Lecture ⋆ Nicolas Seguin, The Finite Element Method for Elliptic Partial Differential Equations

Collector: Nguyen Quan Ba Hong*

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Abstract

This context is the lecture given by Prof. Nicolas Seguin in the course *Finite Element Method*, in the Master 2 Fundamental Mathematics program 2018-2019.

Brief introduction. "This lecture is a numerical counterpart to *Sobolev spaces & elliptic equations*. In the first part, after some reminders on linear elliptic partial differential equations, the approximation of the associated solutions by the finite element methods is investigated. Their construction and their analysis are described in one and two dimensions. The second part of the lectures consists in defining a generic strategy for the implementation of the method based on the variational formulation. A program is written in MATLAB (implementable with MATLAB or OCTAVE)."

*Master 2 student at UFR mathématiques, Université de Rennes 1, Beaulieu - Bâtiment 22 et 23, 263 avenue du Général Leclerc, 35042 Rennes CEDEX, France.
E-mail: nguyenquanbahong@gmail.com
Blog: www.nguyenquanbahong.com

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1 Introduction

1.1 Finite Differences for the Poisson Equation

We look at the following 1-D problem with Dirichlet boundary conditions

$$\begin{cases} -u'' = f, \text{ on } (0,1), \\ u(0) = u_l, \\ u(1) = u_r, \end{cases} \tag{1.1}$$

where f is the source term, $u_l, u_r \in \mathbb{R}$.

If we define $F(x) := \int_0^x f(s)\,ds$, then the unique solution of (1.1) can be written as

$$u(x) = -\int_0^x F(s)\,ds + u_l + x\left(u_r - u_l + \int_0^1 F(s)\,ds\right), \quad \forall x \in [0,1]. \tag{1.2}$$

Proof of (1.2). Integrating both sides of (1.1) yields

$$u'(x) = -\int_0^x f(s)\,ds + u'(0) = -F(x) + u'(0), \quad \forall x \in [0,1]. \tag{1.3}$$

Integrating both sides of (1.3) and then using the left Dirichlet boundary condition give us

$$u(x) = -\int_0^x F(s)\,ds + u_l + u'(0)\,x, \quad \forall x \in [0,1]. \tag{1.4}$$

Plugging $x = 1$ into (1.4) yields

$$u_r = -\int_0^1 F(s)\,ds + u_l + u'(0). \tag{1.5}$$

A combination of (1.4) and (1.5) implies (1.2). □

The analysis of convergence of numerical methods depends on the smoothness of the solution u, which is obtained from the smoothness of the source term f.

1.1.1 Finite Difference Method

Definition 1.1 (Discretization)**.** *Let* $N \in \mathbb{N}^*$[1]*, we define the* N-space step *as* $h := \frac{1}{N+1}$*, and a* uniform discretization *or a* uniform mesh *of* $[0,1]$ *as* $(x_i)_{0\le i\le N+1}$*, where* $x_i := ih$ *for* $i = 0, \dots, N+1$.

We want to approximate $u(x_i)$ by some number u_i, knowing $u_0 = u(0) = u_l$, and $u_{N+1} = u(1) = u_r$. Unknown: $u_h := (u_i)_{1\le i\le N} \in \mathbb{R}^N$.

[1] $\mathbb{N}^*$ denotes the set of positive integers.

Taylor expansion. We have

$$u\left(x_{i+1}\right)=u\left(x_i\right)+hu'\left(x_i\right)+\frac{h^2}{2}u''\left(x_i\right)+O\left(h^3\right), \tag{1.6}$$

$$u\left(x_{i-1}\right)=u\left(x_i\right)-hu'\left(x_i\right)+\frac{h^2}{2}u''\left(x_i\right)+O\left(h^3\right), \tag{1.7}$$

and so

$$u''\left(x_i\right)=\frac{u\left(x_{i+1}\right)-2u\left(x_i\right)+u\left(x_{i-1}\right)}{h^2}+O\left(h\right). \tag{1.8}$$

We deduce the difference formula

$$\text{(FD)}\quad\begin{cases}-\dfrac{u_{i+1}-2u_i+u_{i-1}}{h^2}=f\left(x_i\right), & 1\le i\le N,\\ u_0=u_l, & \\ u_{N+1}=u_r, & \end{cases}$$

Denote A_h the following tridiagonal matrix

$$A_h=\frac{1}{h^2}\begin{bmatrix}2 & -1 & \cdots & 0\\ -1 & 2 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 2\end{bmatrix}, \tag{1.9}$$

for all $h>0$, then

$$\text{(FD)}\Leftrightarrow A_h u_h=b_h, \tag{1.10}$$

where the vector $b_h\in\mathbb{R}^N$ is defined by

$$b_h:=\begin{bmatrix}f\left(x_1\right)+\frac{1}{h^2}u_l\\ f\left(x_2\right)\\ \vdots\\ f\left(x_N\right)+\frac{1}{h^2}u_r\end{bmatrix}. \tag{1.11}$$

Proposition 1.1. *A_h is symmetric, positive definite, and so invertible. Then there exists one and only one solution u_h of* (FD).

QUESTION. $u_h\to u$ as $h\to 0$?

1.1.2 Consistency

Definition 1.2 (Error of consistency, discretization operator)**.** *The error of consistency is defined as*

$$\varepsilon_h\left(u\right):=A_h\Pi_h\left(u\right)-b_h, \tag{1.12}$$

where

$$\Pi_h\left(u\right) := \begin{bmatrix} u\left(x_1\right) \\ \vdots \\ u\left(x_N\right) \end{bmatrix}, \tag{1.13}$$

which is called the discretization operator.

We then have directly from Definition 1.2

$$A_h \Pi_h\left(u\right) = b_h + \varepsilon_h\left(u\right). \tag{1.14}$$

Proposition 1.2 (Consistency)**.** *The scheme* (FD) *is consistent, i.e.,*

$$\left\|\varepsilon_h\left(u\right)\right\|_\infty \to 0 \ \text{as } h \to 0, \tag{1.15}$$

where u is the solution of the Poisson problem (1.1)*. Furthermore,*

$$\left\|\varepsilon_h\left(u\right)\right\|_\infty \le Ch^2, \ \ \forall h > 0. \tag{1.16}$$

Proof. Make more rigorous the Taylor expansion to obtain

$$\left\|\varepsilon_h\left(u\right)\right\|_\infty \le \frac{h^2}{12} \sup_{x\in[0,1]} \left|u^{(4)}\left(x\right)\right|. \tag{1.17}$$

This completes our proof. □

Remark 1.1. *This requires (lots of) smoothness of the solution.*

1.1.3 Stability

Given an error δb_h on b_h, one has

$$A_h\left(u_h + \delta u_h\right) = b_h + \delta b_h. \tag{1.18}$$

Thus,

$$A_h \delta u_h = \delta b_h. \tag{1.19}$$

Stability. Control of δu_h by δb_h. Here, $\delta u_h = A_h^{-1} \delta b_h$.

Proposition 1.3 (Stability)**.**

$$\left\|A_h^{-1}\right\| \le \frac{1}{8}. \tag{1.20}$$

As a consequence, one has

$$\left\|\delta u_h\right\|_\infty \le \frac{1}{8} \left\|\delta b_h\right\|_\infty. \tag{1.21}$$

1.1.4 Convergence

One has

$$A_h\left(u_h - \Pi_h\left(u\right)\right) = -\varepsilon_h\left(u\right). \tag{1.22}$$

Combining (1.17) and (1.20) yields

$$\left\|u_h - \Pi_h\left(u\right)\right\|_\infty \leq \frac{h^2}{96} \sup_{x\in[0,1]} \left|u^{(4)}\left(x\right)\right|. \tag{1.23}$$

Theorem 1.1 (Convergence)**.** *If $u \in C^4\left([0,1]\right)$ then the scheme* (FD) *converges to u with quadrature rate.*

Remark 1.2.
- *Same analysis in $\mathbb{R}^n$, $n \geq 1$, for the Dirichlet problem for the Poisson's equation*

$$\begin{cases} -\Delta u = f \ \text{ on } \Omega := (0,1)^n, \\ u|_{\partial\Omega} = g. \end{cases} \tag{1.24}$$

- *Problem with more complex domain due to the discretization of $\partial\Omega$, we would like to use more complex discretization.*
- *The solution has to be smooth (e.g., C^4 above), which is in contradiction with the theory of Lax-Milgram, i.e., $f \in L^2\left(\Omega\right) \Rightarrow u \in H^2\left(\Omega\right)$ and $f \in H^{-1}\left(\Omega\right) \Rightarrow u \in H^1\left(\Omega\right)$.*

1.2 Analysis of Elliptic PDEs

1.2.1 Sobolev Spaces

Definition 1.3 (Distribution)**.** *Given $\Omega \subset \mathbb{R}^n$, T is a distribution on Ω if T is a linear form on $C_c^\infty\left(\Omega\right)$, such that $\forall K \subset \Omega$ compact, $\exists k \in \mathbb{N}$, $C_K > 0$ such that*

$$\forall \phi \in \mathcal{D}\left(\Omega\right), \ \ \text{supp } \phi \subset K \ \ , \left|\langle T, \phi\rangle\right| \leq C_K \max_{|\alpha|\leq k} \left\|\partial^\alpha \phi\right\|_\infty. \tag{1.25}$$

And k is called the order of T.

