Duong Minh Duc, Calculus of Variations

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Abstract

I retype and correct some errors in [1].

*Typer.

Contents

1 Integrals of Vector-Valued Mappings

Definition 1.1. Let $a_0, \ldots, a_n; c_1, \ldots, c_n$ be $2n+1$ real numbers such that

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b \tag{1.1}$$
$$c_k \in [a_{k-1}, a_k], \quad k = 1, \ldots, n \tag{1.2}$$

We call $P = \{a_0, \ldots, a_n; c_1, \ldots, c_n\}$ a *partition* of the interval $[a, b]$ and put

$$|P| = \max\{a_1 - a_0, a_2 - a_1, \ldots, a_n - a_{n-1}\} \tag{1.3}$$

Define $\mathcal{P}([a, b])$ to be the *set of all partitions of the interval* $[a, b]$.

Definition 1.2. Let $(E, \|\cdot\|)$ be a Banach (normed) space. Let u be a continuous mapping from a closed interval $[a, b]$ into E, and $P = \{a_0, \ldots, a_n; c_1, \ldots, c_n\}$ be a partition of the interval $[a, b]$. We put

$$S(u, P) = \sum_{k=1}^{n} u(c_k)(a_k - a_{k-1}) \tag{1.4}$$

and call $S(u, P)$ a *Riemann sum* with respect to the partition P.

Using uniform continuity of u in $[a, b]$ and completeness of E, we can prove that there exists a vector w such that

$$\lim_{|P| \to 0} S(u, P) = w \tag{1.5}$$

We call w the *integral* of u in $[a, b]$ and denote it by $\int_a^b u(t)\,dt$.

Problem 1.3. *Let $(E, \|\cdot\|)$ be a Banach space. Let u be a continuous mapping from a closed interval $[a, b]$ into E and T be a linear mapping from E into $\mathbb{R}$. Prove that*

$$\int_a^b (T \circ u)(t)\,dt = T\left(\int_a^b u(t)\,dt\right) \tag{1.6}$$

Hint. Notice that

$$S((T \circ u), P) = T(S(u, P)) \tag{1.7}$$

for all partitions P of $[a, b]$. □

Problem 1.4. *Let $(E, \|\cdot\|)$ be a Banach space. Let u, v be two continuous mappings from a closed interval $[a, b]$ into E and α be a positive real number. Prove that*

$$\int_a^b (u + \alpha v)(t)\,dt = \int_a^b u(t)\,dt + \alpha \int_a^b v(t)\,dt \tag{1.8}$$

Hint. Prove

$$T\left(\int_a^b (u + \alpha v)(t)\,dt\right) = T\left(\int_a^b u(t)\,dt + \alpha \int_a^b v(t)\,dt\right) \tag{1.9}$$

for all $T \in L(E, \mathbb{R})$. $\square$

Problem 1.5. *Let $(E, \|\cdot\|)$ be a Banach space. Let u be a continuous mapping from a closed interval $[a, b]$ into E. Prove that*

$$\left\| \int_a^b u(t)\, dt \right\| \leq \int_a^b \|u(t)\|_E dt \tag{1.10}$$

Hint. Choose $T \in L(E, \mathbb{R})$ such that $\|T\| = 1$ and

$$\left\| \int_a^b u(t)\, dt \right\| = T\left(\int_a^b u(t)\, dt \right) \tag{1.11}$$

then applied Problem 1.3 and Problem 1.4. $\square$

2 Differential Calculus in Normed Spaces

Let E and F be two normed spaces equipped norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively, U be an open set in E, and e be a vector in E and $x \in U$.

Problem 2.1. *Prove that there exists a positive real number α such that the open interval $(-\alpha, \alpha)$ is contained in the following set*

$$I_{x,e} = \{t : t \in R, x + te \in U\} \tag{2.1}$$

Define

$$U_{x,h} = \{y : y = x + te \in U, t \in \mathbb{R}\} \tag{2.2}$$

Definition 2.2. Let f be a mapping from U into F, e be a vector in E and $x \in U$. We say

1. f has a *partial derivative with respect to direction* (*directional derivative*) e at the point x if the following limit exists

$$\lim_{t \to 0} \frac{f(x + te) - f(x)}{t} \tag{2.3}$$

   Then we denote this limit by $\frac{\partial f}{\partial e}(x)$.

