My notes on ⋆ Nguyễn Văn Mậu, Lê Ngọc Lăng, Phạm Thế Long, Nguyễn Minh Tuấn, Các đề thi Olympic toán sinh viên toàn quốc

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Abstract

I take some notes when I learn the book [1].

Contents

1 Selected VMC problems

Problem 1. (VMC 1993, Day 2, Problem 3).
Let $p(x)$ be a nonconstant polynomial with real coefficients. Prove that if the system of equations

$$\begin{cases} \int\limits_0^x p(t)\sin t dt = 0 \\ \int\limits_0^x p(t)\cos t dt = 0 \end{cases} \tag{1.1}$$

has real roots then the number of real roots must be finite.

HINT 1. Let

$$\begin{aligned} U_k &= \int\limits_0^x p^{(k)}(t)\sin t dt \\ V_k &= \int\limits_0^x p^{(k)}(t)\cos t dt \end{aligned} \tag{1.2}$$

Suppose that $\deg p = n$, then we have $U_k = 0, V_k = 0, \forall k > n$. Use integration by parts formula, get

$$\begin{cases} U_k = -p^{(k)}(t)\cos t\Big|_0^x + \int\limits_0^x p^{(k+1)}(t)\cos t dt \\ V_k = p^{(k)}(t)\sin t\Big|_0^x - \int\limits_0^x p^{(k+1)}(t)\sin t dt \end{cases} \tag{1.3}$$

Deduce that

$$\begin{cases} U_k = -p^{(k)}(t)\cos t\Big|_0^x + V_{k+1} \\ V_k = p^{(k)}(t)\sin t\Big|_0^x - U_{k+1} \end{cases} \tag{1.4}$$

$$\begin{cases} U_k = -p^{(k)}(t)\cos t\Big|_0^x + p^{(k+1)}(t)\sin t\Big|_0^x - U_{k+2} \\ V_k = p^{(k)}(t)\sin t\Big|_0^x + p^{(k+1)}(t)\cos t\Big|_0^x - V_{k+2} \end{cases}, \forall k \in \mathbb{N} \tag{1.5}$$

$$\begin{cases} U_0 = -\sum\limits_{k=0}^{2k\le n} p^{(2k)}(t)\cos t\Big|_0^x + \sum\limits_{k=0}^{2k+1\le n} p^{(2k+1)}(t)\sin t\Big|_0^x \\ V_k = \sum\limits_{k=0}^{2k\le n} p^{(2k)}(t)\sin t\Big|_0^x + \sum\limits_{k=0}^{2k+1\le n} p^{(2k+1)}(t)\cos t\Big|_0^x \end{cases} \tag{1.6}$$

Put

$$\begin{cases} p_1(t) = \sum\limits_{k=0}^{2k\le n} p^{(2k)}(t) \\ p_2(t) = \sum\limits_{k=0}^{2k+1\le n} p^{(2k+1)}(t) \end{cases} \tag{1.7}$$

Consider the case n is even (the case n odd is similar). Since n is even, easy to get $\deg p_1 = n, \deg p_2 = n-1$. Rewrite the formula (1.3) in the form

$$\begin{cases} U_0 = -p_1\left(t\right)\,\cos t|_0^x + p_2\left(t\right)\,\sin t|_0^x \\ V_0 = p_1\left(t\right)\,\sin t|_0^x + p_2\left(t\right)\,\cos t|_0^x \end{cases} \tag{1.8}$$

Call X is the solution of the given system of equations

$$\begin{cases} U_0 = 0 \\ V_0 = 0 \end{cases} \tag{1.9}$$

For $\forall x \in X$ we have

$$\begin{cases} -p_1\left(t\right)\,\cos t|_0^x + p_2\left(t\right)\,\sin t|_0^x = 0 \\ p_1\left(t\right)\,\sin t|_0^x + p_2\left(t\right)\,\cos t|_0^x = 0 \end{cases} \tag{1.10}$$

Put $p_1\left(0\right) = a, p_2\left(0\right) = b$. Then

$$\begin{cases} p_2\left(x\right)\sin x - p_1\left(x\right)\cos x = -a \\ p_2\left(x\right)\cos x + p_1\left(x\right)\sin x = b \end{cases} \tag{1.11}$$

Deduce that

$$\left(p_2\left(x\right)\sin x - p_1\left(x\right)\cos x\right)^2 + \left(p_2\left(x\right)\cos x + p_1\left(x\right)\sin x\right)^2 = a^2 + b^2 \tag{1.12}$$

