Optimization Algorithms Assignment 001

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Abstract

This assignment aims at solving some selected problems for the midterm exam of the course *Optimization Algorithms*.

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Contents

1 Problems

Problem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a convex set, $k \in \mathbb{N}^\star$, $x_i \in \Omega$, $\lambda_i \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. Prove that $\sum_{i=1}^k \lambda_i x_i \in \Omega$.*

Proof. The case $k = 2$ can be deduced directly from the definition of convex sets. Given $x_i \in \Omega$ for $i = 1, \ldots, k$, we suppose, for some $k > 2$, that

$$\left(x_i \in \Omega,\ \lambda_i \geq 0,\ i = 1, \ldots, k-1, \text{ and } \sum_{i=1}^{k-1} \lambda_i = 1 \right) \Rightarrow \sum_{i=1}^{k-1} \lambda_i x_i \in \Omega. \tag{1.1}$$

Then for any k-tuple $(\lambda_1, \ldots, \lambda_k)$ satisfying $\lambda_i \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. If $\lambda_k = 1$, then $\lambda_i = 0$, for $i = 1, \ldots, k-1$. Thus, $\sum_{i=1}^k \lambda_i x_i = x_k \in \Omega$. If $\lambda_k < 1$, we have $\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} = 1$. Then (1.1) implies that $\sum_{i=1}^{k-1} \frac{\lambda_i x_i}{1-\lambda_k} \in \Omega$. Hence,

$$\sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k-1} \lambda_i x_i + \lambda_k x_k \tag{1.2}$$

$$= (1-\lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} x_i + \lambda_k x_k \in \Omega. \tag{1.3}$$

By the principle of mathematical induction, we deduce that (1.1) holds for all $k \in \mathbb{N}^\star$. □

Problem 1.2. *Use characterizations of convex functions, check whether the following functions is convex or not.*[1]

1. $f(x) = e^{\alpha x} - x$ *in the domain* $\mathbb{R}$.
2. $f(x) = x^q$ *for* $q > 1$, *in the domain* $\mathbb{R}_+$.
3. $f(x) = -\ln x$ *in the domain* $\mathbb{R}_{++}$.
4. $f(x) = x \ln x$ *in the domain* $\mathbb{R}_{++}$.
5. $f(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2 + x_1 - 2x_2$ *in the domain* $\mathbb{R}^2$.
6. $f(x_1, x_2) = x_1 x_2$ *in the domain* $\mathbb{R}^2_{++}$.
7. $f(x_1, x_2) = \frac{x_1^2}{x_2}$ *in the domain* $\mathbb{R} \times \mathbb{R}_{++}$.
8. $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ *for* $0 \leq \alpha \leq 1$, *in the domain* $\mathbb{R}^2_{++}$.
9. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1$ *in the domain* $\mathbb{R}^3$.

Solution. We mainly use the result "f is convex (resp. concave) on a convex set C if and only if the Hessian matrix $\nabla^2 f(x)$ is positive (resp. negative) semi-definite for all $x \in C$" to check the convexity of the given functions.

1. The second derivative of f is given by $f''(x) = \alpha^2 e^{\alpha x} \geq 0$. Thus, f is convex.

[1] Notation: $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{R}_{++} = \{x \in \mathbb{R}; x > 0\}$.

2. The second derivative of f is given by $f''(x) = q(q-1)x^{q-2} \geq 0$ for all $x \in \mathbb{R}_+$. Thus, f is convex.

3. The second derivative of f is given by $f''(x) = \frac{1}{x^2} > 0$ for all $x \in \mathbb{R}_{++}$. Thus, f is convex.

4. The second derivative of f is given by $f''(x) = \frac{1}{x} > 0$ for all $x \in \mathbb{R}_{++}$. Thus, f is convex.

5. The Hessian matrix of f is given by

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \text{ for all } (x_1, x_2) \in \mathbb{R}^2, \tag{1.4}$$

whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$. Thus, $\nabla^2 f$ is positive definite and f is convex.

6. Similarly, the Hessian matrix of f is given by

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ for all } (x_1, x_2) \in \mathbb{R}^2_{++}, \tag{1.5}$$

whose eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$. Thus, f is non-convex.

7. The Hessian matrix of f is given by

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}, \text{ for all } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_{++}, \tag{1.6}$$

whose eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \frac{2\left(x_1^2 + x_2^2\right)}{x_2^3} > 0$. Thus, $\nabla^2 f$ is semi-positive definite and f is convex.

