Finite Volume Method in 2D

Anh Ha Le, Huong Lan Tran

University of Sciences

April 8, 2014

Let $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$. We will use the finite volume method to discretize the following Poisson equation

$$-\Delta u = f(x, y) \quad \text{in } \Omega \tag{2.1}$$

subject to a Dirichlet boundary condition:

$$u(x, y) = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

The Laplacian Δ is defined in Cartesian coordinates by

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

When $f(x, y) = 0$, (2.1) is called Laplace equation.
The existence and uniqueness of the solution to equation (2.1) is proved. Our purpose is to find the discrete solution.

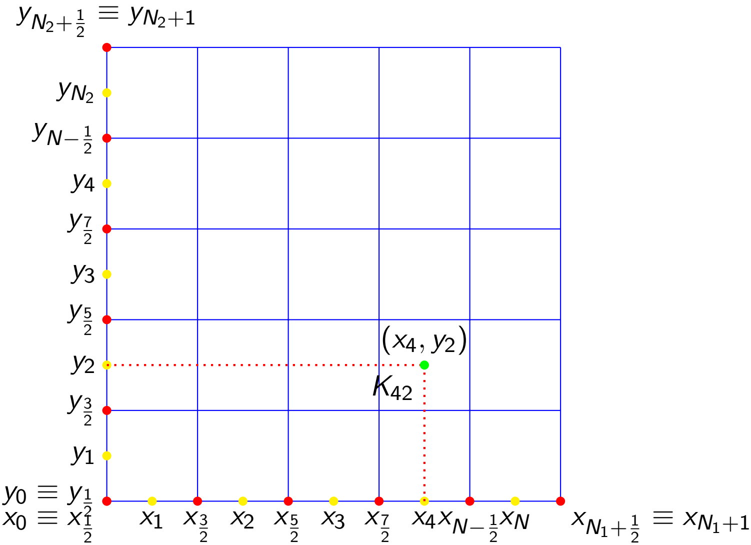

Figure: Rectangular mesh

Consider $\Omega = (0,1) \times (0,1)$. On interval $[0,1]$, we make two partition $(x_{i+\frac{1}{2}})_{i\in\overline{0,N_1}}$, $(y_{j+\frac{1}{2}})_{j\in\overline{0,N_2}}$ such that

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_1-\frac{1}{2}} < x_{N_1+\frac{1}{2}} = 1,$$
$$0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_2-\frac{1}{2}} < x_{N_2+\frac{1}{2}} = 1.$$

Let $\mathcal{T} = (T_{ij})_{i\in\overline{1,N_1}, j\in\overline{1,N_2}}$ be an admissible mesh of $(0,1)\times(0,1)$ such that

$$T_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$$

T_{ij} is called a **control volume** of $\mathcal{T}$. The points $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ are called **mesh points**.

Choosing the sequences $(x_i)_{i\in\overline{0,N_1+1}}$ and $(y_j)_{j\in\overline{0,N_2+1}}$ such that

$$x_0 \equiv x_{\frac{1}{2}}, \quad x_i = \frac{1}{2}\left(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\right), \quad x_{N_1+1} \equiv x_{N_1+\frac{1}{2}},$$
$$y_0 \equiv y_{\frac{1}{2}}, \quad y_j = \frac{1}{2}\left(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}\right), \quad x_{N_2+1} \equiv y_{N_2+\frac{1}{2}}.$$

The point (x_i, y_j) is the **control point** of control volume $T_{i,j}$.
Let

$$h_i = |x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}|, \quad k_j = |y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}| \quad \text{for all } i \in \overline{1,N_1} \text{ and } j \in \overline{1,N_2}$$

and

$$h_{i+\frac{1}{2}} = |x_{i+1} - x_i|, \quad k_{j+\frac{1}{2}} = |y_{j+1} - y_j| \quad \text{for all } i \in \overline{0,N_1} \text{ and } j \in \overline{0,N_2}.$$

Then, the **area** of control volume $|T_{ij}| = h_i k_j$.
$h = \max\{h_i, k_j\}$ is the **mesh size**.

The finite volume scheme is found by integrating the first equation of (2.1) over each control volume $T_{i,j}$, which gives

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} -\Delta u(x,y)dxdy = \frac{1}{|T_{ij}|}\int_{T_{ij}} f(x,y)dxdy \qquad (3.1)$$

Following the definition of $-\Delta$ operator, we have

$$-\frac{1}{|T_{ij}|}\int_{T_{ij}} u_{xx}(x,y)dxdy - \frac{1}{|T_{ij}|}\int_{T_{ij}} u_{yy}(x,y)dxdy = \frac{1}{|T_{ij}|}\int_{T_{ij}} f(x,y)dxd \qquad (3.2)$$

We can rewrite clearer that

$$\frac{-1}{|T_{ij}|}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} u_{xx}(x,y)dxdy - \frac{1}{|T_{ij}|}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}} u_{yy}(x,y)dydx$$
$$\frac{1}{|T_{ij}|}\int_{T_{ij}} f(x,y)dxdy \qquad (3.3)$$

Applying the integral formulation, we obtain:

$$\int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} u_{xx}(x,y)dxdy = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}}, y)dy - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}}, y)dy \tag{3.4}$$

and

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} u_{yy}(x,y)dydx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x, y_{j+\frac{1}{2}})dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x, y_{j-\frac{1}{2}})dy \tag{3.5}$$

Then, we get

$$\frac{-1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}},y)dy + \frac{1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}},y)dy$$
$$-\frac{1}{|T_{ij}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j+\frac{1}{2}})dx + \frac{1}{|T_{ij}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j-\frac{1}{2}})dy$$
$$=\frac{1}{|T_{ij}|}\int_{T_{ij}} f(x,y)dxdy \tag{3.6}$$

The following approximations are made:

$$\frac{1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}},y)dy = \frac{k_j u_x(x_{i+\frac{1}{1}},y_j)}{|T_{ij}|} = \frac{u_{i+1,j}-u_{i,j}}{h_i h_{i+\frac{1}{2}}},$$

$$\frac{1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}},y)dy = \frac{k_j u_x(x_{i-\frac{1}{1}},y_j)}{|T_{ij}|} = \frac{u_{i,j}-u_{i-1,j}}{h_i h_{i-\frac{1}{2}}},$$

$$\frac{1}{|T_{ij}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j+\frac{1}{2}})dy = \frac{h_i u_y(x_i,y_{j+\frac{1}{2}})}{|T_{ij}|} = \frac{u_{i,j+1}-u_{i,j}}{k_j k_{j+\frac{1}{2}}},$$

$$\frac{1}{|T_{ij}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j-\frac{1}{2}})dy = \frac{h_i u_y(x_i,y_{j-\frac{1}{2}})}{|T_{ij}|} = \frac{u_{i,j}-u_{i,j-1}}{k_j k_{j-\frac{1}{2}}}.$$