Example 1.1. *For any $f \in L^1_{\text{loc}}\left(\Omega\right)$, $T : \phi \mapsto \int_\Omega f\phi dx$ is a distribution.*

Definition 1.4 (Convergence in the sense of distribution)**.** *We say u_n converges to u in the sense of distribution, denoted as $u_n \rightharpoonup u$ as $n \to \infty$ in $\mathcal{D}'\left(\Omega\right)$, if*

$$\langle u_n, \phi\rangle \to \langle u, \phi\rangle \ \ \text{as } n \to \infty, \ \ \forall \phi \in \mathcal{D}\left(\Omega\right) := C^\infty\left(\Omega\right). \tag{1.26}$$

Definition 1.5 (Distributional derivative)**.** *Given $u \in \mathcal{D}'\left(\Omega\right)$, for $1 \leq i \leq n$, we denote $\frac{\partial u}{\partial x_i}$ the distribution defined by*

$$\left\langle \frac{\partial u}{\partial x_i}, \phi \right\rangle = -\left\langle u, \frac{\partial \phi}{\partial x_i} \right\rangle, \ \ \forall \phi \in \mathcal{D}\left(\Omega\right). \tag{1.27}$$

More generally, for all $\alpha \in \mathbb{N}^n$, we denote $\partial^\alpha u$ the distribution defined by

$$\langle \partial^\alpha u, \phi\rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi\rangle, \ \ \forall \phi \in \mathcal{D}\left(\Omega\right). \tag{1.28}$$

Definition 1.6 (Sobolev spaces)**.**

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) ; \frac{\partial u}{\partial x_i} \in L^2(\Omega), \ \ 1 \leq i \leq n \right\}, \tag{1.29}$$

$$H^m(\Omega) := \left\{ u \in L^2(\Omega) ; \partial^\alpha u \in L^2(\Omega), \ \ \forall \alpha : |\alpha| \leq m \right\}, \tag{1.30}$$

$$W^{m,p}(\Omega) := \{ u \in L^p(\Omega) ; \partial^\alpha u \in L^p(\Omega), \ \ \forall \alpha : |\alpha| \leq m \}, \tag{1.31}$$

and

$$\|u\|_{H^m(\Omega)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{1.32}$$

$$H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}\Big|_{H^1(\Omega)}. \tag{1.33}$$

Theorem 1.2 (Poincaré inequality)**.** *If Ω is bounded, $\exists C_\Omega > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}, \ \ \forall u \in H_0^1(\Omega). \tag{1.34}$$

Definition 1.7 (Dual space of Sobolev spaces)**.**

$$H^{-1}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) ; u = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \ \ f_i \in L^2(\Omega), \ \ 0 \leq i \leq n \right\}. \tag{1.35}$$

Given $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ is bounded and of class C^1, The trace operator $\gamma_0 : \mathcal{D}\left(\overline{\Omega}\right) \to C^0(\partial\Omega)$ can be extended to a linear continuous map from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

Theorem 1.3 (Green's formula)**.** *For all $u, v \in H^1(\Omega)$,*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} \gamma_0 u \cdot \gamma_0 v \mathrm{n}_i d\sigma, \tag{1.36}$$

where $\overrightarrow{\mathrm{n}} = (n_i)_{1 \leq i \leq n}$ is the unit normal vector to $\partial\Omega$ going outside Ω.

Proposition 1.4. *Let $u \in H^1(\Omega)$, one has*

$$u \in H_0^1(\Omega) \Leftrightarrow \widetilde{u}(x) \in H_0^1(\mathbb{R}^n) \Leftrightarrow \gamma_0 u = 0 \ \textit{a.e.} \ \partial\Omega, \tag{1.37}$$

where

$$\widetilde{u}(x) := \begin{cases} u(x), & \textit{if } x \in \Omega, \\ 0, & \textit{if } x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.38}$$

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded and of class C^1. The canonical injection from $H^1(\Omega)$ to $L^2(\Omega)$ is compact.*

1.2.2 Variational Formulation

Let $\Omega \subset \mathbb{R}^N$ whose $\partial\Omega$ is of class C^1 and bounded. We search for the solution of the following problem

$$\text{(P)} \quad \begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.39}$$

where the source term f is given.

The *variational formulation* associated with (P) is defined as

$$\text{(VF)} \quad \text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \ \ \forall v \in H_0^1(\Omega), \tag{1.40}$$

This gives a "definition" of solutions of (P). Moreover, under some smoothness assumptions, (P) $\Leftrightarrow$ (VF) by the Green's formula.

Definition 1.8 (Variational formulation)**.** *A* variational formulation *is composed by*

- *V: a Hilbert space;*
- *a: a bilinear continuous form on $V \times V$;*
- *l: a linear continuous form on V;*

and corresponds to a problem

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ such that } a(u,v) = l(v), \ \ \forall v \in V, \tag{1.41}$$

Theorem 1.5 (Lax-Milgram)**.** *Given a variational formulation and a is coercive, i.e.,*

$$\exists \alpha > 0, \ \ \forall u \in V, \ \ a(u,u) \geq \alpha \|u\|_V^2, \tag{1.42}$$

Then, (VF) *admits one and only one solution $u \in V$.*

Remark 1.3. 1. *This applies to the problem* (P)*, assuming $f \in H^{-1}(\Omega)$.*

2. *The coercivity of a is obtained by the Poincaré's inequality.*

3. *The notion of variational formulation and Lax-Milgram theorem 1.5 can be generalized to more general elliptic PDEs, e.g.,*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + c(x) u = f, & \textit{in } \Omega, \\ u = g, & \textit{on } \partial\Omega, \end{cases} \tag{1.43}$$

with nonhomogeneous Dirichlet boundary condition, where $g \in H^{\frac{1}{2}}(\partial\Omega) := \gamma_0\left(H^1(\Omega)\right)$. And this Dirichlet boundary conditions can be replace by the following Neumann boundary condition

$$\frac{\partial u}{\partial \mathrm{n}} = 0, \ \ \textit{on } \partial\Omega, \tag{1.44}$$

where $\mathbf{n}$ *is the unit normal vector to $\partial\Omega$.*

4. *The Poisson's equation* (1.39) *can be interpreted as "$u = (-\Delta)^{-1} f$".*
5. *If Ω is bounded and of class C^2, u is the solution of* (P) *and $f \in L^2(\Omega)$ then $u \in H^2(\Omega)$ and there exists a positive constant C such that the following inequality holds*
$$\|u\|_{H^2} \le C\|f\|_{L^2}, \tag{1.45}$$
whereas there is a priori non-control of the derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i \neq j$.

1.2.3 The FEM for the 1-D Dirichlet Problem

We want to approximate the solution of (P) defined by (1.39). To do this, we set $V = H_0^1(\Omega)$ and
$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in V, \tag{1.46}$$
$$l(v) := \int_\Omega f v dx, \quad \forall v \in V, \tag{1.47}$$
Assume that we have a finite dimensional subspace $V_h \subset V$. The *internal* (or *conformal*) *approximation* of (VF) is defined as
$$(\mathrm{VF}_h) \quad \text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h, \tag{1.48}$$
Lemma 1.1. *If a is coercive on V, then* (VF_h) *admits one and only one solution. Moreover,* (VF_h) *is equivalent to a linear system to solve.*

Proof. The existence and uniqueness of the solution, say $u_h \in V_h$, of (VF_h) is deduced by the Lax-Milgram theorem 1.5. Since V_h is finite dimensional vector space, we can introduce a basis of V_h as $(\phi_i)_{1\le i\le N}$ with $N = \dim V_h$. Write $u_h(x) = \sum_{i=1}^N u_i \phi_i(x)$, we arrive at a linear system $A_h U_h = F_h$ where $A_h := (a(\phi_i, \phi_j))_{1\le i,j\le N}$, $U_h := (u_i)_{1\le i\le N}$, and $F_h := (l(\phi_i))_{1\le i\le N}$. Lastly, the coercivity of a implies the positive definiteness of A_h. □

Remark 1.4. *For finite differences, the invertibility of the matrix has to be proved for all problems and for all discretizations used. However, for Finite Elements (FEs, for short), it is direct.*

Lemma 1.2 (Céa's lemma)**.** *Under the same assumptions, the following inequality holds*
$$\|u - u_h\|_V \le \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V, \tag{1.49}$$
where M, α are the continuity and coercivity constants of a, respectively.

Proof. Let $w_h \in V_h$ be a test function. (VF_h) implies $a(u_h, w_h) = l(w_h)$ while (VF) gives us $a(u, w_h) = l(w_h)$ since $V_h \subset V$. Subtracting these yields $a(u - u_h, w_h) = 0$. Put $v_h := u_h - w_h \in V_h$, one has
$$\alpha \|u - u_h\|_V^2 \le a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \le M\|u - u_h\|_V \|u - v_h\|_V, \tag{1.50}$$
which implies the desired inequality. □

In particular, we want that

- $d(u, V_h) \to 0$ as $h \to 0^+$ "fast", and
- the linear system $A_h U_h = F_h$ is "easy" to solve.

