# Finite Volume Method
# Practical Assignment

Doan Tran Nguyen Tung
Student ID: 1411352

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## Contents

# Practical Assignment

Given a 1D Poisson problem on $\Omega = (0, 1)$

$$-u''(x) = f(x), x \in \Omega \tag{1}$$

1. Dirichlet boundary condition

   a) Solve the equation (1) subjected to homogeneous Dirichlet boundary condition

   $$u(0) = a, u(1) = b \tag{2}$$

   by finite volume method on a regular grid and the control points are the midpoints of the control volumes $\left(x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}\right)$.

   b) Solve the equation (1) with regular grid and each control point is $\dfrac{1}{3}$ from the left of each control volume $\left(x_i = \frac{2}{3}x_{i-\frac{1}{2}} + \frac{1}{3}x_{i+\frac{1}{2}}\right)$

   c) How to approximate the mean value of $f$ over the control volumes $T_i$ and compare some ways of approximation.

   d) Solve the equation (1) with a singular grid.

2. Neumann boundary condition
   Solve the equation (1) subjected to Neumann boundary condition

   $$u'(0) = 0, u'(1) = 0 \text{ with } \int_0^1 f(x)dx = 0 \text{ and } \int_0^1 u(x)dx = 0 \tag{3}$$

   by finite volume method on a regular grid and a singular grid and the control points are the midpoints of the control volumes $\left(x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}\right)$.

# 1 Dirichlet boundary condition

In this section, we are going to test the Finite Volume Method for the 1-dimensional Poisson equation subjected to Dirichlet boundary condition with several test cases using uniform grid and singular grid, varying positions of control points, ways of integration approximation.

## 1.1 Test cases

- Case 1:

$$f(x) = \frac{1}{2} - x$$
$$u(0) = \frac{1}{24}$$
$$u(1) = -\frac{1}{24}$$
$$u(x) = \frac{x^2\ (2\,x-3)}{12} + \frac{1}{24}$$

- Case 2:

$$f(x) = -20000\,x^3 + \frac{957200\,x^2}{33} - \frac{1240655\,x}{99} + \frac{916835}{594}$$
$$u(0) = -\frac{175}{44}$$
$$u(1) = \frac{2660}{297}$$
$$u(x) = (1000\,x-50)\left(x-\frac{3}{11}\right)\left(x-\frac{7}{9}\right)\left(x-\frac{5}{12}\right)\left(x-\frac{9}{10}\right)$$

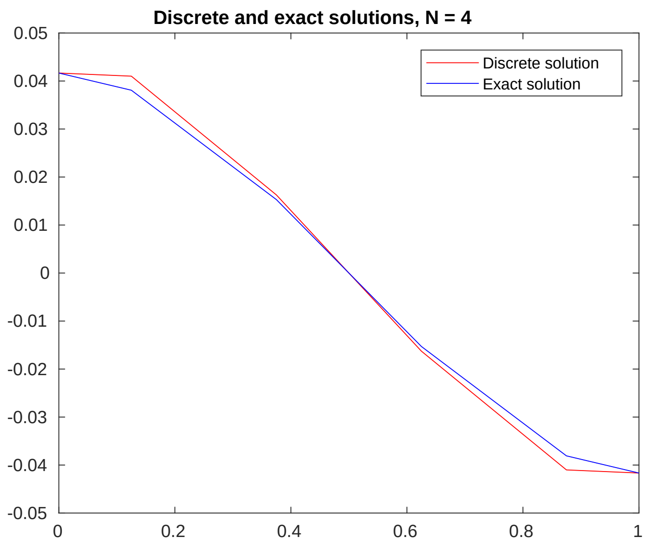
- Case 3:

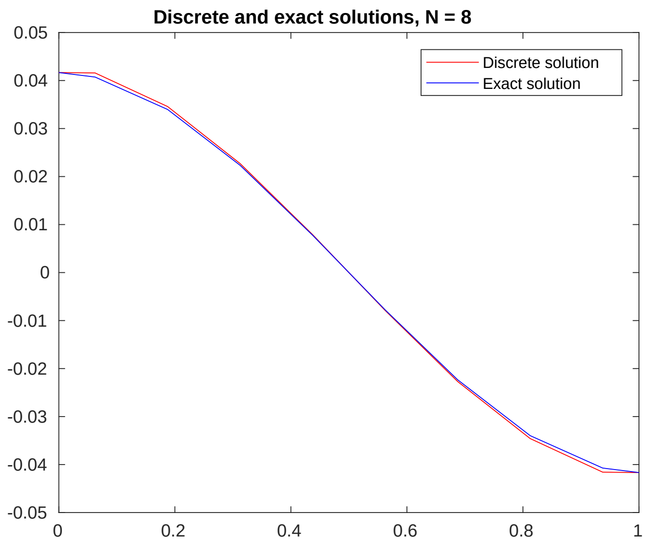
$$\begin{aligned} f(x) =& 3\cos\left(x+\frac{1}{2}\right)^3 \sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^5 + 12500\pi^2\cos\left(x+\frac{1}{2}\right)^3 \sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^5\left(x+\frac{1}{2}\right)^8 \\ & -6\,\cos\left(x+\frac{1}{2}\right)\sin\left(x+\frac{1}{2}\right)^2\sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^5 \\ & -50000\pi^2, \cos\left(x+\frac{1}{2}\right)^3\cos\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^2\sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^3\left(x+\frac{1}{2}\right)^8 \\ & -1000\,\pi\cos\left(x+\frac{1}{2}\right)^3\cos\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)\sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^4\left(x+\frac{1}{2}\right)^3 \\ & +1500\,\pi\cos\left(x+\frac{1}{2}\right)^2\cos\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)\sin\left(x+\frac{1}{2}\right)\sin\left(10\,\pi\left(x+\frac{1}{2}\right)^5\right)^4\left(x+\frac{1}{2}\right)^4 \\ u(0) =& \sin\left(\frac{5\pi}{16}\right)^5\cos\left(\frac{1}{2}\right)^3 \\ u(1) =& \sin\left(\frac{\pi}{16}\right)^5\cos\left(\frac{3}{2}\right)^3 \\ u(x) =& \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^5\cos\left(x+\frac{1}{2}\right)^3 \end{aligned}$$

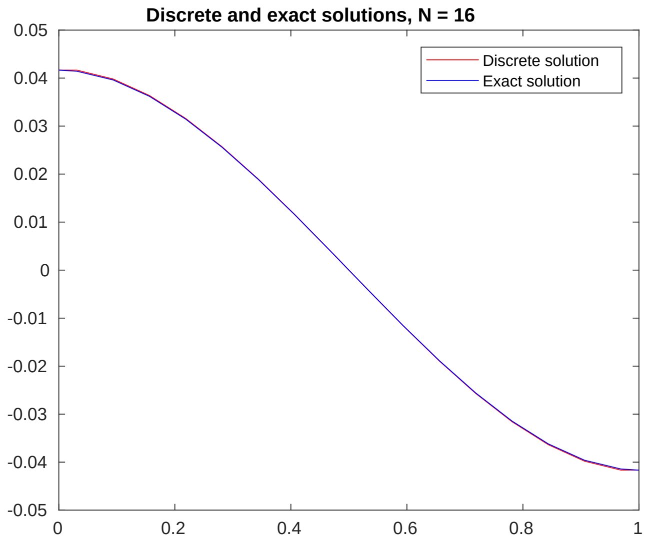
## 1.2 Homogeneous Dirichlet boundary condition, regular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

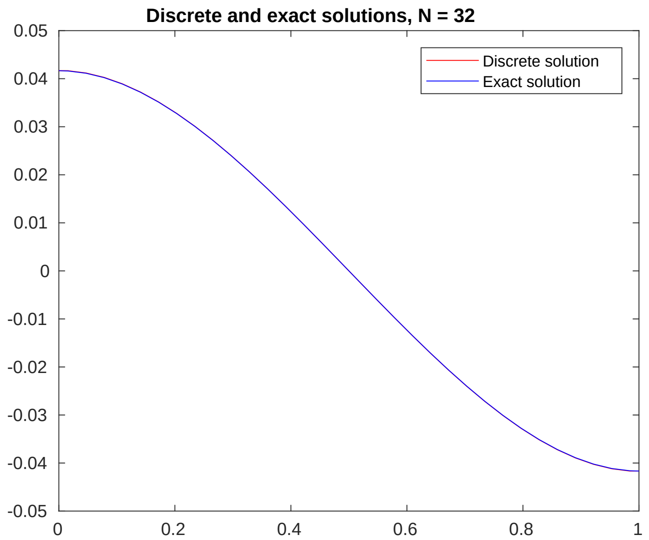
### 1.2.1 Figures of results

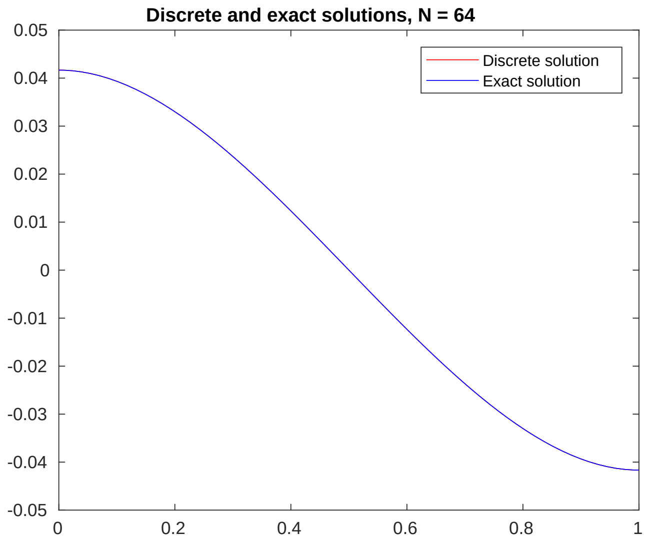
- Case 1:

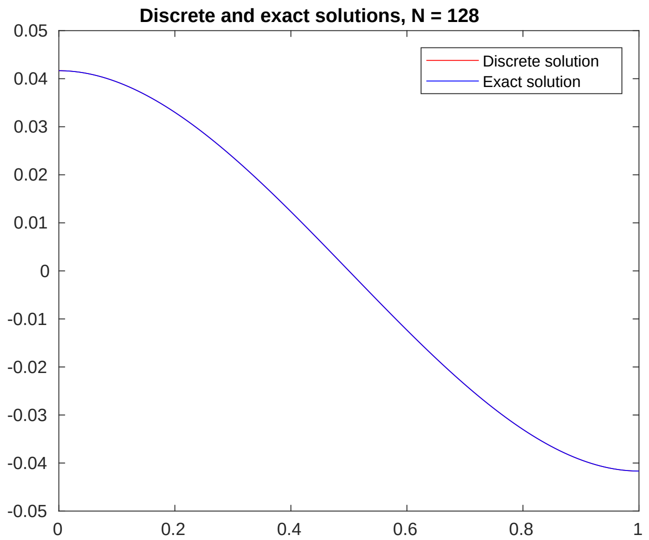


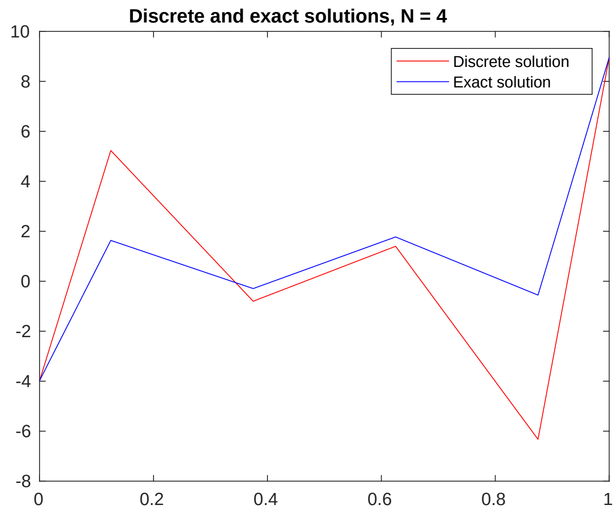

Discrete and exact solutions, N = 16
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

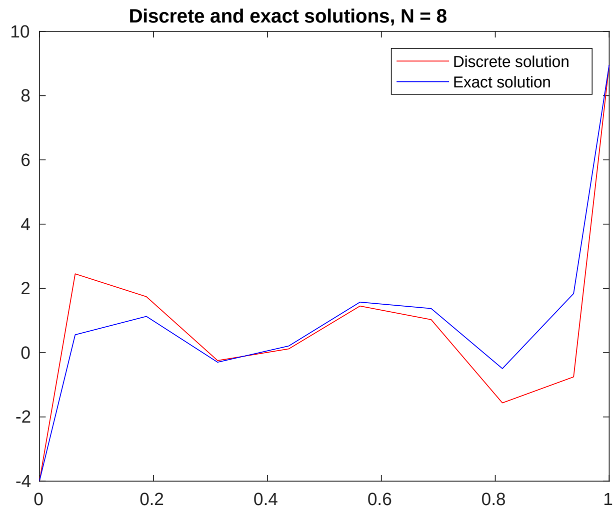

Discrete and exact solutions, N = 32
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

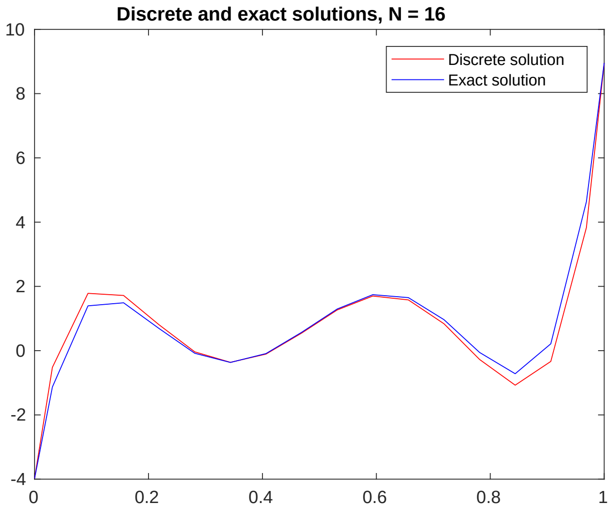

Discrete and exact solutions, N = 64
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

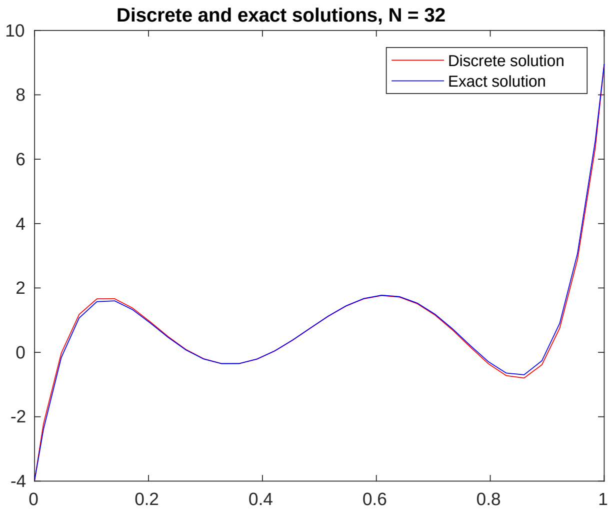

Discrete and exact solutions, N = 128
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

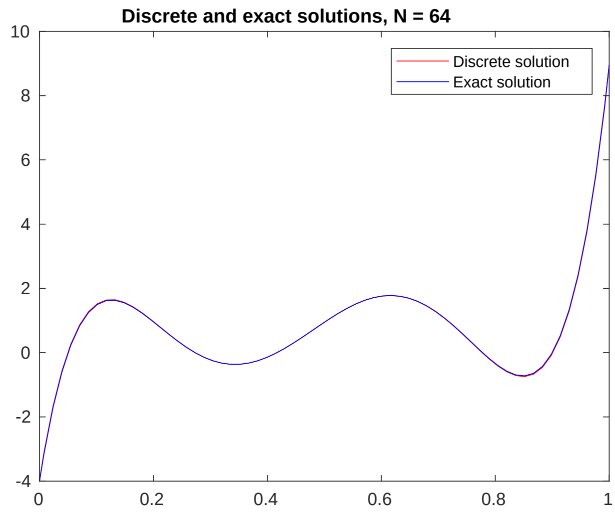
- <u>Case 2:</u>

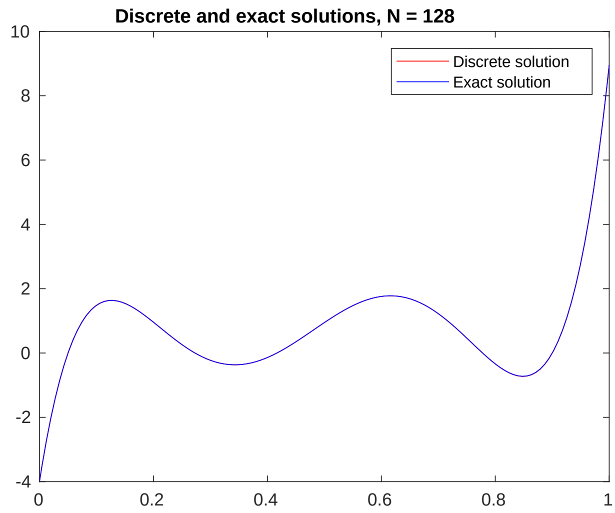

Discrete and exact solutions, N = 4
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
-6
-8
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 8
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1

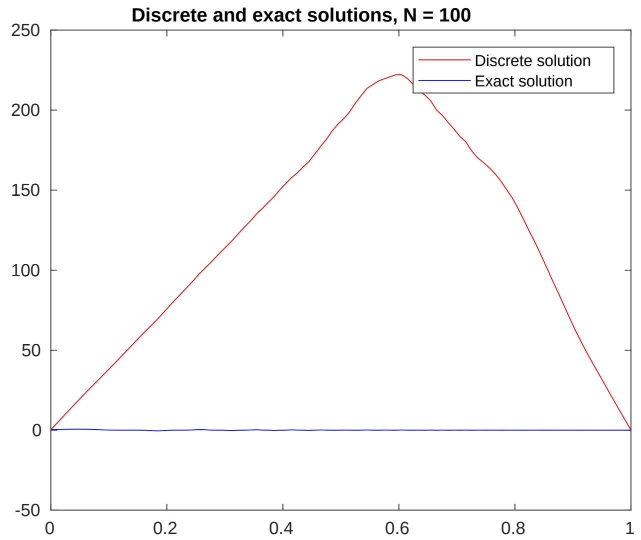

Discrete and exact solutions, N = 16
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1

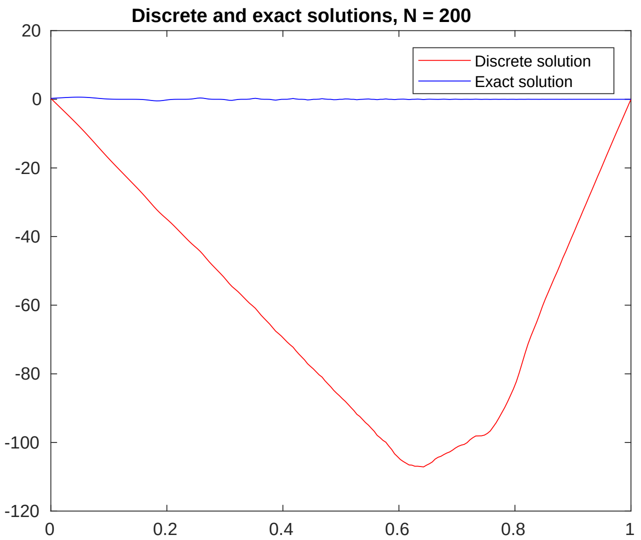

Discrete and exact solutions, N = 32
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1

