Differential Geometry Assignment 002

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Abstract

This context contains my solutions to **Problems 4, 8, 17**, Chapter 2, [1].

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1 Problems

Problem 1.1 (Exercise 4, p.49, [1]). *A regular curve between two points p, q in $\mathbb{R}^n$ with minimal length is necessarily the line segment from p to q.*

Hint. Consider the Schwarz inequality $\langle X, Y \rangle \leq \|X\| \cdot \|Y\|$ for the tangent vector and the difference vector $p - q$.

Proof. Let $c : [0, 1] \to \mathbb{R}^n$ be a regular curve between two given points p, q in $\mathbb{R}^n$, i.e., $c(0) = p$, $c(1) = q$. The length of this curve is

$$L(c) = \int_0^1 \|\dot{c}\| \, dt. \tag{1.1}$$

Applying the Schwarz inequality $\langle X, Y \rangle \leq \|X\| \cdot \|Y\|$ for the tangent vector $\dot{c}$ and the difference vector $q - p$ gives

$$\langle \dot{c}, q - p \rangle \leq \|\dot{c}\| \cdot \|q - p\|. \tag{1.2}$$

Combining (1.1) and (1.2) yields

$$\begin{align} L(c) &= \int_0^1 \|\dot{c}\| \, dt \tag{1.3} \\ &\geq \int_0^1 \frac{\langle \dot{c}, q - p \rangle}{\|q - p\|} dt \tag{1.4} \\ &= \frac{\left\langle \int_0^1 \dot{c} dt, q - p \right\rangle}{\|q - p\|} \tag{1.5} \\ &= \frac{\langle c(b) - c(a), q - p \rangle}{\|q - p\|} \tag{1.6} \\ &= \frac{\langle q - p, q - p \rangle}{\|q - p\|} \tag{1.7} \\ &= \|q - p\|, \tag{1.8} \end{align}$$

where we have used the following lemma to deduce the equality between (1.4) and (1.5).

Lemma 1.2. *Let $f : [0, 1] \to R^n$ and $\alpha \in \mathbb{R}^n$ be an integrable vector-valued function and a fixed vector, respectively. Then the following equality holds*

$$\int_0^1 \langle f(t), \alpha \rangle \, dt = \left\langle \int_0^1 f(t) \, dt, \alpha \right\rangle. \tag{1.9}$$

Proof of Lemma 1.2. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $f(t) = (f_1(t), \ldots, f_n(t))$, we have

$$\begin{align} \int_0^1 \langle f(t), \alpha \rangle \, dt &= \int_0^1 \sum_{i=1}^n \alpha_i f_i(t) dt \tag{1.10} \\ &= \sum_{i=1}^n \alpha_i \int_0^1 f_i(t) \, dt \tag{1.11} \end{align}$$

$$= \left\langle \int_0^1 f(t)\, dt, \alpha \right\rangle . \tag{1.12}$$

Hence, (1.9) holds. □

Return to our problem, the equality holds if and only if there exists $\lambda \in \mathbb{R}$ such that $\dot{c} = \lambda (q - p)$. This is equivalent to $c(t) = \lambda (q-p)t + C$ where C is a constant. Using $c(0) = p$, $c(1) = q$ for this parametrization of c yields

$$c(t) = (q-p)t + p = (1-t)p + tq, \tag{1.13}$$

which is the line segment from p to q. This completes our proof. □

Problem 1.3 (Exercise 8, p.50, [1]). *The Frenet two-frame of a plane curve with given curvature function $\kappa(s)$ can be described by the exponential series for the matrix*

$$\begin{pmatrix} 0 & \int_0^s \kappa(t)\, dt \\ -\int_0^s \kappa(t)\, dt & 0 \end{pmatrix}. \tag{1.14}$$

It follow that

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & \int_0^s \kappa \\ -\int_0^s \kappa & 0 \end{pmatrix}^i. \tag{1.15}$$

Proof. "*Not only does every plane curve uniquely determine its curvature function $\kappa(s)$, but also conversely, the curvature function κ also determines the curve, up to Euclidean motions, i.e., up to the prescription of a point on the curve and the tangent of the curve at that point.*", see p.15, [1].

Let the curvature function $\kappa(s)$ be given. Then one can set

$$e_1(s) = (\cos(\alpha(s)), \sin(\alpha(s))), \tag{1.16}$$

with a function $\alpha(s)$ which is to be found. Necessarily one has

$$e_2(s) = (-\sin(\alpha(s)), \cos(\alpha(s))). \tag{1.17}$$

The Frenet equation says that $\kappa e_2 = {e_1}' = \alpha' e_2$, hence $\kappa = \alpha'$. By a judicious choice of adapted coordinate system we can assume that for $s = 0$, the curve passes through the origin with $e_1(0) = (1, 0)$; then $\alpha(0) = 0$, and hence

$$\alpha(s) = \int_0^s \kappa(t)\, dt. \tag{1.18}$$

Then (1.16) and (1.17) becomes

$$e_1(s) = \left(\cos\left(\int_0^s \kappa(t)\, dt \right), \sin\left(\int_0^s \kappa(t)\, dt \right) \right), \tag{1.19}$$

$$e_2(s) = \left(-\sin\left(\int_0^s \kappa(t)\, dt \right), \cos\left(\int_0^s \kappa(t)\, dt \right) \right). \tag{1.20}$$

We now need the following lemma, which is the equality appeared in p.31, [1].

