Duong Minh Duc, Sobolev Spaces

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Abstract

I retype [1], which is used to teach the course *Calculus of Variations* in Ho Chi Minh University of Sciences. Share it!

Contents

1 Sobolev Spaces

Definition 1.1. Let f be a real function on an open subset D of $\mathbb{R}^n$, $x = (x_1, \ldots, x_n) \in D$ and $i \in \{1, \ldots, n\}$. We define

$$\frac{\partial f}{\partial x_i}(x) \tag{1.1}$$
$$= \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t} \tag{1.2}$$
$$= \lim_{t \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{t} \tag{1.3}$$

provided the limit exists, and $\frac{\partial f}{\partial x_i}(x)$ is called the *partial derivative of f at x with respect to the variable x_i*.

If $\frac{\partial f}{\partial x_i}(x)$ exists for any i in $\{1, \ldots, n\}$, we say f is *differentiable at x* and has *derivative*

$$Df(x) = \nabla f(x) \tag{1.4}$$
$$= \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right) \tag{1.5}$$

Definition 1.2. Let f be a real function on an open subset D of $\mathbb{R}^n$. We say

1. f is *differentiable on D* if $\nabla f(x)$ exists for any x in D.
2. f *is of class $C^1(D)$* if f is differentiable on D and ∇f is a continuous from D into $\mathbb{R}^n$.
3. f *is of class $C_c^1(D)$* if f is of class $C^1(D)$ and $f(x) = 0$ for any x in $D \backslash K_f$, where K_f is a compact set contained in D.
4. f *is of class $C^1(\overline{D})$* if f is of class $C^1(D_f)$, where D_f is a open set containing D.

Definition 1.3. Let f be a real differentiable function on an open subset D of $\mathbb{R}^n$ and $x \in D$. Put $g_j = \frac{\partial f}{\partial x_j}$, then g_j is a real function on D for any j in $\{1, \ldots, n\}$. Let i be in $\{1, \ldots, n\}$. We say

1. f has the *second-order partial derivative* $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ at x if g_j has the partial derivative $\frac{\partial g}{\partial x_i}(x)$ at x.
2. f has the *second-order partial derivative at x* if $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ exists for any i, j in $\{1, \ldots, n\}$. In this case the second-order derivative $D^2 f(x)$ of f at x is the $n \times n$*-matrix*

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]_{i,j=1,2,\ldots,n} \tag{1.6}$$

Definition 1.4. Let f be a real function on an open subset D of $\mathbb{R}^n$. We say

1. f is *differentiable 2-times on* D if $D^2 f(x)$ exists for any x in D.
2. f *is of class* $C^2(D)$ if f is differentiable 2-times on D and $D^2 f$ is a continuous from D into $\mathbb{R}^{n\times n}$.
3. f *is of class* $C_c^2(D)$ if f is of class $C^2(D)$ and $f(x) = 0$ for any x in $D \backslash K_f$, where K_f is a compact set contained in D.
4. f *is of class* $C^2\left(\overline{D}\right)$ if f is of class $C^2(D_f)$, where D_f is a open set containing D.

Similarly, we can define the classes $C^r(D), C_c^r(D)$ and $C^r\left(\overline{D}\right)$ for any integer $r >$. We put

$$C^\infty(D) = \bigcap_{n=1}^{\infty} C^r(D) \tag{1.7}$$

$$C_c^\infty(D) = \bigcap_{n=1}^{\infty} C_c^r(D) \tag{1.8}$$

$$C^\infty\left(\overline{D}\right) = \bigcap_{n=1}^{\infty} C^r\left(\overline{D}\right) \tag{1.9}$$

Theorem 1.5. *Let D be an open subset of $\mathbb{R}^n$, $p \in [1, +\infty)$ and f be in $L^p(D)$. Assume*

$$\int_D fg dx = 0, \quad \forall g \in C_c^\infty(D) \tag{1.10}$$

Then $f = 0$ a.e. on$D$.

Theorem 1.6. *Let D be an open subset of $\mathbb{R}^n$ with smooth boundary ∂D, $i \in \{1, \ldots, n\}$ and $f \in C^1\left(\overline{D}\right)$. Then*

$$\int_D f \frac{\partial g}{\partial x_i} dx = \int_{\partial D} fg ds - \int_D \frac{\partial f}{\partial x_i} g dx, \quad \forall g \in C^1\left(\bar{D}\right) \tag{1.11}$$

$$\int_D f \frac{\partial g}{\partial x_i} dx = -\int_D \frac{\partial f}{\partial x_i} g dx, \quad \forall g \in C_c^1(D) \tag{1.12}$$

where ds is the measure on the boundary ∂D.

