Mathematical Analysis & Numerical Analysis
Giải Tích Toán Học & Giải Tích Số

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Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
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1. *Mathematical Analysis – Giải Tích Toán Học.*
PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.pdf.
TEX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.tex.

Mục lục

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Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

1 Wikipedia

Resources – Tài nguyên.

1. [LL01]. Elliott Lieb, Michael Loss. *Analysis.*
2. [Rud76]. Walter Rudin. *Principles Principles of Mathematical Analysis.*
3. [Rud73; Rud87]. Walter Rudin. *Real & Complex Analysis.*
4. [Tao22a]. Terence Tao. *Analysis I.*
5. [Tao22b]. Terence Tao. *Analysis II.*

"Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones." – [LL01, Chap. 1, p. 1]

1.1 Wikipedia/physics-informed neural networks PINNs

"*Physics-informed neural networks* (PINNs), also referred as as *Theory-Trained Neural Networks* (TTNs), are a type of universal function approximators that can be embed the knowledge of any physical laws that govern a given data-set in the learning process, & can be described by PDEs. They overcome the low data availability of some biological & engineering systems that makes most state-of-the-art machine learning techniques lack robustness, rendering them ineffective in these scenarios. The prior knowledge of general physical laws acts in the training of neural networks (NNs) as a regularization agent that limits the space of admissible solutions, increasing the correctness of the function approximation. This way, embedding this prior information into a neural network results in enhancing the information content of the available data, facilitating the learning algorithm to capture the right solution & to generalize well even with a low amount of training examples. Physics-informed neural networks for solving NSEs.

1.1.1 Function approximation

Most of the physical laws that govern the dynamics of a system can be described by PDEs. E.g., the NSEs are a set of PDEs derived from the conservation laws (i.e., conservation of mass, momentum, & energy) that govern fluid mechanics. The solution of the NSEs with appropriate initial & boundary conditions allows the quantification of flow dynamics in a precisely defined geometry. However, these equations cannot be solved exactly & therefore numerical methods must be used (e.g. FDs, FEs, & FVs). In this setting, these governing equations must be solved while accounting for prior assumptions, linearization, & adequate time & space discretization.

Recently, solving the governing PDEs of physical phenomena using deep learning has emerged as a new field of scientific machine learning (SciML), leveraging the universal approximation theorem & high expressivity of neural networks. In general, deep neural networks could approximate any high-dimensional function given that sufficient training data are supplied. However, such networks do not consider the physical characteristics underlying the problem, & the level of approximation accuracy provided by them is still heavily dependent on careful specifications of the problem geometry as well as the initial & boundary conditions. Without this preliminary information, the solution is not unique & may lose physical correctness. On the other hand, physics-informed neural networks (PINNs) leverage governing physical equations in neural network training. Namely, PINNs are designed to be trained to satisfy the given training data as well as the imposed governing equations. In this fashion, a neural network can be guided with training data that do not necessarily need to be large & complete. Potentially, an accurate solution of PDEs can be found without knowing the boundary conditions. Therefore, with some knowledge about the physical characteristics of the problem & some form of training data (even sparse & incomplete), PINN may be used for finding an optimal solution with high fidelity.

PINNs allow for addressing a wide range of problems in computational science & represent a pioneering technology leading to the development of new classes of numerical solvers for PDEs. PINNs can be thought of as a meshfree alternative to traditional approaches (e.g., CFD for fluid dynamics), & new data-driven approaches for model inversion & system identification. Notably, the trained PINN network can be used for predicting the values on simulation grids of different resolutions without the need to be retrained. In addition, they allow for exploiting automatic differentiation (AD) to compute the required derivatives in the PDEs, a new class of differentiation techniques widely used to derive neural networks assessed to be superior to numerical differentiation or symbolic differentiation.

1.1.2 Modeling & computation

A general nonlinear PDE can be:

$$u_t + N[u;\lambda] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T],$$

where $u(t,\mathbf{x})$ denotes the solution, $N[\cdot;\lambda]$: a nonlinear operator parametrized by λ, & $\Omega \subset \mathbb{R}^d$. This general form of governing equations summarizes a wide range of problems in mathematical physics, e.g. conservative laws, diffusion process, advection-diffusion systems, & kinetic equations. Given noisy measurements of a generic dynamic system described by the equation above, PINNs can be designed to solve 2 classes of problems:

- data-driven solution
- data-driven discovery of PDEs.

Data-driven solution of PDEs. The *data-driven solution of PDE* computes the hidden state $u(t,\mathbf{x})$ of the system given boundary data &/or measurements z, & fixed model parameters λ. Solve:

$$u_t + N[u] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T].$$

By defining the residual $f(t,\mathbf{x})$ as $f \coloneqq u_t + N[u] = 0$, & approximating $u(t,\mathbf{x})$ by a deep neural network. This network can be differentiated using automatic differentiation. The parameters of $u(t,\mathbf{x})$ & $f(t,\mathbf{x})$ can be then learned by minimizing the following loss function $L_{\rm tot} \coloneqq L_u + L_f$ where $L_u \coloneqq \|u - z\|_\Gamma$ is the error between the PINN $u(t,\mathbf{x})$ & the set of boundary conditions & measured data on the set of points Γ where the boundary conditions & data are defined, & $L_f \coloneqq \|f\|_\Gamma$ is the mean-squared error of the residual function. This 2nd term encourages the PINN to learn the structural information expressed by the PDE during the training process.

This approach has been used to yield computationally efficient physics-informed surrogate models with applications in the forecasting of physical processes, model predictive control, multi-physics & multi-scale modeling, & simulation. It has been shown to converge to the solution of the PDE.

Data-driven discovery of PDEs. Given noisy & incomplete measurements z of the state of the system, the *data-driven discovery of PDE* results in computing the unknown state $u(t,\mathbf{x})$ & learning model parameters λ that best describe the observed data & it reads as follows:

$$u_t + N[u;\lambda] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T]. \tag{1}$$

By defining $f(t,\mathbf{x}) \coloneqq u_t + N[u;\lambda] = 0$, & approximating $u(t,\mathbf{x})$ by a deep neural network, $f(t,\mathbf{x})$ results in a PINN. This network can be derived using automatic differentiation. The parameters of $u(t,\mathbf{x}), f(t,\mathbf{x})$, together with the parameter λ of the differential operator can be then learned by minimizing the following loss function $L_{\rm tot} \coloneqq L_u + L_f$ where $L_u \coloneqq \|u - z\|_\Gamma$, with u, z: state solutions & measurements at sparse location Γ, respectively & $L_f \coloneqq \|f\|_\Gamma$ residual function. This 2nd term requires the structured information represented by the PDEs to be satisfied in the training process.

This strategy allows for discovering dynamic models described by nonlinear PDEs assembling computationally efficient & fully differentiable surrogate models that may find application in predictive forecasting, control, & data assimilation.

1.1.3 Physics-informed neural networks for piecewise function approximation

PINN is unable to approximate PDEs that have strong nonlinearity or sharp gradients that commonly occur in practical fluid flow problems. Piecewise approximation has been an old practice in the field of numerical approximation. With the capability of approximating strong nonlinearity extremely light weight PINNs are used to solve PDEs in much larger discrete subdomains that increases accuracy substantially & decreases computational load as well. DPINN (Distributed physics-informed neural networks) & DPIELM (Distributed physics-informed extreme learning machines) are generalizable space-time domain discretization for better approximation. DPIELM is an extremely fast & lightweight approximator with competitive accuracy. Domain scaling on the top has a special effect. Another school of thought is discretization for parallel computation to leverage usage of available computational resources.

XPINNs is a generalized space-time domain decomposition approach for the physics-informed neural networks (PINNs) to solve nonlinear PDEs on arbitrary complex-geometry domains. The XPINNs further pushes the boundaries of both PINNs as well as Conservative PINNs (cPINNs), which is a spatial domain decomposition approach in the PINN framework tailored to conservation laws. Compared to PINN, the XPINN method has large representation & parallelization capacity due to the inherent property of deployment of multiple neural networks in the smaller subdomains. Unlike cPINN, XPINN can be extended to any type of PDEs. Moreover, the domain can be decomposed in any arbitrary way (in space & time), which is not possible in cPINN. Thus, XPINN offers both space & time parallelization, thereby reducing the training cost more effectively. The XPINN is particularly effective for the large-scale problems (involving large data set) as well as for the high-dimensional problems where single network based PINN is not adequate. The rigorous bounds on the errors resulting from the approximation of the nonlinear PDEs (incompressible NSEs) with PINNS & XPINNs are proved. However, DPINN debunks the use of residual (flux) matching at the domain interfaces as they hardly seem to improve the optimization.

1.1.4 Physics-informed neural networks & functional interpolation

X-TFC framework scheme for PDE solution learning. In the PINN framework, initial & boundary conditions are not analytically satisfied, thus they need to be included in the loss function of the network to be simultaneously learned with the differential equation (DE) unknown functions. Having competing objectives during the network's training can lead to unbalanced gradients while using gradient-based techniques, which causes PINNs to often struggle to accurately learn the underlying DE solution. This drawback is overcome by using functional interpolation techniques such as the Theory of Functional Connections (TFC)'s constrained expression, in the Deep-TFC framework, which reduces the solution search space of constrained problems to the subspace of neural network that analytically satisfies the constraints. A further improvement of PINN & functional interpolation approach is given by the Extreme Theory of Functional Connections (X-TFC) framework, where a single-layer Neural Network & the extreme learning machine training algorithm are employed. X-TFC allows to improve the accuracy & performance of regular PINNs, & its robustness & reliability are proved for stiff problems, optimal control, aerospace, & rarefied gas dynamics applications.

1.1.5 Physics-informed PointNet (PIPN) for multiple sets of irregular geometries

Regular PINNs are only able to obtain the solution of a forward or inverse problem on a single geometry. It means that for any new geometry (computational domain), one must retrain a PINN. This limitation of regular PINNs imposes high computational costs, specifically for a comprehensive investigation of geometric parameters in industrial designs. Physics-informed PointNet (PIPN) is fundamentally the result of a combination of PINN's loss function with PointNet. In fact, instead of using a simple fully connected neural network, PIPN uses Pointnet as the core of its neural network. PointNet has been primarily designed for deep learning of 3D object classification & segmentation by the research group of Leonidas J. Guibas. PointNet extracts geometric features of input computational domains in PIPN. Thus, PIPN is able to solve governing equations on multiple computational domains (rather than only a single domain) with irregular geometries, simultaneously. The effectiveness of PIPN has been shown for incompressible flow, heat transfer, & linear elasticity.

1.1.6 Physics-informed neural networks (PINNs) for inverse computations

Physics-informed neural networks (PINNs) have proven particularly effective in solving inverse problems within differential equations, demonstrating their applicability across science, engineering, & economics. They have shown useful for solving inverse problems in a variety of fields, including nano-optics, topology optimization/characterization, multiphase flow in porous media, & high-speed fluid flow. PINNs have demonstrated flexibility when dealing with noisy & uncertain observation datasets. They also demonstrated clear advantages in the inverse calculation of parameters for multi-fidelity datasets, meaning datasets with different quality, quantity, & types of observations. Uncertainties in calculations can be evaluated using ensemble-based or Bayesian-based calculations.

1.1.7 Physics-informed neural networks (PINNs) with backward stochastic differential equation

Deep backward stochastic differential equation method is a numerical method that combines deep learning with Backward stochastic differential equation (BSDE) to solve high-dimensional problems in financial mathematics. By leveraging the powerful function approximation capabilities of deep neural networks, deep BSDE addresses the computational challenges faced by traditional numerical methods like FDMs or Monte Carlo simulations, which struggle with the curse of dimensionality. Deep BSDE methods use neural networks to approximate solutions of high-dimensional PDEs, effectively reducing the computational burden. Additionally, integrating Physics-informed neural networks (PINNs) into the deep BSDE framework enhances its capability by embedding the underlying physical laws into the neural network architecture, ensuring solutions adhere to governing stochastic differential equations, resulting in more accurate & reliable solutions.

1.1.8 Physics-informed neural networks for biology

An extension for adaptation of PINNs are Biologically-informed neural networks (BINNs). BINNs introduce 2 key adaptations to the typical PINN framework:

(i) the mechanistic terms of the governing PDE are replaced by neural networks, &

(ii) the loss function $L_{\rm tot}$ is modified to include $L_{\rm constr}$, a term used to incorporate domain-specific knowledge that helps enforce biological applicability.

For (i), this adaptation has the advantage of relaxing the need to specify the governing differential equation a priori, either explicitly or by using a library of candidate terms. Additionally, this approach circumvents the potential issue of misspecifying regularization terms in stricter theory-informed cases.

A natural example of BINNs can be found in cell dynamics, where the cell density $u(t,\mathbf{x})$ is governed by a reaction-diffusion equation with diffusion & growth functions $D(u), G(u)$, respectively:

$$u_t = \nabla\cdot(D(u)\nabla u) + G(u)u,\ \mathbf{x}\in\Omega,\ t\in[0,T].$$

In this case, a component of $L_{\rm constr}$ could be $\|D\|_\Gamma$ for $D < D_{\min}$, $D > D_{\max}$, which penalizes values of D that fall outside a biologically relevant diffusion range defined by $D_{\min}\le D\le D_{\max}$. Furthermore, the BINN architecture, when utilizing multiplayer-perceptrons (MLPs), would function as follows: an MLP is used to construct $u_{\rm MLP}(t,\mathbf{x})$ from model inputs $(t,\mathbf{x})$, serving as a surrogate model for the cell density $u(t,\mathbf{x})$. This surrogate is then fed into the 2 additional MLPs, $D_{\rm MLP}(u_{\rm MLP}), G_{\rm MLP}(u_{\rm MLP})$, which model the diffusion & growth functions. Automatic differentiation can then be applied to compute the necessary derivatives of $u_{\rm MLP}, D_{\rm MLP}, G_{\rm MLP}$ to form the governing reaction-diffusion equation.

Note that since $u_{\rm MLP}$ is a surrogate for the cell density, it may contain errors, particularly in regions where the PDE is not fully satisfied. Therefore, the reaction-diffusion equation may be solved numerically, e.g. using a method-of-lines approach.

1.1.9 Limitations

Translation & discontinuous behavior are hard to approximate using PINNs. They fail when solving differential equations with slight advective dominance & hence asymptotic behavior causes the method to fail. Such PDEs could be solved by scaling variables. This difficulty in training of PINNs in advection-dominated PDEs can be explained by the Kolmogorov n-width of the solution. They also fail to solve a system of dynamical systems & hence have not been a success in solving chaotic equations. 1 of the reasons behind the failure of regular PINNs is soft-constraining of Dirichlet & Neumann boundary conditions which pose a multi-objective optimization problem which requires manually weighing the loss terms to be able to optimize. More generally, posing the solution of a PDE as an optimization problem brings with it all the problems that are faced in the world of optimization, the major one being getting stuck in local optima." – Wikipedia/physics-informed neural networks PINNs

2 C_0 Semigroup – Nửa Nhóm C_0

Resources – Tài nguyên.

1. [AK16]. Cung Thế Anh, Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng.*

"In mathematical analysis, a C_0-*semigroup*, also known as a *strongly continuous 1-parameter semigroup*, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant ODEs, strongly continuous semigroups provide solutions of linear constant coefficient ODEs in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations & PDEs. Formally, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+,+)$ on some Banach space X that is continuous in the strong toperator topology." -Wikipedia/C_0-semigroup

3 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

1. [Car16]. Manfredo P. do Carmo. *Differential Geometry of Curves & Surfaces.*
2. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries.*
3. [Küh15]. Wolfgang Kühnel. *Differential Geometry.*
4. [Wal15]. Shawn W. Walker. *The Shapes of Things.*

"Differential geometry is the detailed study of the *shape* of a surface (manifold), including *local* & *global* properties. A plane in $\mathbb{R}^3$ is a very simple surface & does not require many tools to characterize. An "arbitrarily" shaped surface, e.g., hood of a car, has many distinguished geometric features (e.g., highly curved regions, regions of near flatness, etc.). Characterizing these features quantitatively & qualitatively requires the tools of differential geometry. Geometric details are important in many physical & biological processes, e.g., surface tension, biomembranes.

