On the smallest constant for a Gagliardo-Nirenberg functional inequality

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Abstract

The main objective of this paper is to present a relationship between the best constant for a classical interpolation inequality due to Nirenberg and Gagliardo, and the ground state solution of the equation

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0.$$

Contents

1 Solution of a variational problem

We begin by studying

$$J^{\sigma,N}(f) := \frac{\|\nabla f\|_2^{\sigma N}\|f\|_2^{2+\sigma(2-N)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}, \tag{1.1}$$

the nonlinear functional naturally associated with the interpolation estimate

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leq C_{\sigma,N}^{2\sigma+2} \|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}, \text{ if } 0<\sigma<\frac{2}{N-2},\ N \geq 2. \tag{1.2}$$

By estimate (1.2), $J^{\sigma,N}$ is defined on $H^1(\mathbb{R}^N)$ for $0<\sigma<\frac{2}{N-2}$[1].

Theorem 1.1. *For $0<\sigma<\frac{2}{N-2}$,*

$$\alpha := \inf_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$$

is attained at a function ψ with the following properties:

1. *ψ is positive and a function of $|x|$ alone.*
2. *$\psi \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.*
3. *ψ is a solution of the following equation*

$$\frac{\sigma N}{2}\Delta\psi - \left(1+\frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0, \tag{1.3}$$

of minimal L^2 norm (the ground state).

In addition,

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}.$$

In the proof of Theorem 1.1, we follow Strauss [5] in using a compactness property of functions in $H^1_{\text{radial}}(\mathbb{R}^N)$.

1.1 Strauss's estimate

Proposition 1.1 (Proposition 1.7.1, [2], p. 20)**.** *Let $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ be a bounded sequence of spherically symmetric functions. If $N \geq 2$ or if $u_n(x)$ is a nonincreasing function of $|x|$ for every $n \geq 0$, then there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \to u$ as $k \to \infty$ in $L^p(\mathbb{R}^N)$ for every $2<p<\frac{2N}{N-2}$ ($2<p\leq\infty$ if $N=1$).*

Proposition 1.1 is an immediate consequence of the Lemma 1.1 and 1.2.

Proof. If $N \geq 2$, we apply the first estimate (1.4) in Lemma 1.2 to each spherically symmetric functions $u_n \in H^1(\mathbb{R}^N)$ to obtain

$$|u_n(x)| \leq \frac{C\|u_n\|_{L^2}^{\frac{1}{2}} \|\nabla u_n\|_{L^2}^{\frac{1}{2}}}{|x|^{\frac{N-1}{2}}} \leq \frac{C\|u_n\|_{H^1}}{|x|^{\frac{N-1}{2}}} \leq \frac{C}{|x|^{\frac{N-1}{2}}},\quad \forall x \in \mathbb{R}^N,\ \forall n \geq 0.$$

[1] Indeed, let $f \in H^1(\mathbb{R}^N)$ and suppose the interpolation estimate (1.2) holds. Since $\|f\|_2 < \infty$ and $\|\nabla f\|_2 < \infty$, (1.2) implies that $f \in L^{2\sigma+2}(\mathbb{R}^N)$ for all $0<\sigma<\frac{2}{N-2}$. Thus, the nonlinear function $J^{\sigma,N}$ defined by (1.1) makes a sense for all $f \in H^1(\mathbb{R}^N)$ for all $0<\sigma<\frac{2}{N-2}$.

where the last inequality is deduced from the boundedness of u_n's.

If $u_n(x)$ is a nonincreasing function of $|x|$ for every $n \geq 0$, we applying the second estimate (1.5) in Lemma 1.2 to u_n to obtain

$$|u_n(x)| \leq \frac{C\|u_n\|_{L^2}}{|x|^{\frac{N}{2}}} \leq \frac{C}{|x|^{\frac{N}{2}}}, \ \ \forall x \in \mathbb{R}^N, \ \forall n \geq 0.$$

In both cases, these estimates imply that $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$. Now, we can apply Lemma 1.1 to obtain the desired result. □

Lemma 1.1 (Lemma 1.7.2, [2], p. 20)**.** *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. Suppose $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$. It follows that there exist a subsequence $u_{n_k} \to u$ as $k \to \infty$ in $L^p(\mathbb{R}^N)$ for every $2 < p < \frac{2N}{N-2}$ ($2 < p \leq \infty$ if $N = 1$).*

Remark 1.1 (Remark 1.3.1(iii), [2], p.7)**.** *Assume $m \geq 1$ and $1 < p \leq \infty$. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{m,p}(\Omega)$, then there exist $u \in W^{m,p}(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. as $k \to \infty$, and*

$$\|u\|_{W^{m,p}} \leq \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}.$$

If $p < \infty$, then also $u_{n_k} \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in N} \subset W_0^{m,p}(\Omega)$, then $u \in W_0^{m,p}(\Omega)$.

Applying this remark for a bounded sequence in $H^1(\mathbb{R}^N)$, there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. as $k \to \infty$, $\|u\|_{H^1(\mathbb{R}^N)} \leq \liminf \|u_n\|_{H^1(\mathbb{R}^N)}$ and $u_{n_k} \rightharpoonup u$ in $H^1(\mathbb{R}^N)$.

