

Machine Learning & Deep Learning – Học Máy & Học Sâu

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Tóm tắt nội dung

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PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/machine_learning/NQBH_machine_learning.pdf.

TeX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/machine_learning/NQBH_machine_learning.tex.

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Question 1. Compare recurrent neural network with recursive neural network.

Community – Cộng đồng.

1. FRANCIS BACH.
 - Website: <https://francisbach.com/>.
 - ENS Homepage: <https://www.di.ens.fr/~fbach/>.
2. PHẠM HY HIẾU.
3. PHẠM TUẤN HUY.
4. VÔ VĂN HUY.
5. YAN LECUN.
6. NGUYỄN PHAN MINH.

1 Journal

1.1 Journal of Machine Learning Research [JMLR]

- Website: <https://www.jmlr.org/>.
Journal of Machine Learning Research [JMLR], established in 2000, provides an international forum for electronic & paper publication of high-quality scholarly articles in all areas of ML. All published papers are freely available online.
JMLR has a commitment to rigorous yet rapid reviewing. Final versions are published electronically (ISSN 1533-7928) immediately upon receipt.

1.2 Machine Learning

Netherlands. Subject area & category: Computer Science [AI, Software]. SJR 2023: 1.72.

Overview. Machine Learning is an international forum focusing on computational approaches to learning.

- Reports substantive results on a wide range of learning methods applied to various learning problems.
- Provides robust support through empirical studies, theoretical analysis, or comparison to psychological phenomena.
- Demonstrates how to apply learning methods to solve significant application problems.
- Improves how ML research is conducted.
- Prioritizes verifiable & replicable supporting evidence in all published papers.

Aims & scope. *Machine Learning* is an international forum for research on computational approaches to learning. Journal publishes articles reporting substantive results on a wide range of learning methods applied to a variety of learning problems, including but not limited to:

- **Learning Problems.** Classification, regression, recognition, & prediction; Problem solving & planning; Reasoning & inference; Data mining; Web mining; Scientific discovery; Information retrieval; Natural language processing; Design & diagnosis; Vision & speech perception; Robotics & control; Combinatorial optimization; Game playing; Industrial, financial, & scientific applications of all kinds.

- **Learning Methods.** Supervised & unsupervised learning methods (including learning decision & regression trees, rules, connectionist networks, probabilistic networks & other statistical models, inductive logic programming, case-based methods, ensemble methods, clustering, etc.); Reinforcement learning; Evolution-based methods; Explanation-based learning; Analogical learning methods; Automated knowledge acquisition; Learning from instruction; Visualization of patterns in data; Learning in integrated architectures; Multistrategy learning; Multi-agent learning.

Papers describe research on problems & methods, applications research, & issues of research methodology. Papers making claims about learning problems (e.g., inherent complexity) or methods (e.g., relative performance of alternative algorithms) provide solid support via empirical studies, theoretical analysis, or comparison to psychological phenomena. Applications papers show how to apply learning methods to solve important applications problems. Research methodology papers improve how ML research is conducted. All papers must state their contributions clearly & describe how contributions are supported. All papers must describe supporting evidence in ways that can be verified or replicated by other researchers. All papers must describe learning component clearly, & must discuss assumptions regarding knowledge representation & performance task. All papers must place their contribution clearly in context of existing work in ML. Variations from these prototypes, e.g. comprehensive surveys of active research areas, critical reviews of existing work, & books reviews, will be considered provided they make a clear contribution to the field.

2 Machine Learning

2.1 [Bac24]. FRANCIS BACH. Learning Theory from 1st Principles

Amazon review. A comprehensive & cutting-edge introduction to foundations & modern applications of learning theory.

Research has exploded in field of machine learning resulting in complex mathematical arguments that are hard to grasp for new comers. In this accessible textbook, FRANCIS BACH presents foundations & latest advances of learning theory for graduate students as well as researchers who want to acquire a basic mathematical understanding of most widely used machine learning architectures. Taking position that learning theory does not exist outside of algorithms that can be run in practice, this book focuses on theoretical analysis of learning algorithms as it relates to their practical performance. BACH provides simplest formulations that can be derived from 1st principles, constructing mathematically rigorous results & proofs without overwhelming students.

- Provides a balanced & unified treatment of most prevalent machine learning methods
- Emphasizes practical application & features only commonly used algorithmic frameworks
- Covers modern topics not found in existing texts, e.g. overparametrized models & structured prediction
- Integrates coverage of statistical theory, optimization theory, & approximation theory
- Focuses on adaptivity, allowing distinctions between various learning techniques
- Hands-on experiments, illustrative examples, & accompanying code link theoretical guarantees to practical behaviors

About the Author. FRANCIS BACH is a researcher at Inria where he leads the machine learning team which is part of the Computer Science department at Ecole Normale Supérieure. His research focuses on machine learning & optimization.

- **Preface. Why study learning theory?** Data have become ubiquitous in science, engineering, industry, & personal life, leading to need for automated processing. Machine learning is concerned with making predictions from training examples & is used in all of these areas, in small & large problems, with a variety of learning models, ranging from simple linear models to deep neural networks. It has now become an important part of algorithmic toolbox.

How can we make sense of these practical successes? Can we extract a few principles to understand current learning methods & guide design of new techniques for new applications or to adapt to new computational environments? This is precisely goal of learning theory. Beyond being already mathematically rich & interesting (as it imports from many mathematical fields), most behaviors seen in practice can, in principle, be understood with sufficient effort & idealizations. In return, once understood, appropriate modifications can be made to obtain even greater success.

Why read this book? Goal of this textbook: to present old & recent results in learning theory for most widely used learning architectures. Doing so, a few principles are laid out to understand overfitting & underfitting phenomena, as well as a systematic exposition of 3 types of components in their analysis, estimation, approximation, & optimization errors. Moreover, goal: not only to show: learning methods can learn given sufficient amounts of data but also to understand how quickly (or slowly) they learn, with a particular eye toward adaptivity to specific structures that make learning faster (e.g. smoothness of prediction functions or dependence on low-dimensional subspaces).

This book is geared toward theory-oriented students, as well as students who want to acquire a basic mathematical understanding of algorithms used throughout machine learning & associated fields that are significant users of learning methods (e.g. computer vision & natural language processing). Moreover, it is well suited to students & researchers coming from other areas of applied mathematics or computer science who want to learn about theory behind machine learning. Finally, since many simple proofs have been put together, it can serve as a reference for researchers in theoretical machine learning.

A particular effort will be made to prove *many results from 1st principles* while keeping exposition as simple as possible. This will naturally lead to a choice of key results showcasing essential concepts in learning theory in simple but relevant instances. A few general results will also be presented without proof. Of course, concept of 1st principles is subjective, & I will assume readers have a good knowledge of linear algebra, probability theory, & differential calculus.

Moreover, focus on part of learning theory that deals with algorithms that can be run in practice, & thus, all algorithmic frameworks described in this book are routinely used. Since many modern learning methods are based on optimization, Chap. 5 is dedicated to that topic. For most learning methods, present some simple *illustrative experiments* with accompanying code (MATLAB & Python for moment, & Julia in future) so students can see for themselves that algorithms are simple & effective in synthetic experiments. Exercises currently come with no solutions & are meant to help students understand related material.

Finally, 3rd part of book provides an in-depth discussion of *modern special topics* e.g. online learning, ensemble learning, structured prediction, & overparametrized models.

Note: this is not an introductory textbook on machine learning. There are already several good ones in several languages (see, e.g., Alpaydin, 2020; Lindholm et al., 2022; Azencott, 2019; Alpaydin, 2022). This textbook focuses on learning theory – i.e., deriving mathematical guarantees for most widely used learning algorithms & characterizing what makes a particular algorithmic framework successful. In particular, given that many modern methods are based on optimization algorithms, put a significant emphasis on gradient-based methods & their relation with machine learning.

A key goal: to look at simplest results to make them easier to understand, rather than focusing on material that is more advanced but potentially too hard at 1st & provides only marginally better understanding. Throughout book, propose references to more modern work that goes deeper.

Book organization. Book comprises 3 main parts: an introduction, a core part, & special topics. Readers are encouraged to read 1st 2 parts to understand main concepts fully & can pick & choose among special topic chapters in a 2nd reading or if used in a 2-semester class.

All chapters start with a summary of main concepts & results that will be covered. All simulation experiments are available at <https://www.di.ens.fr/~fbach/ltpf/> as MATLAB & Python code. Many exercises are proposed & are embedded in text with dedicated paragraphs, with a few mentioned within text (e.g., as “proof left as an exercise”). These exercises are meant to deepen understanding of nearby material, by proposing extensions or applications.

Many topics are not covered at all, & many others are not covered in depth. There are many good textbooks on learning theory that go deeper or wider (e.g., Christmann & Steinwart, 2008; Koltchinskii, 2011; Mohri et al., 2018; Shalev-Shwartz & Ben-David, 2014). See also the nice notes from Alexander Rakhlin & Karthik Sridharan¹, as well as from Michael Wolf.

In particular, book focuses primarily on real-valued prediction functions, as it has become the de facto standard for modern machine learning techniques, even when predicting discrete-valued outputs. Thus, although its historical importance & influence are crucial, choose not to present Vapnik-Chervonenkis dimension (see, e.g., Vapnik & Chervonenkis, 2015), & instead based my generic bounds on Rademacher complexities. This focus on real-valued prediction functions makes least-squares regression a central part of theory, which is well appreciated by students. Moreover, this allows for drawing links with related statistical literature.

Some areas, e.g. online learning or probabilistic methods, are described in a single chapter to draw links with classical theory & encourage readers to learn more about them through dedicated books. Have also included Chap. 12 on overparametrized models & Chap. 13 on structured prediction, which present modern topics in machine learning. More generally, goal in 3rd part of book (special topics) was, for each chapter, to introduce new concepts, while remaining a few steps away from core material & using unified notations.

A book is always a work in process. In particular, there are still typos & almost surely places where more details are needed. Convinced: more straightforward mathematical arguments are possible in many places in book. Let me know if you have any elegant & simple ideas I have overlooked.

Mathematical notations. Throughout textbook, provide unified notations:

- Random variables: given a set \mathcal{X} , will use lowercase notation for a random variable with values in \mathcal{X} , as well as for its observations. Probability distributions will be denoted μ or p & expectations as $\mathbb{E}[f(x)] = \int_{\mathcal{X}} f(x) dp(x)$: slightly ambiguous but will not cause major problems (& is standard in research papers). In this book, following most of learning theory literature, will gloss over measurability issues to avoid overformalizations. For a detailed treatment, see Devroye et al. (1996) & Christmann & Steinwart (2008).
- Norms on \mathbb{R}^d : will consider usual l_p -norms on \mathbb{R}^d , defined through $\|x\|_p^p = \sum_{i=1}^d |x_i|^p$ for $p \in [1, \infty)$, with $\|x\|_{\infty} = \max_{i \in \{1, \dots, d\}} |x_i|$.
- For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $A \succeq 0$ means A is positive semidefinite (i.e., all of its eigenvalues are nonnegative), & for 2 symmetric matrices A, B , $A \succeq B$ means $A - B \succeq 0$. For a vector $\lambda \in \mathbb{R}^n$, $\text{Diag}(\lambda)$: diagonal matrix with diagonal vector λ .
- For a differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, its gradient at \mathbf{x} is denoted $f'(\mathbf{x}) \in \mathbb{R}^d$, & if it is twice differentiable, its Hessian is denoted as $f''(\mathbf{x}) \in \mathbb{R}^{d \times d}$.

¹<http://www.mit.edu/~rakhlin/notes.html>.

How to use this book? 1st 9 chapters (in sequence, without diamond parts) are adapted for a 1-semester upper-undergraduate or graduate class, if possible, after an introductory course on machine learning. Following 6 chapters can be read mostly in any order & are here to deepen understanding of some special topics; they can be read as homework assignments (using exercises) or taught within a longer (e.g., 2-semester) class. Book is intended to be adapted to self-study, with 1st 9 chapters being read in sequence & last 6 in random order. In all situations, Chap. 1, on mathematical preliminaries, can be read quickly & studied in more detail when relevant notions are needed in subsequent chapters.

- **1. Mathematical Preliminaries.** Chapter Summary: *Linear algebra*: A bag of tricks to avoid lengthy & faulty computations. *Concentration inequalities*: For n independent random variables, derivation between empirical average & expectation is of order $O(\frac{1}{\sqrt{n}})$. What is in big O , & how does it depend explicitly on problem parameters?

Mathematical analysis & design of ML algorithms require specialized tools beyond classic linear algebra, differential calculus, & probability. In this chapter, review these nonelementary mathematical tools used throughout book: 1st, linear algebra tricks, & then concentration inequalities. Chapter can be safely skipped for readers familiar with linear algebra & concentration inequalities since relevant results will be referenced when needed.

- **Linear Algebra & Differential Calculus.** Review basic linear algebra & differential calculus results that will be used throughout book. Using these usually greatly simplifies computations. Matrix notations will be used as much as possible.

- * **Minimization of Quadratic Forms.** Given a positive-definite (& hence invertible) symmetric matrix $A \in \mathbb{R}^{n \times n}$ & vector $\mathbf{b} \in \mathbb{R}^n$, minimization of quadratic forms with linear terms can be done in closed form:

$$\inf_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2} \mathbf{x}^\top A \mathbf{x} - \mathbf{b}^\top \mathbf{x} = -\frac{1}{2} \mathbf{b}^\top A^{-1} \mathbf{b}, \quad (1)$$

with minimizer $\mathbf{x}_* = A^{-1} \mathbf{b}$ obtained by zeroing gradient $f'(\mathbf{x}) = A \mathbf{x} - \mathbf{b}$ of function $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top A \mathbf{x} - \mathbf{b}^\top \mathbf{x}$. Moreover, have $\frac{1}{2} \mathbf{x}^\top A \mathbf{x} - \mathbf{b}^\top \mathbf{x} = \frac{1}{2} (\mathbf{x} - \mathbf{x}_*)^\top A (\mathbf{x} - \mathbf{x}_*) - \frac{1}{2} \mathbf{b}^\top A^{-1} \mathbf{b}$. If A were not invertible (simply positive semidefinite) & \mathbf{b} were not in column space of A , then infimum would be $-\infty$.

Note this result is often used in various forms, e.g.

$$\mathbf{b}^\top \mathbf{x} \leq \frac{1}{2} \mathbf{b}^\top A^{-1} \mathbf{b} + \frac{1}{2} \mathbf{x}^\top A \mathbf{x}, \quad \mathbf{b}^\top \mathbf{x} = \frac{1}{2} \mathbf{b}^\top A^{-1} \mathbf{b} + \frac{1}{2} \mathbf{x}^\top A \mathbf{x} \Leftrightarrow \mathbf{b} = A \mathbf{x}. \quad (2)$$

This form is exactly Fenchel–Young inequality (see [Wikipedia/convex conjugate](#)) for quadratic forms & often used in 1D in form $ab \leq \frac{a^2}{2\eta} + \frac{\eta b^2}{2}$, $\forall \eta \geq 0$ & equality iff $\eta = \frac{a}{b}$.

- * **Inverting a 2×2 Matrix.** Solving small systems happens frequently, as well as inverting small matrices. This can be easily done in 2D. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. If $ad - bc \neq 0$, then may invert it as

$$M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (3)$$

This can be checked by multiplying 2 matrices or using [Cramer's rule](#), & it can be generalized to matrices defined by blocks.

- * **Inverting Matrices Defined by Blocks, Matrix Inversion Lemma.**

Above example may be generalized to matrices of form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with blocks of consistent sizes (note: A, D have to be square matrices). Inverse of M may be obtained by applying directly Gaussian elimination in block form. Given 2 matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, N = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, may linearly combine lines (with same coefficients for 2 matrices). Once M has been transformed into identity matrix, N has been transformed to inverse of M .

Make simplifying assumption that A is invertible, use notation $M/A = D - DA^{-1}B$ for Schur complement of block A & also assume that M/A is invertible. Thus get by Gaussian elimination, referring to $L_i, i = 1, 2$ as 2 lines of blocks, so for 1st matrix $M = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$: ***

- **2. Introduction to Supervised Learning.** Chapter Summary: *Decision theory (loss, risk, optimal predictors)*: What is optimal prediction & performance given infinite data & infinite computational resources? *Statistical learning theory*: When is an algorithm “consistent”? “*No free lunch*” theorems: Learning is impossible without making assumptions.

Present supervised learning problem: main object of study in this book. After a short introduction highlighting main motivating practical examples in Sect. 2.1, decision-theoretic probabilistic framework set forth in Sect. 2.2 provides traditional mathematical formalization, with notion of loss, risk, & optimal predictor. This will precisely define goals & evaluation standards of machine learning that will be applied to learning algorithms presented throughout this book. Sect. 2.3 presents 2 main classes of learning algorithms: local averaging techniques, & methods based on empirical risk minimization. Notions of statistical consistencies are described in Sect. 2.4; studying consistency of learning methods: main objective in this book: as shown in Sect. 2.5 on “no free lunch” theorems, no method can perform uniformly well, & assumptions have to be made to obtain meaningful quantitative results, as shown in Sect. 2.6. Sect. 2.7: present classical extensions to basic supervised supervised learning frameworks, & in Sect. 2.8: a summary & an outline of subsequent chapters of this book.

- **2.1: From Training Data to Predictions. Main goal.** Give some observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}, i = 1, \dots, n$, of inputs/outputs, features/labels, covariates/responses (which are referred to as “training data”), main goal of supervised learning is to predict a new $y \in \mathcal{Y}$ given a new previously unseen $x \in \mathcal{X}$. Unobserved data are usually referred to as “testing data.”

– **Mục tiêu chính.** Đưa ra 1 số quan sát $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}, i = 1, \dots, n$, của các đầu vào/đầu ra, các nhân/tính năng, các phản hồi của biến phụ thuộc/(được gọi là “dữ liệu đào tạo”), mục tiêu chính của học có giám sát là dự đoán 1 $y \in \mathcal{Y}$ mới khi biết trước 1 $x \in \mathcal{X}$ mới chưa từng thấy. Dữ liệu chưa quan sát thường được gọi là “dữ liệu thử nghiệm.”

There are few fundamental differences between ML & branch of statistics dealing with regression & its various extensions, particularly when providing theoretical guarantees. Focus on algorithms & computational scalability is arguably stronger within ML (but also exists in statistics). At same time, emphasis on models & their interpretability beyond their predictive performance is more prominent within statistics (but also exists in ML).

Examples. Supervised learning is used in many areas of science, engineering, & industry. There are thus many examples where \mathcal{X}, \mathcal{Y} can be very diverse:

- * **Inputs** $x \in \mathcal{X}$: They can be images, sounds, videos, text documents, proteins, sequences of DNA bases, web pages, social network activities, sensors from industry, financial time series, etc. Set \mathcal{X} may thus have a variety of structures that can be leveraged. All learning methods presented in this textbook will use at 1 point a vector space representation of inputs, either by building an explicit mapping from \mathcal{X} to a vector space, e.g., \mathbb{R}^d , or implicitly by using a notion of pairwise dissimilarity or similarity between pairs of inputs. Choice of these representations is highly domain-dependent. However, note:

- common topologies are encountered in many diverse areas (e.g. sequences or 2D or 3D objects), & thus common tools are used, &
- learning these representations is an active area of research (Chaps. 7 & 9).

In this textbook, will primarily consider that inputs are d -dimensional vectors, with d potentially large, up to 10^6 or 10^9 .

- * **Outputs** $y \in \mathcal{Y}$. Most classical examples are binary labels $\mathcal{Y} = \{0, 1\}$ or $\mathcal{Y} = \{\pm 1\}$, multiclass classification problems with $\mathcal{Y} = \{1, \dots, k\}$, & classical regression with real responses/outputs $\mathcal{Y} = \mathbb{R}$. These will be main examples examined in most of book. Note, however: most of concepts extend to more general *structured prediction* setup, where more general structured outputs (e.g., graph prediction, visual scene analysis, source separation, ranking) can be considered (Chap. 13).

Why difficult? Supervised learning is difficult (& thus interesting) for a variety of reasons:

- * Label y may not be a deterministic function of x : Given $x \in \mathcal{X}$, outputs are noisy, i.e., y is a random function of x . When $y \in \mathbb{R}$, will often make simplifying “additive noise” assumption: $y = f(x) + \varepsilon$ with some zero-mean noise ε , but in general, only assume: there is a conditional distribution of y given x . This stochasticity is typically due to diverging views between labelers or dependence on random external unobserved quantities (i.e., $y = f(x, z)$, with z random & not observed, which is common, e.g., in medical applications, where need to predict a future occurrence of a disease based on limited information about patients).
- * Prediction function f may be quite complex, highly nonlinear when \mathcal{X} is a vector space, & even hard to define when \mathcal{X} is not a vector space.
- * Only a few x ’s are observed: thus need interpolation & potentially extrapolation (diagram for an illustration for $\mathcal{X} = \mathcal{Y} = \mathbb{R}$), & therefore overfitting (predicting well on training data but not as well on testing data) is always a possibility. Moreover, training observations may not be uniformly distributed in \mathcal{X} . In this book, they will be assumed to be random, but some analyzes will rely on deterministically located inputs to simplify some theoretical arguments.
- * Input space \mathcal{X} may be very large (i.e., with high dimension when this is a vector space). This leads to both computational issues (scalability) & statistical issues (generalization to unseen data). One usually refers to this problem as *curse of dimensionality*.
- * There may be a weak link between training & testing distributions. I.e., data at training time can have different characteristics than data at testing time.
- * Criterion for performance is not always well defined.

Main formalization. Most modern theoretical analyzes of supervised learning rely on a probabilistic formulation, i.e., see (x_i, y_i) as a realization of random variables. Criterion: to maximize expectation of some performance measure w.r.t. distribution of test data (in this book, *maximizing* performance will be obtained by *minimizing* a loss function). Main assumption: random variables (x_i, y_i) are independent & identically distributed (i.i.d.) with same distribution as testing distribution. In this book, ignore potential mismatch between train & test distributions (although this is an important research topic, as in most applications, training data are not i.i.d. from same distribution as test data).

A ML algorithm \mathcal{A} is then a function that goes from a dataset (i.e., an element of $(\mathcal{X} \times \mathcal{Y})^n$) to a function from \mathcal{X} to \mathcal{Y} . I.e., output of a ML algorithm is itself an algorithm.

Practical performance evaluation. In practice, do not have access to test distribution but samples from it. In most cases, data given to ML user are split into 3 parts:

- * *Training set*, on which learning models will be estimated.
- * *Validation set*, to estimate hyperparameters (all learning techniques have some) to optimize performance measure.
- * *Testing set*, to evaluate performance of final chosen model.

In theory, test set can be used only once. In practice, this is unfortunately only sometimes the case. If test data are seen multiple times, estimation of performance on unseen data is overestimated.

Cross-validation is often preferred, to use a maximal amount of training data & reduce variability of validation procedure: available data are divided into k folds (typically $k = 5$ or 10), & all models are estimated k times, each time choosing a different fold as validation data, & averaging k obtained error measures. Cross-validation can be applied to any learning method, & its detailed theoretical analysis is an active area of research (see Arlot & Celisse, 2010, & many references therein).

“Debugging” a ML implementation is often an art: on top of commonly found bugs, learning method may not predict well enough with testing data. This is where theory can be useful to understand when a method is supposed to work or not: primary goal of this book.

Model selection. Most ML models have hyperparameters (e.g., regularization weight, size of model, number of parameters). To estimate them from data, common practical approach is to use validation approaches like those highlighted thus far. Also possible to use penalization techniques based on generalization bounds. These 2 approaches are analyzed in Sect. 4.6.

Random design vs. fixed design. What have described is often referred to as “random design” setup in statistics, where both x, y are assumed to be random & sample i.i.d. Common to simplify analysis by considering: input data x_1, \dots, x_n are deterministic, either because they are actually deterministic (e.g., equally spaced in input space \mathcal{X}) or by conditioning on them if they are actually random. This will be referred to as “fixed design” setting & studied precisely in context of least-squares regression in Chap. 3.

In context of fixed design analysis, error is evaluated “within-sample” (i.e., for same input points x_1, \dots, x_n , but over new associated outputs). This explicitly removes difficulty of extrapolating to new inputs, hence a simplification in mathematical analysis.

- **2.2: Decision Theory. Main question.** Tackle question: What is optimal performance, regardless of finiteness of training data? I.e., what should be done if we have a perfect knowledge of underlying probability distribution of data? Will thus introduce concepts of *loss function*, *risk*, & *Bayes predictor*.

Consider a fixed (testing) distribution $p_{(x,y)}$ on $\mathcal{X} \times \mathcal{Y}$, with marginal distribution $p_{(x)}$ on \mathcal{X} . Note: make no assumptions at this point on input space \mathcal{X} .

Will almost always use overload notation p , to denote $p_{(x,y)}$ & $p_{(x)}$, where context can always make definition unambiguous. E.g., when $f : \mathcal{X} \rightarrow \mathbb{R}, g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$, have $\mathbb{E}[f(x)] = \int_{\mathcal{X}} f(x) dp(x), \mathbb{E}[g(x, y)] = \int_{\mathcal{X} \times \mathcal{Y}} g(x, y) dp(x, y)$.

Ignore measurability issues on purpose. Instead reader can look at Christmann & Steinwart (2008): *Support Vector Machines* for a more formal presentation.

- * **Supervised Learning Problems & Loss Functions.** Consider a loss function $l : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ (often \mathbb{R}_+), where $l(y, z)$: loss of predicting z while true label is y .

Some authors swap y, z in def of loss. Some related research communities (e.g., economics) use concept of “utility,” which is then maximized.

Loss function only concerns output space \mathcal{Y} independent of input space \mathcal{X} . Main examples: each corresponding to a particular supervised learning problem (note: for each problem, different losses may be considered):

- **Binary classification.** $\mathcal{Y} = \{0, 1\}$ (or often $\mathcal{Y} = \{\pm 1\}$, or, less, often, when seen as a subcase of multicategory situation below, $\mathcal{Y} = \{1, 2\}$); “0–1 loss” defined as $l(y, z) = 1_{y \neq z}$ is most commonly used, i.e., 0 if $y = z$ (no mistake), & 1 otherwise (mistake).
Very common to mix 2 conventions $\mathcal{Y} = \{0, 1\}$ & $\mathcal{Y} = \{\pm 1\}$: double-check which convention is used when using toolboxes.
- **Multicategory classification.** $\mathcal{Y} = \{1, \dots, k\}, l(y, z) = 1_{y \neq z}$ (0–1 loss).
- **Regression:** $\mathcal{Y} = \mathbb{R}, l(y, z) = (y - z)^2$ (square loss). Absolute loss $l(y, z) = |y - z|$ is often used for robust estimation (since penalty for large errors is smaller).
- **Structured prediction.** while this textbook focuses primarily on 3 examples above, there are many practical problems where \mathcal{Y} is more complicated, with associated algorithms & theoretical results. E.g., when $\mathcal{Y} = \{0, 1\}^k$ (leading to multilabel classification), Hamming loss $l(y, z) = \sum_{j=1}^k 1_{y_j \neq z_j}$ is commonly used; also, ranking problems involve losses on permutations, see Chap. 13 for a detailed treatment.

Throughout this textbook, will assume: loss function is given to us. Note: in practice, final user imposes loss function, as this is how models will be evaluated. Clearly, a single real number may not be enough to characterize entire prediction behavior. E.g., in binary classification, there are 2 types of errors, false positives & false negatives, which can be considered simultaneously. Since now have 2 performance measures, typically need a curve to characterize performance of a prediction function. This is precisely what receiver operating characteristic (ROC) curves are achieving (see, e.g., Bach et al., 2006, & references therein). For simplicity, stick to a single loss function l in this book.

While loss function l will be used to define generalization performance in Sect. 2.2.2, for computational reasons, learning algorithms may explicitly minimize a different (but related) loss function, with better computational properties. This loss function used in training is often called a “surrogate.” This will be studied in context of binary classification in Sect. 4.1, & more generally for structured prediction in Chap. 13.

- * **Risks.** Given loss function $l : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, can define *expected risk* (also referred to as *generalization error*, or *testing error*) of a function $f : \mathcal{X} \rightarrow \mathcal{Y}$, as expectation of loss function between output y & prediction $f(x)$.

Definition 1 (Expected risk). Given a prediction function $f : \mathcal{X} \rightarrow \mathcal{Y}$, a loss function $l : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, & a probability distribution p on $\mathcal{X} \times \mathcal{Y}$, expected risk of f is defined as

$$\mathcal{R}(f) = \mathbb{E}[l(y, f(x))] = \int_{\mathcal{X} \times \mathcal{Y}} l(y, f(x)) dp(x, y). \quad (4)$$

Risk depend on distribution p on (x, y) . Sometimes use notation $\mathcal{R}_p(f)$ to make it explicit. Expected risk is our main performance criterion in this textbook.

Be careful with randomness, or lack thereof, of f : when performing learning from data, f will depend on random training data, not on testing data, & thus $\mathcal{R}(f)$ is typically random because of dependence on training data. However, as a function on functions, expected risk \mathcal{R} is deterministic.

Note: sometimes consider random predictions, i.e., for any x , output a distribution on y , & then risk is taken as expectation over randomness of outputs.

Averaging loss on training data defines *empirical risk* or *training error*.

Definition 2 (Empirical risk). Given a prediction function $f : \mathcal{X} \rightarrow \mathcal{Y}$, a loss function $l : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, & data $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}, i = 1, \dots, n$, empirical risk of f is defined as

$$\hat{\mathcal{R}}(f) = \frac{1}{n} \sum_{i=1}^n l(y_i, f(x_i)). \quad (5)$$

Note: $\hat{\mathcal{R}}$ is a random function on functions (& is often applied to random functions, with dependent randomness as both will depend on training data).

Special cases. For classical losses defined earlier, expected & empirical risks have specific formulations:

- **Binary classifications.** $\mathcal{Y} = \{0, 1\}$ (or often $\mathcal{Y} = \{\pm 1\}$), & $l(y, z) = 1_{y \neq z}$ (0-1 loss). Can express risk as $\mathcal{R}(f) = \mathbb{P}(f(x) \neq y)$. This is simply probability of making a mistake on testing data (error rate), while empirical risk is proportion of mistakes on training data.

In practice, *accuracy*, which is 1 minus error rate, is often reported.

- **Multicategory classification.** $\mathcal{Y} = \{1, \dots, k\}$, & $l(y, z) = 1_{y \neq z}$ (0-1 loss). Can also express risk as $\mathcal{R}(f) = \mathbb{P}(f(x) \neq y)$. This is also probability of making a mistake (error rate).

- **Regression.** $\mathcal{Y} = \mathbb{R}, l(y, z) = (y - z)^2$ (square loss). Risk is then equal to $\mathcal{R}(f) = \mathbb{E}[(y - f(x))^2]$, often referred to as “mean squared error.”

* **Bayes Risk & Bayes Predictor.** Now have defined performance criterion for supervised learning (expected risk), main question tackle here: What is best prediction function f (regardless of training data)?

Using conditional expectation & its associated law of total expectation, have

$$\mathcal{R}(f) = \mathbb{E}[l(y, f(x))] = \mathbb{E}[\mathbb{E}[l(y, f(x))|x]], \quad (6)$$

which can be rewritten, for a fixed $x' \in \mathcal{X}$:

$$\mathcal{R}(f) = \mathbb{E}_{x' \sim p}[\mathbb{E}[l(y, f(x'))|x = x']] = \int_{\mathcal{X}} \mathbb{E}[l(y, f(x'))|x = x'] dp(x'). \quad (7)$$

To distinguish between random variable x & a value it may take, use notation x' .

From conditional distribution given any $x' \in \mathcal{X}$ (i.e., $y|x = x'$), can define *conditional risk* for any $z \in \mathcal{Y}$ (it is a deterministic function of z & x'):

$$r(z|x') = \mathbb{E}[l(y, z)|x = x'], \quad (8)$$

which leads to

$$\mathcal{R}(f) = \int_{\mathcal{X}} r(f(x')|x') dp(x'). \quad (9)$$

To find a minimizing function $f : \mathcal{X} \rightarrow \mathcal{Y}$, 1st assume: set \mathcal{X} is finite: in this situation, risk can be expressed as a sum of functions that depends on a *single* value of f , i.e., $\mathcal{R}(f) = \sum_{x' \in \mathcal{X}} r(f(x')|x') \mathbb{P}(x = x')$. Therefore, can minimize w.r.t. each $f(x')$ *independently*. Therefore, a minimizer of $\mathcal{R}(f)$ can be obtained by considering for any $x' \in \mathcal{X}$, function value $f(x')$ to be equal to a minimizer $z \in \mathcal{Y}$ of $r(z|x') = \mathbb{E}[l(y, z)|x = x']$. This extends beyond finite sets.

Minimizing expected risk w.r.t. a function f in a restricted set does not lead to such decoupling.

Proposition 1 (Bayes predictor & Bayes risk). Expected risk is minimized at a Bayes predictor $f_\star : \mathcal{X} \rightarrow \mathcal{Y}$, satisfying $\forall x' \in \mathcal{X}$, (2.1)

$$f_\star(x') \in \arg \min_{z \in \mathcal{Y}} \mathbb{E}[l(y, z)|x = x'] = \arg \min_{z \in \mathcal{Y}} r(z|x'). \quad (10)$$

Bayes risk \mathcal{R}^\star is risk for all Bayes predictors & is equal to

$$\mathcal{R}^\star = \mathbb{E}_{x' \sim p} \left[\inf_{z \in \mathcal{Y}} \mathbb{E}[l(y, z)|x = x'] \right]. \quad (11)$$

Chứng minh. Have $\mathcal{R}(f) - \mathcal{R}^\star = \mathcal{R}(f) - \mathcal{R}(f_\star) = \int_{\mathcal{X}} [r(f(x')|x') - \min_{z \in \mathcal{Y}} r(z|x')] dp(x')$. □

Note:

1. Bayes predictor is not always unique, but that all lead to same Bayes risk (e.g., in binary classification when $\mathbb{P}(y = 1|x) = \frac{1}{2}$)
2. Bayes risk is usually nonzero (unless dependence between x, y is deterministic). Given a supervised learning problem, Bayes risk is optimal performance; define excess risk as deviation w.r.t. optimal risk.

Definition 3 (Excess risk). Excess risk of a function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is equal to $\mathcal{R}(f) - \mathcal{R}^*$ (always nonnegative).

\Rightarrow ML could be seen trivial: given distribution $y|x$ for any x , optimal predictor is known & given by (2.1). Difficulty: this distribution is unknown.

Special cases. For usual set of losses, can compute Bayes predictors in closed forms as follows:

- **Binary classification.** Bayes predictor for $\mathcal{Y} = \{\pm 1\}$ & $l(y, z) = 1_{y \neq z}$ is s.t.

$$f_*(x') \in \arg \min_{z \in \{\pm 1\}} \mathbb{P}(y \neq z | x = x') = \arg \min_{z \in \{\pm 1\}} 1 - \mathbb{P}(y = z | x = x') = \arg \min_{z \in \{\pm 1\}} \mathbb{P}(y = z | x = x'). \quad (12)$$

Optimal classifier will select most likely class given x' . Using notation $\eta(x') = \mathbb{P}(y = 1 | x = x')$, then, if $\eta(x') > \frac{1}{2}$, $f_*(x') = 1$, while if $\eta(x') < \frac{1}{2}$, $f_*(x') = -1$. What happens for $\eta(x') = \frac{1}{2}$ is irrelevant, as expected error is same for 2 potential predictions.

Bayes risk is then equal to $\mathcal{R}^* = \mathbb{E}[\min\{\eta(x), 1 - \eta(x)\}]$, which in general is strictly positive (unless $\eta(x) \in \{0, 1\}$ almost surely – i.e., y is a deterministic function of x).

This extends directly to multiple categories $\mathcal{Y} = \{1, \dots, k\}$, for $k \geq 2$, where have $f_*(x') \in \arg \max_{i \in \{1, \dots, k\}} \mathbb{P}(y = i | x = x')$.

These Bayes predictors & risks are valid only for 0–1 loss. Less symmetric losses are common in applications (e.g., for spam detection) & would lead to different formulas (Exercise 2.1 & Chap. 13).

- **Regression.** Bayes predictor for $\mathcal{Y} = \mathbb{R}$ & $l(y, z) = (y - z)^2$ is s.t.²

$$f_*(x') \in \arg \min_{z \in \mathbb{R}} \mathbb{E}[(y - z)^2 | x = x'] = \arg \min_{z \in \mathbb{R}} \{ \mathbb{E}[(y - \mathbb{E}[y | x = x'])^2 | x = x'] + (z - \mathbb{E}[y | x = x'])^2 \}. \quad (13)$$

This leads to conditional expectation $f_*(x') = \mathbb{E}[y | x = x']$, with a Bayes risk equal to expected conditional variance.

Problem 1. Consider binary classification with $\mathcal{Y} = \{\pm 1\}$ with loss function $l(-1, -1) = l(1, 1) = 0$ & $l(-1, 1) = c_- > 0$ (cost of a false positive), $l(1, -1) = c_+ > 0$ (cost of a false negative). Compute a Bayes predictor at x as a function of $\mathbb{E}[y|x]$.

Problem 2. Consider a learning problem on $\mathcal{X} \times \mathcal{Y}$, with $\mathcal{Y} = \mathbb{R}$ & absolute loss defined as $l(y, z) = |y - z|$. Compute a Bayes predictor $f_* : \mathcal{X} \rightarrow \mathbb{R}$.

Problem 3. Consider a learning problem $\mathcal{X} \times \mathcal{Y}$, with $\mathcal{Y} = \mathbb{R}$ & “pinball” loss $l(y, z) = \alpha(y - z)_+ + (1 - \alpha)(z - y)_+$, for $\alpha \in (0, 1)$. Compute a Bayes predictor $f_* : \mathcal{X} \rightarrow \mathbb{R}$. Provide an interpretation in terms of quantiles.

Problem 4. Characterize Bayes predictors for regression with “ ε -insensitive” loss defined as $l(y, z) = \max\{0, |y - z| - \varepsilon\}$. If for each x, y is supported in an interval of length $< 2\varepsilon$, what are Bayes predictors?

Problem 5 (Inverting predictions). Consider binary classification problem with $\mathcal{Y} = \{\pm 1\}$ & 0–1 loss. Relate risk of a prediction f & to that of its opposite $-f$.

Problem 6 (“Chance” predictions). Consider binary classification problems with 0–1 loss. What is risk of a random prediction rule where predict 2 classes with equal probabilities independent of input x ? Address same question with multiple categories.

Problem 7. Consider a random prediction rule where predict from probability distribution of y given x . When is this achieving Bayes risk?

- 2.3. Learning from Data. Decision theory framework outlined in Sect. 2.2, with notations summarized in Table 2.1: Summary of notions & notations presented in this chapter & used throughout book [Bac24]:

- * \mathcal{X} : Input space
- * \mathcal{Y} : Output space
- * p : Joint distribution on $\mathcal{X} \times \mathcal{Y}$
- * $(x_1, y_1, \dots, x_n, y_n)$: Training data
- * $f : \mathcal{X} \rightarrow \mathcal{Y}$: Prediction function
- * $l(y, z)$: Loss function between output y & prediction z
- * $\mathcal{R}(f) = \mathbb{E}[l(y, f(x))]$: Expected risk of prediction function f
- * $\widehat{\mathcal{R}}(f) = \frac{1}{n} \sum_{i=1}^n l(y_i, f(x_i))$: Empirical risk of prediction function f
- * $f_*(x') \arg \min_{z \in \mathcal{Y}} \mathbb{E}[l(y, z) | x = x']$: Bayes prediction at x'
- * $\mathcal{R}^* = \mathbb{E}_{x' \sim p} \inf_{z \in \mathcal{Y}} \mathbb{E}[l(y, z) | x = x']$: Bayes risk

gives a test performance criterion & optimal predictors, but it depends on full knowledge of test distribution p . Now briefly review how we can obtain good prediction functions from training data, i.e., data sampled i.i.d. from same distribution.

2 main classes of prediction algorithms will be studied in this textbook: (1) Local averaging (Chap. 6). (2) Empirical risk minimization (Chaps. 3, 4, 7–9, 11–13).

²Use law of total variance: $\mathbb{E}[(y - a)^2] = \text{var}(y) + (\mathbb{E}[y] - a)^2$ for any random variable y & constant $a \in \mathbb{R}$, which can be shown by expanding square.

Note: there are prediction algorithms that do not fit precisely into 1 of these 2 categories, e.g. boosting or ensemble classifiers (which perform several empirical risk minimizations, in series or parallel, see Chap. 10). Moreover, some situations do not fit classical i.i.d. framework, e.g. in online learning (see Chap. 11). Finally, consider probabilistic methods in Chap. 14, which rely on a different principle.

* **2.3.1. Local Averaging.** Goal: to approximate/emulate (bắt chước) Bayes predictor (e.g., $f_*(x') = \mathbb{E}[y|x = x']$ for least-squares regression, or $f_*(x') = \arg \max_{z \in \mathcal{Y}} \mathbb{P}(y = z|x = x')$ for classification with 0–1 loss) from empirical data. This is often done by explicit or implicit estimation of conditional distribution by *local averaging* (k -nearest neighbors, which is used as primary example for this chapter; Nadaraya–Watson estimators; or decision trees). Briefly outline here main properties for 1 instance of these algorithms, see Chap. 6 for details.

k -nearest-neighbor classifier. Given n observations $(x_1, y_1), \dots, (x_n, y_n)$ where \mathcal{X} is a metric space, $\mathcal{Y} \in \{\pm 1\}$, a new point x^{test} is classified by a majority vote among k -nearest neighbors of x^{test} .

Consider 3-nearest-neighbor classifier on a particular testing point (which will be predicted as 1):

- Pros: (1) no optimization or training, (2) often easy to implement, & (3) can get very good performance in low dimensions (in particular for nonlinear dependences between x, y).
- Cons: (1) slow at query time: must pass through all training data at each testing point (there are algorithmic tools to reduce complexity; see Chap. 6); (2) bad for high-dimensional data (because of curse of dimensionality; more on this in Chap. 6); (3) choice of local distance function is crucial; (4) choice of width hyperparameters (or k) has to be performed.
- Plot of training errors & testing errors as functions of k for a typical problem. When k is too large, there is *underfitting* (learned function is too close to a constant, which is too simple), while for k too small, there is *overfitting* (there is a strong discrepancy between testing & training errors).

Problem 8. How would curve move when n increases (assuming same balance between classes)?

* **2.3.2. Empirical Risk Minimization.** Consider a parametrized family of prediction functions (often referred to as *models*) $f_\theta : \mathcal{X} \rightarrow \mathcal{Y}$ for $\theta \in \Theta$ (typically a subset of a vector space). This class of learning methods aims at minimizing empirical risk w.r.t. $\theta \in \Theta$:

$$\widehat{\mathcal{R}}(f_\theta) = \frac{1}{n} \sum_{i=1}^n l(y_i, f_\theta(x_i)). \quad (14)$$

This defines an estimator $\hat{\theta} \in \arg \min_{\theta \in \Theta} \widehat{\mathcal{R}}(f_\theta)$, & thus a prediction function $f_{\hat{\theta}} : \mathcal{X} \rightarrow \mathcal{Y}$.

Most classic example: linear least-squares regression (studied thoroughly in Chap. 3), where minimize $\frac{1}{n} \sum_{i=1}^n (y_i - \theta^\top \varphi(x_i))^2$, & f is linear in some feature vector $\varphi(x) \in \mathbb{R}^d$ (there is no need for \mathcal{X} to be a vector space). Vector $\varphi(x)$ can be quite large (or even implicit, like in kernel methods, see Chap. 7). Other examples include neural networks (Chap. 9).

- Pros: (1) can be relatively easy to optimize (e.g., least-squares with its simple derivation & numerical algebra; see Chap. 3), many algorithms are available (primarily based on gradient descent; see Chap. 5); & (2) can be applied in any dimension (if a suitable feature vector is available).
- Cons: (1) can be relatively hard to optimize when optimization formulation is not convex (e.g., neural networks); (2) need a suitable feature vector for linear methods; (3) dependence on parameters can be complex (e.g., neural networks); (4) need some capacity control to avoid overfitting; & (5) require to parameterize functions with values in $\{0, 1\}$ (see Chap. 4 for use of convex surrogates).

Risk decomposition. Material in this section will be studied further in more detail in Chap. 4.

- Risk decomposition in estimation error + approximation error: given any $\hat{\theta} \in \Theta$, can write excess risk of $f_{\hat{\theta}}$ as

$$\mathcal{R}(f_{\hat{\theta}}) - \mathcal{R}^* = \left\{ \mathcal{R}(f_{\hat{\theta}}) - \inf_{\theta' \in \Theta} \mathcal{R}(f_{\theta'}) \right\} + \left\{ \inf_{\theta' \in \Theta} \mathcal{R}(f_{\theta'}) - \mathcal{R}^* \right\} = \text{estimation error} + \text{approximation error}. \quad (15)$$

Approximation error $\{\inf_{\theta' \in \Theta} \mathcal{R}(f_{\theta'}) - \mathcal{R}^*\}$ is always nonnegative, does not depend on chosen $f_{\hat{\theta}}$, & depends only on class of functions parametrized by $\theta \in \Theta$. It is thus always a deterministic quantity, which characterizes modeling assumptions made by chosen class of functions. When Θ grows, approximation error goes down to 0 if arbitrary functions can be approximated arbitrarily well by functions f_θ . It is also independent of number n of observations.

Estimation error $\{\mathcal{R}(f_{\hat{\theta}}) - \inf_{\theta' \in \Theta} \mathcal{R}(f_{\theta'})\}$ is also always nonnegative & is typically random because function $f_{\hat{\theta}}$ is random. It typically decreases in n & increases when Θ grows.

Overall, typical error curves look like this Fig: Size of Θ –Errors plot.

- Typically, see in later chaps: estimation error is often decomposed as follows, for θ' a minimizer on Θ of expected risk $\mathcal{R}(f_{\theta'})$:

$$\mathcal{R}(f_{\hat{\theta}}) - \mathcal{R}(f_{\theta'}) = \{\mathcal{R}(f_{\hat{\theta}}) - \widehat{\mathcal{R}}(f_{\hat{\theta}})\} + \{\widehat{\mathcal{R}}(f_{\hat{\theta}}) - \widehat{\mathcal{R}}(f_{\theta'})\} + \{\widehat{\mathcal{R}}(f_{\theta'}) - \mathcal{R}(f_{\theta'})\} \leq 2 \sup_{\theta \in \Theta} |\widehat{\mathcal{R}}(f_\theta) - \mathcal{R}(f_\theta)| + \text{empirical optimization error}, \quad (16)$$

where empirical optimization error is $\sup_{\theta \in \Theta} \{\widehat{\mathcal{R}}(f_\theta) - \widehat{\mathcal{R}}(f_{\theta'})\}$ (it is equal to 0 for exact empirical risk minimizers, but it is not when using optimization algorithms from Chap. 5 in practice). Uniform deviation defined as $\sup_{\theta \in \Theta} |\widehat{\mathcal{R}}(f_\theta) - \mathcal{R}(f_\theta)|$ grows with “size” of Θ (e.g., number or norm of parameters), & usually decays with n . See more details in Chap. 4.

Capacity control. To avoid overfitting, need to make sure: set of allowed functions is not too large by typically reducing number of parameters or by restricting norm of predictors (thus by lowering “size” of Θ): this leads to constrained optimization & still allows for risk decompositions as done previously.

Capacity control can also be done by *regularization*, i.e., by minimizing

$$\widehat{\mathcal{R}}(f_\theta) + \lambda\Omega(\theta) = \frac{1}{n} \sum_{i=1}^n l(y_i, f_\theta(x_i)) + \lambda\Omega(\theta), \quad (17)$$

where $\Omega(\theta)$ controls complexity of f_θ . Main example: ridge regression

$$\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (y_i - \theta^\top \varphi(x_i))^2 + \lambda \|\theta\|_2^2. \quad (18)$$

Regularization is often easier for optimization but harder to analyze (see Chaps. 4–5).

There is a difference between parameters (e.g., θ) learned on training data & hyperparameters (e.g., λ) estimated on validation data.

Examples of approximations by polynomials in 1D regression. Consider $(x, y) \in \mathbb{R}^2$, with prediction functions that are polynomials of order k , from $k = 0$ (constant functions) to $k = 14$ (this corresponds to linear regression with $f_\theta(x)$ of form $\theta^\top \varphi(x)$, where $\varphi(x) = (1, x, \dots, x^k)^\top \in \mathbb{R}^{k+1}$). For each k , model has $k + 1$ parameters. Training error (using square loss) is minimized with $n = 20$ observations. Data were generated with inputs uniformly distributed on $[-1, 1]$ & outputs as quadratic function $f(x) = x^2 - \frac{1}{2}$ of inputs plus some independent additive noise (Gaussian with standard deviation $\frac{1}{4}$). As shown in Fig. 2.1: Polynomial regression with increasing orders k . Plots of estimated functions in red, with training & testing errors. Bayes prediction function $f_*(x) = \mathbb{E}[y|x]$ is plotted in blue (same for all plots). Fig. 2.2: Polynomial regression with increasing orders. Plots of training & testing errors with error bars (computed as standard deviations obtained from 32 replications), together with Bayes error. Note: variance is increasing with order k , training error monotonically decreases in k while testing error goes down & then up. Note: strong overfitting when k is large (3d row in Fig. 2.1).

- **2.4. Statistical Learning Theory.** Goal of learning theory: to provide some guarantees of performance on unseen data given some properties of learning problem. A common assumption: data $\mathcal{D}_n(p) = \{(x_1, y_1), \dots, (x_n, y_n)\}$ are obtained as i.i.d. observations from some unknown distribution p from some family \mathcal{P} . Family \mathcal{P} of probability distributions on (x, y) encapsulates properties of learning problem & may consider conditions on distributions of inputs or on conditional distributions of outputs given inputs.

As seen earlier, algorithm \mathcal{A} is a mapping from $\mathcal{D}_n(p)$ (for any n) to a function from $\mathcal{X} \rightarrow \mathcal{Y}$. Expected risk depends on probability distribution $p \in \mathcal{P}$, as $\mathcal{R}_p(f)$. Goal: to find \mathcal{A} s.t. excess expected risk $\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p))) - \mathcal{R}_p^*$ is small, where \mathcal{R}_p^* : Bayes risk (which depends on joint distribution p), assuming: $\mathcal{D}_n(p)$ is sampled from p , but without knowing which $p \in \mathcal{P}$ is considered. Moreover, risk is random because $\mathcal{D}_n(p)$ is random.

- * **2.4.1. Measures of Performance.** There are several ways of dealing with randomness of expected risk of estimator to obtain a criterion:

- *Expected error:* measure performance as $\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))]$, where expectation is w.r.t. training data. Algorithm \mathcal{A} is called *consistent in expectation* for distribution p , if $\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))] - \mathcal{R}_p^* \rightarrow 0$ when $n \rightarrow \infty$. In this book, primarily use this notion of consistency.
- *Probably approximately correct (PAC) learning:* for a given $\delta \in (0, 1), \varepsilon > 0$: $\mathbb{P}(\mathcal{P}_p(\mathcal{A}(\mathcal{D}_n(p))) - \mathcal{R}_p^* \leq \varepsilon) \geq 1 - \delta$. Goal of learning theory in this framework: to find an ε as small as possible (typically as a function of δ, n). Notion of PAC consistency corresponds, for any $\varepsilon > 0$, to have such an inequality for each n & a sequence $\delta_n \rightarrow 0$.

- * **2.4.2. Notions of Consistency over Classes of Problems.** An algorithm is called *universally consistent* (in expectation) if for all probability distributions $p = p_{(x,y)}$ on (x, y) , algorithm \mathcal{A} is consistent in expectation for distribution p .

Be careful with order of quantifiers: convergence speed of excess risk toward 0 will depend on p . See “no free lunch” theorem in Sect. 2.5 that highlights: a uniform rate over all distributions is hopeless.

Most often, want to study uniform consistency within a class \mathcal{P} of distributions satisfying some regularity properties (e.g., inputs live in a compact space or dependence between y, x has at most some complexity, e.g., linear in some feature vector or with a certain number of bounded derivatives).

Thus aim at finding algorithm \mathcal{A} s.t. $\sup_{p \in \mathcal{P}} \{\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))] - \mathcal{R}_p^*\}$ is as small as possible. So-called *minimax risk* is equal to $\inf_{\mathcal{A}} \sup_{p \in \mathcal{P}} \{\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))] - \mathcal{R}_p^*\}$. This is typically a function of sample size n & parameters that are characteristic of \mathcal{X}, \mathcal{Y} & allowed set of problems \mathcal{P} (e.g., dimension of \mathcal{X} , model size). To compute estimates of minimax risk, several techniques exist:

- Upper-bounding optimal excess risk: 1 given algorithm with a convergence proof provides an upper bound: Main focus of this book.
- Lower-bounding optimal excess risk: in some setups, possible to show: infimum over all algorithms is $>$ a certain quantity. See Chap. 15 for a description of techniques to obtain such lower bounds. Machine learners are happy when upper bounds & lower bounds match (up to constant factors).

Nonasymptotic vs. asymptotic analysis. Theoretical results in learning theory can be *nonasymptotic*, with an upper bound with explicit dependence on all quantities; bound is then valid for all n , even if it is sometimes vacuous (e.g., a bound > 1 for a loss uniformly bounded by 1).

Analysis can also be *asymptotic*, where, e.g., $n \rightarrow \infty$ & limits are taken. Alternatively, several quantities can be made to grow simultaneously, which is common in random matrix theory, where dimension d of features & number n of observations both $\rightarrow \infty$, with a ratio tending to a constant (see, e.g., Potters & Bouchaud, 2020). See also discussion in Sect. 4.7.

Key aspect here is (arguably) how these rates depend on problem. Specifically, choice of in expectation vs. in high probability, or asymptotic vs. nonasymptotic, does not really matter as long as problem parameters explicitly appear.

- 2.5. “No Free Lunch” Theorem. Although it may be tempting to define optimal learning algorithm that works optimally for all distributions, this is impossible. I.e., learning is only possible with assumptions. See Chap. 7 of Devroye et al. (1996) for more details.

Prop. 2.2 shows: for any algorithm, for a fixed n , there is a data distribution that makes algorithm useless (with a risk that is the same as chance level).

Proposition 2 (No free lunch–fixed n). *Consider binary classification with 0–1 loss & \mathcal{X} infinite. Let \mathcal{P} denote set of all probability distributions on $\mathcal{X} \times \{0, 1\}$. For any $n > 0$ & any learning algorithm \mathcal{A} , $\sup_{p \in \mathcal{P}} \{\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))] - \mathcal{R}_p^*\} \geq \frac{1}{2}$.*

Main ideas of proof: (1) to construct a probability distribution supported on k elements in \mathbb{N} , where k is large compared to n (which is fixed), & to show: knowledge of n labels does not imply doing well on all k elements, & (2) to choose parameters of this distribution (binary vector r defined next) with largest possible expected risk & compare this worst performance to performance obtained by a random choice of parameters.

A caveat (cảnh báo) of Prop. 2.2: hard distribution used in proof above may depend on n (from proof, it takes k values, with $k \rightarrow \infty$ fast enough compared with n). Following Prop. (Thm. 7.2 from Devroye et al., 1996) is much “stronger,” as it more convincingly shows: learning can be arbitrarily slow without assumption (note: earlier one is not a corollary of later one).

Proposition 3 (No free lunch–sequence of errors). *Consider a binary classification problem with 0–1 loss, with \mathcal{X} infinite. Let \mathcal{P} denote set of all probability distributions on $\mathcal{X} \times \{0, 1\}$. For any decreasing sequence $a_n \rightarrow 0$ & s.t. $a_1 \leq \frac{1}{16}$, for any learning algorithm \mathcal{A} , there exists $p \in \mathcal{P}$ s.t. $\mathbb{E}[\mathcal{R}_p(\mathcal{A}(\mathcal{D}_n(p)))] - \mathcal{R}_p^* \geq a_n$, $\forall n \geq 1$.*

- 2.6. Quest for Adaptivity. As seen in Sect. 2.5, no method can be universal & achieve a good convergence rate on all problems. However, such negative results consider classes of problems that are arbitrarily large. In this textbook, consider reduced sets of learning problems by considering $\mathcal{X} = \mathbb{R}^d$ & putting restrictions on target function f_* based on smoothness &/or dependence on an unknown low-dimensional projection. I.e., most general set of functions will be set of Lipschitz-continuous functions, for which optimal rate will be essentially proportional to $O(n^{-\frac{1}{d}})$, typical of curse of dimensionality (as required number n of observations to reach a given precision is exponential in d). No method can beat this—not k -nearest-neighbors, not kernel methods, & not even neural networks (see lower bounds on performance in Chap. 15).

When target function is smoother (i.e., with all derivatives up to order m bounded), then will see: kernel methods (Chap. 7) & neural networks (Chap. 9), with proper choice of regularization parameter, will lead to optimal rate of $O(n^{-\frac{m}{d}})$.

When target function moreover depends only on a r -dimensional linear projection, neural networks (if optimization problem is solved correctly) will have extra ability to lead to rates of form $O(n^{-\frac{m}{r}})$ instead of $O(n^{-\frac{m}{d}})$. This is not the case for kernel methods (see Chap. 9).

Note: another form of adaptivity, which is often considered, may apply in situations where input data lie on a submanifold of \mathbb{R}^d (e.g., an affine subspace), where for most methods presented in this textbook, adaptivity is obtained. In convergence rate, d can be replaced by dimension of subspace (or submanifold) where data live. For more, see Kpotufe (2011) for k -nearest neighbors, & Hamm & Steinwart (2021) for kernel methods. See more details in <https://francisbach.com/quest-for-adaptivity/>, as well as Chaps. 7 & 9 for detailed results regarding adaptivity for kernel methods & neural networks.

- 2.7. Beyond Supervised Learning. This textbook focuses primarily on traditional supervised learning paradigm, with i.i.d. data & where training & testing distributions match. Many applications require extensions to this basic framework, which also lead to many interesting theoretical developments that are out of scope. Next, present briefly some of these extensions, with references for further reading.

Unsupervised learning. While in supervised learning, both inputs & outputs (e.g., labels) are observed, & main goal: to model how output depends on input, in unsupervised learning only inputs are given. Goal: to find some structure within data – e.g., an affine subspace around which data live for principal component analysis (PCA, studied in Sect. 3.9), separation of data in several groups (for clustering), or identification of an explicit latent variable model (e.g. with matrix factorization). New representation of data is typically either used for visualization (then, with 2D or 3D), or for reducing dimension before applying a supervised learning algorithm.

While supervised learning relied on an explicit decision-theoretic framework, not always clear how to characterize performance & perform evaluation in unsupervised learning; each method typically has an ad hoc empirical criterion, e.g. reconstruction of data, full or partial (like in self-supervised learning); or log-likelihood when probabilistic models are used (see Chap. 14), in particular graphical models (Bishop, 2006; Murphy, 2012). Often, immediate representations are used for subsequent processing (see, e.g., Goodfellow et al., 2016).

Theoretical guarantees can be obtained for sampling behavior & recovery of specific structures when assumed (e.g., for clustering or dimension reduction), with a variety of results in manifold learning, matrix factorization methods e.g. K-means, PCA, or sparse dictionary learning (Mairal et al., 2014), outlier/novelty detection (Pimentel et al., 2014), or independent component analysis (Hyvärinen et al., 2001).

Semisupervised learning. Intermediate situation between supervised & unsupervised, with typically a few labeled examples & typically many unlabeled examples. Several frameworks exist based on various assumptions (Chapelle et al., 2010; van Engelen & Hoos, 2020).

Active learning. A similar setting as semisupervised learning, but user can choose which unlabeled point to label to maximize performance over new labels are obtained. Selection of samples to label is often done by computing some form of uncertainty estimation on unlabeled data points (see, e.g., Settles, 2009).

Online learning. Mostly in a supervised setting, this framework allows us to go beyond training/testing splits, where data are acquired & predictions are made on fly, with a criterion that takes into account sequential nature of learning. See Cesa-Bianchi & Lugosi (2006), Hazan (2022), & Chap. 11.

Reinforcement learning. On top of sequential nature of learning already present in online learning, predictions may influence future sampling distributions; e.g., in situations where some agents interact with an environment (Sutton & Barto, 2018), with algorithms relying on similar concepts than optimal control (Liberzon, 2011).

Generative modeling. A key task in computer vision or natural language processing is to generate images or text documents based on simple “prompts.” Goal: often to given an output that minimizes some loss, but rather to sample from a distribution that reflects natural variability of images & text, given prompt. Sampling from such high-dimensional distributions is a practical & theoretical challenge, where diffusion models prove particularly useful (see, e.g., Chan, 2024, & references therein).

- 2.8. Summary – Book Outline. Introduced main concepts, can give an outline of chapters of this book, separated into 3 parts.

Part I: Preliminaries contains Chap. 1 on mathematical preliminaries, this introductory chapter, & Chap. 3, on linear least-squares regression. Start with least-squares, as it allows introduction of main concepts of book, e.g. underfitting, overfitting, regularization, using only simple linear algebra, without need for more advanced analytic or probabilistic tools.

Part II: Generalization bounds for learning algorithms is dedicated to core concepts in learning theory & should be studied sequentially.

- * **Empirical risk minimization.** Chap. 4 is dedicated to methods based on minimization of potentially regularized or constrained regularized risk, with introduction of key concept of Rademacher complexity, which analyzes estimation errors efficiently. Convex surrogates for binary classification are also introduced to allow use of only real-valued prediction functions.
- * **Optimization.** Chap. 5 shows how gradient-based techniques can be used to approximately minimize empirical risk &, through stochastic gradient descent (SGD), obtain generalization bounds for finitely-parameterized linear models (which are linear in their parameters), leading to convex objective functions.
- * **Local averaging methods.** Chap. 6 is 1st chapter dealing with so-called “nonparametric” methods that can potentially adapt to complex prediction functions. This class of methods explicitly builds a prediction function mimicking Bayes predictor (without any optimization algorithm), e.g., k -nearest-neighbor methods. These methods are classically subject to curse of dimensionality.
- * **Kernel methods.** Chap. 7 presents most general class of linear models that can be infinite-dimensional & adapt to complex prediction functions. They are made computationally feasible using “kernel trick,” & they still rely on convex optimization, so they lead to strong theoretical guarantees, particularly by adapting to smoothness of target prediction function.
- * **Sparse methods.** While Chap. 7 focused on Euclidean or Hilbertian regularization techniques for linear models, Chap. 8 considers regularization by sparsity-inducing penalties e.g. l_1 -norm or l_0 -penalty, leading to high-dimensional phenomenon that learning is possible even with potentially exponentially many irrelevant variables.
- * **Neural networks.** Chap. 9 presents a class of prediction functions that are not linearly parameterized, leading to nonconvex optimization problems, where obtaining a global optimum is not certain. Chap studies approximation & estimation errors, showing adaptivity of neural networks to smoothness & linear latent variables (in particular for nonlinear variable selection).

Part III: Special topics presents a series of chapters on special topics that can be read in essentially any order.

- * **Ensemble learning.** Chap. 10 presents a class of techniques aiming at combining several predictors obtained from same model class but learned on slightly modified datasets. This can be done in parallel, e.g. in bagging techniques, or sequentially, e.g. in boosting methods.
- * **From online learning to bandits.** (bạn cướp) Chap. 11 considers sequential decision problems within regret framework, focusing 1st on online convex optimization, then on 0th-order optimization (without access to gradients), & finally multiarmed bandits.
- * **Overparameterized models.** Chap. 12 presents a series of results related to models with a large number of parameters (enough to fit training data perfectly) & trained with gradient descent (GD). Present implicit bias of GD in linear models toward minimum Euclidean norm solutions & then double descent phenomenon, before looking at implicit biases & global convergence for nonconvex optimization problems.
- * **Structured prediction.** Chap. 13 goes beyond traditional regression & binary classification frameworks by 1st considering multiclass classification & then general framework of structured prediction, where output spaces can be arbitrarily complex.
- * **Probabilistic methods.** Chap. 14 presents a collection of results related to probabilistic modeling, highlighting: probabilistic interpretations can sometimes be misleading but also naturally lead to model selection frameworks through Bayesian inference & PAC–Bayesian analysis.

- * **Lower bounds on generalization & optimization errors.** While most of book is dedicated to obtaining upper bounds on generalization or optimization errors of our algorithms, Chap. 15 considers lower bounds on such errors, showing how many algorithms presented in this book are, in fact, optimal for a specific class of learning or optimization problems.
- **3. Linear Least-Squares Regression.** Chapter Summary: *Ordinary least-squares estimator:* Least-squares regression with linearly parameterized predictors leads to a linear system of size d (number of predictors). *Guarantees in fixed design setting with no regularization:* When inputs are assumed deterministic & $d < n$, excess risk = $\frac{\sigma^2 d}{n}$, where σ^2 : prediction noise variance. *Ridge regression:* With l_2 -regularization, excess risk bounds become dimension independent & allow high-dimensional feature vectors where $d > n$. *Guarantees in random design setting:* Although they are harder to show, they have a similar form. *Lower bound of generalization error:* Under well-specification, rate $\frac{\sigma^2 d}{n}$ cannot be improved.
- **3.1. Introduction.** Introduce & analyze linear least-squares regression, a tool that can be traced to Legendre (1805) & Gauss (1809). See https://en.wikipedia.org/wiki/Least_squares for an interesting discussion & claim: GAUSS had known about it already in 1795. *Why should we study linear least-squares regression? Has there not been any progress since 1805?* A few reasons:
 - * It already captures many of concepts in learning theory, e.g. bias-variance trade-off, as well as dependence of generalization performance on underlying dimension of problem with no regularization, or on dimensionless quantities when regularization is added.
 - * Because of its simplicity, many results can be easily derived without need for complicated mathematics, both in terms of algorithms & statistical analysis (simple linear algebra for simplest linear algebra for simplest results in fixed design setting).
 - * Using nonlinear features, it can lead to arbitrary nonlinear predictions (see discussion of kernel methods in Chap. 7).

In subsequent chapters, will extend many of these results beyond least-squares regression with proper additional mathematical tools.

- **3.2. Least-Squares Framework.** Recall goal of supervised ML from Chap. 2: given some training data composed of observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}, i = 1, \dots, n$, which are pairs of inputs/outputs, sometimes referred to as features/responses. Given $x \in \mathcal{X}$, goal: to predict $y \in \mathcal{Y}$ (testing data) with a *regression* function f s.t. $y \approx f(x)$. Assume $\mathcal{Y} = \mathbb{R}$ & use square loss $l(y, z) = (y - z)^2$, known from Chap. 2: optimal predictor is $f_*(x) = \mathbb{E}[y|x]$ (see Sect. 2.2.3).

In Chap. 3, consider empirical risk minimization for regression problems. Choose a parameterized family of prediction functions (often referred to as “models”) $f_\theta : \mathcal{X} \rightarrow \mathcal{Y} = \mathbb{R}$ for some parameter $\theta \in \Theta$ & minimize empirical risk $\frac{1}{n} \sum_{i=1}^n (y_i - f_\theta(x_i))^2$, leading to estimator $\hat{\theta} \in \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n (y_i - f_\theta(x_i))^2$. Note: in most cases, Bayes predictor f_* does not belong to class of functions $\{f_\theta, \theta \in \Theta\}$, i.e., model is said to be *misspecified*.

Least-squares regression can be carried out with parameterizations of function f_θ that may be nonlinear in parameter θ (e.g. for neural networks in Chap. 9). In this chapter, will consider only situations where $f_\theta(x)$ is linear in θ , which is thus assumed to live in a vector space, taken to be \mathbb{R}^d for simplicity.

Being linear in x or linear in θ is different!

While assume linearity in parameter θ , nothing forces $f_\theta(x)$ to be linear in input x . In fact, even concept of linearity may be meaningless if \mathcal{X} is not a vector space. If $f_\theta(x)$ is linear in $\theta \in \mathbb{R}^d$, then it has to be a linear combination of form $f_\theta(x) = \sum_{i=1}^d \alpha_i(x) \theta_i$, where $\alpha_i : \mathcal{X} \rightarrow \mathbb{R}, i = 1, \dots, d$, are d functions. By concatenating them in a vector $\varphi(x) \in \mathbb{R}^d$ where $\varphi(x)_i = \alpha_i(x)$, get representation $f_\theta(x) = \varphi(x)^\top \theta$. Vector $\varphi(x) \in \mathbb{R}^d$ is typically called *feature vector*, which assume to be known (i.e., given to us & can be computed explicitly when needed). Thus consider minimizing empirical risk:

$$\widehat{\mathcal{R}}(\theta) = \frac{1}{n} \sum_{i=1}^n (y_i - \varphi(x_i)^\top \theta)^2. \quad (19)$$

When $\mathcal{X} \subset \mathbb{R}^d$, can make extra assumptions: f_θ is an affine function in x , which can be obtained through $\varphi(x) = \begin{pmatrix} x \\ 1 \end{pmatrix} = (x^\top, 1)^\top \in \mathbb{R}^{d+1}$. Another classical assumption is to consider vectors $\varphi(x)$ composed of monomials (so that prediction functions are polynomials, as done in experiments in Sect. 3.5.2). See in Chap. 7: *Kernel methods*: can consider infinite-dimensional features.

Matrix notation. Cost function (19) can be rewritten in matrix notation. Let $y = (y_1, \dots, y_n)^\top \in \mathbb{R}^n$: vector of outputs (sometimes called *response vector*), & $\Phi \in \mathbb{R}^{n \times d}$: matrix of inputs, whose rows are $\varphi(x_i)^\top$, called *design matrix* or *data matrix*. In this notation, empirical risk is:

$$\widehat{\mathcal{R}}(\theta) = \frac{1}{n} \|y - \Phi \theta\|_2^2, \quad (20)$$

where $\|\alpha\|_2^2 = \sum_{j=1}^d \alpha_j^2$: squared l_2 -norm of α .

Sometimes tempting at 1st to avoid matrix notation. Strongly advise against it, as it leads to lengthy & error-prone formulas.

- **3.3. Ordinary Least-Squares Estimator.** Assume: matrix $\Phi \in \mathbb{R}^{n \times d}$ has full column rank (i.e., rank of Φ is d). In particular, problem is said to be “overdetermined,” & must have $d \leq n$, i.e., more observations than feature dimension. Equivalently, assume: $\Phi^\top \Phi \in \mathbb{R}^{d \times d}$ is invertible.

Definition 4 (OLS). When Φ has full column rank, minimizer of (20) is unique \mathcal{E} called ordinary least-squares (OLS) estimator.

* 3.3.1. Closed-Form Solution. Since objective function is quadratic, gradient will be linear, & zeroing it will lead to a closed-form solution through a linear system.

Proposition 4. When Φ has full column rank, OLS estimator exists \mathcal{E} is unique, \mathcal{E} is given by $\hat{\theta} = (\Phi^\top \Phi)^{-1} \Phi^\top y$. Denote noncentered³ empirical covariance matrix as $\hat{\Sigma} = \frac{1}{n} \Phi^\top \Phi \in \mathbb{R}^{d \times d}$; have $\hat{\theta} = \frac{1}{n} \hat{\Sigma}^{-1} \Phi^\top y$.

Coercive = going to ∞ at ∞ . Condition $\hat{\mathcal{R}}'(\hat{\theta}) = 0$ gives normal equation $\Phi^\top \Phi \hat{\theta} = \Phi^\top y$. Multidimensional linear normal equations has a unique solution: $\hat{\theta} = (\Phi^\top \Phi)^{-1} \Phi^\top y$, which shows uniqueness of minimizer of $\hat{\mathcal{R}}$, as well as its closed-form expression. Another way to show uniqueness of minimizer is by showing: $\hat{\mathcal{R}}$ is strongly convex since Hessian $\hat{\mathcal{R}}''(\theta) = 2\hat{\Sigma}$ is invertible $\forall \theta \in \mathbb{R}^d$ (convexity is studied in Chap. 5). For readers worried about carrying a factor of 2 in gradients, will sue an additional factor $\frac{1}{2}$ in chaps on optimization.

* 3.3.2. Geometric Interpretation. OLS estimator has a natural geometric interpretation.

Proposition 5. Vector predictions $\Phi \hat{\theta} = \Phi(\Phi^\top \Phi)^{-1} \Phi^\top y$ is orthogonal projection of $y \in \mathbb{R}^n$ onto $\text{im} \Phi \subset \mathbb{R}^n$, column space of Φ .

Can thus interpret OLS estimation as doing following: 1. Compute projection \bar{y} of y onto image of Φ . 2. Solve linear system $\Phi \theta = \bar{y}$, which has a unique solution.

* 3.3.3. Numerical Resolution. While closed-form $\hat{\theta} = (\Phi^\top \Phi)^{-1} \Phi^\top y$ is convenient for analysis, inverting $\Phi^\top \Phi$ is sometimes unstable & has a large computational cost when d is large. Following methods are usually preferred.

QR factorization. QR decomposition factorizes matrix Φ as $\Phi = QR$, where $Q \in \mathbb{R}^{n \times n}$ has orthonormal columns, i.e., $Q^\top Q = I$, $R \in \mathbb{R}^{d \times d}$ is upper triangular (see Golub & Loan, 1996). Computing a QR decomposition is faster & more stable than inverting a matrix. Then have $\Phi^\top \Phi = R^\top Q^\top QR = R^\top R$, & R : Cholesky factor of positive semidefinite matrix $\Phi^\top \Phi \in \mathbb{R}^d$. Since R is invertible, one then has

$$(\Phi^\top \Phi) \hat{\theta} = \Phi^\top y \Leftrightarrow R^\top Q^\top QR \hat{\theta} = R^\top Q^\top y \Leftrightarrow R^\top R \hat{\theta} = R^\top Q^\top y \Leftrightarrow R \hat{\theta} = Q^\top y.$$

Only remains to solve a triangular linear system, which is easy. Overall running time complexity remains $O(d^3)$. Conjugate gradient algorithm can also be used (see Golub & Loan, 1996, for details).

Gradient descent. Can bypass need for matrix inversion or factorization using gradient descent (GD). It consists in approximately minimizing $\hat{\mathcal{R}}$ by taking an initial point $\theta_0 \in \mathbb{R}^d$ & iteratively going toward minimizer by following opposite of gradient: $\theta_t = \theta_{t-1} - \gamma \hat{\mathcal{R}}'(\theta_{t-1})$ for $t \geq 1$, where $\gamma > 0$: step size. When these iterates converge, it is toward OLS estimator since a fixed-point θ satisfies $\hat{\mathcal{R}}'(\theta) = 0$. Study such algorithms in Chap. 5, with running-time complexities going down to linear in d , e.g., $O(nd)$.

◦ 3.4. Statistical Analysis of Ordinary Least-Squares.

• 4. Empirical Risk Minimization.

• 5. Optimization for Machine Learning.

• 6. Local Averaging Models.

• 7. Kernel Methods.

• 8. Sparse Methods.

◦ 8.6. Conclusion. In this chapter, considered sparse methods based on penalization by l_0 - or l_1 -penalties of weight vector of a linear model. For square loss, l_0 -penalties led to an excess risk proportional to $\frac{\sigma^2 k \log d}{n}$, with a price of adaptivity of $\log d$, with few conditions on problem but no provably computationally efficient procedures. On contrary, l_1 -norm penalization can be solved efficiently with appropriate convex optimization algorithms (e.g. proximal methods), but it only obtained a slow rate proportional to $\sqrt{\frac{\log d}{n}}$, exhibiting a high-dimensional phenomenon, but a worse dependence in n . Fast rates can be obtained only with stronger assumptions on covariance matrix of features.

This chapter was limited to linear models. In Chap. 9, on neural networks, will see how models that are nonlinear in their parameters can lead to nonlinear variable selection, still exhibiting a high-dimensional phenomenon but at expense of harder optimization. This will be obtained by an l_1 -norm on an infinite-dimensional space, & studied further in context of gradient boosting in Sect. 10.3.

• 9. Neural Networks. Chapter Summary:

◦ Neural networks are flexible models for nonlinear predictions. They can be studied in terms of 3 errors usually related to empirical risk minimization: optimization, estimation, & approximation errors. In this chapter, focus primarily on single hidden-layer neural networks, which are linear combinations of simple affine functions with additional nonlinearities.

³ Centered covariance matrix would be $\frac{1}{n} \sum_{i=1}^n [\varphi(x_i) - \hat{\mu}][\varphi(x_i) - \hat{\mu}]^\top$, where $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n \varphi(x_i) \in \mathbb{R}^d$ is empirical mean, while consider $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n \varphi(x_i) \varphi(x_i)^\top$.

- **Optimization error:** As prediction functions are nonlinearly dependent on their parameters, obtain nonconvex optimization problems with guaranteed convergence only to stationary points.
- **Estimation error:** Number of parameters is not driver of estimation error, as norms of various weights play an important role, with explicit rates in $O(\frac{1}{\sqrt{n}})$ obtained from Rademacher complexity tools.
- **Approximation error:** For rectified linear unit (ReLU) activation function, universal approximation properties can be characterized & are superior to those of kernel methods because they are adaptive to linear latent variables. In particular, neural networks can efficiently perform nonlinear variable selection.
- **9.1. Introduction.** In supervised learning, main focus has been put on methods to learn from n observations $(x_i, y_i), i = 1, \dots, n$, with $x_i \in \mathcal{X}$ (input space) & $y_i \in \mathcal{Y}$ (output/label space). As presented in Chap. 4, a large class of methods relies on minimizing a regularized empirical risk w.r.t. a function $f : \mathcal{X} \rightarrow \mathbb{R}$, where following cost function is minimized:

$$\frac{1}{n} \sum_{i=1}^n l(y_i, f(x_i)) + \Omega(f), \quad (21)$$

where $l : \mathcal{Y} \times \mathbb{R} \rightarrow \mathbb{R}$: a loss function, $\Omega(f)$: a regularization term. Typical examples were:

- * **Regression.** $\mathcal{Y} = \mathbb{R}, l(y_i, f(x_i)) = \frac{1}{2}(y_i - f(x_i))^2$.
- * **Classification.** $\mathcal{Y} = \{\pm 1\}, l(y_i, f(x_i)) = \Phi(y_i f(x_i))$, where Φ is convex, e.g., $\Phi(u) = \max\{1 - u, 0\}$ (hinge loss leading to support vector machine) or $\Phi(u) = \log(1 + e^{-u})$ (leading to logistic regression). See more examples in Sect. 4.1.1.

Class of prediction functions considered so far were as follows, with their pros & cons:

- * **Linear functions in some explicit features.** Given a feature map $\varphi : \mathcal{X} \rightarrow \mathbb{R}^d$, consider $f(x) = \theta^\top \varphi(x)$, with parameters $\theta \in \mathbb{R}^d$, as analyzed in Chap. 3 (for least-squares regression) & Chap. 4 (for Lipschitz-continuous losses).
 - Pros: Simple to implement, as they lead to convex optimization with gradient descent (GD) algorithms, with running time complexity in $O(nd)$, as shown in Chap. 5. They come with theoretical guarantees that are not necessarily scaling badly with dimension d if regularizers are used (l_2 - or l_1 -norm).
 - Cons: They only apply to linear functions on explicit (& fixed feature spaces), so they can underfit data. Moreover, feature vector φ is not learned from data.
- * **Linear functions in some implicit features through kernel methods.** Feature map can have arbitrarily large dimension, i.e., $\varphi(x) \in \mathcal{H}$ where \mathcal{H} : a Hilbert space, accessed through kernel function $k(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$, as presented in Chap. 7.
 - Pros: Nonlinear flexible predictions, simple to implement, & can be used with convex optimization algorithms with strong guarantees. They provide adaptivity to regularity of target function, allowing higher-dimensional applications than local averaging methods from Chap. 6.
 - Cons: Running-time complexity goes up to $O(n^2)$ with algorithms from Sect. 7.4 (but this scaling can be improved with appropriate techniques discussed in same section, e.g. column sampling or random features). Method may still suffer from curse of dimensionality for target functions that are not smooth enough.

Aim: to explore another class of functions for nonlinear predictions – namely, neural networks, which come with additional benefits, e.g. more adaptivity to linear latent variables, but also have some potential drawbacks, e.g. a harder optimization problem.

- **9.2. Single Hidden-Layer Neural Network.** Consider $\mathcal{X} = \mathbb{R}^d$ & set of prediction functions that can be written as

$$f(\mathbf{x}) = \sum_{j=1}^m \eta_j \sigma(\mathbf{w}_j^\top \mathbf{x} + b_j), \quad (22)$$

where $\mathbf{w}_j \in \mathbb{R}^d, b_j \in \mathbb{R}, j = 1, \dots, m$: input weights, $\eta_j \in \mathbb{R}, j = 1, \dots, m$: output weights, & σ : an activation function. Often represented as **graph**. Same architecture can also be considered with $\boldsymbol{\eta}_j \in \mathbb{R}^k$, for $k > 1$ to deal with multiclass classification (see Sect. 13.1).

Activation function is typically chosen from 1 of following examples (see plot):

- * Sigmoid $\sigma(u) = \frac{1}{1+e^{-u}}$.
- * Step function $\sigma(u) = 1_{u>0}$, which is not continuous & with zero derivative everywhere (& thus not amenable to gradient-based optimization).
- * Rectified linear unit (ReLU) $\sigma(u) = (u)_+ = \max\{u, 0\}$, which will be main focus of this chapter.
- * Hyperbolic tangent $\sigma(u) = \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}$.

Function f is defined as linear combination of m functions $\mathbf{x} \mapsto \sigma(\mathbf{w}_j^\top \mathbf{x} + b_j)$, which are hidden neurons. See <https://playground.tensorflow.org/> for a nice interactive illustrative of this architecture. If input weights are fixed, obtain a linear model with m hidden neurons as features. A key benefit of neural networks: they perform feature learning by optimizing w.r.t. input weights.

Constant terms b_j are sometimes referred to as “biases,” which is unfortunate in a statistical context, as that word already has a precise meaning within bias/variance trade-off (see Chap. 3 & Sect. 7.3).

Do not be confused by name “neural network” & its biological inspiration. This inspiration is not a proper justification for its behavior on ML problems.

Cross-entropy loss & sigmoid activation function for last layer. Following standard practice, we are not adding a nonlinearity to last layer; note: if were to use an additional sigmoid activation & consider cross-entropy loss for binary classification, would exactly be using logistic loss on output without an extra activation function.

Indeed, if consider $g(x) = \frac{1}{1+e^{-f(x)}} \in [0, 1]$, & given an output variable $y \in \{\pm 1\}$, so-called “cross-entropy loss,” an instance of maximum likelihood (see more details in Chap. 14), is equal to

$$-1_{y=1} \log g(x) - 1_{y=-1} \log(1 - g(x)) = 1_{y=1} \log(1 + e^{-f(x)}) + 1_{y=-1} \log(1 + e^{f(x)}) \quad (23)$$

which is exactly logistic loss $\log(1+e^{-yf(x)})$ defined in Sect. 4.1.1 applied to prediction function $f(x)$. Practitioners sometimes refer to cross-entropy loss without mentioning: a sigmoid is applied beforehand (they, in fact, mean logistic loss). Such a discussion applies as well as multicategory classification & softmax loss (see Sect. 13.1.1).

Theoretical analysis of neural networks. As with any method based on empirical risk minimization, have to study 3 classical aspects:

1. optimization error (convergence properties of algorithms for minimizing risk),
2. estimation error (effect of having a finite amount of data on prediction performance),
3. approximation error (effect of having a finite number of parameters or a constraint on norm of these parameters).

* 9.2.1. **Optimization.** To find parameters $\theta = \{(\eta_j), (\mathbf{w}_j), (b_j)\} \in \mathbb{R}^{m(d+2)}$, empirical risk minimization can be applied & following optimization problem has to be solved: (9.2)

$$\min_{\theta \in \mathbb{R}^{m(d+2)}} \frac{1}{n} \sum_{i=1}^n l \left(y_i, \sum_{j=1}^m \eta_j \sigma(\mathbf{w}_j^\top \mathbf{x}_i + b_j) \right), \quad (24)$$

with potentially additional regularization (often squared l_2 -norm of all weights).

Note (as discussed in Chap. 5): true objective is to perform on unseen data, & optimization problem in (9.2) is just a means to an end.

This is a nonconvex optimization problem where GD algorithms from Chap. 5 can be applied without a strong guarantee beyond obtaining a vector with a small gradient norm (Sect. 5.2.6). See following discussion for recent results when providing qualitative global convergence guarantees when m is large.

While stochastic gradient descent (SGD) remains an algorithm of choice (also with a good generalization behavior, as discussed in Sect. 5.4), several algorithmic improvements have been observed to lead to better stability & performance: specific step-size decay schedules, preconditioning as presented in Sect. 5.4.2 (Duchi et al., 2011), momentum (Kingma & Ba, 2014), batch normalization (Ioffe & Szegedy, 2015), & layer normalization (Ba et al., 2016) to make optimization better behaved. However, overall, objective function is nonconvex, & it remains challenging to understand precisely why gradient-based methods perform well in practice, particularly with deeper networks (some elements are presented next & in Chap. 12). See also boosting procedures in Sect. 10.3 & Chap. 12, which learn neuron weights incrementally.

Global convergence of GD for infinite widths. Turn out: global convergence can be shown for this nonconvex optimization problem (Chizat & Bach, 2018; Bach & Chizat, 2022), with tools that go beyond the scope of this book & are partially described in Chap. 12.⁴

Simply show some experimental evidence for a simple 1D setup, where compare several runs of SGD when observations are seen only once (so no overfitting is possible) & with random initializations, on a regression problem with deterministic outputs, thus with optimal testing error (Bayes rate) equal to 0. Show in Fig. 9.1: Comparison of optimization behavior for different numbers m of neurons for ReLU activations $m = 5, 20, 100$. To generate data, also used a neural network with ReLU activations & 3 hidden neurons. Top: examples of final prediction functions at convergence; bottom: plot of test errors vs. number of iterations. estimated predictors & corresponding testing errors with 20 different initializations. Can observe: small errors are never achieved when $m = 5$ (sufficient to have zero testing errors). With $m = 20$ neurons, SGD finds optimal predictor for most restarts. When $m = 100$, all restarts have desired behaviors, highlighting benefits of overparametrization (see more details in Sect. 12.3).

* 9.2.2. **Rectified Linear Units & Homogeneity.** From now on, will mostly focus on ReLU activation $\sigma(u) = u_+$. Main property: will employ its “positive homogeneity”, i.e., for $\alpha > 0$, $(\alpha u)_+ = \alpha u_+$. This implies: in def of prediction function as sum of terms $\eta_j (\mathbf{w}_j^\top \mathbf{x} + b_j)_+$, can freely multiply $\eta_j \in \mathbb{R}$ by a positive scalar α_j & divide $(\mathbf{w}_j, b_j) \in \mathbb{R}^{d+1}$ by same α_j without changing prediction function, since then $\eta_j (\mathbf{w}_j^\top \mathbf{x} + b_j)_+ = (\alpha_j \eta_j) \left(\left(\frac{\mathbf{w}_j}{\alpha_j} \right)^\top \mathbf{x} + \frac{b_j}{\alpha_j} \right)_+$.

This has a particular effect when using a squared l_2 -regularizer on all weights, which is standard, either explicitly (by adding a penalty to cost function) or implicitly (see Sect. 12.1). Indeed, consider penalizing $\eta_j^2 + \|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2}$ for each $j \in \{1, \dots, m\}$, where have added factor R^2 to constant term for unit homogeneity reasons between slope \mathbf{w}_j & constant term b_j (R will be a bound on l_2 -norm of input data). Dealing with unit homogeneity between η_j & $(\mathbf{w}_j, \frac{b_j}{R})$ does not matter because of invariance by rescaling described next.

⁴See also <https://francisbach.com/gradient-descent-neural-networks-global-convergence/> for more details.

Optimizing w.r.t. a scaling factor α_j (which affects only regularizer), have to minimize $\alpha_j^2 \eta_j^2 + \frac{\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2}}{\alpha_j^2}$ with $\alpha_j^2 = \frac{(\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2})^{\frac{1}{2}}}{|\eta_j|}$ as a minimizer & with optimal value of penalty equal to $2|\eta_j|(\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2})^{\frac{1}{2}}$ (note: this leads to an l_1 -norm penalty, thus with potentially sparsifying effects) (setting some of output weights η_j to 0), & robustness to large number of neurons (as shown in Sect. 9.2.3); for other relationship between l_2 -regularization in neural networks & sparse estimation, see Sect. 12.1.3.

Therefore, for theoretical analysis (study of approximation & estimation errors), because of homogeneity, can choose to normalize each (\mathbf{w}_j, b_j) to have unit norm $\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2} = 1$, & use penalty $|\eta_j|$ for each $j \in \{1, \dots, m\}$, & thus use an overall l_1 -norm penalty on η , i.e., $\|\eta\|_1$ (will consider other normalizations for input weights, either to ease exposition or to induce another behavior; e.g., by using l_1 -norms on \mathbf{w}_j 's). Focus on this choice of regularization in following sections. In this chapter, R denotes an almost sure upper bound on x directly, not on a feature map $\varphi(x)$ (as done in earlier chapters).

- * **9.2.3. Estimation Error.** To study estimation error, will consider: parameters of network are constrained, i.e., $\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2} = 1$ for each $j \in \{1, \dots, m\}$ & $\|\eta\|_1 \leq D$. This defines a set Φ of allowed parameters $\theta = \{(\eta_j), (\mathbf{w}_j), (b_j)\}$. Defining class \mathcal{F} of neural network models f_θ with parameters $\theta \in \Theta$, can compute its Rademacher complexity using tools from Chap. 4 (Sect. 4.5). Assume: almost surely, $\|\mathbf{x}\|_2 \leq R$, i.e., input data are bounded in l_2 -norm by R . Following developments of Sect. 4.5 on Rademacher averages, denote by $\mathcal{G} = \{(x, y) \mapsto l(y, f(x)), f \in \mathcal{F}\}$ set of loss functions for a prediction function $f \in \mathcal{F}$. Note: following Sect. 4.5.3, consider a constraint on $\|\eta\|_1$, but could also penalize, which is more common to practice & can be tackled with tools from Sect. 4.5.5. Have, by def of Rademacher complexity $R_n(\mathcal{G})$ of \mathcal{G} , & taking expectations w.r.t. data $(x_i, y_i), i = 1, \dots, n$, which are assumed to be independent & identically distributed (i.i.d.), & independent Rademacher random variables $\varepsilon_i \in \{\pm 1\}, i = 1, \dots, n$:

$$R_n(\mathcal{G}) = \mathbb{E} \left[\sup_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \varepsilon_i l(y_i, f_\theta(x_i)) \right]. \quad (25)$$

This quantity is known to provide an upper bound on estimation error, as, using symmetrization from Prop. 4.2 & (4.10) from Sect. 4.4, when \hat{f} is a minimizer of empirical risk over \mathcal{F} , have

$$\mathbb{E} \left[\mathcal{R}(\hat{f}) - \inf_{f \in \mathcal{F}} \mathcal{R}(f) \right] \leq 4R_n(\mathcal{G}). \quad (26)$$

Can now use properties of Rademacher complexities presented in Sect. 4.5, particularly their nice handling of nonlinearities. Assuming: loss is G -Lipschitz-continuous w.r.t. 2nd variable, using Prop. 4.3 from Chap. 4, which allows getting rid of loss, get bound: **[skipped complicated estimates & techniques]**.

Since ReLU activation function is 1-Lipschitz continuous & satisfies $(0)_+ = 0$, get, this time using extension of Prop. 4.3 from Chap. 4 to Rademacher complexities defined with an absolute value (i.e., Prop. 4.4), which adds an extra factor of 2 **[skipped complicated estimates & techniques]**.

Thus, get Prop. 9.1, with a bound proportional to $\frac{1}{\sqrt{n}}$ with no explicit dependence in number of parameters.

Proposition 6 (Estimation error). *Let \mathcal{F} be class of neural networks defined in (9.1), with constraint that $\|\eta\|_1 \leq D$ & $\|\mathbf{w}_j\|_2^2 + \frac{b_j^2}{R^2} = 1, \forall j \in \{1, \dots, m\}$, with ReLU activation function. If loss function is G -Lipschitz-continuous, then, for \hat{f} a minimizer of empirical risk over \mathcal{F} ,*

$$\mathbb{E} \left[\mathcal{R}(\hat{f}) - \inf_{f \in \mathcal{F}} \mathcal{R}(f) \right] \leq \frac{16GDR}{\sqrt{n}}. \quad (27)$$

Prop. 9.1 will be combined with a study of approximation properties in Sect. 9.3, with a summary provided in Sect. 9.4. Will see in Chap. 12 some recent results showing how optimization algorithms add an implicit regularization that leads to provable generalization in overparameterized neural networks (i.e., networks with many hidden units).

For estimation error, number of parameters is irrelevant! What counts is overall norm of weights.

Problem 9. *Provide a bound similar to Prop. 9.1 for alternative constraint $\|\mathbf{w}_j\|_1 + \frac{|b_j|}{R} = 1$, where R denotes supremum of $\|\mathbf{x}\|_\infty$ over all \mathbf{x} in support of its distribution.*

Before moving on to approximation properties of neural networks, note: reasoning given here for computing Rademacher complexity can be extended by recursion to deeper networks & other activation functions, as Exercise 9.2 shows (see, e.g., Neyshabur et al., 2015, for further results).

Problem 10. *Consider a 1-Lipschitz-continuous activation function σ s.t. $\sigma(0) = 0$, & classes of functions defined recursively as $\mathcal{F}_0 = \{\mathbf{x} \mapsto \boldsymbol{\theta}^\top \mathbf{x}, \|\boldsymbol{\theta}\|_2 \leq D_0\}$, & for $i = 1, \dots, M, \mathcal{F}_i = \{\mathbf{x} \mapsto \sum_{j=1}^{m_i} \theta_j \sigma(f_j(\mathbf{x})), f_j \in \mathcal{F}_{i-1}, \|\theta\|_1 \leq D_i\}$, corresponding to a neural network with M layers. Assuming $\|\mathbf{x}\|_2 \leq R$ almost surely, show by recursion: Rademacher complexity satisfies $R_n(\mathcal{F}_M) \leq 2^M \frac{R}{\sqrt{n}} \prod_{i=0}^M D_i$.*

- o **9.3. Approximation Properties.** As seen in Sect. 9.2.3, estimation error for constrained output weights grows as $\frac{\|\eta\|_1}{\sqrt{n}}$, where η : vector of output weights & is independent of number m of neurons. Several important questions will be tackled in following sects:

- * *Universality*: Can we approximate any prediction function with a sufficiently large number of neurons?
- * *Bound on approximation error*: What is associated approximation error so that we can derive generalization bounds? How can we use control of l_1 -norm $\|\boldsymbol{\eta}\|_1$, particularly when number of neurons m is allowed to tend to ∞ ?
- * *Finite number of neurons*: What is number of neurons required to reach such a behavior?

To do this, need to understand space of functions that neural networks span & how they relate to smoothness properties of function (as did for kernel methods in Chap. 7).

Focus on ReLU activation function, note: universal approximation results exist as soon as activation function is not a polynomial (Leshno et al., 1993). Start with a simple nonquantitative argument to show universality in 1D (& then in all dimensions) before formalizing function space obtained by letting number of neurons go to ∞ .

- * 9.3.1. Universal Approximation Property in 1D. Start with a number of simple, nonquantitative arguments.

Approximation of continuous piecewise affine functions. Since each individual function $x \mapsto \eta_j(w_j x + b_j)_+$ is continuous piecewise affine, output of a neural network has to be continuous piecewise affine as well. Turn out: all continuous piecewise affine functions with $m-2$ kinks in open interval $(-R, R)$ can be represented by m neurons on $[-R, R]$. Indeed, as illustrated here with $m = 8$, if assume: function f is s.t. $f(-R) = 0$, with kinks $a_1 < \dots < a_{m-2}$ on $(-R, R)$, can approximate it on $[-R, a_1]$ by function $v_1(x + R)_+$ where v_1 : slope of f on $[-R, a_1]$. Approximation is tight on $[-R, a_1]$. To have a tight approximation on $[a_1, a_2]$ without perturbing approximation on $[-R, a_1]$, can add to approximation $v_2(x - a_1)_+$, where v_2 is exactly what is needed to compensate for change in slope of f . By pursuing this reasoning, can present function on $[-R, R]$ exactly with $m - 1$ neurons Fig.

To remove constraint $f(-R) = 0$, can simply notice: $\frac{1}{2R}(x + R)_+ + \frac{1}{2R}(-x + R)_+$ is equal to 1 on $[-R, R]$. Thus, with 1 additional neuron (only 1 since $(x + R)_+$ has already been used), can represent any piecewise-affine function with $m - 2$ kinks using m neurons. This argument will be made more quantitative in Sect. 9.3.3 by looking at slopes of piecewise affine function.

Universal approximation properties. Now can represent precisely all continuous piecewise affine functions on $[-R, R]$, can use classical approximation theorems for functions on $[-R, R]$. They come in different flavors depending on norm used to characterize approximation. E.g., continuous functions can be approximated by piecewise affine functions with arbitrary precision in L_∞ -norm (defined as maximal value of $|f(x)|$ for $x \in [-R, R]$) by simply taking piecewise interpolant from a grid (see quantitative arguments in Sect. 9.3.3). With a weaker criterion e.g. L^2 -norm (w.r.t. Lebesgue measure), can approximate any function in L^2 (see, e.g., [Rud87]). This can be extended to any dimension d by using Fourier transform representation as $f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega}^\top \mathbf{x}} d\boldsymbol{\omega}$ & approximating 1D functions sine & cosine as linear superpositions of ReLUs. See a more formal quantitative argument in Sect. 9.3.4.

To obtain precise bounds in all dimensions in terms of number of kinks or l_1 -norm of output weights, 1st need to define limit when number of neurons diverges.

- * 9.3.2. Infinitely Many Neurons & Variation Norm. In this section, consider neural networks of form $f(x) = \sum_{j=1}^m \eta_j(\mathbf{w}_j^\top \mathbf{x} + b_j)_+$, where input weights are constrained, i.e., $(\mathbf{w}_j, \frac{b_j}{R}) \in K$, for K a compact subset of \mathbb{R}^{d+1} , e.g. unit l_2 -sphere (but will consider a slightly different set at end of this sect). Goal: to define set of functions that can be approximated by neural networks, while defining a norm on them that extends l_1 -norm of output weights. Consider \mathcal{X} d -dimensional l_2 -ball of radius R & center 0 (but construction applies to any compact subset of \mathbb{R}^d).

Formulation through measures. Can write a neural network with finitely many neurons $f(x) = \sum_{j=1}^m \eta_j(\mathbf{w}_j^\top \mathbf{x} + b_j)_+$ as integral (9.4)

$$f(x) = \int_K (\mathbf{w}^\top \mathbf{x} + b)_+ d\nu(\mathbf{w}, b), \quad (28)$$

for ν being signed measure $\nu = \sum_{j=1}^m \eta_j \delta_{(\mathbf{w}_j, b_j)}$, where $\delta_{(\mathbf{w}_j, b_j)}$: Diract measure at (\mathbf{w}_j, b_j) . Penalty can be written as $\|\boldsymbol{\eta}\|_1 = \int_K |d\nu(\mathbf{w}, b)|$, which is total variation of ν .⁵

Since want to have a norm $\|\boldsymbol{\eta}\|_1$ which is as small as possible, among all representations of f as in (9.4), look for the one for which $\int_K |d\nu(\mathbf{w}, b)|$ is smallest, i.e., for $f \in \tilde{\mathcal{F}}_1$ set of neural networks with arbitrary (finite) width, define

$$\tilde{\gamma}_1(f) = \inf_{\nu \in \tilde{\mathcal{M}}(K)} \int_K |d\nu(\mathbf{w}, b)| \text{ s.t. } \forall x \in \mathcal{X}, f(x) = \int_K (\mathbf{w}^\top \mathbf{x} + b)_+ d\nu(\mathbf{w}, b), \quad (29)$$

where $\tilde{\mathcal{M}}(K)$: set of signed measures on K with *finite* support. This happens to define a norm on $\tilde{\mathcal{F}}_1$. In order to extend beyond set $\tilde{\mathcal{F}}_1$ (which is equal to set of continuous piecewise affine functions for $d = 1$), simply relax constraint of finite support for measure ν . I.e., for $f : \mathcal{X} \rightarrow \mathbb{R}$, define (9.5)

$$\gamma_1(f) = \inf_{\nu \in \mathcal{M}(K)} \int_K |d\nu(\mathbf{w}, b)| \text{ s.t. } \forall x \in \mathcal{X}, f(x) = \int_K (\mathbf{w}^\top \mathbf{x} + b)_+ d\nu(\mathbf{w}, b), \quad (30)$$

where $\mathcal{M}(K)$: set of signed measures on K with finite total variation, with convention: if no measure can be found to represent f , then $\gamma_1(f) = +\infty$. Prop. 9.2 shows: γ_1 defines a norm on set \mathcal{F}_1 of functions s.t. $\gamma_1(f)$ is finite.

Proposition 7. Assume $K \subset \mathbb{R}^{d+1}$ & $\mathcal{X} \subset \mathbb{R}^d$ are compact sets. Set \mathcal{F}_1 of functions s.t. $\gamma_1(f)$ defined in (9.5) is finite is a vector space, a subset of set of Lipschitz-continuous functions on \mathcal{X} . Moreover, γ_1 is a norm on \mathcal{F}_1 .

⁵When ν has density $\frac{d\nu}{d\tau}$ w.r.t. a base measure τ with full support in K , then total variation is defined as integral $\int_K |\frac{d\nu}{d\tau(\mathbf{w}, b)}| d\tau(\mathbf{w}, b)$ & is independent of choice of τ . See https://en.wikipedia.org/wiki/Total_variation for more details.

Obtain a Banach space \mathcal{F}_1 of functions (proof of completeness is left as a technical exercise), with a norm γ_1 that is often referred to as “variation norm” (Kurková & Sanguinetti, 2001). This characterizes set of functions that can be asymptotically reached by neural networks with a bounded l_1 -norm of output weights, regardless of number of neurons. Index 1 in γ_1 will become natural when we compare with positive-definite kernels in Sect. 9.5. Note: although defined it for ReLU activation, same argument applies to all continuous activation functions. Finally, in order to obtain upper bounds on $\gamma_1(f)$, it suffices to represent f as an integral of neurons as in (9.5), & compute corresponding total variation, e.e.g, for a single neuron $f(\mathbf{x}) = (\mathbf{w}^\top \mathbf{x} + b)_+$ for $(\mathbf{w}, b) \in K$, $\gamma_1(f) \leq 1$, a property that will be used several times in Sect. 9.3.3.

Note: due to positive homogeneity of ReLU activation function, norm γ_1 does not change if replace compact set K with $\bigcup_{c \in [0,1]} cK$ (i.e., union of all segments $[0, v]$ for $v \in K$), with a proof left as an exercise. Therefore, choosing unit l_2 -sphere or unit l_2 -ball for K gives same results. (Will make a slightly different choice below.)

Studying approximation properties of \mathcal{F}_1 . Have characterized function space \mathcal{F}_1 through (9.5), need to describe set of functions with finite norm & relate this norm to classical smoothness properties (as done for kernel methods in Chap. 7). To do so, as illustrated below, consider a smaller set K than unit l_2 -ball, i.e., set K of $(\mathbf{w}, \frac{b}{R})$ s.t. $\|\mathbf{w}\|_2 = \frac{1}{\sqrt{2}}, |b| \leq \frac{R}{\sqrt{2}}$, which is enough to obtain upper bounds on approximation errors. For simplicity, & losing a factor of $\sqrt{2}$, consider normalization $K = \{(\mathbf{w}, \frac{b}{R}) \in \mathbb{R}^{d+1}, \|\mathbf{w}\|_2 = 1, |b| \leq R\}$ & norm γ_1 defined in (9.5) with this set K . Note: for $d = 1$, have $K = \{(\mathbf{w}, \frac{b}{R}) \in \mathbb{R}^d, w \in \{\pm 1\}, |b| \leq R\}$, as illustrated below for $d = 1$ (with new set $\bigcup_{c \in [0,1]} cK$ in dark gray, & old one in light gray). Could stick to l_2 -sphere, but our particular choice of K leads to simpler formulas.

* **9.3.3. Variation Norm in 1D.** ReLU activation function is specific & leads to simple approximation properties in interval $[-R, R]$. As already qualitatively described in Sect. 9.3.1, start with continuous piecewise affine functions, which, given shape of ReLU activation, should be easy to approximate (& immediately lead to universal approximation results as all reasonable functions can be approximated by piecewise affine functions). See more details by Breiman (1993) & Barron & Klusowski (2018).

Continuous piecewise affine functions. Can make reasoning in Sect. 9.3.1 quantitative. Consider a continuous piecewise affine function on $[-R, R]$ with specific knots at each $-R = a_0 < a_1 < \dots < a_{m-2} < a_{m-1} = R$, so on $[a_j, a_{j+1}]$, f is affine with slope v_j , for $j \in \{0, \dots, m-2\}$. [Technical details]

$$\gamma_1(f) \leq \frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] + v_0 \right| + \frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] - v_{m-2} \right| + \sum_{j=1}^{m-2} |v_j - v_{j-1}|. \quad (31)$$

Norm is thus upper-bounded by values of f & its derivatives at boundaries of interval & sums of changes in slope.

Twice continuously differentiable functions. Now consider a twice continuously differentiable function f on $[-R, R]$, & would like to express it as a continuous linear combination of functions $x \mapsto (\pm x + b)_+$. Will consider 2 arguments: one through approximation by piecewise affine functions & one through Taylor’s formula with integral remainder.

Piecewise-affine approximation. Consider equally spaced knots $a_j = -R + \frac{j}{s}R$ for $j \in \{0, \dots, 2s\}$, & piecewise affine interpolation \hat{f} from values $a_j, f(a_j)$ (& slopes v_j on $[a_j, a_{j+1}]$), with $j \in \{0, \dots, 2s\}$, for s that will tend to ∞ (see following illustration, where have $m-1 = 2s$) [Technical details]

Thus, approximant \hat{f} has a γ_1 -norm $\gamma_1(\hat{f})$ upper-bounded asymptotically by

$$\frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] + f'(-R) \right| + \frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] - f'(R) \right| + \frac{R}{s} \sum_{j=1}^{2s-1} \left| f'' \left(-R + \frac{j}{s}R \right) \right|. \quad (32)$$

Last term $\frac{R}{s} \sum_{j=1}^{2s-1} \left| f'' \left(-R + \frac{j}{s}R \right) \right| \rightarrow \int_{-R}^R |f''(x)| dx$. Thus, letting $s \rightarrow \infty$, get (informally, as reasoning given next will make it more formal)

$$\gamma_1(f) \leq \frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] + f'(-R) \right| + \frac{1}{2} \left| \frac{1}{2R} [f(-R) + f(R)] - f'(R) \right| + \int_{-R}^R |f''(x)| dx. \quad (33)$$

This notably shows: although number of neurons is allowed to grow, l_1 -norm of weights remains bounded by quantity in (9.8).

Direct proof through Taylor’s formula. (9.8) can be extended to continuously differentiable functions, which are only twice differentiable a.e. with integrable 2nd-order derivatives. In this sect, assume: function f is twice continuously differentiable but could extend to only integrable 2nd derivatives by a density argument (see, e.g., [Rud87]). For such a function, using Taylor’s formula with integral remainder, have, for $x \in [-R, R]$, using fact $(x - b)_+ = 0$ as soon as $b \geq x$ [Technical details]

Will also use a simpler upper bound, obtained from triangle inequality:

$$\gamma_1(f) \leq \frac{1}{2R} |f(-R) + f(R)| + \frac{1}{2} |f'(R)| + \frac{1}{2} |f'(-R)| + \int_{-R}^R |f''(x)| dx. \quad (34)$$

Problem 11. Assume $-R = x_1 < \dots < x_n = R, y_1, \dots, y_n \in \mathbb{R}$, show: piecewise-affine interpolant on $[-R, R]$ is a minimum norm interpolant.

- * **9.3.4. Variation Norm in an Arbitrary Dimension.** In order to extend to larger dimensions than $d = 1$, will use Fourier transforms. This requires to consider functions on \mathcal{X} ball with center 0 & radius R as restrictions of functions defined on \mathbb{R}^d with compact support (so that they belong to $L^2(\mathbb{R}^d)$, space of square-integrable functions for Lebesgue measure, & $L^1(\mathbb{R}^d)$ space of integrable functions); this can be done in a number of ways (see [Rud87] & end of Sect. 7.5.2). [Technical details]

Obtain (9.14)

$$\gamma_1(f) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\omega)| \gamma_1(\mathbf{x} \mapsto e^{i\omega^\top \mathbf{x}}) d\omega \leq \frac{2}{(2\pi)^d R} \int_{\mathbb{R}^d} |\hat{f}(\omega)| (1 + 2R^2 \|\omega\|_2^2) d\omega. \quad (35)$$

Given function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, $\int_{\mathbb{R}^d} |\hat{g}(\omega)| d\omega$ is a measure of smoothness of g , so $\gamma_1(f)$ being finite imposes: f & all 2nd-order derivatives of f have this form of smoothness. RHS of (9.14) is often referred to as “Barron norm,” which is named after Barron (1993, 1994). See Klusowski & Barron (2018) for more details.

To relate norm γ_1 to other function spaces e.g. Sobolev spaces, will consider further upper bounds (& relate them to another norm γ_2 , described in Sect. 9.5).

Problem 12 (Step activation function). *Consider step activation function defined as $\sigma(u) = 1_{u>0}$. Show: corresponding variation norm can be upper-bounded by a constant times $\int_{\mathbb{R}^d} |\hat{f}(\omega)| (1 + R \|\omega\|_2) d\omega$.*

- * **9.3.5. Precise Approximation Properties.**
 - * **9.3.6. From Variation Norm to a Finite Number of Neurons.**
 - o **9.4. Generalization Performance for Neural Networks.**
 - o **9.5. Relationship with Kernel Methods.**
 - * **9.5.1. From a Banach Space \mathcal{F}_1 to a Hilbert Space \mathcal{F}_2 .**
 - * **9.5.2. Kernel Function.**
 - * **9.5.3. Upper Bound on RKHS Norm.**
 - o **9.6. Experiments.** Consider same experimental setup as Sect. 7.7, i.e., 1D problems to highlight adaptivity of neural methods to regularity of target function, with smooth targets & nonsmooth targets. Consider several values for number m of hidden neurons & a neural network with ReLU activation functions & an additional global constant term. Training is done by SGD with a small constant step size & random initialization.
- Note: for small m , while a neural network with same number of hidden neurons could fit data better, optimization is unsuccessful (SGD gets trapped in a bad local minimum). Moreover, between $m = 32$ & $m = 100$, do not see any overfitting, highlighting potential underfitting behavior of neural networks. See also Stewart et al. (2023) for a formulation of regression through classification that alleviates some of these issues, as well as <https://francisbach.com/quest-for-adaptivity/>.
- o **9.7. Extensions.** Fully connected, single-hidden-layer neural networks are far from what is used in practice, particularly in computer vision, & natural language processing. Indeed, state-of-the-art performance is typically achieved with following extensions:

- * **Going deep with multiple layers.** Most simple form of deep neural network is a multilayer, fully connected neural network. Ignoring constant terms for simplicity, it is of form $f(\mathbf{x}^{(0)}) = \mathbf{y}^{(L)}$, with input $\mathbf{x}^{(0)}$ & output $\mathbf{y}^{(L)}$ given by

$$\mathbf{y}^{(k)} = (W^{(k)})^\top \mathbf{x}^{(k-1)}, \quad (36)$$

$$\mathbf{x}^{(k)} = \sigma(\mathbf{y}^{(k)}), \quad (37)$$

where $W^{(l)}$: matrix of weights for layer l . For these models, obtaining simple & powerful theoretical results is still an active area of research in terms of approximation, estimation, & optimization errors. See, e.g., Lu et al. (2021), Ma et al. (2020), & Yang & Hu (2021). Among these results, so-called “neural tangent kernel” provides another link between neural networks & kernel methods beyond the one described in Sect. 9.5, & that applies more generally (see Sect. 12.4 & e.g., Jacot et al., 2018; Chizat et al., 2019).

- * **Residual networks.** An alternative to stacking layers 1 after the other as before is to introduce a different architecture of form:

$$\mathbf{y}^{(k)} = (W^{(k)})^\top \mathbf{x}^{(k-1)}, \quad (38)$$

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \sigma(\mathbf{y}^{(k)}). \quad (39)$$

Direct modeling of $\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}$ instead of $\mathbf{x}^{(k)}$ through an extra nonlinearity, originating from He et al. (2016), can be seen as a discretization of an ODE (see Chen et al., 2018).

- * **Convolutional neural networks.** To tackle large data & improve performances, important to use prior knowledge about typical data structure to process. E.g., for signals, images, & videos, important to take into account translation invariance (up to boundary issues) of domain. Done by constraining linear operators involved in linear part of neural networks to respect some form of translation invariance, & thus to use convolutions. See Goodfellow et al. (2016) for details. This can be extended beyond grids to topologies expressed in terms of graphs, leading to graph neural networks (see, e.g., Bronstein et al., 2021).

- * **Transformers.** 1 approach to capture long-range dependencies in sequential data $X = (x_1, \dots, x_L) \in \mathbb{R}^{L \times d}$, is to learn query $Q = W^{(Q)}X$, key $K = W^{(K)}X$, & value $V = W^{(V)}X$ matrices obtained by linear operators on X of compatible sizes, which are combined together to form an attention mapping (Bahdanau et al., 2014):

$$\text{attention}(Q, K, V) = \text{softmax}\left(\frac{QK^\top}{\sqrt{d}}\right)V. \quad (40)$$

Such a mapping is capable of capturing a variety of semantic relationships over sequences of data (e.g., grammatical relationships between query & key tokens within a corpus of text). Transformer (Vaswani et al., 2017) is an architecture that consists of stacked blocks made up of attention mappings, fully-connected layers & residual connections. Transformer architecture & its variants have a multitude of applications in fields e.g. natural language processing, audio, & computer vision.

- 9.8. Conclusions. In this chapter, have focused primarily on neural networks with 1 hidden layer & provided guarantees on approximation & estimation errors, which show: this class of models, if empirical risk minimization can be performed, leads to a predictive performance that improves on kernel methods from Chap. 7 by being adaptive to linear latent variables (e.g., dependence on an unknown linear projection of data). In particular, highlight: having a number of neurons in order of number of observations is not detrimental to good generalization performance, so long as norm of weights is controlled.
 - Việc có số lượng nơ-ron theo thứ tự số lượng quan sát không gây bất lợi cho hiệu suất tổng quát tốt, miễn là chuẩn mực của trọng số được kiểm soát.
- Pursue study of overparameterized models in Chap. 12, where show how optimization algorithms both globally converge & lead to implicit biases.

- 10. Ensemble Learning.
- 11. From Online Learning to Bandits.
- 12. Overparametrized Models.
- 13. Structured Prediction.
- 14. Probabilistic Methods.
- 15. Lower Bounds.
- Conclusion.

2.2 [BHH06]. FRANCIS R. BACH, DAVID HECKERMAN, ERIC HORVITZ. Considering Cost Asymmetry in Learning Classifiers

[188 citations]

- **Abstract.** Receiver Operating Characteristics (ROC) curves are a standard way to display performance of a set of binary classifiers \forall feasible ratios of costs associated with false positives & false negatives. For linear classifiers, set of classifiers is typically obtained by training once, holding constant estimated slope & then varying intercept to obtain a parameterized set of classifiers whose performances can be plotted in ROC plane. Consider alternative of varying asymmetry of cost function used for training. Show: ROC curve obtained by varying both intercept & asymmetry, & hence slope, always outperforms ROC curve obtained by varying only intercept. In addition, present a path-following algorithm for support vector machine (SVM) that can compute efficiently entire ROC curve, & that has the same computational complexity as training a single classifier. Finally, provide a theoretical analysis of relationship between asymmetric cost model assumed when training a classifier & cost model assumed in applying classifier. In particular, show: mismatch between step function used for testing & its convex upper bounds, usually used for training, leads to a provable & quantifiable difference around extreme asymmetries.

Keywords. SVM, receiver operating characteristic (ROC) analysis, linear classification

- 1. Introduction. Receiver Operating Characteristic (ROC) analysis has seen increasing attention in recent statistics & ML literature (Pepe, 2000; Provost & Fawcett, 2001; Flach, 2003). ROC is a representation of choice for displaying performance of a classifier when costs assigned by end users to false positives & false negatives are not known at time of training. E.g., when training a classifier for identifying cases of undesirable unsolicited email, end users may have different preferences about likelihood of a false negative & false positive. ROC curve for such a classifier reveals ratio of false negatives & positives at different probability thresholds for classifying an email message as unsolicited or normal email.

In this paper, consider linear binary classification of points in an Euclidean space – noting: it can be extended in a straightforward manner to nonlinear classification problems by using Mercer kernels (Schölkopf & Smola, 2002). I.e., given data $\mathbf{x} \in \mathbb{R}^d$, $d \geq 1$, consider classifiers of form $f(\mathbf{x}) = \text{sign}(\mathbf{w}^\top \mathbf{x} + b)$, where $\mathbf{w} \in \mathbb{R}^d, b \in \mathbb{R}$ are referred to as *slope* & *intercept*. To date, ROC curves have been usually constructed by training once, holding constant estimated slope & varying intercept to obtain curve. In this paper, show: while that procedure appears to be most practical thing to do, it may lead to classifiers with poor performance in some parts of ROC curve.

Crux of our approach: allow asymmetry of cost function to vary, i.e., vary ratio of cost of a false positive & cost of a false negative. For each value of ratio, obtain a different slope & intercept, each optimized for this ratio. In a naive implementation, varying asymmetry would require a retraining of classifier for each point of ROC curve, which would be computational expensive. In Sect. 3.1, present an algorithm that can compute solution of a support vector machine (SVM) (see, e.g., Schölkopf & Smola, 2002; Shawe-Taylor & Cristianini, 2004) for all possible costs of false positives & false negatives, with same computational complexity as obtaining solution for only 1 cost function. Algorithm extends to asymmetric costs algorithm of Hastie et al. (2005) & is based on path-following techniques that take advantage of piecewise linearity of path of optimal solutions. In Sect. 3.2, show how path-following algorithm can be used to obtain ROC curves. In particular, by allowing both asymmetry & intercept to vary, can obtain better ROC curves than by methods that simply vary intercept.

In Sect. 4, provide a theoretical analysis of relationship between asymmetry of costs assumed in training a classifier & asymmetry desired in its application. In particular, show: even in population (i.e., infinite sample) case, use of a training loss function which is a convex upper bound on true or testing loss function (a step function) creates classifiers with sub-optimal accuracy. Quantify this problem around extreme asymmetries for several classical convex-upper-bound loss functions, including square loss & *erf loss*, an approximation of logistic loss based on normal cumulative distribution functions (also referred to as “error function”, & usually abbreviated as *erf*). Analysis is carried through for Gaussian & mixture of Gaussian class-conditional distributions (see Sect. 4 for more details). Main result of this analysis: given an extreme user-defined testing asymmetry, training asymmetry should almost always be chosen to be less extreme.

Consequences of potential mismatch between cost functions assumed in testing vs. training underscore value of using algorithm that we introduce in Sect. 4.3. Even when costs are known (i.e., when only 1 point on ROC curve is needed), classifier resulting from our approach, which builds entire ROC curve, is never less accurate & can be more accurate than one trained with known costs using a convex-upper-bound loss function. Indeed, show in Sect. 4.3: computing entire ROC curve using our algorithm can lead to substantial gains over simply training once.

Paper is organized as follows:

- Sect. 2: give an introduction of linear classification problem & review ROC framework.
- Sect. 3: contains algorithm part of paper, while
- Sect. 4 provide a theoretical analysis of discrepancy between testing & training asymmetries, together with empirical results.

This paper is an extended version of previously published work (Bach et al., 2005a).

- 2. Problem Overview. Given data $\mathbf{x} \in \mathbb{R}^d$ & labels $y \in \{\pm 1\}$, consider linear classifiers of form $f(\mathbf{x}) = \text{sign}(\mathbf{w}^\top \mathbf{x} + b)$, where \mathbf{w} : *slope* of classifier & b : *intercept*. A classifier is determined by parameters $(\mathbf{w}, b) \in \mathbb{R}^{d+1}$. In Sect. 2.1, introduce notation & defs; in Sect. 2.2: lay out necessary concepts of ROC analysis, while in Sect. 2.3, describe how these classifiers & ROC curves are typically obtained from data.
- 2.1. Asymmetric Cost & Loss Functions. Positive (resp. negative) examples are those for which $y = 1$ (resp. $y = -1$). 2 types of misclassification, false positives & false negatives, are assigned 2 different costs. Let C_+ denote cost of a false negative & C_- cost of a false positive. Total *expected* cost is equal to

$$R(C_+, C_-, \mathbf{w}, b) = C_+ P\{\mathbf{w}^\top \mathbf{x} + b < 0, y = 1\} + C_- P\{\mathbf{w}^\top \mathbf{x} + b \geq 0, y = -1\}.$$

In context of large margin classifiers (see, e.g., Bartlett et al., 2004), expected cost is usually expressed in terms of *0-1 loss function*; indeed, if let $\phi_{0-1}(u) = 1_{u < 0}$ be 0-1 loss, can write expected cost as

$$R(C_+, C_-, \mathbf{w}, b) = C_+ E\{1_{y=1} \phi_{0-1}(\mathbf{w}^\top \mathbf{x} + b)\} + C_- E\{1_{y=-1} \phi_{0-1}(-\mathbf{w}^\top \mathbf{x} - b)\},$$

where E denotes expectation w.r.t. joint distribution of (\mathbf{x}, y) .

Expected cost defined using 0-1 loss is cost that end users are usually interested in during use of classifier, while other cost functions that define below are used solely for training purposes. Convexity of these cost functions makes learning algorithms convergent without local minima, & leads to attractive asymptotic properties (Bartlett et al., 2004).

A traditional set-up for learning linear classifiers from labeled data: consider a convex upper bound ϕ on 0-1 loss ϕ_{0-1} , & to use expected ϕ -cost:

$$R_\phi(C_+, C_-, \mathbf{w}, b) = C_+ E\{1_{y=1} \phi(\mathbf{w}^\top \mathbf{x} + b)\} + C_- E\{1_{y=-1} \phi(-\mathbf{w}^\top \mathbf{x} - b)\}.$$

Refer to ratio $\frac{C_-}{C_+ + C_-}$ as *asymmetry*. Shall use *training asymmetry* to refer to asymmetry used for training a classifier using a ϕ -cost, & *testing asymmetry* to refer to asymmetric cost characterizing testing situation (reflecting end user preferences) with actual cost based on 0-1 loss. In Sect. 4, show: these may be different in general case.