Hence

$$p_1^2\left(x\right) + p_2^2\left(x\right) - \left(a^2 + b^2\right) = 0 \tag{1.13}$$

Call Y is the set of solutions of poynomial

$$Q\left(x\right) = p_1^2\left(x\right) + p_2^2\left(x\right) - \left(a^2 + b^2\right) \tag{1.14}$$

Deduce that $X \subset Y$. Since $\deg Q\left(x\right) = 2n$, $|X| \le |Y| \le 2n$, that means X has finite elements. $\square$

Hint 2. Rewrite the system of equations in the form

$$F\left(x\right) := \int\limits_0^x p\left(t\right)e^{it}dt = 0 \tag{1.15}$$

We have

$$F'\left(x\right) = p\left(x\right)e^{it} \tag{1.16}$$

so the equation $F'\left(x\right) = 0$ has finite solutions. Deduce that the equation $F\left(x\right) = 0$ has finite solutions. $\square$

Problem 2. (VMC 1994, Algebra, Problem 6).

Let $A \in \mathcal{M}_2(K)$ for which $A^2 = A$. Prove that the necessary and sufficient condition for $AX - XA = 0_2$ (where $X \in \mathcal{M}_2(K)$) is that there exists a $X_0 \in \mathcal{M}_2(K)$ for which

$$X = AX_0 + X_0A - X_0 \tag{1.17}$$

HINT. $X_0 = 2AX - X$. $\square$

Problem 3. (VMC 1994, Analysis, Problem 3b).
Let a function $f(x)$ which is differentiable on $[a,b]$ and $\forall x \in [a,b], |f'(x)| \leq |f(x)|$. Prove that

$$f(x) = 0, \forall x \in [a,b] \tag{1.18}$$

HINT. Suppose that x_0 is the solution of the equation $f(x) = 0$ for $x_0 \in [a,b]$. Use Taylor's expansion of f with x_0, get

$$f(x) = f(x_0) + f'(c)(x - x_0) = f'(c)(x - x_0) \tag{1.19}$$

Consider the close interval $G := \left[x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right] \cap [a,b]$. Since $f(x)$ is differentiable in $[a,b]$, $f(x)$ has maximum in G. Suppose that

$$|f(x_m)| = \max_{x \in G} |f(x)|, x_m \in G \tag{1.20}$$

Deduce that

$$|f(x_m)| = |f'(c_m)| |x_m - x_0| \leq |f(c_m)| |x_m - x_0| \leq \frac{1}{2} |f(c_m)| \leq \frac{1}{2} |f(x_m)| \tag{1.21}$$

So $f(x) = 0, \forall x \in G$. We have proved that
If for a point x in $[a,b]$ for which $f(x) = 0$, then $f(x) = 0$ in the neighborhood of x with radius $\frac{1}{2}$.
By considering different points x_0 for which $f(x_0) = 0$, tends to a and b, after finite steps, we have $f(x) = 0, \forall x \in [a,b]$. $\square$

Problem 4. (VMC 1995, Algebra, Problem 5).
Let $A = (a_{ij})_{n \times n}$ ($n > 1$) have rank r. Consider $A^* = (A_{ij})$ where A_{ij} is cofactor of element a_{ij} in A. Find the rank of A^*.

HINT. Divide into three cases

- **Case $r \leq n - 2$.** A_{ij} is the determinant of $(n-1) \times (n-1)$ matrix have rank $\leq n - 2$, so $A_{ij} = 0, \forall i, j$ and $A^* = 0$. Deduce that $rank A^* = 0$. In addition, $\det A = 0, AA^* = 0$.

- **Case $r = n - 1$.** A have a sub-determinant degree $n - 1$ is nonzero, so $A^* \neq 0$. Deduce that $rank A^* \geq 1$. Conclude that $rank A^* = 1$. (?)

- **Case** $r = n$**.** $\det A \neq 0$. Since $A^t A^* = |A|\, E$, $|A^*| \neq 0$ and $rank\,(A^*) = n$.