The framework of differential geometry is built by 1st defining a local map (i.e., surface parameterization) which defines the surface. Then, a calculus framework is built up on the surface analogous to the standard "Euclidean calculus". Other approaches are also possible, e.g., those with implicit surfaces defined by level sets & distance functions. But parameterizations, though arbitrary, are quite useful in a variety of settings ⇒ stick mostly with those. Emphasize: The geometry of a surface does not depend on a particular parameterization. Otherwise, we will emphasize the distinction between `object 1` & `object 2`.

We will use this "abuse" of notation when there is no possibility of ambiguity.

Open set. The concept of open set is critical in multivariate calculus to properly define differentiability. The notation for referencing boundaries of sets, as well as the closure of sets, is practical for referencing geometric details of solid objects & their surfaces.

Compactness. Compact support is useful for ignoring boundary effects. This concept is needed to keep the "action of a function" away from the boundary of a set, or to localize the function in a region of interest. 1 reason is to avoid potential difficulties with differentiating a function at its boundary of definition. Or, more commonly, we wish to ignore a quantity depending on the value of a function at a boundary point, e.g., $\int_{\partial S} f = 0$ if f has compact support in S.

Topological mapping/homeomorphism. A bijective, continuous mapping $\mathbf{\Phi}$ whose inverse $\mathbf{\Phi}^{-1}$ is also continuous is called a *topological mapping* or *homeomorphism*. Point sets that can be topologically mapped onto each other are said to be *homeomorphic*. Sets that are homeomorphic have the "same topology", i.e., their connectedness is the same; they have the same kinds of "holes". See [Wal15, Sect. 2.3.1] for what can happen when a mapping is not a homeomorphism.

Rigid motion mapping. A mapping $\mathbf{\Phi}$ is called a *rigid motion* if any pair of points $\mathbf{a},\mathbf{b}$ are the same distance apart as the corresponding pair $\mathbf{\Phi}(\mathbf{a}),\mathbf{\Phi}(\mathbf{b})$.

Orthogonal Transformations. Define the (affine) linear map $\mathbf{\Phi}$ (transformation)

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{2}$$

If A satisfies the properties $A^{-1} = A^\top$, $\det A = 1$ then $\mathbf{\Phi}$ represents a rigid motion. Basically, $\mathbf{\Phi}$ consists of a rotation represented by A followed by a translation represented by $\mathbf{b}$. A rigid motion can be used to transition from 1 Cartesian coordinate system to another. If $\mathbf{b} = \mathbf{0}$ & $A^{-1} = A^\top$, $\det A = 1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is a linear map known as a *direct orthogonal transformation*, which is nothing more than a rotation of the coordinate system with the origin as the center. If $A^{-1} = A^\top$, $\det A = 1$ is replaced by $A^{-1} = A^\top$, $\det A = -1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is called an *opposite orthogonal transformation*, which consists of a rotation about the origin & a reflection in a plane. Both $A^{-1} = A^\top$, $\det A = \pm 1$ are examples of *orthogonal matrices*.

Interpretation of transformations. Can interpret $\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in 2 different ways. Consider a point $P \in \mathbb{R}^3$ with coordinates $\mathbf{x}$:

- *Alias* (Euler perspective). Viewing (2) as a transformation of coordinates, it appears that $\mathbf{x},\widetilde{\mathbf{x}}$ are the coordinates of the same point w.r.t. 2 different coordinate systems, equivalently, the point is referenced by 2 different "names" (sets of coordinates).
- *Alibi* (Lagrange perspective). Viewing (2) as a mapping of sets, it appears that $\mathbf{x},\widetilde{\mathbf{x}}$ are the coordinates of 2 different points w.r.t. the same coordinate system, equivalently, the point at $\widetilde{\mathbf{x}}$ "was previously" at $\mathbf{x}$ before applying the map.

The concept of material point is directly related to the alibi viewpoint. One can think of a "particle" of material, i.e., *material point*, initially located at $\mathbf{x}$, that then moves to $\widetilde{\mathbf{x}}$ because of some physical process. The transformation (2) simply represents the kinematic outcome of that physical process, which is a standard concept in deformable continuum mechanics, especially nonlinear elasticity.

General transformations. In general, transformation may not be linear. The alias viewpoint yields a *curvilinear* coordinate system. The alibi viewpoint implies that the set S is *deformed* into the set $S' = \mathbf{\Phi}(S)$.

Parametric approach – what is a surface? A *surface* is a set of points in space that is "regular enough". A random scattering of points in space does not match our intuitive notion of what a surface is, i.e., it is not regular enough. The boundary of a sphere does match our notion of a surface, i.e., regular enough to be a surface because a sphere is "smooth". *Intuition*: Can think of creating a surface as deforming a flat rubber sheet into a curved sheet. Let $U \subset \mathbb{R}^2$ be a "flat" domain & let $\mathbf{X} : U \to \mathbb{R}^3$ be this deforming transformation, i.e., for each point $(s_1, s_2)^\top \in U$ there is a corresponding point $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ s.t. $\mathbf{x} = \mathbf{X}(s_1, s_2)$. Let $\Gamma = \mathbf{X}(U)$ denote the surface obtained from "deforming" U. Call $\mathbf{x} = \mathbf{X}(s_1, s_2)$ a *parametric representation* of the surface Γ, where s_1, s_2 are called the *parameters* of the representation. Refer to U as a *reference* domain.

Allowable parameterization/immersion. If use $\mathbf{x} = \mathbf{X}(s_1, s_2)$ to define surfaces, then we must place assumptions on $\mathbf{X}$ to guarantee that $\Gamma = \mathbf{X}(U)$ is a valid surface. At the bare minimum, $\mathbf{X}$ must be continuous to avoid "tearing" the rubber sheet. But if want to perform calculus on Γ, need more:

Assumption 1 (Regularity assumptions on $\mathbf{X}$)**.** *An allowable parameterization/immersion is a parameterization of the form* $\mathbf{x} = \mathbf{X}(s_1, s_2)$ *satisfying:*

(A1) The function $\mathbf{X}(s_1, s_2) \in C^\infty(U)$ *& each point* $\mathbf{x} = \mathbf{X}(s_1, s_2) \in \Gamma$ *corresponds to just 1 point* $(s_1, s_2) \in U$*, i.e.,* $\mathbf{X}$ *is injective.*

(A2) The Jacobian matrix $J = [\partial_{s_1}\mathbf{X}, \partial_{s_2}\mathbf{X}]$ *is of rank* 2 *on* U*, i.e., the columns of* J *are linearly independent.*

Regular surface. The fundamental property that makes a set of points in $\mathbb{R}^3$ a surface is that it *locally looks like a plane* at every point. If you "zoom into" a surface, it should look flat. Definition defining a surface in terms of a parameterization is inadequate. Want to define a set in $\mathbb{R}^3$ that is "intrinsically" 2D & is smooth enough so we can perform calculus on it, without regard to a specific parameterization.

Definition 1 (Regular surface)**.**

Remark 1 (Local chart)**.**

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

3.1 Calculus on Surfaces

Goal. Define & develop the fundamental tools of calculus on a regular surface. Start with the notion of differentiability of functions defined only on a surface. Define the concept of vector fields in a surface. Then proceed to develop the gradient & Laplacian operators w.r.t. a surface. These operators allow for alternative expressions of the summed & Gaussian curvatures. Derive integration by parts on surfaces, i.e., the domain of integration is a surface. Conclude with some useful identities & inequalities. Always take Γ: a regular surface, either with or without a boundary.

4 Functional Analysis – Giải Tích Hàm

Resources – Tài nguyên.

1. [Rud91]. Walter Rudin. *Functional Analysis.*
2. Yosida.

5 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

1. [ABT18]. Richard Aster, Brian Borchers, Clifford H. Thurber. *Parameter Estimation & Inverse Problems.*
2. [Kir21]. Andreas Kirsch. *An Introduction to The Mathematical Theory of Inverse Problems.*
3. [IJ15]. Kazufumi Ito, Bangti Jin. *Inverse Problems.*

6 Measure & Integration – Độ Đo & Tích Phân

Resources – Tài nguyên.

1. [EG15]. Lawrence C. Evans, Ronald F. Gariepy. *Measure Theory & Fine Properties of Functions.*

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

"The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$-dimensional volume 'in the other direction' by 1st computing the n-dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y, which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*" – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

7 Mean-Field Game Theory – Lý Thuyết Trò Chơi Trường Trung Bình

Community – Cộng đồng. Nicholetta Tchou (French), Đào Mạnh Khang (Vietnamese), Michael Hintermüller (Austrian), Steven-Marian Stengl (German).

7.1 Wikipedia/mean-field game theory

"*Mean-field game theory* is the study of strategic decision making by small interacting agents in very large populations. It lies at the intersection of game theory with stochastic analysis & control theory. The use of the term "mean field" is inspired by mean-field theory in physics, which considers the behavior of systems of large numbers of particles where individual particles have negligible impacts upon the system. In other words, each agent acts according to his minimization or maximization problem taking into account other agents' decisions & because their population is large we can assume the number of agents goes to infinity & a representative agent exists.

In traditional game theory, the subject of study is usually a game with 2 players & discrete time space, & extends the results to more complex situations by induction. However, for games in continuous time with continuous states (differential games or stochastic differential games) this strategy cannot be used because of the complexity that the dynamic interactions generate. On the other hand with MFGs we can handle large numbers of players through the mean representative agent & at the same time describe complex state dynamics.

This class of problems was considered in the economics literature by Boyan Jovanovic & Robert W. Rosenthal, in the engineering literature by Minyi Huang, Roland Malhame, & Peter E. Caines & independently & around the same time by mathematicians Jean-Michel Lasry & Pierre-Louis Lions.

In continuous time a mean-field game is typically composed of a Hamilton-Jacobi-Bellman equation that describes the optimal control problem of an individual & a Fokker–Planck equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions it can be proved that a class of mean-field games is the limit as $N \to \infty$ of an N-player Nash equilibrium.

A related concept to that of mean-field games is "mean-field-type control". In this case, a social planner controls the distribution of states & chooses a control strategy. The solution to a mean-field-type control problem can typically be expressed as a dual adjoint Hamilton-Jacobi-Bellman equation coupled with Kolmogorov equation. Mean-field-type game theory is the multi-agent generalization of the single-agent mean-field-type control.

7.1.1 General Form of a Mean-field Game

The system of equations

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, m, Du) = 0, \\ \partial_t m - \nu\Delta m - \nabla\cdot(D_p H(x, m, Du)m) = 0, \\ m(0) = m_0, \\ u(T, x) = G(x, m(T)), \end{cases}$$

can be used to model a typical Mean-field game. The basic dynamics of this set of equations can be explained by an average agent's optimal control problem. In a mean-field game, an average agent can control their movement α to influence the population's overall location by

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t,$$

where ν: a parameter, B_t: a standard Brownian motion. By controlling their movement, the agent aims to minimize their overall expected cost C throughout the time period $[0, T]$:

$$C = \mathbb{E}\left[\int_0^T L(X_s, \alpha_s, m(s))\,\mathrm{d}s + G(X_T, m(T))\right],$$

where $L(X_s, \alpha_s, m(s))$ is the running cost at time s & $G(X_T, m(T))$ is the terminal cost at time T. By this definition, at time t & position x, the value function $u(t, x)$ can be determined as

$$u(t, x) = \inf_\alpha \mathbb{E}\left[\int_t^T L(X_s, \alpha_s, m(s))\,\mathrm{d}s + G(X_T, m(T))\right].$$

Given the definition of the value function $u(t, x)$, it can be tracked by the Hamilton-Jacobi equation. The optimal action of the average players $\alpha^*(t, x)$ can be determined as $\alpha^*(t, x) = D_p H(x, m, Du)$. As all agents are relatively small & cannot single-handedly change the dynamics of the population, they will individually adapt the optimal control & the population would move in that way. This is similar to a Nash Equilibrium, in which all agents act in response to a specific set of others' strategies. The optimal control solution then leads to the Kolmogorov-Fokker-Planck equation $\partial_t m - \nu\Delta m - \nabla\cdot(D_p H(x, m, Du)m) = 0$.

7.1.2 Finite State Games

A prominent category of mean field is games with a finite number of states & a finite number of actions per player. For those games, the analog of the Hamilton-Jacobi-Bellman equation is the Bellman equation, & the discrete version of the Fokker-Planck equation is the Kolmogorov equation. Specifically, for discrete-time models, the players' strategy is the Kolmogorv equation's probability matrix. In continuous time models, players have the ability to control the transition rate matrix.

A discrete mean field game can be defined by a tuple $\mathcal{G} = (\mathcal{E}, \mathcal{A}, \{Q_a\}, \mathbf{m}_0, \{c_a\}, \beta)$ where $\mathcal{E}$ is the state space, $\mathcal{A}$ the action set, Q_a the transition rate matrices, $\mathbf{m}_0$ the initial state, $\{c_a\}$ the cost functions & $\beta \in \mathbb{R}$ a discount factor. Furthermore, a mixed strategy is a measurable function $\pi : \mathbb{R}^+ \times \mathbb{E} \to \mathcal{P}(\mathcal{A})$, that associates to each state $i \in \mathcal{E}$ & each time $t \geq 0$ a probability measure $\pi_i(t) \in \mathcal{P}(\mathcal{A})$ on the set of possible actions. Thus $\pi_{i,a}(t)$ is the probability that, at time t a player in state i takes action a, under strategy π. Additionally, rate matrices $\{Q_a(\mathbf{m}^\pi(t))\}_{a\in\mathcal{A}}$ define the evolution over the time of population distribution, where $\mathbf{m}^\pi(t) \in \mathcal{P}(\mathcal{E})$ is the population distribution at time t.

7.1.3 Linear-quadratic Gaussian game problem

From Caines (2009), a relatively simple model of large-scale games is the linear-quadratic Gaussian model. The individual agent's dynamics are modeled as a stochastic differential equation

$$dX_i = (a_i X_i + b_i u_i)dt + \sigma_i dW_i, \; i = 1, \ldots, N,$$

where X_i: the state of the ith agent, u_i: control of the ith agent, W_i: independent Wiener processes $\forall i = 1, \ldots, N$. The individual agent's cost is

$$J_i(u_i, \nu) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}[(X_i - \nu)^2 + r u_i^2]\,\mathrm{d}t\right], \; \nu = \Phi\left(\frac{1}{N}\sum_{k\neq i}^N X_k + \eta\right).$$

The coupling between agents occurs in the cost function.

7.1.4 General & Applied Use

The paradigm of Mean Field Games has become a major connection between distributed decision-making & stochastic modeling. Starting out tin the stochastic control literature, it is gaining rapid adoption across a range of applications, including:

1. **Financial market.** Carmona reviews applications in financial engineering & economics that can be cast & tackled within the framework of the MFG paradigm. Carmona argues that models in macroeconomics, contract theory, finance, ..., greatly benefit from the switch to continuous time from the more traditional discrete-time models. He considers only continuous time models in his review chapter, including systemic risk, price impact, optimal execution, models for bank runs, high-frequency trading, & cryptocurrencies.

2. **Crowd motions.** MFG assumes that individuals are smart players which try to optimize their strategy & path w.r.t. certain costs (equilibrium with rational expectations approach). MFG models are useful to describe the anticipation phenomenon: the forward part describes the crowd evolution while the backward gives the process of how the anticipations are built. Additionally, compared to multi-agent microscopic model computations, MFG only requires lower computational costs for the macroscopic simulations. Some researchers have turned to MFG in order to model the interaction between populations & study the decision-making process of intelligent agents, including aversion & congestion behavior between 2 groups of pedestrians, departure time choice of morning commuters, & decision-making processes for autonomous vehicle.

