{3.22}$$

and

$$W^{2,p}(I) \cap W_0^{1,p}(I) = \left\{u \in W^{2,p}(I)\,; u = 0 \text{ on } \partial I\right\} \tag{3.23}$$

The Dual Space of $W_0^{1,p}(I)$

Notation. The dual space of $W_0^{1,p}(I)$ $(1 \leq p < \infty)$ is denoted by $W^{-1,p'}(I)$ and the dual space of $H_0^1(I)$ is denoted by $H^{-1}(I)$.

Following Remark 3 of Chapter 5, [1], *we identify L^2 and its dual, but we do not identify H_0^1 and its dual.* We have the inclusions (Why?)

$$H_0^1 \subset L^2 \subset H^{-1} \tag{3.24}$$

where these injections are continuous and dense (i.e., they have dense ranges).

If I is a bounded interval we have

$$W_0^{1,p} \subset L^2 \subset W^{-1,p'}, \quad \forall 1 \leq p \leq 2 \tag{3.25}$$

with continuous injections (see Remark 1.30).

The elements of $W^{-1,p'}$ can be represented with the help of functions in $L^{p'}$; to be precise, we have the following

Proposition 1.44. *Let $F \in W^{-1,p'}(I)$. Then there exist two functions $f_0, f_1 \in L^{p'}(I)$ such that*

$$\langle F, u\rangle = \int_I f_0 u + \int_I f_1 u', \quad \forall u \in W_0^{1,p}(I) \tag{3.26}$$

and

$$\|F\|_{W^{-1,p'}} = \max\left\{\|f_0\|_{p'}, \|f_1\|_{p'}\right\} \tag{3.27}$$

When I is ***bounded*** *we can take $f_0 = 0$.*

Proof. Consider the product space $E = L^p(I) \times L^p(I)$ equipped with the norm

$$\|h\| = \|h_0\|_p + \|h_1\|_p \text{ where } h = [h_0, h_1] \tag{3.28}$$

The map $T : u \in W_0^{1,p}(I) \mapsto [u, u'] \in E$ is an isometry from $W_0^{1,p}(I)$ into E. Set $G = T\left(W_0^{1,p}(I)\right)$ equipped with the norm of E and $S = T^{-1} : G \to W_0^{1,p}(I)$. The map $h \in G \mapsto \langle F, Sh\rangle$ is a continuous linear functional on G. By the Hahn-Banach theorem, it can be extended to a continuous linear function Φ on all of E with $\|\Phi\|_{E^\star} = \|F\|$. By the Riesz representation theorem we know that there exist two functions $f_0, f_1 \in L^{p'}(I)$ such that

$$\langle \Phi, h\rangle = \int_I f_0 h_0 + \int_I f_1 h_1, \quad \forall h = [h_0, h_1] \in E \tag{3.29}$$

It is easy to check that (Why?) $\|\Phi\|_{E^\star} = \max\left\{\|f_0\|_{p'}, \|f_1\|_{p'}\right\}$. Also, we have

$$\langle \Phi, Tu\rangle = \langle F, u\rangle = \int_I f_0 u + \int_I f_1 u', \quad \forall u \in W_0^{1,p}(I) \tag{3.30}$$

When I is *bounded* the space $W_0^{1,p}(I)$ may be equipped with the norm $\|u'\|_p$ (see Proposition 1.41). We repeat (Why?) the same argument with $E = L^p(I)$ and $T : u \in W^{1,p}(I) \mapsto u' \in L^p(I)$. $\square$

Remark 1.45. The functions f_0 and f_1 are not uniquely determined by F.

Remark 1.46. The element $F \in W^{-1,p'}(I)$ is usually identified with the distribution $f_0 - f_1'$ (by definition, the distribution $f_0 - f_1'$ is the linear functional $u \mapsto \int_I f_0 u + \int_I f_1 u'$, on $C_c^\infty(I)$).

Remark 1.47. (Why?)The first assertion of Proposition 1.44 also holds for continuous linear functionals on $W^{1,p}$ $(1 \le p < \infty)$, i.e., every continuous linear functional F on $W^{1,p}$ may be represented as

$$\langle F, u\rangle = \int_I f_0 u + \int_I f_1 u', \quad \forall u \in W^{1,p} \tag{3.31}$$

for some functions $f_0, f_1 \in L^{p'}$.

4 Some Examples of Boundary Value Problems

Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \tag{4.1}$$
$$u(0) = u(1) = 0 \tag{4.2}$$

where f is a given function (for example in $C\left(\bar{I}\right)$ or more generally in $L^2(I)$). *The boundary condition* $u(0) = u(1) = 0$ *is called the (homogeneous) Dirichlet boudnary condition.*

Definition 1.48. A *classical solution* of (4.1)-(4.2) is a function $u \in C^2\left(\bar{I}\right)$ satisfying (4.1)-(4.2) in the usual sense. A *weak solution* of (4.1)-(4.2) is a function $u \in H_0^1(I)$ satisfying

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.3}$$

Let us "put in action" the program outlined in Section 1.1:

Step A. Every classical solution is a weak solution. This is obvious by integration by parts (as justified in Corollary 1.32). Indeed, multiplying both sides of (4.1) with $v \in H_0^1(I)$ gives

$$-\int_I u''v + \int_I uv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.4}$$

Using integration by parts formula (2.337) gives

$$-\int_I u''v = -u'(1)\,v(1) + u'(0)\,v(0) + \int_I u'v' \tag{4.5}$$

$$= \int_I u'v', \quad \forall v \in H_0^1(I) \tag{4.6}$$

since $v(0) = v(1) = 0$. Combining (4.4) with (4.5)-(4.6) yields (4.3).

Step B. Existence and uniqueness of a weak solution. This is the content of the following result.

Proposition 1.49. *Given any $f \in L^2(I)$ there exists a unique solution $u \in H_0^1$ to* (4.3)*. Furthermore, u is obtained by*

$$\min_{v\in H_0^1}\left\{\frac{1}{2}\int_I \left(v'^2 + v^2\right) - \int_I fv\right\} \tag{4.7}$$

this is Dirichlet's principle.

Proof. We apply Lax-Milgram's theorem (Corollary 5.8, [1]) in the Hilbert space $H = H_0^1(I)$ with the bilinear form

$$a(u,v) = \int_I u'v' + \int_I uv = (u,v)_{H^1} \tag{4.8}$$

and with the linear functional $\varphi : v \mapsto \int_I fv$. Indeed, it is easy to check that $a(\cdot,\cdot)$ is a bilinear form on H_0^1. It suffices to prove that $a(\cdot,\cdot)$ is continuous, coercive and symmetric. To this end, we have

$$|a(u,v)| \le \|u'v'\|_{L^1(I)} + \|uv\|_{L^1(I)} \tag{4.9}$$

$$\le \|u'\|_{L^2(I)}\|v'\|_{L^2(I)} + \|u\|_{L^2(I)}\|v\|_{L^2(I)} \tag{4.10}$$

$$\le \Big(\|u'\|_{L^2(I)} + \|u\|_{L^2(I)}\Big)\Big(\|v'\|_{L^2(I)} + \|v\|_{L^2(I)}\Big) \tag{4.11}$$

$$= \|u\|_{H^1(I)}\|v\|_{H^1(I)}, \quad \forall u,v \in H_0^1(I) \tag{4.12}$$

which proves the continuity of $a(\cdot,\cdot)$, and

$$a(u,u) = \|u'\|_{L^2(I)}^2 + \|u\|_{L^2(I)}^2 \tag{4.13}$$

$$\ge \frac{1}{2}\Big(\|u'\|_{L^2(I)} + \|u\|_{L^2(I)}\Big)^2 \tag{4.14}$$

$$= \frac{1}{2}\|u\|_{H^1(I)}^2, \quad \forall u \in H_0^1(I) \tag{4.15}$$

which proves the coercivity of $a(\cdot,\cdot)$. Then, applying Lax-Milgram theorem to $a(\cdot,\cdot)$ and the above functional $\varphi \in H^{-1}(I)$ yields that there exists a unique element $u \in H_0^1(I)$ such that

$$a(u,v) = \langle \varphi, v\rangle, \quad \forall v \in H_0^1(I) \tag{4.16}$$

which is exactly (4.3).