2. f is *directional differentiable* at the point x if f has partial derivatives with respect to all directions in E and there exists a linear mapping $Df(x)$ from E into F such that

$$\frac{\partial f}{\partial e}(x) = Df(x)\, e, \;\; \forall e \in E \tag{2.4}$$

Definition 2.3. Let f be a mapping from U into F and $x \in U$. We say

1. f is *Gâteaux differentiable at the point* x if f is directional differentiable at x and $Df(x)$ belongs to $L(E, F)$.

2. f is *Gâteaux differentiable in* U if f is Gâteaux differentiable at all points x in U.

Problem 2.4. *Let* $(H, \|\cdot\|)$ *be a Hilbert space. Put*

$$f(x) = \|x\|^2, \quad \forall x \in H \tag{2.5}$$

Prove that f *is Gâteaux differentiable in* H.

Hint. Let $\langle \cdot, \cdot \rangle$ be the scalar product equipped for H. Notice that

$$\|x\|^2 = \langle x, x \rangle, \quad \forall x \in H \tag{2.6}$$

Problem 2.5. *Let* f *be a continuous linear mapping from a normed space* $(E, \|\cdot\|_E)$ *into a normed space* $(F, \|\cdot\|_F)$. *Prove that* f *is Gâteaux differentiable in* E.

Hint. Use linearity and continuity of f. □

Problem 2.6. *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be two normed spaces,* B *be a continuous bilinear mapping from* $E \times E$ *into* F. *Put* $f(x) = B(x, x)$ *for all* x *in* E. *Prove that* f *is Gâteaux differentiable in* E.

Hint. Use the bilinearity and continuity of B. □

Let E and F be two normed spaces and U be a open set in E. Let $x \in U$. There exists a positive real number r such that $B(x, r) \subset U$.

Definition 2.7. Let f be a mapping from U into F, and $x \in U$. We say

1. f is *Fréchet differentiable at the point* x if there exists a mapping $Df(x) \in L(E, F)$ and ϕ from $B(0, r) \subset E$ into F such that

$$f(x+h) - f(x) = Df(x)(h) + \|h\|_E \phi(h), \quad \forall h \in B(0, r) \tag{2.7}$$

where $\phi(h) \to 0$ as $h \to 0$.

2. f is *Fréchet differentiable in* U if f is Fréchet differentiable at all points x in U.

Problem 2.8. *Let* $(H, \|\cdot\|)$ *be a Hilber space. Put*

$$f(x) = \|x\|^2, \quad \forall x \in H \tag{2.8}$$

Prove that f *is Fréchet differentiable in* H.

Hint. Let $\langle \cdot, \cdot \rangle$ be the scalar product equipped for H. Notice that

$$\|x\|^2 = \langle x, x \rangle, \quad \forall x \in H \tag{2.9}$$

Done. □

Problem 2.9. *Let* f *be a continuous linear mapping from a normed space*

$(E, \|\cdot\|_E)$ *into a normed space* $(F, \|\cdot\|_F)$. *Prove that* f *is Fréchet differentiable in* E.

HINT. Use the linearity and continuity of f. □

Problem 2.10. *Let* f *be a continuous bilinear mapping from a normed space* $(E, \|\cdot\|_E)$ *into a normed space* $(F, \|\cdot\|_F)$. *Prove that* f *is Fréchet differentiable in* E.

HINT. use the bilinearity and continuity of f. □

Problem 2.11. *Let* E *and* F *be two normed spaces,* U *be an open set in* E *and* $x \in U$. *Let* f *be a mapping from* U *into* F *which is Fréchet differentiable at* x. *Prove that* f *is continuous at* x.