The approximate equation (3.6) becomes

$$-\frac{u_{i+1,j}-u_{i,j}}{h_i h_{i+\frac{1}{2}}}+\frac{u_{i,j}-u_{i-1,j}}{h_i h_{i-\frac{1}{2}}}-\frac{u_{i,j+1}-u_{i,j}}{k_j k_{j+\frac{1}{2}}}+\frac{u_{i,j}-u_{i,j-1}}{k_j k_{j-\frac{1}{2}}}=f_{ij} \tag{3.7}$$

where $f_{ij}=\dfrac{1}{|T_{ij}|}\displaystyle\int_{T_{ij}} f(x,y)dx$ is mean-value of f on T_{ij}.
Rearranging equation (3.7) gives

$$-\frac{1}{h_i h_{i-\frac{1}{2}}}u_{i-1,j}-\frac{1}{h_i h_{i+\frac{1}{2}}}u_{i+1,j}-\frac{1}{k_j k_{j-\frac{1}{2}}}u_{i,j-1}-\frac{1}{k_j k_{j+\frac{1}{2}}}u_{i,j+1} \tag{3.8}$$

$$+\left(\frac{1}{h_i h_{i-\frac{1}{2}}}+\frac{1}{h_i h_{i+\frac{1}{2}}}+\frac{1}{k_j k_{j-\frac{1}{2}}}+\frac{1}{k_j k_{j+\frac{1}{2}}}\right)u_{i,j}=f_{ij}$$

By setting

$$a_i = -\frac{1}{h_i h_{i-\frac{1}{2}}}, \; b_i = -\frac{1}{h_i h_{i+\frac{1}{2}}}$$

$$c_j = -\frac{1}{k_j k_{j-\frac{1}{2}}}, \; d_j = -\frac{1}{k_j k_{j+\frac{1}{2}}}$$

$$s_{i,j} = a_i + b_i + c_j + d_j$$

At cell (i,j) for $i \in [1, N_1]$ and $j \in [1, N2]$ the discrete equation is written as

$$-a_i u_{i-1,j} - b_i u_{i+1,j} - c_j u_{i,j-1} - d_j u_{i,j+1} + s_{i,j} u_{i,j} = f_{ij} \qquad (3.9)$$

The system is closed with boundary conditions

$$u_{0,j} = u_{N_1+1,j} = 0, \quad j \in \overline{1, N_2}$$

$$\text{and} \quad u_{i,0} = u_{i,N_2+1} = 0, \quad i \in \overline{1, N_1} \qquad (3.10)$$

Matrix form of the discrete equation

We arrange the discrete unknowns $(u_{i,j}),\ i = 1, ..., N_1,\ j = 1, ..., N_2$ in the following form

$$u = (u_{1,1}, u_{1,2}, ..., u_{1,N_2};\ u_{2,1}, u_{2,2}, ..., u_{2,N_2}; ...; u_{N_1,1}, u_{N_1,2}, ..., u_{N_1,N_2})^T$$

and

$$f = (f_{1,1}, f_{1,2}, ..., f_{1,N_2};\ f_{2,1}, f_{2,2}, ..., f_{2,N_2}; ...; f_{N_1,1}, f_{N_1,2}, ..., f_{N_1,N_2})^T$$

Then the discrete equation is written in the matrix form $Au = f$ where

$$A = \begin{pmatrix} A_1 & D_2 & 0 & \dots & 0 & 0 \\ C_1 & A_2 & D & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{N_1-1} & D_{N_1-1} \\ 0 & 0 & 0 & \dots & C_{N_1} & A_{N_1} \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} s_{i,1} & -b_1 & 0 & \dots & 0 & 0 \\ -a_2 & s_{i,2} & -b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & s_{i,N_2-1} & -b_{N_2-1} \\ 0 & 0 & 0 & \dots & -a_{N_2} & s_{i,N_2} \end{pmatrix}$$

$$C_i = \begin{pmatrix} -c_i & 0 & \dots & 0 \\ 0 & -c_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -c_i \end{pmatrix}, \qquad D_i = \begin{pmatrix} -d_i & 0 & \dots & 0 \\ 0 & -d_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -d_i \end{pmatrix}$$

The scheme is set as a system of $N_1 \times N_2$ equations with $N_1 \times N_2$ unknowns $(u_{i,j})_{i\in[1,N_1],j\in[1,N_2]}$.
In $\mathbb{R}^{N_1\times N_2}$, existence and uniqueness are equivalent for square systems.
Uniqueness of the solution: We prove that if $f_{ij} = 0$ for $i \in [1, N_1]$ and $j \in [1, N_2]$ then $u_{i,j} = 0$ for $i \in [1, N_1]$ and $j \in [1, N_2]$.
Multiplying two sides of (3.7) by $u_{i,j}$ and taking sum over $i \in [1, N_1]$ and $j \in [1, N_2]$

$$\sum_{i,j=1}^{N_1,N_2}\left[-\frac{(u_{i+1,j}-u_{i,j})k_j}{h_{i+\frac{1}{2}}}+\frac{(u_{i,j}-u_{i-1,j})k_j}{h_{i-\frac{1}{2}}}\right]$$

$$+\sum_{i,j=1}^{N_1,N_2}\left[-\frac{(u_{i,j+1}-u_{i,j})h_i}{k_{j+\frac{1}{2}}}+\frac{(u_{i,j}-u_{i,j-1})h_i}{k_{j-\frac{1}{2}}}\right]u_{i,j}=0$$

Or

$$\sum_{i,j=1}^{N_1,N_2}\left[-\frac{(u_{i+1,j}-u_{i,j})u_{i,j}k_j}{h_{i+\frac{1}{2}}}+\frac{(u_{i,j}-u_{i-1,j})u_{i,j}k_j}{h_{i-\frac{1}{2}}}\right]$$
$$\sum_{i,j=1}^{N_1,N_2}\left[-\frac{(u_{i,j+1}-u_{i,j})u_{i,j}h_i}{k_{j+\frac{1}{2}}}+\frac{(u_{i,j}-u_{i,j-1})u_{i,j}h_i}{k_{j-\frac{1}{2}}}\right]=0$$

Changing index and using boundary condition, we get

$$\sum_{i=0,j=1}^{N_1,N_2}\frac{(u_{i+1,j}-u_{i,j})^2k_j}{h_{i+\frac{1}{2}}}+\sum_{i=1,j=0}^{N_1,N_2}\frac{(u_{i,j+1}-u_{i,j})^2h_i}{k_{j+\frac{1}{2}}}=0$$

From this equality and combining with boundary condition, we get $u_{i,j}=0$ for all $i\in[1,N_1]$ and $j\in[1,N_2]$