Moreover, a is symmetric by (4.8). By the second part of Lax-Milgram's theorem, u is then characterized by the property

$$u \in H_0^1(I) \tag{4.17}$$

$$\frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v \in H_0^1(I)} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v\rangle \right\} \tag{4.18}$$

which is exactly (4.7). □

Remark 1.50. Given $F \in H^{-1}(I)$ we know from the Riesz-Fréchet representation theorem (Theorem 5.5, [1]) that there exists a unique $u \in H_0^1(I)$ such that

$$(u,v)_{H^1} = \langle F, v\rangle_{H^{-1},H_0^1}, \quad \forall v \in H_0^1 \tag{4.19}$$

The map $F \mapsto u$ is the Riesz-Fréchet isomorphism from H^{-1} onto H_0^1. The function u coincides with the weak solution of (4.1)-(4.2) in the sense of (4.3).

Step C and D. Regularity of Weak Solutions. Recovery of Classical Solutions

First, note that if $f \in L^2$ and $u \in H_0^1$ is the weak solution of (4.1)-(4.2) , then $u \in H^2$. Indeed, by (4.3)

$$\int_I u'v' = \int_I (f-u)\,v, \quad \forall v \in H_0^1(I) \tag{4.20}$$

In particular, we have

$$\int_I u'v' = \int_I (f-u)\,v, \quad \forall v \in C_c^1(I) \tag{4.21}$$

and thus $u' \in H^1$ (by definition of H^1 and since $f - u \in L^2$), i.e., $u \in H^2$.

Furthermore, if we assume that $f \in C(\bar{I})$, then the weak solution u belongs to $C^2(\bar{I})$. Indeed, $(u')' \in C(\bar{I})$[22] and thus $u' \in C^1(\bar{I})$ (see Remark 1.12). The passage from a weak solution $u \in C^2(\bar{I})$ to a classical solution has been carried out in Section 1.1.

Remark 1.51. If $f \in H^k(I)$, with k an integer ≥ 1, it is easily verified (by induction) that the solution u of (4.3) belongs to $H^{k+2}(I)$.

Proof of Remark 1.51. Suppose $u \in H_0^1$ is the solution of (4.3).

[22] By (4.21), $(u')' = u - f$, where $f \in C(\bar{I})$ and $u \in H_0^1(I)$. By Theorem 1.11, $u \in C(\bar{I})$ (u admits a continuous representation on $\bar{I}$). Hence, $(u')' \in C(\bar{I})$ as stated above.

For $k = 1$, suppose $f \in H^1(I)$, in particular, $f \in L^2$. Hence, $u \in H^2(I)$ as proved above.

Now, we suppose that

$$f \in H^{k-1}(I) \Rightarrow u \in H^{k+1}(I) \tag{4.22}$$

Given $f \in H^k(I)$, by definition, we have $f \in H^{k-1}(I)$. Hence, we can apply (4.22) to f to obtain $u \in H^{k+1}(I)$. Now, by (4.21) again, we have $(u')' = u - f \in H^k(I)$. Combining this with the fact that $u' \in H^k(I)$, which is deduced from $u \in H^{k+1}(I)$, yields $u' \in H^{k+1}(I)$. Combing $u' \in H^{k+1}(I)$ with $u \in H^{k+1}(I)$ once more yields $u \in H^{k+2}(I)$. By the induction principle, we have (4.22) holds for all integers $k \geq 1$. $\square$

The method described above is extremely flexible and can be adapted to a multitude of problems. We indicate several examples frequently encountered. *In each problem it is essential to specify precisely the function space and to find the appropriate weak formulation.*

Example 1.52 (inhomogeneous Dirichlet condition). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \tag{4.23}$$

$$u(0) = \alpha, u(1) = \beta \tag{4.24}$$

with $\alpha, \beta \in \mathbb{R}$ given and f a given function.

Proposition 1.53. *Given $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(I)$ there exists a unique function $u \in H^2(I)$ satisfying* (4.23)-(4.24). *Furthermore, u is obtained by*

$$\min_{v \in H^1(I), v(0)=\alpha, v(1)=\beta} \left\{ \frac{1}{2} \int_I \left(v'^2 + v^2\right) - \int_I fv \right\} \tag{4.25}$$

If, in addition, $f \in \left(\bar{I}\right)$ then $u \in C^2\left(\bar{I}\right)$.

Proof. We give two possible approaches:

Method 1. Fix any smooth function[23] u_0 such that $u_0(0) = \alpha$ and $u_0(1) = \beta$, for instance $u_0(x) = (\beta - \alpha)x + \alpha$. Introduce as new unknown $\tilde{u} = u - u_0$. Then $\tilde{u}$ satisfies

$$-\tilde{u}'' + \tilde{u} = f + u_0'' - u_0 \text{ on } I \tag{4.26}$$

$$\tilde{u}(0) = \tilde{u}(1) = 0 \tag{4.27}$$

We are reduced to the preceding problem for $\tilde{u}$.

Method 2. Consider in the space $H^1(I)$ the closed convex set

$$K = \left\{v \in H^1(I); v(0) = \alpha, v(1) = \beta\right\} \tag{4.28}$$

[23] Choose, for example, u_0 to be affine.

If u is a classical solution of (4.23)-(4.24), by multiplying both sides of (4.23) with $v-u$, where $v \in K$, and then integrating, we have

$$-\int_I u''(v-u) + \int_I u(v-u) = \int_I f(v-u), \quad \forall v \in K \tag{4.29}$$

Integrating by parts the first integral in the left hand side of (4.29) and noticing that $v(0) - u(0) = 0, v(1) - u(1) = 0$, yields

$$\int_I u'(v-u)' + \int_I u(v-u) = \int_I f(v-u), \quad \forall v \in K \tag{4.30}$$

Then in particular,

$$\int_I u'(v-u)' + \int_I u(v-u) \geq \int_I f(v-u), \quad \forall v \in K \tag{4.31}$$

We may now invoke Stampacchia's theorem (Theorem 5.6, [1]): there exists a unique function $u \in K$ satisfying (4.31) and, moreover, u is obtained by

$$\min_{v \in K} \left\{ \frac{1}{2} \int_I \left(v'^2 + v^2\right) - \int_I fv \right\} \tag{4.32}$$

Proof of (4.32). We now consider

$$a(u,v) = \int_I u'v' + \int_I uv, \quad \forall u, v \in H^1(I) \tag{4.33}$$

and $\varphi \in \left(H^1(I)\right)^\star$ defined by

$$\varphi : v \mapsto \int_I fv, \quad \forall v \in H^1(I) \tag{4.34}$$

We have proved that $a(\cdot,\cdot)$ is a symmetric continuous coercive bilinear form on $H^1(I)$. The chosen set $K \subset H^1(I)$, which is defined by (4.28), is a nonempty closed and convex subset. Hence, the hypotheses of Stampacchia's theorem are met. Applying Stampacchia's theorem to $a(\cdot,\cdot)$ and $\varphi \in \left(H^1(I)\right)^\star$, which are defined by (4.33) and (4.34). respectively, yields that there exists a unique element $u \in K$ such that

$$a(u, v-u) \geq \langle \varphi, v-u \rangle, \quad \forall v \in K \tag{4.35}$$

which is exactly (4.31).