Three case are true. Done. $\square$

Problem 5. (VMC 1995, Analysis, Problem 5).
Let a continuous function $f(x)$ on $[a, b]$. Prove that

$$\max_{x\in[a,b]} |f\,(x)| = \lim_{p\to\infty} \left(\int_0^1 |f\,(x)|^p dx \right)^{\frac{1}{p}} \tag{1.22}$$

Hint. Since $f(x)$ is continuous on the compact set $[a, b]$ so $\exists c \in [a, b]$ for which

$$|f\,(c)| = \max_{x\in[0,1]} |f\,(x)| = M \tag{1.23}$$

Suppose that $a < c < b$, have

$$\forall \varepsilon > 0, \exists \delta > 0, |x - c| < \delta \Rightarrow |f\,(x) - f\,(c)| < \frac{\varepsilon}{4} \tag{1.24}$$

Then

$$\left(\int_{c-\delta}^{c+\delta} \left(M - \frac{\varepsilon}{4}\right) dx \right)^{\frac{1}{n}} < \left(\int_{c-\delta}^{c+\delta} |f^n\,(x)| dx \right)^{\frac{1}{n}} \tag{1.25}$$

$$< \left(\int_a^b |f^n\,(x)| dx \right)^{\frac{1}{n}} \tag{1.26}$$

$$< \left(\int_a^b M^n dx \right)^{\frac{1}{n}} \tag{1.27}$$

hence

$$(2\delta)^{\frac{1}{n}} \left(M - \frac{\varepsilon}{4}\right) < \left(\int_a^b |f^n\,(x)| dx \right)^{\frac{1}{n}} < M(b-a)^{\frac{1}{n}} \tag{1.28}$$

Have

$$\lim_{n\to\infty} (2\delta)^{\frac{1}{n}} = 1, \lim_{n\to\infty} (b-a)^{\frac{1}{n}} = 1 \tag{1.29}$$

so

$$\left(\int_a^b |f^n\,(x)| dx \right)^{\frac{1}{n}} = M \tag{1.30}$$

The case $c = a$, we have

$$\forall \varepsilon > 0, \exists \delta > 0, a < x < a + \delta \Rightarrow |f(x) - f(a)| < \varepsilon \tag{1.31}$$

then we proceeds similarly the first case. The case $c = b$ is similar, too. $\square$

Problem 6. (VMC 1998, Algebra, Problem 3).
Suppose that A is $n+1 \times n+2$ matrix defined by

$$A = \begin{pmatrix} C_0^0 & C_1^0 & C_2^0 & \cdots & C_n^0 & C_{n+1}^0 \\ 0 & C_1^1 & C_2^1 & \cdots & C_n^1 & C_{n+1}^1 \\ 0 & 0 & C_2^2 & \cdots & C_n^2 & C_{n+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_n^n & C_{n+1}^n \end{pmatrix} \tag{1.32}$$

where $C_n^k = \begin{pmatrix} n \\ k \end{pmatrix}$ (the left one is used in Vietnam, the right is used internationally).
Call D_k is the determinant which is obtained from A by deleting the k-th column ($k = 1, 2, ..., n+2$).
Prove that $D_k = C_{n+1}^{k-1}$

Hint. Add the last row

$$1, x, x^2, \ldots, x^{n+1} \tag{1.33}$$

to matrix A, we obtain the square matrix degree $n+2$. Call D_{n+2} is the determinant of the 'new matrix'. Transform D_{n+2}: Take the k-th, multiply with -1, then add to $k+1$-th column, for each $k = n+1, n, \ldots, 1$, we obtain

$$D_{n+2} = (x-1) D_{n+1} = \cdots = (x-1)^{n+1} \tag{1.34}$$

Deduce that

$$D_{n+2} = \sum_{k=1}^{n+2} (-1)^{n-k} C_{n+1}^{k-1} x^{k-1} \tag{1.35}$$

On the other hand, if we expand D_{n+2} respect to the last row, we have

$$D_{n+2} = \sum_{k=1}^{n+2} (-1)^{n-k} D_k x^{k-1} \tag{1.36}$$

Conclude that $D_k = C_{n+1}^{k-1}$. $\square$

Problem 7. (VMC 1998, Analysis, Problem 1)
Let $f(x) \in C^1([0,1])$ and $f(0) = 0$. Prove that

$$\int_0^1 |f(t) f'(t)| dt \le \frac{1}{2} \int_0^1 (f'(t))^2 dt \tag{1.37}$$

Hint. Consider

$$F(x) = \int_0^x |f(t)|\,|f'(t)|dt \tag{1.38}$$

$$G(x) = \frac{x}{2}\int_0^x (f'(t))^2 dt \tag{1.39}$$

Then

$$F'(x) = |f(x)|\,|f'(x)| \tag{1.40}$$

$$G'(x) = \frac{x}{2}(f'(x))^2 + \frac{1}{2}\int_0^x (f'(t))^2 dt \tag{1.41}$$

On the other hand

$$f(x) = \int_0^x f'(t)dt, \forall x \in [0,1] \tag{1.42}$$

Use Cauchy inequality

$$|f(x)| = \left|\int_0^x f'(t)dt\right| \le \left(\int_0^x dx\right)^{\frac{1}{2}} \left(\int_0^x (f'(t))^2 dt\right)^{\frac{1}{2}} \tag{1.43}$$

