# * OPTIMAL CONTROL

## I. Optimal Control of ODE

$$(CP)\quad \boxed{\begin{cases} x'(t) = f(X(t), u(t)), & \text{a.e. } t \in [0,T], \\ X(0) = X_0 \end{cases}}$$

+ $X : [0,T] \to \mathbb{R}^n$ is the solution of the ODE.
+ $u : [0,T] \to U$ is control, measurable function.

$$f : \mathbb{R}^n \times \underset{\mathbb{R}^k}{\underset{\cap}{U}} \longrightarrow \mathbb{R}^n$$

+ Meaning of the solution:

$$\boxed{X(\cdot) \text{ solution of } (CP) \text{ iff } \left| \begin{array}{l} X(\cdot) \in W^{1,1}([0,T], \mathbb{R}^n) = \text{absolutely continuous} \\ X(t) = X_0 + \int_0^t f(X(s), u(s))\, ds. \end{array}\right.}$$

$(H_f)$ $\begin{cases} f \text{ continuous, } (\overset{t}{loc}) \text{ Lipschitz in } X, \text{ uniformly} / U \to \exists \text{ solution locally} \\ |f(X,u)| \le a(1 + |X|), \ \forall X \in \mathbb{R}^n, \ \forall u \in U \to \text{solution globally} \end{cases}$

$$(CP)\quad \boxed{\begin{cases} X'(t) = f(X(t), u(t)) & \text{a.e. } t \in [t_0, T], \\ X(t_0) = x_0 \end{cases}}$$

+ The unique solution of (CP): Notes: $t \longmapsto X_t^{t_0, x_0, u}$

$\mathcal{U}(t_0) = \{u : [t_0, T] \longrightarrow U, \text{ measurable}\}.$

$$\boxed{\begin{pmatrix} t_0 \\ x_0 \\ u(\cdot) \end{pmatrix} \longmapsto J(t_0, x_0, u(\cdot)) := \int_{t_0}^{T} \ell(X_s^{t_0,x_0,u}, u(s))\, ds + g(X_T^{t_0,x_0,u}).}$$

$\left.\begin{array}{l} \ell : \mathbb{R}^n \times U \longrightarrow \mathbb{R} \text{ given} \\ g : \mathbb{R}^n \longrightarrow \mathbb{R} \end{array}\right\}$ bounded Lipschitz

$$\boxed{V(t_0, x_0) := \operatorname*{Inf}_{u(\cdot) \in \mathcal{U}} J(t_0, x_0, u(\cdot))}$$

V: value function of the optimal control problem

→ Important issue: Direct study of V (V satisfies a PDE Hamilton Jacobi Bellmann equation)

$$(CP)\quad \begin{cases} X'(t) = f(X(t), u(t)), & \text{a.e. } t \in [0,T] \\ X(0) = x_0 \end{cases}$$

flow in time $t$: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x_0 \longmapsto X_T^{x_0, u}$

* $1^{st}$ motivation: The initial ~~is~~ is not perfectly Known

Ex: • $x_0 \in \{a, b, c\} \subset \mathbb{R}^n$. Probability: $\frac{1}{3}$ a, $\frac{1}{3}$ b, $\frac{1}{3}$ c $\quad \Big| \quad \mu_0 = \frac{1}{3}\delta_a + \frac{1}{3}\delta_b + \frac{1}{3}\delta_c$

• $x_0 \in B_{\mathbb{R}^n}(0,1)$ equi-probability $\dfrac{d\lambda}{\lambda(B(0,1))} = \mu_0 \Big\}$ ?

Instead of Knowing exactly $x_0 \in \mathbb{R}^n$, we Know $\mu_0 \in \mathcal{P}(\mathbb{R}^n)$.

$\underline{l=0}$ $\quad u(\cdot) \longrightarrow \dfrac{g(X_T^{a,u}) + g(X_T^{b,u}) + g(X_T^{c,u})}{3} = \displaystyle\int_{\mathbb{R}^n} g(X_T^{x,u})\, d\left(\tfrac{1}{3}\delta_a + \tfrac{1}{3}\delta_b + \tfrac{1}{3}\delta_c\right)$

$$u \longrightarrow J(\mu_0, u) := \int_{\mathbb{R}^n} g(X_T^{x_0,u})\, d\mu_0(x_0).$$

$$V(t_0, \mu_0) := \inf_u J(\mu_0, u).$$

$$\mu_0 \longrightarrow \mu_t = X_T^{0,u} \# \mu_0$$

$u(\cdot) \longrightarrow \varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\qquad$ Image measure of $\mu$ by $\varphi$ : $\widehat{\varphi \# \mu}$

$\mu \in \mathcal{P}(\mathbb{R}^n)$

- $\forall A \in \mathcal{B}(\mathbb{R}^n)$, $\quad \varphi \# \mu(A) = \mu\{\varphi^{-1}(A)\} = \mu\{x \in \mathbb{R}^n \mid \varphi(x) \in A\}$.
- $\forall \psi \in C_b(\mathbb{R}^n, \mathbb{R})$, $\quad \boxed{\displaystyle\int_{\mathbb{R}^n} \psi\, d(\varphi \# \mu)(x) = \int_{\mathbb{R}^n} \psi(\varphi(x))\, d\mu(x).}$

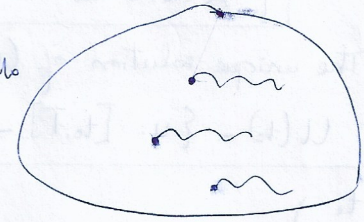
* 2nd motivation $\quad$ Multi agent system

Block of sheeps

$A \in \mathbb{R}^n$ : $\qquad$ Statistical description $:= \dfrac{\text{number of agents in } A}{\text{total number of agents}} \in [0,1]$.

$\mu_0 \in \mathcal{P}(\mathbb{R}^n)$

$$\mu_0 \rightsquigarrow u_t = X_T^{0,u} \# \mu_0.$$




* Ref : → Bardi - Capuzzo Dolcetto, Optimal control (1990)

→ Transport Theory, Villani ($1^{st}$ chapter).

(I) Optimal control of ODE

(II) Optimal control of multiagent system

(III) Conservation law in multiagent system

$$\boxed{\frac{\partial}{\partial t}\mu_t + \nabla \cdot (f \mu_t) = 0}$$

Consider $\begin{cases} X'(t) = f(X(t), u(t)) \\ X(t_0) = x_0. \end{cases}$

$(H_f)$ $\left|\begin{array}{l} f \text{ continuous, Lipschitz in } X, \text{ uniform in } U \\ |f(X,u)| \leq a(1+|X|) \\ \{f(x,u) \mid u \in U\} \text{ is closed, convex.} \end{array}\right.$ $\quad$ | verifies $|f(X,u)| \leq \alpha$.

(1) Notation of Solution

- $x \in W^{1,1}([0,T], \mathbb{R}^n)$

$J(x_0, u) := g(X_T^{t_0,x_0,u}) \longleftarrow$ The controller wants to minimize.

* Differential inclusion $\quad F(x) := \{ f(x,u) \mid u \in U \}$

$x \longrightarrow F(x)$ set-valued map

$$(DI) \quad \boxed{\begin{cases} X'(t) \in F(X(t)) & \text{a.e. } t \in [t_0, T] \\ X(t_0) = x_0 \end{cases}} \quad \textbf{(DI)}$$

* Def: $\boxed{\underline{X(\cdot) \in S_F(t_0, x_0)} \text{ iff } \begin{cases} X(\cdot) \in W^{1,1}([0,T]) \\ X'(t) \in F(X(t)) & \text{a.e. } t \end{cases}}$

If $X(t) = X_t^{t_0, x_0, u} \Rightarrow X'(t) = f(X(t), u(t)) \in F(X(t))$

Conversely: If $X(\cdot)$ is a solution to (DI)

a.e. $t$ $\quad X'(t) \in F(X(t)) = \{ f(X(t), u) \mid u \in U \}$

$\exists\, u(t) : \quad X'(t) = f(X(t), u(t))$

* Theorem $\quad$ If $\forall x \in \mathbb{R}^n$, $F(x)$ is closed, nonempty

$\boxed{X(\cdot) \in S_F(t_0, x_0) \Leftrightarrow \exists\, u(\cdot) \in U \text{ measurable} : X(\cdot) = X_\cdot^{t_0, x_0, u}}$

* Theorem

$\boxed{(H_f) \Rightarrow S_F(t_0, x_0) \text{ is compact for the } \|\cdot\|_\infty .}$

$\Rightarrow x_n \in W^{1,1}([0,T], \mathbb{R}^n) \hookrightarrow C([0,T], \mathbb{R}^n)$

Dem $t_0 = 0$: Let $(x_n(\cdot))_n$ a sequence of $S_F(x_0)$ on $[0,T]$.

$x_n \longrightarrow \exists\, u_n \in U$

$x_n(\cdot) = X_\cdot^{x_0, u_n}$

- $\underset{t>s}{|x_n(t) - x_n(s)|} \le \int_s^t |f(x_n(s), u_n(s))| ds \quad , \quad |f| \le a$

$\le a|t-s| \quad \rightarrow$ equicontinuous.

- $\forall t \in [0,T]$: $\quad (x_n(t))_{n \in \mathbb{N}}$ is bounded

$$|x_n(t)| = \left| x_0 + \int_0^t f(x_n(s), u_n(s)) ds \right| \le |x_0| + \int_0^t a(1 + |X_n(s)|) ds$$

$$|x_n(t)| - a\int_0^t |x_n(s)| ds \le |x_0| + at$$

$$\Rightarrow e^{-at} \left( |x_n(t)| - a\int_0^t |x_n(s)| ds \right) \le e^{-at}(|x_0| + at)$$

$$\frac{d}{dt}\left( e^{-at} \int_0^t |x_n(s)| ds \right) \le e^{-at}(|x_0| + at)$$

$$\rightarrow \quad e^{-at} \int_0^t |x_n(s)| ds - 0 \le \int_0^t (|x_0| + as) e^{-as} ds \le \cancel{M(T)}.\ M(T)$$

- $\exists\, x(\cdot) \in C([0,T], \mathbb{R}^n)$, $\quad \|x_n(\cdot) - x(\cdot)\|_{L^\infty([0,T], \mathbb{R}^n)} \xrightarrow[n]{} 0$ up to a subsequence.

\+ Ascoli-Arzela theorem

- $|x_n'(t)| \le a(1 + |x_n(t)|) \le C(T) \quad , \forall t \in [0,T], \forall n$

$\Rightarrow \|x_n'(\cdot)\|_{L^2[0,T]} \le \cancel{M(T)}.\ M(T).$

Extract $\quad x_n'(\cdot) \xrightarrow[L^2[0,T]]{} y(\cdot)$ up to subsequence.

- $x_n'(t) \in F(x_n(t)) \subset F(x(t)) + k|x_n(t) - x(t)| \overset{\text{closed unit ball}}{B} \subset F(x(t)) + \varepsilon B$ for $n \geq N$.

$x_n'(\cdot) \in A = \{ v(\cdot) \in L^2([0,T]),\ v(t) \in F(x(t)) + \varepsilon B \quad \text{a.e. } t \in [0,T] \} \subset L^2([0,T])$.

$A$ is convex, closed for the $L^2$ strong topology.

+ By the Mazur theorem, $A$ is also closed for the $L^2$ weak topology

$$\left.\begin{array}{l} x_n'(\cdot) \underset{n}{\rightharpoonup} y(\cdot) \\ x_n' \in A \end{array}\right\} \Rightarrow y(\cdot) \in A \Rightarrow y(t) \in F(x(t)) + \varepsilon B,$$
$$\Rightarrow y(t) \in F(x(t)) \quad \text{a.e. } t \in [0,T].$$

- $x_n(t) = x_0 + \int_0^t x_n'(s)ds = x_0 + \int_0^T x_n'(s) 1_{[0,t]}(s)\,ds$

$\downarrow$

$x(t) = \qquad x_0 + \int_0^T y(s) 1_{[0,t]}(s)\,ds$

$\Rightarrow \quad x(t) = x_0 + \int_0^t y(s)ds \Rightarrow y = x'$.

* <u>Theorem</u> $\boxed{\forall K \text{ compact} \subset \mathbb{R}^n, \quad S_F^{[0,T]}/_K \text{ has a compact graph.}}$

$x_n(\cdot) \in S_F^{[0,T]}(\overline{x_n})$ then $\exists$ subsequence $\overline{x_n} \underset{n}{\longrightarrow} \overline{x}$, $\quad x(\cdot) \in S_F(\overline{x})$ ?

$x_n(\cdot) \underset{n}{\longrightarrow} x(\cdot)$

$\|\cdot\|_\infty$

## ② <u>Value function</u>

$(H_g)$: $\quad g: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz bounded.

$$\boxed{V(t_0,x_0) = \inf_{u \in \mathcal{U}} g(X_T^{t_0,x_0,u(\cdot)}) = \inf_{y \in S_F(t_0,x_0)} g(y(T))}$$

* <u>Prop</u> $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$: $\exists \bar{u}(\cdot) \in \mathcal{U}$ : $V(t_0,x_0) = g(X_T^{t_0,x_0,\bar{u}(\cdot)})$.
(Existence of optimal control).

<u>Proof</u>: $\underline{n>0}$ $\quad \exists y_n \in \underbrace{S_F(t_0,x_0)}_{\text{compact}}$ : $\quad V(t_0,x_0) \leq g(y_{n_k}(T)) \leq V(t_0,x_0) + \frac{1}{n_k}$.

$\exists y_{n_k}(\cdot) \underset{\|\cdot\|_\infty}{\longrightarrow} y(\cdot) \in S_F(t_0,x_0) \Rightarrow V(t_0,x_0) \leq g(y(T)) \leq V(t_0,x_0) + 0.$ ✓

* <u>Prop</u> $\boxed{\begin{array}{l}(t,x) \longrightarrow V(t,x) \text{ is Lipschitz continuous \& bounded} \\ [0,T]\times\mathbb{R}^n \longmapsto \mathbb{R}\end{array}}$

<u>Proof</u>: $\cdot\ x_1, x_2$, let $u_2(\cdot) \in \mathcal{U}$ is optimal for $V(t,x_2)$.

$$|V(t,x_1) - V(t,x_2)| \leq g(X_T^{t,x_1,u_2}) - g(X_T^{t,x_2,u_2})$$
$$\leq K|X_T^{t,x_1,u_2} - X_T^{t,x_2,u_2}| \overset{\text{Gronwall}}{\leq} Ke^{KT}|x_1 - x_2|.$$

$$|X_T^{t,x_1,u_2} - X_T^{t,x_2,u_2}| \le |x_1 - x_2| + \int_t^T \underbrace{|f(X_s^{t,x_1,u_2}, u_2(s)) - f(X_s^{t,x_2,u_2}, u_2(s))|}\, ds$$

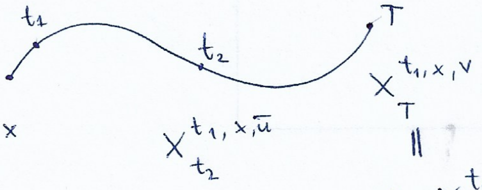
$$\le K \int_t^T |X_s^{t,x_1,u_2} - X_s^{t,x_2,u_2}|\, ds$$

Use Gronwall lemma, get: $|X_T^{t,x_1,u_2} - X_T^{t,x_2,u_2}| \le e^{KT}|x_1 - x_2|$.

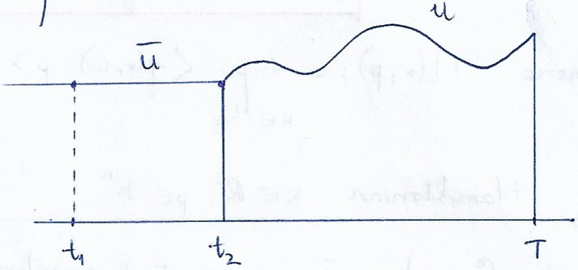
($u_2 \in U$ is optimal for $V(t_2, x)$).

- $t_1 < t_2$

$$I = |V(t_1,x) - V(t_2,x)| \le g(X_T^{t_1,x,v}) - g(X_T^{t_2,x,u_2})$$




$$X_T^{t_1,x,v} \overset{?}{=} X_T^{t_2, X_{t_2}^{t_1,x,\bar u}, u_2}$$




$$v(s) = \begin{cases} \bar u, & s \in [t_1, t_2] \\ u(s), & s \in [t_2, T]. \end{cases}$$

$\exists!$ ODE $\begin{cases} x'(t) = f(x(t), u(t)) \\ x(t_1) = x \end{cases}$

$$\begin{cases} x'(t) = f(x(t), u_2(t)) \\ x(t_2) = X_{t_2}^{t_1,x,\bar u} \end{cases}$$

$$\Rightarrow I \le g\left(X_T^{t_2, X_{t_2}^{t_1,x,\bar u}, u_2}\right) - g\left(X_T^{t_2,x,u_2}\right)$$

$$\le K \left| X_T^{t_2, X_{t_2}^{t_1,x,\bar u}, u_2} - X_T^{t_2,x,u_2} \right|$$

$$\le K e^{KT} \underbrace{|X_{t_2}^{t_1,x,\bar u} - x|} \le K e^{KT} C |t_1 - t_2|$$

$$\le \int_{t_1}^{t_2} |f(X_s^{t_1,x,\bar u}, \bar u)|\, ds \le a|t_1 - t_2|.$$

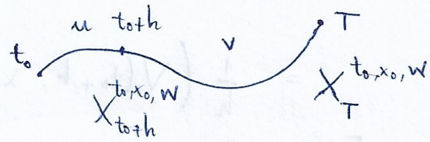
(DPP) $\Rightarrow V(t_0,x_0) = g(X_T^{t_0,x_0,\bar u(\cdot)}) = V(t_0+h, X_{t_0+h}^{t_0,x_0,\bar u})$.

* **Prop:** Dynamic Programming Principle

$0 < t_0 < t_0 + h < T$

$$V(t_0,x_0) = \inf_{u(\cdot)\in U} V(t_0+h, X_{t_0+h}^{t_0,x_0,u})$$ & the inf is achieved for optimal control for $V(t_0,x_0)$.

* **Proof:** Fix $u \in U$, $v \in U$: $w(s) = \begin{cases} u(s), & s \in [t_0, t_0+h] \\ v(s), & s \in [t_0+h, T] \end{cases}$




$$\Rightarrow V(t_0,x_0) \le g(X_T^{t_0,x_0,w}) = g\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,w}, w}\right) = g\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,u}, v}\right)$$

$$\Rightarrow V(t_0,x_0) \le \inf_{v(\cdot)} g\left(X_T^{t_0+h, X_{t_0+h}^{t_0,x_0,u}, v}\right) = V(t_0+h, X_{t_0+h}^{t_0,x_0,u}) \cancel{\le V(t_0+h, X_{t_0+h}^{t_0,x_0,\bar u})}$$

. Let $\bar{u}$ be an optimal control for $V(t_0,x_0)$

$$V(t_0,x_0) = g(X_T^{t_0,x_0,\bar{u}}) = g\left(X_T^{t_0+h,\, X_{t_0+h}^{t_0,x_0,\bar{u}},\, \bar{u}}\right) \geq V(t_0+h, X_{t_0+h}^{t_0,x_0,\bar{u}}).$$

③ PDE : Hamilton-Jacobi-Bellmann equation (H-J-B).

* Prop Suppose that $V$ is $C^1([0,T]\times \mathbb{R}^n, \mathbb{R})$, then $V$ satisfies HJB :

$$\text{(HJB)}\quad \boxed{\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + H\left(x, \dfrac{\partial V}{\partial x}(t,x)\right) = 0, & (t,x)\in ]0,T[\times\mathbb{R}^n, \\ V(T,x) = g(x), & x\in\mathbb{R}^n, \end{cases}}$$

where $H(x,p) := \inf\limits_{u\in\mathcal{U}} \langle f(x,u), p\rangle = \inf\limits_{y\in F(x)} \langle y,p\rangle.$

Hamiltonian $x\in\mathbb{R}^n$, $p\in\mathbb{R}^n$.

* Proof: Consider $\bar{u}$ a control constant.

$\forall h$, $t_0 < t_0+h < T$, DPP $\Rightarrow V(t_0,x_0) \leq V(t_0+h, X_{t_0+h}^{t_0,x_0,\bar{u}})$.

$X_\cdot^{t_0,x_0,\bar{u}}$ is a $C^1$ solution $\begin{cases} X'(t) = f(X(t),\bar{u}) \\ X(t_0) = X_0 \end{cases}$ . (C-S)

Hay xài

$$\boxed{X_{t_0+h}^{t_0,x_0,\bar{u}} = x_0 + hf(x_0,\bar{u}) + h\varepsilon(h).}$$

$$\Rightarrow 0 \leq \frac{V(t_0+h, X_{t_0+h}^{t_0,x_0,\bar{u}}) - V(t_0,x_0)}{h} = \frac{V(t_0+h, x_0 + hf(x_0,\bar{u}) + h\varepsilon(h)) - V(t_0,x_0)}{h}$$

lim $h\to 0^+$

$$0 \leq \frac{\partial V}{\partial t}(t_0,x_0) + \left\langle \frac{\partial V}{\partial x}(t_0,x_0), f(x_0,\bar{u})\right\rangle, \quad \forall \bar{u}.$$

$$\Rightarrow \quad 0 \leq \frac{\partial V}{\partial t} + \inf_{\bar{u}} \left\langle \frac{\partial V}{\partial x}, f(x_0,\bar{u})\right\rangle \Rightarrow 0 \leq \frac{\partial V}{\partial t} + H\left(x_0, \frac{\partial V}{\partial x}\right).$$

. $V(x_0,t_0) + h^2 \geq V(t_0+h, X_{t_0+h}^{t_0,x_0,u_h})$ $(\geq V(t_0,x_0))$, $\exists u_h \in \mathcal{U}$.

$x_h(\cdot) = X_\cdot^{t_0,x_0,u_h}$

$$h \geq \frac{V(t_0+h, X_{t_0+h}^{t_0,x_0,u_h}) - V(t_0,x_0)}{h}$$

$$= \frac{1}{h}\left(V\left(t_0+h, x_0 + h.\frac{1}{h}\int_{t_0}^{t_0+h} f(x_h(s), u_h(s))\,ds\right) - V(t_0,x_0)\right)$$

$\forall s\in[t_0,t_0+h]$, $f(x_h(s),u_h(s)) \in F(x_h(s)) \subset F(x_0) + K|x_h(s) - x_0| B$

$$\subset \underbrace{F(x_0) + K\varepsilon(h) B}_{\text{convex, compact.}}$$

$$v_n = \frac{1}{h_n}\int_{t_0}^{t_0+h_n} f(x_{h_n}(s), u_{h_n}(s))\,ds \xrightarrow{n} v \in F(x_0).$$

$$h_n \geq \frac{V(t_0+h_n,\, x_0 + h_n v + h_n(v_n - v)) - V(t_0,x_0)}{h_n}$$

$$h_n \geq \frac{\partial V}{\partial t}[t_0+\theta_n,\, x_0 + \theta_n v + \theta_n(v_n - v)] + \left\langle \frac{\partial V}{\partial x}[-], (v + (v_n - v))\right\rangle$$

$\exists\, \theta_n \in ]0, h_n[$.

