Differential Geometry Assignment 002

Nguyen Quan Ba Hong

Students at Faculty of Math and Computer Science,

Ho Chi Minh University of Science, Vietnam

email. `nguyenquanbahong@gmail.com`

blog. `www.nguyenquanbahong.com` *

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Abstract

This assignment aims at solving **Exercises 5, 6, 7**, p.168-169, [1].

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1 Problems

Problem 1 (Exercise 5, p.168, [1]). *Consider the parametrized surface (Enneper's surface)*

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right), \tag{1.1}$$

and show that

1. *The coefficients of the first fundamental form are*

$$E = G = \left(1 + u^2 + v^2\right)^2, F = 0. \tag{1.2}$$

2. *The coefficients of the second fundamental form are*

$$e = 2, g = -2, f = 0. \tag{1.3}$$

3. *The principal curvatures are*

$$k_1 = \frac{2}{\left(1 + u^2 + v^2\right)^2}, k_2 = -\frac{2}{\left(1 + u^2 + v^2\right)^2}. \tag{1.4}$$

4. *The lines of curvature are the coordinate curves.*

5. *The asymptotic curves are $u + v = const.$, $u - v = const.$*

Solution.

1. To obtain the first fundamental form, we compute

$$\mathbf{x}_u = \left(1 - u^2 + v^2, 2uv, 2u\right), \tag{1.5}$$
$$\mathbf{x}_v = \left(2uv, 1 - v^2 + u^2, -2v\right), \tag{1.6}$$

and therefore

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle && (1.7)\\
&= \left(1 - u^2 + v^2\right)^2 + 4u^2v^2 + 4u^2 && (1.8)\\
&= \left(1 + u^2 + v^2\right)^2, && (1.9)\\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle && (1.10)\\
&= 2uv\left(1 - u^2 + v^2\right) + 2uv\left(1 - v^2 + u^2\right) - 4uv && (1.11)\\
&= 0, && (1.12)\\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle && (1.13)\\
&= 4u^2v^2 + \left(1 - v^2 + u^2\right)^2 + 4v^2 && (1.14)\\
&= \left(1 + u^2 + v^2\right)^2, && (1.15)
\end{aligned}$$

i.e., (1.2) holds.

2. For the computation of the coefficients e, g, f of the second fundamental form, we need to know $\mathbf{x}_u, \mathbf{x}_v$ (given by (1.5)-(1.6)), N, $\mathbf{x}_{uu}, \mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$:

$$\mathbf{x}_{uu} = (-2u, 2v, 2)\,, \tag{1.16}$$
$$\mathbf{x}_{uv} = (2v, 2u, 0)\,, \tag{1.17}$$
$$\mathbf{x}_{vv} = (2u, -2v, -2)\,. \tag{1.18}$$

Hence,

$$e = \langle N, \mathbf{x}_{uu} \rangle \tag{1.19}$$
$$= \left\langle \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}, \mathbf{x}_{uu} \right\rangle \tag{1.20}$$
$$= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{\sqrt{EG - F^2}} \tag{1.21}$$
$$= \frac{\begin{vmatrix} 1-u^2+v^2 & 2uv & -2u \\ 2uv & 1-v^2+u^2 & 2v \\ 2u & -2v & 2 \end{vmatrix}}{(1+u^2+v^2)^2} \tag{1.22}$$
$$= \frac{2\big(1+u^2+v^2\big)^2}{(1+u^2+v^2)^2} \tag{1.23}$$
$$= 2, \tag{1.24}$$
$$f = \langle N, \mathbf{x}_{uv} \rangle \tag{1.25}$$
$$= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\sqrt{EG - F^2}} \tag{1.26}$$
$$= \frac{\begin{vmatrix} 1-u^2+v^2 & 2uv & 2v \\ 2uv & 1-v^2+u^2 & 2u \\ 2u & -2v & 0 \end{vmatrix}}{(1+u^2+v^2)^2} \tag{1.27}$$
$$= 0, \tag{1.28}$$
$$g = \langle N, \mathbf{x}_{vv} \rangle \tag{1.29}$$
$$= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{\sqrt{EG - F^2}} \tag{1.30}$$
$$= \frac{\begin{vmatrix} 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \\ 2u & -2v & -2 \end{vmatrix}}{(1+u^2+v^2)^2} \tag{1.31}$$
$$= -\frac{2\big(1+u^2+v^2\big)^2}{(1+u^2+v^2)^2} \tag{1.32}$$
$$= -2, \tag{1.33}$$

i.e., (1.3) holds.