3. **Control & mitigation of Epidemics.** Since the epidemic has affected society & individuals significantly, MFG & mean-field controls (MFCs) provide a perspective to study & understand the underlying population dynamics, especially in the context of the Covid-19 pandemic response. MFG has been used to extend the SIR-type dynamics with spatial effects or allowing for individuals to choose their behaviors & control their contributions to the spread of the disease. MFC is applied to design the optimal strategy to control the virus spreading within a spatial domain, control individuals' decisions to limit their social interactions, & support the government's nonpharmaceutical interventions." – Wikipedia/mean-field game theory

8 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

1. [Bre11]. Haïm Brezis. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations.*

2. [Eva10]. Lawrence C. Evans. *Partial Differential Equations.*

3. [GT01]. David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of 2nd Order.*

8.1 Weak solution – Nghiệm yếu

Definition 2 (Weak solution – Nghiệm yếu)**.** *"In mathematics, a* weak solution *(also called a* generalized solution*) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of distributions." – Wikipedia/weak solution*

"Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough." – Wikipedia/weak solution

Example 1 (1st-order wave equation)**.** *The 1st-order wave equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}^2$ with $u = u(t,x)$ has the weak form $\int_{\mathbb{R}^2} u\partial_t\varphi + u\partial_x\varphi\,\mathrm{d}t\,\mathrm{d}x = 0$ has a solution $u(t,x) = |t-x|$ which may be checked by splitting the integrals over region $\{x \ge t\}$ & $\{x \le t\}$ where u is smooth.*

"The notion of weak solution based on distribution is sometimes inadequate. In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. Hamilton-Jacobi equation, there is a very different definition of weak solution called *viscosity solution*." – Wikipedia/weak solution

8.1.1 General idea

When solving a differential equation in u, one can rewrite it using a test function φ s.t. whatever derivatives in u show up in the equation, they are "transferred" via integration by parts to φ, resulting in an equation without derivatives of u. This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x},\partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}),$$

where the multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$ varies over some finite set in $\mathbb{N}^d$ & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x},\partial)u(\mathbf{x} = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

where the differential operator $Q(\mathbf{x}, \partial)$ is given by the formula

$$Q(\mathbf{x}, \partial)\varphi(\mathbf{x}) = \sum (-1)^{|\alpha|}\partial^\alpha[a_\alpha(\mathbf{x})\varphi(\mathbf{x})],$$

which is the formal adjoint of $P(\mathbf{x}, \partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$-times differentiable function u defined on the open set W s.t. $P(\mathbf{x}, \partial)u(\mathbf{x}) = 0$, $\forall \mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x}, \partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$, $\forall \varphi \in C_c^\infty(W)$.

8.2 Viscosity solution – Nghiệm trơn/nhớt

Example 2 (Viscosity solution for Hamilton–Jacobi equation)**.** *Hamilton–Jacobi equation.*

8.3 Very weak solution – Nghiệm rất yếu

Example 3 (Very weak solution of porous medium equation (PME) [Váz07])**.** .

Example 4 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term [DDN20])**.** *Given $f \in L^1_\delta(\Omega), \lambda \ge 0$, a function $u \in L^1_\delta(\Omega)$ is called a* very weak solution *of*

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega,\\ |u|^{m-1}u = 1 & \text{on } \Gamma, \end{cases}$$

if $|u|^{m-1}u \in L^1(\Omega)$ and

$$\int_\Omega u^m\Delta\varphi + \lambda u\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi\,\mathrm{d}\mathbf{x} - \int_\Gamma \partial_{\mathbf{n}}\varphi\,\mathrm{d}\mathbf{x}.$$

Example 5 (Very weak solution of NSEs [Tsa18])**.** .

8.4 Navier–Stokes Equations

Resources – Tài nguyên.

1. [Lad69]. O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.*
2. [Soh01a; Soh01b]. Hermann Sohr. *The NSEs: An Elementary Functional Analytic Approach.*

   Primary objective. To develop an elementary & self-contained approach to the mathematical theory of a viscous incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$, described by NSEs. Formulate the theory for a completely general domain Ω.
3. [Tem77; Tem00]. Roger Temam. *NSES: Theory & Numerical Analysis.*
4. [Tsa18]. Tai-Peng Tsai. *Lectures on NSEs.*

9 Sobolev Spaces – Không Gian Sobolev

Resources – Tài nguyên.

1. [AF03]. Robert A. Adams, John J. F. Fournier. *Sobolev Spaces.*
2. [Gag57]. Emilio Gagliardo. *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.*
3. Nečăs.
4. [Tar06]. Luc Tartar. *An Introduction to Sobolev Spaces & Interpolation Spaces.*

10 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

1. [LeV07]. Randall J. LeVeque. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems.*

11 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

1. [BS08]. SUSANNE C. BRENNER, L. RIDGWAY SCOTT. *The Mathematical Theory of FEMs.*
2. [EG04]. ALEXANDRE ERN, JEAN-LUC GUERMOND. *Theory & Practice of Finite Elements.*
3. [GR86]. VIVETTE GIRAULT, PIERRE-ARNAUD RAVIART. *FEMs for NSEs.*
4. [Gun89]. MAX D. GUNZBURGER. *FEMs for Viscous Incompressible Flows.*
5. [Joh16]. VOLKER JOHN. *FEMs for Incompressible Flow Problems.*

I met VOLKER JOHN, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagonrinsky, k-ϵ & their simulations.

12 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

1. [EGH19]. ROBERT EYMARD, THIERRY GALLOUËT, RAPHAÈLE HERBIN. *Finite Volume Methods.*
2. [LeV02]. RANDALL J. LEVEQUE. *FEMs for Hyperbolic Problems.*

13 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

13.1 Wikipedia/Mathematician

Mathematician. Euclid (holding calipers), Greek mathematician, known as the "Father of Geometry"
Occupation.

- **Occupation type.** Academic

Description.

- **Competencies.** Mathematics, analytical skills and critical thinking skills.
- **Education required.** Doctoral degree, occasionally master's degree.
- **Fields of employment.**
  - universities,
  - private corporations,
  - financial industry,
  - government
- **Related jobs.** statistician, actuary.

A *mathematician* is someone who uses an extensive knowledge of mathematics in their work, typically to solve mathematical problems.

Mathematicians are concerned with numbers, data, quantity, structure, space, models, and change.

13.1.1 History

For broader coverage of this topic, see History of mathematics.

1 of the earliest known mathematicians was Thales of Miletus (c. 624–c.546 BC); he has been hailed as the 1st true mathematician and the 1st known individual to whom a mathematical discovery has been attributed.[Boyer (1991), *A History of Mathematics*, p. 43]

He is credited with the 1st use of deductive reasoning applied to geometry, by deriving 4 corollaries to Thales' Theorem.

The number of known mathematicians grew when Pythagoras of Samos (c. 582–c. 507 BC) established the Pythagorean School, whose doctrine it was that mathematics ruled the universe and whose motto was "All is number".[Boyer 1991, "*Ionia and the Pythagoreans*", p. 49]

It was the Pythagoreans who coined the term "mathematics", and with whom the study of mathematics for its own sake begins.

The 1st woman mathematician recorded by history was Hypatia of Alexandria (AD 350–415).

She succeeded her father as Librarian at the Great Library and wrote many works on applied mathematics.

Because of a political dispute, the Christian community in Alexandria punished her, presuming she was involved, by stripping her naked and scraping off her skin with clamshells (some say roofing tiles).["*Ecclesiastical History*, Bk VI: Chap. 15". Archived from the original on 2014-08-14. Retrieved 2014-11-19.]

Science and mathematics in the Islamic world during the Middle Ages followed various models and modes of funding varied based primarily on scholars.

It was extensive patronage and strong intellectual policies implemented by specific rulers that allowed scientific knowledge to develop in many areas.

Funding for translation of scientific texts in other languages was ongoing throughout the reign of certain caliphs,[Abattouy, M., Renn, J. & Weinig, P., 2001. *Transmission as Transformation: The Translation Movements in the Medieval East and West in a Comparative Perspective*. Science in Context, 14(1-2), 1-12.] and it turned out that certain scholars became experts in the works they translated and in turn received further support for continuing to develop certain sciences.

As these sciences received wider attention from the elite, more scholars were invited and funded to study particular sciences.

An example of a translator and mathematician who benefited from this type of support was al-Khawarizmi.

A notable feature of many scholars working under Muslim rule in medieval times is that they were often polymaths.

Examples include the work on optics, maths and astronomy of Ibn al-Haytham.

The Renaissance brought an increased emphasis on mathematics and science to Europe.

During this period of transition from a mainly feudal and ecclesiastical culture to a predominantly secular one, many notable mathematicians had other occupations: Luca Pacioli (founder of accounting); Niccolò Fontana Tartaglia (notable engineer and bookkeeper); Gerolamo Cardano (earliest founder of probability and binomial expansion); Robert Recorde (physician) and François Viètes (lawyer).

As time passed, many mathematicians gravitated towards universities.

An emphasis on free thinking and experimentation had begun in Britain's oldest universities beginning in the 17th century at Oxford with the scientists Robert Hooke and Robert Boyle, and at Cambridge where Isaac Newton was Lucasian Professor of Mathematics & Physics.

Moving into the 19th century, the objective of universities all across Europe evolved from teaching the "*regurgitation of knowledge*" to "*encourag[ing] productive thinking.*"[Röhrs, "*The Classical Idea of the University*," Tradition and Reform of the University under an International Perspective p. 20]

In 1810, Humboldt convinced the King of Prussia to build a university in Berlin based on Friedrich Schleiermacher's liberal ideas; the goal was to demonstrate the process of the discovery of knowledge and to teach students to "*take account of fundamental laws of science in all their thinking.*"

Thus, seminars and laboratories started to evolve.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 5–6]

British universities of this period adopted some approaches familiar to the Italian and German universities, but as they already enjoyed substantial freedoms and autonomy the changes there had begun with the Age of Enlightenment, the same influences that inspired Humboldt.

The Universities of Oxford and Cambridge emphasized the importance of research, arguably more authentically implementing Humboldt's idea of a university than even German universities, which were subject to state authority.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 12]

Overall, science (including mathematics) became the focus of universities in the 19th and 20th centuries.

Students could conduct research in seminars or laboratories and began to produce doctoral theses with more scientific content.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 13]

According to Humboldt, the mission of the University of Berlin was to pursue scientific knowledge.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 16]

The German university system fostered professional, bureaucratically regulated scientific research performed in well-equipped laboratories, instead of the kind of research done by private and individual scholars in Great Britain and France.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 17–18]

In fact, Rüegg asserts that the *German system is responsible for the development of the modern research university because it focused on the idea of "freedom of scientific research, teaching and study."*[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 31]

13.1.2 Required education

Mathematicians usually cover a breadth of topics within mathematics in their undergraduate education, and then proceed to specialize in topics of their own choice at the graduate level.

In some universities, a qualifying exam serves to test both the breadth and depth of a student's understanding of mathematics; the students, who pass, are permitted to work on a doctoral dissertation.

13.1.3 Activities

Emmy Noether, mathematical theorist and teacher.

Applied mathematics. Main article: Applied mathematics. Mathematicians involved with solving problems with applications in real life are called applied mathematicians.

Applied mathematicians are mathematical scientists who, with their specialized knowledge and professional methodology, approach many of the imposing problems presented in related scientific fields.

With professional focus on a wide variety of problems, theoretical systems, and localized constructs, applied mathematicians work regularly in the study and formulation of mathematical models.

Mathematicians and applied mathematicians are considered to be 2 of the STEM (science, technology, engineering, and mathematics) careers.

The discipline of applied mathematics concerns itself with mathematical methods that are typically used in science, engineering, business, and industry; thus, "applied mathematics" is a mathematical science with specialized knowledge.

The term "applied mathematics" also describes the professional specialty in which mathematicians work on problems, often concrete but sometimes abstract.

As professionals focused on problem solving, *applied mathematicians* look into the *formulation, study*, and *use of mathematical models* in science, engineering, business, and other areas of mathematical practice.

Abstract mathematics. Main article: Pure mathematics. Pure mathematics is mathematics that studies entirely abstract concepts.

From the 18th century onwards, this was a recognized category of mathematical activity, sometimes characterized as *speculative mathematics*,[See for example titles of works by Thomas Simpson from the mid-18th century: *Essays on Several Curious and Useful Subjects in Speculative and Mixed Mathematics, Miscellaneous Tracts on Some Curious and Very Interesting Subjects in Mechanics, Physical Astronomy and Speculative Mathematics. Chisholm*, Hugh, ed. (1911). "Simpson, Thomas". Encyclopædia Britannica. 25 (11th ed.). Cambridge University Press. p. 135.] and at variance with the trend towards meeting the needs of navigation, astronomy, physics, economics, engineering, and other applications.

Another insightful view put forth is that *pure mathematics is not necessarily applied mathematics*: it is possible to study abstract entities w.r.t. their intrinsic nature, and not be concerned with how they manifest in the real world.[Andy Magid, *Letter from the Editor, in Notices of the AMS*, Nov 2005, American Mathematical Society, p. 1173. [1] Archived 2016-03-03 at the Wayback Machine]

Even though the pure and applied viewpoints are distinct philosophical positions, in practice there is much overlap in the activity of pure and applied mathematicians.

To develop accurate models for describing the real world, many applied mathematicians draw on tools and techniques that are often considered to be "pure" mathematics.

On the other hand, many pure mathematicians draw on natural and social phenomena as inspiration for their abstract research.

Mathematics teaching. Many professional mathematicians also engage in the teaching of mathematics.

Duties may include:

- teaching university mathematics courses;
- supervising undergraduate and graduate research; and
- serving on academic committees.

Consulting. Many careers in mathematics outside of universities involve consulting.

E.g., actuaries assemble and analyze data to estimate the probability and likely cost of the occurrence of an event such as death, sickness, injury, disability, or loss of property.

Actuaries also address financial questions, including those involving the level of pension contributions required to produce a certain retirement income and the way in which a company should invest resources to maximize its return on investments in light of potential risk.

Using their broad knowledge, actuaries help design and price insurance policies, pension plans, and other financial strategies in a manner which will help ensure that the plans are maintained on a sound financial basis.

As another example, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to *financial theory*, taking observed market prices as input.

Mathematical consistency is required, not compatibility with *economic theory*.

Thus, e.g., while a financial economist might study the structural reasons why a company may have a certain share price, a financial mathematician may take the share price as a given, and attempt to use stochastic calculus to obtain the corresponding value of derivatives of the stock (see: Valuation of options; Financial modeling).

13.1.4 Occupations

In 1938 in the United States, mathematicians were desired as teachers, calculating machine operators, mechanical engineers, accounting auditor bookkeepers, and actuary statisticians.

According to the Dictionary of Occupational Titles occupations in mathematics include the following.["020 OCCUPATIONS IN MATHEMATICS". *Dictionary Of Occupational Titles*. Archived from the original on 2012-11-02. Retrieved 2013-01-20.]

- Mathematician
- Operations-Research Analyst
- Mathematical Statistician

- Mathematical Technician
- Actuary
- Applied Statistician
- Weight Analyst

13.1.5 Quotations about mathematicians

The following are quotations about mathematicians, or by mathematicians.

> "*A mathematician is a device for turning coffee into theorems.*" - Attributed to both Alfréd Rényi[15] and Paul Erdös

> "*Die Mathematiker sind eine Art Franzosen; redet man mit ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.*"
> (Mathematicians are [like] a sort of Frenchmen; if you talk to them, they translate it into their own language, and then it is immediately something quite different.) - Johann Wolfgang von Goethe[16]

> "*Each generation has its few great mathematicians... and [the others'] research harms no one.*" - Alfred W. Adler (∼1930), "*Mathematics and Creativity*"[17]

> "In short, I never yet encountered the mere mathematician who could be trusted out of equal roots, or one who did not clandestinely hold it as a point of his faith that x squared + px was absolutely and unconditionally equal to q. Say to one of these gentlemen, by way of experiment, if you please, that you believe occasions may occur where x squared + px is not altogether equal to q, and, having made him understand what you mean, get out of his reach as speedily as convenient, for, beyond doubt, he will endeavor to knock you down." - Edgar Allan Poe, *The purloined letter*

> "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas." - G. H. Hardy, *A Mathematician's Apology*

> "*Some of you may have met mathematicians and wondered how they got that way.*" - Tom Lehrer

> "*It is impossible to be a mathematician without being a poet in soul.*" - Sofia Kovalevskaya

> "*There are 2 ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else - but persistent.*" - Raoul Bott

> "*Mathematics is the queen of the sciences and arithmetic the queen of mathematics.*" - Carl Friedrich Gauss [18]

13.1.6 Prizes in mathematics

There is no Nobel Prize in mathematics, though sometimes mathematicians have won the Nobel Prize in a different field, such as economics.

