On the smallest constant for a Gagliardo-Nirenberg functional inequality

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Abstract

The main objective of this paper is to present a relationship between the best constant for a classical interpolation inequality due to Nirenberg and Gagliardo, and the ground state solution of the equation

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0.$$

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1 Solution of a Variational Problem

We begin by studying

$$J^{\sigma,N}(f) := \frac{\|\nabla f\|_2^{\sigma N}\|f\|_2^{2+\sigma(2-N)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}, \tag{1.1}$$

the nonlinear functional naturally associated with the interpolation estimate

$$\|f\|_{2\sigma+2}^{2\sigma+2} \le C_{\sigma,N}^{2\sigma+2}\|\nabla f\|_2^{\sigma N}\|f\|_2^{2+\sigma(2-N)}, \text{ if } 0 < \sigma < \frac{2}{N-2}, N \ge 2. \tag{1.2}$$

By estimate (1.2), $J^{\sigma,N}$ is defined on $H^1(\mathbb{R}^N)$ for $0 < \sigma < \frac{2}{N-2}$.

Theorem 1.1. *For* $0 < \sigma < \frac{2}{N-2}$,

$$\alpha := \inf_{u\in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$$

is attained at a function ψ *with the following properties:*

1. *ψ is positive and a function of $|x|$ alone.*

2. *$\psi \in H^1\left(\mathbb{R}^N\right) \cap C^\infty\left(\mathbb{R}^N\right)$.*

3. *ψ is a solution of the following equation*

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0, \tag{1.3}$$

   of minimal L^2 norm (the ground state).

In addition,

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}.$$

In the proof of Theorem 1.1, we follow Strauss [5] in using a compactness property of functions in $H^1_{\text{radial}}\left(\mathbb{R}^N\right)$.

1.1 Strauss's Estimate

Proposition 1.1 (Proposition 1.7.1, [2], p. 20)**.** *Let $(u_n)_{n\geq 0} \subset H^1\left(\mathbb{R}^N\right)$ be a bounded sequence of spherically symmetric functions. If $N \geq 2$ or if $u_n(x)$ is a nonincreasing function of $|x|$ for every $n \geq 0$, then there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in H^1\left(\mathbb{R}^N\right)$ such that $u_{n_k} \to u$ as $k \to \infty$ in $L^p\left(\mathbb{R}^N\right)$ for every $2 < p < \frac{2N}{N-2}$ ($2 < p \leq \infty$ if $N = 1$).*

Proposition 1.1 is an immediate consequence of the Lemma 1.1 and 1.2.

Proof. If $N \geq 2$, we apply the first estimate (1.4) in Lemma 1.2 to each spherically symmetric functions $u_n \in H^1\left(\mathbb{R}^N\right)$ to obtain

$$|u_n(x)| \leq \frac{C\|u_n\|_{L^2}^{\frac{1}{2}}\|\nabla u_n\|_{L^2}^{\frac{1}{2}}}{|x|^{\frac{N-1}{2}}} \leq \frac{C\|u_n\|_{H^1}}{|x|^{\frac{N-1}{2}}} \leq \frac{C}{|x|^{\frac{N-1}{2}}}, \quad \forall x \in \mathbb{R}^N, \ \forall n \geq 0.$$

where the last inequality is deduced from the boundedness of u_n's.

If $u_n(x)$ is a nonincreasing function of $|x|$ for every $n \geq 0$, we applying the second estimate (1.5) in Lemma 1.2 to u_n to obtain

$$|u_n(x)| \leq \frac{C\|u_n\|_{L^2}}{|x|^{\frac{N}{2}}} \leq \frac{C}{|x|^{\frac{N}{2}}}, \quad \forall x \in \mathbb{R}^N, \ \forall n \geq 0.$$

In both cases, these estimates imply that $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$. Now, we can apply Lemma 1.1 to obtain the desired result. □