3. We recall that the principal curvatures k_1, k_2 are the roots of the following quadratic equation

$$k^2 - 2Hk + K = 0. \tag{1.34}$$

Hence, to obtain k_1, k_2, it suffices to compute the Gaussian curvature K and the mean curvature H:

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} && (1.35)\\ &= -\frac{4}{(1+u^2+v^2)^4}, && (1.36)\\ H &= \frac{1}{2}\cdot\frac{eG - 2fF + gE}{EG - F^2} && (1.37)\\ &= \frac{G - E}{(1+u^2+v^2)^4} && (1.38)\\ &= 0. && (1.39) \end{aligned}$$

The quadratic equation (1.34) then becomes

$$k^2 - \frac{4}{(1+u^2+v^2)^4} = 0, \tag{1.40}$$

i.e., (1.4) holds.

4. We directly apply the following result, which is established in p.161, [1]: "*A necessary and sufficient condition for the coordinate curves of a parametrization to be lines of curvature in a neighborhood of a nonumbilical points is that $F = f = 0$.*" to Enneper's surface. It should be noted that all the points of this surface are nonumbilical since $k_1 \neq k_2$.

5. We recall that a connected regular curve C in the coordinate neighborhood of $\mathbf{x}$ is an asymptotic curve if and only if for any parametrization $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in I$, of C we have $II(\alpha'(t)) = 0$, for all $t \in I$, that is, if and only if

$$e(u')^2 + 2fu'v' + g(v')^2 = 0, \quad t \in I. \tag{1.41}$$

The differential equation of asymptotic curves (1.41), in our situation, becomes (globally)

$$2(u')^2 - 2(v')^2 = 0. \tag{1.42}$$

Hence, $u' + v' = 0$ or $u' - v' = 0$ satisfy (1.42). By integrating these equations with respect to variable t, we conclude that the asymptotic curves are $u + v = \text{const.}$, and $u - v = \text{const.}$ □

Problem 2 (Exercise 6, p.168-169, [1]).
(A Surface with $K \equiv -1$; the Pseudosphere.)

1. *Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the* ***tractrix****); see Fig. 1.*

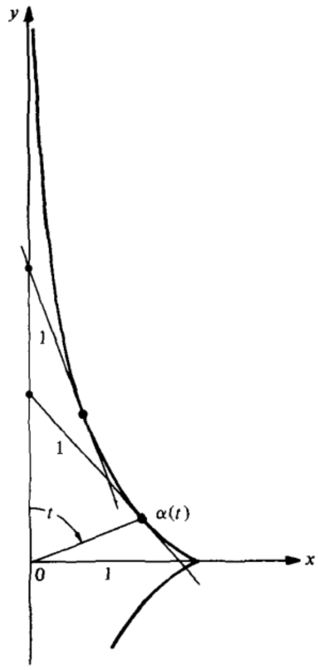

Figure 1: The tractrix.

2. *Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the pseudosphere; see Fig. 2) is regular and find out a parametrization in a neighborhood of a regular point.*

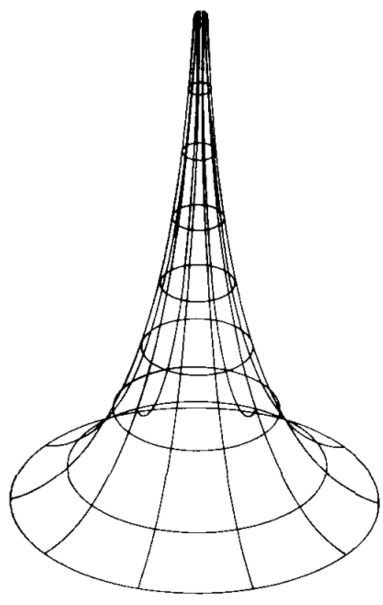

Figure 2: The pseudosphere.

3. *Show that the Gaussian curvature of any regular point of the pseudosphere is* -1.

SOLUTION.

1. By taking the line r as the z axis and a normal to r as the x axis, we have that

$$z' = \frac{\sqrt{1-x^2}}{x}. \tag{1.43}$$

By setting $x = \sin\theta$, we obtain

$$z(\theta) = \int \frac{\cos^2\theta}{\sin\theta} d\theta \tag{1.44}$$

$$= \ln\tan\frac{\theta}{2} + \cos\theta + C. \tag{1.45}$$

If $z\left(\frac{\pi}{2}\right) = 0$, then $C = 0$.