Proof of Lemma 1.1. Since $(u_n)_{n\geq 0}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, applying Remark 1.1 yields that there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \rightharpoonup u$ as $k \to \infty$ in $H^1(\mathbb{R}^N)$. Fix $\varepsilon > 0$ and let $R > 0$ to be chosen later. Given $p \in \left(2, \frac{2N}{N-2}\right)$ ($2 < p \leq \infty$ if $N = 1$), we have[2]

$$\begin{aligned}
\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} &= \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^p(\{|x|\geq R\})} \\
&\leq \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}}.
\end{aligned}$$

We first fix R large enough so that (by uniform convergence)

$$\|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}} \leq \frac{\varepsilon}{2}.$$

[2] Here we use

$$\begin{aligned}
\|u_{n_k} - u\|_{L^p(\{|x|\geq R\})} &= \left(\int_{\{|x|\geq R\}} |u_{n_k} - u|^p\right)^{\frac{1}{p}} \leq \left(\|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{p-2} \int_{\{|x|\geq R\}} |u_{n_k} - u|^2\right)^{\frac{1}{p}} \\
&\leq \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\{|x|\geq R\})}^{\frac{2}{p}} \leq \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}^{\frac{2}{p}}.
\end{aligned}$$

Next, since $\left(u_{n_k}|_{B_R}\right)_{k\geq 0}$ is bounded in $H^1(B_R)$, it follows from Rellich's compactness theorem 2.1 (i) that $u_{n_k}|_{B_R} \to u|_{B_R}$ in $L^p(B_R)$. Therefore for k large enough we have

$$\|u_{n_k} - u\|_{L^p(B_R)} \leq \frac{\varepsilon}{2},$$

and so $\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} \leq \varepsilon$. This proves the result. $\square$

Lemma 1.2 (Lemma 1.7.3, [2], p. 21). *If $u \in H^1(\mathbb{R}^N)$ is a radially symmetric function, then*

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N-1}{2}} |u(x)| \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{1.4}$$

If, in addition, $u(x)$ is a nonincreasing function of $|x|$, then

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N}{2}} |u(x)| \leq C\|u\|_{L^2}. \tag{1.5}$$

Proof. Suppose first $u \in C_c^\infty(\mathbb{R}^N)$. Since u is radially symmetric, there exists a function $\widetilde{u} : \mathbb{R}^+ \to \mathbb{R}$ such that $u(x) = \widetilde{u}(|x|)$ for all $x \in \mathbb{R}^N$. Simple computation gives us $|\nabla u(x)| = |\widetilde{u}'(r)|$ where $r = |x|$. We have

$$\begin{aligned}
\|u\|_{L^2(\mathbb{R}^N)}^2 &= \left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right) = \int_{\partial B_1(0)} \left(\int_0^\infty |u(ry)|^2 r^{N-1} dr\right) dS(y) \\
&= \int_{\partial B_1(0)} \left(\int_0^\infty \widetilde{u}(r)^2 r^{N-1} dr\right) dS(y) = N\alpha_N \int_0^\infty \widetilde{u}(r)^2 r^{N-1} dr, \\
\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 &= \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx\right) = \int_{\partial B_1(0)} \left(\int_0^\infty |\nabla u(ry)|^2 r^{N-1} dr\right) dS(y) \\
&= \int_{\partial B_1(0)} \left(\int_0^\infty \widetilde{u}'(r)^2 r^{N-1} dr\right) dS(y) = N\alpha_N \int_0^\infty \widetilde{u}'(r)^2 r^{N-1} dr,
\end{aligned}$$

and

$$\begin{aligned}
r^{N-1}\widetilde{u}(r)^2 &= -\int_r^\infty \frac{d}{ds}\left(s^{N-1}\widetilde{u}(s)^2\right) ds = -\int_r^\infty \left(\underbrace{(N-1)\,s^{N-2}\widetilde{u}(s)^2}_{\geq 0} + 2s^{N-1}\widetilde{u}(s)\,\widetilde{u}'(s)\right) ds \\
&\leq -2\int_r^\infty s^{N-1}\widetilde{u}(s)\,\widetilde{u}'(s)\,ds \leq 2\left(\int_r^\infty s^{N-1}\widetilde{u}(s)^2 ds\right)^{\frac{1}{2}} \left(\int_r^\infty s^{N-1} u'(s)^2 ds\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_0^\infty s^{N-1}\widetilde{u}(s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^\infty s^{N-1} u'(s)^2 ds\right)^{\frac{1}{2}} \\
&\leq 2\frac{\|u\|_{L^2(\mathbb{R}^N)}}{\sqrt{N\alpha_N}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}}{\sqrt{N\alpha_N}} = \frac{2}{N\alpha_N}\|u\|_{L^2(\mathbb{R}^N)}\|\nabla u\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

where and α_N is the volume of the unit ball in $\mathbb{R}^N$, which is given by $\alpha_N := \frac{2\pi^{\frac{N}{2}}}{N\Gamma\left(\frac{N}{2}\right)}$. That means (1.4) holds for all $u \in C_c^\infty(\mathbb{R}^N)$.

If $u(x)$ is a nonincreasing function of $|x|$, then for all $r \geq 0$,

$$\|u\|_{L^2}^2 = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right) \geq \left(\int_{\{|x|\leq r\}} |u(x)|^2 dx\right) \geq |\{|x| \leq r\}|\, |\widetilde{u}(r)|^2 = \alpha_N \mathbb{R}^N |\widetilde{u}(r)|^2,$$

i.e., (1.5) holds for all $u \in C_c^\infty(\mathbb{R}^N)$.