Prominent prizes in mathematics include the Abel Prize, the Chern Medal, the Fields Medal, the Gauss Prize, the Nemmers Prize, the Balzan Prize, the Crafoord Prize, the Shaw Prize, the Steele Prize, the Wolf Prize, the Schock Prize, and the Nevanlinna Prize.

The American Mathematical Society, Association for Women in Mathematics, and other mathematical societies offer several prizes aimed at increasing the representation of women and minorities in the future of mathematics.

13.1.7 Mathematical autobiographies

Several well known mathematicians have written autobiographies in part to explain to a general audience what it is about mathematics that has made them want to devote their lives to its study.

These provide some of the best glimpses into what it means to be a mathematician.

The following list contains some works that are not autobiographies, but rather essays on mathematics and mathematicians with strong autobiographical elements.

- *The Book of My Life* - Girolamo Cardano[19]
- A Mathematician's Apology - G.H. Hardy[20]
- A Mathematician's Miscellany (republished as Littlewood's miscellany) - J. E. Littlewood[Littlewood, J. E. (1990) [Originally *A Mathematician's Miscellany* published in 1953], Béla Bollobás (ed.), *Littlewood's miscellany*, Cambridge University Press, ISBN 0-521-33702 X]
- *I Am a Mathematician* - Norbert Wiener[Wiener, Norbert (1956), *I Am a Mathematician / The Later Life of a Prodigy*, The M.I.T. Press, ISBN 0-262-73007-3]

- *I Want to be a Mathematician* - Paul R. Halmos

- *Adventures of a Mathematician* - Stanislaw Ulam[Ulam, S. M. (1976), *Adventures of a Mathematician*, Charles Scribner's Sons, ISBN 0-684-14391-7]

- *Enigmas of Chance* - Mark Kac[Kac, Mark (1987), *Enigmas of Chance/An Autobiography*, University of California Press, ISBN 0-520-05986-7]

- *Random Curves* - Neal Koblitz

- *Love and Math* - Edward Frenkel

- *Mathematics Without Apologies* - Michael Harris[Harris, Michael (2015), *Mathematics without apologies/portrait of a problematic vocation*, Princeton University Press, ISBN 978-0-691-15423-7]

13.1.8 See also

- Lists of mathematicians

- Human computer

- Mathematical joke

- A Mathematician's Apology

- Men of Mathematics (book)

- Mental calculator

□

13.2 Wikipedia/Henri Berestycki

Henri Berestycki (born Mar 25, 1951, in Paris)[1] is a French mathematician who obtained his PhD from Université Paris VI - Université Pierre et Marie Curie in 1975.

His Dissertation was titled *Contributions à l'étude des problèmes elliptiques non linéaires*, and his doctoral advisor was Haim Brezis.[2]

He was an L.E. Dickson Instructor in Mathematics at the University of Chicago from 1975–77, after which he returned to France to continue his research.

He has made many contributions in *nonlinear analysis*, ranging from *nonlinear elliptic equations, hamiltonian systems, spectral theory of elliptic operators*, and with applications to the description of *mathematical modelling of fluid mechanics and combustion.*

His current research interests include the mathematical modelling of financial markets, mathematical models in biology and especially in ecology, and modelling in social sciences (in particular, urban planning and criminology).

For these latter topics, he obtained an ERC Advanced grant in 2012.

He worked at the French National Center of Scientific Research (CNRS), then moved to an appointment as Professor at Univ. Paris XIII (1983–1985).

He became a Professor of Mathematics in 1988 at Université Pierre et Marie Curie, Paris VI (1988–2001 of "exceptional class" since 1993), and became Professor at Ecole normale supérieure, Paris (1994–1999), and part-time professor Ecole Polytechnique (1987–1999).

He is also a visiting Professor in the Department of Mathematics at the University of Chicago, and was also co-director of the Stevanovich Center of Financial Mathematics in Chicago.

He is currently the Directeur d'études (Research Professor) at École des hautes études en sciences sociales (EHESS), since 2001.

13.2.1 Services

- National Committee of French universities (1992–1995).

- Since 2002 director of Centre d'analyse et mathématique sociales (CAMS), CNRS -EHESS.

- Vice-president, EHESS (2004–2006).

- Member of the thesis prize committee of the universities of Paris (since 2006).

13.2.2 Awards

- Carrière Prize(1988)
- Prix Sophie Germain of the French Academy of Sciences (2004),
- Humboldt Prize in Germany (2004)
- French Legion of Honor in 2010.
- American Mathematical Society Fellowship (2012).[3]
- Foreign honorary member of the American Academy of Arts and Sciences, 2013.[4]

13.2.3 Articles

- Berestycki, Henri; Roquejoffre, Jean-Michel; Rossi, Luca; The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol.* 66 (2013), no. 4-5, 743–766.
- Barthélemy, Marc; Nadal, Jean-Pierre; Berestycki, Henri Disentangling collective trends from local dynamics. *Proc. Natl. Acad. Sci. USA* 107 (2010), no. 17, 7629–7634.
- Berestycki, Henri; Hamel, François; Nadirashvili, Nikolai Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Comm. Math. Phys.* 253 (2005), no. 2, 451–480.
- Berestycki, Henri; Hamel, François Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* 55 (2002), no. 8, 949–1032.
- Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* 81 (1996), no. 2, 467–494.
- Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S. The principal eigenvalue and maximum principle for 2nd-order elliptic operators in general domains. *Comm. Pure Appl. Math.* 47 (1994), no. 1, 47–92.
- Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345; II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 347–375.
- Bahri, Abbas; Berestycki, Henri A perturbation method in critical point theory and applications. *Trans. Amer. Math. Soc.* 267 (1981), no. 1, 1–32. □

13.3 Wikipedia/Haïm Brezis

Haïm Brezis.

- **Born.** Jun 1m 1944 (age 76). Riom-ès-Montagnes, Cantal, France.
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.**
  - Brezis-Gallouet inequality
  - Bony-Brezis theorem
  - Brezis-Lieb lemma

Scientific career.

- **Fields.** Mathematics.
- **Institutions.** Pierre and Marie Curie University.
- **Doctoral advisor.**
  - Gustave Choquet
  - Jacques-Louis Lions
- **Doctoral students.**
  - Abbas Bahri
  - Henri Berestycki
  - Jean-Michel Coron

- Jesús Ildefonso Díaz
- Pierre-Louis Lions
- Juan Luis Vázquez Suárez

Haïm Brezis (born Jun 1, 1944) is a French mathematician who works in functional analysis and partial differential equations.

13.3.1 Biography

Born in Riom-ès-Montagnes, Cantal, France.

Brezis is the son of a Romanian immigrant father, who came to France in the 1930s, and a Jewish mother who fled from the Netherlands.

His wife, Michal Govrin, a native Israeli, works as a novelist, poet, and theater director.[1]

Brezis received his Ph.D. from the University of Paris in 1972 under the supervision of Gustave Choquet.

He is currently a Professor at the Pierre and Marie Curie University and a Visiting Distinguished Professor at Rutgers University.

He is a member of the Academia Europaea (1988) and a foreign associate of the United States National Academy of Sciences (2003).

In 2012 he became a fellow of the American Mathematical Society.[2]

He holds honorary doctorates from several universities including National Technical University of Athens.[3]

Brezis is listed as an ISI highly cited researcher.[4]

13.3.2 Works

- *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (1973)
- *Analyse Fonctionnelle.* Théorie et Applications (1983)
- *Haïm Brezis. Un mathématicien juif.* Entretien Avec Jacques Vauthier. Collection Scientifiques & Croyants. Editions Beauchesne, 1999. ISBN 978-2-7010-1335-0, ISBN 2-7010-1335-6
- *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer; 1st Edition. edition (November 10, 2010), ISBN 978-0-387-70913-0, ISBN 0-387-70913-4

13.3.3 See also

- Bony-Brezis theorem
- Brezis-Gallouet inequality
- Brezis-Lieb lemma □

13.4 Lawrence Chris Evans

13.5 Wikipedia/Herbert Federer

Herbert Federer (Jul 23, 1920 – Apr 21, 2010)["NAS Membership Directory: Federer, Herbert". National Academy of Sciences. Retrieved Jun 15, 2010.] was an American mathematician.

He is 1 of the creators of geometric measure theory, at the meeting point of differential geometry and mathematical analysis.[Parks, H. (2012) *Remembering Herbert Federer (1920–2010)*, NAMS 59(5), 622–631.]

13.5.1 Career

Federer was born Jul 23, 1920, in Vienna, Austria.

After emigrating to the US in 1938, he studied mathematics and physics at the University of California, Berkeley, earning the Ph.D. as a student of Anthony Morse in 1944.

He then spent virtually his entire career as a member of the Brown University Mathematics Department, where he eventually retired with the title of Professor Emeritus.

Federer wrote more than 30 research papers in addition to his book *Geometric measure theory*.

The Mathematics Genealogy Project assigns him 9 Ph.D. students and well over a hundred subsequent descendants.

His most productive students include the late Frederick J. Almgren, Jr. (1933–1997) a professor at Princeton for 35 years, and his last student, Robert Hardt, now at Rice University.

Federer was a member of the National Academy of Sciences.

In 1987, he and his Brown colleague Wendell Fleming won the American Mathematical Society's Steele Prize "for their pioneering work in *Normal & Integral currents*."

13.5.2 Normal & integral currents

Federer's mathematical work separates thematically into the periods before and after his watershed 1960 paper *Normal and integral currents*, co-authored with Fleming.

That paper provided the *1st satisfactory general solution to Plateau's problem* - the problem of finding a $(k+1)$-dimensional least-area surface spanning a given k-dimensional boundary cycle in n-dimensional Euclidean space.

Their solution inaugurated a new and fruitful period of research on a large class of *geometric variational problems* - especially *minimal surfaces* - via what came to be known as *Geometric Measure Theory.*

13.5.3 Earlier work

During the 15 or so years prior to that paper, Federer worked at the technical interface of geometry and measure theory.

He focused particularly on surface area, rectifiability of sets, and the extent to which one could substitute rectifiability for smoothness in the analysis of surfaces.

His 1947 paper on the *rectifiable subsets of n-space* characterized purely unrectifiable sets by their "invisibility" under almost all projections.

A. S. Besicovitch had proven this for 1-dimensional sets in the plane, but Federer's generalization, valid for subsets of arbitrary dimension in any Euclidean space, was a major technical accomplishment, and later played a key role in *Normal and Integral Currents.*

In 1958, Federer wrote *Curvature Measures*, a paper that took some early steps toward understanding 2nd-order properties of surfaces lacking the differentiability properties typically assumed in order to discuss curvature.

He also developed and named what he called the coarea formula in that paper.

That formula has become a standard analytical tool.

13.5.4 Geometric measure theory

Federer is perhaps best known for his treatise *Geometric Measure Theory*, published in 1969.[Goffman, Casper (1971). "Review: Geometric measure theory, by Herbert Federer" (PDF). Bull. Amer. Math. Soc. 77 (1): 27–35. doi:10.1090/s0002-9904-1971-12603-4.]

Intended as both a text and a reference work, the book is unusually complete, general and authoritative: its nearly 600 pages cover a substantial amount of linear and multilinear algebra, give a profound treatment of measure theory, integration and differentiation, and then move on to rectifiability, theory of currents, and finally, variational applications.

Nevertheless, the book's unique style exhibits a rare and artistic economy that still inspires admiration, respect - and exasperation.

A more accessible introduction may be found in F. Morgan's book listed below.

13.5.5 See also

- Integral current
- Federer-Morse theorem

13.5.6 External links

- Federer's page a Brown □

13.6 Peter Lax

13.7 Jacques-Louis Lions

- **Born.** May 3, 1928. Grasse, Alpes-Maritimes, France.
- **Died.** May 17, 2001 (aged 73).
- **Nationality.** French.
- **Alma mater.** University of Nancy.
- **Known for.** PDEs.
- **Awards.** Japan Prize (1991).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - École Polytechnique

- Collège de France

- **Doctoral advisor.** Laurent Schwartz.
- **Doctoral students.**
  - Alain Bensoussan
  - Jean-Michel Bismut
  - Haïm Brezis
  - Erol Gelenbe
  - Roland Glowinski
  - Roger Temam

Jacques-Louis Lions ([1] 3 May 1928 - May 17, 2001) was a French mathematician who made contributions to the theory of partial differential equations and to stochastic control, among other areas.

He received the SIAM's John von Neumann Lecture prize in 1986 and numerous other distinctions.[2][3]

Lions is listed as an ISI highly cited researcher.[4]

13.7.1 Biography

After being part of the French Résistance in 1943 and 1944, J.-L. Lions entered the École Normale Supérieure in 1947.

He was a professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, and the École polytechnique.

In 1966 he sent an invitation to Gury Marchuk, the soviet mathematician to visit Paris.

This was hand delivered by General De Gaulle during his visit to Akademgorodok in June of that year.[5]

He joined the prestigious Collège de France as well as the French Academy of Sciences in 1973.

In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught and promoted the use of numerical simulations using finite elements integration.

Throughout his career, Lions insisted on the *use of mathematics in industry*, with a particular involvement in the French space program, as well as in domains such as energy and the environment.

This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992.

Lions was elected President of the International Mathematical Union in 1991 and also received the Japan Prize and the Harvey Prize that same year.[3]

In 1992, the University of Houston awarded him an honorary doctoral degree.

He was elected president of the French Academy of Sciences in 1996 and was also a Foreign Member of the Royal Society (ForMemRS)[6] and numerous other foreign academies.[2][3]

He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English and Russian, and major contributions to several collective works, including the 4000 pages of the monumental *Mathematical analysis and numerical methods for science and technology* (in collaboration with Robert Dautray), as well as the *Handbook of numerical analysis* in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994.[7]

Both father and son have received honorary doctorates from Heriot-Watt University in 1986 and 1995 respectively.[8]

13.7.2 Books

- with Enrico Magenes: *Problèmes aux limites non homogènes et applications*. 3 vols., 1968, 1970
- *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. 1968
- with L. Cesari: *Quelques méthodes de résolution des problèmes aux limites non linéaires*. 1969
- with Roger Dautray: *Mathematical analysis and numerical methods for science and technology*. 9 vols., 1984/5
- with Philippe Ciarlet: *Handbook of numerical analysis*. 7 vols.
- with Alain Bensoussan, Papanicolaou: *Asymptotic analysis of periodic structures*. North Holland 1978
- *Controlabilité exacte, perturbations et stabilisation de systèmes distribués*[9]
- with John E. Lagnese: *Modelling Analysis and Control of Thin Plates*. □

13.8 Andrew Joseph Majda

Resources – Tài nguyên.

1. [Eng+23]. BJORN ENGQUIST, PANAGIOTIS SOUGANIDIS, SAMUEL N. STECHMANN, VLAD VICOL. *In memory of Andrew J. Majda*.

   "He was hard working until the end even though he suffered from serious health issues for quite some time."

   "He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, and observed and experimental data ... Andy Majda's modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis and numerical modeling, (iii) observed phenomena and experimental data, and (iv) qualitative and/or asymptotic modeling [Maj00]."

   "Andy's legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored."

   "The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed "Majda's model" for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived "Beale-Kato-Majda criterion," which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84]."

   "At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science."

13.9 Vladimir Mazya

13.10 Jindřich Nečas

13.11 Louis Nirenberg

Resources – Tài nguyên.