$\downarrow n \to 0^+$

$$0 \geq \frac{\partial V}{\partial t}(t_0,x_0) + \left\langle \frac{\partial V}{\partial x}(t_0,x_0), v\right\rangle$$

$$0 \geq \frac{\partial V}{\partial t}(t_0,x_0) + H\left(x_0, \frac{\partial V}{\partial x}(t_0,x_0)\right).$$

* Theorem de verification

Suppose that $W \in C^1$ satisfies (HJB) then $W = V$

if $\hat{u}(t,x)$ s.t. $\left\langle f(x,\hat{u}), \frac{\partial W}{\partial x}(t,x)\right\rangle = H\left(x, \frac{\partial W}{\partial x}(t,x)\right)$.

is such that $\begin{cases} x'(t) = f(X(t), \hat{u}(t,x(t))) \\ X(t_0) = x_0 \end{cases}$ has a $W^{1,1}$ solution $\hat{x}$

+ $u(\cdot) \in \mathcal{U}$ : $t \longmapsto \varphi(t) := W(t, X_t^{t_0,x_0,u})$.

$$\varphi'(t) = \frac{\partial W}{\partial t}(t, X_t^{t_0,x_0,u}) + \underbrace{\left\langle \frac{\partial W}{\partial x}(t, X_t^{t_0,x_0,u}), f(X_t^{t_0,x_0,u}, u(t))\right\rangle}_{\geq H\left(X_t^{t_0,x_0,u}, \frac{\partial W}{\partial x}\right)} \quad \text{a.e. } t.$$

$\Rightarrow \varphi'(t) \geq 0$ a.e. $t$ + $\varphi$ is continuous

$$\varphi(T) - \varphi(t_0) \geq 0$$

$$\Rightarrow g(X_T^{t_0,x_0,u}) - W(t_0,x_0) \geq 0.$$

$$V(t_0,x_0) = \inf_u g(X_T^{t_0,x_0,u}) \geq W(t_0,x_0).$$

- $\frac{\partial W}{\partial t} + H\left(X_t^{t_0,x_0,u}, \frac{\partial W}{\partial x}\right) = 0$
- $X_t^{t_0,x_0,u} \in W^{1,1}([0,T], \mathbb{R}^n) \hookrightarrow C([0,T], \mathbb{R}^n)$
- $(X_t^{t_0,x_0,u})' = f(X_t^{t_0,x_0,u}, u(t))$

$t \to \psi(t) := W(t, \hat{x}(t))$, $\psi'(t) = 0 \Rightarrow \psi(T) = \psi(t_0)$.

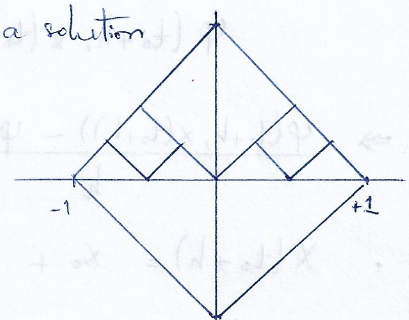
$$\psi'(t) = \frac{\partial W}{\partial t} + \left\langle \frac{\partial W}{\partial x}, f(x,\hat{u})\right\rangle = \frac{\partial W}{\partial t} + H\left(x, \frac{\partial W}{\partial x}\right) = 0.$$

$\Rightarrow V(x_0,t_0) \leq g(\hat{x}(T)) = W(t_0,x_0).$

③ Viscosity Solution

$V: [-1,1] \longrightarrow \mathbb{R}$ $\quad \begin{cases} |\nabla V(x)| = 1 \\ V(-1) = V(1) = 0 \end{cases}$

- no $C^1$
- There exists a solution




*Problem: Too many solution.

* **Def**: [Viscosity solution] $V$: bounded uniformly $C([0,T]\times\mathbb{R}^n, \mathbb{R})$ is a viscosity solution of (HJB) if $V(T,\cdot) = g(\cdot)$. [Viscosity solution]
&

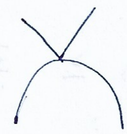
Super solution

$\forall (t,x) \in ]0,T[\times\mathbb{R}^n$, $\forall \varphi \in C^1([0,T]\times\mathbb{R}^n)$

$$\left\{\begin{array}{l}\varphi(t,x) = V(t,x),\\ \varphi(s,y) \le V(s,y), \quad \forall (s,y) \text{ in a neighborhood of } (t,x).\end{array}\right.$$

$V-\varphi$ has a local minimum at $(t,x)$.

$$\Rightarrow \frac{\partial\varphi}{\partial t}(t,x) + H\left(x, \frac{\partial\varphi}{\partial x}(t,x)\right) \le 0.$$



$\Rightarrow V-\varphi$ has a local minimum at point $(t,x)$

* Sub solution

$\forall (t,x) \in (0,T)\times\mathbb{R}^n$, $\forall \varphi \in C^1$:

$$\left.\begin{array}{l}\varphi(t,x) = V(t,x)\\ \varphi(s,y) \ge V(s,y), \ \forall (s,y)\end{array}\right\} \Rightarrow \frac{\partial\varphi(t,x)}{\partial t} + H\left(x, \frac{\partial\varphi}{\partial x}(t,x)\right) \ge 0.$$

$\Rightarrow V-\varphi$ has a local maximum at $(t,x)$.

* **Prop** The value function $V$ satisfies the (HJB) in viscosity sense.

* **Proof** . $(t_0,x_0)$, $\varphi$: $(t_0,x_0) \in \operatorname{ArgMax}(V-\varphi)$.

Consider a constant control $\bar u$: $\bar x(\cdot) = X_\cdot^{t_0,x_0,\bar u}$.

For $h$ small enough,

$$\varphi(t_0+h, \bar x(t_0+h)) \ge V(t_0+h, \bar x(t_0+h)) \underset{\text{DPP}}{\ge} V(t_0,x_0) = \varphi(t_0,x_0).$$

$$\frac{\varphi(t_0+h,\bar x(t_0+h)) - \varphi(t_0,x_0)}{h} \ge 0$$

$\downarrow h\to 0^+$

$$\inf_{\bar u\in\mathcal{U}}\left(\frac{\partial\varphi}{\partial t}(t_0,x_0) + \left\langle\frac{\partial\varphi}{\partial x}(t_0,x_0), f(x_0,\bar u)\right\rangle\right) \ge 0$$

$$\Rightarrow \frac{\partial\varphi}{\partial t}(t_0,x_0) + H\left(x_0, \frac{\partial\varphi}{\partial x}(t_0,x_0)\right) \ge 0$$

. $(t_0,x_0)$, $\varphi$: $(t_0,x_0) \in \operatorname{ArgMin}(V-\varphi)$.

Consider $u$ an optimal control for $V(t_0,x_0)$, $x(\cdot) = X_\cdot^{t_0,x_0,u}$.

$$\varphi(t_0+h, x(t_0+h)) \le V(t_0+h, x(t_0+h)) \underset{\text{DPP}}{=} V(t_0,x_0) = \varphi(t_0,x_0).$$

$$\Rightarrow \frac{\varphi(t_0+h,x(t_0+h)) - \varphi(t_0,x_0)}{h} \le 0$$

. $X(t_0+h) = x_0 + h\left(\frac{1}{h}\int_{t_0}^{t_0+h} f(x(s),u(s))\,ds\right).$

$\exists y \in F(x_0)$: $\quad \frac{\partial \varphi}{\partial t}(t_0,x_0) + \left\langle \frac{\partial \varphi}{\partial x}(t_0,x_0), y \right\rangle \leq 0.$

$$\Downarrow$$

$$\Rightarrow \quad \frac{\partial \varphi}{\partial t}(t_0,x_0) + H\left(x_0, \frac{\partial \varphi}{\partial x}(t_0,x_0)\right) \leq 0.$$

- Subdifferential of $\mathbb{R}^n \to \mathbb{R}$

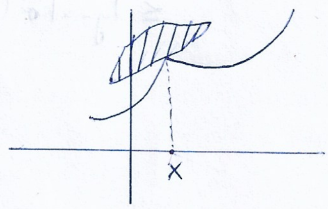
$$\boxed{\partial^- W(x) = \left\{ p \in \mathbb{R}^n \;\middle|\; \liminf_{y \to x} \frac{W(y) - W(x) - \langle p, y-x \rangle}{\|x-y\|} \geq 0 \right\}.}$$

- Superdifferential of $\mathbb{R}^n \to \mathbb{R}$

$$\boxed{\partial^+ W(x) = \left\{ p \in \mathbb{R}^n \;\middle|\; \limsup_{y \to x} \frac{W(y) - W(x) - \langle p, y-x \rangle}{\|x-y\|} \leq 0 \right\}.}$$

$$A(x) = \left\{ \nabla\varphi(x) \;\middle|\; \varphi \in C^1:\ W(x) = \varphi(x),\ W(y) \leq \varphi(y),\ \forall y \text{ near } x. \right\}$$

the hyperplane $y \mapsto W(x) + \langle p, x-y \rangle$ is tangent from above to the graph of $W$ at point $X$




$$B(x) = \left\{ \nabla\psi(x) \;\middle|\; \psi \in C^1:\ W(x) = \psi(x),\ W(y) \geq \psi(y),\ \forall y \text{ near } x. \right\}$$

- Theorem: $\boxed{W \text{ continuous.} \quad A(x) = \partial^+ W(x),\ B(x) = \partial^- W(x)}$

Super solution: $\forall (p_t, p_x) \in \partial^- V(t,x)$: $\quad p_t + H(x, p_x) \leq 0.$

Sub solution: $\forall (p_t, p_x) \in \partial^+ V(t,x)$: $\quad p_t + H(x, p_x) \geq 0.$

- Proof: $A(x) \subset \partial^+ W(x)$.

Take $\varphi \in C^1$: $W(y) \leq \varphi(y)$, $W(x) = \varphi(x)$.

$$\varphi(y) = \varphi(x) + \langle \nabla\varphi(x), y-x \rangle + |x-y|\,\varepsilon(|x-y|)$$

$$\Rightarrow \quad \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y-x \rangle = \varepsilon(|x-y|)\,|x-y|$$

$$\Rightarrow \quad \frac{\varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y-x \rangle}{|y-x|} = \varepsilon(|x-y|)$$

$$\Rightarrow \quad \frac{W(y) - W(x) - \langle \nabla\varphi(x), y-x \rangle}{|y-x|} \leq \varepsilon(|x-y|)$$

$$\Rightarrow \quad \limsup_{y \to x} \frac{W(y) - W(x) - \langle \nabla\varphi(x), y-x \rangle}{|y-x|} \leq 0 \quad \Rightarrow \quad \nabla\varphi(x) \in \partial^+ W(x). \checkmark$$

- $\partial^+ W(x) \subset A(x)$.

Let $p \in \partial^+ W(x)$, $\quad \alpha: [0,s] \to \mathbb{R}$

$$r \longrightarrow \alpha(r) := \begin{cases} \sup\left\{ \left( \frac{W(y) - W(x) - \langle p, y-x \rangle}{|y-x|} \right)^+ ;\ |y-x| \leq r \right\}. \\ 0 \text{ if } r = 0. \end{cases}$$

- $\sigma$ is continuous in $0$.
- $\sigma$ is nondecreasing.
- $\sigma$ is continuous.

- $\tilde{p}(r) = \int_0^r \sigma(t)\,dt \in C^1([0,s])$, $\quad \tilde{p}(0) = \tilde{p}'(0) = 0$.

$r \in [0,t]$

$\boxed{\tilde{p}(2r) \geq r\sigma(r)}$ since $\tilde{p}(2r) = \int_0^{2r} \sigma(t)\,dt \geq \int_r^{2r} \sigma(t)\,dt \geq r\sigma(r)$. $\quad (\sigma \uparrow)$.

+ $\varphi(y) := W(x) + \langle \underset{\mathbb{R}^n}{\overset{\cap}{p}}, y-x \rangle + \tilde{p}(2|y-x|) \in C^1$ since:

$$\varphi'(y) = p + \sigma(2|y-x|)\left(\dots, \frac{2(y_i - x_i)}{|y-x|}, \dots\right)$$

$$\left|\sigma(2|y-x|)\cdot \frac{2(y_i-x_i)}{|y-x|}\right| = \underbrace{\sigma(2|y-x|)\cdot \frac{2|y_i-x_i|}{|y-x|}}_{} \leq 2\sigma(2|y-x|) \xrightarrow{y\to x} 0$$

$$\Rightarrow \lim_{y\to x} \varphi'(y) = p$$

$\nabla\varphi(x) = p$.

$$W(y) - \varphi(y) = \underbrace{W(y) - W(x) - \langle p, y-x\rangle} - \tilde{p}(2|y-x|)$$
$$\leq |y-x|\,\sigma(|y-x|) - \tilde{p}(2|y-x|) \leq 0,$$

* **Prop**
$$\boxed{\begin{array}{l} i)\ H(x_1,p) - H(x_2,p) \leq K|x_1 - x_2||p| \\ ii)\ H(x,p_1) - H(x,p_2) \leq a(1+|x|)|p_1 - p_2| \end{array}}$$

* **Proof** i) $H(x_1,p) - H(x_2,p) \leq \langle f(x_1,u_2), p\rangle - \langle f(x_2,u_2), p\rangle$ : $u_2$ achieves the $\underset{u\in U}{\mathrm{Min}} \langle f(x,u), p\rangle$

$$\leq \| f(x_1,u_2) - f(x_2,u_2)\|\cdot|p|$$
$$\leq K|x_1 - x_2|\cdot|p|,$$

ii) $H(x,p_1) - H(x,p_2) \leq \langle f(x,v_2), p_1\rangle - \langle f(x,v_2), p_2\rangle$ : $v_2$ achieves the $\underset{u\in U}{\mathrm{Min}} \langle f(x,u), p\rangle$.

$$\leq |f(x,v_2)|\,|p_1 - p_2|$$
$$\leq a(1+|x|)\,|p_1 - p_2|.$$

$\square$

$$\boxed{\begin{array}{l} \partial^-_* V(x) = \{\nabla\varphi(x) : \varphi\in C^1 : x\in \mathrm{Arg\,Min}\ V-\varphi\} = \left\{p\in\mathbb{R}^n \,\middle|\, \liminf_{y\to x} \dfrac{V(y) - V(x) - \langle p, y-x\rangle}{|y-x|} \geq 0\right\} \\ \partial^+_* V(x) = \{\nabla\psi(x) : \psi\in C^1 : x\in \mathrm{Arg\,Max}\ V-\psi\} = \left\{p\in\mathbb{R}^n \,\middle|\, \limsup_{y\to x} \dfrac{V(y) - V(x) - \langle p, y-x\rangle}{|y-x|} \leq 0\right\} \end{array}}$$

$V : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous at $x_0$ (l.s.c.) $\Leftrightarrow \liminf_{y\to x_0} V(y) \geq V(x_0)$.

+ $V$ is l.s.c. on $\mathbb{R}^n \Leftrightarrow \underset{\{(x,\lambda) : V(x)\leq\lambda\}}{\overset{}{\mathrm{epi}\,V}}$ is closed.

$\mathrm{epi}\,V = \{(x,\lambda) : V(x) \leq \lambda\}$.

* **Prop** $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous.

i) $\partial^- V(x)$, & $\partial^+ V(x)$ are closed, convex sets (or empty).

ii) If $V$ is differentiable at $x \Rightarrow \partial^- V(x) = \partial^+ V(x) = \{\nabla V(x)\}$

iii) If $\left.\begin{array}{l}\partial^- V(x) \neq \emptyset \\ \partial^+ V(x) \neq \emptyset\end{array}\right\} \Rightarrow V$ is differentiable at $x$.

* **Proof** i) $\partial^+ V(x)$ is closed. $(p_k)_k \subset \partial^+ V(x)$, $p_k \xrightarrow[k]{} p$. $\overset{?}{\Rightarrow}$ $p \in \partial^+ V(x)$.

By contradiction, supposes that : $p \notin \partial^+ V(x) \Rightarrow \exists \alpha > 0$, $\exists (y_n)_n$: $y_n \longrightarrow x$.

$$\frac{V(y_n) - V(x) - \langle p, y_n - x\rangle}{|y_n - x|} > \alpha > 0$$

Take $k$ large enough s.t. $|p_k - p| < \frac{\alpha}{2}$.

$$\frac{V(y_n) - V(x) - \langle p_k, y_n - x\rangle - \langle p - p_k, y_n - x\rangle}{|y_n - x|} > \alpha > 0.$$

$$\Rightarrow \frac{V(y_n) - V(x) - \langle p_k, y_n - x\rangle}{|y_n - x|} \geq \frac{\alpha}{2}.$$

$$\Rightarrow \limsup_{y\to x}\left\{\frac{V(y) + V(x) - \langle p_k, y - x\rangle}{|y-x|}\right\} \geq \limsup_n \frac{V(y_n) - V(x) - \langle p_k, y_n - x\rangle}{|y_n - x|} > \frac{\alpha}{2}$$

contradiction $p_k \in \partial^+ V(x)$. ✓

iii) Consider $p \in \partial^- V(x)$ & $q \in \partial^+ V(x)$ then we want to show that $p = q$.

$$\frac{V(y) - V(x) - \langle q, y - x\rangle}{|y-x|} - \frac{V(y) - V(x) - \langle p, y - x\rangle}{|y-x|} = \frac{\langle p - q, y - x\rangle}{|y-x|}$$

$$y_n := x + \frac{1}{n}(p - q).$$

$$\Rightarrow \limsup_n \frac{V(y_n) - V(x) - \langle q, y_n - x\rangle}{|y_n - x|} - \frac{V(y_n) - V(x) - \langle p, y_n - x\rangle}{|y_n - x|} = |p - q|.$$

$$\Rightarrow \underbrace{\limsup_{y\to x} \frac{V(y) - V(x) - \langle q, y - x\rangle}{|y-x|}}_{\leq 0} - \underbrace{\liminf_{y\to x} \frac{V(y) - V(x) - \langle p, y - x\rangle}{|y-x|}}_{\geq 0} \geq |p - q|$$

$\Rightarrow |p - q| \leq 0 \Rightarrow p = q$. ✓

iv) $\boxed{\begin{array}{l}\{x \in \mathbb{R}^n \mid \partial^- V(x) \neq \emptyset\} \text{ is dense in } \mathbb{R}^n \\ \{x \in \mathbb{R}^n \mid \partial^+ V(x) \neq \emptyset\} \text{ is dense in } \mathbb{R}^n\end{array}}$ "V: Lipschitz $\rightarrow$ V: differentiable a.e."

$\boxed{\text{Lipschitz} \Rightarrow \text{differentiable a.e}}$

$\bar{x} \in \mathbb{R}^n$, $R > 0$, $B(\bar{x}, R)$

$\varepsilon>0$: Consider $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}|x-\bar{x}|^2 \in C^1$.

Let $x_\varepsilon \in \underset{B(\bar{x},R)}{\text{Arg Max}}\ V - \varphi_\varepsilon \Rightarrow V(x_\varepsilon) - \varphi_\varepsilon(x_\varepsilon) \geq V(\bar{x}) - \varphi_\varepsilon(\bar{x})$

$$V(x_\varepsilon) - \frac{1}{\varepsilon}|x_\varepsilon - \bar{x}|^2 \geq V(\bar{x})$$

$$\Rightarrow |x_\varepsilon - \bar{x}|^2 \leq \varepsilon\, 2 \sup_{B(\bar{x},R)} |V|$$

I choose $\varepsilon>0$ small enough s.t. $|x_\varepsilon - \bar{x}| < R \Rightarrow x_\varepsilon \in \mathring{B}(\bar{x},R)$.

$x_\varepsilon \in \text{Arg Max}\ V - \varphi_\varepsilon \Rightarrow 0 \in \partial_+(V-\varphi_\varepsilon)(x_\varepsilon)$

$$\Rightarrow 0 \in \partial_+ V(x_\varepsilon) - \nabla\varphi_\varepsilon(x_\varepsilon)$$

$$\Rightarrow \nabla\varphi_\varepsilon(x_\varepsilon) \in \partial_+ V(x_\varepsilon) \Rightarrow \underset{\downarrow\ \bar{x}}{x_\varepsilon} \in \{x;\ \partial_+ V(x) \neq \emptyset\}.$$

* **Theorem (Comparison)**

Let $V_1, V_2 : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}$ be the Lipschitz continuous, bounded function by M.

$$\text{If}\ \left.\begin{array}{l} V_1: \text{super sol HJB} \\ V_2: \text{sub sol HJB} \\ V_1(T,x) = g(x),\ x\in\mathbb{R}^n \\ V_2(T,x) = g(x),\ x\in\mathbb{R}^n \end{array}\right\} \Rightarrow V_1(t,x) \geq V_2(t,x),\ \forall (t,x)\in[0,T]\times\mathbb{R}^n.$$

Super sol $\geq$ Sub Sol.

* Consequence: $V$ is the unique viscosity solution of (HJB).

* Prop

$\forall (t,x) \in ]0,T[\times\mathbb{R}^n$, $\forall \varphi \in C^1([0,T]\times\mathbb{R}^n, \mathbb{R})$

$(t,x) \in \text{Arg Min}\ V-\varphi \Rightarrow \frac{\partial\varphi}{\partial t}(t,x) + H(x, \frac{\partial\varphi}{\partial x}(t,x)) \leq 0.$

$\Leftrightarrow \forall (t,x) \in ]0,T[\times\mathbb{R}^n$, $\forall\varphi\in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$

$(t,x)$ is a local strict minimum of $V-\varphi \Rightarrow \frac{\partial\varphi}{\partial t}(t,x) + H(x, \frac{\partial\varphi}{\partial x}(t,x)) \leq 0.$

$(t_0,x_0)$: $\varphi(t_0,x_0) = V(t_0,x_0)$

$\varphi(t,x) \leq V(t,x)$, $\forall (t,x)$ near $(t_0,x_0)$.