Then

$$|f(x)\,f'(x)| \le \sqrt{x}\left(\int_0^x (f'(t))^2 dt\right)^{\frac{1}{2}} |f'(x)| \tag{1.44}$$

Deduce that

$$|f(x)\,f'(x)| \le \frac{1}{2}\int_0^x (f'(t))^2 dt + \frac{x}{2}(f'(x))^2 \tag{1.45}$$

That means

$$F'(x) \le G'(x), \forall x \in [0,1] \tag{1.46}$$

$$F(1) - F(0) \le G(1) - G(0) \tag{1.47}$$

and

$$\left|\int_0^1 f(t)\,f'(t)dt\right| \le \frac{1}{2}\int_0^1 (f'(t))^2 dt \tag{1.48}$$

Done. $\square$

Problem 8. (VMC 2000, Algebra, Problem 4).
Let A be a square matrix degree n which has all elements in the main diagonal equal to 0, and other elements are equal to 1 or 2000.
Prove that $rankA = n$ or $rankA = n-1$.

Hint. Consider the matrix $B \in \mathcal{M}_n$ which has all elements equals to 1. Then

$$A - B = \{c_{ij}\}, c_{ij} \in \{-1, 0, 1999\} \tag{1.49}$$

Then

$$\det(A - B) = (-1)^n \,(\mathrm{mod} 1999) \tag{1.50}$$

Deduce that $\det(A-B) \neq 0$ and $n = rank\,(A-B) \leq rankA + rank\,(-B) = rankA + 1$. $\square$

Problem 9. (VMC 2001, Algebra, Problem 4).
Denote $\langle a, b\rangle$ for scalar product of two vectors $a, b \in \mathbb{R}$. Let $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$. Put

$$A = \begin{pmatrix} \langle a_1, a_1\rangle & \langle a_1, a_2\rangle & \cdots & \langle a_1, a_{k-1}\rangle & \langle a_1, a_k\rangle \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle & \cdots & \langle a_2, a_{k-1}\rangle & \langle a_2, a_{k-1}\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle a_{k-1}, a_1\rangle & \langle a_{k-1}, a_2\rangle & \cdots & \langle a_{k-1}, a_{k-1}\rangle & \langle a_{k-1}, a_k\rangle \\ \langle a_k, a_1\rangle & \langle a_k, a_2\rangle & \cdots & \langle a_k, a_{k-1}\rangle & \langle a_k, a_k\rangle \end{pmatrix} \tag{1.51}$$

Prove that

1. $\det A \geq 0$

2. A is a symmetric matrix and its all eigenvalues are nonzero.

Hint.
1. Write a_s as

$$a_s = (a_{s1}, a_{s2}, \ldots, a_{sn}), s = 1, 2, \ldots, k \tag{1.52}$$

Then

$$A = \begin{pmatrix} \sum\limits_{j=1}^n a_{1j}^2 & \sum\limits_{j=1}^n a_{1j}a_{2j} & \cdots & \sum\limits_{j=1}^n a_{1j}a_{kj} \\ \sum\limits_{j=1}^n a_{2j}a_{1j} & \sum\limits_{j=1}^n a_{2j}^2 & \cdots & \sum\limits_{j=1}^n a_{2j}a_{kj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{j=1}^n a_{kj}a_{1j} & \sum\limits_{j=1}^n a_{kj}a_{2j} & \cdots & \sum\limits_{j=1}^n a_{kj}^2 \end{pmatrix} \tag{1.53}$$

Deduce that

$$\det A = \sum_{1 \le j_1, j_2, \ldots, j_k \le n} a_{1j_1} a_{2j_2} \ldots a_{kj_k} \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_k} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{kj_1} & a_{kj_2} & \cdots & a_{kj_k} \end{vmatrix} \tag{1.54}$$

