# Linear Algebra – Đại Số Tuyến Tính

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**Tóm tắt nội dung**

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- *Linear Algebra – Đại Số Tuyến Tính.*
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  TEX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/linear_algebra/NQBH_linear_algebra.tex.

## Mục lục



## 1 Basic

Tôi được giải Nhì Đại số Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Đại số Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Đại số. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Đại số tuyến tính với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

**Resources – Tài nguyên.**

- [Hưn22]. Nguyễn Hữu Việt Hưng. *Đại Số Tuyến Tính.*
- [Tiệ25]. Vũ Hữu Tiệp. *Machine Learning Cơ Bản.*

  Mã nguồn cuốn ebook "Machine Learning Cơ Bản": https://github.com/tiepvupsu/ebookMLCB.

  Phép nhân từng phần/tích Hadamard (Hadamard product) thường xuyên được sử dụng trong ML. Tích Hadamard của 2 ma trận cùng kích thước $A, B \in \mathbb{R}^{m\times n}$, được ký hiệu là $A \odot B = (a_{ij}b_{ij})_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}$.

  Việc chuyển đổi hệ cơ sở sử dụng ma trận trực giao có thể được coi như 1 phép xoay trục tọa độ. Nhìn theo 1 cách khác, đây cũng chính là 1 phép xoay vector dữ liệu theo chiều ngược lại, nếu ta coi các trục tọa độ là cố định.

  Việc phân tích 1 đại lượng toán học ra thành các đại lượng nhỏ hơn mang lại nhiều hiệu quả. Phân tích 1 số thành tích các thừa số nguyên tố giúp kiểm tra 1 số có bao nhiêu ước số. Phân tích đa thức thành nhân tử giúp tìm nghiệm của đa thức. Việc phân tích 1 ma trận thành tích của các ma trận đặc biệt cũng mang lại nhiều lợi ích trong việc giải hệ phương trình tuyến tính, tính lũy thừa của ma trận, xấp xỉ ma trận, ...

  **Phép phân tích trị riêng.** Cách biểu diễn 1 ma trận vuông $A$ với $\mathbf{x}_i \neq \mathbf{0}$ là các vector riêng của 1 ma trận vuông $A$ ứng với các giá trị riêng lặp hoặc phức $\lambda_i$: $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $\forall i = 1, \ldots, n$: $A = X\Lambda X^{-1}$ với $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$.

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**Norm – Chuẩn.** Khoảng cách Euclid chính là độ dài đoạn thẳng nối 2 điểm trong mặt phẳng. Đôi khi, để đi từ 1 điểm này tới 1 điểm kia, không thể đi bằng đường thẳng vì còn phụ thuộc vào hình dạng đường đi nối giữa 2 điểm. Cf. đường trắc địa trong Hình học Vi phân – geodesics in Differential Geometry. Việc đo khoảng cách giữa 2 điểm dữ liệu nhiều chiều rất cần thiết trong ML – chính là lý do khái niệm *chuẩn* (norm) ra đời.

**Trace – Vết.** *Vết* (trace) của 1 ma trận vuông $A$ được ký hiệu là $\operatorname{trace} A$ là tổng tất cả các phần tử trên đường chéo chính của nó. Hàm vết xác định trên tập các ma trận vuông được sử dụng nhiều trong tối ưu vì nó có các tính chất đẹp.

**Kiểm tra gradient.** Việc tính gradient của hàm nhiều biến thông thường khá phức tạp & rất dễ mắc lỗi. Trong thực nghiệm, có 1 cách để kiểm tra liệu gradient tính được có chính xác không. Cách này dựa trên định nghĩa của đạo hàm cho hàm 1 biến.

- [TB97; TB22]. Lloyd N. Trefethen, David Bau III. *Numerical Linear Algebra.*

# 2 Wikipedia

## 2.1 Wikipedia/abstract structure

"An *abstract structure* is an abstraction that might be of the geometric spaces or a set structure, or a hypostatic abstraction that is defined by a set of mathematical theorems & laws, properties, & relationships in a way that is logically if not always historically independent[1] of the structure of contingent experiences, e.g., those involving physical objects. Abstract structures are studied not only in local & mathematics but in the fields that apply them, as computer science & computer graphics, & in the studies that reflect on them, such as philosophy (especially the philosophy of mathematics). Indeed, modern mathematics has been defined in a very general sense as the study of abstract structures (by the Bourbaki group: see discussion there, at algebraic structure & also structure).

An abstract structure may be represented (perhaps with some degree of approximation) by 1 or more physical objects – this is called an implementation or instantiation of the abstract structure. But the abstract structure itself is defined in a way that is not dependent on the properties of any particular implementation.