Lemma 1.1 (Lemma 1.7.2, [2], p. 20)**.** *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $H^1\left(\mathbb{R}^N\right)$. Suppose $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$. It follows that there exist a subsequence $u_{n_k} \to u$ as $k \to \infty$ in $L^p\left(\mathbb{R}^N\right)$ for every $2 < p < \frac{2N}{N-2}$ ($2 < p \leq \infty$ if $N = 1$).*

Remark 1.1 (Remark 1.3.1(iii), [2], p. 7)**.** *Assume $m \geq 1$ and $1 < p \leq \infty$. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{m,p}(\Omega)$, then there exist $u \in W^{m,p}(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. as $k \to \infty$, and*

$$\|u\|_{W^{m,p}} \leq \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}.$$

If $p < \infty$, then also $u_{n_k} \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in N} \subset W_0^{m,p}(\Omega)$, then $u \in W_0^{m,p}(\Omega)$.

Applying this remark for a bounded sequence in $H^1(\mathbb{R}^N)$, there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. as $k \to \infty$, $\|u\|_{H^1(\mathbb{R}^N)} \leq \liminf \|u_n\|_{H^1(\mathbb{R}^N)}$ and $u_{n_k} \rightharpoonup u$ in $H^1(\mathbb{R}^N)$.

Proof of Lemma 1.1. Since $(u_n)_{n\geq 0}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, applying Remark 1.1 yields that there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \rightharpoonup u$ as $k \to \infty$ in $H^1(\mathbb{R}^N)$. Fix $\varepsilon > 0$ and let $R > 0$ to be chosen later. Given $p \in \left(2, \frac{2N}{N-2}\right)$ ($2 < p \leq \infty$ if $N = 1$), we have[1]

$$\begin{aligned}\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} &= \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^p(\{|x|\geq R\})} \\ &\leq \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}}.\end{aligned}$$

We first fix R large enough so that (by uniform convergence)

$$\|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}} \leq \frac{\varepsilon}{2}.$$

Next, since $\left(u_{n_k}|_{B_R}\right)_{k\geq 0}$ is bounded in $H^1(B_R)$, it follows from Rellich's compactness theorem that $u_{n_k}|_{B_R} \to u|_{B_R}$ in $L^p(B_R)$. Therefore for k large enough we have

$$\|u_{n_k} - u\|_{L^p(B_R)} \leq \frac{\varepsilon}{2},$$

and so $\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} \leq \varepsilon$. This proves the result. □

Lemma 1.2 (Lemma 1.7.3, [2], p. 21)**.** *If $u \in H^1(\mathbb{R}^N)$ is a radially symmetric function, then*

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N-1}{2}} |u(x)| \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{1.4}$$

If, in addition, $u(x)$ is a nonincreasing function of $|x|$, then

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N}{2}} |u(x)| \leq C\|u\|_{L^2}. \tag{1.5}$$

[1] Here we use

$$\begin{aligned}\|u_{n_k} - u\|_{L^p(\{|x|\geq R\})} &= \left(\int_{\{|x|\geq R\}} |u_{n_k} - u|^p\right)^{\frac{1}{p}} \leq \left(\|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{p-2} \int_{\{|x|\geq R\}} |u_{n_k} - u|^2\right)^{\frac{1}{p}} \\ &\leq \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\{|x|\geq R\})}^{\frac{2}{p}} \leq \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}}.\end{aligned}$$

Proof. Suppose first $u \in C_c^\infty(\mathbb{R}^N)$. Since u is radially symmetric, there exists a function $\widetilde{u}: \mathbb{R}^+ \to \mathbb{R}$ such that $u(x) = \widetilde{u}(|x|)$ for all $x \in \mathbb{R}^N$. Simple computation gives us $|\nabla u(x)| = |\widetilde{u}'(r)|$ where $r = |x|$. We have