Let $A_{j_1 < j_2 < \cdots < j_k}$ are matrices in the right hand side of the expression (1.54) respect to $(j_1, j_2, \ldots, j_k)$ which is fixed and sorted increasingly. Then, all the terms have the same index set $(j_1, j_2, \ldots, j_k)$ have the sum equals to

$$\begin{aligned} &\sum_{j_1 < j_2 < \cdots < j_k} (-1)^{inv(j_1 < j_2 < \cdots < j_k)} a_{1j_1} a_{2j_2} \ldots a_{kj_k} \det\left(A_{j_1 < j_2 < \cdots < j_k}\right) \\ &= \left(\det\left(A_{j_1 < j_2 < \cdots < j_k}\right)\right)^2 \end{aligned} \tag{1.55}$$

From the expression (1.54), we obtain

$$\det A = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} \left(\det\left(A_{j_1 < j_2 < \cdots < j_k}\right)\right)^2 \ge 0 \tag{1.56}$$

2. Scalar product is a symmetric bi-linear form, that means $\langle a_i, a_j \rangle = \langle a_j, a_i \rangle$. So, A is symmetric matrix and all its eigenvalues are real. By the proof above, all sub-determinants of A are nonzero. Therefore, the characteristic polynomial of A has the form

$$P_A(t) = (-1)^k t^k + (-1)^{k-1} a_1 t^{k-1} + \cdots - a_{k-1} t + a_k \tag{1.57}$$

where coefficients $a_1, a_2, \ldots, a_k$ are nonzero. Deduce that $P_A(t) > 0$ when $t < 0$. So all eigenvalues of A are nonzero. □

Problem 10. (VMC 2001, Analysis, Problem 4).
Given f is defined on $\mathbb{R}$ and have 2-th derivative on $\mathbb{R}$ satisfying $f(x) + f''(x) \ge 0, \forall x \in R$.
Prove

$$f(x) + f(x + \pi) \ge 0, \forall x \in R \tag{1.58}$$

Hint. For each fixed $x \in \mathbb{R}$, consider the function

$$g(y) = f'(y) \sin(y - x) - f(y) \cos(y - x) \tag{1.59}$$

Have

$$g'(y) = \left[f''(y) + f(y)\right] \sin(y - x) \ge 0, \forall y \in [x, x + \pi] \tag{1.60}$$

Deduce that $g(y)$ is monotonically increasing on $[x, x + \pi]$. So $g(x) \le g(x + \pi)$, that means $f(x) + f(x + \pi) \ge 0$. □

Problem 11. (VMC 2003, Algebra, Problem 7).

Given a real coefficients polynomial $P(x)$ have degree n $(n \geq 1)$ have m real roots. Prove that the polynomial

$$Q(x) = \left(x^2 + 1\right) P(x) + P'(x) \tag{1.61}$$

has at least m real roots.

Hint. Consider the function

$$f(x) = e^{\frac{x^3}{3} + x} P(x) \tag{1.62}$$

The set of real roots of $f(x) = 0$ and the set of real roots of $P(x)$ is the same. By Rolle theorem, the equation

$$f'(x) = e^{\frac{x^3}{3} + x} \left[P(x) + \left(x^2 + 1\right) P'(x)\right] = 0 \tag{1.63}$$

has at least $m - 1$ real roots, or the equation

$$P'(x) + \left(x^2 + 1\right) P(x) = 0 \tag{1.64}$$

has as least $m - 1$ real roots. Consider two cases

- **Case 1. m is even.** If n is odd then $P(x)$ has at least $m + 1$ real roots, absurd. So, n must be even. Then

  $$P'(x) + \left(x^2 + 1\right) P(x) \tag{1.65}$$

  has degree equal to $n + 2$ (even number) and has $m - 1$ (odd number) real roots. Deduce that this polynomial must have at least $(m - 1) + 1 = m$ real roots.

- **Case 2. m is odd.** If n is even then $P(x)$ has at least $m + 1$ real roots, absurd. So n must be odd. Then

  $$P'(x) + \left(x^2 + 1\right) P(x) \tag{1.66}$$

  has degree $n + 2$ (odd number) and has $m - 1$ (even number) real roots. Deduce that this polynomial must have at least $(m-1)+1 = m$ real roots.

In both cases, we have desired result. □

Problem 12. (VMC 2004, Analysis, Problem 5).
Let $P(x), Q(x), R(x)$ are real coefficients polynomials which have degree $3, 2, 3$ respectively, satisfying the condition

$$(P(x))^2 + (Q(x))^2 = (R(x))^2 \tag{1.67}$$

How many real roots which the polynomial

$$T(x) = P(x)\, Q(x)\, R(x) \tag{1.68}$$

has at least (including repetition roots)?

