Homework Assignment
Differential Geometry

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Problem: If a circle if rolled along a line (without friction), then a fixed point on that circle has its trajectory as the so-called *cycloid*.

(a) Find a parameterization for the cycloid.

(b) Compute the curvature of the cycloid.

SOLUTION

To find a parameterization for the cycloid, we should have some intuition about it through the following figure.

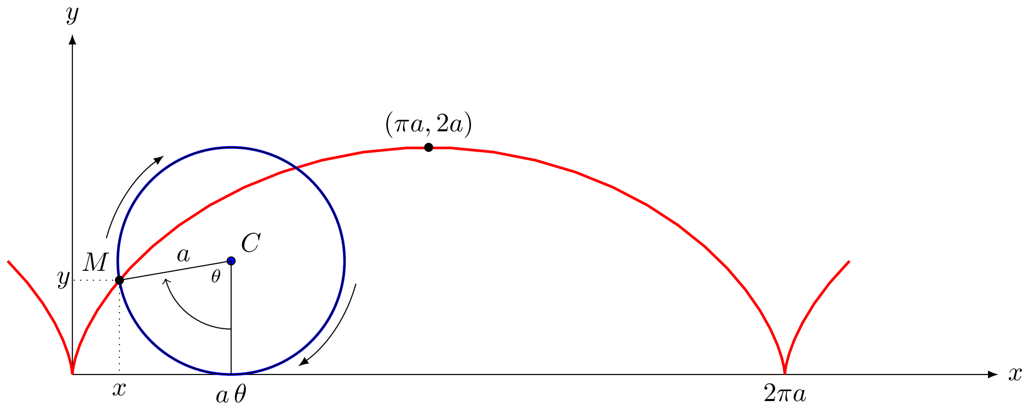

Figure 1: A cycloid, [1]

We need to find the coordinate of the point M with respect to θ.

First, we can see that the longitude of the center of the circle (point C) is the radius of the circle and the latitude is the arc length subtended by the angle θ on the circle.

Therefore, the coordinate of C is $(a\theta, a)$.

As for M, we can denote it's coordinate as

$$r(\theta) = (x(\theta), y(\theta)) \tag{1}$$

Let's consider the following cases.