2. With the above notations; see p.76, [1],

$$x = \sin\theta, z = \ln\tan\frac{\theta}{2} + \cos\theta, \quad 0 < \theta < \pi, \tag{1.46}$$

is a parametrization for the tractrix C and denote by δ the rotation angle about the z axis. Thus, we obtain a map

$$\mathbf{x}(\theta,\delta) = \left(\sin\theta\cos\delta, \sin\theta\sin\delta, \ln\tan\frac{\theta}{2} + \cos\theta\right), \tag{1.47}$$

from the open set $U = \left\{(\theta,\delta) \in \mathbb{R}^2; 0 < \theta < \pi, 0 < \delta < 2\pi\right\}$ into the pseudosphere S.

To show that S is regular, we need to prove that $\mathbf{x}$ is a parametrization of S, i.e., we must check condition 1, 2, and 3 of Def. 1, Sec. 2.2, p.52, [1].

   (a) $\mathbf{x}$ *is differentiable.* This is obvious by (1.47). We write

$$\mathbf{x}(\theta,\delta) = (x(\theta,\delta), y(\theta,\delta), z(\theta,\delta)), \quad (\theta,\delta) \in U, \tag{1.48}$$

   where

$$x(\theta,\delta) = \sin\theta\cos\delta, \tag{1.49}$$

$$y(\theta,\delta) = \sin\theta\sin\delta, \tag{1.50}$$

$$z(\theta,\delta) = \ln\tan\frac{\theta}{2} + \cos\theta, \tag{1.51}$$

   have continuous partial derivatives of all orders in U.

   (b) $\mathbf{x}$ *is a homeomorphism.* To show that $\mathbf{x}$ is a homeomorphism, we first show that $\mathbf{x}$ is one-to-one. In face, since $\left(\sin\theta, \ln\tan\frac{\theta}{2} + \cos\theta\right)$ is a parametrization of C, given z and $x^2 + y^2 = \sin^2\theta$, we can determine θ uniquely. Thus, $\mathbf{x}$ is one-to-one.

We remark that because $\left(\sin\theta, \ln\tan\frac{\theta}{2} + \cos\theta\right)$ is a parametrization of C, θ is a continuous function of z and of $\sqrt{x^2+y^2}$ and thus a continuous function of (x, y, z). [1]

To prove that $\mathbf{x}^{-1}$ is continuous, it remains to show that θ is a continuous function of (x, y, z). To see this, we first observe that if $\theta \neq \pi$, we obtain, since $\sin\theta \neq 0$ $(0 < \theta < \pi)$,

$$\tan\frac{\delta}{2} = \frac{\sin\frac{\delta}{2}}{\cos\frac{\delta}{2}} \tag{1.54}$$

$$= \frac{2\sin\frac{\delta}{2}\cos\frac{\delta}{2}}{2\cos^2\frac{\delta}{2}} \tag{1.55}$$

$$= \frac{\sin\delta}{1+\cos\delta} \tag{1.56}$$

$$= \frac{\frac{y}{\sin\theta}}{1+\frac{x}{\sin\theta}} \tag{1.57}$$

$$= \frac{y}{x+\sqrt{x^2+y^2}}, \tag{1.58}$$

hence,

$$\delta = 2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}. \tag{1.59}$$

Thus, if $\delta \neq \pi$, δ is a continuous function of (x, y, z). By the same token, if δ is in a small interval about π, we obtain

$$\delta = 2\cot^{-1}\frac{y}{-x+\sqrt{x^2+y^2}}. \tag{1.60}$$

Thus, δ is a continuous of (x, y, z). This shows that $\mathbf{x}^{-1}$ is continuous.