The general case then follows by a density argument. □

1.2 Proof of Theorem 1.1

Proof of Theorem 1.1. First note that if we set $u^{\lambda,\mu}(x) := \mu u(\lambda x)$, then $\nabla u^{\lambda,\mu}(x) = \mu\lambda\nabla u(\lambda x)$, and

$$\left\|u^{\lambda,\mu}\right\|_2^2 = \int_{\mathbb{R}^N} \left|u^{\lambda,\mu}(x)\right|^2 dx = \int_{\mathbb{R}^N} |\mu u(\lambda x)|^2 dx = \frac{\mu^2}{\lambda^N}\int_{\mathbb{R}^N} |u(x)|^2 dx = \frac{\mu^2}{\lambda^N}\|u\|_2^2,$$

$$\left\|u^{\lambda,\mu}\right\|_{2\sigma+2}^{2\sigma+2} = \int_{\mathbb{R}^N} \left|u^{\lambda,\mu}(x)\right|^{2\sigma+2} dx = \int_{\mathbb{R}^N} |\mu u(\lambda x)|^{2\sigma+2} dx = \frac{|\mu|^{2\sigma+2}}{\lambda^N}\|u\|_{2\sigma+2}^{2\sigma+2},$$

$$\left\|\nabla u^{\lambda,\mu}\right\|_2^2 = \int_{\mathbb{R}^N} \left|\nabla u^{\lambda,\mu}(x)\right|^2 dx = \int_{\mathbb{R}^N} |\mu\lambda\nabla u(\lambda x)|^2 dx = \frac{\mu^2}{\lambda^{N-2}}\|\nabla u\|_2^2,$$

$$J^{\sigma,N}\left(u^{\lambda,\mu}\right) = \frac{\|\nabla u^{\lambda,\mu}\|_2^{\sigma N}\,\|u^{\lambda,\mu}\|_2^{2\sigma+2-\sigma N}}{\|u^{\lambda,\mu}\|_{2\sigma+2}^{2\sigma+2}} = \frac{\left(\frac{\mu^2}{\lambda^{N-2}}\right)^{\frac{\sigma N}{2}}\|\nabla u\|_2^{\sigma N}\left(\frac{\mu^2}{\lambda^N}\right)^{\frac{2\sigma+2-\sigma N}{2}}\|u\|_2^{2\sigma+2-\sigma N}}{\frac{\mu^{2\sigma+2}}{\lambda^N}\|u\|_{2\sigma+2}^{2\sigma+2}}$$

$$= \frac{\|\nabla u\|_2^{\sigma N}\,\|u\|_2^{2\sigma+2-\sigma N}}{\|u\|_{2\sigma+2}^{2\sigma+2}} = J^{\sigma,N}(u).$$

Since $J^{\sigma,N}(u) \geq 0$, there exists a minimizing sequence $u_v \in H^1(\mathbb{R}^N) \cap L^{2\sigma+2}(\mathbb{R}^N)$, i.e., $\alpha = \inf_{u\in H^1(\mathbb{R}^N)} J^{\sigma,N}(u) = \lim_{v\uparrow\infty} J^{\sigma,N}(u_v) < \infty$. We can assume $u_v > 0$ (since $J^{\sigma,N}(u) = J^{\sigma,N}(-u)$), and by symmetrization we can take $u_v = u_v(|x|)$[3].

Choosing $\lambda_v = \frac{\|u_v\|_2}{\|\nabla u_v\|_2}$, $\mu_v = \frac{\|u_v\|_2^{\frac{N}{2}-1}}{\|\nabla u_v\|_2^{\frac{N}{2}}}$[4], we obtain a sequence $\psi_v(x) := u^{\lambda_v,\mu_v}(x)$ with the following properties:

(a) $\psi_v(x) \geq 0$, $\psi_v = \psi_v(|x|)$,

(b) $\psi_v \in H^1(\mathbb{R}^N)$,

(c) $\|\psi_v\|_2 = 1$, and $\|\nabla\psi_v\|_2 = 1$,

(d) $J^{\sigma,N}(\psi_v) \downarrow \alpha$ as $v \to \infty$.

[3] Indeed, since for any $u \in H^1(\mathbb{R}^N) \cap L^{2\sigma+2}(\mathbb{R}^N)$, its symmetric-decreasing rearrangement u^* satisfies $\|u^*\|_{2\sigma+2} = \|u\|_{2\sigma+2}$, $\|u^*\|_2 = \|u\|_2$, $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$, and thus $J^{\sigma,N}(u^*) \leq J^{\sigma,N}(u)$. Hence, it suffices to consider only radially symmetric functions to minimize $J^{\sigma,N}$.

[4] Solve $\|u_v^{\lambda,\mu}\|_2 = \|\nabla_x u_v^{\lambda,\mu}\|_2 = 1$ to obtain λ_v and μ_v.

Since the sequence ψ_v is bounded in $H^1\left(\mathbb{R}^N\right)$, some subsequence has a weak H^1 limit ψ^*. Since ψ_v are radial and uniformly bounded in $H^1\left(\mathbb{R}^N\right)$, it follows from the compactness lemma that we can take ψ_v strongly convergent to ψ^* in $L^{2\sigma+2}\left(\mathbb{R}^N\right)$ for $0<\sigma<\frac{2}{N-2}$. By weak convergence, $\|\psi^*\|_2 \leq 1$ and $\|\nabla\psi^*\|_2 \leq 1$. Hence,

$$\alpha \leq J^{\sigma,N}\left(\psi^*\right) \leq \frac{1}{\int |\psi^*|^{2\sigma+2}dx} = \lim_{v\uparrow\infty} J\left(\psi_v\right) = \alpha.$$

It follows that $\|\nabla\psi^*\|_2^{\sigma N}\|\psi^*\|_2^{2+\sigma(2-N)} = 1$ and therefore $\|\psi^*\|_2 = \|\nabla\psi^*\|_2 = 1$, so $\psi_v \to \psi^*$ strongly in H^1 [5]. This proves part (1) and (2) of Theorem 1.1.