Introduce: $\tilde{\varphi}(t,x) := \varphi(t,x) - (|x-x_0|^2 + |t-t_0|^2)$

$$\left.\begin{array}{l}\tilde{\varphi}(t_0,x_0) = \varphi(t_0,x_0) = V(t_0,x_0) \\ \tilde{\varphi}(t,x) < \varphi(t,x) \leq V(t,x)\end{array}\right\} \Rightarrow (t_0,x_0) \text{ is a local strict min of } V-\tilde{\varphi}.$$

$$\Rightarrow \frac{\partial\tilde{\varphi}}{\partial t}(t_0,x_0) + H(x_0, \frac{\partial\tilde{\varphi}}{\partial x}(t_0,x_0)) \leq 0$$

$$\Rightarrow \frac{\partial\varphi}{\partial t}(t_0,x_0) + H(x_0, \frac{\partial\varphi}{\partial x}(t_0,x_0)) \leq 0$$

* Prop

$$\boxed{\begin{array}{l} \text{If } V \text{ is a } \overset{\text{(sub sol)}}{\text{super sol}} \text{ then } \forall x \in \mathbb{R}^n : \forall \varphi \in C^1(]-\eta, T[\times \mathbb{R}^n) \\ (0,x) \in \underset{[0,T]\times\mathbb{R}^n}{\overset{\text{(Max)}}{\operatorname{Arg\,Min}}}\, V - \varphi \Rightarrow \frac{\partial \varphi}{\partial t}(0,x) + H\left(x, \frac{\partial \varphi}{\partial x}(0,x)\right) \overset{(\geq)}{\leq} 0 \end{array}}$$

* Proof $\varphi \in C^1$, $(0,x_0)$ is a local strict Min of $V-\varphi$ on $[0,T]\times\mathbb{R}^n$.

$\forall n \in \mathbb{N}^*$ : $\tilde{\varphi}_n(t,x) = \varphi(t,x) - \frac{1}{nt}$.

$(t_n, x_n) \in \underset{[0,T]\times B(x,R)}{\operatorname{Arg\,Min}}\, V - \tilde{\varphi}_n$, clearly $t_n > 0$.

$$\Rightarrow \frac{\partial \tilde{\varphi}_n}{\partial t}(t_n, x_n) + H\left(x_n, \frac{\partial \tilde{\varphi}_n}{\partial x}(t_n, x_n)\right) \leq 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial t}(t_n, x_n) + \frac{1}{nt_n^2} + H\left(x_n, \frac{\partial \tilde{\varphi}_n}{\partial x}(t_n, x_n)\right) \leq 0$$

$$\overset{?}{\Rightarrow} \frac{\partial \varphi}{\partial t}(0,x_0) + H\left(x_0, \frac{\partial \varphi}{\partial x}(0,x_0)\right) \leq 0.$$

$(t_n, x_n) \underset{n}{\longrightarrow} (\bar{t}, \bar{x})$ (up to subsequence).

$$V(t_n, x_n) - \tilde{\varphi}_n(t_n, x_n) \leq V(t,x) - \tilde{\varphi}_n(t,x), \quad \forall (t,x) \in ]0,T]\times\mathbb{R}^n.$$

$$\downarrow$$

$$V(\bar{t},\bar{x}) - \varphi(\bar{t},\bar{x}) \leq V(t,x) - \varphi(t,x) \Rightarrow \underset{(0,x_0)}{\overset{\shortparallel}{(\bar{t},\bar{x})}} \text{ is a local min of } V-\varphi.$$

* Proof of Comparison theorem. Suppose that by contradiction

$$-\xi := \underset{[0,T]\times\mathbb{R}^n}{\operatorname{Inf}}\, V_1 - V_2 < 0.$$

$$\Rightarrow \exists (t_0, x_0) : V_1(t_0, x_0) - V_2(t_0, x_0) < -\frac{\xi}{2}$$

Doubling of variable technique : Fix $\varepsilon > 0$, $\eta > 0$.

$$\boxed{\phi_\varepsilon(s,x,t,y) := V_1(s,x) - V_2(t,y) + \frac{1}{\varepsilon^2}\left(|x-y|^2 + |t-s|^2\right) + \varepsilon\left(|x|^2 + |y|^2\right) + \eta(2T - t - s).}$$

$t, s \in [0,T]$

$x, y \in \mathbb{R}^n$

Let $(s_\varepsilon, x_\varepsilon, t_\varepsilon, y_\varepsilon) \in \underset{[0,T]\times\mathbb{R}^n}{\operatorname{Arg\,Min}}\, \phi_\varepsilon$.

$$\phi_\varepsilon(s_\varepsilon, x_\varepsilon, t_\varepsilon, y_\varepsilon) \leq \phi_\varepsilon(t_0, x_0, t_0, x_0) \leq -\frac{\xi}{2} + \underbrace{2\varepsilon|x_0|^2 + 2\eta(T-t_0)}_{< \frac{\xi}{4}} < -\frac{\xi}{4}.$$

$$\Rightarrow \frac{1}{\varepsilon^2}\left(|x_\varepsilon - y_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2\right) + \varepsilon\left(|x_\varepsilon|^2 + |y_\varepsilon|^2\right) + \eta(2T - t_\varepsilon - s_\varepsilon) \leq V_2(t_\varepsilon, y_\varepsilon) - V_1(s_\varepsilon, x_\varepsilon) \leq 2M$$

$$\Rightarrow \begin{cases} \varepsilon|y_\varepsilon|^2,\ \varepsilon|x_\varepsilon|^2 \le 2M \\ |x_\varepsilon - y_\varepsilon| \le \sqrt{2M}\,\varepsilon \\ |s_\varepsilon - t_\varepsilon| \le \sqrt{2M}\,\varepsilon \qquad (0) \\ \eta(2T - t_\varepsilon - s_\varepsilon) \le 2M \\ V_1(s_\varepsilon, x_\varepsilon) - V_2(t_\varepsilon, y_\varepsilon) \le -\frac{\delta_0}{4} \end{cases}$$

$$\phi_\varepsilon(s_\varepsilon, x_\varepsilon, t_\varepsilon, y_\varepsilon) \le \phi_\varepsilon(t_\varepsilon, x_\varepsilon, t_\varepsilon, x_\varepsilon)$$

$$\Rightarrow \frac{1}{\varepsilon^2}\left(|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2\right) + \varepsilon\big(\underset{-|x_\varepsilon|^2}{\cancel{|x_\varepsilon|^2}} + |y_\varepsilon|^2\big) + \eta\big(\cancel{2T} - \underset{t_\varepsilon}{\cancel{t_\varepsilon}} - s_\varepsilon\big)$$

$$\le V_2(t_\varepsilon, y_\varepsilon) - V_1(s_\varepsilon, x_\varepsilon) + V_1(t_\varepsilon, x_\varepsilon) - V_2(t_\varepsilon, x_\varepsilon)$$

$$\le K|y_\varepsilon - x_\varepsilon| + K|s_\varepsilon - t_\varepsilon| \qquad \le 2K\sqrt{|x_\varepsilon - y_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}$$

$$\Rightarrow \frac{1}{\varepsilon^2}\left(|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2\right) \le \varepsilon|x_\varepsilon - y_\varepsilon||x_\varepsilon + y_\varepsilon| + \eta(s_\varepsilon - t_\varepsilon) + K\sqrt{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2}$$

$$\le (C + \eta + \underline{K})\sqrt{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2}$$

$$\Rightarrow \sqrt{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2} \le (C + \underset{**}{\eta})\varepsilon^2 \quad \Rightarrow \begin{cases} |x_\varepsilon - y_\varepsilon| \le (C + \overset{**}{\eta})\varepsilon^2 \\ |s_\varepsilon - t_\varepsilon| \le (C + \overset{**}{\eta})\varepsilon^2 \end{cases}$$

- $\forall s, x: \quad \phi_\varepsilon(s, x, t_\varepsilon, y_\varepsilon) \ge \phi_\varepsilon(s_\varepsilon, x_\varepsilon, t_\varepsilon, y_\varepsilon)$

$$\underbrace{V_1(s,x) - \Big\{V_1(s_\varepsilon, x_\varepsilon) + \frac{1}{\varepsilon^2}\left(|y_\varepsilon - x|^2 - |x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s|^2 - |t_\varepsilon - s_\varepsilon|^2\right) - \varepsilon\left(|x|^2 - |x_\varepsilon|^2\right) + \eta(t_\varepsilon + s) - \eta(t_\varepsilon + s_\varepsilon)\Big\}}_{\text{Define } \varphi(s,x) :=\ \in C^1} \ge 0$$

$$\varphi(s_\varepsilon, x_\varepsilon) = V_1(s_\varepsilon, x_\varepsilon)$$

$$V_1(s,x) \ge \varphi(s,x).$$

$\Rightarrow (s_\varepsilon, x_\varepsilon) \in \operatorname{ArgMin}(V_1 - \varphi)$ + $V_1$ is a super solution.

$$\Rightarrow \boxed{-\frac{2}{\varepsilon^2}(s_\varepsilon - t_\varepsilon) + \eta + H\left(x_\varepsilon, -\frac{2}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) - 2\varepsilon x_\varepsilon\right) \le 0} \qquad (1)$$

- $\phi_\varepsilon(s_\varepsilon, x_\varepsilon, t_\varepsilon, y_\varepsilon) \le \phi_\varepsilon(s_\varepsilon, x_\varepsilon, t, y)$.

$$\Rightarrow \underbrace{V_2(t,y) - \Big\{V_2(t_\varepsilon, y_\varepsilon) - \frac{1}{\varepsilon^2}\left(|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2 - |s_\varepsilon - t|^2 - |x_\varepsilon - y|^2\right) - \varepsilon\left(|y_\varepsilon|^2 - |y|^2\right) + \eta\left(t_\varepsilon + \cancel{s_\varepsilon} - t - \cancel{s_\varepsilon}\right)\Big\}}_{\psi(t,y) :=\ \in C^1} \le 0$$

$\Rightarrow\ (t_\varepsilon, y_\varepsilon) \in \text{ArgMax}\,(V_2 - \psi)$

$$\Rightarrow\ \boxed{-\frac{2}{\varepsilon^2}(s_\varepsilon - t_\varepsilon) - \eta + H\left(y_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) + 2\varepsilon y_\varepsilon\right) \geq 0} \qquad (2)$$

- $V_1(s_\varepsilon, x_\varepsilon) - V_2(t_\varepsilon, y_\varepsilon) \leq -\frac{\xi}{4}$.

$-V_1(T, x_\varepsilon) + V_2(T, x_\varepsilon) = 0$

$$\Rightarrow\ -K|T - s_\varepsilon| - K(|T - t_\varepsilon| + |x_\varepsilon - y_\varepsilon|) \leq V_1(s_\varepsilon, x_\varepsilon) - V_2(t_\varepsilon, y_\varepsilon) - V_1(T, x_\varepsilon) + V_2(T, x_\varepsilon) \leq -\frac{\xi}{4},$$

$(KM\varepsilon < \frac{\xi}{8})$

$$\Rightarrow\ \underbrace{\frac{\xi}{4} - KM\varepsilon}_{\geq \frac{\xi}{8}} \leq \frac{\xi}{4} - K|x_\varepsilon - y_\varepsilon| \leq K|T - s_\varepsilon| + K|T - t_\varepsilon| \leq 2K|T - \theta_\varepsilon|$$

$$\Rightarrow \begin{cases} s_\varepsilon \neq T \\ t_\varepsilon \neq T \end{cases} \quad (7)$$

where $|T - \theta_\varepsilon| = \max(|T - s_\varepsilon|, |T - t_\varepsilon|)$

$$\Rightarrow\ |T - \theta_\varepsilon| \geq \frac{\xi}{16K} \quad \& \quad |s_\varepsilon - t_\varepsilon| \leq \sqrt{2M}\,\varepsilon.$$

$$(1)+(2) \Rightarrow\ 2\eta \leq H\left(y_\varepsilon, \frac{2}{\varepsilon^2}(y_\varepsilon - x_\varepsilon) + 2\varepsilon y_\varepsilon\right) - H\left(y_\varepsilon, -\frac{2}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) - 2\varepsilon x_\varepsilon\right)$$

$$- H\left(x_\varepsilon, -\frac{2}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) - 2\varepsilon x_\varepsilon\right) + H\left(y_\varepsilon, -\frac{2}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) - 2\varepsilon x_\varepsilon\right)$$

$$\leq a(1 + |y_\varepsilon|)\,2\varepsilon|x_\varepsilon - y_\varepsilon| + K|x_\varepsilon - y_\varepsilon|\underbrace{\left(\frac{2}{\varepsilon^2}|x_\varepsilon - y_\varepsilon| + 2\varepsilon|x_\varepsilon|\right)}_{\text{bounded}}$$

$$\leq KC\Big(C\varepsilon^2 + 2\varepsilon\underbrace{(\varepsilon^2|x_\varepsilon|)}_{\leq C}\Big)$$

$$+\ a(1 + |y_\varepsilon|)\,2\varepsilon|x_\varepsilon - y_\varepsilon| \leq 2a(1 + |y_\varepsilon|)\,\varepsilon C\varepsilon^2 \qquad |x_\varepsilon - y_\varepsilon| < C\varepsilon^2$$

$$\leq 2aC\varepsilon^3 + 2a\varepsilon\underbrace{(\varepsilon^2|y_\varepsilon|)}_{\text{bounded}}$$

$$\Rightarrow\ 2\eta \leq C_{(a,M)}\,\varepsilon.$$

Contradiction with the fact that $\eta > 0$.

$$F(x, V(x), \nabla V(x)) = 0.$$

* **Theorem (Stability)**

Suppose $(\theta_n)_n$ a sequence of $BUC(\mathbb{R}^d)$ ~~satisfying~~: is a viscosity solution

$$F_n(x, V(x), \nabla V(x)) = 0, \quad \mathbb{R}^d.$$

If: $\theta_n$ converge uniformly on every compact of $\mathbb{R}^d$ to $\theta$.

$F_n: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ converge uniformly on compact $F$

then $F(x, \theta(x), \nabla\theta(x)) = 0, \ \mathbb{R}^d$ in viscosity sense.

* **Lemma**

$\psi: \mathbb{R}^d \to \mathbb{R}$ continuous, $\psi_n: \mathbb{R}^d \to \mathbb{R}$ continuous

If $\left|\begin{array}{l} \psi_n \xrightarrow[n]{} \psi \text{ unif on every compact} \\ x \text{ is a strict local maximum of } \psi \end{array}\right.$

then $\exists x_n \in \operatorname{Arg\,Max}_{\mathbb{R}^d} \psi_n$ s.t. $x_n \xrightarrow[n]{} x$

Proof $x \in \operatorname{Arg\,Max}_{\mathring{B}(x,\delta)} \psi$. Define $x_n \in \operatorname{Arg\,Max}_{B(x,\delta)} \psi_n$.

Take $l$ an accumulation point of $(x_n)_n$, $x_{n_p} \xrightarrow[p]{} l$.

$\forall y \in B(x,\delta): \ \psi_{n_p}(y) \le \psi_{n_p}(x_{n_p}) \ \Rightarrow \ \forall y \in B(x,\delta): \ \psi(y) \le \psi(l)$

$\Rightarrow l = x.$ ✓

* Proof of theorem. $\theta$ is a super solution.

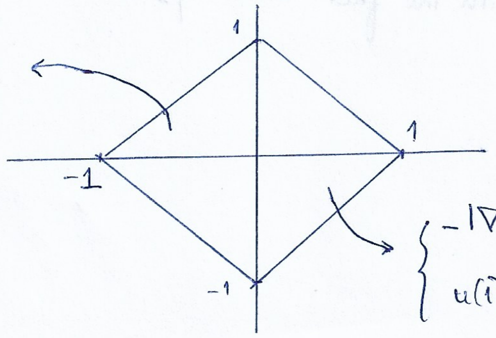
$x \in \mathbb{R}^d, \ \forall \varphi \in C^1: \ x \in \operatorname{Arg\,Max}_{\text{strict}} (\theta - \varphi)$.

We want $F(x, \theta(x), \nabla\varphi(x)) \le 0$.

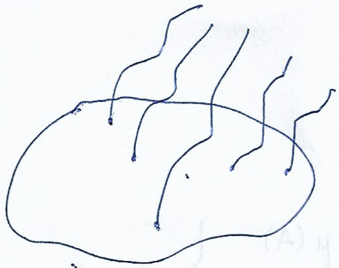
By the lemma, $\exists x_n \in \operatorname{Arg\,Max} (\theta_n - \varphi): \ x_n \longrightarrow x$

$$\Rightarrow \ 0 \ge F_n(x_n, \theta_n(x_n), \nabla\varphi(x_n)) \xrightarrow[n]{} F(x, \theta(x), \nabla\varphi(x)).$$

Exercise: $\left\{\begin{array}{l} |\nabla u(x)| = 1, \ x \in \,]-1,1[\,, \\ u(1) = u(-1) = 0. \end{array}\right.$



$\left\{\begin{array}{l} -|\nabla u(x)| = -1 \\ u(1) = u(-1) = 0. \end{array}\right.$



$\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, $A \subset \mathbb{R}^d$.

$\mu_0(A) =$ fraction of the total population in $A$.

$$\begin{cases} x' = f(x,u) \\ t_0,\ \mu_0 \end{cases}$$

$x \in \operatorname{supp} \mu_0$

$$t \longrightarrow X_t^{x,u}, \quad g(X_T^{t,x,u})$$

$$\boxed{V(t_0,\mu_0) := \inf_{u\in\mathcal{U}} \int_{\mathbb{R}} g(X_T^{t,x,u})\, d\mu_0(x)}$$

$\longrightarrow$ Regularity (Lipschitz)

$\longrightarrow$ DPP

$\longrightarrow$ HJB

$\longrightarrow$ Characterization

$X \subset \mathbb{R}^d$, $\mathcal{P}_2(X) = \{\mu:$ probability measure on $\mathbb{R}^d \mid \operatorname{Supp}\mu \subset X\}$

$$\int_{\mathbb{R}^d} |x|^2\, d\mu(x) < +\infty$$

$(\mu_n)_n$ sequence of measure $\in \mathcal{P}_2(X)$.

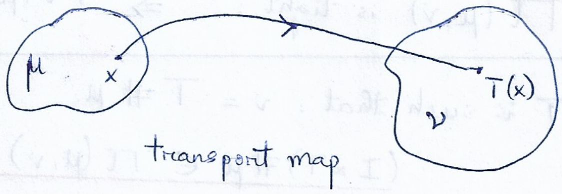
$$\mu_n \longrightarrow \mu \iff \forall \varphi \in C_b(X,\mathbb{R}), \int_X \varphi(x)\, d\mu_n(x) \xrightarrow[n]{} \int_X \varphi(x)\, d\mu(x).$$

* Theorem (Prokhorov) (Billingsley)

$\mathcal{M} \subset \mathcal{P}(X)$, $\mathcal{M}$ is tight. $(\forall \varepsilon > 0, \exists K_\varepsilon \subset X, \mu(K_\varepsilon) \geq 1-\varepsilon, \forall \mu \in \mathcal{M})$

then $\mathcal{M}$ is relatively sequentially compact.

$\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$




(M) Monge problem:

$$\min_{\substack{\nu = T\#\mu \\ T:\ \text{measurable}}} \int_{\mathbb{R}^d} \overbrace{|x - Tx|}^{c(x,Tx)}\, d\mu(x)$$

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Case where there is no solution

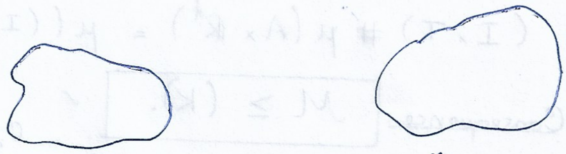
$\mu = \delta_{x_0}$

$\nu = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$

dim 1: $\mu = \frac{1}{n} \mathbb{1}_{[0,n]} dX$

$\nu = \frac{1}{n} \mathbb{1}_{[1,n+1]}$

$T: x \longmapsto x+1.$



$\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ $\qquad \nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$

$$\min_{\sigma\in S_n} \left\{ \frac{1}{n}\sum_{j=1}^n |x_j - y_{\sigma(j)}| \;\middle|\; \sigma \in S_n \right\}$$

## (K) Kantorovitch

$$(K) \quad \boxed{\underset{\gamma \in \Pi(\mu,\nu)}{\text{Min}} \int_{\mathbb{R}^{2d}} \overset{c(x,y)}{|x-y|} \, d\gamma(x,y)}$$

transport plan

$$\Pi(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^{2d}),\ \forall A \in \mathcal{B}(\mathbb{R}^d) : \gamma(A \times \mathbb{R}^d) = \mu(A) \atop \forall B \in \mathcal{B}(\mathbb{R}^d) : \gamma(\mathbb{R}^d \times B) = \nu(B) \right\}$$

→ $c(x,y)$ : continuous bounded function.

→ $\gamma \longrightarrow \int_{\mathbb{R}^{2d}} |x-y| \, d\gamma$ : continuous.

(K) has a solution.

→ $\boxed{\Pi(\mu,\nu) \text{ is relatively compact.}}$

proof: $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\forall \varepsilon > 0$ : $\exists K_\mu$ compact : $\mu(K_\mu) > 1-\varepsilon$

$\exists K_\nu$ compact : $\nu(K_\nu) > 1-\varepsilon$

$\gamma \in \Pi(\mu,\nu)$:

$$\gamma(\underbrace{K_\mu \times K_\nu \cup (\mathbb{R}^d \times K_\nu)^c \cup (K_\mu \times \mathbb{R}^d)^c}_{\cup\, \mathbb{R}^{2d}}) = 1.$$

$$\Rightarrow \gamma(K_\mu \times K_\nu) + \gamma((\mathbb{R}^d \times K_\nu)^c) + \gamma((K_\mu \times \mathbb{R}^d)^c) \geq 1.$$

$$\Rightarrow \gamma(K_\mu \times K_\nu) + 1 - \nu(K_\nu) + 1 - \mu(K_\mu) \geq 1.$$

$$\Rightarrow \gamma(\underbrace{K_\mu \times K_\nu}_{\text{compact}}) \geq 1 - 2\varepsilon.$$

$\Rightarrow \Pi(\mu,\nu)$ is tight : $\Rightarrow \Pi(\mu,\nu)$ is relatively compact. □

---

$\mathcal{H}$ : $T$ is such that : $\nu = T \# \mu$.

$(I \times T) \# \mu \in \Pi(\mu,\nu)$

$\cap$

$\mathcal{P}(\mathbb{R}^{2d})$

$$I \times T : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$$
$$x \longmapsto (x, Tx)$$

$$(I \times T) \# \mu (A \times \mathbb{R}^d) = \mu((I \times T)^{-1}(A \times \mathbb{R}^d)) = \mu(\{x \in \mathbb{R}^d : Ix \in A, Tx \in \mathbb{R}^d\}) = \mu(A).$$

* Consequence: $\boxed{\mathcal{M} \geq (K).}$

• Wasserstein distance on $\mathcal{P}_2(X)$, $X \subset \mathbb{R}^d$, $X$ compact: on $\mathcal{P}_2(X)$?

$$\boxed{W_2(\mu,\nu) := \underset{\gamma \in \Pi(\mu,\nu)}{\text{Min}} \sqrt{\int_{\mathbb{R}^{2d}} |x-y|^2 \, d\gamma(x,y)}}$$ is a metric.