HINT. Use definition. □

Problem 2.12. *Let* U *be an open set in a normed space* E, $\alpha \in \Phi$ *and* f, g *be two mappings from* U *into a normed space* F. *Suppose that* f *and* g *are Gâteaux differentiable (resp. Fréchet differentiable) at some point* x *in* U. *Prove that* $f + g$ *and* αf *are Gâteaux differentiable (resp. Fréchet differentiable) at* x. *In addition,*

$$D(f+g)(x) = Df(x) + Dg(x) \tag{2.10}$$

$$D(\alpha f)(x) = \alpha Df(x) \tag{2.11}$$

HINT. Use definition. □

Theorem 2.13. *Let* U *and* O *be a open subsets in normed spaces* E *and* F, *respectively. Let* $f : U \to O$ *and* $g : O \to G$ *be mappings, where* G *is a normed space. Let* x *be some point in* U. *Suppose that* f *is Fréchet differentiable at* x *and* g *is Fréchet differentiable at* $f(x)$. *Then* $g \circ f$ *is Fréchet differentiable at* x *and*

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x) \tag{2.12}$$

Problem 2.14 (Mean Value Theorem). *Let* f *be a Gâteaux differentiable mapping from an open set* U *in a normed space* E *into a normed space* F. *Let* a *and* b *in* U *such that the set* $[a, b] \equiv \{a + t(b-a) \,|t \in [0, 1]\}$ *contained in* U. *Suppose that* f *is continuous on* $[a, b]$. *Prove that*

$$\|f(b) - f(a)\|_F \leq \|b-a\|_E \sup\{\|Df(y)\| : y \in [a, b]\} \tag{2.13}$$

HINT. Let $T \in L(E, \mathbb{R})$ satisfy $\|T\| = 1$ and

$$\|f(b) - f(a)\|_F = T(f(b) - f(a)) \tag{2.14}$$

Put

$$g(s) = T(f(a + s(b-a))), \quad \forall s \in [0, 1] \tag{2.15}$$

Prove

$$\|(T \circ f)(b) - (T \circ f)(a)\|_F \leq \|b-a\|_E \sup\{\|D(T \circ f)(y)\| \,|y \in [a, b]\} \tag{2.16}$$

Done. □

Problem 2.15 (Mean Value Theorem). *Let f be a Gâteaux differentiable mapping from an open set U in a normed space E into a normed space F. Suppose that $x \mapsto Df(x)(h)$ is a continuous mapping in U for all h in E. Let a and b in U such that the set $[a,b] \equiv \{a + t(b-a) \,|t \in [0,1]\}$ contained in U. Prove that*

$$f(b) - f(a) = (b-a) \int_0^1 Df(a + t(b-a))\, dt \tag{2.17}$$

Hint. Let $T \in L(E, \mathbb{R})$, prove

$$D(T \circ f)(a + t(b-a)) = T(Df(a + t(b-a))), \quad \forall t \in [0,1] \tag{2.18}$$

$$T(f(b) - f(a)) = T\left((b-a)\int_0^1 Df(a + t(b-a))\, dt\right) \tag{2.19}$$

Problem 2.16. *Let f be a Gâteaux differentiable mapping from a open set U in a normed space E into a normed space F. Suppose that $x \mapsto Df(x)$ is a continuous mapping from U into $L(E,F)$. Prove that f is Fréchet differentiable in U.*

Hint. Let $x \in U$ and a positive real number r such that $B(x,r) \subset U$. Put

$$\phi(h) = \int_0^1 (Df(x + th) - Df(x))(h)\, dt, \quad \forall h \in B(0,r) \tag{2.20}$$

Prove

$$\lim_{h \to 0} \frac{\phi(h)}{\|h\|} = 0 \tag{2.21}$$

then use Mean Value Theorem. □

Definition 2.17. Let f be a mapping from U into F. We say

1. f is *continuously Gâteaux differentiable in U* if f is Gâteaux differentiable in U and the mapping $x \mapsto Df(x)$ is continuous from U into $L(E,F)$.
2. f is *continuously Fréchet differentiable in U* if f is Fréchet differentiable in U and the mapping $x \mapsto Df(x)$ is continuous from U into $L(E,F)$. We also say that f *is of class $C^1(U)$*.

Definition 2.18. Let E and F be two normed spaces, U be an open set in E and f be of class $C^1(U)$. Then Df is a mapping from U into the normed space $L(E,F)$. If Df is Fréchet differentiable in U, we say that f is *two-time Fréchet differentiable in U* and denote $D(Df)$ by $D^2 f$ and call it *second derivative of f*.

Using mathematical induction, we can define *n-time Fréchet differentiable* concept in U and denote $D\left(D^{n-1}f\right)$ by $D^n f$ where $D^0 f = f$ for all positive integer n, and call it *nth derivative of f*.