Case 1: $0 < \theta < \frac{\pi}{2}$

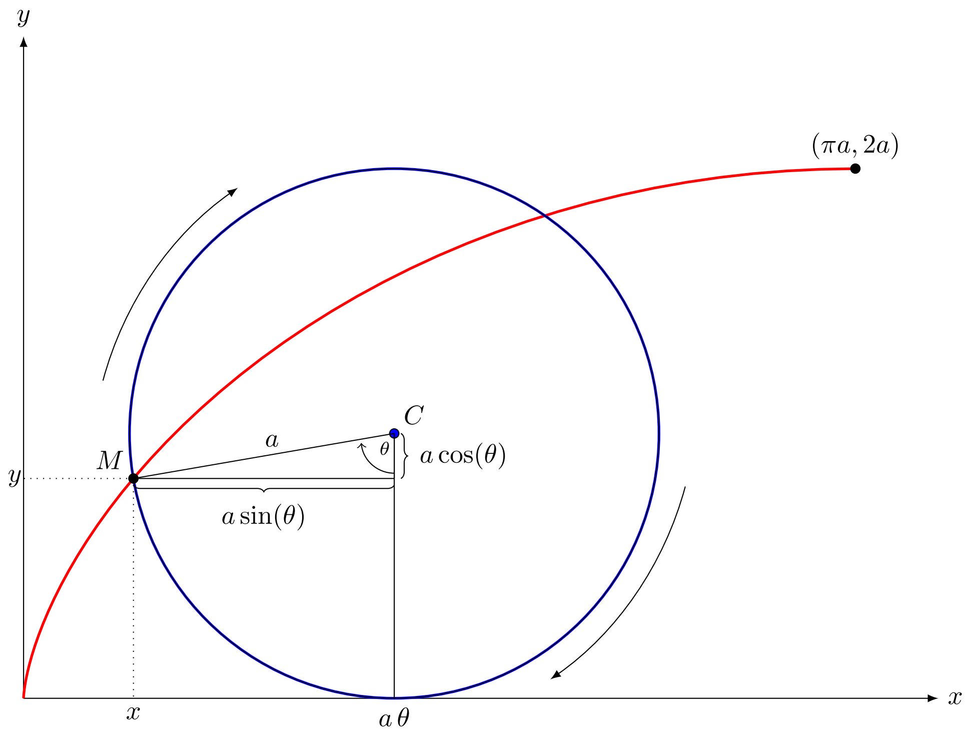

Figure 2: Case 1: $0 < \theta < \frac{\pi}{2}$

From the figure, we can find the relative position of M and C and thus, the coordinate of M.

$$\begin{cases} x(\theta) &= a\theta - a\sin(\theta) \\ &= a\left(\theta - \sin(\theta)\right) \\ y(\theta) &= a - a\cos(\theta) \\ &= a\left(1 - \cos(\theta)\right) \end{cases} \tag{2}$$

Case 2: $\frac{\pi}{2} < \theta < \pi$

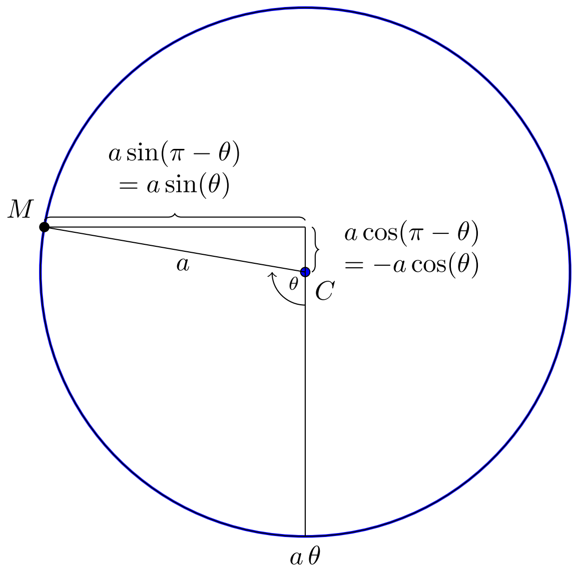

Figure 3: Case 2: $\frac{\pi}{2} < \theta < \pi$

From the figure, we can find the coordinate of M as

$$\begin{cases} x(\theta) & = a\theta - a\sin(\theta) \\ & = a\left(\theta - \sin(\theta)\right) \\ y(\theta) & = a + \left(-a\cos(\theta)\right) \\ & = a\left(1 - \cos(\theta)\right) \end{cases} \tag{3}$$

Case 3: $\pi < \theta < \frac{3\pi}{2}$

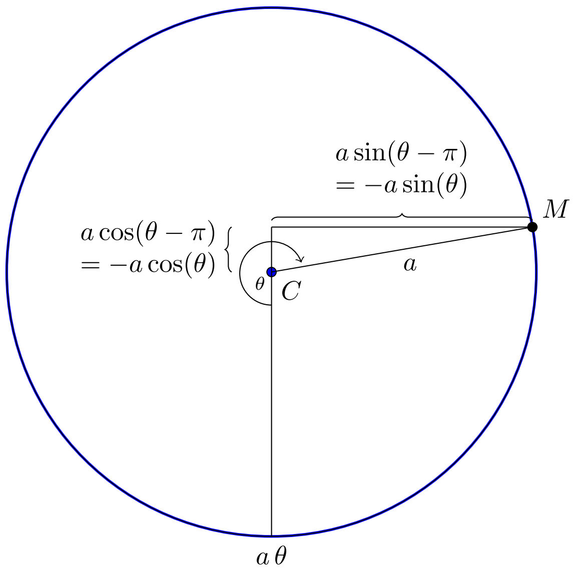

Figure 4: Case 3: $\pi < \theta < \frac{3\pi}{2}$

From the figure, we can find the coordinate of M as

$$\begin{cases} x(\theta) & = a\theta + (-a\sin(\theta)) \\ & = a\,(\theta - \sin(\theta)) \\ y(\theta) & = a\theta + (-a\cos(\theta)) \\ & = a\,(1 - \cos(\theta)) \end{cases} \tag{4}$$