Lemma 1.4. *Given a realnumber K, the following equality holds*

$$\begin{pmatrix} \cos K & \sin K \\ -\sin K & \cos K \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^i. \tag{1.21}$$

Proof of Lemma 1.4. It is easy to prove

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^{2i} = (-1)^i \begin{pmatrix} K^{2i} & 0 \\ 0 & K^{2i} \end{pmatrix} \text{ for } i \in \mathbb{N}* \tag{1.22}$$

and

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^{2i+1} = (-1)^i \begin{pmatrix} 0 & K^{2i+1} \\ -K^{2i+1} & 0 \end{pmatrix} \text{ for } i \in \mathbb{N}, \tag{1.23}$$

by induction.

We recall that the Maclaurin series expansions of $\sin K$ and $\cos K$ are given by

$$\sin K = \sum_{i=0}^{\infty} (-1)^i \frac{K^{2i+1}}{(2i+1)!}, \tag{1.24}$$

$$\cos K = \sum_{i=0}^{\infty} (-1)^i \frac{K^{2i}}{(2i)!}. \tag{1.25}$$

Using (1.22)-(1.25), we can transform the right-hand side of (1.21) as follows.

$$\begin{aligned}
& \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^i && (1.26) \\
= & \sum_{i=0}^{\infty} \frac{1}{(2i)!} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^{2i} + \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^{2i+1} && (1.27) \\
= & \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \begin{pmatrix} K^{2i} & 0 \\ 0 & K^{2i} \end{pmatrix} + \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} \begin{pmatrix} 0 & K^{2i+1} \\ -K^{2i+1} & 0 \end{pmatrix} && (1.28) \\
= & \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} K^{2i} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} K^{2i} \end{pmatrix} && (1.29) \\
& + \begin{pmatrix} 0 & \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} K^{2i+1} \\ -\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} K^{2i+1} & 0 \end{pmatrix} && (1.30) \\
= & \begin{pmatrix} \sum_{i=0}^{\infty} (-1)^i \frac{K^{2i}}{(2i)!} & \sum_{i=0}^{\infty} (-1)^i \frac{K^{2i+1}}{(2i+1)!} \\ -\sum_{i=0}^{\infty} (-1)^i \frac{K^{2i+1}}{(2i+1)!} & \sum_{i=0}^{\infty} (-1)^i \frac{K^{2i}}{(2i)!} \end{pmatrix} && (1.31) \\
= & \begin{pmatrix} \cos K & \sin K \\ -\sin K & \cos K \end{pmatrix}. && (1.32)
\end{aligned}$$

Hence, (1.21) holds. $\square$

Return to our problem, applying Lemma 1.4 for $K = \int_0^s \kappa(t)\,dt$ yields

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\int_0^s \kappa(t)\,dt\right) & \sin\left(\int_0^s \kappa(t)\,dt\right) \\ -\sin\left(\int_0^s \kappa(t)\,dt\right) & \cos\left(\int_0^s \kappa(t)\,dt\right) \end{pmatrix} \tag{1.33}$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & \int_0^s \kappa(t)\,dt \\ -\int_0^s \kappa(t)\,dt & 0 \end{pmatrix}^i. \tag{1.34}$$

This completes our proof. $\square$

Problem 1.5 (Exercise 17, p.52, [1]). *In the orthogonal (but not normal) three-frame $c', c'', c' \times c''$ the Frenet equations of a space curve take the equivalent form*

$$\begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{pmatrix} \begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}. \tag{1.35}$$

Here the entries of the matrix depend in some sense rationally (i.e., without roots) on $\kappa^2 = \langle c'', c'' \rangle$ and τ, because of the relation

$$\frac{\kappa'}{\kappa} = \frac{1}{2}\left(\log\left(\kappa^2\right)\right)'. \tag{1.36}$$

Proof. We recall that the accompanying three-frame of a space curve is given by

$$e_1 = c', \tag{1.37}$$

$$e_2 = \frac{c''}{\|c''\|} = \frac{c''}{\kappa}, \tag{1.38}$$

$$e_3 = e_1 \times e_2 = \frac{c' \times c''}{\kappa}, \tag{1.39}$$

The Frenet equation is given by

$$e_1{}' = \kappa e_2, \tag{1.40}$$

$$e_2{}' = -\kappa e_1 + \tau e_3, \tag{1.41}$$

$$e_3{}' = -\tau e_2, \tag{1.42}$$

Combining (1.37)-(1.39) with (1.41) yields

$$-\kappa c' + \frac{\tau}{\kappa} c' \times c'' = \left(\frac{c''}{\kappa}\right)' \tag{1.43}$$

$$= \frac{c'''\kappa - c''\kappa'}{\kappa^2}, \tag{1.44}$$

i.e.,

$$c''' = -\kappa^2 c' + \frac{\kappa'}{\kappa} c'' + \tau c' \times c'' \tag{1.45}$$

Combining (1.37)-(1.39) with (1.42) yields

$$-\frac{\tau}{\kappa}c'' = \left(\frac{c' \times c''}{\kappa}\right)' \tag{1.46}$$

$$= \frac{(c' \times c'')' \kappa - c' \times c'' \kappa'}{\kappa^2}, \tag{1.47}$$

i.e.,

$$(c' \times c'')' = -\tau c'' + \frac{\kappa'}{\kappa} c' \times c''. \tag{1.48}$$

Then (1.45) and (1.48) give (1.35). And (1.36) is obvious by calculating

$$\frac{1}{2}\left(\log\left(\kappa^2\right)\right)' = \frac{1}{2} \cdot \frac{2\kappa\kappa'}{\kappa^2} = \frac{\kappa'}{\kappa}. \tag{1.49}$$

This completes our proof. □

THE END

References

[1] Wolfgang Kühnel, *Differential Geometry, Curves - Surfaces - Manifolds*, Second Edition, Student Mathematical Library, Volume 16, AMS.