(c) *The regularity condition.* We will prove that for each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one. To this end, we consider the following Jacobian determinants

$$\frac{\partial(x,y)}{\partial(\theta,\delta)} = \begin{vmatrix} x_\theta & x_\delta \\ y_\theta & y_\delta \end{vmatrix} \tag{1.61}$$

$$= \begin{vmatrix} \cos\theta\cos\delta & -\sin\theta\sin\delta \\ \cos\theta\sin\delta & \sin\theta\cos\delta \end{vmatrix} \tag{1.62}$$

$$= \sin\theta\cos\theta\cos^2\delta + \sin\theta\cos\theta\sin^2\delta \tag{1.63}$$

$$= \sin\theta\cos\theta, \tag{1.64}$$

which is nonzero for $\theta \neq \frac{\pi}{2}$,

$$\frac{\partial(y,z)}{\partial(\theta,\delta)} = \begin{vmatrix} y_\theta & y_\delta \\ z_\theta & z_\delta \end{vmatrix} \tag{1.65}$$

[1] Indeed, θ can be represented as

$$\theta = (\ln\circ\tan\circ f + \cos)^{-1}(z), \tag{1.52}$$

$$\theta = \arcsin\sqrt{x^2+y^2}, \tag{1.53}$$

where $f : \theta \mapsto \frac{\theta}{2}$.

$$= \begin{vmatrix} \cos\theta \sin\delta & \sin\theta \cos\delta \\ \dfrac{1}{\sin\theta} & 0 \end{vmatrix} \tag{1.66}$$

$$= -\cos\delta, \tag{1.67}$$

which is nonzero for $\delta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, and

$$\frac{\partial(x,z)}{\partial(\theta,\delta)} = \begin{vmatrix} x_\theta & x_\delta \\ z_\theta & z_\delta \end{vmatrix} \tag{1.68}$$

$$= \begin{vmatrix} \cos\theta \cos\delta & -\sin\theta \sin\delta \\ \dfrac{1}{\sin\theta} & 0 \end{vmatrix} \tag{1.69}$$

$$= \sin\delta, \tag{1.70}$$

which is nonzero for $\delta \neq \pi$.

Combing these facts, we deduce that the two column vectors of the matrix

$$d\mathbf{x}_q = \begin{pmatrix} x_\theta & x_\delta \\ y_\theta & y_\delta \\ z_\theta & z_\delta \end{pmatrix}, \tag{1.71}$$

is linearly independent, i.e., the regularity condition is satisfied.

Therefore, as promised, $\mathbf{x}$ is a parametrization of S. Since S can be entirely covered by similar parametrizations, it follows that S is a regular surface. A parametrization in a neighborhood of a regular point of S is given by (1.47).

3. We shall compute the Gaussian curvature of the regular points of the surface S by the parametrization $\mathbf{x}(\theta,\delta)$ defined by (1.47). To this end, we compute

$$\mathbf{x}_\theta = \left(\cos\theta\cos\delta, \cos\theta\sin\delta, \frac{\cos^2\theta}{\sin\theta}\right), \tag{1.72}$$

$$\mathbf{x}_\delta = \left(-\sin\theta\sin\delta, \sin\theta\cos\delta, 0\right), \tag{1.73}$$

$$\mathbf{x}_{\theta\theta} = \left(-\sin\theta\cos\delta, -\sin\theta\sin\delta, \frac{\cos\theta\left(\cos^2\theta - 2\right)}{\sin^2\theta}\right), \tag{1.74}$$

$$\mathbf{x}_{\theta\delta} = \left(-\cos\theta\sin\delta, \cos\theta\cos\delta, 0\right), \tag{1.75}$$

$$\mathbf{x}_{\delta\delta} = \left(-\sin\theta\cos\delta, -\sin\theta\sin\delta, 0\right), \tag{1.76}$$

From these, we obtain the coefficients of the first fundamental form

$$E = \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle \tag{1.77}$$

$$= \cos^2\theta\cos^2\delta + \cos^2\theta\sin^2\delta + \frac{\cos^4\theta}{\sin^2\theta} \tag{1.78}$$

$$= \cos^2\theta + \frac{\cos^4\theta}{\sin^2\theta} \tag{1.79}$$

$$= \cot^2\theta, \tag{1.80}$$

$$
\begin{aligned}
F &= \langle \mathbf{x}_\theta, \mathbf{x}_\delta \rangle && (1.81)\\
&= -\sin\theta\cos\theta\sin\delta\cos\delta + \sin\theta\cos\theta\sin\delta\cos\delta && (1.82)\\
&= 0, && (1.83)\\
G &= \langle \mathbf{x}_\delta, \mathbf{x}_\delta \rangle && (1.84)\\
&= \sin^2\theta\sin^2\delta + \sin^2\theta\cos^2\delta && (1.85)\\
&= \sin^2\theta, && (1.86)
\end{aligned}
$$