Hint. Without loss of generality (w.l.o.g.), we can suppose that the coefficients respect of the maximal degree terms of polynomials P, Q, R are positive. Firstly, we prove that $Q(x)$ always have 2 real roots. We have $Q^2 = (R - P)(R + P)$. Since $\det P = \det R = 3$, $\det(R + P) = 3$. Since $\deg Q^2 = 4$, $\deg(R - P) = 1$. Therefore, Q^2 has real roots, so Q also has real roots. Since $\deg Q = 2$, Q has exactly 2 roots. Next, we prove that $P(x)$ always have 3 real roots. We have $P^2 = (R - Q)(R + Q)$. Since $\deg(R - Q) = \deg(R + Q) = 3$, $(R - Q)$ and $(R + Q)$ have real roots. If these two roots is distinct, P has 2 distinct real roots and the third root of P is also real. If $(R - Q)$ and $(R + Q)$ have a common real root $x = a$ then $x = a$ is root of R and Q. So,

$$R(x) = (x - a) R_1(x) \tag{1.69}$$

$$Q(x) = (x - a) Q_1(x) \tag{1.70}$$

$$P(x) = (x - a) P_1(x) \tag{1.71}$$

Put these formula into $P^2 = (R - Q)(R + Q)$, we get $P_1^2 = R_1^2 - Q_1^2$, where P_1, R_1 is quadratic, Q_1 is linear. We have

$$Q_1^2 = (R_1 - P_1)(R_1 + P_1) \tag{1.72}$$

Since Q_1^2 is quadratic and $R_1 + P_1$ is quadratic, $R_1 - P_1$ is constant polynomial. So, if $P_1(x) = ax^2 + bx + c, (a > 0), Q_1(x) = dx + e$, then $R_1(x) = ax^2 + bx + c + k$ and

$$k[R_1(x) + P_1(x)] = (dx + e)^2 \tag{1.73}$$

Hence $k > 0$. Take $x = -\frac{e}{d}$ in (1.73), we obtain

$$R_1\left(-\frac{e}{d}\right) + P_1\left(-\frac{e}{d}\right) = 0 \tag{1.74}$$

so $P_1\left(-\frac{e}{d}\right) = -\frac{k}{2} < 0$. Therefore, quadratic $P_1(x)$ has 2 real roots and $P(x)$ has 3 real roots.

Since P has 3 real roots, Q has 2 real roots and R is cubic (has at least 1 real root), the number of roots of $T(x)$ is larger or equal to 6. For example, if we choose

$$P(x) = x^3 + 3x^2 + 2x \tag{1.75}$$

$$Q(x) = 2\left(x^2 + 2x + 1\right) \tag{1.76}$$

$$R(x) = x^3 + 3x^2 + 4x + 2 \tag{1.77}$$

then $P^2 + Q^2 = R^2$ and PQR has exactly 6 real roots. □

Problem 13. (VMC 2005, Analysis, Problem 4).

Let f be a continuous function on $[0, 1]$ satisfying the condition

$$\int_x^1 f(t)dt \geq \frac{1 - x^2}{2}, \forall x \in [0, 1] \tag{1.78}$$

Prove that

$$\int_0^1 (f(x))^2 dx \geq \int_0^1 x f(x) dx \tag{1.79}$$

Hint. We have

$$0 \leq \int_0^1 (f(x) - x)^2 dx = \int_0^1 (f(x))^2 dx - 2\int_0^1 x f(x) dx + \frac{1}{3} \tag{1.80}$$

Hence

$$\int_0^1 (f(x))^2 dx \geq 2\int_0^1 x f(x) dx - \frac{1}{3} \tag{1.81}$$

Put

$$A = \int_0^1 \left(\int_x^1 f(t) dt \right) dx \tag{1.82}$$

We have

$$A = \int_0^1 \left(\int_x^1 f(t) dt \right) dx \geq \int_0^1 \frac{1 - x^2}{2} dx = \frac{1}{3} \tag{1.83}$$

On the other hand

$$A = \int_0^1 \left(\int_x^1 f(t) dt \right) dx = x \int_0^1 f(t) dt \Big|_0^1 + \int_0^1 x f(x) dx = \int_0^1 x f(x) dx \tag{1.84}$$