Moreover, since a is symmetric, u is characterized by the property $u \in K$ and

$$\frac{1}{2} a(u,u) - \langle \varphi, u \rangle = \min_{v \in K} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \right\} \tag{4.36}$$

which is exactly (4.32). □

To recover a classical solution of (4.23)-(4.24), set $v = u \pm w$ in (4.31) with $w \in H_0^1$ and obtain

$$\int_I u'w' + \int_I uw = \int_I fw, \quad \forall w \in H_0^1 \tag{4.37}$$

This implies (as above) that $u \in H^2(I)$. If $f \in C(\bar{I})$ the same argument as in the homogeneous case shows that $u \in C^2(\bar{I})$.

Example 1.54 (Sturm-Liouville problem). Consider the problem

$$-(pu')' + qu = f \text{ on } I = (0,1) \tag{4.38}$$

$$u(0) = u(1) = 0 \tag{4.39}$$

where $p \in C^1(\bar{I})$, $q \in C(\bar{I})$, and $f \in L^2(I)$ are given with

$$p(x) \geq \alpha > 0, \quad \forall x \in I \tag{4.40}$$

If u is a classical solution of (4.38)-(4.39), by multiplying both side of (4.38) by $v \in H_0^1(I)$, we have

$$-\int_I (pu')' v + \int_I quv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.41}$$

Integrating by parts the first integral in the left hand side of (4.41) yields

$$\int_I pu'v' + \int_I quv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.42}$$

We use $H_0^1(I)$ as our function space and

$$a(u,v) = \int_I pu'v' + \int_I quv, \quad \forall u, v \in H_0^1(I) \tag{4.43}$$

as symmetric continuous bilinear form on $H_0^1(I)$. Indeed, $a(\cdot,\cdot)$ is obvious a symmetric bilinear form on $H_0^1(I)$, it suffices to verify the continuity of $a(\cdot,\cdot)$. This can be easily handled by the following estimates

$$\begin{aligned}
|a(u,v)| &= \left|\int_I pu'v' + \int_I quv\right| && (4.44)\\
&\leq \left|\int_I pu'v'\right| + \left|\int_I quv\right| && (4.45)\\
&\leq \|p\|_\infty \|u'v'\|_1 + \|q\|_\infty \|uv\|_1 && (4.46)\\
&\leq \max\{\|p\|_\infty, \|q\|_\infty\}(\|u'v'\|_1 + \|uv\|_1) && (4.47)\\
&\leq \max\{\|p\|_\infty, \|q\|_\infty\}(\|u'\|_2\|v'\|_2 + \|u\|_2\|v\|_2), \text{ by Hölder} && (4.48)\\
&\leq \max\{\|p\|_\infty, \|q\|_\infty\}(\|u\|_2 + \|u'\|_2)(\|v\|_2 + \|v'\|_2) && (4.49)\\
&\leq \max\{\|p\|_\infty, \|q\|_\infty\}\|u\|_{H^1(I)}\|v\|_{H^1(I)}, \quad \forall u, v \in H_0^1(I) && (4.50)
\end{aligned}$$

In addition, if $q \geq 0$ on I this form is coercive by Poincaré's inequality (Proposition 1.41). Indeed,

$$\begin{aligned}
a(v,v) &= \int_I pv'^2 + \int_I qv^2 && (4.51)\\
&\geq \alpha\|v'\|_2, \text{ by (4.40)} && (4.52)\\
&\geq \frac{\alpha}{C}\|v\|_{H^1(I)}, \quad \forall v \in H_0^1(I), \text{ by (3.9)} && (4.53)
\end{aligned}$$

Thus, by Lax-Milgram's theorem, there exists a unique $u \in H_0^1(I)$ such that

$$a(u,v) = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.54}$$

Moreover, u is obtained by

$$\min_{v \in H_0^1(I)} \left\{ \frac{1}{2} \int_I \left(pv'^2 + qv^2 \right) - \int_I fv \right\} \tag{4.55}$$

It is clear from (4.54) that $pu' \in H^1(I)$. Indeed, (4.54) can be rewritten as

$$\int_I pu'v' = \int_I (f - qu)\, v, \quad \forall v \in H_0^1(I) \tag{4.56}$$

In particular,

$$\int_I pu'v' = \int_I (f - qu)\, v, \quad \forall v \in C_c^1(I) \tag{4.57}$$

We also have $f - qu \in L^2(I)$ since $f \in L^2(I), q \in C\left(\bar{I}\right)$ and $u \in H_0^1(I)$. Combining this with (4.57) yields $pu' \in H^1(I)$ as stated.

Thus (by Corollary 1.32) $u' = \frac{1}{p}(pu') \in H^1(I)$ and hence $u \in H^2(I)$. Finally, if $f \in C\left(\bar{I}\right)$, then $pu' \in C^1\left(\bar{I}\right)$, and so $u' \in C\left(\bar{I}\right)$, i.e., $u \in C^2\left(\bar{I}\right)$. Step D carries over and we conclude that u is a classical solution of (4.38)-(4.39). □

Consider now the more general problem

$$-\left(pu'\right)' + ru' + qu = f \text{ on } I = (0,1) \tag{4.58}$$

$$u(0) = u(1) = 0 \tag{4.59}$$

The assumptions on p, q, and f are the same as above, and $r \in C\left(\bar{I}\right)$. If u is a classical solution of (4.58)-(4.59) we have

$$\int_I pu'v' + \int_I ru'v + \int_I quv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{4.60}$$

We use $H_0^1(I)$ as our function space and

$$a(u,v) = \int_I pu'v' + \int_I ru'v + \int_I quv \tag{4.61}$$

as bilinear continuous form. This form is *not* symmetric. In certain cases it is coercive; for example,

1. if $q \geq 1$ and $r^2 < 4\alpha$.
2. or if $q \geq 1$ and $r \in C^1\left(\bar{I}\right)$ with $r' \leq 2$; here we use the fact that

$$\int_I rv'v = -\frac{1}{2} \int_I r'v^2, \quad \forall v \in H_0^1(I) \tag{4.62}$$

Proof of coercivity in the above cases.