Error estimate

Let $\Omega = (0,1) \times (0,1)$ and $f \in L^2(\Omega)$. Let u be the unique solution of (2.1). We assume that there exist $\zeta > 0$ such that $h_i \geq \zeta h$ for $i \in [1, N_1]$ and $k_j \geq \zeta k$ for $j \in [1, N_2]$. Let $(u_{i,j})_{i\in[1,N_1],j\in[1,N_2]}$ be the unique discrete solution of (3.7). There exists $C > 0$ only depending on u, Ω and ζ such that

$$\sum_{i=0,j=1}^{N_1,N_2} \frac{(e_{i+1,j} - e_{i,j})^2}{h_{i+1/2}} k_j + \sum_{i=1,j=0}^{N_1,N_2} \frac{(e_{i,j+1} - e_{i,j})^2}{k_{j+1/2}} h_i \leq Ch^2, \quad (3.11)$$

and

$$\sum_{i,j=1}^{N_1,N_2} (e_{i,j})^2 h_i k_j \leq Ch^2, \quad (3.12)$$

where $e_{i,j} = u(x_i, y_j) - u_{i,j}$ for all $i \in [0, N_1]$ and $j \in [0, N_2]$

Relation (3.11) can be seen as an estimate of a discrete H_0^1 norm of the error, while relation (3.12) gives an estimate of the L^2 norm of the error.
Consider first the case $u \in \mathbb{C}^2(\overline{\Omega})$. We can prove as in 1D case.
Consistency
If $u \in \mathbb{C}^2([0,1] \times [0,1], \mathbb{R})$, there exists $C \in \mathbb{R}_+$ only depending on u such that

$$|R_{i+1/2,j}| = |\frac{1}{k_j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+1/2}, y) dy - \frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h_{i+\frac{1}{2}}}| \leq Ch$$

Proof

Using Taylor series expansion, there exist $\eta_{i+1/2} \in (x_{i+1/2}, x_{i+1})$ and $\theta_{i+1/2} \in (x_i, x_{i+1/2})$ such that

$$\frac{u(x_{i+1}, y_j) - u(x_{i+1/2}, y_j)}{h_{i+\frac{1}{2}}} - \frac{x_{i+1} - x_{i+1/2}}{h_{i+\frac{1}{2}}} u_x(x_{i+1/2}, y_j)$$
$$= \frac{1}{2} \frac{(x_{i+1} - x_{i+1/2})^2}{h_{i+\frac{1}{2}}} u_{xx}(\eta_{i+1/2}, y_j)$$

$$\frac{u(x_{i+1/2}, y_j) - u(x_i, y_j)}{h_{i+\frac{1}{2}}} - \frac{x_{i+1/2} - x_i}{h_{i+\frac{1}{2}}} u_x(x_{i+1/2}, y_j)$$
$$= -\frac{1}{2} \frac{(x_{i+1/2} - x_i)^2}{h_{i+\frac{1}{2}}} u_{xx}(\theta_{i+1/2}, y_j)$$

Taking sum of the two expressions gives

$$R^*_{i+\frac{1}{2},j} = \frac{u(x_{i+1},y_j) - u(x_i,y_j)}{h_{i+\frac{1}{2}}} - u_x(x_{i+1/2},y_j)$$
$$= -\frac{(x_{i+1} - x_{i+\frac{1}{2}})^2}{2(h_{i+\frac{1}{2}})} u_{xx}(\eta_{i+\frac{1}{2}},y_j) + \frac{(x_{i+\frac{1}{2}} - x_i)^2}{2(h_{i+\frac{1}{2}})} u_{xx}(\theta_{i+\frac{1}{2}},y_j)$$

The following inequality holds

$$|R^*_{i+\frac{1}{2},j}| \leq \frac{(x_{i+1} - x_{i+\frac{1}{2}})^2}{2(h_{i+\frac{1}{2}})} |u_{xx}(\eta_{i+\frac{1}{2}},y_j)| + \frac{(x_{i+\frac{1}{2}} - x_i)^2}{2(h_{i+\frac{1}{2}})} |u_{xx}(\theta_{i+\frac{1}{2}},y_j)|$$
$$\leq C\left(\frac{(x_{i+1} - x_{i+\frac{1}{2}})^2}{h_{i+\frac{1}{2}}} + \frac{(x_{i+\frac{1}{2}} - x_i)^2}{h_{i+\frac{1}{2}}}\right)$$
$$\leq C\frac{(h_{i+\frac{1}{2}})^2}{h_{i+\frac{1}{2}}}$$
$$\leq Ch$$

According to mean-value theorem, the following equality holds

$$u_x(x_{i+\frac{1}{2}}, y) - u_x(x_{i+\frac{1}{2}}, y_j) = (y - y_j)u_{xy}(x_{i+\frac{1}{2}}, \zeta), \qquad \zeta \in (y, y_j)$$

Then

$$\begin{aligned} |u_x(x_{i+\frac{1}{2}}, y) - u_x(x_{i+\frac{1}{2}}, y_j)| &= |y - y_j||u_{xy}(x_{i+\frac{1}{2}}, \zeta)| \\ &\leq k_j C \end{aligned}$$

Incorporating with the equality on $|R^*_{i+\frac{1}{2},j}|$ it implies that

$$\begin{aligned} |R_{i+\frac{1}{2},j}| &\leq |\frac{1}{k_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+1/2}, y)dy - u_x(x_{i+\frac{1}{2}}, y_j)| + |R^*_{i+\frac{1}{2},j}| \\ &\leq \frac{1}{k_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} |u_x(x_{i+1/2}, y) - u_x(x_{i+\frac{1}{2}}, y_j)|dy + +|R^*_{i+\frac{1}{2},j}| \\ &\leq k_j C + Ch \\ &\leq 2Ch \end{aligned}$$

Prove the error estimate

1.

$$\sum_{i=0,j=1}^{N_1,N_2} \frac{(e_{i+1,j}-e_{i,j})^2 k_j}{h_{i+1/2}} + \sum_{i=1,j=0}^{N_1,N_2} \frac{(e_{i,j+1}-e_{i,j})^2 h_i}{k_{j+1/2}} \le Ch^2$$

Integrating equation $-\Delta u = f$ over $K_{i,j}$ yields

$$\begin{aligned}
&-\frac{1}{|K_{i,j}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}},y)dy + \frac{1}{|K_{i,j}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}},y)dy \\
&-\frac{1}{|K_{i,j}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j+\frac{1}{2}})dx + \frac{1}{|K_{i,j}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j-\frac{1}{2}})dy \\
&= f_{i,j} \qquad (3.13)
\end{aligned}$$

The approximate solution U satisfies

$$-\frac{u_{i+1,j}-u_{i,j}}{h_i h_{i+\frac{1}{2}}} + \frac{u_{i,j}-u_{i-1,j}}{h_i h_{i-\frac{1}{2}}} - \frac{u_{i,j+1}-u_{i,j}}{k_j k_{j+\frac{1}{2}}} + \frac{u_{i,j}-u_{i,j-1}}{k_j k_{j-\frac{1}{2}}} = f_{ij} \qquad (3.14)$$