* Result: $X$ compact set of $\mathbb{R}^d$

$$\boxed{\mu_n \xrightarrow[n]{} \mu \iff W_2(\mu_n,\mu) \xrightarrow[n]{} 0}$$

$(M)_{(\mu,\nu)} \in (\mathcal{P}(X) \times \mathcal{P}(Y))$ complete separable

$$\underset{\substack{T: X \to Y \\ T\#\mu = \nu}}{\text{Min}} \int_X \overbrace{|x - Tx|}^{c(x,y)} \, d\mu(x)$$

$c: X \times Y \to \mathbb{R}$ continuous (bounded).

$$\boxed{(K) \quad \underset{\pi \in \Pi(\mu,\nu)}{\text{Min}} \int_{X\times Y} c(x,y)\, d\pi(x,y) \quad \Bigg| \Rightarrow \quad \begin{array}{l} T \text{ is a solution of } (M) \\ (I \times T) \text{ is solution of } (K). \end{array}}$$

$$\Pi(\mu,\nu) = \left\{ \pi \in \mathcal{P}(X\times Y) \;\middle|\; \begin{array}{l} \text{the first margin of } \pi \text{ is } \mu \\ \text{the second margin of } \pi \text{ is } \nu \end{array} \right\}$$

(Thm Prokhorov) $\Rightarrow$ $\Pi(\mu,\nu)$ is compact.

$$\boxed{\Pi_0(\mu,\nu) = \{\pi \in \Pi(\mu,\nu) \text{ optimal for } (K)\}}$$

$p \geq 1$:
$$\boxed{W_p(\mu,\nu) = \sqrt[p]{\underset{\pi \in \Pi(\mu,\nu)}{\text{Min}} \int |x-y|^p \, d\pi(x,y)}}$$

$E$: a normed space, $\varphi: E \to \mathbb{R} \cup \{+\infty\}$.

- $\text{epi}\,\varphi = \{(x,\lambda) \in E \times \mathbb{R} \mid \varphi(x) \leq \lambda\}$.
- $\varphi$ convex function iff epi $\varphi$ convex set.
- $\varphi$ l.s.c. (lower semicontinuous) iff epi $\varphi$ closed.

<u>Def</u>: <u>Fenchel transform</u>

$$\varphi^* \left| \begin{array}{ccc} E^* & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ l & \longmapsto & \varphi^*(l) := \sup\limits_{x \in E} \{\langle l, x\rangle - \varphi(x)\} \end{array} \right.$$

- $\varphi^*$ convex
- $\varphi \leq \psi \Rightarrow \varphi^* \geq \psi^*$.
- $(\inf\limits_{i \in I} \varphi_i)^* = \sup\limits_{i \in I} \varphi_i^*$.

* <u>Fenchel Inequality</u>:
$$\boxed{\varphi^*(l) + \varphi(x) \geq \langle l, x\rangle}$$

$$\varphi^{**} \left| \begin{array}{ccc} E & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ x & \longmapsto & \sup\limits_{l \in E^*} \{\langle l, x\rangle - \varphi^*(l)\} \end{array} \right.$$

* <u>Fenchel Thm</u>: $\boxed{\varphi \text{ convex} \Leftrightarrow \varphi^{**} = \varphi.}$

* **Thm** (Rockaffelar Duality)

$f, g : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ convex. $\exists u_0 \in \mathrm{Dom} f \cap \mathrm{Dom}\, g$ s.t. $g$ continuous at $u_0$.

$$\inf_{x \in E} \{ f(x) + g(x) \} = \operatorname*{Max}_{l \in E^*} \{ -f^*(l) - g^*(-l) \}$$

Brezis p. 10-20.

$$\boxed{(K): \operatorname*{Min}_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y)\, d\pi(x,y).}$$

* **Thm** (Duality).

$c$: l.s.c. : $X \times Y \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ continuous, bounded

$$\boxed{\operatorname*{Min}_{\pi \in \Pi(\mu,\nu)} \underbrace{\int c\, d\pi}_{\substack{\parallel \\ I(\pi)}} = \sup_{(\varphi,\psi) \in \Phi_c} \underbrace{\int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y)}_{\substack{\parallel \\ J(\mu,\nu)}}} \quad (DK)$$

where $\Phi_c := \{ (\varphi, \psi) \in L^1_\mu(X) \times L^1_\nu(X) \mid \varphi(x) + \psi(y) \leq c(x,y)$ for a.e. $\mu \otimes \nu$, $(x,y) \in X \times Y \}$

$\in C_b(X) \times C_b(Y)$

Proof: $\pi \in \Pi(\mu,\nu)$, $(\varphi, \psi) \in \Phi_c$.

$$\varphi(x) + \psi(y) \leq c(x,y), \ \forall (x,y) \in X \times Y.$$

$$\Rightarrow \int_{X \times Y} \varphi(x) + \psi(y)\, d\pi(x,y) \leq \int_{X \times Y} c(x,y)\, d\pi(x,y)$$

$$\rightarrow \int_X \varphi\, d\mu + \int_Y \psi\, d\nu \leq \int c\, d\pi$$

$$\sup_{(\varphi,\psi) \in \Phi_c} \int_X \varphi\, d\mu + \int_Y \psi\, d\nu \leq \operatorname*{Min}_{\pi} \int c\, d\pi.$$

$E := C_b(X, Y)$, $E^* = \mathcal{M}(X \times Y)$

$$\theta \left| \begin{array}{l} E \longrightarrow \mathbb{R} \cup \{+\infty\} \\ u \longmapsto \begin{cases} 0 & \text{if } u(x,y) \geq -c(x,y),\ \forall (x,y) \in X \times Y \\ +\infty & \end{cases} \end{array} \right.$$

$$S: \left| \begin{array}{l} E \longrightarrow \mathbb{R} \cup \{+\infty\} \\ u \longmapsto \begin{cases} \int \varphi\, d\mu + \int \psi\, d\nu & \text{if } u(x,y) = \varphi(x) + \psi(y),\ \forall (x,y) \in X \times Y \\ +\infty & \text{elsewhere} \end{cases} \end{array} \right.$$

$+ \quad u(x,y) = \varphi(x) + \psi(y) = \tilde{\varphi}(x) + \tilde{\psi}(y), \forall x, y$

$\exists m \in \mathbb{R}, \forall x, y, \quad \varphi(x) - \tilde{\varphi}(x) = \tilde{\psi}(y) - \psi(y) = m.$

$$\int \varphi \, d\mu - \int \tilde{\varphi} \, d\mu = m = \int \tilde{\psi} \, d\nu - \int \psi \, d\nu$$

$$\Rightarrow \int \varphi \, d\mu + \int \psi \, d\nu = \int \tilde{\varphi} \, d\mu + \int \tilde{\psi} \, d\nu$$

$\theta$ & $S$ are convex l.s.c.

$u_0 = 1$, $\theta$ continuous in $u_0$.

$u_0 \in \mathrm{Dom}\, \theta \cap \mathrm{Dom}\, g, \quad \theta(u_0) = 0.$

$\|u_n - y\|_\infty \xrightarrow[n]{} 0, \quad \exists N > 0, \forall n, \quad u_n \geq \frac{1}{2} \geq 0 \geq -c.$

$\pi \in \mathcal{M}(X \times Y)$ (Radon measure)

$$\theta^*(-\pi) = \sup_{u \in C_b(X \times Y)} \left\{ -\int_{X \times Y} u \, d\pi - \theta(u) \right\}$$

$$= \sup_{u \in E} \left\{ -\int u \, d\pi - 0 ; \ u(x,y) \geq -c(x,y), \forall (x,y) \right\}$$

$$= \sup_{u \in E} \left\{ \int_X u \, d\pi ; \ u(x,y) \leq c(x,y) \right\} = \int_{X \times Y} c \, d\pi \quad \text{if} \quad \pi \in \mathcal{M}_+(X,Y)$$

$\cdot$ If $\pi \notin \mathcal{M}_+(X,Y) \Rightarrow \exists v \in C_b, \ v \leq 0$ & $\int_{X \times Y} v \, d\pi > 0.$

$$\theta^*(-\pi) \geq \int_{X \times Y} t v \, d\pi = t \int v \, d\pi \longrightarrow +\infty, \quad \forall t > 0$$

$$\boxed{\theta^*(-\pi) = \begin{cases} \int_{X \times Y} c \, d\pi & \text{if } \pi \in \mathcal{M}_+(X \times Y) \\ +\infty & \text{else.} \end{cases}}$$

$$S^*(\pi) = \sup_{u \in E} \left\{ \int u \, d\pi - S(u) \right\}$$

$$= \sup_{\substack{u \in E \\ \{\varphi(x) + \psi(y) = u(x,y)}} \left( \int_{X \times Y} u \, d\pi - \int_X \varphi \, d\mu - \int_Y \psi \, d\nu \right)$$

if $\pi \in \Pi(\mu,\nu)$ : $\quad \int u \, d\pi - \int (\varphi + \psi) \, d\pi = 0$

if $\pi \notin \Pi(\mu,\nu)$, $\exists (\varphi, \psi)$: $\int_{X \times Y} (\varphi + \psi) \, d\pi \neq \int_X \varphi \, d\mu + \int_Y \psi \, d\nu.$

$$\boxed{S^*(\pi) = \begin{cases} 0 & \text{if } \pi \in \Pi(\mu,\nu) \\ +\infty & \text{else} \end{cases}}$$

$$\underset{\pi\in E^*}{\mathrm{Max}}\{-S^*(\pi)-\theta^*(-\pi)\} = \underset{\pi\in\Pi(\mu,\nu)}{\mathrm{Max}}\left\{-\int_{X\times Y} c\,d\pi + 0\right\}$$

$$\varphi(x)+\psi(y) = u(x,y) \geq -c(x,y).$$

$$\inf_{u\in E}\{\theta(u)+S(u)\} = \inf_{u\in C_b(X\times Y)}\left\{0+\int\varphi\,d\mu+\int\psi\,d\nu\right\}$$

$$= \inf_{(\varphi,\psi)}\left\{-\int\varphi\,d\mu-\int\psi\,d\nu\ ;\ \varphi(x)+\psi(y)\leq c(x,y)\right\}$$

$$= -\sup_{(\varphi,\psi)}\left\{\int\varphi\,d\mu+\int\psi\,d\nu \mid \varphi+\psi\leq c(x,y)\right\}$$

*Lemma

c bounded unif continuous

$$(\varphi,\psi)\in\Phi_c \Rightarrow \begin{cases}(\varphi^{cc},\varphi^{c})\in\Phi_c & (\varphi^c,\varphi^{cc})\in\Phi_c\\ J(\varphi^{cc},\varphi^{c})\geq J(\varphi,\psi)\end{cases}$$

where
$$\varphi^c(x) := \inf_{y\in Y} c(x,y)-\psi(y)$$
$$\varphi^{cc}(y) = \inf_{x\in X} c(x,y)-\varphi^c(x)$$
$$\varphi^{ccc}(x) := \inf_{y\in Y} c(x,y)-\varphi^{cc}(y) = \varphi^c(x).$$

$$\varphi^{ccc}\equiv\varphi^c$$

$\cdot\ \psi^c(y) = \inf_{x\in X} c(x,y)-\varphi(x).$

$$(DK) = \sup_{(\varphi^{cc},\varphi^c)\in\Phi_c} J(\varphi,\psi).$$

- $\varphi(x)\leq c(x,y)-\psi(y) \Rightarrow \varphi(x)\leq\varphi^c(x)$
  $\Rightarrow J(\varphi,\psi)\leq J(\varphi^c,\psi)$

$$\psi(y)\leq\psi^c(y)$$

$$J(\varphi,\psi)\leq J(\varphi^c,\psi)\leq J(\varphi^c,\psi^c).$$

- $\varphi^c$ continuous at $x$, $(x_n)_n \xrightarrow{n} x$.

Denote $w(\cdot)$ the modulus of continuity of $c$.

$$\inf_y c(x,y)-\psi(y)-w(|x-x_n|)\leq\varphi^c(x_n) = \inf_{y\in Y} c(x_n,y)-\psi(y)$$
$$\leq \inf_y\{c(x,y)-\psi(y)\} + c(y,x_n)-c(y,x).$$

$$\varphi^c_c(x)-w(|x-x_n|)\leq\varphi^c(x_n)\leq\varphi^c(x)+w(|x_n-x|)$$

$$\cdot \inf_{y\in Y}\{c(x,y)-\varphi^{cc}(y)\} = \inf_{y\in Y}\left\{c(x,y)-\inf_{x'\in X}\left[c(x',y)-\varphi^c(x')\right]\right\}$$

$$\geq \inf_{y\in Y}\{\cancel{c(x,y)}-\cancel{c(x,y)}+\varphi^c(x)\} \geq \varphi^c(x).$$

$$= \inf_{y\in Y}\left\{c(x,y)-\inf_{x'\in X}\left[c(x',y)-\inf_{y'\in Y}\left[c(x',y')-\psi(y')\right]\right]\right\}$$

$$= \inf_{y\in Y}\sup_{x'\in X}\inf_{y'\in Y}\{c(x,y)-c(x',y)+c(x',y')-\psi(y')\}$$

$$\leq \inf_{y\in Y}\sup_{x'\in X}\{c(x,y)-\cancel{c(x',y)}+\cancel{c(x',y)}-\psi(y)\} = \varphi^c(x)$$

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Case where $c(x,y)=\underset{\text{distance}}{\underset{\uparrow}{d(x,y)}}$, $X=Y$, $X$ compact.
$d$ bounded

* Thm Kantorovitch - RUBINSTEIN

$$\boxed{\begin{array}{l}\displaystyle\min_{\pi\in\Pi(\mu,\nu)}\int d\,d\pi = \max_{\varphi\in Lip_1(X)}\underbrace{\int_X\varphi(x)\,d\mu(x)-\int_X\varphi(x)\,d\nu(x)}_{\int_X\varphi\,d(\mu-\nu)}. \\[2ex] \displaystyle\varphi^d(x)=\inf_{y\in X} d(y,x)-\psi(y), \qquad \sup_{(\varphi^{dd},\varphi^d)\in\Phi} J(\varphi^{dd},\varphi^d)\end{array}}$$

$\varphi^d\in Lip_1(X)$: $y,z\in X$.

$$\varphi^d(y)-\varphi^d(z) \leq d(y,\tilde z)-\cancel{\psi(\tilde z)}-d(\tilde z,z)+\cancel{\psi(\tilde z)}+\varepsilon$$

$$\leq d(y,z)+\cancel{d(z,\tilde z)}-\cancel{d(\tilde z,z)}+\varepsilon \leq d(y,z).$$

$\varphi^{dd}\in Lip_1(X)$, $\varphi^{ddd}=\varphi^d$. $\quad \underset{\shortparallel}{\varphi^{dd}(y)}$ $\quad\Rightarrow \boxed{\varphi^{dd}(y)=-\varphi^d(y)}$

$$\inf_x d(x,y)-\varphi^d(y)+\varphi^d(y)-\varphi^d(x) = \inf_x d(x,y)-\varphi^d(x) \overset{x=y}{\leq} 0-\varphi^d(y)$$

$$\inf_x d(x,y)-\varphi^d(y)-d(x,y) \leq \sup J(\varphi^d,\varphi^{dd}) = \sup J(\varphi^d,-\varphi^d) \leq \sup_{u\in Lip_1(X)} J(u,-u)$$

$$\shortparallel$$

$$\int u\,d\mu-\int u\,d\nu$$

$\varphi\in Lip_1(X)$, $\pi\in\Pi(\mu,\nu)$.

$$\int_X\varphi(x)\,d(\mu-\nu)(x) = \int_{X\times X}(\varphi(x)-\varphi(y))\,d\pi(x,y) \leq \int_{X^2} d(x,y)\,d\pi(x,y)$$

$$\max_{\varphi\in Lip_1(X)}\int\varphi\,d(\mu-\nu) \leq \inf_\pi\int d(x,y)\,d\pi \underset{\text{Thm D}}{\leq} \max_{(\varphi,\psi)\in\Phi} J(\varphi,\psi) \leq \max_{\varphi\in Lip_1(X)}\int\varphi\,d(\mu-\nu).$$

$$\downarrow \leq \sup J(\varphi^d,\varphi^{dd}) \text{ Lemma}$$

$$\boxed{W_1(\mu,\nu) = \underset{\pi\in\Pi(\mu,\nu)}{\mathrm{Min}} \int_{X^2} |x-y|\, d\pi(x,y) = \underset{u\in \mathrm{Lip}_1(X)}{\mathrm{Sup}} \int u(x)\, d(\mu-\nu)(x).}$$

Investigate $\boxed{W_2(\mu,\nu), \quad c(x,y) = \frac{1}{2}|x-y|^2, \quad X\subset \mathbb{R}^d.}$

$f: E \to \mathbb{R}\cup\{+\infty\}$ a convex function, $a\in \mathrm{dom} f$.

$$\boxed{\partial f(a) = \{ l \in E^* \mid f(x) \geq f(a) + \langle l, x-a\rangle, \ \forall x\in E\}}$$

RK: • If $f$ is differentiable then $\partial f(a) = \{\nabla f(a)\}$.

• $l\in \partial f(a) \iff a \in \mathrm{ArgMin}\,(f-l)$.

• $0 \in \partial f(a) \iff a\in \mathrm{Arg\,Min}\, f$.

* Prop: $f$ convex l.s.c.

$$\boxed{l\in \partial f(a) \iff a\in \partial f^*(l) \iff f(a) + f^*(l) = \langle l, a\rangle.}$$

dem: $l\in \partial f(a) \iff \forall x\in E, \ \langle l, x\rangle - f(x) \leq \langle l, a\rangle - f(a)$.

$\iff f^*(l) \leq \langle l,a\rangle - f(a)$ ~~$\iff f^*(l) = \langle l,a\rangle - f(a)$~~. $\iff f^*(l) = \langle l,a\rangle - f(a)$.

$a\in\partial f^*(l) \iff \forall y\in E^*, \ f^*(y) \geq f^*(l) + \langle y-l, a\rangle$.

$$\iff \underbrace{\sup_{y\in E^*} \left(\langle y, a\rangle - f^*(y)\right)}_{f^{**}(a) = f(a)} \leq \langle l,a\rangle - f^*(l) \iff f(a) = \langle l, a\rangle - f^*(l).$$

* ~~Prop~~: $f$ convex continue in $a\in \mathrm{Int\,Dom} f$, $f: E\to \mathbb{R}\cup\{+\infty\}$

* Thm: $\partial f(a)$ is nonempty compact weakly $(\sigma(E^*, E))$

Proof: Hint: $g(x) := \begin{cases} 0, & \text{if } x = a \\ +\infty & \text{else} \end{cases}$ ?

$\boxed{\text{convex} \Rightarrow \text{locally Lipschitz around } a.}$

• $f$ convex in $a \Rightarrow f$ locally Lipschitz around $a$, $\partial f(a) \subset B_{E^*}(0,K)$.

$\forall x\in B(a,1)$, $\quad l(x-a) \leq f(x) - f(a)$.

$x = a+w \qquad l(w) \leq f(a+w) - f(a) \leq K\|w\| \leq K$.

$\forall w\in B(0,1) \qquad \|l\| = \sup_{w\in B(0,1)} l(w) \leq K$.

---

$$\underset{\pi\in\Pi(\mu,\nu)}{\mathrm{Min}} \int \frac{1}{2}|x-y|^2\, d\pi(x,y) = \sup\left\{ \int\varphi\, d\mu + \int \psi\, d\nu \,;\ \varphi(x) + \psi(y) \leq \frac{1}{2}|x-y|^2\right\}$$

• $\int \frac{1}{2}|x-y|^2\, d\pi(x,y) \leq \int |x|^2 + |y|^2\, d\pi(x,y) = \boxed{\int |x|^2\, d\mu + \int |y|^2\, d\nu =: M}$

$\mu,\nu\in \mathcal{P}_2(X) = \{\mu \mid \int |x|^2 d\mu < M\}$. $\qquad M := \int |x|^2 d\mu + \int |y|^2 d\nu$

$\cdot\ (\varphi,\psi)\in\Phi_c \Leftrightarrow \varphi(x)+\psi(y)\le \frac{1}{2}|x-y|^2$

$\Leftrightarrow x\cdot y \le \underbrace{\left(\frac{1}{2}|x|^2-\varphi(x)\right)}_{\tilde\varphi(x)} + \underbrace{\left(\frac{1}{2}|y|^2-\psi(y)\right)}_{\tilde\psi(y)}$

$\Leftrightarrow (\tilde\varphi,\tilde\psi)\in\tilde\Phi = \{(\tilde\varphi,\tilde\psi) \mid \tilde\varphi(x)+\tilde\psi(y)\ge xy\}.$

$$\operatorname*{Inf}_{\pi\in\Pi(\mu,\nu)} \int \frac{1}{2}|x-y|^2\,d\pi$$
$$\| $$
$$\frac{1}{2}M - \sup_{\pi\in\Pi(\mu,\nu)}\int_{X^2} xy\,d\pi(x,y).$$

$$\operatorname*{Sup}_{(\varphi,\psi)} J(\varphi,\psi) = \sup_{(\tilde\varphi,\tilde\psi)} \frac{1}{2}M - J(\tilde\varphi,\tilde\psi).$$

$$\sup_{\pi\in\Pi(\mu,\nu)}\int_{X^2} xy\,d\pi(x,y) = \operatorname*{Inf}_{\tilde\Phi} J(\tilde\varphi,\tilde\psi). \qquad // \quad \operatorname*{Inf}_{\varphi\in C_b} J(\varphi^{**},\varphi^*).$$

$$(\varphi,\psi)\in\tilde\Phi \Leftrightarrow \psi(y)\ge \sup_{x\in X}\langle x,y\rangle-\varphi(x)=\varphi^*(y).$$

Exercise

$\varphi(x)\ge\varphi^{**}(x)$ $\longrightarrow$ $\varphi^*(y)\ge\langle y,x\rangle-\varphi(x).$
$\rightarrow \varphi^{**}(x)\le\sup_y\{\langle y,x\rangle-\langle y,x\rangle+\varphi(x)\}=\varphi(x).$

$$J(\varphi,\psi)\ge J(\varphi,\varphi^*)\ge J(\varphi^{**},\varphi^*).$$

$$\operatorname*{Inf}_{\tilde\Phi} J(\varphi,\psi) = \operatorname{Inf} J(\varphi^{**},\varphi^*) \hookrightarrow (\varphi^{**},\varphi^*)\in\tilde\Phi.$$

* **Prop** Suppose $\mu$ & $\nu$ has a compact support
$\operatorname*{Inf}_{\varphi\in C_b(X)} J(\varphi^{**},\varphi^*)$ has a Min

dem: $n\in\mathbb{N}$, $\exists\varphi_n$, $\operatorname*{Inf}_{\varphi} J(\varphi^{**},\varphi^*)\le J(\varphi_n^{**},\varphi_n^*) < \operatorname*{Inf}_{\varphi} J(\varphi^{**},\varphi^*)+\frac{1}{n}.$

$\cdot\ \varphi_n^*(y_1)-\varphi_n^*(y_2)=\sup_{z\in X}\left(\langle y_1,z\rangle-\varphi_n(z)\right)-\sup_{z'\in X}\left(\langle y_2,z'\rangle-\varphi_n(z')\right)$

$\cdot\ \varepsilon>0,\ \exists z_\varepsilon\in X:\quad \le \langle y_1,z_\varepsilon\rangle-\cancel{\varphi_n(z_\varepsilon)}+\varepsilon-\langle y_2,z_\varepsilon\rangle+\cancel{\varphi_n(z_\varepsilon)}$

$\le \varepsilon+\|y_1-y_2\|\underbrace{\|z_\varepsilon\|}_{\le M} \le \varepsilon+\|y_1-y_2\|M$ $\qquad z_\varepsilon\in X$ compact.