1. I can only prove (1) for the case $\alpha > 1$.

$$a(v,v) = \int_I pv'^2 + \int_I rv'v + \int_I qv^2 \tag{4.63}$$
$$\geq \alpha \int_I v'^2 - 2\sqrt{\alpha} \int_I |v'v| + \int_I v^2 \tag{4.64}$$
$$\geq \alpha \|v'\|_2^2 - 2\sqrt{\alpha}\|v\|_2\|v'\|_2 + \|v\|_2^2 \tag{4.65}$$
$$= \left(\sqrt{\alpha}\|v'\|_2 - \|v\|_2\right)^2 \tag{4.66}$$

2. (Why?)

One may then apply the Lax-Milgram theorem, but there is no straightforward associated minimization problem. Here is a device that allows us to recover a symmetric bilinear form. Introduce a primitive R of $\frac{r}{p}$ and set $\zeta = e^{-R}$. Equation (4.58) can be written, after multiplication by ζ, as

$$-\zeta pu'' - \zeta p'u' + \zeta ru' + \zeta qu = \zeta f \tag{4.67}$$

or, since

$$\zeta' p + \zeta r = -R'e^{-R}p + e^{-R}r \tag{4.68}$$
$$= -\frac{r}{p}e^{-R}p + e^{-R}r \tag{4.69}$$
$$= 0, \tag{4.70}$$

$$-\left(\zeta pu'\right)' + \zeta qu = \zeta f \tag{4.71}$$

Multiplying both sides of (4.71) by $v \in H_0^1$ gives

$$-\int_I \left(\zeta pu'\right)' v + \int_I \zeta quv = \int_I \zeta fv, \quad \forall v \in H_0^1(I) \tag{4.72}$$

Integrating by parts the first integral in the left hand side of (4.72), as usual, gives

$$\int_I \zeta pu'v' + \int_I \zeta quv = \int_I \zeta fv, \quad \forall v \in H_0^1(I) \tag{4.73}$$

Define on H_0^1 the symmetric continuous bilinear form

$$a(u,v) = \int_I \zeta pu'v' + \int_I \zeta quv \tag{4.74}$$

When $q \geq 0$, this form is coercive. Indeed,

$$a(v,v) = \int_I \zeta pv'^2 + \int_I \zeta qv^2 \tag{4.75}$$
$$\geq \int_I \zeta pv'^2 \tag{4.76}$$
$$\geq \zeta\alpha \|v'\|_2^2 \tag{4.77}$$

$$\geq \frac{\zeta\alpha}{C^2} \|v\|^2_{H^1(I)}, \quad \forall v \in H^1_0(I), \text{ by (3.9)} \tag{4.78}$$

And so there exists a unique $u \in H^1_0$ such that

$$a(u,v) = \int_I \zeta f v, \quad \forall v \in H^1_0(I) \tag{4.79}$$

Furthermore, u is obtained by

$$\min_{v \in H^1_0(I)} \left\{ \frac{1}{2} \int_I \left(\zeta p v'^2 + \zeta q v^2\right) - \int_I \zeta f v \right\} \tag{4.80}$$

It is easily verified that $u \in H^2$, and if $f \in C\left(\bar{I}\right)$ then $u \in C^2\left(\bar{I}\right)$ is a classical solution of (4.58)-(4.59).

Example 1.55 (homogeneous Neumann condition). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \tag{4.81}$$
$$u'(0) = u'(1) = 0 \tag{4.82}$$

Proposition 1.56. *Given $f \in L^2(I)$ there exists a unique function $u \in H^2(I)$ satisfying* (4.81)-(4.82).[24] *Furthermore, u is obtained by*

$$\min_{v \in H^1(I)} \left\{ \frac{1}{2} \int_I \left(v'^2 + v^2\right) - \int_I f v \right\} \tag{4.83}$$

If, in addition, $f \in C\left(\bar{I}\right)$, then $u \in C^2\left(\bar{I}\right)$.

Proof. If u is a classical solution of (4.81)-(4.82), multiplying both sides of (4.81) with $v \in H^1(I)$ gives

$$-\int_I u''v + \int_I uv = \int_I fv, \quad \forall v \in H^1(I) \tag{4.84}$$

Integrating by parts the first integral in the left hand side of (4.84) gives

$$\int_I u''v = u'(1)v(1) - u'(0)v(0) - \int_I u'v', \quad \forall v \in H^1(I) \tag{4.85}$$

Combining (4.85) with the boundary conditions (4.82), (4.84) becomes

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H^1(I) \tag{4.86}$$

We use $H^1(I)$ as our function space: there is no point in working in $H^1_0(I)$ as above since $u(0)$ and $u(1)$ are a priori *unknown*. We apply the Lax-Milgram theorem with the bilinear form

$$a(u,v) = \int_I u'v' + \int_I uv \tag{4.87}$$

[24] Note that $u \in H^2(I) \Rightarrow u \in C^1\left(\bar{I}\right)$ and thus the condition $u'(0) = u'(1) = 0$ *makes sense. It would not make sense if we knew only that $u \in H^1$.*

and the linear functional

$$\varphi : v \mapsto \int_I fv \tag{4.88}$$

In this way we obtain a unique function $u \in H^1(I)$ satisfying (4.86). From (4.86) it follows, as above, that $u \in H^2(I)$. Using (4.86) once more, by integrating by parts the first integral in the left hand side of (4.86), we obtain

$$\int_I (-u'' + u - f)\, v + u'(1)\, v(1) - u'(0)\, v(0) = 0, \ \ \forall v \in H^1(I) \tag{4.89}$$

In (4.89) begin by choosing $v \in H_0^1(I)$ and obtain $-u'' + u = f$ a.e. (use Corollary 4.15). Returning to (4.89), there remains

$$u'(1)\, v(1) - u'(0)\, v(0) = 0, \ \ \forall v \in H^1(I) \tag{4.90}$$

Since $v(0)$ and $v(1)$ are arbitrary, we deduce that $u'(0) = u'(1) = 0$. $\square$

Example 1.57 (inhomogeneous Neumann condition). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \tag{4.91}$$
$$u'(0) = \alpha, u'(1) = \beta \tag{4.92}$$

with $\alpha, \beta \in \mathbb{R}$ given and f a given function.

Proposition 1.58. *Given any $f \in L^2(I)$ and $\alpha, \beta \in \mathbb{R}$ there exists a unique function $u \in H^2(I)$ satisfying (4.91)-(4.92). Furthermore, u is obtained by*

$$\min_{v \in H^1(I)} \left\{ \frac{1}{2} \int_I \left(v'^2 + v^2\right) - \int_I fv + \alpha v(0) - \beta v(1) \right\} \tag{4.93}$$

If, in addition, $f \in C\left(\bar{I}\right)$ then $u \in C^2\left(\bar{I}\right)$.