8. The Hessian matrix of f is given by

$$\nabla^2 f(x_1, x_2) = \alpha(\alpha - 1) \begin{bmatrix} x_1^{\alpha-2} x_2^{1-\alpha} & -\frac{x_1^{\alpha-1}}{x_2^{\alpha}} \\ -\frac{x_1^{\alpha-1}}{x_2^{\alpha}} & \frac{x_1^{\alpha}}{x_2^{\alpha+1}} \end{bmatrix}, \tag{1.7}$$

for all $(x_1, x_2) \in \mathbb{R}^2_{++}$, whose eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \frac{\alpha(\alpha-1)x_1^{\alpha-2}\left(x_1^2+x_2^2\right)}{x_2^{\alpha+1}}$. Since $0 \leq \alpha \leq 1$, we have $\lambda_2 \leq 0$. Thus, f is non-convex.

9. The Hessian matrix of f is given by

$$\nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \text{ for all } (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{1.8}$$

whose eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$. Thus, $\nabla^2 f$ is semi-positive definite, and f is convex. $\square$

Problem 1.3. *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex mappings. Define*

$$g(x) = \min\{f_1(x), f_2(x)\}, \quad h(x) = \max\{f_1(x), f_2(x)\}. \tag{1.9}$$

Which of these functions is convex?

Solution. Take $f_1(x) = x^2$, $f_2(x) = (x-2)^2$ for all $x \in \mathbb{R}$. Since ${f_1}''(x) = {f_2}''(x) = 2$, both f_1 and f_2 are convex. We have $g(x) = \min\left\{x^2, (x-2)^2\right\}$. Then $g(0) = \min\{0,4\} = 0$, $g(1) = \min\{1,1\} = 1$, $g(2) = \min\{4,0\} = 0$. Thus

$$g\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2\right) = 1 > 0 = \frac{g(0)+g(2)}{2}. \tag{1.10}$$

This inequality implies that g is non-convex.

Next, we will prove that h is convex. Indeed, for $i = 1, 2$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $t \in [0,1]$,

$$\begin{aligned} f_i(tx + (1-t)y) &\le t f_i(x) + (1-t) f_i(y) && (1.11)\\ &\le t\max\{f_1(x), f_2(x)\} + (1-t)\max\{f_1(y), f_2(y)\} && (1.12)\\ &= th(x) + (1-t)h(y). && (1.13)\end{aligned}$$

Thus, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $t \in [0,1]$,

$$\begin{aligned} h(tx + (1-t)y) &= \max\{f_1(tx + (1-t)y), f_2(tx + (1-t)y)\} && (1.14)\\ &\le th(x) + (1-t)h(y). && (1.15)\end{aligned}$$

This completes our proof. □

Problem 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and α be an arbitrary real number. The level set α is defined as follows,*

$$L_\alpha := \{x \in \mathbb{R}^n; f(x) \le \alpha\}. \tag{1.16}$$

1. *Prove that if f is convex then for all $\alpha \in \mathbb{R}$, the level set L_α is convex.*
2. *The converse of the above statement is true or false? Why?*

Solution.

1. For $x \in L_\alpha$, $y \in L_\alpha$, we have $f(x) \le \alpha$, $f(y) \le \alpha$. Since f is convex, for all $t \in [0,1]$,

$$\begin{aligned} f(tx + (1-t)y) &\le tf(x) + (1-t)f(y) && (1.17)\\ &\le t\alpha + (1-t)\alpha && (1.18)\\ &= \alpha. && (1.19)\end{aligned}$$

   Hence, $tx + (1-t)y \in L_\alpha$ for all $t \in [0,1]$, which implies that L_α is a convex set.

2. (Counter-example) Consider the function $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) = |x|^{\frac{1}{2}}$ for all $x \in \mathbb{R}$, we have $L_\alpha = \emptyset$ for all $\alpha < 0$ and

$$L_\alpha : = \{x \in \mathbb{R}^n; f(x) \le \alpha\} \tag{1.20}$$

$$= \left\{x \in \mathbb{R}^n; |x|^{\frac{1}{2}} \le \alpha\right\} \tag{1.21}$$

$$= \left[-\alpha^2, \alpha^2\right] \text{ for all } \alpha \ge 0. \tag{1.22}$$

Combining both cases, L_α is convex for all $\alpha \in \mathbb{R}$. Now we check whether f is convex or not. Since

$$2f\left(\frac{1}{2}\right) = \sqrt{2} > 1 = f(0) + f(1), \tag{1.23}$$

the function f chosen is non-convex. Thus, the converse of the first statement fails in general. □

Problem 1.5 (Jensen inequality).) *Let $f : domf \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, $domf$ is a convex set, $k \in \mathbb{N}^\star$, $x_1, \dots, x_k \in domf$ and $\lambda_i \ge 0$, for $i = 1, \dots, k$ satisfying $\sum_{i=1}^k \lambda_i = 1$. Prove that*