Reconsider (1.1) with $u_l = u_r = 0$ and its discretization given by 1.1 , and $f \in H^{-1}(0,1)$, we define $T_j^- := [x_{j-1}, x_j]$, $T_j^+ := [x_j, x_{j+1}]$. Consider the following space

$$W_h := \left\{ v \in C([0,1]) ; v|_{T_j^+} \in \mathbb{P}_1, \ \ j = 0, \ldots, N \right\}, \tag{1.51}$$

Notice that $W_h \not\subset V := H_0^1(0,1)$ but $W_h \subset H^1(0,1)$, and $\dim W_h = N+2$. We take

$$V_h := W_h \cap V = \{ v \in W_h ; v(0) = v(1) = 0 \}, \tag{1.52}$$

then $\dim V_h = N$. A basis[2] of V_h, say $(\phi_j)_{1 \le j \le N}$, is given by $\phi_i(x_j) = \delta_{ij}$ for all $i = 1, \ldots, N$, $j = 0, \ldots, N+1$. A simple calculation give yields the following explicit formula of ϕ_j's as follows,

$$\phi_j(x) = \begin{cases} \dfrac{x - x_j}{h}, & \text{in } T_j^-, \\ \dfrac{x_{j+1} - x}{h}, & \text{in } T_j^+, \\ 0, & \text{elsewhere}, \end{cases} \tag{1.53}$$

Consider the matrix A_h whose the ijth element is defined by $a(\phi_i, \phi_j) = \int_0^1 \phi_i' \phi_j' dx$, if $|i-j| > 1$ then $A_{ij} = 0$ since $\operatorname{supp} \phi_i \cap \operatorname{supp} \phi_j = \emptyset$. If $i = j$, $A_{ij} = \frac{2}{h}$. And if $|i-j| = 1$, $A_{ij} = -\frac{1}{h}$. Hence,

$$A_h = \frac{1}{h} \begin{bmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix}, \tag{1.54}$$

This matrix is sparse since it contains many zero terms.

$\star$ **Convergence and Error Estimates.**

Definition 1.9. *Let $v \in H^1(0,1)$, we define*

$$\Pi_h v(x) := \sum_{j=1}^{N+1} v(x_j) \phi_j(x), \ \ \forall x \in [0,1]. \tag{1.55}$$

Lemma 1.3. *There exists a positive constant being independent of h such that*

$$\|\Pi_h v\|_{H^1} \le C \|v\|_{H^1}, \ \ \forall v \in H^1(0,1). \tag{1.56}$$

[2] The elements of this basis are well-known as *hat functions*.

Proof. For $x \in T_j^+$, one has $(\Pi_h v)'(x) = (v(x_{j+1}) - v(x_j))/h$ and thus

$$\int_{T_j^+} \left((\Pi_h v)'(x)\right)^2 dx = \frac{\left(v(x_{j+1}) - v(x_j)\right)^2}{h} = \frac{1}{h}\left(\int_{T_j^+} v' dx\right)^2 \leq \int_{T_j^+} \left|v'\right|^2 dx. \tag{1.57}$$

Summing this inequality over the index j yields $\left\|(\Pi_h v)'\right\|_{L^2} \leq \|v'\|_{L^2}$.

Moreover,

$$\|\Pi_h v\|_{L^2} \leq \max_{[0,1]} |\Pi_h v| \leq \max_{[0,1]} |v| \leq C\|v\|_{H^1}. \tag{1.58}$$

Combing these inequalities yields the desired result. □

Lemma 1.4. *a) There exists a positive constant C which is independent of h such that*

$$\|v - \Pi_h v\|_{L^2} \leq Ch\left\|v'\right\|_{L^2}, \ \ \forall v \in H^1(0,1). \tag{1.59}$$

b) There exists a positive constant C which is independent of h such that

$$\|v - \Pi_h v\|_{L^2} \leq Ch^2\left\|v''\right\|_{L^2}, \ \text{and} \ \left\|v' - (\Pi_h v)'\right\|_{L^2} \leq Ch\left\|v''\right\|_{L^2}, \ \ \forall v \in H^2(0,1). \tag{1.60}$$

c) One has

$$\lim_{h \to 0} \left\|v' - (\Pi_h v)'\right\|_{L^2} = 0, \ \ \forall v \in H^1(0,1). \tag{1.61}$$

Theorem 1.6. *a) Let $f \in H^{-1}(0,1)$, $u \in H_0^1(0,1)$ is the solution of* (VF)*, $u_h \in V_h$ is the solution of* (VF$_h$)*. Then*

$$\lim_{h \to 0} \|u - u_h\|_{H^1} = 0. \tag{1.62}$$

b) Let $f \in L^2(0,1)$, then $u \in H^2(0,1)$ and

$$\|u - u_h\|_{H^1} \leq Ch\|f\|_{L^2}, \tag{1.63}$$

$$\|u - u_h\|_{L^2} \leq Ch^2\|f\|_{L^2}, \tag{1.64}$$

Proof. The first inequality is deduced from

$$\|u - u_h\|_{H^1} \leq \|u - \Pi_h u\|_{H^1} \to 0 \text{ as } h \to 0. \tag{1.65}$$

For the second inequality, let $f \in L^2(0,1)$, a well-known regularity result gives us $u \in H^2(0,1)$. And then

$$\|u - u_h\|_{H^1} \leq \|u - \Pi_h u\|_{H^1} \leq Ch\left\|u''\right\|_{L^2} = Ch\|f\|_{L^2}. \tag{1.66}$$

For the last one, one has

$$\|u - u_h\|_{L^2} = \sup_{g \in L^2(0,1), g \neq 0} \frac{\int_0^1 (u - u_h)\, g dx}{\|g\|_{L^2}}. \tag{1.67}$$

For any $g \in L^2(0,1)$, we consider the following variational formulation associated with g,

$$(\mathrm{VF}_g) \text{ Find } \phi_g \in H_0^1(0,1) \text{ such that } a(\phi_g, v) = \langle g, v \rangle, \ \forall v \in H_0^1(0,1). \tag{1.68}$$

Then there exists a positive constant C such that $\|\phi_g\|_{H^1} \leq C\|g\|_{L^2}$.

Take $v := u - u_h$, one has $a(u - u_h, \phi_g) = \int_0^1 (u - u_h)\, g dx$. Since $\Pi_h \phi_g \in V_h$, one also has $a(u - u_h, \Pi_h \phi_g) = 0$. Hence,

$$\begin{aligned}\int_0^1 (u-u_h)\, g dx &= a(u - u_h, \phi_g - \Pi_h \phi_g) \leq \left\|(u-u_h)'\right\|_{L^2} \left\|(\phi_g - \Pi_h \phi_g)'\right\|_{L^2} \\ &\leq \left\|(u-u_h)'\right\|_{H^1} \left\|(\phi_g - \Pi_h \phi_g)'\right\|_{H^1} \leq Ch\|f\|_{L^2} \cdot Ch\left\|\phi_g''\right\|_{L^2} \leq Ch^2\|f\|_{L^2}\|g\|_{L^2},\end{aligned}$$

which implies that $\|u - u_h\|_{L^2} \leq Ch^2\|f\|_{L^2}$. □

Remark 1.5. *When $f = 0$, Theorem 1.6 implies that $u = u_h$. Moreover, this is analogous to Proposition 1.3 in FDM.*

1.2.4 Construction of A_h and b_h

Denote $\mathcal{T}_h := \{[x_i, x_{i+1}]\,; i = 0, \ldots, N\}$, then the elements of the matrix A_h can be written as

$$(A_h)_{ij} = \sum_{T \in \mathcal{T}_h, T \subset \text{supp } \phi_i \,\cap\, \text{supp } \phi_j} \int_T {\phi_i}' {\phi_j}' dx. \tag{1.69}$$

So $(A_h)_{ij} \neq 0 \Leftrightarrow i = j-1, j, j+1$.

Let X_α, $\alpha = 1, 2$ be the vertices of T. Then T is only defined by $(X_\alpha)_\alpha$ local or $(x_i)_i$ global.

Definition 1.10 (Elementary matrix)**.** *The* elementary matrix *related to T is defined as*

$$m_T := \left(\int_T \widetilde{\phi}'_\alpha \widetilde{\phi}'_\beta dx\right)_{\alpha,\beta=1,2}, \tag{1.70}$$

where $\widetilde{\phi}_\alpha, \widetilde{\phi}_\beta$ are the local elements of the basis of V_h, associated with X_α, X_β.

