Combinatorics – Tổ Hợp

Nguyễn Quản Bá Hồng*

Ngày 8 tháng 4 năm 2025

Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
URL: `https://nqbh.github.io/advanced_STEM/`.
Latest version:

- *Combinatorics – Tổ Hợp*.
  PDF: URL: `.pdf`.
  TEX: URL: `.tex`.
- .
  PDF: URL: `.pdf`.
  TEX: URL: `.tex`.

Mục lục

1 Wikipedia's

1.1 Wikipedia/extremal combinatorics

"*Extremal combinatorics* is a field of mathematics, which is itself a part of mathematics. Extremal combinatorics studies how large or how small a collection of finite objects (numbers, graphs, vectors, sets, etc.) can be, if it has to satisfy certain restrictions.

Much of extremal combinatorics concerns classes of sets; this is called *extremal set theory*. E.g., in an n-element set, what is the largest number of k-element subsets that can pairwise intersect one another? What is the largest number of subsets of which more contains any other? The latter question is answered by Sperner's theorem, which gave rise to much of extremal set theory.

Another kind of example: How many people can be invited to a party where among each 3 people there are 2 who know each other & 2 who don't know each other? Ramsey theory shows: at most 5 persons can attend such a party (see Theorem on Friends & Strangers). Or, suppose given a finite set of nonzero integers, & are asked to mark as large a subset as possible of this set under the restriction that the sum of any 2 marked integers cannot be marked. It appears that (independent of what the given integers actually are) we can always mark at least $\frac{1}{3}$ of them." – Wikipedia/extremal combinatorics

1.2 Wikipedia/extremal graph theory

"Turán graph $T(n,r)$ is an example of an extremal graph. It has the maximum possible number of edges for a graph on n vertices without $(r+1)$-cliques. This is $T(13,4)$. *Extremal graph theory* is a branch of combinatorics, itself an area of mathematics, that lies at the intersection of extremal combinatorics & graph theory. In essence, extremal graph theory studies how global properties of a graph influence local substructure. Results in extremal graph theory deal with quantitative connections between various graph properties, both global (e.g. number of vertices & edges) & local (e.g. existence of specific subgraphs), & problems in extremal graph theory can often be formulated as optimization problems: how big or small a parameter of a graph can be, given some constraints that the graph has to satisfy? A graph that is an optimal solution to such an optimization problem is called an *extremal graph*, & extremal graphs are important objects of study in extremal graph theory.

Extremal graph theory is closely related to fields e.g. Ramsey theory, spectral graph theory, computational complexity theory, & additive combinatorics, & frequently employs probabilistic method.

*A Scientist & Creative Artist Wannabe. E-mail: `nguyenquanbahong@gmail.com`. Bến Tre City, Việt Nam.

1.2.1 History

> "Extremal graph theory, in its strictest sense, is a branch of graph theory developed & loved by Hungarians." – Bollobás (2004)

Mantel's Theorem (1907) & Turán's Theorem (1941) were some of 1st milestones in stud of extremal graph theory. In particular, Turán's theorem would later on become a motivation for the finding of results e.g. Erdős–Stone theorem (1946). This result is surprising because it connects chromatic number with maximal number of edges in an H-free graph. An alternative proof of Erdős–Stone was given in 1975, & utilized Szemerédi regularity lemma, an essential technique in resolution of extremal graph theory problems.

1.2.2 Topics & concepts

- **Graph coloring.** Main article: Wikipedia/graph coloring. A *proper (vertex) coloring* of a graph G is a coloring of vertices of G s.t. no 2 adjacent vertices have the same color. Minimum number of colors needed to properly color G is called *chromatic number* of G, denoted $\chi(G)$. Determining chromatic number of specific graphs is a fundamental question in extremal graph theory, because many problems in area & related areas can be formulated in terms of graph coloring.