$\cdot\ (\varphi_n^*)_n$ is equiLipschitz.

$\cdot\ (\varphi_n^{**})_n$ is equiLipschitz.

up to subsequence $\varphi_n^*\longrightarrow\varphi$ | Ex: $\psi=\varphi^*$. ?

By the Ascoli theorem $\varphi_n^{**}\longrightarrow\psi$ | $\psi^*=\varphi^{**}$.

* **Thm.**:

$c(x,y) = \frac{1}{2}|x-y|^2$, $\mu \in \mathcal{P}_2(X)$, $\nu \in \mathcal{P}_2(X)$.

① $\pi$ is an optimal for the (K) iff $\exists \varphi$ convex l.s.c. $X \longrightarrow \mathbb{R} \cup \{+\infty\}$.

$\text{Supp}\, \pi \subset \text{Graph}\,(\partial\varphi)$.

Moreover, $J(\varphi, \varphi^*) = \inf\{ J(\varphi, \psi) \mid \varphi(x) + \psi(y) \geq xy, \ \forall x, y\}$

② If $\mu$ is absolutely continuous w.r.t. the Lebesgue measure $\lambda$ ($\mu \ll \lambda$).

then $\pi =' (I \times \nabla\varphi) \# \mu$ is optimal for (K)

③ $\nabla\varphi$ is the unique solution of (M), $\nu = \nabla\varphi \# \mu$.

**Proof**: 1) Assume $\pi$ optimal & $\varphi, \varphi^*$ optimal for the dual problem

$$\int xy \, d\pi = \int \varphi(x) \, d\mu(x) + \int \varphi^*(y) \, d\nu(y) \qquad \pi \text{ a.e. } (x,y).$$

$$\int_{X^2} \underbrace{(\varphi(x) + \varphi^*(y) - x \cdot y)}_{\geq 0 \text{ for def } \varphi^*} d\pi(x,y) = 0 \Rightarrow \underline{\varphi(x) + \varphi^*(y) - x \cdot y = 0}$$

$$\Updownarrow$$

$$y \in \partial\varphi(x) \Leftrightarrow x \in \partial\varphi^*(y).$$

$$\Updownarrow$$

for $\pi$ almost every $(x,y) \in \text{supp}\,\pi$, $(x,y) \in \text{Graph}\,\partial\varphi$.

Conversely, suppose $\exists \varphi$ convex s.t. $\text{supp}\,\pi \subset \text{Graph}\,\partial\varphi$.

$$I(\pi) = \int_{X^2} xy \, d\pi = \int_{\text{Supp}\,\pi} xy \, d\pi = \int_{\text{Graph}\,\partial\varphi} xy \, d\pi = \int_{\text{Graph}\,\partial\varphi} \varphi(x) + \varphi^*(y) \, d\pi \geq \operatorname{Inf} J(\varphi, \varphi^*) = \operatorname{Max} I(\pi)$$

2) Let $\varphi$ be given by 1)

By Rademacher's theorem — Rademacher's theorem: Lipschitz continuity $\Rightarrow$ differentiable a.e.

$\varphi$ convex $\Rightarrow \varphi$ is locally Lipschitz $\Rightarrow \varphi$ is differentiable ($\lambda$) a.e.

$\Rightarrow \lambda$ a.e. $x$, $\partial\varphi(x) = \{\nabla\varphi(x)\}$

$\Rightarrow \mu$ a.e. $x$ $\partial\varphi(x) = \{\nabla\varphi(x)\}$.

$\Rightarrow$ for $\pi$ a.e. $(x,y) \in \text{Graph}\,\partial\varphi \Rightarrow y \in \partial\varphi(x)$, $\mu$. a.e.

$\Rightarrow (x,y) \in \text{Graph}\,\nabla\varphi$. □

* $\mathbb{R}^d$, $p \geq 1$, $W_p(\mu,\nu) = \left( \inf_{\pi \in \Pi(\mu,\nu)} \int |x-y|^p \, d\pi \right)^{\frac{1}{p}}$.

$$\boxed{\mathcal{P}_p(\mathbb{R}^d) = \{ \mu \in \mathcal{P}(\mathbb{R}^d) \mid \int |x|^p \, d\mu(x) < \infty \}.}$$

* $\boxed{\text{convex} \Rightarrow \text{locally Lipschitz} \Rightarrow \text{differentiable a.e.}}$

$\star$ **Thm:** $(\mu_n)_{n\geq 0}$ sequence in $\mathcal{P}_p(\mathbb{R}^d)$, $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $\exists M$, $\int |x|^p \, d\mu_n(x) \leq M$, $\forall n \geq 0$.

(i) $\mu_n \xrightarrow[n]{} \mu$.

$\Updownarrow$

(ii) $W_p(\mu_n, \mu) \xrightarrow[n]{} 0$.

$\mu_n \xrightarrow{n} \mu \Leftrightarrow W_p(\mu_n, \mu) \xrightarrow{n} 0$

**Proof:** $p=1$:

(ii) $\Rightarrow$ (i) $\quad W_1(\mu_n, \mu) \xrightarrow[n]{} 0$ We want to prove.

$$\forall \varphi \in C_b(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi \, d\mu_n \xrightarrow[n]{} \int_{\mathbb{R}^d} \varphi \, d\mu.$$

- if $\varphi \in Lip_1(X)$: $W_1(\mu_n, \mu) \geq \int \varphi(x) \, d(\mu_n - \mu) = \underbrace{\int \varphi \, d\mu_n - \int \varphi \, d\mu}_{\xrightarrow{n} 0}$

since $-\varphi \in Lip_1(X)$, $W_1(\mu_n, \mu) \geq \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu \right|$.

- $\varphi \in Lip_K(X)$.
- $\varphi \in C_b(X)$, $\quad \exists (\varphi_K)_K \in Lip(X) \nearrow_K \varphi$

$\exists (\psi_K)_K \in Lip(X) \searrow_K \varphi \qquad \varphi_K \leq \varphi \leq \psi_K$.

$$\liminf_n \int \varphi \, d\mu_n \geq \liminf_n \int \varphi_K \, d\mu_n = \int_X \varphi_K \, d\mu \xrightarrow[K]{} \int_X \varphi \, d\mu.$$

(i) $\Rightarrow$ (ii) Suppose moreover $\bigcup_n \operatorname{supp} \mu_n \cup \operatorname{supp} \mu \subset K$ compact.

$\mu_n \xrightarrow{n} \mu$.

$f_n(x) = \inf_{y \in X} f(y) + n\|x - y\|$

$f_n \in Lip$, $f_n(x) \leq f(x)$

$f_n \nearrow$

We want to prove

$$\sup_{\substack{u \in Lip_1(X) \\ u(a) = 0}} \int u \, d(\mu_n - \mu) = W_1(\mu_n, \mu) \xrightarrow[n]{} 0.$$

$\exists \varphi_n \in Lip_1(X)$, $\varphi_n(a) = 0$ $\qquad \int \varphi_n \, d(\mu_n - \mu)$ (equal to the sup above)

$\rightarrow$ Ascoli theorem, $\varphi_n \xrightarrow[n]{\|\cdot\|_\infty} \varphi$ up to a subsequence

$$\sup_{\substack{\varphi \in Lip_1(X) \\ \varphi(a) = 0}} \int \varphi \, d(\mu_n - \mu) = \int \varphi_n \, d(\mu_n - \mu) = \underbrace{\int \varphi \, d(\mu_n - \mu)}_{\downarrow n \; 0} + \int (\varphi_n - \varphi) \, d(\mu_n - \mu)$$

$$+ \underbrace{2\|\varphi_n - \varphi\|_\infty}_{\downarrow n \; 0}$$

$\mathcal{J}_2(\mathbb{R}^d)$, $W_2(\mu,\nu) = \sqrt{\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\, d\pi(x,y)}$

$$\begin{cases} X'(t) = f(X(t), u(t)) \\ \mu_0 \in \mathcal{J}_2(\mathbb{R}^d) \end{cases}$$

$$\boxed{V(t_0,\mu_0) = \inf_{u(\cdot)\in\mathcal{U}} \int_{\mathbb{R}^d} g(X_T^{t_0,x_0,u})\, d\mu_0(x_0)}$$

* <u>Prop</u> $\boxed{(t_0,\mu_0)\in[0,T]\times\mathcal{J}_2(\mathbb{R}^d) \longmapsto V(t_0,\mu_0) \text{ is Lipschitz continuous (with respect to } |\cdot,\cdot| + W_2(\cdot,\cdot))}$

* <u>Proof</u> <u>Lipschitz in $t$:</u> Fix $\mu_0\in\mathcal{J}_2(\mathbb{R}^d)$.

$t_1,t_2\in[0,T]$, $t_2>t_1$:

$$|X_T^{t_1,x_0,u} - X_T^{t_2,x_0,u}| = \left|\int_{t_1}^T f(X_s^{t_1,x_0,u})\,ds - \int_{t_2}^T f(X_s^{t_2,x_0,u})\,ds\right|$$

$$= \left|\int_{t_1}^{t_2} f(X_s^{t_1,x_0,u})\,ds\right| + \left|\int_{t_2}^T \Big(f(\underbrace{X_s^{t_1,x_0,u}}_{= X_s^{t_2, X_{t_2}^{t_1,x_0,u}, u}}) - f(X_s^{t_2,x_0,u})\Big)\right|$$

$$\le M|t_1-t_2| + \int_{t_2}^T K\, \underbrace{\left|X_s^{t_2, X_{t_2}^{t_1,x_0,u}, u} - X_s^{t_2,x_0,u}\right|}_{\le e^{KT}|X_{t_2}^{t_1,x_0,u} - x_0| \le (t_2-t_1)Me^{KT}}\, ds$$

$$\le M|t_1-t_2| + Ke^{KT}|t_1-t_2|T.$$

\+ $u$ is $\varepsilon$-optimal for $V(t_2,\mu_0)$

$$V(t_1,\mu_0) - V(t_2,\mu_0) \le \int_{\mathbb{R}^d} g(X_T^{t_1,x_0,u})\,d\mu_0 - \int_{\mathbb{R}^d} g(X_T^{t_2,x_0,u})\,d\mu_0(x_0) + \varepsilon$$

$$\le \int_{\mathbb{R}^d} K|X_T^{t_1,x_0,u} - X_T^{t_2,x_0,u}|\,d\mu_0(x_0) + \varepsilon$$

<u>Lipschitz in $\mu$:</u> $t_0$ fixed $\mu_1,\mu_2$

$u$ $\varepsilon$-optimal for $V(t_0,\mu_2)$

$$V(t_0,\mu_2) \le \int_{\mathbb{R}^d} g(X_T^{t_0,x_0,u})\,d\mu_2(x_0) < V(t_0,\mu_2) + \varepsilon$$

$$V(t_0,\mu_1) - V(t_0,\mu_2) < \int_{\mathbb{R}^d} g(X_T^{t_0,x_1,u})\,d\mu_1(x_1) - \int_{\mathbb{R}^d} g(X_T^{t_0,x_2,u})\,d\mu_2(x_2) + \varepsilon$$

Take $\pi\in\Pi_0(\mu_1,\mu_2)$ $$\le \int_{\mathbb{R}^{2d}} g(X_T^{t_0,x_1,u})\,d\pi(x_1,x_2) - \int_{\mathbb{R}^{2d}} g(X_T^{t_0,x_2,u})\,d\pi(x_1,x_2) + \varepsilon$$

$$\le \int_{\mathbb{R}^{2d}} K\,\underbrace{|X_T^{t_0,x_1,u} - X_T^{t_0,x_2,u}|}_{\le e^{KT}|x_1-x_2|}\,d\pi(x_1,x_2) + \varepsilon$$

$$\leq \varepsilon + \underbrace{\sqrt{\int_{\mathbb{R}^{2d}} |x_1 - x_2|^2 \, d\pi(x_1, x_2)}}_{W_2(\mu_1, \mu_2)} \sqrt{\int_{\mathbb{R}^{2d}} (Ke^{KT})^2 \, d\pi(x_1, x_2)}$$

$$|V(t_0, \mu_1) - V(t_0, \mu_2)| \leq Ke^{KT} W_2(\mu_1, \mu_2).$$

** Thm: Principe de Programation Dynamique*

$$\boxed{\begin{array}{l} 0 \leq t_0 \leq t_0 + h \leq T, \quad \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \\ V(t_0, \mu_0) = \inf\limits_{u(\cdot) \in \mathcal{U}} V(t_0 + h, X^{t_0, \cdot, u}_{t_0+h} \# \mu_0) \end{array}}$$

* *Proof*: Let $u(\cdot)$ given, Fix $\varepsilon > 0$, choose $v(\cdot) \in \mathcal{U}$ $\quad v$ : $\varepsilon$-optimal for $V(t_0+h, X^{t_0, \cdot, u}_{t_0+h} \# \mu_0)$.

$$V(t_0+h, X^{t_0, \cdot, u}_{t_0+h} \# \mu_0) \leq \int_{\mathbb{R}^d} g(X_T^{t_0+h, x_0, v}) \, d(X^{t_0, \cdot, u}_{t_0+h} \# \mu_0)(x_0)$$

$$\leq \varepsilon + V(t_0+h, X^{t_0, \cdot, u}_{t_0+h} \# \mu_0)$$

$$\int \psi \, d(\varphi \# \mu)(x) = \int \psi(\varphi(x)) \, d\mu(x).$$

$$V(t_0+h, X^{t_0, \cdot, \mu_0}_{t_0+h} \# \mu_0) \geq -\varepsilon + \int_{\mathbb{R}^d} g(X_T^{t_0+h, X^{t_0, y, u}_{t_0+h}, v}) \, d\mu_0(y)$$

$$W(t) = \begin{cases} u(t), & [t_0, t_0+h] \\ v(t), & [t_0+h, T] \end{cases} \qquad \geq -\varepsilon + \int_{\mathbb{R}^d} g(X_T^{t_0, y, w}) \, d\mu_0(y)$$

$$\geq -\varepsilon + V(t_0, \mu_0).$$

Let $\bar{u}$ be $\varepsilon$-optimal for $V(t_0, \mu_0)$

$$V(t_0, \mu_0) + \varepsilon \geq \int_{\mathbb{R}^d} g(\underbrace{X_T^{t_0, x_0, \bar{u}}}_{X_T^{t_0+h, X^{t_0, x_0, \bar{u}}_{t_0+h}, \bar{u}}}) \, d\mu_0(x_0)$$

$$= \int_{\mathbb{R}^d} g(X_T^{t_0+h, X^{t_0, x_0, \bar{u}}_{t_0+h}, \bar{u}}) \, d\mu_0(x_0)$$

$$= \int_{\mathbb{R}^d} g(X_T^{t_0+h, z, \bar{u}}) \, d(X^{t_0, \cdot, \bar{u}}_{t_0+h} \# \mu_0)(z)$$

$$\geq V(t_0+h, X^{t_0, \cdot, \bar{u}}_{t_0+h} \# \mu_0) \geq \inf_{u(\cdot) \in \mathcal{U}} V(t_0+h, X^{t_0, \cdot, u}_{t_0+h} \# \mu_0).$$

* **Lemma**: $\mu,\nu \in \mathcal{J}_2(\mathbb{R}^d)$, $\pi \in \Pi_0(\mu,\nu)$

(i) $\exists! p \in \mathbb{L}^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, $\forall \varphi \in \mathbb{L}^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$
$$\int_{\mathbb{R}^{2d}} \langle x-y, \varphi(x)\rangle \, d\pi(x,y) = \int_{\mathbb{R}^d} \langle p(x), \varphi(x)\rangle \, d\mu(x)$$

(ii) $\exists! q \in \mathbb{L}^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$, $\forall \psi \in \mathbb{L}^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$
$$\int_{\mathbb{R}^{2d}} \langle x-y, \psi(y)\rangle \, d\pi(x,y) = \int_{\mathbb{R}^d} \langle q(y), \psi(y)\rangle \, d\nu(y)$$

* **Proof**: (i)
$$L \left| \begin{array}{ccc} \mathbb{L}^2_\mu(\mathbb{R}^d,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \int_{\mathbb{R}^{2d}} \langle x-y, \varphi(x)\rangle \, d\pi(x,y) \end{array}\right.$$

prove: linear continuous.

$$|L(\varphi)| = \int_{\mathbb{R}^{2d}} |\langle x-y, \varphi(x)\rangle| \, d\pi(x,y) \le \underbrace{\sqrt{\int_{\mathbb{R}^{2d}} |x-y|^2 \, d\pi(x,y)}}_{W_2(\mu,\nu)} \underbrace{\sqrt{\int_{\mathbb{R}^{2d}} |\varphi(x)|^2 \, d\pi(x,y)}}_{\|\varphi\|_{\mathbb{L}^2_\mu}}$$

By Riesz representation theorem: $\exists! \, p \in \mathbb{L}^2_\mu$
$$\forall \varphi \in \mathbb{L}^2_\mu, \quad L(\varphi) = \langle \varphi, p\rangle_{\mathbb{L}^2_\mu}. \qquad \square$$

* **Goal**: To give a meaning to $\boxed{\dfrac{\partial V}{\partial t}(t,\mu) + H\left(\mu, \dfrac{\partial V}{\partial \mu}(t,\mu)\right) = 0}$

* **Def**:
$$H(\mu,p) := \inf_{u\in U} \int_{\mathbb{R}^d} \langle f(x,u), p(x)\rangle \, d\mu(x)$$
$\mu \in \mathcal{J}_2(\mathbb{R}^d)$, $p \in \mathbb{L}^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$

* **Prop**: $p \longmapsto H(\mu,p)$ is positively homogeneous
$$|H(\mu,p) - H(\nu,q)| \le K W_2^2(\mu,\nu)$$
for $(p,q) \in \mathbb{L}^2_\mu \times \mathbb{L}^2_\nu$ associated with $(\mu,\nu)$ by the lemma.

* **Proof**:
$$H(\mu,p) = \inf_{u\in U} \int_{\mathbb{R}^{2d}} \langle f(x,u), p(x)\rangle \, d\pi(x,y)$$
$$= \inf_{u\in U} \int_{\mathbb{R}^{2d}} \langle f(x,u), x-y\rangle \, d\pi(x,y)$$
$$= \inf_{u\in U} \int_{\mathbb{R}^{2d}} \langle f(y,u), x-y\rangle \, d\pi(x,y) + \int_{\mathbb{R}^{2d}} \langle f(x,u) - f(y,u), x-y\rangle \, d\pi(x,y)$$
$$\le \underbrace{\inf_{u\in U} \int_{\mathbb{R}^d} \langle f(y,u), q(y)\rangle \, d\nu(y)}_{H(\nu,q)} + \underbrace{K \int_{\mathbb{R}^{2d}} |x-y|^2 \, d\pi(x,y)}_{K(W_2(\mu,\nu))^2}.$$

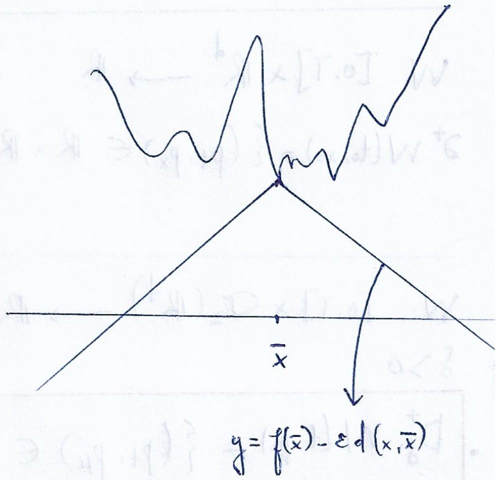
* **Theorem** (Ekeland)

$(X,d)$ a complete metric space.

$f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ (lower semi) continuous, bounded from below.