Case 4: $\frac{3\pi}{2} < \theta < 2\pi$

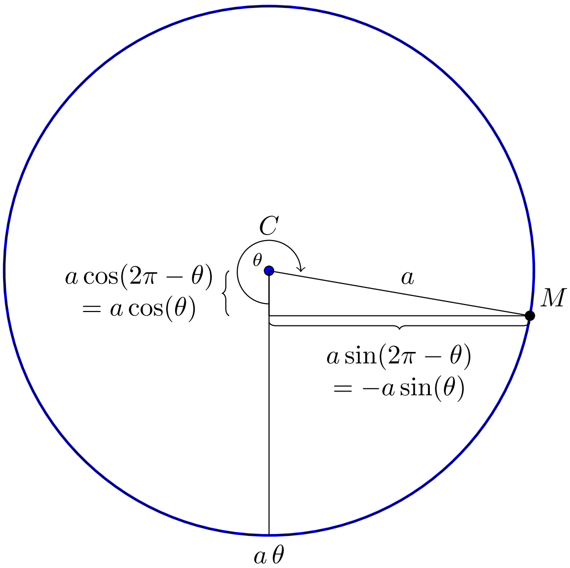

Figure 5: Case 4: $\frac{3\pi}{2} < \theta < 2\pi$

From the figure, we can find the coordinate of M as

$$\begin{cases} x(\theta) & = a\theta + (-a\sin(\theta)) \\ & = a\,(\theta - \sin(\theta)) \\ y(\theta) & = a - a\cos(\theta) \\ & = a\,(1 - \cos(\theta)) \end{cases} \tag{5}$$

Case 5: $\theta = 0\pi$

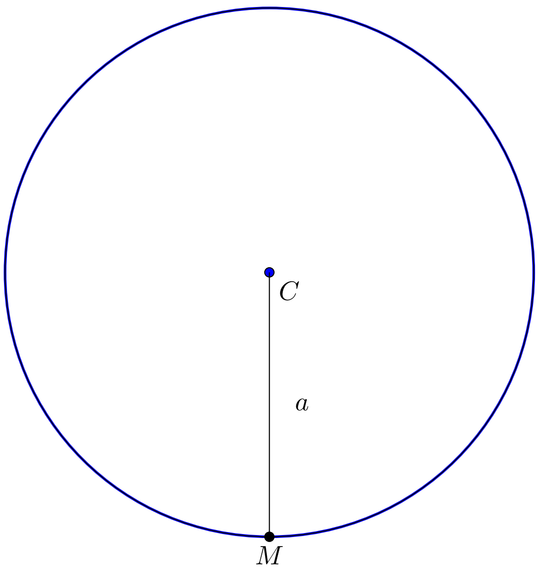

Figure 6: Case 5: $\theta = 0\pi$

In this case, we can see that

$$\begin{cases} \sin(\theta) & = 0 \\ \cos(\theta) & = 1 \end{cases} \tag{6}$$

Therefore, the coordinate of M is

$$\begin{cases} x(\theta) & = 0 = a\,(\theta - \sin(\theta)) \\ y(\theta) & = 0 = a\,(1 - \cos(\theta)) \end{cases} \tag{7}$$