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} \lambda_i f(x_i). \tag{1.24}$$

Applications. *Use the convexity of the function $f(x) = -\ln x$.*

1. *(Cauchy inequality) For $a_i \in \mathbb{R}_+$, $i = 1, \dots, n$,*

$$\frac{a_1 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 \cdots a_n}. \tag{1.25}$$

2. *(Hölder inequality) Let $x, y \in \mathbb{R}^n$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that*

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}. \tag{1.26}$$

Proof. The case $k = 2$ is deduced directly from the definition of convex functions. For some $k > 2$, we suppose that

$$\left(x_i \in \mathrm{dom} f,\ \lambda_i \ge 0,\ i = 1, \dots, k-1,\ \sum_{i=1}^{k-1} \lambda_i = 1\right) \Rightarrow f\left(\sum_{i=1}^{k-1} \lambda_i x_i\right) \le \sum_{i=1}^{k-1} \lambda_i f(x_i). \tag{1.27}$$

Similar to the proof of Problem 1.1, for any k-tuple $(\lambda_1, \dots, \lambda_k)$ satisfying $\lambda_i \ge 0$ for $i = 1, \dots, k$ and $\sum_{i=1}^k \lambda_i = 1$. If $\lambda_k = 1$, then $\lambda_i = 0$, for $i = 1, \dots, k-1$. Thus,

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = f(x_k) = \sum_{i=1}^{k} \lambda_i f(x_i). \tag{1.28}$$

If $\lambda_k < 1$, we have $\sum\limits_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} = 1$. Then (1.27) implies that

$$f\left(\sum_{i=1}^{k-1} \frac{\lambda_i x_i}{1 - \lambda_k}\right) \le \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} f(x_i). \tag{1.29}$$

Hence,

$$f\left(\sum_{i=1}^{k}\lambda_i x_i\right) = f\left((1-\lambda_k)\sum_{i=1}^{k-1}\frac{\lambda_i x_i}{1-\lambda_k}+\lambda_k x_k\right) \tag{1.30}$$

$$\leq (1-\lambda_k)\, f\left(\sum_{i=1}^{k-1}\frac{\lambda_i x_i}{1-\lambda_k}\right)+\lambda_k f(x_k) \tag{1.31}$$

$$\leq (1-\lambda_k)\sum_{i=1}^{k-1}\frac{\lambda_i}{1-\lambda_k} f(x_i)+\lambda_k f(x_k) \tag{1.32}$$

$$= \sum_{i=1}^{k}\lambda_i f(x_i). \tag{1.33}$$

By the principle of mathematical induction, we deduce that (1.24) holds for all $k \in \mathbb{N}^\star$.

Now we consider some applications of Jensen inequality.

1. Applying Jensen inequality for the convex function $f(x) = -\ln x$ for $x \in \mathbb{R}_+$, $a_i \in \mathbb{R}_+$, $\lambda_i = \frac{1}{n}$ for $i = 1, \dots, n$, (the convexity of f has been proved in Problem 1.2-3) yields

$$-\ln\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right) \leq -\frac{1}{n}\sum_{i=1}^{n}\ln a_i, \tag{1.34}$$

   which is equivalent to (1.25).

2. Since $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, we also have $q>1$. It suffices to prove (1.26) for $x_i > 0$, $i = 1, \dots, n$ since the zero terms (if exist) can be removed without affecting the inequality. Since $f(x) = x^q$, $q > 1$ is convex in $\mathbb{R}_+$ (this has been proved in Problem 1.2-2), applying Jensen inequality to f yields

$$\left(x_i > 0,\ \lambda_i > 0,\ i = 1, \dots, n,\ \sum_{i=1}^{n}\lambda_i = 1\right) \Rightarrow \left(\sum_{i=1}^{n}\lambda_i x_i\right)^q \leq \sum_{i=1}^{n}\lambda_i x_i^q. \tag{1.35}$$

   Plugging $\lambda_i = \frac{|x_i|^p}{\sum_{i=1}^n |x_i|^p} > 0$, $x_i = \frac{|x_i||y_i|}{\lambda_i} \geq 0$, for $i = 1, \dots, n$ in (1.35) yields

$$\left(\sum_{i=1}^{n}\frac{|x_i|^p}{\sum\limits_{i=1}^{n}|x_i|^p}\cdot\frac{|x_i|\,|y_i|}{\frac{|x_i|^p}{\sum\limits_{i=1}^{n}|x_i|^p}}\right)^q \leq \sum_{i=1}^{n}\frac{|x_i|^p}{\sum_{i=1}^{n}|x_i|^p}\cdot\frac{|x_i|^q|y_i|^q}{\frac{|x_i|^{pq}}{\left(\sum\limits_{i=1}^{n}|x_i|^p\right)^q}}, \tag{1.36}$$

   which is equivalent to

$$\left(\sum_{i=1}^{n}|x_i|\,|y_i|\right)^q \leq \left(\sum\nolimits_{i=1}^{n}|x_i|^p\right)^{q-1}\sum_{i=1}^{n}|y_i|^q. \tag{1.37}$$