In the same way,

$$b_T := \left(\int_T f \widetilde{\phi}_\beta dx\right)_{\beta=1,2}. \tag{1.71}$$

Algorithm 1 Elementary matrices and RHS.

for $T \in \mathcal{T}_h$ **do**
  for $\beta = 1, 2$ **do**
    Compute $(b_T)_\beta$;
    for $\alpha = 1, 2$ **do**
      Compute $(m_T)_{\alpha,\beta}$;
    end for
  end for
end for
$b_h = 0$; $A_h = 0$;
for $T \in \mathcal{T}_h$ **do**
  for $\beta = 1, 2$ **do**
    $i :=$ global index of X_β;
    $(b_h)_i := (b_h)_i + (b_T)_\beta$;
    for $\alpha = 1, 2$ **do**
      $j :=$ global index of X_α;
      $(A_h)_{ij} := (A_h)_{ij} + (m_T)_{\alpha\beta}$;
    end for
  end for
end for

Remark 1.6 (Quick notes)**.**

- *The algorithm is implemented for loops on the elements $T \in \mathcal{T}_h$, not on the two indices i, j, which will cause double loops.*
- *To compute m_T and b_T, we*
  - *define a reference interval: $\widehat{T} = [0, 1]$;*
  - *define an affine function F_T such that $F_T\left(\widehat{T}\right) = T$.*
  - *Carry out all the computations over the reference interval $\widehat{T}$ and apply F_T (use numerical integrations).*

2 General Construction of the FEM

In this section, we will approximate the solutions of some second-order elliptic PDEs for which their corresponding variational formulations are well-posed.

Definition 2.1 (Triangulation)**.** *Let $\Omega \in \mathbb{R}^n$ be a domain, a* triangulation *(or* mesh*) $\mathcal{T}_h$ of Ω is a partition of Ω into (closed) elements K's such that*

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K. \tag{2.1}$$

A space of polynomial approximation *can be defined by*

$$P_K := \{v : K \to \mathbb{R}; v \in \mathbb{P}_m(K)\}, \ \ \forall K \in \mathcal{T}_h, \tag{2.2}$$

for some positive integer m.

A global space X_h *is defined by*

$$X_h := \{v : \Omega \to \mathbb{R}; v|_K \in P_K, \ \ \forall K \in \mathcal{T}_h\}. \tag{2.3}$$

Remark 2.1. *Define V_h as*

$$V_h := \{v : \Omega \to \mathbb{R}; v|_K \in P_K, \ \ \forall K \in \mathcal{T}_h, \ \ v|_{\partial\Omega} = 0\}, \tag{2.4}$$

which is analogous to its 1-D version (1.52), one has $V_h \subset X_h$. Notice the difference between these spaces, V_h is used to approximate functions in $H_0^1(0,1)$ while X_h is used to do that in $H^1(0,1)$. In addition, the elements of a basis of X_h for small $h > 0$ will have "small" supports.

2.1 Classical Families of FEM

Here are two classical FEs:

- n-simplices Finite Element;
- n-rectangular Finite Element;

Quick note. Points of approximation $\equiv$ unknowns $\equiv$ degrees of freedom (*dof*, for short).

If we use

- points of values, this will lead us to Lagrange Finite Element;
- points values and directional derivatives, this will gives us Hermite Finite Element.

We assume that there exists an affine mapping F_T such that $F_T\left(\widehat{K}\right) = K$, where $\widehat{K}$ is a reference element. This also suggest that Ω is a polyhedron.

2.1.1 Some Basis

We consider the following variational formulation

$$\text{(VF)} \quad \text{Find } u \in V \text{ such that } a(u,v) = f(v), \ \ \forall v \in V, \tag{2.5}$$

where a is the bilinear form associated with some second-order elliptic PDEs for which Lax-Milgram theorem 1.5 can apply.

Galerkin method. For a given finite dimensional subspace $V_h \subset V$, we associate (VF) with the following variational formulation

$$(\text{VF}_h) \quad \text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = f(v_h), \ \ \forall v_h \in V_h. \tag{2.6}$$

Remark 2.2. *If a is symmetric,* (VF) *and* (VF$_h$) *are equivalent to the problem of minimizing the energy*

$$J(u) := \frac{1}{2}a(u,u) - f(u). \tag{2.7}$$

In this case, Galerkin method is also called Ritz method.

$\star$ **Construction of $\mathcal{T}_h$.**

(FEM$_1$):

($\mathcal{T}_1$) $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$;

($\mathcal{T}_2$) $\forall K \in \mathcal{T}_h$, K is closed, $\overset{o}{K} \neq \emptyset$;

($\mathcal{T}_3$) $\forall K_1, K_2 \in \mathcal{T}_h$, $K_1 \neq K_2$, $\overset{o}{K}_1 \cap \overset{o}{K}_2 = \emptyset$;

($\mathcal{T}_4$) $\forall K \in \mathcal{T}_h$, ∂K is Lipschitz continuous.

$\star$ **Construction of X_h.** To construct X_h, we need the following lemma.

Lemma 2.1 (Jênôme Dronion)**.** *If $\partial\Omega$ is bounded and Lipschitz continuous, then*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega); \gamma_0 u = 0\right\}. \tag{2.8}$$

Theorem 2.1. *Assume $P_K \subset H^1\left(\overset{o}{K}\right)$*[3]*, $\forall K \in \mathcal{T}_h$, $X_h \subset C^0\left(\overline{\Omega}\right)$. Then $X_h \subset H^1(\Omega)$ and*

$$X_{0h} := \{v_h \in X_h; \gamma_0 v_h = 0\} \subset H_0^1(\Omega). \tag{2.9}$$

Proof. Let $v \in X_h$. Clearly, $v \in L^2(\Omega)$. We want to find functions $w_i \in L^2(\Omega)$, $i = 1, \ldots, n$ such that

$$\int_\Omega w_i \phi dx = -\int_\Omega v \partial_i \phi dx, \ \ \forall \phi \in \mathcal{D}(\Omega). \tag{2.10}$$

Let's take $w_i|_K = \partial_i v|_K \in L^2(K)$. Green's formula helps us arrive at

$$\int_K \partial_i v|_K \phi dx = -\int_K v|_K \partial_i \phi dx + \int_{\partial K} \gamma_0 v|_K \gamma_0 \phi (\nu_K)_i d\sigma, \ \ \forall K \in \mathcal{T}_h, \ \forall \phi \in \mathcal{D}(\Omega). \tag{2.11}$$

Summing this equality over $K \in \mathcal{T}_h$ yields

$$\int_\Omega w_i \phi dx = -\int_\Omega v \partial_i \phi dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \gamma_0 v|_K \gamma_0 \phi (\nu_K)_i d\sigma, \ \ \forall \phi \in \mathcal{D}(\Omega). \tag{2.12}$$

Either $\partial K \subset \partial\Omega$ we have $\gamma_0 \phi = 0$, or we have two contributions of opposite signs. Thus, (2.12) is equivalent to

$$\langle w_i, \phi \rangle = -\langle v, \partial_i \phi \rangle, \ \ \forall \phi \in \mathcal{D}(\Omega). \tag{2.13}$$

Therefore, $X_h \subset H^1(\Omega)$. The second inclusion is a corollary of Lemma 2.1. $\square$

[3] The choice (2.2) is not used here, P_K is assumed to be more general.

Theorem 2.2. *Assume* $P_K \subset H^2\left(\overset{o}{K}\right)$[4], $\forall K \in \mathcal{T}_h$, $X_h \subset C^1\left(\overline{\Omega}\right)$. *Then* $X_h \subset H^2(\Omega)$, *and*

$$X_{0h} := \{v_h \in X_h; \gamma_0 v_h = 0\} \subset H^2(\Omega) \cap H_0^1(\Omega), \tag{2.14}$$

$$X_{00h} := \{v_h \in X_{0h}; \gamma_0(\partial_\nu v) = 0\} \subset H_0^2(\Omega). \tag{2.15}$$

2.2 Examples of Finite Elements and Finite Element Spaces

For a moment, we do not consider any boundary conditions. Assume that $\Omega \subset \mathbb{R}^n$ is a polyhedron, we reuse the notions of triangulation $\mathcal{T}_h$, reference element $\widehat{K}$, affine mapping F_K as above.

2.2.1 n-Simplices of Type (k)

The n-simplices of type (k) are the most classical ones: "very" general, "easy" to implement, and apply to "many" cases.

Denote $\mathbb{P}_k$ the spaces of polynomial of degree less than or equal to k in $\mathbb{R}^n$, one can verify that $\dim \mathbb{P}_k = \frac{(n+k)!}{n!k!}$. Consider

$$p \in \mathbb{P}_k : \mathbb{R}^n \to \mathbb{R} \tag{2.16}$$

$$x \mapsto p(x) := \sum_{|\alpha| \le k} \gamma_\alpha x^\alpha, \tag{2.17}$$

where $\alpha \in \mathbb{N}^n$ is the multi-index notation.

Definition 2.2 (n-simplex in $\mathbb{R}^n$)**.** *The n-simplex K is defined as the convex hull of $(n+1)$-points $(a_j)_{j=1,\ldots,n+1}$, $a_j \in \mathbb{R}^n$.*

Then,

$$\overset{o}{K} \ne \emptyset \Leftrightarrow A := \begin{bmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ddots & a_{n,n+1} \\ 1 & \cdots & 1 \end{bmatrix} \text{ is not singular,} \tag{2.18}$$

where $a_j = [a_{1j}, \ldots, a_{nj}]^T$, $j = 1, \ldots, n+1$.