Case 6: $\theta = \frac{\pi}{2}$

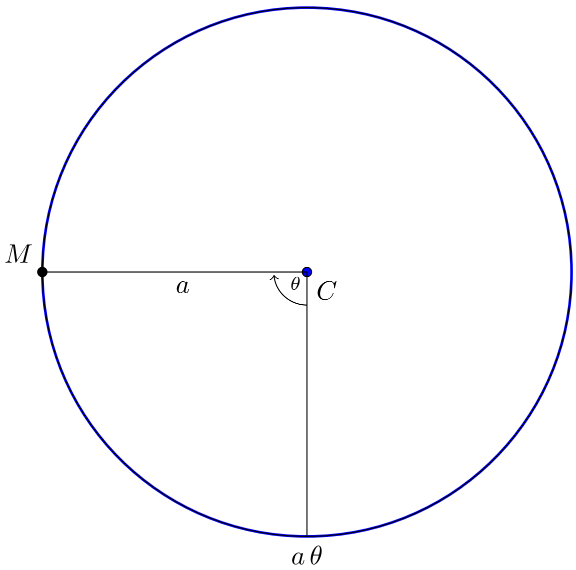

Figure 7: Case 6: $\theta = \frac{\pi}{2}$

In this case, we can see that

$$\begin{cases} \sin(\theta) & = 1 \\ \cos(\theta) & = 0 \end{cases} \tag{8}$$

Therefore, the coordinate of M is

$$\begin{cases} x(\theta) & = a\frac{\pi}{2} - a = a\,(\theta - \sin(\theta)) \\ y(\theta) & = a = a\,(1 - \cos(\theta)) \end{cases} \tag{9}$$

Case 7: $\theta = \pi$

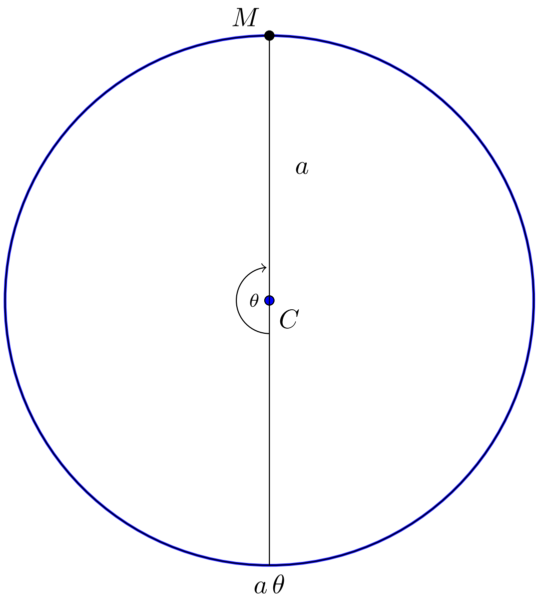

Figure 8: Case 7: $\theta = \pi$

In this case, we can see that

$$\begin{cases} \sin(\theta) & = 0 \\ \cos(\theta) & = -1 \end{cases} \tag{10}$$

Therefore, the coordinate of M is

$$\begin{cases} x(\theta) & = a\pi = a\,(\theta - \sin(\theta)) \\ y(\theta) & = 2a = a\,(1 - \cos(\theta)) \end{cases} \tag{11}$$

Case 8: $\theta = \frac{3\pi}{2}$

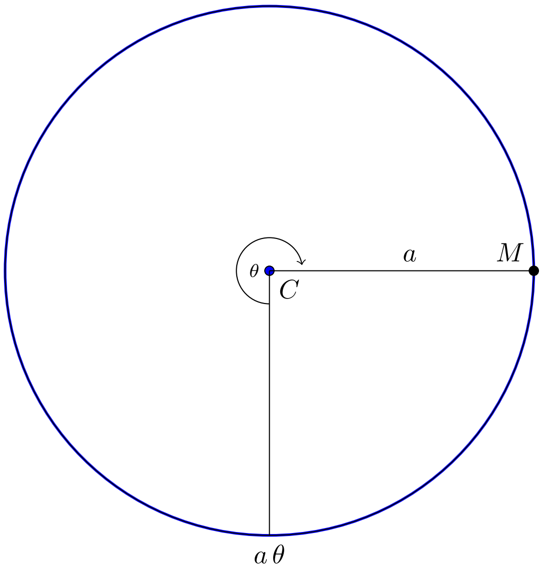

Figure 9: Case 8: $\theta = \frac{3\pi}{2}$

In this case, we can see that

$$\begin{cases} \sin(\theta) & = -1 \\ \cos(\theta) & = 0 \end{cases} \tag{12}$$

Therefore, the coordinate of M is

$$\begin{cases} x(\theta) & = a\pi + a = a\,(\theta - \sin(\theta)) \\ y(\theta) & = a = a\,(1 - \cos(\theta)) \end{cases} \tag{13}$$