We denote by $C^r(U,F)$ the *set of all r-time Fréchet differentiable in U* such that $D^n f$ is continuous in U for all $n \le r$. If f *is of class $C^r(U,F)$*, we say that

f is *continuously r-time Fréchet differentiable in U*. We define

$$C^{\infty}(U,F)=\bigcap_{r=1}^{\infty}C^{r}(U,F) \tag{2.22}$$

Theorem 2.19. *Let U be an open set in a Banach space E, u in $C^2(U,F)$, $x\in U$ and $h,k\in E$. Then*

$$D^2u(x)(h,k)=D^2u(x)(k,h) \tag{2.23}$$

Problem 2.20. *Let A be a bounded measurable set in $\mathbb{R}^n$. Let μ be Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(u)=\int_A u^3 d\mu,\ \ \forall u\in L^3(A) \tag{2.24}$$

Prove that f is of class $C^1\left(L^3(A),\mathbb{R}\right)$.

Hint. Let u and v be in $L^3(A)$ and $t\in(-1,1)\setminus\{0\}$. Prove

$$\frac{f(u+tv)-f(u)}{t}=\int_A\left(3u^2v+3tuv^2+t^2v^3\right)d\mu \tag{2.25}$$

Thus, f is Gâteaux differentiable in $L^3(A)$ and

$$Df(u)(v)=3\int_A u^2v\,d\mu,\ \ \forall u,v\in L^3(A) \tag{2.26}$$

Let $u,v,w\in L^3(A)$. Prove

$$|(Df(u)-Df(w))(v)|\leq 3\|u+w\|_{L^3(A)}\|u-w\|_{L^3(A)}\|v\|_{L^3(A)} \tag{2.27}$$

Problem 2.21. *Let A be a bounded measurable set in $\mathbb{R}^n$. Let μ be the Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(u)=\int_A u^3 d\mu,\ \ \forall u\in L^3(A) \tag{2.28}$$

Prove that f is of class $C^2\left(L^3(A),\mathbb{R}\right)$.

Hint. Let $u,v,w\in L^3(A)$. Prove that

$$\frac{(Df(u+tw)-Df(u))(v)}{t}=3\int_A\left(2uvw+tw^2v\right)d\mu \tag{2.29}$$

Thus,

$$D^2f(u)(v,w)=6\int_A uvw\,d\mu \tag{2.30}$$

Let $u,v,w,z\in L^3(A)$. Prove

$$\left|\left(D^2f(u)-D^2f(z)\right)(v,w)\right|\leq 6\|u-z\|_{L^3(A)}\|v\|_{L^3(A)}\|w\|_{L^3(A)} \tag{2.31}$$

Problem 2.22. *Let A be a bounded measurable in $\mathbb{R}^n$. Let μ be the Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(u) = \int_A u^3 d\mu, \quad \forall u \in L^3(A) \tag{2.32}$$

Prove that f is of class $C^3\left(L^3(A), R\right)$.

HINT. Let $u, v, w, z \in L^3(A)$. Prove

$$\frac{\left(D^2 f(u+tz) - D^2 f(u)\right)(v,w)}{t} = 6\int_A vwz d\mu \tag{2.33}$$

Thus,

$$D^3 f(u)(v,w,z) = 6\int_A vwz d\mu \tag{2.34}$$

Problem 2.23. *Let A be a bounded measurable set in $\mathbb{R}^n$. Let μ be the Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(u) = \int_A \sin(u(t))\, d\mu, \quad \forall u \in L^5(A) \tag{2.35}$$

Prove that f is of class $C^1\left(L^5(A), R\right)$.

HINT. Let $u, v \in L^5(A)$ and $t \in (-1,1)\setminus\{0\}$. Prove

$$\frac{f(u+tv) - f(u)}{t} = \int_A \frac{\sin(u(s) + tv(s)) - \sin(u(s))}{t} d\mu \tag{2.36}$$

Use the Lebesgue dominated convergence theorem, prove

$$\lim_{t\to 0}\frac{f(u+tv) - f(u)}{t} = \int_A \cos(u)\, v d\mu \tag{2.37}$$

Definition 2.24. Let f be a real function from a subset A in a normed space E and a is a point in A. We say

1. f attains *maximum* at a if $f(x) \leq f(a)$ for all $x \in A$. Then a is called a *maximum point* of f.