Let $\varepsilon > 0$, $x_0 \in \mathrm{Dom}\, f$.

$$\exists \bar{x} \in X : \begin{cases} i) & f(\bar{x}) + \varepsilon d(\bar{x}, x_0) \leq f(x_0) \\ ii) & \forall x \neq \bar{x},\ f(\bar{x}) < f(x) + \varepsilon d(x, \bar{x}). \end{cases}$$




* Proof: w.l.o.g. $f \geq 0$

$\forall x \in X$. Define

$$F(x) = \{ y \in X \mid f(y) + \varepsilon d(x,y) \leq f(x) \}$$

* $F(x) \neq \emptyset$ because $x \in F(x)$.
* $F(x)$ closed.
* $\boxed{\left.\begin{array}{l} y \in F(x) \\ z \in F(y) \end{array}\right\} \Rightarrow z \in F(x)}$

$y \in F(x)$, $\quad f(y) + \varepsilon d(x,y) \leq f(x)$

$z \in F(y)$, $\quad f(z) + \varepsilon d(y,z) \leq f(y)$.

$$f(z) + \varepsilon d(x,z) \leq f(z) + \varepsilon (d(x,y) + d(y,z)) \leq f(x).$$

Define $\quad g \,\Big|\, \begin{array}{l} \mathrm{Dom}\, f \longrightarrow \mathbb{R} \\ y \longmapsto g(y) = \inf\limits_{z \in F(y)} f(z). \end{array}$

$$\forall y \in F(x), \quad g(x) + \varepsilon d(x,y) \leq f(y) + \varepsilon d(x,y) \leq f(x).$$

$$y, y' \in F(x), \quad d(y,y') \leq d(y,x) + d(x,y') \leq \frac{2}{\varepsilon}(f(x) - g(x))$$

$$\boxed{\mathrm{diam}\, F(x) \leq \frac{2}{\varepsilon}(f(x) - g(x)).}$$

$x_0$.

$$\exists x_1 \in F(x_0) : \quad g(x_0) \leq f(x_1) < g(x_0) + 1.$$

$$\exists x_2 \in F(x_1) : \quad g(x_1) \leq f(x_2) < g(x_1) + \frac{1}{2}.$$

$$\vdots$$

$$\exists x_{n+1} \in F(x_n) : \quad g(x_n) \leq f(x_{n+1}) < g(x_n) + \frac{1}{2^n}.$$

$$\Rightarrow F(x_{n+1}) \subset F(x_n), \rightarrow g(x_n) \leq g(x_{n+1}).$$

$$g(x_{n+1}) \leq f(x_{n+1}) \leq g(x_n) + \frac{1}{2^n} \leq g(x_{n+1}) + \frac{1}{2^n}.$$

$$\Rightarrow \quad 0 \leq f(x_{n+1}) - g(x_{n+1}) \leq \frac{1}{2^n}.$$

$$\operatorname{diam} F(x_{n+1}) \le \frac{2}{\varepsilon}\left(f(x_{n+1}) - g(x_{n+1})\right) \le \frac{1}{\varepsilon 2^{n-1}}$$

$$\Rightarrow \bigcap_{n\ge 1} F(x_n) = \{\bar{x}\}.$$

- $\bar{x} \in F(x_n) \subset \cdots \subset F(x_0) \Rightarrow f(\bar{x}) + \varepsilon d(\bar{x}, x_0) \le f(x_0) \quad \to i)$
- $\bar{x} \in F(\bar{x}) \subset \bigcap_n F(x_n) = \{\bar{x}\} \Rightarrow F(\bar{x}) = \{\bar{x}\}.$
- $x \ne \bar{x}$ then $x \notin F(\bar{x}) \Rightarrow f(x) + \varepsilon d(\bar{x}, x) > f(\bar{x}).$ □

---

$W: [0,T]\times \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\partial^+ W(t_0,x_0) = \left\{ (p_t, p_x) \in \mathbb{R}\times\mathbb{R}^d \;\middle|\; \limsup_{\substack{t\to t_0\\ x\to x_0}} \frac{W(t,x) - W(t_0,x_0) - p_t(t-t_0) - \langle p_x, x-x_0\rangle}{|t-t_0| + |x-x_0|} \le 0 \right\}$$

---

$W: [0,T]\times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$

$\delta > 0$

- $$\boxed{D_\delta^+ W(t_0,\mu_0) = \left\{ (p_t, p_\mu) \in \mathbb{R}\times \mathbb{L}^2_\mu(\mathbb{R}^d,\mathbb{R}^d),\ \phi\in \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d),\ \limsup_{\substack{\|\phi\|_{L^2_\mu}\to 0\\ |t-t_0|\to 0}} \frac{W(t,(I+\phi)\#\mu_0) - W(t_0,\mu_0) - p_t(t-t_0) - \int_{\mathbb{R}^d} \langle p_\mu(x), \phi(x)\rangle d\mu_0(x)}{|t-t_0| + \|\phi\|_{L^2_\mu}} \le \delta \right\}}$$

- $$\boxed{D_\delta^- W(t_0,\mu_0) = \left\{ (p_t, p_\mu) \in \mathbb{R}\times L^2_\mu(\mathbb{R}^d,\mathbb{R}^d),\ \phi\in \mathcal{C}_b(\mathbb{R}^d,\mathbb{R}^d),\ \liminf_{\substack{\|\phi\|_{L^2_\mu}\to 0\\ |t-t_0|\to 0}} \frac{W(t,(I+\phi)\#\mu_0) - W(t_0,\mu_0) - p_t(t-t_0) - \int_{\mathbb{R}^d} \langle p_\mu(x), \phi(x)\rangle d\mu_0(x)}{|t-t_0| + \|\phi\|_{L^2_\mu}} \ge -\delta \right\}}$$

* **Def** $V: [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ a Lipschitz is a <u>viscosity solution</u> of (HJB)

(HJB) $\boxed{V_t + H(\mu, V_\mu) = 0}$

iff

$\forall\delta > 0,\ \forall (t,\mu)\in ]0,T[\times \mathcal{P}_2(\mathbb{R}^d),\ \forall (p_t,p_\mu)\in D_\delta^+ V(t,\mu)$

(SUB SOLUTION): $\quad p_t + H(\mu, p_\mu) \ge -C\delta$

$\forall\delta > 0,\ \forall (t,\mu)\in ]0,T[\times \mathcal{P}_2(\mathbb{R}^d),\ \forall (p_t,p_\mu)\in D_\delta^- V(t,\mu)$

(SUPER SOLUTION) $\quad p_t + H(\mu, p_\mu) \le +C\delta$

$$V(T,\mu) = \int_{\mathbb{R}^d} g(x)\, d\mu.$$

* Thm (Comparison)

$$\boxed{\begin{array}{l} W_1 \ \& \ W_2 \quad [0,T]\times \mathcal{P}_2(\mathbb{R}^d) \text{ are Lipschitz continuous bounded.} \\ W_1 \text{ SUB SOL} \\ W_2 \text{ SUPER SOL} \end{array} \Rightarrow \inf_{[0,T]\times\mathcal{P}_2(\mathbb{R}^d)} W_2 - W_1 = \inf_{\mathcal{P}_2(\mathbb{R}^d)} W_2(T,\cdot) - W_1(T,\cdot) =: A \overset{\text{wLOG.}}{=} 0}$$

* Corollary 1: $\boxed{W_2(T,\cdot) \equiv W_1(T,\cdot) \Rightarrow \forall (t,\mu),\ W_2(t,\mu) \geq W_1(t,\mu)}$

* Corollary 2: $W_1$ & $W_2$ are solution (HJB) + $W_i(T,\mu) = \int g(x)\,d\mu(x)$

$\Rightarrow W_1 \equiv W_2$ (uniqueness).

* Proof: By contradiction, suppose $\displaystyle -\xi := \inf_{[0,T]\times\mathcal{P}_2} W_2 - W_1 < 0.$

Consider $\varepsilon > 0$, $\eta > 0$, $\delta > 0$.

Take $(t_0,\mu_0)$:

$$(0): \quad \boxed{\begin{array}{l} \eta > 2C\delta + 2K(K+\delta)^2\varepsilon \\ \xi > 2K(K+\delta)\varepsilon + 2\eta T \end{array}}$$

$$W_2(t_0,\mu_0) - W_1(t_0,\mu_0) < -\frac{\xi}{2}.$$

$$\boxed{\phi(s,\mu,t,\nu) := -W_1(s,\mu) + W_2(t,\nu) + \frac{1}{\varepsilon}\left(W_2^2(\mu,\nu) + (t-s)^2\right) - \eta s}$$

By Ekeland thm, $\exists\, (\bar s, \bar t, \bar\mu, \bar\nu)$

E i) $\phi(\bar s, \bar\mu, \bar t, \bar\nu) \leq \phi(t_0,\mu_0,t_0,\mu_0)$

E ii) $\phi(\bar s,\bar\mu,\bar t,\bar\nu) \leq \phi(s,\mu,t,\nu) + \delta\left(\sqrt{W_2^2(\mu,\bar\mu) + (s-\bar s)^2} + \sqrt{W_2^2(\nu,\bar\nu) + (t-\bar t)^2}\right)$

$\forall (s,\mu,t,\nu)$

$\rho^2 = W_2^2(\bar\mu,\bar\nu) + (\bar t - \bar s)^2$

$(s,\mu,t,\nu) \longrightarrow (\bar s,\bar\mu,\bar s,\bar\mu)$

$$-\cancel{W_1(\bar s,\bar\mu)} + W_2(\bar t,\bar\nu) + \frac{1}{\varepsilon}\left(W_2^2(\bar\mu,\bar\nu) + (\bar t - \bar s)^2\right) - \cancel{\eta\bar s}$$

$$\leq -\cancel{W_1(\bar s,\bar\mu)} + W_2(\bar s,\bar\mu) - \cancel{\eta\bar s} + \delta(0 + \rho)$$

$$\frac{1}{\varepsilon}\rho^2 - \delta\rho \leq W_2(\bar s,\bar\mu) - W_2(\bar t,\bar\nu) \leq K\rho.$$

$$\sqrt{W_2^2(\bar\mu,\bar\nu) + (\bar t - \bar s)^2} = \rho \leq \varepsilon(K+\delta).$$

Let $p \in L^2_{\bar\mu}$ & $q \in L^2_{\bar\nu}$ given by the lemma, $\pi \in \Pi_0(\bar\mu,\bar\nu)$.

* Lemma: $\boxed{\begin{array}{l} \left(\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta, \frac{2}{\varepsilon}p\right) \in D^+_\delta W_1(\bar{s},\bar{\mu}) \\ \left(\frac{2}{\varepsilon}(\bar{s}-\bar{t}), \frac{2}{\varepsilon}q\right) \in D^-_\delta W_2(\bar{t},\bar{\nu}) \end{array}}$

$$\forall (s,\mu) : \quad \phi(\bar{s},\bar{\mu},\bar{t},\bar{\nu}) \le \phi(s,\mu,\bar{t},\bar{\nu}) + \delta\sqrt{W_2^2(\mu,\bar{\mu}) + (s-\bar{s})^2}.$$

$$W_1(s,\mu) \le W_1(\bar{s},\bar{\mu}) + \frac{1}{\varepsilon}\left(W_2^2(\mu,\bar{\nu}) - W_2^2(\bar{\mu},\bar{\nu}) + (s-\bar{t})^2 - (\bar{s}-\bar{t})^2\right) - \eta(s-\bar{s})$$
$$+ \delta\sqrt{W_2^2(\mu,\bar{\mu}) + (s-\bar{s})^2}.$$

$$\phi \in L^2_{\bar{\mu}}, \quad \mu = (I+\phi)\#\bar{\mu}, \quad \pi \in \Pi_o(\bar{\mu},\bar{\nu}).$$

Observe $\pi' := ((I+\phi), I)\#\pi \in \Pi(\mu,\bar{\nu})$

$$W_2^2(\mu,\bar{\nu}) - W_2^2(\bar{\mu},\bar{\nu}) \le \int_{\mathbb{R}^{2d}} |x'-y'|^2\, d\pi'(x',y') - \int_{\mathbb{R}^{2d}} |x-y|^2\, d\pi(x,y)$$

$$= \int_{\mathbb{R}^{2d}} |x+\phi(x)-y|^2 - |x-y|^2\, d\pi(x,y) = 2\int \langle x-y, \phi(x)\rangle\, d\pi(x,y) + \int |\phi(x)|^2\, d\pi(x,y)$$
$$= 2\int \langle p(x), \phi(x)\rangle\, d\bar{\mu}(x) + \|\phi\|^2_{L^2_{\bar{\mu}}}.$$

$$\frac{W_1(s,(I+\phi)\#\bar{\mu}) - W_1(\bar{s},\bar{\mu}) - \int_{\mathbb{R}^d}\langle \frac{2}{\varepsilon}p, \phi\rangle\, d\bar{\mu} - \left(\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta\right)(s-\bar{s})}{\sqrt{|s-\bar{s}|^2 + \|\phi\|^2_{L^2_{\bar{\mu}}}}}$$

$$\le \frac{-\frac{1}{\varepsilon}(\bar{s}-s)^2 + \frac{1}{\varepsilon}\|\phi\|^2_{L^2_{\bar{\mu}}} + \delta\sqrt{W_2^2(\mu,\bar{\mu}) + (s-\bar{s})^2}}{\sqrt{|s-\bar{s}|^2 + \|\phi\|^2_{L^2_{\bar{\mu}}}}}$$

$$\le \frac{-\frac{1}{\varepsilon}(\bar{s}-s)^2 + \frac{1}{\varepsilon}\|\phi\|^2_{L^2_{\bar{\mu}}} + \delta\sqrt{\|\phi\|^2_{L^2_{\bar{\mu}}} + (s-\bar{s})^2}}{\sqrt{|s-\bar{s}|^2 + \|\phi\|^2_{L^2_{\bar{\mu}}}}}$$

since $W_2^2(\mu,\bar{\mu}) \le \int |x-y|^2\, d(I+\phi, I)\#\bar{\mu}(x,y) = \int |x-\phi(x)-x|^2\, d\bar{\mu}(x) = \|\phi\|^2_{L^2_{\bar{\mu}}}$.

$W_1$ subsolution: $\quad \frac{2}{\varepsilon}(\bar{s}-\bar{t}) - \eta + H(\bar{\mu}, \frac{2}{\varepsilon}p) \ge -C\delta$

$W_2$ supersolution: $\quad -\frac{2}{\varepsilon}(\bar{s}-\bar{t}) - H(\bar{\nu}, \frac{2}{\varepsilon}q) \ge -C\delta$,

$$-\eta + \frac{2K}{\varepsilon}W_2^2(\bar{\mu},\bar{\nu}) \ge -\eta + \frac{2}{\varepsilon}\left(H(\bar{\mu},p) - H(\bar{\nu},q)\right) \ge -2C\delta$$

$$\Rightarrow \quad 2K\varepsilon(K+\delta)^2 \ge K W_2^2(\bar{\mu},\bar{\nu}) \ge -2C\delta + \eta.$$

$\longrightarrow W_2^2(\bar{\mu},\bar{\nu}) \le \varepsilon^2(K+\delta)^2.$

A contradiction with (0).

$(\bar t,\bar s) \notin \{0,T\} \longrightarrow$ <u>Classical</u> : Def of viscosity $[0,T[$

$\xrightarrow{\text{by } i)}$ $\bar s, \bar t \notin \{T\}$, $\phi(\bar s,\bar\mu,\bar t,\bar v) \le -\frac{\xi}{2} - \eta t_0 \le -\frac{\xi}{2}$.

Suppose by contradiction that $\bar s = T$.

$$-\frac{\xi}{2} \ge -W_1(T,\bar\mu) + W_2(\bar t,\bar v) + \frac{1}{\varepsilon}\rho^2 - \eta T$$

$$\ge \underbrace{-W_1(T,\bar\mu) + W_2(T,\bar\mu)}_{\ge 0} \underbrace{- W_2(T,\bar\mu) + W_2(\bar t,\bar v)}_{\ge -K\rho} + \frac{1}{\varepsilon}\rho^2 - \eta T$$

$$-K\rho + \frac{1}{\varepsilon}\rho^2 - \eta T$$

$$\xi \le 2\eta T + 2K\rho - \frac{2}{\varepsilon}\rho^2 \le 2\eta T + 2K\rho \le 2\eta T + 2K\varepsilon(K+\delta),$$ contradiction. □

* <u>Prop</u>

$$\boxed{\begin{array}{c}\text{V is the unique bounded, Lipschitz continuous viscosity solution of (HJB)}\\ V(T,\mu) = \int_{\mathbb{R}} \bar g(x)\,d\mu(x), \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d)\end{array}}$$

* <u>Proof</u>: $(t_0,\mu_0)$, $\delta>0$, $(p_t, p_\mu) \in D^+_\delta V(t_0,\mu_0)$.

$$0 \underset{\text{DPP}}{\le} V(t_0+h, X^{t_0,\cdot,u}_{t_0+h} \# \mu_0) - V(t_0,\mu_0)$$

<u>Subsolution</u>

Take $u(\cdot) = u \in U$ constant control.

$$X^{t_0,x_0,u}_{t_0+h} = x_0 + \int_{t_0}^{t_0+h} f(X^{t_0,x_0,u}_s, \mu(s))\,ds = x_0 + hf(x_0,u) + h\varepsilon(h)$$

$$x_0 \longrightarrow \phi(x_0) = \int_{t_0}^{t_0+h} f(X^{t_0,x_0,u}_s, \mu(s))\,ds \qquad \phi_h(x_0) := \int_{t_0}^{t_0+h} f(X^{t_0,x_0,u}_s, \mu(s))\,ds.$$

$$V(t_0+h, X^{t_0,\cdot,u}_{t_0+h}\#\mu_0) - V(t_0,\mu_0) - p_t h - \int_{\mathbb{R}^d} \left\langle p_{\mu_0}(x_0), \int_{t_0}^{t_0+h} f(X^{t_0,x_0,u}_s, \mu(s))\,ds \right\rangle d\mu_0(x_0)$$

$$\le \left\{ \delta\left(\sqrt{h^2 + \|\phi\|^2_{L^2}} + \varepsilon\sqrt{h^2 + \|\phi\|^2_{L^2}}\right) \cancel{\sqrt{h^2+\|\phi\|^2_{L^2}}} \right\} \qquad \frac{\|\phi_h\|^2_{L^2_\mu}}{h^2} \le M.$$

$$+ \quad 0 \le \{\} + p_t h + \int_{\mathbb{R}^d}\int_{t_0}^{t_0+h} \langle p_{\mu_0}(x_0), \underbrace{f(X^{t_0,x_0,u}_s, u)}_{f(x_0,u) + h\varepsilon(h)} \rangle\, d\mu_0(x_0)\,ds$$

$$+ \quad 0 \le \frac{\{\}}{h} + p_t h + h\int_{\mathbb{R}^d} \langle p_{\mu_0}(x_0), f(x_0,u)\rangle\, d\mu_0(x) + h^2\varepsilon(h).$$

$$0 \le M\delta + p_t + \int_{\mathbb{R}^d} \langle p_{\mu_0}(x_0), f(x_0,u)\rangle\, d\mu_0(x).$$

$$\Rightarrow \quad 0 \le M\delta + p_t + \inf_{u\in U} \int_{\mathbb{R}^d} \langle p_{\mu_0}(x_0), f(x_0,u)\rangle\, d\mu_0(x) \quad . \; □$$

Consider $\left| \begin{array}{l} X'(s) = f(X(s)) \\ X(0) = x_0 \end{array} \right.$ , $\underset{C^1}{f : \mathbb{R}^d \to \mathbb{R}^d}$ flow $x_0 \longrightarrow X_t^{x_0}$.

$A \in \mathbb{R}^d$, $\mathrm{Vol}(A)$.

$$\boxed{\mathrm{Vol}(X_t^A) \overset{?}{=} \mathrm{Vol}(A) + t\int_A \mathrm{div} f(x) dx + t\varepsilon(t)}$$

Prove:

$$\mu(X_t^A) = \mu(A) + t\int_A \mathrm{div}\, f(x)\, d\mu(x) + t\varepsilon(t).$$

$$\frac{d}{dt}\mu(X_t^A)\Big|_{t=0} = \int_A \mathrm{div} f(x)\, d\mu(x).$$

+ $f: \mathbb{R}^2 \to \mathbb{R}^2$

$y = X_t(x)$

$$dy = (\mathrm{Jac}(X_t)(x))\, dx = \begin{vmatrix} 1 + t\frac{\partial f_1}{\partial x_1} & t\frac{\partial f_1}{\partial x_2} \\ t\frac{\partial f_2}{\partial x_1} & 1 + t\frac{\partial f_2}{\partial t_2} \end{vmatrix} + t^2 \dots dx,$$

$$\mu(X_t(A)) = \int_{X_t(A)} d\mu = \int_A \left[\left(1 + t\frac{\partial f_1}{\partial x_1}\right)\left(1 + t\frac{\partial f_2}{\partial x_2}\right) - t^2\frac{\partial f_2}{\partial x_1}\cdot\frac{\partial f_1}{\partial x_2} + t^2\dots\right] d\mu(x)$$

$$= \mu(A) + t\int \underbrace{\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}}_{\mathrm{div} f} d\mu(x) + t^2\dots \qquad \square$$

---

⊛ How to model for that all the agents is not necessarily the same

$$u(t) \longrightarrow u(t,x)$$

If $(\mu_t)_{0\le t\le 1} \subset \mathcal{P}_2(\mathbb{R}^d) \to \boxed{\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0}$ **(CE)**

$v_t(x) \in F(x)$ for $\lambda\otimes\mu_t$ almost all $(t,x) \in [0,1]\times\mathbb{R}^d$.

* Heuristic Calculation

If $\forall x \in \mathbb{R}^d: v(x) \in F(x)$ & $x \mapsto v(x)$ is Lipschitz,

$\left\{ \begin{array}{l} \dot{x}(t) = v(x(t)) \\ X(0) = x_0 \end{array} \right.$ has a unique solution $X_t^{x_0}$.

If $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, define $\boxed{\forall t:\ \mu_t := X_t^{\cdot}\#\mu_0.}$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\mu_t(\varphi) := \int_{\mathbb{R}^d}\varphi\, d\mu_t = \int_{\mathbb{R}^d}\varphi(X_t^x)\, d\mu_0(x) \approx \int_{\mathbb{R}^d}\varphi(x + tv(x))\, d\mu_0(x)$$

$$\approx \int_{\mathbb{R}^d}\varphi(x) + t\nabla\varphi(x)v(x)\, d\mu_0(x) = \mu_0(\varphi) - t\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{div}(v(x)\mu_0(x))\, dx$$

$$\to \lim_{t\to 0^+} \frac{1}{t}\left(\mu_t(\varphi) - \mu_0(\varphi)\right) = -\int_{\mathbb{R}^d} \varphi(x)\, \mathrm{div}\left(v(x)\,\mu_0(x)\right) dx$$

**Def:**

Let $E$ be a vector measure $\in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$

$\mathrm{div}(E)$ is defined by

$$\forall \varphi \in C_b^\infty(\mathbb{R}^d, \mathbb{R}): \int_{\mathbb{R}^d} \varphi(x)\,(\mathrm{div}\, E)(dx) = -\int_{\mathbb{R}^d} \nabla\varphi(x) \cdot dE(x)$$

* **Hypo**: $v : (t,x) \in [0,T] \times \mathbb{R}^d \longrightarrow v_t(x) \in \mathbb{R}^d$ Borel measurable such that:

$$\boxed{(C_v): \int_0^T \int_{\mathbb{R}^d} |v_t(x)|\, d\mu_t(x)\, dt < +\infty}$$

**Def:** $\begin{array}{l} t \longmapsto \mu_t \\ [0,T] \longmapsto \mathcal{T}_2(\mathbb{R}^d) \end{array}$ is a solution of (CE) iff:

$(\alpha)$ $t \longrightarrow \mu_t$ is continuous

$(\beta)$ $\forall \varphi \in C_c^\infty(]0,T[ \times \mathbb{R}^d, \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}^d} \left\{ \frac{\partial \varphi}{\partial t}(t,x) + v_t(x) \cdot \nabla_x \varphi(t,x) \right\} d\mu_t(x)\, dt = 0$$

$(\gamma)$ $\mu_t|_{t=0} = \mu_0$

* **Prop:** $(\mu_t)_{t\in[0,T]}$ sol (CE) $\Leftrightarrow$ $(\alpha), (\gamma)$: $\forall \psi \in C_c^\infty(\mathbb{R}^d)$:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \psi(x)\, d\mu_t(x) = +\int_{\mathbb{R}^d} v_t(x) \cdot \nabla_x \varphi(t,x)\, d\mu_t(x)$$

$\nabla\psi(x)$

a.e. $t \in [0,T]$

(CE) $\quad \boxed{\begin{array}{l} \partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0 \\ \mu_t|_{t=0} = \mu_0 \\ v_t(x) \in F(x) \end{array}}$ (CE)

* **Proof:** "$\Rightarrow$" Take $\varphi(t,x) = \underset{C_c^\infty(\mathbb{R}^d,\mathbb{R})}{\psi(x)}\, \underset{C_c^\infty(]0,T[,\mathbb{R})}{\eta(t)}$

$$0 = \int_0^T \int_{\mathbb{R}^d} \psi(x)\eta'(t) + v_t(x) \cdot \nabla\psi(x)\,\eta(t)\, d\mu_t(x)\, dt$$

$$= \underbrace{\int_0^T \eta'(t) \int_{\mathbb{R}^d} \psi(x)\, d\mu_t(x)\, dt}_{= -\int_0^T \eta(t) \frac{d}{dt}\left(\int_{\mathbb{R}^d} \psi(x)\, d\mu_t(x)\right) dt} + \int_0^T \eta(t) \int_{\mathbb{R}^d} v_t(x) \cdot \nabla\psi(x)\, d\mu_t(x)\, dt$$

"$\Leftarrow$" We have the result by $\varphi(t,x) = \psi(x)\,\eta(t)$

* $\left\{ \psi\eta \;\middle|\; \begin{array}{l} \psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}) \\ \eta \in C_c^\infty((0,T); \mathbb{R}) \end{array} \right\}$ is dense in $C_c^1((0,T)\times\mathbb{R}^d; \mathbb{R})$.