Thus,

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} |x_i| \, |y_i| \tag{1.38}$$

$$\leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{q-1}{q}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}} \tag{1.39}$$

$$= \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}, \tag{1.40}$$

for all $x, y \in \mathbb{R}^n$. □

Problem 1.6. *Use the first-order necessary condition to find stationary points*[2] *of the following functions.*

1. $f(x_1, x_2) = x_1^2 + 3x_2^2 - 4x_1 + 8x_2$.
2. $f(x_1, x_2, x_3) = 2x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 - 6x_1 - 7x_2 - 8x_3 + 9$.
3. $f(x_1, x_2) = (x_1x_2 - x_1 - 1)^2 + \left(x_2^2 - 1\right)^2$.
4. $f(x_1, x_2, x_3) = x_1x_2x_3e^{-x_1-x_2-x_3}$.
5. $f(x_1, x_2) = \frac{1}{x_1x_2} + x_1 + x_2$ in the domain $\mathbb{R}^2_{++}$.

SOLUTION.

1. The gradient of f is given by

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 - 4 \\ 6x_2 + 8 \end{bmatrix}, \tag{1.41}$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Solving the equation $\nabla f(x_1, x_2) = \mathbf{0}$ yields that $\left(2, -\frac{4}{3}\right)$ is the unique stationary point of f.

2. The gradient of f is given by

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 + x_2 - 6 \\ x_1 + 2x_2 + x_3 - 7 \\ x_2 + 2x_3 - 8 \end{bmatrix}, \tag{1.42}$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Solving the equation $\nabla f(x_1, x_2, x_3) = \mathbf{0}$ yields that $\left(\frac{6}{5}, \frac{6}{5}, \frac{17}{5}\right)$ is the unique stationary point of f.

3. The gradient of f is given by

$$\nabla f(x_1, x_2) = \begin{bmatrix} -2(x_2 - 1)(x_1 - x_1x_2 + 1) \\ 4x_2\left(x_2^2 - 1\right) - 2x_1(x_1 - x_1x_2 + 1) \end{bmatrix}, \tag{1.43}$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Solving the equation $\nabla f(x_1, x_2) = \mathbf{0}$ yields that $(-1, 0)$, $(0, 1)$, and $\left(-\frac{1}{2}, -1\right)$ are the only stationary points of f.

[2] "stationary point", see [2], or "critical points", see [1].

4. The gradient of f is given by

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} (x_2x_3 - x_1x_2x_3)\, e^{-x_1-x_2-x_3} \\ (x_3x_1 - x_1x_2x_3)\, e^{-x_1-x_2-x_3} \\ (x_1x_2 - x_1x_2x_3)\, e^{-x_1-x_2-x_3} \end{bmatrix}, \tag{1.44}$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Solving the equation $\nabla f(x_1, x_2, x_3) = \mathbf{0}$ yields that $(1,1,1)$, $(a,0,0)$, for arbitrary $a \in \mathbb{R}$ and its permutations are the only stationary points of f.

5. The gradient of f is given by

$$\nabla f(x_1, x_2) = \begin{bmatrix} 1 - \frac{1}{x_1^2x_2} \\ 1 - \frac{1}{x_1x_2^2} \end{bmatrix}, \tag{1.45}$$

for all $(x_1, x_2) \in \mathbb{R}^2_{++}$. Solving the equation $\nabla f(x_1, x_2) = 0$ yields that $(1,1)$ is the unique stationary point of f. $\square$

Problem 1.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Gâteaux differentiable and convex function. Consider the following unconstrained minimization problem*

$$\text{Min } f(x) \text{ s.t. } x \in \mathbb{R}^n. \tag{1.46}$$

Prove the following properties.

1. *$\bar{x}$ is a local minimizer of (P) $\Leftrightarrow$ $\nabla f(\bar{x}) = 0$.*
2. *$\bar{x}$ is a local minimizer of (P) $\Leftrightarrow$ $\bar{x}$ is a global minimizer of (P).*
3. *The set of minimizers of (P) is a convex set.*
4. *If f is strictly convex, then (P) has the unique minimizer (if there exists at least one).*

Proof.

1. $(\Rightarrow)$ *First-order necessary condition for a local optimizer.*[3] We have the following theorem, which is more general than our task.