Part (3) follows from the fact that ψ^*, the minimizing function, is in $H^1\left(\mathbb{R}^N\right)$ and satisfies the Euler-Lagrange equation:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J^{\sigma,N}\left(\psi^*+\varepsilon\eta\right) = 0, \;\; \forall\eta \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.6}$$

Taking into account that $\|\psi^*\|_2 = 1$ and $\|\nabla\psi^*\|_2 = 1$, we have

$$\frac{\sigma N}{2}\Delta\psi^* - \left(1+\frac{\sigma}{2}\left(2-N\right)\right)\psi^* + \alpha\left(\sigma+1\right)\left(\psi^*\right)^{2\sigma+1} = 0. \tag{1.7}$$

Indeed, given $\eta \in C_0^\infty\left(\mathbb{R}^N\right)$, we define

$$g_\eta\left(\varepsilon\right) := J^{\sigma,N}\left(\psi^*+\varepsilon\eta\right) = \frac{\left(\int_{\mathbb{R}^N}|\nabla\psi^*+\varepsilon\nabla\eta|^2dx\right)^{\frac{\sigma N}{2}}\left(\int_{\mathbb{R}^N}|\psi^*+\varepsilon\eta|^2dx\right)^{\frac{2+\sigma(2-N)}{2}}}{\int_{\mathbb{R}^N}|\psi^*+\varepsilon\eta|^{2\sigma+2}dx}.$$

then (1.6) is equivalent to

$$g_\eta{}'\left(0\right) = 0, \;\; \forall\eta \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.8}$$

For simplicity, define

$$\begin{aligned} A\left(\varepsilon\right) &:= \int_{\mathbb{R}^N}|\nabla\psi^*+\varepsilon\nabla\eta|^2dx, \\ B\left(\varepsilon\right) &:= \int_{\mathbb{R}^N}|\psi^*+\varepsilon\eta|^2dx, \\ C\left(\varepsilon\right) &:= \int_{\mathbb{R}^N}|\psi^*+\varepsilon\eta|^{2\sigma+2}dx. \end{aligned}$$

In particular,

$$\begin{aligned} A\left(0\right) &= \int_{\mathbb{R}^N}|\nabla\psi^*|^2dx = \|\nabla\psi^*\|_2^2 = 1, \\ B\left(0\right) &= \int_{\mathbb{R}^N}|\psi^*|^2dx = \|\psi^*\|_2^2 = 1, \end{aligned}$$

[5] If $x_n \rightharpoonup x$ in a Hilbert space H, and $\|x_n\|_H \to \|x\|_H$, then x_n converges to x strongly.

$$C(0) = \int_{\mathbb{R}^N} |\psi^*|^{2\sigma+2} dx = \frac{1}{\alpha}.$$

The derivatives of these functions are given by

$$\begin{aligned}
A'(\varepsilon) &= \int_{\mathbb{R}^N} \frac{d}{d\varepsilon} |\nabla\psi^* + \varepsilon\nabla\eta|^2 dx = 2\int_{\mathbb{R}^N} (\nabla\psi^* + \varepsilon\nabla\eta)\cdot\nabla\eta dx,\\
B'(\varepsilon) &= \int_{\mathbb{R}^N} \frac{d}{d\varepsilon}\left(|\psi^* + \varepsilon\eta|^2\right) dx = 2\int_{\mathbb{R}^N} (\psi^* + \varepsilon\eta)\,\eta dx,\\
C'(\varepsilon) &= \int_{\mathbb{R}^N} \frac{d}{d\varepsilon}\left(|\psi^* + \varepsilon\eta|^{2\sigma+2}\right) dx = 2(\sigma+1)\int_{\mathbb{R}^N} \operatorname{sign}(\psi^* + \varepsilon\eta)\,|\psi^* + \varepsilon\eta|^{2\sigma+1}\eta dx.
\end{aligned}$$

In particular,

$$\begin{aligned}
A'(0) &= 2\int_{\mathbb{R}^N} \nabla\psi^*\cdot\nabla\eta dx,\\
B'(0) &= 2\int_{\mathbb{R}^N} \psi^*\eta dx,\\
C'(0) &= 2(\sigma+1)\int_{\mathbb{R}^N} \operatorname{sign}(\psi^*)\,|\psi^*|^{2\sigma+1}\eta dx = 2(\sigma+1)\int_{\mathbb{R}^N} (\psi^*)^{2\sigma+1}\eta dx.
\end{aligned}$$

where the last equality is deduced from the fact that $\psi^* \geq 0$.

Now we compute $g'_\eta(\varepsilon)$. Note that

$$g_\eta(\varepsilon) = \frac{A(\varepsilon)^{\frac{\sigma N}{2}} B(\varepsilon)^{1+\frac{\sigma(2-N)}{2}}}{C(\varepsilon)},$$

its derivative is given by

$$\begin{aligned}
g'_\eta(\varepsilon) = &\frac{\frac{\sigma N}{2} A(\varepsilon)^{\frac{\sigma N}{2}-1} A'(\varepsilon)\, B(\varepsilon)^{1+\frac{\sigma(2-N)}{2}} + \left(1+\frac{\sigma(2-N)}{2}\right) A(\varepsilon)^{\frac{\sigma N}{2}} B(\varepsilon)^{\frac{\sigma(2-N)}{2}} B'(\varepsilon)}{C(\varepsilon)}\\
&- \frac{A(\varepsilon)^{\frac{\sigma N}{2}} B(\varepsilon)^{1+\frac{\sigma(2-N)}{2}} C'(\varepsilon)}{C^2(\varepsilon)},
\end{aligned}$$

and then

$$\begin{aligned}
g'_\eta(0) &= \frac{\frac{\sigma N}{2} A'(0) + \left(1+\frac{\sigma(2-N)}{2}\right) B'(0)}{C(0)} - \frac{C'(0)}{C^2(0)}\\
&= 2\alpha\left[\frac{\sigma N}{2}\int_{\mathbb{R}^N} \nabla\psi^*\cdot\nabla\eta dx + \left(1+\frac{\sigma(2-N)}{2}\right)\int_{\mathbb{R}^N} \psi^*\eta dx - \alpha(\sigma+1)\int_{\mathbb{R}^N} (\psi^*)^{2\sigma+1}\eta dx\right].
\end{aligned}$$