Hypo $(C_{v_t})$: $\boxed{\int_0^T \left( \sup_{\mathbb{R}^d} |v_t| + \mathrm{Lip}(v_t(\cdot)) \right) dt < +\infty}$

* Prop

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ & since then for all $t \in [0,T]$: $\mu_t := X_t^\cdot \# \mu_0$ or

$$t \longmapsto X_t^{x_0} \;\middle|\; \begin{array}{l} x'(s) = v_s(X(s)),\ s \in [0,T] \\ X(0) = x_0 \end{array}$$

so $\begin{array}{l} t \longmapsto \mu_t \\ [0,T] \longrightarrow \mathcal{P}_2(\mathbb{R}^d) \end{array}$ is sol (CE).

* Proof $s_n \to t$. We want $\mu_{s_n} \longrightarrow \mu_t$.

* $\varphi \in C_b^c(\mathbb{R}^d)$: $\quad \mu_{s_n}(\varphi) := \int_{\mathbb{R}^d} \varphi \, d\mu_{s_n} = \int_{\mathbb{R}^d} \underbrace{\varphi(X_{s_n}^x)}_{\forall x \in \mathbb{R}^d \ \downarrow n} d\mu_0(x)$

$$\to \int_{\mathbb{R}^d} \varphi(X_t^x) d\mu_0(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu_t(x).$$

* $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^d, \mathbb{R})$, Define $t \longrightarrow \psi_t(x) := \varphi(t, X_t^x)$

for a.e. $t \in [0,T]$: $\dfrac{d\psi_t(x)}{dt} = \dfrac{\partial\varphi}{\partial t}(t, X_t^x) + \nabla_x \varphi(t, X_t^x)\, v_t(X_t^x)$

$$\Rightarrow \int_0^t \int_{\mathbb{R}^d} \left| \frac{d}{dt} \psi_t(x) \right| d\mu_t\, dt \le T\, \mathrm{Lip}\,\varphi + \mathrm{Lip}\,\varphi \int_0^T \int_{\mathbb{R}^d} \underbrace{|v_t(X_t^x)|}_{\le \sup_{\mathbb{R}^d} |v_t|} d\mu_0(x)\, dt$$

We compute:

$$0 = \mu_T(\varphi(T,\cdot)) - \mu_0(\varphi(0,\cdot)) = \int_{\mathbb{R}^d} (\underbrace{\varphi(T, X_T^x)}_{\psi_T(x)} - \underbrace{\varphi(0,x)}_{\psi_0(x)}) d\mu_0(x)$$

$$= \int_{\mathbb{R}^d} \int_0^T \frac{d\psi_t}{dt}(x)\, dt\, d\mu_0(x) \overset{\text{Fubini}}{=} \int_0^T \int_{\mathbb{R}^d} \left\{ \frac{\partial\varphi}{\partial t}(t, X_t^x) + \nabla_x \varphi(t, X_t^x) v_t(X_t^x) \right\} d\mu_0(x)\, dt.$$

* **Prop** Let $v$ verify $(C_v^+)$.

Consider $(\mu_t)_{t\in[0,T]}$ a sol of (CE) iff $\mu_t = X_t^{\cdot} \# \mu_0$, $\forall t \in [0,T]$.

$X_t$ is the flow of $\begin{cases} X'(s) = v_s(X(s)) \\ X(0) = x_0 \end{cases}$

* **Proof:** It is enough to prove the uniqueness

$$\mu_t := X_t^{\cdot} \# \mu_0.$$

$(\tilde\mu_t)_{t\in[0,T]}$ another sol of (CE)

$$\sigma_t := \mu_t - \tilde\mu_t \ : \text{signed measure.}$$

We will prove : $\sigma_t \leq 0$.

Let $\psi \in C_0^\infty(]0,T]\times\mathbb{R}^d)$. We suppose $0 \leq \psi \leq 1$.

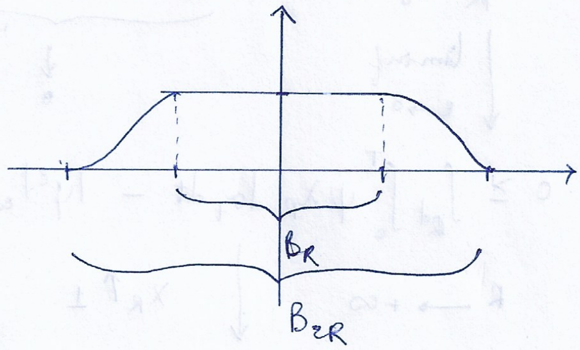
$R>0$:

Consider $\chi_R : \mathbb{R}^d \to \mathbb{R}$ s.t.

$0 \leq \chi_R \leq 1$

$\chi_R \equiv 1$ on $B_R$ $\qquad |\nabla \chi_R| \leq \frac{2}{R}$

$\equiv 0$ on $\mathbb{R}^d \setminus B_{2R}$




$$W_t(\cdot) := \begin{cases} v_t(\cdot) & \text{in } [0,T]\times B_{2R} \\ 0 & \text{else} \end{cases}$$

We consider : $W_t^\varepsilon(x) =$ mollification of $W$

$$W_t^\varepsilon(x) = \rho_\varepsilon(\cdot,\cdot) * W(\cdot)(t,x)$$

$$\sup_{\varepsilon\in]0,1[} \left( \int_0^t \left( \sup_{\mathbb{R}^d} |W_t^\varepsilon| + \mathrm{Lip}(W_t^\varepsilon) \right) dt \right) < +\infty$$

* **Method of characteristic functions**

$\varphi \in C_b^1(]0,T[\times\mathbb{R}^d)$

$\varphi_T \in C_b^1(\mathbb{R}^d)$

$$\boxed{\begin{cases} \partial_t \varphi(t,x) + v_t \cdot \nabla_x \varphi(t,x) = \psi & \text{on } ]0,T[\times\mathbb{R}^d \\ \varphi(x,T) = \varphi_T(x) & \text{on } \mathbb{R}^d \end{cases}}$$

$$\Rightarrow \varphi(t,x) := \varphi_T(X_T^{t,x}) - \int_t^T \psi(s, X_s^{t,x})\,ds.$$

$X_s^{t,x}$ sol of $\begin{cases} X'(s) = v_s(X(s)), & [t,T] \\ X(t) = x. \end{cases}$

Let $\varphi_\varepsilon \in C^1(\mathbb{R}^d \times [0,T])$ the sol of

$$\begin{cases} \partial_t \varphi_\varepsilon(t,x) + W_t^\varepsilon(x) \cdot \nabla_x \varphi(t,x) = \psi, & ]0,T[ \times \mathbb{R}^d \\ \varphi_\varepsilon(x,T) = 0 \end{cases}$$

$\Rightarrow \varphi_\varepsilon(t,x) = \int_t^T \psi(s, X_s^{x,t})\, ds$ where $X_s^{x,t}$ is the flow $W_t^\varepsilon$.

$0 \geq \varphi_\varepsilon \geq -T$, $|\nabla \varphi_\varepsilon|$ is bounded independently of $\varepsilon$ on $t$.

- $(t,x) \longrightarrow \varphi^\varepsilon(t,x)\chi_R(x)$ as a test function.

$$0 = \int_{\mathbb{R}^d} \varphi_\varepsilon(T,x)\chi_R(x)\, d(\mu_0 - \tilde{\mu}_0)(x) = \int_{\mathbb{R}^d}\int_0^T \underbrace{\left(\chi_R \frac{\partial \varphi^\varepsilon}{\partial t} + \left\langle v_t, \chi_R \nabla\varphi^\varepsilon + \varphi^\varepsilon \nabla \chi_R \right\rangle\right)}_{\frac{d}{dt}\left(\chi_R \varphi^\varepsilon(t, X_t^{s,x})\right)} d\sigma_t\, dt$$

$$= \int_{\mathbb{R}^d}\int_0^T \psi\chi_R + \underbrace{\left\langle v_t - w_t^\varepsilon, \chi_R \nabla\varphi^\varepsilon \right\rangle}_{\downarrow\, 0} d\sigma_t\, dt + \int_0^T\int_{\mathbb{R}^d} \varphi^\varepsilon \nabla\chi_R\, v_t\, d\sigma_t\, dt$$

$\downarrow \liminf_{\varepsilon \to 0}$

$$\Rightarrow 0 \geq \int_{\mathbb{R}^d}\int_0^T \psi\chi_R\, d\sigma_t\, dt - |\varphi^\varepsilon|_\infty \left|\frac{2}{R}\right| \int_0^T\int_{\mathbb{R}^d} |v_t|\, d\sigma_t\, dt.$$

$R \longrightarrow +\infty \quad \downarrow \quad \chi_R \uparrow 1.$

$$\Rightarrow \quad 0 \geq \int_{\mathbb{R}^d}\int_0^T \psi\, d\sigma_t\, dt = \int_0^T \left(\int_{\mathbb{R}^d} \psi\, d\sigma_t\right) dt \Rightarrow \sigma_t \leq 0$$

because $0 \leq \psi \leq 1$ arbitrary

---

$\forall A \subset \mathbb{R}^d$. $\sigma_t(A) \leq 0$

$\psi(t,x) = \eta(t)\varphi(x)$ $\qquad 0 \leq \eta \leq 1$, $0 \leq \varphi \leq 1$

$$\Rightarrow 0 \geq \int_0^T \eta(t)\left(\int_{\mathbb{R}^d} \varphi(x)\, d\sigma_t(x)\right) dt$$

$$\Rightarrow \int_{\mathbb{R}^d} \varphi(x)\, d\sigma_t(x) \leq 0 \text{ a.e. } t \in [0,T] \Rightarrow \sigma_t \leq 0$$

$$\boxed{\begin{aligned} \sigma_t \leq 0 &\Leftrightarrow \forall \varphi \geq 0 : \sigma_t(\varphi) = 0 \\ &\Leftrightarrow \forall A: \quad \sigma_t(A) = 0 \end{aligned}}$$

* **Thm: (Super Solution Principle)**

Let $t \in [0,T] \longrightarrow \mu_t$ be continuous s.t. $\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) \, dt < +\infty$

If $(\mu_t)_{t \in [0,T]}$ is a sol of (CE), then: $\exists \eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$ such that:

i) $\operatorname{supp} \eta \subset B := \left\{ (x,\gamma) \in \mathbb{R}^d \times W^{1,1}[0,T] : \begin{cases} \gamma'(s) = v_s(\gamma(s)) \text{ a.e. } s \in [0,T] \\ \gamma(0) = x \end{cases} \right\}$

ii) $\mu_t := e_t \# \eta$ $\left( \text{i.e. } \forall \varphi \in C_b(\mathbb{R}^d) : \int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d \times \Gamma_T} \varphi(\gamma(t)) \, d\eta(x,\gamma) \right)$

* **Notation** $\Gamma_T := C([0,T], \mathbb{R}^d)$

$$e_t : \mathbb{R}^d \times \Gamma_T \longrightarrow \mathbb{R}^d$$
$$(x,\gamma) \longmapsto \gamma(t)$$

$e_t(x,\gamma) = \gamma(t)$

**Conversely**, if $\eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$ s.t. $\begin{cases} \operatorname{supp} \eta \subset B \\ \int_0^T \int_{\mathbb{R}^d \times \Gamma_T} |v_t(\gamma(t))|^2 \, d\eta(x,\gamma) \, dt < +\infty. \end{cases}$

then $\mu_t := e_t \# \eta \quad \forall t$

$(\mu_t)_{t \in [0,T]}$ is a sol of (CE) with $\mu_0 = e_0 \# \eta$.

**Proof**: (2) $\Rightarrow$ (1): Let $\eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$

$$F := \left\{ (t,x,\gamma) \in [0,T] \times \mathbb{R}^d \times \Gamma_T \text{ s.t. } \begin{cases} \nexists \gamma'(t) \\ \text{or } \gamma'(t) \neq v_t(\gamma(t)) \end{cases} \text{ is a } \lambda \otimes \eta \text{ negligeable} \right\}$$

. $t \longmapsto e_t \# \eta$ is continuous $e_s \# \eta$

Indeed, $s \to t$ : $\int_{\mathbb{R}^d} \varphi \, d(e_s \# \eta) = \int_{\mathbb{R}^d} \varphi(\gamma(s)) \, d\eta(x,\gamma)$

$\varphi \in C_b(\mathbb{R}^d)$ $\quad \Big\downarrow s \to t$

$\forall \gamma \in \Gamma_T : \varphi(\gamma(s)) \xrightarrow[s \to t]{} \varphi(\gamma(t))$ $\quad \int_{\mathbb{R}^d \times \Gamma_T} \varphi(\gamma(t)) \, d\eta(x,\gamma) = \int_{\mathbb{R}^d} \varphi \, d(e_t \# \eta)$

. $t \longmapsto \int_{\mathbb{R}^d} \xi \, d\mu_t$ is absolutely continuous.

$\xi \in C_c^\infty(\mathbb{R}^d)$

$s, t \in [0,T]$, $\mu_t = e_t \# \eta$.

$$\left| \int_{\mathbb{R}^d} \xi \, d\mu_s - \int \xi \, d\mu_t \right| = \left| \int_{\mathbb{R}^d \times \Gamma_T} \xi(\gamma(s)) - \xi(\gamma(t)) \, d\eta(x,\gamma) \right|$$

$$= \int_{\mathbb{R}^d \times \Gamma_T} \int_s^t \nabla \xi(\gamma(\tau)) \cdot \underbrace{\gamma'(\tau)}_{v_\tau(\gamma(\tau))} \, d\tau \, d\eta(x,\gamma)$$

$$\leq \|\nabla \xi\|_\infty \int_{\mathbb{R}^d \times \Gamma_T} \int_s^t |v_\tau(\gamma(\tau))| \, d\tau \, d\eta(x,\gamma)$$

$$\leq \|\nabla \xi\|_\infty \int_s^t \int_{\mathbb{R}^d \times \Gamma_T} 1 \cdot |v_\tau(\gamma(\tau))| \, d\eta(x,\gamma) \, d\tau$$

$$\leq \|\nabla \xi\|_\infty \int_s^t \sqrt{\int_{\mathbb{R}^d \times \Gamma_T} |v_\tau(\gamma(\tau))|^2 \, d\eta(x,\gamma)} \, d\tau \quad \left( \int_{A \subset [0,T]} M(\tau) d\tau \xrightarrow[|A| \to 0]{} 0 \right)$$

$$\frac{d}{dt} \int_{\mathbb{R}^d} \xi \, d\mu_t = \frac{d}{dt} \int_{\mathbb{R}^d \times \Gamma_T} \xi(\gamma(t)) \, d\eta(x,\gamma) = \int_{\mathbb{R}^d \times \Gamma_T} \nabla \xi(\gamma(t)) \cdot v_t(\gamma(t)) \, d\eta(x,\gamma)$$

$$= \int_{\mathbb{R}^d} \nabla \xi(x) \cdot v_t(x) \, d\mu_t(x)$$

* Approximation lemma $\varepsilon > 0$. $(\mu_t)_t$ sol of (CE)

$v_t$ satisfies : $\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) dt < +\infty$

$P_\varepsilon(\cdot) \in C^\infty(\mathbb{R}^d)$ s.t. $P_\varepsilon > 0$.

$$\left( P_\varepsilon(x) := \frac{1}{\sqrt{(2\pi\varepsilon)^d}} e^{-\frac{|x|^2}{2\varepsilon}} \right)$$

$$\mu_t^\varepsilon := \mu_t * P_\varepsilon \qquad \left( \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t^\varepsilon(x) = \int_{\mathbb{R}^d} \varphi(x) P_\varepsilon(x-y) \, d\mu_t(y) \, dx \right)$$

$$E_t^\varepsilon := (v_t \mu_t) * P_\varepsilon.$$

$$v_t^\varepsilon := \frac{E_t^\varepsilon}{\mu_t^\varepsilon} \quad \text{(RN derivative)}$$

then $\partial_t \mu_t^\varepsilon + \operatorname{div}(v_t^\varepsilon \mu_t^\varepsilon) = 0$

a) $\int_{\mathbb{R}^d} |v_t^\varepsilon(x)|^2 \, d\mu_t^\varepsilon(x) \leq \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x)$.

b) $E_t^\varepsilon \xrightarrow[\varepsilon \to 0]{} v_t \mu_t$

c) $\lim_{\varepsilon \to 0^+} \|v_t^\varepsilon\|_{L^2(\mu_t^\varepsilon)} = \|v_t\|_{L^2(\mu_t)}$

* Proof: We know that : $\mu_t^\varepsilon = X_t^\varepsilon \# \mu_0^\varepsilon$, where $X_s^{\varepsilon\,\prime} = v_s(X_s^\varepsilon)$.

& $\mu_0^\varepsilon = \mu_0 * P_\varepsilon$.

Define : $\eta_\varepsilon := (I \times X_\cdot^{\cdot,\varepsilon}) \# \mu_0^\varepsilon \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$

$$X_\cdot^{\cdot,\varepsilon} : \mathbb{R}^d \longrightarrow \Gamma_T$$
$$x \longmapsto X_\cdot^{x,\varepsilon}$$

$$I \times X_\cdot^{\cdot,\varepsilon} : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \Gamma_T.$$

The rest of the proof consists in showing:

- $(\eta_\varepsilon)_{\varepsilon \in ]0,1]}$ is tight.

  By Prokhorov thm: $\exists\, \varepsilon_n \to 0$, $\eta_{\varepsilon_n} \xrightarrow[n]{} \eta$

- $\eta$ satisfied the thm.

$$\pi^1 : \begin{array}{l} \mathbb{R}^d \times \Gamma_T \longrightarrow \mathbb{R}^d \\ (x,\gamma) \longmapsto x \end{array} \qquad \cdot\ (\eta_\varepsilon)_{\varepsilon>0} \text{ is tight} \iff \begin{cases} (\pi_1 \# \eta_\varepsilon)_{\varepsilon>0} \text{ is tight } \& \\ (\pi_2 \# \eta_\varepsilon)_{\varepsilon>0} \text{ is tight.} \end{cases}$$

$$\pi^2 : \begin{array}{l} \mathbb{R}^d \times \Gamma_T \longrightarrow \Gamma_T \\ (x,\gamma) \longmapsto \gamma(\cdot) - x \end{array}$$

$\oplus$ $(\pi_1 \# \eta_\varepsilon)_{\varepsilon>0} = (\mu_0^\varepsilon)_{\varepsilon>0} = (\mu_0 * \rho_\varepsilon)_{\varepsilon>0}$ tight

$\exists\, \delta>0$, $\exists\, K_\delta$ compact s.t. $\mu_0(K_\delta^c) < \delta$

$$\uparrow$$

$$\mu_0 * \rho_\varepsilon (K_\delta^c) < 2\delta$$

$\oplus$ $(\pi_2 \# \eta_\varepsilon)_{\varepsilon>0}$ is tight.

$$\int_{\Gamma_T} \int_0^T |\gamma'(t)|^2 \,dt\, d(\pi_2 \# \eta_\varepsilon)(\gamma) = \int_{\mathbb{R}^d} \int_0^T |X_t^{\cdot,\varepsilon} - x|^2 \,dt\, d\mu_0^\varepsilon(x) \le \int_{\mathbb{R}^d} \int_0^T |v_t^\varepsilon(x)|^2 \,d\mu_\varepsilon(x)\, dt$$

$c \in \mathbb{R}$: $A_c := \Big\{ \gamma(\cdot) - x \in \Gamma_T : \underbrace{\| \int_0^T |\gamma'(s)|^2 ds}_{\varphi(\gamma) :=} < c \Big\}$ compact set by Ascoli.

$$\sup_{\varepsilon>0} \int \varphi(\gamma)\, d(\pi^2 \# \eta^\varepsilon)(\gamma) < 0.$$

| |
|---|
| $\star$ **Lemma** $X$ metric, $K \in \mathcal{P}(X)$. <br> $\varphi : X \to \mathbb{R}^d \cup \{+\infty\}$ l.s.c. $\begin{cases} (a)\ \forall \varepsilon > 0, \{x \in X : \varphi(x) \le \varepsilon\} \text{ compact} \\ (b)\ \sup_{\mu \in K} \varphi(\mu) < \infty \end{cases}$ <br> $\iff$ $K$ is tight. |

__Proof:__ "$\Rightarrow$", $\forall n \in \mathbb{N}^*$, $K_n = \{x \in X \mid \varphi(x) \le n\}$ compact

Define $c = \sup_{\mu \in K} \varphi(\mu) < \infty$.

$$\mu(K_n^c) = \mu(\varphi > n) \underset{\substack{\uparrow \\ \text{Markov}}}{\le} \frac{1}{n} \int_X \varphi\, d\mu \le \frac{c}{n}, \ \forall \mu \in K.$$

"$\Leftarrow$" $\forall n$, $\exists K_n^c$, $\mu(K_n^c) < \frac{1}{n}$, $\forall \mu \in K$, where $(K_n)_n$ is an increasing family of compact.