$$
\begin{aligned}
\|u\|_{L^2(\mathbb{R}^N)}^2 &= \left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right) = \int_{\partial B_1(0)} \left(\int_0^\infty |u(ry)|^2 r^{N-1} dr\right) dS(y) \\
&= \int_{\partial B_1(0)} \left(\int_0^\infty \widetilde{u}(r)^2 r^{N-1} dr\right) dS(y) = N\alpha_N \int_0^\infty \widetilde{u}(r)^2 r^{N-1} dr, \\
\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 &= \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx\right) = \int_{\partial B_1(0)} \left(\int_0^\infty |\nabla u(ry)|^2 r^{N-1} dr\right) dS(y) \\
&= \int_{\partial B_1(0)} \left(\int_0^\infty \widetilde{u}'(r)^2 r^{N-1} dr\right) dS(y) = N\alpha_N \int_0^\infty \widetilde{u}'(r)^2 r^{N-1} dr,
\end{aligned}
$$

and

$$
\begin{aligned}
r^{N-1}\widetilde{u}(r)^2 &= -\int_r^\infty \frac{d}{ds}\left(s^{N-1}\widetilde{u}(s)^2\right) ds = -\int_r^\infty \left(\underbrace{(N-1)s^{N-2}\widetilde{u}(s)^2}_{\geq 0} + 2s^{N-1}\widetilde{u}(s)\widetilde{u}'(s)\right) ds \\
&\leq -2\int_r^\infty s^{N-1}\widetilde{u}(s)\widetilde{u}'(s)\, ds \leq 2\left(\int_r^\infty s^{N-1}\widetilde{u}(s)^2 ds\right)^{\frac{1}{2}}\left(\int_r^\infty s^{N-1}u'(s)^2 ds\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_0^\infty s^{N-1}\widetilde{u}(s)^2 ds\right)^{\frac{1}{2}}\left(\int_0^\infty s^{N-1}u'(s)^2 ds\right)^{\frac{1}{2}} \\
&\leq 2\frac{\|u\|_{L^2(\mathbb{R}^N)}}{\sqrt{N\alpha_N}}\frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sqrt{N\alpha_N}} = \frac{2}{N\alpha_N}\|u\|_{L^2(\mathbb{R}^N)}\|\nabla u\|_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

where and α_N is the volume of the unit ball in $\mathbb{R}^N$, which is given by $\alpha_N := \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$. That means (1.4) holds for all $u \in C_c^\infty(\mathbb{R}^N)$.

If $u(x)$ is a nonincreasing function of $|x|$, then for all $r \geq 0$,

$$
\|u\|_{L^2}^2 = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right) \geq \left(\int_{\{|x|\leq r\}} |u(x)|^2 dx\right) \geq |\{|x| \leq r\}|\,|\widetilde{u}(r)|^2 = \alpha_N \mathbb{R}^N |\widetilde{u}(r)|^2,
$$

i.e., (1.5) holds for all $u \in C_c^\infty(\mathbb{R}^N)$.

The general case then follows by a density argument. □

1.2 Proof of Theorem 1.1

Proof of Theorem 1.1. First note that if we set $u^{\lambda,\mu}(x) := \mu u(\lambda x)$, then $\nabla u^{\lambda,\mu}(x) = \mu\lambda\nabla u(\lambda x)$, and

$$
\left\|u^{\lambda,\mu}\right\|_2^2 = \int_{\mathbb{R}^N} \left|u^{\lambda,\mu}(x)\right|^2 dx = \int_{\mathbb{R}^N} |\mu u(\lambda x)|^2 dx = \frac{\mu^2}{\lambda^N}\int_{\mathbb{R}^N} |u(x)|^2 dx = \frac{\mu^2}{\lambda^N}\|u\|_2^2,
$$