Combining this with (1.8) yields

$$\frac{\sigma N}{2}\int_{\mathbb{R}^N} \nabla\psi^*\cdot\nabla\eta dx + \left(1+\frac{\sigma(2-N)}{2}\right)\int_{\mathbb{R}^N} \psi^*\eta dx - \alpha(\sigma+1)\int_{\mathbb{R}^N} (\psi^*)^{2\sigma+1}\eta dx = 0,\ \ \forall\eta\in C_0^\infty\left(\mathbb{R}^N\right),$$

i.e.,

$$\frac{\sigma N}{2}\Delta\psi^* - \left(1+\frac{\sigma}{2}(2-N)\right)\psi^* + \alpha(\sigma+1)(\psi^*)^{2\sigma+1} = 0 \text{ in } \mathcal{D}'.$$

Let $\psi = [\alpha(\sigma+1)]^{\frac{1}{2\sigma}}\psi^*$, then

i) ψ is positive and radially symmetric.

ii) $\psi \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.

iii) ψ satisfies

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0 \text{ in } \mathcal{D}'. \tag{1.9}$$

Now we regularize ψ by a bootstrap argument:

$\star$ *Step 1*: Since $\psi \in H^1_{\text{radial}}(\mathbb{R}^N)$, the Compactness Lemma implies that $\psi \in L^{2\sigma+2}(\mathbb{R}^N)$, and thus $\psi^{2\sigma+1} \in L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^N)$. Since $1 < \frac{2\sigma+2}{2\sigma+1} < 2$, we have implies that $L^2(\mathbb{R}^N) \hookrightarrow L^{\frac{2\sigma+1}{2\sigma+2}}_{\text{loc}}(\mathbb{R}^N)$, and consequently $\psi \in L^{\frac{2\sigma+1}{2\sigma+2}}_{\text{loc}}(\mathbb{R}^N)$. Then (1.9) implies that $\Delta\psi \in L^{\frac{2\sigma+2}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Using elliptic regularity, it follows that $\psi \in W^{2,\frac{2\sigma+2}{2\sigma+1}}_{loc}(\mathbb{R}^N)$.

Similarly, we can prove that[6]

Statement 1: *If* $\psi \in L^q_{\text{loc}}(\mathbb{R}^N)$, *then* $\psi \in W^{2,\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

Put $q_0 := 2\sigma + 2$, we currently have $\psi \in W^{2,\frac{q_0}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. We consider the following cases depending on σ and N:

- *Case* $\frac{2\sigma+1}{q_0} < \frac{2}{N}$: Applying the general Sobolev embedding theorem 2.2 ii) to $(k, N, p) = \left(2, N, \frac{q_0}{2\sigma+1}\right)$ implies $\psi \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.
- *Case* $\frac{2\sigma+1}{q_0} = \frac{2}{N}$: Applying the general Sobolev embedding theorem 2.3 to $(k, N, p) = \left(2, N, \frac{q_0}{2\sigma+1}\right)$ implies $\psi \in L^r_{\text{loc}}(\mathbb{R}^N)$ for all $r \in \left[\frac{N}{2}, +\infty\right)$. In particular, choosing $r = (\sigma+1)N > \frac{N}{2}$, we have $\psi \in L^{(\sigma+1)N}_{\text{loc}}(\mathbb{R}^N)$. Statement 1 then implies $\psi \in W^{2,\frac{(\sigma+1)N}{2\sigma+1}}_{loc}(\mathbb{R}^N)$. Since $\frac{2\sigma+1}{(\sigma+1)N} < \frac{2}{N}$, applying the general Sobolev embedding theorem 2.2 ii) yields $\psi \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.
- *Case* $\frac{2\sigma+1}{q_0} > \frac{2}{N}$: We define $q_1 > 0$ by

$$\frac{1}{q_1} = \frac{2\sigma+1}{q_0} - \frac{2}{N},$$

  and then applying the general Sobolev embedding theorem 2.2 i) yields $\psi \in L^{q_1}_{\text{loc}}(\mathbb{R}^N)$. Statement 1 then implies $\psi \in W^{2,\frac{q_1}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

We continue this treatment for q_1. There are two possibilities: either $\psi \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ or $\psi \in W^{2,\frac{q_2}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$ with $\frac{1}{q_2} = \frac{2\sigma+1}{q_1} - \frac{2}{N}$.

[6]Since $\psi \in L^q_{\text{loc}}(\mathbb{R}^N)$, $\psi^{2\sigma+1} \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. We have $L^q_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$, and thus $\psi \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Then (1.9) implies that $\Delta\psi \in L^{\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$. Using elliptic regularity, it follows that $\psi \in W^{2,\frac{q}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^N)$.

We claim that there exists $n^* \in \mathbb{N}$ such that $\frac{2\sigma+1}{q_{n^*}} \le \frac{2}{N}$. Indeed, suppose for the contrary that the sequence $(q_n)_n$ defined by

$$\begin{cases} q_0 = 2\sigma + 2, \\ \dfrac{1}{q_n} = \dfrac{2\sigma+1}{q_{n-1}} - \dfrac{2}{N}, \quad \forall n \in \mathbb{N}, \end{cases}$$

consists of all positive real terms.