Introducing the values just obtain in the coefficients of the second fundamental form gives

$$
\begin{aligned}
e &= \langle N, x_{\theta\theta} \rangle && (1.87)\\
&= \left\langle \frac{x_\theta \wedge x_\delta}{|x_\theta \wedge x_\delta|}, x_{\theta\theta} \right\rangle && (1.88)\\
&= \frac{(x_\theta, x_\delta, x_{\theta\theta})}{\sqrt{EG-F^2}} && (1.89)\\
&= \frac{\begin{vmatrix} \cos\theta\cos\delta & -\sin\theta\sin\delta & -\sin\theta\cos\delta \\ \cos\theta\sin\delta & \sin\theta\cos\delta & -\sin\theta\sin\delta \\ \frac{\cos^2\theta}{\sin\theta} & 0 & \frac{\cos\theta\left(\cos^2\theta-2\right)}{\sin^2\theta} \end{vmatrix}}{|\cos\theta|} && (1.90)\\
&= -\frac{|\cos\theta|}{\sin\theta}, && (1.91)\\
f &= (N, x_{\theta\delta}) && (1.92)\\
&= \frac{(x_\theta, x_\delta, x_{\theta\delta})}{\sqrt{EG-F^2}} && (1.93)\\
&= \frac{\begin{vmatrix} \cos\theta\cos\delta & -\sin\theta\sin\delta & -\cos\theta\sin\delta \\ \cos\theta\sin\delta & \sin\theta\cos\delta & \cos\theta\cos\delta \\ \frac{\cos^2\theta}{\sin\theta} & 0 & 0 \end{vmatrix}}{|\cos\theta|} && (1.94)\\
&= 0, && (1.95)\\
g &= (N, x_{\delta\delta}) && (1.96)\\
&= \frac{(x_\theta, x_\delta, x_{\delta\delta})}{\sqrt{EG-F^2}} && (1.97)\\
&= \frac{\begin{vmatrix} \cos\theta\cos\delta & -\sin\theta\sin\delta & -\sin\theta\cos\delta \\ \cos\theta\sin\delta & \sin\theta\cos\delta & -\sin\theta\sin\delta \\ \frac{\cos^2\theta}{\sin\theta} & 0 & 0 \end{vmatrix}}{|\cos\theta|} && (1.98)\\
&= |\cos\theta|\sin\theta, && (1.99)
\end{aligned}
$$

Finally, we obtain the Gaussian curvature of the regular point p of the pseudosphere

$$
\begin{aligned}
K &= \frac{eg-f^2}{EG-F^2} && (1.100)\\
&= \frac{-\frac{|\cos\theta|}{\sin\theta} \cdot |\cos\theta|\sin\theta}{\cot^2\theta\sin^2\theta} && (1.101)
\end{aligned}
$$

$$= -1, \tag{1.102}$$

as desired. □

Problem 3 (Exercise 7, p.169, [1]).

(Surfaces of Revolution of Constant Curvature.)

$(\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$ *is given as a surface of revolution with constant Gaussian curvature* K. *To determine the function* φ *and* ψ, *choose the parameter* v *in such a way that*

$$(\varphi')^2 + (\psi')^2 = 1, \tag{1.103}$$

(geometrically, this means that v *is the arc length of the generating curve* $(\varphi(v), \psi(v))$*). Show that*

1. φ *satisfies* $\varphi'' + K\varphi = 0$ *and* ψ *is given by*

$$\psi = \int \sqrt{1 - (\varphi')^2} dv, \tag{1.104}$$

   thus, $0 < u < 2\pi$, *and the domain of* v *is such that the last integral makes sense.*

2. *All surfaces of revolution with constant curvature* $K = 1$ *which intersect perpendicularly the plane* xOy *are given by*

$$\varphi(v) = C\cos v, \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v} dv, \tag{1.105}$$

   where C *is a constant* ($C = \varphi(0)$). *Determine the domain of* v *and draw a rough sketch of the profile of the surface in the* xz *plane for the cases* $C = 1$, $C > 1$, $C < 1$. *Observe that* $C = 1$ *gives a sphere (Fig. 2).*

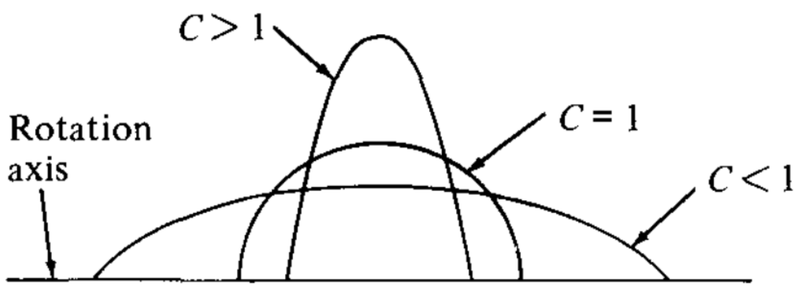

Figure 3: The profile of the surface in the xz plane for the cases $C = 1, C > 1, C < 1$.