$$\left\|u^{\lambda,\mu}\right\|_{2\sigma+2}^{2\sigma+2}=\int_{\mathbb{R}^N}\left|u^{\lambda,\mu}(x)\right|^{2\sigma+2}dx=\int_{\mathbb{R}^N}|\mu u(\lambda x)|^{2\sigma+2}dx=\frac{\mu^{2\sigma+2}}{\lambda^N}\|u\|_{2\sigma+2}^{2\sigma+2},$$
$$\left\|\nabla u^{\lambda,\mu}\right\|_2^2=\int_{\mathbb{R}^N}\left|\nabla u^{\lambda,\mu}(x)\right|^2dx=\int_{\mathbb{R}^N}|\mu\lambda\nabla u(\lambda x)|^2dx=\frac{\mu^2}{\lambda^{N-2}}\|\nabla u\|_2^2,$$
$$J^{\sigma,N}\left(u^{\lambda,\mu}\right)=\frac{\|\nabla u^{\lambda,\mu}\|_2^{\sigma N}\|u^{\lambda,\mu}\|_2^{2\sigma+2-\sigma N}}{\|u^{\lambda,\mu}\|_{2\sigma+2}^{2\sigma+2}}=\frac{\left(\frac{\mu^2}{\lambda^{N-2}}\right)^{\frac{\sigma N}{2}}\|\nabla u\|_2^{\sigma N}\left(\frac{\mu^2}{\lambda^N}\right)^{\frac{2\sigma+2-\sigma N}{2}}\|u\|_2^{2\sigma+2-\sigma N}}{\frac{\mu^{2\sigma+2}}{\lambda^N}\|u\|_{2\sigma+2}^{2\sigma+2}}$$
$$=\frac{\|\nabla u\|_2^{\sigma N}\|u\|_2^{2\sigma+2-\sigma N}}{\|u\|_{2\sigma+2}^{2\sigma+2}}=J^{\sigma,N}(u).$$

Since $J^{\sigma,N}(u)\geq 0$, there exists a minimizing sequence $u_v\in H^1\left(\mathbb{R}^N\right)\cap L^{2\sigma+2}\left(\mathbb{R}^N\right)$, i.e., $a:=\inf_{u\in H^1(\mathbb{R}^N)}J^{\sigma,N}(u)=\lim_{v\uparrow\infty}J^{\sigma,N}(u_v)<\infty$. We can assume $u_v>0$ (since $J^{\sigma,N}(u)=J^{\sigma,N}(-u)$), and by symmetrization we can take $u_v=u_v(|x|)$[2].

Choosing $\lambda_v=\frac{\|u_v\|_2}{\|\nabla u_v\|_2}$, $\mu_v=\frac{\|u_v\|_2^{\frac{N}{2}-1}}{\|\nabla u_v\|_2^{\frac{N}{2}}}$[3], we obtain a sequence $\psi_v(x):=u^{\lambda_v,\mu_v}(x)$ with the following properties:

(a) $\psi_v(x)\geq 0$, $\psi_v=\psi_v(|x|)$,

(b) $\psi_v\in H^1\left(\mathbb{R}^N\right)$,

(c) $\|\psi_v\|_2=1$, and $\|\nabla\psi_v\|_2=1$,

(d) $J^{\sigma,N}(\psi_v)\downarrow\alpha$ as $v\to\infty$.

Since the sequence ψ_v is bounded in $H^1\left(\mathbb{R}^N\right)$, some subsequence has a weak H^1 limit ψ^*. Since ψ_v are radial and uniformly bounded in $H^1\left(\mathbb{R}^N\right)$, it follows from the compactness lemma that we can take ψ_v strongly convergent to ψ^* in $L^{2\sigma+2}\left(\mathbb{R}^N\right)$ for $0<\sigma<\frac{2}{N-2}$. By weak convergence, $\|\psi^*\|_2\leq 1$ and $\|\nabla\psi^*\|_2\leq 1$. Hence,

$$\alpha\leq J^{\sigma,N}(\psi^*)\leq\frac{1}{\int|\psi^*|^{2\sigma+2}dx}=\lim_{v\uparrow\infty}J(\psi_v)=\alpha.$$

It follows that $\|\nabla\psi^*\|_2^{\sigma N}\|\psi^*\|_2^{2+\sigma(2-N)}=1$ and therefore $\|\psi^*\|_2=\|\nabla\psi^*\|_2=1$, so $\psi_v\to\psi^*$ strongly in H^1 [4]. This proves part (1) and (2) of Theorem (1.1).