Put

$$\|f\|_{1,p} = \left(\int_D \left(|f|^p + \|\nabla f\|^p\right) dx\right)^{\frac{1}{p}}, \quad \forall f \in C^1\left(\bar{D}\right) \tag{1.13}$$

$$\|f\|_{2,p} = \left(\int_D \left(|f|^p + \|\nabla f\|^p + \left\|D^2 f\right\|^p\right) dx\right)^{\frac{1}{p}}, \quad \forall f \in C^2\left(\bar{D}\right) \tag{1.14}$$

$$\|f\|_{k,p} = \left(\int_D \left(|f|^p + \sum_{r=1}^{k} \|D^r f\|^p \right) dx \right)^{\frac{1}{p}}, \quad \forall f \in C^k\left(\bar{D}\right) \tag{1.15}$$

We see that $\left(C_c^k\left(D\right), \|\cdot\|_{1,p}\right)$ and $\left(C^k\left(\overline{D}\right), \|\cdot\|_{1,p}\right)$ are norm linear spaces. We denote by $W_0^{k,p}\left(D\right)$ and $W^{k,p}\left(D\right)$ their completions respectively. These Banach spaces are called *Sobolev spaces.*

We see that

$$W_0^{k,p}\left(D\right) \subset W^{k,p}\left(D\right), \quad \forall k \geq 1 \tag{1.16}$$

$$W^{k,p}\left(D\right) \subset W^{k-1,p}\left(D\right) \subset L^p\left(D\right), \quad \forall k > 1 \tag{1.17}$$

Let $p \in [1, +\infty)$ and $u \in W^{1,p}\left(D\right)$. There is a Cauchy sequence $\{u_m\}$ "converges" to u in following sense: $\{u_m\}$ converges to u in $L^p\left(D\right)$, $\left\{\dfrac{\partial u_m}{\partial x_i}\right\}$ is a Cauchy sequence in $L^p\left(D\right)$ for any $i \in \{1, \ldots, n\}$.

We can choose $\{u_m\}$ and $v_1, \ldots, v_n$ in $L^p\left(D\right)$ such that

$$\lim_{m\to\infty} \left\| \frac{\partial u_m}{\partial x_i} - v_i \right\|_p = 0, \quad \forall i \in \{1, \ldots, n\} \tag{1.18}$$

$$u\left(x\right) = \lim_{m\to\infty} u_m\left(x\right) \text{ a.e on } D \tag{1.19}$$

$$v_i\left(x\right) = \lim_{m\to\infty} \frac{\partial u_m}{\partial x_i}\left(x\right) \text{ a.e on } D, \quad \forall i \in \{1, \ldots, n\} \tag{1.20}$$

We have

$$\int_D u_m \frac{\partial \varphi}{\partial x_i} dx = -\int_D \frac{\partial u_m}{\partial x_i} \varphi dx, \quad \forall \varphi \in C_\infty^1\left(D\right), m \in \mathbb{N} \tag{1.21}$$

and

$$\left| \int_D u_m \frac{\partial \varphi}{\partial x_i} dx - \int_D u \frac{\partial \varphi}{\partial x_i} dx \right| \tag{1.22}$$

$$= \left| \int_D \left(u_m - u\right) \frac{\partial \varphi}{\partial x_i} dx \right| \tag{1.23}$$

$$\leq \int_D \left| \left(u_m - u\right) \frac{\partial \varphi}{\partial x_i} \right| dx \tag{1.24}$$

$$\leq \left(\int_D |u_m - u|^p dx \right)^{\frac{1}{p}} \left(\int_D \left| \frac{\partial \varphi}{\partial x_i} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \to 0 \text{ as } m \to \infty \tag{1.25}$$

similarly,

$$\left| \int_D \frac{\partial u_m}{\partial x_i} \varphi dx - \int_D v_i \varphi dx \right| \tag{1.26}$$

$$= \left| \int_D \left(\frac{\partial u_m}{\partial x_i} - v_i \right) \varphi dx \right| \tag{1.27}$$

$$\leq \int_D \left| \left(\frac{\partial u_m}{\partial x_i} - v_i \right) \varphi \right| dx \tag{1.28}$$

$$\leq \left(\int_D \left| \frac{\partial u_m}{\partial x_i} - v_i \right|^p dx \right)^{\frac{1}{p}} \left(\int_D |\varphi|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \to 0 \text{ as } m \to \infty \tag{1.29}$$

Combining (1.21), (1.25) and (1.29) yields

$$\int_D u \frac{\partial \varphi}{\partial x_i} dx = - \int_D v_i \varphi dx, \ \ \forall \varphi \in C^1_\infty (D), i \in \{1, \ldots, n\} \tag{1.30}$$

We say v_i is the generalized partial derivative of u with respect to x_i and denote it by $\dfrac{\partial u}{\partial x_i}$.