Proof. If u is a classical solution of (4.91)-(4.92) we have

$$\int_I u'v' + \int_I uv = \int_I fv - \alpha v(0) + \beta v(1), \ \ \forall v \in H^1(I) \tag{4.94}$$

We use $H^1(I)$ as our function space and we apply the Lax-Milgram theorem with the bilinear form

$$a(u,v) = \int_I u'v' + \int_I uv \tag{4.95}$$

and the linear functional

$$\varphi : v \mapsto \int_I fv - \alpha v(0) + \beta v(1) \tag{4.96}$$

This linear functional is continuous (by Theorem 1.26). Indeed, we have

$$|\varphi(u) - \varphi(v)| = \left| \int_I f(u - v) - \alpha(u(0) - v(0)) + \beta(u(1) - v(1)) \right| \tag{4.97}$$

$$\leq \left| \int_I f(u-v) \right| + |\alpha| \, |u(0) - v(0)| + |\beta| \, |u(1) - v(1)| \quad (4.98)$$
$$\leq \|f\|_2 \|u-v\|_2 + (|\alpha| + |\beta|) \, \|u-v\|_\infty \quad (4.99)$$
$$\leq \|f\|_2 \|u-v\|_{H^1(I)} + C(|\alpha| + |\beta|) \, \|u-v\|_{H^1(I)} \quad (4.100)$$
$$= (\|f\|_2 + C(|\alpha| + |\beta|)) \, \|u-v\|_{H^1(I)} \quad (4.101)$$

Then proceed as in Example 1.55 to prove that $u \in H^2(I)$ and that $u'(0) = \alpha, u'(1) = \beta$. (Why?) □

Example 1.59 (mixed boundary condition). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \quad (4.102)$$
$$u(0) = 0, u'(1) = 0 \quad (4.103)$$

If u is a classical solution of (4.102)-(4.103) we have

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H^1(I) \text{ with } v(0) = 0 \quad (4.104)$$

The appropriate space to work in is

$$H = \left\{ v \in H^1(I) ; v(0) = 0 \right\} \quad (4.105)$$

equipped with the H^1 scalar product. The rest (Why?). □

Example 1.60 (Robin, or "third type", boundary condition). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \quad (4.106)$$
$$u'(0) = ku(0), u(1) = 0 \quad (4.107)$$

where $k \in \mathbb{R}$ is given.[25]

If u is a classical solution of (4.106)-(4.107) we have

$$\int_I u'v' + \int_I uv + ku(0)v(0) = \int_I fv, \quad \forall v \in H^1(I) \text{ with } v(1) = 0 \quad (4.110)$$

The appropriate space for applying Lax-Milgram is the Hilbert space

$$H = \left\{ v \in H^1(I) ; v(1) = 0 \right\} \quad (4.111)$$

equipped with the $H^1(I)$ scalar product. The bilinear form

$$a(u,v) = \int_I u'v' + \int_I uv + ku(0)v(0) \quad (4.112)$$

[25] More generally, one can handle the boundary condition

$$\alpha_0 u'(0) + \beta_0 u(0) = 0 \quad (4.108)$$
$$\alpha_1 u'(1) + \beta_1 u(1) = 0, \quad (4.109)$$

with appropriate conditions on the constants $\alpha_0, \beta_0, \alpha_1$, and β_1.

is symmetric and continuous. It is coercive if $k \geq 0$.[26]

Example 1.61 (periodic boundary conditions). Consider the problem

$$-u'' + u = f \text{ on } I = (0,1) \tag{4.113}$$
$$u(0) = u(1), u'(0) = u'(1) \tag{4.114}$$

If u is a classical solution of (4.113)-(4.114) we have

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H^1(I) \text{ with } v(0) = v(1) \tag{4.115}$$

The appropriate setting for applying Lax-Milgram is the Hilbert space

$$H = \left\{ v \in H^1(I); v(0) = v(1) \right\} \tag{4.116}$$

with the bilinear form

$$a(u,v) = \int_I u'v' + \int_I uv \tag{4.117}$$

When $f \in L^2(I)$ we obtain a solution $u \in H^2(I)$ of (4.113)-(4.114). If, in addition, $f \in C(I)$ then the solution is classical.

Example 1.62 (a boundary value problem on $\mathbb{R}$.) Consider the problem

$$-u'' + u = f \text{ on } \mathbb{R} \tag{4.118}$$
$$u(x) \to 0 \text{ as } |x| \to \infty \tag{4.119}$$

with f given in $L^2(\mathbb{R})$. A *classical solution* of (4.118)-(4.119) is a function $u \in C^2(\mathbb{R})$ satisfying (4.118)-(4.119) in the usual sense. A *weak solution* of (4.118)-(4.119) is a function $u \in H^1(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} u'v' + \int_{\mathbb{R}} uv = \int_{\mathbb{R}} fv, \quad \forall v \in H^1(\mathbb{R}) \tag{4.120}$$

We have first to prove that any classical solution u is a weak solution; let us check in the first place that $u \in H^1(\mathbb{R})$. Choose a sequence (ζ_n) of cut-off functions as in the proof of Theorem 1.23. Multiplying (4.118) by $\zeta_n u$ and then integrating gives

$$-\int_{\mathbb{R}} \zeta_n u u'' + \int_{\mathbb{R}} \zeta_n u^2 = \int_{\mathbb{R}} \zeta_n u f \tag{4.121}$$

Integrating by parts the first integral in the left hand side of (4.121), we obtain

$$\int_{\mathbb{R}} u' \left(\zeta_n u' + \zeta_n{}' u\right) + \int_{\mathbb{R}} \zeta_n u^2 = \int_{\mathbb{R}} \zeta_n f u \tag{4.122}$$

[26] If $k < 0$ with $|k|$ small enough the form $a(u,v)$ is still coercive. (Why?) On the other hand, an explicit calculation shows that there exist a negative value of k and (smooth) functions f for which (4.106)-(4.107) has no solution (see Exercise 8.21, [1]). (Why?)

from which we deduce

$$\int_{\mathbb{R}} \zeta_n \left(u'^2 + u^2\right) = \int_{\mathbb{R}} \zeta_n f u + \frac{1}{2} \int_{\mathbb{R}} \zeta_n{}'' u^2 \tag{4.123}$$

since[27]

$$\int_{\mathbb{R}} \zeta_n{}' u u' = -\frac{1}{2} \int_{\mathbb{R}} \zeta_n{}'' u^2 \tag{4.124}$$

But

$$\frac{1}{2} \int_{\mathbb{R}} \zeta_n{}'' u^2 \leq \frac{C}{n^2} \int_{n<|x|<2n} u^2 \text{ with } C = \|\zeta''\|_{L^\infty(\mathbb{R})} \tag{4.125}$$

and

$$\frac{1}{n^2} \int_{n<|x|<2n} u^2 \to 0 \text{ as } n \to \infty \tag{4.126}$$

since $u(x) \to 0$ as $|x| \to \infty$. Inserting the inequality

$$\int_{\mathbb{R}} \zeta_n f u \leq \frac{1}{2} \int_{\mathbb{R}} \zeta_n u^2 + \frac{1}{2} \int_{\mathbb{R}} \zeta_n f^2 \tag{4.127}$$

in (4.123) gives

$$\int_{\mathbb{R}} \zeta_n u'^2 + \frac{1}{2} \int_{\mathbb{R}} \zeta_n u^2 \leq \frac{1}{2} \int_{\mathbb{R}} \zeta_n f^2 + \frac{1}{2} \int_{R} \zeta_n{}'' u^2 \tag{4.128}$$

Hence,

$$\frac{1}{2} \int_{\mathbb{R}} \zeta_n \left(u'^2 + u^2\right) \leq \int_{\mathbb{R}} \zeta_n u'^2 + \frac{1}{2} \int_{\mathbb{R}} \zeta_n u^2 \tag{4.129}$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} \zeta_n f^2 + \frac{1}{2} \int_{\mathbb{R}} \zeta_n{}'' u^2, \text{ by (4.128)} \tag{4.130}$$

We also notice that $\frac{1}{2} \int_{\mathbb{R}} \zeta_n f^2$ is bounded since $0 \leq \zeta_n \leq 1$ and $f \in L^2(\mathbb{R})$ and

$$\frac{1}{2} \int_{\mathbb{R}} \zeta_n{}'' u^2 \to 0 \text{ as } n \to \infty \tag{4.131}$$

as proved above. Thus, $\int_{\mathbb{R}} \zeta_n \left(u'^2 + u^2\right)$ remains bounded as $n \to \infty$ and therefore $u \in H^1(\mathbb{R})$.