$\varphi(x) = \inf\{n \geq 0 \mid x \in K_n\}$ , $\forall c \in \mathbb{R}, \exists n \in \mathbb{N}, n \leq c < n+1.$

$\{x \in X \mid \varphi(x) \leq c\} \subset \{x \in X \mid \varphi(x) \leq n+1\} = K_{n+1}$ compact.

$$\boxed{\begin{cases} \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \\ v_t(x) \in F(x), \quad \mu_t \text{ a.e. } x \in \mathbb{R} \\ \mu_t|_{t=0} = \mu_0 \end{cases}}$$

$(C_v)$ $\int_0^T \int_{\mathbb{R}^d} |v_t(x)| \, d\mu(x) \, dt < \infty$

$t \longrightarrow (\mu_t)_t$

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Def: 1) $\begin{vmatrix} t \longmapsto \mu_t \\ [0,T] \longrightarrow \mathcal{P}_2(\mathbb{R}^d) \end{vmatrix}$ Absolutely continuous

$\forall \varphi \in C_c^{\infty}(]0,T[ \times \mathbb{R}^d)$, $\int_0^T \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial t}(t,x) + \langle v_t(x), \nabla_x \varphi(t,x) \rangle \, d\mu_t(x) \, dt = 0$

Def 2) $t \longrightarrow (\mu_t)$ AC

$\forall \psi \in C_c^{\infty}(\mathbb{R}^d)$, $\frac{d}{dt} \int_{\mathbb{R}^d} \psi(x) \, d\mu_t(x) = \int_{\mathbb{R}^d} \langle v_t, \nabla \psi(x) \rangle \, d\mu_t(x)$

for a.e. $t \in [0,T]$.

Def 1) $\Rightarrow$ Def 2) $\varphi(t,x) = \eta(t) \psi(x)$.

$\rightarrow$ $v_t(\cdot)$ is regular enough (Lipschitz).

$(\mu_t)_{t \in [0,T]}$ solves (CE) iff $\mu_t = X_t^{\cdot} \# \mu_0$.

where $t \longrightarrow X_t^{x_0}$ is sol of $\begin{cases} X'(s) = v_s(X(s)) \\ X(0) = x_0 \end{cases}$

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* Thm (Superposition thm)

$(\mu_t)_{t \in [0,T]}$ is a sol of (CE) $\Leftrightarrow$ $\exists \eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$, $\Gamma_T = C([0,T], \mathbb{R}^d)$

i) $\operatorname{supp} \eta \subset \left\{(x,\gamma) \in \mathbb{R}^d \times AC([0,T]) \;\middle|\; \begin{cases} \gamma'(s) \in F(\gamma(s)) \text{ for a.e. } s \in [0,T] \\ \gamma(0) = x \end{cases} \right\}$

ii) $\mu_t = e_t \# \eta$, $e_t : \begin{vmatrix} \mathbb{R}^d \times \Gamma_T \longrightarrow \mathbb{R}^d \\ (x, \gamma(\cdot)) \longrightarrow \gamma(t) \end{vmatrix}$

$\Rightarrow$ : 
$$\left.\begin{aligned} \mu_t^{\varepsilon} &= \mu_t * \rho_\varepsilon \\ E_t^{\varepsilon} &= (v_t \mu_t) * \rho_\varepsilon \\ v_t^{\varepsilon} &= \frac{E_t^{\varepsilon}}{\mu_t^{\varepsilon}} \end{aligned}\right\} \text{ Approximation lemma}$$

$$\eta^{\varepsilon} := (I \times \underbrace{X_{\cdot}^{\cdot \varepsilon}}_{v_t^{\varepsilon}(\cdot)}) \# \mu_0^{\varepsilon} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$$

$\pi^1: (x,\gamma) \longrightarrow x$

$\pi^2: (x,\gamma) \longrightarrow \gamma - x \in \Gamma_T$

<u>Exercise:</u> $\left.\begin{array}{l}(\pi^1 \# \eta^\varepsilon)_\varepsilon \text{ tight} \\ (\pi^2 \# \eta^\varepsilon)_{\varepsilon>0} \text{ tight}\end{array}\right\} \Leftrightarrow (\eta_\varepsilon)_{\varepsilon>0} \text{ tight.}$

---

$$\int_{\Gamma_T} \int_0^T |\gamma'(t)|^2 \, dt \, d(\pi^2 \# \eta^\varepsilon)(\gamma) < C < +\infty$$

$$\varphi: \left| \begin{array}{l} \Gamma_T \longrightarrow \mathbb{R}^+ \cup \{+\infty\} \\ \gamma \longrightarrow \begin{cases} \int_0^T |\gamma'(t)|^2 \, dt & \text{if } \gamma \text{ is AC} \\ +\infty & \text{else.} \end{cases} \end{array}\right.$$

- <u>$\forall c > 0$</u> $\quad A_c = \{\gamma(1) - \gamma(0) \in \Gamma_T \mid \int_0^T |\gamma'(s)|^2 \, ds \leq C\}$

- $\sup_{\varepsilon>0} \int_\gamma \varphi(\gamma) \, d(\pi^2 \# \eta^\varepsilon)(\gamma) < C.$

$$\boxed{V(t_0, \mu_0) = \inf_{\begin{cases}(\mu_t)_t \text{ sol (CE)} \\ \mu_t|_{t=t_0} = \mu_0\end{cases}} G(\mu_T)}$$

$\rightarrow$ $V$ is regular (Lipschitz)

$\rightarrow$ $V$ DPP

$\rightarrow$ HJB satisfied by $V$

$\rightarrow$ $V$ is the unique sol of the HJB.

* <u>Prop</u> <u>(Gronwall type lemma)</u>

$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $(\mu_t)_{t \in [0,T]} \in \mathcal{P}(\mu) \leftarrow$ (sol of (CE) starting from $\mu$ at $t = 0$).

there exist $(\nu_t)_{t \in [0,T]} \in \mathcal{P}(\nu)$ s.t.

$$W_2(\mu_t, \nu_t) \leq C\, W_2(\mu, \nu), \; \forall t \in [0,T]$$

* <u>Proof:</u> By superposition thm, $\exists \eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_t)$, $\mu_t = e_t \# \eta$ a.e. $t \in [0,T]$.

We disintegrate the measure $\eta$ w.r.t. $\mu$, $\eta = \mu \otimes \eta_x$ (product measure).

$(\eta_x)_{x \in \mathbb{R}^d} \in \mathcal{P}(\Gamma_T)$, $\quad \forall \varphi \in C_b(\mathbb{R}^d \times \Gamma_T)$, $\int_{\mathbb{R}^d \times \Gamma_T} \varphi \, d\eta = \int_{\mathbb{R}^d} \int_{\Gamma_T} \varphi(x,\gamma) \, d\eta_x(\gamma) \, d\mu(x).$

$x \longrightarrow \eta_x$ is Borel

Let $\pi \in \Pi(\mu, \nu)$, $\quad W_2^2(\mu,\nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x,y).$

Let us define $\bar{\pi} := \pi \otimes \eta_x \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \times \Gamma_T).$

For every $(x,\gamma) \in \mathbb{R}^d \times \Gamma_T$. Define

$$H(x,\gamma) := \left\{ u \in L^1([0,T], U),\ \gamma(t) = x + \int_0^t f(\gamma(s), \mu(s)) \, ds,\ \forall t \in [0,T] \right\}.$$

* <u>Rk:</u> if $(x,\gamma) \in \operatorname{supp} \eta$ then $H(x,\gamma) \neq \emptyset.$

So there exists $\left|\begin{array}{l} \operatorname{supp}\eta \longrightarrow u(\cdot) \\ (x,\gamma) \longrightarrow u_{x,\gamma} \in H(x,\gamma) \end{array}\right.$ which is Borel measurable (not proved).

$y \in \mathbb{R}^d$, $\hat{\gamma} \in \Gamma_T$, $\hat{\gamma}'(s) \in F(\hat{\gamma}(s))$ for a.e $s \in [0,T]$

$\tau(y,\hat{\gamma}) \in \Gamma_T$ is the unique solution of $\left|\begin{array}{l} \gamma'(s) = f(\gamma(s), u_{\hat{\gamma}(0)}, \hat{\gamma}(s)) \\ \gamma(0) = y. \end{array}\right.$

Gronwall $x_1, x_2 \in \mathbb{R}^d$

$|X_t^{t_0,x_1,u} - X_t^{t_0,x_2,u}| \leq e^{(\mathrm{Lip} f)T} |x_1 - x_2|$

Rk:
- $\tau(y,\hat{\gamma})(0) = y$.
- $\frac{d}{dt}\tau(y,\hat{\gamma})(t) \in F(\tau(y,\hat{\gamma})(t))$ a.e.
- $|\tau(y,\hat{\gamma})(t) - \hat{\gamma}(t)| \leq \exp(\mathrm{Lip} f \cdot t)|y - \hat{\gamma}(0)|$

* Def of $(\mu_t)_{t\in[0,T]}$: for all $t \in [0,T]$, $\nu_t$ is given,

$$\forall \varphi \in C_0(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi \, d\nu_t = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} \varphi \circ e_t(y, \tau(y,\gamma)) \, d\bar{\pi}(x,y,\gamma)$$

or equivalently $g_t \left|\begin{array}{l} \mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T \longrightarrow \mathbb{R}^d \\ (x,y,\gamma) \longrightarrow e_t(y,\tau(y,\gamma)) \end{array}\right.$ $\boxed{\nu_t = g_t \# \bar{\pi}.}$

What is $\nu_0$? $\int_{\mathbb{R}^d} \varphi \, d\nu_0 = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} \varphi \circ e_0(y,\tau(y,\gamma))\, d\bar{\pi}(x,y,\gamma)$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\Gamma_T} \varphi(y)\, d\eta_x(\gamma)\, d\pi(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y)\, d\pi(x,y)$$

$$= \int_{\mathbb{R}^d} \varphi(y)\, d\nu(y).$$

$$\mu_t = e_t \# \eta = e_t \# (\mu \otimes \eta_x) = \boxed{e_t(y,\gamma) \# \bar{\pi} = \mu_t}$$

$(x,y,\gamma) \longrightarrow e_t(y,\gamma)$

$$W_2^2(\mu_t,\nu_t) \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y| \, d\pi(x,y), \quad \forall \pi \in \Pi(\mu_t,\nu_t)$$

$$\leq \int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} |\gamma(t) - \tau(y,\gamma)(t)|^2 \, d\bar{\pi}(x,y,\gamma).$$

$$\leq \int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} |\gamma(0) - y|^2 e^{2(\mathrm{Lip} f)(t)} \, d\bar{\pi}(x,y,\gamma)$$

$$\leq e^{2(\mathrm{Lip} f)t} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} |x-y|^2 \, d\bar{\pi}(x,y,\gamma)$$

$$= e^{2(\mathrm{Lip} f)t} \times \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, d\pi(x,y)$$

$$= e^{2(\mathrm{Lip} f)t} W_2^2(\mu,\nu).$$

* Prop $\boxed{(t,\mu) \longrightarrow V(t,\mu) \text{ is Lipschitz continuous}}$ $\exists (\mu_t)_{t\in[0,T]} \in \mathcal{P}(\mu)$

* Proof Proof of $\mu \longrightarrow V(0,\mu)$ is Lipschitz, $V(0,\mu) \le G(\mu_T) < V(0,\mu)+\varepsilon$

Fix $\varepsilon>0$, $\exists (\nu_t) \in S(\nu)$, $W_2(\mu_t,\nu_t) \le c\, W_2(\mu,\nu)$.

$$
\begin{aligned}
V(0,\nu) - V(0,\mu) &< V(0,\nu) - G(\mu_T) + \varepsilon \\
&\le G(\nu_T) - G(\mu_T) + \varepsilon \le \mathrm{Lip}(G)\big(W_2(\nu_T,\mu_T)\big) + \varepsilon \\
&\le (\mathrm{Lip}\, G)\, c\, W_2(\mu,\nu).
\end{aligned}
$$

QED.

* Prop DPP $s,\mu \in [0,T]\times \mathcal{P}_2(\mathbb{R}^d)$,

$$\boxed{\tau \in [0,T], \quad V(s,\mu) = \inf_{(\mu_t)_t \in S(s,\mu)} V(\tau,\mu_\tau).}$$

Proof: $\boxed{\ge}$ $\displaystyle \inf_{(\mu_t)\in S} V(\tau,\mu_\tau) \le V(\tau,\bar\mu_\tau) < \inf_{(\mu_t)_t\in S(s,\mu)} V(\tau,\mu_\tau) + \varepsilon$

$$V(\tau,\bar\mu_\tau) < G(\tilde\mu_T) < V(\tau,\bar\mu_\tau) + \varepsilon, \qquad \exists (\tilde\mu_t)_t \in S(\tau,\bar\mu_\tau)$$

$$\hat\mu_t := \begin{cases} \bar\mu_t & \text{if } t\in[s,\tau] \\ \tilde\mu_t & \text{if } t\in[\tau,T] \end{cases} \in S(s,\mu)$$

$$\inf_{(\mu_t)_t\in S(s,\mu)} V(\tau,\mu_\tau) + 2\varepsilon \ge V(\tau,\bar\mu_\tau) + \varepsilon \ge G(\hat\mu_T) = G(\tilde\mu_T) \ge V(s,\mu).$$

$$\forall (s,\mu_t)_{t\in[s,T]}, \quad V(s,\mu) \le G(\mu_T^{s,\mu}) = G\left(\mu_T^{\tau,\mu_\tau^{s,\mu}}\right)$$

$$(\mu_t)_{t\in[s,T]} = (\mu_t)_{t\in[s,\tau]} \odot (\mu_t)_{t\in[\tau,T]}$$

$$V(s,\mu) \le \inf_{\substack{(\mu_t)_{t\in[\tau,T]} \\ \in S(\tau,\mu_\tau)}} G(\mu_T^{\tau,\mu_\tau})$$

$$
\begin{array}{ccl}
V(s,\mu) \le & G\left(\bar\mu_T^{s,\mu}\right) & \le V(s,\mu) + \varepsilon \\
& \| & \\
\displaystyle\inf_{\mu\in S(\tau,\bar\mu_\tau)} G\left(\mu_T^{\tau,\bar\mu_\tau}\right) \le & G\Big(\underbrace{\bar\mu_T^{\tau,\bar\mu_\tau^{s,\mu}}}_{\in S(\tau,\bar\mu_\tau)}\Big) &
\end{array}
$$

$$\begin{cases} \dfrac{\partial}{\partial t} V(t,\mu) + \mathcal{H}\left(\mu, \dfrac{\partial V}{\partial \mu}(t,\mu)\right) = 0 \\ V(T,\mu) = G(\mu) \end{cases}$$

$$\mathcal{H}(\mu, p) = \inf_{\substack{v(\cdot) \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d) \\ v(x) \in F(x) \text{ for } \mu \text{ a.e. } x \in \mathbb{R}^d}} \int_{\mathbb{R}^d} v(x)\, p(x)\, d\mu(x)$$

$p \in L^2_\mu(\mathbb{R}^d)$

optimal displacement

* **Def**: $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$

$p$ is an optimal displacement from $\mu$.

iff $\quad W_2(\mu, (I-p)\#\mu) \leq \int |x - (x - p(x))|^2 d\mu(x) = \int |p(x)|^2 d\mu(x)$

iff $\quad \exists \nu \in \mathcal{P}_2(\mathbb{R}^d), \ \forall \pi \in \Pi_0(\mu, \nu)$

$$\forall \phi \in L^2(\mu), \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \phi(x), x - y \rangle d\pi(x,y) = \int \langle \phi(x), p(x) \rangle d\mu(x).$$

(i) $\Rightarrow$ (ii) $\quad \pi := (I \times (I-p))\#\mu \in \Pi_0(\mu, (I-p)\#\mu)$

**Def**: $W : [0,T] \times \mathcal{J}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ Lipschitz bounded

$(\bar{t}, \bar{\mu}) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$

$\delta > 0$: $\quad (p_t, p_\mu) \in \mathbb{R} \times L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$

$(p_t, p_\mu) \in D^+_\delta W(\bar{t}, \bar{\mu})$ iff

a) $p_\mu$ is an optimal displacement from $\bar{\mu}$.

b)
$$\frac{W(t,\nu) - W(\bar{t}, \bar{\mu}) - p_t(t - \bar{t}) - \int_{\mathbb{R}} \langle p_\mu(x), y - x \rangle d\pi(x,y)}{\sqrt{|t - \bar{t}|^2 + W_2^2(\nu, \bar{\mu})}}$$

$$\leq \delta + \varepsilon \sqrt{|t - \bar{t}|^2 + W_2^2(\nu, \bar{\mu})}$$

$\forall (t, \nu), \quad \forall \pi \in \Pi_0(\bar{\mu}, \nu)$

Similarly, $\quad (p_t, p_\mu) \in D^-_\delta W(\bar{t}, \bar{\mu})$

iff $\quad (p_t, p_\mu) \in -D^+_\delta(-W)(\bar{t}, \bar{\mu})$.

**RK**: $\forall (\mu, \nu), \quad |H(\mu, \lambda p_{\mu,\nu}) - H(\nu, \lambda q_{\mu,\nu})| \leq \lambda K W_2^2(\mu, \nu)$

- $p_{\mu,\nu}$ is such that for $\pi \in \Pi_0(\mu, \nu)$

$$\forall \varphi \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d), \quad \int \langle \varphi(x), x - y \rangle d\pi(x,y) = \int \langle \varphi(x), p_{\mu,\nu}(x) \rangle d\mu(x).$$

- W is a subsolution iff $\exists C\ \forall (t,\mu)\ \forall \delta$
$$\forall (p_t, p_\mu) \in D^+_\delta W(t,\mu)$$
$$p_t + H(\mu, p_\mu) \geq -C\delta$$

* Thm (Comparison)

$W_1, W_2$ Lipschitz bounded $[0,T] \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$

such that $\left| \begin{array}{l} W_1 \text{ viscosity subsolution of (HJB)} \\ W_2 \text{ viscosity supersolution of (HJB)} \end{array} \right.$

$$W_1(T,\mu) = W_2(T,\mu) = G(\mu),\ \forall \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

then $W_1(t,\mu) \leq W_2(t,\mu),\ \forall (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$.

* Corollary: $\boxed{\text{Uniqueness of solution to (HJB)}}$

* Prop: $\boxed{\text{The value function is a solution of (HJB)}}$

* Thm $\boxed{\text{Value function is the unique viscosity solution.}}$

Proof: Assume by contradiction $0 > -\xi = \inf\limits_{[0,T]\times \mathbb{R}^d} W_2 - W_1$

$\exists (t_0, \mu_0)$, $\quad W_2(t_0,\mu_0) - W_1(t_0,\mu_0) < -\frac{\xi}{2}$

$t_0 \neq 0$
$t_0 \neq T$

$\sigma > 0$, $\quad \frac{2\sigma}{t_0} < \frac{\xi}{8}$.

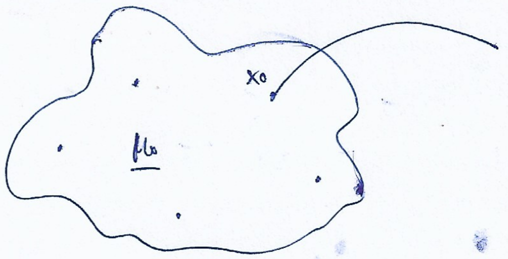
$$\phi_{\varepsilon\eta}(s,\mu,t,\nu) = \begin{cases} W_2(t,\nu) - W_1(s,\mu) + \frac{1}{2\varepsilon}\left(|s-t|^2 + W_2^2(\mu,\nu)\right) - \eta s + \frac{\sigma}{s} + \frac{\sigma}{t} & \text{if } st \neq 0 \\ +\infty & \text{else} \end{cases}$$

By Ekeland's principle: $\exists z_{\varepsilon\eta\delta} = (s_{\varepsilon\eta_\varepsilon}, \mu_{\varepsilon\eta_\varepsilon}, t_{\varepsilon\eta_\varepsilon}, \nu_{\varepsilon\eta_\varepsilon})$, $z_0 = (t_0, \mu_0, t_0, \nu_0)$

$$\left| \begin{array}{l} \phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) + \delta d(z_0, z_{\varepsilon\eta\delta}) \leq \phi_{\varepsilon\eta}(z_0) \\ \phi_\varepsilon(z_{\varepsilon\eta\delta}) \leq \phi(z) + \delta d(z, z_{\varepsilon\eta\delta}) \end{array} \right.$$

$\forall z = (s,\mu,t,\nu) \in [0,T]^2 \times \mathcal{P}_2^2(\mathbb{R}^d)$

$$d((s,\mu,t,\nu),(s',\mu',t',\nu')) = \sqrt{|s-s'|^2 + W_2^2(\mu,\mu')} + \sqrt{|t-t'|^2 + W_2^2(\nu,\nu')}$$




$$\left|\begin{array}{l} x'(t) = v_t(x(t)) \\ x(0) = x_0 \end{array}\right. , \quad v_t(\cdot) \in F(\cdot)$$

$$\left|\begin{array}{l} X'(t) \in F(\mu_t, X(t)) \\ X(0) = x_0 . \end{array}\right.$$

$$(CE) \quad \begin{cases} \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \\ v_t(x) \in F(\mu_t, x) \qquad \mu_t \quad \text{a.e. } x \in \mathbb{R}^d \end{cases}$$

F Lipschitz in both $x$ & $\mu_t$.

* <u>Thm</u>

$\exists (\mu_t)_{t\in[0,T]}$ of (CE),

$\exists \eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$,

$$\mu_t = e_t \# \eta$$

$$\operatorname{supp} \eta = \left\{ (x,\gamma) \,\middle|\, \begin{array}{l} \gamma'(t) \in F(e_t \# \eta, \gamma(t)) \quad \text{a.e. } t \in [0,T] \\ \gamma(0) = x \end{array} \right\}$$

$\mu_0$ :

$$\left|\begin{array}{l} \mu_t + \operatorname{div}(v_t \mu_t) = 0 \\ v_t(x) \in F(\mu_0, x) \quad \text{a.e. } x \in \mathbb{R}^d \\ \mu|_{t=0} = \mu_0 \end{array}\right. \longrightarrow \eta_0, \ (\mu_t^0)_{t>0}$$

$$\left|\begin{array}{l} \mu_t + \operatorname{div}(v_t \mu_t) = 0 \\ v_t(x) \in F(\mu_0^t, x) \quad \mu_1^t \text{ a.e. } x \in \mathbb{R}^d \end{array}\right. \longrightarrow \eta_1, \ (\mu_t^1)_{t\geq 0}$$

$$\left| v_t(x) \in F(\mu_n^t, x), \quad \mu_{n+1}^t \text{ a.e. } x \in \mathbb{R}^d \right. \longrightarrow \eta_{n+1}, \ (\mu_t^{n+1})_{t\geq 0}$$

$(\eta_n)_{n\geq 0}$ is tight, $\eta_n \longrightarrow \eta$ up to subsequences.



*<u>How to model for that all the agents are not the same</u>.

$$(CE) \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$