1. [**Vazquez2020**]. JUAN LUIS VÁZQUEZ. *Remembering Louis Nirenberg & His Mathematics.*

Abstract

The article is dedicated to recalling the life and mathematics of Louis Nirenberg, a distinguished Canadian mathematician who recently died in New York, where he lived.

An emblematic figure of analysis and PDEs in the last century, he was awarded the Abel Prize in 2015.

From this watchtower at the Courant Institute in New York, he was for many years a global teacher and master.

He was a good friend of Spain.

> *1 of the wonders of mathematics is you go somewhere in the world & you meet other mathematicians, and it is like 1 big family.*
> *This large family is a wonderful joy.*[1]

13.11.1 Introduction

This article is dedicated to remembering the life and work of the prestigious Canadian mathematician Louis Nirenberg, born in Hamilton, Ontario, in 1925, who died in New York on Jan 26, 2020, at the age of 94.

Professor for much of his life at the mythical Courant Institute of New York University, he was considered 1 of the best mathematical analysts of the 20th century, a specialist in the analysis of PDEs.

When the news of his death was received, it was a very sad moment for many mathematicians, but it was also the opportunity of reviewing an exemplary life and underlining some of its landmarks.

His work unites diverse fields between what is considered Pure Mathematics and Applied Mathematics, and in particular he was cult figure in the discipline of PDEs, a key theory and tool in the mathematical formulation of many processes in science, in engineering, and in other branches of mathematics.

His work is a prodigy of sharpness and logical perfection, and at the same time its applications span today multiple scientific areas.

In recognition of his work, in 2015 he received the Abel Prize along with the another great mathematician, John Nash.

The Abel Prize is 1 of the greatest awards in Mathematics, comparable to the Nobel prizes in other sciences.

At that time, the Courant Institute, where he was for so many decades a renowned professor, published an article called *Beautiful Minds*[2] which is quite enjoyable reading.

He was a distinguished member of the AMS (American Mathematical Society).

Throughout his life, he received many other honors and awards, e.g. the AMS Bôcher Memorial Prize (1959), the Jeffery-Williams Prize (1987), the Steele Prize for Lifetime Achievement (1994 and 2014), the National Medal of Science (1995), the inaugural Crafoord Prize from the Royal Swedish Academy (1982), and the 1st Chern medal at the 2010 International Congress of Mathematicians, awarded by the International Mathematical Union and the Chern Foundation.

[1] From an interview with Louis Nirenberg appeared in *Notices of the AMS*, 2002, [43]

[2] Beautiful Minds: Courant's Nirenberg, Princeton's John Nash Win Abel Prize in Mathematics .

He was a plenary speaker at the International Congress of Mathematicians held in Stockholm in Aug 1962; the title of the conference was *"Some Aspects of Linear & Nonlinear PDE"*.

In 1969 he was elected Member of the U.S. National Academy of Sciences.

It was not honors that concerned him most, but rather his profession and the mathematical community that surrounded him.

In his long career at the Courant Institute he discovered many mathematical talents and collaborated in numerous relevant works with distinguished colleagues.

A wise man in science and life, he was 1 of the most influential and beloved mathematicians of the last century, and the current century too.

His teaching extended 1st to the international centers that he loved to visit, and then to the entire world.

Indeed, we live at this height of time in a world-wide scientific society whose close connection brings so many benefits to the pursuit of knowledge.

Many of his articles are among the most cited in the world.[3]

13.11.2 Starting

In order to start the tour of his mathematics, nothing better than to quote a few paragraphs from the mention of the Abel Prize Committee in 2015:[4]

Fig. Louis Nirenberg receiving the Abel Prize from King Harald V of Norway in the presence of John Nash (photo: Berit Roald/NTB scanpix).

Mathematical giants:

> *Nash and Nirenberg are 2 mathematical giants of the 20th century.*
>
> *They are being recognized for their contributions to the field of PDEs, which are equations involving rates of change that originally arose to describe physical phenomena but, as they showed, are also helpful in analyzing abstract geometrical objects.*

The Abel committee writes:

> *"Their breakthroughs have developed into versatile and robust techniques that have become essential tools for the study of nonlinear PDEs.*
>
> *Their impact can be felt in all branches of the theory."*

About Louis they say:

> *"Nirenberg has had 1 of the longest and most fêted careers in mathematics, having produced important results right up until his 70s.*
>
> *Unlike Nash, who wrote papers alone, Nirenberg preferred to work in collaboration with others, with more than 90% of his papers written jointly.*
>
> *Many results in the world of elliptic PDEs are named after him and his collaborators, e.g. the Gagliardo–Nirenberg inequalities, the John–Nirenberg inequality and the Kohn–Nirenberg theory of pseudo-differential operators."*

They conclude:

> *"Far from being confined to the solutions of the problems for which they were devised, the results proven by Nash and Nirenberg have become very useful tools and have found tremendous applications in further contexts."*

To be precise, Nirenberg made fundamental contributions to both linear and nonlinear PDEs, functional analysis, and their application in geometry and complex analysis.

Among the most famous contributions we will discuss are the Gagliardo-Nirenberg interpolation inequality, which is important in solving the elliptic PDEs that arise within many areas of mathematics; the formalization of the BMO spaces of bounded mean oscillation, and others that we will be seeing.

A work of utmost relevance was the work with Luis Caffarelli and Robert Kohn aimed at solving the big open problem of existence and smoothness of the solutions of the Navier-Stokes system of fluid mechanics.

This work was described by the AMS in 2002 as "1 of the best ever done."

The problem is on the Millennium Problems List (the list compiled by the Clay Foundation), and is 1 of the most appealing open problems of mathematical physics, raised nearly 2 centuries ago.

Fermat's Last Theorem and the Poincaré Conjecture have been defeated at the turn of the century, but the Navier-Stokes enigma (and in some sense its companion about the Euler's system) keep defying us.

We will deal with the issue in detail in Section 4.

[3]Topic 35, PDEs, from the mathematical database Mathscinet, includes 3 articles by L. Nirenberg among the 10 most cited ever.

[4]John F. Nash, Jr. and Louis Nirenberg share the Abel Prize.

The beginnings. From Canada to New York. Louis Nirenberg grew up in Montréal, where his fatehr was a Hebrew teacher.

After graduating[5] in 1945 at McGill University, Montréal, Louis found a summer job at the National Research Council of Canada, where he met the physicist Ernest Courant, the son of Richard Courant, a famous professor at New York University.

Ernest mentioned to Nirenberg that he was going to New York to see his father and Louis begged him for advice on a good place to apply for a master study in physics.

He returned with Richard Courant's invitation for Louis to go to New York University (NYU) for a master's degree in mathematics, after which he would be prepared for a physics program.

But once Louis began studying Mathematics at NYU, he never changed.

He defended his doctoral thesis under James Stoker in 1949, solving a problem in differential geometry.

The dice were cast.

We reach a crucial moment in Louis's life.

Breaking with the golden rule[6] according to which "a recent doctor should move to a different environment", Richard Courant kept his best students around him, including Louis Nirenberg, and he thus created the NYU Mathematical Institute, the famous Courant Institute, which has become a world benchmark for high mathematics, comparable only to the Princeton Institute for Advanced Study on the East Coast of the USA.

Louis was 1st a postdoc and then a permanent member of the faculty.

There he thrived and spent his life.

Equations & Geometry. The problem Stoker gave to Louis for his thesis, entitled *"The Determination of a Closed Convex Surface Having Given Line Elements"*, is called "the embedding problem" or "Weyl Problem".

It can be stated as follows: Given a 2D sphere with a Riemannian metric s.t. the Gaussian curvature is positive everywhere, the question is whether a surface can be constructed in 3D space so that the Riemannian distance function coincides with the distance inherited from the usual Euclidean distance in the 3D space (in other words, whether there is an isometric embedding as a convex surface in $\mathbb{R}^3$).

The great German mathematician Hermann Weyl had taken a significant 1st step to solve the problem in 1916, and Nirenberg, as a student, completed Weyl's construction.

The work to do was to solve a system of nonlinear PDEs of the so-called "elliptic type".

It is the kind of equation and application that Louis Nirenberg has been working on ever since.

Progress has been slow but continued over time and is impressive at this moment.[7]

13.11.3 The power & beauty of inequalities

Focus on 1 of the most relevant topics in Louis Nirenberg's broad legacy, at the same time closest to our mathematical interest.

(Almost) every career in PDEs begin with the study of linear elliptic equations.

These form nowadays a well-established theory which combines Functional Analysis, Calculus of Variations, and explicit representations to produce solutions in suitable functional spaces.

For the classical equilibrium equations in the mechanics of continuous media, known as Laplace's and Poisson's equations, in symbols $\Delta u = f$, there is a classical "maximum principle" that provides the necessary estimates that guarantee the existence and uniqueness of solutions.

When combined with skillful tricks of the trade, it makes possible to obtain finer estimates, e.g. regularity and other properties.

Let us mention the estimates known under the names Harnack and Schauder, cf. [Eva10; GT01].

In this regard, Nirenberg is quoted as saying, either jokingly or seriously,

> *"I made a living off the maximum principle."*[8]

Many of the interesting problems that are proposed in Physics and other sciences and involve PDEs are **nonlinear**, e.g. the fluid equations or the curvature problems in geometry.

These nonlinear problems can seldom be solved by explicit formulas.

Because of that difficulty, the mathematical study of these problems has attracted increasing attention from the best mathematical minds of the past century, with remarkable success stories.

The usual approach goes as follows: the solution has to be obtained by some kind of approximation, and an essential technical point is usually to show that the proposed approximation procedure (or procedures) converge to a solution[9].

A complicated topology and functional analysis machinery has been developed over time and is available to test such convergence, provided certain estimates are fulfilled; their role is to allow for the approximation to be controlled.

See in this sense the book that many of us have studied as young people [Bre11].

[5] With a degree in mathematics and physics, also in mathematics being bilingual counts.

[6] which is an essential part of the American professional practice.

[7] Isometrically embedding low dimensional manifolds into higher dimensional Euclidean spaces is the contents of a famous paper by J. Nash in 1956.

[8] Curiously, it applies to Vázquez too.
Vázquez's most read article deals with the "Strong Maximum Principle", [74].

[9] Taken in some sense acceptable to physics, e.g., the solution in the weak sense or the solution in the distributional sense.

Much of the work of an "EDP Analyst"[10] consists in finding estimates that control the passage to the limit that has to be applied, or to find a convenient fixed point theorem.

A common saying in our trade goes as follows: *Existence theorems come from a priori estimates and suitable functional analysis.*

Estimate, this is the key word in the world that Louis Nirenberg and his colleagues bequeathed us.

"Estimate" means the same thing as "inequality", and here Vázquez refers of course to a functional or numerical inequality.

It may look surprising to the reader, even weird, to find it so clearly stated: Inequalities, and not equalities (or identities), are the technical core of such a central theory of mathematics as PDEs.

However, this is precisely the mathematical revolution that was in the making when Louis was young.

Indeed, when he arrived at NYU, the most active and renowned researcher was probably Kurt Otto Friedrichs, who decisively influenced Nirenberg's future research career.

Friedrichs loved inequalities, as Louis put it:

> *"Friedrichs was a great lover of inequalities and that affected me very much.*
> *The point of view was that the inequalities are more interesting than the equalities."*

Carrying forward on that idea, Nirenberg has been unanimously recognized as a world master of inequalities".

Here is another saying by Louis:

> *"I love inequalities.*
> *So if somebody shows me a new inequality, I say: "Oh, that's beautiful, let me think about it," and I may have some ideas connected to it."*

For many years, mathematicians from all over the world came to the Courant Institute to seek his advice on issues involving inequalities.

And there we are.

We do not reject or despise the beauty of the exact solution if there is one, but functional inequalities are our firm support in an uncertain world that is yet to be discovered and described.

The key technical point of modern PDE theory is to establish the most needed and appropriate estimates in the strongest possible way.

Sobolev, Gagliardo & Nirenberg. There are many types of estimates the researcher needs in the study of nonlinear PDEs, but some have turned out to be much more relevant than others.

Vázquez will talk here about a type that has become particularly famous and useful.

They are often collectively called "Sobolev estimates" in honor of the great Russian mathematician Sergei Lvovich Sobolev because of his seminal work [68], 1938.

Briefly stated, they estimate the norms of functions belonging to the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, in terms of their (weak) derivatives of various orders.

In 1959 Emilio Gagliardo [35] and Louis Nirenberg [59] gave an independent and very simple proof of the following inequality:

Fig. The Talenti profile for different values of the parameters.

Theorem 1 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *Let $1 \le p < n$. There exists a constant $C > 0$ s.t. the following inequality*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \; p^* := \frac{np}{n-p},$$

holds true for all functions $u \in C_c^1(\mathbb{R}^n)$. The constant C depends only on p and n. The exponent p^* is called the Sobolev conjugate *of p. Du denotes the gradient vector.*

Gagliardo and Nirenberg included as their starting point the important case of exponent $p = 1$, left out by Sobolev.

The inequality implies the continuous inclusion of the Banach space called $W^{1,p}(\mathbb{R}^n)$ into $L^{p^*}(\mathbb{R}^n)$ (immersion theorem).

Versions for functions defined in bounded open sets $\mathbb{R}^n$ followed naturally.

This inequality soon attracted multiple applications and a wide array of variants and improvements.

Very interesting versions deal with functions defined on Riemannian manifolds.

Vázquez comments below 4 additional aspects that he finds appropriate for the curious reader.

(i) Thierry Aubin [3] and Giorgio Talenti [72] obtained in 1976 the *best constant* in this inequality, finding the functions that exhibit the *worst behavior*[11]

Indeed, when $1 < p < n$ the maximum quotient $\frac{\|u\|_{L^{p^*}(\mathbb{R}^n)}}{\|Du\|_{L^p(\mathbb{R}^n)}}$ is optimally realized by the function

$$U(\mathbf{x}) = \left(a + b|\mathbf{x}|^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}},$$

[10] *Analysis of PDEs* is an area of Mathematics in the US that perfectly describes our specialty which is neither pure nor applied, and does not need to declare itself in either direction.

Such a denomination is not much used in Spain and other countries; i.e., in Vázquez's opinion, the source of some persistent malfunctions.

[11] This is an apparent grammatical contradiction that gives rise to beautiful functions.

where $a, b > 0$ are arbitrary constants.[12]

It is the famous *Talenti profile*.

Note that $\frac{n-p}{p} = \frac{n}{p^*}$.

It happens that U is a probability density (integrable) if $\frac{n-p}{p-1} > n$, i.e., if $1 < p < p_c = \frac{2n}{n+1}$.

The U profile and its power appear recurrently in PDEs.

Thus, in nonlinear diffusion we find it as a power of the Barenblatt profile in fast diffusion, see Chap. 11 of [75], and the curiously critical exponent p_c also appears, but with consequences that go in the converse direction.

- Gagliardo and Nirenberg's work extends to the famous *Gagliardo-Nirenberg interpolation inequality*, a result in Sobolev's theory of spaces that estimates a certain norm of a function in terms of a product of norms of functions and derivatives thereof.

  We enter here a realm of higher complexity.[13]

  See details in [10].

(iii) In 1984 Luis Caffarelli, Bob Kohn and Louis Nirenberg needed inequalities of the previous type in functional Lebesgue spaces but with the novelty of including so-called *weights*, and this motivated the article [18], on the famous *CKN estimates originated* for spaces with power weights.

This was the beginning of an extensive literature.

A very striking effect arose in those studies: unlike GNS inequalities, there exists a phenomenon of symmetry breaking in the CKN inequalities, i.e., minimizers of such inequalities need not be symmetric functions, even when posed in the whole space or in balls.

The exact range of parameters for the symmetry breaking was found by J. Dolbeault, M. J. Esteban and M. Loss in [29].

(iv) In 2004 D. Cordero-Erausquin, B. Nazareth and C. Villani [24] used mass transport methods to obtain sharp versions of the Sobolev-Gagliardo-Nirenberg inequalities.

Mass transport is 1 of the most powerful new instruments used in PDE research.