Therefore,

$$
\begin{aligned}
&-\frac{1}{|K_{i,j}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}},y)dy + \frac{u_{i+1,j}-u_{i,j}}{h_i h_{i+\frac{1}{2}}}\\
&+\frac{1}{|K_{i,j}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}},y)dy - \frac{u_{i,j}-u_{i-1,j}}{h_i h_{i-\frac{1}{2}}}\\
&-\frac{1}{|K_{i,j}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j+\frac{1}{2}})dx + \frac{u_{i,j+1}-u_{i,j}}{k_j k_{j+\frac{1}{2}}}\\
&+\frac{1}{|K_{i,j}|}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x,y_{j-\frac{1}{2}})dy - \frac{u_{i,j}-u_{i,j-1}}{k_j k_{j-\frac{1}{2}}} = 0
\end{aligned}
$$

Setting

$$-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i+\frac{1}{2}}, y)dy + \frac{(u_{i+1,j} - u_{i,j})k_j}{h_{i+\frac{1}{2}}} = -k_j R_{i+1/2,j} - \frac{(e_{i+1,j} - e_{i,j})k_j}{h_{i+\frac{1}{2}}}$$

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_x(x_{i-\frac{1}{2}}, y)dy - \frac{(u_{i,j} - u_{i-1,j})k_j}{h_{i-\frac{1}{2}}} = k_j R_{i-1/2,j} + \frac{(e_{i,j} - e_{i-1,j})k_j}{h_{i-\frac{1}{2}}}$$

$$-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x_i, y_{j+\frac{1}{2}})dy + \frac{(u_{i,j+1} - u_{i,j})h_i}{k_{j+\frac{1}{2}}} = -h_i R_{i,j+1/2} - \frac{(e_{i,j+1} - e_{i,j})h_i}{k_{j+\frac{1}{2}}}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_y(x_i, y_{j-\frac{1}{2}})dy - \frac{(u_{i,j} - u_{i,j-1})h_i}{k_{j-\frac{1}{2}}} = h_i R_{i,j-1/2} + \frac{(e_{i,j} - e_{i,j-1})h_i}{k_{j-\frac{1}{2}}}$$

Then

$$-\frac{(e_{i+1,j} - e_{i,j})k_j}{h_{i+\frac{1}{2}}} + \frac{(e_{i,j} - e_{i-1,j})k_j}{h_{i-\frac{1}{2}}} - \frac{(e_{i,j+1} - e_{i,j})h_i}{k_{j+\frac{1}{2}}} + \frac{(e_{i,j} - e_{i,j-1})h_i}{k_{j-\frac{1}{2}}}$$

$$= k_j R_{i+1/2,j} - k_j R_{i-1/2,j} + h_i R_{i,j+1/2} - h_i R_{i,j+1/2}$$

Multiplying by $e_{i,j}$ and summing over $i = 1, ..., N_1$; $j = 1, ..., N_2$ gives

$$
\begin{aligned}
&-\sum_{i=1,j=1}^{N_1,N_2} \frac{(e_{i+1,j} - e_{i,j})e_{i,j}k_j}{h_{i+\frac{1}{2}}} + \sum_{i=1,j=1}^{N_1,N_2} \frac{(e_{i,j} - e_{i-1,j})e_{i,j}k_j}{h_{i-\frac{1}{2}}} \\
&-\sum_{i=1,j=1}^{N_1,N_2} \frac{(e_{i,j+1} - e_{i,j})e_{i,j}h_i}{k_{j+\frac{1}{2}}} + \sum_{i=1,j=1}^{N_1,N_2} \frac{(e_{i,j} - e_{i,j-1})e_{i,j}h_i}{k_{j-\frac{1}{2}}} \\
&= \sum_{i=1,j=1}^{N_1,N_2} R_{i+1/2,j}e_{i,j}k_j - \sum_{i=1,j=1}^{N_1,N_2} R_{i-1/2,j}e_{i,j}k_j + \sum_{i=1,j=1}^{N_1,N_2} R_{i,j+1/2}e_{i,j}h_i \\
&- \sum_{i=1,j=1}^{N_1,N_2} R_{i,j-1/2}e_{i,j}h_i
\end{aligned}
$$

Changing the index gives

$$
\begin{aligned}
&-\sum_{i=1,j=1}^{N_1,N_2}\frac{(e_{i+1,j}-e_{i,j})e_{i,j}k_j}{h_{i+\frac{1}{2}}}+\sum_{i=0,j=1}^{N_1-1,N_2}\frac{(e_{i+1,j}-e_{i,j})e_{i+1,j}k_j}{h_{i+\frac{1}{2}}}\\
&-\sum_{i=1,j=1}^{N_1,N_2}\frac{(e_{i,j+1}-e_{i,j})e_{i,j}h_i}{k_{j+\frac{1}{2}}}+\sum_{i=1,j=0}^{N_1,N_2-1}\frac{(e_{i,j+1}-e_{i,j})e_{i,j+1}h_i}{k_{j+\frac{1}{2}}}\\
&=\sum_{i=1,j=1}^{N_1,N_2}R_{i+1/2,j}e_{i,j}k_j-\sum_{i=0,j=1}^{N_1-1,N_2}R_{i+1/2,j}e_{i+1,j}k_j+\sum_{i=1,j=1}^{N_1,N_2}R_{i,j+1/2}e_{i,j}h_i\\
&-\sum_{i=1,j=0}^{N_1,N_2-1}R_{i,j+1/2}e_{i,j+1}h_i
\end{aligned}
$$

Reordering and using $e_{0,j} = e_{N_1,j} = e_{i,0} = e_{i,N_2} = 0$ yields

$$\sum_{i=0,j=1}^{N_1,N_2} \frac{(e_{i+1,j} - e_{i,j})^2 k_j}{h_{i+\frac{1}{2}}} + \sum_{i=1,j=0}^{N_1,N_2} \frac{(e_{i,j+1} - e_{i,j})^2 h_i}{k_{j+\frac{1}{2}}}$$

$$= \sum_{i=0,j=1}^{N_1,N_2} R_{i+1/2,j}(e_{i,j} - e_{i+1,j})k_j + \sum_{i=1,j=0}^{N_1,N_2} R_{i,j+1/2}(e_{i,j} - e_{i,j+1})h_i$$

Using the consistency property, it implies

$$\sum_{i=0,j=1}^{N_1,N_2} \frac{(e_{i+1,j} - e_{i,j})^2 k_j}{h_{i+\frac{1}{2}}} + \sum_{i=1,j=0}^{N_1,N_2} \frac{(e_{i,j+1} - e_{i,j})^2 h_i}{k_{j+\frac{1}{2}}}$$

$$\leq Ch \sum_{i=0,j=1}^{N_1,N_2} |(e_{i,j} - e_{i+1,j})k_j| + Ch \sum_{i=1,j=0}^{N_1,N_2} |(e_{i,j} - e_{i,j+1})h_i| \tag{3.15}$$

Applying Cauchy-Schwarz inequality

$$\sum_{j=1}^{N_2} k_j \sum_{i=0}^{N_1} |(e_{i,j}-e_{i+1,j})| \leq \sum_{j=1}^{N_2} k_j \left(\sum_{i=0}^{N_1} \frac{(e_{i+1,j}-e_{i,j})^2}{h_{i+\frac{1}{2}}} \right)^{1/2} (\sum_{i=0}^{N_1} h_{i+\frac{1}{2}})^{1/2}$$

e.g.