Thus, let u be in $W^{1,p}(D)$, then u has its generalized partial derivatives $\dfrac{\partial u}{\partial x_i} \in L^p(D)$ such that

$$\int_D u \frac{\partial \varphi}{\partial x_i} dx = - \int_D \frac{\partial u}{\partial x_i} \varphi dx, \ \ \forall \varphi \in C^1_\infty (D), i \in \{1, \ldots, n\} \tag{1.31}$$

Thus, let u be in $W^{1,p}(D)$, then u has its generalized partial derivatives $\dfrac{\partial u}{\partial x_i} \in L^p(D)$ such that

$$\int_D u \frac{\partial \varphi}{\partial x_i} dx = - \int_D \frac{\partial u}{\partial x_i} \varphi dx, \ \ \forall \varphi \in C^1_c (D), i \in \{1, \ldots, n\} \tag{1.32}$$

Example 1.7. Let η be in $W^{1,p}_0(D)$. We can choose a sequence $\{\varphi_m\}$ in $C^1_c(D)$, which converges to η in $W^{1,p}_0(D)$. Arguing as above, we get

$$\int_D u \frac{\partial \eta}{\partial x_i} dx = - \int_D \frac{\partial u}{\partial x_i} \eta dx, \ \ \forall \eta \in W^{1,p}_0 (D), i \in \{1, \ldots, n\} \tag{1.33}$$

Let $D = (-1, 1)$ and $u(x) = |x|$ for any x in D. Put

$$u_m(x) = \sqrt{x^2 + \frac{1}{m}}, \ \ \forall x \in D, m \in \mathbb{N}^* \tag{1.34}$$

We have

$$|u_m(x)| \leq \sqrt{2} \tag{1.35}$$

$$\lim_{m \to \infty} u_m(x) = \sqrt{x^2} = u(x), \ \ \forall x \in D \tag{1.36}$$

$$|u_m{}'(x)| = \left| \frac{x}{\sqrt{x^2 + \frac{1}{m}}} \right| \leq 1, \ \ \forall x \in D \backslash \{0\} \tag{1.37}$$

$$\lim_{m \to \infty} u_m{}'(x) = \frac{x}{\sqrt{x^2}} = \text{sign} x, \ \ \forall x \in D \backslash \{0\} \tag{1.38}$$

By the Lebesgue dominated convergence theorem, u is in $W^{1,2}(D)$ and its generalized derivative is $u'(x) = \text{sign} x$. □

Example 1.8. Let $D = (-1, 1)$. Put

$$u(x) = \begin{cases} 1, \forall x \in (-1, 0] \\ 0, \forall x \in (0, 1) \end{cases} \tag{1.39}$$

We see that $u \in L^2(D)$.

Now assume there is $v \in L^2(D)$ such that

$$\int_D u\varphi' dx = -\int_D v\varphi dx, \quad \forall \varphi \in C_c^1(D) \tag{1.40}$$

We have

$$\int_D u\varphi' dx = \int_{-1}^{0} \varphi' dx \tag{1.41}$$

$$= \varphi(0) - \varphi(-1) \tag{1.42}$$

$$= \varphi(0), \quad \forall \varphi \in C_c^1(D) \tag{1.43}$$

By (1.40) and (1.43), we see that

$$\int_D v\varphi dx = 0, \quad \forall \varphi \in C_c^1(D\backslash\{0\}) \tag{1.44}$$

which implies $v = 0$ a.e. on $D\backslash\{0\}$. Thus $v = 0$ a.e. on D or

$$\int_D v\varphi dx = 0, \forall \varphi \in C_c^1(D) \tag{1.45}$$

By (1.43) and (1.45), $\varphi(0) = 0$ for any $\varphi \in C_c^1(D)$.

Therefore, $W^{1,2}(D) \subset Ł^2(D)$, but $W^{1,2}(D) \neq L^2(D)$. □

The following properties of generalized derivatives are proved in Chapter 7 of the book "D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*".