This topic is related to the isoperimetric inequalities of ancient fame.[14] that now live moments of fruitful coincidence with Sobolev theory.

The survey [15] talks about this relationship.

The world of estimates that we have outlined has came to be an enormous space presided over by quite distinguished names, like H. Poincaré, J. Nash, G. H. Hardy, C. Morrey, J. Moser, N. Trudinger and other remarkable figures.

Hardy-Littlewood-Pólya's book [41] had a great influence on generations of analysts.

A commendable book on the importance of inequalities in Physics is the 2nd volume of Elliott Lieb's selected works, [53].

As a representative example chosen from among the numerous recent works, Vázquez mentions the arcile by M. del Pino and Jean Dolbeault [25].

It establishes a new optimal version of the Euclidean Gagliardo-Nirenberg inequalities.

This allows the authors to obtain the convergence rates to the equilibrium profiles of some nonlinear diffusion equations, e.g. those of the "porous media" type, 1 of the leitmotifs of Vázquez's research.

The authors completed the study and application with 2 new articles in 2003.

New functional inequalities based on entropy, maximum principles, and symmetrization processes allowed a group of Vázquez to find convergence rates for very fast diffusion equations in [7], thus solving in 2009 a much studied open problem.

It was almost 3 years of work by a team of 5 people.

Plus the work of previous authors.

Finally, there is a great deal of activity in the world of Sobolev spaces of fractional order (also called *Slobodeckii spaces*), and the associated fractional diffusions, cf. [21, 27].

It is a topic in full swing, a part of Vázquez's current mathematical efforts.[15]

New Spaces. John-Nirenberg space. Go back for a moment to the origins.

The limiting case of the Gagliardo-Nirenberg-Sobolev inequality happens for $p = n$.

Thanks to new inequalities due to C. Morrey, we know that for $p > n$ the resulting functions are Hölder continuous functions, [Eva10].

But the $p = n$ case was bizarre and it was left to Fritz John and Louis Nirenberg to solve the puzzle in 1961 by introducing the new BMO space of functions of *bounded mean oscillation*, see [44].

Actually, BMO is not a function space but rather a space of function classes modulo constants.

For this space there is the appropriate inequality.

[12] Consider the simple case $a = b = 1$, $p = 2$ in dimension $n = 4$.
The function looks a bit like Gaussian but it is not at all.

[13] Vázquez will avoid further details on these inequalities that can be found in the cited references.

[14] See Wikipedia/Isoperimetric Inequality.

[15] There is a wide representation of Spanish mathematicians active in these subjects with remarkable results that would be well worth a review.

Theorem 2 (John-Nirenberg)**.** *If $u \in W^{1,n}(\mathbb{R}^n)$ then u belongs to BMO and*

$$\|u\|_{BMO} \leq C\|Du\|_{L^n(\mathbb{R}^n)},$$

for a constant $C > 0$ depending only on n.[16]

The BMO spaces are once a very popular new object in functional and harmonic analysis, they replace L^∞ when it turns out so.

They were characterized by Charles Fefferman in [32].

The BMO spaces are slightly larger than L^∞.

The possible inequality (and functional immersion) of John-Nirenberg type using L^∞ instead of BMO as image space may seem reasonable but it is false.[17]

We ought to be very careful then with the critical cases, that Louis treated with utmost attention.

The John-Nirenberg spaces are used in analysis, in PDEs, in stochastic processes, and in multiple applications.

The reader may use the references [49] and [8] for some updates to recent work.

13.11.4 Navier-Stokes Equations

The Navier-Stokes system of equations describes the dynamics of an incompressible viscous fluid.

It was proposed in the 19th century to correct Euler's equations of ideal fluids, and adapt them to the more realistic viscous real world, [4].

The system reads **(1)**

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \nu\Delta\mathbf{u} + \dfrac{1}{\rho}\mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

where $\mathbf{u}$ is the *velocity vector*, p is the *pressure*, both variable, while ρ (the *density*) and ν (the *viscosity*) can be taken as positive constants.

It has had a spectacular success in practical science and engineering, but its essential mathematical aspects (existence, uniqueness, and regularity) have offered a stubborn resistance in the physical case of 3 space dimensions (3 or > 3 for the mathematician).

Nirenberg on the blackboard (photo: Courant Institute, NYU).

Fundamental works to cast the theory in a modern functional framework are due to Jean Leray [50, 51], who already in 1934 speaks of weak derivatives in spaces of integrable functions.

Using the new methods of functional analysis, authors soon obtained estimates that proved to be good enough to establish the existence and uniqueness of Leray solutions in 2 space dimensions, $n = 2$.

Furthermore, for regular initial data the solution is classical.

But the advance stopped sharply in higher dimensions, $n \geq 3$.

Vázquez gives the word to Charles Fefferman, of Princeton University, in his description of the open problem as the Clay Foundation Millennium Problem.

It is about proving or refuting the following Conjecture:

> *(A) Existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$.*
> *Take viscosity $\nu > 0$ and $n = 3$.*
> *Let $\mathbf{u}_0(\mathbf{x})$ be any smooth, divergence-free vector field satisfying the regularity and decay conditions (*specified*).*
> *Take external force $f(t, \mathbf{x})$ to be identically zero.*
> *Then there exists smooth functions $p(t, \mathbf{x})$, $u_i(t, \mathbf{x})$ on $[0, \infty) \times \mathbb{R}^3$ that satisfy the Navier-Stokes system with initial conditions in the whole space.*[18]

The most significant advance in this field is in Vázquez's opinion the article [17] in which L. Caffarelli, R. Kohn and L. Nirenberg attached the problem of regularity and showed that if a solution with classical data develops singularities in a finite time, the set of such singularities must be in any case quite small in size.

More specifically, "the 1D measure, in the Hausdorff sense, of the set of possible singularities (located in space-time) is zero."

This implies that if the singular set is not empty, it cannot contain any line or filament.

In 1998 F. H. Lin [54] gave an interesting new proof of this result.

Vázquez is talking about 1 of the milestones of the authors' career; it happened during the stay of a young Luis Caffarelli at the Courant Institute at Louis's invitation, and was published in 1982.

The topic Fluids is completely different from the previous sections, but the functional estimates in Sobolev spaces play an essential role, along with the machinery of geometric measure theory.

The possible presence of these singularities was conjectured by Leray as a possible explanation for the phenomenon of *turbulence*.

According to this hypothesis, even for regular data, solutions in 3 or more dimensions can develop singularities in finite time in the form of points where the so-called *vorticity* becomes infinite.

---

[16] The curious reader will wonder which function optimizes the constant. So?

[17] Find an elementary counterexample.

[18] See full details of the presentation in https://www.claymath.org/sites/default/files/navierstokes.pdf.

In the elapsed time, it has not been possible to prove or refute Conjecture (A).

Many efforts have been invested and Vázquez believes that will bear fruit 1 day.

An account of the state of affairs in the Euler and NSEs around 2008 is due to P. Constantin [23].

At the present moment Vázquez is entertained by a number of trials and false proofs (some of them quite well published).

There are excellent general texts on Navier-Stokes, e.g. [36] and [73].

2 very recent texts are [66] and [67].

13.11.5 Elliptic Equations & the Calculus of Variations

For reasons of selection and space, Vázquez will be quite brief on a subject in which Louis made so many contributions.

Vázquez mentions 1st of all the article [11] by Haim Brezis and Louis Nirenberg, which figures among the most widely read among the works of both authors.

It deals with the existence of solutions of semilinear elliptic equations with critical exponent (once again!)

$$\Delta u + f(\mathbf{x}, u) + u^{\frac{n+2}{n-2}} = 0.$$

2 further articles that had great impact are work in collaboration with Shmuel Agmon and Avron Douglis [1], year 1959, and [2], year 1964.

They are near-the-boundary estimates for solutions of elliptic equations that satisfy general boundary conditions.

Behavior near the boundary of nonlinear or degenerate PDE solutions, or in domains with nonsmooth boundaries, is a really delicate issue.

Indeed, it is a topic of permanent interest in our community, in theory and also because of its practical interest[19].

The article [6]with Henri Berestycki and S. R. S. Varadhan links the study of the 1st eigenvalue with the maximum principle, a subject that Louis enjoyed so much.

In this context Vázquez finds the famous article on the method of the "moving planes" of 1991 [5] in collaboration with Henri Berestycki, which Vázquez consider a gem.

In the Calculus of Variations, Vázquez quotes the article [11] with Haim Brezis, about the difference between local minimizers in the spaces H^1 and C^1. See also [12].

A topic of great interest for Louis was the study of geometric properties e.g. symmetry.

The articles [37, 38] with Vasilis Gidas and Wei-Ming Ni deal with the radial symmetry of certain positive solutions of nonlinear elliptic equations that is imposed by the equation and the shape of the domain.

13.11.6 Other contributions

Vázquez collects here brief comments on important results obtained by Louis and his collaborators on various topics that would deserve a more extensive treatment.

Operator theory. Nirenberg and Joseph J. Kohn[20] introduced of a *pseudo-differential operator* that helped generate a huge amount of later work in the brilliant school of harmonic analysis.

In a 1965 article, [48], they dealt with pseudo-differential operators with a complete and algebraic view.

The operators in question act on the space of tempered distributions at $\mathbb{R}^n$, and are estimated in terms of Fourier transform norms.

The importance of these results is that they take into account all the "lower order terms", difficult to deal with in previous articles.

See also the volume [61] edited by Louis.

Free boundary problems. This is 1 of the favorite topics of this reviewer.

In 1977 Louis published with David Kinderlehrer the article [45] on the regularity of free boundary problems for elliptic equations, at the beginning of an era that was to witness great progress.

To put it clearly, let us assume that u is a solution to the problem

$$\Delta u \leq f, \; u \geq 0, \; (\Delta u - f)u = 0,$$

defined in a domain $D \subset \mathbb{R}^n$.

Boundary data are also given at the fixed boundary ∂D.

These data are intended to determine not only u but also the positivity domain $\Omega = \{\mathbf{x} \in D; u(\mathbf{x}) > 0\}$, or still better the boundary of Ω that lies within D, called the *free boundary*:

$$\Gamma(u) = \partial\Omega \cap D.$$

This is properly called an *obstacle problem*.

[19] Think about the behavior of fluids in domains with corners.

[20] J. J. Kohn is a brilliant Princeton analyst, not to be confused with R. Kohn from Courant. J. J. Kohn speaks perfect Spanish with an Ecuadorian accent.

To get a physical idea, we can imagine a membrane in space $\mathbb{R}^3$ of height $z = U(x,y)$ that is subject to boundary conditions $U = h \geq 0$ in ∂D and must lie above a stable (obstacle) of height $U_{\text{obst}}(x,y) = 0$.

Fig. Free boundaries & obstacles (pictures: X. Ros-Oton).

Often, we want to consider a nontrivial obstacle φ, usually a concave function as in the figure.

This leads to an interesting equivalent formulation.

If we put $u = U + \varphi$, we arrive at the problem

$$\Delta u \leq g, \ u \geq \varphi, \ (\Delta u - g)(U - \varphi) = 0,$$

with *driving term* $g = f + \Delta\varphi$, and then we usually take $g = 0$.

In this formulation, u is constrained to stay above the obstacle $u_{\text{obst}}(\mathbf{x}) = var\phi$.

In any case, in the "free part", $\{\mathbf{x} \in \mathbb{R}^n; U(\mathbf{x}) > 0\} = \{\mathbf{x} \in \mathbb{R}^n; u(\mathbf{x}) > \varphi\}$, an elastic equation $\Delta U = f$ is satisfied, but a priori we do not know where that part could be located.

It is therefore a problem that combines PDEs and Geometry (again!).

This problem was known to have a unique *solution pair*, (u, Γ).

The attentive reader will have observed that once Γ is known, and with it Ω, the PDE problem to find u is rather elementary.

Therefore, the difficulty lies in principle in the geometry.

However, the solution to the puzzle was rather found in nonlinear analysis, [47], which also produces efficient numerical methods.

We then encounter a big theoretical problem: determining how regular is the set Γ, that we have found by abstract methods, and also determining how regular is u near Γ.

Even the simplest question: "is Γ a surface?" has to be answered.

D. Kinderlehrer and L. Nirenberg gave local conditions on f and assumed a certain initial regularity of u to conclude that then Γ is a very regular, even analytical, hyper-surface.

The study of free boundaries extends to problems evolving in time, e.g. the very famous Stefan problem discussed by Louis in [46].

The 1980s were years of great progress in the mathematical understanding of free boundaries, with reference books e.g. [28, 34].

This is a field of very intense activity, both theoretical and applied, in which Vázquez has worked with great delight for decades.

A required reference for in-depth study of the regularity of the free boundaries is the book [20] by L. Caffarelli and S. Salsa, see also A. Petrosyan et al. [64].

A study of tumor growth modeling, seen as a free boundary problem, was done by B. Perthame et al. in [63], it is just an example from a vast literature.

Geometric Equations. The article [55] with Charles Loewner in 1974 deals with PDEs that are invariant under conformal or projective transformations.

The reader will recall in this context the current relevance of PDEs linked to problems of Riemannian geometry, e.g. the Yamabe problem.

Vázquez refers to the lengthy overview [52] due to Yan Yan Li, Louis's doctoral student that has been for many years professor at Rutgers.

Complex geometry. The topic interested Louis a lot in his beginnings.

A Newlander and L. Nirenberg wrote in 1965 an article published in Annals of Mathematics [56] on analytical coordinates in quasi-complex manifolds.

The Newlander-Nirenberg Theorem states that any integrable quasi-complex structure is induced by a complex structure.

Integrability is expressed through a list of differential conditions.

Vázquez puts an end here to the mathematical journey, unfortunately unfair in many aspects due to the brevity of space and his ignorance in so many subjects.

Vázquez hopes that the extensive cited literature will serve as an indication of the profound influence of Louis Nirenberg and his world on the mathematicians and mathematics that have followed him.

For the curious reader, there are excellent articles dealing with the work and life of Louis Nirenberg: a congress in his honor on the occasion of the 75th anniversary was organized by Alice Chang et al. and is collected in [22].

He was interviewed by Allyn Jackson for the *AMS Notices* in 2002, [43], and Simon Donaldson, Fields Medal, wrote about him in the same journal in 2011, [30].

Yan-Yan Li's [52] 2010 article focuses on the analysis of geometric problems.

On the occasion of the Abel Prize, Xavier Cabré wrote a review in Catalan in [14] and Tristan Rivière reviews his work in PDEs in [65].

A mathematical description of the influence of his ideas appeared in 2016 in [69] with contributions of a number of experts: X. Cabré (symmetries of solutions), A. Chang (Gauss curvature problem), G. Seregin (Navier-Stokes problem), E. Carlen and A. Figalli (stability of the GNS inequality), M. T. Wang and S. T. Yau (Weyl problem and general relativity).

Finally, the book [42] presents the laureates of the Abel Prize in the period 2013–2017.

In it Robert V. Kohn devotes to L. Nirenberg the article *"A few of Louis Nirenberg's many contributions to the theory of PDEs"*.

By the way, there is a beautiful quotation from Abel as motto for the book:

> "Au reste il me paraît que si l'on veut faire des progrès dans les mathématiques il faut étudier les maîtres et non pas les écoliers."[21]

Update. the article *"A personal tribute to Louis Nirenberg"*, posted by Prof. Joel Spruck in the Arxiv repository in May 2021, [70].

As a person who met Louis Nirenberg in 1972 and became a Courant Instructor, his detailed report on a selection of Louis's works is a very commendable reading.

He concentrates on the work inspired by geometric problems beginning around 1974, especially the method of moving planes, and implicit fully nonlinear elliptic equations, and makes comments on Louis' personality.

Fig. Nirenberg in Barcelona in 2017 (photo: Jordi Play).

13.11.7 The quiet wise man & Spain

Vázquez's 1st memory of Louis Nirenberg sets them in Lisbon in the spring of 1982.[22]

Louis was already famous and Vázquez was a novice in the art.