It is deduced from the recursion that

$$\frac{1}{q_n} - \frac{1}{\sigma N} = (2\sigma+1)\left(\frac{1}{q_{n-1}} - \frac{1}{\sigma N}\right), \quad \forall n \in \mathbb{N}.$$

Thus,

$$\frac{1}{q_n} - \frac{1}{\sigma N} = (2\sigma+1)^n\left(\frac{1}{q_0} - \frac{1}{\sigma N}\right),$$

or equivalently,

$$\frac{1}{q_n} = \frac{1}{\sigma N} + (2\sigma+1)^n\left(\frac{1}{q_0} - \frac{1}{\sigma N}\right).$$

Since $\frac{1}{q_0} - \frac{1}{\sigma N} = \frac{1}{2\sigma+2} - \frac{1}{\sigma N} = \frac{\sigma(N-2)-2}{\sigma N(2\sigma+2)} < 0$, the RHS of the last equality tends to $-\infty$ as $n \to +\infty$, which contradicts to the assumption $q_n > 0$ for all $n \in \mathbb{N}$.

Therefore, there exists $n^* \in \mathbb{N}$ such that $\frac{2\sigma+1}{q_{n^*}} \le \frac{2}{N}$ and thus we have $\psi \in C^{0,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$ for some $\alpha \in (0,1)$.

Step 2: We can prove that $\psi^{2\sigma+1} \in C^{0,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$. Then Schauder theorem 2.4 implies $\psi \in C^{2,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$. Using bootstrap argument, we can prove that $\psi \in C^{4,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$, etc. So $\psi \in C^{2m,\alpha}_{\text{loc}}\left(\mathbb{R}^N\right)$ for all $m \in \mathbb{N}$, and thus $\psi \in C^{\infty}\left(\mathbb{R}^N\right)$. □

Prove that ψ is a solution of (1.7) *of minimal L^2 norm.* Let φ be an arbitrary solution of (1.7)

$$\frac{\sigma N}{2}\Delta\varphi - \left(1 + \frac{\sigma}{2}(2-N)\right)\varphi + \varphi^{2\sigma+1} = 0. \tag{1.10}$$

Multiply (1.10) by φ and integrate over $\mathbb{R}^N$, we obtain

$$\frac{\sigma N}{2}\|\nabla\varphi\|_2^2 + \left(1 + \frac{\sigma}{2}(2-N)\right)\|\varphi\|_2^2 = \|\varphi\|_{2\sigma+2}^{2\sigma+2}. \tag{1.11}$$

Multiply (1.10) by $x \cdot \nabla\varphi$ and integrate over $\mathbb{R}^N$, we obtain

$$\frac{\sigma N}{2}\int_{\mathbb{R}^N}\Delta\varphi x \cdot \nabla\varphi dx - \left(1 + \frac{\sigma}{2}(2-N)\right)\int_{\mathbb{R}^N}\varphi x \cdot \nabla\varphi dx + \int_{\mathbb{R}^N}\varphi^{2\sigma+1}x \cdot \nabla\varphi dx = 0. \tag{1.12}$$

We have

$$\int_{\mathbb{R}^N}\Delta\varphi x \cdot \nabla\varphi dx = \int_{\mathbb{R}^N}\left(\sum_{i=1}^N \partial_{x_i}^2\varphi\right)\left(\sum_{j=1}^N x_j\partial_{x_j}\varphi\right)dx = \sum_{i=1}^N\sum_{j=1}^N\int_{\mathbb{R}^N}\partial_{x_i}^2\varphi x_j\partial_{x_j}\varphi dx$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \partial_{x_i}^2 \varphi x_i \partial_{x_i} \varphi dx + \sum_{i \neq j} \int_{\mathbb{R}^N} \partial_{x_i}^2 \varphi x_j \partial_{x_j} \varphi dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} \partial_{x_i} \varphi \partial_{x_i} \left(x_i \partial_{x_i} \varphi \right) dx - \sum_{i \neq j} \int_{\mathbb{R}^N} \partial_{x_i} \varphi x_j \partial_{x_i x_j} \varphi dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} (\partial_{x_i} \varphi)^2 dx - \sum_{i=1}^{N} \int_{\mathbb{R}^N} \partial_{x_i} \varphi x_i \partial_{x_i x_i} \varphi dx - \sum_{i \neq j} \int_{\mathbb{R}^N} \partial_{x_i} \varphi x_j \partial_{x_i x_j} \varphi dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} (\partial_{x_i} \varphi)^2 dx - \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \partial_{x_i} \varphi x_j \partial_{x_i x_j} \varphi dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} (\partial_{x_i} \varphi)^2 dx - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^N} x_j \partial_{x_j} \left((\partial_{x_i} \varphi)^2 \right) dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} (\partial_{x_i} \varphi)^2 dx + \frac{N}{2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} (\partial_{x_i} \varphi)^2 dx \\
&= \left(\frac{N}{2} - 1 \right) \|\nabla \varphi\|_2^2 ,
\end{aligned}
$$

$$
\int_{\mathbb{R}^N} \varphi x \cdot \nabla \varphi dx = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \varphi x_i \partial_{x_i} \varphi dx = \frac{1}{2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} x_i \partial_{x_i} \left(\varphi^2 \right) dx = -\frac{1}{2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \varphi^2 dx = -\frac{N}{2} \|\varphi\|_2^2 .
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^N} \varphi^{2\sigma+1} x \cdot \nabla \varphi dx &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \varphi^{2\sigma+1} x_i \partial_{x_i} \varphi dx = \frac{1}{2\sigma+2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \partial_{x_i} \left(\varphi^{2\sigma+2} \right) x_i dx \\
&= -\frac{1}{2\sigma+2} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \varphi^{2\sigma+2} dx = -\frac{N}{2\sigma+2} \|\varphi\|_{2\sigma+2}^{2\sigma+2} .
\end{aligned}
$$