Shall consider several common loss functions. Some of loss functions (square loss, hinge loss) leads to attractive computational properties, while others (square loss, erf loss) are more amenable to theoretical manipulations (see Fig. 1: Loss functions. Left: 0-1 loss, hinge loss, erf loss, square loss. Right: 0-1 loss, probit loss, logistic loss. for plot of loss functions, as they are commonly used & defined below. Note: by rescaling, each of these loss functions can be made to be an upper bound on 0-1 loss which is tight at 0.):

* **square loss**: $\phi_{\text{sq}}(u) = \frac{1}{2}(u - 1)^2$; classifier is equivalent to linear regression on y ,

* **hinge loss**: $\phi_{\text{hi}}(u) = \max\{1 - u, 0\}$; classifier is support vector machine (Shawe-Taylor & Cristianini, 2004),

* **erf loss**: $\phi_{\text{erf}}(u) = [u\Psi(u) - u + \Psi'(u)]$, where Ψ : cumulative distribution of standard normal distribution, i.e.: $\Psi(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^v e^{-\frac{t^2}{2}} dt$ & $\Psi'(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$. Erf loss can be used to provide a good approximation of *logistic loss* $\phi_{\log}(u) = \log(1 + e^{-u})$ as well as its derivative, & is amenable to closed-form computations for Gaussians & mixture of Gaussians (see Sect. 4 for more details). Note: erf loss is different from *probit loss* $-\log \Psi(u)$, which leads to probit regression (Hastie et al., 2001).

- o 2.2. **ROC Analysis**. Aim of ROC analysis: display in a single graph performance of classifiers \forall possible costs of misclassification. Consider sets of classifiers $f_\gamma(x)$, parameterized by a variable $\gamma \in \mathbb{R}$ (γ can either be intercept or training asymmetry). For a classifier $f(x)$, can define a point (u, v) in “ROC plane”, where u : proportion of false positives $u = P(f(x) = 1 | y = -1)$, & v : proportion of true positives $v = P(f(x) = 1 | y = 1)$. When γ is varied, obtain a curve in ROC plane, ROC curve (see, e.g., Fig. 2: Left: ROC curve: regular ROC curve; convex envelope. Points a, c are ROC-consistent & point b is ROC-inconsistent. Right: ROC curve & dashed equi-cost lines: All lines have direction (p_+C_+, p_-C_-) , plain line is optimal & point a is optimal classifier.). Whether γ is intercept or training asymmetry, ROC curve always passes through point $(0, 0), (1, 1)$, which corresponds to classifying all points as negative (resp. positive).

Upper convex envelope of curve is set of optimal ROC points that can be achieved by set of classifiers; indeed, if a point in envelope is not 1 of original points, it must lie in a segment between 2 points $(u(\gamma_0), v(\gamma_0)), (u(\gamma_1), v(\gamma_1))$, & all points in a segment between 2 classifiers can always be attained by choosing randomly between 2 classifiers. Note: this classifier itself is not a linear classifier; its performance, defined by given true positive & false positive rates, can only be achieved by a mixture of 2 linear classifiers. However, if user is only interested in testing cost, which is a weighted linear combination of true positive & false positive rates, lowest testing cost is always achieved by both of these 2 classifiers (Sect. 4.3).

Denoting $p_+ = P(y = 1), p_- = P(y = -1)$, expected (C_+, C_-) -cost for a classifier (u, v) in ROC space, is simply $p_+C_+(1 - v) + p_-C_-u$, & thus optimal classifiers for (C_+, C_-) -cost can be found by looking at lines of slope that are normal to direction $(p_-C_-, -p_+C_+)$, which intersects ROC curve & are as close as point $(0, 1)$ as possible (see Fig. 2).

A point $(u(\gamma), v(\gamma))$ is said to be *ROC-consistent* if it lies on upper convex envelope; in this case, tangent direction $\left(\frac{du}{d\gamma}, \frac{dv}{d\gamma}\right)$ defines a cost $(C_+(\gamma), C_-(\gamma))$ for which classifier is optimal (for testing cost, which is defined using 0-1 loss). Condition introduced earlier, namely that $\left(\frac{du}{d\gamma}, \frac{dv}{d\gamma}\right)$ is normal to direction $(p_-C_-, -p_+C_+)$, leads to:

$$p_-C_- \frac{du}{d\gamma}(\gamma) - p_+C_+ \frac{dv}{d\gamma}(\gamma) = 0.$$

Optimal testing asymmetry $\beta(\gamma)$ defined as ratio $\frac{C_+(\gamma)}{C_+(\gamma) + C_-(\gamma)}$, is thus =

$$\beta(\gamma) := \frac{C_+(\gamma)}{C_+(\gamma) + C_-(\gamma)} = \frac{1}{1 + \frac{p_+ \frac{dv}{d\gamma}(\gamma)}{p_- \frac{du}{d\gamma}(\gamma)}}.$$

If a point $(u(\gamma), v(\gamma))$ is ROC-inconsistent, then quantity $\beta(\gamma)$ has no meaning, & such a classifier is generally useless, because, \forall settings of misclassification cost, that classifier can be outperformed by other classifiers or a combination of classifiers. See Fig. 2 for examples ROC-consistent & ROC-inconsistent points.

In Sect. 4, relate optimal asymmetry of cost in testing or eventual use of a classifier in real world, to asymmetry of cost used to train that classifier; in particular, show: they differ & quantify this difference for extreme asymmetries (i.e., close to corner points $(0, 0), (1, 1)$). This analysis highlights value of generating entire ROC curve, even when only 1 point is needed (Sect. 4.3).

Handling ROC surfaces. Also consider varying both asymmetry of cost function & intercept, leading to a set of points in ROC plane parameterized by 2 real values. Although concept of ROC-consistency could easily be extended to ROC surfaces, for simplicity do not consider it here. In all experiments, those ROC surfaces are reduced to curves by computing their convex upper envelopes.

- o 2.3. **Learning From Data**. Given n labeled data points $(x_i, y_i), i = 1, \dots, n$, *empirical cost* =

$$\hat{R}(C_+, C_-, \mathbf{w}, b) = \frac{C_+}{n} \# \{y_i(\mathbf{w}^\top \mathbf{x}_i + b) < 0, y_i = 1\} + \frac{C_-}{n} \# \{y_i(\mathbf{w}^\top \mathbf{x}_i + b) < 0, y_i = -1\}.$$

Empirical ϕ -cost =

$$\hat{R}_\phi(C_+, C_-, \mathbf{w}, b) = \frac{C_+}{n} \sum_{i \in I_+} \phi(y_i(\mathbf{w}^\top \mathbf{x}_i + b)) + \frac{C_-}{n} \sum_{i \in I_-} \phi(y_i(\mathbf{w}^\top \mathbf{x}_i + b)),$$

where $I_+ = \{i, y_i = 1\}, I_- = \{i, y_i = -1\}$. When learning a classifier from data, a classical setup: minimize sum of *empirical ϕ -cost* & a regularization term $\frac{1}{2n} \|\mathbf{w}\|^2$, i.e., to minimize $\hat{J}_\phi(C_+, C_-, \mathbf{w}, b) = \hat{R}_\phi(C_+, C_-, \mathbf{w}, b) + \frac{1}{2n} \|\mathbf{w}\|^2$.

Note: objective function is no longer homogeneous in (C_+, C_-) ; sum $C_+ + C_-$ is referred to as *total amount of regularization*. Thus, 2 end-user-defined parameters are needed to train a linear classifier: *total amount of regularization* $C_+ + C_- \in \mathbb{R}^+$, & *asymmetry* $\frac{C_+}{C_+ + C_-} \in [0, 1]$. In Sect. 3.1, show how minimum of $\hat{J}_\phi(C_+, C_-, \mathbf{w}, b)$, w.r.t. \mathbf{w} & b , can be computed efficiently

for hinge loss, for many values of (C_+, C_-) , with a computational cost that is within a constant factor of computational cost of obtaining a solution for 1 value of (C_+, C_-) .

Building an ROC curve from data. If a sufficiently large validation set is available, can train on training set & use empirical distribution of validation data to plot ROC curve. If sufficient validation data is not available, then can use several (typically 10 or 25) random splits of data & average scores over those splits to obtain ROC scores. Can also use this approach to obtain confidence intervals (Flach, 2003).

- 3. Building ROC Curves for SVM. Present an algorithm to compute ROC curves for SVM that explores 2D space of cost parameters (C_+, C_-) efficiently. 1st show how to obtain optimal solutions of SVM without solving optimization problems many times for each value of (C_+, C_-) . This method generalizes results of Hastie et al. (2005) to case of asymmetric cost functions. Then describe how space (C_+, C_-) can be appropriately explored & how ROC curves can be constructed.
 - 3.1. Building Paths of Classifiers. Given n data points $x_i, i = 1, \dots, n$ which belongs to \mathbb{R}^d , & n labels $y_i \in \{\pm 1\}$, minimizing regularized empirical hinge loss \Leftrightarrow solving following convex optimization problem (Schölkopf & Smola, 2002):

$$\min_{\mathbf{w}, b, \xi} \sum_i C_i \xi_i + \frac{1}{2} \|\mathbf{w}\|^2 \text{ s.t. } \forall i, \xi_i \geq 0, \xi_i \geq 1 - y_i(\mathbf{w}^\top \mathbf{x}_i + b),$$

where

$$C_i = \begin{cases} C_+ & \text{if } y_i = 1, \\ C_- & \text{if } y_i = -1. \end{cases}$$

Optimality conditions & dual problems. Know derive usual Karush–Kuhn–Tucker (KKT) optimality conditions (Boyd & Vandenberghe, 2003). Lagrangian of problem is (with dual variables $\alpha, \beta \in \mathbb{R}_+^n$):

$$L(\mathbf{w}, b, \xi, \alpha, \beta) = \sum_i C_i \xi_i + \frac{1}{2} \|\mathbf{w}\|^2 + \sum_i \alpha_i (1 - y_i(\mathbf{w}^\top \mathbf{x}_i + b)) - \xi_i - \sum_i \beta_i \xi_i.$$

Derivatives w.r.t. primal variables are

$$\frac{\partial L}{\partial w} = w - \sum_i \alpha_i y_i x_i, \quad \frac{\partial L}{\partial b} = - \sum_i \alpha_i y_i, \quad \frac{\partial L}{\partial \xi_i} = C_i - \alpha_i - \beta_i.$$

1st set of KKT conditions corresponds to nullity of Lagrangian derivatives w.r.t. primal variables, i.e.: (2)

$$w = \sum_i \alpha_i y_i x_i, \quad \sum_i \alpha_i y_i = \alpha^\top \mathbf{y} = 0, \quad \forall i, \quad C_i = \alpha_i + \beta_i.$$

Slackness conditions (Điều kiện lỏng lẻo) are:

$$\forall i, \alpha_i (1 - \xi_i + y_i(\mathbf{w}^\top \mathbf{x}_i + b)) = 0, \quad \beta_i \xi_i = 0.$$

Finally dual problem can be obtained by computing minimum of Lagrangian w.r.t. primal variables. If denote K $n \times n$ Gram matrix of inner products, i.e., defined by $K_{ij} = \mathbf{x}_i^\top \mathbf{x}_j$ & $\tilde{K} = \text{Diag}(\mathbf{y}) K \text{Diag}(\mathbf{y})$, dual problem is:

$$\max_{\alpha \in \mathbb{R}_+^n} -\frac{1}{2} \alpha^\top \tilde{K} \alpha + 1^\top \alpha \text{ s.t. } \alpha^\top \mathbf{y} = 0, \quad \forall i, \quad 0 \leq \alpha_i \leq C_i.$$

In following, only consider primal variables (\mathbf{w}, b, ξ) & dual variables (α, β) that verify 1st set of KKT condition (2), which implies: \mathbf{w}, β are directly determined by α .

Piecewise affine solutions. Following Hastie et al. (2005), for an optimal set of primal-dual variables (\mathbf{w}, b, α) , can separate data points in 3 disjoint sets, depending on values of $y_i(\mathbf{w}^\top \mathbf{x}_i + b) = (\tilde{K}\alpha)_i + by_i$.

- * Margin: $\mathcal{M} = \{i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) = 1\}$,
- * Left of margin: $\mathcal{M} = \{i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) < 1\}$,
- * Right of margin: $\mathcal{M} = \{i, y_i(\mathbf{w}^\top \mathbf{x}_i + b) > 1\}$.

- 3.2. Constructing ROC Curve.
- 4. Training vs. Testing Asymmetry.
 - 4.1. Optimal Solutions for Extreme Cost Asymmetries.
 - 4.2. Expansion of Testing Asymmetries.
 - 4.3. Building Entire ROC Curve for a Single Point.

- 5. Conclusion. Have presented an efficient algorithm to build ROC curves by varying training cost asymmetries for SVMs. Algorithm is based on piecewise linearity of path of solutions when cost of false positives & false negatives vary. Have also provided a theoretical analysis of relationship between potentially different cost asymmetries assumed in training & testing classifiers, showing: they differ under certain circumstances. In particular, in case of extreme asymmetries, our theoretical analysis suggests: training asymmetries should be chosen less extreme than testing asymmetry.

Have characterized key relationships, & have worked to highlight potential practical value of building entire ROC curve even when a single point may be needed. All learning algorithms considered in this paper involve using a convex surrogate (lỗi thay thế) to correct non differentiable non convex loss function. Our theoretical analysis implies: because use a convex surrogate, using testing asymmetry for training leads to non-optimal classifiers. Thus propose to generate all possible classifiers corresponding to all training asymmetries, & select the one that optimizes a good approximation to true loss function on unseen data (i.e., using held out data or cross validation). As shown in Sect. 3, it turns out: this can be done efficiently for SVM. Such an approach can lead to a significant improvement of performance with little added computational cost.

Note: although have focused in this paper on single kernel learning problem, our approach can be readily extended to multiple kernel learning setting (Bach et al., 2005b) with appropriate numerical path following techniques.

- Appendix A. Proof of Expansion of Optimal Solutions.
- Appendix B. Proof of Expansion of Testing Asymmetries.

2.3 WENBO CAO, WEIWEI ZHANG. *ML of PDEs from Noise Data*. **Theoretical & Applied Mechanics Letters**

[12 citations] Keywords. PDE, ML, Sparse regression, Noise data.

Abstract. ML of PDEs from data is a potential breakthrough for addressing lack of physical equations in complex dynamic systems. Recently, sparse regression has emerged as an attractive approach. However, noise presents biggest challenge in sparse regression for identifying equations, as it relies on local derivative evaluations of noisy data. This study proposes a simple & general approach that significantly improves noises robustness by projecting evaluated time derivative & partial differential term into a subspace with less noise. This method enables accurate reconstruction of PDEs involving high-order derivatives, even from data with considerable noise. Additionally, discuss & compare effects of proposed method based on Fourier subspace & POD (proper orthogonal decomposition) subspace. Generally, the latter yields better results since it preserves maximum amount of information.

- 1. Introduction. PDEs are increasingly important in modern science. PDEs are used to describe mathematical laws behind physical systems & play a vital role in analysis, prediction, & control of many systems. In past, PDEs were derived via basic conservation laws, which resulted in many canonical models in physics, engineering, & other fields. However, in modern applications, mechanisms of many complex systems remain unclear, making it difficult to derive PDEs. This is particularly true in fields e.g. multiphase flow, neuroscience, finance, bioscience, & others. In past decade, with rapid development of sensors, computing power & data storage, cost of data collection & computing has been greatly reduced, resulting in a large amount of experimental data. Meanwhile rapid development of ML [1] has also provided a reliable tool to discover potential laws of system from large datasets. Nowadays, ML of differential equations has become a promising new technology to discover physical laws in complex systems.

Among all methods investigated for model discovery [2–9], sparse regression [4–6] has gained most attention in recent studies because of its ability to discover interpretable & generalizable models with a balance of accuracy & efficiency. This approach provides an important model discovery framework, relying on sparse linear regression to select sparse terms to match data from a predefined function library which containing many nonlinear & partial derivative terms. Sparse regression has been applied to many challenging problems to identify potential ODEs or PDEs, including fluid dynamics [10–12], turbulence closures [13–15], vortex-induced vibration [16], subsurface flow equations [17,18], & others. More details on important applications & extensions of sparse regression can be found in [19].

Nowadays, biggest challenge with sparse regression: identify equations from noisy data. Sensitivity of sparse regression to noise seriously damages reliability of results because sparse regression relies on accurate evaluation of derivatives, which is especially challenging for PDEs where noise can be strongly amplified when computing higher-order spatial derivatives. Various approaches have been proposed to improve noise robustness of sparse regression, which can be roughly divided into 3 types. 1st: obtain smooth data by using smooth function to approximate noisy data globally, including use of neural network [8,20–23] or filter program. This approach can filter out part of noise, but usually introduces new approximation errors, which may result in data no longer strictly satisfying original governing equation. A typical demonstration: performing such preprocessing on clean data usually leads to loss of accuracy. 2nd: use accurate derivation approximation methods [5,21], including spline smoothing, polynomial fitting, Gaussian kernel smoothing & Tikhonov differentiation. These methods approximate noisy data locally, & they also introduce approximation errors, but usually small than 1st type of methods. 3rd: avoid numerical differentiation by explicitly or implicitly introducing numerical integration [20,24–26] for ODE discovery or reduce order of derivative by using weak form [27–31] for PDE discovery. These methods can greatly improve robustness of sparse regression against noise. Because they are derived by mathematical methods, they will not lose accuracy or even improve accuracy when used for clean data. However, those methods are complex to use & often contain many hyperparameters.

Present paper significantly improves noise robustness of sparse regression by simply projecting time derivative & library functions into a low noise subspace. Below introduce our method & discuss its similarities & differences with other methods, & then give some representation examples.

- **Methods.** Here, 1st describe general framework of sparse regression, then propose our improved method

- **2.1. Sparse regression.** Consider general form of nonlinear PDE $u_t = N(u, u_x, u_{xx}, \dots, x, \mu)$, where, subscripts denote partial differentiation in either time or space, & $N(\cdot)$: an unknown nonlinear function of state $u(t, x)$. Method builds an over-completed library that contains all terms that may appear in PDE, e.g.

$$\theta(u) = [u, u^2, u_x, uu_x, u^2u_x, u_{xx}, uu_{xx}, u^2u_{xx}, u_{xxx}, uu_{xxx}, u^2u_{xxx}]. \quad (41)$$

Then, time derivative u_t can be expressed as a linear combination of library terms (2)

$$u_t = \sum_{i=1}^K \xi_i f_i(u) = \theta(u)\xi. \quad (42)$$

For given experimental data of a physical field, values of u_t & $f_k(u)$ at many spatiotemporal points are evaluated, which lead to a system of linear equations (3)

$$(\mathbf{U}_t)_c = (\boldsymbol{\Theta}(u))_c \boldsymbol{\xi}, \quad (43)$$

where, $(\cdot)_c$: operator that arranges data into a column vector, $(\boldsymbol{\Theta})_c$ denotes $[(\mathbf{F}_1)_c, (\mathbf{F}_2)_c, \dots, (\mathbf{F}_K)_c]$, $\mathbf{U}_t, \mathbf{F}_k$: matrices whose i, j element holds function values of $u_t, f_k(u)$, resp., at i th time at j th space position, & $\boldsymbol{\xi}$: a vector of unknown coefficients. Each element in $\boldsymbol{\xi}$ is a coefficient corresponding to a term in PDE. Many dynamical systems have relatively few active terms in governing equations. Thus, may employ sparse regression to identify sparse vector $\boldsymbol{\xi}$, which signifies fewest active terms from library that result in a good model fit. This can be represented as (4)

$$\boldsymbol{\xi} = \arg \min_{\boldsymbol{\xi}} \|(\mathbf{U}_t)_c - (\boldsymbol{\Theta}(u))_c \boldsymbol{\xi}\|_2^2 + R(\boldsymbol{\xi}). \quad (44)$$

Regularizer $R(\boldsymbol{\xi})$ is chosen to promote sparsity of $\boldsymbol{\xi}$. E.g., sequentially thresholded least-squares (STLS) [4] uses $R(\boldsymbol{\xi}) = \lambda \|\boldsymbol{\xi}\|_0$, whereas sequentially thresholded ridge regression (STRidge) [5] uses $R(\boldsymbol{\xi}) = \lambda_1 \|\boldsymbol{\xi}\|_0 + \lambda_2 \|\boldsymbol{\xi}\|_2$. Solution of sparse vector $\boldsymbol{\xi}$ reveals hidden PDE of given system.

- **2.2. Subspace projection denoising.** Sparse regression is simple & effective, but difficult to identify PDE from noisy data because local derivative evaluation strongly amplifies noise. In this paper, propose subspace projection denoising (SPD) method to solve thorny problem.

As a basic assumption of sparse regression, any linear or nonlinear PDE can be expressed as a linear combination of library terms (2). Therefore, performing a same linear transformation on u_t & f_k , have

$$L(u_t) = L(\theta)\boldsymbol{\xi}, \quad (45)$$

where, L : a linear operator, $L(\boldsymbol{\theta})$ denotes $[L(f_1), L(f_2), \dots, L(f_K)]$. To filter noise as much as possible, project both time derivative \mathbf{U}_t & library function \mathbf{F}_k into a subspace with less noise

$$\tilde{\mathbf{U}}_t = \boldsymbol{\Theta}_t^\top \mathbf{U}_t \boldsymbol{\Theta}_x, \quad \tilde{\mathbf{F}}_k = \boldsymbol{\Theta}_t^\top \mathbf{F}_k \boldsymbol{\Theta}_x, \quad (46)$$

where, $\boldsymbol{\Theta}_t$: a matrix whose columns represent basis functions related to time, & $\boldsymbol{\Theta}_x$: a matrix whose columns represent basis functions related to space. Since projection is a linear operator, vector $\boldsymbol{\xi}$ can be obtained by solving

$$(\tilde{\mathbf{U}}_t)_c = (\tilde{\boldsymbol{\Theta}})_c \boldsymbol{\xi}, \quad (47)$$

where $(\tilde{\boldsymbol{\Theta}})_c := [(\tilde{\mathbf{F}}_1)_c, \dots, (\tilde{\mathbf{F}}_K)_c]$. Fourier basis can be used as basis function, in which case projection (6) can be easily implemented by fast Fourier Transformation (FFT) on each spatiotemporal axis, then low frequency components are chosen to solve vector $\boldsymbol{\xi}$. On other hand, proper orthogonal decomposition (POD) is a linear method for establishing an optimal basis, or modal decomposition, of an ensemble of continuous or discrete functions. Therefore, POD basis can also be selected to better characterize current data & PDE, in which case $\boldsymbol{\Theta}_t$ & $\boldsymbol{\Theta}_x$ are given by singular value decomposition of \mathbf{U} :

$$\mathbf{U} \cong \mathbf{W}_r \boldsymbol{\Sigma}_r \mathbf{V}_r^* = \boldsymbol{\Theta}_t \boldsymbol{\Sigma}_r \boldsymbol{\Theta}_x^*, \quad (48)$$

where $\boldsymbol{\Theta}_x^*$: transpose of $\boldsymbol{\Theta}_x$, r : a user-specified hyperparameter, which indicates that only 1st r bases with larger energy are retained, & $\boldsymbol{\Sigma}_r = \text{diag} \lambda_i$ is a diagonal matrix with singular values of \mathbf{U} , which represents energy of corresponding POD basis function. Obviously, proposed subspace projection denoising (SPD) method is independent of evaluation method of derivatives. It only projects evaluated time derivatives & library functions into a new space. Therefore, it can still be effectively combined with other robust derivation approximation methods. SPD method is quite simple & requires only minimal additional processing (6), but it significantly enhances robustness to noisy data, which will be verified later. Furthermore, method is suitable for high dimensional spatiotemporal data & PDE, requiring only additional rounds of FFT or projection.

Before presenting results, further illustrate proposed method by comparing it with low-pass filtering. When using Fourier basis, proposed method is similar to low-pass filtering of state $u(t, x)$, but their purposes & effects are different. Low-pass filtering of state $u(t, x)$ may change its distribution so that filtered data no longer satisfies original governing equation. Therefore, it can only filter out little noise to preserve real signal as much as possible. In contrast, SPD method performs low-pass filtering on evaluated time derivative & library functions. It can filter out part of real signal to filter noise as much as possible, & can use lowest frequency component with low-noise data to identify PDE, because (5) is always automatically satisfied for any linear operator L .

- 3. Parameter identification of PDEs. For simplicity, this subsect assumes: equation structure is known, & only parameters in equation are calculated from noisy data to evaluate effectiveness of proposed method. Obviously, accuracy of calculated parameter errors directly affects likelihood of identifying equation from library.

Adopt def of noise given by Rudy [5]

$$u_n = u + \sigma \times \text{std}(u) \times \text{randn}, \quad (49)$$

where u : numerical solution, σ : noise level, $\text{std}(u)$: standard deviation of u , randn : a random variable with standard normal distribution, & u_n represents data with noise level σ . Accuracy of model reconstruction quantified by relative errors

$$\text{error}_i = \left| \frac{\hat{\xi}_i - \xi_i}{\xi_i} \right|, \quad (50)$$

where ξ_i : correct coefficients & $\hat{\xi}_i$: coefficient estimated from noisy data. In all cases, hyperparameter r is set to 5. Therefore, total number of equations in system defined by (7) is 5^2 for 2D PDE & 5^3 for 3D PDE.

- 3.1. Burgers' equation. Burgers' equation arises in many technological contexts, including fluid mechanics, nonlinear acoustics, gas dynamics, & traffic flow. Take form $u_t = -uu_x + \nu u_{xx}$, where $\nu > 0$: diffusion coefficient. Burgers' equation is a nonlinear 2nd-order PDE. Test SPD method on Burgers' equation $u_t = \xi_1 uu_x + \xi_2 u_{xx}$ with unknown coefficients $\xi_1 = -1, \xi_2 = 0.05$. Data set used to identify coefficients is shown in Fig. 1: Solution of Burgers' equation with 0% noise & 20% noise.

1stly, cf relative error distributions of \mathbf{U}_{xx} evaluated from data with 20% noise in physical space, Fourier space & POD space. Fig. 2: Relative error $\log_{10} \left| \frac{\hat{U}_{xx} - U_{xx}}{U_{xx}} \right|$ evaluated from data with 20% noise in (a) physical space, (b) Fourier space, & (c) POD space, k_x, k_t represent corresponding coordinates in Fourier space, which are frequency indexes. n_x, n_t represent corresponding coordinates in POD space, & energy (i.e., singular value) of corresponding basis function decreases as they increase. Number of basis functions for both time & space are limited to 40 to make picture clearer. Fig. 2a shows: error is almost evenly distributed throughout original physical space. In Fourier space, error decreases as frequency decreases (Fig. 2b), while in POD space, error decreases as singular value increases (Fig. 2c). Therefore, projecting each term of library into Fourier subspace with lower frequencies or POD subspace with higher energy can significantly reduce error, as shown in lower left corner of Fig. 2b & c.

After that, PDE-FIND is used as baseline, in which derivative is evaluated by finite difference, & SPD method is used to improve noise robustness. Results for different noise levels are shown in Fig. 3: Coefficient identification errors of (a) uu_x . (b) u_{xx} in Burgers' equation against different noise levels, where local derivatives are evaluated by finite difference., where "initial" refers to identification results of projecting time derivative & library functions into Fourier subspace & POD subspace resp. (7).

As shown in Fig. 3, since noise is strongly amplified when computing derivative by finite difference, identified coefficients error of initial method is $> 10\%$ for only 1% noise, & coefficient error of u_{xx} is even close to 100%, which is actually maximum error because identified coefficient tends to 0 as noise level increases. Compared with initial method, SPD method with either Fourier bases or POD bases has a great improvement, which can decrease error by 2–4 orders of magnitude. Even for clean data, SPD method has less error than baseline. In addition, since time derivative & library functions have larger components in truncated POD subspace than Fourier subspace, former has smaller errors.

In additions, present a result of applying low-pass filtering on $u(t, x)$ with 0% noise, preserving same frequencies as SPD-Fourier method. As shown in Fig. 4: (a) Low-pass filtered data. (b) Error between filtered data & original data., low-pass filtering introduces considerable error because it retains only very few frequencies. As a result, coefficient identification error for clean data is as high as 20%, $\hat{\xi}_1 = -0.7973, \hat{\xi}_2 = 0.0602$. This supports the point mentioned in Sect. 2.2: low-pass filtering of $u(t, x)$ can cause data to no longer satisfy equation, whereas low-pass filtering of evaluated time derivative & function libraries does not.

Furthermore, polynomial fitting is used to evaluate local derivatives due to its robustness to noisy data. Fig. 5: Coefficient identification errors of (a) uu_x . (b) u_{xx} in Burgers' equation against different noise levels, where local derivatives are evaluated by polynomial fitting. shows: polynomial fitting reduces coefficient errors for noisy data compared with finite difference. But at same time, errors for clean data are also increased due to approximation error of polynomial fitting. Therefore, although polynomial fitting can improve robustness to noise, it also inevitably introduces new approximation errors. In contrast, SPD method filters noise without introducing any new errors because (5) is always true for any linear operator L . Although polynomial fitting reduces derivative evaluation error for noisy data, coefficient error of initial method is still $> 40\%$ for 20% noise, while SPD method can reduce errors by 2 orders of magnitude.

- 3.2. Kuramoto–Sivashinsky equation. Kuramoto–Sivashinsky describes chaotic dynamics of laminar flame fronts, reaction-diffusion systems, & coating flows. It takes form $u_t = -uu_x - u_{xx} - u_{xxx}$. This is a notable example of a nonlinear PDE

that involves high-order partial derivatives, which has made it difficult to identify from noisy data accurately. Test SPD method on equation $u_t = \xi_1 uu_x + \xi_2 u_{xx} + \xi_3 u_{xxx}$ with unknown coefficients $\xi_1 = \xi_2 = \xi_3 = -1$. Data set used to identify coefficients is shown in Fig. 6: Solution of Kuramoto–Sivashinsky equation with (a) 0% noise. (b) 20% noise.

Use PDE-FIND as baseline & polynomial fitting is used to evaluate derivatives. Fig. 7: Maximum coefficient error of Kuramoto–Sivashinsky equation against different noise levels. shows maximum coefficient error for different noise levels, which corresponds to 1 of terms in PDE. It shows: even for high-order derivative with 20% noise, SPD method can still reduce coefficient error to about 5%. This result is roughly equivalent to that of weak form [29,30], which is most robust method for noisy data reported. Moreover, SPD method is much simpler than weak form. In addition, identified coefficients have small error for clean data, which is caused by approximation error of polynomial fitting.

- 4. Sparse regression. Finally, as an example of how proposed method could be used in context of sparse regression, consider a numerical example in [29], which applies weak form to discover λ - ω reaction-diffusion system

$$u_t = D\nabla^2 u + \lambda u - \omega v, \quad (51)$$

$$v_t = D\nabla^2 v + \omega u + \lambda v, \quad (52)$$

where $\omega = -(u^2 + v^2)$, $\lambda = 1 - u^2 - v^2$, & $D = 0.1$ is constant. Weak form is 1 of current state-of-art methods for identifying equations from noisy data. It attempts to reduce order of derivatives by multiplying basis functions over terms in library & integrating result by parts over a spatiotemporal domain. In order to compare with weak form, use same data set (as shown in Fig. 8: A typical snapshot of solution of λ - ω reaction-diffusion system with (a) 0% noise. (b) 100% noise.) & library functions (as listed in (12)) as [29]. In total, generalized model involves a total of 20 different terms (2 diffusion terms & 18 polynomial terms). Correspondingly, 20 unknown coefficients need to be determined:

$$\theta_{u_t} = [\nabla^2 u, u, u^2, u^3, v, v^2, v^3, uv, u^2 v, uv^2], \quad \theta_{v_t} = [\nabla^2 v, u, u^2, u^3, v, v^2, v^3, uv, u^2 v, uv^2]. \quad (53)$$

1stly, evaluate accuracy of identified parameter under different noise levels & compare it with weak form [29]. Initial method & SPD method uses finite difference instead of polynomial fitting to evaluate derivative to avoid high computational cost. Fig. 9: Maximum coefficient error of λ - ω system against different noise levels shows: under all noise levels, errors of weak form & SPD method are dramatically reduced compared with baseline. When noise level is $> 3\%$, weak form & SPD method have about same accuracy; when noise level is smaller, SPD method has higher accuracy. In addition, emphasize: implementation of SPD is much simpler than weak form. In this case, error of SPD using Fourier bases is lower than that of POD bases. This can be explained by data's approximate periodicity along each space-time axis.

After that, use SINDy [4] algorithm to determine parsimonious model (mô hình tiết kiệm). Find: for noise levels of up to 10%, PDE was identified correctly for 200 cases with different random noises by SPD method with Fourier bases or POD bases. With 30% noise, model is identified correctly is about 6% of cases, with remaining cases featuring spurious terms that are not present in λ - ω system. For ref, PDE-FIND failed to correctly identify this PDE for as little as 1% noise & weak form correctly identify this PDE in about 95% of cases for 10% noise. Therefore, SPD method has roughly same effect as weak form, with a dramatic improvement over baseline.

- 5. Discussion. This work has developed a simple & effective method that greatly improves robustness & accuracy for model discovery, reducing data requirements & increasing noise tolerance. Proposed subspace projection denoising method projects evaluated time derivative & library terms into low frequency subspace with low noise or POD subspace with high energy so as to greatly reduce influence of noise. Moreover, method can be used in combination with many other methods, including polynomial fitting, neural network smoothing, to further improve robustness to noisy data. Several typical examples show: compared with baseline, SPD method can reduce error by several orders of magnitude, achieving same effect as weak form, while SPD method is simpler & has only 1 hyperparameter. More importantly, this study points out: original PDE automatically satisfied when any linear operator applied to evaluated time derivative & library functions, which provides a general framework for denoising, thus allowing more denoising methods with different linear operators.

2.4 KYUNGHYUN CHO. Machine Learning: A Lecture Note. May 8 2025

- Preface. Prepared this lecture note to teach DS-GA 1003 ML at Center for DS of NYU. 1st course on ML for master's & PhD students in DS, & goal: provide them with a solid foundation on top of which they can continue on to learn more advanced & modern topics in ML, DS + broadly AI. Because of this goal, this lecture note has quite a bit of mathematical derivations of various concepts in ML. This should not deter students from reading through this lecture note, as I have interleaved these deviations with accessible explanations on intuition & insights behind these derivations.

In preparing this lecture note, tried best to constantly remind of “Bitter Lesson” by RICHARD SUTTON [Sutton, 2019]. I forced myself to present various algorithms, models, & theories in ways that support scalable implementations, both for compute & data. All ML algorithms in this lecture are thus presented to work with stochastic gradient descent & its variants. There are other aspects of scalability, e.g. distributed computing, but I expect & hope: other more advanced follow-up courses would teach students with these advanced topics based on foundation this course has equipped those students with.

Despite my intention to cover as much foundational topics as possible in this course, it only became apparent: 1 course is not long enough to dig deeper into all of these topics. Had to make a difficult decision to omit some topics I find foundational, interesting, & exciting, e.g. online learning, kernel methods, & how to handle missing values. There were on other hand some

topics I intentionally omitted, although I believe them to be foundational as well, because they are covered extensively in various other courses, e.g. sequence modeling (or large-scale language modeling). Have furthermore refrained from discussing any particular application, hoping: there are other follow-up courses focused on individual application domains, e.g. computer vision, computational biology, & NLP.

There are a few more modern topics I hoped I could cover but could not due to time. To list a few of those, they include ODE based generative models & contrastive learning for both representation learning & metric learning. Perhaps in future, could create a 2-course series in ML & add these extra materials. Until then, students will have to look for other materials to learn about these topics.

– Có một số chủ đề hiện đại hơn mà tôi hy vọng có thể đề cập đến nhưng không thể vì thời gian. Để liệt kê một số chủ đề trong số đó, chúng bao gồm các mô hình tạo dựa trên ODE & học tương phản cho cả học biểu diễn & học số liệu. Có lẽ trong tương lai, có thể tạo một loạt 2 khóa học trong ML & thêm các tài liệu bổ sung này. Cho đến lúc đó, sinh viên sẽ phải tìm kiếm các tài liệu khác để tìm hiểu về các chủ đề này.

This lecture note is not intended to be a reference book but was created to be a teaching material. This is my way of apologizing in advance: have not been careful at all on extensively & exhaustively citing all relevant past literature.

- 1. An Energy Function. A usual way to teaching ML: go through different problem setups. It often starts with binary classification, when perceptron, logistic regression & SVMs are introduced, & continues with multiclass classification. At this point, usual to introduce regression as a continuous version of classification. Often, at this point, one would learn about kernel methods & neural networks, with focus on backpropagation (a more recent development in terms of teaching ML.) This is also at a point where one would take a detour by learning probabilistic ML, with eventual goal of introducing a Bayesian approach to ML, i.e., marginalization over optimization. Latter half of course would closely resemble contents so far however in an unsupervised setting, where we learn: ML can be useful even when observations are not associated with outcomes (labels). One would learn about a variety of matrix factorization techniques, clustering as well as probabilistic generative modeling. If lecturer were ambitious, they would sneak in 1 or 2 lectures on reinforcement learning at very end.

A main issue of teaching ML in such a conventional way: extremely inconvenient for students to see a common foundation underlying all these different techniques & paradigms. Often challenging for students to see how supervised & unsupervised learning connect with each other. Even more challenging for students to figure out: classification & clustering are simply 2 sides of same coin. In my opinion, simply impossible to make a majority of students see unifying foundation behind all these different techniques & paradigms if stick to enumerating all these paradigms & techniques. In this course, thus try to take a new approach to teaching ML, largely based on & inspired by an earlier tutorial paper authored by YAN LECUN & his colleagues [LeCun et al., 2006]. Other than this tutorial paper, this approach does not yet exist & will take a shape I continue to write this lecture note as course continues.

To begin on this journey, start by defining an energy function, or a negative compatibility score. This energy function e assigns a real value to a pair of an observed instance & a latent instance (x, z) & is parametrized by a multi-dimensional vector θ :
(1.1) $e : \mathcal{X} \times \mathcal{Z} \times \Theta \rightarrow \mathbb{R}$,

- 2. Basic Ideas in ML with Classification.
 - 2.1. Classification.
 - 2.2. Backpropagation.
 - 2.3. Stochastic Gradient Descent.
 - 2.4. Generalization & Model Selection.
 - 2.5. Hyperparameter Tuning: Model Selection.
- 3. Building Blocks of Neural Networks.
 - 3.1. Normalization.
 - 3.2. Convolutional blocks.
 - 3.3. Recurrent blocks.
 - 3.4. Permutation equivariance: attention.
- 4. Probabilistic ML & Unsupervised Learning.
 - 4.1. Probabilistic interpretation of energy function.
 - 4.2. Variational inference & Gaussian mixture models.
 - 4.3. Continuous latent variable models.
- 5. Undirected Generative Models.
 - 5.1. Restricted Boltzmann machines: Product of Experts.
 - 5.2. Energy-based generative adversarial networks.
 - 5.3. Autoregressive models.

- 6. Further Topics.
 - 6.1. Reinforcement Learning.
 - 6.2. Ensemble Methods.
 - 6.3. Meta-Learning.
 - 6.4. Regression: Mixture Density Networks.
 - 6.5. Causality.

2.5 [Cho21]. FRANCOIS CHOLLET. Deep Learning with Python. 2021

[401 Amazon ratings]

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About technology. Recent innovations in DL unlock exciting new software capabilities like automated language translation, image recognition, & more. DL is quickly becoming essential knowledge for every software developer, & modern tools like Keras & TensorFlow put it within your reach – even if have no background in mathematics or DS. This book shows you how to get started.

About book. *DL Learning with Python 2e* introduces field of DL using Python & powerful Keras library. In this revised & expanded new edition, Keras creator FRANCOIS CHOLLET offers insights for both novice & experienced ML practitioners. As move through this book, build understanding through intuitive explanations, crisp color illustrations, & clear examples. Quickly pick up skills need to start developing DL applications.

About reader. For readers with immediate Python skills. No previous experience with Keras, TensorFlow, or ML is required.

About Author. FRANCOIS CHOLLET is a software engineer at Google & creator of Keras DL library.

“CHOLLET explains complex concepts with minimal fuss. A joy to read.” – MARTIN GÖRNER, Google

- **Preface.** If picked up this book, probably aware of extraordinary process that DL has represented for field of AI in recent past. Went from near-unusable computer vision & natural language processing to highly performant systems deployed at scale in products you use every day. Consequences of this sudden progress extend to almost every industry. Already applying DL to an amazing range of important problems across domains as different as medical imaging, agriculture, autonomous driving, education, disaster prevention, & manufacturing.

Believe DL is still in its early days. DL has only realized a small fraction of its potential so far. Over time, DL will make its way to every problem where it can help – a transformation that will take place over multiple decades.

In order to begin deploying DL technology to every problem that it could solve, need to make it accessible to as many people as possible, including non-experts – people who aren’t researchers or graduate students. For DL to reach its full potential, need to radically democratize (dân chủ hóa triệt để) it. & today, believe we are at cusp of a historical transition, where DL is moving out of academic labs & R&D departments of large tech companies to become a ubiquitous part of toolbox of every developer out there – not unlike trajectory of web development in late 1990s. Almost anyone can now build a website or web app for their business or community of a kind that would have required a small team of specialist engineers in 1998. In not-so-distant future, anyone with an idea & basic coding skills will be able to build smart applications that learn from data.

When CHOLLET released 1st version of Keras DL framework in Mar 2015, democratization of AI wasn’t what CHOLLET had in mind. Had been doing research in ML for several years & had built Keras to help with his own experiments. But since 2015, hundreds of thousands of newcomers have entered field of DL; many of them picked up Keras as their tool of choice. As watched scores of smart people use Keras in unexpected, powerful ways, came to care deeply about accessibility & democratization of AI. Realized: further spread these technologies, more useful & valuable they become. Accessability quickly became an explicit goal in development of Keras, & over a few short years, Keras developer community has made fantastic achievements on this front. Put DL into hands of hundreds of thousands of people, who in turn are using it to solve problems that were until recently thought to be unsolvable.

Book is another step on way to making DL available to as many people as possible. Keras had always needed a companion course to simultaneously cover fundamentals of DL, DL best practices, & Keras usage patterns. In 2016 & 2017, did best

to produce such a course, which became 1e of this book, released in Dec 2017. Quickly became a ML best seller that sold > 50000 copies & was translated into 12 languages.

However, field of DL advances fast. Since release of 1e, many important developments have taken place – release of TensorFlow 2, growing popularity of Transformer architecture, & more. & so, in late 2019, set out to update book. Originally thought, quite naively, that it would feature about 50% new content & would end up being roughly same length as 1e. In practice, after 2 years of work, it turned out to be over a 3rd longer, with about 75% novel content. More than a refresh, it is a whole new book.

Wrote it with a focus on making concepts behind DL, & their implementation, as approachable as possible. Doing so didn't require to dumb down anything – strongly believe: there are no difficult ideas in DL. Hope find this book valuable & it will enable you to begin building intelligent applications & solve problems that matter to you.

Over past 6 years, Keras has grown to have hundreds of open source contributors & > 1 million users. Fantastic to see Keras adopted as TensorFlow's high-level API. A smooth integration between Keras & TensorFlow greatly benefits both TensorFlow users & Keras users, & makes DL accessible to most.

- **About this book.** This book was written for anyone who wishes to explore DL from scratch or broaden their understanding of DL. Whether a practicing ML engineer, a software developer, or a college student, find value in these pages.

Explore DL in an approachable way – starting simply, then working up to state-of-art techniques. Find: this book strikes a balance between intuition, theory, & hands-on practice. It avoids mathematical notation, preferring instead to explain code ideas of ML & DL via detailed code snippets & intuitive mental models. Learn from abundant code examples that include extensive commentary, practical recommendations, & simple high-level explanations of everything need to know to start using DL to solve concrete problems.

Code examples use Python DL framework Keras, with TensorFlow 2 as its numerical engine. They demonstrate modern Keras & TensorFlow 2 best practices as of 2021.

After reading this book, have a solid understand of what DL is, when it's applicable, & what its limitations are. Familiar with standard workflow for approaching & solving ML problems, & know how to address commonly encountered issues. Be able to use Keras to tackle real-world problems ranging from computer vision to natural language processing: image classification, image segmentation, timeseries forecasting, text classification, machine translation, text generation, & more.

- **Who should read this book.** This book is written for people with Python programming experience who want to get started with ML & DL. But this book can also be valuable to many different types of readers:

- * If a data scientist familiar with ML, this book will provide with a solid, practical introduction to DL, fastest-growing & most significant subfield of ML.
- * If a DL researcher or practitioner looking to get started with Keras framework, find this book to be ideal Keras crash course.
- * If a graduate student studying DL in a formal setting, find this book to be a practical complement to your education, helping build intuition around behavior of deep neural networks & familiarizing you with key best practices.

Even technically minded people who don't code regularly will find this book useful as an introduction to both basic & advanced DL concepts.

In order to understand code examples, need reasonable Python proficiency. Additionally, familiarity with NumPy library will be helpful, although it isn't required. Don't need previous experience with ML or DL: this book covers, from scratch, all necessary basics. Don't need an advanced mathematics background, either – high school-level mathematics should suffice in order to follow along.

- **About code.** This book contains many examples of source code both in numbered listings, & in line with normal text. In both cases, source code is formatted in a **fixed-width font** to separate it from ordinary text.

In many cases, original source code has been reformatted, added line breaks & reworked indentation to accommodate available page space in book. Additionally, comments in source code have often been removed from listings when code is described in text. Code annotations accompany many of listings, highlighting important concepts.

Jupyter notebooks on GitHub at <https://github.com/fchollet/deep-learning-with-python-notebooks> can be run directly in browser via Google Colaboratory, a hosted Jupyter notebook environment that you can use for free. An internet connection & a desktop web browser are all you need to get started with DL.

- **About Author.** FRANCOIS CHOLLET is creator of Keras, 1 of most widely used DL frameworks. Currently a software engineer at Google, where leads Keras team. In addition, he does research on abstraction, reasoning, & how to achieve greater generality in AI.

- **1. What is DL?** Cover: High-level definitions of fundamental concepts. Timeline of development of ML. Key factors behind DL's rising popularity & future potential.

In past few years, AI has been a subject of intense media hype. ML, DL, & AI come up in countless articles, often outside of technology-minded publications. Promised a future of intelligent chatbots, self-driving cars, & virtual assistants – a future sometimes painted in a grim light & other times as utopian, where human jobs will be scarce & most economic activity will be handled by robots or AI agents. For a future or current practitioner of ML, important to be able to recognize signal amid

noise, so that you can tell world-changing developments from overhyped press releases. Our future is at stake, & it's a future in which you have an active role to play: after reading this book, will be 1 of those who develop those AI systems. So tackle these questions: What has DL achieved so far? How significant is it? Where are we headed next? Should you believe hype?

This chap provides essential context around AI, ML, & DL.

- 1.1. AI, ML, & DL. 1st, need to define clearly what talking about when mention AI. What are AI, ML, & DL. How do they relate to each other?

- * 1.1.1. AI. AI was born in 1950s, when a handful of pioneers from nascent field of CS started asking whether computers could be made to “think” – a question whose ramifications still exploring today.

While many of underlying ideas had been brewing in years & even decades prior, “AI” finally crystallized as a field of research in 1956, when JOHN MCCARTHY, then a young Assistant Professor of Mathematics at Dartmouth College, organized a summer workshop under the following proposal:

Study: proceed on basis of conjecture that every aspect of learning or any other feature of intelligence can in principle be so precisely described that a machine can be made to simulate it. An attempt will be made to find how to make machines use language, form abstractions & concepts, solve kinds of problems now reserved for humans, & improve themselves. Think: a significant advance can be made in 1 or more of these problems if a carefully selected group of scientists work on it together for a summer.

At end of summer, workshop concluded without having fully solved riddle it set out to investigate. Nevertheless, it was attended by many people who would move on to become pioneers in field, & it set in motion an intellectual revolution that is still ongoing to this day.

Concisely, AI can be described as *effort to automate intellectual tasks normally performed by humans*. As such, AI is a general field that encompasses ML & DL, but that also includes many more approaches that may not involve any learning. Consider that until 1980s, most AI textbooks didn't mention “learning” at all! Early chess programs, e.g., only involved hardcoded rules crafted by programmers, & didn't qualify as ML. In fact, for a fairly long time, most experts believed: human-level AI could be achieved by having programmers handcraft a sufficiently large set of explicit rules for manipulating knowledge stored in explicit databases. This approach is known as *symbolic AI*. It was dominant paradigm in AI from 1950s to late 1980s, & it reached its peak popularity during *expert systems* boom o 1980s.

Although symbolic AI proved suitable to solve well-defined, logical problems, e.g. playing chess, it turned out to be intractable to figure out explicit rules for solving more complex, fuzzy problems, e.g. image classification, speech recognition, or natural language translation. A new approach arose to take symbolic AI's place: *ML*.

- * 1.1.2. ML. In Victorian England, Lady ADA LOVELACE was a friend & collaborator of CHARLES BABBAGE, inventor of *Analytical Engine*: 1st-known general-purpose mechanical computer. Although visionary & far ahead of its time, Analytical Engine wasn't meant as a general-purpose computer when it was designed in 1830s & 1840s, because concept of general-purpose computation was yet to be invented. It was merely meant as a way to use mechanical operations to automate certain computations from field of mathematical analysis – hence name Analytical Engine. As such, it was intellectual descendant of earlier attempts at encoding mathematical operations in gear form, e.g. Pascaline, or Leibniz's step reckoner, a refined version of Pascaline. Designed by BLAISE PASCAL in 1642 (at age 19), Pascaline was world's 1st mechanical calculator – it could add, subtract, multiply, or even divide digits.

In 1843, ADA LOVELACE remarked on invention of Analytical Engine,

Analytical Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform ... Its province is to assist us in making available what we've already acquainted with.

Even with 178 years of historical perspective, LADY LOVELACE's observation remains arresting. Could a general-purpose computer “originate” anything, or would it always be bound to dully execute processes we humans fully understand? Could it ever be capable of any original thought? Could it learn from experience? Could it show creativity?

Her remark was later quoted by AI pioneer ALAN TURING as “LADY LOVELACE's objection” in his landmark 1950 paper “Computing Machinery & Intelligence,” which introduced *Turing test* as well as key concepts that would come to shape AI.⁶ Turing was of opinion – highly provocative at time – computers could in principle be made to emulate all aspects of human intelligence.

Usual way to make a computer do useful work: have a human programmer write down *rules* – a computer program – to be followed to turn input data into appropriate answers, just like Lady LOVELACE writing down step-by-step instructions for Analytical Engine to perform. ML turns this around: machine looks at input data & corresponding answers, & figures out what rules should be (Fig. 1.2: ML: a new programming paradigm.). A ML system is *trained* rather than explicitly programmed. It's presented with many examples relevant to a task, & it finds statistical structure in these examples that eventually allows system to come up with rules for automating task. E.g., if wished to automate task of tagging vacation pictures, could present a ML system with many examples of pictures already tagged by humans, & system would learn statistical rules for associating specific pictures to specific tags.

Although ML only started to flourish in 1990s, it has quickly become most popular & most successful subfield of AI, a trend driven by availability of faster hardware & larger datasets. ML is related to mathematical statistics, but it differs from statistics in several important ways, in same sense: medicine is related to chemistry but cannot be reduced to chemistry, as medicine deals with its own distinct systems with their own distinct properties. Unlike statistics, ML

⁶Although Turing test has sometimes been interpreted as a literal test – a goal of field of AI should set out to reach – Turing merely meant it as a conceptual device in a philosophical discussion about nature of cognition.

tends to deal with large, complex datasets (e.g. a dataset of millions of images, each consisting of tens of thousands of pixels) for which classical statistical analysis e.g. Bayesian analysis would be impractical. As a result, ML, & especially DL, exhibits comparatively little mathematical theory – maybe too little – & is fundamentally an engineering discipline. Unlike theoretical physics or mathematics, ML is a very hands-on field driven by empirical findings & deeply reliant on advances in software & hardware.

- * 1.1.3. **Learning rules & representations from data.** To define *DL* & understand difference between DL & other ML approaches, 1st need some idea of what ML algorithms do. Just stated: ML discovers rules for executing a data processing task, given examples of what’s expected. So, to do ML, need 3 things:

- *Input data points* – e.g., if task is speech recognition, these data points could be sound files of people speaking. If task is image tagging, they could be pictures.
- *Examples of expected output* – In a speech-recognition task, these could be human-generated transcripts of sound files. In an image task, expected outputs could be tags e.g. “dog”, “cat”, & so on.
- *A way to measure whether algorithm is doing a good job* – This is necessary in order to determine distance between algorithm’s current output & its expected output. Measurement is used as a feedback signal to adjust way algorithm works. This adjustment step is what we call *learning*.

A ML model transforms its input data into meaningful outputs, a process that is “learned” from exposure to known examples of inputs & outputs. Therefore, central problem in ML & DL: *meaningfully transform data*: i.e., to learn useful *representations* of input data at hand – representations that get us closer to expected output.

Before go any further: what’s a representation? At its core, it’s a different way to look at data – to represent or encode data. E.g., a color image can be encoded in RGB format (red-green-blue) or in HSV format (hue-saturation-value): these are 2 different representations of same data. Some tasks that may be difficult with 1 representation can become easy with another. E.g., task “select all red pixels in image” is simpler in RGB format, whereas “make image less saturated” is simpler in HSV format. ML models are all about finding appropriate representations for their input data – transformations of data that make it more amenable to task at hand.

Make its concrete. Consider an x -axis, a y -axis, & some points represented by their coordinates in (x, y) system, as shown in Fig. 1.3: **Some sample data**.

As can see, have a few white points & a few black points. Say want to develop an algorithm that can take coordinates (x, y) of a point & output whether that point is likely to be black or to be white. In this case,

- Inputs are coordinates of our points.
- Expected outputs are colors of our points.
- A way to measure whether our algorithm is doing a good job could be, e.g., percentage of points that are being correctly classified.

What need here is a new representation of our data that cleanly separates white points from black points. 1 transformation could use, among many other possibilities, would be a coordinate change, illustrated in Fig. 1.4: **Coordinate change**.

In this new coordinate system, coordinates of our points can be said to be a new representation of our data. & it’s a good one! With this representation, black/white classification problem can be expressed as a simple rule: “Black points are s.t. $x > 0$,” or “White points are s.t. $x < 0$.” This new representation, combined with this simple rule, neatly solves classification problem.

In this case defined coordinate change by hand: used our human intelligence to come up with our own appropriate representation of data. This is fine for such an extremely simple problem, but could do same if task were to classify images of handwritten digits? Could you write down explicit, computer-executable image transformations that would illuminate difference between a 6 & an 8, between a 1 & a 7, across all kinds of different handwriting?

This is possible to an extent. Rules based on representations of digits e.g. “number of closed loops” or vertical & horizontal pixel histograms can do a decent job of telling apart handwritten digits. But finding such useful representations by hand is hard work, &, as can imagine, resulting rule-based system is brittle (giòn, mỏng manh) – a nightmare to maintain. Every time you come across a new example of handwriting that breaks your carefully thought-out rules, you will have to add new data transformations & new rules, while taking into account their interaction with every previous rule.

Probably think: if this process is so painful, could automate it? What if tried systematically searching for different sets of automatically generated representations of data & rules based on them, identifying good ones by using a feedback percentage of digits being correctly classified in some development dataset? Would then be doing ML. *Learning*, in context of ML, describes an automatic search process for data transformations that produce useful representations of some data, guided by some feedback signal – representations that are amenable to simpler rules solving task at hand.

These transformations can be coordinate changes (like in 2D coordinates classification example), or taking a histogram of pixels & counting loops (like in digits classification example), but they could also be linear projections, translations, nonlinear operations (e.g. “select all points s.t. $x > 0$ ”), & so on. ML algorithms aren’t usually creative in finding these transformations; they’re merely searching through a predefined a set of operations, called a *hypothesis space*. E.g., space of all possible coordinate changes would be our hypothesis space in 2D coordinates classification example.

So that’s what ML is, concisely: searching for useful representations & rules over some input data, within a predefined space of possibilities, using guidance from a feedback signal. This simple idea allows for solving a remarkably broad range of intellectual tasks, from speech recognition to autonomous driving.

Now understand what mean by *learning*, take a look at what makes *DL* special.

- * 1.1.4. **“Deep” in “DL”.** DL is a specific subfield of ML: a new take on learning representations from data that puts an emphasis on learning successive layers of increasingly meaningful representations. “Deep” in “DL” isn’t a reference to any

kind of deeper understanding achieved by approach; rather, it stands for this idea of successive layers of representations. How many layers contribute to a model of data is called *depth* of model. Other appropriate names for field could have been *layered representations learning* or *hierarchical representations learning*. Modern DL often involves 10s or even hundreds of successive layers of representations, & they're all learned automatically from exposure to training data. Meanwhile, other approaches to ML tend to focus on learning only 1 or 2 layers of representations of data (say, taking a pixel histogram & then applying a classification rule); hence, they're sometimes called *shallow learning*.

In DL, these layered representations are learned via models called *neural networks*, structured in literal layers stacked on top of each other. Term “neural network” refers to neurobiology, but although some of central concepts in DL were developed in part by drawing inspiration from our understanding of brain (in particular, visual cortex), DL models are not models of brain. There's no evidence that brain implements anything like learning mechanisms used in modern DL models. May come across pop-science articles proclaiming: DL works like brain or was modeled after brain, but that isn't case. It would be confusing & counterproductive for newcomers to field to think of DL as being in any way related to neurobiology; don't need that shroud of “just like our minds” mystique & mystery, & you may as well forget anything you may have read about hypothetical links between DL & biology. For our purposes, DL is a mathematical framework for learning representations from data.

What do representations learned by a DL algorithm look like? Examine how a network several layers deep (Fig. 1.5: A deep neural network for digit classification.) transforms an image of a digit in order to recognize what digit it is.

As can see in Fig. 1.6: Data representations learned by a digit-classification model., network transforms digit image into representations that are increasingly different from original image & increasingly informative about final result. Can think of a deep network as a multistage *information-distillation* process, where information goes through successive filters & comes out increasingly *purified* (i.e., useful with regard to some task).

So that's what DL is, technically: a multistage way to learn data representations. It's a simple idea – but, as it turns out, very simple mechanisms, sufficiently scaled, can end up looking like magic.

- * 1.1.5. Understanding how DL works, in 3 figures. At this point, know: ML is about mapping inputs (e.g. images) to targets (e.g. label “cat”), which is done by observing many examples of input & targets. Also know: deep neural networks do this input-to-target mapping via a deep sequence of simple data transformations (layers) & these data transformations are learned by exposure to examples. Look at how this learning happens, concretely.

Specification of what a layer does to its input data is stored in layer's *weights*, which in essence are a bunch of numbers. In technical terms, say: transformation implemented by a layer is *parameterized* by its weights (Fig. 1.7: A neural network is parameterized by its weights. Goal: find right values for these weights.). (Weights are also sometimes called *parameters* of a layer.) In this context, *learning* means finding a set of values for weights of all layers in a network, s.t. network will correctly map example inputs to their associated targets. But here's the thing: a deep neural network can contain 10s of millions of parameters. Finding correct values for all of them may seem like a daunting task, especially given that modifying value of 1 parameter will affect behavior of all the others!

To control sth, 1s need to be able to observe it. To control output of a neural network, need to be able to measure how far this output is from what you expected. This is job of *loss function* of network, also sometimes called *objective function* or *cost function*. Loss function takes predictions of network & true target (what you wanted network to output) & computes a distance score, capturing how well network has done on this specific example (Fig. 1.8: A loss function measures quality of network's output.).

Fundamental trick in DL: use this score as a feedback signal to adjust value of weights a little, in a direction that will lower loss score for current example (Fig. 1.9: Loss score to used as a feedback signal to adjust weights.). This adjustment is job of *optimizer*, which implements what's called *Backpropagation* algorithm: central algorithm in DL. Next chap explains in more detail how backpropagation works.

Initially, weights of network are assigned random values, so network merely implements a series of random transformations. Naturally, its output is far from what it should ideally be, & loss score is accordingly very high. But with every example network processes, weights are adjusted a little in correct direction, & loss score decreases. This is *training loop*, which, repeated a sufficient number of times (typically 10s of iterations over thousands of examples), yields weight values that minimize loss function. A network with a minimal loss is 1 for which outputs are as close as they can be to targets: a trained network. Once again, it's a simple mechanism that, once scaled, ends up looking like magic.

- * 1.1.6. What DL has achieved so far. Although DL is a fairly old subfield of ML, it only rose to prominence in early 2010s. In few years since, it has achieved nothing short of a revolution in field, producing remarkable results on perceptual tasks & even natural language processing tasks – problems involving skills that seem natural & intuitive to humans but have long been elusive for machines.

In particular, DL has enabled following breakthroughs, all in historically difficult areas of ML:

- Near-human-level image classification
- Near-human-level speech transcription
- Near-human-level handwriting transcription
- Dramatically improved machine translation
- Dramatically improved text-to-speech conversion
- Digital assistants e.g. Google Assistant & Amazon Alexa
- Near-human-level autonomous driving
- Improved ad targeting, as used by Google, Baidu, or Bing

- Improved search results on web
- Ability to answer natural language questions
- Superhuman Go playing

Still exploring full extent of what DL can do. Have started applying it with great success to a wide variety of problems that were thought to be impossible to solve just a few years ago – automatically transcribing 10s of thousands of ancient manuscripts held in Vatican’s Apostolic Archive, detecting & classifying plant diseases in fields using a simple smartphone, assisting oncologists or radiologists with interpreting medical imaging data, predicting natural disasters e.g. floods, hurricanes, or even earthquakes, & so on. With every milestone, getting closer to an age where DL assists us in every activity & every field of human endeavor – science, medicine, manufacturing, energy, transportation, software development, agriculture, & even artistic creation.

- * 1.1.7. **Don’t believe short-term hype.** Although DL has led to remarkable achievements in recent years, expectations for what field will be able to achieve in next decade tend to run much higher than what will likely be possible. Although some world-changing applications like autonomous cars are already within reach, many more are likely to remain elusive for a long time, e.g. believable dialogue systems, human-level machine translation across arbitrary languages, & human-level natural language understanding. In particular, talk of human-level general intelligence shouldn’t be taken too seriously. Risk with high expectations for short term: as technology fails to deliver, research investment will dry up, slowing progress for a long time.

This has happened before. Twice in past, AI went through a cycle of intense optimism followed by disappointment & skepticism, with a dearth (sự khan hiếm) of funding as a result. It started with symbolic AI in 1960s. In those early days, projections about AI were flying high. 1 of best-known pioneers & proponents of symbolic AI approach was MARVIN MINSKY, who claimed in 1967, “Within a generation . . . problem of creating ‘artificial intelligence’ will substantially be solved.” 3 years later, in 1970, he made a more precisely quantified prediction: “In from 3–8 years will have a machine with general intelligence of an average human being.” In 2021 such an achievement will appear to be far in future – so far that have no way to predict how long it will take – but in 1960s & early 1970s, several experts believed it to be right around corner (as do many people today). A few years later, as these high expectations failed to materialize, researchers & government funds turned away from field, marking start of 1st *AI winter* (a reference to a nuclear winter, because this was shortly after height of Cold War).

It wouldn’t be last one. In 1980s, a new take on symbolic AI, *expert systems*, started gathering steam among large companies. A few initial success stories triggered a wave of investment, with corporations around world starting their own in-house AI departments to develop expert systems. Around 1985, companies were spending over \$1 billion each year on technology; but by early 1990s, these systems had proven expensive to maintain, difficult to scale, & limited in scope, & interest died down. Thus began 2nd AI winter.

May be currently witnessing 3rd cycle of AI hype & disappointment, & still in phase of intense optimism. Best to moderate our expectations for short term & make sure people less familiar with technical side of field have a clear idea of what DL can & can’t deliver.

- * 1.1.8. **Promise of AI.** Although may have unrealistic short-term expectations for AI, long-term picture is looking bright. Only getting started in applying DL to many important problems for which it could prove transformative, from medical diagnoses to digital assistants. AI research has been moving forward amazingly quickly in past 10 years, in large part due to a level of funding never before seen in short history of AI, but so far relatively little of this progress has made its way into products & processes that form our world. Most of research findings of DL aren’t yet applied, or at least are not applied to full range of problems they could solve across all industries. Your doctor doesn’t yet use AI, & neither does your accountant. Probably don’t use AI technologies very often in day-to-day life. Of course, can ask smartphone simple questions & get reasonable answers, can get fairly useful product recommendations on [Amazon.com](https://www.amazon.com), & can search for “birthday” on Google Photos & instantly find those pictures of daughter’s birthday party from last month. That’s a far cry from where such technologies used to stand. But such tools are still only accessories to our daily lives. AI has yet to transition to being central to way we work, think, & live.

Right now, it may seem hard to believe: AI could have a large impact on world, because it isn’t yet widely deployed (triển khai rộng rãi) – much as, back in 1995, it would have been difficult to believe in future impact of internet. Back then, most people didn’t see how internet was relevant to them & how it was going to change their lives. Same is true for DL & AI today. But make no mistake: AI is coming. In a not-so-distant future, AI will be your assistant, even your friend; it will answer questions, help educate your kids, & watch over your health. It will deliver your groceries to door & drive you from point A to point B. It will be your interface to an increasingly complex & information-intensive world. &, even more important, AI will help humanity as a whole move forward, by assisting human scientists in new breakthrough discoveries across all scientific fields, from genomics to mathematics.

On way, may face a few setbacks & maybe even a new AI winter – in much same way internet industry was overhyped in 1998–99 & suffered from a crash that dried up investment throughout early 2000s. But get there eventually. AI will end up being applied to nearly every process that makes up society & daily lives, much like internet is today.

Don’t believe short-term hype, but do believe in long-term vision. It may take a while for AI to be deployed to its true potential – a potential full extent of which no one has yet dared to dream – but AI is coming, & it will transform world in a fantastic way.

- 1.2. **Before DL: A brief history of ML.** DL has reached a level of public attention & industry investment never before seen in history of AI, but it isn’t 1st successful form of ML. Safe to say: most of ML algorithms used in industry today aren’t DL algorithms. DL isn’t always right tool for job – sometimes there isn’t enough data for DL to be applicable, & sometimes

problem is better solved by a different algorithm. If DL is your 1st contact with ML, may find yourself in a situation where all you have is DL hammer, & every ML problem starts to look like a nail. Only way not to fall into this trap: be familiar with other approaches & practice them when appropriate.

A detailed discussion of classical ML approaches is outside of scope of this book, but briefly go over them & describe historical context in which they were developed. This will allow us to place DL in broader context of ML & better understand where DL comes from & why it matters.

- * **1.2.1. Probabilistic modeling.** *Probabilistic modeling* is application of principles of statistics to data analysis. It is 1 of earliest forms of ML, & still widely used to this day. 1 of best-known algorithms in this category: Naive Bayes algorithm. Naive Bayes is a type of ML classifier based on applying Bayes' theorem while assuming: features in input data are all independent (a strong, or "naive" assumption, which is where name comes from). This form of data analysis predates computers & was applied by hand decades before its 1st computer implementation (most likely dating back to 1950s). Bayes' theorem & foundations of statistics date back to 18th century, & these are all you need to start using Naive Bayes classifiers.

A closely related model is *logistic regression* (logreg for short), which is sometimes considered to be "Hello World" of modern ML. Don't be misled by its name – logreg is a classification algorithm rather than a regression algorithm. Much like Naive Bayes, logreg predates computing by a long time, yet it's still useful to this day, thanks to its simple & versatile nature (bản chất đa dạng). It's often 1st thing a data scientist will try on a dataset to get a feel for classification task at hand.

- * **1.2.2. Early neural networks.** Early iterations of neural networks have been completely supplanted by modern variants covered in these pages, but it's helpful to be aware of how DL originated. Although core ideas of neural networks were investigated in toy forms as early as 1950s, approach took decades to get started. For a long time, missing piece was an efficient way to train large neural networks. This changed in mid-1980s, when multiple people independently rediscovered Backpropagation algorithm – a way to train chains of parametric operations using gradient-descent optimization (precisely define these concepts later) – & started applying it to neural networks.

1st successful practical application of neural nets came in 1989 from Bell Labs, when YANN LECUN combined earlier ideas of convolutional neural networks & backpropagation, & applied them to problem of classifying handwritten digits. Resulting network, dubbed *LeNet*, was used by United States Postal Service in 1990s to automate reading of ZIP codes on mail envelopes.

- * **1.2.3. Kernel methods.** As neural networks started to gain some respect among researchers in 1990s, thanks to this 1st success, a new approach to ML rose to fame & quickly sent neural nets back to oblivion: kernel methods. *Kernel methods* are a group of classification algorithms, best known of which is *Support Vector Machine* (SVM). Modern formulation of an SVM was developed by VLADIMIR VAPNIK & CORINNA CORTES in early 1990s at Bell Labs & published in 1995 [VLADIMIR VAPNIK & CORINNA CORTES, "Support-Vector Networks," *Machine Learning* 20, no. 3 (1995): 273–297.] although an older linear formulation was published by VAPNIK & ALEXEY CHERVONENKIS as early as 1963. [VLADIMIR VAPNIK & ALEXEY CHERVONENKIS, "A Note on 1 Class of Perceptrons," *Automation & Remote Control* 25 (1964).]

SVM is a classification algorithm that works by finding "decision boundaries" separating 2 classes (Fig. 1.10: A decision boundary). SVMs proceed to find these boundaries in 2 steps:

1. Data is mapped to a new high-dimensional representation where decision boundary can be expressed as a hyperplane (if data was 2D, as in Fig. 1.10, a hyperplane would be a straight line).
2. A good decision boundary (a separation hyperplane) is computed by trying to maximize distance between hyperplane & closest data points from each class, a step called *maximizing margin*. This allows boundary to generalize well to new samples outside of training dataset.

Technique of mapping data to a high-dimensional representation where a classification problem becomes simpler may look good on paper, but in practice it's often computationally intractable. That's where *kernel trick* comes in (key idea that kernel methods are named after). Here's gist (ý chính) of it: to find good decision hyperplanes in new representation space, don't have to explicitly compute coordinates of your points in new space; just need to compute distance between pairs of points in that space, which can be done efficiently using a kernel function. A *kernel function* is a computationally tractable operation that maps any 2 points in your initial space to distance between these points in your target representation space, completely bypassing explicit computation of new representation. Kernel functions are typically crafted by hand rather than learned from data – in case of an SVM, only separation hyperplane is learned.

At time they were developed, SVMs exhibited state-of-art performance on simple classification problems & were 1 of few ML methods backed by extensive theory & amenable to serious mathematical analysis, making them well understood & easily interpretable. Because of these useful properties, SVMs became extremely popular in field for a long time.

– Vào thời điểm chúng được phát triển, SVM đã thể hiện hiệu suất tiên tiến trên các vấn đề phân loại đơn giản & là 1 trong số ít phương pháp ML được hỗ trợ bởi lý thuyết mở rộng & có thể thực hiện phân tích toán học nghiêm túc, khiến chúng được hiểu rõ & dễ diễn giải. Nhờ những đặc tính hữu ích này, SVM đã trở nên cực kỳ phổ biến trong lĩnh vực này trong một thời gian dài.

But SVMs proved hard to scale to large datasets & didn't provide good results for perceptual problems e.g. image classification. Because an SVM is a shallow method, applying an SVM to perceptual problems requires 1st extracting useful representations manually (a step called *feature engineering*), which is difficult & brittle. E.g., if want to use an SVM to classify handwritten digits, can't start from raw pixels; should 1st find by hand useful representations that make problem more tractable, like pixel histograms.

- * 1.2.4. Decision trees, random forests, & gradient boosting machines. *Decision trees* are flowchart-like structures that let you classify input data points or predict output values given inputs (Fig. 1.11: A decision tree: parameters that are learned are questions about data. A question could be, e.g., “Is coefficient 2 in data > 3.5 ?). They are easy to visualize & interpret (hình dung & diễn giải). Decision trees learned from data began to receive significant research interest in 2000s, & by 2010 they were often preferred to kernel methods.
In particular, *Random Forest* algorithm introduced a robust, practical take on decision-tree learning that involves building a large number of specialized decision trees & then ensembling their outputs. Random forests are applicable to a wide range of problems – could say: they are almost always 2nd-best algorithm for any shallow ML task. When popular ML competition website Kaggle <http://kaggle.com> got started in 2010, random forests quickly became a favorite on platform – until 2014, when *gradient boosting machines* took over. A gradient boosting machine, much like a random forest, is a ML technique based on ensembling weak prediction models, generally decision trees. It uses *gradient boosting*, a way to improve any ML model by iteratively training new models that specialize in addressing weak points of previous models. Applied to decision trees, use of gradient boosting technique results in models that strictly outperform random forests most of time, while having similar properties. It may be 1 of best, if not *the* best, algorithm for dealing with nonperceptual data today. Alongside DL, it’s 1 of most commonly used techniques in Kaggle competitions.
- * 1.2.5. Back to neural networks. Around 2010, although neural networks were almost completely shunned by scientific community at large, a number of people still working on neural networks started to make important breakthroughs: group of GEOFFREY HINTON at University of Toronto, Yoshua Bengio at the University of Montreal, YANN LECUN at New York University, & IDSIA in Switzerland.
In 2011, DAN CIRESAN from IDSIA began to win academic image-classification competitions with GPU-trained deep neural networks – 1st practical success of modern DL. But watershed moment came in 2012, with entry of HILTON’s group in yearly large-scale image-classification challenge ImageNet (ImageNet Large Scale Visual Recognition Challenge, or ILSVRC for short). ImageNet challenge was notoriously difficult at time consisting of classifying high-resolution color images into 1000 different categories after training on 1.4 million images. In 2011, top-5 accuracy of winning model, based on classical approaches to computer vision, was only 74.3%.⁷ Then, in 2012, a team led by ALEX KRIZHEVSKY & advised by GEOFFREY HINTON was able to achieve a top-5 accuracy of 83.6% – a significant breakthrough. Competition has been dominated by deep convolutional neural networks every year since. By 2015, winner reached an accuracy of 96.4%, & classification task on ImageNet was considered to be a completely solved problem.
Since 2012, deep convolutional neural networks (*convnets*) have become go-to algorithm \forall computer vision tasks; more generally, they work on all perceptual tasks. At any major computer vision conference after 2015, it was nearly impossible to find presentations that didn’t involve convnets in some form. At same time, DL has also found applications in many other types of problems, e.g. natural language processing. It has completely replaced SVMs & decision trees in a wide range of applications. e.g., for several years, European Organization for Nuclear Research, CERN, used decision tree-based methods for analyzing particle data from ATLAS detector at Large Hadron Collider (LHC), but CERN eventually switched to Keras-based deep neural networks due to their highest performance & ease of training on large datasets.
- * 1.2.6. What makes DL different. Primary reason DL took off so quickly: it offered better performance for many problems. But that’s not only reason. DL also makes problem-solving much easier, because it completely automates what used to be most crucial step in a ML workflow: feature engineering.
Previous ML techniques – shallow learning – only involved transforming input data into 1 or 2 successive representation spaces, usually via simple transformations e.g. high-dimensional nonlinear projections (SVMs) or decision trees. But refined representations required by complex problems generally can’t be attained by such techniques. As such, humans had to go to great lengths to make initial input data more amenable to processing by these methods: they had to manually engineer good layers of representations for their data. This is called *feature engineering*. DL, on other hand, completely automates this step: with DL, learn all features in 1 pass rather than having to engineer them yourself. This has greatly simplified ML workflows, often replacing sophisticated multistage pipelines with a single, simple, end-to-end DL model.
May ask, if crux of issue is to have multiple successive layers of representations, could shallow methods be applied repeatedly to emulate effects of DL? In practice, successive applications of shallow-learning methods produce fast-diminishing returns, because optimal 1st representation layer in a 3-layer model isn’t optimal 1st layer in a 1-layer or 2-layer model. What is transformative about DL: DL allows a model to learn all layers of representation *jointly*, at same time, rather than in succession (*greedily*, as it’s called). With joint feature learning, whenever model adjusts 1 of its internal features, all other features that depend on it automatically adapt to change, without requiring human intervention. Everything is supervised by a single feedback signal: every change in model serves end goal. This is much more powerful than greedily stacking shallow models, because it allows for complex, abstract representations to be learned by breaking them down into long series of intermediate spaces (layers); each space is only a simple transformation away from previous one.
– Có thể hỏi, nếu cốt lõi của vấn đề là có nhiều lớp biểu diễn liên tiếp, thì các phương pháp học nông có thể được áp dụng nhiều lần để mô phỏng các hiệu ứng của DL không? Trong thực tế, các ứng dụng liên tiếp của các phương pháp học nông tạo ra lợi nhuận giảm dần nhanh chóng, vì lớp biểu diễn thứ nhất tối ưu trong mô hình 3 lớp không phải là lớp thứ nhất tối ưu trong mô hình 1 lớp hoặc 2 lớp. DL có đặc điểm gì là biến đổi: DL cho phép một mô hình học tất cả các lớp biểu diễn *kết hợp*, cùng một lúc, thay vì tuần tự (*tham lam*, như cách gọi của nó). Với việc học tính năng kết hợp, bất cứ khi nào mô hình điều chỉnh 1 trong các tính năng bên trong của nó, tất cả các tính năng khác phụ thuộc vào nó sẽ tự động thích ứng với sự thay đổi, mà không cần sự can thiệp của con người. Mọi thứ đều được giám sát bởi một tín hiệu phản hồi duy nhất: mọi thay đổi trong mô hình đều phục vụ cho mục tiêu cuối cùng. Điều này mạnh hơn nhiều so

⁷“Top-5 accuracy” measures how often model selects correct answer as part of its top 5 guesses (out of 1000 possible answers, in case of ImageNet).

với việc xếp chồng các mô hình nông một cách tham lam, vì nó cho phép học các biểu diễn trừu tượng, phức tạp bằng cách chia chúng thành một chuỗi dài các không gian trung gian (lớp); mỗi không gian chỉ là một phép biến đổi đơn giản so với không gian trước đó.

These are 2 essential characteristics of how DL learns from data: *incremental, layer-by-layer way in which increasingly complex representations are developed*, & *fact: these intermediate incremental representations are learned jointly*, each layer being updated to follow both representational needs of layer above & needs of layer below. Together, these 2 properties have made DL vastly more successful than previous approaches to ML.

- * **1.2.7. Modern ML landscape.** A great way to get a sense of current landscape ML algorithms & tools: look at ML competitions on Kaggle. Due to its highly competitive environment (some contests have thousands of entrants & million-dollar prizes) & to wide variety of ML problems covered, Kaggle offers a realistic way to assess what works & what doesn't. So what kind of algorithm is reliably winning competitions? What tools do top entrants use?

In early 2019, Kaggle ran a survey asking teams: ended in top 5 of any competition since 2017 which primary software tool they had used in competition (Fig. 1.12: ML tools used by top teams on Kaggle.). Turn out: top teams tend to use either DL methods (most often via Keras library) or gradient boosted trees (most often via LightGBM or XGBoost libraries). It's not just competition champions, either. Kaggle also runs a yearly survey among ML & DS professionals worldwide. With 10s of thousands of respondents, this survey is 1 of our most reliable sources about state of industry. Fig. 1.13: Tool usage across ML & DS industry www.kaggle.com/kaggle-survey-2020 shows percentage of usage of different ML software frameworks.

From 2016–2020, entire ML & DS industry has been dominated by these 2 approaches: DL & gradient boosted trees. Specifically, gradient boosted trees is used for problems where structured data is available, whereas DL is used for perceptual problems e.g. image classification.

Users of gradient boosted trees tend to use Scikit-learn, XGBoost, or LightGBM. Meanwhile, most practitioners of DL use Keras, often in combination with its parent framework TensorFlow. Common point of these tools is they're all Python libraries: Python is by far most widely used language for ML & DS>

These are 2 techniques you should be most familiar with in order to be successful in applied ML today: gradient boosted trees, for shallow-learning problems; & DL, for perceptual problems. In technical terms, this mean: need to be familiar with Scikit-learn, XGBoost, & Keras – 3 libraries that currently dominate Kaggle competitions. With this book in hand, already 1 big step closer.

- o **1.3. Why DL? Why now?** 2 key ideas of DL for computer vision – convolutional neural networks & backpropagation – were already well understood by 1990. Long Short-Term Memory (LSTM) algorithm, which is fundamental to DL for timeseries, was developed in 1997 & has barely changed since. So why did DL only take off after 2012? What changed in these 2 decades?

In general, 3 technical forces are driving advances in ML:

- * Hardware
- * Datasets & benchmarks
- * Algorithmic advances

Because field is guided by experimental findings rather than by theory, algorithmic advances only become possible when appropriate data & hardware are available to try new ideas (or to scale up old ideas, as is often case). ML isn't mathematics or physics, where major advances can be done with a pen & a piece of paper. It's an engineering science.

Real bottlenecks throughout 1990s & 2000s were data & hardware. But here's what happened during that time: internet took off & high-performance graphics chips were developed for needs of gaming market.

- * **1.3.1. Hardware.** Between 1990 & 2010, off-shelf CPUs became faster by a factor of approximately 5000. As a result, nowadays it's possible to run small DL models on laptop, whereas this would have been intractable 25 years ago.

But typical DL models used in computer vision or speech recognition require orders of magnitude more computational power than your laptop can deliver. Throughout 2000s, companies like NVIDIA & AMD invested billions of dollars in developing fast, massively parallel chips (graphical processing units, or GPUs) to power graphics of increasingly photorealistic video games – cheap, single-purpose supercomputers designed to render complex 3D scenes on your screen in real time. This investment came to benefit scientific community when, in 2007, NVIDIA launched CUDA <https://developer.nvidia.com/about-cuda>, a programming interface for its line of GPUs. A small number of GPUs started replacing massive clusters of CPUs in various highly parallelizable applications, beginning with physics modeling. Deep neural networks, consisting mostly of many small matrix multiplications, are also highly parallelizable, & around 2011 some researchers began to write CUDA implementations of neural nets – DAN CIREAN & ALEX KRIZHEVSKY were among 1st: [“Flexible, High Performance Convolutional Neural Networks for Image Classification,” Proceedings of the 22nd International Joint Conference on Artificial Intelligence (2011), www.ijcai.org/Proceedings/11/Papers/210.pdf.] [“ImageNet Classification with Deep Convolutional Neural Networks,” Advances in Neural Information Processing Systems 25 (2012), <http://mng.bz/2286>.]

What happened: gaming market subsidized (được trợ cấp) supercomputing for next generation of AI applications. Sometimes, big things begin as games. Today, NVIDIA Titan RTX, a GPU that cost \$2,500 at end of 2019, can deliver a peak of 16 teraFLOPS in single precision (16 trillion float32 operations per sec). That's about 500 times more computing power than world's fastest supercomputer from 1990s, Intel Touchstone Delta. On a Titan RTX, it takes only a few hours to train an ImageNet model of sort that would have won ILSVRC competition around 2012 or 2013. Meanwhile, large companies train DL models on clusters of hundreds of GPUs.

What's more, DL industry has been moving beyond GPUs & is investing in increasingly specialized, efficient chips for DL. In 2016, at its annual I/O convention, Google revealed its Tensor Processing Unit (TPU) project: a new chip design developed from ground up to run deep neural networks significantly faster & far more energy efficient than top-of-line GPUs. Today, in 2020, 3rd iteration of TPU card represent 420 teraFLOPs of computing power. That's 10000 times more than Intel Touchstone Delta from 1990.

These TPU cards are designed to be assembled into large-scale configurations, called "pods". 1 pod (1024 TPU cards) peaks at 100 petaFLOPs. For scale, that's about 10% of peak computing power of current largest supercomputer, IBM Summit at Oak Ridge National Lab, which consists of 27000 NVIDIA GPUs & peaks at around 1.1 exaFLOPs.

- * 1.3.2. **Data.** AI is sometimes heralded as new industrial revolution. If DL is steam engine of this revolution, then data is its coal: raw material that powers intelligent machines, without which nothing would be possible. When it comes to data, in addition to exponential progress in storage hardware over past 20 years (following Moore's law), game changer has been rise of internet, making it feasible to collect & distribute very large datasets for ML. Today, large companies work with image datasets, video datasets, & natural language datasets that couldn't have been collected without internet. User-generated image tags on Flickr, e.g., have been a treasure trove of data for computer vision. So are YouTube videos. & Wikipedia is a key dataset for natural language processing.

If there's 1 dataset that has been a catalyst for rise of DL, it's ImageNet dataset, consisting of 1.4 million images that have been hand annotated with 1000 image categories (1 category per image). But what makes ImageNet special isn't just its large size, but also yearly competition associated with it. ImageNet Large Scale Visual Recognition Challenge (ILSVRC), www.image-net.org/challenges/LSVRC.

As Kaggle has been demonstrating since 2010, public competitions are an excellent way to motivate researchers & engineers to push envelope. Having common benchmarks that researchers compete to beat has greatly helped rise of DL, by highlighting its success against classical ML approaches.

- * 1.3.3. **Algorithms.** In addition to hardware & data, until late 2000s, were missing a reliable way to train very deep neural networks. As a result, neural networks were still fairly shallow, using only 1 or 2 layers of representations; thus, they weren't able to shine against more-refined shallow methods e.g. SVMs & random forests. Key issue was that of *gradient propagation* through deep stacks of layers. Feedback signal used to train neural networks would fade away as number of layers increased.

This changed around 2009–2010 with advent of several simple but important algorithmic improvements that allowed for better gradient propagation:

- Better *activation functions* for neural layers
- Better *weight-initialization schemes*, starting with layer-wise pretraining, which was then quickly abandoned
- Better *optimization schemes*, e.g. RMSProp & Adam

Only when these improvements began to allow for training models with 10 or more layers did DL start to shine.

Finally, in 2014–2016, even more advanced ways to improve gradient propagation were discovered, e.g. batch normalization, residual connections, & depthwise separable convolutions.

Today, can train models that are arbitrarily deep from scratch. This has unlocked use of extremely large models, which hold considerable representational power, i.e., which encode very rich hypothesis spaces. This extreme scalability is 1 of defining characteristics of modern DL. Large-scale model architectures, which feature 10s of layers & 10s of millions of parameters, have brought about critical advances both in computer vision (e.g., architectures e.g. ResNet, Inception, or Xception) & natural language processing (e.g., large Transformer-based architectures e.g. BERT, GPT-3, or XLNet).

- * 1.3.4. **A new wave of investment.** As DL became new state of art for computer vision in 2012–2013, & eventually for all perceptual tasks, industry leaders took note. What followed was a gradual wave of industry investment far beyond anything previously seen in history of AI (Fig. 1.14: OECD estimate of total investments in AI startup.).

In 2011, right before DL took spotlight, total venture capital investment (tổng vốn đầu tư mạo hiểm) in AI worldwide was < a billion dollars, which went almost entirely to practical applications of shallow ML approaches. In 2015, it had risen to over \$5 billion, & in 2017, to a staggering \$16 billion. Hundreds of startups launched in these few years, trying to capitalize on DL hype. Meanwhile, large tech companies e.g. Google, Amazon, & Microsoft have invested in internal research departs in amounts that would most likely dwarf flow of venture-capital money.

ML – in particular, DL – has become central to product strategy of these tech giants. In late 2015, Google CEO SUNDAR PICHAI stated "ML is a core, transformative way by which we're rethinking how we're doing everything. Thoughtfully applying it across all products, be it search, ads, YouTube, or Play. & we're in early days, but you'll see us – in a systematic way – apply ML in all these areas." [SUNDAR PICHAI, Alphabet earnings call, Oct. 22, 2015.]

As a result of this wave of investment, number of people working on DL went from a few hundred to 10s of thousands in < 10 years, & research progress has reached a frenetic pace.

- * 1.3.5. **Democratization of DL.** 1 of key factors driving this inflow of new facts in DL has been democratization of toolsets in field. In early days, doing DL required significant C++ & CUDA expertise, which few people possessed.

Nowadays, basic Python scripting skills suffice to do advanced DL research. This has been driven most notably by development of now-defunct Theano library, & then TensorFlow library – 2 symbolic tensor-manipulation frameworks for Python that support autodifferentiation, greatly simplifying implementation of new models – & by rise of user-friendly libraries e.g. Keras, which makes DL as easy as manipulating LEGO bricks. After its release in early 2015, Keras quickly became go-to DL solution for large numbers of new startups, graduate students, & researchers pivoting into field.

- * 1.3.6. **Will it last?** Is there anything special about deep neural networks that makes them "right" approach for companies to be investing in & for researchers to flock to? Or is DL just a fad that may not last? Will still be using deep neural

networks in 20 years?

DL has several properties that justify its status as an AI revolution, & it's here to stay. May not be using neural networks 2 decades from now, but whatever we use will directly inherit from modern DL & its core concepts. These important properties can be broadly sorted into 3 categories:

- *Simplicity*: DL removes need for feature engineering, replacing complex, brittle, engineering-heavy pipelines with simple, end-to-end trainable models that are typically built using only 5 or 6 different tensor operations.
- *Scalability*: DL is highly amenable to parallelization (rất dễ dàng để song song hóa) on GPUs or TPUs, so it can take full advantage of Moore's law. In addition, DL models are trained by iterating over small batches of data, allowing them to be trained on datasets of arbitrary size. (Only bottleneck is amount of parallel computational power available, which, thanks to Moore's law, is a fast-moving barrier.)
- *Versatility & reusability*: Unlike many prior ML approaches, DL models can be trained on additional data without restarting from scratch, making them viable for continuous online learning – an important property for very large production models. Furthermore, trained DL models are repurposable & thus reusable: e.g., possible to take a DL model trained for image classification & drop it into a video-processing pipeline. This allows us to reinvest previous work into increasingly complex & powerful models. This also makes DL applicable to fairly small datasets.

DL has only been in spotlight for a few years, & may not yet have established full scope of what it can do. With every passing year, learn about new use cases & engineering improvements that lift previous limitations. Following a scientific revolution, progress generally follows a sigmoid curve: it starts with a period of fast progress, which gradually stabilizes as researchers hit hard limitations, & then further improvements become incremental.

When CHOLLET was writing of this book, in 2016, predicted: DL was still in 1st half of that sigmoid, with much more transformative progress to come in following few years. This has proven true in practice, as 2017 & 2018 have seen rise of Transformer-based DL models for natural language processing, which have been a revolution in field, while DL also kept delivering steady progress in computer vision & speech recognition. Today, in 2021, DL seems to have entered 2nd half of that sigmoid. Should still expect significant progress in years to come, but probably out of initial phase of explosive progress.

Today, extremely excited about deployment of DL technology to every problem it can solve – list is endless. DL is still a revolution in making, & will take many years to realize its full potential.

- 2. Mathematical building blocks of neural networks. Cover: A 1st example of a neural network. Tensors & tensor operations. How neural networks learn via backpropagation & gradient descent.

Understanding DL requires familiarity with many simple mathematical concepts: *tensors*, *tensor operations*, *differentiation*, *gradient descent*, & so on. Goal of this chap: build up your intuition about these notions without getting overly technical. In particular, steer away from mathematical notation, which can introduce unnecessary barriers for those without any mathematics background & isn't necessary to explain things well. Most precise, unambiguous description of a mathematical operation is its executable code.

To provide sufficient context for introducing tensors & gradient descent, begin chap with a practical example of a neural network. Then go over every new concept that's been introduced, point by point. Keep in mind: these concepts will be essential for you to understand practical examples in following chaps.

After reading this chap, have an intuitive understanding of mathematical theory behind DL, & ready to start diving into Keras & TensorFlow in Chap. 3.

- 2.1. A 1st look at a neural network. Look at a concrete example of a neural network that uses Python library Keras to learn to classify handwritten digits. Unless already have experience with Keras or similar libraries, won't understand everything about this 1st example right away. That's fine. In next chap, review each element in example & explain them in detail. So don't worry if some steps seem arbitrary or look like magic to you! Got to start somewhere.

Problem trying to solve: classify grayscale images of handwritten digits 28×28 pixels into their 10 categories (0 through 9). Use MNIST dataset, a classic in ML community, which has been around almost as long as field itself & has been intensively studied. It's a set of 60000 training images, plus 10000 test images, assembled by National Institute of Standards & Technology (NIST in MNIST) in 1980s. Can think of "solving" MNIST as "Hello World" of DL – it's what you do to verify that your algorithms are working as expected. As become a ML practitioner, see MNIST come up over & over again in scientific papers, blog posts, & so on. Can see some MNIST samples in Fig. 2.1: MNIST sample digits.

Note 1. In ML, a category in a classification problem is called a class. Data points are called samples. Class associated with a specific sample is called a label.

Don't need to try to reproduce this example on your machine just now. If wish to, 1st need to set up a DL workspace, covered in Chap. 3.

MNIST dataset comes preloaded in Keras, in form of a set of 4 NumPy arrays. Listing 2.1: Loading MNIST dataset in Keras.

```
from tensorflow.keras.datasets import mnist
(train_images, train_labels), (test_images, test_labels) = mnist.load_data()
```

`train_images`, `train_labels` form training set, data that model will learn from. Model will then be tested on test set, `test_images`, `test_labels`. Images are encoded as NumPy arrays, & labels are an array of digits, ranging from 0 to 9. Images & labels have a 1-1 correspondence. Look at training data:

```
>>> train_images.shape
(60000, 28, 28)
>>> len(train_labels)
60000
>>> train_labels
array([5, 0, 4, ..., 5, 6, 8], dtype=uint8)
```

Test data:

```
>>> test_images.shape
(10000, 28, 28)
>>> len(test_labels)
10000
>>> test_labels
array([7, 2, 1, ..., 4, 5, 6], dtype=uint8)
```

Workflow will be as follows: 1st, feed neural network training data, `train_images`, `train_labels`. Network will then learn to associate images & labels. Finally, ask network to produce predictions for `test_images`, & verify whether these predictions match labels from `test_labels`.

Build network – again, remember: aren’t expected to understand everything about this example yet. Listing 2.2: Network architecture:

```
from tensorflow import keras
from tensorflow.keras import layers
model = keras.Sequential([
    layers.Dense(512, activation="relu"),
    layers.Dense(10, activation="softmax")
])
```

Core building block of neural networks is *layer*. Can think of a layer as a filter for data: some data goes in, & it comes out in a more useful form. Specifically, layers extract *representations* out of data fed into them – hopefully, representations that are more meaningful for problem at hand. Most of DL consists of chaining together simple layers that will implement a form of progressive *data distillation*. A DL model is like a sieve for data processing, made of a succession of increasingly refined data filters – layers.

Here, model consists of a sequence of 2 `Dense` layers, which are densely connected (also called *fully connected*) neural layers. 2nd (& last) layer is a 10-way *softmax classification* layer, i.e. will return an array of 10 probability scores (summing to 1). Each score will be probability: current digit image belongs to 1 of our 10 digit classes.

To make model ready for training, need to pick 3 more things as part of *compilation* step:

- * *An optimizer*: Mechanism through which model will update itself based on training data it sees, so as to improve its performance.
- * *A loss function*: How model will be able to measure its performance on training data, & thus how it will be able to steer itself in right direction.
- * *Metrics to monitor during training & testing*: Here only care about accuracy (fraction of images that were correctly classified).

Exact purpose of loss function & optimizer will be made clear throughout next 2 chaps. Listing 2.3: Compilation step.

```
model.compile(optimizer="rmsprop",
              loss="sparse_categorical_crossentropy",
              metrics=["accuracy"])
```

Before training, preprocess data by reshaping it into shape model expects & scaling it so that all values are in $[0, 1]$ interval. Previously, training images were stored in an array of shape $(60000, 28, 28)$ of type `uint8` with values in $[0, 255]$ interval. Transform it into a `float32` array of shape $(60000, 28 \times 28)$ with values between 0 & 1. Listing 2.4: Preparing image data.

```
train_images = train_images.reshape((60000, 28 * 28))
train_images = train_images.astype("float32") / 255
test_images = test_images.reshape((10000, 28 * 28))
test_images = test_images.astype("float32") / 255
```

Now ready to train model, which in Keras is done via a call to model’s `fit()` method – *fit* model to its training data. Listing 2.5: “Fitting” model.

```
>>> model.fit(train_images, train_labels, epochs=5, batch_size=128)
Epoch 1/5
60000/60000 [=====] - 5s - loss: 0.2524 - acc: 0.9273
Epoch 2/5
51328/60000 [=====>.....] - ETA: 1s - loss: 0.1035 - acc: 0.9692
```

2 quantities are displayed during training: loss of model over training data, & accuracy of model over training data. Quickly reach an accuracy of 0.989 = 98.9% on training data.

Now: have a trained model, can use it to predict class probabilities for *new* digits – images that weren't part of training data, like those from test set. Listing 2.6: Using model to make predictions.

```
>>> test_digits = test_images[0:10]
>>> predictions = model.predict(test_digits)
>>> predictions[0]
array([1.0726176e-10, 1.6918376e-10, 6.1314843e-08, 8.4106023e-06,
       2.9967067e-11, 3.0331331e-09, 8.3651971e-14, 9.9999106e-01,
       2.6657624e-08, 3.8127661e-07], dtype=float32)
```

Each number of index *i* in that array corresponds to probability that digit image `test_digits[0]` belongs to class *i*.

This 1st test digit has highest probability score (0.99999106, almost 1) at index 7, so according to our model, it must be a 7:

```
>>> predictions[0].argmax()
7
>>> predictions[0][7]
0.99999106
```

Can check: test label agrees:

```
>>> test_labels[0]
7
```

On average, how good is our model at classifying such never-before-seen digits? Check by computing average accuracy over entire test set. Listing 2.7: Evaluating model on new data.

```
>>> test_loss, test_acc = model.evaluate(test_images, test_labels)
>>> print(f"test_acc: {test_acc}")
test_acc: 0.9785
```

Test-set accuracy turns out to be 97.8% – that's quite a bit lower than training set accuracy 98.9%. This gap between training accuracy & test accuracy is an example of *overfitting*: fact that ML models tend to perform worse on new data than on their training data. Overfitting is a central topic in Chap. 3.

This concludes 1st example – just saw how can build & train a neural network to classify handwritten digits in < 15 lines of Python code. In this chap & next, go into detail about every moving piece just previewed & clarify what's going on behind scenes. Learn about tensors, data-storing objects going into model; tensor operations, which layers are made of; & gradient descent, which allows your model to learn from its training examples.

- 2.2. Data representations for neural networks. In prev example, started from data stored in multidimensional NumPy arrays, also called *tensors*. In general, all current ML systems use tensors as their basic data structure. Tensors are fundamental to field – so fundamental that TensorFlow was named after them. So what's a tensor?

At its score, a tensor is a container for data – usually numerical data. So, it's a container for numbers. May be already familiar with matrices, which are rank-2 tensors: tensors are a generalization of matrices to an arbitrary number of *dimensions* (note: in context of tensors, a dimension is often called an *axis*).

- * 2.2.1. Scalars (rank-0 tensors). A tensor that contains only 1 number is called a *scalar* (or scalar tensor, or rank-0 tensor, or 0D tensor). In NumPy, a `float32` or `float64` number is a scalar tensor (or scalar array). Can display number of axes of a NumPy tensor via `ndim` attribute; a scalar tensor has 0 axes `ndim == 0`. Number of axes of a tensor is also called its *rank*. A NumPy scalar:

```
>>> import numpy as np
>>> x = np.array(12)
>>> x
array(12)
>>> x.ndim
0
```


- * 2.2.2. **Vectors (rank-1 tensors).** An array of numbers is called a *vector*, or rank-1 tensor, or 1D tensor. A rank-1 tensor is said to have exactly 1 axis. A NumPy vector:

```
>>> x = np.array([12, 3, 6, 14, 7])
>>> x
array([12, 3, 6, 14, 7])
>>> x.ndim
1
```

This vector has 5 entries & so is called a *5-dimensional vector*. Don't confuse a 5D vector with a 5D tensor! A 5D vector has only 1 axis & has 5 dimensions along its axis, whereas a 5D tensor has 5 axes (& may have any number of dimensions along each axis). *Dimensionality* can denote either number of entries along a specific axis (as in case of our 5D vector) or number of axes in a tensor (e.g. a 5D tensor), which can be confusing at times. In latter case, technically more correct to talk about a *tensor of rank 5* (rank of a tensor being number of axes), but ambiguous notation *5D tensor* in common regardless.

- * 2.2.3. **Matrices (rank-2 tensors).** An array of vectors is a *matrix*, or rank-2 tensor, or 2D tensor. A matrix has 2 axes (often referred to as *rows*, *columns*). Can visually interpret a matrix as a rectangular grid of numbers. A NumPy matrix:

```
>>> x = np.array([[5, 78, 2, 34, 0], [6, 79, 3, 35, 1], [7, 80, 4, 36, 2]])
>>> x.ndim
2
```

Entries from 1st axis are called *rows*, & entries from 2nd axis are called *columns*. In prev example, [5, 78, 2, 34, 0] is 1st row of x, & [5, 6, 7] is 1st column.

- * 2.2.4. **Rank-3 & higher-rank tensors.** If pack such matrices in a new array, obtain a rank-3 tensor (or 3D tensor), which can visually interpret as a cube of numbers. Following is a NumPy rank-3 tensor:

```
>>> x = np.array([[[5, 78, 2, 34, 0], [6, 79, 3, 35, 1], [7, 80, 4, 36, 2]], [[5, 78, 2, 34, 0], [6, 79, 3, 35, 1], [7, 80, 4, 36, 2]]])
>>> x.ndim
3
```

By packing rank-3 tensors in an array, can create a rank-4 tensor, & so on. In DL, generally manipulate tensors with ranks 0–4, although may go up to 5 if process video data.

- * 2.2.5. **Key attributes.** A tensor is defined by 3 key attributes:

- *Number of axes (rank).* E.g., a rank-3 tensor has 3 axes, & a matrix has 2 axes. This is also called tensor's **ndim** in Python libraries e.g. NumPy or TensorFlow.
- *Shape:* This is a tuple of integers that describes how many dimensions tensor has along each axis. E.g., prev matrix example has shape (3, 5), & rank-3 tensor example has shape (3, 3, 5). A vector has a shape with a single element, e.g. (5,), whereas a scalar has an empty shape ().
- *Data type (usually called dtype in Python libraries).* This is type of data contained in tensor; e.g., a tensor's type could be float16, float32, float64, uint8, & so on. In TensorFlow, also likely to come across **string** tensors.

To make this more concrete, look back at data processed in MNIST example. 1st, load MNIST dataset:

```
from tensorflow.keras.datasets import mnist
(train_images, train_labels), (test_images, test_labels) = mnist.load_data()
```

Display number of axes of tensor **train_images**, **ndim** attribute:

```
>>> train_images.ndim
3
```

Its shape:

```
>>> train_images.shape
(60000, 28, 28)
```

& this is its data type, **dtype** attribute:

```
>>> train_images.dtype
uint8
```

So what have here is a rank-3 tensor of 8-bit integers. More precisely, it's an array of 60000 matrices of 28×28 integers. Each such matrix is a grayscale image, with coefficients between 0 & 255.

Display 4th digit in this rank-3 tensor, using Matplotlib library (a well-known Python data visualization library, which comes preinstalled in Colab); Fig. 2.2: 4th sample in our dataset. Listing 2.8: Displaying 4th digit.

```
import matplotlib.pyplot as plt
digit = train_images[4]
```

```
plt.imshow(digit, cmap=plt.cm.binary)
plt.show()
```

Naturally, corresponding label is integer 9:

```
>>> train_labels[4]
9
```

- * 2.2.6. **Manipulating tensors in NumPy.** In prev example, selected a specific digit alongside 1st axis using syntax `train_images[i]`. Selecting specific elements in a tensor is called *tensor slicing*. Look at tensor-slicing operations can do on NumPy arrays. Following example selects digits #10 to #100 (#100 isn't included) & puts them in an array of shape (90, 28, 28):

```
>>> my_slice = train_images[10:100]
>>> my_slice.shape
(90, 28, 28)
```

⇔ this more detailed notation, which specifies a start index & stop index for slice along each tensor axis. Note: `:` ⇔ selecting entire axis:

```
>>> my_slice = train_images[10:100, :, :]
>>> my_slice.shape
(90, 28, 28)
>>> my_slice = train_images[10:100, 0:28, 0:28]
>>> my_slice.shape
(90, 28, 28)
```

In general, may select slices between any 2 indices along each tensor axis. E.g., in order to select 14×14 pixels in bottom-right corner of all images, would do this:

```
my_slice = train_images[:, 14:, 14:]
```

Also possible to use negative indices. Much like negative indices in Python lists, they indicate a position relative to end of current axis. In order to crop images to patches of 14×14 pixels centered in middle, do this:

```
my_slice = train_images[:, 7:-7, 7:-7]
```

- * 2.2.7. **Notion of data batches.** In general, 1st axis (axis 0, because indexing starts at 0) in all data tensors come across in DL will be *samples axis* (sometimes called *samples dimension*). In MNIST example, “samples” are images of digits. In addition, DL models don't process an entire dataset at once; rather, they break data into small batches. Concretely, 1 batch of MNIST digits, with a batch size of 128:

```
batch = train_images[:128]
```

Next batch:

```
batch = train_images[128:256]
```

*n*th batch:

```
n = 3
batch = train_images[128 * n:128 * (n + 1)]
```

When considering such a batch tensor, 1st axis (axis 0) is called *batch axis* or *batch dimension*. This is a term frequently encounter when using Keras & other DL libraries.

- * 2.2.8. **Real-world examples of data tensors.** Make data tensors more concrete with a few examples similar to what encounter later. Data you'll manipulate will almost always fall into 1 of following categories:

- *Vector data:* Rank-2 tensors of shape (**samples**, **features**), where each sample is a vector of numerical attributes (“features”)
- *Timeseries data or sequence data:* Rank-3 tensors of shape (**samples**, **timesteps**, **features**), where each sample is a sequence (of length **timesteps**) of feature vectors
- *Images:* Rank-4 tensors of shape (**samples**, **height**, **width**, **channels**), where each sample is a 2D grid of pixels, & each pixel is represented by a vector of values (“channels”)
- *Video:* Rank-5 tensors of shape (**samples**, **frames**, **height**, **width**, **channels**), where each sample is a sequence (of length **frames**) of images

- * 2.2.9. **Vector data.** This is 1 of most common cases. In such a dataset, each single data point can be encoded as a vector, & thus a batch of data will be encoded as a rank-2 tensor (i.e., an array of vectors), where 1st axis is *samples axis* & 2nd axis is *features axis*.

2 examples:

- An actuarial dataset of people, where consider each person's age, gender, & income. Each person can be characterized as a vector of 3 values, & thus an entire dataset of 100000 people can be stored in a rank-2 tensor of shape (100000, 3).
- A dataset of text documents, where represent each document by counts of how many times each word appears in it (out of a dictionary of 20000 common words). Each document can be encoded as a vector of 20000 values (1 count per word in dictionary), & thus an entire dataset of 500 documents can be stored in a tensor of shape (500, 20000).

* **2.2.10. Timeseries data or sequence data.** Whenever time matters in your data (or notion of sequence order), it makes sense to store it in a rank-3 tensor with an explicit time axis. Each sample can be encoded as a sequence of vectors (a rank-2 tensor), & thus a batch of data will be encoded as a rank-3 tensor (Fig. 2.3: A rank-3 timeseries data tensor.).

Time axis is always 2nd axis (axis of index 1) by convention. Examples:

- A dataset of stock prices. Every minute, store current price of stock, highest price in past minute, & lowest price in past minute. Thus, every minute is encoded as a 3D vector, an entire day of trading is encoded as a matrix of shape (390, 3) (there are 390 minutes in a trading day), & 250 days' worth of data can be stored in a rank-3 tensor of shape (250, 390, 3). Here, each sample would be 1 day's worth of data.
 - A dataset of tweets, where encode each tweet as a sequence of 280 characters out of an alphabet of 128 unique characters. In this setting, each character can be encoded as a binary vector of size 128 (an all-0s vector except for a 1 entry at index corresponding to character). Then each tweet can be encoded as a rank-2 tensor of shape (280, 128), & a dataset of 1 million tweets can be stored in a tensor of shape (1000000, 280, 128).
- * **2.2.11. Image data.** Images typically have 3 dimensions: height, width, & color depth. Although grayscale images (like MNIST digits) have only a single color channel & could thus be stored in rank-2 tensors, by convention image tensors are always rank-3, with a 1D color channel for grayscale images. A batch of 128 grayscale images of size 256×256 could thus be stored in a tensor of shape (128, 256, 256, 1), & a batch of 128 color images could be stored in a tensor of shape (128, 256, 256, 3) (Fig. 2.4: A rank-4 image data tensor.).

There are 2 conventions for shapes of image tensors: *channels-last* convention (which is standard in TensorFlow) & *channels-1st* convention (which is increasingly falling out of favor).

Channels-last convention places color-depth axis at end: (samples, height, width, color_depth). Meanwhile, channels-1st convention places color depth axis right after batch axis: (samples, color_depth, height, width). With channels-1st convention, prev examples would become (128, 1, 256, 256), (128, 3, 256, 256). Keras API provides support for both formats.

* **2.2.12. Video data.** Video data is 1 of few types of real-world data for which need a rank-5 tensors. A video can be understood as a sequence of frames, each frame being a color image. Because each frame can be stored in a rank-3 tensor (height, width, color_depth), a sequence of frames can be stored in a rank-4 tensor (frames, height, width, color_depth) & thus a batch of different videos can be stored in a rank-5 tensor of shape (samples, frames, height, width, color_depth). E.g., a 60-sec, 144×256 YouTube video clip sampled at 4 frames per sec would have 240 frames. A batch of 4 such video clips would be stored in a tensor of shape (4, 240, 144, 256, 3): a total of 106,168,320 values! If dtype of tensor was float32, each value would be stored in 32 bits, so tensor would represent 405 MB. Heavy! Videos encounter in real life are much lighter, because they aren't stored in float32, & they're typically compressed by a large factor (e.g. in MPEG format).

- **2.3. Gears (Bánh răng) of neural networks: Tensor operations.** Much as any computer program can be ultimately reduced to a small set of binary operations on binary inputs (AND, OR, NOR, & so on), all transformations learned by deep neural networks can be reduced to a handful of *tensor operations* (or *tensor functions*) applied to tensors of numeric data. E.g., possible to add tensors, multiply tensors, & so on.

In initial example, build model by stacking Dense layers on top of each other. A Keras layer instance looks like this:

```
keras.layers.Dense(512, activation="relu")
```

This layer can be interpreted as a function, which takes as input a matrix & returns another matrix – a new representation for input tensor. Specifically, function is as follows (where W: a matrix & b: a vector, both attributes of layer):

```
output = relu(dot(input, W) + b)
```

Unpack this: Have 3 tensor operations here:

- * A dot product dot between input tensor & a tensor named W
- * An addition + between resulting matrix & a vector b
- * A relu operation: relu(x) is max(x, 0); “relu” stands for “rectified linear unit”

Note 2. Although this sect deals entirely with linear algebra expressions, won't find any mathematical notation here. Found: mathematical concepts can be more readily mastered by programmers with no mathematical background if they're expressed as short Python snippets instead of mathematical equations. So use NumPy & TensorFlow code throughout.

- * **2.3.1. Element-wise operations.** relu operation & addition are element-wise operations: operations that are applied independently to each entry in tensors being considered. I.e. these operations are highly amenable to massively parallel implementations (*vectorized* implementations, a term that comes from *vector processor* supercomputer architecture from

1970–90 period). If want to write a native Python implementation of an element-wise operation, use a `for` loop, as in this naive implementation of an element-wise `relu` operation:

```
def naive_relu(x):
    assert len(x.shape) == 2
    x = x.copy()
    for i in range(x.shape[0]):
        for j in range(x.shape[1]):
            x[i, j] = max(x[i, j], 0)
    return x
```

Could do same for addition:

```
def naive_add(x, y):
    assert len(x.shape) == 2
    assert x.shape == y.shape
    x = x.copy()
    for i in range(x.shape[0]):
        for j in range(x.shape[1]):
            x[i, j] += y[i, j]
    return x
```

On same principle, can do element-wise multiplication, subtraction, & so on.

In practice, when dealing with NumPy arrays, these operations are available as well-optimized built-in NumPy functions, which themselves delegate (đại biểu) heavy lifting to a Basic Linear Algebra Subprograms (BLAS) implementation. BLAS are low-level, highly parallel, efficient tensor-manipulation routines that are typically implemented in Fortran or C.

So, in NumPy, can do following element-wise operation, & it will be blazing fast:

```
import numpy as np
z = x + y # element-wise addition
z = np.maximum(z, 0.) # element-wise relu
```

Actually time difference:

```
import time
x = np.random.random((20, 100))
y = np.random.random((20, 100))
t0 = time.time()
for _ in range(1000):
    z = x + y
    z = np.maximum(z, 0.)
print("Took: {0:.2f} s".format(time.time() - t0))
```

This takes 0.02s. Meanwhile, naive version takes a stunning 2.45 s:

```
t0 = time.time()
for _ in range(1000):
    z = naive_add(x, y)
    z = naive_relu(z)
print("Took: {0:.2f} s".format(time.time() - t0))
```

Likewise, when running TensorFlow code on a GPU, element-wise operations are executed via fully vectorized CUDA implementations that can best utilize highly parallel GPU chip architecture.

- * 2.3.2. **Broadcasting.** Earlier naive implementation of `naive_add` only supports addition of rank-2 tensors with identical shapes. But in `Dense` layer introduced earlier, added a rank-2 tensor with a vector. What happens with addition when shapes of 2 tensors being added differ?

When possible, & if there's no ambiguity, smaller tensor will be *broadcast* to match shape of larger tensor. Broadcasting consists of 2 steps:

1. Axes (called *broadcast axes*) are added to smaller tensor to match `ndim` of larger tensor.
2. Smaller tensor is repeated alongside these new axes to match full shape of larger tensor.

A concrete example. Consider `X` with shape `(32, 10)` & `y` with shape `(10,)`:

```
import numpy as np
X = np.random.random((32, 10))
y = np.random.random((10,))
```

1st, add an empty 1st axis to `y`, whose shape becomes `(1, 10)`:

```
y = np.expand_dims(y, axis=0)
```

Then, repeat y 32 times along this new axis, so that end up with a tensor Y with shape $(32, 10)$, where $Y[i, :] == y$ for i in `range(0, 32)`:

```
Y = np.concatenate([y] * 32, axis=0)
```

At this point, can proceed to add X , Y , because they have same shape.

In terms of implementation, no new rank-2 tensor is created, because that would be terribly inefficient. Repetition operation is entirely virtual: it happens at algorithmic level rather than at memory level. But thinking of vector being repeated 10 times alongside a new axis is a helpful mental model. Here's what a naive implementation would look like:

```
def naive_add_matrix_and_vector(x, y):
    assert len(x.shape) == 2
    assert len(y.shape) == 1
    assert x.shape[1] == y.shape[0]
    x = x.copy()
    for i in range(x.shape[0]):
        for j in range(x.shape[1]):
            x[i, j] += y[j]
    return x
```

With broadcasting, can generally perform element-wise operations that take 2 inputs tensors if 1 tensor has shape $(a, b, \dots n, n + 1, \dots m)$ & other has shape $(n, n + 1, \dots m)$. Broadcasting will then automatically happen for axes a through $n - 1$.

Following example applies element-wise `maximum` operation to 2 tensors of different shapes via broadcasting:

```
import numpy as np
x = np.random.random((64, 3, 32, 10))
y = np.random.random((32, 10))
z = np.maximum(x, y)
```

* 2.3.3. Tensor product. *Tensor product*, or *dot product* (not to be confused with an element-wise product, `*` operator), is 1 of most common, most useful tensor operations.

In NumPy, a tensor product is done using `np.dot` function (because mathematical notation for tensor product is usually a dot):

```
x = np.random.random((32,))
y = np.random.random((32,))
z = np.dot(x, y)
```

In mathematical notation: $z = x \cdot y$. Mathematically, what does dot operation do? Start with dot product of 2 vectors x , y : computed as follows:

```
def naive_vector_dot(x, y):
    assert len(x.shape) == 1
    assert len(y.shape) == 1
    assert x.shape[0] == y.shape[0]
    z = 0.
    for i in range(x.shape[0]):
        z += x[i] * y[i]
    return z
```

Notice: dot product between 2 vectors is a scalar & that only vectors with same number of elements are compatible for a dot product.

Can also take dot product between a matrix x & a vector y , which returns a vector where coefficients are dot products between y & rows of x . Implement it as follows:

```
def naive_matrix_vector_dot(x, y):
    assert len(x.shape) == 2
    assert len(y.shape) == 1
    assert x.shape[1] == y.shape[0]
    z = np.zeros(x.shape[0])
    for i in range(x.shape[0]):
        for j in range(x.shape[1]):
            z[i] += x[i, j] * y[j]
    return z
```

Could also reuse code wrote previously, which highlights relationship between a matrix-vector product & a vector product:

```
def naive_matrix_vector_dot(x, y):
    z = np.zeros(x.shape[0])
    for i in range(x.shape[0]):
        z[i] = naive_vector_dot(x[i, :], y)
    return z
```

Note: as soon as 1 of 2 tensors has an `ndim > 1`, `dot` is no longer *symmetric*, i.e. `dot(x, y)` isn't same as `dot(y, x)`. Of course, a dot product generalizes to tensors with an arbitrary number of axes. Most common applications may be dot product between 2 matrices. Can take dot product of 2 matrices `x, y` (`dot(x, y)`) iff `x.shape[1] == y.shape[0]`. Result is a matrix with shape `(x.shape[0], y.shape[1])`, where coefficients are vector products between rows of `x` & columns of `y`. Naive implementation:

```
def naive_matrix_dot(x, y):
    assert len(x.shape) == 2 # x & y are NumPy matrices.
    assert len(y.shape) == 2
    assert x.shape[1] == y.shape[0]
    z = np.zeros((x.shape[0], y.shape[1]))
    for i in range(x.shape[0]):
        for j in range(y.shape[1]):
            row_x = x[i, :]
            column_y = y[:, j]
            z[i, j] = naive_vector_dot(row_x, column_y)
    return z
```

To understand dot-product shape compatibility, help to visualize input & output tensors by aligning them as shown in Fig. 2.5: Matrix dot-product box diagram.

In figure, `x, y, z` are pictured as rectangles (literal boxes of coefficients). Because rows of `x` & columns of `y` must have same size, it follows: width of `x` must match height of `y`. If go on to develop new ML algorithms, likely be drawing such diagrams often.

More generally, can take dot product between higher-dimensional tensors, following same rules for shape compatibility as outlined earlier for 2D case: $(a, b, c, d) \cdot (d, e) \rightarrow (a, b, c, e)$, $(a, b, c, d) \cdot (d, e) \rightarrow (a, b, c, e)$, & so on.

- * 2.3.4. Tensor reshaping. A 3rd type of tensor operation that's essential to understand is *tensor reshaping*. Although it wasn't used in Dense layers in 1st neural network example, used it when preprocessed digits data before feeding it into our model:

```
train_images = train_images.reshape((60000, 28 * 28))
```

Reshaping a tensor means rearranging its rows & columns to match a target shape. Naturally, reshaped tensor has same total number of coefficients as initial tensor. Reshaping is best understood via simple examples:

```
>>> x = np.array([[0., 1.], [2., 3.], [4., 5.]])
>>> x.shape
(3, 2)
>>> x = x.reshape((6, 1))
>>> x
array([[ 0.], [ 1.], [ 2.], [ 3.], [ 4.], [ 5.]])
>>> x = x.reshape((2, 3))
>>> x
array([[ 0., 1., 2.],
       [ 3., 4., 5.]])
```

A special case of reshaping that's commonly encountered is *transposition*. *Transposing* a matrix means exchanging its rows & its columns, so that `x[i, :]` becomes `x[:, i]`:

```
>>> x = np.zeros((300, 20))
>>> x = np.transpose(x)
>>> x.shape
(20, 300)
```

- * 2.3.5. Geometric interpretation of tensor operations. Because contents of tensors manipulated by tensor operations can be interpreted as coordinates of points in some geometric space, all tensor operations have a geometric interpretation. E.g., consider addition. Start with vector: `A = [0.5, 1]` – Fig. 2.6: A point in 2D space. Common to picture a vector as an arrow linking origin to point Fig. 2.7: A point in a 2D space pictured as an arrow. Consider a new point `B = [1, 0.25]`, which add to prev one. This is done geometrically by changing together vector arrows, with resulting location being vector representing sum of prev 2 vectors (Fig. 2.8: Geometric interpretation of sum of 2 vectors). As can see, adding a vector `B`

to a vector A represents action of copying point A in a new location, whose distance & direction from original point A is determined by vector B . If apply same vector addition to a group of points in plane (an “object”), would be creating a copy of entire object in a new location (Fig. 2.9: 2D translation as a vector addition). Tensor addition thus represents action of *translating an object* (moving object without distorting it) by a certain amount in a certain direction. In general, elementary geometric operations e.g. translation, rotation, scaling, skewing, & so on can be expressed as tensor operations. A few examples:

- *Translation*: As just saw, adding a vector to a point will move point by a fixed amount in a fixed direction. Applied to a set of points (e.g. a 2D object), called a “translation”.
- *Rotation*: A counterclockwise rotation of a 2D vector by an angle θ (Fig. 2.10: 2D rotation (counterclockwise) as a dot product) can be achieved via a dot product with a 2×2 matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- *Scaling*: A vertical & horizontal scaling of image (Fig. 2.11: 2D scaling as a dot product) can be achieved via a dot product with a 2×2 matrix $S = \begin{bmatrix} \text{horizontal_factor} & 0 \\ 0 & \text{vertical_factor} \end{bmatrix}$ (note: such a matrix is called a “diagonal matrix”, because it only has nonzero coefficients in its “diagonal”, going from top left to bottom right).
 - *Linear transform*: A dot product with an arbitrary matrix implements a linear transform. Note: *scaling* & *rotation* are by definition linear transforms.
 - *Affine transform*: An affine transform (Fig. 2.12: Affine transform in plane.) is combination of a linear transform (achieved via a dot product with some matrix) & a translation (achieved via a vector addition). As have probably recognized, that’s exactly $y = W \bullet x + b$ computation implemented by **Dense** layer! A **Dense** layer without an activation function is an affine layer.
 - *Dense layer with relu activation*: An important observation about affine transforms: if apply many of them repeatedly, still end up with an affine transform (so could just have applied that 1 affine transform in 1st place). Try it with 2: $\text{affine2}(\text{affine1}(x)) = W_2 \bullet (W_1 \bullet x + b_1) + b_2 = (W_2 \bullet W_1) \bullet x + (W_2 \bullet b_1 + b_2)$. That’s an affine transform where linear part is matrix $W_2 \bullet W_1$ & translation part is vector $W_2 \bullet b_1 + b_2$. As a consequence, a multilayer neural network made entirely of **Dense** layers without activations would be \Leftrightarrow a single **Dense** layer. This “deep” neural network would just be a linear model in disguise! This is why need activations, like **relu** (seen in action in Fig. 2.13: Affine transform followed by **relu** activation). Thanks to activation functions, a chain of **Dense** layers can be made to implement very complex, nonlinear geometric transformations, resulting in very rich hypothesis spaces for deep neural networks. Cover this idea in more detail in next chap.
- * 2.3.6. A geometric interpretation of DL. Just learned: neural networks consist entirely of chains of tensor operations, & that these tensor operations are just simple geometric transformations of input data. It follows: you can interpret a neural network as a very complex geometric transformation in a high-dimensional space, implemented via a series of simple steps. In 3D, following mental image may prove useful. Imagine 2 sheets of colored paper: 1 red & 1 blue. Put 1 on top of the other. Now crumple (nhàu nát) them together into a small ball. That crumpled paper ball is input data, & each sheet of paper is a class of data in a classification problem. What a neural network is meant to do is figure out a transformation of paper ball that would uncrumple it, so as to make 2 classes cleanly separable again (Fig. 2.14: Uncrumbling a complicated manifold of data). With DL, this would be implemented as a series of simple transformations of 3D space, e.g. those you could apply on paper ball with fingers, 1 movement at a time.