3. *All surfaces of revolution with constant curvature* $K = -1$ *may be given by one of the following types:*

(a) $\varphi(v) = C\cosh v, \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 v}dv$.

(b) $\varphi(v) = C\sinh v, \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 v}dv$.

(c) $\varphi(v) = e^v, \psi(v) = \int_0^v \sqrt{1 - e^{2v}}dv$.

Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane.

4. *The surface of type c in part 3 is the pseudosphere of Exercise 6.*

5. *The only surfaces of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane.*

Solution.

1. Let

$$\mathbf{x}(u, v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)), \quad \varphi(v) \neq 0, \tag{1.106}$$

where the domain of u and v will be determined, be a parametrization of given surface, denoted by S as usual, of revolution. We shall compute the Gaussian curvature of the points of surface S by the parametrization (1.106). To this end, we compute

$$\begin{aligned}
\mathbf{x}_u &= (-\varphi(v)\sin u, \varphi(v)\cos u, 0), &(1.107)\\
\mathbf{x}_v &= (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v)), &(1.108)\\
\mathbf{x}_{uu} &= (-\varphi(v)\cos u, -\varphi(v)\sin u, 0), &(1.109)\\
\mathbf{x}_{uv} &= (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0), &(1.110)\\
\mathbf{x}_{vv} &= (\varphi''(v)\cos u, \varphi''(v)\sin u, \psi''(v)), &(1.111)
\end{aligned}$$

From these, we obtain the coefficients of the first fundamental form

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle &(1.112)\\
&= \varphi^2(v)\sin^2 u + \varphi^2(v)\cos^2 u &(1.113)\\
&= \varphi^2(v), &(1.114)\\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle &(1.115)\\
&= -\varphi(v)\varphi'(v)\sin u\cos u + \varphi(v)\varphi'(v)\sin u\cos u &(1.116)\\
&= 0, &(1.117)\\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle &(1.118)\\
&= (\varphi'(v))^2\cos^2 u + (\varphi'(v))^2\sin^2 u + (\psi'(v))^2 &(1.119)\\
&= (\varphi'(v))^2 + (\psi'(v))^2 &(1.120)\\
&= 1, \text{ by the assumption (1.103).} &(1.121)
\end{aligned}$$

Introducing the values just obtained in the coefficients of the second fundamental form gives

$$\begin{aligned}
e &= \langle N, \mathbf{x}_{uu} \rangle &(1.122)\\
&= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{\sqrt{EG - F^2}} &(1.123)
\end{aligned}$$

$$= \frac{\begin{vmatrix} -\varphi(v)\sin u & \varphi'(v)\cos u & -\varphi(v)\cos u \\ \varphi(v)\cos u & \varphi'(v)\sin u & -\varphi(v)\sin u \\ 0 & \psi'(v) & 0 \end{vmatrix}}{|\varphi(v)|} \tag{1.124}$$

$$= -\psi'(v)\,|\varphi(v)|\,, \tag{1.125}$$

$$f = \langle N, \mathbf{x}_{uv} \rangle \tag{1.126}$$

$$= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\sqrt{EG - F^2}} \tag{1.127}$$

$$= \frac{\begin{vmatrix} -\varphi(v)\sin u & \varphi'(v)\cos u & -\varphi'(v)\sin u \\ \varphi(v)\cos u & \varphi'(v)\sin u & \varphi'(v)\cos u \\ 0 & \psi'(v) & 0 \end{vmatrix}}{|\varphi(v)|} \tag{1.128}$$

$$= 0, \tag{1.129}$$

$$g = \langle N, \mathbf{x}_{vv} \rangle \tag{1.130}$$

$$= \frac{(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{\sqrt{EG - F^2}} \tag{1.131}$$

$$= \frac{\begin{vmatrix} -\varphi(v)\sin u & \varphi'(v)\cos u & \varphi''(v)\cos u \\ \varphi(v)\cos u & \varphi'(v)\sin u & \varphi''(v)\sin u \\ 0 & \psi'(v) & \psi''(v) \end{vmatrix}}{|\varphi(v)|} \tag{1.132}$$

$$= \frac{\varphi(v)}{|\varphi(v)|}\left(\varphi''(v)\,\psi'(v) - \varphi'(v)\,\psi''(v)\right). \tag{1.133}$$