In Lisbon Vázquez listened to 1 of his talks, which brought together the depth of the mathematics, the simplicity of the exposition and a grace to add some comment as timely as it was nice, characteristic features of Louis that delighted the public.

In the fall of that same year Vázquez set foot in the US, headed for the University of Minnesota,[23] to work on free boundary problems with Don Aronson and with Luis Caffarelli, who was back from his visit to Courant Institute.

Then Vázquez saw, through the group of great professor Vázquez had access to, that mathematical research offered a much better way of life.

Among that group of friends Vázquez counts Haim Brezis and Luis Caffarelli who have been Vázquez's masters, Louis Nirenberg, Constantine Dafermos, Donald Aronson, Mike Crandall, Hans Weinberger,... Vázquez will never cease from thanking them for that vision.

A few years later, Vázquez had the honor of participating in the organization of a summer course at the UIMP[24] which included Louis as lecturer along with Don G. Aronson (Minnesota), Philippe Bénilan (Besançon), Luis A. Caffarelli (IAS Princeton) and Constantine Dafermos (Brown Univ.).

These courses were inspired by Luis Caffarelli, close collaborator and friend of Louis, with the support of the Rector of the UIMP, Prof. Ernest Lluch,[25] and somehow they transmitted a certain spirit of mathematics that was being done around the Courant Institute.

The course had a remarkable consequence.

A young mathematician from Barcelona, Xavier Cabré, a student in the course, went to the Courant Institute with Louis Nirenberg and thus began an international mathematical career, like the ones that so many young people crave today.

His thesis, directed by Louis, dealt with "Estimates for Solutions of Elliptic and Parabolic Equations" (NYU, 1994).

Following his stay in New York, he published with Luis Caffarelli the beautiful book [16] on the so-called *completely nonlinear elliptic equations*.

Xavier Cabré is now an ICREA Professor at the UPC in Barcelona.

Louis Nirenberg visited Spain several times, specially Barcelona, and had many Spanish friends and admirers.

Although Vázquez did not become a collaborator of Louis, Vázquez had the opportunity of seeing him and talking to him on several occasions.

Vázquez highlights a stay at the Courant Institute in the winter of 1996 where Vázquez could appreciate the day-to-day life of the "quiet wise man", or a congress in Argentina in 2009 when Louis was already very senior but loved life as the 1st day.

The last event in which Vázquez saw him took place at Columbia University, New York, in May of last year (2019), in a congress in honor of Luis Caffarelli.

He went to some talks in his wheelchair at 94 years old, and, with his proverbial good humor he told them that it was a bit difficult for him to follow the lectures!

Impressed by his personality, the young mathematician David Fernández and Vázquez wrote a portrait of him in 2 entries in the blog *"The Republic of Mathematics"* that they edit in "Investigación y Ciencia" (Spanish partner of "Sciencific American").

They called the essays "Louis Nirenberg, the quiet wise man" (I) and (II).[26]

He was a teacher and master of science as those described by George Steiner in [71], where the relationship between teacher and pupil, master and disciple, is what matters.

Louis had 46 doctoral students, many of them well-known mathematicians.[27]

It was not his style to write long textbooks, he was the author of [60] and the recently published [62].

We will miss the teacher, master and senior friend who always looked gentle and kind, who loved Italy (*il bel paese*), culture, good food and talking about movies and friends, and with whom mathematics was easy and exciting.

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[21] In English: *"Finally, it appears to me that if one wants to make progress in mathematics, one should study the masters, not the students."* Taken from the book.

[22] At the International Symposium in Homage to Prof. J. Sebatião e Silva.

[23] This American university was very popular with young Spanish graduates and doctors for the excellence of its studies in Mathematics & Economics.

[24] Menéndez Pelayo International University, the course took place in 1987 at the Palacio de la Magdalena in Santander.

[25] Scholar of indelible memory, great protector of science and great conversationalist, he died tragically for being a good person at a very turbulent time.

[26] https://www.investigacionyciencia.es/blogs/matematicas/75/posts.

[27] The 1st was Walter Littman (in 1956), whom Vázquez treated so much in Minnesota.

Nirenberg lived in New York since 1949, in the Upper West Side, he was a perfect New Yorker and at the same time a citizen of the wide world.

He worked until the end of his life, frequently visiting "his" Institute.

Lucky soul, how Vázquez envies him, now and here the "elders" seem expendable for public utility.

Vázquez is proud to bear his name Louis = Luis, like Luis Caffarelli or Jacques Louis Lions or Luigi Ambrosio.

He is already a great name in mathematics and it is an honor that carries the burden of working as Louis Nirenberg, only for the best and always in a good mood, and that is not easy.

Rest in eternal peace, beloved Master.

In the Elysian fields you will have time to think about new functional inequalities, the beautiful functions that optimize them, and their surprising fruits.

In our own small way, we also follow them, as in [26]. □

13.12 Stanley Osher

13.13 Laurent Schwartz

Laurent Schwartz.

- **Born.** Mar 5, 1915. Paris, France.
- **Died.** Jul 4, 2002 (aged 87). Paris, France.
- **Nationality.** French.
- **Alma mater.** Ecole Normale Supérieure.
- **Known for.**
  - Theory of Distributions
  - Schwartz kernel theorem
  - Schwartz space
  - Schwartz-Bruhat function
  - Radonifying operator
  - Cylinder set measure
- **Awards.** Fields Medal (1950).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - University of Strashbourg
  - University of Nancy
  - University of Grenoble
  - École Polytechnique
  - Université de Paris VII
- **Doctoral advisor.** Georges Valiron.
- **Doctoral students.**
  - Maurice Audin
  - Georges Glaeser
  - Alexander Grothendieck
  - Jacques-Louis Lions
  - Bernard Malgrange
  - André Martineau
  - Bernard Maurey
  - Leopoldo Nachbin
  - Henri Hogbe Nlend
  - Gilles Pisier

- François Treves

- **Influenced.** Per Enflo.

Laurent-Moïse Schwartz (Mar 5, 1915 - Jul 4, 2002) was a French mathematician.

He pioneered the theory of distributions, which gives a well-defined meaning to objects such as the Dirac delta function.

He was awarded the Fields Medal in 1950 for his work on the theory of distributions.

For several years he taught at the École polytechnique.

13.13.1 Biography

Family Laurent Schwartz came from a Jewish family of Alsatian origin, with a strong scientific background: his father was a well-known surgeon, his uncle Robert Debré (who contributed to the creation of UNICEF) was a famous pediatrician, and his great-uncle-in-law, Jacques Hadamard, was a famous mathematician.

During his training at Lycée Louis-le-Grand to enter the École Normale Supérieure, he fell in love with Marie-Hélène Lévy, daughter of the probabilist Paul Lévy who was then teaching at the École polytechnique.

Later they would have 2 children, Marc-André and Claudine.

Marie-Hélène was gifted in mathematics as well, as she contributed to the geometry of singular analytic spaces and taught at the University of Lille.

Angelo Guerraggio describes "Mathematics, politics and butterflies" as his "3 great loves".[1]

Education According to his teachers, Schwartz was an exceptional student.

He was particularly gifted in Latin, Greek and mathematics.

1 of his teachers told his parents: "*Beware, some will say your son has a gift for languages, but he is only interested in the scientific and mathematical aspect of languages: he should become a mathematician.*"

In 1934, he was admitted at the École Normale Supérieure, and in 1937 he obtained the agrégation (with rank 2).

World War II As a man of Trotskyist affinities and Jewish descent, life was difficult for Schwartz during World War II.

He had to hide and change his identity to avoid being deported after Nazi Germany overran France.

He worked for the University of Strasbourg (which had been relocated in Clermont-Ferrand because of the war) under the name of Laurent-Marie Sélimartin, while Marie-Hélène used the name Lengé instead of Lévy.

Unlike other mathematicians at Clermont-Ferrand such as Feldbau, the couple managed to escape the Nazis.

Later career Schwartz taught mainly at École Polytechnique, from 1958 to 1980.

At the end of the war, he spent one year in Grenoble (1944), then in 1945 joined the University of Nancy on the advice of Jean Delsarte and Jean Dieudonné, where he spent 7 years.

He was both an influential researcher and teacher, with students such as Bernard Malgrange, Jacques-Louis Lions, François Bruhat and Alexander Grothendieck.

He joined the science faculty of the University of Paris in 1952.

In 1958 he became a teacher at the École polytechnique after having at 1st refused this position.

From 1961 to 1963 the École polytechnique suspended his right to teach, because of his having signed the Manifesto of the 121 about the Algerian war, a gesture not appreciated by Polytechnique's military administration.

However, Schwartz had a lasting influence on mathematics at the École polytechnique, having reorganized both teaching and research there.

In 1965 he established the Centre de mathématiques Laurent-Schwartz (CMLS) as its 1st director.

In 1973 he was elected corresponding member of the French Academy of Sciences, and was promoted to full membership in 1975.

13.13.2 Mathematical legacy

In 1950 at the International Congress of Mathematicians, Schwartz was a plenary speaker[Schwartz, Laurent (1950). "*Théorie des noyaux*" (PDF). In: Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., Aug 30–Sep 6, 1950. vol. 1. pp. 220–230.] and was awarded the Fields Medal for his work on distributions.

He was the 1st French mathematician to receive the Fields medal.

Because of his sympathy for Trotskyism, Schwartz encountered serious problems trying to enter the United States to receive the medal; however, he was ultimately successful.

The theory of distributions clarified the (then) mysteries of the Dirac delta function and Heaviside step function.

It helps to extend the theory of Fourier transforms and is now of critical importance to the theory of partial differential equations.

13.13.3 Popular science

Throughout his life, Schwartz actively worked to promote science and bring it closer to the general audience.

Schwartz said:

> *"What are mathematics helpful for? Mathematics are helpful for physics.*
> *Physics helps us make fridges.*
> *Fridges are made to contain spiny lobsters, and spiny lobsters help mathematicians who eat them and have hence better abilities to do mathematics, which are helpful for physics, which helps us make fridges which..."[3]*

13.13.4 Entomology

Clanis schwartzi Paratype MHNT.

His mother, who was passionate about natural science, passed on her taste for entomology to Laurent.

His personal collection of 20,000 Lepidoptera specimens, collected during his various travels was bequeathed to the Muséum national d'histoire naturelle), the Science Museum of Lyon, the Museum of Toulouse and the Museo de Historia Natural Alcide d'Orbigny in Cochabamba (Bolivia).

Several species discovered by Schwartz bear his name.

13.13.5 Personal ideology

Apart from his scientific work, Schwartz was a well-known outspoken intellectual.

As a young socialist influenced by Leon Trotsky, Schwartz opposed the totalitarianism of the Soviet Union, particularly under Joseph Stalin.

Schwartz ultimately rejected Trotskyism for democratic socialism.

On his religious views, Schwartz called himself an atheist.[4]

13.13.6 Books

Research articles

- *Œuvres scientifiques. I.*

  With a general introduction to the works of Schwartz by Claude Viterbo and an appreciation of Schwartz by Bernard Malgrange.

  With 1 DVD.

  Documents Mathématiques (Paris), 9. Société Mathématique de France, Paris, 2011. x+523 pp. ISBN 978-2-85629-317-1

  > the 1st half of his works in analysis and partial differential equations.
  > After a preface by Claude Viterbo, which includes a few photos, one will find a note by Schwartz himself about his works, followed by a few original documents (letters, course notes), a presentation by Bernard Malgrange of the theory of distributions for which Schwartz received the Fields Medal in 1950, and a selection of articles covering the period 1944–1954.

- *Œuvres scientifiques. II.*

  With an appreciation of Schwartz by Alain Guichardet.

  With 1 DVD.

  Documents Mathématiques (Paris), 10.

  Société Mathématique de France, Paris, 2011. x+507 pp. ISBN 978-2-85629-318-8

  > the 2nd half of his works in analysis and partial differential equations.
  > After a note by Alain Guichardet on Schwartz and his seminars, one will find a selection of articles covering the period 1954–1966.

- *Œuvres scientifiques. III.*

  With appreciations of Schwartz by Gilles Godefroy and Michel Émery.

  With 1 DVD.

  Documents Mathématiques (Paris), 11. Société Mathématique de France, Paris, 2011. x+619 pp. ISBN 978-2-85629-319-5

  > his works on Banach space theory (1968–1987), introduced by Gilles Godefroy, and on probability theory (1970–1996), presented by Michel Émery, as well as some articles of a historical nature (1955–1994).

Technical books

- *Analyse hilbertienne*. Collection Méthodes. Hermann, Paris, 1979. ii+297 pp. ISBN 2-7056-5897-1
- *Application of distributions to the theory of elementary particles in quantum mechanics*. Gordon and Breach, New York, NY, 1968. 144pp. ISBN 9780677300900
- *Cours d'analyse. 1*. 2nd edition. Hermann, Paris, 1981. xxix+830 pp. ISBN 2-7056-5764-9
- *Cours d'analyse. 2*. 2nd edition. Hermann, Paris, 1981. xxiii+475+21+75 pp. ISBN 2-7056-5765-7

5 *Étude des sommes d'exponentielles. 2ième éd*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris 1959 151 pp.

- *Geometry and probability in Banach spaces*. Based on notes taken by Paul R. Chernoff. Lecture Notes in Mathematics, 852. Springer-Verlag, Berlin-New York, 1981. x+101 pp. ISBN 3-540-10691-X
- *Lectures on complex analytic manifolds*. With notes by M. S. Narasimhan. Reprint of the 1955 edition. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 4. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1986. iv+182 pp. ISBN 3-540-12877-8
- *Mathematics for the physical sciences*. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966 358 pp.
- *Radon measures on arbitrary topological spaces and cylindrical measures*. Tata Institute of Fundamental Research Studies in Mathematics, No. 6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. xii+393 pp.
- *Semimartingales and their stochastic calculus on manifolds*. Edited and with a preface by Ian Iscoe. Collection de la Chaire Aisenstadt. Presses de l'Université de Montréal, Montreal, QC, 1984. 187 pp. ISBN 2-7606-0660-0
- *Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes*. Lecture Notes in Mathematics, 780. Springer, Berlin, 1980. xv+132 pp. ISBN 3-540-09749-X
- Les tenseurs. *Suivi de "Torseurs sur un espace affine" by Y. Bamberger and J.-P. Bourguignon*. 2nd edition. Hermann, Paris, 1981. i+203 pp. ISBN 2-7056-1376-5

6 *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris 1966 xiii+420 pp.

Seminar notes

- *Séminaire Schwartz in Paris 1953 bis 1961*. Online edition: [1]

Popular books

- *Pour sauver l'université*. Editions du Seuil, 1983. 122 pp. ISBN 2020065878
- *A mathematician grappling with his century*. Translated from the 1997 French original by Leila Schneps. Birkhäuser Verlag, Basel, 2001. viii+490 pp. ISBN 3-7643-6052-6

13.13.7 See also

- Schwartz distribution
- Schwartz kernel theorem
- Schwartz space
- Schwartz-Bruhat function
- Nicolas Bourbaki" – Wikipedia/Laurent Schwartz

13.14 Roger Temam

Roger Meyer Temam.

- **Born.** May 19, 1940 (age 80).
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.** Navier-Stokes equations.

Scientific career.

- **Fields.** Applied mathematics.
- **Institutions.**
  - Paris-Sud University (Orsay)
  - Indiana University
- **Doctoral advisor.** Jacques-Louis Lions.
- **Doctoral students.**
  - Etienne Pardoux
  - Denis Serre

Roger Meyer Temam (born May 19, 1940) is a French applied mathematician working in numerical analysis, nonlinear partial differential equations and fluid mechanics.

He graduated from the University of Paris - the Sorbonne in 1967, completing a doctorate (*thèse d'Etat*) under the direction of Jacques-Louis Lions.

He has published over 400 articles, as well as 12 (authored or co-authored) books.

13.14.1 Scientific work

The 1st work of Temam in his thesis dealt with the *fractional steps method*.