2. f attains *minimum* at a if $f(x) \geq f(a)$ for all $xinA$. Then a is called a *minimum point* of f.

3. f attains *extremity* at a if f attains maximum or minimum at a. Then a is called *extreme point* of f.

4. f attains *local maximum* at a if there exists a positive real number r such that $f(x) \leq f(a)$ for all $x \in A \cap B(a,r)$. Then a is called a *local maximum point* of f.

5. f attains *local minimum* at a if there exists a positive real number r such that $f(x) \geq f(a)$ for all $x \in A \cap B(a,r)$. Then a is called a *local minimum point* of f.

6. f attains *local extremity* at a if f attains local maximum or local minimum at a. Then a is called a *local extreme point* of f.

7. a is called a *critical point* of f if A is an open set and f is directional differentiable at a and $Df(a)(h) = 0$ for all $h \in E$.

Problem 2.25. *Let f be a real function on an open set U in a normed space E which attains extremity at a in U. Let h in E such that f is directional differentiable with respect to direction h at a. Prove that*

$$\frac{\partial f}{\partial h}(a) = 0 \tag{2.38}$$

Problem 2.26. *Let A be a bounded measurable in $\mathbb{R}^n$. Let μ be the Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(u) = \int_A \sin(u(t))^2 d\mu, \quad \forall u \in L^7(A) \tag{2.39}$$

Accept that f is of class $C^1\left(L^7(A), \mathbb{R}\right)$, find some critical points of f without calculating derivative of f.

Hint. Find extremity of f. □

Theorem (Lagrange Multipliers). *Let f and g be two real functions which are continuously Fréchet differentiable on an open set U in a normed space E. Define*

$$M = \{x \in U | g(x) = 0\} \tag{2.40}$$

Suppose that there exists a in M such that $f(a)$ is an extreme value of $f(M)$ and $Dg(a) \not\equiv 0$. Then there exists a real number λ such that

$$Df(a) = \lambda Dg(a) \tag{2.41}$$

3 Lower Semicontinuous Functions

Definition 3.1. Let (M, δ) be a metric space and f be a real function on M. We say that f is *lower semicontinuous in M* if for all sequence $\{x_m\}_{m=1}^{\infty}$ converging to x in M, the following inequality holds

$$f(x) \leq \liminf_{m \to \infty} f(x_m) \tag{3.1}$$

Problem 3.1. *Let (M, δ) be a metric space, f be a real lower semicontinuous function in M, and α be a real number. Prove*

$$\{x \in M | f(x) > \alpha\} \tag{3.2}$$

is an open set in M.

Hint. Prove that

$$\{x \in M | f(x) \leq \alpha\} \tag{3.3}$$

is a closed set in M. $\square$

Problem 3.2. *Let (M, δ) be a metric space, f be a real function in M. Suppose that*

$$\{x \in M | f(x) > \alpha\} \tag{3.4}$$

is an open set in M for all real number α. Prove that f is lower semicontinuous in M.

Hint. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence converging to x in M. Let β be real number for which $\beta < f(x)$. Prove

$$\beta < \liminf_{m\to\infty} f(x_m) \tag{3.5}$$

$\square$

Problem 3.3. *Let (M, δ) be a metric space, f be a real lower semicontinuous function in M, and α in $f(M)$. Suppose that*

$$\{x \in M | f(x) \leq \alpha\} \tag{3.6}$$

is compact in M. Prove that there exists u in M such that $f(u) = \min f(M)$.

Hint. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in M such that

$$\lim_{m\to\infty} f(x_m) = \inf f(M) \tag{3.7}$$

Problem 3.4. *Let (M, δ) be a metric space, f be a real function in M, and α in $f(M)$. Suppose that for all $\beta \leq \alpha$ the set*

$$K_\beta = \{x \in M : f(x) \leq \beta\} \tag{3.8}$$

is compact. Prove that there exists some u in M such that

$$f(u) = \min f(M) \tag{3.9}$$

Hint. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in M such that $\{f(x_m)\}_{m=1}^{\infty}$ converges to $\inf f(M)$. Suppose

$$\beta_m = f(x_m) \leq \alpha \tag{3.10}$$

Prove that there exists a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ of the sequence $\{x_m\}_{m=1}^{\infty}$ which converges to u in the set $\bigcap\limits_{m=1}^{\infty} K_{\beta_m}$. $\square$

Definition 3.5. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in a normed space E. We say that $\{x_m\}_{m=1}^{\infty}$ *weakly converges* to x in E if $\{T(x_m)\}_{m=1}^{\infty}$ converges to $T(x)$ for all $T \in L(E, \mathbb{R})$.