Hence, the Gaussian curvature K is given by

$$K = \frac{eg - f^2}{EG - F^2} \tag{1.134}$$

$$= -\frac{\psi'(v)\,|\varphi(v)| \cdot \frac{\varphi(v)}{|\varphi(v)|}\left(\varphi''(v)\,\psi'(v) - \varphi'(v)\,\psi''(v)\right)}{\varphi^2(v)} \tag{1.135}$$

$$= -\frac{\psi'(v)\left(\varphi''(v)\,\psi'(v) - \varphi'(v)\,\psi''(v)\right)}{\varphi(v)}. \tag{1.136}$$

It is convenient to put the Gaussian curvature in another form. By differentiating (1.103) we obtain

$$\varphi'(v)\,\varphi''(v) = -\psi'(v)\,\psi''(v)\,. \tag{1.137}$$

Thus,

$$K = -\frac{\psi'(v)\left(\varphi''(v)\,\psi'(v) - \varphi'(v)\,\psi''(v)\right)}{\varphi(v)} \tag{1.138}$$

$$= -\frac{\varphi''(v)\left(\psi'(v)\right)^2 + \varphi''(v)\left(\varphi'(v)\right)^2}{\varphi(v)} \tag{1.139}$$

$$= -\frac{\varphi''(v)}{\varphi(v)}. \tag{1.140}$$

Thus, φ satisfies the following equation

$$\varphi'' + K\varphi = 0, \tag{1.141}$$

and, by integrating the equation $\psi'(v) = \sqrt{1 - (\varphi'(v))^2}$, ψ is given by

$$\psi = \int \sqrt{1 - (\varphi')^2} dv, \tag{1.142}$$

Thus, $0 < u < 2\pi$ and the domain of v is such that the last integral makes sense.

2. Plugging $K = 1$ in (1.141) gives

$$\varphi'' + \varphi = 0. \tag{1.143}$$

Solving this homogeneous second-order linear differential equation yields

$$\varphi(v) = C_1 \cos v + C_2 \sin v, \tag{1.144}$$

where C_1 and C_2 are arbitrary constants.

Recall that the generating curve $(\varphi(v), \psi(v))$ is in the xOz plane and the surface of revolution $\mathbf{x}(u, v)$ is generated by rotating the generating curve around the z-axis. Since the surfaces of revolution intersects perpendicularly the plane xOy, the generating curves must intersects perpendicularly with the x-axis.

Assume that the generating curve $(\varphi(v), \psi(v))$ intersects the x axis at the point $(C, 0)$ where $C = \varphi(0)$, since $(\varphi(v), \psi(v))$ intersects the x-axis perpendicularly, the tangent vector of $(\varphi(v), \psi(v))$ is perpendicular to the x-axis at the intersection $(C, 0)$ (this also gives $\varphi(0) = C, \psi(0) = 0$) , i.e.

$$\begin{aligned} \varphi'(0) &= \langle (\varphi'(0), \psi'(0)), (1, 0) \rangle && (1.145) \\ &= 0. && (1.146) \end{aligned}$$

Now, combining (1.144) with $\varphi(0) = C, \varphi'(0) = 0$ yields $C_1 = C, C_2 = 0$, i.e. (1.144) becomes

$$\varphi(v) = C \cos v. \tag{1.147}$$

And (1.142) then becomes, note that $\psi(0) = 0$

$$\psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 \bar{v}} d\bar{v}. \tag{1.148}$$

The domain of v is, therefore, determined by requiring that the integrand in (1.148) makes sense, i.e., the domain of v is given by

$$\left\{ v : 0 < v < \pi, |\sin v| \le \frac{1}{|C|} \right\}. \tag{1.149}$$

In the case when $C = 1$, (1.147)-(1.148) gives a sphere.

3. Plugging $K = -1$ in (1.141) gives

$$\varphi'' - \varphi = 0. \tag{1.150}$$

Solving this homogeneous second-order linear differential equation gives

$$\varphi(v) = C_1 e^v + C_2 e^{-v}, \tag{1.151}$$

where C_1 and C_2 are arbitrary constants.