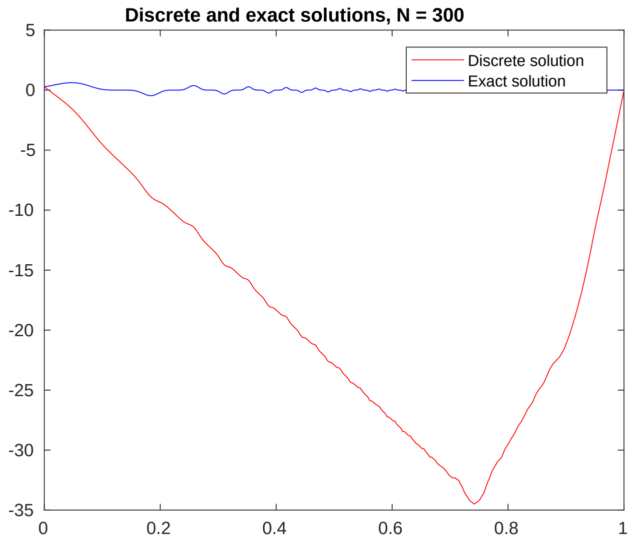

Discrete and exact solutions, N = 64
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 128
Discrete solution
Exact solution
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1

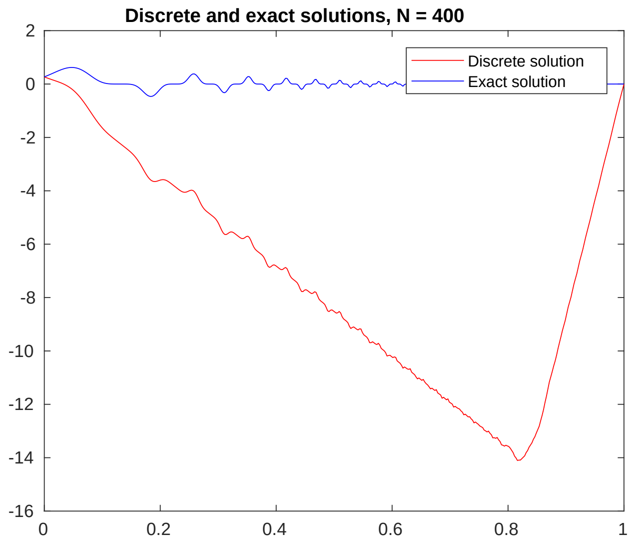
- Case 3:


Discrete and exact solutions, N = 100
250
200
150
100
50
0
-50
0
0.2
0.4
0.6
0.8
1
Discrete solution
Exact solution

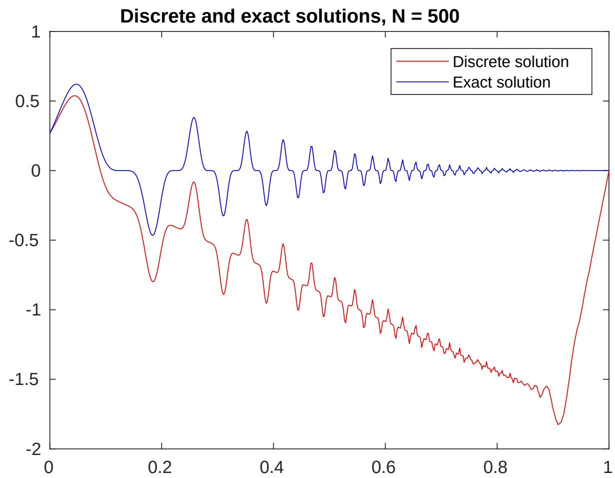

Discrete and exact solutions, N = 200
20
0
-20
-40
-60
-80
-100
-120
0
0.2
0.4
0.6
0.8
1
Discrete solution
Exact solution

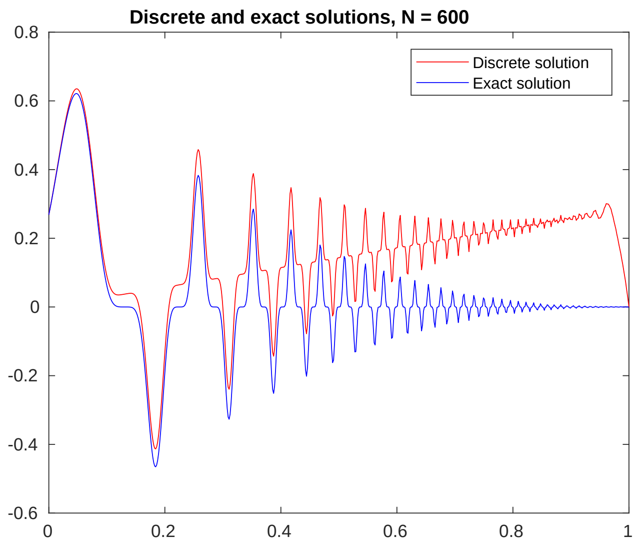


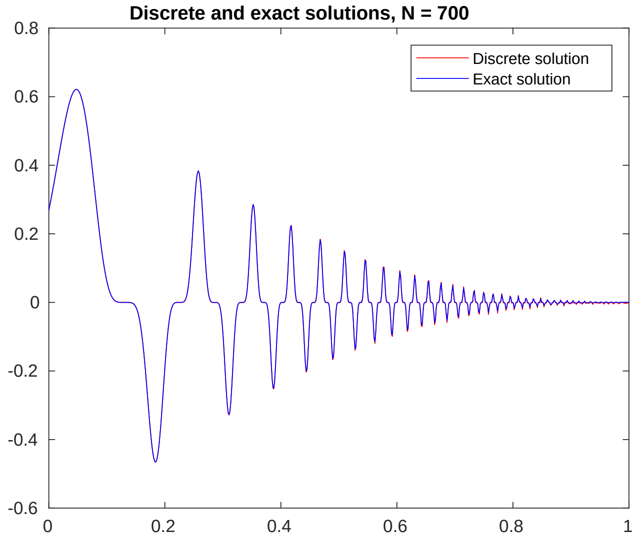
Discrete and exact solutions, N = 300
5
0
-5
-10
-15
-20
-25
-30
-35
0
0.2
0.4
0.6
0.8
1
Discrete solution
Exact solution

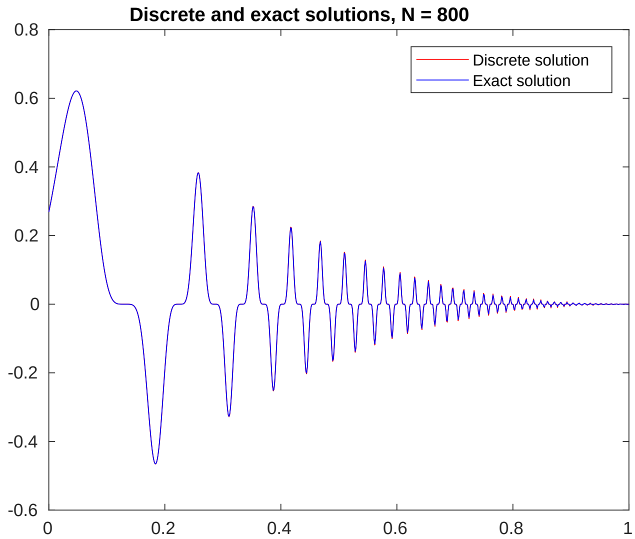
Discrete and exact solutions, N = 400
2
0
-2
-4
-6
-8
-10
-12
-14
-16
0
0.2
0.4
0.6
0.8
1
Discrete solution
Exact solution


Discrete and exact solutions, N = 500
Discrete solution
Exact solution
1
0.5
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 600
Discrete solution
Exact solution
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.2
0.4
0.6
0.8
1



Discrete and exact solutions, N = 700
Discrete solution
Exact solution
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.2
0.4
0.6
0.8
1

Discrete and exact solutions, N = 800
Discrete solution
Exact solution
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.2
0.4
0.6
0.8
1

### 1.2.2 Errors

- Case 1:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 2.183660e-03 | 1.353165e-02 |
| 8 | 5.593964e-04 | 5.167483e-03 |
| 16 | 1.406791e-04 | 1.891105e-03 |
| 32 | 3.522145e-05 | 6.796587e-04 |
| 64 | 8.808591e-06 | 2.422258e-04 |
| 128 | 2.202349e-06 | 8.597891e-05 |

- Case 2:

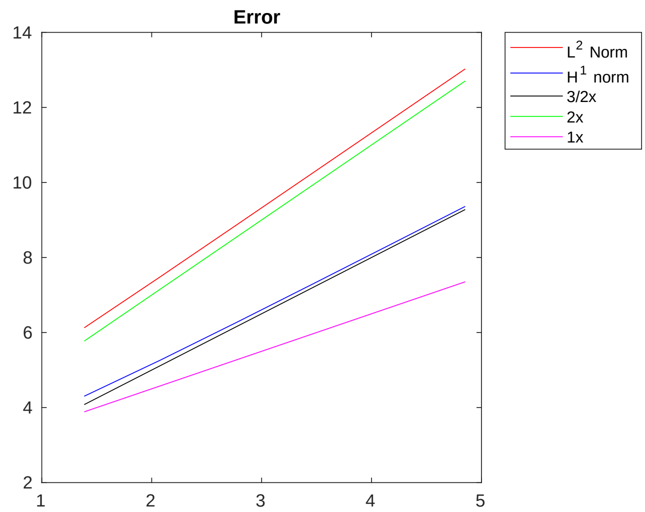
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 3.414525e+00 | 1.723527e-02 |
| 8 | 1.224066e+00 | 1.429127e+01 |
| 16 | 3.297560e-01 | 6.028506e+00 |
| 32 | 8.393195e-02 | 2.295482e+00 |
| 64 | 2.107644e-02 | 8.397813e-01 |
| 128 | 5.274953e-03 | 3.018207e-01 |

- Case 3:

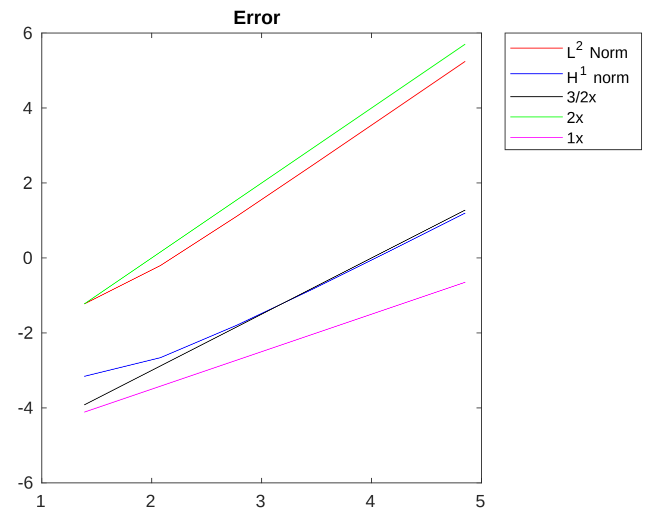
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 1.372029e+02 | 4.718806e+02 |
| 200 | 6.795356e+01 | 2.432018e+02 |
| 300 | 2.068556e+01 | 8.656798e+01 |
| 400 | 8.220398e+00 | 3.814539e+01 |
| 500 | 9.637916e-01 | 7.476626e+00 |
| 600 | 1.613099e-01 | 2.708167e+00 |
| 700 | 1.691419e-03 | 1.378369e+00 |
| 800 | 1.035571e-03 | 1.034955e+00 |