Theorem 1.7.1 (First-order necessary condition for a local optimizer). *Let $f : U \to \mathbb{R}$ be a Gâteaux differentiable function on an open subset $U \subseteq \mathbb{R}^n$. A local optimizer is a critical point, that is,*

$$\bar{x} \text{ a local optimizer } \Rightarrow \nabla f(\bar{x}) = 0. \tag{1.47}$$

Proof of Theorem 1.7.1. Let $\bar{x}$ be a local minimizer of f. Then there exists $\varepsilon > 0$ such that

$$x \in B(\bar{x}, \varepsilon) \Rightarrow f(\bar{x}) \le f(x). \tag{1.48}$$

If $d \in \mathbb{R}^n$, then

$$f'(\bar{x}; d) = \lim_{t \to 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t} = \langle \nabla f(\bar{x}), d \rangle. \tag{1.49}$$

[3]See [1], Theorem 2.7, p. 35.

If $|t|$ is small, then the numerator in the above limit is nonnegative, since $\bar{x}$ is a local minimizer. If $t > 0$, then the difference quotient is nonnegative, so in the limit as $t \downarrow 0$, we have $f'(\bar{x}; d) \geq 0$. However, if $t < 0$, the difference quotient is nonpositive, and we have $f'(\bar{x}; d) \leq 0$. Thus, we conclude that $f'(\bar{x}; d) = \langle \nabla f(\bar{x}), d \rangle = 0$. If $\bar{x}$ is a local maximizer of f, then $\langle \nabla f(\bar{x}), d \rangle = 0$, since $\bar{x}$ is a local minimizer of $-f$. Picking $d = \nabla f(\bar{x})$ gives

$$f'(\bar{x}; \nabla f(\bar{x})) = \|\nabla f(\bar{x})\|^2 = 0, \tag{1.50}$$

that is, $\nabla f(\bar{x}) = 0$. □

($\Leftarrow$) Suppose $\nabla f(\bar{x}) = 0$, we will prove $\bar{x}$ is a local minimizer of (P). Indeed, plugging $d = x - \bar{x}$ for arbitrary $x \in \mathbb{R}^n$ into (1.49) yields

$$f'(\bar{x}; x - \bar{x}) = \lim_{t \to 0} \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} = \langle \nabla f(\bar{x}), x - \bar{x} \rangle = 0, \tag{1.51}$$

for all $x \in \mathbb{R}^n$. Since f is convex, we also have

$$f(\bar{x} + t(x - \bar{x})) = f(tx + (1-t)\bar{x}) \leq tf(x) + (1-t) f(\bar{x}), \tag{1.52}$$

for all $t \in [0, 1]$.

Combining (1.51) and (1.52) yields

$$\begin{aligned} 0 &= \lim_{t \to 0^+} \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} && (1.53) \\ &\leq \lim_{t \to 0^+} \frac{tf(x) + (1-t) f(\bar{x}) - f(\bar{x})}{t} && (1.54) \\ &= f(x) - f(\bar{x}), && (1.55) \end{aligned}$$

or equivalently, $f(\bar{x}) \leq f(x)$ for all $x \in \mathbb{R}^n$. That means x is a global (thus local) minimizer of (P).

2. We have just proved that for any convex function f, $\bar{x}$ is a local minimizer of $(P) \Rightarrow \nabla f(\bar{x}) = 0 \Rightarrow \bar{x}$ is a global minimizer of $(P) \Rightarrow \bar{x}$ is a local minimizer of (P), where the last implication is obvious. Thus, $\bar{x}$ is a local minimizer $\Leftrightarrow \bar{x}$ is a global minimizer.

   Alternative proof. In this proof, we will prove that $\bar{x}$ is a local minimizer of $(P) \Rightarrow \bar{x}$ is a global minimizer of (P). Indeed, let $\bar{x}$ be a local minimizer of (P). Then there exists $\varepsilon > 0$ such that

$$x \in B(\bar{x}, \varepsilon) \Rightarrow f(\bar{x}) \leq f(x). \tag{1.56}$$

   So it remains to prove that $f(\bar{x}) \leq f(x)$ for all $x \in \mathbb{R}^n \backslash B(\bar{x}, \varepsilon)$. Suppose, to get a contradiction, that there exists $x \in \mathbb{R}^n \backslash B(\bar{x}, \varepsilon)$ such that $f(x) < f(\bar{x})$ and consider $z \in \{tx + (1-t)\bar{x}; t \in [0, 1]\} \cap B(\bar{x}, \varepsilon)$. Then z can be expressed as $z = tx + (1-t)\bar{x}$ for some $0 < t < \frac{\varepsilon}{\|x - \bar{x}\|} \leq 1$. Since f is convex, we have successively

$$f(z) = f(tx + (1-t)\bar{x}) \tag{1.57}$$

$$\begin{aligned} &\leq tf(x) + (1-t) f(\bar{x}) &(1.58)\\ &< tf(\bar{x}) + (1-t) f(\bar{x}) &(1.59)\\ &= f(\bar{x}). &(1.60)\end{aligned}$$

But this contradicts (1.56). So $\bar{x}$ is a global minimizer of (P).