  2 simple lower bounds to chromatic number of a graph G is given by clique number $\omega(G)$ – all vertices of a clique must have distinct colors – & by $\frac{|V(G)|}{\alpha(G)}$, where $\alpha(G)$ is independence number, because set of vertices with a given color must form an independent set.

  A greedy coloring gives upper bound $\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is maximum degree of G. When G is not an odd cycle or a clique, Brooks' theorem states: upper bound can be reduced to $\Delta(G)$. When G is a planar graph, 4-color theorem states: G has chromatic number ≤ 4.

  In general, determining whether a given graph has a coloring with a prescribed number of colors is known to be NP-hard.

  In addition to vertex coloring, other types of coloring are also studied, e.g. edge colorings. *Chromatic index* $\chi'(G)$ of a graph G is minimum number of colors in a proper edge-coloring of a graph, & Vizing's theorem states: chromatic index of a graph G is either $\Delta(G)$ or $\Delta(G) + 1$.

- **Forbidden subgraphs.** Main article: Wikipedia/forbidden subgraph problem. *Forbidden subgraph problem* is 1 of central problems in extremal graph theory. Given a graph G, forbidden subgraph problem asks for maximal number of edges $\mathrm{ex}(n, G)$ in an n-vertex graph that does not contain a subgraph isomorphic to G.

  When $G = K_r$ is a complete graph, Turán's theorem gives an exact value for $\mathrm{ex}(n, K_r)$ & characterizes all graphs attaining this maximum; such graphs are known as Turán graphs. For non-bipartite graphs G, Erdős–Stone theorem gives an asymptotic value of $\mathrm{ex}(n, G)$ in terms of chromatic number of G. Problem of determining asymptotics of $\mathrm{ex}(n, G)$ when G is a bipartite graph is open; when G is a complete bipartite graph, this is known as Zarankiewicz problem.

- **Homomorphism density.** Main article: Wikipedia/Homomorphism density. *Homomorphism density* $t(H, G)$ of a graph H in a graph G describes probability that a randomly chosen map from vertex set of H to vertex set of G is also a graph homomorphism. It is closely related to *subgraph density*, which describes how often a graph H is found as a subgraph of G.

  Forbidden subgraph problem can be restated as maximizing edge density of a graph with G-density 0, & this naturally leads to generalization in form of *graph homomorphism inequalities*, which are inequalities relating $t(H, G)$ for various graphs H. By extending homomorphism density to graphons, which are objects that arise as a limit of dense graphs, graph homomorphism density can be written in form of integrals, & inequalities e.g. Cauchy–Schwarz inequality & Hölder's inequality can be used to derive homomorphism inequalities.

  A major open problem relating homomorphism densities is Sidorenko's conjecture, which states a tight lower bound on homomorphism density of a bipartite graph in a graph G in terms of edge density of G.

- **Graph regularity.** Main article: Wikipedia/Szemerédi regularity lemma. Edges between parts in a regular partition behave in a "random-like" fashion. *Szemerédi's regularity lemma* states: all graphs are 'regular' in sense: vertex set of any given graph can be partitioned into a bounded number of parts s.t. bipartite graph between most pairs of parts behave like random bipartite graphs. This partition gives a structural approximation to original graph, which reveals information about properties of original graph.

  Regularity lemma is a central result in extremal graph theory, & also has numerous applications in adjacent fields of additive combinatorics & commputational complexity theory. In addition to (Szemerédi) regularity, closely related notions of graph regularity e.g. strong regularity & Frieze-Kannan weak regularity have also been studied, as well as extensions of regularity to hypergraphs.

  Applications of graph regularity often utilize forms of counting lemmas & removal lemmas. In simplest forms, graph counting lemma uses regularity between pairs of parts in a regular partition to approximate number of subgraphs, & graph removal lemma states: given a graph with few copies of a given subgraph, can remove a small number of edges to eliminate all copies of subgraph." – Wikipedia/extremal graph theory

2 Miscellaneous