Uncrumpling paper balls is what ML is about: finding neat representations for complex, highly folded data *manifolds* in high-dimensional spaces (a manifold is a continuous surface, like crumpled sheet of paper). At this point, should have a pretty good intuition as to why DL excels as this: it takes approach of incrementally decomposing a complicated geometric transformation into a long chain of elementary ones, which is pretty much strategy a human would follow to uncrumple a paper ball. Each layer in a deep network applies a transformation that disentangles data a little, & a deep stack of layers makes tractable an extremely complicated disentanglement process.

– Gỡ rối những quả bóng giấy là mục đích của ML: tìm ra các biểu diễn gọn gàng cho dữ liệu phức tạp, có nhiều nếp gấp *manifolds* trong các không gian có nhiều chiều (một đa tạp là một bề mặt liên tục, giống như tờ giấy nhàu nát). Tại thời điểm này, bạn nên có trực giác khá tốt về lý do tại sao DL lại vượt trội như thế này: nó sử dụng phương pháp phân tích gia tăng một phép biến đổi hình học phức tạp thành một chuỗi dài các phép biến đổi cơ bản, đây gần như là chiến lược mà con người sẽ áp dụng để gỡ rối một quả bóng giấy. Mỗi lớp trong một mạng sâu áp dụng một phép biến đổi giúp gỡ rối dữ liệu một chút, & một chồng lớp sâu giúp quá trình gỡ rối cực kỳ phức tạp trở nên dễ xử lý.

- 2.4. Engine of neural networks: Gradient-based optimization. Each neural layer from 1st model example transforms its input data as follows:

```
output = relu(dot(input, W) + b)
```

In this expression, W , b : tensors that are attributes of layer, called *weights* or *trainable parameters* of layer (*kernel*, *bias* attributes, resp.). These weights contain information learned by model from exposure to training data.

Initially, these weight matrices are filled with small random values (a step called *random initialization*). Of course, there’s no reason to expect: $\text{relu}(\text{dot}(\text{input}, W) + b)$, when W , b are random, will yield any useful representations. Resulting representations are meaningless – but they’re a starting point. What comes next is to gradually adjust these weights, based on a feedback signal. This gradual adjustment, also called *training*, is learning that ML is all about.

This happens within what's called a *training loop*, which works as follows. Repeat these steps in a loop, until loss seems sufficiently low:

1. Draw a batch of training samples \mathbf{x} & corresponding targets $\mathbf{y_true}$.
2. Run model on \mathbf{x} (a step called *forward pass*) to obtain predictions $\mathbf{y_pred}$.
3. Compute loss of model on batch, a measure of mismatch between $\mathbf{y_pred}$, $\mathbf{y_true}$.
4. Update all weights of model in a way that slightly reduces loss on this batch.

Eventually end up with a model that has a very low loss on its training data: a low mismatch between predictions $\mathbf{y_pred}$, & expected targets $\mathbf{y_true}$. Model has “learned” to map its inputs to correct targets. From afar, it may look like magic, but when reduce it to elementary steps, it turns out to be simple.

Step 1 sounds easy enough – just I/O code. Steps 2 & 3 are merely application of a handful of tensor operations, so could implement these steps purely from what learned in prev sect. Difficult part is step 4: updating, model's weights. Given an individual weight coefficient in model, how can compute whether coefficient should be increased or decreased, & by how much?

1 naive solution would be to freeze all weights in model except 1 scalar coefficient being considered, & try different values for this coefficient. Say initial value of coefficient is 0.3. After forward pass on a batch of data, loss of model on batch is 0.5. If change coefficient's value to 0.35 & return forward pass, loss increases to 0.6. But if lower coefficient to 0.25, loss falls to 0.4. In this case, it seems: updating coefficient by -0.05 would contribute to minimizing loss. This would have to be repeated for all coefficients in model.

But such an approach would be horribly inefficient, because need to compute 2 forward passes (which are expensive) for every individual coefficient (of which there are many, usually thousands & sometimes up to millions). Thankfully, there's a much better approach: *gradient descent*.

Gradient descent is optimization technique that powers modern neural networks. Here's gist ý chính) of it. All of functions used in our models (e.g. `dot` or `+`) transform their input in a smooth & continuous way: if look at $\mathbf{z} = \mathbf{x} + \mathbf{y}$, e.g., a small change in \mathbf{y} only results in a small change in \mathbf{z} , & if you know direction of change in \mathbf{y} , can infer direction of change in \mathbf{z} . Mathematically, say these functions are *differentiable*. If chain together such functions, bigger function obtain is still differentiable. In particular, this applies to function that maps model's coefficients to loss of model on a batch of data: a small change in model's coefficients results in a small, predictable change in loss value. This enables to use a mathematical operator called *gradient* to describe how loss varies as you move model's coefficients in different directions. If compute this gradient, can use it to move coefficients (all at once in a single update, rather than one at a time) in a direction that decreases loss.

If already know what *differentiable* means & what a *gradient* is, can skip to Sect. 2.4.3. Otherwise, following 2 sects help understand concepts.

* 2.4.1. What's a derivative? Consider a continuous, smooth function $f(x) = y$, mapping a number x to a new number y . Can use function in Fig. 2.15: A continuous smooth function as an example.

Because function is *continuous*, a small change in x can only result in a small change in y – that's intuition behind *continuity*. Say: increase x by a small factor, `epsilon_x`: this results in a small `epsilon_y` changes to y , as shown in Fig. 2.16: With a continuous function, a small change in x results in a small change in y .

In addition, because function is *smooth* (its curve doesn't have any abrupt angles), when `epsilon_x` is small enough, around a certain point p , possible to approximate f as a linear function of slope a , so that `epsilon_y` becomes $a * \text{epsilon}_x$:

$$f(x + \text{epsilon}_x) = y + a * \text{epsilon}_x$$

Obviously, this linear approximation is valid only when x is close enough to p .

Slope a is called *derivative* of f in p . If a is negative, it means a small increase in x around p will result in a decrease of $f(x)$ (as shown in Fig. 2.17: Derivative of f in p), & if $a > 0$, a small increase in x will result in an increase of $f(x)$. Further, absolute value of a (*magnitude* of derivative) tells how quickly this increase or decrease will happen.

For every differentiable function $f(x)$ (*differentiable* means “can be derived”: e.g., smooth, continuous functions can be derived), there exists a derivative function $f'(x)$, that maps values of x to slope of local linear approximation of f in those points. E.g., derivative of $\cos x$ is $-\sin x$, derivative of $f(x) = ax$ is $f'(x) = a$, ...

Being able to derive functions is a very powerful tool when it comes to *optimization*, task of finding values of x that minimize value of $f(x)$. If trying to update x by a factor `epsilon_x` in order to minimize $f(x)$, & know derivative of f , then your job is done: derivative completely describes how $f(x)$ evolves as you change x . If want to reduce value of $f(x)$, just need to move x a little in opposite direction from derivative.

* 2.4.2. Derivative of a tensor operation: gradient. Function we were just looking at turned a scalar value x into another scalar value y : could plot it as a curve in a 2D plane. Now imagine a function that turns a tuple of scalars (x, y) into a scalar value z : that would be a vector operation. Could plot it as a 2D *surface* in a 3D space (indexed by coordinates \mathbf{x} , \mathbf{y} , \mathbf{z}). Likewise, can imagine functions that take matrices as inputs, functions that take rank-3 tensors as inputs, etc.

Concept of derivation can be applied to any such function, as long as surfaces they describe are continuous & smooth. Derivative of a tensor operation (or tensor function) is called a *gradient*. Gradients are just generalization of concept of derivatives to functions that take tensors as inputs. Remember how, for a scalar function, derivative represents *local slope* of curve of function? In same way, gradient of a tensor function represents *curvature* of multidimensional surface described by function. It characterizes how output of function varies when its input parameters vary.

An example grounded in ML. Consider:

- An input vector \mathbf{x} (a sample in a dataset)
- A matrix \mathbf{W} (weights of a model)
- A target $\mathbf{y_true}$ (what model should learn to associate to \mathbf{x})
- A loss function `loss` (meant to measure gap between model's current predictions & $\mathbf{y_true}$)

Can use \mathbf{W} to compute a target candidate $\mathbf{y_pred}$, & then compute loss, or mismatch, between target candidate $\mathbf{y_pred}$ & target $\mathbf{y_true}$:

```
y_pred = dot(W, x)
loss_value = loss(y_pred, y_true)
```

Like to use gradients to figure out how to update \mathbf{W} so as to make `loss_value` smaller. How do we do that?

Given fixed inputs \mathbf{x} , $\mathbf{y_true}$, preceding operations can be interpreted as a function mapping values of \mathbf{W} (model's weights) to loss values:

```
loss_value = f(W)
```

Say current value of \mathbf{W} is $\mathbf{W0}$. Then derivative of f at point $\mathbf{W0}$ is a tensor `grad(loss_value, W0)`, with same shape as \mathbf{W} , where each coefficient `grad(loss_value, W0)[i, j]` indicates direction & magnitude of change in `loss_value` you observe when modifying $\mathbf{W0}[i, j]$. That tensor `grad(loss_value, W0)` is gradient of function $f(\mathbf{W}) = \text{loss_value}$ in $\mathbf{W0}$, also called "gradient of `loss_value` w.r.t. \mathbf{W} around $\mathbf{W0}$."

Remark 1 (Partial derivatives). *Tensor operation `grad(f(W), W)` (which takes as input a matrix \mathbf{W}) can be expressed as a combination of scalar functions, `grad_ij(f(W), w_ij)`, each of which would return derivative of `loss_value = f(W)` w.r.t. coefficient $\mathbf{W}[i, j]$ of \mathbf{W} , assuming all other coefficients are constant. `grad_ij` is called partial derivative of f w.r.t. $\mathbf{W}[i, j]$.*

Concretely, what does `grad(loss_value, W0)` represent? Saw earlier: derivative of a function $f(x)$ of a single coefficient can be interpreted as slope of curve of f . Likewise, `grad(loss_value, W0)` can be interpreted as tensor describing *direction of steepest ascent* of `loss_value = f(W)` around $\mathbf{W0}$, as well as slope of this ascent. Each partial derivative describes slope of f in a specific direction.

For this reason, in much same way: for a function $f(x)$, can reduce value of $f(x)$ by moving x a little in opposite direction from derivative, with a function $f(\mathbf{W})$ of a tensor, can reduce `loss_value = f(W)` by moving \mathbf{W} in opposite direction from gradient: e.g., $\mathbf{W1} = \mathbf{W0} - \text{step} * \text{grad}(f(\mathbf{W0}), \mathbf{W0})$ (where `step` is a small scaling factor). I.e., going against direction of steepest ascent of f , which intuitively should put you lower on curve. Note: scaling factor `step` is needed because `grad(loss_value, W0)` only approximates curvature when close to $\mathbf{W0}$, so don't want to get too far from $\mathbf{W0}$.

- * 2.4.3. **Stochastic gradient descent.** Given a differentiable function, it's theoretically possible to find its minimum analytically: known: a function's minimum is a point where derivative is 0, so all you have to do is find all points where derivative goes to 0 & check for which of these points function has lowest value.

Applied to a neural network, i.e. finding analytically combination of weight values that yields smallest possible loss function. This can be done by solving equation `grad(f(W), W) = 0` for \mathbf{W} . This is a polynomial equation of N variables, where N : number of coefficients in model. Although it would be possible to solve such an equation for $N = 2$ or $N = 3$, doing so is intractable for real neural networks, where number of parameters is never < a few thousand & can often be several 10s of millions.

Instead, can use 4-step algorithm outlined at beginning of this sect: modify parameters little by little based on current loss value for a random batch of data. Because dealing with a differentiable function, can compute its gradient, which gives you an efficient way to implement Step 4. If update weights in opposite direction from gradient, loss will be a little less every time:

1. Draw a batch of training samples x & corresponding targets $\mathbf{y_true}$.
2. Run model on x to obtain predictions $\mathbf{y_pred}$ (this is called *forwards pass*).
3. Compute loss of model on batch, a measure of mismatch between $\mathbf{y_pred}$, $\mathbf{y_true}$.
4. Compute gradient of loss w.r.t. model's parameters (called *backward pass*).
5. Move parameters a little in opposite direction from gradient – e.g., $\mathbf{W} -= \text{learning_rate} * \text{gradient}$ – thus reducing loss on batch a bit. *Learning rate* (`learning_rate` here) would be a scalar factor modulating "speed" of gradient descent process.

Easy enough! What just described is called *mini-batch stochastic gradient descent* (mini-batch SGD). Term *stochastic* refers to fact: each batch of data is drawn at random (*stochastic* is a scientific synonym of *random*). Fig. 2.18: SGD down a 1D loss curve (1 learnable parameter) illustrates what happens in 1D, when model has only 1 parameter & have only 1 training sample.

Intuitively important to pick a reasonable value for `learning_rate` factor. If it's too small, descent down curve will take many iterations, & it could get stuck in a local minimum. If `learning_rate` is too large, your updates may end up taking you to completely random locations on curve.

Note: a variant of mini-batch SGD algorithm would be to draw a single sample & target at each iteration, rather than drawing a batch of data. This would be *trueSGD* (as opposed to *mini-batchSGD*). Alternatively, going to opposite extreme, could run every step on *all* data available, called *batch gradient descent*. Each update would then be more accurate, but far more expensive. Efficient compromise between these 2 extremes: us mini-batches of reasonable size.

Although Fig. 2.18 illustrates gradient descent in a 1D parameter space, in practice use gradient descent in highly dimensional spaces: every weight coefficient in a neural network is a free dimension in space, & there may be 10s of thousands or even millions of them. To help build intuition about loss surfaces, can also visualize gradient descent along a 2D loss surface, as shown in Fig. 2.19: Gradient descent down a 2D loss surface (2 learnable parameters).. But can't possibly visualize what actual process of training a neural network looks like – can't represent a 1000000-dimensional space in a way that makes sense to humans. As such, good to keep in mind: intuitions develop through these low-dimensional representations may not always be accurate in practice. This has historically been a source of issues in world of DL research.

Additionally, there exist multiple variants of SGD that differ by taking into account prev weight updates when computing next weight update, rather than just looking at current value of gradients. There is, e.g., SGD with momentum, as well as Adagrad, RMSprop, & several others. Such variants are known as *optimization methods* or *optimizers*. In particular, concept of *momentum*, which is used in many of these variants, deserves your attention. Momentum addresses 2 issues with SGD: convergence speedh & local minima. Configure Fig. 2.20: A local minimum & a global minimum, which shows curve of a loss as a function of a model parameter.

Around a certain parameter value, there is a *local minimum*: around that point, moving left would result in loss increasing, but so would moving right. If parameter under consideration were being optimized via SGD with a small learning rate, optimization process could get stuck at local minimum instead of making its way to global minimum.

Can avoid such issues by using momentum, which draws inspiration from physics. A useful mental image here: think of optimization process as a small ball rolling down loss curve. If it has enough momentum, ball won't get stuck in a ravine & will end up at global minimum. Momentum is implemented by moving ball at each step based not only on current slope value (current acceleration) but also on current velocity (resulting from past acceleration). In practice, i.e., updating parameter w based not only on current gradient value but also prev parameter update, e.g. in this naive implementation:

```
past_velocity = 0.
momentum = 0.1
while loss > 0.01:
    w, loss, gradient = get_current_parameters()
    velocity = past_velocity * momentum - learning_rate * gradient
    w = w + momentum * velocity - learning_rate * gradient
    past_velocity = velocity
    update_parameter(w)
```

* 2.4.4. Chaining derivatives: Backpropagation algorithm. In preceding algorithm, casually assumed: because a function is differentiable, can easily compute its gradient. But is that true? How can compute gradient of complex expressions in practice? In 2-layer model started chap with, how can get gradient of loss w.r.t. weights? That's where *Backpropagation algorithm* comes in.

· Chain rule. Backpropagation is a way to use derivatives of simple operations (e.g. addition, relu, or tensor product) to easily compute gradient of arbitrarily complex combinations of these atomic operations. Crucially, a neural network consists of many tensor operations chained together, each of which has a simple, known derivative. E.g., model defined in listing 2.2 can be expressed as a function parameterized by variables $W1$, $b1$, $W2$, $b2$ (belonging to 1st & 2nd Dense layers resp.), involving atomic operations `dot`, `relu`, `softmax`, & `+`, as well as our loss function `loss`, which are all easily differentiable:

```
loss_value = loss(y_true, softmax(dot(relu(dot(inputs, W1) + b1), W2) + b2))
```

Calculus tells: such a chain of functions can be derived using *chain rule*.

Consider 2 functions f, g , as well as composed function fg s.t. $fg(x) == f(g(x))$:

```
def fg(x):
    x1 = g(x)
    y = f(x1)
    return y
```

Then chain rule states: $grad(y, x) == grad(y, x1) * grad(x1, x)$. This enables to compute derivative of fg as long as know derivatives of f, g . Chain rule is named as it is because when add more intermediate functions, it starts looking like a chain:

```
def fghj(x):
    x1 = j(x)
    x2 = h(x1)
    x3 = g(x2)
    y = f(x3)
    return y
```

```
grad(y, x) == (grad(y, x3) * grad(x3, x2) * grad(x2, x1) * grad(x1, x))
```

Applying chain rule to computation of gradient values of a neural network gives rise to an algorithm called *backpropagation*. See how that works, concretely.

• **Automatic differentiation with computation graphs.** A useful way to think about backpropagation is in terms of *computation graphs*. A computation graph is data structure at heart of TensorFlow & DL revolution in general. It's a directed acyclic graph of operations – in our case, tensor operations. E.g., Fig. 2.21: Computation graph representation of 2-layer model shows graph representation of 1st model.

Computation graphs have been an extremely successful abstraction in CS because they enable us to *treat computation as data*: a computable expression is encoded as a machine-readable data structure that can be used as input or output of another program. E.g., could imagine a program that receives a computation graph & returns a new computation graph that implements a large-scale distributed version of same computation – i.e. could distribute any computation without having to write distribution logic yourself. Or imagine a program that receives a computation graph & can automatically generate derivative of expression it represents. Much easier to do these things if your computation is expressed as an explicit graph data structure rather than, say, lines of ASCII characters in a .py file.

To explain backpropagation clearly, look at a really basic example of a computation graph (Fig. 2.22: A basic example of a computation graph.). Consider a simplified version of Fig. 2.21, where only have 1 linear layer & where all variables are scalar. Take 2 scalar variables w , b , a scalar input x , & apply some operations to them to combine them into an output y . Finally, apply an absolute value error-loss function: $\text{loss_val} = \text{abs}(y_true - y)$. Since want to update w , b in a way that will minimize loss_val , interested in computing $\text{grad}(\text{loss_val}, b)$, $\text{grad}(\text{loss_val}, w)$.

Set concrete values for “input nodes” in graph, i.e., input x , target y_true , w , b . Propagate these values to all nodes in graph, from top to bottom, until reach loss_val . This is *forward pass* (Fig. 2.23: Running a forward pass).

“Reverse” graph: for each edge in graph going from A to B, will create an opposite edge from B to A, & ask, how much does B vary when A varies? I.e., what is $\text{grad}(B, A)$? Annotate each inverted edge with this value. This backward graph represents *backward pass* (Fig. 2.24: Running a backward pass). Have:

1. $\text{grad}(\text{loss_val}, x2) = 1$, because as $x2$ varies by an amount ϵ , $\text{loss_val} = \text{abs}(4 - x2)$ varies by same amount.
2. $\text{grad}(x2, x1) = 1$, because as $x1$ varies by an amount ϵ , $x2 = x1 + b = x1 + 1$ varies by same amount.
3. $\text{grad}(x2, b) = 1$, because as b varies by an amount ϵ , $x2 = x1 + b = 6 + b$ varies by same amount.
4. $\text{grad}(x1, w) = 2$, because as w varies by an amount ϵ , $x1 = x * w = 2 * w$ varies by $2 * \epsilon$.

What chain rule says about this backward graph: can obtain derivative of a node w.r.t. another node by *multiplying derivatives for each edge along path linking 2 nodes*.

E.g., $\text{grad}(\text{loss_val}, w) = \text{grad}(\text{loss_val}, x2) * \text{grad}(x2, x1) * \text{grad}(x1, w)$ (Fig. 2.25: Path from loss_val to w in backward graph). By applying chain rule to graph, obtain what were looking for:

$$\begin{aligned}\text{grad}(\text{loss_val}, w) &= 1 * 1 * 2 = 2 \\ \text{grad}(\text{loss_val}, b) &= 1 * 1 = 1\end{aligned}$$

Note 3. If there are multiple paths linking 2 models of interest, a , b , in backward graph, would obtain $\text{graph}(b, a)$ by summing contributions of all paths.

& with that, just saw backpropagation in action! Backpropagation is simply application of chain rule to a computation graph. There's nothing more to it. Backpropagation starts with final loss value & works backward from top layers to bottom layers, computing contribution that each parameter had in loss value. That's where name “backpropagation” comes from: “back propagate” loss contributions of different nodes in a computation graph.

Nowadays people implement neural networks in modern frameworks that are capable to *automatic differentiation*, e.g. TensorFlow. Automatic differentiation is implemented with kind of computation graph just seen. Automatic differentiation makes it possible to retrieve gradients of arbitrary compositions of differentiable tensor operations without doing any extra work besides writing down forward pass. When CHOLLET wrote his 1st neural networks in C in 2000s, had to write gradients by hand. Now, thanks to modern automatic differentiation tools, never have to implement backpropagation yourself. Consider yourself lucky!

• **Gradient tape in TensorFlow.** API through which you can leverage (đòn bẩy) TensorFlow's powerful automatic differentiation capabilities is **GradientTape**. It's a Python scope that will “record” tensor operations that run inside it, in form of a computation graph (sometimes called a “tape”). This graph can then be used to retrieve gradient of any output w.r.t. any variable or set of variables (instances of `tf.Variable` class). A `tf.Variable` is a specific kind of tensor meant to hold mutable state – e.g., weights of a neural network are always `tf.Variable` instances.

```
import tensorflow as tf
x = tf.Variable(0.)
with tf.GradientTape() as tape:
    y = 2 * x + 3
grad_of_y_wrt_x = tape.gradient(y, x)
```

GradientTape works with tensor operations:

```
x = tf.Variable(tf.random.uniform((2, 2)))
with tf.GradientTape() as tape:
    y = 2 * x + 3
```

```
grad_of_y_wrt_x = tape.gradient(y, x)
```

It works works with list of variables:

```
W = tf.Variable(tf.random.uniform((2, 2)))
b = tf.Variable(tf.zeros((2,)))
x = tf.random.uniform((2, 2))
with tf.GradientTape() as tape:
    y = tf.matmul(x, W) + b
grad_of_y_wrt_W_and_b = tape.gradient(y, [W, b])
```

- 2.5. Looking back at 1st example. Should now have a general understanding of what's going on behind scenes in a neural network. What was a magical black box at start of chap has turned into a clearer picture, as illustrated in Fig. 2.26: Relationship between network, layers, loss function, & optimizer.: model, composed of layers that are chained together, maps input data to predictions. Loss function then compares these predictions to targets, producing a loss value: a measure of how well model's predictions match what was expected. Optimizer uses this loss value to update model's weights.

Go back to 1st example in this chap & review each piece of it in light of what learned since. This was input data:

```
(train_images, train_labels), (test_images, test_labels) = mnist.load_data()
train_images = train_images.reshape((60000, 28 * 28))
train_images = train_images.astype("float32") / 255
test_images = test_images.reshape((10000, 28 * 28))
test_images = test_images.astype("float32") / 255
```

Now understand: input images are stored in NumPy tensors, which are here formatted as `float32` tensors of shape (60000, 784) (training data) & (10000, 784) (test data) resp. This was our model:

```
model = keras.Sequential([
    layers.Dense(512, activation="relu"),
    layers.Dense(10, activation="softmax")
])
```

Now understand: this model consists of a chain of 2 `Dense` layers, that each layer applies a few simple tensor operations to input data, & these operations involve weight tensors. Weight tensors, which are attributes of layers, are where *knowledge* of model persists. This was model-compilation step:

```
model.compile(optimizer="rmsprop",
              loss="sparse_categorical_crossentropy",
              metrics=["accuracy"])
```

Now understand: `sparse_categorical_crossentropy` is loss function used as a feedback signal for learning weight tensors, & which training phase will attempt to minimize. Also know that this reduction of loss happens via mini-batch stochastic gradient descent. Exact rules governing a specific use of gradient descent are defined by `rmsprop` optimizer passed as 1st argument. This was training loop:

```
model.fit(train_images, train_labels, epochs=5, batch_size=128)
```

Now understand what happens when call `fit`: model will start to iterate on training data in mini-batches of 128 samples, 5 times over (each iteration over all training data is called an *epoch*). For each batch, model will compute gradient of loss w.r.t. weights (using Backpropagation algorithm, which derives from chain rule in calculus) & move weights in direction that will reduce value of loss for this batch.

After these 5 epochs, model will have performed 2345 gradient updates (469 per epoch), & loss of model will be sufficiently low: model will be capable of classifying handwritten digits with high accuracy.

At this point, already know most of what there is to know about neural networks. Prove it by reimplementing a simplified version of that 1st example “from scratch” in TensorFlow, step by step.

- Summary.

- * *Tensors* form foundation of modern ML systems. They come in various flavors of `dtype`, `rank`, `shape`.
- * You can manipulate numerical tensors via *tensor operations* (e.g. addition, tensor product, or element-wise multiplication), which can be interpreted as encoding geometric transformations. In general, everything in DL is amenable to a geometric interpretation.
- * DL models consist of chains of simple tensor operations, parametrized by *weights*, which are themselves tensor. Weights of a model are where its “knowledge” is stored.

- * *Learning* means finding a set of values for model’s weights that minimizes a *loss function* for a given set of training data samples & their corresponding targets.
- * Learning happens by drawing random batches of data samples & their targets, & computing gradient of model parameters w.r.t. loss on batch. Model parameters are then moved a bit (magnitude of move is defined by learning rate) in opposite direction from gradient. This is called *mini-batch stochastic gradient descent*.
- * Entire learning process is made possible by fact: all tensor operations in neural networks are differentiable, & thus possible to apply chain rule of derivation to find gradient function mapping current parameters & current batch of data to a gradient value. This is called *backpropagation*.
- * 2 key concepts see frequently in future chaps are *loss* & *optimizers*. These are 2 things need to define before begin feeding data into a model.
 - *Loss* is quantity you’ll attempt to minimize during training, so it should represent a measure of success for task you’re trying to solve.
 - *Optimizer* specifies exact way in which gradient of loss will be used to update parameters: e.g., it could be RMSProp optimizer, SGD with momentum, & so on.
- 3. Introduction to Keras & TensorFlow. Cover: A closer look at TensorFlow, Keras, & their relationship. Setting up a DL workspace. An overview of how core DL concepts translate to Keras & TensorFlow.

This chap is meant to give everything need to start doing DL in practice. Give a quick presentation of Keras <https://keras.io> & TensorFlow <https://tensorflow.org>, Python-based DL tools used throughout book. Find out how to set up a DL workspace, with TensorFlow, Keras, & GPU support. Finally, building on top of 1st contact had with Keras & TensorFlow in Chap. 2, review core components of neural networks & how they translate to Keras & TensorFlow APIs.

By end of this chap, ready to move on to practical, real-world applications, which will start with Chap. 4.

- 3.1. What’s TensorFlow? TensorFlow is a Python-based, free, open source ML platform, developed primarily by Google. Much like NumPy, primary purpose of TensorFlow: enable engineers & researchers to manipulate mathematical expressions over numerical tensors. But TensorFlow goes far beyond scope of NumPy in following ways:
 - * It can automatically compute gradient of any differentiable expression, making it highly suitable for ML.
 - * It can run not only on CPUs, but also on GPUs & TPUs, highly parallel hardware accelerators.
 - * Computation defined in TensorFlow can be easily distributed across many machines.
 - * TensorFlow programs can be exported to other runtimes, e.g. C++, JavaScript (for browser-based applications), or TensorFlow Lite (for applications running on mobile devices or embedded devices), etc. This makes TensorFlow applications easy to deploy in practical settings.

Important to keep in mind: TensorFlow is much more than a single library. It’s really a platform, home to a vast ecosystem of components, some developed by Google & some developed by 3rd parties. E.g., there’s TF-Agents for reinforcement-learning research, TFX for industry-strength ML workflow management, TensorFlow Serving for production deployment, & there’s TensorFlow Hub repository of pretrained models. Together, these components cover a very wide range of use cases, from cutting-edge research to large-scale production applications.

TensorFlow scales fairly well: e.g., scientists from Oak Ridge National Lab have used it to train a 1.1. exaFLOPS extreme weather forecasting model on 27000 GPUs of IBM Summit supercomputer. Likewise, Google has used TensorFlow to develop very compute-intensive DL applications, e.g. chess-playing & Go-playing agent AlphaZero. For own models, if have budget, can realistically hope to scale to around 10 petaFLOPS on a small TPU pod or a large cluster of GPUs rented on Google Cloud or AWS. That would still be around 1% of peak compute power of top supercomputer in 2019!

- 3.2. What’s Keras? Keras is a DL API for Python, built on top of TensorFlow, that provides a convenient way to define & train any kind of DL model. Keras was initially developed for research, with aim of enabling fast DL experimentation. Through TensorFlow, Keras can run on top of different types of hardware (Fig. 3.1: Keras & TensorFlow: TensorFlow is a low-level tensor computing platform, & Keras is a high-level DL API) – GPU, TPU, or plain CPU – & can be seamlessly scaled to thousands of machines.

Keras is known for prioritizing developer experience. It’s an API for human beings, not machines. It follows best practices for reducing cognitive load: it offers consistent & simple workflows, it minimizes number of actions required for common use cases, & it provides clear & actionable feedback upon user error. This makes Keras easy to learn for a beginner, & highly productive to use for an expert.

Keras has well over a million users as of late 2021, ranging from academic researchers, engineers, & data scientists at both startups & large companies to graduate students & hobbyists. Keras is used at Google, Netflix, Uber, CERN, NASA, Yelp, Instacart, Square, & hundreds of startups working on a wide range of problems across every industry. YouTube recommendations originate from Keras models. Waymo self-driving cars are developed with Keras models. Keras is also a popular framework on Kaggle, ML competition website, where most DL competitions have been won using Keras.

Because Keras has a large & diverse user base, it doesn’t force you to follow a single “true” way of building & training models. Rather, it enables a wide range of different workflows, from very high level to very low level, corresponding to different user profiles. E.g., have an array to ways to build models & an array of ways to train them, each representing a certain trade-off between usability & flexibility. In Chap. 5, review in detail a good fraction of this spectrum of workflows.

Could be using Keras like would use Scikit-learn – just calling `fit()` & letting framework do its thing – or could be using it like NumPy – taking full control of every little detail.

I.e., everything learning now as getting started will still be relevant once become an expert. Can get started easily & then gradually dive into workflows where writing more & more logic from scratch. Won't have to switch to an entirely different framework as go from student to researcher, or from data scientist to DL engineer.

This philosophy is not unlike that of Python itself! Some languages only offer 1 way to write programs – e.g., OOP or functional programming. Meanwhile, Python is a multiparadigm language: it offers an array of possible usage patterns that all work nicely together. This makes Python suitable to a wide range of very different use cases: system administration, DS, ML engineering, web development ... or just learning how to program. Likewise, can think of Keras as Python of DL: a user-friendly DL language that offers a variety of workflows to different user profiles.

- 3.3. Keras & TensorFlow: A brief history. Keras predates TensorFlow by 8 months. It was released in Mar 2015, & Tensorflow was released in Nov 2015. May ask if Keras is built on top of TensorFlow, how it could exist before TensorFlow was released? Keras was originally built on top of Theano, another tensor-manipulation library that provided automatic differentiation & GPU support – earliest of its kind. Theano, developed at Montréal Institute for Learning Algorithms (MILA) at the Université de Montréal, was in many ways a precursor (tiền thân) of TensorFlow. It pioneered idea of using static computation graphs for automatic differentiation & for compiling code to both CPU & GPU.

In late 2015, after release of TensorFlow, Keras was refactored to a multibackend architecture: it became possible to use Keras with either Theano or TensorFlow, & switching between 2 was as easy as changing an environment variable. By Sep 2016, TensorFlow had reached a level of technical maturity where it became possible to make it default backend option for Keras. In 2017, 2 new additional backend options were added to Keras: CNTK (developed by Microsoft) & MXNet (developed by Amazon). Nowadays, both Theano & CNTK are out of development, & MXNet is not widely used outside of Amazon. Keras is back to being a single-backend API – on top of TensorFlow.

Keras & TensorFlow have had a symbiotic relationship for many years. Throughout 2016 & 2017, Keras became well known as user-friendly way to develop TensorFlow applications, funneling new users into TensorFlow ecosystem. By late 2017, a majority of TensorFlow users were using it through Keras or in combination with Keras. In 2018, TensorFlow leadership picked Keras & TensorFlow's official high-level API. As a result, Keras API is front & center in TensorFlow 2.0, released in Sep 2019 – an extensive redesign of TensorFlow & Keras that takes into account > 4 years of user feedback & technical progress.

By this point, must be eager to start running Keras & TensorFlow code in practice.

- 3.4. Setting up a DL workspace. Before can get started developing DL applications, need to set up your development environment. It's highly recommended, although not strictly necessary, that run DL code on a modern NVIDIA GPU rather than computer's CPU. Some applications – in particular, image processing with convolutional networks – will be excruciatingly slow (chậm kinh khủng) on CPU, even a fast multicore CPU. & even for applications that can realistically be run on CPU, generally see speed increase by a factor of 5 or 10 by using a recent GPU.

To do DL on a GPU, have 3 options:

- * Buy & install a physical NVIDIA GPU on your workstation.
- * use GPU instances on Google Cloud or AWS EC2.
- * Use free GPU runtime on Colaboratory, a hosted notebook service offered by Google.

Colaboratory is easiest way to get started, as it requires no hardware purchase & no software installation – just open a tab in browser & start coding. It's the opinion recommend for running code examples in this book. However, free version of Colaboratory is only suitable for small workloads. If want to scale up, have to use 1st or 2nd option.

If don't already have a GPU that can use for DL (a recent, high-end NVIDIA GPU), then running DL experiments in cloud is a simple, low-cost way to move to larger workloads without having to buy any additional hardware. If develop using Jupyter notebooks, experience of running in cloud is no different from running locally.

But if a heavy user of DL, this setup isn't sustainable in long term (bền vững trong dài hạn) – or even for more than a few months. Cloud instances aren't cheap: pay \$2.48 per hour for a V100 GPU on Google Cloud in mid-2021. Meanwhile, a solid consumer-class GPU will cost you somewhere between \$1500–\$2500 – a price that has been fairly stable over time, even as specs of these GPUs keep improving. If a heavy user of DL, consider setting up a local workstation with 1 or more GPUs.

Additionally, whether running locally or in cloud, better to be using a Unix workstation. Although it's technically possible to run Keras on Windows directly, don't recommend it. If a Windows user wants to do DL on own workstation, simplest solution to get everything running is to set up an Ubuntu dual boot on your machine, or to leverage Windows Subsystem for Linux (WSL), a compatibility layer that enables to run Linux applications from Windows. It may seem like hassle, but it will save you a lot of time & trouble in long run.

- * 3.4.1. Jupyter notebooks: preferred way to run DL experiments. Jupyter notebooks are a great way to run DL experiments – in particular, many code examples in this book. They're widely used in DS & ML communities. A *notebook* is a file generated by Jupyter Notebook app <https://jupyter.org> that can edit in browser. It mixes ability to execute Python code with rich text-editing capabilities for annotating what doing. A notebook also allows to break up long experiments into smaller pieces that can be executed independently, which makes development interactive & means don't have to rerun all of your prev code if sth goes wrong late in an experiment.

Recommend using Jupyter notebooks to get started with Keras, although that isn't a requirement: can also run standalone Python scripts or run code from within an IDE e.g. PyCharm. All code examples in this book are available as open source notebooks; can download: github.com/fchollet/deep-learning-with-python-notebooks.

- * 3.4.2. Using Colaboratory. Colaboratory (or Colab for short) is a free Jupyter notebook service that requires no installation & runs entirely in cloud. Effectively, it's a web page that lets you write & execute Keras scripts right away. It gives access to a free (but limited) GPU runtime & even a TPU runtime, so don't have to buy your own GPU. Colaboratory is what recommend for running code examples in this book.
 - 1st steps with Colaboratory. To get started with Colab, go to <https://colab.research.google.com> & click New Notebook button. See standard Notebook interface shown in Fig. 3.2: A Colab notebook. Notice 2 buttons in toolbar: + Code, + Text. They're for creating executable Python code cells & annotation text cells, resp. After entering code in a code cell, Pressing Shift-Enter will execute it (Fig. 3.3: Creating a code cell)
- 3.5. 1st steps with TensorFlow.
- 4. Getting started with neural networks: Classification & regression.
- 5. Fundamentals of ML.
- 6. Universal workflow of ML.
- 7. Working with Keras: A deep dive.
- 8. Introduction to DL for computer vision.
- 9. Advanced DL for computer vision.
- 10. DL for timeseries.
- 11. DL for text.
- 12. Generative DL.
- 13. Best practices for real world.
- 14. Conclusions.

2.6 [DG24]. SHLOMO DUBNOV, ROSS GREER. **Deep & Shallow: Machine Learning in Music & Audio. 2023**

- “*Deep & Shallow* by SHLOMO DUBNOV & ROSS GREER is an exceptional journey into convergence of music, AI, & signal processing. Seamlessly weaving together intricate theories with practical programming activities, book guides readers, whether novices or experts, toward a profound understanding of how AI can reshape musical creativity. A true gem for both enthusiasts & professionals, this book eloquently bridges gap between foundational concepts of music information dynamics as an underlying basis for understanding music structure & listening experience, & cutting-edge applications, ushering us into future of music & AI with clarity & excitement.” – GIL WEINBERG, Professor & Founding Director, Georgia Tech Center for Music Technology
- “The authors make an enormous contribution, not only as a textbook, but as essential reading on music information dynamics, bridging multiple disciplines of music, information theory, & ML. Theory is illustrated & grounded in plenty of practical information & resources.” – ROGER B. DANNENBERG, Emeritus Professor of Computer Science, Art & Music, Carnegie Mellon University
- “DEEP & SHALLOW by DUBNOV & GREER is ... a great introductory textbook on topic of ML for music & audio. Especially useful are Jupyter Notebooks, providing examples of everything covered in book. Advanced undergrads, graduate students, or working professionals with some experience in statistics & Python should find book perfect for learning basics, & recent/advanced topics e.g. DL.” – PERRY R. COOK, PhD, Professor (Emeritus), Princeton University Computer Science (also Music)

Deep & Shallow. Providing an essential & unique bridge between theories of signal processing, ML, & AI in music, this book provides a holistic overview (tổng quan toàn diện) of foundational ideas in music, from physical & mathematical properties of sound to symbolic representations. Combining signals & language models in 1 place, this book explores how sound may be represented & manipulated by computer systems, & how our devices may come to recognize particular sonic patterns as musically meaningful or creative through lens of information theory.

Introducing popular fundamental ideas in AI at a comfortable pace, more complex discussions around implementations & implications in musical creativity are gradually incorporated as book progresses. Each chap is accompanied by guided programming activities designed to familiarize readers with practical implications of discussed theory, without frustrations of free-form coding.

Surveying state-of-art methods in applications of deep neural networks to audio & sound computing, as well as offering a research perspective that suggests future challenges in music & AI research, this book appeals to both students of AI & music, as well as industry professionals in fields of ML, music, & AI.

SHLOMO DUBNOV is a Professor in Music Department & Affiliate Professor in Computer Science & Engineering at University of California, San Diego. He is best known for his research on poly-spectral analysis of musical timbre & inventing method of Music Information Dynamics with applications in Computer Audition & Machine improvisation. His previous books on *The Structure*

of *Style: Algorithmic Approaches to Understanding Manner & Meaning & Cross-Cultural Multimedia Computing: Semantic & Aesthetic Modeling* were published by Springer.

ROSS GREER is a PhD Candidate in Electrical & Computer Engineering at University of California, San Diego, where he conducts research at intersection of AI & human agent interaction. Beyond exploring technological approaches to musical expression, ROSS creates music as a conductor & orchestrator for instrumental ensembles. ROSS received his B.S. & B.A. degrees in EECS, Engineering Physics, & Music from UC Berkeley, & an M.S. in Electrical & Computer Engineering from US San Diego.

Preface. This book complements a largely missing textbook for a popular & growing in demand field of ML for Music & Audio. 1 of difficulties in covering this subject is interdisciplinary nature of domain knowledge that creates a barrier to entry to this field, which combines background in signal processing, symbolic sequence processing, statistical analysis, classical ML, & more recently quickly growing field of deep neural networks. As a result of breath of required background, students or practitioners who are interested in learning about field have to look for relevant information across multiple disciplines. Moreover, focus of book is on generative methods, which are usually considered as more advanced subjects in ML that are often not explained in introductory or basic-level courses on ML.

Book is intended for upper-division undergraduate- or graduate-level courses, with basic understanding of statistics & Python programming. Music concepts & basic concepts in analysis of audio signals will be covered in book, providing a 1-stop reference to questions of musical representation & feature extraction, without assuming extensive prior musical & acoustics knowledge.

Starting with representation aspects in audio & symbolic music in Chap. 1, 2nd chap goes immediately into stochastic aspects in both music & audio, addressing conceptual premise of thinking about relations between noise & structure. Idea of composition using random dice with lookup tables, or so-called “Mozart Dice Game”, shows early on how concepts are related, also providing historical context & relation to later formal algorithmic methods.

Although symbolic music vs. audio signals are capturing sound data at different timescales & levels of detail, parallels between 2 domains are demonstrated throughout book by considering aspects of anticipation both in perceptual (sound vs. noise) & cognitive (events surprisal) realms. In Chap. 2, introduce a unifying analysis framework called “Music Information Dynamics” that stems from conceptual formulation of music & audio modeling & understanding as a problem of information processing. Introducing Markov models as basic statistical models for text & music, proceed to consider information dynamics for case of Markov models, relating it to questions of what type of random processes generates music that are more or less interesting to a predictive listener.

Accordingly, concepts from information theory & Shannon’s laws are covered in Chap. 3. Central idea of thinking about music as an information source forms basis to way this book approaches & represents musical ML problem. Using “off the shelf” data compression methods, one can build efficient representations of musical style. Demonstrate how style learning & continuation of musical phrases in a particular style can be accomplished by traversing a codebook that was derived from famous Lempel-Ziv compression method.

In Chap. 4, look deeper into properties of audio signals by introducing basic mathematical tools of digital signal processing. After briefly introducing concepts of sampling, talk about frequency analysis & conditions for short-time Fourier transform (STFT) perfect reconstruction. Such signal reconstruction is not possible when only magnitude of STFT is available, so cover Griffin-Lim phase reconstruction that is often used for generating audio (also called vocoding) from magnitude spectrograms. Conclude chap with discussion of source-filter models & their application to speech modeling. Finally, use concept of spectral analysis &, in particular, idea of spectral flatness to distinguish noise vs. structure in audio, relating it to questions of prediction & information dynamics.

More advance information theory concepts are developed in Chap. 5, as part of concatenative & recombinant audio generative methods. Using a symbolization process that renders audio into a sequence of symbolic representation, known as Audio Oracle, demonstrate use of Variable-Memory Markov models & string-matching compression algorithms for creating new sequences of audio features & audio descriptors. These new symbolic sequences are then rendered back into audio using concatenative method. Problem of finding repetitions in audio data & use of recopying for compression & prediction are also linked to questions of computational aesthetics, namely aesthetics measures of Birkhoff & Bense as ratios of order & complexity, as well as ethical aspects of reuse of existing music in training generative models.

Chap. 6 serves as a bridge between signal processing & statistical methods of earlier chaps & materials in later chaps that are dealing with deep neural networks for music & audio. After introducing basic building blocks of artificial neural networks, discuss aspects of representation learning by looking into similarities & differences between auto-encoder & a classical statistical data reduction & representation method known as *principal components analysis* (PCA). Specifically, use of a PCA-like method for finding an optimal transform of audio signals, known as Krahunen-Loeve transform, is demonstrated in context of noise removal.

Chaps. 7–10 cover deep neural network approaches to music, starting in Chap. 7 with question of unsupervised representation learning using sequence & image models that use Recurrent Neural Networks (RNN) & Convolutional Neural Networks (CNN). Idea of tokenizations is basic to ability of capturing complex sequential features in music into a neural recurrent representation. Relations across both time & pitch or frequency are captured using CNN approach.

In Chap. 8, return to fundamental concept of turning noise into structure through notion of neural networks that imagine. Concept of generative models is introduced by extending previously discussed autoencoder into a Variational Autoencoder (VAE) that allows probabilistic approximation & representation. In this view, random seed or input to generative networks results in generation of variable replicas or samples from same statistical model of music that was learned by neural network. This chap also covers Generative Adversarial Networks, typing representation learning of VAE to broader generative tasks, & also introducing conditional generation methods similar to style transfer in images. Such methods are important for specifying constraints on generative processes in order to produce music or audio that correspond to user specifications or other contextual input.

Chap. 9 covers recently extremely successful Transformers models, relating them to problem of finding repetition & memory structures that were encountered earlier in Variable Markov Oracle (VMO) of Chap. 5 & RNN in Chap. 7. Several state-of-art

transformer models are surveyed here both in audio & symbolic representations.

Book concludes by reviewing & summarizing broad problem of viewing music as a communication process in Chap. 10. By relating phenomena of music understanding & creation to models of music information dynamics, able to discuss future of AI in music in terms of building new compositional tools that operate by learning musical structures from examples & generating novel music according to higher level meta-creative specifications, approaching novel levels of man-machine co-creative interaction.

Interactive code exercises are an important feature of this book. Instructor solutions are available by request through publisher's book website. Referenced code will be available as Jupyter Notebooks at <https://github.com/deep-and-shallow/notebooks>.

Authors provide following possible suggestions for curriculum of an introductory & advanced course on ML for Music & Audio. Focus of 1st course is an introduction to basic concept of music & audio representation, modeling music as a probabilistic information source, music information dynamics, & an introduction to representation learning & sequence modeling using neural networks.

- Class 1: Representation of Sound & Music Data (MIDI, audio), Audio features, MFCC, Chroma
- Class 2: Aleatoric music, stochastic processes in music & sound (Mozart Dice Game), Noise & Electronic Music
- Class: Sampling, Spectral Analysis, Fourier transform, FFT, Spectral Flatness
- Class 4: Short-Time Fourier Analysis, Perfect Reconstruction (COLA), Griffith-Lim phase reconstruction
- Class 5: Information Theory & Music, String Matching, Universal Compression & Prediction
- Class 6: Markov Models for Text & Music, Lempel-Ziv Algorithm for Motif Continuation
- Class 7: Introduction to Neural Networks, Neural Network Models of Music
- Class 8: Autoencoder (AE) & PCA, Representation Learning
- Class 9: Neural Language Models, Recurrent Neural Network for Music
- Class 10: Introduction to Generative Models & next course overview

Advanced 2nd course dives deeper into signal processing of speech signals & vocoders, generative neural networks for audio & music with focus on variational methods, & return to information theory for deep musical information dynamics with possible implications toward understanding creativity in neural networks.

- Class 1: Review of Music & Audio Representation, Short-Time Fourier Analysis
- Class 2: Linear Filters & Convolution Theorem, Pole-Zero Plots
- Class 3: History of voder/vocoder, Source-Filter Models, & Linear Prediction
- Class 4: Concatenative Synthesis, Variable Markov Oracle (VMO), Symbolization & Information Dynamics
- Class 5: Review of Neural Networks, AutoEncoders & RNN
- Class 6: Convolutional NN for Music & Audio (U-Net) for Genre Classification & Source Separation
- Class 7: Variational AE, ELBO, Generating Music with VAE
- Class 8: Generative Adversarial Networks, W-GAN, MuseGAN, CycleGAN
- Class 9: Attention & Transformers, MuseNet, Audio Spectrogram Transformer
- Class 10: Information Theory Models of NN, Deep Music Information Dynamics

Including extra time for quizzes, projects, & exams, these course contents should suffice for a semester-long course each, or as 2 quarters of instruction followed by a quarter of practical projects. Overall, estimate: with additional practical & theoretical assignments, materials of book represent about 1 year of academic study that is needed for students with focus on AI & music or a concentration in ML with specialization in music & audio. Hope: current text would help establish this as a field of study.

- Chap. 1. Introduction to Sounds of Music. Music is a particular form of art that is particularly hard to grasp, define, or put in a mold. In many aspects, music deals with shaping air vibrations into meaningful sound perceptions, & each type of music, depending on culture, style, period, or location, has its own particular ways of expression that might be grounded in practices that are close to hearts of some & meaningless to others. In this book, trying to understand music from AI & ML perspectives. To make question about meaning of music even more acute, may ask ourselves – what does it take to make music meaningful to a machine? If desire to build artificial agents that can compose music or engage creatively in real time with musicians in a process of improvisation, hope: these interactions will convey to intelligent machine same significance we ascribe to music (gán cho âm nhạc).

- **1.1. From Sound to Features.** Broadly speaking, meaning of music is established by human musicians themselves who have creative intent of producing particular types of sound structures, & by their listeners who engage with these sounds to experience them in interesting & exciting ways. Music, in many ways, is most abstract art, devoid of a particular reference to outside world. It may excite us or calm us down, affect our mood or be simply regarded as an intricate fabric of sound structures, where pleasure can be found in act of listening itself. Accordingly, asking question of what establishes meaning in sounds is a formidable problem to which we unfortunately will not be able to give a definite answer.

In journey from physical sounds to musical meaning, need to go through several intermediate stages. 1st steps is choosing palette of music sounds at our disposal, which can be later organized in time into a complete musical piece. Since 1 person's music may be another person's noise, challenge for machine is very much same as that of uninitiated listener: to 1st of all distinguish noise from music or find structure in sounds that makes it musical. In journey into creative musical machines, start with sound structures that are most common in music, namely notes, rhythms, & scales. Most elementary musical sounds, often called "tones" (tông màu) comprise duration (length), frequency (pitch), amplitude (loudness), & timbre (âm sắc) (sound quality). Tones are organized in patterns, where times & lengths of their appearances (onsets & durations) are organized into rhythm; selection of allowed frequencies is organized into scales & tonalities; multiple notes playing together create harmonies; & so on. Each of these aspects can be considered as a musical representation; it summarizes a physical phenomena – sound – into few essential properties that are considered musical. Since this is not a music theory book, will not go into elaborate theories of how music is represented or "rules" for musical composition that depend also on musical style & culture, but will rather stay as close as possible to examples in representation related to common Western music notation (Note: many of these techniques can be applied to analysis in other music notation schemes, given appropriate methods of tokenization & abstraction.) & representations that can be extracted from a sound recording before further processing to discover statistical relations by ML methods. Accordingly, may summarize basic sound properties or features as follows:

- * **Pitch** is a perceptual phenomenon which encapsulates relationship between frequency generated by an instrument & frequency perceived by listener. E.g., when musicians in a modern orchestra tune to concert pitch, they typically listen to an oboe play pitch A4. This pitch is produced at a frequency of 440 Hertz (Hertz: unit used to measure frequency, referring to occurrences per sec of any periodic phenomena.). Slight changes in frequency have little effect on perceived pitch. Perception of pitch varies logarithmically with generated frequency; i.e., to hear same pitch at a higher octave, frequency of sound must be doubled.

Tuning systems explore problem of mapping a discrete number of possible notes to continuous frequency space. E.g., a piano has a limited number of keys spanning an octave, but there are infinite number of frequencies which theoretically exist between any set of notes. Western music traditionally uses a 12-tone equal-tempered scale. In an *equal-tempered scale*, every pair of adjacent notes has an identical frequency ratio. Thus, each octave is divided into 12 logarithmically equal parts.

Example 1 (Notes on a Keyboard). *Musical Instrument Digital Interface, or MIDI, is a protocol for recording & playing music on computers. Lowest note on a standard acoustic piano, named A0, is pitch number 21 in MIDI convention. Highest note on a keyboard, named C8, is pitch number 108 in MIDI. By defining note named A4 (MIDI pitch 69) to be reference pitch at 440 Hz, can define following relationship between any MIDI pitch number p & its corresponding frequency:*

$$F_{\text{pitch}}(p) = 2^{\frac{p-69}{12}} \cdot 440.$$

To interpret this equation, note: exponential term effectively counts how many steps of an 12-step-octave input note is from reference pitch, then uses this as (fractional) power due to logarithmic relationship between perceived notes & requisite frequency. This coefficient is then used as a multiple to reference pitch of 440 Hz. Can see: a pitch located 12 half step⁸

- * **Timbre**, also referred to as tone quality or tone color, is quality of musical sound that distinguishes different types of sound production, e.g. different voices or instruments. When hear same note played at same volume & same duration 1st by 1 instrument & then by another, difference you hear is due to timbre. Physically, this property depends on energy distribution in harmonics & relationships between constituents sinusoids, phenomena introduced in Chap. 4.

As a prototype of sound, consider a sinusoidal wave (i.e., a wave with a fixed frequency, phase, & amplitude). Period is time it takes for wave cycle to repeat & can be measured using successive peaks in wave (unit: secs), & is inverse of frequency (unit: Hertz). Amplitude is 1-half pressure difference between high & low points (often measured in micro Pascals).

- * **1.1.1. Loudness.** Loudness is subjective perception of sound pressure, i.e., loudness is a combination of not only physical attributes of a sound but also physiological & psychological phenomena.

Torben Poulsen [1] shows: physical aspect of loudness is related to sound pressure level, frequency content, & sound duration. *Sound pressure level* is used to measure effective pressure of a sound relative to a reference pressure value (often 20 μPa , a canonical threshold for human hearing). Sound pressure level is described by formula

$$20 \log_{10} \frac{p}{p_0},$$

where p : measured sound pressure & p_0 : reference pressure, giving a unit of decibels (dB).

⁸Here must acknowledge some possible confusion in terminology: though octave is divided into 12 portions, musicians refer to these portions as *half steps*. 2 half steps span a *whole step*, & sequential combinations of half & whole steps are used to define different *scales*. Musical term *flat* refers to a pitch 1 half step below reference, while term *sharp* refers to a pitch 1 half step above reference. This provides a variety of possible descriptors for same pitch, referred to as an *enharmonic* relationship; e.g., C \sharp (C-sharp) & D \flat (D-flat) refer to same pitch. Nuances in use of these symbols can provide context to a performer about special tuning to play within a particular modes or harmony, but for sake of our examples, can consider these pitches equivalent & thus must be careful that our schemes for abstracting these pitches do not mistakenly contain 2 possible table entries for same pitch.

Steven's power law, formulated as $\Psi(I) = kI^\alpha$, connects sound pressure levels I to perceived loudness of a single tone; i.e., it defines relationship between a change in stimulus sound pressure level & corresponding increase in perceived sensation. Steven's power law typically uses an exponent α of 0.67, & this exponent & scalar constant k are empirically sourced approximations; accordingly, there exist other methods & models to relate sound pressure levels to perception. Another interesting perceptual phenomena is relationship between stimulus frequency & perceived loudness, portrayed by a so-called *equal-loudness graph* as shown in Fig. 1.1: Equal-loudness graph created by SUZUKI & TAKESHIMA [2] shows: different frequencies create same perceived loudness at different sound pressure levels. Following any 1 curve on graph provides frequency-SPL pairs which were found to create same perceived loudness.

- * 1.1.2. MFCC. In field of timbral analysis, i.e., extracting sound properties that are unrelated to pitch, duration, or loudness, Mel-Frequency Cepstral Coefficients (MFCC) became “go-to” method for capturing overall shape of frequency distribution, also called *spectral envelope*. Spectral envelope may characterize color of sound that is independent of specific note an instrument plays, which is useful in distinguishing between different types of instruments. MFCCs have historically been dominant features in speech recognition since they are invariant to pitch or loudness of speech & rather capture broad shape of energy distribution of voice (that comprises so-called “speech formants”: Voice as a combination of speech formants is introduced in Chap. 4.). Moreover, MFCC analysis has been applied to complex music e.g. polyphonic music (multiple voices or multiple instruments, e.g. orchestra) to characterize distribution of energies at different frequencies. E.g., they were used in music retrieval systems, music summarization, & musical audio analysis toolboxes (Musical audio analysis toolboxes e.g. Librosa are introduced in Appendix B.).

A detailed description of MFCC requires technical terms that assume knowledge of spectral analysis, which is using Fourier transform to discover underlying frequency components in a sound signal. While Fourier analysis is introduced in next chap, what is important to note at this point: each step in process of MFCC computation is motivated by perceptual or computational considerations. Briefly, it 1st converts Fourier analysis, which is linear in frequency, to a Mel-scale that is perceptually scaled according to human judgments of pitches being equal distance from each other. I.e., when comparing distances between any 2 points on Mel scale, same distance should correspond to same perceived change in pitch. Within linear frequency scale, this relationship is not preserved since relationship of frequency to perceived pitch is logarithmic. A common formula for relating Mel (a unit whose name is derived from “melody” [3]) to frequency in Hertz (Hz) is

$$m = 2595 \log_{10} \left(1 + \frac{f}{700} \right).$$

After that, a technique called “cepstral analysis” is applied to scaled spectrum. Idea of cepstral analysis: separate short-time structure vs. longer period repetitions in sounds by assuming a “source-filter” model. By looking at spectral analysis of a sound, one notices: there are harmonics, or partials showing as individual peaks, & an overall “spectral envelope” that shapes spectrum in terms of broad areas of resonances. In source-filter view, spectral peaks of harmonics originate from repetitions of excitation source, e.g. flapping of vocal chords in speech or vibrations of reed or string in a musical instrument. Broad shape of spectrum is attributed to filter, representing resonance of body of instrument or speech formants in a vocal tract. Operation of a filter on another signal amounts to an operation of multiplication in frequency domain between filter's & signal's spectra, resp. In cepstral analysis, a log function is applied to log of amplitude-spectrum, & then a Fourier analysis is applied to split it into low-order & high-order components. Idea of cepstral analysis: log operation, at least approximately, separates aspects of source & filter into 2 additive components, one that contains harmonic without spectral envelope & the other that captures envelope. Then by applying Fourier transform result is rendered back into time-like units, called “Quefreny”. By going back to a time-like domain, spectral envelope is translated into lower cepstral coefficients (low quefreny), while harmonic structure due to pitch shows up peaks at higher quenfrenies. Eventually, only lower cepstral coefficients (filter characteristics) are kept as features, as they capture broad shape of spectrum, removing spectral shape of harmonics due to pitch (source). Formally written, cepstral analysis is defined as follows: (1.3)

$$C(q) = |\mathcal{F}^{-1} \{ \log (|\mathcal{F}\{x(t)\}|^2) \}|$$

with $C(q)$ being cepstrum & different quefrenies, $x(t)$: original time signal, $\mathcal{F}, \mathcal{F}^{-1}$ being Fourier & inverse Fourier transforms, resp. A diagram of MFCC analysis is shown in Fig. 1.2: Main steps in calculation of MFCC. To obtain a frequency scale that better corresponds to human hearing, a mel-scale is used instead of linear frequency spacing. Moreover, since sound properties change over time, MFCC analysis is applied to short instances of sound by applying a windowing function that extracts a portion of a signal in time. It should be noted: total energy or overall loudness of sound is captured by 1st cepstral coefficient & is often disregarded too⁹. An investigation suggests: MFCC are approximately optimal representations, at least for case of speech & song, s they approximate a statistically optimal basis¹⁰, derived by Principal Component Analysis (PCA) applied to a Mel-Spectra correlation matrix. Technical method of PCA will be described in Chap. 6. To summarize, vector of MFCC is a timbral feature that captures overall spectral shape of a sound in its lower coefficients, independent of aspects of pitch or energy, in approximately optimal & independent manner.

- * 1.1.3. Chroma. Chromatic features of sound (Đặc điểm sắc độ của âm thanh) refer to phenomenon that doubling or halving frequency of sound results in a similar perceived sound, just higher or lower. A musical example of this *octave* phenomenon

⁹Since Fourier transform performs & integral operation over a signal multiplied by a phasor at each frequency, Fourier analysis at 0 frequency is simply an integral over whole signal, which in case of (1.3) is an integral over log of energy at all frequencies, thus giving total log-energy of signal.

¹⁰PCA finds basis vectors that optimally represent randomly distributed vectors, e.g. sound features, in a way that is both decorrelated & sorted according to amount of variance they capture in original signal. A basis is a set of vectors that can be combined in multiples to construct any other vector within a defined vector space.

is shown in Fig. 1.3: 3rd movement of Beethoven's 9th Symphony famously begins with octave jumps in strings. Each set of notes begins at a high pitch, then drops an octave. From a physical standpoint, frequency of string's vibration is halved. Listen to a recording of this scherzo to hear remarkable similarity between pitches spaced an octave apart; this similarity is described as *chroma* feature.

This chromatic feature is best visualized with *chromagram*, shown in Fig. 1.4: Helix pattern of chromagram. Projections onto circle at base provide chroma perceived, while height provides reference to general frequency level associated with pitch. There is a property of chromatic invariance between pitches separated by 1 or more octaves; as move up or down helix of chromagram, projection of position onto chromatic wheel provides chroma perceived, while height indicates how “high” or “low” sound is perceived. Distance between any point & point directly above is equal to 1 octave (i.e., next time same chroma will occur). Thus, every frequency can be constructed as a combination of height & chroma:

$$f = 2^{h+c},$$

where height $h \in \mathbb{N}$ & chroma $c \in [0, 1)$. Similarly, chroma itself can be thought of as “carry” after subtracting nearest height level from frequency:

$$c = \log 2f - \lfloor \log 2f \rfloor.$$

Having an understanding of these properties & features of sound will be a useful frame for analysis of music, especially in relation to musical events we perceive as consonant or dissonant, to which may ascribe musical significance we seek to represent & recreate in our models.

– Hiểu được các đặc tính & đặc điểm của âm thanh này sẽ là một khuôn khổ hữu ích để phân tích âm nhạc, đặc biệt là liên quan đến các sự kiện âm nhạc mà chúng ta coi là hòa hợp hay bất hòa, từ đó có thể gán ý nghĩa âm nhạc mà chúng ta muốn thể hiện & tái tạo trong các mô hình của mình.

o 1.2. Representation of Sound & Music Data.

- * 1.2.1. Symbolic Representations. Though may never answer classic question of “What *is* music?”, here hope to make discussion of some ways may *represent* music.

If music is composed of sounds, might consider Western music notation to be an example of *symbolic representation* of sounds. Note: while this system of symbols is capable of representing creative ideas, this music notation is incomplete in its ability to represent every possible sound & is of course subject to interpretation by its time & cultural context. This incompleteness is a property we will see across representations, even in our “best” audio encodings. Always limited by number of bits (A *bit* (binary digit) is smallest unit of information that can be represented by a computer; conceptually, unit is an entity that can take 1 of 2 states, which can be imagined as “on” or “off”, 1 or 0, “True” or “False”, etc.) available to represent sonic information, so a representation which allows for perfect reconstruction of physical sound is a near-impossible task. Of course, can come close, adopting representations which can encode information to resolution & rate of human hearing.

Though a trivial statement, note: representation scheme offered by Western music notation is, in general, sufficient to represent Western music. So, define some of its common structures & features used to describe common musical expressions, as use these structures as building blocks for composing similar music.

- A *note* describes a pitch & duration. A very brief introduction to Western music notation is provided in Fig. 1.5: This excerpt from Tchaikovsky's Overture to Romeo & Juliet illustrates characteristic symbols used to notate Western music. Each note has a **head** which sits on a position within **staff** (name given to each series of 5 horizontal lines). Placement of head on staff indicates pitch relative to some reference, provided by **clef** (symbol sitting in left-most position of staff). Flags, beams, & stems are used to indicate note durations. A brief catalog of some note & rest duration is shown in this excerpt, ranging from whole (entire measure of time) to 16th (1-16th of measure duration). Top staff contains whole notes, 2nd staff half notes, 3rd staff 16th notes, 4th staff 8th notes, & bottom staff quarter notes. Similarly, 3rd staff contains a half rest, 4th staff an 8th rest, & bottom staff a quarter rest. While this fractional notation of duration is in theory unbounded, beyond a certain limit (32nd notes or so), smaller fraction, more rare its occurrence in commonly performed music.
- A *chord* is combination of 2 or more notes sounding simultaneously.
- A *melody* is a sequence of notes.
- *Rhythm* is timing pattern assigned to a set of notes.

Symbolic representation relates abstract symbols to continuous-time sound waves (audio) perceive or imagine, when resp. listening or composing. However, it can be difficult to precisely derive these features from audio. Can explore associated challenges by trying provided *Introduction to Music Representation* programming exercises, where explore a means of converting a symbolic representation to a pseudo-physical representation (audio), then transcribing this generated audio back to a symbolic representation.

Most musicians may be especially familiar with symbolic representation of music as a printed notation system. Consider 1st 5 bars of famous 5th Symphony of Beethoven, shown in Fig. 1.6: Score depicting well-known motif which opens Beethoven's 5th Symphony.¹¹ This is a graphical notation form which indicates which notes are to be played, & for what duration. This representation relies on an agreed upon convention of understanding, & can be considered a set of performance instructions. These instructions are translated by a musician & their instrument into physical vibrations of air, forming acoustic signal we hear. Performance instructions we are used to reading draw from a very limited set of possible sounds

¹¹Technically, missing clarinets (kèn clarinet) from this figure (& many instruments which are resting), but motif is clear & should spark intended musical memory.

when consider noise that surrounds us every day (speaking voices, chirping birds, clanging machinery, etc.). In this sect, introduce some common symbolic notation systems adopted by digital devices.

- 1.2.1.1. MIDI. How might a computer represent symbolic music? Considering 1st violin line of Fig. 1.6, information communicated to performer is sequences of pitches to be played (silence, then a series of 3 Gs, then an E-flat), & duration for these pitches or silence (8th notes, at a tempo of Allegro con brio, a rate of 108 beats per minute; in case of this piece, each beat comprises a measure of 4 8th notes).

There are some challenges to translate this to a computer instruction (Exercise 2). 1st, though 3 notes are given in indirect sequence, there must be some brief (& unmeasured) separation between them, which is conventionally understood by performer; without this, would hear sequential notes performed as a continuous tone. I.e., sound must not be sustained between 3 notes since separation defines note length. MIDI [Musical Instrument Digital Interface] standard requires definition of PPQ (pulses per quarter note) & tempo (measured as microseconds per quarter note). Using these rates, can set durations (measured in ticks) for each note, with appropriate pauses in between, resulting in memory table illustrated in Fig. 1.7: A table representing MIDI instructions for 1st violin of Beethoven 5. In addition to duration of each note, must also communicate which pitch is to be played (in this case, a G4), which map to MIDI index standard (reference pitch C4 = Note Number 60) to determine G4 = Note Number 67. Having covered pitch & duration, remaining information available in a score is relative volume at which notes are played, often reflected in parameter of Velocity in MIDI, which can indicate power at which a note is struck or released [4]. MIDI table is therefore a sequence of these parameters, given as instructions to turn given note ON or OFF for given duration at given velocity.

- 1.2.1.2. Piano Roll. Similar to Western music notation, *piano roll* notation system uses a horizontal time axis & a vertical pitch axis. Notes are represented by activation of a point on piano roll, indicating pitch should be sounded at that time. Original piano rolls were perforated paper (giấy đục lỗ), with perforations read by a player piano to activate a particular keypress. May also be familiar with piano roll as a popular visualization of keyboard music on YouTube (Check out a great variety of piano roll notation examples, 1st example being a true piano roll: <https://www.youtube.com/watch?v=TgrqjkWqhE8>, <https://www.youtube.com/watch?v=IAOKJ14Zaoo>, <https://www.youtube.com/watch?v=8QorZ9fcg3o>.) (though note: sometimes time & pitch axes are interchanged). Fig. 1.8: Notice similarities between notated score, MIDI representation, & now piano roll representation of 1st bars of Beethoven's 5th Symphony. Quantized cells are highlighted to indicate activation; as reader scrolls through piano roll, these pitches will be sounded as cursor (whether a reader's eyes, a computer scan, or a turning mechanism) passes over respective column. shows continued example of opening of Beethoven 5, this time in piano roll notation.

- 1.2.1.3. Note-Tuple Representation. MusPy package reads music in note-tuple representation, where music is framed as a sequence of (pitch, time, duration, velocity) tuples. If want to encode beginning of Beethoven 5 for a violin part, would could use chart in Fig. 1.7: (67,0,55,100), (67,60,55,100), (67,120,55,100), (63,180,240,100). Note: events are represented with an initial onset time & duration, rather than a set of note-on & note-off instructions.

- 1.2.1.4. ABC Notation. ABC notation <https://abcnotation.com/> (stylized “abc”) was designed to notate folk tunes (kỷ hiệu giai điệu dân gian) (i.e., single melodies of 1 stave – giai điệu đơn của 1 khuôn nhạc) in plain text format. Since its inception by CHRIS WALSHAW in 1993 & subsequent popularization, features have expanded to allow for multiple voices, articulation, dynamics, & more. 8th note is given unit length, so Beethoven 5 could be written as `[[zGGG|_E4]]`; check out coding scheme included on ABC notation reference site to understand how different notes, rests, & durations are encoded.

- 1.2.1.5. Music Macro Language. Music Macro Language (MML) is another text-based language, similar to abc notation. Many variations have developed with different use cases; 1 distinguishing feature: use of operators that change parser state while parsing notes (e.g. an “octave up” or “octave down” symbol which reads continually in a new octave until reset). Usual Beethoven example might be notated as “pg8g8g8e-2” (but note: this would be preceded in an MML file by some global parameters e.g. default note length, tempo, etc.). A description of modern MML can be found in Byrd et al.'s reference handbook [5].

- * 1.2.2. Audio Representations. Unlike symbolic representations of music, *audio* does not explicitly specify musical structures & events e.g. note onsets, instead modeling physical sound itself. When considering audio representation, referring to digital representation of physical sonic information, as opposed to coding schemes & data-structures used to format such information into “files”. I.e., when speak of audio, speaking of vector of values that correspond to pressure amplitudes perceived by microphone, & not to specific encoding formats (e.g. “WAV”, “MP3”, etc.) used to represent this information. When record an orchestra playing score from Fig. 1.6, microphones record air pressure patterns in form of digital audio, depicted in Fig. 1.9: Visualization of a recording of opening of Beethoven 5. Horizontal axis is time (secs), & vertical axis is amplitude. Digital audio, at its finest level, represents sound encoded as a series of binary numbers. Wave-like nature of sound is not apparent due to physically compressed viewing window, so a zoomed-in view is offered in Fig. 1.10: Zoomed-in visualization of a recording of 1 of 1st 3 notes of Beethoven 5 (orchestra playing G2, G3, & G4 simultaneously). Horizontal axis is time (secs), & vertical axis is amplitude. As zoom in on audio signal, wave-like nature may be more clear. These visualizations are generated by free audio software Audacity, introduced in Appendix C.

Basic process of capturing an audio file is recording acoustic vibration using some sort of a sound-capturing mechanism, normally a mechanically driven microphone that translates air pressure changes to electrical signals. Will not discuss microphone & recording techniques in this book. Since dealing with computational methods, also skip analog ways of recording & storing sounds, e.g. magnetic tapes or vinyl records (băng từ hoặc đĩa than). In next chap, discuss digitizing sounds, also called analog-to-digital conversion (ADC) which is a process of sampling a continuous sound at fixed, albeit very short or fast-paced intervals. Understanding theory of sampling & reconstruction of analog signals from their digital

version, called digital-to-analog conversion (DAC), is important for assuring: quality of digital sound will be equal to its analog version, i.e., to assure perfect reconstruction of acoustic waveform to prevent distortion due to sampling process. Considering representation of audio from content perspective, there are also multiple levels of analysis that must be addressed. Immediate level is conversion of audio signal information initially represented as a sequence of numbers in time, to a collection of frequencies that change over time. This type of representation change, or transformation of audio data from 1 format to another is purely technical or mathematical & is actually a common technique in audio processing which can be quite revealing. Note: also call a digital sound a waveform, even though it is in fact a time series or an array of numbers.

Sound can be framed as superposition of sinusoids (called *Fourier analysis*). Notes produced by a musical instrument are a complex superposition of pure tones, as well as other noise-like components. *Harmonics* are integer-multiples of fundamental frequency which produces sound.

Example 2 (Harmonics & Overtones). *If begin with a fundamental frequency of 440 Hz, 1st integer multiple of frequency ($2 \times 440 = 880$ Hz) is referred to as 1st harmonic. 2nd integer multiple of frequency ($3 \times 440 = 1320$ Hz) is referred to as 2nd harmonic.*

How are these harmonics generated? When energize a physical oscillator (pluck a string, strike a drum – gảy 1 dây đàn, đánh 1 cái trống), medium has several modes of vibration possible, & usually case: multiple modes will vibrate simultaneously. These additional modes beyond fundamental generate overtones, additional pitches present (usually with lower energy than the fundamental) when an instrument produces a sound. Because these overtones often fall very close to harmonic series, they are often referred to as harmonics.

At next abstracted level of audio representation, more & more complex features can be extracted. 1 immediate caveat of moving up in hierarchy of features: in order to get deeper insight into sound, also need to discard some information. E.g., might want to consider sinusoidal amplitude or energies at different frequency bands, rather than preserve exact detail about timing of sinusoids, thus discarding their phase information. This immediately causes representation to be non-invertible, i.e., original waveform cannot be directly reconstructed from features. Pitch, loudness, MFCC, & Chroma features that were discussed earlier in this chap are considered to be higher level representation of audio. General term applied to such representation is “audio descriptors”, which can vary in granularity (độ chi tiết) in time or extent of sound events that it represents. General study of such features belongs to a relatively young field of *Music Information Retrieval* (MIR) since 1 of immediate uses of such features is classification or recognition of music that can be used for various retrieval applications.

One might consider *transcription*, process of converting music from an audio representation to a symbolic notation, as ultimate representation task. Some musicians transcribe by ear to handwritten notation; likewise, researcher explore methods of computer transcription. Many commercial software applications offer their solutions toward automatic transcription as a tool for composers & performers, but there is still much to be improved in these systems, much like an auto-captioning system may make mistakes while transcribing spoken language to text. To understand complexity of transcription task, in our computer exercises provide an example of matching MIDI piano roll representation to an acoustic representation called Constant-Q Transform (CQT). Difficulty in finding such match depends on complexity of sounds performing MIDI score, i.e., timbre of instrument performing music complicates frequency contents of signal. Today, state of art transcription methods e.g. MT3 system [6] use neural networks especially trained to recognize different musical instruments, significantly outperforming traditional signal-processing-based methods.

- 1.3. Acoustics & Basics of Notes. 1 of basic tenets (nguyên lý) of applying ML to music: acoustic signals that we consider musical have a certain common structure & organization that can be captured by a machine, if it is exposed to a rich variety of examples. There is an inherent trade-off or tension between randomness & rules in music. Expect music to be exciting, changing, & dynamic, but at same time predictable enough to make sense, be recognizable, & provide engagement. On most rudimentary level, might consider musical acoustics from perspective of periodicity vs. noise. After all, all musical cultures have selected from infinite possibilities of sounds 2 large categories – musical tones, which are sound phenomena that have a relatively clear perception of pitch or musical height (i.e., notes), & percussive events (sự kiện gõ) that have sharp transitory sounds, where main structuring element & characteristic would be organization into rhythms. Other types of sounds, e.g. many natural sounds & sound effects (think about sounds of water in a river sound or splashing waves, animal calls in a jungle, or industrial noises) have come into use only recently, with invention of recording technology & electronic sound synthesis. Such sounds are much more difficulty to categorize, organize, & compose. Fascination with such sounds was a driving force behind many modern music experimental practices (a term used to denote music innovations of 20th century), as well as instrumental music innovations till this day.

Keeping in mind broad distinction between structure & randomness (Terence Tao’s book with same title), deal 1st with question of periodicity vs. noise on short term acoustic level. Broadly speaking, distinguish between:

- * periodic or almost periodic signals
- * noise or a-periodic signals

Framework we will use to consider such signals is so-called “random processes”, which considers random signals as signals whose instantaneous values are largely unknown, but can still be predicted with a certain level of probability. Moreover, largely divide random signals into stationary & non-stationary, where stationary signals retain their statistical characteristics in long run, which can be either done by long enough observation of signal statistics that do not change, or by repeated independent observations of signal on different occasions. Strictly speaking, this equivalence of long observation or many short independent observations is called “ergodic” property, which is a common assumption in stochastic analysis. E.g.,

every time pluck a string on a guitar or blow air into a flute, exact vibrations of air waves will be different, but if repeat it many times, will be able to observe (or hear) all such variations. In practice, most audio signals, & formidably music, is non-stationary. Play different notes, & express different sounds when we speak (vowels, consonants, change in intonation – nguyên âm, phụ âm, thay đổi ngữ điệu). Even machinery that keeps running in same manner (think about a car engine), will have different sounds depending on changes in way it works (press gas pedal). Separately deal with non-stationary sounds when try to segment audio recording into frames of short-time sound events, or treat notes as an abstraction of a short-term fixed event, which in practice will be synthesized or rendered into sound slightly differently every time. Each note will be characterized by some typical periodic property, e.g. name or MIDI number of note being played. These tones are periodic, but need to define periodicity 1st.

Periodic signals are considered random signals having some repeated property or phenomena that is same after a particular time. Periodic phenomena do not need to be exact repetitions. Periodic signals return approximately to their previous values after a specified time interval, an effect that can be statistically captured by computing *correlation* or other statistical similarities.

Example 3 (Correlation). *In interest of creating ML models which generate audio that is similar to existing audio, important: have a means of measuring similarity of 2 audio signals.*

Consider a sine wave. Sine wave is offset from cosine wave, unless given a delay of half its period, at which point 2 signals perfectly align. Because of this, difficult to produce exactly 1 number that captures strength of similarity between 2 signals – there is a natural dependence on amount of delay.

Correlation function accounts for this, providing a measure of similarity (or coherence) as a function of signals' relative delay n :

$$(f \star g)[n] = \sum_{m=-\infty}^{\infty} \overline{f[m]}g[m+n],$$

where f, g : 2 signals being compared. Here, bar over $f[m]$ refers to complex conjugate; multiplying a conjugate term with a non-conjugated term gives a value which increases when 2 complex numbers are in near phase angles.

Readers interested in understanding mathematics behind cross-correlation & similar topics of autocorrelation & convolution are encouraged to visit resources in signal processing provided in Chap. 4. Recommend 2 primary takeaways:

- * Similarity between signals is an important property that we seek to quantify as a tool for training ML models.*
- * A product between 2 signal samples, then summed over all possible steps through signal, will be a recurring theme in both signal processing & neural network architectures, giving networks their ability to extract features from signal they act upon.*

Use Fourier Analysis to compare signal to idea repeating signals which are sinusoidal wave-forms (dạng sóng hình sin). When multiple sinusoidal wave-forms can be found in a signal, call these constituents “partials” or “harmonics”. For a signal to be periodic, its partials have to be related by integer ratios, which is a reason for calling them “harmonics”. One can show this applying convolution theorem states: convolution (or filtering operation) in time is multiplication in frequency & vice versa. For a time-domain signal to be periodic, in need to repeat same block of signal every period. Such periodic signal can be expressed as a short basic waveform comprising of a single period waveshape convolved with a Dirac comb. Thus, its Fourier transform will be Fourier transform of basic waveform multiplied by Fourier transform of a Dirac comb, which turn to be another Dirac comb, thus resulting in a signal that is nonzero at discrete frequencies. Another way to prove why periodic signals have discrete frequencies: think about physics of standing waves. Since boundary conditions of a vibrating string or pressure levels for a sound wave propagating along a pipe are fixed, only waves that can exist are standing waves that respect these boundary conditions. From geometric considerations (or by looking at solutions of wave equation) one can see: basic frequency is lowest period that satisfies boundary condition, but then other waves are possible too, but they always will need to be multiples of basic frequency to satisfy boundary conditions.

Sounds can be in-harmonic (e.g. bell sounds) that have multiple periods that sound more like musical chords than a single note with well defined pitch. Period of note is often called “fundamental frequency”; & is closely related to a perception of pitch. To be precise, pitch is a perception of note height, while its fundamental frequency is mathematical property of lowest partial in a harmonic signal (& sometimes perceived pitch is higher than fundamental frequency).

Analysis of sound into its sinusoidal constituents is also known as spectral analysis since individual partials are considered as equivalent of a specific color or spectral line in visual domain. Fig. 1.11: Prism splits lights according to colors schematically demonstrates this effect of splitting white light into its constituent colors by a prism (Hình 1.11: Lăng kính phân tách ánh sáng theo màu sắc minh họa sơ đồ hiệu ứng này để phân tách ánh sáng trắng thành các màu thành phần của nó bằng lăng kính). Light analogy should be carried with some caution since underlying prism phenomena occurs due to physical effect called diffraction (sự nhiễu xạ) that causes bending of light when it passes from 1 material to another. When diffraction depends on frequency, this causes different colors to land at different angles. A much more interesting physical decomposition of sound into constituent frequencies happens in ear, when due to specific geometry of cochlea (ốc tai) standing waves occur at different positions on basilar membrane (màng đáy), which are then captured by hair cells & transmitted to brain according to position & time of firing that contains information about magnitude & relative phase of each frequency. Details of auditory processing (xử lý thính giác) happening in cochlea are beyond scope of this book, & interested readers are referred to a beautiful animation by Howard Hughes Medical Institute <https://www.youtube.com/watch?v=dyenMluFaUw>.

So same way spectral analysis of visual signals or even electromagnetic radiation tries to separate signal into individual spectral lines, Fourier analysis does that to audio signals. On opposite side of periodic signal stands noise. Mathematical speaking, noise does not have spectral lines. Its spectral analysis comprises broad spectral bands, or continuous spectrum.

An interesting relation exists between periodicity & information. 1 common property of signals, musical or not (think about radio waves), is its ability to carry information. Encoding information into a physical signal & later extracting it from signal (a process known as decoding) requires extracting or recognizing salient properties (tính chất nổi bật) of signal. E.g., when fluctuation in air pressure from a sound reaches eardrum, this signal is converted into electrical impulses that are sent to brain, which is later interpreted as something “meaningful” or sth carrying or containing information. Later define concept of information as a relation between incoming signal & expectations or predictions of decoder. This subtle, & somewhat confusing term, is very important. Basically say: information is carried between 2 random phenomena, & ability to decode or extract information is related to reduction in uncertainty about incoming signal done in our brain, or by a model that a computer has learned. A signal’s decoding (as well as perception, from an information-theoretic view) allows information aspects of signal to be removed from any specific physical nature of that signal, e.g., current, voltage, or amplitude of an acoustic wave. In this view, noise is unpredictable or disturbing element that prevents or makes it more difficult to extract useful information from a signal. Mathematical model of signal is some function that often has ability to capture essential structure, e.g. periodicity of signal, & distinguish it from noise. Focus will be not on physical models, but rather predictive or information-theoretical models.

- 1.4. Perception & Cognition: Anticipation & Principles of Music Information Dynamics. (Nhận thức & Nhận thức: Dự đoán & Nguyên tắc của động lực thông tin âm nhạc) Same as in case of representation, question of music perception & cognition exists on various levels: it deals with physical transduction of acoustic phenomena to nerve impulses, translation of these impulses to perception of sound in auditory cortex, segmentation & segregation of audio flow into sound events & their recognition, all way to cognitive models dealing with expectation, emotions, & feelings of surprise vs. familiarity of incoming sound stream – including enculturation & entrainment of specific musical styles & learning of normative sound expressions. Note: dealing with music understanding will often require to consider questions of perception & cognition in rather abstract terms. Unlike speech or natural or man-made sound events, e.g. understanding spoken word, recognizing a bird chirp or detecting a submarine from a clutter of sonar blips, music as an art form often does not have a concrete or specific denotation with a universally agreed meaning. Moreover, different musical cultures use different methods for structuring sounds into music – some may put more emphasis on structuring notes in harmonies, tonalities, & long-term structure, while others may exhibit more sensibilities to subtle variations in sound or emotional inflections of sound production itself, while allowing more freedom or deviations in terms of tuning, or molding voice relations into strict harmonic relations.

– Tương tự như trong trường hợp biểu diễn, câu hỏi về nhận thức âm nhạc & nhận thức tồn tại ở nhiều cấp độ khác nhau: nó liên quan đến sự chuyển đổi vật lý của hiện tượng âm thanh thành xung thần kinh, dịch các xung này thành nhận thức âm thanh trong vỏ não thính giác, phân đoạn & phân tách luồng âm thanh thành các sự kiện âm thanh & nhận dạng chúng, tất cả đều hướng đến các mô hình nhận thức liên quan đến kỳ vọng, cảm xúc, & cảm giác ngạc nhiên so với sự quen thuộc của luồng âm thanh đến – bao gồm cả sự đồng hóa & sự đồng bộ của các phong cách âm nhạc cụ thể & học các biểu đạt âm thanh chuẩn mực. Lưu ý: khi giải quyết vấn đề hiểu âm nhạc thường đòi hỏi phải xem xét các câu hỏi về nhận thức & nhận thức theo những thuật ngữ trừu tượng hơn. Không giống như lời nói hoặc các sự kiện âm thanh tự nhiên hoặc do con người tạo ra, ví dụ như hiểu từ được nói, nhận ra tiếng chim hót hoặc phát hiện ra tàu ngầm từ một mô tín hiệu sonar, âm nhạc như một hình thức nghệ thuật thường không có một biểu thị cụ thể hoặc cụ thể với ý nghĩa được thống nhất chung. Hơn nữa, các nền văn hóa âm nhạc khác nhau sử dụng các phương pháp khác nhau để cấu trúc âm thanh thành âm nhạc – một số có thể nhấn mạnh hơn vào cấu trúc nốt nhạc trong sự hòa âm, âm điệu, & cấu trúc dài hạn, trong khi những nền văn hóa khác có thể thể hiện nhiều nhạy cảm hơn với những thay đổi tinh tế trong âm thanh hoặc sự biến đổi cảm xúc của chính quá trình tạo ra âm thanh, đồng thời cho phép nhiều sự tự do hoặc độ lệch hơn về mặt điều chỉnh hoặc định hình mối quan hệ giọng nói thành mối quan hệ hòa âm chặt chẽ.

Relating computer modeling & ML methods to human perception & cognition is a formidable task that is beyond scope of this book. Way we may relate statistical & ML models to aspects of human perception is through identifying & naming different types of musical data on continuum of acoustics-perception-cognition. As such, make a distinction between musical timbre, or quality of sound itself, musical texture, which is a relatively short-term aggregation of musical elements governed by common statistics, & musical structure that deals with organization of shorted musical elements into longer repetitions, e.g. chorus-verse or other formal arrangements of musical repetitions with variations, comprising so-called musical form.

In our approach, expectancies will be playing a central role in shaping experience of musical structure. An idea put forward many years ago by music theorists e.g. MEYER & NARMOUR states: listening to music consists of forming expectations & continual fulfillment or denial thereof [7,8]. Information-theoretic measures of audio structure have been proposed in attempt to characterize musical contents according to its predictive structure [9–12]. These measures consider statistical relations between past, present & future in signals, e.g. predictive information that measures mutual information between limited past & complete future, information rate that measures information between unlimited past & immediate present, & predictive information rate that tries to combine past & future in view of a known present. Additional models of musical expectancy structure build upon short-long-term memory neural networks introduced in Chap. 7 & predictive properties that are local to a single piece vs. broader knowledge collected through corpus-based analysis. Later models also include use of more elaborate & sophisticated neural models, e.g. WaveNet, SampleRNN, Transformers & more, covered in Chaps. 7 & 9. 1 common property of such models: they assume: some hidden or *latent* (ẩn hoặc tiềm ẩn) representation exists that is capable of capturing a great variety of music in 1 model. Such mapping is often termed encoder-decoder to emphasize information communicated by system, where a mapping of musical notes or acoustic data (sometimes referred to as musical surface or foreground) into its latent representation (also called background or reduction) is done in terms of probabilistic model. When time aspects are taken into account, such musical processes are broadly analyzed in terms of changes in information called “Music Information Dynamics”. Underlying assumption in investigation of Musical Information Dynamics

(MID): changes in information content, which could be measured in terms of statistical properties e.g. entropy & mutual information, correlate with musically significant events, which in parallel could be captured by cognitive processes related to music perception & acquired through exposure & learning of regularities present in a corpus of music in a certain style. These models may provide an explanation for inverted-U relationship proposed by DANIEL BERLYNE that is often found between simple measures of complexity & judgments of aesthetic value [13]. When applied to music & its predictive properties, see: very simple music, though predictable, has less MID effect since relative reduction in complexity through prediction is not significant. Very complex music that is hardly predictable has also small MID since complexity before & after prediction remain high. “Sweet-spot”, according to MID theory is when music appears complex initially, but by careful listening & prediction, complexity is drastically reduced. In other ways, ability of listener to “explain out” a lot of music complexity by active listening has an important effect on their enjoyment of music, which is captured by MID in terms of relative reduction of complexity through prediction.

- **Chap. 2. Noise: Hidden Dynamics of Music.** Noise has always played a central role in music. As sound production, noise often serves as initial source of energy in musical instruments that “carve out” notes from a pluck of a string or from a turbulent air stream blown into a pipe. Have seen in prev chap how an unvoiced hiss or pulses of a glottal buzz are shaped into speech sound via a source-filter voice model (cách một tiếng rít vô thanh hoặc các xung của tiếng vo ve thanh quần được định hình thành âm thanh lời nói thông qua mô hình giọng nói nguồn-bộ lọc). Filters & resonances help turn flat frequency response of an impulses or of a continuous noise into a periodic, more predictable sound structure. As an idea generator, noise is driving musical choices during composition process. As much as music theory tells us how choices of notes need to be done according to certain rules, role of noise is in creating uncertainties & ambiguities that are an essential component in music listening experience. Random methods, often called “aleatoric” music (nhạc ngẫu nhiên) in contemporary practice, have been used historically to trigger & drive process of music creation by generating initial musical choices, which were further constrained & shaped during later stages of composition. Moreover, on a broad cultural & historical perspective, ATTALI’s book *Noise: The Political Economy of Music* examines relations of music to political-economic structure of societies through noise. Although such social-science & economic investigations of music are outside of scope of this book, a consideration of what ATTALI understands by term “Music” might be appropriate for discussion. For ATTALI music is organization of noise, & entirely of evolution of Western knowledge through development of scientific, technical, & market tools is viewed through lens of shaping & controlling noise:

“World is not for beholding, it is for hearing. Our science has desired to monitor, measure, abstract, & castrate meaning, forgetting that life is full of noise & that death alone is silent. Nothing essential happens in absence of noise.”

- **2.1. Noise, Aleatoric Music, & Generative Processes in Music.** Consider generative processes that turn randomness into structure. In prev chaps considered 2 main data formats for music: one is as a sound recording captured into an audio file, the other is a sequence of performance actions captures as notes in a symbolic MIDI or music sheet format. To understand better structures present in these 2 data types, introduced various features that allow more meaningful representations by extracting salient aspects of sound, going back & forth between signals & symbols. Music is a unique human endeavor (nỗ lực độc đáo của con người): it comprises deliberate & intentional selection of sounds from infinite possibilities of air vibrations, narrowed into a much smaller palette of musical instruments producing notes from periodic vibrations. These notes are further organized into scales, tonalities, melodies, & chords, building progressions & repetition of themes arranged into verse, chorus, & other musical forms. Fact: all of these constructions are deliberate human choices is often taken for granted. Different cultures follow different musical rules, & when musical styles evolve & instruments change, type of music expression may drastically change. It takes a while to get used to & find beauty in unfamiliar music, or come up with new ways of sonic expression or break existing musical rules to create artistic novelty & impact.

As listeners, or even as composers, often do not think about such formal aspects of musical evolution or change, but when challenge of finding or generating meaning in musical data is given to machine, need to revisit basic premises (tiền đề cơ bản) of what makes musical sounds meaningful in order to be able to program musical algorithms or build musical learning machines. Music can be defined as *Organization of Sounds to Create an Experience*. In this most abstract & general way of defining music, capture some of fundamental principles for a formal study of music. 1st, a selection of sounds is made, representing them as latent variables or features extracted from sound data. Their organization assumes certain temporal & sequential relations that could be further established by learning dynamics of these latent variables. & finally, creating an experience requires including listener as part of model.

- * **2.1.1. Music as Communication.** Understanding music requires both a sender & a receiver, or encoder & a decoder that communicate information. But unlike in vision, speech, or computer communications, where a typical inference task is 1 of recognizing a particular shape, transcribing speech, or executing a command, effect of musical information is 1 of eliciting an emotional or aesthetic response. Such response assumes an active listening framework that triggers a wide variety of reactions, some of which might be innate or pre-wired, & others learned from exposure to existing musical cultures. Can such structures be learned by a machine in a so-called end-to-end fashion by feeding it with music data paired with a desired target output? Can novel applications & unheard musical style be created by going outside of statistical models of existing styles or emerge through machine interaction with a human?

In journey of ML in music, find concept of noise particularly useful. On signal level, spectral analysis allows distinguishing acoustic noise from periodic sounds. Noise, in its “purest” physical or engineering sense, is a signal characterized by a white spectrum (i.e., composed of equal or almost-equal energies in all frequencies). Mathematically, to qualify as a random or stochastic process, probability density of signal frequency components must have a continuous spectrum, whereas periodic

components that are more predictable present themselves in frequency domain as spectral lines, i.e. sharp & narrow spectral peaks. Moreover, in terms of its musical use, noise does not allow much structural manipulation or organization. In contrast to noise, notes are commonly basic building blocks of larger sound phenomena that we call music, & as such could be considered as elements of sonic “structure”. On symbolic level, noise is more closely related to predictability of a sequence of elements, often encoded into tokens. When applying language models to music, noise refers to uncertainty or variability of next token as possible continuations to past sequence of tokens. Although seemingly unrelated, these 2 aspects of noise have common mathematical foundations that are related by signal information contents measured in terms of residual uncertainty passed in a signal over time. Better our prediction about future becomes, smaller prediction error will be, & overall uncertainty will be reduced. In this chap, take a new look at relations between randomness & structure in both symbolic & audio signals. In order to do so, consider meaning of randomness as a phenomena that has an aspect of uncertainty & surprise to it. This is contrasted with *structure*, an aspect of data that is more predictable, rule-based, or even deterministic.

From listener perspective, acoustic noise belongs to perceptual domain, while perception of musical organization belongs to cognitive domain. 2 domains are bridged by Prediction Coding framework in cognitive science that is centered around idea: organisms represent world by constantly predicting their own internal states. Comparisons between predictions & bottom-up signals occur in a hierarchical manner, where “error signal” corresponds to unpredicted aspect of input that passes across various stages of representation & processing. Core framework posited by such theories is 1 of deep generative models that can be understood both in terms of Bayesian theories of cognition that consider perception as a hypothesis-testing mechanism, & as a simulation model that considers signal representation in terms of statistical models that incorporate signal uncertainty as part of their generative structure. Such fundamental questions about mental representations are outside scope of this book, but similar concerns emerge in design of musical applications where practical issues of structural representation are often given by more historically mainstream accounts of music theories, & later extended through statistical-probabilistic structures that are designed to bear musically relevant similarity to some musical goal or idea. Such similarity can vary from stylistic imitation, sometimes called “deep fake” models of existing music, to experimental practices that explore novel musical ideas & possibilities for creative interaction with machines.

Throughout book, often refer to a particular approach to musical modeling that views anticipation as key to understanding music, planning its structure, & a possible way of understanding effects that music has on listener’s mind. Already mentioned anticipation in prev chap where introduced concept of Musical Information Dynamics as a generic framework for modeling processes in terms of changes in its information contents. But before exploring further statistical tools that are needed in order to extract & quantify information in music & sound, it helps to see anticipation in context of historical developments that led to contemporary ways of music modeling & formalization.

- 2.2. History of Mathematical Theory of Music & Compositional Algorithms. Continuing discussion of noise from historical perspective, important to note: creation of music has been always linked to certain technical tools, namely musical instruments, that enabled creation of sounds & determined their structuring principles. As such, randomness & uncertainty in sound production has always been linked to its production & composition methods.

Ancient Greek philosopher Pythagoras (570–495 B.C.) is generally credited with having discovered mathematical relations of musical intervals which result in a sensation of consonance. Such relations are deeply rooted in acoustics of string & wind instruments, & human voice, which are highly harmonic. In Pythagorean tuning, frequency ratios of notes are all derived from number ratio 3:2, which is called an interval of a *5th* in music theory.

Remark 2. *In musical terms, an interval is difference in pitch between 2 notes. In a physical sense, interval is a ratio of frequencies between 2 notes. Different tuning systems may assign different frequency ratios to intervals of same name. In Western music, intervals are usually described by a number (unison, 2nd, 3rd, 4th, 5th, etc.) & a quality (perfect, major, minor, augmented, or diminished). In general, number indicates spacing, while quality indicates some further tuning on a smaller scale than represented by number.*

When 2 notes are played in a 5th, this music interval produces a combined sound where most of partials overlap, thus creating a seamless blending & a highly coherent precept. Using 5th & moving along circle of 5ths (Fig. 2.1: Circle of 5ths is a helpful construct for musicians to relate patterns in key signatures, harmonic progressions, & other uses of chromatic scale. Each step clockwise around circle represents a note an interval of a perfect 5th above the previous.) results in pentatonic & diatonic scales (taking 5 & 7 steps, accordingly) that are common in Western music. Although this tuning was adjusted later on in history to accommodate for more complex chromatic musical styles, idea: music sensations can be related to mathematical proportions & rules had been rooted deeply in music theory. Moreover, cultures that use different musical instruments, e.g. metallophones (metallic percussion instruments) dominant in Gamelan music, result in different tuning system, e.g. Pelog & Slendro, where preferred intervals are s.t. different tones optimally blend by maximizing overlap between partials of their respective timbres.

In terms of composition practices, probably best known early example of using formal rules for writing music can be attributed to GUIDO D’AREZZO who, around 1026, set text to music by assigning pitches to different vowels. According to this method a melody was created according to vowels present in text. In 14th & 15th centuries isorhythmic techniques were used to create repeated rhythmic cycles (talea), which were often juxtaposed with melodic cycles of different length to create long varying musical structures. DUFAY’s motet Nuper Rosarum Flores (1436) used repetition of talea patterns at different speeds in 4 secs according to length ratios 6:4:2:3, which some claim were taken from proportions of nave, crossing, apse, & height of arch of Florence Cathedral or from dimensions of Temple of Solomon given in Kings 6:120. Formal manipulations e.g. retrograde (backward motion) or inversion (inverting direction of intervals in a melody) are found in music of J.S. BACH

& became basis for 20th century 12-tone (dodecaphonic) serial techniques of ARNOLD SCHOENBERG (1874–1951). Exploiting recombinant structure of harmonic progression (Khái thác cấu trúc tái tổ hợp của tiến trình hài hòa), MOZART devised his *Musikalisches Würfelspiel* (“Musical Dice Game”) (1767: Though this game is attributed to MOZART, earlier examples have been discovered, including JOHANN KIRNBERGER’s 1757 *Der allezeit fertige Menuetten- und Polonaisencomponist* – Nhà soạn nhạc minuet và polonaise luôn sẵn sàng) that uses outcomes of a dice throw & a table of pre-composed musical fragments to create a Minuet. There are 176 possible Minuet measures & 96 possible Trio measures to choose from. 2 6-sided dice are used to determine each of 16 Minuet measures & 1 6-sided die is used to determine each of 16 Trio measures. This provides an early example of a stochastic process driving musical output, a theme in Variational Autoencoder & other examples. In this case, compositional blocks are defined in a discrete codebook, but in later cases, find these constituent pieces to be learned from bodies of musical data.

Search for formal composition techniques took on a life of its own in post-WWII academic music. In a trend which is sometimes attributed to desire to break away from stylistic confines & associations of late Romanticism, mathematical rules e.g. serialism were sought in order to construct musical materials that sounded both new & alien to traditional tonal & rhythmical musical languages. 1 such feeling was expressed by Czech novelist MILAN KUNDERA in his writing about music of Greek composer & microsound theorist IANNIS XENAKIS 1922–2001) – “Music has played an integral & decisive part in ongoing process of sentimentalization (Âm nhạc đã đóng một vai trò không thể thiếu & quyết định trong quá trình tình cảm hóa đang diễn ra). But it can happen at a certain moment (in life of a person or of a civilization that sentimentality) (previously considered as a humanizing force, softening coldness of reason) becomes unmasked as ‘supra-structure of a brutality’. . . . XENAKIS opposes whole of European history of music. His point of departure is elsewhere; not in an artificial sound isolated from nature in order to express a subjectivity, but in an earthly ‘objective’ sound, in a mass of sound which does not rise from human heart, but which approaches us from outside, like raindrops or voice of wind; [4].

Use of automation or process is common thread in above examples. Composers have increasingly allowed portions of their musical decision-making to be controlled according to processes & algorithms. In age of computer technology, this trend has grown exponentially. Before going into more technical aspects of musical modeling with computers, helpful to compare these experimental academic music approaches to those in non-academic musical world. To gain this perspective requires briefly surveying different stylistic approaches for using computers in popular music making.

* 2.2.1. **Generative Music in Popular Music Practices.** 1 particular genre of generative music that has become popular among general listening public is *ambient music*. Governed by recombinant processes of chords & melodies, ambient music emphasizes mood & texture over directionality & contrast, blending with environment in which it is heard. This music often lacks dramatic or sentimental design of classical & popular music. Unlike generative methods in academic experimental genres, e.g. XENAKIS’s exploration of masses of sounds & processes in opposition to Western music tradition, ambient genre centers around familiar & often pleasant sounds drawn either from tonal harmonic language or from recordings of environmental sounds that blend into each other with no particular larger scale form or intention.

Perhaps 1st explicit notion of ambient music comes from ERIK SATIE’s concept of *musique d’ameublement*, translated as “furniture music”. SATIE’s music is often built from large blocks of static harmony, & creates a sense of timelessness & undirected sound, but this was music which was meant not to be listened to, only to be heard as background to some other event [15]. There is arguably no composer more associated with ambient & environmental sound than JOHN CAGE, 20th-century experimentalist who redefined materials & procedures of musical composition. Throughout his career, CAGE explored outer limits of sonic experience, but his most notorious work is famous “silent piece”, 4’33”, composed in 1952. Instead of defining musical materials, CAGE defined a musical structure, & allowed ambient sounds of concert environment to fill structure [16]. His other groundbreaking approaches to music included reliance on chance operations, or so-called indeterminate or “aleatoric techniques” for its random selection of notes. In his *Music of Changes* Chinese I-Ching book technique was used to select note pitches & durations. I-Ching is also used to determine how many layers should there be in a given phrase to create different densities of musical voices. These chance operations were done independently of previous outcome. When selection of next notes are done in ways that are dependent on already generated events, this induces so-called *Markov property* to chance operations. LEJAREN HILLER & LEONARD ISAACSON’s 1957 *Illiac Suite* used Markov chains to selected consecutive intervals for each instrument according to greatest weight to consonant vs. smallest intervals. In 1959, IANNIS XENAKIS wrote several works in which he used many simultaneous Markov chains, which called “screens”, to control successions of large-scale events in which some number of elementary “grains” of sound may occur.

Late 20th century saw an explosion of musical styles which challenged traditional notions of musical experience & experimented with ways in which sound can communicate space, place, & location. “Soundscape” composition & use of field recordings help to create a sense of a real or imagined place, while genres of electronic dance music like chill-out are often associated with specific club environments. Some artists have also bridged gap between contemporary visual & installation art & music, & “sound art” is now commonplace. These artists often deal with psychoacoustic phenomena & relationships between visual & auditory media, often work with multiple forms of media, & take influence from minimalism, electronic music, conceptual art, & other trends in 20th-century art.

Most importantly for our discussion: way in which composers of ambient & related genres often delegate (ủy nhiệm, giao quyền) high-level aspects of their work to algorithmic processes or generative procedures – or in case of field-recording work, to features of sound materials themselves. Techniques which originated in most avant-garde (tiên phong) & experimental traditions, e.g. looping, automated mixing, & complex signal processing, are increasingly used in popular music & are now commonly found in commercial, off-shelf music production software.

BRIAN ENO’s *Ambient 1: Music for Airports* (1978) provides template for many later works: repetitive, with an absence of abrasive or abrupt attacks, & using long decays, harmony & melody seem to continue indefinitely. During compositional

process, ENO constructed tape loops of differing lengths out of various instrumental fragments [17]. As these loops played, sounds & melodies moved in & out of phase, producing shifting textures & patterns. Studio configuration itself became a generative instrument. In 1996 BRIAN ENO released title “Generative Music 1” with SSEYO Koan Software. The software, published in 1994, won 2001 Iterative Entertainment BAFTA (British Academy of Film & Television Arts) Award for Technical Innovation. Koan was discontinued in 2007, & following year a new program was released. Noatikl “Inmo” – or “in the moment” (“inmo”) – Generative Music Lab was released by ntermorphic ltd. as a spiritual successor to Koan. Software had new generative music & lyric engines, & creators of software claimed:

“... learning to sit-back & delegate responsibility for small details to a generative engine is extremely liberating. ... Yes, can focus on details in Noatikl, but can also take more of a gardener’s view to creating music: casting musical idea seeds on ground, & selecting those ideas which blossom & discarding those that don’t appeal.”

Electronic dance music producers, like English duo Autechre, often embrace most avant-garde trends & techniques, while still producing music which remains accessible & maintains a level of popular success. Autechre have often employed generative procedures in their music-making, using self-programmed software & complex hardware routings to produce endless variations on & transformations between thematic materials. In a recent interview, ROB BROWN spoke of attempting to algorithmically model their own style:

“Algorithms are a great way of compressing your style. ... if you can’t go back to that spark or moment where you’re created sth new & reverse-engineering it, it can be lost to that moment ... It has always been important to us to be able to reduce sth that happened manually into sth that is contained in an algorithm.” [18]

Commercial digital audio workstation (DAWs) – computer software for recording, mixing, & editing music & sound – often support various types of generative procedures. E.g., “Follow” function in Ableton Live, a popular DAW, allows composers to create Markov chain-like transitions between clips. 1 of typical uses of this: add naturalness to a clip. Instead of having a single, fixed loop, one records multiple variations or adds effects, e.g. cuts, amplitude envelopes or pitch changes, to a single loop & uses “Follow” to sequence clips randomly. This allows variations both in terms of musical materials contents & their expressive inflections that breaks away from mechanical regularity & synthetic feel often associated with loop-based musical production.

In field of computer & video games, composers & sound designers, with help of custom software (often called *music middle-ware*), frequently make use of generative & procedural music & audio in their work. Advantages of procedural audio include interactivity, variety, & adaptation [19]. KAREN COLLINS provides a good overview of procedural music techniques [20]. COLLINS describes challenges face by game composers, who must create music which both supports & reacts to an unpredictable series of events, often within constraints imposed by video game hardware &/or storage. In fact, COLLINS points out: a contributing factor to use of generative strategies in video games was need to fit a large mount of music into a small amount of storage. Generative techniques allow for creation of large amounts of music with smaller amounts of data. Some strategies for this kind of composition can include:

- Simple transformations: changes in tempo, restructuring of largely pre-composed materials, conditional repeats of material based on in-game activities
- Probabilistic models for organizing large numbers of small musical fragments
- Changes in instrumentation of music based on presence of specific in-game characters
- Generation of new materials within melodic/rhythmic constraints
- Re-sequencing, also known as horizontal remixing, allowing nonlinear playback of pre-recorded music
- Remixing, or vertical remixing, which operates by adding or muting parallel simultaneous musical tracks.

- 2.3. Computer Modeling of Musical Style & Machine Improvisation. Aug 09, 1956 “Illiac Suite” by LEJAREN HILLER & LEONARD ISAACSON, named after computer used to program composition, saw its world premiere at University of Illinois [21]. Work, though performed by a traditional string quartet, was composed using a computer program, & employed a handful of algorithmic composition techniques. This work, together with work of XENAKIS described in *Formalized Music*, marks beginning of modern mathematical & computational approaches to representing, modeling, & generating music. Among most prominent algorithmic composition techniques are stochastic or random modeling approaches, formal grammars, & dynamical systems that consider fractals & chaos theory, as well as genetic algorithms & cellular automata for generation & evolution of musical materials. More recently, advanced ML techniques e.g. neural networks, hidden Markov models, dynamic texture models, & DL are being actively researched for music retrieval, & to a somewhat lesser extent, also for creative or generative applications. With advent of ML methods, models could be trained to capture statistics of existing music, imitate musical styles, or produce novel operations in computer-aided composition systems. With these developments, goals & motivation behind using random operations for composing has changed from thinking about mass events with densities & texture, to capturing fine details of musical style that were not previously amenable to probabilistic modeling.

Markov models are probably 1 of best-established paradigm of algorithmic composition, both in experimental & style learning domains. In this approach, a piece of music is represented by a sequence of events or states, which correspond to symbolic music elements e.g. notes or note combinations, or sequences of signal features extracted from a recording. Markov sources are random processes that produce new symbols depending on a certain number of past symbols. I.e., whole process of music generation from a Markov model has some kind of “memory” whose effect is that new symbols are produced in a manner dependent on its past. I.e., old symbols influence new symbols by determining probability of generating certain continuations. Markov chains are statistical models of random sequences that assume: probability for generating next symbol depends only on a limited past. A Markov model establishes a sequence of states & transition probabilities, which are extracted by counting statistics of events & their continuations from a musical corpus. Length of context (number of prev elements in musical

sequence used for calculation of next continuation) is called *order* of Markov model. Markov sources can have different orders: simple independent random sequences are considered as Markov of order 0, then processes of a memory of a single symbol are Markov of order 1, of a memory of length 2 are Markov of order 2, & so on. Since higher order models produce sequences that are more similar to corpus, but are harder to estimate & may also have disadvantageous effects in terms of model's ability to deal with variations, several researchers have turned their attention to predictive models that use variable memory length models, dictionary models, & style-specific lexicons for music recognition & generation. See [22] for a survey of music generation from statistical models & [23] for comparison of several predictive models for learning musical style & stylistic imitation.

Generative grammars are another well-established & powerful formalism for generation of musical structure. Generative grammars function through specification of rewriting rules of different expressiveness. Study of grammars in music in many ways paralleled that of natural language. CHOMSKY erroneously believed: grammatical sentence production cannot be achieved by finite state methods, because they cannot capture dependencies between non-adjacent words in sentences or model recursive embedding of phrase structure found in natural language [24]. Since such complex grammars could not be easily “learned” by a machine, this belief limited types of music handled by grammar models to those that could be coded by experts in terms of rules & logic operations. This approach is sometimes called an *expert system* or *knowledge engineering*. Although these methods achieve impressive results, they are labor intensive as they require explicit formulation of musical knowledge in terms that are often less than intuitive.

Some more specific models, e.g. Transition Networks [25] & Petri Nets [26,27] have been suggested for modeling musical structure. These are usually used to address higher levels of description e.g. repetition of musical objects, causality relations among them, concurrency among voices, parts, sections, & so on. DAVID COPE's work describes an interesting compromise between formal grammar & pattern-based approaches. Cope uses a grammatical-generation system combined with what he calls “signatures”: melodic micro-gestures typical of individual composers [28]. By identifying & reusing such signatures, COPE reproduced style of past composers & preserved a sense of naturalness in computer-generated music.

In following, describe some of research on learning musical structure that begins by attempting to build a model by discovering phrases or patterns which are idiomatic to a certain style or performer. This is done automatically, & in an *unsupervised* manner. This model is then assigned stochastic generation rules for creating new materials that have similar stylistic characteristics to learning example. 1 of main challenges in this approach is formulating rules for generation of new materials that would obey aesthetic rules or take into account perceptual & cognitive constraints on music making & listening that would render pleasing music materials.

Remark 3. *A learning process is unsupervised if process learns patterns in data without being provided additional expected output. By contrast, supervised methods rely on an expected, defined output (commonly referred to as ground truth) to drive learning. Supervised & unsupervised methods are applied to different types of problems based on what can be learned in absence (or presence) of ground truth (i.e., known labels or answers to question at hand). E.g., problem of classification (predicting class associated with a data instance) can be addressed with supervised learning, given a collection of datum-class pairs. On other hand, without knowing explicit class membership for each data instance, process becomes unsupervised – but can still learn associations between like-data-points, a problem known as clustering.*

Another interesting field of research is machine improvisation, wherein a computer program is designed to function as an improvisational partner to an instrumentalist. Software receives musical input from instrumentalist, in form of MIDI or audio, & responds appropriately according to some stylistic model or other algorithm. An important early work in this style is GEORGE LEWIS's *Voyager* (1988), a “virtual improvising orchestra”. *Voyager* software receives MIDI input derived from an instrumental audio signal, analyzes it, & produces output with 1 of a set of many timbres [29]. Input data is analyzed in terms of average pitch, velocity, probability of note activity, & spacing between notes. In response, *Voyager* is capable of both producing variations on input material & generating completely new material according to musical parameters e.g. tempo (speed), probability of playing a note, spacing between notes, melodic interval width, choice of pitch material based on last several notes received, octave range, microtonal transposition, & volume. Due to coarse statistical nature of both analysis & generative mechanisms, type of music produced by system is limited to abstract free improvisation style.

Another family of “virtual improvisor” software will be introduced in later chap, including PyOracle & VMO <https://github.com/wangsix/vmo>, & musical improvisation systems OMax & SOMax ¹². These programs use sequence matching based on an algorithm called Factor Oracle, to create a model of repetition structure specific to incoming music material both for audio or MIDI input. By recombining carefully selected segments of original recording according to this repetition structure, programs can produce new variations on input, while maintaining close resemblance to original musical style. Although essentially a procedural remixing method, once determination of next musical step is performed by chance operations that are governed by probability of continuations of motifs of variable length as found in a training corpus, this improvisation process can be considered as an extension of Markov approach to variable memory length models. In next sect introduce mathematical foundation for Markov models that are tool of trade for modeling sequences of data, notably in domain of text & natural language modeling. This will allow a more profound look into concepts of entropy, uncertainty & information, paving road to understanding musical creation & listening as a communication process.

- 2.4. Markov Models & Language Models for Music. Assume: have a sequence of notes (a melody). If notes are generated in an independent manner (e.g. in JOHN CAGE's “Music of Changes” mentioned earlier), best description of that music could be in terms of frequencies of appearance of different notes. CLAUDE SHANNON, considered to be father of information theory, suggested: if look at large blocks of symbols (long sequences), this information around frequency of appearance “reveals itself”

¹²<https://forum.ircam.fr/collections/detail/improvisation-et-generativite/>

without any need to calculate probabilities. This property (called *asymptotic equipartition property* or AEP, discussed in next chap) basically says: if look at a long enough sequence, see only “typical” messages. Entropy of source defines amount of typical messages. That is why a completely random (aleatoric) sequence, where all notes appear with equal probability, has maximal entropy: number of typical sequences is equal to all possible sequences. On other hand, more structure a sequence has, less will be number of typical sequences. Exponential rate of growth of number of typical sequences, relative to number of all possible sequences of same length, is *entropy*, introduced in next sect. Thus, looking at longer blocks of symbols approach “true” statistics of source.

* **2.4.1. Entropy, Uncertainty, & Information.** Dealing with noise as a driving force of music, learn: not all noise is created equal. In process of throwing dice or passing an acoustic noise through a filter, shaping it into sounds & music structures by making choices of note sequences, some being more probable than others, distinguishing between more probable & more surprising events. Can we measure such surprises? Formally, *entropy* is defined as a measure of uncertainty of a random variable x when it is drawn from a distribution $x \sim P(x)$:

$$H(x) = - \sum_x P(x) \log_2 P(x),$$

where sum is over all possible outcomes of x . It can be shown: $H(x)$ is nonnegative, with entropy being equal to 0 when x is deterministic, i.e., when only 1 of outcomes x is possible, say x^* , so that $P(x = x^*) = 1, P(x \neq x^*) = 0$ otherwise. Maximal entropy is achieved when all outcomes are equally probable. One can easily show: for x taking values over a set of size N (also understood as alphabet of size N or cardinality of x), then $H(X) = \log_2 N$, which is, up to integer constraints, number of bits needed to represent symbols in alphabet of that size.

Example 4 (Entropy of a Binary Variable). *For a binary variable, with $x \in \{0, 1\}$, with probability $p, 1 - p$ resp., entropy is*

$$H(X) = -p \log_2 p - (1 - p) \log_2 (1 - p).$$

This entropy is shown graphically in Fig. 2.2: Binary entropy as a function of probability p .

Entropy can be considered as a functional, which is a function that accepts another function & returns a number. In our case, function that entropy expression operates on is probability distribution, & it summarizes whole probability distribution into a single number that gives a sense of overall uncertainty of that distribution.

Example 5 (Chromatic Entropy). *As an example of musical application, entropy can be considered as a measure of uncertainty in note distributions. Given 12 notes of a musical chromatic scale, possible to measure uncertainty of notes when assuming: each note is drawn independently from a scale. Tonal profiles were found by musicologists (Krumhansl & Schmuckler) [30] to determine a scale in terms of probability of notes, with most probable notes being 5th (dominant) & 1st (tonic) note of a scale. Of course, basic units of music do not have to be individual notes. These could be pairs of notes, or longer sequences, & growing length of melodies leads to increasingly more complex distributions. A Markov model is 1 such modeling approach that tries to account for such complexity.*

Another important notion that is built “on top” of entropy is *mutual information*. Mutual information measures relative reduction in uncertainty between 2 random variables X, Y , when outcome of 1 of them is known. To be able to define mutual information when need 1st to define *conditional entropy*, which is

$$H(X|Y) = - \sum_{x,y} P(x,y) \log_2 P(x|y) = - \sum_y P(y) \sum_x P(x|y) \log_2 P(x|y).$$

Note: probabilities appearing in 2 components in conditional entropy expression are not same. Logarithm contains conditional probability only, that is averaged both over x, y . I.e., conditional entropy estimates entropy of x for each possible conditioning value of y , averaged over all possible ys .

Thus, mutual information, difference in uncertainty about 1 variable when the other is known, can be written as:

$$I(X, Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X, Y).$$

Mutual information can be visualized as intersection or overlap between entropies of variables X, Y , as represented in Venn diagram in Fig. 2.3: **Mutual information as a cross-section between entropies of 2 variables.**

In following apply notion of mutual information to times series, as a way to quantify amount of information passing between past & present in a musical sequence. Thus, instead of considering long sequences, will focus on entropy of a single symbol in a sequence given its past. This entropy is in fact representing uncertainty of continuation. By exploiting an implicit knowledge of how to complete unfinished phrases, SHANNON played his famous game of having human subjects guess successive characters in a string of text selected at random from various sources [31]. SHANNON recorded probability of taking r guesses until correct letter is guessed & showed: these statistics can be effectively used as a bound on entropy of English. 1 of main problems with application of Markov theory “as is”: true order of process is unknown. Accordingly, when N -gram models are used to capture dependence of next symbol on past N symbols, N needs to being fixed as a model parameter of choice. Main difficulty in Markov modeling is handling large orders, or even more interestingly, dealing with sources of variable-length memory. In music, reasonable to assume: memory length is variable since musical structures sometimes use short melodic figuration, sometimes longer motives, & sometimes rather long phrases can repeat, often with some variations. SHANNON’s experiment establishes several interesting relations between language complexity & its continuation statistics. Since only certain pairs, triplets & higher N -grams tend to appear together, amount of

guesses needed to predict next letter is drastically reduced for longer patterns. Moreover, show: one can obtain an efficient representation of language by encoding only continuation choices. This representation, SHANNON called a *reduced text*, can be used to recover original text, but requiring less storage. Later in book, use text compression schemes that encode letter continuations for larger & larger blocks of symbols as an effective way to capture variable memory Markov models in music & use then for generating novel musical sequences.

- * **2.4.2. Natural Language Models.** Natural Languages are often characterized by long structured relations between their words, arranged into sequences of letters or words arranged into dictionary entries. In following chaps, consider statistical modeling of sequences that is based on information theory & deep neural network learning. In both cases, goal of modeling: create a probability representation of long sequences of symbols or audio units that can be used for inference (finding probability of a sequences for classification purpose) or generation (producing more sequences of same kind). In this way, modeling music becomes a similar task to modeling natural language.

As a motivation & perspective about Markov & future models, discuss briefly general problem of modeling long temporal sequences, known as Neural Language models.

In general, a language model is a probability distribution over sequences of tokens. Traditionally, languages were represented using formal grammars, or learning equivalent automata that can generate specific strings that obey language rules. In modern approaches, problem in language modeling became 1 of learning a language model from examples, e.g. a model of English sentences from a large corpus of sentences.

Mathematically speaking, let $P(w_1, \dots, w_n)$ denote probability of a sequence (w_1, \dots, w_n) of tokens. Aim of learning: learn such probabilities from a training set of sequences.

Consider related problem of predicting next token of a sequence. Model this a $P(w_n|w_1, \dots, w_{n-1})$. Note: if can accurately predict probabilities $P(w_n|w_1, \dots, w_{n-1})$, can chain them together using rules of conditional probability to get

$$P(w_1, \dots, w_n) = P(w_1)P(w_2|w_1)P(w_3|w_1, w_2)P(w_n|w_1, \dots, w_{n-1}).$$

Most learning approaches target language modeling as “next token” prediction problem, where learning system learns a mapping of (w_1, \dots, w_{n-1}) input to w_n output. Such learning can be considered supervised learning, though there is no need to meticulously label datasets (dán nhãn tập dữ liệu một cách tỉ mỉ) since future provides target “label”. When done statistically, output of such system is a probability distribution over various values that w_n could take. Such output could be a high-dimensional probability vector, with hundreds of thousands to millions of options for a language model on English sentences.

Moreover, in case of language, input is not a fixed-length vector. Next token is often determined or largely depends on a long phrase, or in case of music, a motif that comprises a sequence of length n that varies. This phenomena is sometimes called “variable memory length” & is 1 of primary factors that make language, & music modeling, difficult.

Statistical approaches to estimating $P(w_n|w_1, \dots, w_{n-1})$ when n is large must make some assumptions to make problem tractable. Even with an alphabet of just 2 symbols, 0 & 1, minimal training set size to reliably estimate $P(w_n|w_1, \dots, w_{n-1})$ is of order 2^n . Such a big dataset is required in order to reliably visit or sample all possible input combinations in order to be able to predict next token reasonably in each case.

Most widely made assumption in statistical approaches in Markovian. Specifically,

$$P(w_n|w_1, \dots, w_{n-1})$$

is approximated by a limited k -tuple,

$$P(w_n|w_{n-k}, \dots, w_{n-1}).$$

This assumes: next token probabilities depend only on last k tokens. Most often k is a small number of order 1 or 2. This vastly simplifies modeling & estimation problem at cost of not being able to model long-range influences.