3. Suppose that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ are two (global) minimizers of f ($x = y$ is a possibility), we denote $\alpha = f(x) = f(y)$ the minimizer value of f. Since f is convex, we have

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t) f(y) &(1.61)\\ &= t\alpha + (1-t)\alpha &(1.62)\\ &= \alpha, &(1.63)\end{aligned}$$

for all $t \in [0,1]$. Since α is the minimizer value of f, this implies that $f(tx + (1-t)y) = \alpha$ for all $t \in [0,1]$. Thus, the set of minimizers of f is convex.

4. Suppose that there exist two (global) minima $\bar{x}$ and $\bar{y}$ of (P). Since f is strictly convex, we have

$$f\left(\frac{\bar{x}+\bar{y}}{2}\right) < \frac{f(\bar{x}) + f(\bar{y})}{2} = f(\bar{x}), \tag{1.64}$$

which contradicts the fact that $\bar{x}$ is a global minimizer of (P). Thus, if a strictly convex function has a (global) minimizer, then it is unique. □

Problem 1.8. *Find the number of minimizers with respect to m of the following problem*

$$\text{Min } \frac{3}{2}\left(x^2 + y^2\right) + (1+m)xy - x - y + 4 \text{ s.t. } (x,y) \in \mathbb{R}^2. \tag{1.65}$$

Solution. Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{3}{2}\left(x^2 + y^2\right) + (1+m)xy - x - y + 4, \text{ for } (x,y) \in \mathbb{R}^2, \tag{1.66}$$

this is a quadratic function since it can be expressed as

$$f(x,y) = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 3 & 1+m \\ 1+m & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 4, \tag{1.67}$$

for all $(x,y) \in \mathbb{R}^2$. Its gradient is given by

$$\nabla f(x,y) = \begin{bmatrix} 3x + (1+m)y - 1 \\ 3y + (1+m)x - 1 \end{bmatrix}, \text{ for all } (x,y) \in \mathbb{R}^2, \tag{1.68}$$

and its Hessian matrix is

$$\nabla^2 f(x,y) = \begin{bmatrix} 3 & 1+m \\ 1+m & 3 \end{bmatrix}, \text{ for all } (x,y) \in \mathbb{R}^2, \tag{1.69}$$

whose eigenvalues are $\lambda_1 = 2 - m$ and $\lambda_2 = m + 4$. The equation $\nabla f(x, y) = 0$ has the unique root $\left(\frac{1}{m+4}, \frac{1}{m+4}\right)$ when $m \neq -4$ and $m \neq 2$, no root when $m = -4$, and infinite roots with the form $\left(a, \frac{1}{3} - a\right)$ for arbitrary $a \in \mathbb{R}$ when $m = 2$.

We consider the following cases depending on m.

- *Case* $m \leq -4$. Similarly, f can be expressed as

$$f(x, y) = \frac{3}{2}(x - y)^2 + (4 + m)\, xy - x - y + 4, \tag{1.70}$$

  for all $(x, y) \in \mathbb{R}^2$. In particular, $f(x, x) = (4 + m)\, x^2 - 2x + 4 \to -\infty$ as $x \to +\infty$. Thus, f has no minimizers in this case.

- *Case* $m \in (-4, 2)$. In this case, we have $\lambda_1 > 0$, and $\lambda_2 > 0$, which implies that $\nabla^2 f(x, y)$ is positive definite, and thus f is strictly convex. Applying Problem 1.7 to f, we deduce that $\left(\frac{1}{m+4}, \frac{1}{m+4}\right)$, which is the unique stationary point of f, is the unique minimizer of f.

- *Case* $m = 2$. In this case, we have $\lambda_1 = 0$, $\lambda_2 = 6$, which implies that $\nabla^2 f(x, y)$ is semi-positive definite, and thus f is convex. Applying Problem 1.7 to f, we deduce that $\left(a, \frac{1}{3} - a\right)$ for arbitrary $a \in \mathbb{R}$ are minimizers of f. Thus, there are infinite minimizers of f in this case.

- *Case* $m > 2$. We have $f(x, -x) = (2 - m)\, x^2 + 4 \to -\infty$ as $x \to +\infty$. Thus, f has no minimizers in this case. □

Remark 1.8.1. We also give elementary proofs for some cases in the proof above.