An abstract structure has a richer structure than a concept or an idea. An abstract structure must include precise rules of behavior which can be used to determine whether a candidate implementation actually matches the abstract structure in question, & it must be free from contradictions. Thus we may debate how well a particular government fits the concept of democracy, but there is no room for debate over whether a given sequence of moves is or is not a valid game of chess (e.g. Kasparovian approaches).

### 2.1.1 Examples

- A sorting algorithm is an abstract structure, but a recipe is not, because it depends on the properties & quantities of its ingredients.
- A simple melody is an abstract structure, but an orchestration is not, because it depends on the properties of particular instruments.
- Euclidean geometry is an abstract structure, but the theory of continent drift is not, because it depends on the geology of the Earth.
- A formal language is an abstract structure, but a natural language is not, because its rules of grammar & syntax are open to debate & interpretation.

## 2.2 Wikipedia/direct sum

"The *direct sum* is an operation between structures in abstract algebra, a branch of mathematics. It is defined differently, but analogously, for different kinds of structures. E.g., the direct sum of 2 abelian groups $A, B$ is another abelian group $A \oplus B$ consisting of the ordered pairs $(a, b)$ where $a \in A, b \in B$. To add ordered pairs, we define the sum $(a, b) + (c, d) := (a + c, b + d)$, i.e., addition is defined coordinate-wise. E.g., the direct sum $\mathbb{R} \oplus \mathbb{R}$ where $\mathbb{R}$ is real coordinate space, is the Cartesian plane $\mathbb{R}^2$. A similar process can be used to form the direct sum of 2 vector spaces or 2 modules.

We can also form direct sums with any finite number of summands, e.g., $A \oplus B \oplus C$, provided $A, B, C$ are the same kinds of algebraic structures (e.g., all abelian groups, or all vector spaces). This relies on the fact that the direct sum is associative up to isomorphism. I.e., $(A \oplus B) \oplus C \cong A \oplus (B \oplus C)$ for any algebraic structures $A, B, C$ of the same kind. The direct sum is also commutative up to isomorphism, i.e., $A \oplus B \cong B \oplus A$ for any algebraic structures $A, B$ of the same kind.

The direct sum of finitely many abelian groups, vector spaces, or modules is canonically isomorphic to the corresponding direct product. This is false, however, for some algebraic object, like nonabelian groups.

In the case where infinitely many objects are combined, the direct sum & direct product are not isomorphic, even for abelian groups, vector spaces, or modules. E.g., consider the direct sum & direct product of (countably) infinitely many copies of the integers. All element in the direct product is an infinite sequence, e.g., $(1, 2, 3, \dots)$ but in the direct sum, there is a requirement that all but finitely many coordinates be zero, so the sequence $(1, 2, 3, \dots)$ would be an element of the direct product but not of

[1] However historical dependencies are partially considered in event theory as part of the combinatorics theory in Kolmogorov complexity & Kolmogorov-Khinchin equations.

the direct sum, while $(1, 2, 0, 0, \dots)$ would be an element of both. Often, if a + sign is used, all but finitely many coordinates must be zero, while if some form of multiplication is used, all but finitely many coordinates must be 1. In more technical language, if the summands are $(A_i)_{i \in I}$, the direct sum $\bigoplus_{i \in I} A_i$ is defined to be the set of tuples $(a_i)_{i \in I}$ with $a_i \in A_i$ s.t. $a_i = 0$ for all but finitely many $i$. The direct sum $\bigoplus_{i \in I} A_i$ is contained in the direct product $\prod_{i \in I} A_i$, but is strictly smaller when the index set $I$ is infinite, because an element of the direct product can have infinitely many nonzero coordinates.

### 2.2.1 Examples

The $xy$-plane, a 2D vector space, can be thought of as the direct sum of 2 1D vector spaces, namely the $x$ & $y$ axes. In this direct sum, the $x, y$ axes intersect only at the origin (the zero vector). Addition is defined coordinate-wise, i.e., $(x_1, y_1) + (x_2, y_2) \coloneqq (x_1 + x_2, y_1 + y_2)$, which is the same as vector addition.

Given 2 structures $A, B$, their direct sum is written as $A \oplus B$. Given an indexed family of structures $A_i$, indexed with $i \in I$, the direct sum may be written $A = \bigoplus_{i \in I} A_i$. Each $A_i$ is called a *direct summand* of $A$. If the index set is finite, the direct sum is the same as the direct product. In the case of groups, if the group operation is written as + the phrase "direct sum" is used, while if the group operation is written $*$ the phrase "direct product" is used. When the index set is infinite, the direct sum is not the same as the direct product since the direct sum has the extra requirement that all but finitely many coordinates must be 0.