Early neural network approach to learning temporal predictions is so-called Recurrent Neural Network that will be covered in a later chap. Such models combine 2 important aspects that are made possible through end-to-end optimization approaches that are hallmark of DL – they learn distributed representations for input tokens, & at same time they learns dependencies or relations between these representation in time. Distributed representations, more commonly called word *embeddings*, map input onto a d -dimensional numeric space. 1 of motivations of such embeddings: semantically related tokens (words in language, piano-roll slices, or signal vectors), will be near each other in this space. Implicitly, this maps probabilities or likelihoods of appearance to distances in some, often non-euclidean geometry, but will not go into fascinating field of information geometry that actually tries to formalize this connection in rigorous mathematical ways.

In this chap consider a simple Markov transition probability model that begins with a set of discrete events (or states), into which all observations (e.g., notes o sound units or features) can be aggregated. Suppose there are R discrete categories into which all observations can be ordered. Can define a transition matrix $P = [p_{ij}]$ as a matrix of probabilities showing likelihood of a sound token staying unchanged or moving to any of other $R - 1$ categories over a given time horizon. Each element of matrix p_{ij} shows: probability of even i in period $t - 1$ moving or transitioning to event j in period t . Impose a simple Markov structure on transition probabilities, & restrict our attention to 1st-order stationary Markov processes, for simplicity.

Estimating a transition matrix is a relatively straightforward process, if can observe sequence of states for each individual unit of observation (i.e., if individual transitions are observed). E.g., if observe a sequence of musical chords in a corpus of music, then can estimate probability of moving from 1 chord to another. Probability of a chord change is given by simple ratio of number of times a chord appeared in different songs with same chord label & transitioned to another (or same)

chord label. In such case, identify chord labels with states, but more sophisticated state-token mappings could be defined. More generally, can let n_{ij} denote number of chord appearances who were in state i in period $t - 1$ & are in state j in period t . Can estimate probability using following formula:

$$P_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}.$$

Thus probability of transition from any given state i = proportion of chord changes that started in state i & ended in state j as a proportion of all chords that started in state i . In exercises, prove: estimator given in above equation is a maximum-likelihood estimator that is consistent but biased, with bias tending toward 0 as sample size increases. Thus, possible to estimate a consistent transition matrix with a large enough sample. A *consistent* estimator converges in probability to true parameter value, & may be *biased* for small sample size, i.e., estimator's expected value differs from true parameter value.

As a demonstration of concept of Markov chains applied to music, interesting to note: a representation of chord progressions as a state machine is often used for teaching music composition. 1 such example is provided in Fig. 2.4: This diagram illustrates harmonic progression patterns common to Western music, where each Roman numeral indicates a triad built from scale degree corresponding to numeral. A base chord (I or i for major & minor, resp.) can transition to any chord. Subsequent chords tend to follow a pattern toward what are known as "pre-dominant" & "dominant" functions, leading back to base chord (or "tonic"). From a mathematical perspective, such models can be formally considered Finite State Automata, with transition probabilities learned from data. From this, it becomes possible to teach machines to compose chord progressions by creating & using such a transition map. One can think of Markov models as a probabilistic extension of Finite State Automata (FSA) that can be learned from data [32] instead of being manually designed from music theory rules.

- 2.5. Music Information Dynamics. In prev discussion of music perception & cognition in Chap. 1, introduced concept of Music Information Dynamics as a generic framework for analysis of predictive structure in music, relating it to a sense of anticipation by listener who actively performs predictions during audition process. Underlying assumption in MID framework: musical signals, from audio level to higher level symbolic structures, carry changing amounts of information that can be measured using appropriate information theoretical tools. In terms of cognitive or computational processes, information is measured as a relative reduction in uncertainty about music events that is allowed through process of anticipation. As have seen in prev sect, def of mutual information basically says: information is carried between 2 random phenomena, & ability to decode or extract information is related to reduction in uncertainty about 1 signal from another. In MID approach consider musical past as 1 of random signals, & present instance of sound or music sect as 2nd random variable. Information is measured as a difference between entropy of 2nd variable & its conditional entropy give past. Moreover, important to note: conditional information of a random variable given its past \Leftrightarrow *entropy rate* of that variable. Accordingly, information dynamics is a measure of information passing from past to present measured by difference in signal entropy before & after prediction, or equivalently difference between signal entropy & its entropy rate. In next 2 sects show now how MIR can be measured for symbolic sequences modeled by a Markov process, & for an acoustic signal whose structure & predictive properties are captured by its spectrum.

- * 2.5.1. Information Rate in Musical Markov Processes. Equipped with basic knowledge of predictive modeling, interesting to consider case of Markov processes in music in terms of its anticipation structure. More formally, consider notion of information passing from past to present called Information Rate (IR) & characterize reduction in uncertainty when a Markov predictor is applied to reduce uncertainty in next instance of symbolic sequence. Moreover, same principle of information rate can be used for characterizing amount of structure vs. noise in acoustic signals. What differs between IR algorithms for 2 domains are their statistical models which in turn are used to estimate uncertainty, or entropy of signal based on its past.

Assume a Markov chain with a finite state space $S = s_1, \dots, s_N$ & transition matrix $P(s_{t+1} = i | s_t = j) = a_{ij}$, which gives probability of moving from a state i at some point in time t to another state j at next step in time $t + 1$. When a Markov process is started from a random initial state, after many iterations a stationary distribution emerges. Denoting stationary distribution as π , meaning of such stationary state: probability of visiting any state remains unchanged for additional transition steps, which is mathematically expressed as $\pi_{t+1} = A\pi_t$, where A : transitions matrix $A = [a_{ij}]$. Calculating entropy of stationary distribution & entropy rate is given following expressions

$$H(S) = H(\pi) = - \sum_{i=1}^N \pi_i \log_2 \pi_i,$$

$$H_r(S) = H_r(A) = - \sum_{i=1}^N \pi_i \sum_{j=1}^N a_{ij} \log_2 a_{ij}.$$

Using these expressions, IR can be simply obtained as difference between entropy of stationary distribution & entropy rate.

$$IR(S) = I(S_{t+1}; S_t) = H(\pi) - H_r(A).$$

E.g., consider a Markov chain that moves repeatedly from state 1 to N , with negligible probability for jumping between non-adjacent states. Such a situation can be described by matrix A that is nearly diagonal (non-diagonal elements will be very small). One can verify: for such a matrix, $H_r(A) \approx 0$ & IR will be close to entropy of stationary state $IR \approx \log_2 N$.

This is maximal information rate for such process, showing: such a very predictable process has high amount of dependency on its past. If on contrary matrix A is fully mixing, with $a_{ij} \approx \frac{1}{N}$, then stationary distribution will be $\pi \approx \frac{1}{N}$ & entropy & entropy rate will be same $H(S) = H_r(A) = \log_2 N$. This results in IR close to 0, i.e., knowledge of prev step tell close to nothing about next step. Such process is very unpredictable. Applying notion of IR to Markov processes can guide choice of Markov processes in terms of anticipation or average surprisal that such a process creates.

A more sophisticated measure was proposed by [12] that takes a separate consideration of past, present, & next future step. Termed *Predictive Information Rate* (PIR), measure is given as

$$\text{PIR} := I(S_{t+1}; S_t | S_{t-1}) = I(S_{t+1}; (S_t, S_{t-1})) - I(S_{t+1}; S_{t-1}) = H_r(A^2) - H_r(A).$$

1st line is def of PIR. 2nd line is derived by using def of mutual information with a conditional variable & adding a subtracting $H(S_{t+1})$

$$I(S_{t+1}; S_t | S_{t-1}) = \dots = -I(S_{t+1}; S_{t-1}) + I(S_{t+1}; (S_{t-1}, S_{t+1})).$$

Last line is derived from entropy rate expressions of Markov process, where conditional entropy is used 1 time for a single step between S_{t+1} & S_t without S_{t-1} due to Markov property, & 2nd time for 2 steps between S_{t+1}, S_{t-1} . Given a transition matrix $A = [a_{ij}]$

$$\begin{aligned} H(S_t) &= H(\pi) = -\sum_i \pi_i \log \pi_i, \\ H(S_{t+1} | S_t, S_{t-1}) &= H(S_{t+1} | S_t) = H_r(A) = -\sum_i \pi_i \sum_j a_{ij} \log a_{ij}, \\ H(S_{t+1} | S_{t-1}) &= H_r(A^2) = -\sum_i \pi_i \sum_j a_{ij}^2 \log a_{ij}^2. \end{aligned}$$

Considering prev example of a fully sequential repetitive process with $A \approx I$ diagonal matrix, one finds: since $H_r(A) \approx H_r(A^2)$, resulting PIR is approximately 0. So in terms of considering mutual information for 1 step prediction given past since process is nearly deterministic, knowing past will determine both next & following steps, so there is little information passing from present to future when past is known.

A simple example demonstrating principal difference between IR & PIR is shown in Fig. 2.5: A simple example of a nearly repetitive sequence of 5 states represented as a Markov model of a probabilistic finite state machine. Nearly deterministic transitions are marked by continuous lines & nearly zero probability transitions are marked by dashed line. In this example a 5 state probabilistic finite state machine is constructed with nearly deterministic (probability near 1) for transitions each state i to state $i + 1$, cyclically repeating from state 4 back to state 0, with negligible transition probabilities outside of sequence, i.e., negligible probabilities of staying in a state or jumping to a non-consequential states. This generative model can be analyzed as Markov model of 1st order, with transition matrix $A = [a_{ij}]$ & stationary distribution $\pi_{A_{4 \times 4}} = [\text{an } \varepsilon \text{ matrix}]$ & stationary distribution $\pi = [0.25, 0.25, 0.25, 0.25]$. For such border case have $H(\pi) \approx \log_2 5$, $H_r(A) \approx 0$, $H_r(A^2) \approx 0$. Accordingly have maximal IR $\approx \log_2 5$ & minimal PIR ≈ 0 . This raises question which measure best captures our interest in sequence. According to IR, learning that sequence is repetitive & using this knowledge to reduce our uncertainty about future is maximally beneficial to listener, while according to PIR, this sequence is not “interesting” since once past outcome is known, very little information is passed between next present & next sample.

Not clear which measure represents better human sensation of surprise, as it seems: both make sense in different situations. If consider very predictable sequence as informative, then IR captures: but one might as well say: such sequence is dull in terms of its music effect, & listener will favor music that carries a lot of local or immediate information, but without past making it too redundant. In following chaps consider yet another way of computing anticipation (tính toán dự đoán) or average surprisal in case of variable memory processes. In Markov case, all information about future was contained in prev state. Since music builds long motifs that continue, diverge, or recombine in unexpected ways, being able to understand such processes requires models that capture longer term relations in non-Markov ways. Extending Markov models to longer memories is computationally hard since using tuples or N -tuples grows state space exponentially, & resulting models & their estimated transition matrices are difficult to compute or estimate. In next chap describe string matching models & use of compression methods to find motifs & calculate continuations of phrases at different memory lengths.

- Chap. 3. Communicating Musical Information. “Music is greatest communication in world. Even if people don’t understand language that you’re singing in, they still know good music when they hear it”, said great American record producer, singer, & composer Lou Rawls. What is missing from his quote is def of what “knowing good music” means, & why this knowledge amounts to communication. Evidently music is a signal that passes from musician to their audience, so there is a sender-receiver structure to way music operates. & there is an aspect of “knowing” inherent in music listening process, which makes both encoder/musician & decoder/listener share some common understanding. In this chap explore basic premise of this book – ML of music, production, & computer audition are all parts of a communication process.

- 3.1. Music as Information Source. basic 1-directional model of communication assumes: process of musical creation & listening comprises a series of activities that transmit information from a source/musician to a receiver/listener. Research on information theory of music is based on Shannon & Weavers (1949) communication model [33]. This basic communication model comprises an information source that is encoded into messages that are transmitted over a noisy channel, & a decoder that recovers input data from corrupted &/or compressed representation that was received at output of channel. 1st step in this information-theoretical approach to modeling of music & musical style is accepting idea: music can be treated as an information source. Accordingly, process of music generation is based on notion of information itself that essentially

treats any specific message, & in our case any particular musical composition, as an instance of a larger set of possible messages that share common statistical properties. A CLAUDE SHANNON, father of information theory, expressed in his work *The Mathematical Theory of Communication* [33], concept of information concerns possibilities of a message rather than its content: “That is, information is a measure of one’s freedom of choice when one selects a message”. Measure of amount of information contained in a source is done by means of entropy, sometimes also called “uncertainty”, a concept covered in prev chap.

Initial interest in information came from communications: how a message can be transmitted over a communication channel in best possible way. Relevant questions in that setting concerned compression & errors that happen during encoding or transmission process. Model of information source is generative in sense that it assumes an existence of a probability distribution from which messages are randomly drawn. Theory of information, as introduced by SHANNON, encompassed 3 different domains of communication:

1. Lossless compression: any information source can be compressed to its entropy.
2. Error correction: all errors in communication can be corrected if transmission is done at a rate that is \leq channel capacity. Concept of channel capacity is beyond scope of this book, as will focus on Thms 1 & 3 & their relation to music generation.
3. Lossy compression: for any given level of reconstruction error or distortion, a minimal rate exists that will not exceed that error, & vice versa, for a given rate of transmission a coding scheme can be found that minimizes distortion.

An example of a communications channel is shown in Fig. 3.1: Basic model of a communication channel that includes encoder, decoder, & a noisy channel that establishes uncertain relations between source messages & what appears in output at receiver’s end. Since both source & channel errors are described probabilistically, information theory studies terms under which communication can be made more efficient. This includes methods for encoding or compression of source in a lossless or lossy manner, & methods for coding that provide correction or more robustness to channel errors. In our work, mostly focus on compression aspects as a way for finding efficient representation of source data, & investigating tradeoff between compression & reconstruction error in case of reduced representation modeled as lossy encoding.

Not all of Shannon concepts are directly applicable to music generation, so will limit ourselves to Thms. 1 & 3. Main concept borrow from Thm. 1: compression can be used as an efficient way to represent structure of information source. Music generation using compression methods uses concepts from Thm. 1 in a reverse manner to original Shannon design – instead of encoding a known information source, having an efficient encoding can be used for generating novel instances of same information source, & in our case composing novel music in same style. In original information-theoretic framework goal of encoding: find an efficient representation to a known probability distribution of source messages. Use compression as a way to learn distribution, i.e., given an efficient compression scheme can use it to produce more similar instances of same information source, which is our case is generating novel music by sampling from a its compressed representation. A good compression scheme is one which has learned statistical regularities over a set of examples, & can exploit these regularities in novel & creative ways. Moreover, Thm. 3 allows considering reduced musical representation through lossy compression that further reduces amount of information contained in a musical message in order to focus on more robust & essential aspects of musical structure. As such, statistical models derived from applying compression schemes to musical corpora in a particular style became also useful tool for *Stylometrics*, which is study of structures & variations that are characteristic of musical materials that may be broadly defined as “style”.

Initial work on using information theory for music started in late 90’s [34]. Today, problem of generative modeling in music has become 1 of formidable challenges (những thách thức to lớn) facing ML community & DL research in particular. Music is a complex, multidimensional temporal data (dữ liệu thời gian đa chiều) that has long-term & short-term structures, & capturing such structure requires developing sophisticated statistical models. For historical as well as conceptual reasons, thinking about music in terms of a complex information source provides powerful insights to building novel ML methods. Using communication theory in music research not only puts music into a transmitter-receiver communication setting that takes into account both composer & listener, but it also captures dynamics of music entertainment & creativity since co-evolution of musical knowledge that happens over time between musician & listener can be captured by methods of information theory, opening novel possibilities for man-machine co-creative interactions.

* 3.1.1. **Style Modeling.** Style Modeling using information theory started with application of universal sequence models, specifically well-known Lempel-Ziv (LZ) universal compression technique [35] to MIDI files [34]. Universality of LZ compression can be understood in terms of it not assuming any a-priori knowledge of statistics of information source, but rather having its asymptotic compression performance as good as any statistical model, e.g. Markov or finite state model. This is an empirical learning approach where a statistical model is induced through application of compression algorithms to existing compositions. Moreover, it captures statistics of musical sequences of different lengths, a situation that is typical to music improvisation (ngẫu hứng âm nhạc) that creates new musical materials at different time-levels, from short motifs or even instantaneous sonority (âm thanh tức thời), to melodies & complete thematic phrases. Specifically, universal modeling allows application of notion of entropy to compare sequences in terms of similarity between their statistical sources, as well as generation of novel musical sequences without explicit knowledge of their statistical model. One should note: “universality” in universal modeling approach still operates under assumption: musical sequences are realizations of some high-order Markovian sources. Information theoretical modeling method allows following musical operations to be applied to musical data:

- stochastic generation of new sequences that have similar phrase structure as training sequence, creating musical results in between improvisation & original on motivic or melodic level, &

stochastic morphing or interpolation between musical styles, where generation of new sequences is done in a manner where new statistics are obtained by a mixing procedure that creates a “mutual source” between 2 or more training styles. Extent to which new improvisation is close to 1 of original sources can be controlled by mixture parameters, providing a gradual transition between 2 styles which is correct in statistical sense.

Early experiments with style morphing [36] used a joint source algorithm [37], later using more efficient dictionary & string matching methods [38]. In addition to generative applications, universal models are used for music information retrieval, e.g. performing hierarchical classification by repeatedly agglomerating closest sequences (thực hiện phân loại theo thứ bậc bằng cách kết tụ nhiều lần các chuỗi gần nhất), or selecting most significant phrases from dictionary of parsed phrases in a MIDI sequence, selected according to probability of their appearance.

- * 3.1.2. **Stochastic Modeling, Prediction, Compression, & Entropy.** Underlying assumption in information-theoretical approach to machine improvisation: a given musical piece can be produced by an unknown stochastic source, & all musical pieces in that style are generated by same stochastic sources. Finite Markov-order assumptions do not allow capturing arbitrarily complex music structures, e.g. very long structure dependency due to musical form; nevertheless, by allowing for sufficiently long training sequences that capture dependence on past, universal model can capture much of melodic structure of variable length in a musical piece.

This connection between compression & prediction is a consequence of asymptotic equipartition property (AEP) [39], which is information-theoretic analog to law of large numbers in probability theory. AEP tells us: if x_1, x_2, \dots are independent & identically distributed random variables distributed with probability $P(x)$, then

$$-\frac{1}{n} \log_2 P(x_1, x_2, \dots) \rightarrow H(P)$$

where $H(x)$: Shannon Entropy of $x \sim P(x)$ (i.e., $H(x) = -\sum_x P(x) \log_2 P(x)$, where averaging is over all possible occurrences of sequences x).

AEP property is graphically represented in Fig. 3.2: **Asymptotic Equipartition Property of long sequences.** Outer circle represents all possible sequences of length n , which are combinatorial possibilities depending on size of alphabets. Shannon’s theory proves: in view of different probabilities of occurrence of each symbol (in simplest case, these are unbalanced heads or tails, known as Bernoulli distribution, with binary choice which takes value 1 with probability p & value 0 with probability $q = 1 - p$), entropy of probability can be used to define an effectively much smaller set of outcomes whose probability will tend to be 1.

Remark 4 (Bernoulli Distribution). *Bernoulli distribution is a discrete probability distribution that represents outcome of a single trial with 2 possible outcomes. Distribution is named after Swiss mathematician JACOB BERNOULLI, who introduced it in late 1600s. Probability distribution function of Bernoulli distribution is $P(X = k) = p^k(1 - p)^{1-k}$, where $P(X = k)$: probability of getting outcome k (either 0 or 1) & p : probability of outcome 1 on a single trial, with $p \in [0, 1]$. Mean (expected value) & variance of Bernoulli distribution are $E[X] = p$, $\text{Var}[X] = p(1 - p)$. Bernoulli distribution is often used as a building block for more complex distributions, e.g. binomial distribution, which models number of successes in a fixed number of independent Bernoulli trials.*

This set of outcomes is called “Typical Set” & is denoted here as A_n , where n : number of elements in a sequence that needs to be sufficiently large. Moreover, all sequences of typical set are equiprobable, i.e., one can index these events in a way that assigns equal length codes to these events, & zero codes to any remaining events of non typical set. I.e., for long enough observations from a probability source, possible outcomes split into 2 – sequences that deviate from probability statistics, e.g. sequences that do not achieve expected mean number of heads or tails in a Bernoulli probability, will never occur, while those that do occur, happen with equal probability. No further structure exists in typical set, or otherwise additional compression could have been applied to exploit any remaining imbalance. In our case, for generative purposes, this means: one can generate new instances of a data source by accessing strings from its typical set using a uniform random number generator. Conceptually i.e.: sampling from a typical set turns white noise into outcomes that have desired statistical structure of that information source. For stationary ergodic processes, & in particular, finite order Markov processes, generalization of AEP is called Shannon-McMillan-Breiman theorem [39]. Connection with compression: for long x lower limit on compressibility is $H(x)$ bits per symbol. Thus, if can find a good algorithm that reaches entropy, then dictionary of phrases it creates can be used for generating new instances from that source. When algorithm has compressed phrase dictionary to entropy, any excess compressible structures have been eliminated. I.e.: dictionary is very efficient, so can sample from source by random selection from that dictionary.

- * 3.1.3. **Generative Procedure.** Dictionary-based predictions methods sample data so that a few selected phrases represent most of information. Below described 2 dictionary based generative approaches: *incremental parsing* (IP) (phân tích gia tăng) & *prediction suffix trees* (PST). IP is based on universal prediction, a method derived from information theory. PST is a learning technique that initially was developed to statistically model complex sequences, & has found applications also in linguistics & computational biology. Both IP & PST methods belong to general class of dictionary-based prediction methods. These methods operate by parsing an existing musical text into a lexicon of phrases or patterns, called *motifs*, & provide an inference rule for choosing next musical object that best follows a current past context. Parsing scheme must satisfy 2 conflicting constraints: on 1 hand, maximally increasingly dictionary helps to achieve a better prediction, but on other, enough evidence must be gathered before introducing a new phrase to allow obtaining a reliable estimate of conditional probability for generation of next symbol. “Trick” of dictionary-based prediction (& compression) methods: they cleverly sample data so that only a few selected phrases reliably represent most of information. In contrast to dictionary-based methods, fixed-order Markov models build potentially large probability tables for appearance of a next

symbol at every possible context entry. To avoid this pitfall, more advanced “selective prediction” methods build more complex variable memory Markov models. Although it may seem: Markov & dictionary-based methods operate in a different manner, they both stem from similar statistical insights.

Use dictionary-based methods to model musical (information) source in terms of a lexicon of motifs & their associated prediction probabilities. To generate new instances (messages), these models “stochastically browse” prediction tree in following manner:

- Given a current context, check if it appears as a motif in tree. If found, choose next symbol according to prediction probabilities.
- If context is not found, shorten it by removing oldest (leftmost) symbol & go back to prev step.

These steps iterate indefinitely, producing a sequence of symbols that presumably corresponds to a new message originating from same source. In some cases, this procedure might fail to find an appropriate continuation & end up with an empty context, or it might tend to repeat same sequence over & over again in an infinite loop. Specific aspects of how dictionary is created & how continuity is being maintained between random draws will be discussed next.

- 3.2. Lempel-Ziv Algorithm & Musical Style. In *A universal algorithm for sequential data compression* [40], LEMPEL & ZIV take a series of symbols from a finite alphabet as input, & build a tree of observed continuations of combinations of input symbols. This tree grows dynamically as input is parsed using what is called *incremental parsing* (IP) method. If input is a sample of a stationary stochastic process, LZ asymptotically achieves an optimal description of input in sense: resulting tree can be used to encode input at lowest possible bit rate (entropy of process). This implies: coding tree somehow encodes law of process; e.g., if one uses same tree (as it stands after a sufficiently large input) to encode another string, obeying a different law, resulting bit rate is higher than entropy.

- * 3.2.1. Incremental Parsing.

- * 3.2.2. Generative Model Based on LZ.

- 3.3. Lossy Prediction Using Probabilistic Suffix Tree.

- 3.4. Improved Suffix Search Using Factor Oracle Algorithm.

- Chap. 4. Understanding & (Re)Creating Sound.

- Chap. 5. Generating & Listening to Audio Information.

- Chap. 6. Artificial Musical Brains.

- 6.1. Neural Network Models of Music. In this chap, describe musical applications of ML with neural networks, with emphasis on representation learning or feature learning (methods by which a system automatically discovers representations needed for feature detection or classification from original data).

Musical neural networks are not a new research direction; a 1991 book by Peter M. Todd & Gareth Loy, *Music & Connectionism*, explores the topic. At time, *connectionism* referred to movement in cognitive science that explained intellectual abilities using artificial neural networks ANNs. Now, see neural networks, especially DL techniques, applied widespread among many domains of interest. Naturally, this has renewed interest in musical applications, with research in other domains spilling over to music as well, providing interesting & important results. With technical advances of algorithm optimization, computing power, & widespread data, possible for many to enjoy creating & experimenting with their own musical neural networks.

Before visiting these applications, begin with a preliminary tour of vocabulary, algorithms, & structures associated with neural networks.

- * 6.1.1. Neural Networks. Consider a simplified model of brain as a network of interconnected neurons. Structure of linked neurons allows information (in form of electrical & chemical impulses) to propagate through brain. Organizing a series of mathematical operations & instructions as a *neural network* allows computer to function analogously to a brain in certain tasks.

In a standard circuit, may expect an electronic signal to be passed forward, either as-is or in an amplified or reduced form. In a neuron, this message passes through an additional barrier; if its excitement passes a particular threshold, it will emit an action potential, or “fire”, & otherwise, signal will not move forward. In this simplified model, can imagine a collection of these neurons to act like a collection of interconnected light switches; if right sequence are left “on”, signals may pass through in a combination s.t. a correct destination “bulb” may light.

- * 6.1.2. Training Neural Networks.

Definition 5. A neural network is composed of a collection of scalar values which are used in mathematical operations to achieve a desired output. These scalar values are referred to as network weights or parameters. Can measure network’s performance on a task using a function which returns a value called loss.

Goal of training a neural network: modify network weights s.t. forward passes of data through network lead to minimal loss. Achieve this through combining processes known as backpropagation [82] (used to calculate gradients of weights in neural network) & optimization (methods by which weights are modified to reduce loss). Some popular optimization methods include gradient descent (& its stochastic & mini-batch variants) & Adaptive Moment Estimation (Adam) [83].

6.1.2.1. **Loss Functions: Quantifying Mistakes.** Imagine multiple pedagogical environments in which a person might learn to play piano. In 1 classroom, this person may have an excellent teacher who explains a technique in such great, concrete, retainable detail that the person is immediately able to execute performance technique after lesson. In another classroom, this teacher may briefly instruct student on intended technique, then ask student to perform. Student gives a suboptimal performance, & teacher points out student's mistakes. From these corrections (repeated as often as necessary), student then (eventually) learns proper performance technique.

When it comes to neural networks, learning process is much like that of our 2nd piano student. Teacher (in this case, expected labels or *ground truth* associated with our data) must be cleverly turned into a quantifiable metric which can be used by learning algorithm to adjust network parameters. Call such a metric a *loss function*.

Utilized loss functions will vary from problem to problem. 2 notably recurring loss functions are Mean Squared Error (MSE) & Categorical Cross-Entropy.

1. Mean Squared Error is typically used when performing regression tasks (i.e., possible output exists on a continuum, like real numbers). For a given vector of length N , MSE generally follows formula

$$\text{MSE} = \frac{1}{N} \sum_{n=1}^N (Y_{\text{output}_n} - Y_{\text{truth}_n})^2. \quad (54)$$

2. Categorical Cross-Entropy is typically used when performing classification tasks (i.e.g, assigning of input to 1 of C possible classes). Cross-Entropy generally follows formula

$$\text{CCE} = - \sum_{c=1}^C \mathbf{1}(c = y_{\text{truth}}) \log P(c), \quad (55)$$

where $\mathbf{1}(c = y_{\text{truth}})$ is an indicator function with value 1 when c is equal to true class, & 0 otherwise. Because classification problems expect as output a vector of class probabilities, this function essentially leverages properties of logarithm function to create strong values when model fails to predict known class (i.e., a log value approaching negative infinity, inverted by negative sign in front of equation) & weak, near-0 values when model correctly (or near correctly) predicts known class since $\log 1 = 0$.

- 6.1.2.2. **Gradient Descent.** After quantifying value of loss in form of a loss function, next desire to modify weights of our neural network s.t. loss is reduced. A common method to achieve this is gradient descent:

$$W_{\text{new}} = W - \alpha \nabla L(W). \quad (56)$$

For a fixed input x , value of loss function varies only w.r.t. weights W . Can compute a gradient by considering derivative of loss function w.r.t. each weight w_i in W . This gradient describes direction & magnitude of steepest ascent toward maximum value of loss; therefore, negative of gradient points in steepest direction toward minimum, with magnitude proportional to rate of change of loss in associated direction. Therefore, gradient descent equation adjusts weights in a direction toward minimal loss for input, scaled by hyperparameter α , also referred to as learning rate.

6.1.2.2.1. Chain Rule for MLP. In a feedforward neural network, input is passed through an initial layer where it interacts with network weights, & in case of a multi-layer network, these resulting values (often referred to as features) are passed to a next layer, & so on. Mathematically, a *multilayer perceptron* (our classic feedforward network) is essentially a composite of functions. To compute gradient, make use of chain rule. Consider basic example network shown in Fig. 6.1: An example multi-layer perceptron, consisting of input, network parameters (weights, bias), intermediate nodes, & output., consisting of a single perceptron unit.

For some loss function L , desire to compute components of gradient $\frac{\partial L}{\partial w_i}$ for each weight w_i . However, in a multi-layer perceptron, each weight is effectively buried under a composition of functions which eventually lead to network output (& loss). Chain rule allows us to compute desired gradient: $\frac{\partial L}{\partial w_i} = \frac{\partial L}{\partial y_j} \frac{\partial y_j}{\partial w_i}$, where y_j is an output in a preceding layer directly used in computing loss. This rule can be iterated over $\frac{\partial y_j}{\partial w_i}$ using preceding nodes, until input layer of network is reached.

As an example, consider network shown in Fig. 6.1. Provide to this network some training sample $\vec{x} = \{x_1 = 1, x_2 = 2\}$, associated with expected training output $z_{\text{truth}} = 5$. Assume: network parameters have been initialized s.t. all $w_i = 1, b_i = 1$. When network weights are applied to input, reach $z = w_5 y_1 + w_6 y_2 + b_3 = \dots = 9$.

Using squared-error as a measure of deviation from expected value, there is a loss (our term for this deviation) of 16 associated with this sample. Now, would like to adjust weights to improve network performance. Rewrite loss expression as a function of weights.

$$L = (z - z_{\text{truth}})^2 = (w_5 y_1 + w_6 y_2 + b_3 - z_{\text{truth}})^2. \quad (57)$$

Considering influence of just 1 of our parameters, w_1 , on network performance, chain rules states

$$\frac{\partial L}{\partial w_1} = \frac{\partial (z - z_{\text{truth}})^2}{\partial w_1} = \frac{\partial (z - z_{\text{truth}})^2}{\partial z} \frac{\partial z}{\partial w_1} = \frac{\partial (z - z_{\text{truth}})^2}{\partial z} \frac{\partial z}{\partial y_1} \frac{\partial y_1}{\partial w_1}. \quad (58)$$

Backpropagation is recursive algorithm by which compute derivative of final function value (in our case, L) w.r.t. an intermediate value in computation (e.g. any w_i) using recursive calls to compute values $\frac{\partial L}{\partial w_j} \forall w_j$ computed directly from w .

Now with possibility to compute a complete gradient over loss for each network parameter w, b , how do we use training data to improve network performance?

Consider 1st training sample. Loss value associated with this sample is $L = (9 - 5)^2 = 16$. Compute an exact value for our gradient term $\frac{\partial L}{\partial w_1}$:

$$\frac{\partial L}{\partial w_1} = \frac{\partial(z - z_{\text{truth}})^2}{\partial z} \frac{\partial z}{\partial y_1} \frac{\partial y_1}{\partial w_1} = 2(z - z_{\text{truth}})w_5x_1 = \dots = 8. \quad (59)$$

Compute $\nabla L = \left[\frac{\partial L}{\partial w_1}, \dots, \frac{\partial L}{\partial w_6}, \frac{\partial L}{\partial b_1}, \dots, \frac{\partial L}{\partial b_3} \right]$: this is where gradient descent comes into play. Adjust weights according to formula given in (6.3), employing a learning rate of 0.1 for our example. $W_{\text{new}} = \dots$. Our loss drops from 16 to $L = (3.72 - 5)^2 = 1.6384$ exactly like we wanted! Could continue our descent until approximately reach a local minimum (or reach suitable performance for particular problem).

While this example illustrates a method for calculating useful gradient, note: these gradients are computed in efficient, vectorized fashion in DL software.

6.1.2.3. **A Note on Initialization.** Before we can begin training, network weights must be initialized to some value so that a gradient can eventually be computed. Saw: our “set all weights to 1” approach in previous example may not have been too effective, especially since both halves of network effectively learned symmetrically, making them redundant. A better option: initialize weights randomly. This is a very important step, because if network weights are too small, corresponding nodes will never be activated by data, & weights may be too small for loss gradient to enact meaningful changes to weights. Similarly, if network weights begin too large, values may “explode”, creating too strong of a response from corresponding nodes, & consume all of gradient descent effects to prevent learning on other network weights (especially when values exceed limits of your programming language’s numeric representations & become NaN or infinite).

Xavier Glorot & Yoshua Bengio show: using an initialization randomly sampled from a uniform distribution bounded by $\pm \frac{\sqrt{6}}{\sqrt{n_{\text{in}} + n_{\text{out}}}}$, now popularly termed “Xavier initialization”, maintains near identical variances of weight gradients across network layers during backpropagation [84]. In this equation, n_{in} : number of edges into layer, & n_{out} : number of edges out of layer.

6.1.2.4. **Modeling Nonlinearity.** Our simplified network example is missing 1 major component which mimics “firing” behavior we seek from our neurons. This ability to propagate a signal when a certain threshold is passed is a *nonlinear* behavior, as opposed to structured in our simplified example, which can effectively be reduced to a series of linear equations (matrix multiplications). *Activation functions* are applied after such standard multiplication layers to introduce thresholding. Importantly, activation functions are (mostly¹³) differentiable, which allows us to continue to use backpropagation to learn network parameters. This is exactly why binary activation function shown in Fig. 6.2: **Binary activation function** cannot serve as our activation function; given input 0, then unclear whether weights should face no change from associated derivative, or *infinite* change.

However, there exist plenty of other functions which share an ability to operate as a thresholding function while still maintaining near differentiability. Some common activation functions include:

1. Sigmoid (or Logistic) Function:

$$s(x) = \frac{1}{1 + e^{-x}} \quad (60)$$

this function tends to 0 on left & 1 on right, just like binary step. Critically, it is smooth & differentiable between these limits Fig. 6.3: **Sigmoid activation function**.

2. Hyperbolic Tangent: function $\tanh x$ is very similar to sigmoid function in shape, but its larger range allows for a wider range of output values to propagate forward in network Fig. 6.4: **Hyperbolic tangent activation function**.

3. Rectified Linear Unit (ReLU):

$$\text{ReLU}(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases} \quad (61)$$

With these tools in place, now possible to construct neural network architectures which model real-world phenomenon (i.e., patterns which are not necessarily linear) through connections of matrix-multiplication-like layers (often fully-connected tapestries of nodes) followed by activation functions, with successive layers repeated, modified, & tailored to content of task at hand.

o 6.2. **Viewing Neural Networks in a Musical Frame.** 1 application of neural networks for music: field of music information retrieval (tìm kiếm thông tin âm nhạc). Information retrieval deals with ability to search & find content by some query. A few examples of problems in music information retrieval include

1. Music Classification: sorting music based on properties e.g. genre, style, mood, etc.,
2. Music Recommendation: determining musical similarity between samples & corresponding suggestion of music based on user history,
3. Music Source Separation & Instrument Identification: separating a mixed audio signal to its original signals, as well as recognizing & extracting a particular instrument within a mixed signal,
4. Music Transcription: converting an audio recording to a symbolic notation, &

¹³Some activation functions have points of non-differentiability; in practice, these points are assigned derivative associated with nearest point. E.g., ReLU (rectified linear unit) is not differentiable at 0, so a value of 0 (left derivative) or 1 (right derivative) may be assigned at this point.

5. Music Generation: automatic creation of musical compositions or audio signals.

Many of above problems require an output used for making a prediction over multiple predefined possibilities. E.g., in classifying music, may be predicting 1 genre out of n possibilities. In generation, may be predicting next note, given n possible notes in instrument's range. Typically represent such outputs using a vector of size n , where each entry of vector represents confidence of model in selecting particular element. Often beneficial to treat these values way we treat probabilities, so that their numeric values can be used to generate output that matches some learned distribution. However, this requires values in vector to be between 0 & 1 (inclusive), with sum of all entries = 1. May wonder, why not just select element of greatest value, without normalizing vector? Such an *argmax* function is non-differentiable, so network will lose ability to learn from this structure. To overcome this, *softmax* function is used, notable for its normalization property, its ability to preserve numeric ranking among entries, & its differentiability.

Example 6 (Softmax Learning). *Softmax equation is*

$$\text{softmax}(v)_i = \frac{\exp v_i}{\sum_{j=0}^{n-1} \exp v_j}. \quad (62)$$

Note: output of softmax is another vector of length n . Recall: vector output by softmax can be imagined as a probability distribution over classes.

Sometimes, want distribution to be very sharp, with nearly all probability associated with 1 particular value, indicating a strong confidence for a particular class. Other times, particularly with generating so-called creative output, may want a softer distribution, where values still preserve order but may introduce some ambiguity & lack of confidence (quantified by magnitude of probability). Put differently: there is a tradeoff between distribution flatness & novelty of repeated sampling. If repeatedly sample a uniform distribution to predict next note of a sequence, will have a very random (aleotoric – ngẫu nhiên) sequence. By contrast, if repeatedly sample a distribution with probability 1 for note G & 0 for all other notes, our piece will certainly not contain variance we come to expect in music.

Mathematical mechanism we can use to influence these distributions is referred as as temperature. Temperature is reflected in coefficient T , using which we rewrite softmax with temperature:

$$\text{softmax}_T(v)_i = \frac{\exp \frac{v_i}{T}}{\sum_{j=0}^{n-1} \exp \frac{v_j}{T}}. \quad (63)$$

Can think of T as a scaling factor which increases (or decreases) distance between points of v on number line; concavity of exponential function then amplifies these differences, meaning: points pushed further apart on number line have even greater relative values to their neighbors, & points pushed inward have less strength. Accordingly, $0 < T < 1$ will create a sharper distribution, while $1 < T < \infty$ creates a softer distribution. In exercises at end of chap, show: distribution approaches uniformly when T approaches infinity.

Consider again our equation for categorical cross-entropy for classification:

$$\text{CCE} = - \sum_{c=1}^C \mathbf{1}(c = y_{\text{truth}}) \log P(c). \quad (64)$$

Using properties of logarithms, can reformulate this as:

$$\text{CCE} = - \log \prod_{c=1}^C P(c)^{(c=y_{\text{truth}})}. \quad (65)$$

In spirit of temperature, can reformulate this expression for cases when goal is not to learn a single-probability result, but rather a distribution over outcomes. Replace indicator function $\mathbf{1}(c = y_{\text{truth}})$ with target probability for class c, t_c :

$$\text{CCE} = - \sum_{c=1}^C t_c \log P(c). \quad (66)$$

Using properties of logarithms, can reformulate this as:

$$\text{CCE} = - \log \prod_{c=1}^C P(c)^{t_c}. \quad (67)$$

Term inside logarithm is in fact likelihood function of network parameters θ given observation $X = (x, y_{\text{truth}})$. This is stylized as $\mathcal{L}(\theta|X)$ & it = $P(X|\theta)$.

Our measure of error depends on these output probabilities $P(c)$, which will refer to as output vector y . Also clear: these probabilities depend on input provided to softmax layer (values that become “scaled” into probabilities). Refer to these inputs as vector z . A natural question arises in order to backpropagate: how does error change w.r.t. z ? I.e., what is gradient $\frac{\partial L}{\partial z}$? For any component,

$$\frac{\partial L}{\partial z_i} = - \sum_{j=1}^c \frac{\partial t_c \log y_i}{\partial z_j} = - \sum_{j=1}^C t_c \frac{\partial \log y_i}{\partial z_j} = - \sum_{j=1}^C t_c \frac{1}{y_j} \frac{\partial y_i}{\partial z_j}. \quad (68)$$

Show: $\frac{\partial y_i}{\partial z_j} = y_i(1 - y_i)$ when $i = j$, & $\frac{\partial y_i}{\partial z_j} = -y_i y_j$ when $i \neq j$. Use this result in gradient computation:

$$\frac{\partial L}{\partial z_i} = \dots = -t_i + y_i \sum_{j=1}^C t_j = y_i - t_i. \quad (69)$$

With this gradient available, our network can learn a distribution associated with a sample using softmax function.

Now: have shown a way to learn by tuning weights, think through some immediate architectural ideas & consequences. Begin by considering 2D of contraction or expansion of our network. Can grow vertically (i.e., adding or subtracting neurons to a particular layer) or horizontally (adding additional layers to network).

- * What happens when we shrink vertically? In this case, may *underparameterize* our model; i.e., trying to represent a complex problem or pattern with too few variables. When we do so, model may succeed in generalizing & perform well in most typical cases, but may fail for nuanced, less frequent cases.
- * What happens when we grow vertically? In this case, may *overparameterize* our model; have created so many variables that each sample in our data may be fit to its own subset of variables. Model may fail to generalize to underlying pattern in data, so while it may perform perfectly in training, it will be unlikely to perform on unseen data.
- * What happens when we shrink horizontally (i.e., make network more “shallow”)? In this case, model will lose its ability to combine preceding pieces of information, a process refer to as generating higher-level features. In this type of abstract reasoning, may begin to observe a numerical piece & reason about its individual notes at early layers (e.g., hearing an F#, then another F#, then a G ...). In a middle layer, might combine some of these notes together to form an idea of a small motif (like a scale ascending a minor 3rd). At even deeper layers, motifs may be combined to form a phrase (a scale ascending a minor 3rd, then descending a perfect 5th ... ah, perhaps this is Ode to Joy!). Our model’s ability to combine features & learn on these combinations is limited by depth of network.
- * Might think: a solution to this shallow learning challenge: make network even deeper. But, what happens when a network is made too deep? This leads to what is commonly referred to as *vanishing gradient problem*.

- 6.3. Learning Audio Representations.
- 6.4. Audio-Basis Using PCA & Matrix Factorization.
- 6.5. Representation Learning with Auto-Encoder.
- 6.6. Exercises.
- Chap. 7. Representing Voices in Pitch & Time. Music & Audio AI is an interdisciplinary field between computer science & music. In comparison to more established AI fields e.g. natural language processing (NLP) & computer vision (CV), it is still in its infancy & contains a greater number of unexplored topics. As a result of this, DL models from natural language processing & computer vision are frequently applied to music & audio processing tasks. Take neural network architectures as an example. Convolutional neural networks (CNNs) have been extensively used in audio classification, separation, & melody extraction tasks, whereas recurrent neural networks (RNNs) & transformer models are widely used in music generation & recommendation tasks. Have seen various models of RNN, CNN, & transformer gradually outperform traditional methods to achieve new state-of-arts in recent years.

In this chap, introduce 2 basic neural network architectures that are used to learn musical structures in time & pitch or time & frequency, namely Recurrent Neural Network (RNN) & Convolutional Neural Network (CNN). However, extent to which these architectures are capable of handling music & audio tasks remains debatable. In this chap, illustrate similarities of music & audio problems with NLP & CV tasks by comparing music generation with text generation, & music melody extraction with image semantic segmentation. This provides an overview of how RNN, CNN, & transformer architectures can be applied to these problems. Then, examine uniqueness of music & audio problems in comparison to tasks in other fields. Previous research has demonstrated that certain designs for these architectures, both in terms of representation & multitaskstructure, can result in improved performance in music & audio tasks.

RNN caught attention of music research as early as 1990s, after opening up field of neural networks to modeling language & sequences. In [91], PETER TODD describes a connectionist approach to algorithmic composition based on Elman & Jordan networks.¹⁴ Early approaches were mostly monophonic (đơn âm) & were difficulty to train. With advent of long short-term memory networks (LSTMs) & success of modern RNNs in Natural Lagrange Processing, these tools became 1st go-to neural architecture for modeling sequential structure in music. Inspired by success of CNNs in computer vision, & thinking about music as a time-frequency 2D structure, CNNs were also applied to music. Big difference between 2 architectures is in their treatment of relations between frequencies & times. In RNN model is purely sequential; all temporal relations are summarized into tokens that are used to represent either single or multiple voices. Thus, tokenization will be our 1st topic of study. In CNN, convolutional filters are applied both in time & across notes (in MIDI pianoroll representation) or frequencies (in audio spectral representation). Main difficulty in using CNNs for music: limited span or “receptive field” of convolutional kernel (filter). While RNN has a decaying memory that is arbitrarily long, CNNs field of view is limited to size of filter. Using dilated (giãn nở) CNNs (kernels that skip samples) & hierarchical structures, e.g. Wavenet, became possible ways to overcome limitation of short-time scope of filter response. Some hybrid variants were proposed as well, including biaxial-RNNs that apply RNNs both to time & frequency/notes axis, or combinations of CNN-RNN that try to combine local feature extraction aspect of CNNs with long term memory aspect of RNNs.

¹⁴Elman network, which became modern RNN, uses feedback connections from hidden units, rather than from output in Jordan design.

- 7.1. Tokenization. How do we pass musical input to a neural network? This is question addressed by tokenization (phân loại), process of converting musical information into smaller units called *tokens*.

Tokenization is used widely in Natural Language Processing. To illustrate variety of tokenization approaches, consider 2 possible tokenizations of Einstein’s sentence, “If I were not a physicist, I would probably be a musician.” If 1 such letter-based tokenization, create a representation for each token {I,f,w,e,r,n,o,t,a,p,h,y,s,c,u,l,d,b,m}. In a different word-based tokenization, create a representation for each token {If, I, were, not, a, physicist, would, probably, be, musician}. What are possible benefits of different tokenization methods? If have a model with which we would like to learn how to spell or construct words, letter-based tokenization will be much more useful. On other hand, if would like to learn relationship between words in a sentence for proper grammatical & semantic usage, word-based tokenization is more appropriate. Similarly, way we tokenize musical data has a direct impact on what our models can efficiently learn from data, & types of output they can produce.

Now thinking in terms of input & output, if want to learn end-to-end a complete MIDI file, why is MIDI not best representation? Network would need to learn formatting specifications of MIDI encoding, which are not related to musical information itself. So, can tokenize musical information itself to help our network learn musical patterns – in essence, extract & translate musical building blocks from MIDI data.

MidiTok [92] is a Python package which implements multiple popular MIDI tokenization schemes:

1. MIDI-Like [93]: This scheme contains 413 tokens, including 128 note-on & note-off events (1 per MIDI pitch), 125 time-shift events (increments of 8ms to 1 sec), & 32 velocity events. MidiTok added additional quantization parameters in their implementation to allow for different numbers of tokens to accommodate different memory constraints or precision goals.
2. Revamped (Đã được cải tiến) MIDI-Derived Events (REMI) [94]: REMI maintains note-on & note velocity events from MIDI-like. Note-off events are instead replaced with Note Duration events; a single note is therefore represented by 3 consecutive tokens (note-on, note velocity, & note duration). Authors propose this to be advantageous in modeling rhythm than placing a series of time-shift events between a note-on & note-off in MIDI-like. Additionally, problem of a “dangling” note-on event is resolved, should model fail to learn that note-on & note-off events must (naturally) appear in pairs.
3. Structured [95]: Unlike MIDI-Like, with Structured tokenization, tokens are provided in standardized order of pitch, velocity, duration, time shift. This encoding cannot accommodate rests or chords, so it is only suitable for modeling particular musical examples.
4. Compound Word [96]: At this point, may have considered an odd artifact of MIDI encoding. Oftentimes, there are events which are intended to be simultaneous, but due to 1-instruction-at-a-time nature of MIDI, they are provided as a sequence. E.g., a combination of note-on instruction & its associated note velocity would be provided as a sequence of 2 sequential instructions in MIDI, but these instructions are intended to describe same event. This idiosyncrasy (tính cách lập dị) is corrected in Compound Word. These compound words are supertokens – i.e., combinations of tokens for pitch (vocabulary size 86), duration (17), velocity (24), chord (133), position (17), tempo (58), track (2), & family (4). Including “ignore” tokens for each category, there are a total of 338 vocabulary items to form a Compound Word supertoken. “Family” token allows for distinction between track events, note events, metric events, & end-of-sequence events. Similar encodings include Octuple [97] & MuMidi [98].

MidiTok uses 4 parameters to tokenize for each of implemented schemes:

1. Pitch Range: minimum & maximum pitch values (integers in MIDI).
2. Number of Velocities: how many MIDI velocity levels (range 0 to 127) to quantize to.
3. Beat Resolution: sample rate per beat, & beat ranges for which sample rate should be applied. Specifying a high resolution for notes with a short number of beats can improve precision of time shifts & note durations.
4. Additional Tokens: opportunity to specify additional tokens to be included in addition to standard set. This includes:
 - (a) Chord: a token which indicates a chord is being played at particular timestep.
 - (b) Rest: explicit definition of a time-shift event. Helpful to insert a tokenized rest for understanding input/output, as opposed to simply placing next note at a later timestep since rests are an important, notated feature of most music.
 - (c) Tempo: a token to represent current tempo; a range of possible tempos & number of tempos to quantize within this range should be provided to MidiTok.

* 7.1.1. Music Generation vs. Text Generation. What is relation between Music Generation vs. Text? Music generation’s objective: create musical content from input data or machine representations. Music content can be in form of symbolic music notes or audio clips. Typically, input data serves as generative condition, e.g. previous musical context, motivation, or other musical information (e.g., chord progression, structure indication, etc.). Text generation’s objective: generate natural language in response to a condition. This condition could be a couple of initial words, conclusion of a sequence, or change in mode (e.g., professional, neural, creative, etc.). By & large, music & text generation have a similar conditional probability of decomposing formats:

$$\mathbf{y} = y(y_1, y_2, \dots, y_T), p(\mathbf{y}|\mathbf{c}) = \prod_t p(y_t | \mathbf{y}_{1:t-1}, \mathbf{c}), \quad (70)$$

where y denotes token in different sequential tasks, & c denotes different conditions as discussed above. In symbolic music generation, this token can be a single musical note or a latent embedding of a measure/phrase of music. In text generation, this token can be a single character, a word embedding, or even a sentence embedding with prompt learning.

Music & text generation both follow a similar probability model. As a result, successful RNN & transformer models for text generation can be easily transferred to music generation tasks. Earlier models of music generation, e.g. MidiNet [99] & Performance-RNN, directly applied RNN-family architectures (LSTM, GRU) to melody generation task.

- * 7.1.2. **Representation Design of Music Data.** While text & music generation share a time sequential model, music generation is distinguished by 2 characteristics: (1) spatiality of musical content; & (2) hierarchy of musical conditions. Musical note is fundamental element of music. As illustrated in Fig. 7.1: Music score of Mozart Sonata K331, 1st variation (top), & text data from internet (bottom). Black part is given context & red one is text generation., music sequences contain information beyond temporal dimension – vertical or spatial relationship of notes. Each time step in a text sequence contains a single word or letter; in a music sequence, each time step contain multiple notes. Whereas piano piece depicted in figure is a relatively straightforward form of music composition, multi-track compositions (e.g., symphony, concerto, polyphony) contain more vertical relationships between notes & even different instruments. This data violates 1 of fundamental assumptions of RNN & transformer architectures (based on positional encoding) – sequence naturally contains sequential temporal information for each step of input. Simply encoding music data results in many steps of input containing same temporal information. As a result, architecture introduces erroneous conditional dependencies on various music notes mathematically.

2nd, as illustrated in Fig. 7.2: Music structure analysis of a jazz composition – Autumn Leaves, by BILL EVANS., music has a more complex structure than text. A piano piece is composed of paragraphs, phrases, sub-phrases, harmonic functions, melodies, & textures. It becomes even more complex when expressiveness structure is added (volume changes, rhythmic changes). While such a structure may exist in a typical novel’s text generation (prologue, scenes, characters, paragraphs, & content), frequently absent from a large number of text generation samples. More importantly, music’s structure is dynamic as well as hierarchical in nature. As illustrated in Fig. 7.2, melody & texture change with each beat; harmonic functions change with certain intervals; phrases & sub-phrases undergo even greater changes. A significant flaw in current music generation model is absence of long-term structure in generated samples (e.g., repetition & variation of musical motives). In following secs, discuss some of designs of music data representations & discuss how they affect performance of music generation models.

- 7.1.2.1. **3-state Melody Tokens.** Basic representation of music data is shown in Fig. 7.3: 3-state melody tokens of music data. It encodes music into 3 types of tokens:

1. Pitch, musical pitch of note, range from [A0, C8].
2. Hold, duration of note, to maintain last pitch tone.
3. Rest, rest note.

MidiNet [100], Performance-RNN [101], & Folk-RNN generated music directly from tokenized representation. Consequently, MusicVAE [102], Music InpaintNet [103], EC2-VAE [104], & Music SketchNet [105] combined it with variational autoencoder (VAE) to generate latent variables to handle a piece of music as basic generation unit. This is an intuitive representation of music that retains essential musical information. Model’s inputs & outputs are directly representation itself, eliminating need for additional structures to decode. Disadvantage of this representation is also self-evident: it cannot describe polyphonic music (i.e., multiple notes played simultaneously), but only monophonic melodies, because its temporal information has already been bound to sequence’s position, as is case with text generation.

- 7.1.2.2. **Simultaneous Note Group.** To encode polyphonic music as input, must consider overlapping notes structure. To address this issue, Pianotree-VAE [106] proposes a vertical representation – simultaneous note group. As illustrated in Fig. 7.4: Simultaneous note group encoding of music data, by Pianotree-VAE [106], it groups simultaneous notes by grouping multiple note events that share same onset, where each note has several attributes e.g. pitch & duration. Model’s input is further encoded using RNN with simultaneous group as fundamental unit, which is further transformed into hidden states by GRU in Pitch-Duration Encoder; in decoder, model is expanded with another GRU layer for predicting pitches of each group & an final fully connected layer for predicting duration of each note. This results in final latent z vector representation, that captures structure of music for every 2 bars without overlap. Paper compares midi-event tokens, simultaneous note structure & latent z for a downstream music generative applications, showing: simultaneous notes & latent vector representation are superior to midi-event encoding.

- 7.1.2.3. **MIDI Message.** Although above 3 representations have their own ways to process music data, they miss a crucial part of encoding other music information besides notes. MIDI message encoding, from Music Transformer [107], breaks this limitation by introducing more music content information to music generation task. As shown in Fig. 7.5: MIDI message encoding of music data, MIDI message is a 4-state token:

1. NOTE-ON (128 pitch classes): starting signal of 1 note.
2. NOTE-OFF (128 pitch classes): ending signal of 1 note.
3. TIME-SHIFT (32 interval classes): duration mark to move timeline.
4. VEL (128 volume classes): velocity changes.

In REMI [108], MIDI message is expanded into 6 states, including note onset, tempo, bar line, phrase, time, & volume. With this representation, model is able to acquire additional musical data & discover new patterns, resulting better well-structure & varied musical composition.

- 7.1.2.4. **Music Compound Word.** Music Compound Word [109] proposes a new structural design for MIDI messages, complete with an improved input & output network module. As illustrated in Fig. 7.6: Compound word encoding of music data, cf. MIDI-message encoding., state of music input in a compound word transformer is comparable to that of a MIDI message, but input & output mechanisms are entirely different.

Music compound word can be adapted into different scenarios of music generation by changing types of states along with decoding layers. Fig. 7.7: Compound word encoding of 3 music generation scenarios. shows 3 types of music compound words for

1. music generation with performance labels
2. multi-track music generations with multiple instruments
3. music generation with structural labels

Besides that, because music compound word contains position state (e.g., beat, bar, etc.), transformer architecture can be used without any positional encoding. This begins to differentiate music generation task from text generation task when same architecture is used.

7.1.2.5. Discussion: Signal Embedding vs. Symbolic Embedding. In previous chap, saw an embedding of an audio signal that can be used to obtain salient audio components, e.g. audio basis, compression, or noise removal. Used analogy between PCA & linear AE to get an intuition that embedding is a subspace where meaningful part of signal resides. In particular, showed for sinusoid+noise example: this embedding is very similar to signal transformation using Fourier methods. Projection to a lower dimensional space of sinusoids not only allows discarding noise, but can also be used as a representation of “meaningful” part of signal. Representation is latent space found by AE & representation found by Fourier transform share similar basic vectors.

But can such embedding also be applied to MIDI? Autoencoding of categorical variables is a poorly defined problem – after all, neural network embeddings assume: similar concepts are placed nearby in embedding space. These proximities can be used to understand music concepts based on cluster categories. Then, they can also serve as an input to a ML model for a supervised task & for visualization of concepts & relations between categories.

Natural to represent categorical variables as 1-hot vectors. Operation of 1-hot encoding categorical variables is actually a simple embedding where each category is mapped to a different vector. This process takes discrete entities & maps each observation to a vectors of 0s & a single 1 signaling specific category. 1-hot encoding technique has 2 main drawbacks:

1. For high-cardinality variables – those with many unique categories – dimensionality of transformed vector becomes unmanageable.
2. Mapping is completely uninformed: “similar” categories are not placed closer to each other in embedding space.

So embeddings could be obtained using either a manually designed representation, or could try to obtain it from data. This type of contextual embedding is what commonly refer to as “representation learning”. In case of temporal sequences, might want to obtain a representation of a complete sequence. In many musical applications, RNN is used to pre-process sequences before representation learning happens.

◦ 7.2. Recurrent Neural Network for Music.

- * 7.2.1. Sequence Modeling with RNNs. RNNs are a way to model time sequences, an early neural network application. They are used to represent a function where output depends on input as well as previous output. I.e., model must have memory for what has happened before, maintaining some representation of interval state.

Can imagine RNN to be learning probability of a sequence of tokens, $P(w_1, \dots, w_n)$, from a collection of example sequences used in training.

Once this language model is learned, can solve problem of predicting next token of a sequence (i.e., computing $P(w_n|w_1, \dots, w_{n-1})$). This can be framed as a supervised learning problem, in which sequence (w_1, \dots, w_{n-1}) acts as input which corresponds to output w_n .

Essentially, an RNN is 2 neural networks: one which acts on input, & one which acts on hidden state.

$$h_t = f(h_{t-1}, x_t) \text{ (general form),} \quad (71)$$

$$h_t = \tanh(W_{hh}h_{t-1} + W_{hx}x_t) \text{ (NN with activation),} \quad (72)$$

$$o_t = W_{hy}h_t \text{ (output)} \quad (73)$$

In these equations, h represents hidden state & x represents input.

Example 7 (Hello World from an RNN). *To introduce RNN architecture, build an example which learns to say “Hello World”. More accurately, this model will learn to output a sequence of letters that match sequences it has been trained on (in this case, “Hello World”). Given a starting letter (or set of letters), model generates next character of output. Continue building this example through sec as cover foundations of RNNs.*

1st, how shall we represent input to this network? Because mathematical operations in neural network must operate on numbers, must covert our letter data (“h”, “e”, “l”, “l”, “o”) into numerical data.

For simplicity, work in a hypothetical world with a reduced alphabet: E-D-H-L-O-R-W.

1 scheme for encoding letter “h” would be to replace letter with its positional index in alphabet (2, in case of our reduced alphabet [E using 0-indexing]). However, this introduces an issue as value of different numbers will carry different apparent weights; mathematically, recognize $5 > 2$, but would not say: letter R has a different implicit significance than letter H.

Solution for this is to use previously introduced 1-hot encoding. In this schema, letters are treated as categorical (rather than quantitative) variables, so any 1 letter is replaced by a 7-vector of 0s – with exception of index associated with letter we represent, where we find a one. In this way, every letter has an equal magnitude in representation, while information distinguishing between letters is preserved.

Important: this method fails for large (or expanding) vocabulary sizes. To overcome these particular challenges (or simply to reduce input dimensionality), technique of embedding can be used, a concept recurring through our discussion in prior E following chaps.

But even though we've learned to encode single letters to vectors, input is still not ready for RNN. RNNs are meant to operate on sequences rather than singular input. So, input length may actually vary depending on use cases. In Hello World example, use an input length of 3 (so a 3×7 matrix when considering encoding for each sequence element), but in other problems this length can grow to 1000s of values. Explore challenges associated with large input sequence lengths in this chap.

What about output? Like other classification problems, our network will output a fixed-length probability vector, representing probability that next token is a particular word from a predefined vocabulary set. Think about how dimensionality of this vector may change between domains; for forming English words, have 26 letters, but for forming English language, there are > 171000 words in dictionary alone (excluding all sorts of proper nouns). For Western music, have 12 chroma, but for a single instrument, vocabulary may span entire playable range, & for some instruments (e.g. piano), our vocabulary grows exponentially with number of possible simultaneous notes possible in a chord.

Start RNN example by preparing 1-hot encoding scheme. 1st, import necessary packages:

```
import numpy as np
import tensorflow as tf
from tensorflow import keras
from tensorflow.keras import layers
import tensorflow.keras.backend as K
```

Define our letter encoding:

```
def letter_to_encoding(letter):
    letters = ['e', 'd', 'h', 'l', 'o', 'r', 'w']
    vec = np.zeros((7), dtype="float32")
    vec[letters.index(letter)] = 1
    return vec
```

Can see effects of this encoding with this sample:

```
for letter in "hello":
    print(letter_to_encoding(letter))
```

which gives output

```
[0. 0. 1. 0. 0. 0. 0.]
[1. 0. 0. 0. 0. 0. 0.]
[0. 0. 0. 1. 0. 0. 0.]
[0. 0. 0. 1. 0. 0. 0.]
[0. 0. 0. 0. 1. 0. 0.]
```

Remark 5 (NQBH). If use `import keras.backend as K` then get error

```
Traceback (most recent call last):
```

```
File "/home/nqbh/advanced_STEM_beyond/machine_learning/Python/hello_world.py", line 53, in <module>
    print_output = K.eval(out)
```

```
~~~~~
```

```
AttributeError: module 'keras.backend' has no attribute 'eval'
```

then change it to `import tensorflow.keras.backend as K`. Link: <https://stackoverflow.com/questions/58047454/how-to-fix-module-keras-backend-tensorflow-backend-has-no-attribute-is-tf>.

For our example, use a single-layer RNN. Network will take an input with shape $3 \times 7 \times N$, where 3 represents input sequence length, 7 represents size of data vector associated with each input, & N : number of samples in a training batch. RNN layer is a Keras SimpleRNN; each of 3 nodes of this RNN will create a 7-dimensional output (again used to represent likelihood associated with a single character in our alphabet). Though each length-3 input will map to a length-3 output, take only 3rd (final) output to represent predicted “next character” of our phrase. Had we left off `return_sequences` parameter, network would output only this final character (but for sake of example, it will be interesting to see entire predicted sequence). To normalize raw output into a probability-like representation, apply a softmax activation function prior to output.

Model definition:

```
model = keras.Sequential()
model.add(keras.Input(shape=(3,7)))
model.add(layers.SimpleRNN(7, activation="softmax", return_sequences=True))
model.summary()
```

Exercises at end of chap asks you to think about why this network has 105 trainable parameters. As a hint, recall: input to RNN node is made up of input vector, plus hidden state, plus a bias term, & these inputs are fully connected to generated output.

In below code, create our training data. In our hypothetical world, only phrase that exists is “helloworld”, so train network on fragments of this phrase. For those following along by running code, can see in below example: for each 3-letter portion, output matches next predicted letter following each letter of input.

```
train_text = "helloworld"*30
def generate_train_set(train_text, as_words=False):
    x_train = []
    y_train = []
    for i in range(len(train_text) - 4):
        if as_words:
            x_train += [[train_text[i:i+3]]]
            y_train += [[train_text[i+1:i+4]]]
        else:
            x_train += [[letter_to_encoding(letter) for letter in train_text[i:i+3]]]
            y_train += [[letter_to_encoding(letter) for letter in train_text[i+1:i+4]]]
    if as_words:
        print(x_train[0][:5])
        print(y_train[0][:5])
    else:
        print(np.array(x_train)[0,:5])
        print(np.array(x_train).shape)
    return np.array(x_train), np.array(y_train)

generate_train_set(train_text, True)
x_train, y_train = generate_train_set(train_text)
```

So what happens when we generate an output phrase using untrained network?

```
# What happens when generate an output phrase using untrained network?
letters = ['e','d','h','l','o','r','w']

seed = "hel"
result = "hel"
input_data = np.array([[letter_to_encoding(letter) for letter in seed]])
model.get_weights()

for i in range(7):
    out = model(input_data)
    print_output = K.eval(out)
    for row in print_output[0]:
        next_letter = letters[np.argmax(row)]
    result += next_letter
    print(result)
    input_data = np.array([[letter_to_encoding(letter) for letter in result[-3:]])
```

After running above code, reach sth like this:

```
hele
helee
heleee
heleeee
heleeeee
heleeeeee
heleeeeeee
```

Model weights are not yet optimized; train with following lines:

```
model.compile(optimizer=keras.optimizers.Adam(learning_rate=0.01), loss=tf.keras.losses.CategoricalCr
model.fit(x_train, y_train, batch_size=24, epochs=300, verbose=0)
```

After training, try again to produce output:

```
input_data = np.array([[letter_to_encoding(letter) for letter in seed]])
model.get_weights()
result = "hel"
for i in range(7):
    out = model(input_data)
    print_output = K.eval(out)
```

```

for row in print_output[0]:
    next_letter = letters[np.argmax(row)]
    result += next_letter
    print(result)
    input_data = np.array([[letter_to_encoding(letter) for letter in result[-3:]])

```

Give following output from our simple RNN:

```

hell
hello
hellow
hellowo
hellowor
helloworld
helloworld

```

Is a recurrent neural model of music appropriate for generating music that “sounds like” other music by a particular composer or within a genre? An ideal statistical model would learn from all possible sequence inputs to create an expected musical output, but size of possible inputs is v^n , where v : vocabulary size & n : input sequence length. Even considering just 88 piano keys & 10 notes in a sequence creates 2.785^{19} combinations!

To overcome this, think back to Markov assumption introduced in Chap. 2. If approximate: a prediction depends only on a small number of preceding tokens, problem becomes much more tractable to model. But, in doing so, lose ability to model long-range effects or earlier tokens.

So, how do neural models maintain long-range effects?

A 1st attempt to solve this problem could consider combining sequence of input tokens into 1 mega-token, for which a corresponding prediction can be made. A neural network can be applied over this input matrix, with dimensions of vocabulary size or embedding vector length by number of elements in input sequence. Note: have a built-in limitation with such a model: tokens further away than number of elements in input sequence cannot influence model output.

Pros of RNNs. Recurrent neural networks overcome this constraint; i.e., recurrent neural networks can learn extensive long-range influences despite limited-length input sequences. It does so by encoding sequential history into a hidden state, which will refer to at time t by h_t in this discussion (can imagine “h” to stand for “history”). This state h_t is updated with input x_t , then used in predicting output y_t . Encasing this history state in a black box, RNN appears to receive a sequence of vectors x_0, x_1, \dots, x_T as input, & produces a sequence of vectors o_0, o_1, \dots, o_T as output. Internally, can imagine process to begin with computation of history state:

$$h_t = F(h_{t-1}, x_t), \quad (74)$$

then prediction of output using updated history:

$$o_t = G(h_t). \quad (75)$$

Relationships between inputs, multi-dimensional hidden history state, & outputs can be learned with neural network training procedures, as studied in previous chaps. These relationships are represented by functions F, G , which may be composed as:

$$F(h, x) = A_1(W_{hh} * h + W_{xh} * x), \quad (76)$$

$$G(h) = A_2(W_{ho} * h). \quad (77)$$

In this notation, A_i stands for an **activation function**, & W_{IO} stands for a matrix of learnable **weights** which map from dimensionality of matrix’s input vector class to matrix’s output vector class. W_{xh} describes influence of input on next state, W_{hh} describes influence of current state on next state, & W_{ho} describes influence of next state on next output. An illustration of relationship between input, hidden state, & output are shown in Fig. 7.8: In its compact form (left), & RNN consists of a learned hidden state h , which is driven by both current input & past values, & itself drives output o , as described in RNN equations. To make explicit development of state h over time & its relationship to input & output, can imagine RNN in its “unfolded” form (right).

If ignore intermediate outputs o_1, \dots, o_{N-1} , this is really just a feedforward network, with initial inputs h_0 & x_1 , & with additional inputs concatenated to outputs of particular intermediate layers. However, in a traditional feedforward network, weights between layers are uniquely learned. In case of an RNN, these weights are *shared*, i.e., same weights will act on input x_t regardless of time, & same weights will act on 1 state h_t to create next state (or to create output) regardless of time.

How do we learn in this shared-weight scenario? In previous backpropagation example, derivative of loss w.r.t. a particular weight could be calculated, & weights adjusted accordingly. Now, for same weight, have multiple possible derivatives since there are multiple inputs influenced by that parameter!

Without diving too deep into theory, consider an approach by which we take average of these gradient contributions. I.e., for any of N instances of w_i in network, which can call w_{i_n} , can compute $\frac{\partial L}{\partial w_{i_n}}$. Then, when deciding how to modify

shared parameter w_i , use

$$\frac{\partial L}{\partial w_i} = \frac{1}{N} \sum_{n=1}^N \frac{\partial L}{\partial w_{i_n}}. \quad (78)$$

In principle, average should move us closer toward an optimal solution since largest amount of “mistake” explained by weight applied at a particular layer will have largest amount of influence on this averaged gradient component.

– Về nguyên tắc, giá trị trung bình sẽ đưa chúng ta đến gần hơn với giải pháp tối ưu vì lượng “sai sót” lớn nhất được giải thích bằng trọng lượng áp dụng cho một lớp cụ thể sẽ có ảnh hưởng lớn nhất đến thành phần gradient trung bình này.

There is 1 critical issue with this approach (& in fact, with any networks sufficiently deep), referred to as *vanishing gradient problem*, as discussed in Chap. 6. This problem is particularly notable for modeling sequential data since it effectively limits how far back in time a model can learn to “reach” to influence its output; while RNN is capable of representing long sequences of data, it loses ability to learn at length.

– Có 1 vấn đề quan trọng với cách tiếp cận này (thực tế là với bất kỳ mạng nào đủ sâu), được gọi là *sự cố gradient biến mất*, như đã thảo luận trong Chương 6. Sự cố này đặc biệt đáng chú ý đối với việc mô hình hóa dữ liệu tuần tự vì nó hạn chế hiệu quả khoảng thời gian mà một mô hình có thể học để “tiếp cận” nhằm tác động đến đầu ra của nó; trong khi RNN có khả năng biểu diễn các chuỗi dữ liệu dài, thì nó lại mất khả năng học theo chiều dài.

Methods to mitigate problem of long-range influence have been developed, including Gated Recurrent Unit (GRU) & Long Short-Term Memory (LSTM) network.

Example 8 (Gated Architectures: Long Short-Term Memory Network & Gated Recurrent Unit). *A Long Short-Term Memory network, or LSTM, is an RNN architecture designed to deal with vanishing gradient problem that plagues traditional RNNs, causing difficulty in learning long-term dependencies commonly found in music & audio.*

LSTM architecture introduces a new type of unit called memory cell that can store information for long periods of time. This represents “long” component of LSTM. Memory cell is controlled by 3 gates:

1. *input gate,*
2. *forget gate,*
3. *output gate.*

Each gate is typically activated using a sigmoid function which determines how much of input should be let through to memory cell, how much of memory cell’s contents should be forgotten, & how much of memory cell’s contents should be output.

“Short-term” component of LSTM is accounted for by addition of a hidden state at each time t , similar to an RNN. Hidden state is updated based on current input, previous hidden state, & (a key distinction) memory cell output. This state can help to carry time forward from each step of input sequence as well as call upon long-term information.

All 3 gates of memory cell receive current input & previous hidden state to inform their evaluation. By using memory cells & gates to control flow of information, LSTMs are better able to learn long-term dependencies in input sequence.

Like classic RNN & LSTM, a Gated Recurrent Unit architecture, or GRU, processes input 1 sequence element at a time & maintains a hidden state updated by a combination of both current input & previous hidden state. What differentiates GRUs is addition of new types of gating mechanisms to control flow of information through network. GRUs use 2 types of gates:

1. *an update gate, to decide how much information from previous state should be retained & incorporated into new hidden state, &*
2. *a reset gate, to decide how much of previous hidden state should be discarded.*

Benefit of GRU is its simpler architecture, with fewer parameters than a comparable LSTM. This helps to avoid overfitting on small datasets, a common problem to overparameterized models. Both LSTMs & GRUs offer a gated mechanism for modeling long-term dependencies, & choice between architectures ultimately falls to considerations related to task at hand, e.g. dataset size, expected time dependencies, & computational constraints.

- * **7.2.2. Sequence-to-Sequence Modeling with RNNs.** Sometimes seek not to learn only an estimated distribution for a sequential pattern, but a mapping from 1 sequence to another. E.g., pretend you are a jazz saxophonist, & you attend a friend’s classical piano recital (buổi độc tấu piano cổ điển). A striking melody from 1 of their pieces catches your attention, & you’d like to improvise on it when you practice later that day. In this case, you are taking an input sequence of melody in a classical context, & translating it to a jazz melody (which may have different styles of ornamentation). Such a problem is solved using sequence-to-sequence (often abbreviated seq2seq) modeling.

Formally, can describe such problems this way: Given vocabularies V_z, V_x & training data pairs (z_n, x_n) independently & identically distributed from some distribution p , can we estimate distribution $p(x|z)$?

Framing above problem in this notation, V_z are possible notes from piano, & V_x are possible notes from saxophone. Can collect samples of same melody played once by classical pianist z_n & again by jazz saxophonist x_n , & use these to learn a distribution of what we expect jazz saxophone to sound like x for a given piano melody z . There are numerous other problems we can frame as **seq2seq**, including problem of transcription (given an audio sequence, returning same represented music in symbolic notation).

- * **7.2.3. Performance RNN.** Now, extend our RNN exploration to a musical example. Performance RNN [101] is a network which generates performance-quality MIDI piano output¹⁵ In fact, Performance RNN was project which introduced MIDI-

¹⁵Performance RNN is a product of Google Magenta, introduced in Appendix A. Can learn more about Performance RNN on authors’ blog <https://magenta.tensorflow.org/performance-rnn>, & run code yourself with instructions available in their GitHub Repository https://github.com/magenta/magenta/tree/main/magenta/models/performance_rnn.

Like tokenization scheme introduced in previous sect. To understand Performance RNN, examine its components:

- Data: Performance RNN is trained on Yamaha e-Piano Competition dataset (introduced in Appendix D). This is a collection of ≈ 1400 MIDI files, generated by recording keypresses of highly skilled classical pianists. Accordingly, music is expressive in both timing (rubato) & velocity (dynamics).
- Tokenization: MIDI-Like tokenization is employed. At each step, input to RNN is a single 1-hot 413-dimensional vector representing 413 possible note-on, note-off, velocity, & time-shift events.
- Model Architecture: 3 hidden layers of LSTMs with 512 cells.
- Training & Loss: Model is trained using RNN technique of *teacher forcing*.

Example 9. Teacher forcing helps models reach stable convergence during training by providing correct (rather than predicted) output as input to each sequential step. E.g., imagine a model meant to learn a C-major scale (C-D-E-F-G-A-B-C). In a non-forced training, if model is given input C to 1st cell, & incorrectly predicts an output of E, E would propagate as input to 2nd sequential is forced to D to match ground truth. While this is beneficial to efficiently learn given pattern, it should be noted: this reduces model’s ability to learn from variations on pattern which may not be represented in training data (but may exist otherwise). For generative models which seek to perform & innovate (rather than replicate), this could be detrimental in some cases.

Log loss (i.e. categorical cross-entropy) is loss function used to drive training backpropagation.

- Output Generation: beam search.

Example 10 (Beam Search). Beam search is a heuristic search algorithm used to find most likely sequence of output given an input sequence. It is a variant of breadth-1st search algorithm (thuật toán tìm kiếm theo chiều rộng), which attempts to balance exploration & exploitation, a problem common to search algorithms as well as AI subfield of reinforcement learning.¹⁶

Basic idea behind beam search: generate multiple possible outputs at each step, but keep only top-K most probable candidates. This value K is referred to as beam width. Algorithm then proceeds to next step, generating a new set of candidate outputs for each of previous K candidates, continuing until a stopping criterion is met.

In theory, beam search algorithm eliminates low-probability candidates early on to speed up & improve quality of output sequence search. However, beam search can also suffer from problem of getting stuck in local optima, where most likely candidates at each step do not necessarily lead to overall most likely output sequence or reflect desired long-term behavior. E.g., consider a diatonic melody (giai điệu diatonic) which starts on C, & will choose next note from C major scale. Predictive model may output probabilities for each of these notes: (C, 0.1), (D, 0.1), (E, 0.1), (F, 0.2), (G, 0.3), (A, 0.15), (B, 0.05). If use a beam width of 3, select top 3 most likely possible sequences, CF, CG, CA, & associate these with respective probabilities 0.2, 0.3, 0.15.

Now, next 3 sets of next sequence options are generated, conditioned on each of these selections; this may be sth like

1. CF: (C, 0.05), (D, 0.025), (E, 0.25), (F, 0.3), (G, 0.3), (A, 0.05), (B, 0.025)
2. CG: (C, 0.3), (D, 0.025), (E, 0.25), (F, 0.05), (G, 0.3), (A, 0.05), (B, 0.025)
3. CA: (C, 0.2), (D, 0.025), (E, 0.2), (F, 0.3), (G, 0.2), (A, 0.05), (B, 0.025)

Score for each of resulting possibilities is probability of prior sequence times probability of next token. E.g., considering 1st row above, would have as possibilities (CFC, 0.01), (CFD, 0.005), (CFE, 0.05), (CFF, 0.06), (CFG, 0.06), (CFA, 0.01), (CFB, 0.0025). Repeat this $\forall 3$ beams, & from resulting scores, select again top 3 candidate sequences – these are only sequences which will continue propagating forward. Once desired sequence length is reached, beam search is concluded & available sequence assigned most probability can be selected.

- * 7.2.4. SampleRNN. In introduction of *SampleRNN: An Unconditioned End-to-End Neural Audio Generation Model* [64], Mehri et al. highlight primary challenges associated with audio generation:

1. A large discrepancy (sự khác biệt, sự tương phản) between dimensionality of audio signal & semantically meaningful unit (e.g., 1 word is, on average, around 6000 samples at a 16 kHz sample rate).
2. Structure occurs at various scales (correlations between neighboring samples as well as samples thousands of units apart).
3. Compression of raw audio to meaningful spectral (or other handcrafted) features.
4. Necessary correction of audio after decompression (e.g., consider complexity of Griffin-Lim for reconstructing phase).

SampleRNN seeks to reduce this complicated, engineering-heavy pipeline by allowing a neural network to learn these compression & reconstruction processes as well as relationships between many samples comprising meaningful audio signals.

SampleRNN approaches this task using a hierarchical framework, where lower-level modules operate on individual samples, but higher-level modules operate at a larger timescale & lower temporal resolution. This is 1 example approach toward learning audio (or musical) trends which operate at different scales to overcome limitations fo a rudimentary sample-by-sample RNN framework.

How are model outputs of SampleRNN evaluated (&, in general, how can we evaluate performance of generative audio models)? In this case, **AB Testing** with human rates is employed. Pairwise combinations of SampleRNN with other comparative models (including WaveNet) are presented to human raters, who then express preference for “A” over “B”

¹⁶While reinforcement learning is outside scope of this book, readers are encouraged to consider: generation in a creative or imaginative sense requires this same consideration of exploring new ideas vs. utilizing known patterns.

(or vice versa). Then, one can compute a number of statistics, most common being percentage of times that samples from model in question were preferred over alternative. Subjectivity of evaluating artistic or preference-based generative models creates difficulties in presenting conclusive metrics, but AB preference testing is 1 such approach that is generally accepted & useful.

- 7.3. Convolutional Neural Networks for Music & Audio. In this sec, introduce Convolutional Neural Networks (CNN) for 2D representation of music & audio. CNNs are popular for image processing & use an implementation of a local filtering approach over neighboring pixels. Idea of CNN, in a way similar to adaptive filtering approaches encountered in DSP part of book: data is used to estimate or train filter parameters to perform some desired task. Thinking about CNN as a filter, must note:

- * CNNs implement multiple local filters,
- * They are often trained for feature extraction & inference, i.e. some classification tasks & not as an audio filter,
- * Local properties (finite impulse response) of CNNs are often implemented in 2D, which in our case will span both time & frequency, or time & voices in spectral or piano-roll representations, resp.

One must note: 1 of big advantages of CNNs in visual domain: many of features in images are local & translation invariant. I.e., want to detect lines, corners, or part of faces & so on, regardless of exact position where these visual elements appear in picture. In music, local relations might exist between notes or frequency elements in time, & also across frequencies or notes, e.g. in case of harmonic or polyphonic (multi-voice) relations. Big difficulty: such relations are usually much longer or distant in time than what a normal visual patch would be. Talk about capturing long term relation when discuss attention in RNN & transformers in a later chap. Regardless of limitations, CNNs have been found to be very useful tools for analysis of time-frequency structures, i.e., representing voices in time.

- * 7.3.1. What is a CNN? CNNs are defined by presence of *convolutional* layers. Many signal processing textbooks explain mathematical details of convolution, & here present material in a conceptual manner. Convolution is result of iterating through & operating upon portions of a signal using another signal (referred to as a convolutional filter or kernel); nature of this interaction is typically a sum of elementwise multiplications. An example is shown in Fig. 7.9: 2D convolution operation. A kernel (middle) is applied to input image (left) in a dot-product operation where each grid cell of kernel is multiplied by its corresponding cell in input image, then added to a running sum for a given kernel position. This sum becomes output value in feature map (right). Kernel then slides across image to next position, until every position of image has been covered. During convolution, filter is applied to every sub-region of signal that it passes over, generating a scalar value for each position it assumes. Pseudo-image created from mosaic of these scalar values is called a *feature map*. When defining convolution operation, possible to vary spatial pacing of filter as it passes through image (referred to as *stride*) as well as extent to which filter is used at edges of signal (commonly addressed via *padding*). Stride, filter size, & number of filters are sufficient to define a convolutional layer within a CNN architecture.

These filters become a powerful tool for extracting meaningful information from a signal. Readers who have studied signal processing are likely familiar with a basic ensemble of filters (low-pass, high-pass, bandpass, etc.) which allow certain frequency components to be extracted from a composite signal. When it comes to 2D convolution, filters can be used for many informational & transformational purposes; as 2 examples of many possibilities, some filters can extract meaningful qualities e.g. edges (location in image with intense transitions from dark to light or vice versa), & others can transform an image by taking local averages (referred to as image smoothing or blurring).

In above cases, image filter is strictly defined, then applied to image. E.g., a Laplacian operator, which acts as a great detector of strong intensity changes by approximating a local 2nd derivative, always features a strong, positive value at central location, surrounded by a ring of weaker, negative values. Other times, it may be useful to use filters which adapt to problem you intend to solve. Such is case with object template matching, where a filter is designed by selecting values that exhibit a strong value when convolved with template object, & weak values elsewhere. Will reserve details about implementation & theory of such filters for a digital image processing textbook, but driving principle should be remembered: filters can be tuned to highlight particular types of information from an image. Selecting or handcrafting these filters often leads to 2 situations:

- Selected theoretical filter is so general that it can only find particular low-level features (e.g., local intensity changes in a particular direction).
- Handcrafted filter is so specific that it can only find exact matches to particular high-level entities (e.g., a very specific pattern in a spectrogram from a violinist playing a phrase, which would not have same strength of response if a different violinist played same phrase).

This is where *convolutional neural networks* come into play. Instead of selecting or crafting convolutional filters, each filter's parameters are *learned* during training. Each convolutional layer consists of a bank of n filters, defined by their height & width. These filters are eventually optimized for task at hand through usual methods of backpropagation & gradient descent. Convolutional layers are stacked (creating *deep* neural networks), & through this stacking mechanism, features are extracted & combined at low, mid, & high levels of abstraction (e.g., edges combine to become geometric shapes, which combine to become recognizable object patterns).

- 7.3.1.1. Pooling. Consider effect of a bank of $10 \times 3 \times 3$ filters acting on a 20×20 pixel single-channel image. This layer would require 90 parameters. After this convolutional layer (& assuming appropriate padding), output feature map would maintain a spatial dimension of 20×20 , but with a depth of 10 features. If this convolutional layer is comprised of an additional 10 filters, each filter would now need $3 \times 3 \times 10$ parameters, requiring 900 parameters for layer.

Following this trend, can see: requisite parameters quickly explodes deeper data propagates through network. As a means to reduce this issue, can use technique of *pooling*. Pooling provides a means of reducing spatial dimensionality of a feature map, operating on assumption: reduced feature map will still exhibit useful, informative patterns available at full scale.

In a pooling scheme, a square region of an image is replaced by a single value intended to represent region. Example selected values may be region's maximum (*maxpooling*) or average (*averagepooling*).

As convolutional & pooling layers progress (with padding reduced), eventually reach a spatial dimension of 1 with a deep set of features; from this point, network is typically concluded with a series of fully connected layers.

- * 7.3.2. CNN Audio Example: WavenNet. Slightly contrary to above discussion, begin examples of CNNs with WaveNet, a model which actually does not act on image-like 2D representations (piano roll, spectrogram, etc.); WaveNet is a deep generative model used to process raw audio waveforms.

WaveNet has been shown to generate speech mimicking human voice¹⁷ & synthesize musical audio signals. WaveNet directly models raw audio waveform sample-by-sample, in direct contrast to historical methods of speech synthesis where phonemes (âm vị) from a bank can be concatenated to create words, or vocoders can produce synthetic sound fragments. This allows for improved realism & flexibility in model, but why has this approach been a long-standing challenge?

Large issue at play: rate of prediction necessary to support sample-level modeling; while phonemes may change on scale of milliseconds, audio files change value at a rate of 16000 samples (or more!) per sec. To make matters worse, have seen from our earlier discussion on deep neural networks that long-term patterns become increasingly hard to model further back they exist in time (relative to current predictive position); when we are moving at 16000 samples per second, there is a huge gap between our current position & information we may need to effectively model sound!

Researchers at Google DeepMind had a solid basis for taking leap of faith to model massive sequential lengths; previously, research team developed PixelCNN & PixelRNN. These neural networks acted on images, but there are themes drawn from this work that translate to audio domain:

- Recurrent Layers, as studied in previous sect, parameterize series of conditional distributions.
- Residual Connections are used to share information between different network depths.
- Generated samples are crisp, varied, & globally coherent, 3 properties which are of equal concern when generating audio.

After seeing promising results that an image could be generated pixel-by-pixel (i.e., row-wise sequentially, as opposed to a more expensive unsampling method), similar techniques were applied to audio data, which is naturally sequential.

Dilation (Sự giãn nở) of WaveNet architecture varies at each layer; this allows WaveNet to maintain an enormous *receptive field*, referring to range of time from original signal subtended by network to generate its next output. This WaveNet architecture has become a standard tool to other works in audio processing. Considering original goals of authors, it proved effective in modeling realistic voices for text-to-speech translation, & even showed promise in learning patterns across different speaking voices. Authors additionally trained WaveNet on classical piano recordings, & results are unique piano recordings which sound realistic, with minute amounts of noise seemingly an artifact of a phantom recording process rather than spurious musical decisions.

– Sự giãn nở của kiến trúc WaveNet thay đổi ở mỗi lớp; điều này cho phép WaveNet duy trì một *trường tiếp nhận* khổng lồ, đề cập đến phạm vi thời gian từ tín hiệu gốc được mạng lưới bao bọc để tạo ra đầu ra tiếp theo của nó. Kiến trúc WaveNet này đã trở thành một công cụ tiêu chuẩn cho các tác phẩm khác trong xử lý âm thanh. Xem xét các mục tiêu ban đầu của các tác giả, nó đã chứng minh được hiệu quả trong việc mô hình hóa giọng nói thực tế để dịch văn bản thành giọng nói, & thậm chí còn cho thấy triển vọng trong các mẫu học tập trên các giọng nói khác nhau. Các tác giả cũng đã đào tạo WaveNet về các bản ghi âm piano cổ điển, & kết quả là các bản ghi âm piano độc đáo nghe có vẻ thực tế, với lượng tiếng ồn nhỏ dường như là sản phẩm của quá trình ghi âm mà thay vì các quyết định âm nhạc giả tạo.

- * 7.3.3 CNN Audio Example: UNet for Source Separation. Earlier introduced problem of *source separation*, isolation of individual sounds (or sources) in an audio mixture¹⁸ Techniques e.g. active source estimation can help to solve for mixture of source energies between spectral frames of audio signal, & can even be extended with information from an audio-aligned written score for enhanced estimation [111]. CNNs have also become a popular approach toward this task, as their convolution & transpose (reverse) convolution operations can be used to generate masks to filter spectral representations of audio into individual sources.

Such a *mask* layer can be overlaid on original spectrogram (or mathematically speaking, element-wise multiplied) to allow only information associated with a particular source to pass through. From this masked spectrogram, sound of a single source can be approximately reconstructed. To do this, size of output of CNN must match size of input (i.e., mask must be same size as spectrogram so that it perfectly covers). This naturally requires an encoder-decoder structure to network, so that learned parameters can be structured in a way that rebuilds downsampled input to its original size. UNet architecture [112] is a popular architecture for such problems due to its symmetric U-shape, use of residual “skip connections” to pass information between spatial resolutions, & inclusion of layers of parameters at decoding side to assist in reconstruction.

2 examples (of many) which use this technique can be found in works by Geng et al. [113] & Kong et al. [114]. Geng et al. use a UNet variant called “gated nested” UNet (GNUnet), where there are nested series of layers “filling in” center of U. Original U “backbone” has gating units applied, a mechanism originally proposed for RNNs which control information flow throughout network to allow for modeling more sophisticated interactions. Outputs of GNUnet are used to create a time-frequency spectral mask to generate 2 masks: 1 for singing, 1 for accompaniment – sự đệm đàn. 2 masks can be

¹⁷ Examples are available from DeepMind at <https://www.deepmind.com/blog/wavenet-a-generative-model-for-raw-audio>.

¹⁸ A fantastic resource to learn about this problem & its applications comes from a tutorial by ETHAN MANILOW, PREM SEETHARAMAN, & JUSTIN SALAMON at 2020 International Society for Music Information Retrieval conference [110].

multiplied by magnitude & phase spectra of mixture, then transformed back into time-domain signals simultaneously, as opposed to networks which isolate & extract only 1 source. Also using a UNet architecture, Kong et al. additionally estimate complex ideal ratio masks (i.e., decoupling masks into a mask for magnitude & a mask for phase) to decrease reconstruction error. Result is a system effective at separating vocal tracks, bass, drums, & more.

- 7.4. Pretrained Audio Neural Networks.
- 7.5. Exercises.
- Chap. 8. Noise Revisited: Brains that Imagine.
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 - 8.3. Variational Methods: Autoencoder, Evidence Lower Bound.
 - 8.4. Generating Music & Sound with Variational Autoencoder.
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- Chap. 9. Paying (Musical) Attention.
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 - 10.1. Music Communication Revisited: Information Theory of VAE.
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 - 10.3. Future of AI in Music & Man-Machine Creative Interaction.
- APPENDIX A. INTRODUCTION TO NEURAL NETWORK FRAMEWORKS: KERAS, TENSORFLOW, PYTORCH. Throughout this book & accompanying exercises, have shared examples of code drawing from a variety of neural network frameworks. In this Appendix, briefly introduce these programming tools, as well as some common pitfalls & recommended practices.

3 libraries imported in this book are Keras, TensorFlow, & PyTorch.

 - Keras is an open-source neural network library written in Python. Designed to be user-friendly, flexible, & modular, which makes it easy for users to build & experiment with different types of neural networks. Keras has a high-level API that allows users to quickly build & train neural networks using pre-built layers & models. This makes it a popular choice for beginners who are just getting started with neural networks.
 - TensorFlow is another open-source ML library that was developed by Google Brain. Also written in Python & is designed for large-scale ML projects. In fact, Keras was built on top of a TensorFlow backend, allowing for abstraction of certain implementation details while still maintaining access to powerful features of TensorFlow. TensorFlow has a low-level API that allows users to build custom models & operations, as well as a high-level API that is similar to Keras. TensorFlow has become 1 of most popular DL libraries because of its scalability, flexibility, & robustness.
 - PyTorch is another popular DL library that was developed by Facebook (now Meta). Also written in Python & is known for its dynamic computation graph, which allows users to build & modify neural networks on fly. PyTorch has a user-friendly API that is similar to Keras, which makes it easy for beginners to get started. PyTorch is also known for its ease of debugging, which makes it a popular choice for researchers who are developing new DL models.

A note on libraries: As you may have seen from programming examples & notebooks, Python libraries are changing all time, & many scripts we write draw on multiple such libraries. Sometimes a change made to 1 version of a library affects others which depend on it. For this reason, important for users to be aware of which versions of a library they should install to run a program, & try to provide this information when possible.

A frequent problem: have to run many programs with conflicting dependencies; while 1 program might want to use TensorFlow version 1, another may need TensorFlow version 2. As a possible tool, virtual environments provide a way for us to manage which libraries our python interpreters “sees” when we execute program. These can be created & managed through **virtualenv** package, or alternatively through Anaconda environment manager. There are plenty of other virtual environment tools that exist, & most serve same purpose. In these setups, have our default (base) environment, & then can additionally create virtual environments. These virtual environments are blank slates until one installs desired libraries within environment. Recommended practice to make a virtual environment to run your code, then install your associated libraries to this virtual environment before running program. This will help you to avoid package conflicts between programming examples, problems, & projects.

In general, recommend readers consult documentation of any library used in case of errors introduced during implementation.

- **APPENDIX B. SUMMARY OF PROGRAMMING EXAMPLES & EXERCISES.** Accompanying this book are a set of programming examples & exercises, available at <https://github.com/deep-and-shallow/notebooks> & formatted as Jupyter notebooks. Here, provide a brief description of each notebook & a recommended chap of content for which notebook could be paired.
 - Chap. 1: **Introduction to Music Representation** lets readers convert same music through a variety of representations & visualizations (MIDI, Piano Roll, Audio) & observe strengths, weaknesses, & challenges associated with such representations & conversions.
 - Chap. 2: **Probability** invites readers to implement Mozart’s Dice Game & experiment with random number generation to create music from noise.
 - Chap. 3: **Markov Chain & LZify** guides readers in exploring Markov Models to generate pop music chord progressions, then provides an exercise in implementing & using Lempel-Ziv algorithm to parse, analyze, & generate a musical melody.
 - Chap.4: **Discrete Fourier Transform** lets readers construct & use DFT matrix to quickly compute Fourier transform of audio signals.

Spectrograms STFT & Griffin Lim will introduce readers to generation of Short-Time Fourier Transform plots, & readers can implement Griffin Lim algorithm to reconstruct phase when discarded from frequency representations of signals. Readers are invited to record their own short audio samples for these exercises.

Speech Formants & LPC lets readers play with source-filter model of human voice, investigating combinations of frequencies to create vowel sounds.

 - Chap. 5: **VMO Audio Oracle** uses VMO model to construct a recombinant model from a given saxophone target recording. Then, readers will use VMO query function to “drive” this model using another sound file. Query sound file will contain an accompaniment part of a song, so result will be a new saxophone improvisation over that song.
 - Chap. 6: **PCA with Linear Autoencoder** walks readers through training an autoencoder for purpose of comparing its output to that of PCA, using basic sinusoidal audio input as training data.
 - Chap. 7: **Generating Music with RNN** helps readers to practice with Keras to create a generative model based on music created by stochastic Mozart Dice Game.

Parallel CNN–RNN explores use of CNN-RNN for genre classification, comparing its performance to similar CNN-only for RNN-only models.

 - Chap. 8: **GAN, chroma (MIDI) & pix2pix** explores a style transfer application that tries to change musical texture of a piece while maintaining harmonic structure. In order to do so, readers train a pix2pix type of model that learns relations between chroma & musical texture (distribution of note). For this purpose readers extract chroma from MIDI data & learn a generator that complements notes from a given texture.
 - Chap. 9: **Attention & Transformers** introduces readers to effects of attention in Transformer model, visualizing effects of models with & without attention on weight applied to various aspects of input, drawing on examples from both NLP & symbolic music.- **APPENDIX C. SOFTWARE PACKAGES FOR MUSIC & AUDIO REPRESENTATION & ANALYSIS.** In this Appendix, briefly introduce some of programming frameworks that are used for loading, processing & analysis of audio & symbolic music data, often as a 1st step before proceeding with statistical modeling using deep or shallow methods. Some of these packages are used in programming exercises accompanying this book as a 1st step in preparing data for fitting into AI models.
 - **Librosa.** Librosa is a Python package for audio analysis, designed mostly for music information retrieval. It includes a range of commonly used functions, broadly falling into 4 categories: audio & time-series operations, spectrogram calculation, time & frequency conversion, & pitch operations. Audio & time-series operations include functions for reading audio from disk `load`, resampling a signal at a desired rate `resample`, stereo to mono conversion `to_mono`, time-domain bounded auto-correlation `autocorrelate`, zero-crossing detection `zero_crossings`, & estimate of dominant frequency of STFT bins `piptrack` & pitch tracking `yin`. Spectrogram operations include short-time Fourier transform `stft`, inverse STFT `istft`, & instantaneous frequency spectrogram `ifgram`, & constant-Q transform `cqt`. It provides also mapping between different time representations: seconds, frames, or samples; & frequency representations: hertz, constantQ basis index, Fourier basis index, Mel basis index, MIDI note number, or note in scientific pitch notation. Since many of spectral manipulation operations are performed on STFT magnitude, it offers phase recovery from magnitude spectrum using Griffin-Lim algorithm `griffinlim`. Beat module provides functions to estimate global tempo & positions of beat events. Higher level structural analyses are provided using recurrence or self-similarity plots `segment.recurrence_matrix`. Decompose module factors spectrograms, or general feature arrays, into components & activations. By default, this is done with nonnegative matrix factorization NMF, but any `sklearn.decomposition`-type object will work. Additionally, it provides several signal transformation & signal generation functions.
 - **Pretty Midi** provides functions for representing & handling MIDI data. At top is `PrettyMIDI` class, which contains global information e.g. tempo changes & MIDI resolution. It also contains a list of `Instrument` class instances, with each `Instrument` specified by a MIDI program number that correspond to names of General MIDI instruments convention. `Instrument` class instances contain 3 lists for `Note`, `Pitchbend`, & `ControlChange`. `Note` class is a container for MIDI notes, with velocity, pitch, & start & end time attributes. `PitchBend` & `ControlChange` classes contain attributes for bend or control change’s time & value.

Functions for performing analysis are defined in `PrettyMIDI` class & their a corresponding Instrument class. Some of implemented functions include extracting MIDI tempo change event `get_tempo_changes`, tempo estimate according to inner-onset intervals `estimate_tempo`, beats for every quarter note for 3/4 or 4/4 time signature & every 3rd denominator note for 6/8 or 6/16 time `get_beats`, list of MIDI note onsets `start_times` & downbeats `get_downbeats`. A particularly useful & intuitive representation is a piano roll `get_piano_roll` that returns a matrix representation of MIDI notes, with each column spaced apart by fixed amount of seconds as provided by user. Called “sampling frequency” this should not be confused with audio sampling, but it is rather a conversion of MIDI events from tempo dependent “ticks relative to previous event, to absolute time in seconds. Another useful aggregation is in terms of pitch classes independent of octave, also known as MIDI chroma `get_chroma`. Manipulating & synthesizing MIDI file from a piano roll matrix is not provided in original package, as this requires some fine tuning of user to determine best sampling rate & other properties that are missing in order to invert matrix representation to MIDI. These inversion functions `examples/reverse_pianoroll` are provided as examples on a private github by CHRIS RAFFAEL (1 of Pretty Midi authors) <https://github.com/craffel/pretty-midi>. See also <https://github.com/craffel/pretty-midi/blob/main/Tutorial.ipynb> for basic functions summarized in a Jupyter notebook.

- **Music21** (21 in the title comes from designation for MIT’s classes in Music, Course 21M) is a Python-based toolkit for computer-aided musicology. Core music21 object is Stream object with its subclasses (Score, Part, Measure) designed as containers for music21 objects e.g. Note, Chord, Clef, & TimeSignature objects. Streams can store other Streams, permitting a wide variety of nested, ordered, & time structures. Elements within Streams are accessed with methods e.g. `getElementById()`, similar to Document Object Model (DOM) for retrieving elements in XML & HTML documents. 1 of most important music21 classes is a Note class that has both a `.pitch` attribute, which contains a Pitch object, & a `.duration` attribute, which contains a Duration object. 1 of convenient utilities of Music21: Streams can be easily visualized in Lilypond music notation software or with other programs that support MusicXML (such as MuseScore, Finale or Sibelius). Other important objects are Chords that combine multiple Pitch objects on a single stem & Interval Object that calculates distance in semitones between 2 notes, with or without taking octave into account. Another important object is Durations object that represents a span of musical time for Note & all Music21Objects, e.g. TimeSignature objects. Containers e.g. Stream & Score also have durations which are equal to position of ending of last object in Stream. Other notable tools in music21 are Chordify which is a process of making chords out of non-chords. Chordify is used for reducing a complex score with multiple parts to a succession of chords, which can be further queried for various chord properties, e.g. checking for presence of a specific chord in a polyphonic score. TinyNotation is a lightweight notation syntax, Calling a `converter.parse` function on a simple text string preceded by a “tinyNotation:” tag results in a Stream. Part subclass that is meant for designating music that is considered a single part.
- **jchord** is a Python package for working with chord progressions. It has object representations for notes, chords, & progressions designed for Western 12-tone system. 1 of notable utilities of package is its ability to convert between different naming conventions for chords, & can convert back & forth between objects & names. Also it provides simple command line tools for converting between progression represented as a string in `.txt`, `.xlsx` or `.midi` format to an output file in strings `.txt`, `.xlsx`, `.midi` or `.pdf` formats allows a quick & easy viewing or sonification of strings representing sequences of chords. Another useful functionality: **jchord** can parse MIDI files to sequence of block chord by approximate grouping of notes into chords allowing some imperfections.
- **Mingus** is a music theory & notation package for Python with MIDI file & playback support. It can also be used to create sheet music with LilyPond. Musical object include notes, intervals, chords, scales, keys & meters. Chords include natural diatonic triads, 7th chords, augmented chords, suspended chords, 6ths, 9ths, 11ths, 13ths, altered chords, & special: “5”, “NC”, “hendrix” & absolute chords represented from shorthand (min7, m/M7, etc.). **mingus** also supports inversions, slashed chords & polychords, & refering to chords by their diatonic function (tonic, subtonic, etc. or I, ii, iii, IV, etc.). Substitution algorithms, diatonic scales & their modes (ionian, mixolydian, etc.), minor (natural, harmonic, & melodic) & chromatic or whole note scales are represented. Library can recognize intervals, scales, & hundreds of chords from lists of notes & their harmonic functions. To compose music with Mingus a Composition class is used to organize Tracks, to which notes are added.
- **MusPy** is an open source Python library for management of symbolic music datasets for DL research by providing interfaces between music data & PyTorch & TensorFlow. It supports working with common symbolic music formats e.g. MIDI, MusicXML, & ABC, & interfaces to other symbolic music libraries e.g. music21, mido, **pretty_midi**, & Pypianoroll. It also provides evaluation tools for music generation systems, including audio rendering, score, & piano-roll visualizations & objective metrics.
- **APPENDIX D. FREE MUSIC & AUDIO-EDITING SOFTWARE.**
 - **Audacity** is a free & open-source digital audio editor. Users can import or record audio in Audacity. Download Audacity & learn more at <https://www.audacityteam.org/>.
 - **LMMS** (originally named Linux MultiMedia Studio, but now available on multiple platforms) is a free, open-source digital audio workstation with an array of features to synthesize & mix sounds. Download LMMS & learn more at lmmms.io.
 - **MuseScore** is a free music notation software suite & growing online community for writing & sharing music. MuseScore can be used to notate & export scores as PDF or MIDI.
 - **Magenta** <https://magenta.tensorflow.org/> is an open-source Google project built in TensorFlow to explore role of ML as a tool in creative process.

Readers are encouraged to visit their demo page <https://magenta.tensorflow.org/demos> to play with existing tools & find inspiration for further ML projects. For those interested in technical details, Magenta blogposts are an excellent starting point with references to & explanations of associated research works.

- APPENDIX E. DATASETS.
- APPENDIX F. FIGURE ATTRIBUTIONS.

2.7 [DFO24]. MARC PETER DEISENROTH, A. ALDO FAISAL, CHENG SOON ONG. **Mathematics for Machine Learning. 2024**

[849 Amazon ratings]

Amazon review. Fundamental mathematical tools needed to understand ML include linear algebra, analytic geometry, matrix decompositions, vector calculus, optimization, probability & statistics. These topics are traditionally taught in disparate courses, making it hard for DS or CS students, or professionals, to efficiently learn mathematics. This self-contained textbook bridges gap between mathematical & ML texts, introducing mathematical concepts with a minimum of prerequisites. It uses these concepts to derive 4 central ML methods: linear regression, principal component analysis, Gaussian mixture models & support vector machines. For students & others with a mathematical background, these derivations provide a starting point to ML texts. For those learning mathematics for 1st time, methods help build intuition & practical experience with applying mathematical concepts. Every chap includes worked examples & exercises to test understanding. Programming tutorials are offered on book's web site.

Editorial Reviews.

- “This book provides great coverage of all basic mathematical concepts for ML. I’m looking forward to sharing it with students, colleagues, & anyone interested in building a solid understanding of fundamentals.” – JOELLE PINEAU, McGill University, Montreal
- “The field of ML has grown dramatically in recent years, with an increasingly impressive spectrum of successful applications. This comprehensive text covers key mathematical concepts that underpin modern ML, with a focus on linear algebra, calculus, & probability theory. It will prove valuable both as a tutorial for newcomers to field, & as a reference text for ML researchers & engineers.” – CHRISTOPHER BISHOP, Microsoft Research Cambridge
- “This book provides a beautiful exposition of mathematics underpinning modern ML. Highly recommended for anyone wanting a 1-stop-shop to acquire a deep understanding of ML foundations.” – PIETER ABBEEL, University of California, Berkeley
- “Really successful are numerous explanatory illustrations, which help to explain even difficult concepts in a catch way. Each chap concludes with many instructive exercises. An outstanding feature of this book is additional material presented on website ...” – VOLKER H. SCHULZ, SIAM Review

Book Description. Distills key concepts from linear algebra, geometry, matrices, calculus, optimization, probability & statistics that are used in ML.

About the Author.

Foreword. ML is latest in a long line of attempts to distill human knowledge & reasoning into a form that is suitable for constructing machines & engineering automated systems. As ML becomes more ubiquitous & its software packages become easier to use, natural & desirable: low-level technical details are abstracted away & hidden from practitioner. However, this brings with it danger that a practitioner becomes unaware of design decisions &, hence, limits of ML algorithms.

Enthusiastic practitioner who is interested to learn more about magic behind successful ML algorithms currently faces a daunting set of pre-requisite knowledge:

- Programming languages & data analysis tools
- Large-scale computation & associated frameworks
- Mathematics & statistics & how ML builds on it

At universities, introductory courses on ML tend to spend early parts of course covering some of these pre-requisites. For historical reasons, courses in ML tend to be taught in CS department, where students are often trained in 1st 2 areas of knowledge, but not so much in mathematics & statistics.

Current ML textbooks primarily focus on ML algorithms & methodologies & assume: reader is competent in mathematics & statistics. Therefore, these books only spend 1 or 2 chaps on background mathematics, either at beginning of book or as appendices. Have found many people who want to delve into foundations of basic ML methods who struggle with mathematical knowledge required to read a ML textbook. Having taught undergraduate & graduate courses at universities, find: gap between high school mathematics & mathematics level required to read a standard ML textbook is too big for many people.

This book brings mathematical foundations of basic ML concepts to fore & collects information in a single place so that this skills gap is narrowed or even closed.

- **Why Another Book on ML?** ML builds upon language of mathematics to express concepts that seem intuitively obvious but that are surprisingly difficult to formalize. Once formalized properly, can gain insights into task we want to solve. 1 common complain of students of mathematics around globe: topics covered seem to have little relevance to practical problems. Believe: ML is an obvious & direct motivation for people to learn mathematics.

“Math is linked in popular mind with phobia & anxiety. You’d think we’re discussing spiders.” Strogatz, 2014, p. 281

This book is intended to be a guidebook to vast mathematical literature that forms foundations of modern ML. Motivate need for mathematical concepts by directly pointing out their usefulness in context of fundamental ML problems. In interest of keeping book short, many details & more advanced concepts have been left out. Equipped with basic concepts presented here, & how they fit into larger context of ML, reader can find numerous resources for further study, provided at end of respective chaps. For readers with a mathematical background, this book provides a brief but precisely stated glimpse of ML. In contrast to other books that focus on methods & models of ML (MacKay, 2003; Bishop, 2006; Alpaydin, 2010; Barber, 2012; Murphy, 2012; Shalev-Shwartz & Ben-David, 2014; Rogers & Girolami, 2016) or programmatic aspects of ML (Müller & Guido, 2016; Raschka & Mirjalili, 2017; Chollet & Allaire, 2018), provide only 4 representative examples of ML algorithms. Instead, focus on mathematical concepts behind models themselves. Hope readers will be able to gain a deeper understanding of basic questions in ML & correct practical questions arising from use of ML with fundamental choices in mathematical model.

Do not aim to write a classical ML book. Instead, intention: provide mathematical background, applied to 4 central ML problems, to make it easier to read other ML textbooks.

- **Who Is Target Audience?** As applications of ML become widespread in society, believe: everybody should have some understanding of its underlying principles. This book is written in an academic mathematical style, which enables us to be precise about concepts behind ML. Encourage readers unfamiliar with this seemingly terse style to persevere & to keep goals of each topic in mind. Sprinkle comments & remarks throughout text, in hope: it provides useful guidance w.r.t. big picture.

Book assumes reader to have mathematical knowledge commonly covered in high school mathematics & physics. E.g., reader should have seen derivatives & integrals before, & geometric vectors in 2D or 3D. Starting from there, generalize these concepts. Therefore, target audience of book includes undergraduate university students, evening learners & learners participating in online ML courses.

In analogy to music, there are 3 types of interaction that people have with ML:

- **Astute Listener.** (Người nghe tinh ý): Democratization of ML by provision of open-source software, online tutorials & cloud-based tools allows users to not worry about specifics of pipelines. Users can focus on extracting insights from data using off-the-shelf tools. This enables non-tech-savvy domain experts to benefit from ML. This is similar to listening to music; user is able to choose & discern between different types of ML, & benefits from it. More experienced users are like music critics, asking important questions about application of ML in society e.g. ethics, fairness, & privacy of individual. Hope: this book provides a foundation for thinking about certification & risk management of ML systems, & allows them to use their domain expertise to build better ML systems.
 - **Astute Listener.** (Người nghe tinh ý): Dân chủ hóa ML bằng cách cung cấp phần mềm nguồn mở, hướng dẫn trực tuyến & các công cụ dựa trên đám mây cho phép người dùng không phải lo lắng về các chi tiết cụ thể của đường ống. Người dùng có thể tập trung vào việc trích xuất thông tin chi tiết từ dữ liệu bằng các công cụ có sẵn. Điều này cho phép các chuyên gia trong lĩnh vực không am hiểu công nghệ được hưởng lợi từ ML. Điều này tương tự như việc nghe nhạc; người dùng có thể lựa chọn & phân biệt giữa các loại ML khác nhau, & hưởng lợi từ nó. Những người dùng có kinh nghiệm hơn giống như các nhà phê bình âm nhạc, đặt ra những câu hỏi quan trọng về ứng dụng ML trong xã hội, ví dụ như đạo đức, công bằng, & quyền riêng tư của cá nhân. Hy vọng: cuốn sách này cung cấp nền tảng để suy nghĩ về chứng nhận & quản lý rủi ro của các hệ thống ML, & cho phép họ sử dụng chuyên môn trong lĩnh vực của mình để xây dựng các hệ thống ML tốt hơn.
- **Experienced Artist.** (Nghệ sĩ giàu kinh nghiệm): Skilled practitioners of ML can plug & play different tools & libraries into an analysis pipeline. Stereotypical practitioner would be a data scientist or engineer who understands ML interfaces & their use cases, & is able to perform wonderful feats of prediction from data. This is similar to a virtuoso playing music, where highly skilled practitioners can bring existing instruments to life & bring enjoyment to their audience. Using mathematics presented here as a primer, practitioners would be able to understand benefits & limits of their favorite method, & to extend & generalize existing ML algorithms. Hope this book provides impetus for more rigorous & principled development of ML methods.
 - **Nghệ sĩ giàu kinh nghiệm.** (Nghệ sĩ giàu kinh nghiệm): Những người hành nghề ML có kỹ năng có thể cắm & chạy các công cụ & thư viện khác nhau vào 1 đường ống phân tích. Người hành nghề theo khuôn mẫu sẽ là 1 nhà khoa học dữ liệu hoặc kỹ sư hiểu các giao diện ML & các trường hợp sử dụng của chúng, & có thể thực hiện những kỳ công dự đoán tuyệt vời từ dữ liệu. Điều này tương tự như 1 nghệ sĩ chơi nhạc điều luyện, nơi những người hành nghề có kỹ năng cao có thể thổi hồn vào các nhạc cụ hiện có & mang lại niềm vui cho khán giả của họ. Sử dụng toán học được trình bày ở đây như 1 tài liệu tham khảo, những người hành nghề sẽ có thể hiểu được lợi ích & giới hạn của phương pháp yêu thích của họ, & mở rộng & khái quát hóa các thuật toán ML hiện có. Hy vọng cuốn sách này sẽ cung cấp động lực cho sự phát triển & có nguyên tắc chặt chẽ hơn của các phương pháp ML.
- **Fledgling Composer.** (Nhà soạn nhạc trẻ): As ML is applied to new domains, developers of ML need to develop new methods & extend existing algorithms. They are often researchers who need to understand mathematical basis of ML & uncover relationships between different tasks. This is similar to composers of music who, within rules & structure of musical

theory, create new & amazing pieces. Hope this book provides a high-level overview of other technical books for people who want to become composers of ML. There is a great need in society for new researchers who are able to propose & explore novel approaches for attacking many challenges of learning from data.

– **Fledgling Composer.** (Nhà soạn nhạc trẻ): Khi ML được áp dụng vào các lĩnh vực mới, các nhà phát triển ML cần phát triển các phương pháp mới & mở rộng các thuật toán hiện có. Họ thường là các nhà nghiên cứu cần hiểu cơ sở toán học của ML & khám phá mối quan hệ giữa các nhiệm vụ khác nhau. Điều này tương tự như các nhà soạn nhạc, những người, trong các quy tắc & cấu trúc của lý thuyết âm nhạc, tạo ra các tác phẩm mới & tuyệt vời. Hy vọng cuốn sách này cung cấp 1 cái nhìn tổng quan cấp cao về các cuốn sách kỹ thuật khác cho những người muốn trở thành nhà soạn nhạc của ML. Xã hội có nhu cầu lớn đối với các nhà nghiên cứu mới có khả năng đề xuất & khám phá các phương pháp tiếp cận mới để giải quyết nhiều thách thức của việc học từ dữ liệu.

- **Acknowledgments.** Grateful to many people who looked at early drafts of book & suffered through painful expositions of concepts. Tried to implement their ideas that we did not vehemently disagree with. Have been lucky to benefit from generosity of online community, who have suggested improvements via GitHub, which greatly improved book. Following people have found bugs, proposed clarifications & suggested relevant literature, either via GitHub or personal communication.

PART I: MATHEMATICAL FOUNDATIONS.

- **1. Introduction & Motivation.** ML is about designing algorithms that automatically extract valuable information from data. Emphasis here is on “automatic”, i.e., ML is concerned about general-purpose methodologies that can be applied to many datasets, while producing sth that is meaningful. There are 3 concepts that are at core of ML: data, a model, & learning.

Since ML is inherently data driven, *data* is at core of ML. Goal of ML: design general-purpose methodologies to extract valuable patterns from data, ideally without much domain-specific expertise. E.g., given a large corpus of documents (e.g., books in many libraries), ML methods can be used to automatically find relevant topics that are shared across documents (Hoffman et al., 2010). To achieve this goal, design *models* that are typically related to process that generates data, similar to dataset given. E.g., in a regression setting, model would describe a function that maps inputs to real-valued outputs. To paraphrase Mitchell (1997): A model is said to learn from data if its performance on a given task improves after data is taken into account. Goal: find good models that generalize well to yet unseen data, which we may care about in future. *Learning* can be understood as a way to automatically find patterns & structure in data by optimizing parameters of model.

While ML has seen many success stories, & software is readily available to design & train rich & flexible ML systems, believe: mathematical foundations of ML are important in order to understand fundamental principles upon which more complicated ML systems are built. Understanding these principles can facilitate creating new ML solutions, understanding & debugging existing approaches, & learning about inherent assumptions & limitations of methodologies we are working with.

- **1.1. Find Words for Intuitions.** A challenge we face regularly in ML: concepts & words are slippery, & a particular component of ML system can be abstracted to different mathematical concepts. E.g., word “algorithm” is used in ≥ 2 different senses in context of ML. In 1st sense, use phrase “ML algorithm” to mean a system that makes predictions based on input data. Refer to these algorithms as *predictors*. In 2nd sense, use exact same phrase “ML algorithm” to mean a system that adapts some internal parameters of predictor so that it performs well on future unseen input data. Here refer to this adaptation as *training* a system.

– Một thách thức mà chúng ta thường xuyên gặp phải trong ML: các khái niệm & từ ngữ rất khó nắm bắt, & 1 thành phần cụ thể của hệ thống ML có thể được trừu tượng hóa thành các khái niệm toán học khác nhau. Ví dụ, từ “thuật toán” được sử dụng theo ≥ 2 nghĩa khác nhau trong bối cảnh của ML. Theo nghĩa thứ nhất, sử dụng cụm từ “thuật toán ML” để chỉ 1 hệ thống đưa ra dự đoán dựa trên dữ liệu đầu vào. Tham khảo các thuật toán này là *predictors*. Theo nghĩa thứ hai, sử dụng chính xác cụm từ “thuật toán ML” để chỉ 1 hệ thống điều chỉnh 1 số tham số nội bộ của bộ dự đoán để nó hoạt động tốt trên dữ liệu đầu vào chưa từng thấy trong tương lai. Ở đây, hãy gọi sự điều chỉnh này là *training* 1 hệ thống.

This book will not resolve issue of ambiguity, but want to highlight upfront that, depending on context, same expressions can mean different things. However, attempt to make context sufficiently clear to reduce level of ambiguity.

1st part of this book introduces mathematical concepts & foundations needed to talk about 3 main components of a ML system: data, models, & learning. Briefly outline these components here & revisit them in Chap. 8 once have discussed necessary mathematical concepts.

While not all data is numerical, often useful to consider data in a number format. In this book, assume: *data* has already been appropriately converted into a numerical representation suitable for reading into a computer program. Therefore, think of data as vectors. As another illustration of how subtle words are, there are (at least) 3 different ways to think about vectors: a vector as an array of numbers (a CS view), a vector as an arrow with a direction & magnitude (a physics view), & a vector as an object that obeys addition & scaling (a mathematical view).

A *model* is typically used to describe a process for generating data, similar to dataset at hand. Therefore, good models can also be thought of as simplified versions of real (unknown) data-generating process, capturing aspects that are relevant for modeling data & extracting hidden patterns from it. A good model can then be used to predict what would happen in real world without performing real-world experiments.

Now come to crux of matter, *learning* component of ML. Assume given a dataset & a suitable model. *Training* model means to use data available to optimize some parameters of model w.r.t. a utility function that evaluates how well model predicts training data. Most training methods can be thought of as an approach analogous to climbing a hill to reach its peak. In

this analogy, peak of hill corresponds to a maximum of some desired performance measure. However, in practice, interested in model to perform well on unseen data. Performing well on data that we have already seen (training data) may only mean that we found a good way to memorize data. However, this may not generalize well to unseen data, & in practical applications, often need to expose ML system to situations that it has not encountered before.

Summarize main concepts of ML covered in this book:

- * Represent data as vectors.
 - * Choose an appropriate model, either using probabilistic or optimization view.
 - * Learn from available data by using numerical optimization methods with aim: model performs well on data not used for training.
- o 1.2. 2 Ways to Read This Book. Can consider 2 strategies for understanding mathematics for ML:
- * **Bottom-up.** Building up concepts from foundational to more advanced: Often preferred approach in more technical fields, e.g. mathematics. This strategy has advantage that reader at all times is able to rely on their previously learned concepts. Unfortunately, for a practitioners many of foundational concepts are not particularly interesting by themselves, & lack of motivation means: most foundational defs are quickly forgotten.
 - * **Top-down.** Drilling down from practical needs to more basic requirements. This goal-driven approach has advantage that readers know at all times why they need to work on a particular concept, & there is a clear path of required knowledge. Downside of this strategy: knowledge is built on potentially shaky foundations, & readers have to remember a set of words that they do not have any way of understanding.

Decided to write this book in a modular way to separate foundational (mathematical) concepts from applications so that this book can be read in both ways. Book is split into 2 parts, where Part I lays mathematical foundations & Part II applies concepts from Part I to a set of fundamental ML problems, which form 4 pillars of ML as illustrated in Fig. 1.1: **Foundations & 4 pillars of ML**: regression, dimensionality reduction, density estimation, & classification. Chaps in Part I mostly build upon previous ones, but possible to skip a chap & work backward if necessary. Chaps in Part II are only loosely coupled & can be read in any order. There are many pointers forward & backward between 2 parts of book to link mathematical concepts with ML algorithms.

Of course there are > 2 ways to read this book. Most readers learn using a combination of top-down & bottom-up approaches, sometimes building up basic mathematical skills before attempting more complex concepts, but also choosing topics based on applications of ML.

Part I Is About Mathematics. 4 pillars of ML covered in this book require a solid mathematical foundation, which is laid out in Part I.

- * Chap. 2: Represent numerical data as vectors & represent a table of such data as a matrix. Study of vectors & matrices is called *linear algebra*. Describe collection of vectors as a matrix.
- * Chap. 3: Given 2 vectors representing 2 objects in real world, want to make statements about their similarity. Idea: vectors that are similar should be predicted to have similar outputs by ML algorithm (our predictor). To formalize idea of similarity between vectors, need to introduce operations that take 2 vectors as input & return a numerical value representing their similarity. Construction of similarity & distances is central to *analytic geometry*.
- * Chap. 4: Introduce some fundamental concepts about matrices & *matrix decomposition*. Some operations on matrices are extremely useful in ML, & they allow for an intuitive interpretation of data & more efficient learning. Often consider data to be noisy observations of some true underlying signal. Hope: by applying ML, can identify signal from noise. This requires us to have a language for quantifying what “noise” means. Often would also like to have predictors that allows us to express some sort of uncertainty, e.g., to quantify confidence we have about value of prediction at a particular test data point. Quantification of uncertainty is realm of *probability theory* & is covered in Chap. 6. To train ML models, typically find parameters that maximize some performance measure. Many optimization techniques require concept of a gradient, which tells us direction in which to search for a solution. Chap. 5 is about *vector calculus* & details concept of gradients, which subsequently use in Chap. 7, where talk about *optimization* to find maxima/minima of functions.

Part II is about ML. 2nd part of book introduces 4 *pillars of ML*. Illustrate how mathematical concepts introduced in 1st part of book are foundation for each pillar. Broadly speaking, chaps are ordered by difficulty (in ascending order).

- * Chap. 8: restate 3 components of ML (data, models, & parameter estimation) in a mathematical fashion. In addition, provide some guidelines for building experimental set-ups that guard against overly optimistic evaluations of ML systems. Recall: goal: build a predictor that performs well on unseen data.
- * Chap. 9: Have a close look at *linear regression*, where objective: find functions that map inputs $\mathbf{x} \in \mathbb{R}^d$ to corresponding observed function values $y \in \mathbb{R}$, which can interpret as labels of their respective inputs. Discuss classical model fitting (parameter estimation) via maximum likelihood & maximum a posteriori estimation, as well as Bayesian linear regression, where integrate parameters out instead of optimizing them.
- * Chap. 10 focuses on *dimensionality reduction*, 2nd pillar in Fig. 1.1, using principal component analysis. Key objective of dimensionality reduction: find a compact, lower-dimensional representation of high-dimensional data $\mathbf{x} \in \mathbb{R}^d$, often easier to analyze than original data. Unlike regression, dimensionality reduction is only concerned about modeling data – there are no labels associated with a data point \mathbf{x} .

- * Chap. 11: Move to 3rd pillar: *density estimation*. Objective of density estimation: find a probability distribution that describes a given dataset. Focus on Gaussian mixture models for this purpose, & discuss an iterative scheme to find parameters of this model. As in dimensionality reduction, there are no labels associated with data points $\mathbf{x} \in \mathbb{R}^d$. However, do not seek a low-dimensional representation of data. Instead, interested in a density model that describes data.
- * Chap. 12 concludes book with an in-depth discussion of 4th pillar: *classification*. Discuss classification in context of support vector machines. Similar to regression (Chap. 9), have inputs \mathbf{x} & corresponding labels y . However, unlike regression, where labels were real-valued, labels in classification are integers, which requires special care.
- 1.3. Exercises & Feedback. Provide some exercises in Part I, which can be done mostly by pen & paper. For Part II, provide programming tutorials (jupyter notebooks) to explore some properties of ML algorithms discussed.
- 2. Linear Algebra.
 - 2.1. Systems of Linear Equations.
 - 2.2. Matrices.
 - 2.3. Solving Systems of Linear Equations.
 - 2.4. Vector Spaces.
 - 2.5. Linear Independence.
 - 2.6. Basis & Rank.
 - 2.7. Linear Mappings.
 - 2.8. Affine Spaces.
 - 2.9. Further Reading. There are many resources for learning linear algebra, including textbooks by Strang (2003), Golan (2007), Axler (2015), & Liesen & Mehrmann (2015). There are also several online resources mentioned in introduction to this chap. Only covered Gaussian elimination here, but there are many other approaches for solving systems of linear equations, & refer to numerical linear algebra textbooks by Stoer & Burlirsch (2002), Golub & Van Loan (2012), & Horn & Johnson (2013) for an in-depth discussion.

In this book, distinguish between topics of linear algebra (e.g., vectors, matrices, linear independence, basis) & topics related to geometry of a vector space. In Chap. 3, introduce inner product, which induces a norm. These concepts allow us to define angles, lengths & distances, which will use for orthogonal projections. Projections turn out to be key in many ML algorithms, e.g. linear regression & principal component analysis, both of which covered in Chaps. 9–10, resp.
- 3. Analytic Geometry.
 - 3.1. Norms.
 - 3.2. Inner Products.
 - 3.3. Lengths & Distances.
 - 3.4. Angles & Orthogonality.
 - 3.5. Orthonormal Basis.
 - 3.6. Orthogonal Complement.
 - 3.7. Inner Product of Functions.
 - 3.8. Orthogonal Projections.
 - 3.9. Rotations.
 - 3.10. Further Reading.
- 4. Matrix Decompositions.
 - 4.1. Determinant & Trace.
 - 4.2. Eigenvalues & Eigenvectors.
 - 4.3. Cholesky Decomposition.
 - 4.4. Eigendecomposition & Diagonalization.
 - 4.5. Singular Value Decomposition.
 - 4.6. Matrix Approximation.
 - 4.7. Matrix Phylogeny.
 - 4.8. Further Reading.
- 5. Vector Calculus.
 - 5.1. Differentiation of Univariate Functions.
 - 5.2. Partial Differentiation & Gradients.

- 5.3. Gradients of Vector-Valued Functions.
- 5.4. Gradients of Matrices.
- 5.5. Useful Identities for Computing Gradients.
- 5.6. Backpropagating & Automatic Differentiation. A good discussion about backpropagation & chain rule is available at a blog by TIM VIEIRA at <http://timvieira.github.io/blog/post/2017/08/18/backprop-is-not-just-the-chain-rule/>. In many ML applications, find good model parameters by performing gradient descent (Sect. 7.1), which relies on fact that can compute gradient of learning objective w.r.t. parameters of model. For a given objective function, can obtain gradient w.r.t. model parameters using calculus & applying chain rule, see Sect. 5.2.2. Already had a taste in Sect. 5.3 when looked at gradient of a squared loss w.r.t. parameters of a linear regression model.

Consider function

$$f(x) = \sqrt{x^2 + e^{x^2}} + \cos(x^2 + e^{x^2}). \quad (79)$$

By application of chain rule, & noting that differentiation is linear, compute gradient:

$$\frac{df}{dx} = 2x \left(\frac{1}{2\sqrt{x^2 + e^{x^2}}} - \sin(x^2 + e^{x^2}) \right) (1 + e^{x^2}). \quad (80)$$

Writing out gradient in this explicit way is often impractical since it often results in a very lengthy expression for a derivative. In practice, it means: if not careful, implementation of gradient could be significantly more expensive than computing function, which imposes unnecessary overhead. For training deep neural network models, *backpropagation* algorithm (Kelley, 1960; Bryson, 1961; Dreyfus, 1962; Rumelhart et al., 1986) is an efficient way to compute gradient of an error function w.r.t. parameters of model.

- * 5.6.1. Gradients in a Deep Network. An area where chain rule is used to an extreme is DL, where function value \mathbf{y} is computed as a many-level function composition

$$\mathbf{y} = (f_K \circ f_{K-1} \circ \dots \circ f_1)(\mathbf{x}) = f_K(f_{K-1}(\dots(f_1(\mathbf{x}))\dots)), \quad (81)$$

where \mathbf{x} : inputs (e.g., images), \mathbf{y} : observations (e.g., class labels), & every function $f_i, i = 1, \dots, K$, possesses its own parameters.

In neural networks with multiple layers, have functions $f_i(\mathbf{x}_{i-1}) = \sigma(\mathbf{A}_{i-1}\mathbf{x}_{i-1} + \mathbf{b}_{i-1})$ in i th layer. Here \mathbf{x}_{i-1} is output of layer $i-1$ & σ an activation function, e.g. logistic sigmoid $\frac{1}{1+e^{-x}}$, tanh or a rectified linear unit (ReLU). In order to train these models, require gradient of a loss function L w.r.t. all model parameters $\mathbf{A}_i, \mathbf{b}_i$ for $i = 1, \dots, K$. This also requires us to compute gradient of L w.r.t. inputs of each layer. E.g., if have inputs \mathbf{x} & observations \mathbf{y} & a network structure defined by

$$\mathbf{f}_0 = \mathbf{x}, \quad (82)$$

$$\mathbf{f}_i = \sigma_i(\mathbf{A}_{i-1}\mathbf{f}_{i-1} + \mathbf{b}_{i-1}), \quad i = 1, \dots, K, \quad (83)$$

see Fig. 5.8: Forward pass in a multi-layer neural network to compute loss L as a function of inputs \mathbf{x} & parameters $\mathbf{A}_i, \mathbf{b}_i$. for a visualization, may be interested in finding $\mathbf{A}_i, \mathbf{b}_i$ for $i = 0, \dots, K-1$ s.t. squared loss

$$L(\boldsymbol{\theta}) = \|\mathbf{y} - \mathbf{f}_K(\boldsymbol{\theta}, \mathbf{x})\|^2 \quad (84)$$

is minimized, where $\boldsymbol{\theta} = \{\mathbf{A}_0, \mathbf{b}_0, \dots, \mathbf{A}_{K-1}, \mathbf{b}_{K-1}\}$.

To obtain gradients w.r.t. parameter set $\boldsymbol{\theta}$, require partial derivatives of L w.r.t. parameters $\boldsymbol{\theta}_i = \{\mathbf{A}_i, \mathbf{b}_i\}$ of each layer $i = 0, \dots, K-1$. Chain rule allows us to determine partial derivatives as

$$\frac{\partial L}{\partial \boldsymbol{\theta}_{K-1}} = \frac{\partial L}{\partial \mathbf{f}_K} \frac{\partial \mathbf{f}_K}{\partial \boldsymbol{\theta}_{K-1}}, \quad (85)$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_{K-2}} = \frac{\partial L}{\partial \mathbf{f}_K} \frac{\partial \mathbf{f}_K}{\partial \mathbf{f}_{K-1}} \frac{\partial \mathbf{f}_{K-1}}{\partial \boldsymbol{\theta}_{K-2}}, \quad (86)$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \frac{\partial L}{\partial \mathbf{f}_K} \frac{\partial \mathbf{f}_K}{\partial \mathbf{f}_{K-1}} \dots \frac{\partial \mathbf{f}_{i+2}}{\partial \mathbf{f}_{i+1}} \frac{\partial \mathbf{f}_{i+1}}{\partial \boldsymbol{\theta}_i}, \quad (87)$$

$$(88)$$

Orange terms are partial derivatives of output of a layer w.r.t. its inputs, whereas blue terms are partial derivatives of output of a layer w.r.t. its parameters. Assuming, have already computed partial derivatives $\frac{\partial L}{\partial \boldsymbol{\theta}_{i+1}}$, then most of computation can be reused to compute $\frac{\partial L}{\partial \boldsymbol{\theta}_i}$. Additional terms that we need to compute are indicated by boxes. Fig. 5.9: Backward pass in a multi-layer neural network to compute gradients of loss function visualizes: gradients are passed backward through network.

- * 5.6.2. Automatic Differentiation. Turn out: Backpropagation is a special case of a general technique in numerical analysis called *automatic differentiation*. Can think of automatic differentiation as a set of techniques to numerically (in contrast to symbolically) evaluate exact (up to machine precision) gradient of a function by working with intermediate variables & applying chain rule. Automatic differentiation applies a series of elementary arithmetic operations, e.g., addition &

multiplication & elementary functions, e.g., sin, cos, exp, log. By applying chain rule to these operations, gradient of quite complicated functions can be computed automatically. Automatic differentiation applies to general computer programs & has forward & reverse modes. Baydin et al. (2018) give a great overview of automatic differentiation in ML.

Fig. 5.10: Simple graph illustrating flow of data from x to y via some intermediate variables a, b shows a simple graph representing data flow from inputs x to outputs y via some intermediate variables a, b . If were to compute derivative $\frac{dy}{dx}$, would apply chain rule & obtain

$$\frac{dy}{dx} = \frac{dy}{db} \frac{db}{da} \frac{da}{dx}. \quad (98)$$

Intuitively, forward & reverse mode differ in order of multiplication. Due to associativity of matrix multiplication, can choose between

$$\frac{dy}{dx} = \left(\frac{dy}{db} \frac{db}{da} \right) \frac{da}{dx} = \frac{dy}{db} \left(\frac{db}{da} \frac{da}{dx} \right). \quad (99)$$

1st eqn would be *reverse mode* because gradients are propagated backward through graph, i.e., reverse to data flow. 2nd eqn would be *forward mode*, where gradients flow with data from left to right through graph.

In general case, work with Jacobians, which can be vectors, matrices, or tensors.

Automatic differentiation is different from symbolic differentiation & numerical approximations of gradient, e.g., by using finite differences.

In following, focus on reverse mode automatic differentiation, which is backpropagation. In context of neural networks, where input dimensionality is often much higher than dimensionality of labels, reverse mode is computationally significantly cheaper than forward mode. Start with an instructive example.

Example 11. Consider function

$$f(x) = \sqrt{x^2 + e^{x^2}} + \cos(x^2 + e^{x^2}). \quad (100)$$

If were to implement a function f on a computer, would be able to save some computation by using intermediate variables:

$$a = x^2, b = \exp a, c = a + b, d = \sqrt{c}, e = \cos c, f = d + e. \quad (101)$$

This is same kind of thinking process that occurs when applying chain rule. Note: preceding set of equations requires fewer operations than a direct implementation of function $f(x)$. Corresponding computation graph in Fig. 5.11: Computation graph with inputs x , function values f , & intermediate variables a, b, c, d, e . shows flow of data & computations required to obtain function value f .

Set of equations that include intermediate variables can be thought of as a computation graph, a representation that is widely used in implementations of neural network software libraries. Can directly compute derivatives of intermediate variables w.r.t. their corresponding inputs by recalling definition of derivative of elementary functions. Obtain:

$$\frac{\partial a}{\partial x} = 2x, \frac{\partial b}{\partial a} = e^a, \frac{\partial c}{\partial a} = 1 = \frac{\partial c}{\partial b}, \frac{\partial d}{\partial c} = \frac{1}{2\sqrt{c}}, \frac{\partial e}{\partial c} = -\sin c, \frac{\partial f}{\partial d} = 1 = \frac{\partial f}{\partial e}. \quad (102)$$

By looking at computation graph in Fig. 5.11, can compute $\partial_x f$ by working backward from output & obtain

$$\frac{\partial f}{\partial c} = \frac{\partial f}{\partial d} \frac{\partial d}{\partial c} + \frac{\partial f}{\partial e} \frac{\partial e}{\partial c}, \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} \frac{\partial c}{\partial b}, \frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} \frac{\partial b}{\partial a} + \frac{\partial f}{\partial c} \frac{\partial c}{\partial a}, \frac{\partial f}{\partial x} = \frac{\partial f}{\partial a} \frac{\partial a}{\partial x}. \quad (103)$$

Note: implicitly applied chain rule to obtain $\frac{\partial f}{\partial x}$. By substituting results of derivatives of elementary functions, get

$$\frac{\partial f}{\partial c} = 1 \cdot \frac{1}{2\sqrt{c}} + 1 \cdot (-\sin c), \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} \cdot 1, \frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} e^a + \frac{\partial f}{\partial c} \cdot 1, \frac{\partial f}{\partial x} = \frac{\partial f}{\partial a} \cdot 2x. \quad (104)$$

By thinking of each of derivatives above as a variable, observe: computation required for calculating derivative is of similar complexity as computation of function itself. This is quite counterintuitive since mathematical expression for derivative $\frac{\partial f}{\partial x}$ is significantly more complicated than mathematical expression of function $f(x)$.

Automatic differentiation is a formalization of Example 5.14. Let x_1, \dots, x_d : input variables to function x_{d+1}, \dots, x_{D-1} be intermediate variables, & x_D : output variable. Then computation graph can be expressed as follows: (5.143)

$$\text{For } i = d+1, \dots, D : x_i = g_i(x_{\text{Pa}(x_i)}), \quad (105)$$

where $g_i(\cdot)$: elementary functions & $x_{\text{Pa}(x_i)}$: parent nodes of variable x_i in graph. Given a function defined in this way, can use chain rule to compute derivative of function in a step-by-step fashion. Recall by def $f = x_D$ & hence

$$\frac{\partial f}{\partial x_D} = 1. \quad (106)$$

For other variables x_i , apply chain rule (5.145)

$$\frac{\partial f}{\partial x_i} = \sum_{x_j: x_i \in \text{Pa}(x_j)} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i} = \sum_{x_j: x_i \in \text{Pa}(x_j)} \frac{\partial f}{\partial x_j} \frac{\partial g_j}{\partial x_i}, \quad (107)$$

where $\text{Pa}(x_j)$: set of parent nodes of x_j in computation graph. Equation (5.143) is forward propagation of a function, whereas (5.145) is backpropagation of gradient through computation graph. For neural network training, backpropagate error of prediction w.r.t. label.

Auto-differentiation in reverse mode requires a parse tree.

Automatic differentiation approach above works whenever have a function that can be expressed as a computation graph, where elementary functions are differentiable. In fact, function may not even be a mathematical function but a computer program. However, not all computer programs can be automatically differentiated, e.g., if cannot find differential elementary functions. Programming structures, e.g. `for` loops & `if` statements, require more care as well.

- 5.7. Higher-Order Derivatives.
- 5.8. Linearization & Multivariate Taylor Series.
- 5.9. Further Reading. Further details of matrix differentials, along with a short review of required linear algebra, can be found in Magnus & Neudecker (2007). Automatic differentiation has had a long history, & refer to Griewank & Walther (2003), Griewank & Walther (2008), & Elliott (2009) & the references therein.

In ML (& other disciplines), often need to compute expectations, i.e., need to solve integrals of form (5.181)

$$\mathbb{E}_{\mathbf{x}}[f(\mathbf{x})] = \int f(\mathbf{x})p(\mathbf{x}) d\mathbf{x}. \quad (99)$$

Even if $p(\mathbf{x})$ is in convenient form (e.g., Gaussian), this integral generally cannot be solved analytically. Taylor series expansion of f is 1 way of finding an approximate solution: Assuming $p(\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is Gaussian, then 1st-order Taylor series expansion around $\boldsymbol{\mu}$ locally linearizes nonlinear function f . For linear functions, can compute mean (& covariance) exactly if $p(\mathbf{x})$ is Gaussian distributed (Sect. 6.5). This property is heavily exploited by *extended Kalman filter* (Maybeck, 1979) for online state estimation in nonlinear dynamical systems (also called “state-space models”). Other deterministic ways to approximate integral in (5.181) are *unscented transform* (biến đổi không mùi) (Julier & Uhlmann, 1997), which does not require any gradients, or *Laplace approximation* (MacKay, 2003; Bishop, 2006; Murphy, 2012), which uses a 2nd-order Taylor series expansion (requiring Hessian) for a local Gaussian approximation of $p(\mathbf{x})$ around its mode.

- 6. Probability & Distributions. Probability, loosely speaking, concerns study of uncertainty. Probability can be thought of as fraction of times an event occurs, or as a degree of belief about an event. Then would like to use this probability to measure chance of sth occurring in an experiment. Often quantify uncertainty in data, uncertainty in ML model, & uncertainty in predictions produced by model. Quantifying uncertainty requires idea of a *random variable*, which is a function that maps outcomes of random experiments to a set of properties that we are interested in. Associated with random variable is a function that measures probability that a particular outcome (or set of outcomes) will occur; this called *probability distribution*.

Probability distributions are used as a building block for other concepts, e.g. probabilistic modeling (Sect. 8.4), graphical models (Sect. 8.5), & model selection (Sect. 8.6). In next sect, present 3 concepts that define a probability space (sample space, events, & probability of an event) & how they are related to a 4th concept called random variable. Presentation is deliberately slightly hand wavy since a rigorous presentation may occlude intuition behind concepts. An outline of concepts presented in this chap are shown in Fig. 6.1: A mind map of concepts related to random variables & probability distributions.

- 6.1. Construction of a Probability Space. Theory of probability aims at defining a mathematical structure to describe random outcomes of experiments. E.g., when tossing a single coin, cannot determine outcome, but by doing a large number of coin tosses, can observe a regularity in average outcome. Using this mathematical structure of probability, goal: perform automated reasoning, & in this sense, probability generalizes logical reasoning (Jaynes, 2003).

* 6.1.1. Philosophical Issues. When constructing automated reasoning systems, classical Boolean logic does not allow us to express certain forms of plausible reasoning. Consider scenario: observe A is false. Find B becomes less plausible (có thể xảy ra), although no conclusion can be drawn from classical logic. Observe: B is true. Seem A becomes more plausible. use this form of reasoning daily. Waiting for a friend, & consider 3 possibilities: H1, she is on time; H2, she has been delayed by traffic; & H3, she has been abducted by aliens. When observe friend is late, must logical rule out H1. Also tend to consider H2 to be more likely, though not logically required to do so. Finally, may consider H3 to be possible, but continue to consider it quite unlikely. How do conclude H2 is the most plausible answer? Seen in this way, probability theory can be considered a generalization of Boolean logic. In context of ML, often applied in this way to formalize design of automated reasoning systems. Further arguments about how probability theory is foundation of reasoning systems can be found in Pearl (1988).

“For plausible reasoning it is necessary to extend discrete true & false values of truth to continuous plausibilities” (Jaynes, 2003).

Philosophical basis of probability & how it should be somehow related to what we think should be true (in logical sense) was studied by Cox (Jaynes, 2003). Another way to think about it: if we are precise about our common sense end up constructing probabilities. E. T. JAYNES (1922–1998) identified 3 mathematical criteria, which must apply to all plausibilities:

1. Degrees of plausibility are represented by real numbers.
2. These numbers must be based on rules of common sense.
3. Resulting reasoning must be consistent, with 3 following meanings of word “consistent”:
 - (a) Consistency or non-contradiction: When same result can be reached through different means, same plausibility value must be found in all cases.
 - (b) Honesty: All available data must be taken into account.

- (c) Reproducibility: If our state of knowledge about 3 problems are same, then must assign same degree of plausibility to both of them.

Cox–Jaynes theorem proves these plausibilities to be sufficient to define universal mathematical rules that apply to plausibility p , up to transformation by an arbitrary monotonic function. Crucially, these rules *are* rules of probability.

Remark 6. In ML & statistics, there are 2 major interpretations of probability: Bayesian & frequentist interpretations (Bishop, 2006; Efron & Hastie, 2016). Bayesian interpretation uses probability to specify degree of uncertainty that user has about an event. Sometimes referred to as “subjective probability” or “degree of belief”. Frequentist interpretation considers relative frequencies of events of interest to total number of events that occurred. Probability of an event is defined as relative frequency of event in limit when one has infinite data.

Some ML texts on probabilistic models use lazy notation & jargon (thuật ngữ chuyên ngành), which is confusing. This text is no exception. Multiple distinct concepts are all referred to as “probability distribution”, & reader has to often disentangle meaning from context. 1 trick to help make sense of probability distributions: check whether we are trying to model sth categorical (a discrete random variable) & something continuous (a continuous random variable). Kinds of questions tackle in ML are closely related to whether we are considering categorical or continuous models.

– 1 số văn bản ML về các mô hình xác suất sử dụng ký hiệu lười biếng & thuật ngữ chuyên ngành, điều này gây nhầm lẫn. Văn bản này cũng không ngoại lệ. Nhiều khái niệm riêng biệt đều được gọi là “phân phối xác suất”, & người đọc thường phải tách nghĩa ra khỏi ngữ cảnh. 1 mẹo giúp hiểu được phân phối xác suất: kiểm tra xem chúng ta có đang cố gắng mô hình hóa cái gì đó theo phạm trù (một biến ngẫu nhiên rời rạc) & cái gì đó liên tục (một biến ngẫu nhiên liên tục) hay không. Các loại câu hỏi giải quyết trong ML có liên quan chặt chẽ đến việc chúng ta đang xem xét các mô hình theo phạm trù hay liên tục.

- * 6.1.2. Probability & Random Variables. There are 3 distinct ideas that are often confused when discussing probabilities. 1st is idea of a probability space, which allows us to quantify idea of a probability. However, mostly do not work directly with this basic probability space. Instead, work with random variables (2nd idea), which transfers probability to a more convenient (often numerical) space. 3rd idea is idea of a distribution or law associated with a random variable. Introduce 1st 2 ideas in this sect & expand on 3rd idea in Sect. 6.2.

Modern probability is based on a set of axioms proposed by KOLMOGOROV (Grinstead & Snell, 1997; Jaynes, 2003) that introduce 3 concepts of sample space, event space, & probability measure. Probability space models a real-world process (referred to as an experiment) with random outcomes.

- **Sample space Ω .** *Sample space* is set of all possible outcomes of experiment, usually denoted by Ω . E.g., 2 successive coin tosses have a sample space of $\{hh, tt, ht, th\}$, where “h” denotes “heads” & “t” denotes “tails”.
- **Event space \mathcal{A} .** *Event space* is space of potential results of experiment. A subset A of sample space Ω is in event space \mathcal{A} if at end of experiment can observe whether a particular outcome $\omega \in \Omega$ is in A . Event space \mathcal{A} is obtained by considering collection of subsets of Ω , & for discrete probability distributions (Sect. 6.2.1) \mathcal{A} is often power set of Ω .
- **Probability P .** With each event $A \in \mathcal{A}$, associate a number $P(A)$ that measures probability or degree of belief that event will occur. $P(A)$ is called *probability* of A .

Probability of a single event must lie in interval $[0, 1]$, & total probability over all outcomes in sample space Ω must be 1, i.e., $P(\Omega) = 1$. Given a probability space (Ω, \mathcal{A}, P) , want to use it to model some real-world phenomenon. In ML, often avoid explicitly referring to probability space, but instead refer to probabilities on quantities of interest, which denote by \mathcal{T} . In this book, refer to \mathcal{T} as *target space* & refer to elements of \mathcal{T} as states. Introduce a function $X : \Omega \rightarrow \mathcal{T}$ that takes an element of Ω (an outcome) & returns a particular quantity of interest x , a value in \mathcal{T} . This association/mapping from Ω to \mathcal{T} is called a *random variable*. E.g., in case of tossing 2 coins & counting number of heads, a random variable X maps to 3 possible outcomes: $X(hh) = 2, X(ht) = 1, X(th) = 1, X(tt) = 0$. In this particular case, $\sqcup = \{0, 1, 2\}$, & we are interested in probabilities on elements of \mathcal{T} . For a finite sample space Ω & finite \mathcal{T} , function corresponding to a random variable is essentially a lookup table. For any subset $S \subseteq \sqcup$, associated $P_X(S) \in [0, 1]$ (probability) to a particular event occurring corresponding to random variable X . Example 6.1: This toy example is essentially a biased coin flip example. provides a concrete illustration of terminology.

Name “random variable” is a great source of misunderstanding as it is neither random nor is it a variable. It is a function.

Remark 7. Aforementioned sample space Ω unfortunately is referred to by different names in different books. Another common name for Ω is “state space” (Jacod & Protter, 2004), but state space is sometimes reserved for referring to states in a dynamical system (Hasselblatt & Katok, 2003). Other names sometimes used to describe Ω : “sample description space”, “possibility space,” & “event space”.

In calculation, equated 2 different concepts, probability of output of X & probability of samples in Ω . E.g., say $P(X = 0) = P((\mathcal{L}, \mathcal{L}))$. Consider random variable $X : \Omega \rightarrow \mathcal{T}$ & a subset $S \subseteq \mathcal{T}$ (e.g., a single element of \mathcal{T} , e.g. outcome that 1 head is obtained when tossing 2 coins). Let $X^{-1}(S)$ be pre-image of S by X , i.e., set of elements of Ω that map to S under X ; $\{\omega \in \Omega : X(\omega) \in S\}$. 1 way to understand transformation of probability from events in Ω via random variable X : associate it with probability of pre-image of S (Jacod & Protter, 2004). For $S \subseteq \mathcal{T}$, have notation (6.8)

$$P_X(S) = P(X \in S) = P(X^{-1}(S)) = P(\{\omega \in \Omega : X(\omega) \in S\}).$$

LHS is probability of set of possible outcomes (e.g., number of \$ = 1) that interested in. Via random variable X , which maps states to outcomes, see in RHS: this is probability of set of states (in Ω) that have property (e.g., \$£, £\$). Say a random variable X is distributed according to a particular probability distribution P_X , which defines probability mapping

between event & probability of outcome of random variable. I.e., function P_X or equivalently $P \circ X^{-1}$ is *law* or *distribution* of random variable X .

Remark 8. Target space, i.e., range \mathcal{T} of random variable X , is used to indicate kind of probability space, i.e., a \mathcal{T} random variable. When \mathcal{T} is finite or countably infinite, this is called a discrete random variable (Sect. 6.2.1). For continuous random variables (Sect. 6.2.2), only consider $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{R}^D$.

- * 6.1.3. **Statistics.** Probability theory & statistics are often presented together, but they concern different aspects of uncertainty. 1 way of contrasting them is by kinds of problems that are considered. Using probability, can consider a model of some process, where underlying uncertainty is captured by random variables, & use rules of probability to derive what happens. In statistics, observe: sth has happened & try to figure out underlying process that explains observations. In this sense, ML is close to statistics in its goals to construct a model that adequately represents process generated data. Can use rules of probability to obtain a “best-fitting” model for some data.

Another aspect of ML systems: interested in generalization error (see Chap. 8). I.e., actually interested in performance of our system on instances that we will observe in future, which are not identical to instances that have seen so far. This analysis of future performance relies on probability & statistics, most of which is beyond what will be presented in this chap. Interested reader is encouraged to look at books by Boucheron et al. (2013) & Shalev-Shwartz & Ben-David (2014). See more about statistics in Chap. 8.

- 6.2. **Discrete & Continuous Probabilities.** Focus attention on ways to describe probability of an event as introduced in Sect. 6.1. Depending on whether target space is discrete or continuous, natural way to refer to distributions is different. When target space \mathcal{T} is discrete, can specify probability that a random variable X takes a particular value $x \in \mathcal{T}$, denoted as $P(X = x)$. Expression $P(X = x)$ for a discrete random variable X is known as *probability mass function*. When target space \mathcal{T} is continuous, e.g., real line \mathbb{R} , more natural to specify probability that a random variable X is in an interval, denoted by $P(a \leq X \leq b)$ for $a < b$. By convention, specify probability that a random variable X is \leq a particular value x , denoted by $P(X \leq x)$. Expression $P(X \leq x)$ for a continuous random variable X is known as *cumulative distribution function*. Discuss continuous random variables in Sect. 6.2.2. Revisit nomenclature & contrast discrete & continuous random variables in Sect. 6.2.3.

Remark 9. Use phrase *univariate distribution* to refer to distributions of a single random variable (whose states are denoted by non-bold x). Refer to distributions of > 1 random variable as *multivariate distributions*, & will usually consider a vector of random variables (whose states are denoted by bold \mathbf{x}).

- * 6.2.1. **Discrete Probabilities.** When target space is discrete, can imagine probability distribution of multiple random variables as filling out a (multidimensional) array of numbers. Fig. 6.2: Visualization of a discrete bivariate probability mass function, with random variables X, Y . This diagram is adapted from Bishop (2006). shows an example. Target space of joint probability is Cartesian product of target spaces of each of random variables. Define *joint probability* as entry of both values jointly (6.9)

$$P(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

where n_{ij} : number of events with state x_i, y_j , N : total number of events. Joint probability is probability of intersection of both events, i.e., $P(X = x_i, Y = y_j) = P(X = x_i \cap Y = y_j)$. Fig. 6.2 illustrates *probability mass function* (pmf) of a discrete probability distribution. For 2 random variables X, Y , probability that $X = x, Y = y$ is (lazily) written as $p(x, y)$ & is called joint probability. One can think of a probability as a function that takes state x, y & returns a real number, which is reason write $p(x, y)$. *Marginal probability* that X takes value x irrespective of value of random variable Y is (lazily) written as $p(x)$. Write $X \sim p(x)$ to denote that random variable X is distributed according to $p(x)$. If consider only instances where $X = x$, then fraction of instances (*conditional probability*) for which $Y = y$ is written (lazily) as $p(y|x)$.

Example 12. Consider 2 random variables X, Y , where X has 5 possible states & Y has 3 possible states, as shown in Fig. 6.2. Denote by n_{ij} number of events with state $X = x_i, Y = y_j$, & denote by N total number of events. Value c_i is sum of individual frequencies for i th column, i.e., $c_i = \sum_{j=1}^3 n_{ij}$. Similarly, value r_j is row sum, $r_j = \sum_{i=1}^5 n_{ij}$. Using these defs, can compactly express distribution of X, Y .

Probability distribution of each random variable, marginal probability, can be seen as sum over a row or column

$$P(X = x_i) = \frac{c_i}{N} = \frac{\sum_{j=1}^3 n_{ij}}{N},$$

$$P(Y = y_j) = \frac{r_j}{N} = \frac{\sum_{i=1}^5 n_{ij}}{N},$$

where c_i, r_j are i th column & j th row of probability table, resp. By convention, for discrete random variables with a finite number of events, assume: probabilities sum up to 1, i.e.,

$$\sum_{i=1}^5 P(X = x_i) = 1, \quad \sum_{j=1}^3 P(Y = y_j) = 1.$$

Conditional probability is fraction of a row or column in a particular cell. E.g., conditional probability of Y given X is

$$P(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i},$$

\mathcal{E} conditional probability of X given Y is

$$P(X = x_i | Y = y_j) = \frac{n_{ij}}{r_j}.$$

In ML, use discrete probability distributions to model *categorical variables*, i.e., variables that take a finite set of unordered values. They could be categorical features, e.g. degree taken at university when used for predicting salary of a person, or categorical labels, e.g. letters of alphabet when doing handwriting recognition. Discrete distributions are also often used to construct probabilistic models that combine a finite number of continuous distributions (Chap. 11).

- * 6.2.2. **Continuous Probabilities.** Consider real-valued random variables in this sect, i.e., consider target spaces that are intervals of real line \mathbb{R} . In this book, pretend: can perform operations on real random variables as if have discrete probability spaces with finite states. However, this simplification is not precise for 2 situations: when repeat sth infinitely often, & when want to draw a point from an interval. 1st situation arises when discuss generalization errors in ML (Chap. 8). 2nd situation arises when want to discuss continuous distributions, e.g. Gaussian (Sect. 6.5). For our purposes, lack of precision allows for a briefer introduction to probability.

Remark 10. In continuous spaces, there are 2 additional technicalities, which are counterintuitive. 1st, set of all subsets (used to define event space \mathcal{A} in Sect. 6.1) is not well behaved enough. \mathcal{A} needs to be restricted to behave well under set complements, set intersections, & set unions. 2nd, size of a set (which in discrete spaces can be obtained by counting elements) turns out to be tricky. Size of a set is called its measure. E.g., cardinality of discrete sets, length of an interval in \mathbb{R} , & volume of a region in \mathbb{R}^d are all measures. Sets that behave well under set operations & additionally have a topology are called a Borel σ -algebra. Betancourt details a careful construction of probability spaces from set theory without being bogged down in technicalities; see https://betanalpha.github.io/assets/case_studies/probability_theory.html. For a more precise construction, refer to Billingsley (1995) & Jacod & Protter (2004).

In this book, consider real-valued random variables with their corresponding Borel σ -algebra. Consider random variables with values in \mathbb{R}^D to be a vector of real-valued random variables.

Definition 6 (Probability Density Function). A function $f : \mathbb{R}^D \rightarrow \mathbb{R}$ is called a probability density function (pdf) if

1. $\forall \mathbf{x} \in \mathbb{R}^D: f(\mathbf{x}) \geq 0$.
2. Its integral exists & (6.15)

$$\int_{\mathbb{R}^D} f(\mathbf{x}) d\mathbf{x} = 1.$$

For probability mass function (pmf) of discrete random variables, integral in (6.15) is replaced with a sum (6.12).

Observe: probability density function is any function f that is nonnegative & integrates to 1. Associate a random variable X with this function f by

$$P(a \leq X \leq b) = \int_a^b f(x) dx,$$

where $a, b \in \mathbb{R}$ & $x \in \mathbb{R}$ are outcomes of continuous random variable X . States $\mathbf{x} \in \mathbb{R}^D$ are defined analogously by considering a vector of $x \in \mathbb{R}$. This association (6.16) is called *law* or *distribution* of random variable X .

Remark 11. In contrast to discrete random variables, probability of a continuous random variable X taking a particular value $P(X = x)$ is 0. This is like trying to specify an interval in (6.16) where $a = b$.

$\{X = x\}$ is a set of measure 0.

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- * 6.2.3. **Contrasting Discrete & Continuous Distributions.**
- 6.3. Sum Rule, Product Rule, & Bayes' Theorem.
- 6.4. Summary Statistics & Independence.
- 6.5. Gaussian Distribution.
- 6.6. Conjugacy & Exponential Family.
- 6.7. Change of Variables/Inverse Transform.
- 6.8. Further Reading.
- 7. **Continuous Optimization.** Since ML algorithms are implemented on a computer, mathematical formulations are expressed as numerical optimization methods. This chap describes basic numerical methods for training ML models. Training a ML model often boils down to finding a good set of parameters. Notion of “good” is determined by objective function or probabilistic model, which we will see examples of in 2nd part of this book. Given an objective function, finding best value is done using optimization algorithms.

Since consider data & models in \mathbb{R}^D , optimization problems we face are *continuous* optimization problems, as opposed to *combinatorial* optimization problems for discrete variables.

This chap covers 2 main branches of continuous optimization (Fig. 7.1: A mind map of concepts related to optimization, as presented in this chap. There are 2 main ideas: gradient descent & convex optimization.): unconstrained & constrained optimization. Assume in this chap: objective function is differentiable (Chap. 5), hence have access to a gradient at each location in space to help us find optimum value. By convention, most objective functions in ML are intended to be minimized, i.e., best value

is minimum value. Intuitively finding best value is like finding values of objective function, & gradients point us uphill. Idea: move downhill (opposite to gradient) & hope to find deepest point. For unconstrained optimization, this is only concept we need, but there are several design choices, discussed in Sect. 7.1. For constrained optimization, need to introduce other concepts to manage constraints (Sect. 7.2). Will also introduce a special class of problems (convex optimization problems in Sect. 7.3) where we can make statements about reaching global optimum.

Consider function in Fig. 7.2: Example objective function. Negative gradients are indicated by arrows, & global minimum is indicated by dashed blue line. Function has a *global minimum* around $x = -4.5$, with a function value of approximately -47 . Since function is “smooth,” gradients can be used to help find minimum by indicating whether should take a step to right or left. This assumes that are in correct bowl, as there exists another *local minimum* around $x = 0.7$. Recall: can solve \forall stationary points of a function by calculating its derivative & setting it to 0. For $l(x) = x^4 + 7x^3 + 5x^2 - 17x + 3$, obtain corresponding gradient as $\frac{dl(x)}{dx} = 4x^3 + 21x^2 + 10x - 17$.

Stationary points are real roots of derivative, i.e., points that have zero gradient.

Since this is a cubic equation, it has in general 3 solutions when set to 0. In example, 2 of them are minimums & one is a maximum (around $x = -1.4$). To check whether a stationary point is a minimum or maximum, need to take derivative a 2nd time & check whether 2nd derivative is positive or negative at stationary point. In our case, 2nd derivative is $\frac{d^2l(x)}{dx^2} = 12x^2 + 42x + 10$. By substituting our visually estimated values of $x = -4.5, -1.4, 0.7$, observe: as expected middle point is a maximum $\frac{d^2l(x)}{dx^2} < 0$ & the other 2 stationary points are minimums.

Note: have avoided analytically solving for values of x in previous discussion, although for low-order polynomials e.g. preceding, could do so. In general, unable to find analytic solutions, & hence need to start at some value, say $x_0 = -6$, & follow negative gradient. Negative gradient indicates: should go right, but not how far (this is called *step-size*). Furthermore, if had started at right side (e.g., $x_0 = 0$) negative gradient would have led us to wrong minimum. Fig. 7.2 illustrates fact: for $x > -1$, negative gradient points toward minimum on right of figure, which has a larger objective value.

According to Abel–Ruffini theorem, there is in general no algebraic solution for polynomials of degree ≥ 5 (Abel, 1826).

In Sect. 7.3, will learn about a class of functions called *convex functions*, that do not exhibit this tricky dependency on starting point of optimization algorithm. For convex functions, all local minimums are global minimum. Turn out: many ML objective functions are designed s.t. they are convex, & see an example in Chap. 12.

For convex functions all local minima are global minimum.

Discussion in this chap so far was about a 1D function, where able to visualize ideas of gradients, descent directions, & optimal values. In rest of this chap, develop same ideas in high dimensions. Unfortunately, can only visualize concepts in 1D, but some concepts do not generalize directly to higher dimensions, therefore some care needs to be taken when reading.

- 7.1. Optimization Using Gradient Descent.
- 7.2. Constrained Optimization & Lagrange Multipliers.
- 7.3. Convex Optimization.
- 7.4. Further Reading.

PART II: CENTRAL MACHINE LEARNING PROBLEMS.

- 8. When Models Meet Data. In 1st part of book, introduced mathematics that form foundations of many ML methods. Hope: a reader would be able to learn rudimentary forms (hình thức thô sơ) of language of mathematics from 1st part, which we will now use to describe & discuss ML. 2nd part of book introduces 4 pillars of ML:
 - Chap. 9: Regression
 - Chap. 10: Dimensionality reduction
 - Chap. 11: Density estimation
 - Chap. 12: Classification

Main aim of this part of book: illustrate how mathematical concepts introduced in 1st part of book can be used to design ML algorithms that can be used to solve tasks within remit of 4 pillars (nhiệm vụ của 4 trụ cột). Do not intend to introduce advanced ML concepts, but instead to provide a set of practical methods that allow reader to apply knowledge they gained from 1st part of book. It also provides a gateway to wider ML literature for readers already familiar with mathematics.

- 8.1. Data, Models, & Learning. Worth at this point, to pause & consider problem that a ML algorithm is designed to solve. There are 3 major components of a ML system: data, models, & learning. Main question of ML: “What do we mean by good models?”. Word *model* has many subtleties, & revisit it multiple times in this chap. Also not entirely obvious how to objectively define word “good”. 1 of guiding principles of ML: good models should perform well on unseen data. This requires us to define some performance metrics, e.g. accuracy or distance from ground truth, as well as figuring out ways to do well under these performance metrics. This chap covers a few necessary bits & pieces of mathematical & statistical

language that are commonly used to talk about ML models. By doing so, briefly outline current best practices for training a model s.t. resulting predictor does well on data that we have not yet seen.

There are 2 different senses in which use phrase “ML algorithm”: training & prediction. Describe these ideas in this chap, as well as idea of selecting among different models. Introduce framework of empirical risk minimization in Sect. 8.2, principle of maximum likelihood in Sect. 8.3 & idea of probabilistic models in Sect. 8.4. Briefly outline a graphical language for specifying probabilistic models in Sect. 8.5 & finally discuss model selection in Sect. 8.6. Rest of this sect expands upon 3 main components of ML: data, models, & learning.

* **8.1.1. Data as Vectors.** Assume: our data can be read by a computer, & represented adequately in a numerical format. Data is assumed to be tabular Fig. 8.1: **Examples data from a fictitious human resource database that is not in a numerical format**, where think of each row of table as representing a particular instance or example, & each row of table to be a particular feature. In recent years, ML has been applied to many types of data that do not obviously come in tabular numerical format, e.g. genomic sequences, text, & image contents of a webpage, & social media graphs. Do not discuss important & challenging aspects of identifying good features. Many of these aspects depend on domain expertise & require careful engineering, &, in recent years, they have been put under umbrella of DS (Stray, 2016; Adhikari & DeNero, 2018).

Data is assumed to be in a tidy format (Wickham, 2014; Codd, 1990).

Even when have data in tabular format, there are still choices to be made to obtain a numerical representation. E.g., in Table 8.1: **Example data from a fictitious human resource database that is not in a numerical format.**, gender column (a categorical variable) may be converted into numbers 0 representing “Male” & 1 representing “Female”. Alternatively, gender could be represented by numbers ± 1 , resp., as shown in Table 8.2: **Example data from a fictitious human resource database, converted to a numerical format.** Furthermore, often important to use domain knowledge when constructing representation, e.g. knowing that university degrees progress from bachelor’s to master’s to PhD or realizing: postcode provided is not just a string of characters but actually encodes an area in London. In Table 8.2, converted data from Table 8.1 to a numerical format, & each postcode is represented as 2 numbers, a latitude & longitude. Even numerical data that could potentially be directly read into a ML algorithm should be carefully considered for units, scaling, & constraints. Without additional information, one should shift & scale all columns of dataset s.t. they have an empirical mean of 0 & an empirical variance of 1. For purposes of this book, assume: a domain expert already converted data appropriately, i.e., each input \mathbf{x}_n is a d -dimensional vector of real numbers, which are called *features*, *attributes*, or *covariates* (các tính năng, thuộc tính hoặc biến phụ thuộc). Consider a dataset to be of form as illustrated by Table 8.2. Observe: have dropped Nam column of Table 8.1 in new numerical representation. There are 2 main reasons why this is desirable:

1. Do not expect identifier (Name) to be informative for a ML task;
2. May wish to anonymize data to help protect privacy of employees.

In this part of book, use N to denote number of examples in a dataset & index examples with lowercase $n = 1, \dots, N$. Assume: are given a set of numerical data, represented as an array of vectors. Each row is a particular individual \mathbf{x}_n , often referred to as an *example* or *data point* in ML. Subscript n refers to fact: this is n th example out of a total of N examples in dataset. Each column represents a particular feature of interest about example, & index features as $d = 1, \dots, D$. Recall data is represented as vectors, i.e., each example (each data point) is a D -dimensional vector. Orientation of table originates from database community, but for some ML algorithms, more convenient to represent examples as column vectors.

Consider problem of predicting annual salary from age, based on data in Table 8.2. This is called a *supervised learning problem* where have a label y_n (salary) associated with each example \mathbf{x}_n (age). Label y_n has various other names, including *target*, *response variable*, & *annotation*. A dataset is written as a set of example-label pairs $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n), \dots, (\mathbf{x}_N, y_N)\}$. Table of examples $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ is often concatenated, & written as $\mathbf{X} \in \mathbb{R}^{N \times D}$. Fig. 8.1: **Toy data for linear regression.** Training data in (x_n, y_n) pairs from rightmost 2 columns of Table 8.2. Interested in salary of a person aged 60 ($x = 60$) illustrated as a vertical dashed red line, which is not part of training data. illustrates dataset consisting of 2 rightmost columns of Table 8.2, where $x = \text{age}$ & $y = \text{salary}$.

Use concepts introduced in 1st part of book to formalize ML problems e.g. that in previous paragraph. Representing data as vectors \mathbf{x}_n allows us to use concepts from linear algebra. In many ML algorithms, need to additionally be able to compare 2 vectors. As see in Chaps. 9 & 12, computing similarity or distance between 2 examples allows us to formalize intuition that examples with similar features should have similar labels. Comparison of 2 vectors requires: construct a geometry & allows us to optimize resulting learning problem using techniques from Chap. 7.

Since have vector representations of data, can manipulate data to find potentially better representations of it. Discuss finding good representations in 2 ways: finding lower-dimensional approximations of original feature vector, & using nonlinear higher-dimensional combinations of original feature vector. In Chap. 10, see an example of finding a low-dimensional approximation of original data space by finding principal components. Finding principal components is closely related to concepts of eigenvalue & singular value decomposition. For high-dimensional representation, see an explicit *feature map* $\phi(\cdot)$ that allows us to represent inputs \mathbf{x}_n using a higher-dimensional representation $\phi(\mathbf{x}_n)$. Main motivation for higher-dimensional representations: can construct new features as nonlinear combinations of original features, which in turn may make learning problem easier. Discuss feature map in Sect. 9.2 & show how this feature map leads to a *kernel* in Sect. 12.4. In recent years, DL methods (Goodfellow et al., 2016) have shown promise in using data itself to learn new good features & have been very successful in areas, e.g. computer vision, speech recognition, & natural language processing. Will not cover neural networks in this part of book, but reader is referred to Sect. 5.6 for mathematical description of backpropagation, a key concept for training neural networks.

* **8.1.2. Models as Functions.** Once have data in an appropriate vector representation, can get to business of constructing a predictive function (known as a *predictor*). In Chap. 1, did not yet have language to be precise about models. Using concepts from 1st part of book, can now introduce what “model” means. Present 2 major approaches in this book: a predictor as a function, & a predictor as a probabilistic model. Describe former here & latter in next subsection. A *predictor* is a function that, when given a particular input example (in our case, a vector of features), produces an output. For now, consider output to be a single number, i.e., a real-valued scalar output. This can be written as $f : \mathbb{R}^D \rightarrow \mathbb{R}$, where input vector \mathbf{x} : D -dimensional (has D features), & function f then applied to it (written as $f(\mathbf{x})$) returned a real number. Fig. 8.2: Example function (black solid diagonal line) & its prediction at $x = 60$, i.e., $f(60) = 100$. illustrates a possible function that can be used to compute value of prediction for input values x .

In this book, do not consider general case of all functions, which would involve need for functional analysis. Instead, consider special case of linear functions

$$f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x} + \theta_0 \quad (100)$$

for unknown $\boldsymbol{\theta}, \theta_0$. This restriction means: contents of Chaps. 2–3 suffice for precisely stating notion of a predictor for non-probabilistic (in contrast to probabilistic view described next) view of ML. Linear functions strike a good balance between generality of problems that can be solved & amount of background mathematics that is needed.

* **8.1.3. Models as Probability Distributions.** Often consider data to be noisy observations of some true underlying effect, & hope: by applying ML, can identify signal from noise. This requires us to have a language for quantifying effect of noise. Often would also like to have predictors that express some sort of uncertainty, e.g., to quantify confidence we have about value of prediction for a particular test data point. As seen in Chap. 6, probability theory provides a language for quantifying uncertainty. Fig. 8.3: Example function (black solid diagonal line) & its predictive uncertainty at $x = 60$ (drawn as a Gaussian) illustrates predictive uncertainty of function as a Gaussian distribution.

Instead of considering a predictor as a single function, could consider predictors to be probabilistic models, i.e., models describing distribution of possible functions. Limit ourselves in this book to special case of distributions with finite-dimensional parameters, which allows us to describe probabilistic models without needing stochastic processes & random measures. For this special case, can think about probabilistic models as multivariate probability distributions, which already allow for a rich class of models.

Introduce how to use concepts from probability (Chap. 6) to define ML models in Sect. 8.4, & introduce a graphical language for describing probabilistic models in a compact way in Sect. 8.5.

* **8.1.4. Learning is Finding Parameters.** Goal of learning: find a model & its corresponding parameters s.t. resulting predictor will perform well on unseen data. There are conceptually 3 distinct algorithmic phases when discussing ML algorithms:

1. Prediction or inference – Dự đoán hoặc suy luận
2. Training or parameter estimation
3. Hyperparameter tuning or model selection – Điều chỉnh siêu tham số hoặc lựa chọn mô hình

Prediction phase is when use a trained predictor on previously unseen test data. I.e., parameters & model choice is already fixed & predictor is applied to new vectors representing new input data points. As outlined in Chap. 1 & previous subsection, will consider 2 schools of ML in this book, corresponding to whether predictor is a function or a probabilistic model. When have a probabilistic model (discussed further in Sect. 8.4) prediction phase is called *inference*.

Remark 12. Unfortunately, there is no agreed upon naming for different algorithmic phases. Word “inference” is sometimes also used to mean parameter estimation of a probabilistic model, & less often may be also used to mean prediction for non-probabilistic models.

Training or parameter estimation phase is when adjust our predictive model based on training data. Would like to find good predictors given training data, & there are 2 main strategies for doing so: finding best predictor based on some measure of quality (sometimes called *finding a point estimate*), or using Bayesian inference. Finding a point estimate can be applied to both types of predictors, but Bayesian inference requires probabilistic models.

For non-probabilistic model, follow principle of *empirical risk minimization*, described in Sect. 8.2. Empirical risk minimization directly provides an optimization problem for finding good parameters. With a statistical model, principle of *maximum likelihood* is used to find a good set of parameters (Sect. 8.3). Can additionally model uncertainty of parameters using a probabilistic model (Sect. 8.4).

Use numerical methods to find good parameters that “fit” data, & most training methods can be thought of as hill-climbing approaches to find maximum of an objective, e.g. maximum of a likelihood. To apply hill-climbing approaches us gradients described in Chap. 5 & implement numerical optimization approaches from Chap. 7.

Convention in optimization: minimize objectives. Hence, there is often an extra minus sign in ML objectives.

Interested in learning a model based on data s.t. it performs well on future data. Not enough for model to only fit training data well, predictor needs to perform well on unseen data. Simulate behavior of our predictor on future unseen data using *cross-validation* (Sect. 8.2.4). To achieve goal of performing well on unseen data, need to balance between fitting well on training data & finding “simple” explanations of phenomenon. This trade-off is achieved using regularization (Sect. 8.2.3) or by adding a prior (Sect. 8.3.2). In philosophy, this is considered to be neither induction nor deduction, but is called *abduction* (bắt cóc, dụ dỗ). According to *Stanford Encyclopedia of Philosophy*, abduction is process of inference to best explanation (Douven, 2017).

Often need to make high-level modeling decisions about structure of predictor, e.g. number of components to use or class of probability distributions to consider. Choice of number of components is an example of a *hyperparameter*, & this

choice can affect performance of model significantly. Problem of choosing among different models is called *model selection*, described in Sect. 8.6. For non-probabilistic models, model selection is often done using *nested cross-validation*, described in Sect. 8.6.1. Also use model selection to choose hyperparameters of our model.

Remark 13. *Distinction between parameters & hyperparameters is somewhat arbitrary, \mathcal{E} is mostly driven by distinction between what can be numerically optimized vs. what needs to use search techniques. Another way to consider distinction: consider parameters as explicit parameters of a probabilistic model, \mathcal{E} to consider hyperparameters (higher-level parameters) as parameters that control distribution of these explicit parameters.*

In following sects, look at 3 flavors of ML: empirical risk minimization (Sect. 8.2), principle of maximum likelihood (Sect. 8.3), & probabilistic modeling (Sect. 8.4).

- 8.2. Empirical Risk Minimization. After having all mathematics under our belt, now in a position to introduce what it means to learn. “Learning” part of ML boils down to estimating parameters based on training data.

In this sect, consider case of a predictor that is a function, & consider case of probabilistic models in Sect. 8.3. Describe idea of empirical risk minimization, which was originally popularized by proposal of support vector machine, described in Chap. 12. However, its general principles are widely applicable & allow us to ask question of what is learning without explicitly constructing probabilistic models. There are 4 main design choices:

1. Sect. 8.2.1: What is set of functions we allow predictor to take?
2. Sect. 8.2.2: How do we measure how well predictor performs on training data?
3. Sect. 8.2.3: How do we construct predictors from only training data that performs well on unseen test data?
4. Sect. 8.2.4: What is procedure for searching over space of models?

- * 8.2.1. Hypothesis Class of Functions. Assume given N examples $\mathbf{x}_n \in \mathbb{R}^D$ & corresponding scalar labels $y_n \in \mathbb{R}$. Consider supervised learning setting, where obtain pairs $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)$. Given this data, would like to estimate a predictor $f(\cdot, \boldsymbol{\theta}) : \mathbb{R}^D \rightarrow \mathbb{R}$, parametrized by $\boldsymbol{\theta}$. Hope to be able to find a good parameter $\boldsymbol{\theta}^*$ s.t. fit data well, i.e.,

$$f(\mathbf{x}_n, \boldsymbol{\theta}^*) \approx y_n, \quad \forall n = 1, \dots, N. \quad (101)$$

Use notation $\hat{y}_n := f(\mathbf{x}_n, \boldsymbol{\theta}^*)$ to represent output of predictor.

Remark 14. *For ease of presentation, describe empirical risk minimization in terms of supervised learning (where have labels). This simplifies definition of hypothesis class & loss function. Also common in ML to choose a parameterized class of functions, e.g. affine functions.*

Affine functions are often referred to as linear functions in ML.

Example 13. *Introduce problem of ordinary least-squares regression to illustrate empirical risk minimization. A more comprehensive account of regression is given in Chap. 9. When label y_n is real-valued, a popular choice of function class for predictors is set of affine functions. Choose a more compact notation for an affine function by concatenating an additional unit feature $x^{(0)} = 1$ to \mathbf{x}_n , i.e., $\mathbf{x}_n = [1, x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(d)}]^\top$. Parameter vector is correspondingly $\boldsymbol{\theta} = [\theta_0, \theta_1, \dots, \theta_D]^\top$, allowing us to write predictor as a linear function*

$$f(\mathbf{x}_n, \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x}_n. \quad (102)$$

This linear predictor is equivalent to affine model

$$f(\mathbf{x}_n, \boldsymbol{\theta}) = \theta_0 + \sum_{d=1}^D \theta_d x_n^{(d)}. \quad (103)$$

Predictor takes vector of features representing a single example \mathbf{x}_n as input & produces a real-valued output, i.e., $f : \mathbb{R}^{D+1} \rightarrow \mathbb{R}$. Previous figures in this chap had a straight line as a predictor, i.e., have assumed an affine function.

Instead of a linear function, may wish to consider nonlinear functions as predictors. Recent advances in neural networks allow for efficient computation of more complex nonlinear function classes.

Given class of functions, want to search for a good predictor. Now move on to 2nd ingredient of empirical risk minimization: how to measure how well predictor fits training data.

- * 8.2.2. Loss Function for Training. Consider label y_n for a particular example; & corresponding prediction \hat{y}_n that we make based on \mathbf{x}_n . To define what it means to fit data well, need to specify a *loss function* $l(y_n, \hat{y}_n)$ that takes ground truth label & prediction as input & produces a nonnegative number (referred to as loss) representing how much error we have made on this particular prediction. Goal for finding a good parameter vector $\boldsymbol{\theta}^*$: minimize average loss on set of N training examples.

Expression “error” is often used to mean loss.

1 assumption commonly made in ML: set of example $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)$ is *independent & identically distributed*. Word independent (Sect. 6.4.5) means: 2 data points $(\mathbf{x}_i, y_i), (\mathbf{x}_j, y_j)$ do not statistically depend on each other, meaning: empirical mean is a good estimate of population mean (Sect. 6.4.1). This implies: can use empirical mean of loss on training data. For a given *training set* $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)\}$, introduce notation of an example matrix $\mathbf{X} := [\mathbf{x}_1, \dots, \mathbf{x}_N]^\top \in \mathbb{R}^{N \times D}$ & a label vector $\mathbf{y} := [y_1, \dots, y_N]^\top \in \mathbb{R}^N$. Using this matrix notation, average loss is given by (8.6)

$$\mathbf{R}_{\text{emp}}(f, \mathbf{X}, \mathbf{y}) = \frac{1}{N} \sum_{n=1}^N l(y_n, \hat{y}_n), \quad (104)$$

where $\hat{y}_n = f(\mathbf{x}_n, \boldsymbol{\theta})$. (8.6) is called *empirical risk* & depends on 3 arguments, predictor f & data \mathbf{X}, \mathbf{y} . This general strategy for learning is called *empirical risk minimization*.

Example 14 (Least-Square Loss). *Continuing example of least-squares regression, specify: measure cost of making an error during training using squared loss $l(y_n, \hat{y}_n) = (y_n - \hat{y}_n)^2$. Wish to minimize empirical risk (8.6), which is average of losses over data*

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^D} \frac{1}{N} \sum_{n=1}^N (y_n - f(\mathbf{x}_n, \boldsymbol{\theta}))^2, \quad (105)$$

where substituted predictor $\hat{y}_n = f(\mathbf{x}_n, \boldsymbol{\theta})$. By using our choice of a linear predictor $f(\mathbf{x}_n, \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x}_n$, obtain optimization problem

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^D} \frac{1}{N} \sum_{n=1}^N (y_n - \boldsymbol{\theta}^\top \mathbf{x}_n)^2. \quad (106)$$

This equation can be equivalently expressed in matrix form

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^D} \frac{1}{N} \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|^2. \quad (107)$$

This is known as least-squares problem. There exists a closed-form analytic solution for this by solving normal equations, discussed in Sect. 9.2.

Not interested in a predictor that only performs well on training data. Instead, seek a predictor that performs well (has low risk) on unseen test data. More formally, interested in finding a predictor f (with parameters fixed) that minimizes *expected risk*

$$\mathbf{R}_{\text{true}}(f) = \mathbb{E}_{\mathbf{x}, y} [l(y, f(\mathbf{x}))], \quad (108)$$

where y : label, $f(\mathbf{x})$: prediction based on example \mathbf{x} . Notation $\mathbf{R}_{\text{true}}(f)$ indicates: this is the true risk if had access to an infinite amount of data (NQBH: still not sure? may be all data). Expectation is over (infinite) set of all possible data & labels. There are 2 practical questions that arise from our desire to minimize expected risk, which address in following 2 subsects:

1. How should we change our training procedure to generalize well?
2. How do we estimate expected risk from (finite) data?

Another phrase commonly used for expected risk is “population risk”.

Remark 15. *Many ML tasks are specified with an associated performance measure, e.g., accuracy of prediction or root mean squared error. Performance measure could be more complex, be cost sensitive, & capture details about particular application. In principle, design of loss function for empirical risk minimization should correspond directly to performance measure specified by ML task. In practice, there is often a mismatch between design of loss function & performance measure. This could be due to issues e.g. ease of implementation or efficiency of optimization.*

- * **8.2.3. Regularization to Reduce Overfitting.** This sect describes an addition to empirical risk minimization that allows it to generalize well (approximately minimizing expected risk). Recall: aim of training a ML predictor is so that we can perform well on unseen data, i.e., predictor generalizes well. Simulate this unseen data by holding out a proportion of whole dataset. This hold out set is referred to as *test set*. Given a sufficiently rich class of functions for predictor f , can essentially memorize training data to obtain zero empirical risk. While this is great to minimize loss (& therefore risk) on training data, would not expect predictor to generalize well to unseen data. In practice, have only a finite set of data, & hence split our data into a training & a test set. Training set is used to fit model, & test set (not seen by ML algorithm during training) is used to evaluate generalization performance. Important for user to not cycle back to a new round of training after having observed test set. Use subscripts $\text{train}, \text{test}$ to denote training & test sets, resp. Revisit this idea of using a finite dataset to evaluate expected risk in Sect. 8.2.4.

Even knowing only performance of predictor on test set leaks information (Blum & Hardt, 2015).

Turn out: empirical risk minimization can lead to *overfitting*, i.e., predictor fits too closely to training data & does not generalize well to new data (Mitchell, 1997). This general phenomenon of having very small average loss on training set but large average loss on test set tends to occur when have little data & a complex hypothesis class. For a particular predictor f (with parameters fixed), phenomenon of overfitting occurs when risk estimate from training data $\mathbf{R}_{\text{emp}}(f, \mathbf{X}_{\text{train}}, \mathbf{y}_{\text{train}})$ underestimates expected risk $\mathbf{R}_{\text{true}}(f)$. Since estimate expected risk $\mathbf{R}_{\text{true}}(f)$ by using empirical risk on test set $\mathbf{R}_{\text{emp}}(f, \mathbf{X}_{\text{test}}, \mathbf{y}_{\text{test}})$ if test risk is much larger than training risk, this is an indication of overfitting. Revisit idea of overfitting in Sect. 8.3.3.

Therefore, need to somehow bias search for minimizer of empirical risk by introducing a penalty term, which makes it harder for optimizer to return an overly flexible predictor. In ML, penalty term is referred to as *regularization*. Regularization is a way to compromise between accurate solution of empirical risk minimization & size or complexity of solution.

Example 15 (Regularized Least Squares). *Regularization is an approach that discourages complex or extreme solutions to an optimization problem. Simplest regularization: replace least-squares problem*

$$\min_{\boldsymbol{\theta}} \frac{1}{N} \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|^2, \quad (109)$$

in previous example with “regularized” problem by adding a penalty term involving only θ :

$$\min_{\theta} \frac{1}{N} \|\mathbf{y} - \mathbf{X}\theta\|^2 + \lambda \|\theta\|^2. \quad (110)$$

Additional term $\|\theta\|^2$ is called *regularizer*, & parameter λ : *regularization parameter*. Regularization parameter trades off minimizing loss on training set & magnitude of parameters θ . Often happens: magnitude of parameter values become relatively large if run into overfitting (Bishop, 2006).

Regularization term is sometimes called *penalty term*, which biases vector θ to be closer to origin. Idea of regularization also appears in probabilistic models as prior probability of parameters. Recall from Sect. 6.6: for posterior distribution to be of same form as prior distribution, prior & likelihood need to be conjugate. Revisit this idea in Sect. 8.3.2. See in Chap. 12: idea of regularizer is equivalent to idea of a large margin.

- * **8.2.4. Cross-Validation to Assess Generalization Performance.** Mentioned in previous sect: measure generalization error by estimating it by applying predictor on test data. This data is also sometimes referred to as *validation set*. Validation set is a subset of available training data that we keep aside. A practical issue with this approach: amount of data is limited, & ideally we would use as much of data available to train model. This would require us to keep our validation set \mathcal{V} small, which then would lead to a noisy estimate (with high variance) of predictive performance. 1 solution to these contradictory objectives (large training set, large validation set): use *cross-validation*. K -fold cross-validation effectively partitions data into K chunks, $K - 1$ of which form training set \mathcal{R} , & last chunk serves as validation set \mathcal{V} (similar to idea outlined previously). Cross-validation iterates through (ideally) all combinations of assignments of chunks to \mathcal{R}, \mathcal{V} ; see Fig. 8.4: K -fold cross-validation. Dataset is divided into $K = 5$ chunks, $K - 1$ of which serve as training set (blue) & 1 as validation set (orange hatch). This procedure is repeated $\forall K$ choices for validation set, & performance of model from K runs is averaged.

Partition our dataset into 2 sets $\mathcal{D} = \mathcal{R} \cup \mathcal{V}$, s.t. they do not overlap $\mathcal{R} \cap \mathcal{V} = \emptyset$, where \mathcal{V} is validation set, & train our model on \mathcal{R} . After training, assess performance of predictor f on validation set \mathcal{V} (e.g., by computing root mean square error (RMSE) of trained model on validation set). More precisely, for each partition k training data $\mathcal{R}^{(k)}$ produces a predictor $f^{(k)}$, which is then applied to validation set $\mathcal{V}^{(k)}$ to compute empirical risk $R(f^{(k)}, \mathcal{V}^{(k)})$. Cycle through all possible partitionings of validation & training sets & compute average generalization error of predictor. Cross-validation approximates expected generalization error

$$\mathbb{E}_{\mathcal{V}}[R(f, \mathcal{V})] \approx \frac{1}{K} \sum_{k=1}^K R(f^{(k)}, \mathcal{V}^{(k)}), \quad (111)$$

where $R(f^{(k)}, \mathcal{V}^{(k)})$ is risk (e.g., RMSE) on validation set $\mathcal{V}^{(k)}$ for predictor $f^{(k)}$. Approximation has 2 sources: 1st, due to finite training set, which results in not best possible $f^{(k)}$; & 2nd, due to finite validation set, which results in an inaccurate estimation of risk $R(f^{(k)}, \mathcal{V}^{(k)})$. A potential disadvantage of K -fold cross-validation is computational cost of training model K times, which can be burdensome if training cost is computationally expensive. In practice, often not sufficient to look at direct parameters alone. E.g., need to explore multiple complexity parameters (e.g., multiple regularization parameters), which may not be direct parameters of model. Evaluating quality of model, depending on these hyperparameters, may result in a number of training runs that is exponential in number of model parameters. One can use nested cross-validation (Sect. 8.6.1) to search for good hyperparameters.

However, cross-validation is an *embarrassingly parallel* problem, i.e., little effort is needed to separate problem into a number of parallel tasks. Given sufficient computing resources (e.g., cloud computing, server farms), cross-validation does not require longer than a single performance assessment.

In this sect, saw: empirical risk minimization is based on following concepts: hypothesis class of functions, loss function & regularization. In Sect. 8.3, see effect of using a probability distribution to replace idea of loss functions & regularization.

- * **8.2.5. Further Reading.** Due to fact: original development of empirical risk minimization (Vapnik, 1998) was couched in heavily theoretical language, many of subsequent developments have been theoretical. Area of study is called *statistical learning theory* (Vapnik, 1999; Evgeniou et al., 2000; Hastie et al., 2001; von Luxburg & Schölkopf, 2011). A recent ML textbook that builds on theoretical foundations & develops efficient learning algorithms is Shalev-Shwartz & Ben-David (2014).

Concept of regularization has its roots in solution of ill-posed inverse problems (Neumaier, 1998). Approach presented here is called *Tikhonov regularization*, & there is a closely related constrained version called *Ivanov regularization*. Tikhonov regularization has deep relationships to bias-variance trade-off & feature selection (Bühlmann & Van De Geer, 2011). An alternative to cross-validation: bootstrap & jackknife (Efron & Tibshirani, 1993; Davidson & Hinkley, 1997; Hall, 1992). Thinking about empirical risk minimization (Sect. 8.2) as “probability free” is incorrect. There is an underlying unknown probability distribution $p(\mathbf{x}, y)$ that governs data generation. However, approach of empirical risk minimization is agnostic to that choice of distribution. This is in contrast to standard statistical approaches that explicitly require knowledge of $p(\mathbf{x}, y)$. Furthermore, since distribution is a joint distribution on both examples \mathbf{x} & labels y , labels can be non-deterministic. In contrast to standard statistics, do not need to specify noise distribution for labels y .

- **8.3. Parameter Estimation.** See also [ABT18]. In Sect. 8.2, did not explicitly model our problem using probability distributions. In this sect, see how to use probability distributions to model our uncertainty due to observation process & our uncertainty in parameters of our predictors. In Sect. 8.3.1, introduce likelihood, which is analogous to concept of loss functions (Sect. 8.2.2) in empirical risk minimization. Concept of priors (Sect. 8.3.2) is analogous to concept of regularization (Sect. 8.2.3).

- * 8.3.1. Maximum Likelihood Estimation. Idea behind *maximum likelihood estimation* (MLE): define a function of parameters that enables us to find a model that fits data well. Estimation problem is focused on *likelihood* function, or more precisely its negative logarithm. For data represented by a random variable \mathbf{x} & for a family of probability densities $p(\mathbf{x}|\boldsymbol{\theta})$ parameterized by $\boldsymbol{\theta}$, *negative log-likelihood* is given by

$$\mathcal{L}_{\mathbf{x}}(\boldsymbol{\theta}) = -\log p(\mathbf{x}|\boldsymbol{\theta}). \quad (112)$$

Notation $\mathcal{L}_{\mathbf{x}}(\boldsymbol{\theta})$ emphasizes fact: parameter $\boldsymbol{\theta}$ is varying & data \mathbf{x} is fixed. Very often drop reference to \mathbf{x} when writing negative log-likelihood, as it is really a function of $\boldsymbol{\theta}$, & write it as $\mathcal{L}(\boldsymbol{\theta})$ when random variable representing uncertainty in data is clear from context.

Interpret what probability density $p(\mathbf{x}|\boldsymbol{\theta})$ is modeling for a fixed value of $\boldsymbol{\theta}$. It is a distribution that models uncertainty of data. I.e., once have chosen type of function we want as a predictor, likelihood provides probability of observing data \mathbf{x} . In a complementary view, if consider data to be fixed (because it has been observed), & vary parameters $\boldsymbol{\theta}$, what does $\mathcal{L}(\boldsymbol{\theta})$ tell us? It tells us how likely a particular setting of $\boldsymbol{\theta}$ is for observations \mathbf{x} . Based on this 2nd view, maximum likelihood estimator gives us most likely parameter $\boldsymbol{\theta}$ for set of data.

Consider supervised learning setting, where obtain pairs $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)$ with $\mathbf{x}_n \in \mathbb{R}^D$ & labels $y_n \in \mathbb{R}$. Interested in constructing a predictor that takes a feature vector \mathbf{x}_n as input & produces a prediction y_n (or sth close to it), i.e., given a vector \mathbf{x}_n we want probability distribution of label y_n . I.e., specify conditional probability distribution of labels given examples for particular parameter settings $\boldsymbol{\theta}$.

Example 16. 1st example often used: specify: conditional probability of labels given examples is a Gaussian distribution. I.e., assume: can explain our observation uncertainty by independent Gaussian noise (refer to Sect. 6.5) with zero mean, $\varepsilon_n \sim \mathcal{N}(0, \sigma^2)$. Further assume: linear model $\mathbf{x}_n^\top \boldsymbol{\theta}$ is used for prediction. I.e., specify a Gaussian likelihood for each example label pair \mathbf{x}_n, y_n ,

$$p(y_n|\mathbf{x}_n, \boldsymbol{\theta}) = \mathcal{N}(y_n|\mathbf{x}_n^\top \boldsymbol{\theta}, \sigma^2). \quad (113)$$

An illustration of a Gaussian likelihood for a given parameter $\boldsymbol{\theta}$ is shown in Fig. 8.3. Sect. 9.2: how to explicitly expand preceding expression out in terms of Gaussian distribution.

Assume: set of examples $(x_1, y_1), \dots, (x_N, y_N)$ are *independent & identically distributed* (i.i.d.). Word “independent” (Sect. 6.4.5) implies: likelihood of whole dataset $\mathcal{Y} = \{y_1, \dots, y_N\}$ & $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ factorizes into a product of likelihoods of each individual example (8.16)

$$p(\mathcal{Y}|\mathcal{X}, \boldsymbol{\theta}) = \prod_{n=1}^N p(y_n|\mathbf{x}_n, \boldsymbol{\theta}), \quad (114)$$

where $p(y_n|\mathbf{x}_n, \boldsymbol{\theta})$ is a particular distribution (which was Gaussian in Example 8.4). Expression “identically distributed” means: each term in product (8.16) is of same distribution, & all of them share same parameters. Often easier from an optimization viewpoint to compute functions that can be decomposed into sums of simpler functions. Hence, in ML, often consider negative log-likelihood (8.17)

$$\mathcal{L}(\boldsymbol{\theta}) = -\log p(\mathcal{Y}|\mathcal{X}, \boldsymbol{\theta}) = -\sum_{n=1}^N \log p(y_n|\mathbf{x}_n, \boldsymbol{\theta}). \quad (115)$$

While it is tempting to interpret fact: $\boldsymbol{\theta}$ is on right of conditioning in $p(y_n|\mathbf{x}_n, \boldsymbol{\theta})$ (8.15), & hence should be interpreted as observed & fixed, this interpretation is incorrect. Negative log-likelihood $\mathcal{L}(\boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$. Therefore, to find a good parameter vector $\boldsymbol{\theta}$ that explains data $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)$ well, minimize negative log-likelihood $\mathcal{L}(\boldsymbol{\theta})$ w.r.t. $\boldsymbol{\theta}$.

Remark 16. Negative sign in (8.17) is a historical artifact that is due to convention that we want to maximize likelihood, but numerical optimization literature tends to study minimization of functions.

Example 17. Continuing on our example of Gaussian likelihoods (8.15), negative log-likelihood can be rewritten as

$$\mathcal{L}(\boldsymbol{\theta}) = \dots = \frac{1}{2\sigma^2} \sum_{n=1}^N (y_n - \mathbf{x}_n^\top \boldsymbol{\theta})^2 - \sum_{n=1}^N \log \frac{1}{\sqrt{2\pi\sigma^2}}. \quad (116)$$

As σ is given, 2nd term in the last formula is constant, & minimizing $\mathcal{L}(\boldsymbol{\theta})$ corresponds to solving least-squares problem (cf. (8.8)) expressed in 1st term.

Turn out: for Gaussian likelihoods resulting optimization problem corresponding to maximum likelihood estimation has a closed-form solution. See more details on this in Chap. 9. Fig. 8.5: For given data, maximum likelihood estimate of parameters results in black diagonal line. Orange square shows value of maximum likelihood prediction at $x = 60$. shows a regression dataset & function that is induced by maximum-likelihood parameters. Maximum likelihood estimation may suffer from overfitting (Sect. 8.3.3), analogous to unregularized empirical risk minimization (Sect. 9.2.3). For other likelihood functions, i.e., if model our noise with non-Gaussian distributions, maximum likelihood estimation may not have a closed-form analytic solution. In this case, resort to numerical optimization methods discussed in Chap. 7.

- * 8.3.2. Maximum A Posteriori Estimation. If have prior knowledge about distribution of parameters $\boldsymbol{\theta}$, can multiply an additional term to likelihood. This additional term is a prior probability distribution on parameters $p(\boldsymbol{\theta})$. For a given prior, after observing some data \mathbf{x} , how should we update distribution of $\boldsymbol{\theta}$? I.e., how should we represent fact: have

more specific knowledge of θ after observing data \mathbf{x} ? Bayes' theorem (Sect. 6.3) gives us a principled tool to update our probability distributions of random variables. It allows us to compute a *posterior* distribution $p(\theta|\mathbf{x})$ (more specific knowledge) on parameters θ from general *prior* statements (prior distribution) $p(\theta)$ & function $p(\mathbf{x}|\theta)$ that links parameters θ & observed data \mathbf{x} (called *likelihood*):

$$p(\theta|\mathbf{x}) = \frac{p(\mathbf{x}|\theta)p(\theta)}{p(\mathbf{x})}. \quad (117)$$

Recall: interested in finding parameter θ that maximizes posterior. Since distribution $p(\mathbf{x})$ does not depend on θ , can ignore value of denominator for optimization & obtain

$$p(\theta|\mathbf{x}) \propto p(\mathbf{x}|\theta)p(\theta). \quad (118)$$

Preceding proportion relation hides density of data $p(\mathbf{x})$, which may be difficult to estimate. Instead of estimating minimum of negative log-likelihood, now estimate minimum of negative log-posterior, which is referred to as *maximum a posteriori estimation* (MAP estimation). An illustration of effect of adding a zero-mean Gaussian prior is shown in Fig. 8.6: Comparing predictions with maximum likelihood estimate & MAP estimate at $x = 60$. Prior biases slope to be less steep & intercept to be closer to 0. In this example, bias that moves intercept closer to 0 actually increases slope.

Example 18. In addition to assumption of Gaussian likelihood in previous example, assume: parameter vector is distributed as a multivariate Gaussian with zero mean, i.e., $p(\theta) = \mathcal{N}(\mathbf{0}, \Sigma)$, where Σ : covariance matrix (Sect. 6.5). Note: conjugate prior of a Gaussian is also a Gaussian (Sect. 6.6.1), & therefore expect posterior distribution to also be a Gaussian. See details of maximum a posteriori estimation in Chap. 9.

Idea of including prior knowledge about where good parameters lie is widespread in ML. An alternative view, which saw in Sect. 8.2.3, is idea of regularization, which introduces an additional term that biases resulting parameters to be close to origin. Maximum a posteriori estimation can be considered to bridge non-probabilistic & probabilistic worlds as it explicitly acknowledges need for a prior distribution but it still only produces a point estimate of parameters.

Remark 17. Maximum likelihood estimate θ_{ML} possesses following properties (Lehmann & Casella, 1998; Efron & Hastie, 2016):

- Asymptotic consistency: MLE converges to true value in limit of infinitely many observations, plus a random error that is approximately normal.
- Size of samples necessary to achieve these properties can be quite large.
- Error's variance decays in $\frac{1}{N}$, where N : number of data points.
- Especially, in “small” data regime, maximum likelihood estimation can lead to overfitting.

Principle of maximum likelihood estimation (& maximum a posteriori estimation) uses probabilistic modeling to reason about uncertainty in data & model parameters. However, have not yet taken probabilistic modeling to its full extent. In this sect, resulting training procedure still produces a point estimate of predictor, i.e., training returns 1 single set of parameter values that represent best predictor. In Sect. 8.4, will take view: parameter values should also be treated as random variables, & instead of estimating “best” values of that distribution, use full parameter distribution when making predictions.

- * **8.3.3. Model Fitting.** Consider setting where given a dataset, & interested in fitting a parameterized model to data. When talk about “fitting”, typically mean optimizing/learning model parameters so that they minimize some loss function, e.g., negative log-likelihood. With maximum likelihood (Sect. 8.3.1) & maximum a posteriori estimation (Sect. 8.3.2), already discussed 2 commonly used algorithms for model fitting.

Parametrization of model defines a model class M_θ with which we can operate. E.g., in a linear regression setting, may define relationship between inputs x & (noise-free) observations y to be $y = ax + b$, where $\theta := \{a, b\}$: model parameters. In this case, model parameters θ describe family of affine functions, i.e., straight lines with slope a , which are offset from 0 by b . Assume: data comes from a model M^* , which is unknown to us. For a given training dataset, optimize θ so that M_θ is as close as possible to M^* , where “closeness” is defined by objective function we optimize (e.g., squared loss on training data). Fig. 8.7: Model fitting. In a parametrized class M_θ of models, optimize model parameters θ to minimize distance to true (unknown) model M^* . illustrates a setting where have a small model class (indicated by circle M_θ), & data generation model M^* lies outside set of considered models. Begin our parameter search at M_{θ_0} . After optimization, i.e., when obtain best possible parameters θ^* , distinguish 3 different cases: (i) overfitting, (ii) underfitting, & (iii) fitting well. Give a high-level intuition of what these 3 concepts mean.

1 way to detect overfitting in practice: observe: model has low training risk but high test risk during cross validation (Sect. 8.2.4).

Roughly speaking, *overfitting* refers to situation where parametrized model class is too rich to model dataset generated by M^* , i.e., M_θ could model much more complicated datasets. E.g., if dataset was generated by a linear function, & define M_θ to be class of 7th-order polynomials, could model not only linear functions, but also polynomials of degree 2, 3, etc. Models that overfit typically have a large number of parameters. An observation often make: overly flexible model class M_θ uses all its modeling power to reduce training error. If training data is noisy, it will therefore find some useful signal in noise itself. This will cause enormous problems when predict away from training data. Fig. 8.8(a): Fitting (by maximum likelihood) of different model classes to a regression dataset: (a) Overfitting. (b) Underfitting. (c) Fitting well. gives an example of overfitting in context of regression where model parameters are learned by means of maximum likelihood (Sect. 8.3.1). Discuss overfitting in regression more in Sect. 9.2.2.

When run into *underfitting*, encounter opposite problem where model class M_{θ} is not rich enough. E.g., if our dataset was generated by a sinusoidal function, but θ only parametrizes straight lines, best optimization produce will not get us close to true model. However, still optimize parameters & find best straight line that models dataset. Fig. 8.8(b) shows an example of a model that underfits because it is insufficiently flexible. Models that underfit typically have few parameters. 3rd case is when parametrized model class is about right. Then, our model fits well, i.e., it neither overfits nor underfits. I.e., our mean class is just rich enough to describe dataset given. Fig. 8.8(c) shows a model that fits given dataset fairly well. Ideally, this is model we would want to work with since it has good generalization properties.

In practice, often define very rich model classes M_{θ} with many parameters, e.g. deep neural networks. To mitigate problem of overfitting, can use regularization (Sect. 8.2.3) or priors (Sect. 8.3.2). Discuss how to choose model class in Sect. 8.6.

- * 8.3.4. **Further Reading.** When considering probabilistic models, principle of maximum likelihood estimation generalizes idea of least-squares regression for linear models, discussed in detail in Chap. 9. When restricting predictor to have linear form with an additional nonlinear function φ applied to output, i.e.,

$$p(y_n|\mathbf{x}_n, \theta) = \varphi(\theta^\top \mathbf{x}_n), \quad (119)$$

can consider other models for other prediction tasks, e.g. binary classification or modeling count data (McCullagh & Nelder, 1989). An alternative view of this: consider likelihoods that are from exponential family (Sect. 6.6). Class of models, which have linear dependence between parameters & data, & have potentially nonlinear transformation φ (called a *link function*), is referred to as *generalized linear models* (Agresti, 2002, Chap. 4).

Maximum likelihood estimation has a rich history, & was originally proposed by Sir RONALD FISHER in 1930s. Will expand upon idea of a probabilistic model in Sect. 8.4. 1 debate among researchers who use probabilistic models: discussion between Bayesian & frequentist statistics. As mentioned in Sect. 6.1.1, it boils down to def of probability. Recall from Sect. 6.1: one can consider probability to be a generalization (by allowing uncertainty) of local reasoning (Cheeseman, 1985; Jaynes, 2003). Method of maximum likelihood estimation is frequentist in nature, & interested reader is pointed to Efron & Hastie (2016) for a balanced view of both Bayesian & frequentist statistics.

There are some probabilistic models where maximum likelihood estimation may not be possible. Reader is referred to more advanced statistical textbooks, e.g., Casella & Berger (2002), for approaches, e.g. method of moments, M -estimation, & estimating equations.

- o 8.4. **Probabilistic Modeling & Inference.** In ML, frequently concerned with interpretation & analysis of data, e.g., for prediction of future events & decision making. To make this task more tractable (dễ uốn nắn/làm/sai khiến), often build models that describe *generative process* that generates observed data.

E.g., can describe outcome of a coin-flip experiment (“heads” or “tails”) in 2 steps. 1st, define a parameter μ , which describes probability of “heads” as parameter of a Bernoulli distribution (Chap. 6); 2nd, can sample an outcome $x \in \{\text{head}, \text{tail}\}$ from Bernoulli distribution $p(x|\mu) = \text{Ber}(\mu)$. Parameter μ gives rise to a specific dataset \mathcal{X} & depends on coin used. Since μ is unknown in advance & can never be observed directly, need mechanisms to learn sth about μ given observed outcomes of coin-flip experiments. In following, discuss how probabilistic modeling can be used for this purpose.

- * 8.4.1. **Probabilistic models.**

A probabilistic model is specified by joint distribution of all random variables.

Probabilistic models represent uncertain aspects of an experiment as probability distributions. Benefit of using probabilistic models: they offer a unified & consistent set of tools from probability theory (Chap. 6) for modeling, inference, prediction, & model selection.

In probabilistic modeling, joint distribution $p(\mathbf{x}, \theta)$ of observed variables \mathbf{x} & hidden parameters θ of observed variables \mathbf{x} & hidden parameters θ is of central importance: It encapsulates information from following:

- Prior & likelihood (product rule, Sect. 6.3).
- Marginal likelihood $p(\mathbf{x})$, which will play an important role in model selection (Sect. 8.6), can be computed by taking joint distribution & integrating out parameters (sum rule, Sect. 6.3).
- Posterior, which can be obtained by dividing joint by marginal likelihood.

Only joint distribution has this property. Therefore, a probabilistic model is specified by joint distribution of all its random variables.

- * 8.4.2. **Bayesian Inference.**

Parameter estimation can be phrased as an optimization problem.

A key task in ML: take a model & data to uncover values of model’s hidden variables θ given observed variables \mathbf{x} . In Sect. 8.3.1, already discussed 2 ways for estimating model parameters θ using maximum likelihood or maximum a posteriori estimation. In both cases, obtain a single-best value for θ so that key algorithmic problem of parameter estimation is solving an optimization problem. Once these point estimates θ^* are known, use them to make predictions. More specifically, predictive distribution will be $p(\mathbf{x}|\theta^*)$, where use θ^* in likelihood function.

As discussed in Sect. 6.3, focusing solely on some statistics of posterior distribution (e.g. parameter θ^* that maximizes posterior) leads to loss of information, which can be critical in a system that uses prediction $p(\mathbf{x}|\theta^*)$ to make decisions. These decision-making systems typically have different objective functions than likelihood, a squared-error loss or a mis-classification error. Therefore, having full posterior distribution around can be extremely useful & leads to more robust decisions. *Bayesian inference* is about finding this posterior distribution (Gelman et al., 2004). For a dataset \mathcal{X} , a

parameter prior $p(\boldsymbol{\theta})$ & a likelihood function, posterior

$$p(\boldsymbol{\theta}|\mathcal{X}) = \frac{p(\mathcal{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{X})}, \quad p(\mathcal{X}) = \int p(\mathcal{X}|\boldsymbol{\theta})p(\boldsymbol{\theta}) d\boldsymbol{\theta}, \quad (120)$$

is obtained by applying Bayes' theorem. Key idea: exploit Bayes' theorem to invert relationship between parameters $\boldsymbol{\theta}$ & data \mathcal{X} (given by likelihood) to obtain posterior distribution $p(\boldsymbol{\theta}|\mathcal{X})$.

Bayesian inference is about learning distribution of random variables.

Bayesian inference inverts relationship between parameters & data.

Implication of having a posterior distribution on parameters: it can be used to propagate uncertainty from parameters to data. More specifically, with a distribution $p(\boldsymbol{\theta})$ on parameters, our predictions will be (8.23)

$$p(\mathbf{x}) = \int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta}) d\boldsymbol{\theta} = \mathbb{E}_{\boldsymbol{\theta}}[p(\mathbf{x}|\boldsymbol{\theta})], \quad (121)$$

& they no longer depend on model parameters $\boldsymbol{\theta}$, which have been marginalized/integrated out. (8.23) reveals: prediction is an average over all plausible parameter values $\boldsymbol{\theta}$, where plausibility is encapsulated by parameter distribution $p(\boldsymbol{\theta})$.

Having discussed parameter estimation in Sect. 8.3 & Bayesian inference here, compare these 2 approaches to learning. Parameter estimation via maximum likelihood or MAP estimation yields a consistent point estimate $\boldsymbol{\theta}^*$ of parameters, & key computational problem to be solved is optimization. In contrast, Bayesian inference yields a (posterior) distribution, & key computational problem to be solved is integration. Predictions with point estimates are straightforward, whereas predictions in Bayesian framework require solving another integration problem; see (8.23). However, Bayesian inference gives us a principled way to incorporate prior knowledge, account for side information, & incorporate structural knowledge, all of which is not easily done in context of parameter estimation. Moreover, propagation of parameter uncertainty to prediction can be valuable in decision-making systems for risk assessment & exploration in context of data-efficient learning (Deisenroth et al., 2015; Kamthe & Deisenroth, 2018).

While Bayesian inference is a mathematically principled framework for learning about parameters & making predictions, there are some practical challenges that come with it because of integration problems need to solve; see (8.22) & (8.23). More specifically, if do not choose a conjugate prior on parameters (Sect. 6.6.1), integrals in (8.22) & (8.23) are not analytically tractable, & cannot compute posterior, predictions, or marginal likelihood in closed form. In these cases, need to resort to approximations. Here, can use stochastic approximations, e.g. Markov chain Monte Carlo (MCMC) (Gilks et al., 1996), or deterministic approximations, e.g. Laplace approximation (Bishop, 2006; Barber, 2012; Murphy, 2012), variational inference (Jordan et al., 1999; Blei et al., 2017), or expectation propagation (Minka, 2001a).

Despite these challenges, Bayesian inference has been successfully applied to a variety of problems, including large-scale topic modeling (Hoffman et al., 2013), click-through-rate prediction (Graepel et al., 2010), data-efficient reinforcement learning in control systems (Deisenroth et al., 2015), online ranking systems (Herbrich et al., 2007), & large-scale recommender systems. There are generic tools, e.g. Bayesian optimization (Brochu et al., 2009; Snoek et al., 2012; Shahriari et al., 2016), that are very useful ingredients for an efficient search of meta parameters of models or algorithms.

Remark 18. *In ML literature, there can be a somewhat arbitrary separation between (random) “variables” & “parameters”. While parameters are estimated (e.g., via maximum likelihood), variables are usually marginalized out. In this book, not so strict with this separation because, in principle, can replace a prior on any parameter & integrate it out, which would then turn parameter into a random variable according to aforementioned separation.*

* **8.4.3. Latent-Variable Models.** In practice, sometimes useful to have additional *latent variables* \mathbf{z} (besides model parameters $\boldsymbol{\theta}$) as part of model (Moustaki et al., 2015). These latent variables are different from model parameters $\boldsymbol{\theta}$ as they do not parameterize model explicitly. Latent variables may describe data-generating process, thereby contributing to interpretability of model. They also often simplify structure of model & allow us to define simpler & richer model structures. Simplification of model structure often goes hand in hand with a smaller number of model parameters (Paquet, 2008; Murphy, 2012). Learning in latent-variable models (at least via maximum likelihood) can be done in a principled way using expectation maximization (EM) algorithm (Dempster et al., 1977; Bishop, 2006). Examples, where such latent variables (biến tiềm ẩn) are helpful, are principle component analysis for dimensionality reduction (Chap. 10), Gaussian mixture models for density estimation (Chap. 11), hidden Markov models (Maybeck, 1979) or dynamical systems (Ghahramani & Roweis, 1999; Ljung, 1999) for time-series modeling, & meta learning & task generalization (Hausman et al., 2018; Sæmundsson et al., 2018). Although introduction of these latent variables may make model structure & generative process easier, learning in latent-variable models is generally hard, as see in Chap. 11.

Since latent-variable models also allow us to define process that generates data from parameters, have a look at this generative process. Denoting data by \mathbf{x} , model parameters by $\boldsymbol{\theta}$ & latent variables by \mathbf{z} , obtain conditional distribution $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$ that allows us to generate data for any model parameters & latent variables. Given that \mathbf{z} are latent variables, place a prior $p(\mathbf{z})$ on them.

As models discussed previously, models with latent variables can be used for parameter learning & inference within frameworks discussed in Sects. 8.3 & 8.4.2. To facilitate learning (e.g., by means of maximum likelihood estimation or Bayesian inference), follow a 2-step procedure. 1st, compute likelihood $p(\mathbf{x}|\boldsymbol{\theta})$ of model, which does not depend on latent variables. 2nd, use this likelihood for parameter estimation or Bayesian inference, where use exactly same expressions as in Sects. 8.3 & 8.4.2, resp.

Since likelihood function $p(\mathbf{x}|\boldsymbol{\theta})$ is predictive distribution of data given model parameters, need to marginalize out latent

variables so that (8.25)

$$p(\mathbf{x}|\boldsymbol{\theta}) = \int p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})p(\mathbf{z}) d\mathbf{z}, \quad (122)$$

where $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$ is given in (8.24) & $p(\mathbf{z})$ is prior on latent variables. Note: likelihood must not depend on latent variables \mathbf{z} , but it is only a function of data \mathbf{x} & model parameters $\boldsymbol{\theta}$.

Likelihood is a function of data & model parameters, but is independent of latent variables.

Likelihood in (8.25) directly allows for parameter estimation via maximum likelihood. MAP estimation is also straightforward with an additional prior on model parameters $\boldsymbol{\theta}$ as discussed in Sect. 8.3.2. Moreover, with likelihood (8.25) Bayesian inference (Sect. 8.4.2) in a latent-variable model works in usual way: Place a prior $p(\boldsymbol{\theta})$ on model parameters & use Bayes' theorem to obtain a posterior distribution (8.26)

$$p(\boldsymbol{\theta}|\mathcal{X}) = \frac{p(\mathcal{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{X})} \quad (123)$$

over model parameters given a dataset \mathcal{X} . Posterior in (8.26) can be used for predictions within a Bayesian inference framework; see (8.23).

1 challenge have in this latent-variable model: likelihood $p(\mathcal{X}|\boldsymbol{\theta})$ requires marginalization of latent variables according to (8.25). Except when choose a conjugate prior $p(\mathbf{z})$ for $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$, marginalization in (8.25) is not analytically tractable, & need to resort to approximations (Bishop, 2006; Paquet, 2008; Murphy, 2012; Moustaki et al., 2015).

Similar to parameter posterior (8.26), can compute a posterior on latent variables according to

$$p(\mathbf{z}|\mathcal{X}) = \frac{p(\mathcal{X}|\mathbf{z})p(\mathbf{z})}{p(\mathcal{X})}, \quad p(\mathcal{X}|\mathbf{z}) = \int p(\mathcal{X}|\mathbf{z}, \boldsymbol{\theta})p(\boldsymbol{\theta}) d\boldsymbol{\theta}, \quad (124)$$

where $p(\mathbf{z})$: prior on latent variables & $p(\mathcal{X}|\mathbf{z})$ requires us to integrate out model parameters $\boldsymbol{\theta}$.

Given difficulty of solving integrals analytically, clear: marginalizing out both latent variables & model parameters at same time is not possible in general (Bishop, 2006; Murphy, 2012). A quantity that is easier to compute is posterior distribution on latent variables, but conditioned on model parameters, i.e.,

$$p(\mathbf{z}|\mathcal{X}, \boldsymbol{\theta}) = \frac{p(\mathcal{X}|\mathbf{z}, \boldsymbol{\theta})p(\mathbf{z})}{p(\mathcal{X}|\boldsymbol{\theta})}, \quad (125)$$

where $p(\mathbf{z})$: prior on latent variables & $p(\mathcal{X}|\mathbf{z}, \boldsymbol{\theta})$ is given i (8.24).

In Chaps. 10–11, derive likelihood functions for PCA & Gaussian mixture models, resp. Moreover, compute posterior distributions (8.28) on latent variables for both PCA & Gaussian mixture models.

Remark 19. In following chaps, may not be drawing such a clear distinction between latent variables \mathbf{z} & uncertain model parameters $\boldsymbol{\theta}$ & call model parameters “latent” or “hidden” as well because they are unobserved. In Chaps. 10–11, where use latent variables \mathbf{z} , will pay attention to difference as will have 2 different types of hidden variables: model parameters $\boldsymbol{\theta}$ & latent variables \mathbf{z} .

Can exploit fact: all elements of a probabilistic model are random variables to define a unified language for representing them. In Sect. 8.5, see a concise graphical language for representing structure of probabilistic models. Use this graphical language to describe probabilistic models in subsequent chaps.

- * 8.4.4. **Further Reading.** Probabilistic models in ML (Bishop, 2006; Barber, 2012; Murphy, 2012) provide a way for users to capture uncertainty about data & predictive models in a principled fashion. Ghahramani (2015) presents a short review of probabilistic models in ML. Given a probabilistic model, may be lucky enough to be able to compute parameters of interest analytically. However, in general, analytic solutions are rare, & computational methods e.g. sampling (Gilks et al., 1996; Brooks et al., 2011) & variational inference (Jordan et al., 1999; Blei et al., 2017) are used. Moustaki et al. (2015) & Paquet (2008) provide a good overview of Bayesian inference in latent-variable models.

In recent years, several programming languages have been proposed that aim to treat variables defined in software as random variables corresponding to probability distributions. Objective: be able to write complex functions of probability distributions, while under hood compiler automatically takes care of rules of Bayesian inference. This rapidly changing field is called *probabilistic programming*.

- o 8.5. **Directed Graphical Models.** In this sect, introduce a graphical language for specifying a probabilistic model, called *directed graphical model*. It provides a compact & succinct way to specify probabilistic models, & allows reader to visually parse dependencies between random variables. A graphical model visually captures way in which joint distribution over all random variables can be decomposed into a product of factors depending only on a subset of these variables. In Sect. 8.4, we identified joint distribution of a probabilistic model as key quantity of interest because it comprises information about prior, likelihood, & posterior. However, joint distribution by itself can be quite complicated, & it does not tell us anything about structural properties of probabilistic model. E.g, joint distribution $p(a, b, c)$ does not tell us anything about independence relations. This is point where graphical models come into play. This sect relies on concepts of independence & conditional independence, as described in Sect. 6.4.5.

Directed graphical models are also known as Bayesian networks.

It a *graphical model*, nodes are random variables. In Fig. 8.9: Examples of directed graphical models: (a) Fully connected. (b) Not fully connected(a), nodes represent random variables a, b, c . Edges represent probabilistic relations between variables, e.g., conditional probabilities.

Remark 20. *Not every distribution can be represented in a particular choice of graphical model. A discussion of this can be found in Bishop (2006).*

Probabilistic graphical models have some convenient properties:

- * they are a simple way to visualize structure of a probabilistic model.
- * Inspection of graph alone gives us insight into properties, e.g., conditional independence.
- * Complex computations for inference & learning in statistical models can be expressed in terms of graphical manipulations.
- * **8.5.1. Graph Semantics.** (Ngữ nghĩa đồ thị) *Directed graphical models/Bayesian networks* are a method for representing conditional dependencies in a probabilistic model. They provide a visual description of conditional probabilities, hence, providing a simple language for describing complex interdependence. Modular description also entails computational simplification. Directed links (arrows) between 2 nodes (random variables) indicate conditional probabilities. E.g., arrow between a, b in Fig. 8.9(a) gives conditional probability $p(b|a)$ of b given a .

With additional assumptions, arrows can be used to indicate causal relationships (Pearl, 2009).

Directed graphical models can be derived from joint distributions if know sth about their factorization.

Example 19. *Consider joint distribution $p(a, b, c) = p(c|a, b)p(b|a)p(a)$ (8.29) of 3 random variables a, b, c . Factorization of joint distribution in (8.29) tells us sth about relationship between random variables: c depends directly on a & b , b depends directly on a , a depends neither on b nor on c .*

For factorization in (8.29), obtain directed graphical model in Fig. 8.9(a).

In general, can construct corresponding directed graphical model from a factorized joint distribution as follows:

1. Create a node \forall random variables.
2. For each conditional distribution, add a directed link (arrow) to graph from nodes corresponding to variables on which distribution is conditioned.

Graph layout depends on factorization of joint distribution.

Graph layout depends on choice of factorization of joint distribution.

Discussed how to get from a known factorization of joint distribution to corresponding directed graphical model. Now, will do exactly opposite & describe how to extract joint distribution of a set of random variables from a given graphical model.

Example 20. *Looking at graphical model in Fig. 8.9(b), exploit 2 properties: (i) Joint distribution $p(x_1, \dots, x_5)$ we seek is product of a set of conditionals, one for each node in graph. In this particular example, need 5 conditionals. (ii) Each conditional depends only on parents of corresponding node in graph. E.g., x_4 will be conditioned on x_2 .*

These 2 properties yield desired factorization of joint distribution $p(x_1, x_2, x_3, x_4, x_5) = p(x_1)p(x_5)p(x_2|x_5)p(x_3|x_1, x_2)p(x_4|x_2)$.

In general, joint distribution $p(\mathbf{x}) = p(x_1, \dots, x_K)$ is given as

$$p(\mathbf{x}) = \prod_{k=1}^K p(x_k | \mathbf{Pa}_k), \quad (126)$$

where \mathbf{Pa}_k means “parent nodes of x_k ”. Parent nodes of x_k are nodes that have arrows pointing to x_k .

Conclude this subsection with a concrete example of coin-flip experiment. Consider a Bernoulli experiment (Example 6.8) where probability that outcome x of this experiment is “heads” is

$$p(x|\mu) = \text{Ber}(\mu). \quad (127)$$

Now repeat this experiment N times & observe outcomes x_1, \dots, x_N so that obtain joint distribution

$$p(x_1, \dots, x_N | \mu) = \prod_{n=1}^N p(x_n | \mu). \quad (128)$$

Expression on RHS is a product of Bernoulli distributions on each individual outcome because experiments are independent. Recall from Sect. 6.4.5: statistical independence means: distribution factorizes. To write graphical model down for this setting, make distinction between unobserved/latent variables & observed variables. Graphically, observed variables are denoted by shaded nodes so that obtain graphical model in Fig. 8.10(a): Graphical models for a repeated Bernoulli experiment: (a) Versions with x_n explicit. (b) Version with plate notation. (c) Hyperparameters α, β on latent μ . See: single parameter μ is same $\forall x_n, n = 1, \dots, N$ as outcomes x_n are identically distributed. A more compact, but equivalent, graphical model for this setting is given in Fig. 8.10(b), where use *plate* notation. Plate (box) repeats everything inside (in this case, observations x_n) N times. Therefore, both graphical models are equivalent, but plate notation is more compact. Graphical models immediately allow us to place a hyperprior on μ . A *hyperprior* is a 2nd layer of prior distributions on parameters of 1st layer of priors. Fig. 8.10(c) places a $\text{Beta}(\alpha, \beta)$ prior on latent variable μ . If treat α, β as deterministic parameters, i.e., not random variables, omit circle around it.

- * **8.5.2. Conditional Independence & d -Separation.** Directed graphical models allow us to find conditional independence (Sect. 6.4.5) relationship properties of joint distribution only by looking at graph. A concept called *d -separation* (Pearl, 1988) is key to this.

Consider a general directed graph in which $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are arbitrary nonintersecting sets of nodes (whose union may be smaller than complete set of nodes in graph). Wish to ascertain whether a particular conditional independence statement, “ \mathcal{A} is conditionally independent of \mathcal{B} given \mathcal{C} ”, denoted by $\mathcal{A} \perp\!\!\!\perp \mathcal{B} | \mathcal{C}$, is implied by a given directed acyclic graph. To do so, consider all possible trails (paths that ignore direction of arrows) from any node in \mathcal{A} to any nodes in \mathcal{B} . Any such path is said to be *blocked* if it includes any node s.t. either of following are true:

- Arrows on path meet either head to tail or tail to tail at node, & node is in set \mathcal{C} .
- Arrows meet head to head at node, & neither node nor any of its descendants is in set \mathcal{C} .

If all paths are blocked, then \mathcal{A} is said to be *d-separated* from \mathcal{B} by \mathcal{C} , & joint distribution over all of variables in graph will satisfy $\mathcal{A} \perp\!\!\!\perp \mathcal{B} | \mathcal{C}$.

Example 21 (Conditional Independence). *Consider graphical model in Fig. 8.11: D-separation example. Visual inspection gives us $b \perp\!\!\!\perp d | a, c$, $a \perp\!\!\!\perp c | b$, $b \perp\!\!\!\perp d | c$, $a \perp\!\!\!\perp c | b, e$.*

Directed graphical models allow a compact representation of probabilistic models, & will see examples of directed graphical models in Chaps. 9–11. Representation, along with concept of conditional independence, allows us to factorize respective probabilistic models into expressions that are easier to optimize.

Graphical representation of probabilistic model allows us to visually see impact of design choices we have made on structure of model. Often need to make high-level assumptions about structure of model. These modeling assumptions (hyperparameters) affect prediction performance, but cannot be selected directly using approaches have seen so far. Discuss different ways to choose structure in Sect. 8.6.

- * **8.5.3. Further Reading.** An introduction to probabilistic graphical models can be found in Bishop (2006, Chap. 8), & an extensive description of different applications & corresponding algorithmic implications can be found in book by Koller & Friedman (2009). There are 3 main types of probabilistic graphical models

- *Directed graphical models (Bayesian networks)*; see Fig. 8.12(a): 3 types of graphical models: (a) Directed graphical models (Bayesian networks); (b) Undirected graphical models (Markov random fields); (c) Factor graphs.
- *Undirected graphical models (Markov random fields)*; see Fig. 8.12(a)
- *Factor graphs*; see Fig. 8.12(c)

Graphical models allow for graph-based algorithms for inference & learning, e.g., via local message passing. Application range from ranking in online games (Herbrich et al., 2007) & computer vision (e.g., image segmentation, semantic labeling, image denoising, image restoration (Kittler & Föglein, 1984; Sucar & Gillies, 1994; Shotton et al., 2006; Szeliski et al., 2008)) to coding theory (McEliece et al., 1998), solving linear equation systems (Shental et al., 2008), & iterative Bayesian state estimation in signal processing (Bickson et al., 2007; Deisenroth & Mohamed, 2012).

1 topic particularly important in real applications that we do not discuss in this book: idea of structured prediction (Bakir et al., 2007; Nowozin et al., 2014), which allows ML models to tackle predictions that are structured, e.g. sequences, trees, & graphs. Popularity of neural network models has allowed more flexible probabilistic models to be used, resulting in many useful applications of structured models (Goodfellow et al., 2016, Chap. 16). In recent years, there has been a renewed interest in graphical models due to their applications to causal inference (Pearl, 2009; Imbens & Rubin, 2015; Peters et al., 2017; Rosenbaum, 2017).

- **8.6. Model Selection.** In ML, often need to make high-level modeling decisions that critically influence performance of model. Choices we make (e.g., functional form of likelihood) influence number & type of free parameters in model & thereby also flexibility & expressivity of model. More complex models are more flexible in sense that they can be used to describe more datasets. E.g., a polynomial of degree 1 (a line $y = a_0 + a_1x$) can only be used to describe linear relations between inputs x & observations y . A polynomial of degree 2 can additionally describe quadratic relationships between inputs x & observations y . A polynomial of degree 2 can additionally describe quadratic relationships between inputs & observations.

One would now think: very flexible models are generally preferable to simple models because they are more expressive. A general problem: at training time can only use training set to evaluate performance of model & learn its parameters. However, performance on training set is not really what we are interested in. In Sect. 8.3, have seen: maximum likelihood estimation can lead to overfitting, especially when training dataset is small. Ideally, our model (also) works well on test set (which is not available at training time). Therefore, need some mechanisms for assessing how a model *generalizes* to unseen test data. *Model selection* is concerned with exactly this problem.

A polynomial $y = a_0 + a_1x + a_2x^2$ can also describe linear functions by setting $a_2 = 0$, i.e., strictly more expressive than a 1st-order polynomial.

- **8.6.1. Nested Cross-Validation.** Have already seen an approach (cross-validation in Sect. 8.2.4) that can be used for model selection. Recall: cross-validation provides an estimate of generalization error by repeatedly splitting dataset into training & validation sets. Can apply this idea 1 more time, i.e., for each split, can perform another round of cross-validation. This is sometimes referred to as *nested cross-validation*; see Fig. 8.13: Nested cross-validation. Perform 2 levels of *K-fold cross-validation*. Inner level is used to estimate performance of a particular choice of model or hyperparameter on a internal validation set. Outer level is used to estimate generalization performance for best choice of model chosen by inner loop. Can test different model & hyperparameter choices in inner loop. To distinguish 2 levels, set used to estimate generalization performance is often called *test case* & set used for choosing best model is called *validation set*. Inner loop estimates expected value of generalization error for a given model (8.39), by approximating it using empirical error on validation set, i.e.,

$$\mathbb{E}_{\mathcal{V}}[\mathbf{R}(\mathbf{V}|M)] \approx \frac{1}{K} \sum_{k=1}^K \mathbf{R}(\mathcal{V}^{(k)}|M), \quad (129)$$

where $\mathbf{R}(\mathcal{V}|M)$ is empirical risk (e.g., root mean square error) on validation set \mathcal{V} for model M . Repeat this procedure \forall models & choose model that performs best. Note: cross-validation not only gives us expected generalization error, but can also obtain high-order statistics, e.g., standard error, an estimate of how uncertain mean estimate is. Once model is chosen, can evaluate final performance on test set.

- 8.6.2. Bayesian Model Selection.
- 8.6.3. Bayes Factors for Model Comparison.
- 8.6.4. Further Reading. Mentioned at start of sect: there are high-level modeling choices that influence performance of model. Examples include:
 - * Degree of a polynomial in a regression setting
 - * Number of components in a mixture model
 - * Network architecture of a (deep) neural network
 - * Type of kernel in a support vector machine
 - * Dimensionality of latent space in PCA
 - * Learning rate (schedule) in an optimization algorithm

In parametric models, number of parameters is often related to complexity of model class.

Rasmussen & Ghahramani (2001) showed: automatic Occam’s razor does not necessarily penalize number of parameters in a model, but it is active in terms of complexity of functions. Also showed: automatic Occam’s razor also holds for Bayesian nonparametric models with many parameters, e.g., Gaussian processes.

If focus on maximum likelihood estimate, there exist a number of heuristics for model selection that discourage overfitting. They are called *information criteria*, & choose model with largest value. *Akaike information criterion* (AIC) (Akaike, 1974) $\log p(\mathbf{x}|\theta) - M$ corrects for bias of maximum likelihood estimator by addition of a penalty term to compensate for overfitting of more complex models with lots of parameters. Here, M : number of model parameters. AIC estimates relative information lost by a given model.

Bayesian information criterion (BIC) (Schwarz, 1978)

$$\log p(\mathbf{x}) = \log \int p(\mathbf{x}|\theta)p(\theta) d\theta \approx \log p(\mathbf{x}|\theta) - \frac{1}{2}M \log N \quad (130)$$

can be used for exponential family distributions. Here, N : number of data points & M : number of parameters. BIC penalizes model complexity more heavily than AIC.

- 9. Linear Regression. In following, will apply mathematical concepts from Chaps. 2, 5, 6, & 7 to solve linear regression (curve fitting) problems. In *regression*, aim: find a function f that maps input $\mathbf{x} \in \mathbb{R}^D$ to corresponding function values $f(\mathbf{x}) \in \mathbb{R}$. Assume: given a set of training inputs \mathbf{x}_n & corresponding noisy observations $y_n = f(\mathbf{x}_n) + \varepsilon$, where ε is an i.i.d. random variable that describes measurement/observation noise & potentially unmodeled processes (which will not consider further in this chap). Throughout this chap, assume zero-mean Gaussian noise. Our task: find a function that not only models training data, but generalizes well to predicting function values at input locations that are not part of training data (see Chap. 8). An illustration of such a regression problem is given in Fig. 9.1: (a) Dataset: Regression problem: observed noisy function values from which we wish to infer underlying function that generated data. (b) Possible solution to regression problem: Regression solution: possible function that could have generated data (blue) with indication of measurement noise of function value at corresponding inputs (orange distributions). A typical regression setting is given in Fig. 9.1(a): For some input values x_n , observe (noisy) function values $y_n = f(x_n) + \varepsilon$. Task: infer function f that generated data & generalizes well to function values at new input locations. A possible solution is given in Fig. 9.1(b), where we also show 3 distributions centered at function values $f(x)$ that represent noise in data.

Regression is a fundamental problem in ML, & regression problems appear in a diverse range of research areas & applications, including time-series analysis (e.g., system identification), control & robotics (e.g, reinforcement learning, forward/inverse model learning), optimization (e.g., line searches, global optimization), & DL applications (e.g., computer games, speech-to-text translation, image recognition, automatic video annotation). Regression is also a key ingredient of classification algorithms. Finding a regression function requires solving a variety of problems, including following:

- **Choice of model (type) & parametrization** of regression function. Given a dataset, what function classes (e.g., polynomials) are good candidates for modeling data, & what particular parametrization (e.g., degree of polynomial) should we choose? Model selection, as discussed in Sect. 8.6, allows us to compare various models to find simplest model that explains training data reasonably well.
- **Finding good parameters.** Having chosen a model of regression function, how to we find good model parameters? Here, need to look at different loss/objective functions (they determine what a “good” fit is) & optimization algorithms that allow us to minimize this loss.
- **Overfitting & model selection.** Overfitting is a problem when regression function fits training data “too well” but does not generalize to unseen test data. Overfitting typically occurs if underlying model (or its parameterization) is overly flexible & expressive; see Sect. 8.6. Look at underlying reasons & discuss ways to mitigate effect of overfitting in context of linear regression.

- **Relationship between loss functions & parameter priors.** Loss functions (optimization objectives) are often motivated & included by probabilistic models. Look at connection between loss functions & underlying prior assumptions that induce these losses.
- **Uncertainty modeling.** In any practical setting, have access to only a finite, potentially large, amount of (training) data for selecting model class & corresponding parameters. Given that this finite amount of training data does not cover all possible scenarios, may want to describe remaining parameter uncertainty to obtain a measure of confidence of model's prediction at test time; smaller training set, more important uncertainty modeling. Consistent modeling of uncertainty equips model predictions with confidence bounds.

In following, will be using mathematical tools from Chaps. 3, 5, 6, & to solve linear regression problems. Discuss maximum likelihood & maximum a posteriori (MAP) estimation to find optimal model parameters. Using these parameter estimates, have a brief look at generalization errors & overfitting. Toward end of this chap, will discuss Bayesian linear regression, which allows us to reason about model parameters at a high level, thereby removing some of problems encountered in maximum likelihood & MAP estimation.

- **9.1. Problem Formulation.** Because of presence of observation noise, adopt a probabilistic approach & explicitly model noise using a likelihood function. More specifically, throughout this chap, consider a regression problem with likelihood function (9.1)

$$p(y|\mathbf{x}) = \mathcal{N}(y|f(\mathbf{x}), \sigma^2). \quad (131)$$

Here $\mathbf{x} \in \mathbb{R}^D$: inputs & $y \in \mathbb{R}$: noisy function values (targets). With (9.1), functional relationship between \mathbf{x} & y is given as

$$y = f(\mathbf{x}) + \epsilon, \quad (132)$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ is independent, identically distributed (i.i.d.) Gaussian measurement noise with mean 0 & variance σ^2 . Objective: find a function that is close (similar) to unknown function f that generated data & that generalizes well.

In this chap, focus on parametric models, i.e., choose a parametrized function & find parameters $\boldsymbol{\theta}$ that “work well” for modeling data. For time being, assume: noise variance σ^2 is known & focus on learning model parameters $\boldsymbol{\theta}$. In linear regression, consider special case: parameters $\boldsymbol{\theta}$ appear linearly in our model. An example of linear regression is given by (9.3)–(9.4)

$$p(y|\mathbf{x}, \boldsymbol{\theta}) = \mathcal{N}(y|\mathbf{x}^\top \boldsymbol{\theta}, \sigma^2) \Leftrightarrow y = \mathbf{x}^\top \boldsymbol{\theta} + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2), \quad (133)$$

where $\boldsymbol{\theta} \in \mathbb{R}^D$: parameters we seek. Class of functions described by (9.4) are straight lines that pass through origin. In (9.4), chose a parametrization $f(\mathbf{x}) = \mathbf{x}^\top \boldsymbol{\theta}$.

Likelihood in (9.3) is probability density function of y evaluated at $\mathbf{x}^\top \boldsymbol{\theta}$. Note: only source of uncertainty originates from observation noise (as \mathbf{x} & $\boldsymbol{\theta}$ are assumed known in (9.3)). Without observation noise, relationship between \mathbf{x} & y would be deterministic & (9.3) would be a Dirac delta.

A Dirac delta (delta function) is zero everywhere except at a single point, & its integral is 1. It can be considered a Gaussian in limit of $\sigma^2 \rightarrow 0$.

Example 22. For $x, \theta \in \mathbb{R}$ linear regression model in (9.4) describes straight lines (linear functions), & parameter θ : slope of line. Fig. 9.2(a) Linear regression example. (a) Example functions that fall into this category: Example functions (straight lines) that can be described using linear model in (9.4). (b) training set. (c) maximum likelihood estimate. shows some example functions for different values of θ .

Linear regression refers to models that are linear in parameters.

Linear regression model in (9.3)–(9.4) is not only linear in parameters, but also linear in inputs x . Fig. 9.2(a) shows examples of such functions. $y = \boldsymbol{\phi}^\top(\mathbf{x})\boldsymbol{\theta}$ for nonlinear transformations $\boldsymbol{\phi}$ is also a linear regression model because “linear regression” refers to models that are “linear in parameters”, i.e., models that describe a function by a linear combination of input features. Here, a “feature” is a representation $\boldsymbol{\phi}(\mathbf{x})$ of inputs \mathbf{x} .

In following, discuss in more detail how to find good parameters $\boldsymbol{\theta}$ & how to evaluate whether a parameter set “works well”. For time being, assume: noise variation σ^2 is known.

- **9.2. Parameter Estimation.** Consider linear regression setting (9.4) & assume given a *training set* $\mathcal{D} := \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)\}$ consisting of N inputs $\mathbf{x}_n \in \mathbb{R}^D$ & corresponding observations/targets $y_n \in \mathbb{R}$, $n = 1, \dots, N$. Corresponding graphical model is given in Fig. 9.3: Probabilistic graphical model for linear regression. Observed random variables are shaded, deterministic/known values are without circles.. Note: y_i & y_j are conditionally independent given their respective inputs $\mathbf{x}_i, \mathbf{x}_j$ so that likelihood factorizes according to

$$p(\mathcal{Y}|\mathcal{X}, \boldsymbol{\theta}) = p(y_1, \dots, y_N|\mathbf{x}_1, \dots, \mathbf{x}_N, \boldsymbol{\theta}) = \prod_{i=1}^N p(y_n|\mathbf{x}_n, \boldsymbol{\theta}) = \prod_{n=1}^N \mathcal{N}(y_n|\mathbf{x}_n^\top \boldsymbol{\theta}, \sigma^2), \quad (134)$$

where defined $\mathcal{X} := \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ & $\mathcal{Y} := \{y_1, \dots, y_N\}$ as sets of training inputs & corresponding targets, resp. Likelihood & factors $p(y_n|\mathbf{x}_n, \boldsymbol{\theta})$ are Gaussian due to noise distribution; see (9.3).

In following, discuss how to find optimal parameters $\boldsymbol{\theta}^* \in \mathbb{R}^D$ for linear regression model (9.4). Once parameters $\boldsymbol{\theta}^*$ are found, can predict function values by using this parameter estimate in (9.4) so that at an arbitrary test input \mathbf{x}_* distribution of corresponding target y_* is

$$p(y_*, \mathbf{x}_*, \boldsymbol{\theta}^*) = \mathcal{N}(y_*|\mathbf{x}_*^\top \boldsymbol{\theta}^*, \sigma^2). \quad (135)$$

In following, have a look at parameter estimation by maximizing likelihood, a topic that already covered to some degree in Sect. 8.3.

* **9.2.1. Maximum Likelihood Estimation.** A widely used approach to finding desired parameters θ_{ML} is *maximum likelihood estimation*, where find parameters θ_{ML} that maximize likelihood (9.5b). Intuitively, maximizing likelihood means maximizing predictive distribution of training data given model parameters. Obtain maximum likelihood parameters as

$$\theta_{\text{ML}} = \arg \max_{\theta} p(\mathcal{Y}|\mathcal{X}, \theta). \quad (136)$$

Remark 21. *Likelihood $p(\mathbf{y}|\mathbf{x}, \theta)$ is not a probability distribution in θ : It is simply a function of parameters θ but does not integrate to 1 (i.e., it is unnormalized), & may not even be integrable w.r.t. θ . However, likelihood in (9.7) is a normalized probability distribution in \mathbf{y} .*

Maximizing likelihood means maximizing predictive distribution of (training) data given parameters.

Likelihood is not a probability distribution in parameters.

To find desired parameters θ_{ML} that maximize likelihood, typically perform gradient ascent (or gradient descent on negative likelihood). In case of linear regression considered here, however, a closed-form solution exists, which makes iterative gradient descent unnecessary. In practice, instead of maximizing likelihood directly, apply log-transformation to likelihood function & minimize negative log-likelihood.

Since logarithm is a (strictly) monotonically increasing function, optimum of a function f is identical to optimum of $\log f$.

Remark 22 (Log-Transformation). *Since likelihood (9.5b) is a product of N Gaussian distributions, log-transformation is useful since (a) it does not suffer from numerical underflow, & (b) differentiation rules will turn out simpler. More specifically, numerical underflow will be a problem when multiply N probabilities, where N : number of data points, since cannot represent very small numbers, e.g. 10^{-256} . Furthermore, log-transform will turn product into a sum of log-probabilities s.t. corresponding gradient is a sum of individual gradients, instead of a repeated application of product rule to compute gradient of a product of N terms.*

To find optimal parameters θ_{ML} of our linear regression problem, minimize negative log-likelihood (9.8)

$$-\log p(\mathcal{Y}|\mathcal{X}, \theta) = -\log \prod_{n=1}^N p(y_n|\mathbf{x}_n, \theta) = -\sum_{n=1}^N \log p(y_n|\mathbf{x}_n, \theta), \quad (137)$$

where exploited: likelihood (9.5b) factorizes over number of data points due to our independence assumption on training set.

In linear regression model (9.4), likelihood is Gaussian (due to Gaussian additive noise term), s.t. arrive at

$$\log p(y_n|\mathbf{x}_n, \theta) = -\frac{1}{2\sigma^2}(y_n - \mathbf{x}_n^\top \theta)^2 + \text{const}, \quad (138)$$

where constant includes all terms independent of θ . Using (9.9) in negative log-likelihood (9.8), obtain (ignoring constant terms)

$$\mathcal{L}(\theta) := \frac{1}{2\sigma^2} \sum_{n=1}^N (y_n - \mathbf{x}_n^\top \theta)^2 = \frac{1}{2\sigma^2} (\mathbf{y} - \mathbf{X}\theta)^\top (\mathbf{y} - \mathbf{X}\theta) = \frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{X}\theta\|^2, \quad (139)$$

where define *design matrix* $\mathbf{X} := [\mathbf{x}_1, \dots, \mathbf{x}_N]^\top \in \mathbb{R}^{N \times D}$ as collection of training inputs & $\mathbf{y} := [y_1, \dots, y_N]^\top \in \mathbb{R}^N$ as a vector that collects all training targets. Note: n th row in design matrix \mathbf{X} corresponds to training input \mathbf{x}_n . In (9.10b), used fact: sum of squared errors between observations y_n & corresponding model prediction $\mathbf{x}_n^\top \theta$ equals squared distance between \mathbf{y} & $\mathbf{X}\theta$.

Negative log-likelihood function is also called *error function*.

Squared error is often used as a measure of distance. Recall from Sect. 3.1: $\|\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{x}$ if choose dot product as inner product.

With (9.10b), have now a concrete form of negative log-likelihood function we need to optimize. Immediately see: (9.10b) is quadratic in θ . I.e., can find a unique global solution θ_{ML} for minimizing negative log-likelihood \mathcal{L} . Can find global optimum by computing gradient of \mathcal{L} , setting it to $\mathbf{0}$ & solving for θ .

Using results from Chap. 5, compute gradient of \mathcal{L} w.r.t. parameters as

$$\frac{d\mathcal{L}}{d\theta} = \dots = \frac{1}{\sigma^2} (-\mathbf{y}^\top \mathbf{X} + \theta^\top \mathbf{X}^\top \mathbf{X}) \in \mathbb{R}^{1 \times D}. \quad (140)$$

Maximum likelihood estimator θ_{ML} solves $\frac{d\mathcal{L}}{d\theta} = \mathbf{0}^\top$ (necessary optimality condition) & obtain

$$\frac{d\mathcal{L}}{d\theta} = \mathbf{0}^\top \Leftrightarrow \dots \Leftrightarrow \theta_{\text{ML}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}. \quad (141)$$

Could right-multiply 1st equation by $(\mathbf{X}^\top \mathbf{X})^{-1}$ because $\mathbf{X}^\top \mathbf{X}$ is positive definite if $\text{rk}(\mathbf{X}) = D$, where $\text{rk}(\mathbf{X})$ denotes rank of \mathbf{X} .

Ignoring possibility of duplicate data points, $\text{rk}(\mathbf{X}) = D$ if $N \geq D$, i.e., do not have more parameters than data points.

Remark 23. Setting gradient to $\mathbf{0}^\top$ is a necessary & sufficient condition, & obtain a global minimum since Hessian $\nabla_{\theta}^2 \mathcal{L}(\theta) = \mathbf{X}^\top \mathbf{X} \in \mathbb{R}^{D \times D}$ is positive definite.

Remark 24. Maximum likelihood solution in (9.12c) requires us to solve a system of linear equations of form $\mathbf{A}\theta = \mathbf{b}$ with $\mathbf{A} = (\mathbf{X}^\top \mathbf{X})$ & $\mathbf{b} = \mathbf{X}^\top \mathbf{y}$.

Example 23 (Fitting Lines). Look at Fig. 9.2, where we aim to fit a straight line $f(x) = \theta x$, where θ is an unknown slope, to a dataset using maximum likelihood estimation. Examples of functions in this model class (straight lines) are shown in Fig. 9.2(a). For dataset shown in Fig. 9.2(b), find maximum likelihood estimate of slope parameter θ using (9.12c) & obtain maximum likelihood linear function in Fig. 9.2(c).

• Maximum Likelihood Estimation with Features.

Linear regression refers to “linear-in-the-parameters” regression models, but inputs can undergo any nonlinear transformation.

So far, considered linear regression setting described in (9.4), which allowed us to fit straight lines to data using maximum likelihood estimation. However, straight lines are not sufficiently expressive when it comes to fitting more interesting data. Fortunately, linear regression offers us a way to fit nonlinear functions within linear regression framework: Since “linear regression” only refers to “linear in parameters”, can perform an arbitrary nonlinear transformation $\phi(\mathbf{x})$ of inputs \mathbf{x} & then linearly combine components of this transformation. Corresponding linear regression model is

$$p(y|\mathbf{x}, \theta) = \mathcal{N}(y|\phi^\top(\mathbf{x})\theta, \sigma^2) \Leftrightarrow y = \phi^\top(\mathbf{x})\theta + \epsilon = \sum_{k=0}^{K-1} \theta_k \phi_k(\mathbf{x}) + \epsilon, \quad (142)$$

where $\phi : \mathbb{R}^D \rightarrow \mathbb{R}^K$: a (nonlinear) transformation of inputs \mathbf{x} & $\phi_k : \mathbb{R}^D \rightarrow \mathbb{R}$: k th component of feature vector ϕ . Note: model parameters θ still appear only linearly.

Example 24 (Polynomial Regression). Concerned with a regression problem $y = \theta^\top x\theta + \epsilon$, where $x \in \mathbb{R}$, $\theta \in \mathbb{R}^K$. A transformation often used in this context is $\phi(x) = [\phi_0(x), \phi_1(x), \dots, \phi_{K-1}(x)]^\top = [1, x, x^2, \dots, x^{K-1}] \in \mathbb{R}^K$ (9.14). I.e., “lift” original 1D input space into a K -dimensional feature space consisting of all monomials x^k for $k = 0, \dots, K-1$. With these features, can model polynomials of degree $\leq K-1$ within framework of linear regression: A polynomial of degree $K-1$ is

$$f(x) = \sum_{k=0}^{K-1} \theta_k x^k = \theta^\top(x)\theta, \quad (143)$$

where θ is defined in (9.14) & $\theta = [\theta_0, \dots, \theta_{K-1}]^\top \in \mathbb{R}^K$ contains (linear) parameters θ_k .

Now have a look at maximum likelihood estimation of parameters θ in linear regression model (9.13). Consider training inputs $\mathbf{x}_n \in \mathbb{R}^D$ & targets $y_n \in \mathbb{R}$, $n = 1, \dots, N$, & define feature matrix (design matrix) is $\Phi := [\phi^\top(\mathbf{x}_1), \dots, \phi^\top(\mathbf{x}_N)]^\top = \dots \in \mathbb{R}^{N \times K}$, where $\Phi_{ij} = \phi_j(\mathbf{x}_i)$ & $\phi_j : \mathbb{R}^D \rightarrow \mathbb{R}$.

Example 25 (Feature Matrix for 2nd-order Polynomials). For a 2nd-order polynomial & N training points $x_n \in \mathbb{R}$, $n = 1, \dots, N$, feature matrix is

$$\Phi = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 \end{bmatrix}. \quad (144)$$

With feature matrix Φ defined in (9.16), negative log-likelihood for linear regression model (9.13) can be written as (9.18)

$$-\log p(\mathcal{Y}|\mathcal{X}, \theta) = \frac{1}{2\sigma^2} (\mathbf{y} - \Phi\theta)^\top (\mathbf{y} - \Phi\theta) + \text{const.} \quad (145)$$

Comparing (9.18) ***

- 9.3. Bayesian Linear Regression.
- 9.4. Maximum Likelihood as Orthogonal Projection.
- 9.5. Further Reading.
- 10. Dimensionality Reduction with Principal Component Analysis.
 - 10.1. Problem Setting.
 - 10.2. Maximum Variance Perspective.
 - 10.3. Projection Perspective.
 - 10.4. Eigenvector Computation & Low-Rank Approximations.
 - 10.5. PCA in High Dimensions.
 - 10.6. Key Steps of PCA in Practice.

- 10.7. Latent Variable Perspective.
- 10.8. Further Reading.
- 11. Density Estimation with Gaussian Mixture Models.
 - 11.1. Gaussian Mixture Model.
 - 11.2. Parameter Learning via Maximum Likelihood.
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 - 11.4. Latent-Variable Perspective.
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- 12. Classification with Support Vector Machines.
 - 12.1. Separating Hyperplanes.
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 - 12.5. Numerical Solution.
 - 12.6. Further Reading.

2.8 FATEMEH JAMSHIDI, GARY PIKE, AMIT DAS, RICHARD CHAPMAN. Machine Learning Techniques in Automatic Music Transcription: A Systematic Survey. 2024

[2 citations]

- **Abstract.** In domain of Music Information Retrieval (MIR), Automatic Music Transcription (AMT) emerges as a central challenge, aiming to convert audio signals into symbolic notations like musical notes or sheet music. This systematic review accentuates (nhấn mạnh) pivotal role (vai trò then chốt) of AMT in music signal analysis, emphasizing its importance due to intricate & overlapping spectral structure of musical harmonies. Through a thorough examination of existing ML techniques utilized in AMT, explore progress & constraints of current models & methodologies. Despite notable advancements, AMT systems have yet to match accuracy of human experts, largely due to complexities of musical harmonies & need for nuanced interpretation (diễn giải sắc thái). This review critically evaluates both fully automatic & semi-automatic AMT systems, emphasizing importance of minimal user intervention & examining various methodologies proposed to date. By addressing limitations of prior techniques & suggesting avenues for improvement, objective: steer (lái) future research towards fully automated AMT systems capable of accurately & efficiently translating intricate audio signals into precise symbolic representations. This study not only synthesizes latest advancements but also lays out a road-map for overcoming existing challenges in AMT, providing valuable insights for researchers aiming to narrow gap between current systems & human-level transcription accuracy.
- **1. Introduction.** Automatic Music Transcription (AMT) is process of converting an acoustic signal (tín hiệu âm thanh) into its equivalent notation, pitch, duration, onset & offset time, musical score or sheet, or any other musical representation [1–5]. Applications of AMT include music education (e.g., through systems for automatic instrument tutoring), music creation (e.g., dictating improvised musical ideas & automatic music accompaniment), music production (e.g., music content visualization & intelligent content-based editing), music search (e.g., indexing & recommendation of music by melody, bass, rhythm, or chord progression), & musicology (e.g., analyzing jazz improvisations & other annotated music) [4].
AMT problem can be divided into several sub-tasks [2, 6]: Multi-pitch detection, Note onset/offset detection, Loudness estimation & quantization (lượng tử hóa), Instrument recognition, Extraction of rhythmic information, Time quantization, Extraction of velocity & dynamic.

Fig. 1: Automatic music transcription system [7] illustrates data representations in an AMT system. AMT system takes an audio waveform as input, computes a time-frequency representation of audio, outputs a representation of pitches over time in a spectrogram, & generates a typeset music score [3]. Previous studies have tackled Automatic Music Transcription (AMT) using 2 main approaches: Nonnegative Matrix Factorization (NMF) [8], & Neural Networks [2, 9]. NN techniques typically involve processing spectrograms with various neural network architectures, e.g. long short-term memory layers or Convolutional Neural Networks (CNNs). Many AMT studie rely on NNs, particularly in context of polyphonic piano transcription. 1 notable model is Google Magenta Onsets & Frames (OaF) [10], which comprises 2 components: onsets head & frames head. Recent advancements introduce alternative methods aimed at enhancing transcription accuracy by reconstructing input spectrogram [11].

Alternative approaches to Automatic Music Transcription (AMT) involve processing mixed signals using multitask deep learning techniques. This entails separating signal sources before conducting AMT, which necessitates incorporating a source separation component into network architecture. E.g., Cerberus [12] employs 3 components – source separation, deep clustering, & transcription heads – within its multitask DL model for AMT. Furthermore, researchers are exploring application of AMT in multi-instrument music transcript, wherein instrument identification is performed as a sub-task using self-attention mechanisms [13].

- **2. Frame-Level Transcription.** Frame-level transcription, also referred to as Multi-Pitch Estimation (MPE), estimates number & pitch of notes concurrently present in each time frame, typically with a latency of approximately 10 milliseconds [4]. Each frame is usually processed independently, although contextual information may be considered through filtering frame-level pitch estimates in a post-processing stage. However, this method does not explicitly model musical notes or encompass high-level musical structures. Numerous existing Automatic Music Transcription (AMT) techniques operate at this level, including traditional signal processing methods [14, 15], probabilistic modeling [16], Bayesian approaches [17], Nonnegative Matrix Factorization NMF [18–21], & neural networks [22, 23]. Each of these methods presents distinct advantages & limitations, & research has not yet converged on a singular approach. E.g., traditional signal processing methods are characterized by their simplicity & speed, & they demonstrate good generalization across different instruments. Conversely, deep neural network methods generally achieve higher accuracy on specific instruments [9].

In single-instrument AMT, process commences (bắt đầu) with frequency estimation, encompassing 2 subcategories: Fundamental Frequency Estimation (f0 estimations) & Multi-f0 estimation. Fundamental frequency estimation involves identifying fundamental frequency of notes in each time frame. Various methodologies have been proposed, including template matching, probabilistic algorithms, & salience function techniques (kỹ thuật hàm nổi bật). Some approaches are available for estimating f0 in monophonic signals, including SWIPE [24], which matches spectrum of a waveform with a template, & a probabilistic variant of YIN [25] that decodes a pitch value sequence by using a Hidden Markov Model (HMM). Latest & best-performing methods, e.g. CREPE [26], employ DL techniques, converting input signals into spectrograms & processing them through CNNs.

On other hand, Multi-f0 estimation, tackles challenge of discerning multiple fundamental frequencies present in a polyphonic signal, where several notes coexist within each time frame. Earlier studies in multi-f0 estimation either model spectral peaks or utilize CNNs with constant Q-Transform as inputs to learn salience representations for estimating fundamental frequencies [27].

- **3. Note-Level Transcription in Polyphonic Music.** Piano stands out as most thoroughly examined instrument for polyphonic music transcription. This prominence owes much to accessibility of comprehensive datasets & its percussive onset characteristics (đặc điểm khởi đầu của tiếng gõ), akin to other percussion instruments. Consequently, primary focus of study for nearly all contemporary [28] end-to-end models, leveraging Convolutional Neural Networks (CNNs) & Long Short-Term Memory networks (LSTMs), lies in multi-f0 estimation for piano transcription.

End-to-end models, employing deep neural networks, are pivotal in facilitating Automatic Music Transcription (AMT). Nonetheless, certain models necessitate discrete sub-tasks, like utilizing waveform domain signals as input for deep neural networks or mandating a pre-processing phase to convert wave-forms into time-frequency representations.

1 noteworthy model developed to tackle polyphonic piano & drums transcription is Onset & Frames (OaF) [10]. This model excels in detecting note onsets & predicting pitches, adaptable for transcribing piano pieces using MAESTRO dataset [29] or drums employing Expanded MIDI Groove (E-GMD) [30] dataset, contingent upon dataset used for training.

Spectrograms emerge as preferred input for these models, showcasing remarkable AMT outcomes for piano datasets. This preference is underscored by architecture’s ability to enhance transcription quality through accurate onset predictions tailored for piano compositions. However, performance of these models remains untested with instruments characterized by disparate timbres & onset envelopes.

- **4. Stream-Level Transcription.** Multipitch streaming (MPS), alternatively referred to as stream-level transcription, is a technique that groups estimated pitches or notes into streams, where each stream typically representing an individual instrument or musical voice. This technique is closely related to instrument source separation. Distinguished from note-level transcription, MPS entails a higher degree of complexity due to elongated pitch contours of each stream, encompassing multiple discontinuities arising from silent intervals, non-pitched sounds, & abrupt frequency shifts.

Timbre (Âm sắc) emerges as a crucial factor in MPS that is not explored in MPE & note tracking. This is because notes within same stream tend to share similar timbral characteristics, distinguishing them from those in separate streams. Despite significance of this approach, existing literature remains somewhat sparse, with only a handful of examples e.g. [31,20,32].

As transcription task becomes more complex from frame level to note level to stream level, it necessitates incorporation of additional musical structures & cues. Despite increasing complexity, transcription outputs at these 3 levels are all parametric transcriptions, which provide a parametric description of audio content. A prime instance of such transcription is MIDI piano roll, serving as an abstraction of musical audio that has not yet attained level of detail found in traditional music notation. Although it provides pitch & timing information, it still lacks fundamental concepts e.g. beats, bars, meter, key, & stream delineation (phân định dòng suối).

- **5. Notation-Level Transcription.** Notation-level transcription aims to convert audio file into a human-readable musical score, necessitating a comprehensive grasp of musical structures e.g. harmonic, rhythmic, & stream structures. Prior research in this domain has predominantly focused on timing quantization & employed DL techniques. Some models have endeavored to address this challenge through end-to-end models that take either a signal or time-frequency representation as input & output score of musical piece. Conversely, other methodologies concentrate on transcribing from MIDI files to musical scores.

2 primary methodologies exist for notation-level transcription: 1st perform f0 detection, while 2nd encompasses multi-f0 detection coupled with note-tracking algorithms. CREPE [26] pre-trained weights are used for f0 estimation, & model output is then used for note tracking. Proposed methods are tested on multiple musical instruments with different timbres. Results are

compared with state-of-art Onset & Frames (OaF) [10] model, designed for polyphonic piano transcription, to evaluate how timbre affects pitch estimation & note tracking. Following meticulous evaluation, a multi-f0 model is developed to transcribe polyphonic signals emanating from single instruments across a diverse range of timbral variations. – Sau quá trình đánh giá tỉ mỉ, một mô hình đa f0 được phát triển để phiên âm các tín hiệu đa âm phát ra từ các nhạc cụ đơn lẻ trên nhiều biến thể âm sắc khác nhau.

- **6. Music Transcription Dataset for Training ML Models.** MAESTRO (“MIDI & Audio Edited for Synchronous Tracks & Organization”) dataset [29], contains over a week of paired audio & MIDI recordings from 9 years of International Piano-e-Competition events. Dataset includes annotation of isolated notes, duration of notes, chords, & complete piano pieces. MIDI data includes key strikes, velocities & sustain pedal positions. Achieving an alignment accuracy of approximately 3 ms accuracy, audio & MIDI files are segmented into individual musical pieces, each with composer, title, & year of performance. Use set of synthesized pieces of a single train/validation/test split designed to satisfy following criteria:

- No composition should appear in > 1 split.
- Train/validation/test should make up roughly 80/10/10% of dataset (in time), resp. These proportions should be true globally & also within each composer. Maintaining these proportions is not always possible because some composers have few compositions in dataset.
- Validation & test splits should contain various compositions. More popular compositions performed by many performers should be placed in training split.

Table 1: Statistics of MAESTRO dataset replicated [29] contains aggregate statistics (thống kê tổng hợp) of MAESTRO dataset.

1st translate “sustain pedal” controls into longer note duration to process MAESTRO MIDI files for training & evaluation. Sustain will extend a note until either sustain is turned off or same note is played again if that note is active. As a result of this process, note durations are same as those included in dataset’s text files.

Several datasets containing both piano audio & MIDI have been published previously, enabling significant progress in automatic piano transcription. **Table 2: Piano MIDI datasets** represents existing piano MIDI datasets.

MAESTRO differs from existing datasets in several properties that affect model training:

MusicNet (Thickstun et al., 2017) contains recordings of human performances but separately-sourced scores. As Hawthorne et al. (2018) discussed, alignment between audio & score is not entirely accurate. MusicNet offers a greater variety of recording environments & instruments besides piano (not included in Table 2).

MAPS (Emiya et al., 2010) includes synthesized audio created from MIDI files entered via sequencer & Disklavier recordings. As such, “performances” are not as natural as MAESTRO performances captured from live performances. Moreover, synthesized audio accounts for a significant portion of MAPS dataset. Aside from individual notes & chords, MAPS also contains syntheses & recordings.

Saarland Music Data (SMD) (Muller et al., 2011) contains recordings of human performances on Disklaviers, but it is 30 times smaller than MAESTRO.

GiantMIDI-Piano is largest dataset, including 38,700,838 transcribed notes & 10,855 unique solo piano works composed by 2786 composers. **Table 3: Piano transcription evaluation on GiantMIDI-Piano dataset based on [33]** represents alignment performance. Median alignment S_M, D_M, I_M, ER_M on MAESTRO dataset are 0.009, 0.024, 0.021, & 0.061, resp. Median alignment S_G, D_G, I_G, ER_G on GiantMIDI-Piano dataset are 0.015, 0.051, 0.069, & 0.154, resp. [33].

Proposed research endeavors to offer a novel augmentation to dataset by through integration of MAESTRO dataset & GIANT-MIDI piano dataset, standardizing annotation, & applying augmentation. MAESTRO dataset, which contains > 172 hours of virtuosic piano performances captured with fine alignment between note labels & audio waveforms, is combined with GIANT-MIDI piano dataset. Annotation of combined dataset is standardized to produce pairs of audio & MIDI files time-aligned to represent same musical events.

- **6.1. Augmentation.** (Tăng cường) Augmentation serves as a pivotal stage in enhancing dataset diversity & refining ML model generalization. 1 common technique is time stretching, altering audio signal duration while preserving pitch. This manipulation can be executed using formula

$$y(t) = x\left(\frac{t}{\alpha}\right), \quad (146)$$

where $x(t)$: original audio signal, $y(t)$: time-stretched signal, α : time-stretching factor. Another significant method is pitch shifting (thay đổi cao độ), which alters pitch while maintaining original duration. This is achieved through:

$$y(t) = x\left(\frac{t}{\beta}\right), \quad (147)$$

where $x(t)$: original audio, $y(t)$: pitch-shifted signal, & β : pitch-shifting factor. Applying these techniques to dataset increases its diversity & robustness, improving model performance. Our research aims to enhance automatic piano transcription with an expanded dataset, standardized annotation, & augmented data, facilitating more precise ML models. Aim: generate piano transcriptions containing perceptually relevant performance information without considering recording environment. To achieve this, need a numerical measure. Poor-quality transcriptions can still score high due to short, spurious, & repeated

notes. High note-with-offset scores capture perceptual information from onsets & durations, ensuring dynamics capture. This study improves note-with-offset scores, achieves state-of-art results for frame & note scores, & extends model to transcribe velocity scores (ghi chép điểm tốc độ).

- 7. Feature Representations Computational Models.
- 8. Training.
- 9. Evaluation of Automatic Piano Transcription.

2.9 [Kut23]. GITTA KUTYNIOK. **Mathematics of Artificial Intelligence**

[35 citations]

- **Abstract.** Currently witness spectacular success of AI in both science & public life. However, development of a rigorous mathematical foundation is still at an early stage. In this survey article, which is based on an invited lecture at ICM2022, in particular focus on current “workhorse” of AI, namely deep neural networks. Present main theoretical directions along with several exemplary results & discuss key open problems.
- **Keywords.** applied harmonic analysis, approximation theory, DL, inverse problems, PDEs
- **1. Introduction.** AI is currently leading to 1 breakthrough after another, both in public with, e.g., autonomous driving & speech recognition, & in sciences in areas e.g. medical diagnostics or molecular dynamics. In addition, research on AI &, in particular, on its theoretical foundations is progressing at an unprecedented rate. One can envision: corresponding methodologies will in future drastically change way we live in numerous respects.
 - **1.1. Rise of AI.** AI is, however, not a new phenomenon. In fact, already in 1943, McCulloch & Pitts started to develop algorithmic approaches to learning by mimicking functionality of human brain, through artificial neurons which are connected & arranged in several layers to form artificial neural networks. Already at that time, they had a vision for implementation of AI. However, community did not fully recognize potential of neural networks. Therefore, this 1st wave of AI was not successful & vanished. Around 1980, ML became popular again, & several highlights can be reported from that period. Real breakthrough & with it a new wave of AI came around 2010 with extensive application of *deep* neural networks. Today, this model might be considered “workhorse” of AI, & in this article, focus predominantly on this approach. Structure of deep neural networks is precisely structure McCulloch & Pitts introduced, namely numerous consecutive layers of artificial neurons. Today 2 main obstacles from previous years have also been eliminated; due to drastic improvement of computing power, training of neural networks with hundreds of layers in sense of *deep* neural networks is feasible, & we are living in age of data, hence vast amounts of training data are easily available.
 - **1.2. Impact on mathematics.** Rise of AI also had a significant impact on various fields of mathematics. Maybe 1st area which embraced these novel methods was area of inverse problems, in particular, imaging science, where such approaches have been used to solve highly ill-posed problems e.g. denoising, inpainting, superresolution, or (limited-angle) computed tomography, to name a few. One might note: due to lack of a precise mathematical model of what an image is, this area is particularly suitable for learning methods. Thus, after a few years, a change of paradigm could be observed, & novel solvers are typically at least to some extent based on methods from AI. Area of PDEs was much slower to embrace these new techniques, reason being: it was not per se evident what advantage of methods from AI for this field would be. Indeed, there seems to be no need to utilize learning-type methods, since a PDE is a rigorous mathematical model. But, lately, observation: deep neural networks are able to beat curse of dimensionality in high-dimensional settings led to a change of paradigm in this area as well. Research at intersection of numerical analysis of PDEs & AI therefore accelerated since about 2017. Will delve further into this topic in Sect. 4.2.
 - **1.3. Problems of AI.** However, as promising as all these developments seem to be, a word of caution is required. besides fact: practical limitations of methods e.g. deep neural networks have not been explored at all & at present neural networks are still considered a “jack-of-all-trades,” even more worrisome that a comprehensive theoretical foundation is completely lacking. This was very prominently stated during major conference on AI & ML, which is NIPS (today called NeurIPS) in 2017, when ALI RAHIMI from Google received Test of Time Award & during his plenary talk stated: “ML has become a form of alchemy.” (Học máy đã trở thành một hình thức thuật giả kim). This raised a heated discussion to which extent a theoretical foundation does exist & is necessary at all. From a mathematical viewpoint, crystal clear: a fundamental mathematical understanding of AI is inevitably necessary, & one has to admit: its development is currently in a preliminary state at best. This lack of mathematical foundations, e.g., in case of deep neural networks, results in a time-consuming search for a suitable network architecture, a highly delicate trial-&-error-based (training) process, & missing error bounds for performance of trained neural network. One needs to stress: in addition, such approaches also sometimes unexpectedly fail dramatically when a small perturbation of input data causes a drastic change of output leading to radically different – & often wrong – decisions. Such adversarial examples are a well-known problem, which becomes severe in sensitive applications e.g. when minor alterations of traffic signs, e.g., placement of stickers, cause autonomous vehicles to suddenly reach an entirely wrong decision. Evident: such robustness problems can only be tackled by a profound mathematical approach.

- 1.4. **A need for mathematics.** These considerations show: there is a tremendous need for mathematics in area of AI. One can currently witness: numerous mathematicians move to this field, bringing in their own expertise. Indeed, discussed in Sect. 2.4, basically all areas of mathematics are required to tackle various difficult, but exciting challenges in area of AI. One can identify 2 different research directions at intersection of mathematics & AI:
 - * *Mathematical Foundations for AI.* This direction aims for deriving a deep mathematical understanding. Based on this, it strives to overcome current obstacles e.g. lack of robustness or places entire training process on a solid theoretical foundation.
 - * *AI for Mathematical Problems.* This direction focuses on mathematical problem settings e.g. inverse problems & PDEs with goal of employing methodologies from AI to develop superior solvers.
- 1.5. **Outline.** Both research directions will be discussed in this survey paper, showcasing some novel results & pointing out key future challenges for mathematics. Start with an introduction into mathematical setting, stating main defs & notations (Sect. 2). Next, in Sect. 3, delve into 1st main direction, namely mathematical foundations for AI, & discuss research threads of expressivity, optimization, generalization, & explainability. Sect. 4 is then devoted to 2nd main direction, which is AI for mathematical problems, & highlight some exemplary results. Finally, Sect. 5 states 7 main mathematical problems & concludes.
- 2. **Mathematical Setting of AI.** Get into more details on precise def of a deep neural network, which is after all a purely mathematical object. Also touch upon typical application setting & training process, as well as current key mathematical directions.

◦ 2.1. **Def of deep neural networks.** Core building blocks are artificial neurons. For their def, recall structure & functionality of a neuron in human brain. Basic elements of such a neuron are dendrites, through which signals are transmitted to its soma while being scaled/amplified due to structural properties of respective dendrites. In soma of neuron, those incoming signals are accumulated, & a decision is reached whether to fire to other neurons or not, & also with which strength.

– Các khối xây dựng cốt lõi là các tế bào thần kinh nhân tạo. Đối với định nghĩa của chúng, hãy nhớ lại cấu trúc & chức năng của một tế bào thần kinh trong não người. Các thành phần cơ bản của một tế bào thần kinh như vậy là các nhánh cây, qua đó các tín hiệu được truyền đến thân tế bào của nó trong khi được mở rộng/khuếch đại do các đặc tính cấu trúc của các nhánh cây tương ứng. Trong thân tế bào thần kinh, các tín hiệu đến đó được tích lũy, & một quyết định được đưa ra là có nên kích hoạt đến các tế bào thần kinh khác hay không, & cũng như với cường độ nào.

This forms basis for a mathematical def of an artificial neuron.

Definition 7. An artificial neuron with weights $w_1, \dots, w_n \in \mathbb{R}$, bias $b \in \mathbb{R}$, & activation function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ is defined as function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$f(x_1, \dots, x_n) = \rho \left(\sum_{i=1}^n x_i w_i - b \right) = \rho(\langle \mathbf{x}, \mathbf{w} \rangle - b), \quad (148)$$

where $\mathbf{w} = (w_1, \dots, w_n)$, $\mathbf{x} = (x_1, \dots, x_n)$.

By now, there exists a zoo of activation functions with most well-known ones being as follows:

1. Heaviside function

$$\rho(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases} \quad (149)$$

2. Sigmoid function $\rho(x) = \frac{1}{1+e^{-x}}$.

3. Rectifiable Linear Unit (ReLU) $\rho(x) = \max\{0, x\}$.

Remark that of these examples, by far most extensively used activation function is ReLU due to its simple piecewise-linear structure, which is advantageous in training process & still allows superior performance.

Similar to structure of a human brain, these artificial neurons are now being concatenated & arranged in layers, leading to an (artificial feed-forward) neural network. Due to particular structure of artificial neurons, such a neural network consists of compositions of affine linear maps & activation functions. Traditionally, a deep neural network is then defined as resulting function. From a mathematical standpoint, this bears difficulty that different arrangements lead to same function. Therefore, sometimes a distinction is made between architecture of a neural network & corresponding realization function (see, e.g., [6]). For this article, will, however, avoid such technical delicacies & present most standard def.

Definition 8. Let $d \in \mathbb{N}$ be dimension of input layer, L number of layers, $N_0 := d$, $N_l, l = 1, \dots, L$, dimensions of hidden & last layer, $\rho : \mathbb{R} \rightarrow \mathbb{R}$ a (nonlinear) activation function, & for $l = 1, \dots, L$, let T_l be affine functions

$$T_l : \mathbb{R}^{N_{l-1}} \rightarrow \mathbb{R}^{N_l}, \quad T_l \mathbf{x} = W^{(l)} \mathbf{x} + b^{(l)}, \quad (150)$$

with $W^{(l)} \in \mathbb{R}^{N_l \times N_{l-1}}$ being weight matrices & $b^{(l)} \in \mathbb{R}^{N_l}$ bias vectors of l th layer. Then $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{N_L}$, given by $\Phi(x) = T_L \rho(T_{L-1} \rho(\dots \rho(T_1(x))))$, $x \in \mathbb{R}^d$, is called a (deep) neural network of depth L .

Weights & biases are free parameters which will be learned during training process. An illustration of multilayered structure of a deep neural network can be found in Fig. 1: Deep neural network $\Phi : \mathbb{R}^4 \rightarrow \mathbb{R}$ with depth 5.

- 2.2. **Application of a deep neural network.** Aiming to identify main mathematical research threads, 1st have to understand how a deep neural network is used for a given application setting.

- * *Step 1 (Train-test split of dataset)*. Assume given samples $(\mathbf{x}^{(i)}, y^{(i)})_{i=1}^{\tilde{m}}$ of inputs & outputs. Task of deep neural network is then to identify relation between those. E.g., in a classification problem, each output $y^{(i)}$ is considered to be label of respective class to which input $\mathbf{x}^{(i)}$ belongs. One can also take viewpoint: $(\mathbf{x}^{(i)}, y^{(i)})_{i=1}^{\tilde{m}}$ arise as samples from a function e.g. $g : \mathcal{M} \rightarrow \{1, 2, \dots, K\}$, where \mathcal{M} might be a lower-dimensional manifold of \mathbb{R}^d , in sense of $y^{(i)} = g(\mathbf{x}^{(i)}) \forall i = 1, \dots, \tilde{m}$. Set $(\mathbf{x}^{(i)}, y^{(i)})_{i=1}^{\tilde{m}}$ is then split into a training data set $(\mathbf{x}^{(i)}, y^{(i)})_{i=1}^m$ & a test data set $(\mathbf{x}^{(i)}, y^{(i)})_{i=m+1}^{\tilde{m}}$. Training data set is – as name indicates – used for training, whereas test data set will later on be solely exploited for testing performance of trained network. Emphasize: neural network is not exposed to test data set during entire training process.
- * *Step 2 (Choice of architecture)*. For preparation of learning algorithm, architecture of neural network needs to be decided upon, i.e., number of layers L , number of neurons in each layer $(N_l)_{l=1}^L$, & activation function ρ have to be selected. Known: a fully connected neural network is often difficult to train, hence, in addition, one typically preselects certain entries of weight matrices $(W^{(l)})_{l=1}^L$ to already be set to 0 at this point. For later purposes, define selected class of deep neural networks by \mathcal{NN}_θ with θ encoding this chosen architecture.
- * *Step 3 (Training)*. Next step is actual training process, which consists of learning affine functions $(T_l)_{l=1}^L = (W^{(l)} \cdot + b^{(l)})_{l=1}^L$. This is accomplished by minimizing *empirical risk* (2.1)

$$\widehat{\mathcal{R}}(\Phi_{(W^{(l)}, b^{(l)})_l}) := \frac{1}{m} \sum_{i=1}^m (\Phi_{(W^{(l)}, b^{(l)})_l}(\mathbf{x}^{(i)}) - y^{(i)})^2. \quad (151)$$

A more general form of optimization problem: (2.2)

$$\min_{(W^{(l)}, b^{(l)})_l} \sum_{i=1}^m \mathcal{L}(\Phi_{(W^{(l)}, b^{(l)})_l}(\mathbf{x}^{(i)}), y^{(i)}) + \lambda \mathcal{P}((W^{(l)}, b^{(l)})_l), \quad (152)$$

where \mathcal{L} : a loss function to determine a measure of closeness between network evaluated in training samples & (known) values $y^{(i)}$, with \mathcal{P} being a penalty/regularization term to impose additional constraints on weight matrices & bias vectors. 1 common algorithmic approach is gradient descent. Since, however, m is typically very large, this is computationally not feasible. This problem is circumvented by randomly selecting only a few gradients in each iteration, assuming: they constitute a reasonable average, which is coined *stochastic gradient descent*.

Solving optimization problem then yields a network $\Phi_{(W^{(l)}, b^{(l)})_l} : \mathbb{R}^d \rightarrow \mathbb{R}^{N_L}$, where

$$\Phi_{(W^{(l)}, b^{(l)})_l}(\mathbf{x}) = T_L \rho(T_{L-1} \rho(\dots \rho(T_1(\mathbf{x}))). \quad (153)$$

- * *Step 4 (Testing)*. Finally, performance (often also called *generalization ability*) of trained neural network is tested using test data set $(\mathbf{x}^{(i)}, y^{(i)})_{i=m+1}^{\tilde{m}}$ by analyzing whether

$$\Phi_{(W^{(l)}, b^{(l)})_l}(\mathbf{x}^{(i)}) \approx y^{(i)}, \quad \forall i = m+1, \dots, \tilde{m}. \quad (154)$$

- o 2.3. Relation to a statistical learning problem. From procedure above, can already identify selection of architecture, optimization problem, & generalization ability as key research directions for mathematical foundations of deep neural network. Considering entire learning process of a deep neural network as a statistical learning problem reveals those 3 research directions as indeed natural ones for analyzing overall error.

For this, assume: there exists a function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ s.t. training data $(\mathbf{x}^{(i)}, y^{(i)})_{i=1}^m$ is of form $(\mathbf{x}^{(i)}, g(\mathbf{x}^{(i)}))_{i=1}^m$ & $\mathbf{x}^{(i)} \in [0, 1]^d \forall i = 1, \dots, m$. A typical continuum viewpoint to measure success of training: consider *risk* of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$\mathcal{R}(f) := \int_{[0,1]^d} (f(\mathbf{x}) - g(\mathbf{x}))^2 d\mathbf{x}, \quad (155)$$

where used L^2 -norm to measure distance between f, g . Error between trained deep neural network $\Phi^0 := \Phi_{(W^{(l)}, b^{(l)})_l} \in \mathcal{NN}_\theta$ & optimal function g can then be estimated by (\leq Optimization error + Generalization error + Approximation error)

$$\mathcal{R}(\Phi^0) \leq [\widehat{\mathcal{R}}(\Phi^0) - \inf_{\Phi \in \mathcal{NN}_\theta} \widehat{\mathcal{R}}(\Phi)] + 2 \sup_{\Phi \in \mathcal{NN}_\theta} |\mathcal{R}(\Phi) - \widehat{\mathcal{R}}(\Phi)| + \inf_{\Phi \in \mathcal{NN}_\theta} \mathcal{R}(\Phi). \quad (156)$$

These considerations lead to main research threads described in:

- o 2.4. Main research threads. Can identify 2 conceptually different research threads, 1st being focused on developing mathematical foundations of AI & 2nd aiming to use methodologies from AI to solve mathematical problems. Intriguing to see how both have already led to some extent to a paradigm shift in some mathematical research areas, most prominently area of numerical analysis.
- * 2.4.1. Mathematical foundations for AI. Following up on discussion in Sect. 2.3, can identify 3 research directions which are related to 3 types of errors which one needs to control in order to estimate overall error of entire training process:
 - *Expressivity*. This direction aims to derive a general understanding whether & to which extent aspects of a neural network architecture affect best case performance of deep neural networks. More precisely, goal: analyze approximation error $\inf_{\Phi \in \mathcal{NN}_\theta} \mathcal{R}(\Phi)$ from (2.4), which estimates approximation accuracy when approximating g by hypothesis class \mathcal{NN}_θ of deep neural networks of a particular architecture. Typical methods for approaching this problem are from applied harmonic analysis & approximation theory.

- *Learning/Optimization.* Main goal of this direction is analysis of training algorithm e.g. stochastic gradient descent, in particular, asking why it usually converges to suitable local minima even though problem itself is highly nonconvex. This requires analysis of optimization error, which is $\widehat{\mathcal{R}}(\Phi^0) - \inf_{\Phi \in \mathcal{NN}_\theta} \widehat{\mathcal{R}}(\Phi)$ (cf. (2.4)) & which measures accuracy with which learnt neural network Φ^0 minimizes empirical risk (2.1), (2.2). Key methodologies for attacking such problems come from areas of algebraic/differential geometry, optimal control, & optimization.
- *Generalization.* This direction aims to derive an understanding of out-of-sample error, namely $\sup_{\Phi \in \mathcal{NN}_\theta} |\mathcal{R}(\Phi) - \widehat{\mathcal{R}}(\Phi)|$ from (2.4), which measures distance of empirical risk (2.1), (2.2), & actual risk (2.3). Predominantly, learning theory, probability theory, & statistics provide required methods for this research thread.

A very exciting & highly relevant new research direction has recently emerged, coined explainability. At present, it is from standpoint of mathematical foundations still a wide open field.

- *Explainability.* This direction considers deep neural networks, which are already trained, but no knowledge about training is available; a situation one encounters numerous times in practice. Goal: derive a deep understanding of how a given trained deep neural network reaches decisions in sense of which features of input data are crucial for a decision. Range of required approaches is quite broad, including areas e.g. information theory or uncertainty quantification.
- * 2.4.2. **AI for mathematical problems.** Methods of AI have also turned out to be extremely effective for mathematical problem settings. In fact, area of inverse problems, in particular, in imaging sciences, has already undergone a profound paradigm shift. & area of numerical analysis of PDEs seems to soon follow same path, at least in very high dimensional regime. Briefly characterize those 2 research threads similar to previous subject on mathematical foundations of AI.
 - *Inverse Problems.* Research in this direction aims to improve classical model-based approaches to solve inverse problems by exploiting methods of AI. In order to not neglect domain knowledge e.g. physics of problem, current approaches aim to take best out of both worlds in sense of optimally combining model- & data-driven approaches. This research direction requires a variety of techniques, foremost from areas e.g. imaging science, inverse problems, & microlocal analysis, to name a few.
 - *PDEs.* Similar to area of inverse problems, goal: improve classical solvers of PDEs by using ideas from AI. A particular focus is on high-dimensional problems in sense of aiming to beat curse of dimensionality. This direction obviously requires methods from areas e.g. numerical mathematics & PDEs.
- 3. **Mathematical Foundations for AI.** This sect shall serve as an introduction into main research threads aiming to develop a mathematical foundation for AI. Introduce problem settings, showcase some exemplary results, & discuss open problems.

- 3.1. **Expressivity.** Expressivity is maybe richest area at present in terms of mathematical results. General question can be phrased as follows: Given a function class/space \mathcal{C} & a class of deep neural networks \mathcal{NN}_θ , how does approximation accuracy when approximating elements of \mathcal{C} by networks $\Phi \in \mathcal{NN}_\theta$ relate to complexity of such Φ ? Making this precise thus requires introduction of a complexity measure for deep neural networks. In sequel, choose canonical one, which is complexity in terms of memory requirements. Notice: certainly various other complexity measures exist. Further, recall: $\|\cdot\|_0$ -“norm” counts number of nonzero components.

Definition 9. *Retaining same notation for deep neural networks as in Def. 2.2, complexity $C(\Phi)$ of a deep neural network Φ is defined by*

$$C(\Phi) := \sum_{l=1}^L (\|W^{(l)}\|_0 + \|b^{(l)}\|_0). \quad (157)$$

Most well-known – & maybe even 1st – result on expressivity is universal approximation theorem [8,13]. State: each continuous function on a compact domain can be approximated up to an arbitrary accuracy by a shallow neural network.

- 3.2. **Optimization.**
- 3.3. **Generalization.**
- 3.4. **Explainability.**
- 4. **AI for Mathematical Problems.** Turn to research direction of AI for mathematical problems, with 2 most prominent problems being inverse problems & PDEs. As before, introduce problem settings, showcase some exemplary results, & also discuss open problems.

- 4.1. **Inverse problems.** Methods of AI, in particular, deep neural networks, have a tremendous impact on area of inverse problems. 1 current major trend: optimally combine classical solvers with DL in sense of taking best out of model- & data-world.

To introduce such results, start by recalling some basics about solvers of inverse problems. For this, assume: given an (ill-posed) inverse problem

$$Kf = g, \quad (158)$$

where $K : X \rightarrow Y$: an operator & X, Y : e.g., Hilbert spaces. Drawing from area of imaging science, examples include denoising, deblurring, or inpainting (recovery of missing parts of an image). Most classical solvers are of form (which includes Tikhonov regularization)

$$f^\alpha := \arg \min_f (\|Kf - g\|^2 + \alpha \mathcal{P}(f)), \quad (159)$$

where $\|Kf - g\|^2$: data fidelity term, $\mathcal{P}(f)$: penalty/regularization term, $\mathcal{P} : X \rightarrow \mathbb{R}$, $f^\alpha \in X, \alpha > 0$: an approximate solution of inverse problem (4.1). 1 very popular & widely applicable special case is *sparse regularization*, where \mathcal{P} is chosen by $\mathcal{P}(f) := \|(\langle f, \varphi_i \rangle)_{i \in I}\|_1$ & $(\varphi_i)_{i \in I}$: a suitably selected orthonormal basis or a frame for X .

Turn to DL approaches to solve inverse problems, which might be categorized into 3 classes:

- * *Supervised approaches*. An ad hoc approach in this regime is given in [14], which 1st applies a classical solver followed by a neural network to remove reconstruction artifacts. More sophisticated approaches typically replace parts of classical solver by a custom-built neural network [26] or a network specifically trained for this task [1].
- * *Semisupervised approaches*. These approaches encode regularization as a neural network with an example being adversarial regularizers [20].
- * *Unsupervised approaches*. A representative of this type of approaches: technique of deep image prior [29]. This method interestingly shows: structure of a generator network is sufficient to capture necessary statistics of data prior to any type of learning.

Aiming to illustrate superiority of approaches from AI for inverse problems, now focus on inverse problem of computed tomography (CT) from medical imaging. Forward operator K in this setting is *Radon transform*, defined by

$$\mathcal{R}f(s, \vartheta) = \int_{\mathbb{R}} f(s\omega(\vartheta) + t\omega(\vartheta)^\perp) dt, \quad (s, \vartheta) \in \mathbb{R} \times (0, \pi). \quad (160)$$

Here $\omega(\vartheta) := (\cos \vartheta, \sin \vartheta)$: unitary vector with orientation described by angle ϑ w.r.t. x_1 -axis & $\omega(\vartheta)^\perp := (-\sin \vartheta, \cos \vartheta)$. Often, only parts of so-called sinogram $\mathcal{R}f$ can be acquired due to physical constraints as in, e.g., electron tomography. Resulting, more difficult problem is termed *limited-angle CT*. Notice: this problem is even harder than problem of low-dose CT, where not an entire block of measurements is missing, but angular component is “only” undersampled.

Most prominent features in images f are edge structures. This is also due to fact: human visual system reacts most strongly to those. These structures in turn can be accurately modeled by microlocal analysis, in particular, by notion of wavefront sets $WF(f) \subseteq \mathbb{R}^2 \times \mathbb{S}$, which – coarsely speaking – consist of singularities together with their direction. Basing in this sense application of a deep neural network on microlocal considerations, in particular, also using a DL-based wavefront set detector [2] in regularization term, reconstruction performance significantly outperforms classical solvers e.g. sparse regularization with shearlets (see Fig. 6: CT reconstruction from Radon measurements with a missing angle of 40° , also refer to [3] for details). Notice: this approach is of a hybrid type & takes best out of both worlds in sense of combining model- & AI-based approaches.

Finally, DL-based wavefront set extraction itself is yet another evidence of improvements on state-of-art now possible by AI. Fig. 7: Wavefront set detection by a model-based & a hybrid approach. shows a classical result from [23], whereas [2] uses shearlet transform as a coarse edge detector, which is subsequently combined with a deep neural network.

- 4.2. PDE.s 2nd main range of mathematical problem settings, where methods from AI are very successfully applied to, are PDEs. Although benefit of such approaches was not initially clear, both theoretical & numerical results show their superiority in high-dimensional regimes.

Most common approach aims to approximate solution of a PDE by a deep neural network, which is trained according to this task by incorporating PDE into loss function. More precisely, given a PDE $\mathcal{L}(u) = f$, train a neural network Φ s.t. $\mathcal{L}(\Phi \approx f$. Since 2017, research in this general direction has significantly accelerated. Some of highlights are Deep Ritz method [10] & Physics Informed Neural Networks [22], or a very general approach for high-dimensional parabolic PDEs [12].

Note: most theoretical results in this regime are of an expressivity type & also study phenomenon whether & to which extent deep neural networks are able to beat curse of dimensionality. In sequel, briefly discuss 1 such result as an example. In addition, notice: there already exist contributions – though very few – which analyze learning & generalization aspects.

Let $\mathcal{L}(u_y, y) = f_y$ denote a parametric PDE with y being a parameter from a high-dimensional parameter space $\mathcal{Y} \subset \mathbb{R}^p$ & u_y associated solution in a Hilbert space \mathcal{H} . After a high-fidelity discretization, let $b_y(u_y^h, v) = f_y(v)$ be associated variational form with u_y^h now belonging to associated high-dimensional space U^h , where set $D := \dim U^h$. Denote coefficient vector of u_y^h w.r.t. a suitable basis of U^h by \mathbf{u}_y^h . Of key importance in this area: *parametric map* given by

$$\mathbb{R}^p \supseteq \mathcal{Y} \ni y \mapsto \mathbf{u}_y^h \in \mathbb{R}^D \text{ s.t. } b_y(u_y^h, v) = f_y(v), \quad \forall v, \quad (161)$$

which in multiquery situations e.g. complex design problems needs to be solved several times. If p is very large, curse of dimensionality could lead to an exponential computational cost.

Aim to analyze whether parametric map can be solved by a deep neural network, which would provide a very efficient & flexible method, hopefully also circumventing curse of dimensionality in an automatic manner. From an expressivity viewpoint, one might ask whether, for each $\varepsilon > 0$, there exists a neural network Φ s.t. (4.2)

$$\|\Phi(y) - \mathbf{u}_y^h\| \leq \varepsilon \quad \forall y \in \mathcal{Y}. \quad (162)$$

Ability of this approach to tackle curse of dimensionality can then be studied by analyzing how complexity of Φ depends on p, D . A result of this type was proven in [18].

Theorem 1. *There exists a neural network Φ which approximates parametric map, i.e., which satisfies (4.2), & dependence of $C(\Phi)$ on p, D can be (polynomially) controlled.*

Analyzing learning procedure & generalization ability of neural network in this setting is currently out of reach. Aiming to still determine whether a trained neural networks does not suffer from curse of dimensionality as well in [11] extensive numerical experiments were performed, which indicate: indeed, curse of dimensionality is also beaten in practice.

- **5. Conclusion: 7 Mathematical Key Problems.** Conclude with 7 mathematical key problems of AI as they were stated in [6]. Those constitute main obstacles in *Mathematical Foundations for AI* with its subfields being expressivity, optimization, generalization, & explainability, as well as in *AI for Mathematical Problems*, which focus on application to inverse problems & PDEs.

1. What is role of depth?
2. Which aspects of a neural network architecture affect performance of DL?
3. Why does stochastic gradient descent converge to good local minima despite nonconvexity of problem?
4. Why do large neural networks not overfit?
5. Why do neural networks perform well in very high-dimensional environments?
6. Which features of data are learned by deep architectures?
7. Are neural networks capable of replacing highly specialized numerical algorithms in natural sciences?

2.10 GABRIEL PEYRÉ. The Mathematics of Artificial Intelligence. 2025

- **Abstract.** This overview article highlights critical role of mathematics in AI, emphasizing: mathematics provides tools to better understand & enhance AI systems. Conversely, AI raises new problems & drives development of new mathematics at intersection of various fields. This article focuses on application of analytical & probabilistic tools to model neural network architectures & better understand their optimization. Statistical questions (particularly generalization capacity of these networks) are intentionally set aside, though they are of crucial importance. Also shed light on evolution of ideas that have enabled significant advances in AI through architectures tailored to specific tasks, each echoing distinct mathematical techniques. Goal: encourage more mathematicians to take an interest in & contribute to this exciting field.
- **1. Supervised Learning.** Recent advancements in AI have mainly stemmed from development of neural networks, particularly deep networks. 1st significant successes, after 2010, came from supervised learning, where training pairs $(x^i, y^i)_i$ are provided, with x^i representing data (e.g., images or text) & y^i corresponding labels (typically, classes describing content of data). More recently, spectacular progress has been achieved in unsupervised learning, where labels y_i are unavailable, thanks to techniques known as generative AI. These methods will be discussed in Sect. 4.

Empirical Risk Minimization. In supervised learning, objective: construct a function $f_\theta(x)$, dependent on parameters θ , s.t. it approximates data well: $y^i \approx f_\theta(x^i)$. Dominant paradigm involves finding parameters θ by minimizing an empirical risk function, defined as

$$\min_{\theta} E(\theta) := \sum_i l(f_\theta(x^i), y^i), \quad (163)$$

where l : a loss function, typically $l(y, y') = \|y - y'\|^2$ for vector-valued $(y^i)_i$. Optimization of $E(\theta)$ is performed using gradient descent:

$$\theta_{t+1} = \theta_t - \tau \nabla E(\theta_t), \quad (164)$$

where τ : step size. In practice, a variant known as *stochastic gradient descent* [23] is used to handle large datasets by randomly sampling a subset at each step t . A comprehensive theory for convergence of this method exists in cases where f_θ depends linearly on θ , as $E(\theta)$ is convex. However, in case of deep neural networks, $E(\theta)$ is non-convex, making its theoretical analysis challenging & still largely unresolved. In some instances, partial mathematical analyses provide insights into observed practical successes & guide necessary modifications to improve convergence. For a detailed overview of existing results, refer to [Bac24].

Automatic Differentiation. Computing gradient $\nabla E(\theta)$ is fundamental. FDMs or traditional differentiation of composite functions would be too costly, with complexity on order of $O(DT)$, where D : dimensionality of θ , T : computation time of f_θ . Advances in deep network optimization rely on use of automatic differentiation in reverse mode [13], often referred to as “backpropagation of gradient”, with complexity on order of $O(T)$.

- **2. 2-Layer Neural Networks.**

Multi-Layer Perceptrons. A multi-layer network, or multi-layer perceptron (MLP) [25], computes function $f_\theta(x) = x_L$ in L steps (or layers) recursively starting from $x_0 = x$:

$$x_{l+1} = \sigma(W_l x_l + b_l), \quad (165)$$

where W_l : a weight matrix, b_l : a bias vector, & σ : a nonlinear activation function, e.g. sigmoid $\sigma(s) = \frac{e^s}{1+e^s}$, which is bounded, or ReLU (Rectified Linear Unit), $\sigma(s) = \max(s, 0)$, which is unbounded. Parameters to optimize are weights & biases $\theta = (W_l, b_l)_l$. If σ were linear, then f_θ would remain linear regardless of number of layers. Nonlinearity of σ enriches set of functions representable by f_θ , by increasing both width (dimension of intermediate vectors x_k) & depth (number L of layers).

2-Layer Perceptrons & Universality. Mathematically best-understood case is that of $L = 2$ layers. Denoting $W_1 = (v_k)_{k=1}^n$ & $W_2^\top = (u_k)_{k=1}^n$ as n rows & columns of 2 matrices (where n : network width), can write f_θ as a sum of contributions from its n neurons: (4)

$$f_\theta(x) = \frac{1}{n} \sum_{k=1}^n u_k \sigma(\langle v_k, x \rangle + b_k). \quad (166)$$

Parameters are $\theta = (u_k, v_k, b_k)_{k=1}^n$. Here, added a normalization factor $\frac{1}{n}$ (to later study case $n \rightarrow \infty$) & ignored nonlinearity of 2nd layer. A classical result by Cybenko [8] shows: these functions f_θ can uniformly approximate any continuous function on a compact domain. This result is similar to Weierstrass theorem for polynomial approximation, except (4) defines, for a fixed n , a nonlinear function space. Elegant proof by Hornik [15] relies on Stone–Weierstrass theorem, which implies result for $\sigma = \cos$ (since [4] defines an algebra of functions when n is arbitrary). Proof is then completed by uniformly approximating a cosine in 1D using a sum of sigmoids or piecewise affine functions (e.g., for $\sigma = \text{ReLU}$).

Mean-Field Representation. This result is, however, disappointing, as it does not specify how many neurons are needed to achieve a given approximation error. Impossible without adding assumptions about regularity of function to approximate. Barron’s fundamental result [4] introduces a Banach space of regular functions, defined by semi-norm $\|f\|_B := \int |\hat{f}(\omega)| |\omega| d\omega$, where \hat{f} denotes Fourier transform of f . Barron shows: in L^2 norm, for any n , there exists a network f_θ with n neurons s.t. approximation error on a compact Ω of radius R is of order $\|f - f_\theta\|_{L^2(\Omega)} = O\left(\frac{R\|f\|_B}{\sqrt{n}}\right)$. This result is remarkable because it avoids “curse of dimensionality”: unlike polynomial approximation, error does not grow exponentially with dimension d (although dimension affects constant $\|f\|_B$).

Barron’s proof relies on a “mean-field” generalization of [4], where a distribution (a probability measure) ρ is considered over parameter space (u, v, b) , & network is expressed as: (5)

$$F_\rho(x) := \int u \sigma(\langle v, x \rangle + b) d\rho(u, v, b). \quad (167)$$

A finite-size network (4), $f_\theta = F_{\hat{\rho}}$, is obtained with a discrete measure $\hat{\rho} = \frac{1}{n} \sum_{k=1}^n \delta_{(u_k, v_k, b_k)}$. An advantage of representation (5): F_ρ depends linearly on ρ , which is crucial for Barron’s proof. This proof uses a probabilistic Monte Carlo-like method (where error decreases as $\frac{1}{\sqrt{n}}$ as desired): it involves constructing a distribution ρ from f , then sampling a discrete measure $\hat{\rho}$ whose parameters $(u_k, v_k, b_k)_k$ are distributed according to ρ .

Wasserstein Gradient Flow. In general, analyzing convergence of optimization (2) $\theta_{t+1} = \theta_t - \tau \nabla E(\theta_t)$ is challenging because function E is non-convex. Recent analyses have shown: when number of neurons n is large, dynamics are not trapped in a local minimum. Fundamental result by Chizat & Bach [7] is based on fact: distribution ρ_t defined by gradient descent (2), as $\tau_t \rightarrow 0$, follows a gradient flow in space of probability distributions equipped with optimal transport distance. These Wasserstein gradient flows, introduced by [16] & studied extensively in book [1], satisfy a McKean-Vlasov-type PDE:

$$\partial_t \rho_t + \nabla \cdot (\rho_t \mathcal{V}(\rho)) = 0, \quad (168)$$

where $x \rightarrow \mathcal{V}(\rho)(x)$ is a vector field depending on ρ , which can be computed explicitly from data $(x^i, y^i)_i$. Result by Chizat & Bach can be viewed both as a PDE result (convergence to an equilibrium of a class of PDEs with a specific vector field $\mathcal{V}(\rho)$) & as a ML result (successful training of a 2-layer network via gradient descent when number of neurons is sufficiently large).

- **3. Very Deep Networks.** Unprecedented recent success of neural networks began with work of [17], which demonstrated: deep networks, when trained on large datasets, achieve unmatched performance. 1st key point: to achieve these performances, necessary to use weight matrices W_k that exploit structure of data. For images, this means using convolutions [18]. However, this approach is not sufficient for extremely deep networks, with number of layers L reaching into hundreds.

Residual Networks. Major breakthrough that empirically demonstrated that network performance increases with L was introduction of residual connections, known as ResNet [14]. Main idea: ensure: for most layers x_l , dimensions of x_l & x_{l+1} are identical, & to replace (3) with L steps: (7)

$$x_{l+1} = x_l + \frac{1}{L} U_l^\top \sigma(V_l x_l + b_l), \quad (169)$$

where $U_l, V_l \in \mathbb{R}^{n \times d}$: weight matrices, with n : number of neurons per layer (which, as in (4), can be increased to enlarge function space). Intuition for success of (7): this formula allows, unlike (3), for steps that are small deformations near identity mappings. This makes network $f_\theta(x_0) = x_L$ obtained after L steps wellposed even when L is large, & it can be rigorously proven [20]: this well-posedness is preserved during optimization via gradient descent (2).

Neural Differential Equation. As L approaches inf, (7) can be interpreted as a discretization of an ODE: (8)

$$\frac{dx_s}{ds} = U_s^\top \sigma(V_s x_s + b_s), \quad (170)$$

where $s \in [0, 1]$ indexes network depth. Network $f_\theta(x_0) := x_1$ maps initialization $x_{s=0}$ to solution $x_{s=1}$ of (8) at time $s = 1$. Parameters are $\theta = (U_s, V_s, b_s)_{s \in [0, 1]}$. This formalization, referred to as a *neural ODE*, was initially introduced in [6] to leverage tools from adjoint equation theory to reduce memory cost of computing gradient $\nabla E(\theta)$ during backpropagation. It also

establishes a connection to control theory, as training via gradient descent (2) computes an *optimal control* θ that interpolates between data x^i & labels y^i . However, specificity of learning theory compared to control theory lies in goal of computing such control using gradient descent (2). To date, no detailed results exist on this. Nonetheless, in his thesis, RAPHAEL BARBONI [3] demonstrated: if network is initialized near an interpolating network, gradient descent converges to it.

- 4. Generative AI for Vector Data.

Self-Supervised Pre-Training. Remarkable success of large generative AI models for vector data, e.g. images (often referred to as “diffusion models”), & for text (large language models for LLMs), relies on use of new large-scale network architectures & creation of new training tasks known as “self-supervised”. Manually defining labels y^i through human intervention is too costly, so these are calculated automatically by solving simple tasks. For images, these involve denoising tasks, while for text, they involve predicting next word. A key advantage of these simple tasks (known as pretraining tasks): it becomes possible to use a pre-trained network f_θ generatively. Starting from an image composed of pure noise & iterating on network, one can randomly generate a realistic image [27]. Similarly, for text, starting from a prompt & sequentially predicting next word, possible to generate text, e.g., to answer questions [5]. 1st describe case of vector data generation, e.g. images, & address LLMs for text in following sect.

Sampling as an Optimization Problem. Generative AI models aim to generate (or “sample”) random vectors x according to a distribution β , which is learned from a large training dataset $(x^i)_i$. This distribution is obtained by “pushing forward” a reference distribution α (most commonly an isotropic Gaussian distribution $\alpha = \mathcal{N}(0, \text{Id})$) through a neural network f_θ . Specifically, if $X \sim \alpha$ is distributed according to α , then $f_\theta(X) \sim \beta$ follows law β , which is denoted as $(f_\theta)_\# \alpha = \beta$, where $\#$ represents pushforward operator (also known as image measure in probability theory).

Early approaches to generative AI, e.g. “Generative Adversarial Networks” (GANs) [12] attempted to directly optimize a distance D between probability distributions (e.g., an optimal transport distance [22]): (9)

$$\min_\theta D((f_\theta)_\# \alpha, \beta). \quad (171)$$

This problem is challenging to optimize because computing D is expensive, & β must be approximated from data $(x^i)_i$.

Flow-Based Generation. Recent successes, particularly in generation of vector data, involve neural networks f_θ that are computed iteratively by integrating a flow [21], similar to a neural differential equation (8): (10)

$$f_0(\mathbf{x}_0) := \mathbf{x}_1 \text{ where } \frac{d\mathbf{x}_s}{ds} = g_\theta(\mathbf{x}_s, s), \quad (172)$$

where $g_\theta : (\mathbf{x}, s) \in \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$. Input space includes an additional temporal dimension s , compared to f_θ . Most effective neural networks for images are notably U-Nets (UNet) [24].

Central mathematical question is how to replace (9) with a simpler optimization problem when f_θ is computed by integrating a flow (10). An extremely elegant solution was 1st proposed in context of diffusion models [27] (corresponding to specific case $\alpha = \mathcal{N}(0, \text{Id})$) & later generalized under name “flow matching” [19].

Main idea: if $x_0 \sim \alpha$ is initialized randomly, then x_s , solution at time s of (10), follows a distribution α_s that interpolates between $\alpha_0 = \alpha$ & α_1 , which is desired to match β . This distribution satisfies a conservation equation: (11)

$$\partial_s \alpha_s + \nabla \cdot (\alpha_s \mathbf{v}_s) = 0, \quad (173)$$

where vector field is defined as $\mathbf{v}_s(\mathbf{x}) := g_\theta(\mathbf{x}, s)$. This equation is similar to evolution of neuron distributions during optimization (6), but it is simpler because \mathbf{v}_s does not depend on distribution α_s , making equation linear in α_s .

Denoising Pre-Training. Question: how to find a vector field $\mathbf{v}_s(\mathbf{x}) = g_\theta(\mathbf{x}, s)$ s.t. $\alpha_1 = \beta$, i.e., final distribution matches desired one. If \mathbf{v}_s is known, distribution α_s is uniquely determined. Key idea: reason in reverse: starting from an interpolation α_s satisfying $\alpha_0 = \alpha$, $\alpha_1 = \beta$, how can we compute a vector field \mathbf{v}_s s.t. (11) holds? There are infinitely many possible solutions (since \mathbf{v}_s can be modified by a conservative field), but for certain specific interpolations α_s , remarkably simple expressions can be derived.

1 example: interpolation obtained via barycentric averaging: take a pair $x_0 \sim \alpha$, $x_1 \sim \beta$, & define α_s as distribution of $(1-s)x_0 + sx_1$. It can be shown [19]: an admissible vector field is given by a simple conditional expectation: (12)

$$v_s(\mathbf{x}) = \mathbb{E}_{x_0 \sim \alpha, x_1 \sim \beta} (x_1 - x_0 | (1-s)x_0 + sx_1 = x). \quad (174)$$

Key advantage of this formula: conditional mean corresponds to a linear regression, which can be approximated using a neural network $v_s \approx g_\theta(\cdot, s)$. Expectation over $x_1 \sim \beta$ can be replaced by a sum over training data $(x^i)_i$, leading to following optimization problem:

$$\min_\theta E(\theta) := \int_0^1 \mathbb{E}_{x_0 \sim \alpha} \sum_i \|x^i - x_0 - g_\theta((1-s)x_0 + sx^i, s)\|^2 ds. \quad (175)$$

This function $E(\theta)$ is an empirical risk function (1), & similar optimization techniques are used to find an optimal θ . In particular, stochastic gradient descent efficiently handles both integral \int_0^1 & expectation $\mathbb{E}_{x_0 \sim \alpha}$.

Problem (12) corresponds to an unsupervised pre-training task: there are no labels y^i , but an artificial supervision task is created by adding random noise x_0 to data x^i . This task is called *denoising*. For textual data, a different pre-training task is used:

- 5. Generative AI for Text.

Tokenization & Next-Word Prediction. Generative AI methods for text differ from those used for vector data generation. Neural network architectures are different (they involve transformers, as will describe), & pre-training method is based not on denoising but on next-word prediction [28]. Worth noting: these transformer neural networks are now also used for image generation [9], but specific aspects related to causal nature of text remain crucial. 1st preliminary step, called “tokenization”, consists of transforming input text into a sequence of vectors $X = (x[1], \dots, x[P])$, where number P is variable (& may increase, e.g., when generating text from a prompt). Each token $x[p]$ is a vector that encodes a group of letters, generally at level of a syllable. Neural network $x = f_\theta(X)$ is applied to all tokens & predicts a new token x . During training, a large corpus of text $(X^i)_i$ is available. If denote by \tilde{X}^i text X^i with last token x^i removed, minimize an empirical risk for prediction:

$$\min_{\theta} E(\theta) := \sum_i l(f_\theta(\tilde{X}^i), x^i), \quad (176)$$

which is exactly similar to (1). When using a pre-trained network f_θ for text generation, one starts with a prompt X & iteratively adds a new token in an auto-regressive manner: $X \leftarrow [X, f_\theta(X)]$.

Transformers & Attention. Large networks used for text generation tasks are Transformer networks [10]. Unlike ResNet (7), these networks f_θ no longer operate on a single vector \mathbf{x} , but on a set of vectors $X = (x[1], \dots, x[P])$ of size P . In Transformers, ResNet layer (7), operating on a single vector, is replaced by an *attention* layer, where all tokens interact through a barycentric combination of vectors $(Vx[q])_q$ where V is a matrix: (13)

$$A_\omega(X)_p := \sum_q M_{p,q}(Vx[q]) \text{ where } M_{p,q} := \frac{e^{\langle Qx[p], Kx[q] \rangle}}{\sum_l e^{\langle Qx[p], Kx[l] \rangle}} \text{ with } \omega := (Q, K, V). \quad (177)$$

Coefficients $M_{p,q}$ are computed by normalized correlation between $x[p], x[q]$, depending on 2 parameter matrices Q, K . A Transformer $f_\theta(X)$ is then defined similarly to ResNet (7), iterating L attention layers with residual connections: (14)

$$X_{l+1} = X_l + \frac{1}{L} A_{\omega_l}(X_l). \quad (178)$$

Parameters θ of Transformer $f_\theta(X_0) = X_L$ with L layers are $\theta = (\omega_l = (Q_l, K_l, V_l))_0^{L-1}$.

This description is simplified: in practice, a Transformer network also integrates normalization layers & MLPs operating independently on each token. For text applications, attention must be causal, imposing $M_{i,j} = 0$ for $j > i$. This constraint is essential for recursively generating text & ensuring next-word prediction task is meaningful.

Mean-Field Representation of Attention. Attention mechanism (13) can be viewed as a system of interacting particles, & (14) describes evolution of tokens across depth. As with ResNet (8), in limit $L \rightarrow +\infty$, one can consider a system of coupled ODEs: (15)

$$\frac{dX_s}{ds} = A_{\omega_s}(X_s). \quad (179)$$

A crucial point: for non-causal Transformers, system $X_s = (x_s[p])_p$ is invariant under permutations of indices. Thus, this system can be represented as a probability distribution $\mu_s := \frac{1}{P} \sum_p \delta_{x_s[p]}$ over token space. This perspective was adopted in MICHAEL SANDER’s thesis [26], which rewrites attention as:

$$\mathcal{A}_\omega(x) := \frac{\int e^{\langle Qx, Kx' \rangle} Vx' d\mu(x')}{\int e^{\langle Qx, Kx' \rangle} d\mu(x')} \text{ where } \omega := (Q, K, V). \quad (180)$$

Particle system (15) then becomes a conservation equation for advection by vector field $\mathcal{A}_{\omega_s}(\mu)$: (16)

$$\partial_s \mu_s + \nabla \cdot (\mu_s \mathcal{A}_{\omega_s}(\mu_s)) = 0. \quad (181)$$

Surprisingly, this yields a McKean-Vlasov-type equation, similar to the one describing training of 2-layer MLPs (6), but with velocity field $\mathcal{A}_\omega(\mu)(x)$ replacing $\mathcal{V}(\rho)(u, v, b)$. However, here evolution occurs in token space x rather than in neuron space (u, v, b) , & evolution variable corresponds to depth s , not optimization time t .

Unlike (6), evolution (16) is not a gradient flow in Wasserstein metric [26]. Nonetheless, in certain cases, this evolution can be analyzed, & it can be shown: measure μ_s converges to a single Dirac mass [11] as $s \rightarrow \infty$: [tokens tend to cluster]. Better understanding these evolutions, as well as optimization of parameters $\theta = (Q_s, K_s, V_s)_{s \in [0,1]}$ via gradient descent (2), remains an open problem. This problem can be viewed as a control problem for PDE (16).

- **Conclusion.** Mathematics plays a critical role in understanding & improving performance of deep architectures while presenting new theoretical challenges. Emergence of Transformers & generative AI raises immense mathematical problems, particularly for better understanding training of these networks & exploring structure of “optimal” networks. 1 essential question remains: whether LLM merely interpolates training data or is capable of genuine reasoning. Moreover, issues of resource efficiency & privacy in AI system development demand significant theoretical advancements, where mathematics will play a pivotal role (đóng vai trò then chốt). Whether designing resource-efficient models, ensuring compliance with ethical standards, or exploring fundamental limits of these systems, mathematics is poised to be an indispensable tool for future of AI.

2.11 THANG NGUYEN, DUNG NGUYEN, KHA PHAM, TRUYEN TRAN. MP-PINN: A Multi-Phase Physics-Informed Neural Network for Epidemic Forecasting

Abstract. Forecasting temporal processes e.g. virus spreading in epidemics often requires more than just observed time-series data, especially at beginning of a wave when data is limited. Traditional methods employ mechanistic models like SIR family, which make strong assumptions about underlying spreading process, often represented as a small set of compact differential equations. Data-driven methods e.g. deep neural networks make no such assumptions & can capture generative process in more detail, but fail in long-term forecasting due to data limitations. Propose a new hybrid method called MP-PINN (Multi-Phase Physics-Informed Neural Network) to overcome limitations of these 2 major approaches. MP-PINN instills spreading mechanism into a neural network, enabling mechanism to update in phases over time, reflecting dynamics of epidemics due to policy interventions. Experiments on COVID-19 waves demonstrate: MP-PINN achieves superior performance over pure data-driven or model-driven approaches for both short-term & long-term forecasting.

Keywords. Time-series forecasting; Physics-Informed Neural Network; Epidemiological Models; COVID-19.

- **1. Introduction.** COVID-19 pandemic has claimed > 7 million lives for just 4 years. Pause for a moment & consider a hard *counterfactual* question: Could majority of these lives have been saved if we had predicted spread better at onset of pandemic & acted more effectively? World did all it could: modeling, forecasting, implementing lockdowns, developing vaccines, & much more. In absence of proper understanding of viruses' nature & with limited data available when a wave just started, epidemiologists had to make assumptions in *model-driven* methods, e.g. those in mechanistic SIR family [10] or in detailed agent-based simulations [11]. When some data became available, e.g., after a month, *data-driven* methods, as preferred by DS community, could be employed to detect trends in time-series [25,21]. A key open challenge: complex interplay of evolving interventions, human factors & technological advances driving epidemic waves [1].

If anything, extremely high death toll has profoundly demonstrated 1 thing: Have failed to predict spread of COVID-19 virus variants. Fig. 1: A representative case of forecasting COVID-19 at 35 days of wave. Our hybrid multiphase method MP-PINN strikes a balance between model-driven & data-driven approaches, & hence is more accurate in both short/long-term forecasting. clearly illustrates this failure. As seen, model-driven methods e.g. mechanistic SIR models capture overall shape of wave but are inadequate in reflecting current data & changing reality on a daily basis. This might be due to rigid & strong assumptions made at modeling time. Data-driven methods, e.g. those using deep neural networks, fit new evidence better but fail to capture long-term underlying mechanisms. Clearly, a better approach is needed to (a) capture both short-term & long-term processes [22,18], & (b) dynamically calibrate models in face of new evidence [19].

To this end, propose MP-PINN (which stands for Multi-Phase Physics-Informed Neural Network) to overcome these limitations. MP-PINN employs a recent powerful approach known as Physics-Informed Neural Network (PINN), which trains a neural network to agree with both empirical data & epidemiological models. However, PINNs alone are not sufficient to reflect reality: Must account for complex interplay of evolving factors driving epidemic waves, e.g. changing regulations, emerging information, & shifting public sentiment, all of which influence pandemic's trajectory. This is where MP-PINN comes in: Instead of assuming a single set of parameters fore entire wave, allow model to vary across *multiple distinct phases*, each represented by a set of SIR parameters. This brings adaptability into MP-PINN.

Demonstrate MP-PINN on COVID-19 data sets collected from COVID-19 data from 21 regions in Italy in 1st half of 2020. Results show: MP-PINN achieves superior performance in both short-term & long-term forecasting of COVID-19 spread. In particular, MP-PINN outperforms traditional SIR models, pure data-driven approaches (MLP), & single-phase PINNs. See Fig. 1 for a representative case demonstrating efficacy of MP-PINN.

- **2. Related Works.**
 - **Epidemic Forecasting.** Briefly review literature in epidemic forecasting most relevant to our work as literature has exploded since COVID-19 outbreak in late 2019:
 - * *Model-driven approach.* Compartmental models (Mô hình ngăn), e.g. Susceptible-Infectious-Recovered (SIR) model, are foundational in epidemic modeling due to their simplicity & reliance on mechanistic understanding of disease spread. They are particularly effective for long-term forecasting because they incorporate known epidemiological dynamics [10]. However, their fixed parameters often lead to less accurate short-term predictions, as they cannot easily adapt to rapid changes in transmission dynamics. Have been a plethora of SIR extensions with sophisticated assumptions e.g. SIRD [13], SEIR [4], & SEIRM [24].
 - * *Data-driven approach.* ML models, particularly DL techniques, have gained prominence for short-term forecasting due to their ability to capture complex patterns in large datasets [3]. Techniques e.g. Long Short-Term Memory (LSTM) networks excel in identifying trends & making predictions over short periods. However, their performance deteriorates over longer horizons due to their lack of incorporation of epidemiological knowledge, making them less reliable for long-term predictions [28]. In [12], authors studied forecasting influenza outbreaks, e.g. training model with data in 4 years to predict in future outbreaks, hence training data is much larger than ours which consists of only 1 outbreak.

Most existing data-driven approaches make short-term predictions (< 1 month). E.g., [9] used 2 months to train & predict next months with prediction windows are 3/7/14 days. Likewise, model in [30] trained on almost 10 months & test on 1 month with prediction window is 7/14/21/28 days. In [14,26], models were trained on 377 days, but window of forecasting in both works is next 7 days (observed previous 21 days). In contrast, train model only on data collected for 35 days, & forecast rest of outbreak (97 days). Thus out setting is much more challenging & most impossible to achieve without utilizing epidemic models & prior knowledge.

- **Physics-Informed Neural Networks (PINNs).** Recently, PINNs have emerged as a framework to incorporate known physical rules to train deep neural networks [23]. It has been demonstrated to be effective in solving forwards & inverse problems involving PDEs. Since then, numerous studies have explored application of PINNs in various domains, e.g. fluid dynamics [2], material science [20], & epidemic modelling [24].

PINNs for epidemic modeling. PINNs have been used to build hybrid forecasting models, integrating model-driven & data-driven methods [7,16,24,27]. [24] proposed to regularize embeddings from both time-dependent model which can be regularized via physical law & exogenous features extractor which obtain information from multiple sources for making better predictions. Wang et al. [31] proposed a physics-informed neural network (PINN) framework for learning parameters of a COVID-19 compartment model from observed data. [5] used epidemic model to compute ahead unobserved data points then augment training process directly with prediction loss. In [5,29], although parameters of epidemic model, e.g., infection rate & recovery rate, are generated by a trainable NNs-based module, potential impact of data instability on learned epidemic parameters is not explicitly addressed. E.g., if case counts change significantly & differ from historical patterns, it can be challenging for [5,29] models to learn valid & stable transmission & recovery rates. In contrast, our work introduces distinct phases within SIR model to capture long-term dynamics of an outbreak, which may not be apparent in short-term or noisy data.

Building on strengths & limitations of these approaches, our proposed MP-PINN framework aims to address gaps in both model-driven & data-driven methods.

• 3. Preliminaries.

- **3.1. Mechanistic Compartment Models.** In epidemiology & other fields where behavior of large populations is studied, compartment models are models in which population are divided into a discrete set of qualitatively-distinct states/types/classes/groups, so-called *compartments* (ngăn). These models also define transition between compartments. Focus on SIR (Susceptible-Infected-Recovered), most popular compartments model used to model spread of infectious diseases in epidemiology [10]. SIR contains 3 compartments: (1) Susceptible (Dễ bị tổn thương), (2) Infected (Bị lây nhiễm), (3) Recovered/removed populations (Dân số được phục hồi/loại bỏ) which are denoted as $S(t), I(t), R(t)$ as a function of time t , resp. At time t , an individual in population is classified into 1 of these compartments, typically transiting from being susceptible to infected & finally recovered (or removed). Hence, size of population N is sum of number of susceptible, infectious, & recovered persons, i.e., $N = S(t) + I(t) + R(t)$. Here, considered a *single outbreak* SIR model comprises a set of ODEs that describe transitions between 3 compartments:

$$\frac{dS(t)}{dt} = -\frac{\beta}{N}S(t)I(t), \quad (182)$$

$$\frac{dI(t)}{dt} = \frac{\beta}{N}S(t)I(t) - \gamma I(t), \quad (183)$$

$$\frac{dR(t)}{dt} = \gamma I(t), \quad (184)$$

where parameters of model $\beta > 0, \gamma \in (0, 1)$: *infection rate* & *recovery rate*, resp. In real-world & more complex models, these parameters can also vary over time or depend on different factors e.g. policy or other properties of population. Initial condition of ODEs are $S(0) > 0, I(0) > 0, R(0) \geq 0$.

An important assumption of SIR model: all recovered individuals (in R group) are completely immune & cannot return to susceptible S or infected I groups, & total population N remains constant. However, N does not always represent entire population, especially in real-world scenarios. E.g., 1 assumption of SIR model [8] : population mixes homogeneously, meaning everyone has same level of interaction with others. However, during early stages of COVID-19 outbreak in 2020, it was impractical to consider entire population as susceptible. Instead, N might only represent a fraction of population. Moreover, population could therefore be divided into 2 distinct groups [32]: those with inherited immunity & those without. These distinctions imply: total number of susceptible individuals $S(t)$ may not always correspond to entire population N , but rather to a specific portion of it, depending on factors e.g. inherited immunity or other epidemiological considerations.

- **3.2. Physics-Informed Neural Networks (PINNs).** PINNs [23] are neural networks equipped with physical constraints of domain, either through network architecture or as a regularization term during training process. These physical laws often involve parameters that need to be estimated. During training of a PINN, both neural network weights & physical parameters are optimized to achieve best fit to observed data while simultaneously satisfying physical constraints.

More formally, given a training dataset \mathcal{D} , learning searches for a neural network $f \in \mathcal{H}$ by solving an optimization problem:

$$f^* = \min_{f \in \mathcal{H}} \mathcal{L}(f; \mathcal{D}) + \lambda \Omega(f), \quad (185)$$

where $\mathcal{L}(f; \mathcal{D})$: usual data loss function, $\Omega(f)$: a regularization term that introduces physical prior knowledge into learning process, & $\lambda > 0$: a balancing weight. When prior is specified as PDEs, regularization typically takes form of PDE residual loss:

$$\Omega(f) = \frac{1}{L} \sum_{i=1}^L (\partial f_{\text{NN}}(x_i) - \partial f_{\text{PDE}}(x_i))^2, \quad (186)$$

where $\partial f_{\text{NN}}(x_i)$ denotes partial derivative of neural network evaluated at x_i & $\partial f_{\text{PDE}}(x_i)$ denotes corresponding partial derivative specified by PDEs. Evaluation points $i = 1, 2, \dots, L$ are sampled so that function f & its derivative are well supported.

- 4. Methods. Present main contributions of developing PINNs for epidemic forecasting. Overall framework is depicted in Fig. 2: Multi-phase Physics-Informed Neural Network (MP-PINN) Framework for Epidemic Forecasting. Framework illustrates integration of expert knowledge & data-driven approaches to estimate parameters in a multi-phase scenario, where key parameters e.g. infection rate β & recovery rate γ vary across phases. Start from a single-phase assumption, then advance into multi-phase model. Both single-phase & multi-phase models integrate expert knowledge & a data-driven parameter estimation process. In multi-phase model, however, parameters e.g. infection rate β & recovery rate γ vary between phases, enhancing model's ability to adapt to complex real-world scenarios. Single-phase approach is described in Sect. 4.1 & multi-phase approach is detailed in Sect. 4.2. More concretely, using SIR model described in Sect. 3.1 as physics prior, build a PINN framework with separate neural networks $f_{\psi_1}^S(t), f_{\psi_2}^S(t)$ where f_{ψ_i} that takes t as input to predict Susceptible $S(t)$ & Infected $I(t)$, resp. For clarity, present case where input is only time t , but any other features, static or dynamic can be applicable. Note: do not need to model Recovered $R(t)$ because of constraint $S(t) + I(t) + R(t) = N$.
 - 4.1. Single-Phase PINN (SP-PINN).
 - 4.2. Multi-Phase PINN (MP-PINN).
- 5. Experiments.
 - 5.1. Settings.
 - 5.2. Results.
- 6. Conclusions. Introduced MP-PINN (Multi-Phase Physics-Informed Neural Network), a novel approach to epidemic forecasting that addresses limitations of both traditional model-driven & data-driven methods. By integrating mechanistic understanding of SIR models with flexibility of neural networks & allowing for multiple SIR parameters across phases, MP-PINN achieves superior performance in both short-term & long-term forecasting of COVID-19 spread. Our experiments on COVID-19 data from 21 regions in Italy demonstrate: MP-PINN outperforms traditional SIR models, pure data-driven approaches (MLP), & single-phase PINNs. Ability to capture evolving dynamics through multiple phases proves crucial in reflecting impact of changing interventions & public behaviors throughout course of epidemic. MP-PINN's success highlights potential of hybrid approaches that combine domain knowledge with data-driven learning. By allowing for incorporation of expert insights & prior knowledge about parameter ranges, our method provides a flexible framework that can adapt to complex, evolving nature of real-world epidemics. Future work could explore automatic detection of phase transition points & incorporation of additional epidemiological factors. Finally, emphasize: MP-PINN framework is generally applicable to any outbreaks where underlying dynamics may shift over time.

2.12 [RLM22]. SEBASTIAN RASCHKA, YUXI (HAYDEN) LIU, VAHID MIRJALILI. **Machine Learning with PyTorch & Scikit-Learn: Develop ML & DL Models with Python. 2023**

PyTorch book of bestselling & widely acclaimed *Python ML* series. Foreword by DMYTRO DZHULGAKOV – PyTorch Core Maintainer

- Foreword. “Over recent years, ML methods, with their ability to make sense of vast amounts of data & automate decisions, have found widespread applications in healthcare, robotics, biology, physics, consumer products, internet services, & various other industries.

Giant leaps in science usually come from a combination of powerful ideas & great tools. ML is no exception. Success of data-driven learning methods is based on ingenious ideas of thousands of talented researchers over field's 60-year history. But their recent popularity is also fueled by evolution of hardware & software solutions that make them scalable & accessible. Ecosystem of excellent libraries for numeric computing, data analysis, & ML built around Python like NumPy & scikit-learn gained wide adoption in research & industry. This has greatly helped propel Python to be most popular programming language.

Massive improvements in computer vision, text, speech, & other tasks brought by recent advent of DL techniques exemplify this theme. Approaches draw on neural network theory of last 4 decades that started working remarkably well in combination with GPUs & highly optimized compute routines.

Goal with building PyTorch over past 5 years has been to give researchers most flexible tool for expressing DL algorithms while taking care of underlying engineering complexities. benefited from excellent Python ecosystem. In turn, have been fortunate to see community of very talented people build advanced DL models across various domains on top of PyTorch. Authors of this book were among them.

DMYTRO DZHULGAKOV has known SEBASTIAN within this tight-knit community for a few years now. He has unmatched talent in easily explaining information & making complex accessible. SEBASTIAN contributed to many widely used ML software packages & authored dozens of excellent tutorials on DL & data visualization.

Mastery of both ideas & tools is also required to apply ML in practice. Getting started might feel intimidating, from making sense of theoretical concepts to figuring out which software packages to install.

Luckily, this book does a beautiful job of combining ML concepts & practical engineering steps to guide you in this journey. You're in for a delightful ride from basics of data-driven techniques to most novel DL architectures. Within every chap, find concrete code examples applying introduced methods to a practical task.

When 1e came out in 2015, it set a very high bar for *ML & Python* book category. But excellence didn't stop there. With every edition, SEBASTIAN & team kept upgrading & refining material as DL revolution unfolded in new domains. In this new PyTorch edition, find new chaps on transformer architectures & graph neural networks. These approaches are on cutting edge of DL & have taken fields of text understanding & molecular structure by storm in last 2 years. You will get to practice them using new yet widely popular software packages in ecosystem like Hugging Face, PyTorch Lightning, & PyTorch Geometric.

Excellent balance of theory & practice this book strikes is no surprise given authors' combination of advanced research expertise & experience in solving problems hands-on. SEBASTIAN RASCHKA & VAHID MIRJALILI draw from their background in DL research for computer vision & computational biology. HAYDEN LIU brings experience of applying ML methods to event prediction, recommendation systems, & other tasks in industry. All of authors share a deep passion for education, & it reflects in approachable way book goes from simple to advanced.

Confident: find this book invaluable both as a broad overview of exciting field of ML & as a treasure of practical insights. Hope it inspires you to apply ML for greater good in your problem area, whatever it might be." DMYTRO DZHULGAKOV, PyTorch Core Maintainer

- **Preface.** Through exposure to news & social media, probably are familiar with fact: ML has become 1 of most exciting technologies of our time & age. Large companies, e.g. Microsoft, Google, Meta, Apple, Amazon, IBM, & many more, heavily invest in ML research & applications for good reasons. While it may seem: ML has become buzzword (từ thông dụng) of our time & age, certainly not hype (sự cường điệu). This exciting field opens way to new possibilities & has become indispensable to our daily lives. Talking to voice assistant on smartphones, recommending right product for our customers, preventing credit card fraud, filtering out spam from e-mail inboxes, detecting & diagnosing medical diseases, list goes on & on.

If want to become a ML practitioner, a better problem solver, or even consider a career in ML research, then this book is for you! However, for a novice (cho người mới bắt đầu), theoretical concepts behind ML can be quite overwhelming. Yet, many practical books that have been published in recent years will help you get started in ML by implementing powerful learning algorithms.

Getting exposed to practical code examples & working through example applications of ML is a great way to dive into this field. Concrete examples help to illustrate broader concepts by putting learned material directly into action. However, remember: with great power comes great responsibility! In addition to offering hands-on experience with ML using Python & Python-based ML libraries, this book also introduces mathematical concepts behind ML algorithms, which is essential for using ML successfully. Thus, this book is different from a purely practical book; it is a book that discusses necessary details regarding ML concepts, offers intuitive yet informative explanations on how ML algorithms work, how to use them, & most importantly, how to avoid most common pitfalls (cạm bẫy).

In this book, embark (tham gia) on exciting journey that covers all essential topics & concepts to give you a head start in this field. If find that your thirst for knowledge is not satisfied, this book references many useful resources that you can use to follow up on essential breakthroughs in this field.

- **Who this book is for.** This book is ideal companion for learning how to apply ML & DL to a wide range of tasks & datasets. If you are a programmer who wants to keep up with recent trends in technology, this book is definitely for you. Also, if you are a student or considering a career transition, this book will be both your introduction & a comprehensive guide to world of ML.
- **What this book covers.**
 - * *Chap. 1: Giving Computers Ability to Learn from Data*, introduces to main subareas of ML to tackle various problem tasks. In addition, it discusses essential steps for creating a typical ML model building pipeline that will guide us through following chaps.
 - * *Chap. 2: Training Simple ML Algorithms for Classification*, goes back to origins of ML & introduces binary perceptron classifiers & adaptive linear neurons. This chap is a gentle introduction to fundamentals of pattern classification & focuses on interplay of optimization algorithms & ML.
 - * *Chap. 3: A Tour of ML Classifiers Using Scikit-Learn*, describes essential ML algorithms for classification & provides practical examples using 1 of most popular & comprehensive open-source ML libraries, **scikit-learn**.
 - * *Chap. 4: Building Good Training Datasets – Data Preprocessing*, discusses how to deal with most common problems in unprocessed datasets, e.g. missing data. Also discuss several approaches to identify most informative features in datasets & teach how to prepare variables of different types as proper inputs for ML algorithms.
 - * *Chap. 5: Compressing Data in Dimensionality Reduction*, describes essential techniques to reduce number of features in a dataset to smaller sets while retaining most of their useful & discriminatory information. Discuss standard approach to dimensionality reduction via principal component analysis & compare it to supervised & nonlinear transformation techniques.
 - * *Chap. 6: Learning Best Practices for Model Evaluation & Hyperparameter Tuning*, discusses the do's & don'ts for estimating performances of predictive models. Moreover, discuss different metrics for measuring performance of our models & techniques to fine-tune ML algorithms.
 - * *Chap. 7: Combining Different Models for Ensemble Learning*, introduces to different concepts of combining multiple learning algorithms effectively. Teach how to build ensembles of experts (làm thế nào để xây dựng các nhóm chuyên gia) to overcome weaknesses of individual learners, resulting in more accurate & reliable predictions.

- * *Chap. 8: Applying ML to Sentiment Analysis*, discusses essential steps to transform textual data into meaningful representations for ML algorithms to predict opinions of people based on their writing.
 - * *Chap. 9: Predicting Continuous Target Variables with Regression Analysis*, discusses essential techniques for modeling linear relationships between target & response variables to make predictions on a continuous scale. After introducing different linear models, also talks about polynomial regression & tree-based approaches.
 - * *Chap. 10: Working with Unlabeled Data – Clustering Analysis*, shifts focus to a different subarea of ML, unsupervised learning. Apply algorithms from 3 fundamental families of clustering algorithms to find groups of objects that share a certain degree of similarity.
 - * *Chap. 11: Implementing a Multilayer Artificial Neural Network from Scratch*, extends concept of gradient-based optimization, which 1st introduced in Chap. 2, to build powerful, multilayer neural networks based on popular backpropagation algorithm in Python.
 - * *Chap. 12: Parallelizing Neural Network Training with PyTorch*, builds upon knowledge from previous chap to provide you with a practical guide for training neural networks more efficiently. Focus of this chap is on PyTorch, an open-source Python library that allows us to utilize multiple cores of modern GPUs & construct deep neural networks from common building blocks via a user-friendly & flexible API.
 - * *Chap. 13: Going Deeper – Mechanics of PyTorch*, picks up where previous chap left off & introduces more advanced concepts & functionality of PyTorch. PyTorch is an extraordinary vast & sophisticated library, & this chap walks you through concepts e.g. dynamic computation graphs & automatic differentiation. Also learn how to use PyTorch's object-oriented API to implement complex neural networks & how PyTorch Lightning helps with best practices & minimizing boilerplate code (giảm thiểu mã mẫu).
 - * *Chap. 14: Classifying Images with Deep Convolutional Neural Networks*, introduces *convolutional neural networks (CNNs)*. A CNN represents a particular type of deep neural network architecture that is particularly well-suited for working with image datasets. Due to their superior performance compared to traditional approaches, CNNs are now widely used in computer vision to achieve state-of-art results for various image recognition tasks. Throughout this chap, learn how convolutional layers can be used as powerful feature extractors for image classification.
 - * *Chap. 15: Modeling Sequential Data Using Recurrent Neural Networks*, introduces another popular neural network architectures for DL that is especially well suited for working with text & other types of sequential data & time series data. As a warm-up exercises, this chap introduces recurrent neural networks for predicting sentiment of movie reviews. Then, teach recurrent networks to digest information from books in order to generate entirely new text.
 - * *Chap. 16: Transformers – Improving Natural Language Processing with Attention Mechanisms*, focuses on latest trends in natural language processing & explains how attention mechanisms help with modeling complex relationships in long sequences. In particular, this chap describes influential transformer architecture & state-of-art transformer models e.g. BERT & GPT.
 - * *Chap. 17: Generative Adversarial Networks for Synthesizing New Data*, introduces a popular adversarial training regime for neural networks that can be used to generate new, realistic-looking images. Chap starts with a brief introduction to autoencoders, which is a particular type of neural network architecture that can be used for data compression. Chap then shows how to combine decoder part of an autoencoder with a 2nd neural network that can distinguish between real & synthesized images. By letting 2 neural networks compete with each other in an adversarial training approach, will implement a generative adversarial network that generates new handwritten digits.
 - * *Chap. 18: Graph Neural Networks for Capturing Dependencies in Graph Structured Data*, goes beyond working with tabular datasets, images, & text. Introduces graph neural networks that operate on graph-structured data, e.g. social media networks & molecules. After explaining fundamentals of graph convolutions, this chap includes a tutorial showing how to implement predictive models for molecular data.
 - * *Chap. 19: Reinforcement Learning for Decision Making in Complex Environments*, covers a subcategory of ML that is commonly used for training robots & other autonomous systems. Chap starts by introducing basics of *reinforcement learning (RL)* to become familiar with agent/environment interactions, reward process of RL systems, & concept of learning from experience. After learning about main categories of RL, will implement & train an agent that can navigate in a grid world environment using Q-learning algorithm. Finally, this chap introduces deep Q-learning algorithm – a variant of Q-learning that uses deep neural networks.
- **To get most out of book.** Ideally, already comfortable with programming in Python to follow along with code examples provided to both illustrate & apply various algorithms & models. To get most out of this book, a firm grasp of mathematical notation will be helpful as well.
A common laptop or desktop computer should be sufficient for running most of code in this book, & provide instructions for your Python environment in 1st chap. Later chaps will introduce additional libraries & installation recommendations when need arises.
A recent *graphical processing unit (GPU)* can accelerate code runtimes in later DL chaps. However, a GPU is not required, & also provide instructions for using free cloud resources.
 - **Download example code files.** All code examples are available for download through <https://github.com/rasbt/machine-learning>. Also have other code bundles from rich catalog of books & videos available at <https://github.com/PacktPublishing/>. While recommend using Jupyter Notebook for executing code interactively, all code examples are available in both a Python script (e.g., `ch02/ch02.py`) & a Jupyter Notebook format (e.g., `ch02/ch02.ipynb`). Furthermore, recommend viewing `README.md` file that accompanies each individual chap for additional information & updates.

- **Chap. 1: Giving Computers Ability to Learn from Data.** ML, application & science of algorithms that make sense of data, is most exciting field of all computer sciences! Living in an age where data comes in abundance; using self-learning algorithms from field of ML, can turn this data into knowledge. Thanks to many powerful open-source libraries that have been developed in recent years, there has probably never been a better time to break into ML field & learn how to utilize powerful algorithms to spot patterns in data & make predictions about future events.

In this chap, learn about main concepts & different types of ML. Together with a basic introduction to relevant terminology, lay groundwork for successfully using ML technique for practical problem solving.

In this chap, cover topics:

- General concepts of ML
- 3 types of learning & basic terminology
- Building blocks for successfully designing ML systems
- Installing & setting up Python for data analysis & ML
- Building intelligence machines to transform data into knowledge. In this age of modern technology, there is 1 resource that we have in abundance: a large amount of structured & unstructured data. In 2nd half of 20th century, ML evolved as a subfield of AI involving self-learning algorithms that derive knowledge from data to make predictions.

Instead of requiring humans to manually derive rules & build models from analyzing large amounts of data, ML offers a more efficient alternative for capturing knowledge in data to gradually improve performance of predictive models & make data-driven decisions.

Not only is ML becoming increasingly important in CS research, but it is also playing an ever-greater role in our everyday lives. Thanks to ML, enjoy robust email spam filters, convenient text & voice recognition software, reliable web search engines, recommendations on entertaining movies to watch, mobile check deposits, estimated meal delivery times, & much more. Hopefully, soon, will add safe & efficient self-driving cars to this list. Notable progress has been made in medical applications; e.g., researchers demonstrated: DL models can detect skin cancer with near-human accuracy <https://www.nature.com/articles/nature21056>. Another milestone was recently achieved by researchers at DeepMind, who used DL to predict 3D protein structures, outperforming physics-based approaches by substantial margin <https://deepmind.com/blog/article/alphafold-a-solution-to-a-50-year-old-grand-challenge-in-biology>. While accurate 3D protein structure prediction plays an essential role in biological & pharmaceutical research, there have been many other important applications of ML in healthcare recently. E.g., researchers designed systems for predicting oxygen needs of COVID-19 patients up to 4 days in advance to help hospitals allocate resources for those in need <https://ai.facebook.com/blog/new-ai-research-to-help-predict-covid-19-resource-needs-from-a-series-of-x-rays/>. Another important topic of our day & age is climate change, which presents 1 of biggest & most critical challenges. Today, many efforts are being directed toward developing intelligent systems to combat it <https://www.forbes.com/sites/robtoews/2021/06/20/these-are-the-startups-applying-ai-to-tackle-climate-change>. 1 of many approaches to tackling climate change is emergent field of precision agriculture. Researchers aim to design computer vision-based ML systems to optimize resource deployment to minimize use & waste of fertilizers.

- **3 different types of ML.** Take a look at 3 types of ML: *supervised learning*, *unsupervised learning*, & *reinforcement learning*. Learn about fundamental differences between 3 different learning types &, using conceptual examples, develop an understanding of practical problem domains where they can be applied: Fig. 1.1: 3 different types of ML.

1. Supervised Learning: labeled data, direct feedback, predict outcome/future
2. Unsupervised learning: no labels/targets, no feedback, find hidden structure in data
3. Reinforcement learning: decision process, reward system, learn series of actions

* **Making predictions about future with supervised learning.** Main goal in supervised learning: learn a model from labeled training data that allows us to make predictions about unseen or future data. Term “supervised” refers to a set of training examples (data inputs) where desired output signals (labels) are already known. Supervised learning is then process of modeling relationship between data inputs & labels. Can also think of supervised learning as “label learning”.

Fig. 1.2: Supervised learning process summarizes a typical supervised learning workflow, where labeled training data is passed to a ML algorithm for fitting a predictive model that can make predictions on new, unlabeled data inputs.

Considering example of email spam filtering, can train a model using a supervised ML algorithm on a corpus (ngữ liệu) of labeled emails, which are correctly marked as spam or non-spam, to predict whether a new email belongs to either of 2 categories. A supervised learning task with discrete class labels, e.g. in previous email spam filtering example, is also called a *classification task*. Another subcategory of supervised learning is *regression*, where outcome signal is a continuous value.

• **Classification for predicting class labels.** Classification is a subcategory of supervised learning where goal: predict categorical class labels of new instances or data points based on past observations. Those class labels are discrete, unordered values that can be understood as group memberships of data points. Previously mentioned example of email spam detection represents a typical example of a binary classification task, where ML algorithm learns a set of rules to distinguish between 2 possible classes: spam & non-spam emails.

Fig. 1.3: Classifying a new data point illustrates concept of a binary classification task given 30 training examples; 15 training examples are labeled as class A & 15 training examples are labeled as class B. In this scenario, our dataset

is 2D, i.e., each example has 2 values associated with it: x_1, x_2 . Can use a supervised ML algorithm to learn a rule – decision boundary represented as a dashed line – that can separate those 2 classes & classify new data into each those 2 categories given its x_1, x_2 values.

However, set of class labels does not have to be of a binary nature. Predictive model learned by a supervised learning algorithm can assign any class label that was presented in training dataset to a new, unlabeled data point or instance. A typical example of a *multiclass classification* task is handwritten character recognition. Can collect a training dataset that consists of multiple handwritten examples of each letter in alphabet. Letters (“A”, “B”, “C”, ...) will represent different unordered categories or class labels that we want to predict. Now, if a user provides a new handwritten character via an input device, our predictive model will be able to predict correct letter in alphabet with certain accuracy. However, our ML system will be unable to correctly recognize any of digits between 0 & 9, e.g., if they were not part of training dataset.

- **Regression for predicting continuous outcomes.** Learned in prev sec: task of classification: to assign categorical, unordered labels to instances. A 2nd type of supervised learning: prediction of continuous outcomes, also called *regression analysis*. In regression analysis, given a number of predictor (*explanatory*) variables & a continuous response variable (*outcome*), & try to find a relationship between those variables that allows us to predict an outcome.

Note: in field of ML, predictor variables are commonly called “features”, & response variables are usually referred to as “target variables”. Adopt these conventions throughout this book.

E.g., assume: interested in predicting math SAT scores of students. (SAT is a standardized test frequently used for college admissions in US.) If there is a relationship between time spent studying for test & final scores, could use it as training data to learn a model that uses study time to predict test scores of future students who are planning to take this test.

Remark 25 (Regression toward mean). *Term “regression” was devised by FRANCIS GALTON in his article Regression towards Mediocrity in Hereditary Stature in 1886. GALTON described biological phenomenon that variance of height in a population does not increase over time.*

Observed: height of parents is not passed on to their children, but instead, their children’s height regresses toward population mean.

Fig. 1.4: A linear regression example illustrates concept of linear regression. Given a feature variable x & a target variable y , fit a straight line to this data that minimizes distance – most commonly averaged squared distance – between data points & fitted line. Can now use intercept & slope learned from this data to predict target variable of new data. – Bây giờ có thể sử dụng giá trị chặn & độ dốc học được từ dữ liệu này để dự đoán biến mục tiêu của dữ liệu mới.

- * **Solving interactive problems with reinforcement learning.** Another type of ML is *reinforcement learning*. In reinforcement learning, goal: develop a system (*agent*) that improves its performance based on interactions with environment. Since information about current state of environment typically also includes a so-called *reward signal*, can think of reinforcement learning as a field related to supervised learning. However, in reinforcement learning, this feedback is not correct ground truth label or value, but a measure of how well action was measured by a reward function. Through its interaction with environment, an agent can then use reinforcement learning to learn a series of actions that maximizes this reward via an exploratory trial-&-error approach or deliberative planning.

A popular example of reinforcement learning is a chess program. Here, agent decides upon a series of moves depending on state of board (environment), & reward can be defined as *win* or *lose* at end of game: Fig. 1.5: Reinforcement learning process.

There are many different subtypes of reinforcement learning. However, a general scheme: agent in reinforcement learning tries to maximize reward through a series of interactions with environment. Each state can be associated with a positive or negative reward, & a reward can be defined as accomplishing an overall goal, e.g. winning or losing a game of chess. E.g., in chess, outcome of each move can be thought of as a different state of environment.

To explore chess example further, think of visiting certain configuration on chessboard as being associated with states that will more likely lead to winning – e.g., removing an opponent’s chess piece from board or threatening queen. Other positions, however, are associated with states that will more likely result in losing game, e.g. losing a chess piece to opponent in following turn. Now, in game of chess, reward (either positive for winning or negative for losing game) will not be given until end of game. In addition, final reward will also depend on how opponent plays. E.g., opponent may sacrifice queen but eventually win game.

In sum, reinforcement learning is concerned with learning to choose a series of actions that maximizes total reward, which could be earned either immediately after taking an action or via *delayed* feedback.

- * **Discovering hidden structures with unsupervised learning.** In supervised learning, know right answer (label or target variable) beforehand when train a model, & in reinforcement learning, define a measure of reward for particular actions carried out by agent. In unsupervised learning, however, dealing with unlabeled data or data of an unknown structure. Using unsupervised learning techniques, able to explore structure of our data to extract meaningful information without guidance of a known outcome variable or reward function.

- **Finding subgroups with clustering.** *Clustering* is an exploratory data analysis or pattern discovery technique that allows us to organize a pile of information into meaningful subgroups (*clusters*) without having any prior knowledge of their memberships. Each cluster that arises during analysis defines a group of objects that share a certain degree of similarity but are more dissimilar to objects in other clusters, which is why clustering is also sometimes called *unsupervised classification*. Clustering is a great technique for structuring information & deriving meaningful relationships from data. E.g., it allows marketers to discover customer groups based on their interests, in order to develop distinct marketing

programs.

Fig. 1.6: How clustering works illustrates how clustering can be applied to organizing unlabeled data into 3 distinct groups or clusters (A, B, & C, in arbitrary order) based on similarity of their features, x_1, x_2 .

• **Dimensionality reduction for data compression.** Another subfield of unsupervised learning is *dimensionality reduction*. Often, working with data of high dimensionality – each observation comes with a high number of measurements – that can present a challenge for limited storage space & computational performance of ML algorithms. Unsupervised dimensionality reduction is a commonly used approach in feature preprocessing to remove noise from data, which can degrade predictive performance of certain algorithms. Dimensionality reduction compresses data onto a smaller dimensional subspace while retaining most of relevant information.

Sometimes, dimensionality reduction can also be useful for visualizing data; e.g., a high-dimensional feature set can be projected onto 1D, 2D, or 3D feature spaces to visualize it via 2D or 3D scatterplots or histograms. Fig. 1.7: An example of dimensionality reduction from 3D to 2D. shows an example where nonlinear dimensionality reduction was applied to compress a 3D Swiss roll onto a new 2D feature subspace.

- **Introduction to basic terminology & notations.** Discussed 3 broad categories of ML – supervised, unsupervised, & reinforcement learning – have a look at basic terminology that we will be using throughout this book. Following subsect covers common terms we will be using when referring to different aspects of a dataset, as well as mathematical notation to communicate more precisely & efficiently.

As ML is a vast field & very interdisciplinary, guaranteed to encounter many different terms that refer to same concepts sooner rather than later. 2nd subsect collects many of most commonly used terms that are found in ML literature, which may be useful as a ref sect when reading ML publications.

* **Notation & conventions used in this book.** Fig. 1.8: Iris dataset depicts an excerpt of Iris dataset, which is a classic example in field of ML <https://archive.ics.uci.edu/ml/datasets/iris>. Iris dataset contains measurements of 150 Iris flowers from 3 different species – Setosa, Versicolor, & Virginica.

Here, each flower example represents 1 row in our dataset, & flower measurements in centimeters are stored as columns, which also called *features* of dataset.

To keep notation & implementation simple yet efficient, make use of some of basics of linear algebra. In following chaps, use a matrix notation to refer to our data. Follow common convention to represent each example as a separate row in a feature matrix, \mathbf{X} , where each feature is stored as a separate column.

Iris dataset, consisting of 150 examples & 4 features, can then be written as a 150×4 matrix, formally denoted as $\mathbf{X} \in \mathbb{R}^{150 \times 4}$.

Remark 26 (Notational conventions). *For most parts of this book, unless noted otherwise, use superscript i to refer to i th training example, & subscript j to refer to j th dimension of training dataset.*

Use lowercase, bold-face letters to refer to vectors $\mathbf{x} \in \mathbb{R}^{n \times 1}$ & supcase, bold-face letters to refer to matrices $\mathbf{X} \in \mathbb{R}^{n \times m}$.

To refer to single elements in a vector or matrix, write letters in italics $x^{(n)}$ or $x_m^{(n)}$, resp.

E.g., $x_1^{(150)}$ refers to 1st dimension of flow example 150, sepal length (chiều dài lá dài). Each row in matrix \mathbf{X} represents 1 flower instance & can be written as a 4-dimensional row vector $\mathbf{x}^{(i)} \in \mathbb{R}^{1 \times 4}$: $\mathbf{X}^{(i)} = [x_1^{(i)} \ x_2^{(i)} \ x_3^{(i)} \ x_4^{(i)}]$. & each feature dimension is a 150-dimensional column vector $\mathbf{X}^{(i)} \in \mathbb{R}^{150 \times 1}$, e.g.:

$$\mathbf{x}_j = \begin{bmatrix} x_j^{(1)} \\ x_j^{(2)} \\ \vdots \\ x_j^{(150)} \end{bmatrix} \quad (187)$$

Similarly, can represent target variables (here, class labels) as a 150-dimensional column vector:

$$\mathbf{y} = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(150)} \end{bmatrix} \quad \text{where } y^{(i)} \in \{\text{Setosa, Versicolor, Virginica}\} \quad (188)$$

* **ML terminology.** ML is a vast field & also very interdisciplinary as it brings together many scientists from other areas of research. As it happens, many terms & concepts have been rediscovered or redefined & may already be familiar to you but appear under different names. For your convenience, in following list, can find a selection of commonly used terms & their synonyms that you may find useful when reading this book & ML literature in general:

- *Training example:* A row in a table representing dataset & synonymous with an observation, record, instance, or sample (in most contexts, sample refers to a collection of training examples).
- *Training:* Model fitting, for parametric models similar to parameter estimation.
- *Feature, abbr. x :* A column in a data table or data (design) matrix. Synonymous with predictor, variable, input, attribute, or covariate.
- *Target, abbr. y :* Synonymous with outcome, output, response variable, dependent variable, (class) label, & ground truth.
- *Loss function:* Often used synonymously with a *cost* function. Sometimes loss function is also called an *error* function. In some literature, term “loss” refers to loss measured for a single data point, & cost is a measurement that computes loss (averaged or summed) over entire dataset.

- A roadmap for building ML systems. In prev sects, discussed basic concepts of ML & 3 different types of learning. In this sect, discuss other important parts of a ML system accompanying learning algorithm.

Fig. 1.9: Predictive modeling workflow shows a typical workflow for using ML in predictive modeling, which will discuss in following subjects.

- * **Preprocessing – getting data into shape.** Begin by discussing roadmap for building ML systems. Raw data rarely comes in form & shape that is necessary for optimal performance of a learning algorithm. Thus, preprocessing of data is 1 of most crucial steps in any ML application.

If take Iris flower dataset from previous sect as an example, can think of raw data as a series of flower images from which want to extract meaningful features. Useful features could be centered around color of flowers or height, length, & width of flowers.

Many ML algorithms also require: selected features are on same scale for optimal performance, which is often achieved by transforming features in range $[0, 1]$ or a standard normal distribution with zero mean & unit variance.

Some of selected features may be highly correlated & therefore redundant to a certain degree. In those cases, dimensionality reduction techniques are useful for compressing features onto a lower-dimensional subspace. Reducing dimensionality of our feature space has advantage that less storage space is required, & learning algorithm can run much faster. In certain cases, dimensionality reduction can also improve predictive performance of a model if dataset contains a large number of irrelevant features (or noise); i.e., if dataset has a low signal-to-noise ratio.

To determine whether our ML algorithm not only performs well on training dataset but also generalizes well to new data, also want to randomly divide dataset into separate training & test datasets. Use training dataset to train & optimize our ML model, while keep test dataset until very end to evaluate final model.

- * **Training & selecting a predictive model.** Many different ML algorithms have been developed to solve different problem tasks. An important point that can be summarized from DAVID WOLPERT's famous *No free lunch theorems*: we can't get learning "for free" (*Lack of A Priori Distinctions Between Learning Algorithms*, D.H. Wolpert, 1996; *No free lunch theorems for optimization*, D.H. Wolpert & W.G. Macready, 1997). Can relate this concept to popular saying, *I suppose it is tempting, if only tool you have is a hammer, to treat everything as if it were a nail* (Tôi cho rằng thật hấp dẫn nếu công cụ duy nhất bạn có là một cái búa, để coi mọi thứ như thể chúng là một cái đinh.) (Abraham Maslow, 1966). E.g., each classification algorithm has its inherent biases, & no single classification model enjoys superiority if we don't make any assumptions about task. In practice, therefore essential to compare at least a handful of different learning algorithms in order to train & select best performing model. But before can compare different models, 1st have to decide upon a metric to measure performance. 1 commonly used metric is classification accuracy, which is defined as proportion of correctly classified instances.

1 legitimate question to ask is this: how do we know which model performs well on final test dataset & real-world data if we don't use this test dataset for model selection, but keep it for final model evaluation? To address issue embedded in this question, different techniques summarized as "cross-validation" can be used. In cross-validation, further divide a dataset into training & validation subsets in order to estimate generalization performance of model.

Finally, also cannot expect: default parameters of different learning algorithms provided by software libraries are optimal for our specific problem task. Therefore, will make frequent use of hyperparameter optimization techniques that help us to fine-tune performance of our model in later chaps.

Can think of those hyperparameters as parameters that are not learned from data but represent knobs (núm vặn) of a model that we can turn to improve its performance. This will become much clearer in later chaps when see actual examples.

- * **Evaluating models & predicting unseen data instances.** After have selected a model that has been fitted on training dataset, can use test dataset to estimate how well it performs on this unseen data to estimate so-called *generalization error*. If satisfied with its performance, can now use this model to predict new, future data. Important to note: parameters for previously mentioned procedures, e.g. feature scaling & dimensionality reduction, are solely obtained from training dataset, & same parameters are later reapplied to transform test dataset, as well as any new data instances – performance measured on test data may be overly optimistic otherwise.

- **Using Python for ML.** Python is 1 of most popular programming languages for DS, & thanks to its very active developer & open-source community, a large number of useful libraries for scientific computing & ML have been developed.

Although performance of interpreted languages, e.g. Python, for computation-intensive tasks is inferior to lower-level programming languages, extension libraries e.g. NumPy & SciPy have been developed that build upon lower-layer Fortran & C implementations for fast vectorized operations on multidimensional arrays.

For ML programming tasks, mostly refer to `scikit-learn` library, which is currently 1 of most popular & accessible open-source ML libraries. In later chaps, when focus on a subfield of ML called *DL*, use latest version of PyTorch library, which specializes in training so-called *deep neural network* models very efficiently by utilizing graphics cards.

- * **Installing Python & package from Python Package Index.** Python is available for all 3 major OS – Microsoft Windows, macOS, & Linux – & installer, as well as documentation, can be downloaded from official Python website: <https://www.python.org>.

Code examples provided in this book have been written for & tested in Python 3.9, generally recommend: use most recent version of Python 3 that is available. Some of code may also be compatible with Python .7, but as official support for Python 2.7 ended in 2019, & majority of open-source libraries have already stopped supporting Python 2.7 <https://www.python.org>.

[//python3statement.org](http://python3statement.org), strongly advise use Python 3.9 or newer. Can check Python version by executing `python -version` or `python3 -version` in terminal (or PowerShell if using Windows).

Additional packages that we will be using throughout this book can be installed via `pip` installer program, which has been part of Python Standard Library since Python 3.3. More information about `pip`: <https://docs.python.org/3/installing/index.html>.

After have successfully installed Python, can execute `pip` from terminal to install additional Python packages: `pip install SomePackage`. Already installed packages can be updated via `-upgrade` flag:

```
pip install SomePackage --upgrade
```

- * Using Anaconda Python distribution & package manager. A highly recommended open-source package management system for installing Python for scientific computing contexts is `conda` by Continuum Analytics. Conda is free & licensed under a permissive open-source license. Its goal: help with installation & version management of Python packages for DS, math, & engineering across different OSs. If want to use conda, it comes in different flavors, namely Anaconda, Miniconda, & Miniforge:

- Anaconda comes with many scientific computing packages pre-installed. Anaconda installer can be downloaded at <https://docs.anaconda.com/anaconda/install/>, & an Anaconda quick start guide is available at <https://docs.anaconda.com/anaconda/user-guide/getting-started/>.
- Miniconda is a leaner alternative (thay thế gầy hơn) to Anaconda <https://docs.conda.io/en/latest/miniconda.html>. Essentially, similar to Anaconda but without any packages pre-installed, which many people (including authors) prefer.
- Miniforge is similar to Miniconda but community-maintained & uses a different package repository `conda-forge` from Miniconda & Anaconda. Found: Miniforge is a great alternative to Miniconda. Download & installation instructions can be found in GitHub repository at <https://github.com/conda-forge/miniforge>.

After successfully installing `conda` through either Anaconda, Miniconda, or Miniforge, can install new Python packages using command:

```
conda install SomePackage
```

Existing packages can be updated using command:

```
conda update SomePackage
```

Packages that are not available through official conda channel might be available via community-supported `conda-forge` project <https://conda-forge.org>, which can be specified via `-channel conda-forge` flag. E.g.:

```
conda install SomePackage --channel conda-forge
```

Packages that are not available through default conda channel or conda-force can be installed via `pip`. E.g.:

```
pip install SomePackage
```

- * Packages for scientific computing, DS, & ML. Throughout 1st half of this book, mainly use NumPy's multidimensional arrays to store & manipulate data. Occasionally, make use of `pandas`, which is a library built on top of NumPy that provides additional higher-level data manipulation tools that make working with tabular data even more convenient. To augment your learning experience & visualize quantitative data, which is often extremely useful to make sense of it, will use very customizable Matplotlib library.

Main ML library used in this book is `scikit-learn` (Chaps. 3–11). *Chap. 12: Parallelizing Neural Network Training with PyTorch*, will then introduce PyTorch library for DL.

Version numbers of major Python packages that were used to write this book are mentioned in following list. Make sure: version numbers of your installed packages are, ideally, equal to these version numbers to ensure code examples run correctly:

- NumPy 1.21.2
- SciPy 1.7.0
- Scikit-learn 1.0
- Matplotlib 3.4.3
- pandas 1.3.2

After installing these packages, can double-check installed version by importing package in Python & accessing its `__version__` attribute, e.g.:

```
>>> import numpy
>>> numpy.__version__
'1.26.4'
```

For your convenience, included a `python-environment-check.py` script in this book's complimentary code repository at <https://github.com/rasbt/machine-learning-book> so that you can check both Python version & package versions by executing this script.