Elementary proof for the case $m = 2$. When $m = 2$,

$$f(x, y) = \frac{3}{2}\left(x^2 + y^2\right) + 3xy - x - y + 4 \tag{1.71}$$

$$= \frac{3}{2}(x + y)^2 - (x + y) + 4, \text{ for all } (x, y) \in \mathbb{R}^2. \tag{1.72}$$

Put $t = x + y$, we have

$$g(t) := \frac{3}{2}t^2 - t + 4 \tag{1.73}$$

$$= \frac{3}{2}\left(t - \frac{1}{3}\right)^2 + \frac{23}{6} \geq \frac{23}{6}, \text{ for all } t \in \mathbb{R}. \tag{1.74}$$

Hence, f attains its minimum value $\frac{23}{6}$ at the points (x, y) satisfying $x + y = \frac{1}{3}$, i.e., $\left(a, \frac{1}{3} - a\right)$ for arbitrary $a \in \mathbb{R}$. □

In fact, the cases $m = 2$ and $m \in (-4, 2)$ can be merged as in the following elementary proof.

Elementary proof for the case $m \in (-4, 2]$. We rewrite f as

$$f(x, y) = -\frac{1 + m}{2}(x - y)^2 + \frac{m + 4}{2}\left(x^2 + y^2\right) - x - y + 4 \tag{1.75}$$

$$= -\frac{1+m}{2}(x-y)^2 + \frac{m+4}{2}\left(x - \frac{1}{m+4}\right)^2 \tag{1.76}$$

$$+\frac{m+4}{2}\left(y - \frac{1}{m+4}\right)^2 + \frac{4m+15}{m+4}, \tag{1.77}$$

for all $(x, y) \in \mathbb{R}^2$. Applying the inequality $2\left(a^2+b^2\right) \geq (a+b)^2$, whose the equality holds if and only if $a = b$ to $a = x - \frac{1}{m+4}$ and $b = \frac{1}{m+4} - y$ yields

$$\left(x - \frac{1}{m+4}\right)^2 + \left(y - \frac{1}{m+4}\right)^2 \geq \frac{1}{2}(x-y)^2, \tag{1.78}$$

for all $(x, y) \in \mathbb{R}^2$. Thus,

$$f(x,y) \geq -\frac{1+m}{2}(x-y)^2 + \frac{m+4}{4}(x-y)^2 + \frac{4m+15}{m+4} \tag{1.79}$$

$$= \frac{2-m}{4}(x-y)^2 + \frac{4m+15}{m+4} \tag{1.80}$$

$$\geq \frac{4m+15}{m+4}, \tag{1.81}$$

for all $(x, y) \in \mathbb{R}^2$. For $m \in (-4, 2]$, the equality $f(x, y) = \frac{4m+15}{m+4}$ holds if and only if

$$\begin{cases} x - \frac{1}{m+4} = \frac{1}{m+4} - y \\ (2-m)(x-y) = 0 \end{cases}, \tag{1.82}$$

which is equivalent to

$$\begin{cases} x + y = \frac{2}{m+4} \\ \left[\begin{array}{l} m = 2 \\ x = y \end{array}\right. \end{cases}, \tag{1.83}$$

or,

$$\left[\begin{array}{r} m = 2 \text{ and } x + y = \frac{1}{3} \\ m \in (-4, 2) \text{ and } x = y = \frac{1}{m+4} \end{array}\right. . \tag{1.84}$$

Thus $\left(\frac{1}{m+4}, \frac{1}{m+4}\right)$ is the unique minimizer of f when $m \in (-4, 2)$, and $\left(a, \frac{1}{3} - a\right)$ for arbitrary $a \in \mathbb{R}$ are minimizers of f when $m = 2$. □

Problem 1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Gâteaux differentiable function and $\bar{x} \in \mathbb{R}^n$, $d \in \mathbb{R}^n$. Prove the following properties.*

1. *If $\nabla f(\bar{x})^T d < 0$ then d is a descent direction at $\bar{x}$ for f.*
2. *If f is convex then: d is a descent direction at $\bar{x}$ of f $\Leftrightarrow$ $\nabla f(\bar{x})^T d < 0$.*

Proof.

1. Assume $\nabla f(\bar{x})^T d < 0$ for some $d \in \mathbb{R}^n$, we have

$$\nabla f(\bar{x})^T d = \lim_{t \to 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t} < 0. \tag{1.85}$$

Thus, for $\varepsilon > 0$ small enough,

$$t \in (0, \varepsilon] \Rightarrow f(\bar{x} + td) < f(\bar{x}), \tag{1.86}$$

that means d is a descent direction at $\bar{x}$ of f.