More explicitly, the n-simplex K can be represented as

$$K := \left\{ x \in \mathbb{R}^n; x = \sum_{j=1}^{n+1} \lambda_j a_j, \sum_{j=1}^{n+1} \lambda_j = 1, 0 \le \lambda_j \le 1, j = 1, \ldots, n+1 \right\}, \tag{2.19}$$

[4]The choice (2.2) is also not used here, P_K is assumed to be more general.

i.e., each point $x \in K$ can be written as a convex combination of the vertices a_j's. The $(n+1)$-tuple λ_j's is the barycentric coordinates of the corresponding vector $x \in \mathbb{R}^n$. Denote $\Lambda = [\lambda_1, \dots \lambda_{n+1}]^T$, one has $A\Lambda = \begin{bmatrix} x \\ 1 \end{bmatrix}$.

Moreover, if $B = A^{-1}$, then

$$\lambda_i(x) = \sum_{j=1}^{n} b_{ij} x_j + b_{i,n+1}. \tag{2.20}$$

Note that λ_i is an affine function in the variable x, i.e., $\lambda_i \in \mathbb{P}_1(K)$, and

$$\lambda_i(a_j) = \delta_{ij}, \ \ 1 \le i, j \le n+1. \tag{2.21}$$

$\star$ *Type* (1) *($k = 1$).* Any $p \in \mathbb{P}_1(K)$ is uniquely defined by its values at the vertices $(a_j)_{j=1,\dots,n+1}$,

$$p(x) = \sum_{i=1}^{n+1} p(a_j) \lambda_j(x). \tag{2.22}$$

These FEs are also called $\mathbb{P}_1$ Lagrange FEs. Let K be the n-simplex, $P_K := \mathbb{P}_1$, then the set of dof of K is defined by

$$\Sigma_K := \{p(a_j); j = 1, \dots, n+1\}. \tag{2.23}$$

$\star$ *Type* (2) *($k = 2$).* Let $\overline{a_{ij}} := \frac{1}{2}(a_i + a_j) \in \partial K$ be the middle of the edge whose vertices are a_i and a_j.

Lemma 2.2. *One has for all $1 \le i, j, k \le n+1$,*

$$\lambda_k(\overline{a_{ij}}) = \frac{1}{2}(\delta_{ki} + \delta_{kj}), \tag{2.24}$$

and

$$p(x) = \sum_{i=1}^{n+1} p(a_i) \lambda_i(x)(2\lambda_i(x) - 1) + 4 \sum_{1 \le i < j \le n+1} p(\overline{a_{ij}}) \lambda_i(x) \lambda_j(x), \ \ \forall p \in \mathbb{P}_2(K). \tag{2.25}$$

These FEs are called $\mathbb{P}_2$ Lagrange FEs. Let K be the n-simplex, $P_K := \mathbb{P}_2$, $\dim \mathbb{P}_2 = \frac{1}{2}(n+1)(n+2)$, the set of dof of K is defined by

$$\Sigma_K := \{p(a_j), p(\overline{a_{ij}}); 1 \le i, j \le n+1\}. \tag{2.26}$$

The polynomial $p \in \mathbb{P}_2$ can be rewritten in terms of elements of Σ_K as

$$p(x) = \sum_{\mathrm{dof} \in \Sigma_{\mathrm{K}}} \phi_{\mathrm{dof}}(x)\, \mathrm{dof}. \tag{2.27}$$

Theorem 2.3. *For all positive integer k, $p \in \mathbb{P}_k$ is uniquely determined by its values on the set*

$$\Sigma_K := \left\{ p\left(x\right); x = \sum_{j=1}^{n+1} l_j a_j,\ \sum_{j=1}^{n+1} l_j = 1,\ \ l_j \in \left\{0, \frac{1}{n}, \ldots, 1\right\} \right\}. \tag{2.28}$$

⋆ **From Simplices to Triangulation.** Recall $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$, we need to our setting to be in $C^0\left(\overline{\Omega}\right)$ to be able to go from $H^1\left(\overset{o}{K}\right)$ to $H^1\left(\Omega\right)$. Consider the following additional assumption on the triangulation $\mathcal{T}_h$,

($\mathcal{T}_5$) Any face of a n-simplex $K_1 \in \mathcal{T}_h$ is either a face of another simplex $K_2 \in \mathcal{T}_h$ or a subset of $\partial\Omega$.

(FEM$_2$) The spaces P_K's are spaces of polynomials and the global space of approximation X_h is defined by

$$X_h := \left\{ v \in C^0\left(\overline{\Omega}\right); v|_K \in P_K \right\}. \tag{2.29}$$

Hence, for all $K \in \mathcal{T}_h$, the spaces P_K's should correspond to a polynomial space of the same order.

Definition 2.3 (Global set of degree of freedoms)**.** *The* global set of dof *is defined by*

$$\Sigma_h := \bigcup_{K\in\mathcal{T}_h} \Sigma_K. \tag{2.30}$$

Theorem 2.4. *Let X_h be the FE space for the n-simplices of degree $k \geq 1$. Then $X_h \subset C^0\left(\overline{\Omega}\right) \cap H^1\left(\Omega\right)$.*

(FEM$_3$) X_h admits a basis of elements whose supports are "small".

After renumeration, let us write

$$\Sigma_h = \left\{ p\left(b_l\right); 1 \leq l \leq M \right\}. \tag{2.31}$$

Definition 2.4 (Canonical basis)**.** *The canonical basis of X_h is defined by $w_l \in X_h$, $w_l\left(b_l'\right) = \delta_{ll'}$, $1 \leq l, l' \leq M$.*

Lemma 2.3.

$$\operatorname{supp} w_l = \bigcup_{b_l\in K\in\mathcal{T}_h} K,\ \ \forall l = 1, \ldots, M. \tag{2.32}$$

2.2.2 n-Rectangles of Type (k)

Given $n \geq 2$, we denote by $\mathbb{Q}_k$ the space of polynomials of degree $\leq k$ for each variable, independently. More explicitly, for any $p \in \mathbb{Q}_k$, p can be represented as

$$p\left(x\right) = \sum_{\{\alpha;\alpha_i\leq k, 1\leq i\leq n\}} \gamma_\alpha x^\alpha = \sum_{\{\alpha;\alpha_i\leq k, 1\leq i\leq n\}} \gamma_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}. \tag{2.33}$$

For example, $k=1$, $p(x)=\gamma_{00}+\gamma_{10}x_1+\gamma_{01}x_2+\gamma_{11}x_1x_2\in\mathbb{Q}_1$. Moreover, $\mathbb{Q}_1\subsetneq\mathbb{P}_2$.

In general,

$$\mathbb{P}_k\subsetneq\mathbb{Q}_k\subsetneq\mathbb{P}_{nk}. \tag{2.34}$$

Lemma 2.4. $\dim\mathbb{Q}_k=(k+1)^n$.

To go from the reference element $\widehat{K}$ to K, we define a diagonal affine mapping F_K such that $F_K\left(\widehat{K}\right)=K$ and $F_K(x)=B_Kx+b_k$, where B_K is a diagonal matrix, and $b_k\in\mathbb{R}^n$.

Proposition 2.1. *Any polynomial $p\in\mathbb{Q}_k\left(\widehat{K}\right)$ is uniquely determined by its value on the set*

$$M_{\widehat{K}}:=\left\{x=\left(\frac{i_1}{k},\dots,\frac{i_n}{k}\right)\in\widehat{K};i_j\in\{0,\dots,k\},1\le j\le n\right\}, \tag{2.35}$$

and then its set of dof is given by

$$\Sigma_{\widehat{K}}:=\left\{p(x);x\in M_{\widehat{K}}\right\}. \tag{2.36}$$

$\star$ **From n-Rectangles to the Triangulation/Rectangulation.** Same as before.

2.2.3 Hermite FE on n-Simplices

In addition to point values, we will use the values of the partial derivatives.

Theorem 2.5. *Let K be the n-simplex whose vertices are a_i, $i=1,\dots,n+1$. Let*

$$\overline{a_{ijk}}:=\frac{a_i+a_j+a_k}{3},\ \ 1\le i,j,k\le n+1. \tag{2.37}$$

Then a polynomial $p\in\mathbb{P}_3$ is uniquely determined by its values and the values of its partial derivatives at the vertices a_i's and its values at the points $\overline{a_{ijk}}$'s.

