Method of Subsolutions and Supersolutions for a Nonlinear Poisson Equation

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Overview

Nonlinear Poisson equation

Consider the following BVP for the nonlinear Poisson equation

$$\begin{cases} -\Delta u = f(u), & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth, with

$$\left| f'(z) \right| \leq C_f, \ \ \forall z \in \mathbb{R}, \tag{2}$$

for some constant C_f.

Weak subsolution, supersolution, & solution

Definition (Weak sub- & supersolutions)

(i) We say that $\overline{u} \in H^1(U)$ is a *weak supersolution* of (1) if

$$\int_U D\overline{u} \cdot Dv dx \geq \int_U f(\overline{u}) v dx, \quad \forall v \in H_0^1(U), \ v \geq 0 \text{ a.e.} \quad (3)$$

(ii) Similarly, $\underline{u} \in H^1(U)$ is a *weak subsolution* of (1) provided

$$\int_U D\underline{u} \cdot Dv dx \leq \int_U f(\underline{u}) v dx, \quad \forall v \in H_0^1(U), \ v \geq 0 \text{ a.e.} \quad (4)$$

(iii) We say $u \in H_0^1(U)$ is a *weak solution* of (1) if

$$\int_U Du \cdot Dv dx = \int_U f(u) v dx, \quad \forall v \in H_0^1(U). \quad (5)$$

Quick notes

Question. *Why does "$v \geq 0$ a.e." appear in the definitions of weak subsolution and supersolution?*

Remark. If $\overline{u}, \underline{u} \in C^2(U)$, then from (3) & (4) it follows that

$$-\Delta\overline{u} \geq f(\overline{u}), \ -\Delta\underline{u} \leq f(\underline{u}), \ \text{in } U. \tag{6}$$

When $f = 0$, this gives the notions of subharmonicity and superharmonicity.

Method of subsolutions & supersolutions

Theorem (Existence of a solution between sub- and supersolutions)

Assume there exists a weak supersolution $\overline{u}$ and a weak subsolution $\underline{u}$ of (1) satisfying

$$\underline{u} \leq 0,\ \overline{u} \geq 0 \text{ on } \partial U \text{ in the trace sense},\ \underline{u} \leq \overline{u} \text{ a.e. in } U. \tag{7}$$

Then there exists a weak solution u of (1), such that

$$\underline{u} \leq u \leq \overline{u} \text{ a.e. in } U. \tag{8}$$

Outline of the proof

Proof. The proof in [Evans, 2010], p. 543 consists of 5 main steps:

1. Fix a number $\lambda > 0$ so large that $h_\lambda(z) := f(z) + \lambda z$ is nondecreasing. Set $u_0 = \underline{u}$, $u_k \in H_0^1(U)$, $k \in \mathbb{Z}^+$ is defined inductively to be the unique weak solution of the linear BVP

$$(\mathrm{P}_{k+1}) \quad \begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f(u_k) + \lambda u_k, & \text{in } U, \\ u_{k+1} = 0, & \text{on } \partial U. \end{cases} \tag{9}$$

2. Prove $\underline{u} = u_0 \leq u_1 \leq \ldots \leq u_k \leq \ldots$ a.e. in U.
3. Prove $u_k \leq \overline{u}$ a.e. in U, $\forall k \in \mathbb{N}$.

Outline of the proof (cont.)

4 Combining Step 2 & 3 yields

$$\underline{u} \leq \ldots \leq u_k \leq u_{k+1} \leq \ldots \overline{u} \text{ a.e. in } U. \tag{10}$$

Thus, $u(x) := \lim_{k\to\infty} u_k(x)$ exists for a.e. $x \in U$, and $u_k \to u$ in $L^2(U)$ by Dominated Convergence Theorem.
Prove $\sup_{k\in\mathbb{N}} \|u_k\|_{H_0^1(U)} < \infty$. Hence there is a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ which converges weakly in $H_0^1(U)$ to $u \in H_0^1(U)$.

5 Passing to limit $j \to \infty$, u is a weak solution of problem (1).□

References

Lawrence C. Evans (2010) Partial Differential Equations, 2e
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