2. For a convex function f, it suffices to prove that d is a descent direction at $\bar{x}$ of $f \Rightarrow \nabla f(\bar{x})^T d < 0$. Assume that $d \in \mathbb{R}^n$ is a descent direction at $\bar{x}$ of f, there exists $\varepsilon > 0$ such that (1.86) holds. Since f is convex, we have

$$f(\bar{x} + td) \geq f(\bar{x}) + t\nabla f(\bar{x})^T d. \tag{1.87}$$

Combining (1.86) and (1.87) yields

$$t \in (0, \varepsilon] \Rightarrow \nabla f(\bar{x})^T d \leq \frac{f(\bar{x} + td) - f(\bar{x})}{t} < 0. \tag{1.88}$$

This ends our proof. □

Problem 1.10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$, $d \in \mathbb{R}^n$. Prove that if $d \neq \mathbf{0}$ and $\|\nabla f(\bar{x}) + d\|^2 \leq \|\nabla f(\bar{x})\|^2$ then d is a descent direction at $\bar{x}$ of f.*

Proof. We have

$$\begin{aligned} \|\nabla f(\bar{x})\|^2 &\geq \|\nabla f(\bar{x}) + d\|^2 && (1.89) \\ &= \|\nabla f(\bar{x})\|^2 + 2\nabla f(\bar{x})^T d + \|d\|^2, && (1.90) \end{aligned}$$

which implies that $\nabla f(\bar{x})^T d \leq -\frac{\|d\|^2}{2} < 0$ (since $d \neq \mathbf{0}$). Due to Problem 1.9, d is a descent direction at $\bar{x}$ of f. □

Problem 1.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ satisfying $f(y) < f(\bar{x})$. Prove that $d = y - \bar{x}$ is a descent direction at $\bar{x}$ of f.*

Proof. Since $f(y) < f(\bar{x})$, we have $\bar{x} \neq y$. The convexity of f gives us

$$\nabla f(\bar{x})^T (y - \bar{x}) \leq f(y) - f(\bar{x}) < 0. \tag{1.91}$$

Thus, $d = y - \bar{x}$ is a descent direction at $\bar{x}$ of f. □

Problem 1.12. *Given $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$, find a descent direction d at $\bar{x}$ of f in the following cases.*

1. $f(x, y) = x^2 + y^2 - xy - x + 2y - 3$ and $\bar{x} = (0, 0)$.
2. $f(x, y) = 2x^2 + y^2 - 2xy + 2x^3 + x^4$ and $\bar{x} = (-1, 0)$.
3. $f(x, y) = \frac{1}{2}(x - 2y)^2 + x^4$ and $\bar{x} = (2, 1)$.
4. $f(x, y, z) = x^2 + y^2 + z^2 - xy - zx + 2x - 4y - 2z$ and $\bar{x} = (0, 0, 1)$

Solution. We mainly use the result "$d = -\nabla f(\bar{x})$ is a descent direction at $\bar{x}$ for f if $\nabla f(\bar{x}) \neq \mathbf{0}$".

1. The gradient of f is given by
$$\nabla f(x,y) = \begin{bmatrix} 2x-y-1 \\ 2y-x+2 \end{bmatrix}, \text{ for all } (x,y) \in \mathbb{R}^2. \tag{1.92}$$
In particular, $\nabla f(0,0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Thus, $d = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a descent direction at $\bar{x}$ for f.

2. The gradient of f is given by
$$\nabla f(x,y) = \begin{bmatrix} 4x^3+6x^2+4x-2y \\ 2y-2x \end{bmatrix}, \text{ for all } (x,y) \in \mathbb{R}^2. \tag{1.93}$$
In particular, $\nabla f(-1,0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Thus, $d = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is a descent direction at $\bar{x}$ for f.

3. The gradient of f is given by
$$\nabla f(x,y) = \begin{bmatrix} 4x^3+x-2y \\ 4y-2x \end{bmatrix}, \text{ for all } (x,y) \in \mathbb{R}^2. \tag{1.94}$$
In particular, $\nabla f(2,1) = \begin{bmatrix} 32 \\ 0 \end{bmatrix}$. Thus, $d = \begin{bmatrix} -32 \\ 0 \end{bmatrix}$ is a descent direction at $\bar{x}$ for f.

4. The gradient of f is given by
$$\nabla f(x,y,z) = \begin{bmatrix} 2x-y-z+2 \\ 2y-x-4 \\ 2z-x-2 \end{bmatrix}, \text{ for all } (x,y,z) \in \mathbb{R}^3. \tag{1.95}$$
In particular, $\nabla f(0,0,1) = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$. Thus, $d = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ is a descent direction at $\bar{x}$ for f. □

THE END

References

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