### 1.2.3 Convergence rate

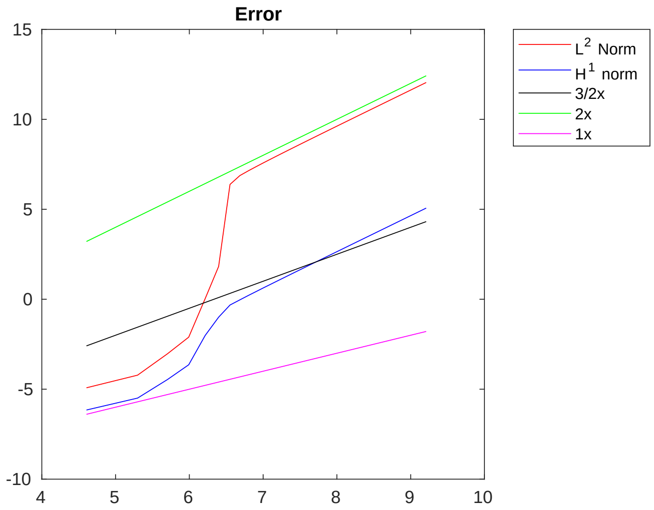
- Case 1:




- Case 2:




- Case 3:




**Remark:** We can see that in case 1 and 2, the finite volume method has convergence rate of order 2 in $L^2$ norm and order $\frac{3}{2}$ in energy norm. But in case 3, finite volume method has convergence rate of order 2 in both $L^2$ and energy norm. More ever, the convergence rate in case 3 only become consistent after the grid refinement reached a certain degree.

### 1.3 Homogeneous Dirichlet boundary condition, regular grid, each control point is 1/3 from the left of corresponding control volume, integration using midpoint rule

#### 1.3.1 Errors

- Case 1:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 4.337732e-03 | 1.723527e-02 |
| 8 | 1.976011e-03 | 7.696433e-03 |
| 16 | 9.604458e-04 | 3.489791e-03 |
| 32 | 4.766545e-04 | 1.633552e-03 |
| 64 | 2.378772e-04 | 7.855739e-04 |
| 128 | 1.188822e-04 | 3.845069e-04 |

- Case 2:

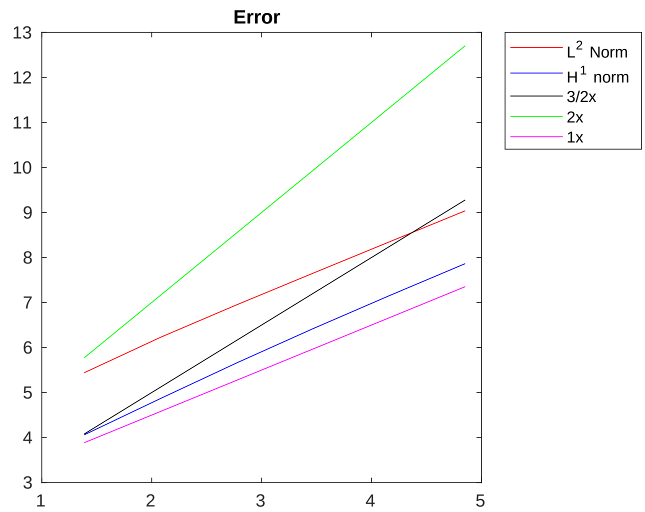
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 6.063755e+00 | 2.989757e+01 |
| 8 | 3.072003e+00 | 1.824323e+01 |
| 16 | 1.453481e+00 | 8.493182e+00 |
| 32 | 7.060401e-01 | 3.804418e+00 |
| 64 | 3.484227e-01 | 1.738566e+00 |
| 128 | 1.731597e-01 | 8.193746e-01 |

- Case 3:

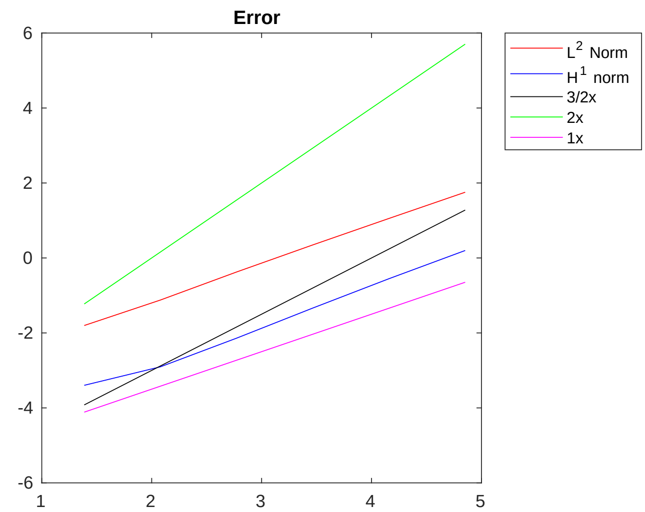
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 1.374387e+02 | 4.724692e+02 |
| 200 | 6.805998e+01 | 2.434741e+02 |
| 300 | 2.074894e+01 | 8.685629e+01 |
| 400 | 8.237658e+00 | 3.826618e+01 |
| 500 | 9.680948e-01 | 7.883261e+00 |
| 600 | 1.628207e-01 | 3.371928e+00 |
| 700 | 4.196379e-03 | 2.193966e+00 |
| 800 | 3.381889e-03 | 1.786729e+00 |

### 1.3.2 Convergence rate

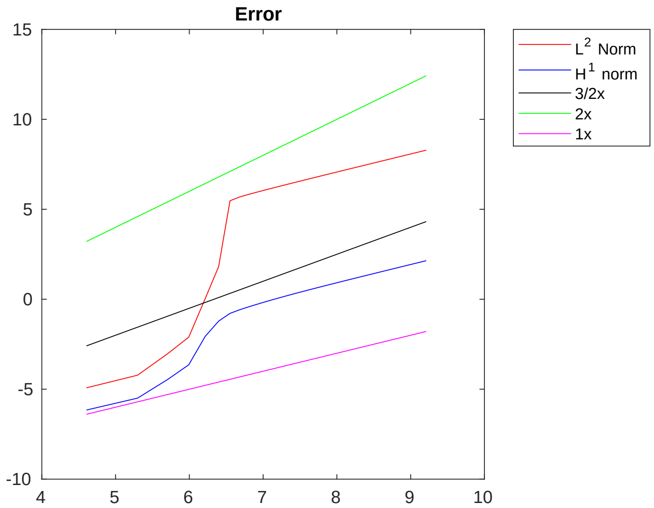
Case 1:




- Case 2:




- Case 3:




**Remark:** Unlike the results in previous section, in all 3 cases, the finite volume method has convergence rate of order 1 in both $L^p$ and energy norm. However, in case 3, similar to before, the convergence rate in case 3 only become consistent after the grid refinement reached a certain degree.

## 1.4 Ways to approximate the mean value of the function f over control volumes

### 1.4.1 Somes way to approximate integral

There are many ways to approximate integrals, both with equally spaced integration points and unequally spaced integration points. However, we are going to test only a few way of approximation with equally spaced integration points.
We need to approximate $\frac{1}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)$, with $T_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ being the control volume on which we need to approximate the mean value of $f$. For the sake of convenience, let $a = x_{i-\frac{1}{2}}$ and $b = x_{i+\frac{1}{2}}$.
**Midpoint rule**

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x) &= \frac{1}{b-a}(b-a)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\frac{(b-a)^3}{24}f^{(2)}(\xi) \\
&= f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f^{(2)}(\xi)
\end{aligned}$$

For some $\xi$ in $[a,b]$.
**Trapezoidal rule**

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x) &= \frac{1}{b-a}\frac{b-a}{2}(f(a)+f(b)) - \frac{1}{b-a}\frac{1}{12}(b-a)^3 f^{(2)}(\xi) \\
&= \frac{f(a)+f(b)}{2} - \frac{1}{12}(b-a)^2 f^{(2)}(\xi)
\end{aligned}$$

For some $\xi$ in $[a,b]$.
**Simpson's rule**
Let $h = \frac{b-a}{2}$, $t_1 = a$, $t_2 = a+h = \frac{b-a}{2}$, $t_3 = a+2h = b$.

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x) &= \frac{1}{2h}\frac{1}{3}\left(f(t_1)+4f(t_2)+f(t_3)\right) + \frac{1}{2h}\frac{1}{90}h^5 f^{(4)}(\xi) \\
&= \frac{1}{6h}\left(f(t_1)+4f(t_2)+f(t_3)\right) + \frac{1}{180}h^4 f^{(4)}(\xi)
\end{aligned}$$

For some $\xi$ in $[a,b]$.
**Boole's rule**
Let $h = \frac{b-a}{4}$, $t_1 = a$, $t_2 = a+h$, $t_3 = a+2h$, $t_4 = a+3h$, $t_5 = a+4h = b$.

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x) &= \frac{1}{4h}\frac{2}{45}h\left(7f(t_1)+32f(t_2)+12f(t_3)+32f(t_4)+7f(t_5)\right) - \frac{8}{945}h^7 f^{(6)}(\xi) \\
&= \frac{1}{90}h\left(7f(t_1)+32f(t_2)+12f(t_3)+32f(t_4)+7f(t_5)\right) - \frac{8}{945}h^7 f^{(6)}(\xi)
\end{aligned}$$

For some $\xi$ in $[a,b]$.