Problem 3.6. *Let $\{x_m\}_{m=1}^{\infty}$ be a sequence which weakly converges to x in a Banach space $(E, \|\cdot\|)$. Prove that $\{\|x_m\|\}_{m=1}^{\infty}$ is bounded in $\mathbb{R}$.*

Hint. Define

$$\Lambda_m(S) = S(x_m), \quad \forall m \in \mathbb{Z}_+, \forall S \in F := L(E, \mathbb{R}) \tag{3.11}$$

and

$$|||S||| = \sup\{|S(u)|\,|u \in E, \|u\| \leq 1\} \tag{3.12}$$
$$|||\Lambda_m||| = \sup\{|\Lambda(T)|\,|T \in L(E, \mathbb{R}), |||T||| \leq 1\} \tag{3.13}$$

Use Hahn-Banach theorem, prove $|||\Lambda_m||| = \|x_m\|$. Then use Banach-Steinhaus theorem, prove that $\{|||\Lambda_m|||\}_{m=1}^{\infty}$ is bounded. □

Definition 3.7. Let M be a nonempty subset in a Banach space E and f be a real function in M. We say that f is *weakly lower semicontinuous in M* if for all sequences $\{x_m\}_{m=1}^{\infty}$ converging to x in M, the following inequality holds

$$f(x) \leq \liminf_{m\to\infty} f(x_m) \tag{3.14}$$

Definition 3.7. Let M be a nonempty subset in a Banach space $(E, \|\cdot\|)$ and f be a real function in M. We say that f is *coercive* in M if for all sequences $\{x_m\}_{m=1}^{\infty}$ in M such that $\{\|x_m\|\}_{m=1}^{\infty}$ converges to ∞, then $\{f(x_m)\}_{m=1}^{\infty}$ converges to ∞.

Definition 3.8. Let M be a subset in a Banach space E. We say that M is *weakly closed in E* if for all sequences $\{x_m\}_{m=1}^{\infty}$ in M which weakly converges to x in E, then $x \in M$.

Problem 3.9. *Let E be a Banach space. Suppose that all each bounded sequence $\{x_m\}_{m=1}^{\infty}$ in E always has a weakly convergent subsequence in E. Let M be a weakly closed subset in E. Let f be a real coercive lower semicontinuous function in M. Prove that there exists some u in M such that*

$$f(u) = \min f(M) \tag{3.15}$$

Hint. Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in M such that $\{f(u_m)\}_{m=1}^{\infty}$ converges to $\inf f(M)$. Prove that $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence whose a subsequence weakly converges to u in M. □

Theorem 3.10. *Let Ω be an open set in $\mathbb{R}^n$ and F be a real function in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Suppose*

1. *For all $(s, z) \in \mathbb{R} \times \mathbb{R}^n$, the function $x \mapsto F(x, s, z)$ is measurable in Ω.*
2. *For all $x \in \Omega$, the function $(s, z) \mapsto F(x, s, z)$ is continuous in $\mathbb{R} \times \mathbb{R}^n$.*
3. *For all $(x, s) \in \Omega \times \mathbb{R}$, the function $z \mapsto F(x, s, z)$ is convex in $\mathbb{R}^n$.*
4. *There exists an integrable function ϕ in Ω such that*

$$F(x, s, z) \geq \phi(x), \quad \forall (x, s, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{3.16}$$

Define

$$J(u) = \int_\Omega F(x, u(x), \nabla u(x))\, dx, \quad \forall u \in W^{1,1}(\Omega) \tag{3.17}$$

Then J is weakly lower semicontinuous in $W^{1,1}(\Omega)$.

THE END

References

[1] Duong Minh Duc, *Lectures: Real Analysis*, Faculty of Math and Computer Science, Ho Chi Minh University of Science.