**Internal & external direct sums.** A distinction is made between internal & external direct sums, though the 2 are isomorphic. If the summands are defined 1st, & then the direct sum is defined in terms of the summands, we have an external direct sum. E.g., if we define the real numbers $\mathbb{R}$ & then define $\mathbb{R} \oplus \mathbb{R}$ the direct sum is said to be *external*.

If, on the other hand, 1st define some algebraic structure $S$ & then write $S$ as a direct sum of 2 substructures $V, W$, then the direct sum is said to be internal. In this case, each element of $S$ is expressible uniquely as an algebraic combination of an element of $V$ & an element of $W$. For an example of an internal direct sum, consider $\mathbb{Z}_6$ (the integers modulo 6), whose elements are $\{0, 1, 2, 3, 4, 5\}$. This is expressible as an internal direct sum $\mathbb{Z}_6 = \{0, 2, 4\} \oplus \{0, 3\}$.

### 2.2.2 Types of direct sum

[...]

### 2.2.3 Homomorphisms

The direct sum $\bigoplus_{i \in I} A_i$ comes equipped with a *projection* homomorphism $\pi_j : \bigoplus_{i \in I} A_i \to A_j$ for each $j \in I$ & a *coprojection* $\alpha_j : A_j \to \bigoplus_{i \in I} A_i$ for each $j \in I$. Given another algebraic structure $B$ (with the same additional structure) & homomorphisms $g_j : A_j \to B$, $\forall j \in I$, there is a unique homomorphism $g : \bigoplus_{i \in I} A_i \to B$, called the sum of the $g_j$, s.t. $g\alpha_j = g_j$, $\forall j$. Thus the direct sum is the coproduct in the appropriate category." – Wikipedia/direct sum

## 2.3 Wikipedia/mathematical structure

"In mathematics, a structure on a set (or on some sets) refers to providing it (or them) with certain additional features (e.g. an operation, relation, metric, or topology). The additional features are attached or related to the set (or to the sets), so as to provide it (or them) with some additional meaning or significance.

A partial list of possible structures are measures, algebraic structures (groups, fields, etc.), topologies, metric structures (geometries), orders, graphs, events, equivalence relations, differential structures, & categories.

Sometimes, a set is endowed with $> 1$ feature simultaneously, which allows mathematicians to study the interaction between the different structures more richly. E.g., an ordering imposes a rigid form, shape, or topology on the set, & if a set has both a topology feature & a group feature, s.t. these 2 features are related in a certain way, then the structure becomes a topological group.

Map between 2 sets with the same type of structure, which preserve this structure [morphism: structure in the domain is mapped properly to the (same type) structure in the codomain] is of special interest in many fields of mathematics. E.g.: homomorphisms, which preserve algebraic structures; continuous functions, which preserve topological structures; & differential functions, which preserve differential structures.

### 2.3.1 History

In 1939, the French group with the pseudonym Nicolas Bourbaki saw structures as the root of mathematics. They 1st mentioned them in their "Fascicule" of *Theory of Sets* & expanded it into Chap. IV of the 1957 edition. They identified 3 *mother structures*: algebraic, topological, & order.

### 2.3.2 Example: the real numbers

"The set of real numbers has several standard structures:

- An order: each number is either less than or greater than any other number.
- Algebraic structure: there are operations of addition & multiplication, the 1st of which makes it into a group & the pair of which together make it into a field.

- A measure: intervals of the real line have a specific length, which can be extended to the Lebesgue measure on many of its subsets.
- A metric: there is a notion of distance between points.
- A geometry: it is equipped with a metric & is flat.
- A topology: there is a notion of open sets.

There are interfaces among these:

- Its order &, independently, its metric structure induce its topology.
- Its order & algebraic structure make it into an ordered field.
- Its algebraic structure & topology make it into a Lie group, a type of topological group." – Wikipedia/mathematical structure

# 3 Miscellaneous

## Tài liệu


[Hưn22] Nguyễn Hữu Việt Hưng. *Đại Số Tuyến Tính*. Tái bản lần thứ 4. Nhà Xuất Bản Đại Học Quốc Gia Hà Nội, 2022, p. 335.

[TB22] Lloyd N. Trefethen and David Bau III. *Numerical linear algebra*. 25th anniversary edition [of 1444820], With a foreword by James G. Nagy. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2022] ©2022, pp. xvi+370. ISBN: 978-1-611977-15-8; [9781611977165].

[TB97] Lloyd N. Trefethen and David Bau III. *Numerical linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997, pp. xii+361. ISBN: 0-89871-361-7. DOI: 10.1137/1.9780898719574. URL: https://doi.org/10.1137/1.9780898719574.

[Tiệ25] Vũ Khắc Tiệp. *Machine Learning Cơ Bản*. 2025, p. 422.