$$\sum_{i=0,j=1}^{N_1,N_2} |(e_{i,j}-e_{i+1,j})k_j| \leq \sum_{j=1}^{N_2} k_j \left(\sum_{i=0}^{N_1} \frac{(e_{i+1,j}-e_{i,j})^2}{h_{i+\frac{1}{2}}} \right)^{1/2}$$

$$\sum_{i=1}^{N_1} h_i \sum_{j=0}^{N_2} |(e_{i,j}-e_{i,j+1})| \leq \sum_{i=1}^{N_1} h_i \left(\sum_{j=0}^{N_2} \frac{(e_{i,j+1}-e_{i,j})^2}{k_{j+\frac{1}{2}}} \right)^{1/2} (\sum_{j=0}^{N_2} k_{j+\frac{1}{2}})^{1/2}$$

e.g.

$$\sum_{i=1,j=0}^{N_1,N_2} |(e_{i,j}-e_{i,j+1})h_i| \leq \sum_{i=1}^{N_1} h_i \left(\sum_{j=0}^{N_2} \frac{(e_{i,j+1}-e_{i,j})^2}{k_{j+\frac{1}{2}}} \right)^{1/2}$$

By setting

$$P_j = \sum_{i=0}^{N_1} \frac{(e_{i+1,j} - e_{i,j})^2}{h_{i+\frac{1}{2}}}, \quad Q_i = \sum_{j=0}^{N_2} \frac{(e_{i,j+1} - e_{i,j})^2}{k_{j+\frac{1}{2}}}$$

we have

$$\sum_{j=1}^{N_2} k_j P_j + \sum_{i=1}^{N_1} h_i Q_i \leq Ch \left(\sum_{j=1}^{N_2} k_j \sqrt{P_j} + \sum_{i=1}^{N_1} h_i \sqrt{Q_i} \right) \qquad (3.16)$$

Applying Cauchy-Schwarz inequality for the RHS of (3.16) it yields

$$\sum_{j=1}^{N_2} k_j P_j + \sum_{i=1}^{N_1} h_i Q_i \leq Ch \left(\sum_{j=1}^{N_2} k_j + \sum_{i=1}^{N_1} h_i \right)^{1/2} \left(\sum_{j=1}^{N_2} k_j P_j + \sum_{i=1}^{N_1} h_i Q_i \right)^{1/2}$$

It implies that

$$\sum_{j=1}^{N_2} k_j P_j + \sum_{i=1}^{N_1} h_i Q_i \leq 2C^2 h^2$$

2. $\sum_{i,j=1}^{N_1,N_2}(e_{i,j})^2 h_i k_j \leq Ch^2$

We have

$$e_{i,j} = \sum_{i_1=0}^{i-1,j}(e_{i_1+1,j} - e_{i_1-1,j})$$

Then

$$|e_{i,j}| \leq \sum_{i_1=0}^{i-1}|e_{i_1+1,j} - e_{i_1-1,j}|$$

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|e_{i,j}| &\leq \left(\sum_{i_1=0}^{i-1}\frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}}\right)^{1/2}\left(\sum_{i_1=0}^{i-1}h_{i_1+1/2}\right)^{1/2} \\
&\leq \left(\sum_{i_1=0}^{N_1}\frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}}\right)^{1/2}
\end{aligned}$$

It is similar, we have

$$|e_{i,j}| \leq \left(\sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{k_{j_1+1/2}} h_i \right)^{1/2}$$

Multiplying two previous inequality, there holds

$$|e_{i,j}|^2 \leq \left(\sum_{i_1=0}^{N_1} \frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}} \right)^{1/2} \left(\sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{h_{j_1+1/2}} \right)^{1/2}$$

Applying Cauchy-Schwarz inequality, we have

$$|e_{i,j}|^2 \leq \frac{1}{2} \sum_{i_1=0}^{N_1} \frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}} + \frac{1}{2} \sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{h_{j_1+1/2}}$$

Or

$$|e_{i,j}|^2 h_i k_j \le \frac{h_i}{2} \sum_{i_1=0}^{N_1} \frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}} k_j + \frac{k_j}{2} \sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{h_{j_1+1/2}} h_i$$

Summing over i and j, we have

$$\begin{aligned} \sum_{i,j=1}^{N_1,N_2} |e_{i,j}|^2 h_i k_j \le{}& \frac{1}{2} \sum_{i=1}^{N_1} h_i \sum_{j=1}^{N_2} \sum_{i_1=0}^{N_1} \frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}} k_j \\ &+ \frac{1}{2} \sum_{j=1}^{N_2} k_j \sum_{i=1}^{N_1} \sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{h_{j_1+1/2}} h_i \end{aligned}$$

Putting

$$P = \sum_{j=1}^{N_2} \sum_{i_1=0}^{N_1} \frac{|e_{i_1+1,j} - e_{i_1,j}|^2}{h_{i_1+1/2}} k_j \qquad Q = \sum_{i=1}^{N_1} \sum_{j_1=0}^{N_2} \frac{|e_{i,j_1+1} - e_{i,j_1}|^2}{h_{j_1+1/2}} h_i$$

Then

$$\sum_{i,j=1}^{N_1,N_2} |e_{i,j}|^2 h_i k_j \leq \frac{1}{2}\sum_{i=1}^{N_1} h_i P + \frac{1}{2}\sum_{j=1}^{N_2} k_j Q$$

Since

$$\sum_{i=1}^{N_1} h_i = 1 \qquad \sum_{j=1}^{N_2} k_j = 1$$

Thus

$$\sum_{i,j=1}^{N_1,N_2} |e_{i,j}|^2 h_i k_j \leq \frac{P+Q}{2} \leq C^2 h^2$$

Let $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$. We will use the finite volume method to discretize the following Poisson equation

$$-\Delta u = f(x, y) \quad \text{in } \Omega \tag{4.1}$$

subject to a Dirichlet boundary condition:

$$u(x, y) = u_d(x, y) \quad \text{on } \Gamma = \partial\Omega. \tag{4.2}$$

Let Ω be a polygonal domain covered by the elements $(T_i)_{i\in[1,I]}$ of a mesh.

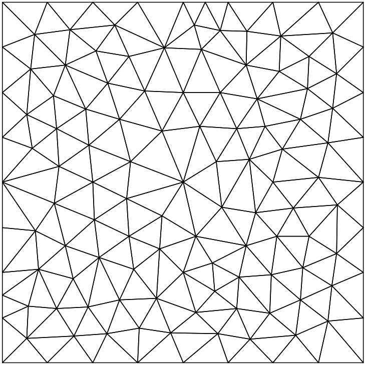

Figure: Mesh T_i cover Ω

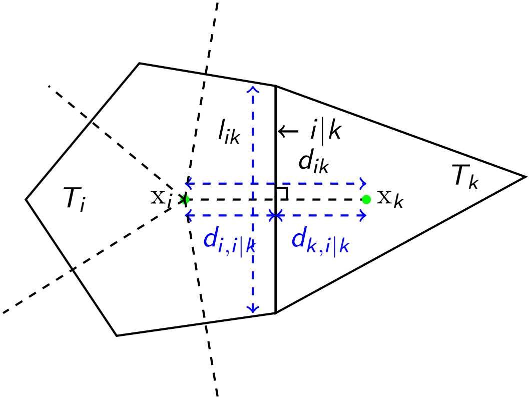

Figure: Two neighboring cells of admissible mesh.