Assuming that u is a classical solution of (4.118)-(4.119), by multiplying both side of (4.118) and then integrating by parts, we have

$$\int_{\mathbb{R}} u'v' + \int_{\mathbb{R}} uv = \int_{\mathbb{R}} fv, \ \ \forall v \in C_c^1(\mathbb{R}) \tag{4.132}$$

By density (and since $u \in H^1(\mathbb{R})$) this holds for every $v \in H^1(\mathbb{R})$. Therefore u is a weak solution of (4.118)-(4.119).

[27] Compare (4.124) with (4.62).

To obtain existence and uniqueness of a weak solution it suffices to apply Lax-Milgram in the Hilbert space $H^1(\mathbb{R})$. One easily verifies that the weak solution u belongs to $H^2(\mathbb{R})$. One easily verifies that (Why?) the weak solution u belongs to $H^2(\mathbb{R})$ and if furthermore $f \in C(\mathbb{R})$ then $u \in C^2(\mathbb{R})$. We conclude (using Corollary 1.31) that given $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, problem (4.118)-(4.119) has a unique classical solution (which furthermore belongs to $H^2(\mathbb{R})$). (Why?)

Remark 1.63. The problem

$$-u'' = f \text{ on } \mathbb{R} \tag{4.133}$$

$$u(x) \to 0 \text{ as } |x| \to \infty \tag{4.134}$$

cannot be attacked by the preceding technique because the bilinear form

$$a(u,v) = \int_{\mathbb{R}} u'v' \tag{4.135}$$

is *not coercive* in $H^1(\mathbb{R})$[28]. In fact, this problem need not have a solution even if f is smooth with compact support (why? (Why?)).

Remark 1.64. On the other hand, the same method applies to the problem

$$-u'' + u = f \text{ on } I = (0, +\infty) \tag{4.136}$$

$$u(0) = 0 \text{ and } u(x) \to 0 \text{ as } x \to +\infty \tag{4.137}$$

with f given in $L^2(0, +\infty)$.

5 The Maximum Principle

Here is a very useful property called the maximum principle.

Theorem 1.65. *Let $f \in L^2(I)$ with $I = (0,1)$ and let $u \in H^2(I)$ be the solution of the Dirichlet problem*

$$-u'' + u = f \text{ on } I \tag{5.1}$$

$$u(0) = \alpha, u(1) = \beta \tag{5.2}$$

Then we have, for every $x \in I$,[29]

$$\min\left\{\alpha, \beta, \inf_f f\right\} \le u(x) \le \max\left\{\alpha, \beta, \sup_f f\right\} \tag{5.5}$$

Proof (using Stampacchia's truncation method). We have

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H_0^1(I) \tag{5.6}$$

Fix any function $G \in C^1(\mathbb{R})$ such that

[28] Notice that we can not apply Poincaré's inequality for $\mathbb{R}$.

[29] $\sup f$ and $\inf f$ refer respectively to the essential sup (possibly $+\infty$) and the essential inf of f (possibly $-\infty$). Recall that

$$\text{ess } \sup f = \inf\{C; f(x) \le C \text{ a.e.}\} \tag{5.3}$$

$$\text{ess } \inf f = -\text{ess } \sup(-f) \tag{5.4}$$

1. G is strictly increasing on $(0, +\infty)$,
2. $G(t) = 0$ for $t \in (-\infty, 0]$.

Set $K = \max\left\{\alpha, \beta, \sup_f f\right\}$ and suppose that $K < \infty$. We shall show that $u \leq K$ on I. The function $v = G(u - K)$ belongs to $H^1(I)$ and even to $H_0^1(I)$, since

$$u(0) - K = \alpha - K \leq 0 \tag{5.7}$$
$$u(1) - K = \beta - K \leq 0 \tag{5.8}$$

Plugging v into (5.6), we obtain

$$\int_I u'^2 G'(u - K) + \int_I uG(u - K) = \int_I fG(u - K) \tag{5.9}$$

that is,

$$\int_I u'^2 G'(u - K) + \int_I (u - K) G(u - K) = \int_I (f - K) G(u - K) \tag{5.10}$$

But $f - K \leq 0$ and $G(u - K) \geq 0$, from which it follows that $(f - K) G(u - K) \leq 0$. Combining this with $\int_I u'^2 G'(u - K) \geq 0$ (since $G' \geq 0$ in $\mathbb{R}$), (5.10) gives

$$\int_I (u - K) G(u - K) \leq 0 \tag{5.11}$$

Since $tG(t) \geq 0, \quad \forall t \in \mathbb{R}$, the preceding inequality implies $(u - K) G(u - K) = 0$ a.e. It follows that $u \leq K$ a.e., and consequently everywhere on I, since u is continuous. The lower bound for u is obtained by applying this upper bound to $-u$. $\square$

Remark 1.66. When $f \in C(\bar{I})$, then $u \in C^2(\bar{I})$ and one can establish (5.5) by a different method: *the classical approach to the maximum principle*. Let $x_0 \in \bar{I}$ be the point where u attains its maximum on $\bar{I}$. If $x_0 = 0$ or if $x_0 = 1$ the conclusion is obvious. Otherwise, $0 < x_0 < 1$ and then $u'(x_0) = 0, u''(x_0) \leq 0$. From equation (5.5) it follows that

$$u(x_0) = f(x_0) + u''(x_0) \leq f(x_0) \leq K \tag{5.12}$$

and therefore $u \leq K$ on I.

Here are some immediate consequences of Theorem 1.65.

Corollary 1.67. *Let u be a solution of* (5.6).

1. *If $u \geq 0$ on ∂I and if $f \geq 0$ on I, then $u \geq 0$ on I.*
2. *If $u = 0$ on ∂I and if $f \in L^\infty(I)$, then $\|u\|_{L^\infty(I)} \leq \|f\|_{L^\infty(I)}$.*
3. *If $f = 0$ on I, then $\|u\|_{L^\infty(I)} \leq \|u\|_{L^\infty(\partial I)}$.*

We have a similar result for the case of Neumann condition.

Proposition 1.68. *Let* $f \in L^2(I)$ *with* $I = (0,1)$ *and let* $u \in H^2(I)$ *be the solution of the problem*

$$-u'' + u = f \text{ on } I \tag{5.13}$$

$$u'(0) = u'(1) = 0 \tag{5.14}$$

Then we have, for every $x \in \bar{I}$,

$$\inf_I f \le u(x) \le \sup_I f \tag{5.15}$$

Proof. We have

$$\int_I u'v' + \int_I uv = \int_I fv, \quad \forall v \in H^1(I) \tag{5.16}$$

Plug $v = G(u-K)$ into (5.16) with $K = \sup_I f$ and the same function G as above.