Part (3) follows from the fact that ψ^*, the minimizing function, is in $H^1\left(\mathbb{R}^N\right)$ and satisfies the Euler-Lagrange equation:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}J^{\sigma,N}(\psi^*+\varepsilon\eta)=0,\ \ \forall\eta\in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.6}$$

[2]Indeed, since for any $u\in H^1\left(\mathbb{R}^N\right)\cap L^{2\sigma+2}\left(\mathbb{R}^N\right)$, its symmetric-decreasing rearrangement u^* satisfies $\|u^*\|_{2\sigma+2}=\|u\|_{2\sigma+2}$, $\|u^*\|_2=\|u\|_2$, $\|\nabla u^*\|_2\leq\|\nabla u\|_2$, and thus $J^{\sigma,N}(u^*)\leq J^{\sigma,N}(u)$. Hence, it suffices to consider only radially symmetric functions to minimize $J^{\sigma,N}$.

[3]Solve $\left\|u_v^{\lambda,\mu}\right\|_2=\left\|\nabla_x u_v^{\lambda,\mu}\right\|_2=1$ to obtain λ_v and μ_v.

[4]If $x_n\rightharpoonup x$ in a Hilbert space H, and $\|x_n\|_H\to\|x\|_H$, then x_n converges to x strongly.

Taking into account that $\|\psi^*\|_2 = 1$ and $\|\nabla\psi^*\|_2 = 1$, we have

$$\frac{\sigma N}{2}\Delta\psi^* - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi^* + \alpha(\sigma+1)(\psi^*)^{2\sigma+1} = 0 \text{ in } \mathcal{D}'. \tag{1.7}$$

Let $\psi = [\alpha(\sigma+1)]^{\frac{1}{2\sigma}}\psi^*$, then

i) ψ is positive and radially symmetric.

ii) $\psi \in H^1(\mathbb{R}^N)$.

iii) ψ satisfies

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0 \text{ in } \mathcal{D}'. \tag{1.8}$$

Now we regularize ψ by a bootstrap argument:

$\star$ *Step 1*: Since $\psi \in H^1_{\text{radial}}(\mathbb{R}^N)$, the Compactness Lemma implies that $\psi \in L^{2\sigma+2}(\mathbb{R}^N)$, and thus $\psi^{2\sigma+1} \in L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^N)$. Since $1 < \frac{2\sigma+2}{2\sigma+1} < 2$, we have implies that $L^2(\mathbb{R}^N) \hookrightarrow L^{\frac{2\sigma+1}{2\sigma+2}}_{\text{loc}}(\mathbb{R}^N)$, and consequently $\psi \in L^{\frac{2\sigma+1}{2\sigma+2}}_{\text{loc}}(\mathbb{R}^N)$. Then (1.8) implies that $\Delta\psi \in L^{\frac{2\sigma+2}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Using elliptic regularity, it follows that $\psi \in W^{2,\frac{2\sigma+2}{2\sigma+1}}_{loc}(\mathbb{R}^N)$.

Similarly, we can prove that[5]

Statement 1: *If* $\psi \in L^q_{\text{loc}}(\mathbb{R}^N)$, *then* $\psi \in W^{2,\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

Put $q_0 := 2\sigma + 2$, we currently have $\psi \in W^{2,\frac{q_0}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. We consider the following cases depending on σ and N:

- *Case* $\frac{2\sigma+1}{q_0} < \frac{2}{N}$: Applying the general Sobolev embedding theorem to $(k, N, p) = \left(2, N, \frac{q_0}{2\sigma+1}\right)$ implies $\psi \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

- *Case* $\frac{2\sigma+1}{q_0} = \frac{2}{N}$: Applying the general Sobolev embedding theorem to $(k, N, p) = \left(2, N, \frac{q_0}{2\sigma+1}\right)$ implies $\psi \in L^r_{\text{loc}}(\mathbb{R}^N)$ for all $r \in \left[\frac{N}{2}, +\infty\right)$. In particular, choosing $r = (\sigma+1)N > \frac{N}{2}$, we have $\psi \in L^{(\sigma+1)N}_{\text{loc}}(\mathbb{R}^N)$. Statement 1 then implies $\psi \in W^{2,\frac{(\sigma+1)N}{2\sigma+1}}_{loc}(\mathbb{R}^N)$. Since $\frac{2\sigma+1}{(\sigma+1)N} < \frac{2}{N}$, applying the general Sobolev embedding theorem ii) yields $\psi \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