With each T_i, we associate a point $\mathrm{x}_i \in \overset{\circ}{T}_i$. We will denote by $i|k$ the common edge of T_i and T_k when these two elements are neighbors.

The mesh is said to be admissible mesh if $[\mathrm{x}_i \mathrm{x}_k]$ is orthogonal to $i|k$ for any couple (T_i, T_k) of neighboring elements, and if, for any element T_i which has an edge on Γ, the orthogonal projection of the associated point x_i on the straight line going over the considered edge belongs to this edge.

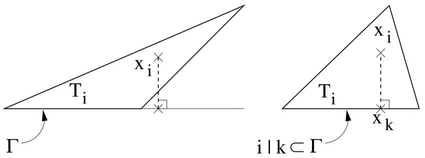

Figure: Non admissibility (left) and admissibility (right) at the boundary

In that case, the orthogonal projections on the edge is still denote by x_k with $k \in [I+1, I+I^\Gamma]$, where I^Γ denotes the number of boundary edges of the mesh, and edge is still denoted by $i|k$.

For $i \in [1, I]$, we denote by $V(i) \subset [1, I + I^{\Gamma}]$ the set of the neighboring indexes of the element T_i, by $|T_i|$ the area of T_i. Let h_i denote the diameter of T_i and ρ_i denote the diameter of the largest ball inscribed in T_i.

We make the following shape regularity assumption on the mesh.
Assumption (shape-regularity of the meshes). There exist the a constant positive θ independent of mesh such that $\min_{i \in [1,I]} h_i/\rho_i \leq \theta$.

We next denote by E the set of all edges in the mesh, by E^{int} the set of interior, by E^{ext} the set of exterior. Let l_{ik} be the length of $i|k$ (note that $l_{ik} = l_{ki}$), and let n_{ik} be the unit vector orthogonal to $i|k$ pointing from T_i to T_k (note that $n_{ik} = -n_{ki}$). We will denote by $d_{i,i|k} = d(\mathrm{x}_i, i|k)$ (resp $d_{k,i|k}$) is the distance between point x_i (resp x_k) and edge $i|k$, and $d_{ik} = d_{ki} = \|\overrightarrow{\mathrm{x}_i \mathrm{x}_k}\|$.

We associate with any finite volume T_i of the mesh an unknown denoted by u_i, which will approximate the value $u(\mathrm{x}_i)$, and we integrate the first equation in (4.1) over T_i, there holds

$$\frac{-1}{|T_i|}\int_{T_i}\Delta u(\mathrm{x})d\mathrm{x} = \frac{1}{|T_i|}\int_{T_i} f(\mathrm{x})d\mathrm{x}. \tag{4.3}$$

The left-hand side of (4.3) may be evaluated thanks to Green formula

$$\frac{-1}{|T_i|}\int_{T_i}\Delta u(\mathrm{x})d\mathrm{x} = \frac{-1}{|T_i|}\sum_{k\in V(i)}\int_{i|k}\nabla u(\sigma)\cdot n_{ik}d\sigma \tag{4.4}$$

When the mesh is admissible, a reasional approximation of $\nabla u \cdot n_{ik}$ on the edge $i|k$ is given by:

$$\nabla u\cdot n_{ik} \approx \frac{u(\mathrm{x}_k)-u(\mathrm{x}_i)}{d_{ik}}. \tag{4.5}$$

Thus as soon as $k \in [1, I]$ (i.e if $i|k$ is interior edge), we can approach

$$\int_{i|k} \nabla u(\sigma) \cdot n_{ik} d\sigma = \frac{l_{ik}}{d_{ik}}(u_k - u_i) \quad \forall k \in [1, I]. \tag{4.6}$$

If $k \in [I+1, I+I^{\Gamma}]$ (i.e $i|k$ is exterior edge), the formula (4.5) is still a good approximation of the gradient in the normal direction. However, the value $u(\mathrm{x}_k)$ is known since $k \in \Gamma$. We may approach

$$\int_{i|k} \nabla u(\sigma) \cdot n_{ik} d\sigma = \frac{l_{ik}}{d_{ik}}(u_d(\mathrm{x}_k) - u_i) \quad \forall k \in \Gamma. \tag{4.7}$$

Finally, the i equation of the scheme thus writes

$$\frac{-1}{|T_i|}\sum_{i|k\notin\Gamma}\frac{l_{ik}}{d_{ik}}(u_k-u_i)-\frac{1}{|T_i|}\sum_{i|k\in\Gamma}\frac{l_{ik}}{d_{ik}}(u_d(\mathrm{x}_k)-u_i)=f_i, \qquad (4.8)$$

where f_i is the mean-value of the function f over T_i it mean that

$$f_i=\frac{1}{T_i}\int_{T_i}f(\mathrm{x})d\mathrm{x}.$$

Divergence and Gradient operators

Let E be the all edges in the mesh and N_E be the number in the set E (N_b be number of boundary edges). We recall that I is the number of elements in the mesh. We define the following discrete divergence operator

$$\begin{aligned} d : \mathbb{R}^{N_E} &\rightarrow \mathbb{R}^I \\ (v_{ik})_{ik\in E} &\mapsto (dv)_i := \frac{1}{|T_i|}\sum_{k\in V(i)} l_{ik} v_{ik} \end{aligned} \tag{4.9}$$

We also introduce the discrete gradient operator

$$\begin{aligned} g : \mathbb{R}^{I+I^\Gamma} &\rightarrow \mathbb{R}^{N_E} \\ (u_i)_{i\in I} &\mapsto (gu)_{ik} := \frac{u_k - u_i}{d_{ik}} \end{aligned} \tag{4.10}$$

From two operator and our scheme (4.8), we get

$$-d(gu)_i = f_i \ \forall \ i \in [1, I] \text{ and } u_k = u_d(x_k) \ \ \forall x_k \in \Gamma \tag{4.11}$$