Theorem 1.9. *Let D be an open subset of $\mathbb{R}^n$, p and q be in $(1, +\infty)$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.46}$$

Let $u \in W^{1,p}(D)$ and $v \in W^{1,q}(D)$. Then $uv \in W^{1,1}(D)$ and

$$\frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}, \forall i \in \{1, \ldots, n\} \tag{1.47}$$

Theorem 1.10. *Let $a_1 < a_2 < \ldots < a_k$ be k real numbers, D be an open subset of $\mathbb{R}^n$. Put $B = \{a_1, \ldots, a_k\}$. Let f be a real function on $\mathbb{R}$ of class $C(\mathbb{R}) \cap C^1(\mathbb{R}\backslash B)$ such that f' is discontinuous at every point of B, and $f' \in L^\infty(\mathbb{R}\backslash B)$. Let $u \in W^{1,p}(D)$ with $p \in [1, +\infty)$. Then $v = f \circ u$ belongs to $W^{1,p}(D)$ and*

$$\frac{\partial v}{\partial x_i}(x) = \begin{cases} f'(u(x)) \dfrac{\partial u}{\partial x_i}, \text{ if } u(x) \in R\backslash B \\ 0, \text{ if } u(x) \in B \end{cases} \tag{1.48}$$

Theorem 1.11. *Let D be an open subset of $\mathbb{R}^n$ and $u \in W^{1,p}(D)$ with $p \in [1,\infty)$. Put*

$$u^+ = \max\{0, u\} \tag{1.49}$$
$$u^- = \max\{0, -u\} \tag{1.50}$$

Then u^+, u^- and $|u|$ belong to $W^{1,p}(D)$ and

$$\frac{\partial u^+}{\partial x_i}(x) = \begin{cases} 0, \text{ if } u(x) \leq 0 \\ \frac{\partial u}{\partial x_i}(x), \text{ if } u(x) > 0 \end{cases} \tag{1.51}$$
$$\frac{\partial u^+}{\partial x_i}(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x), \text{ if } u(x) < 0 \\ 0, \text{ if } u(x) \geq 0 \end{cases} \tag{1.52}$$
$$\frac{\partial |u|}{\partial x_i}(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x), \text{ if } u(x) > 0 \\ 0, \text{ if } u(x) = 0 \\ -\frac{\partial u}{\partial x_i}(x), \text{ if } u(x) < 0 \end{cases} \tag{1.53}$$

We see that

$$W_0^{k,p}(D) \subset W^{k,p}(D), \forall k \geq 1 \tag{1.54}$$
$$W^{k,p}(D) \subset W^{k-1,p}(D) \subset L^p(D), \forall k > 1 \tag{1.55}$$
$$W_0^{1,p}(D) \subset W^{1,p}(D) \subset L^p(D) \tag{1.56}$$

Theorem 1.12 (Sobolev embedding). *Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{1,p}(D)$ with $p \in [1, +\infty)$. Then*

1. *$u \in L^q(D)$ where $q = \frac{np}{n-p}$ if $p < n$.*
2. *u is of class $C^r(\overline{D})$ if $0 \leq r < 1 - \frac{p}{n}$.*

Theorem 1.13 (Sobolev embedding). *Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1, +\infty)$. Then*

1. *$u \in L^q(D)$ where $q = \frac{np}{n-kp}$ if $kp < n$.*
2. *u is of class $C^r(\overline{D})$ if $0 \leq r < k - \frac{p}{n}$.*

The proof of this theorem is in the book of Adams.

Theorem 1.14 (Sobolev embedding). *Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1, +\infty)$. Then $u \in L^q(D)$ if $q \in \left[p, \frac{np}{n-kp}\right]$ and $kp < n$.*

Theorem 1.15 (Sobolev embedding). *Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1, +\infty)$. Then $u \in L^q(D)$ if $q \in \left[1, \frac{np}{n-kp}\right]$ and $kp < n$.*

Theorem 1.16 (Sobolev inequality). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, n and k be positive integers and $p \in [1, +\infty)$ such that $kp < n$. Then for any $q \in \left[1, \frac{np}{n-kp}\right]$ there is a positive real number C such that*

$$\|u\|_q \leq C\|u\|_{k,p}, \quad \forall u \in W^{k,p}(D) \tag{1.57}$$

Theorem 1.17 (Poincare inequality). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, n be a positive integer, $p \in [1, \infty)$ such that $p < n$. Then for any $q \in \left[1, \frac{np}{n-kp}\right]$ there is a positive real number C such that*

$$\|u\|_q \leq C\|\nabla u\|_p, \;\; \forall u \in W_0^{1,p}(D) \tag{1.58}$$

Theorem 1.18. *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, n be a positive integer, $p \in [1, \infty)$ such that $p < n$. Put*

$$|||u|||_{1,p} = \left(\int_D \|\nabla u\|^p dx\right)^{\frac{1}{p}}, \;\; \forall u \in W_0^{1,p}(D) \tag{1.59}$$

