

Mathematical Analysis & Numerical Analysis

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1 Basic

Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng

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chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

Resources – Tài nguyên.

- [Rud76]. WALTER RUDIN. *Principles Principles of Mathematical Analysis*.
- [Rud73; Rud87]. WALTER RUDIN. *Real & Complex Analysis*.
- [Rud91]. WALTER RUDIN. *Functional Analysis*.
- [LL01]. ELLIOTT LIEB, MICHAEL LOSS. *Analysis*.
- [EG15]. LAWRENCE C. EVANS, RONALD F. GARIEPY. *Measure Theory & Fine Properties of Functions*.

“Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones.” – [LL01, Chap. 1, p. 1]

2 Measure & Integration – Độ Đo & Tích Phân

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

“The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$ -dimensional volume ‘in the other direction’ by 1st computing the n -dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y , which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*” – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

3 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

- [Bre11]. HAÏM BREZIS. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations*.
- [Eva10]. LAWRENCE C. EVANS. *Partial Differential Equations*.
- [GT01]. DAVID GILBARG, NEIL S. TRUDINGER. *Elliptic Partial Differential Equations of 2nd Order*.

3.1 Weak solution – Nghiệm yếu

Definition 1 (Weak solution – Nghiệm yếu). “In mathematics, a weak solution (also called a generalized solution) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of *distributions*.” – [Wikipedia/weak solution](#)

“Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the *weak formulation*, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough.” – [Wikipedia/weak solution](#)

Example 1 (1st-order wave equation). The 1st-order *wave equation* $\partial_t u + \partial_x u = 0$ in \mathbb{R}^2 with $u = u(t, x)$ has the weak form $\int_{\mathbb{R}^2} u \partial_t \varphi + u \partial_x \varphi \, dt \, dx = 0$ has a solution $u(t, x) = |t - x|$ which may be checked by splitting the integrals over region $\{x \geq t\}$ & $\{x \leq t\}$ where u is smooth.

“The notion of weak solution based on distribution is sometimes inadequate. In the case of *hyperbolic systems*, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. *Hamilton-Jacobi equation*, there is a very different definition of weak solution called *viscosity solution*.” – [Wikipedia/weak solution](#)

3.1.1 General idea

When solving a differential equation in u , one can rewrite it using a **test function** φ s.t. whatever derivatives in u show up in the equation, they are “transferred” via integration by parts to φ , resulting in an equation without derivatives of u . This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x}, \partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}), \quad (1)$$

where the multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ varies over some finite set in \mathbb{N}^d & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x}, \partial)u(\mathbf{x}) = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) \, d\mathbf{x} = 0, \quad (2)$$

where the differential operator $Q(\mathbf{x}, \partial)$ is given by the formula

$$Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) = \sum (-1)^{|\alpha|} \partial^\alpha [a_\alpha(\mathbf{x})\varphi(\mathbf{x})], \quad (3)$$

which is the **formal adjoint** of $P(\mathbf{x}, \partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$ -times differentiable function u defined on the open set W s.t. $P(\mathbf{x}, \partial)u(\mathbf{x}) = 0, \forall \mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) \, d\mathbf{x} = 0, \forall \varphi \in C_c^\infty(W)$.

3.2 Viscosity solution – Nghiệm trơn/nhớt

Example 2 (Viscosity solution for Hamilton–Jacobi equation). *Hamilton–Jacobi equation.*

3.3 Very weak solution – Nghiệm rất yếu

Example 3 (Very weak solution of porous medium equation (PME)). *[Váz07].*

Example 4 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term). *[DDN20].* Given $f \in L^1_\delta(\Omega), \lambda \geq 0$, a function $u \in L^1_\delta(\Omega)$ is called a very weak solution of

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega, \\ |u|^{m-1}u = 1 & \text{on } \Gamma, \end{cases} \quad (4)$$

if $|u|^{m-1}u \in L^1(\Omega)$ and

$$\int_\Omega u^m \Delta \varphi + \lambda u \varphi \, d\mathbf{x} = \int_\Omega f \varphi \, d\mathbf{x} - \int_\Gamma \partial_{\mathbf{n}} \varphi \, d\mathbf{x}. \quad (5)$$

Example 5 (Very weak solution of NSEs). *[Tsa18].*

4 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

- [ABT18]. RICHARD ASTER, BRIAN BORCHERS, CLIFFORD H. THURBER. *Parameter Estimation & Inverse Problems.*
- [Kir21]. ANDREAS KIRSCH. *An Introduction to The Mathematical Theory of Inverse Problems.*
- [IJ15]. KAZUFUMI ITO, BANGTI JIN. *Inverse Problems.*

5 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

- [Wal15]. SHAWN W. WALKER. *The Shapes of Things.*
- [Car16]. MANFREDO P. DO CARMO. *Differential Geometry of Curves & Surfaces.*
- [Küh15]. WOLFGANG KÜHNEL. *Differential Geometry.*

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

6 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

- [LeV07]. RANDALL J. LEVEQUE. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems*.

7 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

- [BS08]. SUSANNE C. BRENNER, L. RIDGWAY SCOTT. *The Mathematical Theory of FEMs*.
- [EG04]. ALEXANDRE ERN, JEAN-LUC GUERMOND. *Theory & Practice of Finite Elements*.
- [GR86]. VIVETTE GIRAULT, PIERRE-ARNAUD RAVIART. *FEMs for NSEs*.
- [Gun89]. MAX D. GUNZBURGER. *FEMs for Viscous Incompressible Flows*.
- [Joh16]. VOLKER JOHN. *FEMs for Incompressible Flow Problems*.

I met VOLKER JOHN, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagorinsky, k - ϵ & their simulations.

8 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

- [EGH19]. ROBERT EYMARD, THIERRY GALLOUËT, RAPHAËLE HERBIN. *Finite Volume Methods*.
- [LeV02]. RANDALL J. LEVEQUE. *FEMs for Hyperbolic Problems*.

9 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

9.1 Haïm Brezis

9.2 Lawrence Chris Evans

9.3 Peter Lax

9.4 Andrew Joseph Majda

Resources – Tài nguyên.

[Eng+23]

- “He was hard working until the end even though he suffered from serious health issues for quite some time.”
- “He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, and observed and experimental data ... Andy Majda’s modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis and numerical modeling, (iii) observed phenomena and experimental data, and (iv) qualitative and/or asymptotic modeling [Maj00].”
- “Andy’s legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored.”
- “The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed “Majda’s model” for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived “Beale-Kato-Majda criterion,” which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84].”
- “At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science.”

9.5 Vladimir Mazya

9.6 Jindřich Nečas

9.7 Louis Nirenberg

[Váz20]

10 Miscellaneous

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