Mathematical Optimization – Toán Tối Ưu

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Tóm tắt nội dung

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- *Mathematical Optimization – Toán Tối Ưu.*
  PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/optimization/NQBH_mathematical_optimization.pdf.
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Mục lục

1 Basic

2 Optimal Control – Điều Khiển Tối Ưu

3 Shape Optimization – Tối Ưu Hình Dạng

Resources – Tài nguyên.

1. [AH01]. Grégoire Allaire, Antoine Henrot. *On some recent advances in shape optimization.*
2. [Aze20]. Hideyuki Azegami. *Shape Optimization Problems.*
3. [BW23]. Catherine Bandle, Alfred Wagner. *Shape Optimization: Variations of Domains & Applications.*
4. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries.*
5. [HM03]. J. Haslinger, R. A. E. Mäkinen. *Introduction to Shape Optimization.*
6. [MP10]. Bijan Mohammadi, Olivier Pironneau. *Applied Shape Optimization for Fluids.*
7. [MZ06]. Marwan Moubachir, Jean-Paul Zolésio. *Moving Shape Analysis & Control.*
8. Stephan Schmidt. Master course: *Shape & Geometry.* Humboldt University of Berlin. [written in German, taught in English & German].
9. [SZ92]. Jan Sokołowski, Jean-Paul Zolésio. *Introduction to Shape Optimization.*

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10. [Wal15]. Shawn W. Walker. *The Shapes of Things.*

Differential equations on surfaces. Differential geometry is useful for understanding mathematical models containing geometric PDEs, e.g., surface/manifold version of the standard Laplace equation, which requires the development of the surface gradient & surface Laplacian operators – the usual gradient ∇ & Laplacian $\Delta = \nabla \cdot \nabla$ operators defined on a surface (manifold) instead of standard Euclidean space $\mathbb{R}^n$. *Advantage*: provide alternative formulas for geometric quantities, e.g., the summed (mean) curvature, that are much clearer than the usual presentation of texts on differential geometry.

Differentiating w.r.t. Shape. The approach to differential geometry is advantageous for developing the framework of *shape differential calculus* – the study of how quantities change w.r.t. changes of independent "shape variable".

"The framework of shape differential calculus provides the tools for developing the equations of mean curvature flow & Willmore flow, which are geometric flows occurring in many applications such as fluid dynamics & biology." – [Wal15, p. 2]

The shape perturbation $\delta J(\Omega; V)$ is similar to the gradient operator, which is a directional derivative, analogous to $V \cdot \nabla f$ where V is a given direction, providing information about the local slope, or the sensitivity of a quantity w.r.t. some parameters.

It takes only 2 or 3 numbers to specify a point (x, y) in 2D & a point (x, y, z) in 3D, whereas an "infinite" number of coordinate pairs is needed to specify a domain Ω. V is a 2D/3D vector in the scalar function setting; for a shape functional, V is a full-blown function requiring definition at every point in Ω. This "infinite dimensionality" is the reason for using the notation $\delta J(\Omega; V)$ to denote a shape perturbation. $\delta J(\Omega; V)$ indicates how we should change Ω to decrease J, similarly to how $\nabla f(x, y)$ indicates how the coordinate pair (x, y) should change to decrease f, which opens up the world of shape optimization.

3 schools of shape optimization. Cf. engineering shape optimization vs. applied shape optimization [MP10] vs. theoretical shape optimization [SZ92; DZ11].

Example 1 ([Wal15], Sect. 1.2.1, pp. 1–2)**.** *Let $f = f(r, \theta)$ be a smooth function defined on the disk $B_{R,2}(0, 0)$ of radius R in terms of polar coordinates. The integral of f over $B_{2,R}(0, 0)$ $J := \int_{B_{2,R}(0,0)} f \, \mathrm{d}\mathbf{x} = \int_0^{2\pi} \int_0^R f(r, \theta) \, \mathrm{d}r \, \mathrm{d}\theta$ depends on R. Assume f also depends on R, i.e., $f = f(r, \theta, R)$ with a physical example: J is the* net flow rate *of liquid through a pipe with cross-section Ω, then f is the flow rate per unit area & could be the solution of a PDE defined on Ω, e.g., a Navier–Stokes fluid flowing in a circular pipe. Advantageous to know the* sensitivity *of J w.r.t. R, e.g., for optimization purposes. Differentiate J w.r.t. R:*

$$\frac{d}{dR} J = \int_0^{2\pi} \left(\frac{d}{dR} \int_0^R f(r, \theta; R) r \, \mathrm{d}r \right) \mathrm{d}\theta = \int_0^{2\pi} \int_0^R f'(r, \theta; R) r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} f(R, \theta; R) \, \mathrm{d}\theta.$$

The dependence of f on R can more generally be viewed as dependence on $B_{R,2}(0, 0)$, i.e., $f(\cdot; R) \equiv f(\cdot; B_{R,2}(0, 0))$. Rewriting d/dRJ using Cartesian coordinates $\mathbf{x}$:

$$\frac{d}{dR} J = \int_{B_{R,2}(0,0)} f'(\mathbf{x}; \Omega) \, \mathrm{d}\mathbf{x} + \int_{S_{R,2}(0,0)} f(\mathbf{x}; \Omega) \, \mathrm{d}S(\mathbf{x}), \tag{1}$$

where $\mathrm{d}\mathbf{x}$ is the volume measure, $\mathrm{d}S(\mathbf{x})$ is the surface area measure.

Example 2 (Surface height function of a hill)**.** *Let $f = f(x, y)$ be a function describing the surface height of the hill, where (x, y) are the coordinates of our position. Then, by using basic multivariate calculus, finding a direction that will move us downhill is equivalent to computing the gradient (vector) of f & moving in the opposite direction to the gradient. In this sense, we do not need to "see" the whole function. We just need to* locally *compute the gradient ∇f, analogous to feeling the ground beneath.*

4 Topology Optimization – Tối Ưu Tôpô

5 Miscellaneous

Tài liệu

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