### 1.4.2 Effects in the Finite Volume Method

In this section, we will test case 3 with various methods of integration. **Errors**
• Case 3:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 1.372029e+02 | 4.718806e+02 |
| 200 | 6.795356e+01 | 2.432018e+02 |
| 300 | 2.068556e+01 | 8.656798e+01 |
| 400 | 8.220398e+00 | 3.814539e+01 |
| 500 | 9.637916e-01 | 7.476626e+00 |
| 600 | 1.613099e-01 | 2.708167e+00 |
| 700 | 1.691419e-03 | 1.378369e+00 |
| 800 | 1.035571e-03 | 1.034955e+00 |

Error table - Midpoint rule

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 3.773644e+01 | 2.412211e+02 |
| 200 | 8.391778e+01 | 3.030967e+02 |
| 300 | 2.038450e+01 | 8.486880e+01 |
| 400 | 8.220498e+00 | 3.799774e+01 |
| 500 | 9.637879e-01 | 7.212178e+00 |
| 600 | 1.612986e-01 | 2.251598e+00 |
| 700 | 1.234248e-03 | 8.617033e-01 |
| 800 | 5.799255e-04 | 6.097104e-01 |

Error table - Trepozoidal rule

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 1.014161e+02 | 3.530643e+02 |
| 200 | 1.742818e+01 | 6.491145e+01 |
| 300 | 6.995754e+00 | 2.947754e+01 |
| 400 | 2.740099e+00 | 1.278054e+01 |
| 500 | 3.212656e-01 | 2.611736e+00 |
| 600 | 5.377526e-02 | 1.087361e+00 |
| 700 | 7.420314e-04 | 6.371791e-01 |
| 800 | 4.998059e-04 | 4.889503e-01 |

Error table - Simpson's rule

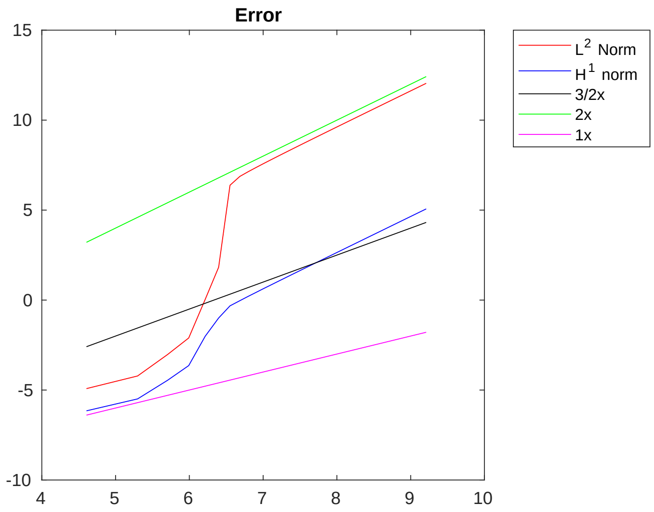
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 2.543855e+01 | 9.312152e+01 |
| 200 | 1.936562e+00 | 1.267931e+01 |
| 300 | 5.423098e-01 | 3.614971e+00 |
| 400 | 1.670127e-01 | 1.894852e+00 |
| 500 | 3.154194e-02 | 1.063986e+00 |
| 600 | 1.971232e-03 | 8.160713e-01 |
| 700 | 6.508166e-04 | 5.954282e-01 |
| 800 | 4.865046e-04 | 4.672134e-01 |

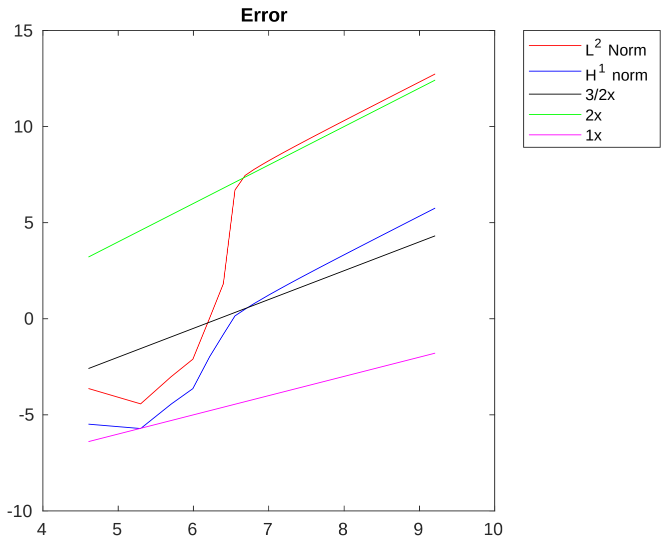
Error table - Boole's rule

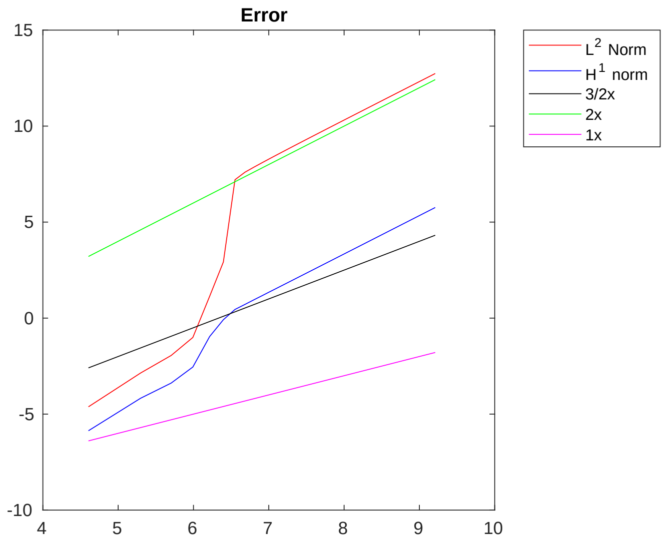
**Remark:** We can see that there are some variation in the accuracy when using different integration methods.

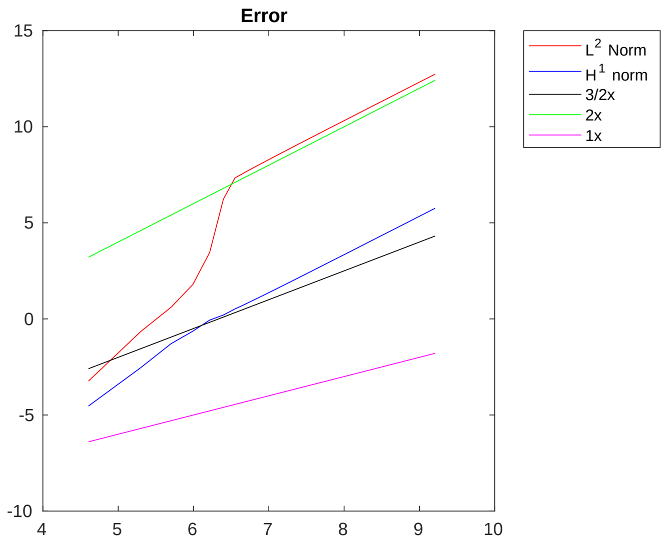
**Convergence rate**

• <u>Case 3:</u> $M = 100$




Error
L2 Norm
H1 norm
3/2x
2x
1x
15
10
5
0
-5
-10
4
5
6
7
8
9
10


Error
L2 Norm
H1 norm
3/2x
2x
1x
15
10
5
0
-5
-10
4
5
6
7
8
9
10

### 1.5 Homogeneous Dirichlet boundary condition, singular grid, each control point is the midpoint of corresponding control volume

In this section, we will consider the grid $x_i = 1 - \cos\left(\frac{\pi i}{2(N+1)}\right)$

#### 1.5.1 Errors

- Case 1:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 2.572594e-03 | 1.408303e-02 |
| 8 | 8.312800e-04 | 6.052092e-03 |
| 16 | 2.366590e-04 | 2.399498e-03 |
| 32 | 6.309364e-05 | 9.021918e-04 |
| 64 | 1.628265e-05 | 3.291959e-04 |
| 128 | 4.135350e-06 | 1.182551e-04 |

- Case 2:

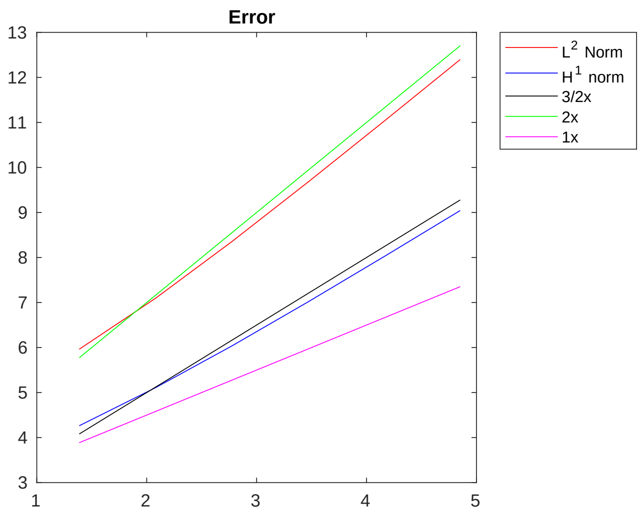
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 5.147535e+00 | 2.212485e+01 |
| 8 | 2.426569e+00 | 1.848306e+01 |
| 16 | 7.617805e-01 | 9.789540e+00 |
| 32 | 2.084740e-01 | 4.495505e+00 |
| 64 | 5.417277e-02 | 2.011451e+00 |
| 128 | 1.378284e-02 | 9.183423e-01 |

- Case 3:

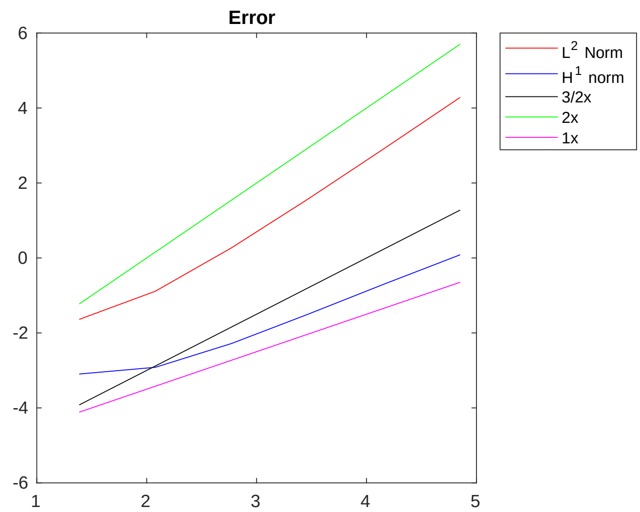
| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 100 | 2.382533e+02 | 7.816445e+02 |
| 200 | 1.021832e+02 | 3.524253e+02 |
| 300 | 6.461806e+01 | 2.324792e+02 |
| 400 | 3.011621e+01 | 1.225957e+02 |
| 500 | 2.319667e+01 | 9.540720e+01 |
| 600 | 1.878345e+00 | 1.109190e+01 |
| 700 | 2.656605e+00 | 1.539149e+01 |
| 800 | 2.128635e+00 | 1.401007e+01 |
| 900 | 2.328867e-01 | 3.282076e+00 |
| 1000 | 1.798795e-02 | 1.705853e+00 |
| 1100 | 1.554054e-03 | 1.344722e+00 |
| 1200 | 1.011087e-03 | 1.052726e+00 |

### 1.5.2 Convergence rate

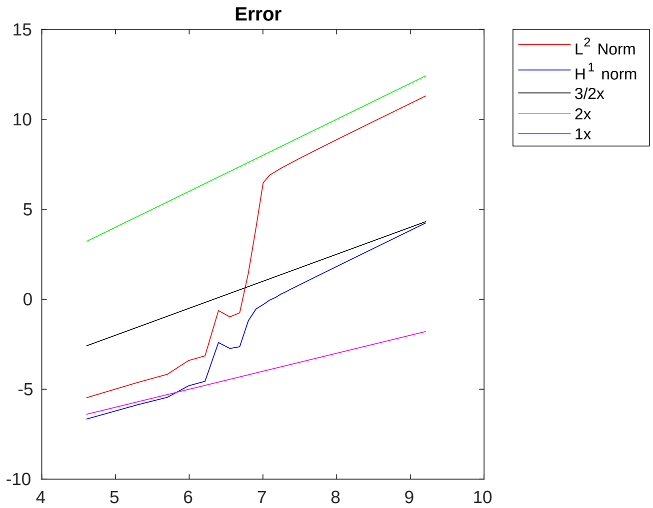
- Case 1:




- Case 2:




- Case 3:




**Remark:**
In case 1, the method appears to have convergence rate of order 2 in $L^2$ norm and order $\frac{3}{2}$ in energy norm.
In case 2, the method appears to have convergence rate of order 2 in $L^2$ norm and order 1 in energy norm.
In case 3, the method appears to have convergence rate of order 2 in both $L^2$ energy norm and there are some "irregularities" in the convergence rate before the grid refinement reach a certain degree.

# 2 Neumann boundary condition

In this section, we are going to test the Finite Volume Method for the 1-dimensional Poisson equation subjected to Neumann boundary condition with several test cases using uniform grid and singular grid.

## 2.1 Main idea

We now review the problem we need to solve

$$\begin{aligned} -u''(x) &= f(x), x \in \Omega && (4)\\ u'(0) &= 0 && (5)\\ u'(1) &= 0 && (6)\\ \int_0^1 f(x)dx &= 0 && (7)\\ \int_0^1 u(x)dx &= 0 && (8) \end{aligned}$$

We note that if we omit the condition (8), this problem will have infinitely many solutions of the form $u_1(x) + c$, with $u_1(0)$ being a solution of the problem.
Let $[U_0, U_1, U_2, \ldots, U_N, U_{N+1}]^\intercal$ be the discrete solution of the problem.
When using derivative approximation with first order convergence rate, the boundary condition can be "discretized" as $U_0 = U_1$ and $U_N = U_{N+1}$. Then, we need to use the linear system to solve for $U = [U_1, U_2, \ldots, U_N]^\intercal$
We also note that the discretized linear system has 1 independent variable. Therefore, we can choose to solve for one solution $\bar{U}$ such that $U_K = 0$ for some $K$ and then use the condition $\int_0^1 u(x)dx = 0$ to find the constant $C$ such that $U = \bar{U} + C$ satisfies the condition.

## 2.2 Test cases

- Case 1:

$$\begin{aligned} f(x) &= \frac{1}{2} - x\\ \int_0^1 f(x) &= 0\\ u'(0) &= 0\\ u'(1) &= 0\\ u(x) &= \frac{x^2\,(2\,x-3)}{12} + \frac{1}{24}\\ \int_0^1 u(x) &= 0 \end{aligned}$$

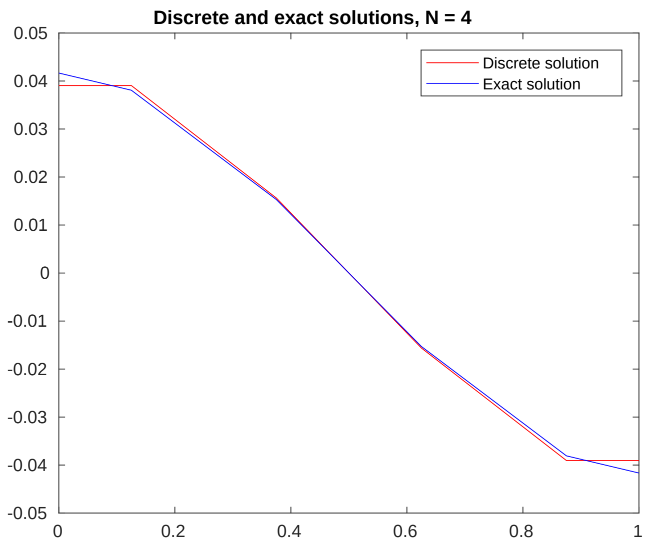
- Case 2:

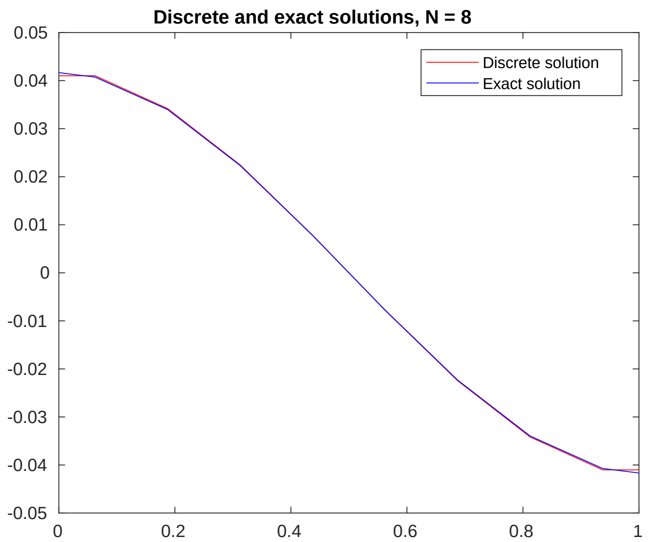
$$\begin{aligned} f(x) &= 400\pi^2\cos(20\pi x)\\ \int_0^1 f(x) &= 0\\ u'(0) &= 0\\ u'(1) &= 0\\ u(x) &= \sin\left(20\pi x + \frac{\pi}{2}\right)\\ \int_0^1 u(x) &= 0 \end{aligned}$$

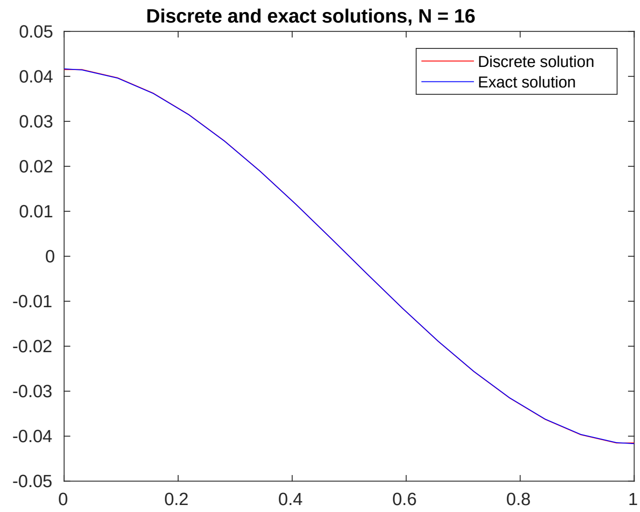
## 2.3 Homogeneous Neumann boundary condition, regular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

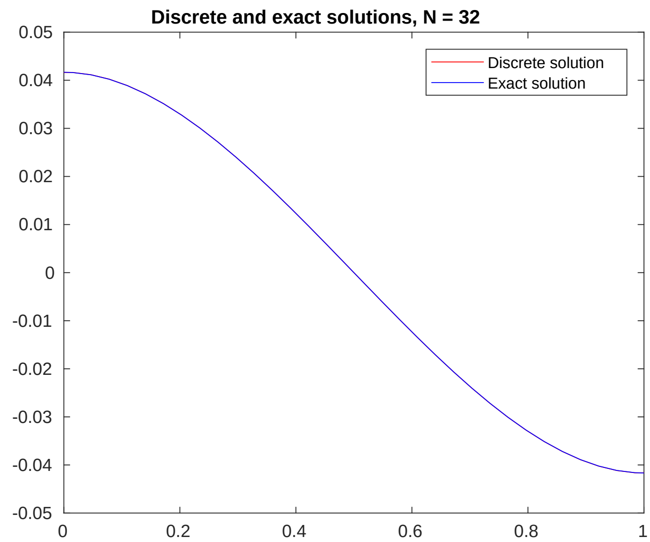
### 2.3.1 Figures of results

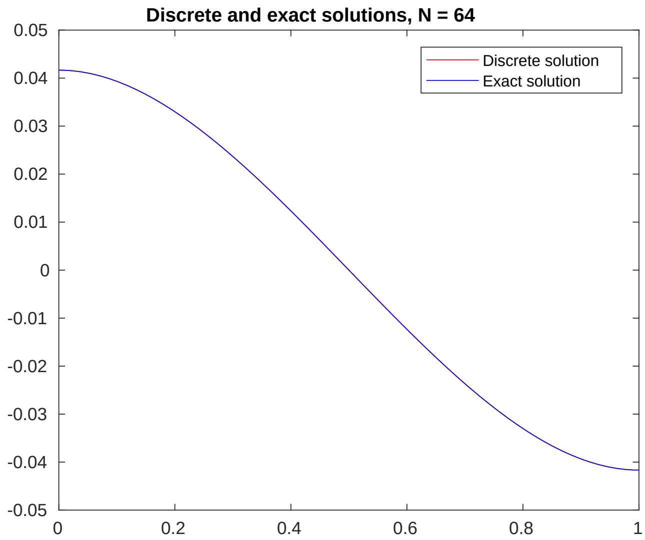
- Case 1:

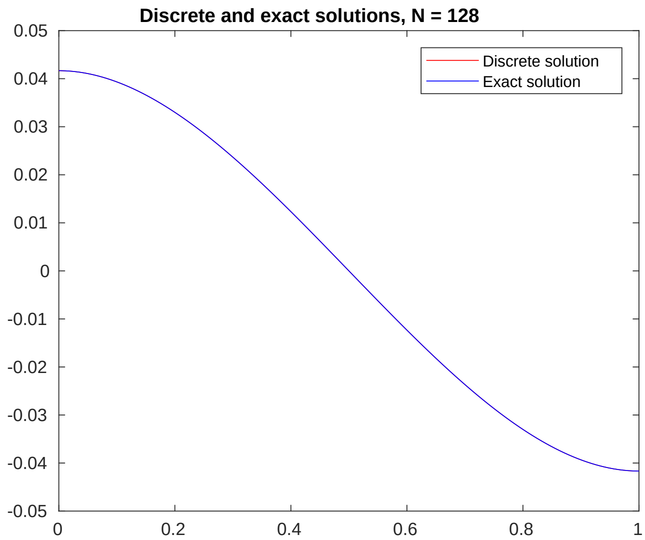


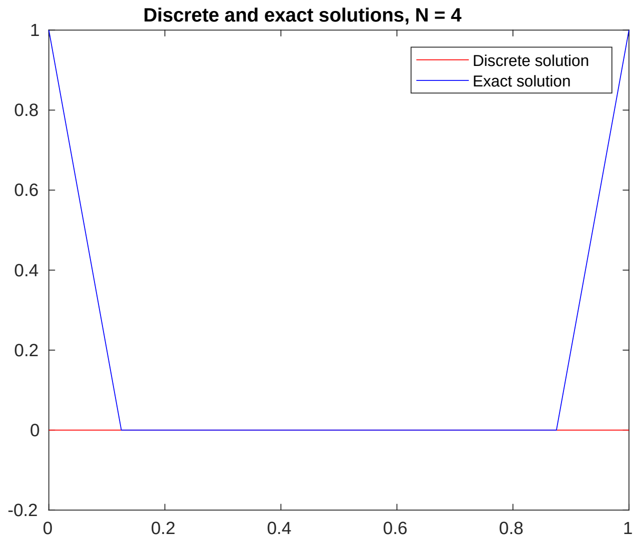

Discrete and exact solutions, N = 16
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

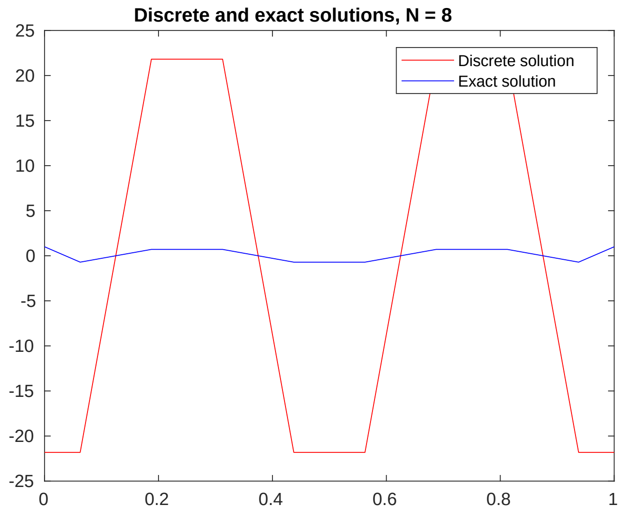

Discrete and exact solutions, N = 32
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

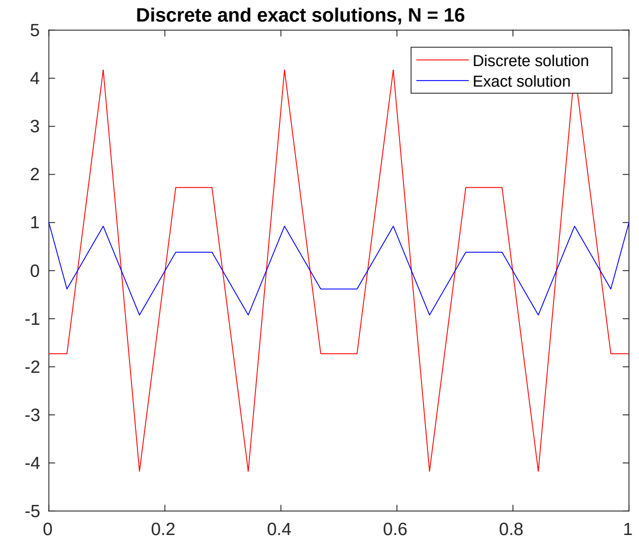

Discrete and exact solutions, N = 64
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

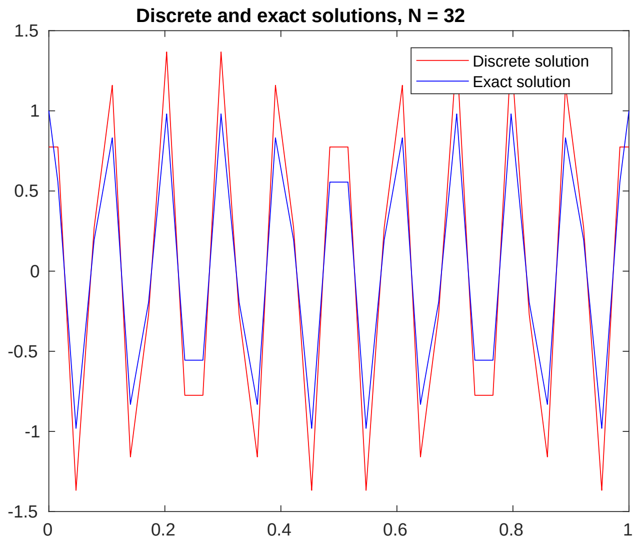

Discrete and exact solutions, N = 128
Discrete solution
Exact solution
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1

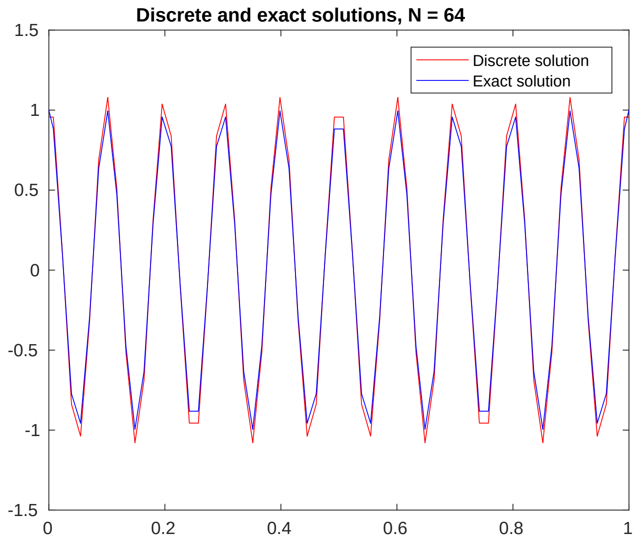
- Case 2:

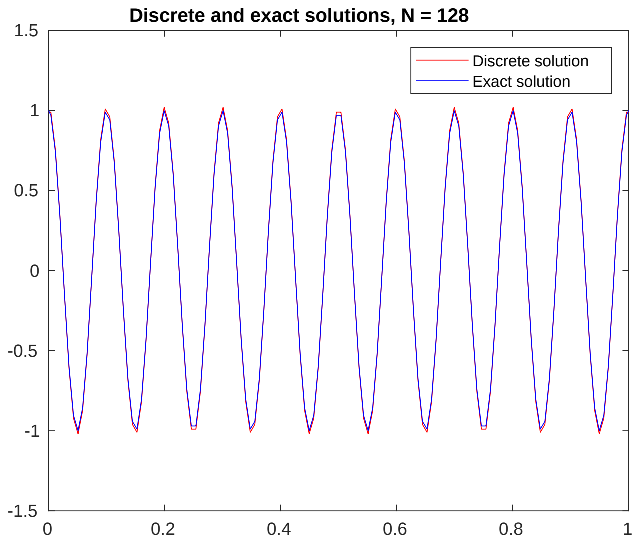

Discrete and exact solutions, N = 4
Discrete solution
Exact solution
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1

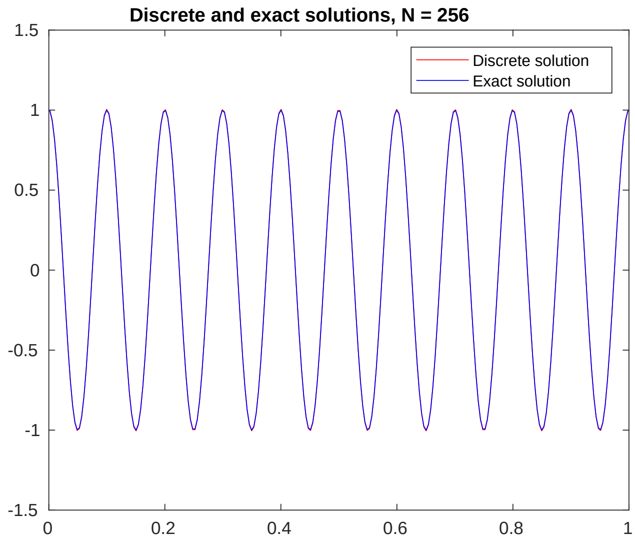

Discrete and exact solutions, N = 8
Discrete solution
Exact solution
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
0.2
0.4
0.6
0.8
1