Then there are a positive real number c such that

$$c\|u\|_{1,p} \leq |||u|||_{1,p} \leq \|u\|_{1,p}, \forall u \in W_0^{1,p}(D) \tag{1.60}$$

Theorem 1.19. $\left(W_0^{1,2}(D), |||\cdot|||\right)$ *is a Hilbert space with the following inner product*

$$\langle u, v\rangle = \int_D \nabla u \cdot \nabla v dx, \;\; \forall u, v \in W_0^{1,2}(D) \tag{1.61}$$

Theorem 1.20. $W^{1,2}(D)$ *is a Hilbert space with the following inner product*

$$\langle u, v\rangle = \int_D (uv + \nabla u \cdot \nabla v)\, dx, \;\; \forall u, v \in W^{1,2}(D) \tag{1.62}$$

Theorem 1.21 (Rellich-Kondrachov). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, k be positive integer, and $p \in [1, +\infty)$ such that $kp < n$. Let $q \in \left[1, \frac{np}{n-kp}\right]$ and put*

$$T(u) = u, \;\; \forall u \in W^{k,p}(D) \tag{1.63}$$

Then T is a bounded linear mapping from $W^{k,p}(D)$ into $L^q(D)$, and the closure $T(A)$ in $L^q(D)$ is compact in $L^q(D)$ for any bounded subset A in $W^{k,p}(D)$.

Theorem 1.22 (Sobolev embedding). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}$, and $u \in W^{1,p}(D)$ with $p \in (1, +\infty)$. Then $u \in L^q(D)$ for any $q \in [1, +\infty)$.*

Theorem 1.23 (Sobolev inequality). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}$, and $p \in (1, +\infty)$. Then for any $q \in [1, +\infty)$, there is a positive real number C such that*

$$\|u\|_q \leq C\|u\|_{1,p}, \;\; \forall u \in W^{1,p}(D) \tag{1.64}$$

Theorem 1.24 (Rellich-Kondrachov). *Let D be a bounded open subset with smooth boundary in $\mathbb{R}$, $p \in (1, +\infty)$ and $q \in [1, +\infty)$. Put*

$$T(u) = u, \;\; \forall u \in W^{1,p}(D) \tag{1.65}$$

Then T is a bounded linear mapping from $W^{1,p}(D)$ into $L^q(D)$, and the closure $T(A)$ in $L^q(D)$ is compact in $L^q(D)$ for any bounded subset A in $W^{1,p}(D)$.

Theorem 1.25. *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, $p \in (1, +\infty)$, and T be a linear mapping from $W^{1,p}(D)$ into $\mathbb{R}$. Then T is continuous on $W^{1,p}(D)$ if and only if there are $g, g_1, \ldots, g_n$ in $L^{\frac{p}{p-1}}(D)$ such that*

$$T(u) = \int_D \left(ug + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} g_i \right) dx, \quad \forall u \in W^{1,p}(D) \tag{1.66}$$

Theorem 1.26. *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, and T be a linear mapping from $W_0^{1,2}(D)$ into $\mathbb{R}$. Then T is continuous on $W_0^{1,2}(D)$ if and only if there is g in $W_0^{1,2}(D)$ such that*

$$T(u) = \int_D \left(\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial g}{\partial x_i} \right) dx, \quad \forall u \in W_0^{1,2}(D) \tag{1.67}$$

Definition 1.27. Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, $p \in (1, +\infty)$, $v \in W^{1,p}(D)$ and $\{v_m\}$ be a sequence in $W^{1,p}(D)$. Then we say $\{v_m\}$ weakly converges to v in $W^{1,p}(D)$ if $\{T(v_m)\}$ converges to $T(v)$ for any bounded linear mapping T from $W^{1,p}(D)$ into $\mathbb{R}$.

Theorem 1.28. *Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, $p \in (1, +\infty)$ and $\{u_m\}$ be a bounded sequence in $W^{1,p}(D)$. Then there are $u \in W^{1,p}(D)$ and a subsequence $\{u_{m_k}\}$ such that $\{u_{m_k}\}$ weakly converges to u.*

THE END

References

[1] Duong Minh Duc, *Sobolev Spaces*, Ho Chi Minh University of Sciences.