Then (1.12) becomes

$$
\frac{\sigma N}{2} \left(\frac{N}{2} - 1 \right) \|\nabla \varphi\|_2^2 + \frac{N}{2} \left(1 + \frac{\sigma}{2} (2 - N) \right) \|\varphi\|_2^2 = \frac{N}{2\sigma+2} \|\varphi\|_{2\sigma+2}^{2\sigma+2} . \tag{1.13}
$$

Combining (1.11) and (1.13) yields the following system

$$
\begin{aligned}
\frac{\sigma N}{2} \|\nabla \varphi\|_2^2 - \|\varphi\|_{2\sigma+2}^{2\sigma+2} &= -\left(1 + \frac{\sigma}{2} (2 - N) \right) \|\varphi\|_2^2 , \\
\frac{\sigma N}{2} \left(\frac{N}{2} - 1 \right) \|\nabla \varphi\|_2^2 - \frac{N}{2\sigma+2} \|\varphi\|_{2\sigma+2}^{2\sigma+2} &= -\frac{N}{2} \left(1 + \frac{\sigma}{2} (2 - N) \right) \|\varphi\|_2^2 .
\end{aligned}
$$

Solving $\|\nabla \varphi\|_2^2$ and $\|\varphi\|_{2\sigma+2}^{2\sigma+2}$ in terms of $\|\varphi\|_2^2$ gives us

$$
\|\nabla \varphi\|_2^2 = \|\varphi\|_2^2 , \quad \|\varphi\|_{2\sigma+2}^{2\sigma+2} = (\sigma + 1) \|\varphi\|_2^2 ,
$$

and then

$$J^{\sigma,N}(\varphi)=\frac{\|\nabla\varphi\|_2^{\sigma N}\|\varphi\|_2^{2+\sigma(2-N)}}{\|\varphi\|_{2\sigma+2}^{2\sigma+2}}=\frac{\|\varphi\|_2^{\sigma N}\|\varphi\|_2^{2+\sigma(2-N)}}{(\sigma+1)\|\varphi\|_2^2}=\frac{\|\varphi\|_2^{2\sigma}}{\sigma+1}.$$

In particular, $J^{\sigma,N}(\psi)=\frac{\|\psi\|_2^{2\sigma}}{\sigma+1}$. Since $J^{\sigma,N}(\varphi)\geq J^{\sigma,N}(\psi)$, it follows that $\|\varphi\|_2\geq\|\psi\|_2$. □

Corollary 1.1. *The best (smallest) constant for which the interpolation estimate (1.2) holds is given by the expression*

$$C_{\sigma,N}:=\left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}},$$

where ψ is the ground state of equation (1.3).

Proof. The best constant $C_{\sigma,N}$ is given by

$$C_{\sigma,N}=\left(\inf_{u\in H^1(\mathbb{R}^N)}J^{\sigma,N}(u)\right)^{-\frac{1}{2\sigma+2}}=\alpha^{-\frac{1}{2\sigma+2}}=\left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}.$$

□

Corollary 1.2. *Let $0<\sigma<\frac{2}{N-2}$. Then, the following equation*

$$\Delta u-u+u^{2\sigma+1}=0 \tag{1.14}$$

has a positive, radial solution of class $H^1(\mathbb{R}^N)$.

Proof. Let ψ be the solution of (1.3). Set $u(x):=\frac{1}{\mu}\psi\left(\frac{x}{\lambda}\right)$, or $\psi(x)=\mu u(\lambda x)$. By Theorem (1.1), u is positive, radial, of class $H^1(\mathbb{R}^N)$ and then satisfies the equation

$$\frac{\sigma N}{2}\lambda\mu\Delta u-\left(1+\frac{\sigma}{2}(2-N)\right)\mu u+\mu^{2\sigma+1}u^{2\sigma+1}=0. \tag{1.15}$$

Choosing

$$\lambda=\frac{2}{\sigma N}\left(1+\frac{\sigma}{2}(2-N)\right),$$
$$\mu=\left(1+\frac{\sigma}{2}(2-N)\right)^{\frac{1}{2\sigma}},$$

then (1.15) gives us

$$\Delta u-u+u^{2\sigma+1}=0.$$

This completes the proof. □

2 Appendix

2.1 Rellich's compactness theorem

Theorem 2.1 (Rellich's compactness theorem, [2], pp. 8-9)**.** *If Ω is bounded and has a Lipschitz continuous boundary, then the following properties hold:*

(i) If $1 \leq p \leq N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in \left[p, \frac{Np}{N-p}\right)$.

(ii) If $p > N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.

If we assume further that Ω has a uniformly Lipschitz continuous boundary, then:

(iii) If $p > N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow \left(\overline{\Omega}\right)$ is compact for every $\lambda \in \left(0, \frac{p-N}{p}\right)$.

2.2 Interpolation estimate

Lemma 2.1 (Interpolation estimate)**.** *If $1 \leq p \leq q \leq r$, then $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ and*

$$\|f\|_q \leq \|f\|_p^\theta \|f\|_r^{1-\theta},$$

where $0 \leq \theta \leq 1$ is given by

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

2.3 Sobolev embedding

Theorem 2.2 (General Sobolev inequalities, [3], pp. 284-285)**.** *Let U be a bounded open subset of $\mathbb{R}^N$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.*

(i) If $k < \frac{N}{p}$, then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

the constant C depending only on k, p, N and U.