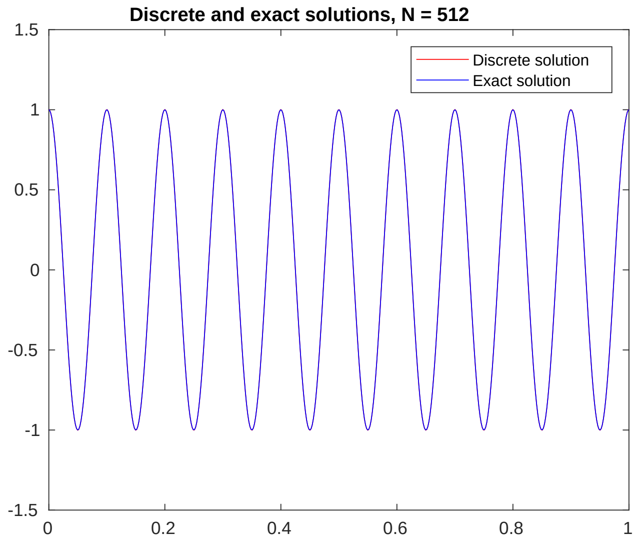

Discrete and exact solutions, N = 16
Discrete solution
Exact solution
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.2
0.4
0.6
0.8
1

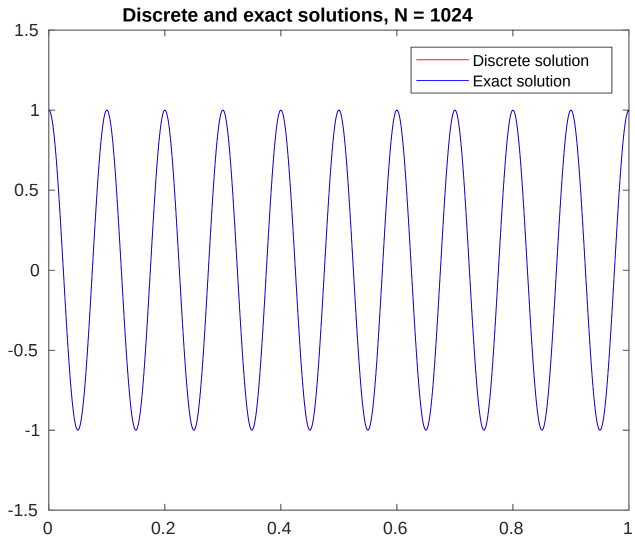

Discrete and exact solutions, N = 32
Discrete solution
Exact solution
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 64
Discrete solution
Exact solution
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 128
Discrete solution
Exact solution
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 256
Discrete solution
Exact solution
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1


Discrete and exact solutions, N = 512
Discrete solution
Exact solution
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1




### 2.3.2 Errors

- Case 1:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 7.278867e-04 | 1.449939e-02 |
| 8 | 1.864655e-04 | 5.329006e-03 |
| 16 | 4.689303e-05 | 1.918917e-03 |
| 32 | 1.174048e-05 | 6.845135e-04 |
| 64 | 2.936197e-06 | 2.430787e-04 |
| 128 | 7.341164e-07 | 8.612922e-05 |

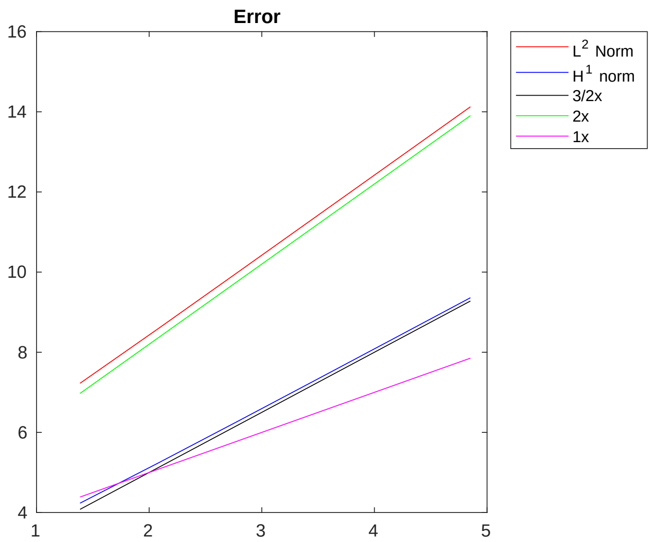
- Case 2:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 3.446145e-13 | 4.000000e+00 |
| 8 | 2.110184e+01 | 2.389353e+02 |
| 16 | 2.486740e+00 | 7.434583e+01 |
| 32 | 2.787005e-01 | 1.565996e+01 |
| 64 | 5.963946e-02 | 4.064360e+00 |
| 128 | 1.437125e-02 | 1.122033e+00 |
| 256 | 3.560351e-03 | 3.281879e-01 |
| 512 | 8.880771e-04 | 1.017972e-01 |
| 1024 | 2.218939e-04 | 3.318688e-02 |

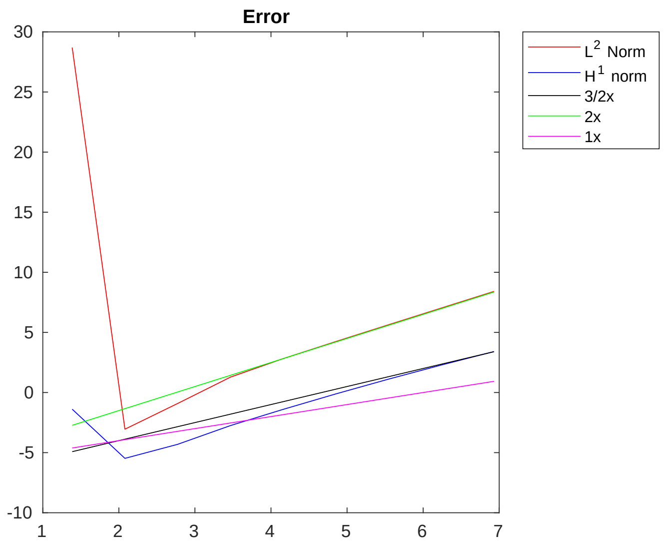
**Remark:**

### 2.3.3 Convergence rate

•<u>Case 1:</u>




•<u>Case 2:</u>




**Remark:** In both case 1 and 2, the method appears to have convergence rate of order 2 in $L^2$ norm and order $\frac{3}{2}$ in energy norm.

## 2.4 Homogeneous Neumann boundary condition, singular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

Again, we will consider the grid $x_i = 1 - \cos\left(\frac{\pi i}{2(N+1)}\right)$ in this section.

### 2.4.1 Errors

- Case 1:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 4.300279e-03 | 2.361702e-02 |
| 8 | 4.473365e-04 | 7.452580e-03 |
| 16 | 7.886546e-05 | 2.530245e-03 |
| 32 | 1.995312e-05 | 9.154423e-04 |
| 64 | 5.129577e-06 | 3.310069e-0 |
| 128 | 1.302358e-06 | 1.185516e-04 |

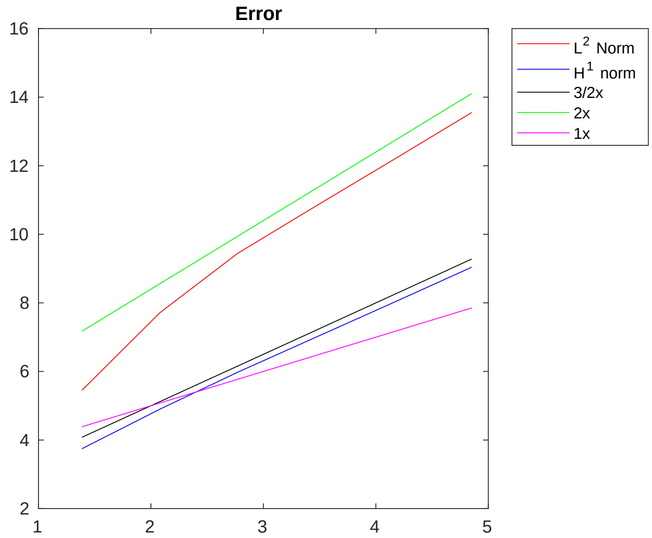
- Case 2:

| N | $\|u_{discrete} - u_{exact}\|_{L^2}$ | $\|u_{discrete} - u_{exact}\|_{H^1}$ |
|---|---|---|
| 4 | 2.518759e+02 | 8.229286e+02 |
| 8 | 2.924364e+01 | 2.378888e+02 |
| 16 | 9.888043e+01 | 3.764880e+02 |
| 32 | 5.525841e-01 | 2.551420e+01 |
| 64 | 1.099226e-01 | 6.825063e+00 |
| 128 | 2.584269e-02 | 1.840590e+00 |
| 256 | 6.388832e-03 | 5.196868e-01 |
| 512 | 1.595856e-03 | 1.549021e-01 |
| 1024 | 3.992680e-04 | 4.880113e-02 |

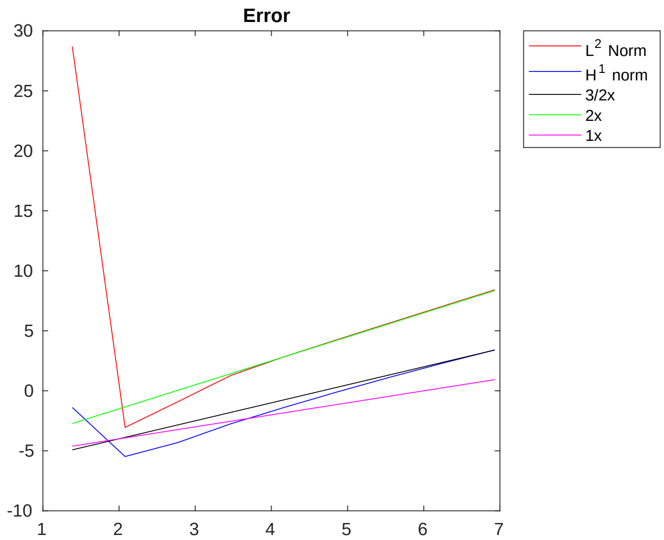
**Remark:**

### 2.4.2 Convergence rate

•<u>Case 1:</u>




•<u>Case 2:</u>




**Remark:** In both case 1 and 2, the method appears to have convergence rate of order 2 in $L^2$ norm and order $\frac{3}{2}$ in energy norm.