- *Case* $\frac{2\sigma+1}{q_0} > \frac{2}{N}$: We define $q_1 > 0$ by

$$\frac{1}{q_1} = \frac{2\sigma+1}{q_0} - \frac{2}{N},$$

  and then applying the general Sobolev embedding theorem yields $\psi \in L^{q_1}_{\text{loc}}(\mathbb{R}^N)$. Statement 1 then implies $\psi \in W^{2,\frac{q_1}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

[5] Since $\psi \in L^q_{\text{loc}}(\mathbb{R}^N)$, $\psi^{2\sigma+1} \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. We have $L^q_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$, and thus $\psi \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Then (1.8) implies that $\Delta\psi \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Using elliptic regularity, it follows that $\psi \in W^{2,\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

We continue this treatment for q_1. There are two possibilities: either $\psi \in C^{0,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$ for some $\alpha \in (0,1)$ or $\psi \in W^{2,\frac{q_2}{2\sigma+1}}_{\text{loc}}\left(\mathbb{R}^N\right)$ with $\frac{1}{q_2} = \frac{2\sigma+1}{q_1} - \frac{2}{N}$.

We claim that there exists $n^* \in \mathbb{N}$ such that $\frac{2\sigma+1}{q_{n^*}} \leq \frac{2}{N}$. Indeed, suppose for the contrary that the sequence $(q_n)_n$ defined by

$$\begin{cases} q_0 = 2\sigma + 2, \\ \dfrac{1}{q_n} = \dfrac{2\sigma+1}{q_{n-1}} - \dfrac{2}{N}, \quad \forall n \in \mathbb{N}, \end{cases}$$

consists of all positive real terms.

It is deduced from the recursion that

$$\frac{1}{q_n} - \frac{1}{\sigma N} = (2\sigma+1)\left(\frac{1}{q_{n-1}} - \frac{1}{\sigma N}\right), \quad \forall n \in \mathbb{N}.$$

Thus,

$$\frac{1}{q_n} - \frac{1}{\sigma N} = (2\sigma+1)^n\left(\frac{1}{q_0} - \frac{1}{\sigma N}\right),$$

or equivalently,

$$\frac{1}{q_n} = \frac{1}{\sigma N} + (2\sigma+1)^n\left(\frac{1}{q_0} - \frac{1}{\sigma N}\right).$$

Since $\frac{1}{q_0} - \frac{1}{\sigma N} = \frac{1}{2\sigma+2} - \frac{1}{\sigma N} = \frac{\sigma(N-2)-2}{\sigma N(2\sigma+2)} < 0$, the RHS of the last equality tends to $-\infty$ as $n \to +\infty$, which contradicts to the assumption $q_n > 0$ for all $n \in \mathbb{N}$.

Therefore, we must have $\psi \in C^{0,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$ for some $\alpha \in (0,1)$.

Step 2: We can prove that $\psi^{2\sigma+1} \in C^{0,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$. Then Schauder theorem implies $\psi \in C^{2,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$. Using bootstrap argument, we can prove that $\psi \in C^{4,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$, etc. So $\psi \in C^{2m,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$ for all $m \in \mathbb{N}$, and thus $\psi \in C^\infty\left(\mathbb{R}^N\right)$. $\square$

Corollary 1.1. *The best (smallest) constant for which the interpolation estimate (1.2) holds is given by the expression*

$$C_{\sigma,N} := \left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}},$$

where ψ is the ground state of equation (1.3).

Proof. The best constant $C_{\sigma,N}$ is given by

$$C_{\sigma,N} = \left(\inf_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)\right)^{-\frac{1}{2\sigma+2}} = \alpha^{-\frac{1}{2\sigma+2}} = \left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}. \tag{1.9}$$

$\square$

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