```
(base) nqbh@nqbh-dell:~/advanced_STEM_beyond/machine_learning/Python$ python python_environment_check.py
/home/nqbh/advanced_STEM_beyond/machine_learning/Python/python_environment_check.py:4: DeprecationWarning:
if LooseVersion(sys.version) < LooseVersion('3.8'):
[OK] Your Python version is 3.12.7 | packaged by Anaconda, Inc. | (main, Oct 4 2024, 13:27:36) [GCC 11.2.0]
/home/nqbh/advanced_STEM_beyond/machine_learning/Python/python_environment_check.py:39: DeprecationWarning:
actual_ver, suggested_ver = LooseVersion(actual_ver), LooseVersion(suggested_ver)
[OK] numpy 1.26.4
[OK] scipy 1.13.1
/home/nqbh/advanced_STEM_beyond/machine_learning/Python/python_environment_check.py:40: DeprecationWarning:
if pkg_name == "matplotlib" & actual_ver == LooseVersion("3.8"):
[OK] matplotlib 3.9.2
[OK] sklearn 1.5.1
[OK] pandas 2.2.2
```

Certain chaps will require additional packages & will provide information about installations. E.g., do not worry about installing PyTorch at this point. Chap. 12 will provide tips & instructions when need them.

If encounter errors even though your code matches code in chap exactly, recommend 1st check version numbers of underlying packages before spending more time on debugging or reaching out to publisher or authors. Sometimes, newer versions of libraries introduce backward-incompatible changes that could explain these errors.

if do not want to change package version in your main Python installation, recommend using a virtual environment for installing packages used in this book. If use Python without conda manager, can use `venv` library to create a new virtual environment. E.g., can create & activate virtual environment via 2 commands:

```
python3 -m venv /Users/sebastian/Desktop/pyml-book
source /Users/sebastian/Desktop/pyml-book/bin/activate
```

Note: need to activate virtual environment every time you open a new terminal or PowerShell. More information about `venv` at <https://docs.python.org/3/library/venv.html>.

If using Anaconda with `conda` package manager, can create & activate a virtual environment as follows:

```
conda create -n pyml python=3.9
conda activate pyml
```

- **Summary.** In this chap, explored ML at a very high level & familiarized with big picture & major concepts that going to explore in following chaps in more detail. Learned: supervised learning is composed of 2 important subfields: classification & regression. While classification models allow to categorize objects into known classes, can use regression analysis to predict continuous outcomes of target variables. Unsupervised learning not only offers useful techniques for discovering structures in unlabeled data, but it can also be useful for data compression in feature preprocessing steps.

Briefly went over typical roadmap for applying ML to problem tasks, which will use as a foundation for deeper discussions & hands-on examples in following chaps. Finally, set up our Python environment & installed & updated required packages to get ready to see ML in action.

Later in this book, in addition to ML itself, introduce different techniques to preprocess a dataset, which will help you to get best performance out of different ML algorithms. While cover classification algorithms quite extensively throughout book, also explore different techniques for regression analysis & clustering.

Have an exciting journey ahead, covering many powerful techniques in vast field of ML. However, approach ML 1 step at a time, building upon our knowledge gradually throughout chaps of this book. In following chap, start this journey by implementing 1 of earliest ML algorithms for classification, which will prepare for *Chap. 3: A Tour of ML Classifiers Using Scikit-Learn*, where cover more advanced ML algorithms using `scikit-learn` open-source ML library.

- **Chap. 2: Training Simple ML Algorithms for Classification.** In this chap, make use of 2 of 1st algorithmically described ML algorithms for classification: perceptron & adaptive linear neurons. Start by implementing a perceptron step by step in Python & training it to classify different flower species in Iris dataset. This will help us to understand concept of ML algorithms for classification & how they can be efficiently implemented in Python.

Discussing basics of optimization using adaptive linear neurons will then lay groundwork for using more sophisticated classifiers via `scikit-learn` ML library in *Chap. 3: A Tour of ML Classifiers Using Scikit-Learn*.

Topics covered in this chap:

- Building an understanding of ML algorithms
- Using pandas, NumPy, & Matplotlib to read in, process, & visualize data
- Implementing linear classifiers for 2-class problems in Python
- Artificial neurons – a brief glimpse into early history of ML. Before discuss perceptron & related algorithms in more detail, take a brief tour of beginnings of ML. Trying to understand how biological brain works in order to design an AI, WARREN

MCCULLOCH & WALTER PITTS published 1st concept of a simplified brain cell, so-called *McCulloch-Pitts (MCP)* neuron, in 1943 (*A Logical Calculus of Ideas Immanent in Nervous Activity* by W.S. MCCULLOCH & W. PITTS, Bulletin of Mathematical Biophysics, 5(4): 115-133, 1943).

Biological neurons are interconnected nerve cells in brain that are involved in processing & transmitting of chemical & electrical signals, illustrated in Fig. 2.1: A neuron processing chemical & electrical signals.

MCCULLOCH & PITTS described such a nerve cell as a simple logic gate with binary outputs; multiple signal arrive at dendrites (các nhánh cây, a short branch at the end of a nerve cell that receives signals from other cells – 1 nhánh gắn ở cuối tế bào thần kinh nhận tín hiệu từ các tế bào khác), they are then integrated into cell body, &, if accumulated signal exceeds a certain threshold, an output signal is generated that will be passed on by axon (sợi trục).

Only a few years later, FRANK ROSENBLATT published 1st concept of perceptron learning rule based on MCP neuron model (*The Perceptron: A Perceiving & Recognizing Automaton* by F. ROSENBLATT, Cornell Aeronautical Laboratory, 1957). With his perceptron rule, ROSENBLATT proposed an algorithm that would automatically learn optimal weight coefficients that would then be multiplied with input features in order to make decision of whether a neuron fires (transmits a signal) or not. In context of supervised learning & classification, e.g. algorithm could then be used to predict whether a new data point belongs to 1 class or the other.

* **Formal def of an artificial neuron.** More formally, can put idea behind *artificial neurons* into context of a binary classification task with 2 classes: 0 & 1. Can then define a decision function $\sigma(z)$ that takes a linear combination of certain input values x & a corresponding weight vector w where z : so-called *net input* $z = \mathbf{w} \cdot \mathbf{x} = w_1x_1 + \dots + w_mx_m$. Now, if net input of a particular example $x^{(i)}$ is greater than a defined threshold θ , predict class 1, & class 0 otherwise. In perceptron algorithm, decision function $\sigma(\cdot)$ is a variant of a *unit step function*:

$$\sigma(z) = \begin{cases} 1 & \text{if } z \geq \theta, \\ 0 & \text{otherwise.} \end{cases} \quad (189)$$

To simplify code implementation later, can modify this setup via a couple of steps. 1st, move threshold θ to left side of equation: $z \geq \theta, z - \theta \geq 0$. 2nd, define a so-called *bias unit* as $b = -\theta$ & make it part of net input: $z = w_1x_1 + \dots + w_mx_m + b = \mathbf{w}^\top \mathbf{x} + b$. 3rd, given introduction of bias unit & redefinition of net input z above, redefine decision function as follows:

$$\sigma(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (190)$$

Fig. 2.2: A threshold function producing a linear decision boundary for a binary classification problem illustrates how net input $z = \mathbf{w}^\top \mathbf{x} + b$ is squashed into a binary output (0 or 1) by decision function of perceptron & how it can be used to discriminate between 2 classes separable by a linear decision boundary.

* **Perceptron learning rule.** Whole idea behind MCP neuron & ROSENBLATT's *thresholded* perceptron model: use a reductionist approach to mimic how a single neuron in brain works: it either *fires* or it doesn't. Thus, ROSENBLATT's classic perceptron rule is fairly simple, & perceptron algorithm can be summarized by following steps:

1. Initialize weights & bias unit to 0 or small random numbers
2. For each training example, $x^{(i)}$:
 - (a) Compute output value $\hat{y}^{(i)}$
 - (b) Update weights & bias unit

Here, output value is class label predicted by unit step function defined earlier, & simultaneous update of bias unit & each weight w_j in weight vector \mathbf{w} can be more formally written as

$$w_i = w_i + \Delta w_i, \quad (191)$$

$$b = b + \Delta b. \quad (192)$$

Update values ("deltas") are computed as follows:

$$\Delta w_j = \eta(y^{(i)} - \hat{y}^{(i)})x_j^{(i)}, \quad (193)$$

$$\Delta b = \eta(y^{(i)} - \hat{y}^{(i)}). \quad (194)$$

Note: unlike bias unit, each weight w_j corresponds to a feature x_j in dataset, which is involved in determining update value Δw_j defined above. Furthermore, η is *learning rate* (typically a constant between 0.0 & 1.0), $y^{(i)}$: *true class label* of i th training example, & $\hat{y}^{(i)}$: *predicted class label*. Important to note: bias unit & all weights in weight vector are being updated simultaneously, i.e., don't recompute predicted labels $\hat{y}^{(i)}$ before bias unit & all of weights are updated via respective update values Δw_j & Δb . Concretely, for a 2D dataset, write update as:

$$\Delta w_1 = \eta(y^{(i)} - \text{output}^{(i)})x_1^{(i)}, \quad (195)$$

$$\Delta w_2 = \eta(y^{(i)} - \text{output}^{(i)})x_2^{(i)}, \quad (196)$$

$$\Delta b = \eta(y^{(i)} - \text{output}^{(i)}). \quad (197)$$

Before implement perceptron rule in Python, go through a simple thought experiment to illustrate how beautifully simple this learning rule really is. In 2 scenarios where perceptron predicts class label correctly, bias unit & weights remain unchanged, since update values are 0:

$$\dots y^{(i)} = \hat{y}^{(i)} \Leftrightarrow \dots \Leftrightarrow \Delta w_j = \Delta b = 0. \quad (198)$$

However, in case of a wrong prediction, weights are being pushed toward direction of positive or negative target class $\Delta b = \pm \eta$.

To get a better understanding of feature value as a multiplicative factor $x_j^{(i)}$, go through another simple example where $y^{(i)} = 1, \hat{y}^{(i)} = 0, \eta = 1$. Assume $x_j^{(i)} = 1.5$ misclassified as class 0: would increase corresponding weight by 2.5 in total so that net input $z = x_j^{(i)} w_j + b$ would be more positive next time we encountered this example, & thus be more likely to be above threshold of unit step function to classify example as class 1.

Weight update Δw_j is proportional to value of $x_j^{(i)}$. E.g., $x_j^{(i)} = 2$ incorrectly classified as class 0, will push decision boundary by an even larger extent to classify this example correctly next time.

Important to note: convergence of perceptron is only guaranteed if 2 classes are linearly separable, i.e., 2 classes can be perfectly separated by a linear decision boundary. Convergence proof in his lecture notes: https://sebastianraschka.com/pdf/lecture-notes/stat453ss21/L03_perceptron_slides.pdf. Fig. 2.3: Examples of linearly & nonlinearly separable classes. (a) Linearly separable: A linear decision boundary that separates 2 classes exist. (b) Not linearly separable: No linear decision boundary that separates 2 classes perfectly exists. shows visual examples of linearly separable & linear inseparable scenarios.

If 2 classes can't be separated by a linear decision boundary, can set a maximum number of passes over training dataset (**epochs** – thời đại) &/or a threshold for number of tolerate misclassifications – perceptron would never stop updating weights otherwise. Later in this chap, cover Adaline algorithm that produces linear decision boundaries & converges even if classes are not perfectly linearly separable. In Chap. 3, learn about algorithms that can produce nonlinear decision boundaries.

Remark 27 (Download example code). *Can download all code examples & datasets directly from <https://github.com/rasbt/machine-learning-book>.*

Before jump into implementation in next sect, what just learned can be summarized in a simple diagram that illustrates general concept of perceptron: Fig. 2.4: **Weights & bias of model are updated based on error function**. Preceding diagram illustrates how perceptron receives inputs of an example x & combines them with bias unit b & weights \mathbf{w} to compute net input. Net input is then passed on to threshold function, which generates a binary output of 0 or 1 – predicted class label of example. During learning phase, this output is used to calculate error of prediction & update weights & bias unit.

- **Implementing a perceptron learning algorithm in Python.** In prev sec, learned how Rosenblatt's perceptron rule works, implement it in Python & apply it to Iris dataset.
- * **An object-oriented perceptron API.** Take an object-oriented approach to defining perceptron interface as a Python class, which will allow us to initialize new **Perceptron** objects that can learn from data via a **fit** method & make predictions via a separate **predict** method. As a convention, append an underscore **_** to attributes that are not created upon initialization of object, but do this by calling object's other methods, e.g., **self.w_**.

Remark 28 (Additional resources for Python's scientific computing stack). *If not yet familiar with Python's scientific libraries or need a refresher, see resources:*

- NumPy: <https://sebastianraschka.com/blog/2020/numpy-intro.html>
- pandas: https://pandas.pydata.org/pandas-docs/stable/user_guide/10min.html
- Matplotlib: <https://matplotlib.org/stable/tutorials/introductory/usage.html>

Implementation of a perceptron in Python: see **perceptron.py**.

Using this perceptron implementation, can now initialize new **Perceptron** objects with a given learning rate **eta** η & number of epochs **n_iter** (passes over training dataset).

Via **fit** method, initialize bias **self.b_** to an initial value 0 & weights in **self.w_** to a vector \mathbb{R}^m where m : number of dimensions (features) in dataset.

Notice: initial weight vector contains small random numbers drawn from a normal distribution with a standard deviation of 0.01 via **rgen.normal(loc=0.0, scale=0.01, size=1 + X.shape[1])** where **rgen** is a NumPy random number generator that we seeded with a user-specified random seed so that we can reproduce previous results if desired.

Technically, could initialize weights to 0 (in fact, this is done in original perceptron algorithm). However, if did that, then learning rate η **eta** would have no effect on decision boundary. If all weights are initialized to 0, learning rate parameter **eta** affects only scale of weight vector, not direction. If familiar with trigonometry, consider a vector $\mathbf{v}_1 = [1, 2, 3]$, where angle between \mathbf{v}_1 & a vector $\mathbf{v}_2 = 0.5\mathbf{v}_1$, would be exactly 0, as demonstrated by code snippet:

```
>>> v1 = np.array([1, 2, 3])
>>> v2 = 0.5 * v1
>>> np.arccos(v1.dot(v2) / (np.linalg.norm(v1) *
...           np.linalg.norm(v2)))
0.0
```

i.e., $(\mathbf{v}_1, \mathbf{v}_2) = \arccos \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|}$. Here, `np.arccos`: trigonometric inverse cosin, `np.linalg.norm` is a function that computes length of a vector. (Our decision to draw random numbers from a random normal distribution – e.g., instead of from a uniform distribution – & to use a standard deviation of 0.01 was arbitrary: remember, just interested in small random values to avoid properties of all-zero vectors.)

As an optional exercise after reading this chap, can change `self.w_ = rgen.normal(loc=0.0, scale=0.01, size=X.shape[1])` to `self.w_ = np.zeros(X.shape[1])` & run perceptron training code presented in next sect with different values for `eta`. Observe: decision boundary does not change.

Remark 29 (NumPy array indexing). *NumPy indexing for 1D arrays works similarly to Python lists using square-bracket [] notation. For 2D arrays, 1st indexer refers to row number & 2nd indexer to column number. E.g., would see `X[2, 3]` to select 3rd row & 4th column of a 2D array X.*

After weights have been initialized, `fit` method loops over all individual examples in training dataset & updates weights according to perceptron learning rule discussed in prev sect.

Class labels are predicted by `predict` method, called in `fit` method during training to get class label for weight update; but `predict` can also be used to predict class labels of new data after we have fitted our model. Furthermore, also collect number of misclassifications during each epoch in `self.errors_` list so that can later analyze how well our perceptron performed during training. `np.dot` function that is used in `net_input` method simply calculates vector dot product $\mathbf{w}^T \mathbf{x} + b$.

Remark 30 (Vectorization: Replacing for loops with vectorized code). *Instead of using NumPy to calculate vector dot product between 2 arrays `a`, `b`, via `a.dot(b)` or `np.dot(a, b)`, could also perform calculation in pure Python via `sum([i * j for i, j in zip(a, b)])`. However, advantage of using NumPy over classic Python for loop structures: its arithmetic operations are vectorized. Vectorizations means: an elemental arithmetic operation is automatically applied to all elements in an array. By formulating our arithmetic operations as a sequence of instructions on an array, rather than performing a set of operations for each element at a time, can make better use of modern central processing unit (CPU) architectures with single instruction, multiple data (SIMD) support. Furthermore, NumPy uses highly optimized linear algebra libraries, e.g. Basic Linear Algebra Subprograms (BLAS) & Linear Algebra Package (LAPACK), that have been written in C or Fortran. Lastly, NumPy also allows us to write our code in a more compact & intuitive way using basics of linear algebra, e.g. vector & matrix dot products.*

- * **Training a perceptron model on Iris dataset.** To test our perceptron implementation, restrict following analyses & examples in remainder of this chap to 2 feature variables (dimensions). Although perceptron rule is not restricted to 2D, considering only 2 features, sepal length & petal length, will allow to visualize decision regions of trained model in scatterplot for learning purposes.

Note: also only consider 2 flower classes, setosa & versicolor, from Iris dataset for practical reasons – remember, perceptron is a binary classifier. However, perceptron algorithm can be extended to multi-class classification – e.g., 1-vs-all (OvA) technique.

Remark 31 (OvA method for multi-class classification). *OvA, sometimes also called 1-vs-rest (OvR), is a technique that allows us to extend & binary classifier to multi-class problems. Using OvA, can train 1 classifier per class, where particular class is treated as positive class & examples from all other classes are considered negative classes. If were to classify a new, unlabeled data instance, would use our n classifiers, where n : number of class labels, & assign class label with highest confidence to particular instance we want to classify. In case of perceptron, would use OvA to choose class label that is associated with largest absolute net input value.*

1st, use `pandas` library to load Iris dataset directly from UCI ML Repository into a `DataFrame` object & print last 5 lines via `tail` method to check: data was loaded correctly:

```
import os
import pandas as pd

try:
    s = 'https://archive.ics.uci.edu/ml/machine-learning-databases/iris/iris.data'
    print('From URL:', s)
    df = pd.read_csv(s, header=None, encoding='utf-8')
except HTTPError:
    s = 'iris.data'
    print('From local Iris path:', s)
    df = pd.read_csv(s, header=None, encoding='utf-8')

df.tail()
```

After executing prev code, should see following output, which shows last 5 lines of Iris dataset:

```
(base) nqbh@nqbh-dell:~/advanced_STEM_beyond/machine_learning/Python$ python perceptron_Iris_dataset.py
From URL: https://archive.ics.uci.edu/ml/machine-learning-databases/iris/iris.data
0    1    2    3    4
145  6.7  3.0  5.2  2.3  Iris-virginica
146  6.3  2.5  5.0  1.9  Iris-virginica
```

```

147  6.5  3.0  5.2  2.0  Iris-virginica
148  6.2  3.4  5.4  2.3  Iris-virginica
149  5.9  3.0  5.1  1.8  Iris-virginica

```

Remark 32 (Loading Iris dataset). Can find a copy of Iris dataset (& all other dataset used in this book) in code bundle of this book, which can use if working offline or if UCI server at <https://archive.ics.uci.edu/ml/machine-learning-databases/iris/iris.data> is temporarily unavailable. E.g., to load Iris dataset from a local directory, can replace this line,

```

df = pd.read_csv(
    'https://archive.ics.uci.edu/ml/'
    'machine-learning-databases/iris/iris.data',
    header=None, encoding='utf-8')

```

with

```

df = pd.read_csv(
    'your/local/path/to/iris.data',
    header=None, encoding='utf-8')

```

Next, extract 1st 100 class labels that correspond to 50 Iris-setosa & 50 Iris-versicolor flowers & convert class labels into 2 integer class labels, 1 (versicolor) & 0 (setosa), that we assign to a vector y where `values` method of a pandas DataFrame yields corresponding NumPy representation.

Similarly, extract 1st feature column (sepal length) & 3rd feature column (petal length) of those 100 training examples & assign them to a feature matrix X which we can visualize via a 2D scatterplot:

```

import numpy as np
import matplotlib.pyplot as plt
y = df.iloc[0:100, 4].values
y = np.where(y == 'Iris-setosa', 0, 1)

# extract sepal length & petal length
X = df.iloc[0:100, [0, 2]].values

# plot data
plt.scatter(X[:50, 0], X[:50, 1],
            color='red', marker='o', label='Setosa')
plt.scatter(X[50:100, 0], X[50:100, 1],
            color='blue', marker='s', label='Versicolor')

plt.xlabel('Sepal length [cm]')
plt.ylabel('Petal length [cm]')
plt.legend(loc='upper left')

# plt.savefig('images/02_06.png', dpi=300)
plt.show()

```

After executing preceding code example, should see following scatterplot Fig. 2.6: Scatterplot of setosa & versicolor flowers by sepal & petal length: show distribution of flower examples in Iris dataset along 2 feature axes: petal length & sepal length (measured in centimeters). In this 2D feature subspace, can see: a linear decision boundary should be sufficient to separate setosa from versicolor flowers. Thus, a linear classifier e.g. perceptron should be able to classify flowers in this dataset perfectly.

Time to train our perceptron algorithm on Iris data subset just extracted. Plot misclassification error for each epoch to check whether algorithm converged & found a decision boundary that separates 2 Iris flower classes:

```

ppn = Perceptron(eta=0.1, n_iter=10)

ppn.fit(X, y)

plt.plot(range(1, len(ppn.errors_) + 1), ppn.errors_, marker='o')
plt.xlabel('Epochs')
plt.ylabel('Number of updates')

# plt.savefig('images/02_07.png', dpi=300)
plt.show()

```

Note: number of misclassification errors & number of updates is the same, since perceptron weights & bias are updated each time it misclassifies an example. An executing preceding code, should see plot of misclassification errors vs. number of

epochs, as shown in Fig. 2.7: A plot of misclassification errors against number of epochs. Our perceptron converged after 6th epoch & should now be able to classify training examples perfectly. Implement a small convenience function to visualize decision boundaries for 2 datasets. [code]

1st, define a number of `colors`, `makers` & create a colormap from list of colors via `ListedColormap`. Then, determine minimum & maximum values for 2 features & use those feature vectors to create a pair of grid arrays `xx1`, `xx2` via NumPy `meshgrid` function. Since trained our perceptron classifier on 2 feature dimensions, need to flatten grid arrays & create a matrix that has same number of columns as Iris training subset so that can use `predict` method to predict class labels `lab` of corresponding grid points.

After reshaping predicted class labels `lab` into a grid with same dimensions as `xx1`, `xx2`, can now draw a contour plot via Matplotlib's `contourf` function, which maps different decision regions to different colors for each predicted class in grid array:

```
>>> plot_decision_regions(X, y, classifier=ppn)
>>> plt.xlabel('Sepal length [cm]')
>>> plt.ylabel('Petal length [cm]')
>>> plt.legend(loc='upper left')
>>> plt.show()
```

After executing preceding code example, should now see a plot of decision regions, as shown in Fig. 2.8: A plot of perceptron's decision regions. As can see in plot, perceptron learned a decision boundary that can classify all flower examples in Iris training subset perfectly.

Remark 33 (Perceptron convergence). *Although perceptron classified 2 Iris flower classes perfectly, convergence is 1 of biggest problems of perceptron. ROSENBLATT proved mathematically: perceptron learning rule converges if 2 classes can be separated by a linear hyperplane. However, if classes cannot be separated perfectly by such a linear decision boundary, weights will never stop updating unless we set a maximum number of epochs. Interested readers can find a summary of proof in lecture notes at https://sebastianraschka.com/pdf/lecture-notes/stat453ss21/L03_perceptron_slides.pdf.*

- o Adaptive linear neurons & convergence of learning. In this sect, take a look at another type of single-layer *neural network (NN)* : *ADaptive LInear NEuron (Adaline)*. Adaline was published by BERNARD WIDROW & his doctoral student TEDD HOFF only a few years after Rosenblatt's perceptron algorithm, & it can be considered an improvement on latter: *An Adaptive "Adaline" Neuron Using Chemical "Memistors"*, Technical Report Number 1553-2 by B. Widrow & colleagues, Stanford Electron Labs, Stanford, CA, October 1960).

Adaline algorithm is particularly interesting because it illustrates key concepts of defining & minimizing continuous loss functions. This lays groundwork for understanding ML algorithms for classification, e.g. logistic regression, support vector machines, & multilayer neural networks, as well as linear regression models.

Key difference between Adaline rule (also known as *Widrow-Hoff rule*) & Rosenblatt's perceptron: weights are updated based on a linear activation function rather than a unit step function like in perception. In Adaline, this linear activation function $\sigma(z)$ is simply identity function of net input, so that $\sigma(z) = z$.

While linear activation function is used for learning weights, still us a threshold function to make final prediction, which is similar to unit step function covered earlier.

Main differences between perceptron & Adaline algorithm are highlighted in Fig. 2.9: A comparison between a perceptron & Adaline algorithm., indicates: Adaline algorithm compares true class labels with linear activation function's continuous valued output to compute model error & update weights. In contrast, perceptron compares true class labels to predicted class labels.

- * Minimizing loss functions with gradient descent. 1 of key ingredients of supervised ML algorithms is a defined *objective function* that is to be optimized during learning process. This objective function is often a loss or cost function that we want to minimize. In case of Adaline, can define loss function L to learn model parameters as *mean squared error (MSE)* between calculated outcome & true class label:

$$L(\mathbf{w}, b) = \frac{1}{n} \sum_{i=1}^n (y^{(i)} - \sigma(z^{(i)}))^2. \quad (199)$$

Main advantage of this continuous linear activation function, in contrast to unit step function: loss function becomes differentiable. Another nice property of this loss function: it is convex; thus, can use a very simple yet powerful optimization algorithm called *gradient descent* to find weights that minimize our loss function to classify examples in Iris dataset.

As illustrated in Fig. 2.10: How gradient descent works, can describe main idea behind gradient descent as *climbing down a hill* until a local or global loss minimum is reached. In each iteration, take a step in opposite direction of gradient, where step size is determined by value of learning rate, as well as slope of gradient (for simplicity, following figure visualizes this only for a single weight w).

Using gradient descent, can now update model parameters by taking a step in opposite direction of gradient $\nabla L(\mathbf{w}, b)$ of our loss function $L(\mathbf{w}, b)$:

$$\mathbf{w} := \mathbf{w} + \Delta \mathbf{w}, \quad b := b + \Delta b. \quad (200)$$

Parameter changes, $\Delta \mathbf{w}$, Δb , are defined as negative gradient multiplied by learning rate η :

$$\Delta \mathbf{w} = -\eta \nabla_{\mathbf{w}} L(\mathbf{w}, b), \quad \Delta b = -\eta \nabla_b L(\mathbf{w}, b). \quad (201)$$

To compute gradient of loss function, need to compute partial derivative of loss function w.r.t. each weight w_j :

$$\frac{\partial L}{\partial w_j} = -\frac{2}{n} \sum_i (y^{(i)} - \sigma(z^{(i)})) x_j^{(i)}. \quad (202)$$

Similarly, compute partial derivative of loss w.r.t. bias as:

$$\frac{\partial L}{\partial b} = -\frac{2}{n} \sum_i (y^{(i)} - \sigma(z^{(i)})). \quad (203)$$

Note: 2 in numerator is merely a constant scaling factor, & could omit it without affecting algorithm. Removing scaling factor has same effect as changing learning rate by a factor of 2. Following information box explains where this scaling factor originates.

So can write weight update as:

$$\Delta w_j = -\eta \frac{\partial L}{\partial w_j}, \quad \Delta b = -\eta \frac{\partial L}{\partial b}. \quad (204)$$

Since update all parameters simultaneously, Adaline learning rule becomes:

$$\mathbf{w} := \mathbf{w} + \Delta \mathbf{w}, \quad b := b + \Delta b. \quad (205)$$

Remark 34 (Mean squared error derivative). *If familiar with calculus, partial derivative of MSE loss function w.r.t. j th weight can be obtained as follows: [...] Same approach can be used to find partial derivative $\frac{\partial L}{\partial b}$ except that $\frac{\partial}{\partial b}(y^{(i)} - \sum_i (w_j^{(i)} x_j^{(i)} + b)) = -1$ & thus last step simplifies to $-\frac{2}{n} \sum_i (y^{(i)} - \sigma(z^{(i)}))$.*

Although Adaline learning rule looks identical to perceptron rule, note: $\sigma(z^{(i)})$ with $z^{(i)} = \mathbf{w}^\top \mathbf{x}^{(i)} + b \in \mathbb{R}$ & not an integer class label. Furthermore, weight update is calculated based on all examples in training dataset (instead of updating parameters incrementally after each training example), which is why this approach is also referred to as *batch gradient descent* (giảm dần độ dốc hàng loạt). To be more explicit & avoid confusion when talking about related concepts later in this chap & this book, refer to this process as *full batch gradient descent* (giảm dần độ dốc hàng loạt đầy đủ).

* **Implementing Adaline in Python.** Since perceptron rule & Adaline are very similar, take perceptron implementation defined earlier & change `fit` method so that weight & bias parameters are now updated by minimizing loss function via gradient descent: [code].

Instead of updating weights after evaluating each individual training example, as in perceptron, calculate gradient based on whole training dataset. For bias unit, this is done via `self.eta * 2.0 * errors.mean()`, where `errors` is an array containing partial derivative values ∂_b . Similarly, update weights. However note: weight updates via partial derivatives $\frac{\partial L}{\partial w_j}$ involve feature values x_j , which can compute by multiplying `errors` with each feature value for each weight:

```
for w_j in range(self.w_.shape[0]):
    self.w_[w_j] += self.eta * (2.0 * (X[:, w_j]*errors)).mean()
```

To implement weight update more efficiently without using a `for` loop, can use a matrix-vector multiplication between our feature matrix & error vector instead:

```
self.w_ += self.eta * 2.0 * X.T.dot(errors) / X.shape[0]
```

Note: `activation` method has no effect on code since it is simply an identity function. Here, added activation function (computed via `activation` method) to illustrate general concept with regard to how information flows through a single-layer NN: features from input data, net input, activation, & output.

In next chap, learn about a logistic regression classifier that uses a non-identity, nonlinear activation function. See: a logistic regression model is closely related to Adaline, with only difference being its activation & loss function.

now, similar to previous perceptron implementation, collect loss values in a `self.losses_` list to check whether algorithm converged after training.

Remark 35 (Matrix multiplication). *Performing a matrix multiplication is similar to calculating a vector dot-product where each row in matrix is treated as a single row vector. This vectorized approach represents a more compact notation & results in a more efficient computation using NumPy.*

In practice, often require some experimentation to find a good learning rate η for optimal convergence. So choose 2 different learning rates $\eta = 0.1, \eta = 0.0001$ to start with & plot loss functions vs. number of epochs to see how well Adaline implementation learns from training data.

Remark 36 (Hyperparameters). *Learning rate η eta as well as number of epochs `n_iter`, are so-called hyperparameters (or tuning parameters) of perceptron & Adaline learning algorithms. In Chap. 6, take a look at different techniques to automatically find values of different hyperparameters that yield optimal performance of classification model.*

Plot loss against number of epochs for 2 different learning rates:

```
>>> fig, ax = plt.subplots(nrows=1, ncols=2, figsize=(10, 4))
>>> ada1 = AdalineGD(n_iter=15, eta=0.1).fit(X, y)
>>> ax[0].plot(range(1, len(ada1.losses_) + 1),
```



```

...         np.log10(ada1.losses_), marker='o')
>>> ax[0].set_xlabel('Epochs')
>>> ax[0].set_ylabel('log(Mean squared error)')
>>> ax[0].set_title('Adaline - Learning rate 0.1')
>>> ada2 = AdalineGD(n_iter=15, eta=0.0001).fit(X, y)
>>> ax[1].plot(range(1, len(ada2.losses_) + 1),
...           ada2.losses_, marker='o')
>>> ax[1].set_xlabel('Epochs')
>>> ax[1].set_ylabel('Mean squared error')
>>> ax[1].set_title('Adaline - Learning rate 0.0001')
>>> plt.show()

```

As can see in resulting loss function plots, encountered 2 different types of problems. Left chart shows what could happen if choose a learning rate that is too large. Instead of minimizing loss function, MSE becomes larger in every epoch, because *overshoot* global minimum. On other hand, can see: loss decreases on right plot, but chosen learning rate $\eta = 0.0001$ is so small: algorithm would require a very large number of epochs to converge to global loss minimum: Fig. 2.11: Error plots for suboptimal learning rates.

Fig. 2.12: A comparison of a well-chosen learning rate & a learning rate that is too large illustrates what might happen if change value of a particular weight parameter to minimize loss function L . Left subfigure illustrates case of a well-chosen learning rate, where loss decreases gradually, moving in direction of global minimum.

Subfigure on right, however, illustrates what happens if choose a learning rate that is too large – overshoot global minimum.

- * **Improving gradient descent through feature scaling.** Many ML algorithms that will encounter throughout this book require some sort of feature scaling for optimal performance, discussed in Chaps. 3–4.

Gradient descent is 1 of many algorithms that benefit from feature scaling. In this sect, use a feature scaling method called *standardization*. This normalization produce helps gradient descent learning to converge more quickly; however, it does not make original dataset normally distributed. Standardization shifts mean of each feature so that it is centered at 0 & each feature has a standard deviation of 1 (unit variance). E.g., to standardize j th feature, can simply subtract sample mean μ_j from every training example & divide it by its standard deviation σ_j :

$$x'_j = \frac{x_j - \mu_j}{\sigma_j}. \quad (206)$$

Here x_j : a vector consisting of j th feature values of all training examples n , & this standardization technique is applied to each feature j in our dataset.

1 of reasons why standardization helps with gradient descent learning: easier to find a learning rate that works well for all weights (& bias). If features are on vastly different scales, a learning rate that works well for updating 1 weight might be too large or too small to update other weight equally well. Overall, using standardized features can stabilize training s.t. optimizer has to go through fewer steps to find a good or optimal solution (global loss minimum). Fig. 2.13: A comparison of unscaled & standardized features on gradient updates illustrates possible gradient updates with unscaled features (left) & standardized features (right), where concentric circles represent loss surface as a function of 2 model weights in a 2D classification problem.

Standardization can easily be achieved by using built-in NumPy methods `mean`, `std`:

```

>>> X_std = np.copy(X)
>>> X_std[:,0] = (X[:,0] - X[:,0].mean()) / X[:,0].std()
>>> X_std[:,1] = (X[:,1] - X[:,1].mean()) / X[:,1].std()

```

After standardization, train Adaline again & see: it now converges after a small number of epochs using a learning rate of $\eta = 0.5$:

```

>>> ada_gd = AdalineGD(n_iter=20, eta=0.5)
>>> ada_gd.fit(X_std, y)
>>> plot_decision_regions(X_std, y, classifier=ada_gd)
>>> plt.title('Adaline - Gradient descent')
>>> plt.xlabel('Sepal length [standardized]')
>>> plt.ylabel('Petal length [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
>>> plt.plot(range(1, len(ada_gd.losses_) + 1),
...         ada_gd.losses_, marker='o')
>>> plt.xlabel('Epochs')
>>> plt.ylabel('Mean squared error')
>>> plt.tight_layout()
>>> plt.show()

```

After executing this code, should see a figure of decision regions, as well as a plot of declining loss, as shown in Fig. 2.14: Plots of Adaline's decision regions & MSE by number of epochs.

As can see in plots, Adaline has now converged after training on standardized features. However, note: MSE remains non-zero even though all flower examples were classified correctly.

- * **Large-scale ML & stochastic gradient descent.** In prev sect, learned how to minimize a loss function by taking a step in opposite direction of loss gradient that is calculated from whole training dataset; this is why this approach is sometimes also referred to as full batch gradient descent. Now imagine: have a very large dataset with millions of data points, which is not uncommon in many ML applications. Running full batch gradient descent can be computationally quite costly in such scenarios, since need to reevaluate whole training dataset each time take 1 step toward global minimum.

A popular alternative to batch gradient descent algorithm is *stochastic gradient descent (SGD)*, which is sometimes also called *iterative- or online gradient descent*. Instead of updating weights based on sum of accumulated errors over all training examples $x^{(i)}$:

$$\Delta w_j = \frac{2\eta}{n} \sum_i (y^{(i)} - \sigma(z^{(i)})) x_j^{(i)} \quad (207)$$

update parameters incrementally for each training example, e.g.:

$$\Delta w_j = \eta(y^{(i)} - \sigma(z^{(i)})) x_j^{(i)}, \quad \Delta b = \eta(y^{(i)} - \sigma(z^{(i)})). \quad (208)$$

Although SGD can be considered as an approximation of gradient descent, it typically reaches convergence much faster because of more frequent weight updates. Since each gradient is calculated based on a single training example, error surface is noisier than in gradient descent, which can also have advantage: SGD can escape shallow local minima more readily if working with nonlinear loss functions, as seen in Chap. 11. To obtain satisfying results via SGD, important to present training data in a random order; also, want to shuffle training dataset for every epoch to prevent cycles.

Remark 37 (Adjusting learning rate during training). In SGD implementations, fixed learning rate η is often replaced by an adaptive learning rate that decreases over time, e.g.: $\frac{c_1}{[\text{number of iterations}] + c_2}$ where c_1, c_2 : constants. Note: SGD does not reach global loss minimum but an area very close to it. Using an adaptive learning rate, can achieve further annealing to loss minimum.

– Lưu ý: SGD không đạt đến mức tổn thất tối thiểu toàn cục mà là một khu vực rất gần với mức đó. Sử dụng tốc độ học thích ứng, có thể đạt được quá trình tối ưu tiếp theo đến mức tổn thất tối thiểu.

Another advantage of SGD: can use it for *online learning*. In online learning, our model is trained on fly as new training data arrives. This is especially useful if accumulating large amounts of data, e.g., customer data in web applications. Using online learning, system can immediately adapt to changes, & training data can be discarded after updating model if storage space is an issue.

Remark 38 (Mini-batch gradient descent). A compromise between full batch gradient descent & SGD is so-called mini-batch gradient descent. Mini-batch gradient descent can be understood as applying full batch gradient descent to smaller subsets of training data, e.g., 32 training examples at a time. Advantage over full batch gradient descent: convergence is reached faster via mini-batches because of more frequent weight updates. Furthermore, mini-batch learning allows us to replace for loop over training examples in SGD with vectorized operations leveraging concepts from linear algebra (e.g., implementing a weighted sum via dot product), which can further improve computational efficiency of our learning algorithm.

Since already implemented Adaline learning rule using gradient descent, only need to make a few adjustments to modify learning algorithm to update weights via SGD. Inside `fit` method, now update weights after each training example. Furthermore, implement an additional `partial_fit` method, which does not reinitialize weights, for online learning. In order to check whether our algorithm converged after training, calculate loss as average loss of training examples in each epoch. Furthermore, add an option to shuffle training data before each epoch to avoid repetitive cycles when optimizing loss function; via `random_state` parameter, allow specification of a random seed for reproducibility [code].

`_shuffle` method that now using in `AdalineSGD` classifier works as follows: via `permutation` function in `np.random`, generate a random sequence of unique numbers in range 0 to 100. Those numbers can then be used as indices to shuffle our feature matrix & class label vector.

Can then use `fit` method to train `AdalineSGD` classifier & use `plot_decision_regions` to plot our training results:

```
>>> ada_sgd = AdalineSGD(n_iter=15, eta=0.01, random_state=1)
>>> ada_sgd.fit(X_std, y)
>>> plot_decision_regions(X_std, y, classifier=ada_sgd)
>>> plt.title('Adaline - Stochastic gradient descent')
>>> plt.xlabel('Sepal length [standardized]')
>>> plt.ylabel('Petal length [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
>>> plt.plot(range(1, len(ada_sgd.losses_) + 1), ada_sgd.losses_, marker='o')
>>> plt.xlabel('Epochs')
>>> plt.ylabel('Average loss')
>>> plt.tight_layout()
```

```
>>> plt.show()
```

2 plots obtained from executing preceding code example: Fig. 2.15: Decision regions & average loss plots after training an Adaline model using SGD.

Average loss goes down pretty quickly, & final decision boundary after 15 epochs looks similar to batch gradient descent Adaline. If want to update model, e.g., in an online learning scenario with streaming data, could simply call `partial_fit` method on individual training examples – e.g., `ada_sgd.partial_fit(X_std[0, :], y[0])`.

- **Summary.** In this chap, gained a good understanding of basic concepts of linear classifiers for supervised learning. After implemented a perceptron, saw how can train adaptive linear neurons effectively via a vectorized implementation of gradient descent & online learning via SGD.

Have seen how to implement simple classifiers in Python, ready to move on to next chap, where use Python `scikit-learn` ML library to get access to more advanced & powerful ML classifiers, commonly used in academia as well as in industry.

Object-oriented approach used to implement perceptron & Adaline algorithms will help with understanding scikit-learn API, which is implemented based on same core concepts used in this chap: `fit`, `predict` methods. Based on these core concepts, learn about logistic regression for modeling class probabilities & support vector machines for working with nonlinear decision boundaries. In addition, introduce a different class of supervised learning algorithms, tree-based algorithms, commonly combined into robust ensemble classifiers.

- **Chap. 3: A Tour of ML Classifiers Using Scikit-Learn.** In this chap, take a tour of a selection of popular & powerful ML algorithms commonly used in academia as well as in industry. While learning about differences between several supervised learning algorithms for classification, also develop an appreciation of their individual strengths & weaknesses. Take 1st steps with scikit-learn library, which offers a user-friendly & consistent interface for using those algorithms efficiently & productively.

Topics covered:

- An introduction to robust & popular algorithms for classification, e.g. logistic regression, support vector machines, decision trees, & k -nearest neighbors
- Examples & explanations using scikit-learn ML library, which provides a wide variety of ML algorithms via a user-friendly Python API
- Discussions about strengths & weaknesses of classifiers with linear & nonlinear decision boundaries
- **Choosing a classification algorithm.** Choosing an appropriate classification algorithm for a particular problem task requires practice & experience; each algorithm has its own quirks & is based on certain assumptions. To paraphrase *no free lunch theorem* by DAVID H. WOLPERT, no single classifier works best across all possible scenarios *The Lack of A Priori Distinctions Between Learning Algorithms*, Wolpert, David H, Neural Computation 8.7 (1996): 1341-1390. In practice, always recommended: compare performance of at least a handful of different learning algorithms to select best model for particular problem; these may differ in number of features or examples, amount of noise in a dataset, & whether classes are linearly separable.

Eventually, performance of a classifier – computational performance as well as predictive power – depends heavily on underlying data that is available for learning. 5 main steps involved in training a supervised ML algorithm can be summarized as follows:

1. Selecting features & collecting labeled training examples
2. Choosing a performance metric
3. Choosing a learning algorithm & training a model
4. Evaluating performance of model
5. Changing settings of algorithm & tuning model.

Since approach of this book is to build ML knowledge step by step, mainly focus on main concepts of different algorithms in this chap & revisit topics e.g. feature selection & preprocessing, performance metrics, & hyperparameter tuning for more detailed discussions later in book.

- **1st steps with scikit-learn – training a perceptron.** In Chap. 2, learned about 2 related learning algorithms for classification, *perceptron* rule & *Adaline*, which implemented in Python & NumPy by ourselves. Now take a look at scikit-learn API, which combines a user-friendly & consistent interface with a highly optimized implementation of several classification algorithms. Scikit-learn library offers not only a large variety of learning algorithms, but also many convenient functions to preprocess data & to fine-tune & evaluate our models. Discuss this in more detail, together with underlying concepts, in Chaps. 4–5.

To get started with scikit-learn library, train a perceptron model similar to one implemented in Chap. 2. For simplicity, use already familiar *Iris dataset* throughout following sects. Conveniently, Iris dataset is already available via scikit-learn, since it is a simple yet popular dataset that is frequently used for testing & experimenting with algorithms. Similar to prev chap, only use 2 features from Iris dataset for visualization purposes.

Assign petal length & petal width of 150 flower examples to feature matrix `X` & corresponding class labels of flower species to vector array `y`:

```
>>> from sklearn import datasets
>>> import numpy as np
>>> iris = datasets.load_iris()
>>> X = iris.data[:, [2, 3]]
>>> y = iris.target
>>> print('Class labels:', np.unique(y))
Class labels: [0 1 2]
```

`np.unique(y)` function returned 3 unique class labels stored in `iris.target`, & Iris flower class names `Iris-setosa`, `Iris-versicolor`, `Iris-virginica`, are already stored as integers (here: 0, 1, 2). Although many scikit-learn functions & class methods also work with class labels in string format, using integer labels is a recommended approach to avoid technical glitches (tránh trục trặc kỹ thuật) & improve computational performance due to a smaller memory footprint; furthermore, encoding class labels as integers is a common convention among most ML libraries.

To evaluate how well a trained model performs on unseen data, further split dataset into separate training & test datasets. In Chap. 6, discuss best practices around model evaluation in more detail. Using `train_test_split` function from scikit-learn's `model_selection` module, randomly split `X`, `y` arrays into 30% test data (45 examples) & 70% training data (105 examples):

```
>>> from sklearn.model_selection import train_test_split
>>> X_train, X_test, y_train, y_test = train_test_split(
...     X, y, test_size=0.3, random_state=1, stratify=y
... )
```

Note: `train_test_split` function already shuffles training datasets internally before splitting; otherwise, all examples from class 0 & class 1 would have ended up in training datasets, & test dataset would consist of 45 examples from class 2. Via `random_state` parameter, provided a fixed random seed `random_state=1` for internal pseudo-random number generator that is used for shuffling datasets prior to splitting. Using such a fixed `random_state` ensures: our results are reproducible. Lastly, took advantage of built-in support for stratification (sự phân tầng) via `stratify=y`. In this context, stratification means: `train_test_split` method returns training & test subsets that have same proportions of class labels as input dataset. Can use NumPy's `bincount` function, which counts number of occurrences of each value in an array, to verify that this is indeed the case:

```
>>> print('Labels counts in y:', np.bincount(y))
Labels counts in y: [50 50 50]
>>> print('Labels counts in y_train:', np.bincount(y_train))
Labels counts in y_train: [35 35 35]
>>> print('Labels counts in y_test:', np.bincount(y_test))
Labels counts in y_test: [15 15 15]
```

Many ML & optimization algorithms also require feature scaling for optimal performance, as saw in *gradient descent* example in Chap. 2. Here, standardize features using `StandardScaler` class from scikit-learn's `preprocessing` module:

```
>>> from sklearn.preprocessing import StandardScaler
>>> sc = StandardScaler()
>>> sc.fit(X_train)
>>> X_train_std = sc.transform(X_train)
>>> X_test_std = sc.transform(X_test)
```

Using preceding code, loaded `StandardScaler` class from `preprocessing` module & initialized a new `StandardScaler` object assigned to `sc` variable. Using `fit` method, `StandardScaler` estimated parameters, μ (sample mean) & σ (standard deviation), for each feature dimension from training data. By calling `transform` method, then standardized training data using those estimated parameters μ, σ . Note: used same scaling parameters to standardize test dataset so that both values in training & test dataset are compatible with 1 another.

Having standardized training data, can now train a perceptron model. Most algorithms in scikit-learn already support multiclass classification by default via *1-vs-rest (OvR)* method, which allows to feed 3 flower classes to perceptron all at once. Code:

```
>>> from sklearn.linear_model import Perceptron
>>> ppn = Perceptron(eta0=0.1, random_state=1)
>>> ppn.fit(X_train_std, y_train)
```

Scikit-learn interface will remind of our perceptron implementation in Chap. 2. After loading `Perceptron` class from `linear_model` module, initialized a new `Perceptron` object & trained model via `fit` method. Here, model parameter `eta0` is equivalent to learning rate `eta` that we used in our own perceptron implementation.

As in Chap. 2, finding an appropriate learning rate requires some experimentation. If learning rate is too large, algorithm will overshoot global loss minimum. If learning rate is too small, algorithm will require more epochs until convergence, which can make learning slow – especially for large datasets. Also, used `random_state` parameter to ensure reproducibility of initial shuffling of training dataset after each epoch.

Having trained a model in scikit-learn, can make predictions via `predict` method, just like in our own perceptron implementation in Chap. 2. Code:

```
>>> y_pred = ppn.predict(X_test_std)
>>> print('Misclassified examples: %d' % (y_test != y_pred).sum())
Misclassified examples: 1
```

Executing code, can see: perceptron misclassifies 1 out of 45 flower examples. Thus, misclassification error on test dataset is $\frac{1}{45} \approx 0.022 = 2.2\%$.

Remark 39 (Classification error vs. accuracy). *Instead of misclassification error, many ML practitioners report classification accuracy of a model, which is simply calculated as follows: $1 - \text{error} = 0.978 = 97.8\%$. Whether use classification error or accuracy is merely a matter of preference.*

Note: scikit-learn also implements a large variety of different performance metrics that are available via `metrics` module. E.g., can calculate classification accuracy of perceptron on test dataset as follows:

```
>>> from sklearn.metrics import accuracy_score
>>> print('Accuracy: %.3f' % accuracy_score(y_test, y_pred))
Accuracy: 0.978
```

Here `y_test`: true labels, `y_pred`: class labels predicted previously. Alternatively, each classifier in scikit-learn has a `score` method, which computes a classifier's prediction accuracy by combining `predict` call with `accuracy_score`, as shown here:

```
>>> print('Accuracy: %.3f' % ppn.score(X_test_std, y_test))
Accuracy: 0.978
```

Remark 40 (Overfitting). *Note: will evaluate performance of our models based on test dataset in this chap. In Chap. 6, learn about useful techniques, including graphical analysis, e.g. learning curves, to detect & prevent overfitting. Overfitting means: model captures patterns in training data well but fails to generalize well to unseen data.*

Finally, can use our `plot_decision_regions` function from Chap. 2 to plot *decision regions* of our newly trained perceptron model & visualize how well it separates different flower examples. However, add a small modifications to highlight data instances from test dataset via small circles: [code].

With slight modification that we made to `plot_decision_regions` function, can now specify indices of examples that we want to mark on resulting plots. Code:

```
>>> X_combined_std = np.vstack((X_train_std, X_test_std))
>>> y_combined = np.hstack((y_train, y_test))
>>> plot_decision_regions(X=X_combined_std, y=y_combined, classifier=ppn, test_idx=range(105, 150))
>>> plt.xlabel('Petal length [standardized]')
>>> plt.ylabel('Petal width [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
```

In resulting plot, 3 flower classes can't be perfectly separated by a linear decision boundary: Fig. 3.1: Decision boundaries of a multi-class perceptron model fitted to Iris dataset. However, remember from discussion in Chap. 2: perceptron algorithm never converges on datasets that aren't perfectly linearly separable, which is why use of perceptron algorithm is typically not recommended in practice. In following sects, look at more powerful linear classifiers that converge to a loss minimum even if classes are not perfectly linearly separable.

Remark 41 (Additional perceptron settings). *Perceptron, as well as other scikit-learn functions & classes, often has additional parameters that we omit for clarity. Can read more about those parameters using `help` function in Python (e.g., `help(Perceptron)`) or by going through excellent scikit-learn online documentation at <http://scikit-learn.org/stable/>.*

- Modeling class probabilities via logistic regression. Although perceptron rule offers a nice & easy-going introduction to ML algorithms for classification, its biggest disadvantage: it never converges if classes are not perfectly linearly separable. Classification task in prev sect would be an example of such a scenario. Reason for this: weights are continuously being updated since there is always at least 1 misclassified training example present in each epoch. Of course, can change learning rate & increase number of epochs, but be warned: perceptron will never converge on this dataset.

To make better use of our time, take a look at another simple, yet more powerful, algorithm for linear & binary classification problems: *logistic regression*. Note: despite its name, logistic regression is a model for classification, not regression.

* **Logistic regression & conditional probabilities.** Logistic regression is a classification model that is very easy to implement & performs very well on linearly separable classes. It is 1 of most widely used algorithms for classification in industry. Similar to perceptron & Adaline, logistic regression model in this chap is also a linear model for binary classification.

Remark 42 (Logistic regression for multiple classes). *Note: logistic regression can be readily generalized to multi-class settings, which is known as multinomial logistic regression, or softmax regression. More detailed coverage of multinomial logistic regression is outside scope of this book, but interested reader can find more information in lecture note at https://sebastianraschka.com/pdf/lecture-notes/stat453ss21/L08_logistic_slides.pdf or <https://www.youtube.com/watch?v=LOFU8NFpx4E>.*

Another way to use logistic regression in multiclass settings is via OvR technique, discussed previously.

To explain main mechanics behind logistic regression as a probabilistic model for binary classification, 1st introduce the odds: odds in favor of a particular event. The odds can be written as $\frac{p}{1-p}$, where p stands for probability of positive event. Term “positive event” does not necessarily mean “good”, but refers to event that we want to predict, e.g., probability that a patient has a certain disease given certain symptoms; can think of positive event as class label $y = 1$ & symptoms as features x . Hence, for brevity, can define probability p as $p := p(y = 1|x)$, conditional probability that a particular example belongs to a certain class 1 given its features x .

Can then further define *logit* function, which is simply logarithm of odds (log-odds):

$$\text{logit}(p) = \log \frac{p}{1-p}. \quad (209)$$

Note: log refers to natural logarithm, as it is common convention in CS. Logit function takes input values in range 0 to 1 & transforms them into values over entire real-number range.

Under logistic model, assume: there is a linear relationship between weighted inputs (referred to as net inputs in Chap. 2) & log-odds:

$$\text{logit}(p) = w_1x_1 + \cdots + w_mx_m + b = \sum_{i=1}^m w_ix_i + b = \mathbf{w}^\top \mathbf{x} + b. \quad (210)$$

While preceding describes an assumption we make about linear relationship between log-odds & net inputs, what actually interested in is probability p , class-membership probability of an example given its features. While logit function maps probability to a real-number range, can consider inverse of this function to map real-number range back to a $[0, 1]$ range for probability p .

This inverse of logit function is typically called *logistic sigmoid function*, sometimes simply abbreviated to *sigmoid function* due to its characteristic S-shape:

$$\sigma(z) = \frac{1}{1 + e^{-z}}. \quad (211)$$

Here z : net input, linear combination of weights, & inputs (i.e., features associated with training examples): $z = \mathbf{w}^\top \mathbf{x} + b$. Simply plot sigmoid function for some values in range $[-7, 7]$ to see how it works:

```
>>> import matplotlib.pyplot as plt
>>> import numpy as np
>>> def sigmoid(z):
...     return 1.0 / (1.0 + np.exp(-z))
>>> z = np.arange(-7, 7, 0.1)
>>> sigma_z = sigmoid(z)
>>> plt.plot(z, sigma_z)
>>> plt.axvline(0.0, color='k')
>>> plt.ylim(-0.1, 1.1)
>>> plt.xlabel('z')
>>> plt.ylabel('$\sigma(z)$')
>>> # y axis ticks & gridline
>>> plt.yticks([0.0, 0.5, 1.0])
>>> ax = plt.gca()
>>> ax.yaxis.grid(True)
>>> plt.tight_layout()
>>> plt.show()
```

As a result of executing previous code example, should now see S-shaped (sigmoidal) curve: Fig. 3.2: A plot of logistic sigmoid function. Can see: $\sigma(z)$ approaches 1 if $z \rightarrow \infty$ since e^{-z} becomes very small for large values of z . Similarly, $\sigma(z) \rightarrow 0$ for $z \rightarrow -\infty$ as a result of an increasingly large denominator. Thus, can conclude: this sigmoid function takes real-number values as input & transforms them into values in range $[0, 1]$ with an intercept at $\sigma(0) = 0.5$.

To build some understanding of logistic regression model, can relate it to Chap. 2. In Adaline, used identity function $\sigma(z) = z$ as activation function. In logistic regression, this activation function simply becomes sigmoid function defined earlier.

Difference between Adaline & logistic regression is illustrated in Fig. 3.3: Logistic regression compared to Adaline, where only difference is activation function.

Output of sigmoid function is then interpreted as probability of a particular example belonging to class 1, $\sigma(z) = p(y = 1|\mathbf{x}; \mathbf{w}, b)$, given its features \mathbf{x} & parametrized by weights \mathbf{w} & bias b . E.g., if compute $\sigma(z) = 0.8$ for a particular flower example, it means: chance this this example is an **Iris-versicolor** flower is 80%. Therefore, probability that this flower is an **Iris-setosa** flower can be calculated as $p(y = 0|\mathbf{x}; \mathbf{w}, b) = 1 - p(y = 1|\mathbf{x}; \mathbf{w}, b) = 0.2 = 20\%$.

Predicted probability can then simply be converted into a binary outcome via a threshold function:

$$\hat{y} = \begin{cases} 1 & \text{if } \sigma(z) \geq 0.5, \\ 0 & \text{otherwise.} \end{cases} \quad (212)$$

If look at preceding plot of sigmoid function, this \Leftrightarrow

$$\hat{y} = \begin{cases} 1 & \text{if } \sigma(z) \geq 0.0, \\ 0 & \text{otherwise.} \end{cases} \quad (213)$$

In fact, there are many applications where not only interested in predicted class labels, but where estimation of class-membership probability is particularly useful (output of sigmoid function prior to applying threshold function). Logistic regression is used in weather forecasting, e.g., not only to predict whether it will rain on a particular day, but also to report change of rain. Similarly, logistic regression can be used to predict chance that a patient has a particular disease given certain symptoms, which is why logistic regression enjoys great popularity in field of medicine.

- * **Learning model weights via logistic loss function.** Have learned how we can use logistic regression model to predict probabilities & class labels; now, briefly talk about how we fit parameters of model, e.g., weights & bias unit \mathbf{w}, b . In prev chap, defined mean squared error loss functions as follows:

$$L(\mathbf{w}, b|\mathbf{x}) = \sum_i \frac{1}{2} (\sigma(z^{(i)}) - y^{(i)})^2. \quad (214)$$

Minimized this function in order to learn parameters for our Adaline classification model. To explain how we can derive loss function for logistic regression, 1st define likelihood \mathcal{L} that we want to maximize when we build a logistic regression model, assuming: individual examples in our dataset are independent of 1 another. Formula is as follows:

$$\mathcal{L}(\mathbf{w}, b|\mathbf{x}) = p(y|\mathbf{x}; \mathbf{w}, b) = \prod_{i=1}^n p(y^{(i)}|\mathbf{x}^{(i)}; \mathbf{w}, b) = \prod_{i=1}^n (\sigma(z^{(i)}))^{y^{(i)}} (1 - \sigma(z^{(i)}))^{1-y^{(i)}}. \quad (215)$$

In practice, easier to maximize (natural) log of this equation, called *log-likelihood* function:

$$l(\mathbf{w}, b|\mathbf{x}) = \log \mathcal{L}(\mathbf{w}, b|\mathbf{x}) = \sum_{i=1}^n \left[y^{(i)} \log(\sigma(z^{(i)})) + (1 - y^{(i)}) \log(1 - \sigma(z^{(i)})) \right]. \quad (216)$$

1stly, applying log function reduces potential for numerical underflow, which can occur if likelihoods are very small. 2ndly, can convert product of factors into a summation of factors, which makes it easier to obtain derivative of this function via addition trick, as may remember from calculus.

Remark 43 (Deriving likelihood function). *Can obtain expression for likelihood of model given data, $\mathcal{L}(\mathbf{w}, b|\mathbf{x})$ as follows. Given: have a binary classification problem with class labels 0 & 1, can think of label 1 as a Bernoulli variable – it can take on 2 values, 0 & 1, with probability p of being 1: $Y \sim \text{Bern}(p)$. For a single data point, can write this probability as $P(Y = 1|X = x^{(i)}) = \sigma(z^{(i)})$ & $P(Y = 0|X = x^{(i)}) = 1 - \sigma(z^{(i)})$.*

Putting these 2 expressions together, & using shorthand $P(Y = y^{(i)}|X = x^{(i)}) = p(y^{(i)}|x^{(i)})$, get probability mass function of Bernoulli variable:

$$p(y^{(i)}|x^{(i)}) = (\sigma(z^{(i)}))^{y^{(i)}} (1 - \sigma(z^{(i)}))^{1-y^{(i)}}. \quad (217)$$

Can write likelihood of training labels given assumption: all training examples are independent, using multiplication rule to compute probability that all events occur, as follows:

$$\mathcal{L}(\mathbf{w}, b|\mathbf{x}) = \prod_{i=1}^n p(y^{(i)}|\mathbf{x}^{(i)}; \mathbf{w}, b). \quad (218)$$

Now, substituting probability mass function of Bernoulli variable, arrive at expression of likelihood, which we attempt to maximize by changing model parameters:

$$\mathcal{L}(\mathbf{w}, b|\mathbf{x}) = \prod_{i=1}^n (\sigma(z^{(i)}))^{y^{(i)}} (1 - \sigma(z^{(i)}))^{1-y^{(i)}}. \quad (219)$$

Could use an optimization algorithm e.g. gradient ascent to maximize this log-likelihood function. (Gradient ascent works exactly same way as gradient descent explained in Chap. 2, except: gradient ascent maximizes a function instead of

minimizing it.) Alternatively, rewrite log-likelihood as a loss function L that can be minimized using gradient descent as in Chap. 2:

$$L(\mathbf{w}, b) = \sum_{i=1}^n \left[-y^{(i)} \log(\sigma(z^{(i)})) - (1 - y^{(i)}) \log(1 - \sigma(z^{(i)})) \right]. \quad (220)$$

To get a better grasp of this loss function, take a look at loss that we calculate for 1 single training example:

$$L(\sigma(z), y; \mathbf{w}, b) = -y \log(\sigma(z)) - (1 - y) \log(1 - \sigma(z)). \quad (221)$$

1st term becomes 0 if $y = 0$, & 2nd term becomes 0 if $y = 1$:

$$L(\sigma(z), y; \mathbf{w}, b) = \begin{cases} -\log(\sigma(z)) & \text{if } y = 1, \\ -\log(1 - \sigma(z)) & \text{if } y = 0. \end{cases} \quad (222)$$

Write a short code snippet to create a plot that illustrates loss of classifying a single training example for different values of $\sigma(z)$:

```
>>> def loss_1(z):
...     return - np.log(sigmoid(z))
>>> def loss_0(z):
...     return - np.log(1 - sigmoid(z))
>>> z = np.arange(-10, 10, 0.1)
>>> sigma_z = sigmoid(z)
>>> c1 = [loss_1(x) for x in z]
>>> plt.plot(sigma_z, c1, label='L(w, b) if y=1')
>>> c0 = [loss_0(x) for x in z]
>>> plt.plot(sigma_z, c0, linestyle='--', label='L(w, b) if y=0')
>>> plt.ylim(0.0, 5.1)
>>> plt.xlim([0, 1])
>>> plt.xlabel('$\sigma(z)$')
>>> plt.ylabel('L(w, b)')
>>> plt.legend(loc='best')
>>> plt.tight_layout()
>>> plt.show()
```

Resulting plot shows sigmoid activation on x axis in range 0 to 1 (inputs to sigmoid function were $z \in [-10, 10]$) & associated logistic loss on y axis: Fig. 3.4: A plot of loss function used in logistic regression. Can see: loss approaches 0 (continuous line) if correctly predict that an example belongs to class 1. Similarly, can see on y axis: loss also approaches 0 if correctly predict $y = 0$ (dashed line). However, if prediction is wrong, loss $\rightarrow \infty$. Main point: penalize wrong predictions with an increasingly larger loss.

- * Converting an Adaline implementation into an algorithm for logistic regression. If were to implement logistic regression ourselves, could simply substitute loss function L in our Adaline implementation from Chap. 2 with new loss function:

$$L(\mathbf{w}, b) = \frac{1}{n} \sum_{i=1}^n \left[-y^{(i)} \log(\sigma(z^{(i)})) - (1 - y^{(i)}) \log(1 - \sigma(z^{(i)})) \right]. \quad (223)$$

Use this to compute loss of classifying all training examples per epoch. Also, need to swap linear activation function with sigmoid. If make those changes to Adaline code, will end up with a working logistic regression implementation. Following is an implementation for full-batch gradient descent (but note: same changes could be made to stochastic gradient descent version as well): [code].

When fit a logistic regression model, have to keep in mind: it only works for binary classification tasks.

So, consider only setosa & versicolor flowers (classes 0 & 1) & check: our implementation of logistic regression works:

```
>>> X_train_01_subset = X_train_std[(y_train == 0) | (y_train == 1)]
>>> y_train_01_subset = y_train[(y_train == 0) | (y_train == 1)]
>>> lrgd = LogisticRegressionGD(eta=0.3, n_iter=1000, random_state=1)
>>> lrgd.fit(X_train_01_subset, y_train_01_subset)
>>> plot_decision_regions(X=X_train_01_subset, y=y_train_01_subset, classifier=lrgd)
>>> plt.xlabel('Petal length [standardized]')
>>> plt.ylabel('Petal width [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
```

Resulting decision region plot looks as Fig. 3.5: Decision region plot for logistic regression model.

Remark 44 (Gradient descent learning algorithm for logistic regression). If compared `LogisticRegressionGD` in prev code with `AdalineGD` code from Chap. 2, may have noticed: weight & bias update rules remained unchanged (except for scaling factor 2). Using calculus, can show: parameter updates via gradient descent are indeed similar for logistic regression & Adaline. However, note: following derivation of gradient descent learning rule is intended for readers who are interested in mathematical concepts behind gradient descent learning rule for logistic regression. Not essential for following rest of this chap.

Fig. 3.6: Calculating partial derivative of log-likelihood function summarizes how can calculate partial derivative of log-likelihood function w.r.t. j th weight: $\frac{\partial L}{\partial w_j} = \frac{\partial L}{\partial a} \frac{da}{dz} \frac{\partial z}{\partial w_j}$ where $a = \sigma(z) = \frac{1}{1+e^{-z}}$ Note: omitted averaging over training examples for brevity.

Remember from Chap. 2: take steps in opposite direction of gradient. Hence, flip $\frac{\partial L}{\partial w_j} = -(y - a)x_j$ & update j th weight as follows, including learning rate η : $w_j := w_j + \eta(y - a)x_j$. While partial derivative of loss function w.r.t. bias unit is not shown, bias derivation follows same overall concept using chain rule, resulting in following update rule: $b := b + \eta(y - a)$. Both weight & bias unit updates are equal to the ones for Adaline in Chap. 2.

- * Training a logistic regression model with scikit-learn. Just went through useful coding & math exercises in prev subset, which helped to illustrate conceptual differences between Adaline & logistic regression. Learn how to use scikit-learn's more optimized implementation of logistic regression, which also supports multiclass settings off shelf. Note: in recent versions of scikit-learn, technique used for multiclass classification, multinomial, or OvR, is chosen automatically. In following code example, use `sklearn.linear_model.LogisticRegression` class as well as familiar `fit` method to train model on all 3 classes in standardized flower training dataset. Also, set `multi_class='ovr'` for illustration purposes. May want to compare results with `multi_class='multinomial'`. Note: `multinomial` setting is now default choice in scikit-learn's `LogisticRegression` class & recommended in practice for mutually exclusive classes, e.g. those found in Iris dataset. Here, "mutually exclusive" means: each training example can only belong to a single class (in contrast to multilabel classification, where a training example can be a member of multiple classes).

Code example:

```
>>> from sklearn.linear_model import LogisticRegression
>>> lr = LogisticRegression(C=100.0, solver='lbfgs', multi_class='ovr')
>>> lr.fit(X_train_std, y_train)
>>> plot_decision_regions(X_combined_std, y_combined, classifier=lr, test_idx=range(105, 150))
>>> plt.xlabel('Petal length [standardized]')
>>> plt.ylabel('Petal width [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
```

After fitting model on training data, plotted decision regions, training examples, & test examples, as shown in Fig. 3.7: Decision regions for scikit-learn's multiclass logistic regression model.

Remark 45 (Algorithms for convex optimization). Note: there exist many different algorithms for solving optimization problems. For minimizing convex loss functions, e.g. logistic regression loss, recommended to use more advanced approaches than regular stochastic gradient descent (SGD). In fact, scikit-learn implements a whole range of such optimization algorithms, which can be specified via `solver` parameter, namely, 'newton-cg', 'lbfgs', 'liblinear', 'sag', 'saga'. While logistic regression loss is convex, most optimization algorithms should converge to global loss minimum with ease. However, there are certain advantages of using 1 algorithm over other. E.g., in prev versions (e.g., v 0.21), scikit-learn used 'liblinear' as a default, which cannot handle multinomial loss & is limited to OvR scheme for multiclass classification. However, in scikit-learn v 0.22, default solver was changed to 'lbfgs', which stands for limited-memory Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm https://en.wikipedia.org/wiki/Limited-memory_BFGS & is more flexible in this regard.

Looking at preceding code used to train `LogisticRegression` model, might now be wondering, "What is this mysterious parameter C?" Will discuss this parameter in next subset, where introduce concepts of overfitting & regularization. However, before move on to those topics, finish discussion of class membership probabilities.

Probability that training examples belong to a certain class can be computed using `predict_proba` method. E.g., can predict probabilities of 1st 3 examples in test dataset as follows:

```
>>> lr.predict_proba(X_test_std[:3, :])
```

This code snippet returns following array:

```
array([[3.81527885e-09, 1.44792866e-01, 8.55207131e-01],
       [8.34020679e-01, 1.65979321e-01, 3.25737138e-13],
       [8.48831425e-01, 1.51168575e-01, 2.62277619e-14]])
```

1st row corresponds to class membership probabilities of 1st flower, 2nd row corresponds to class membership probabilities of 2nd flower, & so forth. Notice: column-wise sum in each row is 1, as expected. (Can confirm this by executing `lr.predict_proba(X_test_std[:3, :]).sum(axis=1)`.)

Highest value in 1st row is ≈ 0.85 , i.e., 1st example belongs to class 3 *Iris-virginica* with a predicted probability of 85%. So, can get predicted class labels by identifying largest column in each row, e.g., using NumPy's `argmax` function:

```
>>> lr.predict_proba(X_test_std[:3, :]).argmax(axis=1)
```

Returned class indices are shown here (they correspond to *Iris-virginica*, *Iris-setosa*, *Iris-setosa*) (?):

```
array([2, 0, 0])
```

In preceding code example, computed conditional probabilities & converted these into class labels manually by using NumPy's `argmax` function. In practice, more convenient way of obtaining class labels when using scikit-learn: call `predict` method directly:

```
>>> lr.predict(X_test_std[:3, :])
array([2, 0, 0])
```

Lastly, a word of caution if want to predict class label of a single flower example: scikit-learn expects a 2D array as data input; thus, have to convert a single row slice into such a format 1st. 1 way to convert a single row entry into a 2D data array: use NumPy's `reshape` method to add a new dimension:

```
>>> lr.predict(X_test_std[0, :].reshape(1, -1))
array([2])
```

- * **Tackling overfitting via regularization.** Overfitting is a common problem in ML, where a model performs well on training data but does not generalize well to unseen data (test data). If a model suffers from overfitting, also say: model has a high variance, which can be caused by having too many parameters, leading to a model that is too complex given underlying data. Similarly, our model can also suffer from *underfitting* (high bias), i.e., our model is not complex enough to capture pattern in training data well & therefore also suffers from low performance on unseen data.

Although have only encountered linear models for classification so far, problems of overfitting & underfitting can be best illustrated by comparing a linear decision boundary to more complex, nonlinear decision boundaries, as shown in Fig. 3.8: Examples of underfitted, well-fitted, & overfitted models.

Remark 46 (Bias-variance tradeoff). *Often, researchers use terms “bias” & “variance” or “bias-variance tradeoff” to describe performance of a model – i.e., may stumble upon (vấp phải) talks, books, or articles where people say: a model has a “high variance” or “high bias.” So, what does that mean? In general, might say: “high variance” is proportional to overfitting & “high bias” is proportional to underfitting.*

In context of ML models, variance measures consistency (or variability) of model prediction for classifying a particular example if train model multiple times, e.g., on different subsets of training dataset. Can say: model is sensitive to randomness in training data. In contrast, bias measures how far off predictions are from correct values in general if rebuild model multiple times on different training datasets; bias is measure of systematic error that is not due to randomness.

If interested in technical specification & derivation of “bias” & “variance” terms, see lecture notes https://sebastianraschka.com/pdf/lecture-notes/stat451fs20/08-model-eval-1-intro_notes.pdf.

1 way of finding a good bias-variance tradeoff: tune complexity of model via regularization. Regularization is a very useful method for handling collinearity (high correlation among features), filtering out noise from data, & eventually preventing overfitting.

Concept behind regularization: introduce additional information to penalize extreme parameter (weight) values. Most common form of regularization is so-called *L2 regularization* (sometimes also called L2 shrinkage or weight decay), which can be written as follows:

$$\frac{\lambda}{2n} \|\mathbf{w}\|^2 = \frac{\lambda}{2n} \sum_{i=1}^m w_i^2. \quad (224)$$

Here λ : so-called *regularization parameter*. Note: 2 in denominator is merely a scaling factor, s.t. it cancels when computing loss gradient. Sample size n is added to scale regularization term similar to loss.

Remark 47 (Regularization & feature normalization). *Regularization is another reason why feature scaling e.g. standardization is important. For regularization to work properly, need to ensure: all our features are on comparable scales.*

Loss function for logistic regression can be regularized by adding a simple regularization term, which will shrink weights during model training:

$$L(\mathbf{w}, b) = \frac{1}{n} \sum_{i=1}^n \left[-y^{(i)} \log(\sigma(z^{(i)})) - (1 - y^{(i)}) \log(1 - \sigma(z^{(i)})) \right] + \frac{\lambda}{2n} \|\mathbf{w}\|^2. \quad (225)$$

Partial derivative of unregularized loss is defined as:

$$\frac{\partial L(\mathbf{w}, b)}{\partial w_j} = \frac{1}{n} \sum_{i=1}^n \left(\sigma(\mathbf{w}^\top \mathbf{x}^{(i)}) - y^{(i)} \right) x_j^{(i)}. \quad (226)$$

Adding regularization term to loss changes partial derivative to following form:

$$\frac{\partial L(\mathbf{w}, b)}{\partial w_j} = \left(\frac{1}{n} \sum_{i=1}^n \left(\sigma(\mathbf{w}^\top \mathbf{x}^{(i)}) - y^{(i)} \right) x_j^{(i)} \right) + \frac{\lambda}{n} w_j. \quad (227)$$

Via regularization parameter λ , can then control how closely we fit training data, while keeping weights small. By increasing value of λ , increase regularization strength. Note: bias unit, which is essentially an intercept term or negative threshold, as learned in Chap. 2, is usually not regularized.

Parameter C , implemented for `LogisticRegression` class in scikit-learn comes from a convention in support vector machines. Term C is inversely proportional to regularization parameter λ . Consequently, decreasing value of inverse regularization parameter C means: increasing regularization strength, which can visualize by plotting L2 regularization path for 2 weight coefficients:

```
>>> weights, params = [], []
>>> for c in np.arange(-5, 5):
...     lr = LogisticRegression(C=10.**c,
... multi_class='ovr')
...     lr.fit(X_train_std, y_train)
...     weights.append(lr.coef_[1])
...     params.append(10.**c)
>>> weights = np.array(weights)
>>> plt.plot(params, weights[:, 0], label='Petal length')
>>> plt.plot(params, weights[:, 1], linestyle='--', label='Petal width')
>>> plt.ylabel('Weight coefficient')
>>> plt.xlabel('C')
>>> plt.legend(loc='upper left')
>>> plt.xscale('log')
>>> plt.show()
```

By executing preceding code, fitted 10 logistic regression models with different values for inverse-regularization parameter C . For illustration purposes, only collected weight coefficients of class 1 (here, 2nd class in dataset: **Iris-versicolor**) vs. all classifiers – remember: we are using OvR technique for multiclass classification.

As can see in resulting plot, weight coefficients shrink if decrease parameter C , i.e., if increase regularization strength: Fig. 3.9: Impact of inverse regularization strength parameter C on L2 regularized model results.

Increasing regularization strength can reduce overfitting, so might ask why don't strongly regularize all models by default. Reason: have to be careful when adjusting regularization strength. E.g., if regularization strength is too high & weights coefficients approach 0, model can perform very poorly due to underfitting, as illustrated in Fig. 3.8.

Remark 48 (An additional resource on logistic regression). *Since in-depth coverage of individual classification algorithms exceeds scope of this book, Logistic Regression: From Introductory to Advanced Concepts & Applications, Dr. SCOTT MENARD, Sage Publications, 2009, is recommended to readers who want to learn more about logistic regression.*

- **Maximum margin classification with support vector machines.** Another powerful & widely used learning algorithm is *support vector machine (SVM)*, which can be considered an extension of perceptron. Using perceptron algorithm, minimized misclassification errors. However, in SVMs, our optimization objective: maximize margin. Margin is defined as distance between separating hyperplane (decision boundary) & training examples that are closest to this hyperplane, which are so-called *support vectors*. Illustrated in Fig. 3.10: SVM maximizes margin between decision boundary & training data points.

* **Maximum margin intuition.** Rationale behind having decision boundaries with large margins: they tend to have a lower generalization error, whereas models with small margins are more prone to overfitting.

Unfortunately, while main intuition behind SVMs is relatively simple, mathematics behind them is quite advanced & would require sound knowledge of constrained optimization.

Hence, details behind maximum margin optimization SVMs are beyond scope of this book. However, recommend following resources if interested in learning more:

- CHRIS J.C. BURGESS's excellent explanation in *A Tutorial on Support Vector Machines for Pattern Recognition* (Data Mining & Knowledge Discovery, 2(2): 121-167, 1998)
- Vladimir Vapnik's book *The Nature of Statistical Learning Theory*, Springer Science+Business Media, Vladimir Vapnik, 2000
- ANDREW NG's very detailed lecture notes at <https://see.stanford.edu/materials/aimlcs229/cs229-notes3.pdf>.

* **Dealing with a nonlinearly separable case using slack variables.** (biến lỏng lẻo) Although don't want to dive much deeper into more involved mathematical concepts behind maximum-margin classification, briefly mention so-called *slack variable*, which was introduced by VLADIMIR VAPNIK in 1995 & led to so-called *soft-margin classification*. Motivation for introducing slack variable was: linear constraints in SVM optimization objective need to be relaxed for nonlinearly separable data to allow convergence of optimization in presence of misclassifications, under appropriate loss penalization.

Use of slack variable, in turn, introduces variable, which is commonly referred to as C in SVM contexts. Can consider C as a hyperparameter for controlling penalty for misclassification. Large values of C corresponds to large error penalties, whereas less strict about misclassification errors if choose smaller values for C . Can then use C parameter to control width of margin & therefore tune bias-variance tradeoff, as illustrated in Fig. 3.11: Impact of large & small values of inverse regularization strength C on classification.

This concept is related to regularization, discussed in prev sect in context of regularized regression, where decreasing value of C increases bias (underfitting) & lowers variance (overfitting) of model.

Have learned basic concepts behind a linear SVM, train an SVM mode to classify different flowers in Iris dataset:

```
>>> from sklearn.svm import SVC
>>> svm = SVC(kernel='linear', C=1.0, random_state=1)
>>> svm.fit(X_train_std, y_train)
>>> plot_decision_regions(X_combined_std, y_combined, classifier=svm, test_idx=range(105, 150))
>>> plt.xlabel('Petal length [standardized]')
>>> plt.ylabel('Petal width [standardized]')
>>> plt.legend(loc='upper left')
>>> plt.tight_layout()
>>> plt.show()
```

3 decision regions of SVM, visualized after training classifier on Iris dataset by executing preceding code example, are shown in Fig. 3.12: SVM's decision regions.

Remark 49 (Logistic regression vs. SVMs). *In practical classification tasks, linear logistic regression & linear SVMs often yield very similar results. Logistic regression tries to maximize conditional likelihoods of training data, which makes it more prone to outliers than SVMs, which mostly care about points that are closest to decision boundary (support vectors). On other hand, logistic regression has advantage of being a simpler model & can be implemented more easily, & is mathematically easier to explain. Furthermore, logistic regression models can be easily updated, which is attractive when working with streaming data.*

- * Alternative implementations in scikit-learn. scikit-learn library's `LogisticRegression` class can make use of LIBLINEAR library by setting `solver='liblinear'`. LIBLINEAR is a highly optimized C/C++ library developed at National Taiwan University <http://www.csie.ntu.edu.tw/~cjlin/liblinear/>.

Similar, SVC class used to train an SVM makes use of LIBSVM, which is an equivalent C/C++ library specialized for SVMs <http://www.csie.ntu.edu.tw/~cjlin/libsvm/>.

Advantage of using LIBLINEAR & LIBSVM over, e.g., native Python implementations: they allow extremely quick training of large amounts of linear classifiers. However, sometimes our datasets are too large to fit into computer memory. Thus, scikit-learn also offers alternative implementations via `SGDClassifier` class, which also supports online learning via `partial_fit` method. Concept behind `SGDClassifier` class is similar to stochastic gradient algorithm implemented in Chap. 2 for Adaline.

Could initialize SGD version of perceptron (`loss='perceptron'`), logistic regression (`loss='log'`), & an SVM with default parameters (`loss='hinge'`) (bản lề), as follows:

```
>>> from sklearn.linear_model import SGDClassifier
>>> ppn = SGDClassifier(loss='perceptron')
>>> lr = SGDClassifier(loss='log')
>>> svm = SGDClassifier(loss='hinge')
```

- Solving nonlinear problems using a kernel SVM. Another reason why SVMs enjoy high popularity among ML practitioners: they can be easily *kernelized* to solve nonlinear classification problems. Before discuss main concept behind so-called *kernel SVM*, most common variant of SVMs, 1st create a synthetic dataset to see what such a nonlinear classification problem may look like.
- * Kernel methods for linearly inseparable data.
- * Using kernel trick to find separating hyperplanes in a high-dimensional space.
- Decision tree learning.
- K-nearest neighbors – a lazy learning algorithm.
- Summary.
- Chap. 4: Building Good Training Datasets – Data Preprocessing.
 - Dealing with missing data.
 - Handling categorical data.
 - Partitioning a dataset into separate training & test datasets.
 - Bringing features onto same scale.
 - Selecting meaningful features.
 - Assessing feature importance with random forests.
 - Summary.
- Chap. 5: Compressing Data via Dimensionality Reduction.
 - Unsupervised dimensionality reduction via principal component analysis.
 - Supervised data compression via linear discriminant analysis.

- Nonlinear dimensionality reduction & visualization.
- Summary.
- Chap. 6. Learning Best Practices for Model Evaluation & Hyperparameter Tuning.
 - Streamlining workflows with pipelines.
 - Using k -fold cross-validation to assess model performance.
 - Debugging algorithms with learning & validation curves.
 - Fine-tuning ML models via grid search.
 - Looking at different performance evaluation metrics.
 - Summary.
- Chap. 7: Combining Different Models for Ensemble Learning.
 - Learning with ensembles.
 - Combining classifiers via majority vote.
 - Bagging – building an ensemble of classifiers from bootstrap samples.
 - Leveraging weak learners via adaptive boosting.
 - Gradient boosting – training an ensemble based on loss gradients.
 - Summary.
- Chap. 8: Applying ML to Sentiment Analysis.
 - Preparing IMDb movie review data for text processing.
 - Introducing bag-of-words model.
 - Training a logistic regression model for document classification.
 - Working with bigger data – online algorithms & out-of-core learning.
 - Topic modeling with latent Dirichlet allocation.
 - Summary.
- Chap. 9: Predicting Continuous Target Variables with Regression Analysis.
 - Introducing linear regression.
 - Exploring Ames Housing dataset.
 - Implementing an ordinary least squares linear regression model.
 - Fitting a robust regression model using RANSAC.
 - Evaluating performance of linear regression models.
 - Using regularized methods for regression.
 - Turning a linear regression model into a curve – polynomial regression.
 - Dealing with nonlinear relationships using random forests.
 - Summary.
- Chap. 10: Working with Unlabeled Data – Clustering Analysis.
 - Grouping objects by similarity using k -means.
 - Organizing clusters as a hierarchical tree.
 - Locating regions of high density via DBSCAN.
 - Summary.
- Chap. 11: Implementing a Multilayer Artificial Neural Network from Scratch. DL is getting a lot of attention from press & is, without doubt, hottest topic in ML field. DL can be understood as a subfield of ML that is concerned with training artificial neural networks (NNs) with many layers efficiently. In this chap, learn basic concepts of artificial NNs so that you are well equipped for following chaps, which will introduce advanced Python-based DL libraries & *deep neural network* (DNN) architectures that are particularly well suited for image & text analyses. Topics covered:
 - Gaining a conceptual understanding of multilayer NNs
 - Implementing fundamental backpropagation algorithm for NN training from scratch
 - Training a basic multilayer NN for image classification

- **Modeling complex functions with artificial neural networks.** At beginning of this book, started journey through ML algorithms with artificial neurons in Chap. 2. Artificial neurons represent building blocks of multilayer artificial NNs. Basic concept behind artificial NNs was built upon hypotheses & models of how human brain works to solve complex problem tasks. Although artificial NNs have gained a lot of popularity in recent years, early studies of NNs go back to 1940s, when WARREN McCULLOCH & WALTER PITTS 1st described how neurons could work. *A logical calculus of the ideas immanent in nervous activity* W. S. McCulloch & W. Pitts, The Bulletin of Mathematical Biophysics, 5(4):115–133, 1943.

However, in decades that followed 1st implementation of *McCulloch-Pitts neuron* model – Rosenblatt’s perceptron in 1950s – many researchers & ML practitioners slowly began to lose interest in NNs since no one had a good solution for training an NN with multiple layers. Eventually, interest in NNs was rekindled in 1986 when D.E. RUMELHART, G.E. HINTON, R.J. WILLIAMS were involved in (re)discovery & popularization of backpropagation algorithm to train NNs more efficiently: *Learning representations by backpropagating errors*, by D.E. RUMELHART, G.E. HINTON, & R.J. WILLIAMS, Nature, 323 (6088): 533–536, 1986. Readers who are interested in history of AI, ML, & NNs are also encouraged to read Wikipedia article on so-called *AI winters*, which are periods of time where a large portion of research community lost interest in study of NNs https://en.wikipedia.org/wiki/AI_winter.

However, NNs are more popular today than ever thanks to many breakthroughs that have been made in prev decade, which resulted in what now call DL algorithms & architectures – NNs that are composed of many layers. NNs are a hot topic not only in academic research but also in big technology companies, e.g. Facebook, Microsoft, Amazon, Uber, Google, & many more that invest heavily in artificial NNs & DL research.

As of today, complex NNs powered by DL algorithms are considered state-of-art solutions for complex problem solving e.g. image & voice recognition. Some of recent applications include:

- * Predicting COVID-19 resource needs from a series of X-rays <https://arxiv.org/abs/2101.04909>
- * Modeling virus mutations <https://science.sciencemag.org/content/371/6526/284>
- * Leveraging data from social media platforms to manage extreme weather events <https://onlinelibrary.wiley.com/doi/abs/10.1111/1468-5973.12311>
- * Improving photo descriptions for people who are blind or visually impaired <https://tech.fb.com/how-facebook-is-using-ai->
- * **Single-layer neural network recap.** This chap is all about multilayer NNs, how they work, & how to train them to solve complex problems. However, before dig deeper into a particular multilayer NN architecture, briefly reiterate some of concepts of single-layer NNs introduced in Chap. 2, namely, *ADaptive LInear NEuron (Adaline)* algorithm, shown in Fig. 11.1: Adaline algorithm.

In Chap. 2, implemented Adaline algorithm to perform binary classification, & used gradient descent optimization algorithm to learn weight coefficients of model. In every epoch (pass over training dataset), updated weight vector \mathbf{w} & bias unit b using following update rule: $\mathbf{w} := \mathbf{w} + \Delta\mathbf{w}, b := b + \Delta b$ where $\Delta w_i = -\eta \frac{\partial L}{\partial w_i}, \Delta b = -\eta \frac{\partial L}{\partial b}$ for bias unit & each weight w_i in weight vector \mathbf{w} .

I.e., computed gradient based on whole training dataset & updated weights of model by taking a step in opposite direction of loss gradient $\nabla L(\mathbf{w})$. (For simplicity, focus on weights & omit bias unit in following paragraphs; however, as remember from Chap. 2, same concepts apply.) In order to find optimal weights of model, optimized objective function defined as *mean of square errors (MSE)* loss function $L(\mathbf{w})$. Furthermore, multiplied gradient by a factor, *learning rate* η , which had to choose carefully to balance speed of learning against risk of overshooting global minimum of loss function.

In gradient descent optimization, updated all weights simultaneously after each epoch, & defined partial derivative for each weight w_i in weight vector \mathbf{w} as follows:

$$\frac{\partial L}{\partial w_j} \frac{1}{n} \sum_i (y^{(i)} - a^{(i)})^2 = -\frac{2}{n} \sum_i (y^{(i)} - a^{(i)}) x_j^{(i)}. \quad (228)$$

Here $y^{(i)}$: target class label of a particular sample $x^{(i)}$, & $a^{(i)}$: activation of neuron, which is a linear function in special case of Adaline.

Furthermore, defined activation function $\sigma(\cdot)$ as follows $\sigma(\cdot) = z = a$. Here, net input z is a linear combination of weights that are connecting input layer to output layer: $z = \sum_i w_i x_i + b = \mathbf{w}^\top \mathbf{x} + b$. While used activation $\sigma(\cdot)$ to compute gradient update, implemented a threshold function to squash continuous-valued output into binary class labels for prediction:

$$\hat{y} = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (229)$$

Remark 50 (Single-layer naming convention). *Although Adaline consists of 2 layer, 1 input layer & 1 output layer, it is called a single-layer network because of its single link between input & output layers.*

Also, learned about a certain *trick* to accelerate model learning, so-called *stochastic gradient-descent (SGD)* optimization. SGD approximates loss from a single training sample (online learning) or a small subset of training examples (mini-batch learning). Make use of this concept later in this chap when implement & train a *multilayer perceptron (MLP)*. Apart from faster learning – due to more frequent weight updates compared to gradient descent – its noisy nature is also regarded as beneficial when training multilayer NNs with nonlinear activation functions, which do not have a convex loss function. Here, added noise can help to escape local loss minima.

* **Introducing multilayer neural network architecture.** In this sect, learn how to connect multiple single neurons to a multilayer feedforward NN, this special type of *fully connected* network is also called *MLP*.

Fig. 11.2: A 2-layer MLP illustrates concept of an MLP consisting of 2 layers. Next to data input, MLP depicted in Fig. 11.2 has 1 hidden layer & 1 output layer. Units in hidden layer are fully connected to input features, & output layer is fully connected to hidden layer. If such a network has > 1 hidden layer, also call it a *deep NN*. (Note: in some contexts, inputs are also regarded as a layer. However, in this case, it would make Adaline model, which is a single-layer neural network, a 2-layer neural network, which may be counterintuitive.)

Remark 51 (Adding additional hidden layers). *Can add any number of hidden layers to MLP to create deeper network architectures. Practically, can think of number of layers & units in an NN as additional hyperparameters that we want to optimize for a given problem task using cross-validation technique, discussed in Chap. 6.*

However, loss gradients for updating network's parameters, which will calculate later via backpropagation, will become increasingly small as more layers are added to a network. This vanishing gradient problem makes model learning more challenging. Therefore, special algorithms have been developed to help train such DNN structures; known as deep learning.

As shown in Fig. 11.2, denote i th activation unit in l th layer as $a_i^{(l)}$. To make math & code implementations a bit more intuitive, will not use numerical indices to refer to layers, but will use *in* superscript for input features, h superscript for hidden layer, & *out* superscript for output layer. E.g., $x_i^{(in)}$ refers to i th input feature value, $a_i^{(h)}$ refers to i th unit in hidden layer, & $a_i^{(out)}$ refers to i th unit in output layer. Note: \mathbf{b} 's in Fig. 11.2 denote bias units. In fact, $\mathbf{b}^{(h)}$, $\mathbf{b}^{(out)}$ are vectors with number of elements = number of nodes in layer they correspond to. E.g., $\mathbf{b}^{(h)}$ stores d bias units, where d : number of nodes in hidden layer. If this sounds confusing, don't worry. Looking at code implementation later, where initialize weight matrices & bias unit vectors, will help clarify these concepts.

Each node in layer l is connected to all nodes in layer $l + 1$ via a weight coefficient. E.g., connection between k th unit in layer l to j th unit in layer $l + 1$ will be written as $w_{j,k}^{(l+1)}$. Referring back to Fig. 11.2, denote weight matrix that connects input to hidden layer as $\mathbf{W}^{(h)}$, & write matrix that connects hidden layer to output layer as $\mathbf{W}^{(out)}$.

While 1 unit in output layer would suffice for a binary classification task, saw a more general form of an NN in preceding figure, which allows us to perform multiclass classification via a generalization of *1-vs-all* (*OvA*) technique. To better understand how this works, remember *1-hot* representation of categorical variables introduced in Chap. 4.

E.g., can encode 3 class labels in familiar Iris dataset (0=Setosa, 1 = Versicolor, 2=Virginica) as follows:

$$0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad 1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad 2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (230)$$

This 1-hot vector representation allows us to tackle classification tasks with an arbitrary number of unique class labels present in training dataset.

If new to NN representations, indexing notation (subscripts & superscripts) may look a little bit confusing at 1st. What may seem overly complicated at 1st will make much more sense in later sects when vectorize NN representation. Summarize weights that connect input & hidden layers by a $d \times m$ dimensional matrix $\mathbf{W}^{(h)}$, where d : number of hidden units & m : number of input units.

* **Activating a neural network via forward propagation.** Describe process of *forward propagation* to calculate output of an MLP model. To understand how it fits into context of learning an MLP model, summarize MLP learning procedure in 3 simple steps:

1. Starting at input layer, forward propagate patterns of training data through network to generate an output.
2. Based on network's output, calculate loss that want to minimize using a loss function described later.
3. Backpropagate loss, find its derivative w.r.t. each weight & bias unit in network, & update model.

After repeat these 3 steps for multiple epochs & learn weight & bias parameters of MLP, use forward propagation to calculate network output & apply a threshold function to obtain predicted class labels in 1-hot representation, described in prev sect.

Walk through individual steps of forward propagation to generate an output from patterns in training data. Since each unit in hidden layer is connected to all units in input layers, 1st calculate activation unit of hidden layer $a_1^{(h)}$ as follows:

$$\begin{aligned} z_1^{(h)} &= x_1^{(in)}w_{1,1}^{(h)} + x_2^{(in)}w_{1,2}^{(h)} + \cdots + x_m^{(in)}w_{1,m}^{(h)}, \\ a_1^{(h)} &= \sigma(z_1^{(h)}). \end{aligned}$$

Here $z_1^{(h)}$: net input & $\sigma(\cdot)$: activation function, which has to be differentiable to learn weights that connect neurons using a gradient-based approach. To be able to solve complex problems e.g. image classification, need nonlinear activation functions in our MLP model, e.g., sigmoid (logistic) activation function that remember from sect about logistic regression in Chap. 3:

$$\sigma(z) = \frac{1}{1 + e^{-z}}. \quad (231)$$

Sigmoid function is an *S-shaped* curve that maps net input z onto a logistic distribution in range 0 to 1, which cuts y axis at $z = 0$, as shown in Fig. 11.3: Sigmoid activation function.

MLP is a typical example of a feedforward artificial NN. Term *feedforward* refers to fact: each layer serves as input to next layer without loops, in contrast to recurrent NNs – an architecture discussed in this chap & discussed in more detail in Chap. 15. Term *multilayer perceptron* may sound a little bit confusing since artificial neurons in this network architecture are typically sigmoid units, not perceptrons. Can think of neurons in MLP as logistic regression units that return values in continuous range between 0 & 1.

For purposes of code efficiency & readability, now write activation in a more compact form using concepts of basic linear algebra, which will allow us to vectorize our code implementation via NumPy rather than writing multiple nested & computationally expensive Python `for` loops:

$$\mathbf{z}^{(h)} = \mathbf{x}^{(in)} \mathbf{W}^{(h)\top} + \mathbf{b}^{(h)}, \quad (232)$$

$$a^{(h)} = \sigma(\mathbf{z}^{(h)}). \quad (233)$$

Here $\mathbf{x}^{(in)}$: our $1 \times m$ dimensional feature vector. $\mathbf{W}^{(h)}$: a $d \times m$ dimensional weight matrix where d : number of units in hidden layer; consequently, transposed matrix $\mathbf{W}^{(h)\top}$: $m \times d$ dimensional. Bias vector $\mathbf{b}^{(h)}$ consists of d bias units (1 bias unit per hidden node).

After matrix-vector multiplication, obtain $1 \times d$ dimensional net input vector $\mathbf{z}^{(h)}$ to calculate activation $\mathbf{a}^{(h)} \in \mathbb{R}^{1 \times d}$. Furthermore, can generalize this computation to all n examples in training dataset:

$$\mathbf{Z}^{(h)} = \mathbf{X}^{(in)} \mathbf{W}^{(h)\top} + \mathbf{b}^{(h)}. \quad (234)$$

Here $\mathbf{X}^{(in)}$ is now an $n \times m$ matrix, & matrix multiplication will result in an $n \times d$ dimensional net input matrix $\mathbf{Z}^{(h)}$. Finally, apply activation function $\sigma(\cdot)$ to each value in net input matrix to get $n \times d$ activation matrix in next layer (here, output layer):

$$\mathbf{A}^{(h)} = \sigma(\mathbf{Z}^{(h)}). \quad (235)$$

Similarly, can write activation of output layer in vectorized form for multiple examples:

$$\mathbf{Z}^{(out)} = \mathbf{A}^{(h)} \mathbf{W}^{(out)\top} + \mathbf{b}^{(out)}. \quad (236)$$

Here, multiply transpose of $t \times d$ matrix $\mathbf{W}^{(out)}$ (t : number of output units) by $n \times d$ dimensional matrix $\mathbf{A}^{(h)}$ & add t dimensional bias vector $\mathbf{b}^{(out)}$ to obtain $n \times t$ dimensional matrix $\mathbf{Z}^{(out)}$. (Rows in this matrix represent outputs for each example.)

Lastly, apply sigmoid activation function to obtain continuous-valued output of network:

$$\mathbf{A}^{(out)} = \sigma(\mathbf{Z}^{(out)}). \quad (237)$$

Similar to $\mathbf{Z}^{(out)}$, $\mathbf{A}^{(out)}$ is an $n \times t$ dimensional matrix.

- **Classifying handwritten digits.** In prev sect, covered a lot of theory around NNs, which can be a little bit overwhelming if new to this topic. Before continue with discussion of algorithm for learning weights of MLP model, backpropagation, take a short break from theory & see an NN in action.

Remark 52 (Additional resources on backpropagation). *NN theory can be quite complex; thus, want to provide readers with additional resources that cover some of topics discussed in this chap in more detail or from a different perspective:*

- * *Chap. 6: Deep Feedforward Networks, DL*, by I. GOODFELLOW, Y. BENGIO, A. COURVILLE, MIT Press, 2016 (manuscripts freely accessible at <http://www.deeplearningbook.org>).
- * *Pattern Recognition & ML* by C. M. BISHOP, Springer New York, 2006.
- * *Lecture video slides from SEBASTIAN RASCHKA's DL course*: <https://sebastianraschka.com/blog/2021/dl-course.html#l08-multinomial-logistic-regression--softmax-regression> <https://sebastianraschka.com/blog/2021/dl-course.html#l09-multilayer-perceptrons-and-backpropagation>

In this sect, implement & train 1st multilayer NN to classify handwritten digits from popular *Mixed National Institute of Standards & Technology (MNIST)* dataset that has been constructed by YANN LECUN & others & serves as a popular benchmark dataset for ML algorithms *Gradient-Based Learning Applied to Document Recognition* by Y. LECUN, L. BOTTOU, Y. BENGIO, & P. HAFFNER Proceedings of the IEEE, 86(11): 2278-2324, 1998.

- * **Obtaining & preparing MNIST dataset.** MNIST dataset is publicly available at <http://yann.lecun.com/exdb/mnist/> & consists of following 4 parts:

1. *Training dataset images*: `train-images-idx3-ubyte.gz` (9.9 MB, 47 MB unzipped, & 60,000 examples)
2. *Training dataset labels*: `train-labels-idx1-ubyte.gz` (29 KB, 60 KB unzipped, & 60,000 labels)
3. *Test dataset images*: `t10k-images-idx3-ubyte.gz` (1.6 MB, 7.8 MB unzipped, & 10,000 examples)
4. *Test dataset labels*: `t10k-labels-idx1-ubyte.gz` (5 KB, 10 KB unzipped, & 10,000 labels)

MNIST dataset was constructed from 2 datasets of US *National Institute of Standards & Technology (NIST)*. Training dataset consists of handwritten digits from 250 different people, 50% high school students, & 50% employees from Census Bureau. Note: test dataset contains handwritten digits from different people following same split.

Instead of downloading abovementioned dataset files & preprocessing them into NumPy arrays ourselves, use scikit-learn's new `fetch_openml` function, which allows us to load MNIST dataset more conveniently:

```
>>> from sklearn.datasets import fetch_openml
>>> X, y = fetch_openml('mnist_784', version=1, return_X_y=True)
>>> X = X.values
>>> y = y.astype(int).values
```

In scikit-learn, `fetch_openml` function downloads MNIST dataset from OpenML <https://www.openml.org/d/554> as pandas `DataFrame` & `Series` objects, which is why use `.values` attribute to obtain underlying NumPy arrays. (If using a scikit-learn version older than 1.0, `fetch_openml` downloads NumPy arrays directly so can omit using `.values` attribute.) $n \times m$ dimensional `X` array consists of 70000 images with 784 pixels each, & `y` array stores corresponding 70000 class labels, which can confirm by checking dimensions of arrays as follows:

```
>>> print(X.shape)
(70000, 784)
>>> print(y.shape)
(70000,)
```

Images in MNIST dataset consist of 28×28 pixels, & each pixel is represented by a grayscale intensity value. Here `fetch_openml` already unrolled 28×28 pixels into 1D row vectors, which represent rows in our `X` array (784 per row or image) above. 2nd array `y` returned by `fetch_openml` function contains corresponding target variable, & class labels (integers 0-9) of handwritten digits.

Next, normalize pixels values in MNIST to range -1 to 1 (originally 0 to 255) via following code line:

```
>>> X = ((X / 255.) - .5) * 2
```

Reason behind this: gradient-based optimization is much more stable under these conditions, as discussed in Chap. 2. Note: scaled images on a pixel-by-pixel basis, which is different from feature-scaling approach that took in prev chaps. Previously, derived scaling parameters from training dataset & used these to scale each column in training dataset & test dataset. However, when working with image pixels, centering them at 0 & rescaling them to a $[-1, 1]$ range is also common & usually works well in practice. To get an idea of how those images in MNIST look, visualize examples of digits 0-9 after reshaping 784-pixel vectors from our feature matrix into original 28×28 image that we can plot via Matplotlib's `imshow` function:

```
>>> import matplotlib.pyplot as plt
>>> fig, ax = plt.subplots(nrows=2, ncols=5, sharex=True, sharey=True)
>>> ax = ax.flatten()
>>> for i in range(10):
...     img = X[y == i][0].reshape(28, 28)
...     ax[i].imshow(img, cmap='Greys')
>>> ax[0].set_xticks([])
>>> ax[0].set_yticks([])
>>> plt.tight_layout()
>>> plt.show()
```

Should now see a plot of 2×5 subfigures showing a representative image of each unique digit: Fig. 11.4: A plot showing 1 randomly chosen handwritten digit from each class.

In addition, also plot multiple examples of same digit to see how different handwriting for each really is:

```
>>> fig, ax = plt.subplots(nrows=5, ncols=5, sharex=True, sharey=True)
>>> ax = ax.flatten()
>>> for i in range(25):
...     img = X[y == 7][i].reshape(28, 28)
...     ax[i].imshow(img, cmap='Greys')
>>> ax[0].set_xticks([])
>>> ax[0].set_yticks([])
>>> plt.tight_layout()
>>> plt.show()
```

After executing code, should see 1st 25 variants of digit 7: Fig. 11.5: Different variants of handwritten digit 7.

Finally, divide dataset into training, validation, & test subsets. Following code will split dataset s.t. 55000 images are used for training, 5000 images for validation, & 10000 images for testing:

```
>>> from sklearn.model_selection import train_test_split
>>> X_temp, X_test, y_temp, y_test = train_test_split(X, y, test_size=10000, random_state=123, stratify=y)
>>> X_train, X_valid, y_train, y_valid = train_test_split(X_temp, y_temp, test_size=5000, random_state=123, stratify=y_temp)
```

* Implementing a multilayer perceptron. Now implement an MLP from scratch to classify images in MNIST dataset. To keep things simple, implement an MLP with only 1 hidden layer. Since approach may seem a little bit complicated at 1st,

encouraged to download sample code for this chap from Packt Publishing website or from GitHub <https://github.com/rasbt/machine-learning-book> so that can view this MLP implementation annotated with comments & syntax highlighting for better readability.

If not running code from accompanying Jupyter Notebook file or don't have access to internet, copy `NeuralNetMLP` code from this chap into a Python script file in current working directory (e.g., `neuralnet.py`), which can then import into current Python session via following command:

```
from neuralnet import NeuralNetMLP
```

Code will contain parts that we have not talked about yet, e.g. backpropagation algorithm. Do not worry if not all code makes immediate sense; will follow up on certain parts later in this chap. However, going over code at this stage can make it easier to follow theory later.

Look at following implementation of an MLP, starting with 2 helper functions to compute logistic sigmoid activation & to convert integer class label arrays to 1-hot encoded labels:

```
import numpy as np
def sigmoid(z):
    return 1. / (1. + np.exp(-z))
def int_to_onehot(y, num_labels):
    ary = np.zeros((y.shape[0], num_labels))
    for i, val in enumerate(y):
        ary[i, val] = 1
    return ary
```

Below, implement main class for our MLP, called `NeuralNetMLP`. There are 3 class methods `__init__()`, `forward()`, `backward()` will be discussed 1 by 1, starting with `__init__` constructor [code]. `__init__` constructor instantiates weight matrices & bias vectors for hidden & output layer. Next, see how these are used in `forward` method to make predictions:

```
def forward(self, x):
    # Hidden layer
    # input dim: [n_examples, n_features] dot [n_hidden, n_features].T
    # output dim: [n_examples, n_hidden]
    z_h = np.dot(x, self.weight_h.T) + self.bias_h
    a_h = sigmoid(z_h)

    # Output layer
    # input dim: [n_examples, n_hidden] dot [n_classes, n_hidden].T
    # output dim: [n_examples, n_classes]
    z_out = np.dot(a_h, self.weight_out.T) + self.bias_out
    a_out = sigmoid(z_out)
    return a_h, a_out
```

`forward` method takes in 1 or more training examples & returns predictions. In fact, it returns both activation values from hidden layer & output layer `a_h, a_out`. While `a_out` represents class-membership probabilities that can convert to class labels, which we care about, also need activation values from hidden layer `a_h` to optimize model parameters; i.e., weight & bias units of hidden & output layers.

Finally, talk about `backward` method, which updates weight & bias parameters of neural network [code].

`backward` method implements so-called *backpropagation* algorithm, which calculates gradients of loss w.r.t. weight & bias parameters. Similar to Adaline, these gradients are then used to update these parameters via gradient descent. Note: multilayer NNs are more complex than their single-layer siblings, & will go over mathematical concepts of how to compute gradients in a later sect after discussing code. For now, just consider `backward` method as a way for computing gradients that are used for gradient descent updates. For simplicity, loss function this derivation is based on is same MSE loss used in Adaline. In later chaps, look at alternative loss functions, e.g. multi-category cross-entropy loss, which is a generalization of binary logistic regression loss to multiple classes.

Looking at this code implementation of `NeuralNetMLP` class, may have noticed: this object-oriented implementation differs from familiar scikit-learn API that is centered around `.fit()`, `.predict()` methods. Instead, main methods of `NeuralNetMLP` class are `.forward()`, `.backward()` methods. 1 of reasons behind this: it makes a complex neural network a bit easier to understand in terms of how information flows through networks.

Another reason: this implementation is relatively similar to how more advanced DL libraries e.g. PyTorch operate, which will introduce & use in upcoming chaps to implement more complex neural networks.

After have implemented `NeuralNetMLP` class, use following code to instantiate a new `NeuralNetMLP` object:

```
>>> model = NeuralNetMLP(num_features=28*28, num_hidden=50, num_classes=10)
```

`model` accepts MNIST images reshaped into 784-dimensional vectors (in format of `X_train`, `X_valid`, `X_test`, defined previously) for 10 integer classes (digits 0-9). Hidden layer consists of 50 nodes. Also, as may be able to tell from looking

at previously defined `.forward()` method, use a sigmoid activation function after 1st hidden layer & output layer to keep things simple. In later chaps, learn about alternative activation functions for both hidden & output layers.

Fig. 11.6: NN architecture for labeling handwritten digits summarizes neural network architecture instantiated above. In next subsect, implement training function that we can use to train network on mini-batches (lô nhỏ) of data via backpropagation.

- * Coding neural network training loop.
- * Evaluating neural network performance.
- Training an artificial neural network.
 - * Computing loss function.
 - * Developing your understanding of backpropagation.
 - * Training neural networks via backpropagation.
- About convergence in neural networks.
- A few last words about neural network implementation.
- Summary. In this chap, have learned basic concepts behind multilayer artificial NNs – currently hottest topic in ML research. In Chap. 2, started our journey with simple single-layer NN structures & now have connected multiple neurons to a powerful NN architecture to solve complex problems e.g. handwritten digit recognition. Demystified popular backpropagation algorithm, which is 1 of building blocks of many NN models used in DL. After learning about backpropagation algorithm in this chap, we are well equipped for exploring more complex DNN architectures. In remaining chaps, will cover more advanced DL concepts & PyTorch, an open source library that allows us to implement & train multilayer NNs more efficiently.
- Chap. 12: Parallelizing Neural Network Training with PyTorch. Move on from mathematical foundations of ML & DL focus on PyTorch. PyTorch is 1 of most popular DL libraries currently available, & it lets us implement *neural networks (NNs)* much more efficiently than any of our prev NumPy implementations. In this chap, start using PyTorch & see how it brings significant benefits to training performance.

This chap will begin next stage of our journey into ML & DL, & will explore following topics:

- How PyTorch improves training performance
- Working with PyTorch's `Dataset`, `DataLoader` to build input pipelines & enable efficient model training
- Working with PyTorch to write optimized ML code
- Using `torch.nn` module to implement common DL architectures conveniently
- Choosing activation functions for artificial NNs
- PyTorch & training performance. PyTorch can speed up our ML tasks significantly. To understand how it can do this, begin by discussing some of performance challenges typically run into when execute expensive calculations on our hardware. Then, take a high-level look at what PyTorch is & what our learning approach will be in this chap.
- * Performance challenges. Performance of computer processors has, of course, been continuously improving in recent years. That allows us to train more powerful & complex learning systems, i.e., can improve predictive performance of our ML models. Even cheapest desktop computer hardware that's available right now comes with processing units that have multiple cores.

In prev chaps, saw: many functions in scikit-learn allow us to spread those computations over multiple processing units. However, by default, Python is limited to execution on 1 core due to *global interpreter lock (GIL)*. So, although indeed take advantage of Python's multiprocessing library to distribute our computations over multiple cores, still have to consider: most advanced desktop hardware rarely comes with > 8 or 16 such cores.

Recall from Chap. 11: implemented a very simple *multilayer perceptron (MLP)* with only 1 hidden layer consisting of 100 units. Had to optimize ≈ 80000 weight parameters $[784 \cdot 100 + 100] + [100 \cdot 10] + 10 = 79510$ for a very simple image classification task. Images in MNIST are rather small 28×28 , & can only imagine explosion in number of parameters if wanted to add additional hidden layers or work with images that have higher pixel densities. Such a task would quickly become unfeasible (không khả thi) for a single processing unit. Question then becomes, how can we tackle such problems more effectively.

Obvious solution to this problem: use *graphics processing units (GPUs)*, which are real workhorses. Can think of a graphics card as a small computer cluster inside your machine. Another advantage: modern GPUs are great value compared to state-of-the-art *central processing units (CPUs)*, as you can see in following overview: Fig. 12.1: Comparison of a state-of-the-art CPU & GPU. Sources for information in Fig. 12.1 are following websites:

- <https://ark.intel.com/content/www/us/en/ark/products/215570/intel-core-i9-11900kb-processor-24m-cache-up-to-65w.html>
- <https://www.nvidia.com/en-us/geforce/graphics-cards/30-series/rtx-3080-3080ti/>

At 2.2 times price of a modern CPU, can get a GPU that has 640 times more cores & is capable of around 46 times more floating-point calculations per sec. So, what is holding us back from utilizing GPUs for our ML tasks? Challenge: writing code to target GPUs is not as simple as executing Python code in our interpreter. There are special packages, e.g. CUDA & OpenCL, that allow us to target GPU. However, writing code in CUDA or OpenCL is probably not most convenient way to implement & run ML algorithms. Good news: this is what PyTorch was developed for!

- * **What is PyTorch?** PyTorch is a scalable & multiplatform programming interface for implementing & running ML algorithms, including convenience wrappers for DL. PyTorch was primarily developed by researchers & engineers from *Facebook AI Research (FAIR)* lab. Its development also involves many contributions from community. PyTorch was initiated released in Sep 2016 & is free & open source under modified BSD license. Many ML researchers & practitioners from academia & industry have adapted PyTorch to develop DL solutions, e.g. Tesla Autopilot, Uber's Pyro, & Hugging Face's Transformers <https://pytorch.org/ecosystem/>.

To improve performance of training ML models, PyTorch allows execution on CPUs, GPUs, & XLA devices e.g. TPUs. However, its greatest performance capabilities can be discovered when using GPUs & XLA devices. PyTorch supports CUDA-enabled & ROCm GPUs officially. PyTorch's development is based on Torch library www.torch.ch. As its name implies, Python interface is primary development focus of PyTorch.

PyTorch is built around a computation graph composed of a set of nodes. Each node represents an operation that may have 0 or more inputs or outputs. PyTorch provides an imperative programming environment that evaluates operations, executes computation, & returns concrete values immediately. Hence, computation graph in PyTorch is defined implicitly, rather than constructed in advance & executed after.

Mathematically, tensors can be understood as a generalization of scalars, vectors, matrices, & so on. More concretely, a scalar can be defined as a rank-0 tensor, a vector can be defined as a rank-1 tensor, a matrix can be defined as a rank-2 tensor, & matrices stacked in a 3rd dimension can be defined as rank-3 tensors. Tensors in PyTorch are similar to NumPy's arrays, except: tensors are optimized for automatic differentiation & can run on GPUs.

To make concept of a tensor clearer, consider Fig. 12.2: Different types of tensor in PyTorch, which represents tensors of ranks 0 & 1 in 1st row, & tensors of ranks 2 & 3 in 2nd row. Now know what PyTorch is, see how to use it.

- * **How we will learn PyTorch.** 1st, cover PyTorch's programming model, in particular, creating & manipulating tensors. Then, see how to load data & utilize `torch.utils.data` module, which will allow us to iterate through a dataset efficiently. In addition, discuss existing, ready-to-use datasets in `torch.utils.data.Dataset` submodule & learn how to use them. After learning about these basics, PyTorch neural network `torch.nn` module will be introduced. Then, move forward to building ML models, learn how to compose & train models, & learn how to save trained models on disk for future evaluation.
- o **1st steps with PyTorch.** Take 1st steps in using low-level PyTorch API. After installing PyTorch, cover how to create tensors in PyTorch & different ways of manipulating them, e.g. changing their shape, data type, & so on.
- * **Installing PyTorch.** To install PyTorch, recommend consulting latest instructions on official <https://pytorch.org> website. Below, outline basic steps that will work on most systems. Depending on how your system is set up, can typically just use Python's `pip` installer & install PyTorch from PyPI by executing following from terminal:

```
pip install torch torchvision
```

This will install latest *stable* version, which is 1.9.0 at time of writing. To install 1.9.0 version, which is guaranteed to be compatible with following code examples, can modify preceding command as follows:

```
pip install torch==1.9.0 torchvision==0.10.0
```

If want to use GPUs (recommended), need a compatible NVIDIA graphics card that supports CUDA & cuDNN. If machine satisfies these requirements, can install PyTorch with GPU support, as follows:

```
pip install torch==1.9.0+cu111 torchvision==0.10.0+cu111 -f https://download.pytorch.org/whl/torch_stable.html
```

for CUDA 11.1 or:

```
pip install torch==1.9.0 torchvision==0.10.0\
whl/torch_stable.html
```

for CUDA 10.2 as of time of writing.

As macOS binaries don't support CUDA, can install from source <https://pytorch.org/get-started/locally/#mac-from-sources>. For more information about installation & setup process, see official recommendations at <https://pytorch.org/get-started/locally/>.

Note: PyTorch is under active development; therefore, every couple of months, new versions are released with significant changes. Can verify PyTorch version from your terminal as follows:

```
python -c 'import torch; print(torch.__version__)'
```

Remark 53 (Troubleshooting installation of PyTorch). *If experience problems with installation procedure, read more about system- & platform-specific recommendations provided at <https://pytorch.org/get-started/locally/>. Note: all code in this chap can be run on your CPU; using a GPU is entirely optional but recommended if want to fully enjoy benefits of PyTorch. E.g., while training some NN models on a CPU could take a week, same models could be trained in just a few hours on a modern GPU. If have a graphics card, refer to installation page to set it up appropriately. In addition, may find this setup guide helpful, which explains how to install NVIDIA graphics card drivers, CUDA, & cuDNN on Ubuntu (not required but recommended requirements for running PyTorch on a GPU): <https://sebastianraschka.com/requirements/01-installing-driver/>.*

- * Creating tensors in PyTorch. Consider a few different ways of creating tensors, & then see some of their properties & how to manipulate them. 1stly, can simply create a tensor from a list or a NumPy array using `torch.tensor` or `torch.from_numpy` function as follows:

```
>>> import torch
>>> import numpy as np
>>> np.set_printoptions(precision=3)
>>> a = [1, 2, 3]
>>> b = np.array([4, 5, 6], dtype=np.int32)
>>> t_a = torch.tensor(a)
>>> t_b = torch.from_numpy(b)
>>> print(t_a)
tensor([1, 2, 3])
>>> print(t_b)
tensor([4, 5, 6], dtype=torch.int32)
```

This resulted in tensors `t_a`, `t_b`, with their properties, `shape=(3,)` & `dtype=int32`, adopted from their source. Similar to NumPy arrays, can also see these properties:

```
>>> t_ones = torch.ones(2, 3)
>>> t_ones.shape
torch.Size([2, 3])
>>> print(t_ones)
tensor([[1., 1., 1.],
        [1., 1., 1.]])
```

Finally, creating a tensor of random values can be done as follows:

```
>>> rand_tensor = torch.rand(2,3)
>>> print(rand_tensor)
tensor([[0.1409, 0.2848, 0.8914],
        [0.9223, 0.2924, 0.7889]])
```

- * Manipulating data type & shape of a tensor. Learning ways to manipulate tensors is necessary to make them compatible for input to a model or an operation. In this sect, learn how to manipulate tensor data types & shapes via several PyTorch functions that cast, reshape, transpose, & squeeze (remove dimensions).

`torch.to()` function can be used to change data type of a tensor to a desired type:

```
>>> t_a_new = t_a.to(torch.int64)
>>> print(t_a_new.dtype)
torch.int64
```

See https://pytorch.org/docs/stable/tensor_attributes.html for all other data types.

Certain operations require: input tensors have a certain number of dimensions (i.e., rank) associated with a certain number of elements (shape). Thus, might need to change shape of a tensor, add a new dimension, or squeeze an unnecessary dimension. PyTorch provides useful functions (or operations) to achieve this, e.g. `torch.transpose()`, `torch.reshape()`, `torch.squeeze()`. Some examples:

- Transposing a tensor:

```
>>> t = torch.rand(3, 5)
>>> t_tr = torch.transpose(t, 0, 1)
>>> print(t.shape, ' --> ', t_tr.shape)
torch.Size([3, 5]) --> torch.Size([5, 3])
```

- Reshaping a tensor (e.g., from a 1D vector to a 2D array):

```
>>> t = torch.zeros(30)
>>> t_reshape = t.reshape(5, 6)
>>> print(t_reshape.shape)
torch.Size([5, 6])
```

- Removing unnecessary dimensions (dimensions that have size 1, which are not needed):

```
>>> t = torch.zeros(1, 2, 1, 4, 1)
>>> t_sqz = torch.squeeze(t, 2)
>>> print(t.shape, ' --> ', t_sqz.shape)
torch.Size([1, 2, 1, 4, 1]) --> torch.Size([1, 2, 4, 1])
```


- * **Applying mathematical operations to tensors.** Applying mathematical operations, in particular linear algebra operations, is necessary for building most ML models. In this subject, will cover some widely used linear algebra operations, e.g. element-wise product, matrix multiplication, & computing norm of a tensor.

1st, instantiate 2 random tensors, one with uniform distribution in range $[-1, 1]$ & the other with a standard normal distribution:

```
>>> torch.manual_seed(1)
>>> t1 = 2 * torch.rand(5, 2) - 1
>>> t2 = torch.normal(mean=0, std=1, size=(5, 2))
```

Note: `torch.rand` returns a tensor filled with a random numbers from a uniform distribution in range of $[0, 1)$.

Notice: `t1`, `t2` have same shape. To compute element-wise product of `t1`, `t2`, can use following:

```
>>> t3 = torch.multiply(t1, t2)
>>> print(t3)
tensor([[ 0.4426, -0.3114],
        [ 0.0660, -0.5970],
        [ 1.1249,  0.0150],
        [ 0.1569,  0.7107],
        [-0.0451, -0.0352]])
```

To compute mean, sum, & standard derivation along a certain axis (or axes), can use `torch.mean()`, `torch.sum()`, `torch.std()`. E.g., mean of each column in `t1` can be computed as follows:

```
>>> t4 = torch.mean(t1, axis=0)
>>> print(t4)
tensor([-0.1373,  0.2028])
```

Matrix-matrix product between `t1`, `t2` (i.e., $t_1 \times t_2^T$, where superscript T is for transpose) can be computed using `torch.matmul()` function as follows:

```
>>> t5 = torch.matmul(t1, torch.transpose(t2, 0, 1))
>>> print(t5)
```

Computing $t_1^T \times t_2$ is performed by transposing `t1`, resulting in an array of size 2×2 :

```
>>> t6 = torch.matmul(torch.transpose(t1, 0, 1), t2)
>>> print(t6)
tensor([[ 1.7453,  0.3392],
        [-1.6038, -0.2180]])
```

Finally, `torch.linalg.norm()` function is useful for computing L^p norm of a tensor. E.g., can calculate L^2 norm of `t1` as follows:

```
>>> norm_t1 = torch.linalg.norm(t1, ord=2, dim=1)
>>> print(norm_t1)
tensor([0.6785, 0.5078, 1.1162, 0.5488, 0.1853])
```

To verify: this code snippet computes L^2 norm of `t1` correctly, can compare results with following NumPy function `np.sqrt(np.sum(np.square(t1.numpy()), axis=1))`.

- * **Split, stack, & concatenate tensors.** In this subject, will cover PyTorch operations for splitting a tensor into multiple tensors, or reverse: stacking & concatenating multiple tensors into a single one.

Assume: have a single tensor, & want to split it into 2 or more tensors. For this, PyTorch provides a convenient `torch.chunk()` function, which divides an input tensor into a list of equally sized tensors. Can determine desired number of splits as an integer using `chunks` argument to split a tensor along desired dimension specified by `dim` argument. In this case, total size of input tensor along specified dimension must be divisible by desired number of splits. Alternatively, can provide desired sizes in a list using `torch.split()` function. Have a look at an example of both these options:

- Provide number of splits:

```
>>> torch.manual_seed(1)
>>> t = torch.rand(6)
>>> print(t)
tensor([0.7576, 0.2793, 0.4031, 0.7347, 0.0293, 0.7999])
>>> t_splits = torch.chunk(t, 3)
>>> [item.numpy() for item in t_splits]
[array([0.758, 0.279], dtype=float32),
 array([0.403, 0.735], dtype=float32),
 array([0.029, 0.799], dtype=float32)]
```

```
array([0.029, 0.8 ], dtype=float32)]
```

In this example, a tensor of size 6 was divided into a list of 3 tensors each with size 2. If tensor size is not divisible by `chunks` value, last chunk will be smaller.

· Providing sizes of different splits: Alternatively, instead of defining number of splits, can also specify sizes of output tensors directly. Here, splitting a tensor of size 5 into tensors of sizes 3 & 2:

```
>>> torch.manual_seed(1)
>>> t = torch.rand(5)
>>> print(t)
tensor([0.7576, 0.2793, 0.4031, 0.7347, 0.0293])
>>> t_splits = torch.split(t, split_size_or_sections=[3, 2])
>>> [item.numpy() for item in t_splits]
[array([0.758, 0.279, 0.403], dtype=float32),
 array([0.735, 0.029], dtype=float32)]
```

Sometimes, working with multiple tensors & need to concatenate or stack them to create a single tensor. In this case, PyTorch functions e.g. `torch.stack()`, `torch.cat()` come in handy. E.g., create a 1D tensor A, containing 1s with size 3, & a 1D tensor B, containing 0s with size 2, & concatenate them into a 1D tensor C of size 5:

```
>>> A = torch.ones(3)
>>> B = torch.zeros(2)
>>> C = torch.cat([A, B], axis=0)
>>> print(C)
tensor([1., 1., 1., 0., 0.])
```

If create 1D tensors A, B, both with size 3, then can stack them together to form a 2D tensor S:

```
>>> A = torch.ones(3)
>>> B = torch.zeros(3)
>>> S = torch.stack([A, B], axis=1)
>>> print(S)
tensor([[1., 0.],
        [1., 0.],
        [1., 0.]])
```

PyTorch API has many operations that can use for building a model, processing data, & more. However, covering every function is outside scope of book, where will focus on most essential ones. For full list of operations & functions, can refer to documentation page of PyTorch at <https://pytorch.org/docs/stable/index.html>.

- Building input pipelines in PyTorch. When training a deep NN model, usually train model incrementally using an iterative optimization algorithm e.g. stochastic gradient descent, as have seen in prev chaps.

`torch.nn` is a module for building NN models. In cases where training dataset is rather small & can be loaded as a tensor into memory, can directly use this tensor for training. In typical use cases, however, when dataset is too large to fit into computer memory, will need to load data from main storage device (e.g., hard drive or solid-state drive) in chunks, i.e., batch by batch. (Note: use of term “batch” instead of “mini-batch” in this chap to stay close to PyTorch terminology.) In addition, may need to construct a data-processing pipeline to apply certain transformations & preprocessing steps to our data, e.g. mean centering, scaling, or adding noise to augment training procedure & to prevent overfitting.

Applying preprocessing functions manually every time can be quite cumbersome. Luckily, PyTorch provides a special class for constructing efficient & convenient preprocessing pipelines. In this sect, see an overview of different methods for constructing a PyTorch `Dataset`, `DataLoader` & implementing data loading shuffling, & batching.

- * Creating a PyTorch `DataLoader` from existing tensors. If data already exists in form of a tensor object, a Python list, or a NumPy array, can easily create a dataset loader using `torch.utils.data.DataLoader()` class. It returns an object of `DataLoader` class, which can use to iterate through individual elements in input dataset. As a simple example, consider following code, which creates a dataset from a list of values from 0 to 5:

```
>>> from torch.utils.data import DataLoader
>>> t = torch.arange(6, dtype=torch.float32)
>>> data_loader = DataLoader(t)
```

Can easily iterate through a dataset entry by entry as follows:

```
>>> for item in data_loader:
...     print(item)
tensor([0.])
tensor([1.])
tensor([2.])
```