Product scalars

We also define the discrete product scalar $(.,.)_T$ by

$$(u_i)_{i\in[1,I]}, (w_i)_{i\in[1,I]} \mapsto (u,w)_T = \sum_{i=1}^{I} T_i u_i w_i \tag{4.12}$$

and $\|u\|^2_{0,T} = (u,u)_T$

With each edge, we define discrete product scalar $(.,.)_D$

$$(a_{ik})_{ik\in E}, (b_{ik})_{ik\in E} \mapsto (a,b)_D = \sum_{ik\in E} \frac{l_{ik} d_{ik}}{2} a_{ik} b_{ik} \tag{4.13}$$

and $\|v\|^2_{0,D} = (v,v)_D$, $|u|^2_{1,D} = \|g(u)\|^2_{0,D} = (g(u), g(u))$

Finaly, we define boundary discrete product scalar on Γ

$$(a_{ik})_{ik\in \Gamma}, (b_{ik})_{ik\in \Gamma} \mapsto (a,b)_\Gamma = \sum_{ik\in \Gamma} l_{ik} a_{ik} b_{ik} \tag{4.14}$$

Proposition

Let $(u_i)_{i\in[I+I^\Gamma]}$ *and* $(v_{ik})_{ik\in E}$ *be given. There holds*

$$(dv,u)_T = -2(v,gu)_D + (v,\gamma u)_\Gamma \tag{4.15}$$

where the discrete trace operator γ *is defined following way:*

$$\begin{aligned} \gamma : \mathbb{R}^{I+I^\Gamma} &\to \mathbb{R}^{I^\Gamma} \\ (u_i)_{i\in I+I^\Gamma} &\mapsto (\gamma u)_{ik} = u_k \text{ when } x_k \in \Gamma \end{aligned} \tag{4.16}$$

The discrete Green formula is discrete equivalent of

$$(\nabla\cdot v,u)_{L^2(\Omega)} = -(v,\nabla u)_{L^2(\Omega)} + (v\cdot n,u)_{L^2(\Gamma)}$$

Discrete variational formulation

Consider any $(w_i)_{i\in[1,I+i^\Gamma]}$ with $w_k = 0$ for all $k \in [I+1, I+I^\Gamma]$. There holds

$$2(g(u), g(w))_D = (f, w)_T \tag{4.17}$$

Proof: Let us start from (4.11), multiple by $|T_i|w_i$ and sum over $i \in [1, I]$, we obtain

$$-(dgu, w)_T = (f, w)_T \tag{4.18}$$

Combining this with the discrete Green formula (4.15), we have

$$2(gu, gw)_D - (gu, \gamma w)_\Gamma = (f, w)_T \tag{4.19}$$

Now, on Γ, γw is vanish. Therefore (4.19) implies (4.17).

Existence and uniqueness of discrete solution

The finite volume scheme may be written as system of $I + I^{\Gamma}$ and $I + I^{\Gamma}$ unknowns given by (4.11)/ Therefore, the existence and uniqueness are equivalent. If $f_i = 0$ for all $i \in [1, I]$ and $u_k = 0$ for all $k \in [I + 1, I + I^{\Gamma}]$, then we can choose $w = u$, there holds, thanks to (4.17),

$$(g(u), g(u))_D = 0 = \sum_{ik \in E} \frac{l_{ik} d_{ik}}{2} (gu)_{ik}^2$$

This implies $(gu)_{ik} = 0$ for all $ik \in E$. By the definition of $(gu)_{ik}$, $u_i = u_k$ for all $i|k$ (including the boundary edges). Combining with boundary condition, $u_i = 0$ for all $i \in [1, I + I^{\Gamma}]$

Discrete maximum principle

Proposition

We suppose that f is possitive on Ω and that $u_k = 0$ for all $k \in [I+1, I+I^\Gamma]$. There holds

$$u_i \geq 0 \qquad \forall i \in [1, I] \tag{4.20}$$

We shall give error estimate for both in discrete $H_0^1(\Omega)$ norm and discrete $L^2(\Omega)$ norm. We shall define mesh size $h = \max_{i\in[1,I]} diam(T_i)$. We suppose that u, exact solution of (4.1) with $u_d = 0$, belong to $C^2(\bar{\Omega})$. We condition the pointwise projection of u as follows

$$(\Pi u)_k = u(x_k) \quad \text{for all } k \in [1, I + I^{\Gamma}]$$

and this implies that

$$(\Pi u)_k = 0 \text{ for all } k \in [I+1, I+I^{\Gamma}] \tag{4.21}$$

We shall consider the diffence between the projection $((\Pi u)_i)_{i\in[1,I+I^{\Gamma}]}$ the value $(u_i)_{i\in[1,I+I^{\Gamma}]}$ obtained by from the finite volune scheme in (4.11) with boundary condition

$$u_k = 0 \quad \text{for all } k \in [I+1, I+I^{\Gamma}] \tag{4.22}$$

Energy norm

We would like to estimate the norm of the discrete gradient of error

$$|u - \Pi u|_{1,D}^2 = (g(u) - g(\Pi u), g(u) - g(\Pi u))_D \tag{4.23}$$

For first, we will define averaged normal gradient on each edge $i|k$

$$(\delta u)_{ik} = \frac{1}{l_{ik}} \int_{i|k} \nabla u \cdot n_{ik}(\sigma) d\sigma \tag{4.24}$$

Note that this implies that $(\delta u)_{ik} = -(\delta u)_{ki}$

Lemma

For any $(w_i)_{i\in[1,I+I^\Gamma]}$ *with* $w_k = 0$ *for all* $k \in [I+1, I+I^\Gamma]$, *there holds*

$$(f, w)_T = 2(\delta u, g(w))_D \tag{4.25}$$

Thanks to (4.17) and (4.25), we have

$$(\delta u, g(w))_D = (g(u), g(w))_D \tag{4.26}$$

for all vanishing on the boundary. Setting $w = u - \Pi u$. Thanks to (4.26), we have

$$\begin{aligned}
|w|_{1,D}^2 = |u - \Pi u|_{1,D}^2 &= (g(u - \Pi u), g(w))_D \\
&= (g(u), g(w))_D - (g(\Pi u), g(w))_D \\
&= (\delta u, g(w))_D - (g(\Pi u), g(w))_D \\
&= (\delta u - g(\Pi u), g(w))_D \le \|\delta u - g(\Pi u)\|_{0,D} |w|_{1,D}
\end{aligned}$$

Thus, we have

$$|u - \Pi u|_{1,D} \leq |\delta u - g(\Pi u)|_{0,D} \tag{4.27}$$

By definition of $|.|_{0,D}$, we have

$$\|\delta u - g(\Pi u)\|_{0,D}^2 = \sum_{ik \in E} \frac{l_{ik} d_{ik}}{2} ((\delta u)_{ik} - (g(\Pi u))_{ik})^2$$

Setting

$$\begin{aligned} e_{ik} &= (\delta u)_{ik} - (g(\Pi u))_{ik} = \frac{1}{l_{ik}} \int_{l_{ik}} \nabla u \cdot n_{ik}(\sigma) d\sigma - \frac{u_k - u_i}{d_{ik}} \\ &= \frac{1}{l_{ik}} \int_{l_{ik}} \nabla u \cdot n_{ik}(\sigma) d\sigma - \frac{1}{d_{ik}} \int_{[x_k, x_i]} \nabla u \cdot n_{ik}(\sigma) d\sigma \end{aligned}$$