We consider the following three cases for the constants C_1, C_2.[2]

(a) *Case* $C_1 = C_2 = \frac{C}{2}$. In this case, (1.151) becomes

$$\begin{aligned} \varphi(v) &= \frac{C}{2}\left(e^v + e^{-v}\right) & (1.152)\\ &= C\cosh v, & (1.153)\\ \psi(v) &= \int_0^v \sqrt{1 - C^2 \sinh^2 \bar{v}}\, d\bar{v}. & (1.154) \end{aligned}$$

where the domain of v is chosen for which the last integral makes sense.

(b) *Case* $C_1 = \frac{C}{2}, C_2 = -\frac{C}{2}$. In this case, (1.151) becomes

$$\begin{aligned} \varphi(v) &= \frac{C}{2}\left(e^v - e^{-v}\right) & (1.155)\\ &= C\sinh v, & (1.156)\\ \psi(v) &= \int_0^v \sqrt{1 - C^2 \cosh^2 \bar{v}}\, d\bar{v}. & (1.157) \end{aligned}$$

where the domain of v is chosen for which the last integral makes sense.

(c) *Case* $C_1 = 1, C_2 = 0$. In this case, (1.151) becomes

$$\begin{aligned} \varphi(v) &= e^v, & (1.158)\\ \psi(v) &= \int_0^v \sqrt{1 - e^{2\bar{v}}}\, d\bar{v}. & (1.159) \end{aligned}$$

where the domain of v is chosen for which the last integral makes sense.

4. We claim that the surface of type 3 in (3) is the pseudosphere described in Problem 1. To this end, we check the following generating curve

$$\varphi(v) = e^v, \psi(v) = \int_0^v \sqrt{1 - e^{2\bar{v}}}\, d\bar{v}, \tag{1.160}$$

is the tractrix. We turn back to equation (1.43), but now we set $x = e^v$ instead of setting $x = \sin\theta$ as before. Then integrating (1.43) with respect to v yields

$$z = \int \frac{\sqrt{1 - e^{2v}}}{e^v} e^v dv \tag{1.161}$$

[2] Is it true that all surfaces of revolution with constant curvature $K = -1$ may be given by one of the given types?

$$= \int \sqrt{1 - e^{2v}} dv. \tag{1.162}$$

Hence, (1.160) yields a parametrization for the tractrix. Finally, since the tractrix is the generating curve for the pseudosphere, the surface

$$\left(e^v \cos u, e^v \sin u, \int_0^v \sqrt{1 - e^{2\bar{v}}} d\bar{v} \right), \tag{1.163}$$

with suitable domains of u and v, is exactly the pseudosphere.

5. Plugging $K = 0$ in (1.141) yields $\varphi'' = 0$. Hence,

$$\varphi(v) = C_1 v + C_2, \tag{1.164}$$

where C_1 and C_2 are arbitrary constants. Plugging $\varphi'(v) = C_1$ in (1.142) gives

$$\psi(v) = \int_0^v \sqrt{1 - C_1^2} d\bar{v} \tag{1.165}$$

$$= v\sqrt{1 - C_1^2}, \tag{1.166}$$

where $-1 \leq C_1 \leq 1$.

We consider the following three cases with respect to C_1.

(a) *Case* $|C_1| = 1$. In this case, the generating curve becomes

$$(\varphi(v), \psi(v)) = (\pm v + C_2, 0), \tag{1.167}$$

which is a line orthogonal to the z-axis.

(b) *Case* $C_1 = 0$. In this case, the generating curve becomes

$$(\varphi(v), \psi(v)) = (C_2, v), \tag{1.168}$$

which is a line orthogonal to the x-axis. Therefore, the surface of revolution in this case is a right circular cylinder.

(c) *Case* $0 < |C_1| < 1$. In this case, the generating curve becomes

$$(\varphi(v), \psi(v)) = \left(C_1 v + C_2, v\sqrt{1 - C_1^2} \right), \tag{1.169}$$

which is a line intersecting the z-axis. Therefore, the surface of revolution in this case is a right circular cone. □

THE END

References

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[2] Wolfgang Kühnel, *Differential Geometry, Curves Surfaces Manifolds*, Second Edition, Student Mathematical Library, Volume 16, AMS.

[3] `http://mathworld.wolfram.com/Pseudosphere.html`