Therefore

$$\int_0^1 x f(x) dx \geq \frac{1}{3} \tag{1.85}$$

(1.81) and (1.85)complete the proof. □

Problem 15. (VMC 2005, Analysis, Problem 5).
Let $f(x)$ be a function which has continuous 2-th derivative on $\mathbb{R}$, satisfying the condition $f(\alpha) = f(\beta) = a$. Prove that

$$\max_{x \in [\alpha, \beta]} \{f''(x)\} \geq \frac{8(a - b)}{(\alpha - \beta)^2} \tag{1.86}$$

where $b = \min\limits_{x \in [\alpha, \beta]} \{f(x)\}$

HINT. Since f is continuous on compact set $[\alpha,\beta]$, f attains its minimum on $[\alpha,\beta]$, that means there exists a $c \in (\alpha,\beta)$ for which $f'(c)=0$ and $f(c)=\min\limits_{x\in[\alpha,\beta]}\{f(x)\}=b$. Use the Taylor expanding of function $f(x)$ respect to c

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(\theta(x))}{2}(x-c)^2 \tag{1.87}$$

Take $x=\alpha$ and $x=\beta$ into the above equality, we obtain

$$a=b+\frac{f''(\theta(\alpha))}{2}(\alpha-c)^2 \tag{1.88}$$

$$a=b+\frac{f''(\theta(\beta))}{2}(\beta-c)^2 \tag{1.89}$$

That is

$$\begin{array}{l} f''(\theta(\alpha))=\frac{2(a-b)}{(\alpha-c)^2} \\ f''(\theta(\beta))=\frac{2(a-b)}{(\beta-c)^2} \end{array} \tag{1.90}$$

Multiply two inequalities, we obtain

$$f''(\theta(\alpha))f''(\theta(\beta))=\frac{4(a-b)^2}{(\alpha-c)^2(\beta-c)^2}\geq\frac{64(a-b)^2}{(\alpha-\beta)^2} \tag{1.91}$$

This inequality completes our proof. □.

2 Training problems

Problem 1. *Suppose that functional equation*

$$f(ax+y)=Af(x)+f(y),(aA\neq 0),\forall x,y\in\mathbb{R} \tag{2.1}$$

has non-constant solution. Prove that if a (or A) is algebraic number with minimal polynomial $P_a(t)$ (respectively, $P_A(t)$, then A is algebraic number and

$$P_a(t)\equiv P_A(t) \tag{2.2}$$

HINT. $f(0)=0$ and $f(ax)=Af(x)$ and by induction:

$$f\left(a^k x\right)=A^k f(x),k\in\mathbb{N} \tag{2.3}$$

Suppose that

$$P_a(t)=t^n+\sum_{i=0}^{n-1}r_i t^i,(r_0,\ldots,r_{n-1}\in\mathbb{Q}) \tag{2.4}$$

Then, by (2.3)

$$f\left[\left(a^n+\sum_{i=0}^{n-1}r_ia^i\right)x\right]=f\left(a^nx\right)+\sum_{i=0}^{n-1}r_if\left(a^ix\right) \tag{2.5}$$

$$=\left(A^n+\sum_{i=0}^{n-1}r_iA^i\right)f\left(x\right) \tag{2.6}$$

Since $f(x)$ is non-constant,

$$A^n+\sum_{i=0}^{n-1}r_iA^i=0 \tag{2.7}$$

Hence A is algebraic number. Deduce that $P_a(t)$ is divisor of $P_A(t)$ and since $P_A(t)$ is minimal polynomial, we obtain

$$P_a\left(t\right)\equiv P_A\left(t\right) \tag{2.8}$$

Conversely, if A is algebraic number satisfy (2.7) then do the reverse process, we obtain

$$a^n+\sum_{i=0}^{n-1}r_ia^i=0 \tag{2.9}$$

hence (2.2). □

Problem 2. *Do there exist functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, where g is periodically function satisfying*

$$x^3=f\left(\lfloor x\rfloor\right)+g\left(\lfloor x\rfloor\right),\forall x\in\mathbb{R} \tag{2.10}$$

Hint. Suppose for the contrary, f and g satisfy all above conditions. Let $T>0$ denote the periodic of g. We have

$$\left(x+T\right)^3=f\left(\lfloor x+T\rfloor\right)+g\left(\lfloor x+T\rfloor\right) \tag{2.11}$$

Hence

$$f\left(\lfloor x+T\rfloor\right)-f\left(\lfloor x\rfloor\right)\equiv T^3+3T^2x+3Tx^2 \tag{2.12}$$

For $x\in[0,\lfloor T\rfloor+1-T)$, the LHS of (2.12) is constant, so it give a quadratic which has infinity roots. Therefore, $T=0$, absurd. □

Problem 3. *Let α_1,α_2 and β_1,β_2 satisfy*

$$\frac{\alpha_2}{\alpha_1}\in\mathbb{Q},\frac{\alpha_2}{\alpha_1}\leq\frac{\beta_2}{\beta_1} \tag{2.13}$$

If $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} f\left(x+\alpha_1\right)\leq f\left(x\right)+\beta_1 \\ f\left(x+\alpha_2\right)\geq f\left(x\right)+\beta_2 \end{cases},\forall x\in\mathbb{R} \tag{2.14}$$

then $g\left(x\right)=f\left(x\right)-\frac{\beta_2}{\alpha_1}x$ is periodical function.