We shall show that $u \le K$ on I. The function $v = G(u-K)$ belongs to $H^1(I)$. Plugging v into (5.16) gives

$$\int_I u'^2 G'(u-K) + \int_I uG(u-K) = \int_I fG(u-K) \tag{5.17}$$

that is,

$$\int_I u'^2 G'(u-K) + \int_I (u-K)G(u-K) = \int_I (f-K)G(u-K) \tag{5.18}$$

But $f - K \le 0$ and $G(u-K) \ge 0$, from which it follows that $(f-K)G(u-K) \le 0$. Combining this with $\int_I u'^2 G'(u-K) \ge 0$ as before, (5.18) gives

$$\int_I (u-K)G(u-K) \le 0 \tag{5.19}$$

Since $tG(t) \ge 0, \ \forall t \in \mathbb{R}$, the preceding inequality implies $(u-K)G(u-K) = 0$ a.e. It follows that $u \le K$ a.e., and consequently everywhere on I, since u is continuous. The lower bound for u is obtained by applying this upper bound to $-u$. $\square$

Remark 1.69. If $f \in C(\bar{I})$, then $u \in C^2(\bar{I})$ and we can establish (5.15) along the same lines as in Remark 1.66 as follows. Let $x_0 \in \bar{I}$ be the point where u attains its maximum on $\bar{I}$. If $1 < x_0 < 1$ then $u'(x_0) = 0, u''(x_0) \le 0$. From equation (5.13) it follows that

$$u(x_0) = f(x_0) + u''(x_0) \le f(x_0) \le K \tag{5.20}$$

with $K = \sup_I f$. Otherwise, if u achieves its maximum on ∂I, i.e., $x_0 = 0$ or $x_0 = 1$. Suppose that $x_0 = 0$ (the case $x_0 = 1$ is handled similarly), then $u''(0) \le 0$ (extending u by reflection to the left of 0 and using the fact that $u'(0) = 0$). We extend u into $[-1,1]$ by the following function

$$\tilde{u} : [-1,1] \to \mathbb{R} \tag{5.21}$$

$$\tilde{u}(x) = u(|x|), \quad \forall v \in [-1,1] \tag{5.22}$$

We have

$$\tilde{u}(x) = \tilde{u}''(x) + f(x) \leq \tilde{u}''(x) + K, \quad \forall x \in I \tag{5.23}$$

Since $u \in C^2\left(\bar{I}\right)$, $\tilde{u} \in C^2([-1,1])$. Letting $x \to 0$ in (5.23) yields $\tilde{u}(0) \leq \tilde{u}''(0) + K$. Hence, $u(0) = u''(0) + K \leq K$, and therefore $u \leq \sup\limits_I f$.

Remark 1.70. Let $f \in L^2(\mathbb{R})$ and let $u \in H^2(\mathbb{R})$ be the solution of

$$-u'' + u = f \text{ on } R \tag{5.24}$$

$$u(x) \to 0 \text{ as } |x| \to \infty \tag{5.25}$$

discussed in Example 1.62. Then we have, for all $x \in \mathbb{R}$,

$$\inf_{\mathbb{R}} f \leq u(x) \leq \sup_{\mathbb{R}} f \tag{5.26}$$

6 Eigenfunctions and Spectral Decomposition

The following is a basic result.

Theorem 1.71. *Let $p \in C^1\left(\bar{I}\right)$ with $I = (0,1)$ and $p \geq \alpha > 0$ on I; let $q \in C\left(\bar{I}\right)$. Then there exist a sequence (λ_n) of real numbers and a Hilbert basis (e_n) of $L^2(I)$ such that $e_n \in C^2\left(\bar{I}\right)$ $\forall n$ and*

$$-\left(p{e_n}'\right)' + qe_n = \lambda_n e_n \text{ on } I \tag{6.1}$$

$$e_n(0) = e_n(1) = 0 \tag{6.2}$$

Furthermore, $\lambda_n \to +\infty$ as $n \to +\infty$.

One says that the (λ_n) are the *eigenvalues* of the differential operator $Au = -(pu')' + qu$ with Dirichlet boundary condition and that the (e_n) are the associated *eigenfunctions*.

Proof. We can always assume $q \geq 0$, for if not, pick any constant C such that $q + C \geq 0$, which amounts to replacing λ_n by $\lambda_n + C$ in (6.1). For every $f \in L^2(I)$ there exists a unique $u \in H^2(I) \cap H_0^1(I)$ satisfying

$$-(pu')' + qu = f \text{ on } I \tag{6.3}$$

$$u(0) = u(1) = 0 \tag{6.4}$$

Denote by T the operator $f \mapsto u$ *considered as an operator from $L^2(I)$ into* $L^2(I)$.[30]

We claim that T is self-adjoint and compact. First, the compactness. Because of (6.3)-(6.4) we have

$$\int_I pu'^2 + \int_I qu^2 = \int_I fu \tag{6.5}$$

[30] We could also envisage T as an operator from H_0^1 into H_0^1 (see Section 9.8, Remark 28, [1]).

and thus $\alpha \|u'\|^2_{L^2(I)} \le \|f\|_{L^2(I)}\|u\|_{L^2(I)}$. It follows that $\|u\|_{H^1(I)} \le C\|f\|_{L^2(I)}$, where C is a constant depending only on α. Indeed, since $u \in H^1_0(I)$, we can apply Poincaré's inequality to $u \in H^1_0(I)$ to obtain

$$\|u\|_{H^1(I)} \le (|I| + 1) \|u'\|_{L^2(I)} \tag{6.6}$$

i.e., $\|u\|_{L^2(I)} \le \|u'\|_{L^2(I)}$. If $\|u\|_{L^2(I)} = 0$, then $\|u'\|_{L^2(I)} = 0$ since $\alpha \|u'\|^2_{L^2(I)} \le \|f\|_{L^2(I)}\|u\|_{L^2(I)}$. And $\|u\|_{H^1(I)} \le C\|f\|_{L^2(I)}$ is obvious for all positive constant C. We now suppose that $\|u\|_{L^2(I)} > 0$, then

$$\|f\|_{L^2(I)} \ge \frac{\alpha \|u'\|^2_{L^2(I)}}{\|u\|_{L^2(I)}} \tag{6.7}$$

$$\ge \alpha\|u'\|_{L^2(I)} \tag{6.8}$$

Hence,

$$\|u\|_{H^1(I)} = \|u\|_{L^2(I)} + \|u'\|_{L^2(I)} \tag{6.9}$$

$$\le \|u\|_{L^2(I)} + \|u'\|_{L^2(I)} \tag{6.10}$$

$$\le 2\|u'\|_{L^2(I)} \tag{6.11}$$

$$\le \frac{2}{\alpha}\|f\|_{L^2(I)} \tag{6.12}$$

This can be written as

$$\|Tf\|_{H^1(I)} \le \frac{2}{\alpha}\|f\|_{L^2(I)}, \ \ \forall f \in L^2(I) \tag{6.13}$$

Since the injection of $H^1(I)$ into $L^2(I)$ is compact (because I is bounded, see Sobolev embedding theorem), we deduce that T is a compact operator from $L^2(I)$ into $L^2(I)$. Next, we show that T is self-adjoint, i.e.,

$$\int_I (Tf)\, g = \int_I f\,(Tg), \ \ \forall f, g \in L^2(I) \tag{6.14}$$

Indeed, setting $u = Tf$ and $v = Tg$, we have

$$-(pu')' + qu = f \tag{6.15}$$

and

$$-(pv')' + qv = g \tag{6.16}$$

Multiplying (6.15) by v and (6.16) by u and then integrating, we obtain

$$\int_I pu'v' + \int_I quv = \int_I fv = \int_I gu \tag{6.17}$$

which is the desired conclusion.