Case 9: $\theta = 2\pi$

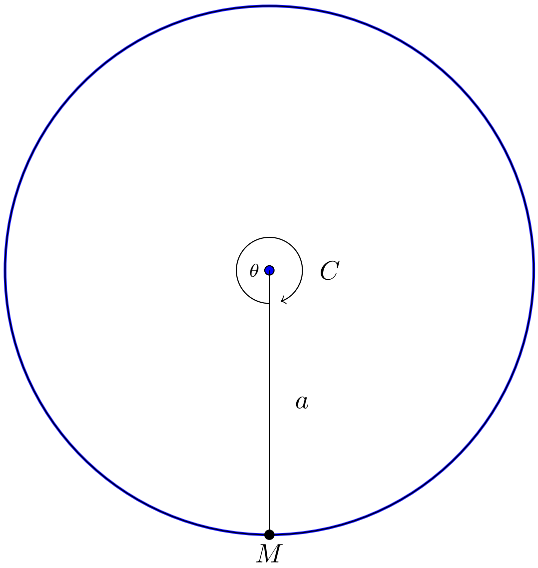

Figure 10: Case 9: $\theta = 2\pi$

In this case, we can see that

$$\begin{cases} \sin(\theta) &= 0 \\ \cos(\theta) &= 1 \end{cases} \tag{14}$$

Therefore, the coordinate of M is

$$\begin{cases} x(\theta) &= a2\pi = a\left(\theta - \sin(\theta)\right) \\ y(\theta) &= 0 = a\left(1 - \cos(\theta)\right) \end{cases} \tag{15}$$

Now, we have a parameterization of the cycloid

$$\begin{aligned} r(\theta) &= (x(\theta), y(\theta)) && (16)\\ &= (a(\theta - \sin(\theta)), a(1 - \cos(\theta))) && (17) \end{aligned}$$

We can proceed to compute the curvature of the cycloid with $0 < \theta < 2\pi$. We have

$$\begin{cases} x' = \dfrac{dx}{d\theta} & = a - a\cos(\theta) \\ x'' = \dfrac{d^2x}{d\theta^2} & = a\sin(\theta) \end{cases} \tag{18}$$

and

$$\begin{cases} y' = \dfrac{dy}{d\theta} & = a\sin(\theta) \\ y'' = \dfrac{d^2y}{d\theta^2} & = a\cos(\theta) \end{cases} \tag{19}$$

The curvature of the cycloid can be computed as

$$\begin{aligned} \kappa &= \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}} && (20)\\ &= \frac{(a - a\cos(\theta))a\cos(\theta) - a\sin(\theta)a\sin(\theta)}{((a - a\cos(\theta))^2 + (a\sin(\theta))^2)^{\frac{3}{2}}} && (21)\\ &= \frac{a^2\cos(\theta) - a^2\cos^2(\theta) - a^2\sin^2(\theta)}{\left(a^2 - 2a^2\cos(\theta) + a^2\cos^2(\theta) + a^2\sin^2(\theta)\right)^{\frac{3}{2}}} && (22)\\ &= \frac{a^2\cos(\theta) - a^2}{(a^2 - 2a^2\cos(\theta) + a^2)^{\frac{3}{2}}} && (23)\\ &= \frac{a^2(\cos(\theta) - 1)}{a^3(2 - 2\cos(\theta))^{\frac{3}{2}}} && (24)\\ &= -\frac{1 - \cos(\theta)}{2^{\frac{3}{2}}a(1 - \cos(\theta))^{\frac{3}{2}}} && (25)\\ &= -\frac{1}{2^{\frac{3}{2}}a(1 - \cos(\theta))^{\frac{1}{2}}} && (26)\\ &= -\frac{1}{4a\frac{1}{\sqrt{2}}(1 - \cos(\theta))^{\frac{1}{2}}} && (27)\\ &= -\frac{\sqrt{2}}{4a(1 - \cos(\theta))^{\frac{1}{2}}} && (28) \end{aligned}$$

Bibliography

[1] https://tex.stackexchange.com/questions/196957/how-can-i-draw-this-cycloid-diagram-with-tikz