3 Interpolating polynomial

Interpolation 1 (Lagrange interpolating polynomial).
Given $x_1, x_2, \ldots, x_n$ be distinct numbers. Find all polynomials $P(x)$ with $\deg P(x) \leq n-1$ satisfying $P(x_k) = a_k \in \mathbb{R}, k = 1, ..., n$ ($a_1, ..., a_n$ are given).

Hint. The Lagrange interpolating polynomial

$$P\left(x\right) = \sum_{i=1}^{n} P\left(x_i\right) \frac{\prod\limits_{k=1,k\neq i}^{n} \left(x - x_k\right)}{\prod\limits_{k=1,k\neq i}^{n} \left(x_i - x_k\right)} \tag{3.1}$$

Uniqueness is obtained by: *If two polynomials degree less than or equal to $n-1$ have same values at n points, then they are identical.* □

Interpolation 2 (Taylor - Gontcharov expanding or Newton interpolating formula).
Given $(x_0, x_1, \ldots, x_n)$ and $(a_0, a_1, \ldots, a_n)$. Find all polynomials $P(x)$ with $\deg P(x) \leq n$ *satisfying*

$$P^{(k)}\left(x_k\right) = a_k, k \in \{0, 1, \ldots, n\} \tag{3.2}$$

Hint. Easy to prove

$$P\left(x\right) = P\left(x_0\right) + \int\limits_{x_0}^{x} P'\left(t\right) dt \tag{3.3}$$

$$P'\left(x\right) = P'\left(x_1\right) + \int\limits_{x_1}^{t} P''\left(t_1\right) dt_1 \tag{3.4}$$

$$P''\left(x\right) = P''\left(x_2\right) + \int\limits_{x_2}^{t_1} P'''\left(t_2\right) dt_2 \tag{3.5}$$

$$\ldots \tag{3.6}$$

We obtain the desired polynomial

$$P\left(x\right) = a_n \int\limits_{x_0}^{x} \int\limits_{x_0}^{t_1} \int\limits_{x_0}^{t_2} \cdots \int\limits_{x_0}^{t_{n-1}} dt_n dt_{n-1} \ldots dt_1 \tag{3.7}$$

$$+ a_{n-1} \int\limits_{x_0}^{x} \int\limits_{x_0}^{t_1} \int\limits_{x_0}^{t_2} \cdots \int\limits_{x_0}^{t_{n-2}} dt_{n-1} \ldots dt_1 + \cdots + a_1 \int\limits_{x_0}^{x} dt_1 + a_0 \tag{3.8}$$

Done. □

Interpolation 3 (Hermite interpolating formula).
Given two distinct numbers x_0 *and* x_1. *Find all polynomials* $P(x)$ *with* $\deg P(x) \leq n (n \in \mathbb{N}^*)$ *satisfying the conditions*

$$P\left(x_0\right) = 1 \tag{3.9}$$

$$P^{(k)}\left(x_1\right) = 0, k \in \{0, 1, \ldots, n-1\} \tag{3.10}$$

HINT.

$$P\left(x\right) = \frac{\left(x - x_1\right)^n}{\left(x_0 - x_1\right)^n} \tag{3.11}$$

Done. □

Interpolation 4 (Hermite interpolating formula).
Given two distinct numbers x_0 *and* x_1. *Find all polynomials* $P(x)$ *with* $\deg P(x) \leq n + 1 (n \in \mathbb{N}^*)$ *satisfying the conditions*

$$P\left(x_0\right) = 1, P'\left(x_0\right) = 1 \tag{3.12}$$

$$P^{(k)}\left(x_1\right) = 0, k \in \{0, 1, \ldots, n-1\} \tag{3.13}$$

HINT.

$$P\left(x\right) = \left(x - x_1\right)^n \left(\frac{\left(x_0 - x_1 - n\right)x + \left(x_0 - x_1\right)\left(1 - x_0\right) + nx_0}{\left(x_0 - x_1\right)^{n+1}}\right) \tag{3.14}$$

Done. □

References

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