(ii) If $k > \frac{N}{p}$, then $u \in C^{k-\left\lfloor \frac{N}{p} \right\rfloor - 1, \gamma}\left(\overline{U}\right)$, where

$$\gamma = \begin{cases} \left\lfloor \dfrac{N}{p} \right\rfloor + 1 - \dfrac{N}{p}, & \textit{if } \dfrac{N}{p} \textit{ is not an integer} \\ \textit{any positive number} < 1, & \textit{if } \dfrac{N}{p} \textit{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\lfloor \frac{N}{p} \rfloor -1,\gamma}\left(\overline{U}\right)} \leq C\|u\|_{W^{k,p}(U)},$$

the constant C depending only on k, P, N, γ and U.

Theorem 2.3 ([1], pp. 283-284)**.** *Let $m \geq 1$ be an integer and let $p \in [1, +\infty)$. We have*

$$\begin{aligned} &W^{k,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{k}{N} \text{ if } \frac{1}{p} - \frac{k}{N} > 0, \\ &W^{k,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \ \forall q \in [p, +\infty), \text{ if } \frac{1}{p} - \frac{k}{N} = 0, \\ &W^{k,p}\left(\mathbb{R}^N\right) \hookrightarrow L^\infty\left(\mathbb{R}^N\right), \text{ if } \frac{1}{p} - \frac{k}{N} < 0, \end{aligned}$$

and all these injections are continuous. Moreover, if $k - \frac{N}{p} > 0$ is not an integer, set

$$\kappa = \left\lfloor k - \frac{N}{p} \right\rfloor \text{ and } \theta = k - \frac{N}{p} - \kappa, \ \ (0 < \theta < 1).$$

We have, for all $u \in W^{k,p}\left(\mathbb{R}^N\right)$,

$$\|D^\alpha u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^N)}, \ \forall \alpha \text{ with } |\alpha| \leq \kappa$$

and

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C\|u\|_{W^{k,p}(\mathbb{R}^N)}|x-y|^\theta \text{ a.e. } x, y \in \mathbb{R}^N, \ \forall |\alpha| \text{ with } |\alpha| = \kappa.$$

In particular, $W^{k,p}\left(\mathbb{R}^N\right) \hookrightarrow C^k\left(\mathbb{R}^N\right)$.

2.4 Schauder theorem

Theorem 2.4 (Schauder, [1], p. 317)**.** *Suppose that Ω is bounded and of class $C^{2,\alpha}$ with $0 < \alpha < 1$. Then for every $f \in C^{0,\alpha}\left(\overline{\Omega}\right)$ there exists a unique solution $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ of the problem*

$$\begin{cases} -\Delta u + u = f \text{ in } \Omega, \\ \qquad\quad u = 0 \text{ on } \Gamma. \end{cases}$$

Furthermore, if Ω is of class $C^{m+2,\alpha}$ ($m \geq 1$ an integer) and if $f \in C^{m,\alpha}\left(\overline{\Omega}\right)$, then

$$u \in C^{m+2,\alpha}\left(\overline{\Omega}\right) \text{ with } \|u\|_{C^{m+2,\alpha}} \leq C\|f\|_{C^{m,\alpha}}.$$

2.5 Symmetric-decreasing rearrangement

See Sec. 3.3, [4], pp. 80-81.

If $A \subset \mathbb{R}^N$ is a Borel set of finite Lebesgue measure, we define A^*, the *symmetric rearrangement of the set* A, to be the open ball centered at the origin whose volume is that of A. Thus,

$$A^* = B_{\mathbb{R}^N}(0; r) \text{ with } \alpha_N r^N = |A|,$$

where $|A|$ denotes the Lebesgue measure of A, and α_N is the volume of the unit ball in $\mathbb{R}^N$, which is given by $\alpha_N := \frac{2\pi^{N/2}}{N\Gamma(N/2)}$.

This definition, together with the layer cake representation (Theorem 1.13, [4]) allows us to define the *symmetric-decreasing rearrangement,* f^*, of a function f as follows.

The symmetric-decreasing rearrangement of a characteristic function of a set is defined by: $\chi_A^* := \chi_{A^*}$.

If $f : \mathbb{R}^N \to \mathbb{C}$ is a Borel measurable function vanishing at infinity, we define

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}}^*(x)\, dt,$$

which is to be compared with a special case of the layer cake representation theorem

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)\, dt.$$

Some properties of the rearrangement f^* which are used for our purpose:

1. $f^*(x)$ is nonnegative.
2. $f^*(x)$ is radially symmetric and nonincreasing, i.e.,

   $$f^*(x) = f^*(y) \text{ if } |x| = |y|,$$

   and

   $$f^*(x) \geq f^*(y) \text{ if } |x| \leq |y|,$$

3. For $f \in L^p(\mathbb{R}^N)$,

   $$\|f\|_p = \|f^*\|_p, \ \forall 1 \leq p \leq \infty.$$

The following lemma (Lemma 7.17, [4], pp. 188-189) is the most important applications of the concept of symmetric-decreasing rearrangement.

Lemma 2.2 (Symmetric decreasing rearrangement decreases kinetic energy)**.** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative measurable function that vanishes at infinity and let f^* denotes its symmetric-decreasing rearrangement. Assume that ∇f, in the sense of distributions, is a function that satisfies $\|\nabla f\|_2 < \infty$. Then ∇f^* has the same property and*

$$\|\nabla f\|_2 \geq \|\nabla f^*\|_2.$$

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