Proof. Check that $\dim\mathbb{P}_3=\operatorname{card}\Sigma_K$ where

$$\Sigma_K:=\left\{p(a_j),p\left(\overline{a_{ijk}}\right),\partial_l p(a_i);1\le i\le n+1,\ 1\le i<j<k\le n+1,\ 1\le l\le n\right\}. \tag{2.38}$$

Check that for any $p\in\mathbb{P}_3$, we have

$$p=\sum_{i=1}^{n+1}\left(-2\lambda_i^3+3\lambda_i^2-7\lambda_i\sum_{j<k,j\neq i,k\neq i}\lambda_j\lambda_k\right)p(a_i)+27\sum_{i<j<k}\lambda_i\lambda_j\lambda_k p\left(\overline{a_{ijk}}\right) \tag{2.39}$$

$$+\sum_{i\neq j}\lambda_i\lambda_j\left(2\lambda_i+\lambda_j-1\right)Dp(a_i)(a_j-a_i). \tag{2.40}$$

This completes the proof. □

3 General Properties

Definition 3.1 (Finite element)**.** *A* finite element *is a tuple* (K, P, Σ) *where*

1. *K is a closed subset of $\mathbb{R}^n$, $K \neq \emptyset$, and ∂K Lipschitz-continuous (polyhedra).*
2. *P is a space of real-valued functions defined on K (e.g., polynomials).*
3. *Σ is a finite set of linearly independent linear functions $(\phi_i)_{i=1,\ldots,N}$ defined on P. We assume that Σ is P-unisolvent: for all $\alpha \in \mathbb{R}^N$, there exists a unique $p \in P$ such that $\phi_i(p) = \alpha_i$, $i = 1, \ldots, N$. Therefore, there exist $p_i \in P$, $i = 1, \ldots, N$ such that $\phi_i(p_j) = \delta_{ij}$, $1 \leq i, j \leq N$,*

$$p(x) = \sum_{i=1}^{N} \varphi_i(p)\, p_i(x), \ \ \forall p \in P, \ \forall x \in K. \tag{3.1}$$

We have $\dim P = \mathrm{card}\Sigma$, and

$$\text{Duality:} \quad \begin{cases} (p_i)_i : & \text{basis of } (K, P, \Sigma)\,, \\ (\phi_i)_i : & \text{dof of } (K, P, \Sigma)\,. \end{cases} \tag{3.2}$$

Definition 3.2 (P-interpolation operator)**.** *The* P-interpolation operator Π *of a smooth function v is defined as*

$$\Pi v := \sum_{i=1}^{N} \phi_i(v)\, p_i. \tag{3.3}$$

Equivalently, we have $\Pi v \in P$ and $\phi_i(\Pi v) = \overline{\phi_i}(v)$, $\forall 1 \leq i \leq N$.

Note. $\phi_i : P \to \mathbb{R}$, $\overline{\phi_i} : C^s(K) \to \mathbb{R}$ where s: $P \subset C^s$.

Remark 3.1. *"Smooth"? $v \in C^s$, where s is the maximal order of derivatives of p which appears in the $(\phi_i)_i$. Moreover, Dom $\Pi = C^s(K)$, and $\Pi p = p$, $\forall p \in P$.*

Proposition 3.1. *Each FE presented before forms an affine family. Let $\left(\widehat{K}, \widehat{P}, \widehat{\Sigma}\right)$ be the reference element. For all $K \in \mathcal{T}_h$, there exist a unique affine and invertible mapping F_K such that*

$$F_K\left(\widehat{K}\right) = K, \ p_K = \widehat{p_K} \circ F_K^{-1}. \tag{3.4}$$

$F_K(\widehat{a}_i) = a_i$, where (a_i) and $(\widehat{a}_i)$ are the points where the dof are computed (as for Hermite FE).

Corollary 3.1. *The interpolant operator and the affine mapping commute.*

$$(\Pi_K v) \circ F_K(v) = \Pi_{\widehat{K}}(v \circ F_K)(x)\,, \ \ \forall v \in C^s(K)\,, \ \forall x \in \widehat{K}. \tag{3.5}$$

In the notion of affine equivalence of $\left(\widehat{K}, \widehat{P}, \widehat{\Sigma}\right)$ and (K, P, Σ), it is easier to use this commutation property instead of the correspondence of the $\left(\widehat{\phi}_i\right)_i$, and $(\phi_i)_i$.

3.1 The Global Setting

$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$, $X_h := \left\{ v \in C^0\left(\overline{\Omega}\right); v|_K \in \mathbb{P}_k \right\}$, and $\Sigma_h := \left\{ \left(\varphi_{i,K}\right)_{i=1,\ldots,N; K \in \mathcal{T}_h} \right\}$.

In Σ_h, no repetition of the dof associated with points in $K_1 \cap K_2 \neq \emptyset$, so we change the notation as $\left(\phi_{h,i}\right)_{i=1,\ldots,M}$. Set

$$\Pi_h\left(v\right) := \sum_{j=1}^{M} \varphi_{h,i}\left(v\right) p_{h,i}, \tag{3.6}$$

where the $p_{h,i}$ are defined by the $\left(p_{K,i}\right)_{i=1,\ldots,N}$ and extended by zero.

4 Analysis of the FEM

We want to look at the convergence of the FEM and obtain error estimates. Let u be the solution of (VF) in V, u_h be the solution of (VF$_h$) in V_h, where $V_h \subset V$, $\dim V_h < +\infty$, $V_h \subset X_h$.

Lemma 4.1 (Céa's lemma)**.**

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{4.1}$$

Idea here: Use $v_h := \Pi_h u \in V_h$ to deduce

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \|u - \Pi_h u\|_V. \tag{4.2}$$

"Just" a problem of interpolation.

Lemma 4.2.

$$\|u - \Pi_h u\|_{H^1(\Omega)} = \left(\sum_{K \in \mathcal{T}_h} \|u - \Pi_h u\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}}. \tag{4.3}$$

As a result, local interpolation results are sufficient.

In the following, we have in mind the problem

$$\text{Find } u \in H_0^1\left(\Omega\right) \text{ s.t. } -\Delta u = f \text{ in } \Omega, \tag{4.4}$$

and Lagrange FE. The error in K depends on the shape of K.

Let $\widehat{K}$ be the reference element, i.e.,

$$\widehat{K} := \left\{ x \in \mathbb{R}^n; x_i \geq 0, \ \forall i = 1, \ldots, n, \ \sum_{i=1}^{n} x_i \leq 1 \right\}. \tag{4.5}$$

Proposition 4.1. *Let F_K be the affine mapping such that $F_K\left(\widehat{K}\right) = K$ and denote $F_K\left(\widehat{x}\right) = B_K\widehat{x} + a_0$, $\forall \widehat{x} \in \widehat{K}$. Then we have the following properties on matrix B_K.*

1. $|\det B_K| = n!\,|K|$ *(where $|K|$: surface when $n = 2$, volume when $n = 3$);*
2. $\|B_K\|_2 \le h_K/\rho_{\widehat{K}}$*;*
3. $\left\|B_K^{-1}\right\|_2 \le h_{\widehat{K}}/\rho_K$*;*

where h_K and $h_{\widehat{K}}$ are the diameters of K and $\widehat{K}$, ρ_K and $\rho_{\widehat{K}}$ are the diameter of the largest ball included in K and $\widehat{K}$.

Remark 4.1.
- *If $h_K = const$ and $\rho \to 0$ then K becomes flat. More generally, it becomes flat when $\frac{h_K}{\rho_K} \to +\infty$.*
- *If K' is homoketic to K then*

$$\frac{h_K}{\rho_K} = \frac{h_{K'}}{\rho_{K'}}. \tag{4.6}$$

Theorem 4.1. *Let $v : K \to \mathbb{R}$ then $v \in H^m(K)$ iff $\widehat{v} = v \circ F_K \in H^m\left(\widehat{K}\right)$.*

Moreover, for all $0 \le k \le m$,

$$|v|_{H^k(K)} \le \frac{C|K|^{\frac{1}{2}}}{\rho_K^k}|\widehat{v}|_{H^k(\widehat{K})}, \tag{4.7}$$

$$|\widehat{v}|_{H^k(\widehat{K})} \le \frac{Ch_K^k}{|K|^{\frac{1}{2}}}|v|_{H^k(K)}, \tag{4.8}$$

where

$$|v|_{H^k(K)} := \left(\sum_{|\alpha|=k} \|\partial^\alpha v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{4.9}$$

Proof. For $m = 1$, $k = 0$ (L^2),

$$\|v\|^2_{L^2(K)} = \int_K |v(x)|^2 dx = \int_{\widehat{K}} |\widehat{v}(\widehat{x})|^2 |\det B_K|\, d\widehat{x} \le C\,\|\widehat{v}\|^2_{L^2(\widehat{K})}. \tag{4.10}$$

For $k = 1$, estimate $\|\nabla v\|_{L^2}$: $\nabla \widehat{v} = B_K^T \nabla v \circ F_K$. Since $\widehat{v} = v \circ F_K$ (assume $a_0 = 0$, where $F_K(\widehat{x}) = B_K\widehat{x} + a_0$),

$$\widehat{v}(\widehat{x}) = v\left(\sum_j b_{1j}\widehat{x}_j, \ldots, \sum_j b_{nj}\widehat{x}_j\right), \tag{4.11}$$

$$\frac{\partial \widehat{v}}{\partial \widehat{x}_i}(\widehat{x}) = b_{1i}\frac{\partial v}{\partial x_1}(F_K(\widehat{x})) + \cdots + b_{ni}\frac{\partial v}{\partial x_n}(F_K(\widehat{x})). \tag{4.12}$$

Hence, $\nabla \widehat{v}\left(\widehat{x}\right) = B_K^T \nabla v\left(F_K\left(\widehat{x}\right)\right)$, and

$$\|\nabla \widehat{v}\|_{L^2(\widehat{K})} \leq \left\|B_K^T\right\|_2 \left\|\widehat{\nabla v}\right\|_{L^2(\widehat{K})}, \tag{4.13}$$

where $\widehat{\nabla v} := \nabla v \circ F_K$.

$$\|\nabla \widehat{v}\|_{L^2(\widehat{K})}^2 \leq \frac{h_K^2}{\rho_K^2} \left\|\widehat{\nabla v}\right\|_{L^2(\widehat{K})}^2 \leq \frac{1}{|K|} \frac{C h_K^2}{\rho_{\widehat{K}}^2} \|\nabla v\|_{L^2(K)}^2, \tag{4.14}$$

where the last inequality is obtained from the previous inequality for $k = 0$.