Thereafter, "he has continually explored and developed new directions and techniques":[2]

- calculus of variations, and the notion of *duality* (book #7), developing the mathematical framework for *discontinuous* (in displacement) *solutions*; a concept later used for his works on the *mathematical theory of plasticity* (book #5);
- mathematical formulation of the equilibrium of a plasma in a cavity, expressed as a nonlinear free boundary problem;[R. Temam, A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma, *Arch. Rational Mech. Anal.*, 60, 1975, 51–73.]
- Korteweg-de Vries equation;[R. Temam, Sur un problème non linéaire, *J. Math. Pures Appl.*, 48, 1969, 159–172.]
- Kuramoto-Sivashinsky equation;[5]
- Euler equations in a bounded domain;[R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.*, 20, 1975, 32–43.]
- infinite-dimensional dynamical systems theory.

  In particular, he studied the existence of the finite-dimensional global attractor for many dissipative equations of mathematical physics, including the incompressible Navier-Stokes equations.[P. Constantin, C. Foias, O. Manley and R. Temam, Determining modes and fractal dimension of turbulent flows, *J. Fluid Mech.*, 150, 1985, 427–440.][C. Foias, O.P. Manley and R. Temam, Physical estimates of the number of degrees of freedom in free convection, *Phys. Fluids*, 29, 1986, 3101–3103.]

  He was also the co-founder of the notion of *inertial manifolds*[C. Foias, G.R. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Diff. Equ.*, 73, 1988, 309–353.] together with Ciprian Foias and George R. Sell and of exponential attractors[A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations*, Collection Recherches en Mathématiques Appliquées, Masson, Paris, and John Wiley, England, 1994.] together with Alp Eden, Ciprian Foias and Basil Nicolaenko;[2]
- optimal control of the incompressible Navier-Stokes equations as a tool for the *control of turbulence*;[F. Abergel and R. Temam, On some control problems in fluid mechanics, *Theoret. Comput. Fluid Dynamics*, 1, 1990, 303–325.]
- boundary layer phenomena for incompressible flows.[12]

Temam's main activities concern the study of geophysical flows, the atmosphere and oceans.[2]

This started in the 1990s by collaboration with Jacques-Louis Lions and Shouhong Wang.[J.L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of the atmosphere and applications, *Nonlinearity*, 5, 1992, 237–288.][J.L. Lions, R. Temam and S. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5, 1992, 1007–1053.][M. Coti Zelati, M. Frémond, R. Temam and J. Tribbia, Uniqueness, regularity and maximum principles for the equations of the atmosphere with humidity and saturation, *Physica D*, 264, 2013, 49-65, https://doi.org/10.1016/j.physd.2013.08.007][Y. Cao, M. Hamouda, R. Temam, J. Tribbia and X. Wang, The equations of the multi-phase humid atmosphere expressed as a quasi variational inequality, *Nonlinearity*, 31, 2018, 4692-4723, https://doi.org/10.1088/1361-6544/aad525.]

According to the Mathematical Genealogy Project database,[17][18] he holds the first position in the top 50 advisors.

More than 30 of his students are now full professors all over the world, and have themselves many descendants.[19]

13.14.2 Administrative activities

Temam became a professor at the Paris-Sud University at Orsay in 1968.

There, he co-founded the Laboratory of Numerical and Functional Analysis which he directed from 1972 to 1988.

He was also a *Maître de Conférences* at the Ecole Polytechnique in Paris from 1968 to 1986.[20]

In 1983, Temam co-founded the French Société de Mathématiques Appliquées et Industrielles (SMAI), analogous to the Society for Industrial and Applied Mathematics (SIAM), and served as its 1st president.[21]

He was also 1 of the founders of the International Congress on Industrial and Applied Mathematics (ICIAM) series and was the chair of the steering committee of the 1st ICIAM meeting held in Paris in 1987; and the chair of the standing committee of the 2nd ICIAM meeting held in Washington, D.C., in 1991.[22]

He was the Editor-in-Chief of the mathematical journal M2AN[23] from 1986 to 1997.

Temam has been the Director of the Institute for Scientific Computing & Applied Mathematics (ISCAM)[24] at Indiana University since 1986 (co-director with Ciprian Foias from 1986 to 1992).

He is also a College Professor (part-time till 2003) and he has been a Distinguished Professor since 2014.[25]

13.14.3 Books

1. (with G.-M. Gie, M. Hamouda and C.-Y. Jung): *Singular perturbations and boundary layers*, Springer-Verlag, New-York, 2018.

2. (with A. Miranville): *Mathematical Modelling in Continuum Mechanics*, Cambridge University Press, 2001. French Translation, Springer-Verlag France, 2002. Chinese Translation, Tsinghua University Press, 2004. 2nd English Edition 2005. Russian translation, Moskva Linom, 2013.

3. (with T. Dubois and F. Jauberteau): *Dynamic, multilevel methods and the numerical simulation of turbulence*; Cambridge University Press, 1999.

4. *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New-York, Applied Mathematical Sciences Series, vol. 68, 1988. 2nd augmented edition, 1997. Reprinted in China by Beijing World Publishing Corp., 2000.

5. *Mathematical Problems in Plasticity*, Gauthier-Villars, Paris, 1983 (in French). English Transl., Gauthier-Villars, New-York, 1985. Russian Transl., Nauk, Moscow, 1991. "Republished by Dover books in Physics, 2018."

6. *Navier-Stokes Equations*, North-Holland Pub. Company, in English, 1977, 500 pages. Revised editions 1979, 1984 and 1985. Russian Translation, Mir, Moscow, 1981. "Republished in the AMS-Chelsea Series, AMS, Providence, 2001."

7. (with I. Ekeland): *Convex Analysis and Variational Problems*. Dunod, Paris, 1974, 350 pages (in French). English Translation, North-Holland, Amsterdam, 1976. Russian Translation, Mir, Moscow, 1979. "English version republished in the Series 'Classics in Applied Mathematics', SIAM, Philadelphia, 1999."

13.14.4 Awards & honors

- Fellow of the American Academy of Arts and Sciences (2015),[26] of the American Mathematical Society (2013),[27] of the American Association for the Advancement of Science (2011),[28] of the Society for Industrial and Applied Mathematics (2009).[29]

- Knight of the Legion of Honor, France, 2012.[30]

- Member of the French Academy of Sciences since 2007.[31]" – Wikipedia/Roger Temam

13.15 Karl Weierstrass

Karl Weierstrass/Karl Weierstraß.

- **Born.** Oct 31, 1815. Ostenfelde, Province of Westphalia, Kingdom of Prussia.

- **Died.** Feb 19, 1897 (aged 81). Berlin, Province of Brandenburg, Kingdom of Prussia.

- **Nationality.** German.

- **Alma mater.**
  - University of Bonn
  - Münster Academy
- **Known for.**
  - Weierstrass function
  - Weierstrass product inequality
  - (ε, δ)-definition of limit
  - Weierstrass-Erdmann condition
  - Weierstrass theorems
  - Bolzano-Weierstrass theorem
- **Awards.**
  - PhD (Hon): University of Königsberg (1854)
  - Copley Medal (1895)

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - Gewerbeinstitut
  - Friedrich Wilhelm University
- **Academic advisors.** Christoph Gudermann.
- **Doctoral students.**
  - Nikolai Bugaev
  - Georg Cantor
  - Georg Frobenius
  - Lazarus Fuchs
  - Wilhelm Killing
  - Leo Königsberger
  - Sofia Kovalevskaya
  - Mathias Lerch
  - Hans von Mangoldt
  - Eugen Netto
  - Adolf Piltz
  - Carl Runge
  - Arthur Schoenflies
  - Friedrich Schottky
  - Hermann Schwarz
  - Ludwig Stickelberger
  - Ernst Kötter

Karl Theodor Wilhelm Weierstrass (German: *Weierstraß*;[Duden. *Das Aussprachewörterbuch*. 7. Auflage. Bibliographisches Institut, Berlin 2015, ISBN 978-3-411-04067-4] Oct 31, 1815 - Feb 19, 1897) was a German mathematician often cited as the "*father of modern analysis*".

Despite leaving university without a degree, he studied mathematics and trained as a school teacher, eventually teaching mathematics, physics, botany and gymnastics.[Weierstrass, Karl Theodor Wilhelm. (2018). In Helicon (Ed.), *The Hutchinson unabridged encyclopedia with atlas and weather guide*. [Online]. Abington: Helicon. Available from: link [Accessed Jul 8, 2018].]

He later received an honorary doctorate and became professor of mathematics in Berlin.

Among many other contributions, Weierstrass formalized the definition of the continuity of a function, proved the intermediate value theorem and the Bolzano-Weierstrass theorem, and used the latter to study the properties of continuous functions on closed bounded intervals.

13.15.1 Biography

Weierstrass was born in Ostenfelde, part of Ennigerloh, Province of Westphalia.[O'Connor, J. J.; Robertson, E. F. (October 1998). "*Karl Theodor Wilhelm Weierstrass*". School of Mathematics and Statistics, University of St Andrews, Scotland. Retrieved Sep 7, 2014.]

Weierstrass was the son of Wilhelm Weierstrass, a government official, and Theodora Vonderforst.

His interest in mathematics began while he was a gymnasium student at the Theodorianum in Paderborn.

He was sent to the University of Bonn upon graduation to prepare for a government position.

Because his studies were to be in the fields of law, economics, and finance, he was immediately in conflict with his hopes to study mathematics.

He resolved the conflict by paying little heed to his planned course of study but continuing private study in mathematics.

The outcome was that he left the university without a degree.

He then studied mathematics at the Münster Academy (which was even then famous for mathematics) and his father was able to obtain a place for him in a teacher training school in Münster.

Later he was certified as a teacher in that city.

During this period of study, Weierstrass attended the lectures of Christoph Gudermann and became interested in elliptic functions.

In 1843 he taught in Deutsch Krone in West Prussia and since 1848 he taught at the Lyceum Hosianum in Braunsberg.

Besides mathematics he also taught physics, botany, and gymnastics.[3]

Weierstrass may have had an illegitimate child named Franz with the widow of his friend Carl Wilhelm Borchardt.[Biermann, Kurt-R.; Schubring, Gert (1996). "Einige Nachträge zur Biographie von Karl Weierstraß. (German) [Some postscripts to the biography of Karl Weierstrass]". *History of mathematics*. San Diego, CA: Academic Press. pp. 65–91.]

After 1850 Weierstrass suffered from a long period of illness, but was able to publish mathematical articles that brought him fame and distinction.

The University of Königsberg conferred an honorary doctor's degree on him on Mar 31, 1854.

In 1856 he took a chair at the *Gewerbeinstitut* in Berlin (an institute to educate technical workers which would later merge with the *Bauakademie* to form the Technical University of Berlin).

In 1864 he became professor at the Friedrich-Wilhelms-Universität Berlin, which later became the Humboldt Universität zu Berlin.

In 1870, at the age of 55, Weierstrass met Sofia Kovalevsky whom he tutored privately after failing to secure her admission to the University. They had a fruitful intellectual, but troubled personal, relationship that "far transcended the usual teacher-student relationship".

The misinterpretation of this relationship and Kovalevsky's early death in 1891 was said to have contributed to Weierstrass' later ill-health.

He was immobile for the last 3 years of his life, and died in Berlin from pneumonia.[*Dictionary of scientific biography*. Gillispie, Charles Coulston,, American Council of Learned Societies. New York. p. 223. ISBN 978-0-684-12926-6. OCLC 89822.]

13.15.2 Mathematical contributions

Soundness of calculus Weierstrass was interested in the soundness of calculus, and at the time there were somewhat ambiguous definitions of the foundations of calculus so that important theorems could not be proven with sufficient rigor.

Although Bolzano had developed a reasonably rigorous definition of a limit as early as 1817 (and possibly even earlier) his work remained unknown to most of the mathematical community until years later, and many mathematicians had only vague definitions of limits and continuity of functions.

The basic idea behind Delta-epsilon proofs is, arguably, 1st found in the works of Cauchy in the 1820s.

- Grabiner, Judith V. (March 1983), "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" (PDF), *The American Mathematical Monthly*, 90 (3): 185–194, doi:10.2307/2975545, JSTOR 2975545

- Cauchy, A.-L. (1823), "Septième Leçon – Valeurs de quelques expressions qui se présentent sous les formes indéterminées $\frac{\infty}{\infty}, \infty^0, \ldots$ Relation qui existe entre le rapport aux différences finies et la fonction dérivée", *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal*, Paris, archived from the original on 2009-05-04, retrieved 2009-05-01, p. 44.

Cauchy did not clearly distinguish between continuity and uniform continuity on an interval.

Notably, in his 1821 *Cours d'analyse*, Cauchy argued that the (pointwise) limit of (pointwise) continuous functions was itself (pointwise) continuous, a statement interpreted as being incorrect by many scholars.

The correct statement is rather that the uniform limit of continuous functions is continuous (also, the uniform limit of uniformly continuous functions is uniformly continuous).

This required the concept of uniform convergence, which was 1st observed by Weierstrass's advisor, Christoph Gudermann, in an 1838 paper, where Gudermann noted the phenomenon but did not define it or elaborate on it.

Weierstrass saw the importance of the concept, and both formalized it and applied it widely throughout the foundations of calculus.

The formal definition of continuity of a function, as formulated by Weierstrass, is as follows:

$f(x)$ is continuous at $x = x_0$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every x in the domain of f, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
In simple English, $f(x)$ is continuous at a point $x = x_0$ if for each x close enough to x_0, the function value $f(x)$ is very close to $f(x_0)$, where the "close enough" restriction typically depends on the desired closeness of $f(x_0)$ to $f(x)$.

Using this definition, he proved the Intermediate Value Theorem.

He also proved the Bolzano-Weierstrass theorem and used it to study the properties of continuous functions on closed and bounded intervals.

Calculus of variations Weierstrass also made advances in the field of calculus of variations.

Using the apparatus of analysis that he helped to develop, Weierstrass was able to give a complete reformulation of the theory that paved the way for the modern study of the calculus of variations.

Among several axioms, Weierstrass established a necessary condition for the existence of strong extrema of variational problems.

He also helped devise the Weierstrass-Erdmann condition, which gives sufficient conditions for an extremal to have a corner along a given extremum and allows one to find a minimizing curve for a given integral.

Other analytical theorems See also: List of things named after Karl Weierstrass.

- Stone-Weierstrass theorem
- Casorati-Weierstrass-Sokhotski theorem
- Weierstrass's elliptic functions
- Weierstrass function
- Weierstrass M-test
- Weierstrass preparation theorem
- Lindemann-Weierstrass theorem
- Weierstrass factorization theorem
- Enneper-Weierstrass parameterization

13.15.3 Students

- Edmund Husserl
- Sofia Kovalevskaya
- Gösta Mittag-Leffler
- Hermann Schwarz
- Carl Johannes Thomae
- Georg Cantor

13.15.4 Honors & awards

The lunar crater Weierstrass and the asteroid 14100 Weierstrass are named after him.

Also, there is the Weierstrass Institute for Applied Analysis and Stochastics in Berlin.

13.15.5 Selected works

- *Zur Theorie der Abelschen Funktionen* (1854)
- *Theorie der Abelschen Funktionen* (1856)
- *Abhandlungen-1*, Math. Werke. Bd. 1. Berlin, 1894
- *Abhandlungen-2*, Math. Werke. Bd. 2. Berlin, 1895
- *Abhandlungen-3*, Math. Werke. Bd. 3. Berlin, 1903
- *Vorl. ueber die Theorie der Abelschen Transcendenten*, Math. Werke. Bd. 4. Berlin, 1902
- *Vorl. ueber Variationsrechnung*, Math. Werke. Bd. 7. Leipzig, 1927

13.15.6 External links

- O'Connor, John J.; Robertson, Edmund F., "Karl Weierstrass", *MacTutor History of Mathematics archive*, University of St Andrews.
- Digitalized versions of Weierstrass's original publications are freely available online from the library of the *Berlin Brandenburgische Akademie der Wissenschaften.*
- Works by Karl Weierstrass at Project Gutenberg
- Works by or about Karl Weierstrass at Internet Archive" – Wikipedia/Karl Weierstrass

14 Miscellaneous

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