Let x_{ik} be intersection between l_{ik} and $[x_k, x_i]$, then

$$e_{ik} = \frac{1}{l_{ik}} \int_{l_{ik}} (\nabla u(\sigma) - \nabla u(\mathrm{x}_{ik})) \cdot n_{ik}(\sigma) d\sigma$$
$$- \frac{1}{d_{ik}} \int_{[x_k, x_i]} (\nabla u(\sigma) - \nabla u(\mathrm{x}_{ik})) \cdot n_{ik}(\sigma) d\sigma$$

There exists constant $C > 0$(depending on u) such that

$$\left| \frac{1}{l_{ik}} \int_{l_{ik}} (\nabla u(\sigma) - \nabla u(\mathrm{x}_{ik})) \cdot n_{ik}(\sigma) d\sigma \right| \leq C l_{ik} \leq Ch$$
$$\left| \frac{1}{l_{ik}} \int_{l_{ik}} (\nabla u(\sigma) - \nabla u(\mathrm{x}_{ik})) \cdot n_{ik}(\sigma) d\sigma \right| \leq C d_{ik} \leq 2Ch$$

Thus, this implies that

$$|e_{ik}| \leq 3Ch \tag{4.28}$$

Using the estimate of e_{ik}, there holds

$$\|\delta u - g(\Pi u)\|_{0,D}^2 = \sum_{ik \in E} \frac{l_{ik} d_{ik}}{2} (3Ch)^2 \tag{4.29}$$

$$\leq (3Ch)^2 \sum_{ik \in E} \frac{l_{ik} d_{ik}}{2} = (3Ch)^2 |\Omega| \tag{4.30}$$

Then,

$$\|\delta u - g(\Pi u)\|_{0,D} \leq 3C|\Omega|^{1/2} h \tag{4.31}$$

This implies

$$|u - \Pi u|_{1,D} \leq 3C|\Omega|^{1/2} h$$

L^2 norm

We will prove the discrete Poincare inequality: Let $w_{i\in[1,I+I^\Gamma]}$ such that $w_k = 0$ for all $k \in \Gamma$. Then there exists $C = |\Omega|^{1/2}$ such that

$$\|w\|_{0,T} \leq C|w|_{1,D} \tag{4.32}$$

Let us define the following function

$$\begin{aligned}\omega :&\Omega \to \mathbb{R}\\ &x \mapsto \omega(x) = w_i \text{ if } x \in T_i\end{aligned}$$

Let $x \in \Omega$ be given, we define by D_x^1 and D_x^2 the two straight line going through with direction $(1,0)$ and (0,1), respectively.

L^2 norm

For a given $i|k$ in the set of edges E and a given $x \in \Omega$,we also define the function χ_{ik}^j with $j = 1$ and $j = 2$ by

$$\chi_{ik}^j : \Omega \to \mathbb{R}$$

$$x \mapsto \chi_{ik}^j(x) = \begin{cases} 1 & \text{if} \quad i|k \cap D_x^j \neq \emptyset \\ 0 & \text{if} \quad i|k \cap D_x^j = \emptyset \end{cases}$$

Then, for a given T_i and for all $x \in T_i$, there holds

$$\omega(x) = w_i = (w_i - w_{k_1}) + (w_{k_1} - w_{k_2}) + \cdots + (w_{k_{q-1}} - w_{k_q}) + (w_{k_q} - w_k)$$

where the index k is such that $x_k \in \Gamma$ and belongs to an edge of the mesh which intersects D_x^1, so that $w_k = 0$. Since

$$w_{k_l} - w_{k_{l+1}} = d_{k_l k_{l+1}} (gw)_{k_l k_{l+1}}$$

L^2 norm

there holds

$$\omega(x) \leq \sum_{k_1|k_2 \in E} d_{k_1 k_2} (gw)_{k_1 k_2} \chi^1_{k_1 k_2}(x)$$

Performing the same calculation with $j = 2$ instead of $j = 1$, multiplying the two inequalities, there holds

$$\omega^2(x) = \left(\sum_{k_1|k_2 \in E} d_{k_1 k_2} (gw)_{k_1 k_2} \chi^1_{k_1 k_2}(x) \right) \left(\sum_{k_1|k_2 \in E} d_{k_1 k_2} (gw)_{k_1 k_2} \chi^2_{k_1 k_2}(x) \right)$$

L^2 norm

Intergrating over Ω and taking into acount that ω is a constant over each T_i, there holds

$$\sum_{i\in[1,I]} |T_i| w_i^2 \leq \int_\Omega \left(\sum_{k_1|k_2\in E} d_{k_1k_2} (gw)_{k_1k_2} \chi^1_{k_1k_2}(x) \right) \left(\sum_{k_1|k_2\in E} d_{k_1k_2} (gw)_{k_1k_2} \chi^2_{k_1k_2}(x) \right) dx_1 dx_2$$

Now, it is easily seen that $\chi^1_{k_1k_2}(x)$ only depends on x_2 and $\chi^2_{k_1k_2}(x)$ only depends on x_1, so that, setting

$$a := \min\{x_1 \text{ s.t. } (x_1, x_2) \in \Omega\}, \qquad b := \max\{x_1 \text{ s.t. } (x_1, x_2) \in \Omega\}$$
$$\alpha := \min\{x_2 \text{ s.t. } (x_1, x_2) \in \Omega\}, \qquad \beta := \max\{x_2 \text{ s.t. } (x_1, x_2) \in \Omega\}$$

L^2 norm

we get

$$\sum_{i\in[1,I]} |T_i| w_i^2 \leq \int_\alpha^\beta \left(\sum_{k_1|k_2\in E} d_{k_1 k_2} (gw)_{k_1 k_2} \chi^1_{k_1 k_2}(x_2) \right) dx_2$$

$$\int_a^b \left(\sum_{k_1|k_2\in E} d_{k_1 k_2} (gw)_{k_1 k_2} \chi^2_{k_1 k_2}(x_1) \right) dx_1$$

It is easily seen that

$$\int_\alpha^\beta \chi^1_{k_1 k_2}(x_2) dx_2 \leq l_{k_1 k_2} \text{ and } \int_a^b \chi^2_{k_1 k_2}(x_1) dx_1 \leq l_{k_1 k_2}$$

L^2 norm

So that we finally get

$$\sum_{i\in[1,I]} |T_i| w_i^2 \leq \left(\sum_{k_1 k_2 \in E} d_{k_1 k_2} l_{k_1 k_2} (gw)_{k_1 k_2} \right)^2$$

Applying the discrete Cauchy-Schwarz inequality, there holds

$$\sum_{i\in[1,I]} |T_i| w_i^2 \leq 4 \left(\sum_{k_1 k_2 \in E} \frac{d_{k_1 k_2} l_{k_1 k_2}}{2} \right) \left(\sum_{k_1 k_2 \in E} \frac{d_{k_1 k_2} l_{k_1 k_2}}{2} (gw)^2_{k_1 k_2} \right)$$

which is exact the discrete Poincare inequality (4.32).