The other inequality is obtained by changing K and $\widehat{K}$. $\square$

Definition 4.1. *Let $\widehat{v}$ be a smooth function on $\widehat{K}$, we introduce*

$$\Pi_{\widehat{K}} \widehat{v}\left(\widehat{x}\right) := \sum_{i=1}^{\mathrm{card}\Sigma_K} \widehat{v}\left(a_i\right) p_i\left(\widehat{x}\right), \quad \forall \widehat{x} \in \widehat{K}. \tag{4.15}$$

Let v be a smooth function on $\overline{\Omega}$, we introduce

$$\Pi_h v\left(x\right) := \sum_{\alpha \in \mathrm{dof}} v\left(\alpha\right) p_{h,i}\left(x\right), \quad \forall x \in \overline{\Omega}. \tag{4.16}$$

Theorem 4.2 (Deny-Lion)**.** *Let U be an open bounded set of $\mathbb{R}^n$, with ∂U Lipschitz-continuous, and $k \in \mathbb{N}$. There exists a constant $C > 0$ such that*

$$\inf_{\pi \in \mathbb{P}_k(U)} \|u - \pi\|_{H^{k+1}(U)} \leq C |u|_{H^{k+1}(U)}, \quad \forall u \in H^{k+1}\left(U\right). \tag{4.17}$$

In the case $k = 0$, we obtain the Poincaré-Wirtinger inequality

$$\left\|u - \frac{1}{|U|} \int_U u\left(x\right) dx\right\|_{H^1} \leq C \|u\|_{H^1}, \quad \forall u \in H^1\left(U\right). \tag{4.18}$$

Proof. For all $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$, f_α be a linear form defined on $H^{k+1}\left(U\right)$ such that

$$f_\alpha\left(u\right) := \int_U \partial^\alpha u dx, \quad \forall u \in H^{k+1}\left(U\right). \tag{4.19}$$

Define

$$\mathcal{F} : \mathbb{P}_k \to \mathbb{R}^M \tag{4.20}$$

$$\pi \mapsto \left(f_\alpha\left(x\right)\right)_{|\alpha| \leq k}, \tag{4.21}$$

then $\mathcal{F}$ is bijective (injectivity actually is sufficient). Indeed, suppose for the contrary, let $\pi \in \mathbb{P}_k$ such that $\pi \neq 0$, $\mathcal{F}\left(\pi\right) = 0$. Let $\alpha_0 \in \mathbb{N}^n$ such that the monomial x^{α_0} is nonzero and all the monomial x^α with $|\alpha| > |\alpha_0|$ are null. But $f_{\alpha_0}\left(\pi\right)$, contradiction. Therefore, $\mathcal{F}$ is bijective.

Let us check that there exists a constant $C > 0$ such that

$$\|u\|_{H^{k+1}} \leq C \left(|u|_{k+1} + \sum_{|\alpha| \leq k} |f_\alpha(x)| \right), \ \forall u \in H^{k+1}(U), \tag{4.22}$$

where

$$|u|_{k+1} := \left(\sum_{|\alpha|=k+1} \|\partial^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}, \ \forall u \in H^{k+1}(U). \tag{4.23}$$

By contradiction, there exists a sequence $(u_n)_n \subset H^{k+1}(U)$ such that $\|u\|_{H^{k+1}} = 1$, and

$$|u_n|_{k+1} + \sum_{|\alpha| \leq k} |f_\alpha(u_n)| \leq \frac{1}{n}. \tag{4.24}$$

Then there exists a subsequence $\left(u_{\varphi(n)}\right)_n$ such that $u_{\varphi(n)} \rightharpoonup u$ in $H^{k+1}(U)$, and $u_{\varphi(n)} \to u$ in $H^k(U)$ as $n \to \infty$.

Moreover, for all $\alpha \in \mathbb{N}^n$, $|\alpha| = k+1$, $\partial^\alpha u_{\varphi(n)} \rightharpoonup 0 = \partial^\alpha u$ as $n \to \infty$, thus $u \in \mathbb{P}_k$. Also, $f_\alpha\left(u_{\varphi(n)}\right) \to 0$ as $n \to \infty$, $\forall \alpha$, $|\alpha| \leq k$. Therefore, $\mathcal{F}(u) = 0 \Rightarrow u = 0$. This is impossible since $\|u\|_{H^{k+1}} = 1$. For all $u \in H^{k+1}(U)$, by surjectivity of $\mathcal{F}$, there exists a $\widetilde{\pi} \in \mathbb{P}_k$ such that $\mathcal{F}(u) = \mathcal{F}(\widetilde{\pi})$. Hence, for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$, $f_\alpha(u - \widetilde{\pi}) = 0$.

$$\inf_{\pi \in \mathbb{P}_k} \|u - \pi\|_{H^{k+1}(U)} \leq \|u - \widetilde{\pi}\|_{H^{k+1}(U)} \leq C \left(|u - \widetilde{\pi}|_{k+1} + \sum_{|\alpha| \leq k} f_\alpha(u - \widetilde{\pi}) \right) \leq C|u|_{k+1}, \tag{4.25}$$

since $|\widetilde{\pi}|_{k+1} = 0$ (because $\widetilde{\pi} \in \mathbb{P}_k$). □

Note. U has to be convex.

Remark 4.2.
- card $\{\alpha \in \mathbb{N}^n; |\alpha| \leq k\} = \dim P_k =: M$.
- *This theorem can be rephrased by considering the quotient space $H^{k+1}/\mathbb{P}_k$, and*

$$\overset{o}{V} := \left\{ w \in H^{k+1}(U); w - v \in \mathbb{P}_k \right\}. \tag{4.26}$$

Theorem 4.3 (Bramble-Hilbert)**.** *Let U be an open convex subset of $\mathbb{R}^n$, ∂U bounded and Lipschitz-continuous. Let Φ be a linear continuous mapping from $H^{k+1}(U)$ to a Banach space E. If $\Phi = 0$ on $\mathbb{P}_k$, then there exists a constant $C > 0$ such that*

$$\|\Phi u\|_E \leq C|u|_{k+1}, \ \forall u \in H^{k+1}(U). \tag{4.27}$$

Proof. Let $u \in H^{k+1}(U)$, $\pi \in \mathbb{P}_k$, $\|\Phi u\|_E = \|\Phi(u - \pi)\|_E$, then

$$\|\Phi u\| \leq \|\Phi\| \, \|u - \pi\|_{H^{k+1}}. \tag{4.28}$$

Since this is true for all π,

$$\|\Phi u\|_E \leq C \inf_{\pi \in \mathbb{P}_k} \|u - \pi\|_{H^{k+1}} \leq C|u|_{k+1}, \tag{4.29}$$

where the last inequality is deduced from Deny-Lions theorem. □

Now we can take $\Phi : u \mapsto u - \Pi_K u$.

Proposition 4.2. *There exists a constant $C > 0$ such that $\forall K \in \mathcal{T}_h, \forall v \in H^{k+1}(K)$:*

$$|v - \Pi_K v|_{H^k(K)} \leq \frac{C h_K^{k+1}}{\rho_K^k} |v|_{H^{k+1}(K)}, \tag{4.30}$$

$$|v|_k \leq \frac{C |K|^{\frac{1}{2}}}{\rho_K^{h_K}} |\widehat{v}|_k, \tag{4.31}$$

$$|\widehat{v}|_k \leq \frac{C h_K}{|K|^{\frac{1}{2}}}, \tag{4.32}$$

where h_K is $\mathrm{diam} K$*, ρ_K is the diameter of the largest ball included in K.*

Proof. Define

$$\Phi : H^{k+1}\left(\widehat{K}\right) \to H^k\left(\widehat{K}\right) \tag{4.33}$$

$$\widehat{v} \mapsto \widehat{v} - \Pi_{\widehat{K}} \widehat{v}, \tag{4.34}$$

then $\Phi \pi = 0, \forall \pi \in P_k$, and Φ is linear continuous.

Bramble-Hilbert theorem: $\exists C > 0$, $\forall \widehat{v} \in H^{k+1}\left(\widehat{K}\right)$, $\left\|\widehat{v} - \Pi_{\widehat{K}} \widehat{v}\right\|_{H^k} \leq C |\widehat{v}|_{H^{k+1}}$, for any $K \in \mathcal{T}_h$, let F_K the associated affine mapping s.t. $F_K\left(\widehat{K}\right) = K$.

For all $v \in H^{k+1}(K)$, we denote $\widehat{v} = v \circ F_K$, then

$$|v - \Pi_K v|_{H^k(K)} \leq \frac{C |K|^{\frac{1}{2}}}{\rho_K^k} |\widehat{v} - \Pi_{\widehat{K}} \widehat{v}|_{H^k(\widehat{K})} \leq \frac{C |K|^{\frac{1}{2}}}{\rho_K^k} |\widehat{v}|_{H^{k+1}(\widehat{K})} \leq \frac{C h_K^{k+1}}{\rho_K^k} |v|_{H^{k+1}(K)}. \tag{4.35}$$

This completes the proof. □

$\mathbb{P}_k \subset H^1(K)$, $\forall K$ and $X_h \subset C^0(\Omega) \Rightarrow X_h \subset H^1(\Omega)$.