Finally, we note that

$$\int_I (Tf)\, f = \int_I uf \tag{6.18}$$

$$= \int_I pu'^2 + qu^2 \geq 0, \ \ \forall f \in L^2(I) \tag{6.19}$$

and also that $N(T) = \{0\}$, since $Tf = 0$ implies $u = 0$ and so $f = 0$.

Applying Theorem 6.11, [1], we know that $L^2(I)$ admits a Hilbert basis $(e_n)_{n\geq 1}$ consisting of eigenvectors of T with corresponding eigenvalues $(\mu_n)_{n\geq 1}$. We have $\mu_n > 0$ $\forall n$ ($\mu_n \geq 0$ by (6.18)-(6.19) and $\mu_n \neq 0$, since $N(T) = \{0\}$). We also know that $\mu_n \to 0$. Writing that $Te_n = \mu_n e_n$, we obtain

$$-({pe_n}')' + qe_n = \lambda_n e_n \text{ with } \lambda_n = \frac{1}{\mu_n} \tag{6.20}$$

$$e_n(0) = e_n(1) = 0 \tag{6.21}$$

In addition, we have $e_n \in C^2(\bar{I})$, since $f = \lambda_n e_n \in C(\bar{I})$ (in fact $e_n \in C^\infty(\bar{I})$ if $p, q \in C^\infty(\bar{I})$).

Example 1.72. If $p \equiv 1$ and $q \equiv 0$ we have

$$e_n(x) = \sqrt{2}\sin(n\pi x) \tag{6.22}$$

$$\lambda_n = n^2\pi^2 \tag{6.23}$$

for $n = 1, 2, \ldots$.

Remark 1.73. *For the same differential operator the eigenvalues and the eigenfunctions vary with the boundary conditions.* As an exercise (Why?) determine the eigenvalues of the operator $Au = -u''$ with the boundary conditions of Examples 1.55, 1.57, 1.59, 1.60, 1.61.

Remark 1.74. The assumption that I is *bounded* enters in an essential way in showing the *compactness* of the operator T. When I is not bounded the conclusion of Theorem 1.71 is in general false;[31] one encounters instead the very interesting phenomenon of *continuous spectrum.* In Exercise 8.38, [1], we determine the eigenvalues and the spectrum of the operator $T : f \mapsto u$, where $u \in H^2(\mathbb{R})$ is the solution of problem (4.118)-(4.119): T is a self-adjoint bounded operator from $L^2(\mathbb{R})$ into itself, but it is *not compact.*

7 Comments

7.1 Some Further Inequalities

Let us mention some very useful inequalities involving the Sobolev norms.

7.1.1 Poincaré-Wirtinger's Inequality

Let I be a bounded interval. Given $u \in L^2(I)$, set

$$\bar{u} = \frac{1}{|I|}\int_I u \tag{7.1}$$

(this is the mean of u on I). We have

$$\|u - \bar{u}\|_\infty \leq \|u'\|_1, \ \ \forall u \in W^{1,1}(I) \tag{7.2}$$

[31] In certain circumstances, with some appropriate assumptions on p and q, the conclusion of Theorem 1.71 still holds on unbounded intervals (see Problem 51, [1]).

7.1.2 Hardy's Inequality

Let $I = (0, 1)$ and let $u \in W_0^{1,p}(I)$ with $1 < p < \infty$. Then the function

$$v(x) = \frac{u(x)}{x(1-x)} \tag{7.3}$$

belongs to $L^p(I)$ and furthermore,

$$\|v\|_p \leq C_p \|u'\|_p, \quad \forall u \in W_0^{1,p}(I) \tag{7.4}$$

7.1.3 Interpolation Inequalities of Gagliardo-Nirenberg

Let I be a bounded interval and let $1 \leq r \leq \infty, 1 \leq q \leq p \leq \infty$. Then there exists a constant C such that

$$\|u\|_p \leq C \|u\|_q^{1-a} \|u\|_{W^{1,r}}^a, \quad \forall u \in W^{1,r}(I) \tag{7.5}$$

where $0 \leq a \leq 1$ is defined by

$$a\left(\frac{1}{q} - \frac{1}{r} + 1\right) = \frac{1}{q} - \frac{1}{p} \tag{7.6}$$

In particular, it follows from inequality (7.5) that if $p < \infty$ (or even if $p = \infty$ but $r > 1$), then

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 \text{ s.t. } \|u\|_p \leq \varepsilon \|u\|_{W^{1,r}} + C_\varepsilon \|u\|_q, \quad \forall u \in W^{1,r}(I) \tag{7.7}$$

One can also establish (7.7) by a direct *"compactness method"*; see Exercise 8.5, [1]. In particular, we call attention to the inequality

$$\|u'\|_p \leq C \|u\|_{W^{2,r}}^{\frac{1}{2}} \|u\|_q^{\frac{1}{2}}, \quad \forall u \in W^{2,r}(I) \tag{7.8}$$

where p is the *harmonic mean* of q and r, i.e., $\frac{1}{p} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{r}\right)$.

7.2 Hilbert-Schmidt Operators

It can be shown that the operator $T : f \mapsto u$ that associates to each f in $L^2(I)$ the unique solution u of the problem

$$-(pu')' + qu = f \text{ on } I = (0, 1) \tag{7.9}$$
$$u(0) = u(1) = 0 \tag{7.10}$$

(assuming $p \geq \alpha > 0$ and $q \geq 0$) is a Hilbert-Schmidt operator from $L^2(I)$ into $L^2(I)$; see Exercise 8.37, [1].

7.3 Spectral Properties of Sturm-Liouville Operators

Many spectral properties of the Sturm-Liouville operator $Au = -(pu')' + qu$ with Dirichlet condition on a bounded interval I are known. Among these let us mention that:

1. Each eigenvalue has *multiplicity one*: it is then said that each eigenvalue is *simple*.

2. If the eigenvalues (λ_n) are arranged in increasing order, then the eigenfunction $e_n(x)$ corresponding to λ_n possesses exactly $(n-1)$ zeros on I; in particular the *first eigenfunction* $e_1(x)$ has a *constant sign* on I, and usually one takes $e_1 > 0$ on I.

3. The quotient $\frac{\lambda_n}{n^2}$ converges as $n \to \infty$ to a positive limit.

Some of these properties are discussed in Exercises 8.33, 8.42 and Problem 49.

The celebrated Gelfand-Levitan theory deals with an importtant "inverse" problem: what informations on the function $q(x)$ can one retrieve purely from the knowledge of the spectrum of the Sturm-Liouville operator $Au = -u'' + q(x)u$? This question has attracted much attention because of its numerous applications; see Comment 13 in Chapter 9, [1]. □

THE END

References

[1] Haim Brezis, Functional Analysis, *Sobolev Spaces and Partial Differential Equations*, Springer.