⋆ **From the local estimate to the global estimate.**

$$\forall v \in H^{k+1}(\Omega), \quad |v - \Pi_h v|_{H^k(\Omega)} \leq C \sigma_h^k \sup_{K \in \mathcal{T}_h} h_K |v|_{H^{k+1}(\Omega)}, \tag{4.36}$$

where

$$\sigma_h = \sup_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K}. \tag{4.37}$$

→ Need to have $H^{k+1}(\Omega) \subset C^0\left(\overline{\Omega}\right)$, i.e., $k + 1 \geq \frac{d}{2}$ ($d = 1$: $k \geq 1$, $d = 2, 3$: $k \geq 1$).

Definition 4.2. *A family of meshes $(\mathcal{T}_h)_{h \geq 0}$ is called* regular *if there exists $C > 0$ s.t. $\sigma_h \leq C$, $\forall h > 0$.*

→ To avoid flat triangles.

Lemma 4.3. *A family of meshes is regular* $\Leftrightarrow \exists C > 0, \forall h > 0, \forall K \in \mathcal{T}_h, |K| \geq C h_K^n$.

Theorem 4.4. *Let* $(\mathcal{T}_h)_{h \geq 0}$ *be a regular family of meshes and define* $h := \sup_{K \in \mathcal{T}_h} h_K$ *for any* $\mathcal{T}_h$*. Assume* f *in* (VF) *to be in* $L^2(\Omega)$ *(then* $u \in H_0^1(\Omega) \cap H^2(\Omega)$*). Consider* u_h *given by Lagrange FE of degree* k*,*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}, \tag{4.38}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2|u|_{H^2(\Omega)}. \tag{4.39}$$

Remark 4.3. *The error estimates do not depend on* k*,* $\rightarrow$ *only order 1.*

Proof of the last inequality. Aubin-Nitsche Trick. Look at the adjoint problem; For any $v \in L^2(\Omega)$, find $\varphi_v \in H_0^1(\Omega)$ s.t.

$$\text{(AVF)} \quad a(w, \varphi_v) = (v, w)_{L^2}, \ \ \forall w \in H_0^1(\Omega). \tag{4.40}$$

then $\varphi_v \in H^2(\Omega)$, $\|\varphi_v\|_{H^2} \leq C\|v\|_{L^2}$. Define error $e_h := u - u_h$ and take $v = w = e_h$ in (AVF): $a(e_h, \varphi_{e_h}) = \|e_h\|_{L^2}^2$. Therefore, $\|e_h\|_{L^2}^2 = a(e_h, \varphi_{e_h} - \Pi_h \varphi_{e_h})$ because e_h is a-orthogonal to V_h: $a(e_h, v_h) = 0$, $\forall v_h \in V_h$.

$$\begin{aligned}
\|e_h\|_{L^2}^2 &\leq \|a\| \, \|e_h\|_{H^1} \|\varphi_{e_h} - \Pi_h \varphi_{e_h}\|_{H^1} && (4.41)\\
&\leq \|a\| \, \|e_h\|_{H^1} h \|\varphi_{e_h}\|_{H^2} \text{ (interpolation result)} && (4.42)\\
&\leq \|a\| \, \|e_h\|_{H^1} h \|e_h\|_{L^2} \text{ (elliptic smoothness).} && (4.43)
\end{aligned}$$

Thus, $\|e_h\|_{L^2} \leq Ch^2|u|_{H^2}$ (first error estimate). □

5 Practical Part

$-\Delta u + u = f$ and BCs. PDE $\rightarrow$ VF. We take test function $v = 0$ on Γ_D because no dof on Γ_D.

$$u \in \left\{ w \in H^1; w|_{\Gamma_D} = u_D \right\} = V + U_D, \tag{5.1}$$

$$v \in \left\{ w \in H^1; w|_{\Gamma_D} = 0 \right\} = V, \tag{5.2}$$

where $U_D \in H^1$ and $U_D|_{\Gamma_D} = u_D$ (Dirichlet boundary condition).

Discretization. $\Omega \rightarrow \Omega_h = \bigcup_{K \in \mathcal{T}_h} K$, and $V \rightarrow V_h = \text{span}\{\varphi_1, \ldots, \varphi_N\}$: a space of finite dimension.

$$(\text{VF}_h) \quad u_h \in V_h + U_D, \ \ \forall v_h \in V_h: \int_{\Omega_h} \nabla u_h \cdot \nabla v_h + \int_{\Omega_h} u_h v_h = \int_{\Omega_h} f_h v_h + \int_{\Gamma_{N,h}} g_{N,h} v_h.$$

$u_h = U_D + \sum\limits_{J=1}^{N} u_J \varphi_J$, take $v = \varphi_I$, $I = 1, \ldots, N$.

$$\sum_{J=1}^{N} u_J \left(\int_{\Omega_h} \nabla \varphi_J \cdot \nabla \varphi_I + \int_{\Omega_h} \varphi_J \varphi_I \right) = \int_{\Omega_h} f \varphi_I + \int_{\Gamma_N} g_N \varphi_I - \int_{\Omega_h} \nabla u_D \cdot \nabla \varphi_I - \int_{\Omega_h} u_D \varphi_I.$$

obtain $AX = B$.

$H^1 \overset{\text{F.E}}{\to} V_h$. The F.E is conform if $V_h \subset H^1$: $\mathbb{P}_1$ Lagrange F.E. H^1 conform and $\mathbb{P}_1$ Lagrange F.E. nonconform.

$\mathbb{P}^k$ F.E., $\to$ Nodes no_I, $I = 1, \dots, M$, φ_I defined on no_I: $\varphi_I \in V_h$, $\varphi_I\left(no_J\right) = \delta_{IJ}$.

$$\int_{\Omega_h} \varphi_I \varphi_J = \sum_{K \in \mathcal{T}_h} \int_K \varphi_I \varphi_J = \sum_{K \in \mathcal{T}_h,\; no_I \in K,\; no_J \in K} \int_K \varphi_I \varphi_J. \tag{5.3}$$

$\exists i, j \in \{1,2,3\}$, $\varphi_I = \varphi_i^K$, $\varphi_J = \varphi_j^K$, $\varphi_i^K \in \mathbb{P}^k(K)$, $\varphi_i^K\left(no_j^K\right) = \delta_{ij}$.

The integral $\int_K \varphi_I \varphi_J$ can be calculated from the knowledge of $\left(\int_K \varphi_i^K \varphi_j^K\right)_{i,j=1,2,3}$ (elementary matrix related to K).

FE Calculate of A and B. Calculation of the elementary matrices and the elementary RHS. $\exists! F_K \in P_1 \times P_1$ s.t. $F_K\left(\widehat{no}_i\right) = no_i^K$.

$$\int_K \varphi_i^K(x)\, \varphi_j^K(x)\, dx = \int_{\widehat{K}} \underbrace{\varphi_i^K\left(F_K(\widehat{x})\right)}_{\widehat{\varphi}_i(\widehat{x})} \underbrace{\varphi_j^K\left(F_K(\widehat{x})\right)}_{\widehat{\varphi}_j(\widehat{x})} \left|JF_K(\widehat{x})\right| d\widehat{x} \tag{5.4}$$

$$= \int_{\widehat{K}} \widehat{\varphi}_i(\widehat{x})\, \widehat{\varphi}_j(\widehat{x}) \left|JF_K(\widehat{x})\right| d\widehat{x}. \tag{5.5}$$

More precise:

$$F_K(\widehat{x}) = \sum_{i=1}^{3} no_i^K \widehat{\psi}_i(\widehat{x}). \tag{5.6}$$

Isoparametric F.E. $\widehat{\psi}_i = \widehat{\varphi}_i$. Non-isoparametric F.E. $\left(\widehat{\varphi}_i\right)_i \neq \left(\widehat{\psi}_i\right)_i$.

$$\widehat{\varphi}_1 \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \widehat{x}_1, \;\; \widehat{\varphi}_2 \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \widehat{x}_2, \;\; \widehat{\varphi}_1 \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = 1 - \widehat{x}_1 - \widehat{x}_2. \tag{5.7}$$

We have $\varphi_i^K \circ F_K = \widehat{\varphi}_i$, $i = 1,2,3$.

Integration over $\widehat{K}$: use Gauss quadrature formula for the reference triangle.

Some useful identities:

$$\int_0^1 x^k (1-x)^k dx = \frac{k!l!}{(k+l+1)!}, \tag{5.8}$$

$$\int_K \left(\lambda_1^K\right)^{k_1} \left(\lambda_2^K\right)^{k_2} = 2\left|K\right| \frac{k_1! k_2!}{(k_1+k_2+2)!}, \tag{5.9}$$

$$\nabla\lambda_1 \cdot \nabla\lambda_1 = \frac{1}{4|K|^2} \left\| no_2^K no_3^K \right\|, \tag{5.10}$$

$$\nabla\lambda_2 \cdot \nabla\lambda_3 = -\frac{1}{4|K|^2} \overrightarrow{no_1 no_2} \cdot \overrightarrow{no_1 no_3}. \tag{5.11}$$