```

tensor([3.])
tensor([4.])
tensor([5.])

```

If want to create batches from this dataset, with a desired batch size of 3, can do this with `batch_size` argument as follows:

```

>>> data_loader = DataLoader(t, batch_size=3, drop_last=False)
>>> for i, batch in enumerate(data_loader, 1):
...     print(f'batch {i}:', batch)
batch 1: tensor([0., 1., 2.])
batch 2: tensor([3., 4., 5.])

```

This will create 2 batches from this dataset, where 1st 3 elements go into batch #1, & remaining elements go into batch #2. Optimal `drop_last` argument is useful for cases when number of elements in tensor is not divisible by desired batch size. Can drop last non-full batch by setting `drop_last` to `True`. Default value for `drop_last` is `False`.

Can always iterate through a dataset directly, but as just saw, `DataLoader` provides an automatic & customizable batching to a dataset.

- * **Combining 2 tensors into a joint dataset.** Often, may have data in 2 (or possibly more) tensors. E.g., could have a tensor for features & a tensor for labels. In such cases, need to build a dataset that combines these tensors, which will allow us to retrieve elements of these tensors in tuples.

Assume: have 2 tensors `t_x, t_y`. Tensor `t_x` holds our feature values, each of size 3, & `t_y` stores class labels. For this example, 1st create these 2 tensors as follows:

```

>>> torch.manual_seed(1)
>>> t_x = torch.rand([4, 3], dtype=torch.float32)
>>> t_y = torch.arange(4)

```

Now, want to create a joint dataset from these 2 tensors. 1st need to create a `Dataset` class as follows: [code].

A custom `Dataset` class must contain following methods to be used by data loader later on:

- `__init__()`: This is where initial logic happens, e.g. reading existing arrays, loading a file, filtering data, & so forth.
- `__getitem__()`: This returns corresponding sample to given index.

Then create a joint dataset of `t_x, t_y` with custom `Dataset` class as follows:

```

>>> from torch.utils.data import TensorDataset
>>> joint_dataset = TensorDataset(t_x, t_y)

```

Finally, can print each example of joint dataset as follows:

```

>>> for example in joint_dataset:
...     print('x: ', example[0], ' y: ', example[1])

```

Can also simply utilize `torch.utils.data.TensorDataset` class, if 2nd dataset is a labeled dataset in form of tensors. So, instead of using our self-defined `Dataset` class, `JointDataset`, can create a joint dataset as follows:

```

>>> joint_dataset = TensorDataset(t_x, t_y)

```

Note: a common source of error could be: element-wise correspondence between original features x & labels y might be lost (e.g., if 2 datasets are shuffled separately). However, once they are merged into 1 dataset, safe to apply these operations. If have a dataset created from list of image filenames on disk, can define a function to load images from these filenames. See an example of applying multiple transformations to a dataset later in this chap.

- * **Shuffle, batch, & repeat.** As was mentioned in Chap. 2, when training an NN model using stochastic gradient descent optimization, important to feed training data as randomly shuffled batches. Have already seen how to specify batch size using `batch_size` argument of a data loader object. Now, in addition to creating batches, will see how to shuffle & reiterate over datasets. Will continue working with prev joint dataset.

1st, create a shuffled version data loader from `joint_dataset` dataset:

```

>>> torch.manual_seed(1)
>>> data_loader = DataLoader(dataset=joint_dataset, batch_size=2, shuffle=True)

```

Here, each batch contains 2 data records x & corresponding labels y . Now iterate through data loader entry by entry as follows:

```

>>> for i, batch in enumerate(data_loader, 1):
...     print(f'batch {i}:', 'x:', batch[0], '\n y:', batch[1])

```

Rows are shuffled without losing 1-to-1 correspondence between entries in x , y .

In addition, when training a model for multiple epochs, need to shuffle & iterate over dataset by desired number of epochs. So, iterate over batched dataset twice:

```
>>> for epoch in range(2):
>>>     print(f'epoch {epoch+1}')
>>>     for i, batch in enumerate(data_loader, 1):
...         print(f'batch {i}:', 'x:', batch[0],
'\n          y:', batch[1])
```

This results in 2 different sets of batches. In 1st epoch, 1st batch contains a pair of values [y=1, y=2], & 2nd batch contains a pair of values [y=3, y=0]. In 2nd epoch, 2 batches contain a pair of values, [y=2, y=0] & [y=1, y=3] resp. For each iteration, elements within a batch are also shuffled.

- * Creating a dataset from files on local storage disk. Will build a dataset from image files stored on disk. There is an image folder associated with online content of this chap. After downloading folder, should be able to see 6 images of cats & dogs in JPEG format.

This small dataset will show how building a dataset from stored files generally works. To accomplish this, going to use 2 additional modules: `Image` in `PIL` to read image file contents & `transforms` in `torchvision` to decode raw contents & resize images.

Remark 54. `PIL.Image` & `torchvision.transforms` modules provide a lot of additional & useful functions, which are beyond scope of book. Encouraged to browse through official documentation to learn more about these functions:

- <https://pillow.readthedocs.io/en/stable/reference/Image.html> for `PIL.Image`
- <https://pytorch.org/vision/stable/transforms.html> for `torchvision.transforms`.

Before start, take a look at content of these files. use `pathlib` library to generate a list of image files:

```
>>> import pathlib
>>> imgdir_path = pathlib.Path('cat_dog_images')
>>> file_list = sorted([str(path) for path in
... imgdir_path.glob('*.jpg')])
>>> print(file_list)
['cat_dog_images/dog-03.jpg', 'cat_dog_images/cat-01.jpg', 'cat_dog_images/cat-
02.jpg', 'cat_dog_images/cat-03.jpg', 'cat_dog_images/dog-01.jpg', 'cat_dog_
images/dog-02.jpg']
```

Next, visualize these image examples using `Matplotlib`:

```
>>> import matplotlib.pyplot as plt
>>> import os
>>> from PIL import Image
>>> fig = plt.figure(figsize=(10, 5))
>>> for i, file in enumerate(file_list):
...     img = Image.open(file)
...     print('Image shape:', np.array(img).shape)
...     ax = fig.add_subplot(2, 3, i+1)
...     ax.set_xticks([]); ax.set_yticks([])
...     ax.imshow(img)
...     ax.set_title(os.path.basename(file), size=15)
>>> plt.tight_layout()
>>> plt.show()
Image shape: (900, 1200, 3)
Image shape: (900, 1200, 3)
Image shape: (900, 1200, 3)
Image shape: (900, 742, 3)
Image shape: (800, 1200, 3)
Image shape: (800, 1200, 3)
```

Fig. 12.3: Images of cats & dogs show example images.

TECHNICAL p. 383: COME BACK LATER WHEN NEEDED

- * Fetching available datasets from `torchvision.datasets` library.
- Building an NN model in `PyTorch`.
 - * `PyTorch` neural network module `torch.nn`.
 - * Building a linear regression model.
 - * Model training via `torch.nn` & `torch.optim` modules.
 - * Building a multilayer perceptron for classifying flowers in Iris dataset.
 - * Evaluating trained model on test dataset.
 - * Saving & reloading trained model.
- Choosing activation functions for multilayer neural networks.

- * Logistic function recap.
- * Estimating class probabilities in multiclass classification via softmax function.
- * Broadening output spectrum using a hyperbolic tangent.
- * Rectified linear unit activation.
- o Summary.
- Chap. 13: Going Deeper – Mechanics of PyTorch.
 - o Key features of PyTorch.
 - o PyTorch's computation graphs.
 - o PyTorch tensor objects for storing & updating model parameters.
 - o Computing gradients via automatic differentiation.
 - o Simplifying implementations of common architectures via `torch.nn` module.
 - o Project 1 – predicting fuel efficiency of a car.
 - o Project 2 – classifying MNIST handwritten digits.
 - o Higher-level PyTorch APIs: a short introduction to PyTorch-Lightning.
 - o Summary.
- Chap. 14: Classifying Images with Deep Convolutional Neural Networks. In prev chap, looked in depth at different aspects of PyTorch neural network & automatic differentiation modules, became familiar with tensors & decorating functions, & learned how to work with `torch.nn`. In this chap, will now learn about *convolutional neural networks (CNNs)* for image classification. Start by discussing basic building blocks of CNNs, using a bottom-up approach. Then, will take a deeper dive into CNN architecture & explore how to implement CNNs in PyTorch. Covered topics:
 - o Convolution operations in 1D & 2D.
 - o Building blocks of CNN architectures
 - o Implementing deep CNNs in PyTorch
 - o Data augmentation techniques for improving generalization performance
 - o Implementing a facial CNN classifier for recognizing if someone is smiling or not

o Building blocks of CNNs. CNNs are a family of models that were originally inspired by how visual cortex of human brain works when recognizing objects. Development of CNNs goes back to 1990s, when YANN LECUN & his colleagues proposed a novel NN architecture for classifying handwritten digits from images: *Handwritten Digit Recognition with a Back-Propagation Network* by Y. LECUN, & colleagues, 1989, published at NeurIPS conference.

Remark 55 (Human visual cortex). *Original discovery of how visual cortex of our brain functions was made by DAVID H. HUBEL & TORSTEN WIESEL in 1959, when they inserted a microelectrode into primary visual cortex of an anesthetized cat (khi họ đưa một vi điện cực vào vỏ não thị giác chính của một con mèo bị gây mê). They observed: neurons respond differently after projecting different patterns of digit in front of cat. This eventually led to discovery of different layers of visual cortex. While primary layer mainly detects edges & straight lines, higher-order layers focus more on extracting complex shapes & patterns.*

Due to outstanding performance of CNNs for image classification tasks, this particular type of feedforward NN gained a lot of attention & led to tremendous improvements in ML for computer vision. Several years later, in 2019, YANN LECUN received Turing award (most prestigious award in CS) for his contributions to field of AI, along with 2 other researchers, YOSHUA BENGIO & GEOFFREY HINTON, whose names you encountered in prev chaps.

In following sects, discuss broader concepts of CNNs & why convolutional architectures are often described as “feature extraction layers”. Then, will delve into (đào sâu vào) theoretical def of type of convolution operation that is commonly used in CNNs & walk through examples of computing convolutions in 1D & 2D.

* Understanding CNNs & feature hierarchies. Successfully extracting *salient (relevant) features* is key to performance of any ML algorithm, & traditional ML models rely on input features that may come from a domain expert or are based on computational feature extraction techniques.

Certain types of NNs, e.g. CNNs, can automatically learn features from raw data that are most useful for a particular task. For this reason, common to consider CNN layers as feature extractors: early layers (those right after input layer) extract *low-level features* from raw data, & later layers (often *fully connected layers*, as in a *multilayer perceptron (MLP)*) use these features to predict a continuous target value or class label.

Certain types of multilayer NNs, & in particular, deep CNNs, construct a so-called *feature hierarchy* by combining low-level features in a layer-wise fashion to form high-level features. E.g., if dealing with images, then low-level features, e.g. edges & blobs (đốm), are extracted from earlier layers, which are combined to form high-level features. These high-level features can form more complex shapes, e.g. general contours of objects like buildings, cats, or dogs.

As can see in Fig. 14.1: Creating feature maps from an image., a CNN computes *feature maps* from an input image, where each element comes from a local patch of pixels in input image.

This local patch of pixels is referred to as *local receptive field* (trường tiếp nhận cục bộ). CNNs will usually perform very well on image-related tasks, & that’s largely due to 2 important ideas:

- *Sparse connectivity*: A single element in feature map is connected to only a small patch (miếng vá) of pixels. (This is very different from connecting to whole input image, as in case of MLPs. May find it useful to look back & compare how implemented a fully connected network that connected to whole image in Chap. 11.)

- *Parameter sharing*: Same weights are used for different patches of input image.

As a direct consequence of these 2 ideas, replacing a conventional, fully connected MLP with a convolution layer substantially decreases number of weights (parameters) in network, & see an improvement in ability to capture *salient* features (đặc điểm nổi bật). In context of image data, it makes sense to assume: nearby pixels are typically more relevant to each other than pixels that are far away from each other.

Typically, CNNs are composed of several *convolutional* & subsampling layers that are followed by 1 or more fully connected layers at end. Fully connected layers are essentially an MLP, where every input unit i is connected to every output unit j with weight w_{ij} (covered in Chap. 11).

Note: subsampling layers, commonly known as *pooling layers*, do not have any learnable parameters; e.g., there are no weights or bias units in pooling layers. However, both convolutional & fully connected layers have weights & biases that are optimized during training. In following sects, study convolutional & pooling layers in more detail & see how they work. To understand how convolution operations work, start with a convolution in 1D, which is sometimes used for working with certain types of sequence data, e.g. text. After discussing 1D convolutions, work through typical 2D ones that are commonly applied to 2D images.

- * **Performing discrete convolutions.** A *discrete convolution* (or simply *convolution*) is a fundamental operation in a CNN. Therefore, important to understand how this operation works. In this sect, cover mathematical def & discuss some of *naïve* algorithms to compute convolutions of 1D tensors (vectors) & 2D tensors (matrices).

Note: formulas & descriptions in this sect are solely for understanding how convolution operations in CNNs work. Indeed, much more efficient implementations of convolutional operations already exist in packages e.g. PyTorch.

Remark 56 (Mathematical notation). *In this chap, use subscript to denote size of multidimensional array (tensor); e.g., $A_{n_1 \times n_2}$ is a 2D array of size $n_1 \times n_2$. Use brackets $[]$ to denote indexing of a multidimensional array. E.g., $A[9, j]$ refers to element at index i, j of matrix A . Furthermore, note: use a special symbol $*$ to denote convolution operation between 2 vectors or matrices, which is not to be confused with multiplication operator $*$ in Python.*

- * **Discrete convolutions in 1D.** Start with some basic defs & notations that we are going to use. A discrete convolution for 2 vectors \mathbf{x}, \mathbf{w} is denoted by $\mathbf{y} = \mathbf{x} * \mathbf{w}$, in which vector \mathbf{x} : our input (sometimes called *signal*) & \mathbf{w} is called *filter* or *kernel*. A discrete convolution is mathematically defined as follows:

$$\mathbf{y} = \mathbf{x} * \mathbf{w} \rightarrow y[i] = \sum_{k=-\infty}^{+\infty} x[i-k]w[k]. \quad (238)$$

As mentioned earlier, brackets $[]$ are used to denote indexing for vector elements. Index i runs through each element of output vector \mathbf{y} . There are 2 odd things in preceding formula needed to clarify: $-\infty$ to $+\infty$ indices & negative indexing for \mathbf{x} .

COMPUTATION TECHNIQUES

- * Padding inputs to control size of output feature maps.
- * Determining size of convolution output.
- * Performing a discrete convolution in 2D.
- * Subsampling layers.
- Putting everything together – implementing a CNN.
- Implementing a deep CNN using PyTorch.
- Smile classification from face images using a CNN.
- Summary.
- Chap. 15: Modeling Sequential Data Using Recurrent Neural Networks.
 - Introducing sequential data.
 - RNNs for modeling sequences.
 - Implementing RNNs for sequence modeling in PyTorch.
 - Summary.
- Chap. 16: Transformers – Improving Natural Language Processing with Attention Mechanisms.
 - Adding an attention mechanism to RNNs.
 - Introducing self-attention mechanism.
 - Attention is all we need: introducing original transformer architecture.
 - Building large-scale language models by leveraging unlabeled data.
 - Fine-tuning a BERT model in PyTorch.

- Summary.
- Chap. 17: Generative Adversarial Networks for Synthesizing New Data.
 - Introducing generative adversarial networks.
 - Implementing a GAN from scratch.
 - Improving quality of synthesized images using a convolutional & Wasserstein GAN.
 - Other GAN applications.
 - Summary.
- Chap. 18: Graph Neural Networks for Capturing Dependencies in Graph Structured Data.
 - Introduction to graph data.
 - Understanding graph convolutions.
 - Implementing a GNN in PyTorch from scratch.
 - Implementing a GNN using PyTorch Geometric library.
 - Other GNN layers & recent developments.
 - Summary.
- Chap. 19: Reinforcement Learning for Decision Making in Complex Environments.
 - Introduction – learning from experience.
 - Theoretical foundations of RL.
 - Reinforcement learning algorithms.
 - Implementing 1st RL algorithm.
 - A glance at deep Q-learning.
 - Chap & book summary.

2.13 [RHP21]. RISHIKESH RANADE, CHRIS HILL, JAY PATHAK. *DiscretizationNet: A ML Based Solver for NSEs using FV Discretization*

Journal of Computer Methods in Applied Mechanics & Engineering. [139 citations]

Abstract. Over last few decades, existing PDE solvers have demonstrated a tremendous success in solving complex, nonlinear PDEs. Although accurate, these PDE solvers are computationally costly. With advances in ML technologies, there has been a significant increase in research of using ML to solve PDEs. Goal: develop an ML-based PDE solver, that couples' important characteristics of existing PDE solvers with ML technologies. 2 solver characteristics that have been adopted in this work are:

1. use of discretization-based schemes to approximate spatio-temporal partial derivatives &
2. use of iterative algorithms to solve linearized PDEs in their discrete form.

In presence of highly nonlinear, coupled PDE solutions, these strategies can be very important in achieving good accuracy, better stability & faster convergence. Our ML-solver, DiscretizationNet, employs a generative CNN-based encoder-decoder model with PDE variables as both input & output features. During training, discretization schemes are implemented inside computational graph to enable faster GPU computation of PDE residuals, which are used to update network weights that result into converged solutions. A novel iterative capability is implemented during network training to improve stability & convergence of ML-solver. ML-Solver is demonstrated to solve steady, incompressible NSEs in 3D for several cases e.g., lid-driven cavity, flow past a cylinder & conjugate heat transfer.

Keywords. PDEs; ML; Discretization Methods; Physics-Informed Learning

- 1. Introduction. Coupling of physics & DL to solve problems in engineering simulation space has drawn interest in recent years. In context of neural networks, this has been achieved by constraining network optimization by embedding physical constraints in loss formulation. These physics-based constraints ensure: solution space is bounded & obeys physical laws. Although this idea was proposed back in 90s [1,2], it has started to have a big impact in recent times due to rapid advances in computational sciences & DL.

Within context of physics-based DL there are 2 types of methods used to approximate PDEs governing physical processes: data-driven & data-free. Data-driven methods use simulation or experimental data to construct models while enforcing physical laws. These models heavily depend on fidelity of data & hence are limited in accuracy & generalizability. Data-free methods use neural networks to generate solutions by rigorously constraining partial differential governing equations through loss formulation. In this respect, Raissi et al. [3,4] recently introduced Physics Informed Neural Network (PINN) framework which approximates partial derivatives of solution variables w.r.t. space & time using automatic differentiation (AD) [5]. Partial derivatives are used to estimate PDE losses which are back-propagated to neural network for weight updates. In addition to constraints on PDE residual, boundary & initial conditions are specified on computational domain & constrained in loss

formulation to ensure that a unique solution is obtained at convergence. Based on this work, PINN framework has been extended in different directions to improve effectiveness of learning PDEs & to develop a theoretical understanding of PINNs. Recently, Kharzami et al. [6] developed Variational PINNs to solve PDE in its weak form. Similarly, Khodayi-Mehr et al. [7] proposed VarNet, where loss formulation was based on integral form of PDE as opposed to differential form. In recent effects a distributed version of PINN has been explored, where learning problem is decomposed into smaller regions of computational domain & a physical compatibility condition is enforced in between neighboring domains [8–11]. Other variations of PINN as well as convergence & stability of PINN-based algorithms has been studied in [12–18].

PINN methodology has been demonstrated to solve a number of canonical PDEs as well as to different applications involving vastly different physics [19–34]. A number of these studies have used PINN framework to solve more complex PDEs e.g. NSEs, which is focus of this work. Dwivedi et al. [35] developed a Distributed-PINN to address some of issues with PINN & demonstrated it to solve NSE in a lid-driven cavity at low Reynolds numbers. Sun et al. [36] demonstrated PINN methodology for surrogate modeling of fluid flow at low Reynolds numbers. Zhu et al. [37] implemented physical constraints on an encoder–decoder network in conjunction with flow based conditional generative models for stochastic modeling of fluid flows. Rao et al. [38] demonstrated PINN approach to solve NSEs at low Reynolds numbers for a 2D flow over a cylinder. Very recently, Jin et al. [39] proposed PIN approach for solving NSEs in both laminar & turbulent regimes. Gao et al. [40] proposed PhysGeoNet to solve NSEs using finite difference discretizations of PDE residuals in neural network loss formulation. Methodology was demonstrated in irregular domains using elliptic coordinate mappings to transform spatial coordinates.

Computation of PDE loss & choice of network architecture used for network optimization become crucial when solving for highly nonlinear, multidimensional, stiff, coupled PDEs e.g. system of NSEs. Highly nonlinear solution space accessed by NSEs may be challenging to resolve due to presence of sharp local gradients in a broad computational domain. As a result, methodology used for computing gradients as well as approach of network training can be very important in achieving accurate solutions, better stability & faster convergence in training process.

Traditional PDE solver technologies developed over last decades have primarily relied on solving discretized formulations of PDEs using methods e.g. FD, FE, or FV. Exact or approximate forms of linearized discrete equations are used, in combination with linear equation solvers, to improve solutions iteratively. Discretization method allows access to higher order & advanced numerical approximations for partial derivatives which can be useful in resolving highly nonlinear parts of PDE solution & also add artificial dissipation to improve solver stability. Additionally, these schemes coupled with iterative solution strategy have proved to be robust in terms of solver stability & convergence. Recently, ML based models have been developed to either learn new discretization schemes from solution data [41,42] or to mimic these schemes through novelties in neural network architectures [43,44]. On other hand, Ansys suite of software already has access to a large number of advanced discretization schemes that can capture complexities over a wide range of physics. Coupling of these discretization schemes with ML algorithms, along with iterative solution algorithm, can provide same benefits in ML-based solvers as observed with traditional solvers.

Main goal: introduce a new ML-solver, DiscretizationNet, which is a framework that couples solver characteristics with generative networks to solve highly nonlinear, multidimensional, stiff, coupled PDEs. Solver does not require any training data but generates PDE solutions & simultaneously learns them during training process. Different FV based numerical schemes are implemented inside computational graph to enable fast, vectorized operations on GPU & a modified encoder–decoder network architecture is proposed to solve PDEs in an iterative manner. Finally, discretization based iterative ML solver is used to solve steady, incompressible, NSE in a 3D for a several cases e.g., lid-driven cavity flow at a high Reynolds number, flow past a cylinder in laminar regime & conjugate heat transfer. Remainder of paper is organized as follows. Sect. 2: introduce solution methodology adopted in DiscretizationNet. Sect. 3: discuss numerical results followed with conclusions. Sect. 4: future work.

- **2. Solution methodology.** Discuss methodology for solving system of steady, incompressible NSEs using DiscretizationNet. System of NSEs consists of continuity equation & momentum equation for each directional velocity component. Non-dimensional equations described in a vectorized form:

$$\nabla \cdot \mathbf{v} = 0, \quad (239)$$

$$(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p - \frac{1}{\text{Re}} \Delta \mathbf{v} = 0, \quad (240)$$

where \mathbf{v} : non-dimensional velocity vector, $\mathbf{v} = (u, v, w)$, p : non-dimensional pressure, ∇ : divergence operator, Re : Reynolds number.

When solving using neural networks, PDE described in (1) are used to compute residuals in loss formulation. Some of reasons that can result in a stiff formulation of PDE residual-based loss term include, complicated geometries, strong coupling of PDE variables in large system of coupled PDEs, presence of multiple domains with different PDE formulations & material properties (e.g. conjugate heat transfer between fluid & solid domains) & reasonably large Reynolds numbers, where nonlinear, convective component, $(\mathbf{v} \cdot \nabla)\mathbf{v}$, is dominant. Automatic differentiation (AD) [5] allows computation of partial derivatives within computational graph using back-propagation, but stiffness of computed gradients may affect accuracy, stability & convergence of neural network training, & may require a large number of training epochs, use of regularization techniques, as well as deep network architectures, which have their own set of problem, e.g. vanishing gradients. Considering these challenges, it becomes increasingly important to resolve such PDEs using advanced numerical schemes & develop novel strategies of neural network training. Existing solver methodologies have solved some of these problems & in this work, draw from vast pool of knowledge to develop our ML-based solver. Next, introduce network architecture used in work followed by loss formulation & training mechanics.

- **2.1. DiscretizationNet architecture.** Network proposed in this work is a generative Convolutional Neural Network (CNN) based encoder–decoder whose input features are flow variables, $\mathbf{v} = (u, v, w, p)$, initialized with random uniform solution fields, boundary condition encoding b , & level set of geometry h . Level sets are real valued functions which depict geometry s.t. regions inside geometry are flagged with -1 , region outside it with 1 & regions representing surface to 0 [45]. Objective of this network, as shown in Fig. 1: DiscretizationNet for Navier–Stokes solution, is to compress input features $(\underline{u}, \underline{v}, \underline{w}, \underline{p}, \underline{b}, \underline{h})$ into a lower dimensional space, (η) , using a convolutional encoder & to decode latent vector encoding to new solutions, $(\hat{u}, \hat{v}, \hat{w}, \hat{p})$, which are closer to actual Navier–Stokes solutions. It may be observed: boundary conditions as well as geometry level sets are used to enrich encoded latent space & also in computation of coupled PDE loss terms, (R_c, R_u, R_v, R_w) , corresponding to continuity & momentum equations in each spatial direction. Enrichment of latent space with geometry & boundary information is crucial as it ensures that network outputs are conditioned upon them. Geometry & boundary encoder networks are pre-trained on similar samples & their weights are used to perform encoding in this network. Encoding geometry & boundary is crucial in this approach, because, in their original form, these representations can be very sparse. This may have an adverse effect on learning & generalizability of DiscretizationNet. Additionally, network proposed here can be used to compute a large batch of solutions at different boundary & geometry conditions in a single training session. As a result, conditioning of solutions with boundary & geometry conditions enables generalization of PDE solutions for a large set of problems. Purpose of designing DiscretizationNet as an encoder–decoder network: obtain a legitimate lower-dimensional encoding of PDE solution space. Encoded solution space can be useful in developing reduced-order models (ROM) on top of DiscretizationNet.
- **2.2. Training mechanics.** It was suggested previously: input vectors to encoder–decoder network are randomly initialized solution fields of velocity & pressure. This has 2 implications:
 1. PDE solution encoding (η) does not have a physical meaning &
 2. decoding solutions, which are functions of random noise, is a difficult task & may slow down convergence of network & provide poor stability.

In order to tackle these challenges, an iterative approach is followed, where inputs to network are replaced with newly generated solutions, every time PDE residuals reduce by an order of magnitude. At any given point during training, solutions generated are dependent on solutions from a previous iteration & not initial random solutions used in beginning. This allows network to converge from partially converged solutions to fully converged solutions in an iterative fashion & improves stability & convergence as compared to other ML methods in this space. At convergence, when L^2 norm of PDE residuals have dropped to a reasonably low level, input & output solutions are very similar & effectively turns network into a conditional autoencoder. It is a conditional autoencoder, because decoder network is conditioned on solution encoding as well as encoding of geometry & boundary. This iterative strategy provides a physical meaning to reduced dimensional solution latent space, which can now describe flow solutions at given boundary, geometry or flow conditions. Moreover, PDE solutions are independent of spatial dimension & depend only on solutions at previous iterations. This is analogous to how traditional solvers function & additionally provides an opportunity to operate this network under transient conditions. Important to note: training is completely data free & goal: generate solutions by minimizing PDE residuals & simultaneously learn them into an encoded latent space.

- **2.3. Geometry & boundary encoder.** In this sect, elaborate on geometry & boundary encoders used in network architecture in Fig. 1. In this work, use a modified level set approach to represent geometry, s.t. voxels inside geometry are represented by 0 & outside by 1 . Gradient of level set are used to track voxels activated by surfaces of geometry. Level sets for primitive geometries of different shape, size, & orientation are learned using a generative encoder–decoder network & represented in a lower-dimensional space η_h . A schematic of geometry autoencoder can be seen in Fig. 2A: Geometry & boundary autoencoders. In this work, encoder & decoder networks are CNN-based & a binary cross-entropy loss function is employed to update weights of networks. Level sets of different geometries can be generated by parsing through latent space vector of a trained geometry encoder & used to parameterize ML-solver.

On other hand, a separate boundary autoencoder is used to represent different boundary conditions. Boundary encoder is only required if boundary conditions are spatially or temporally varying. Here, propose a generative encoder–decoder network to learn boundary condition encoding but leave choice of network architecture open. In scenarios where boundary condition is constant along different surfaces, as is in all of test cases demonstrated in this work, a neural network based boundary condition encoder is not required. Instead, a custom encoding can be constructed & used as latent vector, η_b . Flow conditions, e.g. Reynolds number or Prandtl number can also be perceived as boundary conditions & be added to this latent vector. E.g., an encoding of $\eta_b = [1, 1, 2, 1, 0.3, 1.2, 0, 3, 40]$ can be perceived as Dirichlet inlet boundary conditions (1) on left, right, & bottom surfaces with specified values of $0.3, 1.2, 3.0$, resp., & a Neumann boundary condition (2) on top surface, where flux of variable equal 0 . Reynolds number (40) or other flow conditions can also be specified in encoding. A similar choice of boundary condition encoding is employed in this work.

- **2.4. Loss formulation.** Loss formulation of network comprises of PDE residual from all governing equations. Each PDE has its own loss formulation given as follows: (2)

$$\lambda(\mathbf{W}, \mathbf{b}) = \|\lambda_c\|_{\Omega} + \|\lambda_u\|_{\Omega} + \|\lambda_v\|_{\Omega} + \|\lambda_w\|_{\Omega}, \quad (241)$$

where $\|\lambda_c\|_{\Omega}$: L^2 norm of continuity residual, $\|\lambda_u\|_{\Omega}$: L^2 norm of x -momentum residual, $\|\lambda_v\|_{\Omega}$: L^2 norm of y -momentum residual, $\|\lambda_w\|_{\Omega}$: L^2 norm of z -momentum residual & Ω : solution space. Moreover, this network can generate & learn a large set of solutions at different Reynolds numbers. Different solutions simply form a part of training samples of encoder–decoder network.

Computation of PDE residuals involves approximation of 1st & 2nd order spatial gradients. Employ traditional FV discretization technique to compute PDE residual loss & moreover, all numerical schemes are implemented inside computational graph to enable fast, vectorized GPU computation. However, this does not limit use of other discretization schemes e.g. FEM, Discontinuous Galerkin (DG) etc., if higher order elements are required.

In FV discretization here, each voxel of PDE solution is considered as cell center of an imaginary, finite control volume (CV) as shown in Fig. 3: FV stencil on interior & boundary pixels in a computational graph. Volume integrals on CV are expressed as surface integrals using Green–Gauss divergence theorem shown below.

$$\int (\nabla \cdot \mathbf{v}) d\mathbf{x} = \sum_i^M v_i n_i A_i \quad (242)$$

where v : a solution variable, M : number of faces on CV, n_i : normal along each face on CV & A_i : area of each face on CV. Hence, a face-based approach is used to compute gradients across all interior & boundary faces of each control volume in computation graph. 2nd order gradients are also computed using (3), with only difference: solution variable is replaced by its gradient. Convective fluxes are discretized using 1st or 2nd-order upwind scheme & diffusive fluxes are evaluated using a central difference approximation. A 2nd order approximation is used for computing gradients of pressure field. Since dealing with an incompressible formulation of Navier–Stokes & discretization is analogous to FV discretization on collocated grids, pressure field is prone to checker-boarding due to a lack of explicit coupling between pressure & velocity & use of 2nd order numerical schemes. In this work, pressure-velocity coupling is achieved using Rhie–Chow interpolation [46], which essentially adds a 4th order dissipation of pressure to continuity equation. Addition of Rhie–Chow flux suffices & more sophisticated schemes e.g. SIMPLE [47] are not required but remain an option. In Rhie–Chow formulation, velocity at faces is interpolated as shown in (4):

$$u_e = \frac{u_i + u_{i+1}}{2} + \frac{1}{a_p} (p_i + \nabla P_i \cdot \bar{r}_i - p_{i+1} - \nabla P_{i+1} \cdot \bar{r}_{i+1}), \quad (243)$$

where u_e : velocity approximation at *east* face of control volume, p : pressure, ∇P : gradient of pressure, a_p : matrix coefficients from momentum equations.

Boundary condition treatment is incorporated through discretization of boundary voxels by enforcing boundary constraints in flux computation & enforces order of accuracy at boundary, as opposed to using 1-sided finite difference schemes. E.g., x -velocity gradients along a boundary as shown in Fig. 3 is represented as follows:

$$\left(\frac{du}{dx} \right)_G = \frac{u_B + u_G}{2\Delta x_B} - \frac{u_B + u_I}{2\Delta x_I}, \quad (244)$$

$$u_G = 2u_b - u_B, \quad (245)$$

where u_b : specified boundary condition, u_G : an imaginary ghost pixel, u_B : boundary voxel, & u_I : interior voxel adjacent to boundary in direction of gradient. In case of unstructured boundaries, e.g., cells adjacent to walls of a cylinder, a stair-step discretization [48] or a cut-cell discretization [49] can be implemented. In this work, have implemented stair-step discretization, where boundary conditions at unstructured boundaries are implemented similarly as (5).

Numerical schemes at both interior & boundary voxels are implemented together in computational graph using vectorized operation for fast computation on GPU. Discretization schemes are implemented through custom hidden layers in Keras [50], where solution variable tensors as well as boundary & geometry condition tensors are used to compute PDE residuals of coupled PDE system at each voxel. As a result, loss formulation in (2) implicitly contains boundary information & a separate loss term is not needed to model it, as in previous studies [3,4], thereby avoiding need to use strategies for multi-objective optimization. Moreover, use of discretization techniques to compute loss, allows access to higher order approximations for higher accuracy & advanced numerical schemes e.g. Rhie–Chow flux [46] etc. that can enhance stability & convergence of solution & neural network training by providing additional physics-based regularization.

- o 2.5. Inference for other geometry & boundary conditions. Network architecture described in Fig. 1 is a generative encoder–decoder network, where, at convergence, input & output samples are essentially actual solutions of given PDEs. It may be understood: model obtained at convergence has learned to encode actual PDE solutions & decode them from solution latent space combined with geometry & boundary encoding. As a result, model in its current form cannot be directly used for inferencing solutions at other geometry & boundary conditions, since actual PDE solutions may be required as inputs to network & these are obviously not available.

Here, propose a novel algorithm that enables inferencing for other geometry & boundary conditions. A schematic diagram of algorithm is shown in Fig. 4: Algorithm for inferencing & important steps are outlined:

1. On a given geometry & boundary condition initialized for inference, geometry & boundary encoding, η_h, η_b are computed using their respective encoder networks.
2. Since, solution encoding η is unknown, it is initialized with a random field drawn from a uniform distribution.
3. Initial solution encoding combined with geometry & boundary encoding is passed through trained weights of CNN decoder to generate a solution field $\hat{u}, \hat{v}, \hat{w}, \hat{p}$.
4. Solution field is encoded to a new solution encoding using trained weights of CNN encoder $\hat{\eta}$.
5. New solution encoding $\hat{\eta}$ replaces solution encoding of previous iteration η & steps (iii) & (iv) are repeated until $\|\eta - \hat{\eta}\|_{L^2} < 1e^{-6}$. Geometry & boundary encoding are fixed during entire process.

6. At convergence, PDE solutions at a given geometry & boundary condition are decoded using most recent η .

May be observed: solution inference happens in encoded latent vector space & goal of iterative procedure: steer solution latent vector to a space that is in close proximity to latent vectors observed in network training. Since geometry & boundary condition encoding are fixed for a given problem, they provide necessary constraints for solution latent vector to iteratively improve itself & generate an accurate PDE solution. Outcome of this algorithm, in terms of generalization, improves with number of different variations of geometry & boundary conditions adopted during training. Generally, starting from weights obtained from a well trained model, inference algorithm converges in < 10 iterations. Although not a scope of our current work, functioning in space of latent vectors may provide an opportunity to explore new solution spaces of a given PDE & construct computationally inexpensive reduced order models.

- 3. Results. Provide detailed numerical experiments to demonstrate ML-solver for several cases of fluid flow e.g. lid-driven cavity, flow past a cylinder & conjugate heat transfer. Proposed ML-solver is validated against ANSYS Fluent 19.3 CFD [51] solver for solving incompressible, steady NSE for these different cases.
 - 3.1. Lid-driven cavity flow.
 - 3.2. Laminar flow past a cylinder.
 - 3.3. Conjugate heat transfer.
- 4. Conclusion. Have presented a novel ML-Solver, which uses important characteristics from existing PDE solvers for solving system of steady, incompressible NSE. ML-solver does not require any training data & instead, generates & learns PDE solutions simultaneously, during training process. It uses discretization techniques to approximate PDE residual at each voxel of a given computational domain & uses L^2 norm of residual to update network weights. Discretization schemes are implemented inside computational graph to enable vectorization on GPU & provide access to numerous higher order & advanced based regularization. In this work, have extended discretizations to unstructured domains by employing stair-step discretizations to provide flexibility in modeling different types of geometries as well as widely varying boundary conditions.

From network architecture perspective, introduce DiscretizationNet, which is a generative CNN-based encoder-decoder network conditioned on geometry & boundary conditions. Separate autoencoders are constructed to learn lower-dimensional vectors (or encodings) for different geometry & boundary conditions. These encodings are used to enrich & parameterize solution latent vector space of generative network & thus allow for simultaneously generating & learning a wide range of solutions at different conditions in same training session. Moreover, employ a novel iterative capability in network to mimic existing PDE solvers. In this implementation, inputs to generative model are replaced with outputs during network training, as network learns to generate better solutions. This strategy is unique & have observed: it provides better stability & faster convergence in comparison to other ML strategies, especially in cases when ground truth solutions are not known. Additionally, have proposed an algorithm for interencing using DiscretizationNet. Algorithm functions in latent space to iteratively infer solutions using trained model weights.

Have validated ML-solver by solving 3D steady, incompressible NSEs on 3 different cases, (i) lid-driven cavity, (ii) laminar flow past a cylinder, & (iii) conjugate heat transfer. Contour & line plot comparisons made with ANSYS Fluent R19.3 [51] in all 3 cases show a good agreement. Additionally, it has been observed: training for a large number of PDE solutions results in a stable convergence within 3×10^4 training epochs.

ML-solver proposed here can be extended to solve unsteady problems using LSTM-type networks [55]. Deficiencies in stair-step discretization, in computing accurate solutions near boundaries can be mitigated by using a cut-cell of unstructured grid discretization. Moreover, ML-solver can be applied to other PDEs with complex physics as well as to develop computationally inexpensive, low-dimensional models.

2.14 [Ric+22]. MICHAEL M. RICHTER, SHEULI PAUL, VETON KĚPUSKA, MARIUS SILAGHI. **Signal Processing & Machine Learning with Applications**

- Amazon review. Signal processing captures, interprets, describes & manipulates physical phenomena. Mathematics, statistics, probability, & stochastic processes are among signal processing languages we use to interpret real-world phenomena, model them, & extract useful information. This book presents different kinds of signals humans use & applies them for human machine interaction to communicate.

Signal Processing & ML with Applications presents methods that are used to perform various ML & AI tasks in conjunction with their applications. Organized in 3 parts: Realms of Signal Processing; ML & Recognition; & Advanced Applications & AI. Comprehensive coverage is accompanied by numerous examples, questions with solutions, with historical notes. Book is intended for advanced undergraduate & postgraduate students, researchers & practitioners who are engaged with signal processing, ML & applications.

- Preface. Learning has always a goal. For this reason, for a given process, question is what can we learn about it? Despite diversities in processes, one can find out common principles underlying them. Therefore, discuss stochastic processes from a general point of view, 1st. As for practical applications, consider them through lens of historical developments by considering following aspects:

1. Invention of numbers & number systems is our starting point. From numbers & their representations, humans invented theory of signals & systems. Signals do not occur isolated but within a process.

2. Naturally occurring processes are not deterministic but rather stochastic. Hence, underlying states are often unknown. I.e., those processes are mostly described with hidden stochastic models (HSM).
3. Signal processes contain information. However, this information is not directly visible or understandable to humans. Information is hidden in properties of processes. Properties of signal processes are mostly not directly measurable, & have to find them out, at least approximately.
4. Information about unknown elements is necessary. One needs methods to capture them, & here lie major challenges. To address these challenges, concentrate on ML techniques.

Real-world problems are conceptualized, explained in applied domains. Book is intended for persons interested in techniques & foundations related to signal processing. As mentioned previously, signals occur within processes & are considered not to be fully observable & predictable; hence, they are ruled by probabilities. Therefore, they are termed *stochastic processes*. Concentrate on properties of signals that are useful for practical applications. One has rather to manipulate information available about processes using computations. Doing this properly is a major challenge. Computations needed depend very much on type of signals & application domain.

Development of practical applications was foremost in our mind when writing this book. Principles where problems & differences are exhibited in their realizations. Today one is deploying robotic systems to perform various tasks. This in turn requires solving communication problems between humans & machines. Communicate. Humans & machines can communicate, e.g., using speech, images, & videos. These all have in common that a physical exchange has to take place. At lowest level, it requires use of elementary signals that could be re-interpreted on higher levels of abstraction. Describe our methods as organized in a ML process. There, one records many examples of signals in order to use them when applying ML techniques. Techniques involve several transformations. A major difficulty: find out which transformation one should apply in which situation.

- 1st, audio typically uses cascaded IIR filters to obtain high performance with minimal latency. Note: each filtering stage propagates errors of prev stages, a high degree of precision in both signal & coefficients is required in order to minimize effects of these propagated errors.
- 2nd, signal accuracy must be maintained, even as its magnitude approaches 0 (this is necessary because of sensitivity of human ear).

Signals occur in sets, & these sets contain information & have special structures. Important structures are governed by logic. Often, signals arise from some events & are not part of a systematic communication. Sometimes they are represented in a linear ordering. However, typically they are not ordered & are often described through general stochastic processes.

Humans are interested in information contained in any process, be it discrete (e.g. digital) or continuous & modeled by deterministic or stochastic processes. One can employ ML methods to find information governing stochastic processes. Here, one needs to employ some combinatorial, probabilistic, & learning algorithms. Learning in particular is needed if signal sets are complex, or signals are corrupted by noise. Is for efficiency reasons very restricted by large number of occurring signals & by noise. Explore systematically these topics. Therefore, show their usefulness in a number of widely reaching application areas. Intend to represent whole picture of signal processing instead of restricting it to special aspects. Then look at recognition problems. For both, ML provides a wealth of important techniques. ML is a wide area. In this book, discuss aspects of it related to signal processes. These are mainly supervised & unsupervised learning methods. Finally, discuss real applications. Noise is 1 of main issues when apply theoretical aspects to solve real-world problems. Take place in a clean (noise free) environment & noise often corrupts it. Therefore, included study & handling of noise. Then, discussed multiple application areas. A very important such area is speech processing.

Book is written for students, scientists, researchers, engineers, & practitioners. Readers should have an understanding of basic computational methods as taught in 1st 3 semesters of CS undergraduate curricula. For usage as a textbook, lectures are supported by a systematic presentation of material with examples & exercises including background information accompanied with literature discussions.

- **Problems, Challenges, & Principal Methods.** A very 1st question is what a signal is. There are different interpretations of this term. Defs are mainly motivated by area in which one is interested. E.g.:
 - Digital signals, e.g. Boolean values, integers, or real numbers
 - Arbitrary objects like words, documents, messages, & images, etc.

In this book, focus is on digital signals. They have no parts & cannot be decomposed, they are simply units. However, several signals, when viewed together, gives rise to a signal process. Signals usually do not occur isolated but rather organized in processes. Mostly, such processes are ruled by probabilities, i.e., they are stochastic processes. These occur in 2 ways: with fixed probabilities (e.g., Markov processes) & with unknown probabilities (Hidden stochastic processes, HSM). Concentrate mainly on latter ones because this is what we get from applications. Signal processes contain information. However, this information is not directly visible & is not directly understandable to humans. Information is hidden in properties of processes. Properties of signal processes are in initial phase mostly unknown, & have to find them out, at least approximately. For this, one does not only consider signal processes as they are given, one has rather to manipulate processes using computations. It is a major challenge to do this properly & depends very much on type of signals & application domain.

View our methods as organized in a ML process. To perform this task, one has to deploy many examples of signals. Deployed signals undergo several transformations as necessary. A major difficulty: find out which transformation one should apply in which situation.

From hardware point of view, a digital signal processor is a specialized microprocessor with its architecture optimized for operational needs of digital signal processing. However, this is not main topic of book.

ML is a wide area. In this book, discuss aspects of it related to signal processes. These are mainly unsupervised learning methods.

Properties of signals lead to abstractions. On a more abstract level, one can define properties interesting to applications. In evolution, living organisms have developed such levels & abstractions over time. Of course, one cannot repeat evolution. However, it can still be studied because many of primitive organisms are still alive today. Here, do not consider evolution, but can still make use of observations & results.

Problem: even a perfect solution to a sub problem is not very useful if it does not help with solutions to other sub problems. Reason: sub problems are not contained in modules that can be treated in isolation. Each solution of a sub problem is affected to a degree by system, or as a whole part.

In a book, one faces therefore following difficulty:

- How to formulate overall perspective?
- How to combine solutions to sub problem in view of overall perspective?

Readers should have understanding of basic computational methods as they are taught in 1st 3 semesters of CS.

◦ **Some History.** Origins of signal processing go back to prev centuries, around 17th century. At that time, it was considered as a part of mathematics, in particular, numerical analysis. In modern times (after 1970) several digital processors have been developed, like in Lincoln Laboratories. In 1980s, 1st chips were developed as Intel 2920, TMS32010 from Texas Instrument, & NEC μ PD7720. In each chapter, there is a sect on background info. It contains mainly historical remarks & references for further reading. Here, discussed choice was “Dynamic Automatic Noisy Speech Recognition System” (DANSR). System was devoted to tackle problem of noises of different kinds that occur simultaneously. This is a major task in dissertation of author SHEULI PAUL.

◦ **A Guide to Different Chaps.** In each chap, ones finds illustrations of different kinds. Illustrations typically are figures. They may be derived from peak period when machine was functional, & also when machine is working at standard level. Sometimes, illustration is not understood by itself. Therefore, added some explanations in figure, e.g. in terms of colors. There will be many such figures in book. For most illustrations, MATLAB was used.

Applications was an important part of book. They are given in 2nd half of book. Part C contains description of several real situations where stochastic processes play a role. In real situations, there is always noise disturbing stochastic process. Noise can be of a different level. Based on noise level, termed noise as mild, strong, & steady-unsteady. Therefore, see probability density function (pdf) of such noisy commands, classified in legend as mild, unsteady, & strong.

Structure of all chaps is essentially same. They are structured as: Overview, Main text, Background info, Exercises, Refs. Overview contains indications of what reader should study for better understanding before reading chap. In addition, this allows finding topics of interest quite fast & it helps a lecturer to select topics for a discussion in class.

Main text contains contribution of chap. In background, give origins to discussed topics as well as suggestion for further reading. Here, also present links to refs. In main text, there are no refs. Observe: each chap contains its own list of refs. I.e., refs are listed at end of each chap, for related work discussed in corresponding chap next.

Exercises do not contain solutions so that a lecturer can use them in class. In a short way, now comment on each chap. This is intended for readers who want to see what is close to their special interests.

PART I: REALMS OF SIGNAL PROCESSING. 1st part consists of 7 chaps, & provides a survey of mathematical & engineering foundations assumed to be known by reader.

1. **Chap. 1: Digital Signal Representation.** Details of digital representations of discrete-time signals are presented in this chap. This includes bridging gap between abstract discrete-time signal notation presented in book, $x[n]$, & its representation in a digital processor.
2. **Chap. 2: Signal Processing Background.** Introduce briefly signal processing discipline. Contain problems & principal solution methods. In addition, some history is given as well as a short description for intended readers.
3. **Chap. 3: Fundamentals of Signal Transformation.** Give an overview of processing & transforming signals. Here, some basic signal processing formulations are introduced. Said how signal processing is an integral part of ML & how these 2 areas are related & how related outcome of these 2 areas generates a successful application. In this chap, deal with decomposition of a signal & techniques used. Signal is decomposed into some basic functions & coefficients. Furthermore, coefficients then can be used as linear combinations of signals representing signal. Transformations e.g. z-transform, Fourier transform, discrete cosine transform, local trigonometric transform, & wavelet transform are briefly discussed.
4. **Chap. 4: Digital Filters.** Digital filters are some mathematical operations to manipulate or analyze signals. Filter coefficients are generally not variable like in adaptive filters. Introduce chap with fundamentals of finite impulse response (FIR) filter & infinite impulse response (IIR) filter, & discuss applications of some advanced filters.

5. **Chap. 5: Estimation & Detection.** Discuss some commonly used filters, methods, & techniques to analyze signal adaptively. Filters use optimization approach in order to analyze signals. In this chap, optimization techniques with examples are discussed. Such techniques are minimum variance unbiased estimation (MVU), best linear unbiased estimator (BLUE), maximum likelihood estimator (MLE), least square estimator (LSE), minimum least squares estimator (LMSE), & Bayesian approach.
6. **Chap. 6: Adaptive Signal Processing.** This chap has general info about signal modeling using parametric & nonparametric types. It introduces adaptive filters mainly Wiener filter, Kalman filter, & Particle filter.
7. **Chap. 7: Spectral Analysis.** Consider problem of spectral content of signals by means of parametric & nonparametric signal modeling. In a parametric signal modeling, a signal is analyzed using only few parameters. A nonparametric method has no such parameters quantification. Techniques e.g. linear prediction analysis using auto-regressive (AR) signal modeling, moving average signal modeling (MA), & auto regressive moving average (ARMA) signal modeling are introduced. Nonparametric signal modeling e.g. subspace signal analysis applying sub-band coding, discrete cosine transform, & discrete Fourier transform for spectral signal analysis are outlined.

Very 1st question to address in a book on signal processing: “What is a signal?” There are different interpretations of this item. Defs are mainly motivated by area in which one is interested. E.g., Digital signals (Boolean values, Integers), or Real Numbers.

These signals occur in 2 ways: with probability distributions that can be learned from finite observations or with probability distributions that are given. Signal processes contain information, however, typically this information is not directly measurable.

Properties of signals lead to abstractions. On a more abstract level, one can define properties of interest to applications. In evolution, living organisms have developed such levels & abstractions over time. Of course, one cannot repeat evolution. However, it can still be studied because many of primitive organisms are still alive today. Here do not consider evolution but can still make use of observations & results.

Book has 3 parts, going from foundations to advanced applications.

- **Generalities About Part I.** 1st part consists of 7 chaps & provides a survey of mathematical & engineering foundations assumed to be known by reader. While part can be described as a concise compendium, with pointers to relevant textbooks, it nevertheless presents examples & questions that can accompany a prerequisites review sect of a class, or that can tune reading engineer to expected level of knowledge needed for later incursion into applications.

Reader should look into these chaps to verify having prior knowledge, or to review concepts involved in subsequent discussions. At a 1st one can skip reading this part in detail as sufficient to check titles of sects while later when concepts are used reader can return to corresponding sect for refreshing one’s comprehensive understanding & corresponding notations in book.

- **Overview of Part I.** Part starts with a review of representations for numbers, as encountered in representation of signals or during processing, & including history clarifying what made representations suitable for specific applications.

Further representation of signals is introduced both in terms of abstract notations used for describing processes in book, & in terms of storage in computers & other electronic devices. This introduction is accompanied by reviewing basic concepts like signal power, as well as basic processing e.g. convolution & correlation.

After a short review of main transform function & of some of their properties, concepts of linear & adaptive filters are surveyed together with major notation conventions & stability conditions.

A somewhat more in-depth review of Maximum A Posteriori & Maximum Likelihood estimation of signal model parameters is subsequently presented together with motivations, examples, & related concepts.

Major adaptive filters are then surveyed, e.g. Kalman & Particle Filters together with a list of their most important equations.

Part I concludes with a review of filters based on spectral analysis & involved techniques e.g. independent component analysis & principal component analysis.

- **Main Topics in Part I.** A list of main topics in Part I contains:

- Digital signal representation based on numbers
- Quantization of Signals
- Common Signal Measurements & Features
- Central Limit Theorem
- Laplace, Fourier, Z, Cosine, & Wavelet Transforms for continuous & discrete signals.
- Linear Filters & Windowing
- Transformation, processing, digital, & adaptive filtering
- Wiener, Kalman, Particle Filters & Sampling
- Estimation, detection & spectral analysis.

- 1. Digital Signal Representation.

- **Overview.** Details of digital representations of discrete-time signals are presented in this chap bridging gap from abstract discrete-time signal notation presented in book, $x[n]$, & its representation in a digital processor, & more specifically a digital signal processor.
- **1.1. Introduction.** Continuous [Continuous signals are referred in literature also as Analog signals. Both terms are used interchangeably unless stated otherwise.] signals are necessarily sampled at discrete time intervals as well as approximated by a finite number of discrete magnitude values to be represented digitally. Because digital processing devices process data at discrete time steps, continuous signals must be sampled at discrete time intervals. Turn out: possible to sample many continuous signals at discrete time intervals, producing discrete-time signals, without any loss or degradation as compared to original analog signal. Converted continuous signals to their discrete-time representation are identical to it if certain conditions are met. Those conditions are described in Sampling Theorem.

An additional limitation of digital processing devices, degree of which is dictated by their architecture, is restriction: data must be represented by a finite number of digits, or more specifically by a finite number of bits. Typically, digital processors are designed to store & process data that have fixed specific minimal & maximal number of bits allocated for each representation. These restrictions impose representations to have a finite-precision. Process of representing a continuous actual value by its discrete representation is known as Quantization. When finite-precision is used to represent actual values following steps control quantization effects on output.

1. Quantize in time & magnitude continuous input value $x(t)$ of signal to obtain discrete-time sequence $x[n]$
2. Quantize actual values of coefficients from $\{A_k, k = 0, \dots, N\}$ to a finite-precision representation $\{a_k, k = 0, \dots, N\}$
3. Consider effects of arithmetic operations using finite-precision representations on output & modify implementation as necessary to obtain an optimal result.

Effects of quantization on continuous signal & finite-precision operations are well studied & understood. Consequently, possible to convert continuous signals to digital, process them, & reconstruct it back to continuous representation with desired quality. Reconstructed signals typically have characteristics that fulfill certain quality criteria that are preferred to analog counterparts. In proceeding sects, all 3 numbered enumerated issues regarding representation of data with finite-precision are discussed. However, also important to understand development of abstract concept of numbers & historical roots of such representation. Discussion of numbers & number systems is introduced from a historical perspective: believed: will shed light into on fundamental concepts of numbers & number systems that shaped current understanding of numbers & how they are represented.

- **1.2. Numbers.** Development of human civilization is closely followed by development of representations of numbers. In English numbers are represented by numerals. In past, there were several kinds of numeral notations, & symbols. In early days, 1 pile of items was considered equivalent to another pile of a different number of items of a different kind. This value system was used for trading of goods. Further development was achieved with standardization of “value”; a fixed number of items of 1 kind, e.g., 5, placed in a special corresponding place, & it was considered equivalent to 1 item of a special kind placed in another place. This correspondence also led to earlier ways to representing numbers in written form. Since early days, way we do arithmetic is intimately related to way we represent numbers.

- * **1.2.1. Numbers & Numerals.** Development of human civilization is closely followed by development of representations of numbers. Numbers are represented by numerals. [skipped]
- * **1.2.2. Types of Numbers.** To understand how numbers, e.g., numerals, are represented in modern digital computing systems, important to know what kind of possible numbers are in use. [skipped]
- * **1.2.3. Positional Number Systems.** In positional notation, value of a number depends on numeral as well as its position within number. Typically, value of position is power of 10. This concept leads to generalization of value represented by a numeral as follows:

$$x = \pm(d_{n-1}B^{n-1} + \dots + d_1B^1 + d_0B^0.d_{-1}B^{-1} + \dots + d_{-m}B^{-m}),$$

where \pm : sign of number, $d_i \in \{0, 1, \dots, B-1\}$ is set of numerals, $.$ is decimal, or in general radix point, B : base of number system. Note: number to left of radix point, called *integral part*, denotes an integer part of number represented by n numerals. Number to right of radix point called a *fractional part* represents a fractional number < 1 represented by m numerals. With this notation, set of real numbers \mathbb{R} can be represented.

Computers can only use a finite subset of numbers due to finite resources available to represent a number. Consequently, only a finite & limited set of numbers can be represented. This set is defined by total number of elements that it can represent as well as range of values that it covers. Most common native representation of a numeral in a computer is in Binary system or base $B = 2$. Numerical value in our accustomed, reference base 10, number system of a base 2 (or binary) number is given by following expression:

$$x = \pm(d_{n-1}2^{n-1} + \dots + d_12^1 + d_02^0.d_{-1}2^{-1} + \dots + d_{-m}2^{-m}),$$

where \pm : sign of number, $d_i \in \{0, 1\}$ takes values from set of binary numerals, $.$: binary point. Range of values & their precision is defined by n , number of bits used to represent integer portion of a number, m : number of bits to represent fractional part of number.

- 1.3. Sampling & Reconstruction of Signals. Typical system interfaces with continuous world via Analog-to-Digital (ADC) & Digital-to-Analog (DAC) Converters as depicted in Fig. 1.1: Interfacing with a continuous signal that is optionally conditioned by sensor conditioner. Analog-to-Digital Conversion: (a) Continuous signal $x(t)$, (b) Sampled signal $x_a(t)$ with sampling period T satisfying Nyquist rate as specified by Sampling Theorem. (c) Digital sequence $x[n]$ obtained after sampling & quantization.

A signal is identically reconstructed from discrete samples only when conditions specified by Sampling Theorem are satisfied. Otherwise, existence of a difference between reconstructed signal & original is called *aliasing* (răng cựa). To satisfy Sampling Theorem requirements, continuous input signal must be ensured to be band-limited. Thus, ADC is preceded by a low-pass filter. This pre-filtering is a critical step in any digital processing system. It ensures: effects of aliasing are minimized to levels that are not perceptible by intended audience. Filter is implemented as an analog low-pass filter. Band-limited signal is then sampled by a fixed sample rate or sampling frequency f_s . Sampling is performed by a sample-&-hold device. This signal is then quantized & represented in a digital form as a sequence of binary digits/bits that have values of 1's & 0's. Quantized representation of data is then converted to a desired digital representation of a DSP to facilitate further processing. Conversion process is depicted in Fig. 1.1.

Example 26. Assume input continuous-time signal is a pure periodic signal represented by expression $x(t) = A \sin(\omega_0 t + \phi) = A \sin(2\pi f_0 t + \phi)$ where A is amplitude of signal, ω_0 is frequency in radians per second rad/sec, ϕ is phase in radians, f_0 is frequency in cycles per second measured in Hertz (Hz). Assuming: continuous-time signal $x(t)$ is sampled every T secs or alternatively with sampling rate of $f_s = \frac{1}{T}$, discrete-time signal $x[n]$ representation obtained by $t = nT$ will be $x[n] = A \sin(\omega_0 nT + \phi) = A \sin(2\pi f_0 nT + \phi)$. Alternative representation of $x[n]$: $x[n] = A \sin\left(2\pi \frac{f_0}{f_s} n + \phi\right) = A \sin(2\pi F_0 n + \phi) = A \sin(\Omega_0 n + \phi)$ reveals additional properties of discrete-time signal. $F_0 = \frac{f_0}{f_s}$ defines **normalized frequency**, & Ω_0 **digital frequency** where $\Omega_0 = 2\pi F_0$, $0 \leq \Omega_0 \leq 2\pi$.

A DSP processor performs a programmed operation, typically a complex algorithm, on suitably represented input signal. Result is obtained as a sequence of digital values. Those values after being converted into an appropriate data representation (e.g., 24 bit signed integers) are converted back into continuous domain via *digital-to-analog* converter: DAC. Procedure is depicted in Fig. 1.2: Digital-to-analog conversion. (a) Processed digital signal $y[n]$. (b) Continuous signal representation $y_a(nT)$. (c) Low-pass filtered continuous signal $y(t)$.

Quantization in time (Lượng tử hóa theo thời gian), via sampling, as well as in amplitude of continuous input signals $x(t)$ to discrete-time signal $x[n]$, as well as coefficients of digital signal processing structures requires also resolving how numbers are represented by a digital signal processor.

Next sect discusses issues of quantization (lượng tử hóa), numbers, & their representations.

- * 1.3.1. Scalar Quantization. Component of system that transforms an input value $x[n]$ into 1 of a finite set of prescribed values $\hat{x}[n]$ is called *scalar quantization*. As depicted in Fig. 1.3: Conceptual representation of ADC. Ideal C/D converter represents sampling performed by sample-&-hold, & quantizer & coder combined represent ADC.

This conceptual abstraction allows us to assume: sequence $x[n]$ is obtained with infinite precision. Those values $x[n]$ are scalar quantized to a set of finite precision amplitudes denoted hereby $\hat{x}_Q[n]$. Furthermore, quantization allows: this finite-precision set of amplitudes to be represented by a corresponding set of (bit) patterns or symbols $\hat{x}[n]$. W.l.o.g., it can be assumed: input signals cover a finite range of values defined by minimal x_{\min} & maximal values x_{\max} , resp. This assumption in turn implies: set of symbols representing $\hat{x}[n]$ is finite. Process of representing a finite set of values to a finite set of symbols is known as *encoding* performed by coder, as in Fig. 1.3: Conceptual representation of ADC. Thus one can view quantization & coding as a mapping of infinite precision value of $x[n]$ to a finite precision representation $\hat{x}[n]$ picked from a finite set of symbols.

p. 12 +++

- 2. Signal Processing Background.
- 3. Fundamental of Signal Transformations.
- 4. Digital Filters.
- 5. Estimation & Detection.
- 6. Adaptive Signal Processing.
- 7. Spectral Analysis.

PART II: ML & RECOGNITION. Ideas of learning with a particular insight into influence of learning goals are discussed in 8 chaps in this part.

1. **Chap. 8: General Learning.** Include general aspects & principles of learning. An important aspect is improvement of learner after learning. In this part, emphasize on aspects related to stochastic processes as applied to learning.
2. **Chap. 9: Signal Processes, Learning, & Recognition.** Learning is a very wide & complex area. Cannot give a complete introduction into subject. Therefore, restrict ourselves to aspects & methods related to signal processes & signal recognition. Learning considers in particular hidden stochastic models. Besides a general introduction into topic, main topic is Bayesian Learning. Fundamentals of ML & its role for signal processing are briefed here.

3. **Chap. 10: Stochastic Processes.** Introduce readers to signal processes, in particular to stochastic signal processes. These are partially ordered sets of signals. In stochastic situation, there is no deterministic rule governing process. This is rather of statistical character. Main topic: hidden stochastic model. Their main characteristic: involved probabilities are unknown.
 4. **Chap. 11: Feature Extraction.** Features are short vectors describing processes. They have to be extracted. This requires techniques described in this chap. Basic techniques include spectral shaping, windowing signals, pre-emphasis, folding operators, unfolding, & spectral analysis & its envelope extraction. These techniques are again used in discussed methods e.g. linear prediction, search, mean squared error minimization, Baum–Welch algorithm, & Viterbi algorithm.
 5. **Chap. 12: Unsupervised Learning.** Unsupervised learning techniques e.g. clustering are introduced here. They are related to learning methods as applied to signal processes. Included in this discussion are K-Means, VQ, EM, & GMM.
 6. **Chap. 13: Markov Model & Hidden Stochastic Model (HSM).** Topics e.g. Gaussian mixture models (GMM), autocorrelation, Yule-Walker, co-variance, & unconstrained least squared (ULS) approach are discussed in this chap.
 7. **Chap. 14: Fuzzy Logic & Rough Sets.** When dealing with signals, uncertainty is unavoidable. 1 general method: use fuzzy sets. In order to obtain decisions that are mostly of binary character, rough sets are introduced. Also discuss fuzzy clustering & fuzzy probabilities.
 8. **Chap. 15: Neural Networks.** Introduce neural networks. In addition to basics, it includes feed forward networks. Simulation of how human brain processes information (biology) is presented briefly, as well as use of neural networks in ML (computers).
- 8. General Learning.
 - 9. Signal Processes, Learning, & Recognition.
 - 10. Stochastic Processes.
 - 11. Feature Extraction.
 - 12. Unsupervised Learning.
 - 13. Markov Model & Hidden Stochastic Model.
 - 14. Fuzzy Logic & Rough Sets.
 - 15. Neural Networks.

PART III. REAL ASPECTS & APPLICATIONS. Reader is informed about many topics that occur in applications. It contains required extensions or modifications of techniques that have been introduced in earlier parts.

1. **Chap. 16: Noisy Signals.** Noise can corrupt message contained in a signal to a large extent. Discuss different kinds & properties of noise. This is strongly connected to some measurements criteria. Signal-to-noise relation is a basic noise property. This is measured by average relation of noise signals to original signal. There are different kind of noises that require individually tailored techniques.
2. **Chap. 17: Reasoning Methods & Noise Removal.** Often, an aim is a noise-free signal, thus a main task: minimize or remove noises. An important part of this is noise removal. Noise removal depends very much on noise types. Restrict ourselves to noise types introduced so far. Each type requires different removal techniques. A major problem occurs when different types of noises occur simultaneously, which requires appropriate removal techniques.
3. **Chap. 18: Audio Signals & Speech Recognition.** Handling of audio signals has motivated many progresses in signal processing. Speech processes have specific aspects. These are introduced in some details in this chap.
4. **Chap. 19: Noisy Speech.** Speech is a major example where noise plays a role. Noise disturbs reception of speech independent of type of receiver, which can be a human being or a machine. If a machine is receiver, some response is expected.
5. **Chap. 20: Aspects of Human hearing.** A main characteristic of hearing is its temporal character. This is quite different from dealing with images. Topic is studied because one can obtain elements of recognition of machines from looking at human ear.
6. **Chap. 21: Speech Features.** Speech features contain a short & compact representation of speech. Main condition: no important info is lost. Term “important” differs from situation to situation & therefore there are several feature extraction algorithms.
7. **Chap. 22: Hidden Stochastic Model for Speech.** Speech is a typical example for hidden stochastic models. If a person speaks same words twice, uttered signals are not same. Probabilities over speech elements are unknown & they need to be determined. This chap delves into mathematical techniques used to model those processes.
8. **Chap. 23: Different Speech Applications – Part A.** Here, one finds some typical applications. Applications have in common: speech receiver is a machine. This machine is supposed to execute a command that is contained in speech. Therefore, machine has to be embedded in another complex structure that is responsible for execution as discussed here.

9. **Chap. 24: Different Speech Applications – Part B.** Different aspects of ML of a few practical examples e.g. Wake-Up-Word (WUW) are introduced. In addition, Speech Analysis & Sound Effects Laboratory (SASE Lab), Wake-Up-Word: Tool Demo, & Elevator Simulator are presented in this chap. Aim: give an essence of understanding (bản chất của sự hiểu biết), namely how humans communicate using spoken language.
 10. **Chap. 25: Biomedical Signals: ECG, EEG.** These signals are generated by human body & contain hidden info about it. A major task: extract such information & derive info they contain. There 2 types of signals that are studied: ECG signals, EEG signals.
 11. **Chap. 26: Seismic Signals.** Source of seismic signals is an event that occurs inside Earth. They contain important info; we can predict future behavior of Earth using digital signal processing techniques. A major aspect: use seismic signals for prediction of earthquakes.
 12. **Chap. 27: Radar Signals.** Radar is an electromagnetic system that detects & locates objects using electromagnetic waves. Transmitter, receiver, & target are basic components of a radar. Radar is categorized in different ways based on areas of application, system types, wave-forms, frequency bands, & antenna types. This chap is an introduction to radar sensing attached with an application.
 13. **Chap. 28: Visual Story Telling** is a visual perception of world. An informed decision about situation can be made through an effective communication following perception. Understanding context, selecting right tool to display, right selection of data visualization by removing redundancy, orientation, & focus are important for data visualization. This chap provides essence of VST.
 14. **Chap. 29: Digital Processes & Multimedia.** At low level, one presents multimedia by using signals. Traditionally, they can be 1D, 2D, or 3D. Images are a specific example of this, where signal corresponds to pixels. A major task: transform signal processes into multimedia representations. Examples of multimedia, techniques, & principles can be seen in this chap.
 15. **Chap. 30: Visualizations for Emergency Operation Center.** A visual story is told to manage an emergency environment. This chap is based on actual emergency situations. Importance of visualization in emergency situations is outlined here. How visualizations & visual analytics are applied in a typical emergency situation management center, e.g. an emergency operation center (EOC), is briefly introduced in this chap.
 16. **Chap. 31: Intelligent Interactive Communications.** Multi-modal communication combines multiple data sources for an effective & meaningful interaction. A concept of multi-modal intelligent interactive machine is proposed in this chap. This applies automation capabilities to communicate interactively applying verbal & non-verbal multimodalities of human-to-human & human-to-machine communication. Modes of communication can be speech, text, gesture, images, graphics, & visualization.
 17. **Chap. 32: Comparisons.** Present main distinctions of presented chaps.
- 16. Noisy Signals.
 - 17. Reasoning Methods & Noise Removal.
 - 18. Audio Signals & Speech Recognition.
 - 19. Noisy Speech.
 - 20. Aspects of Human Hearing.
 - 21. Speech Features.
 - 22. Hidden Stochastic Model for Speech.
 - 23. Different Speech Applications – Part A.
 - 24. Different Speech Applications – Part B.
 - 25. Biomedical Signals: ECG, EEG.
 - 26. Seismic Signal.
 - 27. Radar Signals.
 - 28. Visual Story Telling.
 - 29. Digital Processes & Multimedia.
 - 30. Visualizations of Emergency Operation Center.
 - 31. Intelligent Interactive Communications.
 - 32. Comparisons.
 - Glossary.

3 Neural Network

3.1 AMAL ALPHONSE, MICHAEL HINTERMÜLLER, ALEXANDER KISTER, CHIN HANG LUN. A neural network approach to learning solutions of a class of elliptic variational inequalities

Abstract. Develop a weak adversarial approach to solving obstacle problems using neural networks. By employing (generalized) regularized gap functions & their properties, rewrite obstacle problem (which is an elliptic variational inequality) as a minmax problem, providing a natural formulation amenable to learning. Our approach, in contrast to much of literature, does not require elliptic operator to be symmetric. Provide an error analysis for suitable discretizations of continuous problem, estimating in particular approximation & statistical errors. Parametrizing solution & test function as neural networks, apply a modified gradient descent ascent algorithm to treat problem & conclude paper with various examples & experiments. Our solution algorithm is in particular able to easily handle obstacle problem that feature biactivity (or lack of strict complementarity), a situation that poses difficulty for traditional numerical methods.

- 1. Introduction. Use neural networks to find solutions of variational inequalities (VIs) of type:

$$\text{find } u \in K : \langle Au - f, u - v \rangle_{V^*, V} \leq 0, \quad \forall v \in K, \quad (246)$$

where $V := H^1(\Omega)$: usual Sobolev space on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $\langle \cdot, \cdot \rangle_{V^*, V}$ denotes duality pairing between V & its topological dual V^* , constraint set $K := \{u \in H^1(\Omega) | u \geq \psi \text{ in } \Omega, u = h \text{ on } \partial\Omega\}$, $h \in H^{1/2}(\partial\Omega)$: given boundary data, $\psi \in H^1(\Omega)$: a given obstacle that satisfies $\psi \leq h$ on $\partial\Omega$, & $f \in L^2(\Omega)$: a given source term. Operator $A : K \subset V \rightarrow V^*$ appearing in (246) is assumed to be Lipschitz & coercive, i.e., there exist constant $C_a, C_b > 0$ s.t. (3)

$$\|Au - Av\|_{V^*} \leq C_b \|u - v\|_V, \quad \forall u, v \in K, \quad (247)$$

$$\langle Au - Av, u - v \rangle_{V^*, V} \geq C_a \|u - v\|_V^2, \quad \forall u, v \in K, \quad (248)$$

& for simplicity, focus attention on linear differential operators of form

$$\langle Au, u - v \rangle_{V^*, V} = \int_{\Omega} \nabla u \cdot \nabla(u - v) + \sum_{i=1}^n b_i \partial_{x_i} u(u - v) + ku(u - v), \quad (249)$$

where $k, b_i \geq 0, i = 1, \dots, n$, are constants (which of course have to be s.t. (3) is satisfied), $\partial_{x_i} u$: weak partial derivative of u w.r.t. i th coordinate & ∇u : weak gradient of u . In operator form, have $A = -\Delta + \sum_{i=1}^n b_i \partial_{x_i} + k\text{Id}$ with Δ representing weak Laplacian & Id : identity map. Setting $b_i = 0$ for $i = 1, \dots, n, k = 0, h \equiv 0$: prototypical example of an elliptic VI & is commonly referred to as obstacle problem [39].

Variational inequalities of type (246) have numerous applications in diverse scientific areas; mention in particular contact mechanics, processes in biological cells, ecology, fluid flow, & finance, see e.g. [52, 39, 55]. They are also fundamental objects of study in applied analysis due to their interesting structure. Indeed, VIs are examples of free boundary problems. Obstacle problems sometimes are stated in form (5)

$$0 \leq (Au - f) \perp (u - \psi) \geq 0 \text{ a.e. on } \Omega, \quad (250)$$

$$u = h \text{ a.e. on } \partial\Omega, \quad (251)$$

where $a \perp b$ stands for $ab = 0$. This formulation \Leftrightarrow (246) under sufficient regularity (see Prop. 2.2). Classical methods for solving obstacle problems or VIs include projection methods, multilevel & multigrid methods [40, 30], primal dual active set strategies & path following schemes [27], semismooth Newton schemes [25, 33], shape & topological sensitivity based methods [28, 29], level set methods & discontinuous Galerkin schemes [58]; see also [20, 9, 37, 19] & references therein.

Aim: to formulate, analyze, & implement a deep network approach to compute solutions of obstacle problems like (246) or (5). More specifically, rephrase VI as minmax optimization problem involving minimization over feasible set & maximization over all feasible test functions, both of which are parametrized by neural networks, & use a modified gradient descent ascent scheme to numerically solve for solution. Our motivation stems from fact: **neural networks can efficiently represent nonlinear, nonsmooth functions, & have added advantage of not being intrinsically reliant on a mesh**¹⁹: they provide a naturally global & meshless representation of solution, offering an advantage over other methods e.g. finite elements. Furthermore, they are able to handle complicated geometries & high-dimensional problems without great cost. Our work can also be considered as a 1st step in studying more complicated problems involving e.g. operator learning.

Related papers in the literature addressing solving elliptic variational inequalities or hemivariational inequalities via neural networks include [13, 64, 53, 31, 7]. A typical path taken by many works entails rewriting (246) as a minimization problem. Indeed, if operator A generates a bilinear form which is symmetric, then, as explained, e.g., in [52, §4.3, Remark 3.5, p. 97], VI (246) \Leftrightarrow to minimization problem $\min_{u \in K} \frac{1}{2} \langle Au, u \rangle - \langle f, u \rangle$. This formulation gives rise to a natural loss function that can be tackled via neural networks, as done in [13, 31, 64]. If A is nonsymmetric, this equivalence is not available & one cannot in general pose an associated minimization problem. However, (246) \Leftrightarrow minmax problem

$$\min_{u \in K} \max_{v \in K} \langle Au - f, u - v \rangle - \frac{1}{2\gamma} \|u - v\|_V^2 \quad (252)$$

¹⁹có thêm lợi thế là không phụ thuộc hoàn toàn vào lưới.

for a given parameter $\gamma > 0$, regardless of whether A is symmetric or not. This problem formulation appears very natural since it resembles notion of weak formulations in PDEs, which are well understood.

In order to approximate (252), express u & test function v by deep neural networks. This technique falls into class of weak adversarial network (WAN) problems in spirit of [62]; maximization for test function acts as an adversarial approaches for PDEs & related theory. Moreover, our approach is related to Physics Informed Neural Networks (PINNs); see, e.g., [41, 50]. More specifically it can be viewed as a weak PINNs-type approach with a hard constrained boundary condition.

In this work, provide a theoretical justification for minmax problem, a discretized formulation of problem amenable to computation, an error analysis as well as comprehensive numerical algorithms. A special highlight of our work: can also tackle non-symmetric problems, e.g., A given as in (4) with $b_i \neq 0$ for at least 1 $i = 1, \dots, n$.

- 2. Analysis of continuous problem. Some theoretical results concerning VI (246).
 - 2.1. Basic properties & saddle points. Address existence & uniqueness for (246). Since A is coercive & Lipschitz on $H^1(\Omega)$ & K is nonempty²⁰ closed & convex, well posedness follows from classical Lions–Stampacchia theorem [52, §4:3, Thm. 3.1] (see also [52, §4:2, Thm. 2.3] for the case where $A = -\Delta$).
 - Proposition 8** (H^2 -regularity). *Let $A := -\Delta + \sum_{i=1}^n b_i \partial_{x_i} + c \text{Id}$ with coefficients $b_i, c \in L^\infty(\Omega), c \geq c_0 \geq 0$ for a constant c_0 , s.t. coercivity condition (3b) is satisfied, $f \in L^2(\Omega), h \in H^{3/2}(\partial\Omega), \psi \in H^2(\Omega)$ with $\psi|_{\partial\Omega} \leq h$, & let Ω be convex or $C^{1,1}$. Then solution of (246) has regularity $u \in H^2(\Omega)$. Furthermore, we have a priori estimate $\|u\|_{H^2(\Omega)} \leq C^*$, where C^* is a constant (that depends in particular on f, ψ, h).*
 - 2.2. Minma approach via regularized gap function.
 - 2.3. Relaxations of problem.
- 3. Neural network approach. Wish to compute (approximate) solutions of VI (246) using neural networks. Architecture we use is essentially residual neural network considered in [16] consisting of usual affine transformations & activations combined with skip connections, inspired by original work [22]. Residual networks or ResNets have been empirically observed to be better at training deep networks & they avoid vanishing gradient problem, see e.g. [23, 60] for some analysis.

Describe this special ResNet architecture precisely. Let $\mathfrak{b}, \mathfrak{w} \in \mathbb{N}$ be given positive integers (representing depth & width of network resp.).

TROUBLE IN COMPLICATED NOTATIONS!

3.2 ASHISH VASWANI, NOAM SHAZEER, NIKI PARMAR, JAKOB USZKOREIT, LLION JONES, AIDAN N. GOMEZ, ŁUKASZ KAISER. **Attention Is All You Need.** 2017

[152738 citations]

- **Abstract.** Dominant sequence transduction models are based on complex recurrent or convolutional neural networks that include an encoder & a decoder. Best performing models also connect encoder & decoder through an attention mechanism. Propose a new simple network architecture, Transformer, based solely on attention mechanisms, dispensing with recurrence & convolutions entirely. Experiments on 2 machine translation tasks show these models to be superior in quality while being more parallelizable & requiring significantly less time to train. Our model achieves 28.4 BLEU on WMT 2014 English-to-German translation task, improving over existing best results, including ensembles, by over 2 BLEU. On WMT 2014 English-to-French translation task, our model establishes a new single-model state-of-art BLEU score of 41.0 after training for 3.5 days on 8 GPUs, a small fraction of training costs of best models from literature.
- **Equal contribution.** Listing order is random. JAKOB proposed replacing RNNs with self-attention & started effort to evaluate this idea. ASHISH, ILLIA, designed & implemented 1st Transformer models & has been crucially involved in every aspect of this work. NOAM proposed scaled dot-product attention, multi-head attention & parameter-free position representation & became other person involved in nearly every detail. NIKI designed, implemented, tuned, & evaluated countless model variants in our original codebase & tensor2tensor. LLION also experimented with novel model variants, was responsible for our initial codebase, & efficient inference & visualizations. Łukasz & Aidan spent countless long days designing various parts of & implementing tensor2tensor, replacing our earlier codebase, greatly improving results & massively accelerating our research.
- **1. Introduction.** RNNs, long short-term memory [12], & gated recurrent [7] neural networks in particular, have been firmly established as state of art approaches in sequence modeling & transduction problems e.g. language modeling & machine translation [29,2,5]. Numerous efforts have since continued to push boundaries of recurrent language models & encoder-decoder architectures [31,21,13].

Recurrent models typically factor computation along symbol positions of input & output sequences. Aligning positions to steps in computation time, they generate a sequence of hidden states h_t , as a function of previous hidden state h_{t-1} & input for position t . This inherently sequential nature precludes (ngăn cản) parallelization within training examples, which becomes critical at longer sequence lengths, as memory constraints limit batching across examples. Recent work has achieved

²⁰By properties of trace operator [59, Theorem 8.8, Chapter 1], there exists a function $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = h$, & function $\max\{w, \psi\} \in H^1(\Omega)$ belongs to K .

significant improvements in computational efficiency through factorization tricks [18] & conditional computation [26], while also improving model performance in case of the latter. Fundamental constraint of sequential computation, however, remains.

– Các mô hình truy hồi thường phân tích tính toán theo vị trí ký hiệu của chuỗi đầu vào & đầu ra. Căn chỉnh vị trí theo các bước trong thời gian tính toán, chúng tạo ra một chuỗi các trạng thái ẩn h_t , như một hàm của trạng thái ẩn trước đó h_{t-1} & đầu vào cho vị trí t . Bản chất tuần tự vốn có này ngăn cản (ngăn cản) việc song song hóa trong các ví dụ đào tạo, điều này trở nên quan trọng ở các chuỗi dài hơn, vì các ràng buộc về bộ nhớ giới hạn việc xử lý hàng loạt trên các ví dụ. Các công trình gần đây đã đạt được những cải tiến đáng kể về hiệu quả tính toán thông qua các thủ thuật phân tích [18] & tính toán có điều kiện [26], đồng thời cũng cải thiện hiệu suất mô hình trong trường hợp sau. Tuy nhiên, vẫn còn tồn tại ràng buộc cơ bản của tính toán tuần tự.

Attention mechanisms have become an integral part of compelling sequence modeling & transduction models in various tasks, allowing modeling of dependencies without regard to their distance in input or output sequences [2,16]. In all but a few cases [22], however, such attention mechanisms are used in conjunction with a recurrent network.

– Các cơ chế chú ý đã trở thành một phần không thể thiếu của mô hình hóa trình tự hấp dẫn & các mô hình chuyển đổi trong nhiều tác vụ khác nhau, cho phép mô hình hóa các mối phụ thuộc mà không cần quan tâm đến khoảng cách của chúng trong các trình tự đầu vào hoặc đầu ra [2,16]. Tuy nhiên, trong hầu hết các trường hợp ngoại trừ một số ít trường hợp [22], các cơ chế chú ý như vậy được sử dụng kết hợp với một mạng lưới tuần hoàn.

In this work propose Transformer, a model architecture eschewing recurrence & instead relying entirely on an attention mechanism to draw global dependencies between input & output. Transformer allows for significantly more parallelization & can reach a new state of art in translation quality after being trained for as little as 12 hours on 8 P100 GPUs.

– Trong tác phẩm này, đề xuất Transformer, một kiến trúc mô hình tránh lặp lại & thay vào đó hoàn toàn dựa vào cơ chế chú ý để rút ra các mối phụ thuộc toàn cục giữa đầu vào & đầu ra. Transformer cho phép song song hóa nhiều hơn đáng kể & có thể đạt đến trạng thái nghệ thuật mới về chất lượng dịch sau khi được đào tạo trong thời gian ít nhất là 12 giờ trên 8 GPU P100.

- 2. Background. Goal of reducing sequential computation also forms foundation of Extended Neural GPU [20], ByteNet [15] & 7 ConvS2S [8], all of which use CNNs as basic building block, computing hidden representations in parallel for all input & output positions. In these models, number of operations required to relate signals from 2 arbitrary input or output positions grows in distance between positions, linearly for ConvS2S & logarithmically for ByteNet. This makes it more difficult to learn dependencies between distant positions [11]. In Transformer this is reduced to a constant number of operations, albeit at cost of reduced effective resolution due to averaging attention-weighted positions, an effect we counteract with Multi-Head Attention as described in Sect. 3.2.

Self-attention, sometimes called intra-attention is an attention mechanism relating different positions of a single sequence in order to compute a representation of sequence. Self-attention has been used successfully in a variety of tasks including reading comprehension, abstractive summarization, textual entailment & learning task-independent sentence representations [4, 22, 23, 19].

End-to-end memory networks are based on a recurrent attention mechanism instead of sequence-aligned recurrence & have been shown to perform well on simple-language question answering & language modeling tasks [28].

To best of our knowledge, however, Transformer is 1st transduction model relying entirely on self-attention to compute representations of its input & output without using sequence-aligned RNNs or convolution. In following sects, describe Transformer, motivate self-attention & discuss its advantages over models e.g. [14,15,8].

- 3. Model Architecture. Most competitive neural sequence transduction models have an encoder-decoder structure [5, 2, 29]. Here, encoder maps an input sequence of symbol representations (x_1, \dots, x_n) to a sequence of continuous representations $\mathbf{z} = (z_1, \dots, z_n)$. Given \mathbf{z} , decoder then generates an output sequence (y_1, \dots, y_m) of symbols 1 element at a time. At each step model is auto-regressive [9], consuming previously generated symbols as additional input when generating text.

Transformer follows this overall architecture using stacked self-attention & point-wise, fully connected layers for both encoder & decoder, shown in Fig. 1: Transformer - model architecture.

- 3.1. Encoder & Decoder Stacks. Encoder is composed of a stack of $N = 6$ identical layers. Each layer has 2 sub-layers. 1st is a multi-head self-attention mechanism, & 2nd is a simple, position-wise fully connected feed-forward network. Employ a residual connection [10] around each of 2 sub-layers, followed by layer normalization [1]. I.e., output of each sub-layer is $\text{LayerNorm}(x + \text{Sublayer}(x))$, where $\text{Sublayer}(x)$: function implemented by sub-layer itself. To facilitate these residual connections, all sub-layers in model, as well as embedding layers, produce outputs of dimension $d_{\text{model}} = 512$.

Decoder. Decoder is also composed of a stack of $N = 6$ identical layers. In addition to 2 sub-layers in each encoder layer, decoder inserts a 3rd sub-layer, which performs multi-head attention over output of encoder stack. Similar to encoder, employ residual connections around each of sub-layers, followed by layer normalization. Also modify self-attention sub-layer in decoder stack to prevent positions from attending to subsequent positions. This masking, combined with fact: output embeddings are offset by 1 position, ensures: predictions for position i can depend only on known outputs at positions $< i$.

- 3.2. Attention. An attention function can be described as mapping a query & a set of key-value pairs to an output, where query, keys, values, & output are all vectors. Output is computed as a weighted sum of values, where weight assigned to each value is computed by a compatibility function of query with corresponding key.

- * 3.2.1. Scaled Dot-Product Attention. Call our particular attention “Scaled Dot-Product Attention” (Fig. 2: Left: Scaled Dot-Product Attention. Right: Multi-Head Attention consists of several attention layers running in parallel.). Input consists of queries & keys of dimension d_k , & values of dimension d_v . Compute dot products of query with all keys, divide each by $\sqrt{d_k}$, & apply a softmax function to obtain weights on values. In practice, compute attention function on a set of queries simultaneously, packed together into a matrix Q . Keys & values are also packed together into matrices K, V . Compute matrix of output as:

$$\text{Attention}(Q, K, V) = \text{softmax}$$

- * 3.2.2. Multi-Head Attention.
- * 3.2.3. Applications of Attention in our Model.
- o 3.3. Position-wise Feed-Forward Networks.
- o 3.4. Embeddings & Softmax.
- 4. Why Self-Attention.
- 5. Training.
- 6. Results.
- 7. Conclusion.

4 Recurrent Neural Network

4.1 ZACHARY C. LIPTON, JOHN BERKOWITZ, CHARLES ELKAN. A Critical Review of Recurrent Neural Networks for Sequence Learning

[3525 citations] Abstract. Countless learning tasks require dealing with sequential data. Image captioning, speech synthesis, & music generation all require: a model produce outputs that are sequences. In other domains, e.g. time series prediction, video analysis, & musical information retrieval, a model must learn from inputs that are sequences. Interactive tasks, e.g. translating natural language, engaging in dialogue, & controlling a robot, often demand both capabilities. Recurrent neural networks (RNNs) are connectionist models that capture dynamics of sequences via cycles in network of nodes. Unlike standard feedforward neural networks, recurrent networks retain a state that can represent information from an arbitrarily long context window. Although recurrent neural networks have traditionally been difficult to train, & often contain millions of parameters, recent advances in network architectures, optimization techniques, & parallel computation have enabled successful large-scale learning with them. In recent years, systems based on long short-term memory (LSTM) & bidirectional (BRNN) architectures have demonstrated ground-breaking performance on tasks as varied as image captioning, language translation, & handwriting recognition. In this survey, review & synthesize research that over past 3 decades 1st yielded & then made practical these powerful learning models. When appropriate, reconcile conflicting notation & nomenclature. When appropriate, reconcile conflicting notation & nomenclature. Goal: to provide a self-contained explication of state of art together with a historical perspective & refs to primary research.

- 1. Introduction. Neural networks are powerful learning models that achieve state-of-art results in a wide range of supervised & unsupervised machine learning tasks. They are suited especially well for machine perception tasks, where raw underlying features are not individually interpretable. This success is attributed to their ability to learn hierarchical representations, unlike traditional methods that rely upon hand-engineered features [Farabet et al., 2013]. Over past several years, storage has become more affordable, datasets have grown far larger, & field of parallel computing has advanced considerably. In setting of large datasets, simple linear models tend to under-fit, & often under-utilize computing resources. Deep learning methods, in particular those based on deep belief networks (DBNs), which are greedily built by stacking restricted Boltzmann machines, & convolutional neural networks, which exploit local dependency of visual information, have demonstrated record-setting results on many important applications.

However, despite their power, standard neural networks have limitations. Most notably, they rely on assumption of independence among training & test examples. After each example (data point) is processed, entire state of network is lost. If each example is generated independently, this presents no problem. But if data points are related in time or space, this is unacceptable. Frames from video, snippets of audio, & words pulled from sentences, represent settings where independence assumption fails. Additionally, standard networks generally rely on examples being vectors of fixed length. Thus desirable to extend these powerful learning tools to model data with temporal or sequential structure & varying length inputs & outputs, especially in many domains where neural networks are already state of art. Recurrent neural networks (RNNs) are connectionist models with ability to selectively pass information across sequence steps, while processing sequential data 1 element at a time. Thus they can model input &/or output consisting of sequences of elements that are not independent. Further, recurrent neural networks can simultaneously model sequential & time dependencies on multiple scales.

In following subsections, explain fundamental reasons why recurrent neural networks are worth investigating. To be clear, motivated by a desire to achieve empirical results. This motivation warrants clarification because recurrent networks have roots in both cognitive modeling & supervised machine learning. Owing to this difference of perspectives, many published

papers have different aims & priorities. In many foundational papers, generally published in cognitive science & computational neuroscience journals, e.g. [Hopfield, 1982, Jordan, 1986, Elman, 1990], biologically plausible mechanisms are emphasized. In [Schuster & Paliwal, 1997, Socher et al., 2014, Karpathy & Fei-Fei, 2014], biological inspiration is downplayed in favor of achieving empirical results on important tasks & datasets. This review is motivated by practical results rather than biological plausibility, but where appropriate, draw connections to relevant concepts in neuroscience. Given empirical aim, now address 3 significant questions that one might reasonably want answered before reading further.

- 1.1. Why model sequentiality explicitly? In light of practical success & economic value of sequence-agnostic models, this is a fair question. Support vector machines, logistic regression, & feedforward networks have proved immensely useful without explicitly modeling time. Arguably, precisely assumption of independence that has led to much recent progress in machine learning. Further, many models implicitly capture time by concatenating each input with some number of its immediate predecessors & successors, presenting machine learning model with a sliding window of context about each point of interest. This approach has been used with deep belief nets for speech modeling by Maas et al. [2012].

Unfortunately, despite usefulness of independence assumption, it precludes modeling long-range dependencies. E.g., a model trained using a finite-length context window of length 5 could never be trained to answer simple question, “*what was data point seen 6 time steps ago?*” For a practical application e.g. call center automation, such a limited system might learn to route calls, but could never participate with complete success in an extended dialogue. Since earliest conception of artificial intelligence, researchers have sought to build systems that interact with humans in time. In ALAN TURING’s groundbreaking paper *Computing Machinery & Intelligence*, he proposes an “imitation game” which judges a machine’s intelligence by its ability to convincingly engage in dialogue [Turing, 1950]. Besides dialogue systems, modern interactive systems of economic importance include self-driving cars & robotic surgery, among others. Without an explicit model of sequentiality or time, it seems unlikely that any combination of classifiers or regressors can be cobbled together to provide this functionality.

- 1.2. Why not use Markov models? Recurrent neural networks are not only models capable of representing time dependencies. Markov chains, which model transitions between states in an observed sequence, were 1st described by mathematician ANDREY MARKOV in 1906. Hidden Markov models (HMMs), which model an observed sequence as probabilistically dependent upon a sequence of unobserved states, were described in 1950s & have been widely studied since 1960s [Stratonovich, 1960]. However, traditional Markov model approaches are limited because their states must be drawn from a modestly sized discrete state space S . Dynamic programming algorithm that is used to perform efficient inference with hidden Markov models scales in time $O(|S|^2)$ [Viterbi, 1967]. Further, transition table capturing probability of moving between any 2 time-adjacent states is of size $|S|^2$. Thus, standard operations become infeasible with an HMM when set of possible hidden states grows large. Further, each hidden state can depend only on immediately previous state. While possible to extend a Markov model to account for a larger context window by creating a new state space equal to cross product of possible states at each time in window, this procedure grows state space exponentially with size of window, rendering Markov models computationally impractical for modeling long-range dependencies [Graves et al., 2014].

Given limitations of Markov models, ought to explain why reasonable that connectionist models, i.e., artificial neural networks, should fare better. 1st, recurrent neural networks can capture long-range time dependencies, overcoming chief limitation of Markov models. This point requires a careful explanation. As in Markov models, any state in a traditional RNN depends only on current input as well as on state of network at previous time step.²¹ However, hidden state at any time step can contain information from a nearly arbitrarily long context window. This is possible because number of distinct states that can be represented in a hidden layer of nodes grows exponentially with number of nodes in layer. Even if each node took only binary values, network could represent 2^N states where N is number of nodes in hidden layer. When value of each node is a real number, a network can represent even more distinct states. While potential expressive power of a network grows exponentially with number of nodes, complexity of both inference & training grows at most quadratically.

- 1.3. Are RNNs too expensive? Finite-sized RNNs with nonlinear activations are a rich family of models, capable of nearly arbitrary computation. A well-known result: a finite-sized recurrent neural network with sigmoidal activation functions can simulate a universal Turing machine [Siegelmann & Sontag, 1991]. Capability of RNNs to perform arbitrary computation demonstrates their expressive power, but one could argue: C programming language is equally capable of expressing arbitrary programs. & yet there are no papers claiming: invention of C represents a panacea (thuốc chữa bách bệnh) for ML. A fundamental reason: there is no simple way of efficiently exploring space of C programs. In particular, there is no general way to calculate gradient of an arbitrary C program to minimize a chosen loss function. Moreover, given any finite dataset, there exist countless programs which overfit dataset, generating desired training output but failing to generalize to test examples.

Why then should RNNs suffer less from similar problem? 1st, given any fixed architecture (set of nodes, edges, & activation functions), recurrent neural networks with this architecture are differentiable end to end. Derivative of loss function can be calculated w.r.t. each of parameters (weights) in model. Thus, RNNs are amenable to gradient-based training. 2nd, while Turing-completeness of RNNs is an impressive property, given a fixed-size RNN with a specific architecture, not actually possible to reproduce any arbitrary program. Further, unlike a program composed in C, a recurrent neural network can be regularized via standard techniques that help prevent overfitting, e.g. weight decay, dropout, & limiting degrees of freedom.

- 1.4. Comparison to prior literature. Literature on recurrent neural networks can seem impenetrable to uninitiated. Shorter papers assume familiarity with a large body of background literature, while diagrams are frequently underspecified, failing

²¹While traditional RNNs only model dependence of current state on previous state, bidirectional recurrent neural networks (BRNNs) [Schuster & Paliwal, 1997] extend RNNs to model dependence on both past states & future states.

to indicate which edges span time steps & which do not. Jargon abounds, & notation is inconsistent across papers or overloaded within 1 paper. Readers are frequently in unenviable position of having to synthesize conflicting information across many papers in order to understand just one. E.g., in many papers subscripts index both nodes & time steps. In others, h simultaneously stands for a link function & a layer of hidden nodes. Variable t simultaneously stands for both time indices & targets, sometimes in same equation. Many excellent research papers have appeared recently, but clear reviews of recurrent neural network literature are rare.

Among most useful resources are a recent book on supervised sequence labeling with recurrent neural network [Graves, 2012], & an earlier doctoral thesis [Gers, 2001]. A recent survey covers recurrent neural nets for language modeling [De Mulder et al., 2015]. Various authors focus on specific technical aspects; e.g. Pearlmutter [1995] surveys gradient calculations in continuous time recurrent neural networks. Aim: to provide a readable, intuitive, consistently notated, & reasonably comprehensive but selective survey of research on recurrent neural networks for learning with sequences. Emphasize architectures, algorithms, & results, but aim also to distill intuitions that have guided this largely heuristic & empirical field. In addition to concrete modeling details, offer qualitative arguments, a historical perspective, & comparisons to alternate methodologies where appropriate.

- 2. Background. This sect introduces formal notation & provides a brief background on neural networks in general.

- 2.1. Sequences. Input to an RNN is a sequence, &/or its target is a sequence.

An input sequence can be denoted $(\mathbf{x}^{(1)}\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(T)})$ where each data point $\mathbf{x}^{(t)}$: a real-valued vector. Similarly, a target sequence can be denoted $(\mathbf{y}^{(1)}\mathbf{y}^{(2)}, \dots, \mathbf{y}^{(T)})$. A training set typically is a set of examples where each example is an (input sequence, target sequence) pair, although commonly either input or output may be a single data point. Sequences may be of finite or countably infinite length. When they are finite, maximum time index of sequence is called T . RNNs are not limited to time-based sequences. They have been used successfully on non-temporal sequence data, including genetic data [Baldi & Pollastri, 2003]. However, in many important applications of RNNs, sequences have an explicit or implicit temporal aspect. While often refer to time in this survey, methods described here are applicable to non-temporal as well as to temporal tasks. Using temporal terminology, an input sequence consists of data points $\mathbf{x}^{(t)}$ that arrive in a discrete sequence of *time steps* indexed by t . A target sequence consists of data points $\mathbf{y}^{(t)}$. Use superscripts with parentheses for time, & not subscripts, to prevent confusion between sequence steps & indices of nodes in a network. When a model produces predicted data points, these are labeled $\hat{\mathbf{y}}^{(t)}$.

Time-indexed data points may be equally spaced samples from a continuous real-world process. Examples include still images that comprise frames of videos or discrete amplitudes sampled at fixed intervals that comprise audio recordings. Time steps may also be ordinal, with no exact correspondence to durations. In fact, RNNs are frequently applied to domains where sequences have a defined order but no explicit notion of time. This is case with natural language. In word sequence “John Coltrane plays saxophone”, $\mathbf{x}^{(1)} = \text{John}$, $\mathbf{x}^{(2)} = \text{Coltrane}$, etc.

- 2.2. Neural networks. Neural networks are biologically inspired models of computation. Generally, a neural network consists of a set of *artificial neurons*, commonly referred to as *nodes* or *units*, & a set of directed edges between them, which intuitively represent *synapses* in a biological neural network. Associated with each neuron j is an activation function $l_j(\cdot)$, which is sometimes called a *link function*. Use notation l_j & not h_j , unlike some other papers, to distinguish activation function from values of hidden nodes in a network, which, as a vector, is commonly notated \mathbf{h} in literature.

Associated with each edge from node j' to j is a weight $w_{jj'}$. Following convention adopted in several foundational papers. [Hochreiter & Schmidhuber, 1997, Gers et al., 2000, Gers, 2001, Sutskever et al., 2011], index neurons with j & j' , & $w_{jj'}$ denotes “to-form” weight corresponding to directed edge to node j from node j' . Important to note: in many refs indices are flipped & $w_{j'j} \neq w_{jj'}$ denotes “from-to” weight on directed edge from node j' to node j , as in lecture notes by Elkan [2015] & in [Wikipedia/backpropagation](https://en.wikipedia.org/wiki/Backpropagation).

Value v_j of each neuron j is calculated by applying its activation function to a weighted sum of values of its input nodes (Fig. 1: An artificial neuron computes a nonlinear function of a weighted sum of its inputs):

$$v_j = l_j \left(\sum_{j'} w_{jj'} \cdot v_{j'} \right). \quad (253)$$

For convenience, term weighted sum inside parentheses *incoming activation* & notate it as a_j . Represent this computation in diagrams by depicting neurons as circles & edges as arrows connecting them. When appropriate, indicate exact activation function with a symbol, e.g., σ for sigmoid.

Common choices for activation function include sigmoid $\sigma(z) = \frac{1}{1+e^{-z}}$ & tanh function $\phi(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$. The latter was become common in feedforward neural nets & was applied to recurrent nets by Sutskever et al. [2011]. Another activation function which has become prominent in deep learning research is rectified linear unit (ReLU) whose formula is $l_j(z) = \max\{0, z\}$. This type of unit has been demonstrated to improve performance of many deep neural networks [Nair & Hinton, 2010, Maas et al., 2012, Zeiler et al., 2013] on tasks as varied as speech processing & object recognition, & has been used in recurrent neural networks by Bengio et al. [2013].

Activation function at output nodes depends upon task. For multiclass classification with K alternative classes, apply a softmax nonlinearity in an output layer of K nodes. Softmax function calculates

$$\hat{y}_k = \frac{e^{a_k}}{\sum_{k'=1}^K e^{a_{k'}}}, \quad \forall k = 1, \dots, K. \quad (254)$$

Denominator is a normalizing term consisting of sum of numerators, ensuring: outputs of all nodes sum to 1. For multilabel classification, activation function is simply a pointwise sigmoid, & for regression typically have linear output.

- 2.3. Feedforward networks & backpropagation. With a neural model of computation, one must determine order in which computation should proceed. Should nodes be sampled 1 at a time & updated, or should value of all nodes be calculated at once & then all updates applied simultaneously? Feedforward networks (Fig. 2: A feedforward neural network. An example is presented to network by setting values of blue (bottom) nodes. Values of nodes in each layer are computed successively as a function of prior layers until output is produced at topmost layer.) are a restricted class of networks which deal with this problem by forbidding cycles in directed graph of nodes. Given absence of cycles, all nodes can be arranged into layers, & outputs in each layer can be calculated given outputs from lower layers.

Input \mathbf{x} to a feedforward network is provided by setting values of lowest layer. Each higher layer is then successively computed until output is generated at topmost layer $\hat{\mathbf{y}}$. Feedforward networks are frequently used for supervised learning tasks e.g. classification & regression. Learning is accomplished by iteratively updating each of weights to minimize a loss function, $\mathcal{L}(\hat{\mathbf{y}}, \mathbf{y})$, which penalizes distance between output $\hat{\mathbf{y}}$ & target \mathbf{y} .

Most successful algorithm for training neural networks is backpropagation, introduced for this purpose by Rumelhart et al. [1985]. Backpropagation uses chain rule to calculate derivative of loss function \mathcal{L} w.r.t. each parameter in network. Weights are then adjusted by gradient descent. Because loss surface is non-convex, there is no assurance that backpropagation will reach a global minimum. Moreover, exact optimization is known to be an NP-hard problem. However, a large body of work on heuristic pre-training & optimization techniques has led to impressive empirical success on many supervised learning tasks. In particular, convolutional neural networks, popularized by Le Cun et al. [1990], are a variant of feedforward neural network that holds records since 2012 in many computer vision tasks e.g. object detection [Krizhevsky et al., 2012].

Nowadays, neural networks are usually trained with stochastic gradient descent (SGD) using mini-batches. With batch size = 1, stochastic gradient update equation is

$$\mathbf{w} \leftarrow \mathbf{w} - \eta \nabla_{\mathbf{w}} F_i, \quad (255)$$

where η : learning rate, $\nabla_{\mathbf{w}} F_i$: gradient of objective function w.r.t. parameters \mathbf{w} as calculated on a single example (x_i, y_i) . Many variants of SGD are used to accelerate learning. Some popular heuristics, e.g. AdaGrad [Duchi et al., 2011], AdaDelta [Zeiler, 2012], & RMSprop [Tieleman & Hinton, 2012], tune learning rate adaptively for each feature. AdaGrad, arguably most popular, adapts learning rate by caching sum of squared gradients w.r.t. each parameter at each time step. Step size for each feature is multiplied by inverse of square root of this cached value. AdaGrad leads to fast convergence on convex error surfaces, but because cached sum is monotonically increasing, method has a monotonically decreasing learning rate, which may be undesirable on highly non-convex loss surfaces. RMSprop modifies AdaGrad by introducing a decay factor in cache, changing monotonically growing value into a moving average. Momentum methods are another common SGD variant used to train neural networks. These methods add to each update a decaying sum of previous updates. When momentum parameter is tuned well & network is initialized well, momentum methods can train deep nets & recurrent nets competitively with more computationally expensive methods like Hessian-free optimizer of Sutskever et al. [2013].

To calculate gradient in a feedforward neural network, backpropagation proceeds as follows. 1st, an example is propagated forward through network to produce a value v_j at each node & outputs $\hat{\mathbf{y}}$ at topmost layer. Then, a loss function value $\mathcal{L}(\hat{y}_k, y_k)$ is computed at each output node k . Subsequently, for each output node k , calculate

$$\delta_k = \frac{\partial \mathcal{L}(\hat{y}_k, y_k)}{\partial \hat{y}_k} \cdot l'_k(a_k). \quad (256)$$

Given these values δ_k , for each node in immediately prior layer, calculate

$$\delta_j = l'(a_j) \sum_k \delta_k \cdot w_{kj}. \quad (257)$$

This calculation is performed successively for each lower layer to yield δ_j for every node j given δ values for each node connected to j by an outgoing edge. Each value δ_j represents derivative $\partial_{a_j} \mathcal{L}$ of total loss function w.r.t. that node's incoming activation. Given values v_j calculated during forward pass, & values δ_j calculated during backward pass, derivative of loss \mathcal{L} w.r.t. a given parameter $w_{jj'}$ is

$$\partial_{w_{jj'}} \mathcal{L} = \delta_j v'_{j'}. \quad (258)$$

Other methods have been explored for learning weights in a neural network. A number of papers from 1990s [Belew et al., 1990, Gruau et al., 1994] championed idea of learning neural networks with genetic algorithms, with some even claiming: achieving success on real-world problems only by applying many small changes to weights of a network was impossible. Despite subsequent success of backpropagation, interest in genetic algorithms continues. Several recent papers explore genetic algorithms for neural networks, especially as a means of learning architecture of neural networks, a problem not addressed by backpropagation [Bayer et al., 2009, Harp & Samad, 2013]. By *architecture*, mean number of layers, number of nodes in each, connectivity pattern among layers, choice of activation functions, etc.

1 open question in neural network research is how to exploit sparsity in training. In a neural network with sigmoidal or tanh activation functions, nodes in each layer never take value exactly 0. Thus, even if inputs are sparse, nodes at each hidden layer are not. However, rectified linear units (ReLU) introduce sparsity to hidden layers [Glorot et al., 2011]. In this setting, a promising path may be to store sparsity pattern when computing each layer's values & use it to speed up computation of next layer in network. Some recent work shows: given sparse inputs to a linear model with a standard regularizer, sparsity can be fully exploited even if regularization makes gradient be not sparse [Carpenter, 2008, Langford et al., 2009, Singer & Duchi, 2009, Lipton & Elkan, 2015].

- **3. Recurrent neural networks.** Recurrent neural networks are feedforward neural networks augmented by inclusion of edges that span adjacent time steps, introducing a notion of time to model. Like feedforward networks, RNNs may not have cycles among conventional edges. However, edges that connect adjacent time steps, called *recurrent edges*, may form cycles, including cycles of length 1 that are self-connections from a node to itself across time. At time t , nodes with recurrent edges receive input from current data point $\mathbf{x}^{(t)}$ & also from hidden node values $\mathbf{h}^{(t-1)}$ in network's previous state. Output $\hat{\mathbf{y}}^{(t)}$ at each time t is calculated given hidden node values $\mathbf{h}^{(t)}$ at time t . Input $\mathbf{x}^{(t-1)}$ at time $t-1$ can influence output $\hat{\mathbf{y}}^{(t)}$ at time t & later by way of recurrent connections.

2 equations specify all calculations necessary for computations at each time step on forward pass in a simple recurrent neural network as in Fig. 3: A simple recurrent network. At each time step t , activation is passed along solid edges as in a feedforward network. Dashed edges connect a source node at each time t to a target node at each following time $t+1$:

$$\mathbf{h}^{(t)} = \sigma(W_{\text{hx}}\mathbf{x}^{(t)} + W_{\text{hh}}\mathbf{h}^{(t-1)} + \mathbf{b}_h), \quad (259)$$

$$\hat{\mathbf{y}}^{(t)} = \text{softmax}(W_{\text{yh}}\mathbf{h}^{(t)} + \mathbf{b}_y). \quad (260)$$

Here W_{hx} : matrix of conventional weights between input & hidden layer & W_{hh} : matrix of recurrent weights between hidden layer & itself at adjacent time steps. Vectors $\mathbf{b}_h, \mathbf{b}_y$: bias parameters which allow each node to learn an offset.

Dynamics of network depicted in Fig. 3 across time steps can be visualized by *unfolding* it as in Fig. 4: Recurrent network of Fig. 3 unfolded across time steps. Given this picture, network can be interpreted not as cyclic, but rather as a deep network with 1 layer per time step & shared weights across time steps. Then clear: unfolded network can be trained across many time steps using backpropagation. This algorithm, called *backpropagation through time* (BPTT), was introduced by Werbos [1990]. All recurrent networks in common current use apply it.

- **3.1. Early recurrent network designs.** Foundational research on recurrent networks took place in 1980s. In 1982, HOPFIELD introduced a family of recurrent neural networks that have pattern recognition capabilities [Hopfield, 1982]. They are defined by values of weights between nodes & link functions are simple thresholding at 0. In these nets, a pattern is placed in network by setting values of nodes. Network then runs for some time according to its update rules, & eventually another pattern is read out. Hopfield networks are useful for recovering a stored pattern from a corrupted version & are forerunners of Boltzmann machines & auto-encoders.

An early architecture for supervised learning on sequences was introduced by Jordan [1986]. Such a network (Fig. 5: A recurrent neural network as proposed by Jordan [1986]. Output units are connected to special units that at next time step feed into themselves & into hidden units.) is a feedforward network with a single hidden layer that is extended with special units.²² Output node values are fed to special units, which then feed these values to hidden nodes at following time step. If output values are actions, special units allow network to remember actions taken at previous time steps. Several modern architectures use a related form of direct transfer from output nodes; Sutskever et al. [2014] translates sentences between natural languages, & when generating a text sequence, word chosen at each time step is fed into network as input at following time step. Additionally, special units in a Jordan network are self-connected. Intuitively, these edges allow sending information across multiple time steps without perturbing output at each intermediate time step.

Architecture introduced by Elman [1990] is simpler than earlier Jordan architecture. Associated with each unit in hidden layer is a context unit. Each such unit j' takes as input state of corresponding hidden node j at previous time step, along an edge of fixed weight $w_{j'j} = 1$. This value then feeds back into same hidden node j along a standard edge. This architecture is equivalent to a simple RNN in which each hidden node has a single self-connected recurrent edge. Idea of fixed-weight recurrent edges that make hidden nodes self-connected is fundamental in subsequent work on LSTM networks [Hochreiter & Schmidhuber, 1997].

Elman [1990] trains network using backpropagation & demonstrates: network can learn time dependencies. Paper features 2 sets of experiments. The 1st extends logical operation *exclusive or* (XOR) to time domain by concatenating sequences of 3 tokens. For each 3-token segment, e.g., "011", 1st 2 tokens ("01") are chosen randomly & 3rd ("1") is set by performing xor on 1st 2. Random guessing should achieve accuracy of 50%. A perfect system should perform same as random for 1st 2 tokens, but guess 3rd token perfectly, achieving accuracy of 66.7%. Simple network of Elman [1990] does in fact approach this maximum achievable score.

- **3.2. Training recurrent network designs.** Learning with recurrent networks has long been considered to be difficult. Even for standard feedforward networks, optimization task is NP-complete [Blum & Rivest, 1993]. But learning with recurrent networks can be especially challenging due to difficulty of learning long-range dependencies, as described by Bengio et al. [1994] & expanded upon by Hochreiter et al. [2001]. Problems of *vanishing* & *exploding* gradients occur when backpropagating errors across many time steps. As a toy example, consider a network with a single input node, a single output node, & a single recurrent hidden node (Fig. 7: A simple recurrent net with 1 input unit, 1 output unit, & 1 recurrent hidden unit.). Now consider an input passed to network at time τ & an error calculated at time t , assuming input of 0 in intervening time steps. Typing of weights across time steps means: recurrent edge at hidden node j always has same weight. Therefore, contribution of input at time τ to output at time t will either explode or approach 0, exponentially fast, as $t - \tau$ grows large. Hence derivative of error w.r.t. input will either explode or vanish.

Fig. 6: A recurrent neural network as described by Elman [1990]. Hidden units are connected to context units, which feed back into hidden units at next time step.

²²[Jordan 1986] calls special units "state units" while [Elman 1990] calls a corresponding structure "context units." In this paper simplify terminology by using only "context units".

Which of 2 phenomena occurs depends on whether weight of recurrent edge $|w_{jj}| > 1$ or $|w_{jj}| < 1$ & on activation function in hidden node (Fig. 8: A visualization of vanishing gradient problem, using network depicted in Fig. 7, adapted from Graves [2012]. If weight along recurrent edge < 1 , contribution of input at 1st time step to output at final time step will decrease exponentially fast as a function of length of time interval in between.). Given a sigmoid activation function, vanishing gradient problem is more pressing, but with a rectified linear unit $\max\{0, x\}$, easier to imagine exploding gradient. Pascanu et al. [2012] give a thorough mathematical treatment of vanishing & exploding gradient problems, characterizing exact conditions under which these problems may occur. Given these conditions, they suggest an approach to training via a regularization term that forces weights to values where gradient neither vanishes nor explodes.

Truncated backpropagation through time (TBPTT) is 1 solution to exploding gradient problem for continuously running networks [Williams & Zipser, 1989]. With TBPTT, some maximum number of time steps is set along which error can be propagated. While TBPTT with a small cutoff can be used to alleviate exploding gradient problem, it requires: one sacrifice ability to learn long-range dependencies. LSTM architecture described below uses carefully designed nodes with recurrent edges with fixed unit weight as a solution to vanishing gradient problem.

Issue of local optima is an obstacle to effective training that cannot be dealt with simply by modifying network architecture. Optimizing even a single hidden-layer feedforward network is an NP-complete problem [Blum & Rivest, 1993]. However, recent empirical & theoretical studies suggest: in practice, issue may not be as important as once thought. Dauphin et al. [2014] show: while many critical points exist on error surfaces of large neural networks, ratio of saddle points to true local minima increases exponentially with size of network, & algorithms can be designed to escape from saddle points.

Overall, along with improved architectures explained below, fast implementations & better gradient-following heuristics have rendered RNN training feasible. Implementations of forward & backward propagation using GPUs, e.g. Theano [Bergstra et al., 2010] & Torch [Collobert et al., 2011] packages, have made it straightforward to implement fast training algorithms. In 1996, prior to introduction of LSTM, attempts to train recurrent nets to bridge long time gaps were shown to perform no better than random guessing [Hochreiter & Schmidhuber, 1996]. However, RNNs are now frequently trained successfully.

For some tasks, freely available software can be run on a single GPU & produce compelling results in hours [Karpathy, 2015]. Martens & Sutskever [2011] reported success training recurrent neural networks with a Hessian-free truncated Newton approach, & applied method to a network which learns to generate text 1 character at a time in [Sutskever et al., 2011]. In paper that describes abundance of saddle points on error surfaces of neural networks [Dauphin et al., 2014], authors present a saddle-free version of Newton's method. Unlike Newton's method, which is attracted to critical points, including saddle points, this variant is specially designed to escape from them. Experimental results include a demonstration of improved performance on recurrent networks. Newton's method requires computing Hessian, which is prohibitively expensive for large networks, scaling quadratically with number of parameters. While their algorithm only approximates Hessian, still computationally expensive compared to SGD. Thus authors describe a hybrid approach in which saddle-free Newton method is applied only in places where SGD appears to be stuck.

- 4. Modern RNN architectures. Most successful RNN architectures for sequence learning stem from 2 papers published in 1997. 1st paper, *Long Short-Term Memory* by Hochreiter & Schmidhuber [1997], introduces *memory cell*, a unit of computation that replaces traditional nodes in hidden layer of a network. With these memory cells, networks are able to overcome difficulties with training encountered by earlier recurrent networks. 2nd paper, *Bidirectional Recurrent Neural Networks* by Schuster & Paliwal [1997], introduces an architecture in which information from both future & past are used to determine output at any point in sequence. This is in contrast to previous networks, in which only past input can affect output, & has been used successfully for sequence labeling tasks in natural language processing, among others. Fortunately, 2 innovations are not mutually exclusive, & have been successfully combined for phoneme classification [Graves & Schmidhuber, 2005] & handwriting recognition [Graves et al., 2009]. In this section, explain LSTM & BRNN & describe *neural Turing machine* (NTM), which extends RNNs with an addressable external memory [Graves et al., 2014].

- 4.1. Long short-term memory (LSTM). Hochreiter & Schmidhuber [1997] introduced LSTM model primarily in order to overcome problem of vanishing gradients. This model resembles a standard recurrent neural network with a hidden layer, but each ordinary node (Fig. 1) in hidden layer is replaced by a *memory cell* (Fig. 9: 1 LSTM memory cell as proposed by Hochreiter & Schmidhuber [1997]. Self-connected node is internal state s . Diagonal line indicates: it is linear, i.e., identity link function is applied. Blue dashed line is recurrent edge, which has fixed unit weight. Nodes marked II output product of their inputs. All edges into & from II nodes also have fixed unit weight.). Each memory cell contains a node with a self-connected recurrent edge of fixed weight 1, ensuring: gradient can pass across many time steps without vanishing or exploding. To distinguish refs to a memory cell & not an ordinary node, use subscript c .

Term “long short-term memory” comes from following intuition. Simple recurrent neural networks have *long-term memory* in form of weights. Weights change slowly during training, encoding general knowledge about data. They also have *short-term memory* in form of ephemeral activations, which pass from each node to successive nodes. LSTM model introduces an intermediate type of storage via memory cell. A memory cell is a composite unit, built from simpler nodes in a specific connectivity pattern, with novel inclusion of multiplicative nodes, represented in diagrams by letter II. All elements of LSTM cell are enumerated & described below. Note: when use vector notation, we are referring to values of nodes in an entire layer of cells. E.g., \mathbf{s} : a vector containing value of s_c at each memory cell c in a layer. When subscript c is used, it is to index an individual memory cell.

- * *Input node*: This unit, labeled g_c , is a node that takes activation in standard way from input layer $\mathbf{x}^{(t)}$ at current time step & (along recurrent edges) from hidden layer at previous time step $\mathbf{h}^{(t-1)}$. Typically, summed weighted input is run through a tanh activation function, although in original LSTM paper, activation function is a *sigmoid*.

- * *Input gate*: Gates are a distinctive feature of LSTM approach. A gate is a sigmoidal unit that, like input node, takes activation from current data point $\mathbf{x}^{(t)}$ as well as from hidden layer at previous time step. A gate is so-called because its value is used to multiply value of another node. It is a *gate* in sense that if its value is 0, then flow from other node is cut off. If value of gate is 1, all flow is passed through. Value of *input gate* i_c multiplies value of *input node*.
- * *Internal state*: At heart of each memory cell is a node s_c with linear activation, which is referred to in original paper as “internal state” of cell. Internal state s_c has a self-connected recurrent edge with fixed unit weight. Because this edge spans adjacent time steps with constant weight, error can flow across time steps without vanishing or exploding. This edge is often called *constant error carousel*. In vector notation, update for internal state is $\mathbf{s}^{(t)} = \mathbf{g}^{(t)} \odot \mathbf{i}^{(t)} + \mathbf{s}^{(t-1)}$ where \odot is pointwise multiplication.
- * *Forget gate*: These gates f_c were introduced by Gers et al. [2000]. They provide a method by which network can learn to flush contents of internal state. This is especially useful in continuously running networks. With forget gates, equation to calculate internal state on forward pass is:

$$\mathbf{s}^{(t)} = \mathbf{g}^{(t)} \odot \mathbf{i}^{(t)} + \mathbf{f}^{(t)} \odot \mathbf{s}^{(t-1)}. \quad (261)$$

- * *Output gate*: Value v_c ultimately produced by a memory cell is value of internal state s_c multiplied by value of *output gate* o_c . Customary: internal state 1st be run through a tanh activation function, as this gives output of each cell same dynamic range as an ordinary tanh hidden unit. However, in other neural network research, rectified linear units, which have a greater dynamic range, are easier to train. Thus it seems plausible: nonlinear function on internal state might be omitted.

In original paper & in most subsequent work, input node is labeled g . Adhere to this convention but note: may be confusing as g does not stand for *gate*. In original paper, gates are called $y_{\text{in}}, y_{\text{out}}$ but this is confusing because y generally stands for output in ML literature. Seeking comprehensibility, break with this convention & use i, f, o to refer to input, forget, & output gates resp., as in Sutskever et al. [2014].

Since original LSTM was introduced, several variations have been proposed. Forget gates described above were proposed in 2000 & were not part of original LSTM design. However, they have proven effective & are standard in most modern implementations. That same year, Gers & Schmidhuber [2000] proposed peephole connections that pass from internal state directly to input & output gates of that same node without 1st having to be modulated by output gate. They report: these connections improve performance on timing tasks where network must learn to measure precise intervals between events. Intuition of peephole connection can be captured by following example. Consider a network which must learn to count objects & emit some desired output when n objects have been seen. Network might learn to let some fixed amount of activation into internal state after each object is seen. This activation is trapped in internal state s_c by constant error carousel, & is incremented iteratively each time another object is seen. When n th object is seen, network needs to know to let out content from internal state so that it can affect output. To accomplish this, output gate o_c must know content of internal state s_c . Thus s_c should be an input to o_c .

Put formally, computation in LSTM model proceeds according to following calculations, which are performed at each time step. These equations give full algorithm for a modern LSTM with forget gates: ***

4.2 [MC01]. DANILO MANDIC, JONATHON CHAMBERS. Recurrent Neural Networks for Prediction: Learning Algorithms, Architectures & Stability

Preface. New technologies in engineering, physics, & biomedicine are creating problems in which nonstationarity, nonlinearity, uncertainty, & complexity play a major role. Solutions to many of these problems require the use of nonlinear processors, among which neural networks are 1 of the most powerful. Neural networks are appealing because they learn by example & are strongly supported by statistical & optimization theories. They not only complement conventional signal processing techniques, but also emerge as a convenient alternative to expand signal processing horizons.

The use of recurrent neural networks as identifiers & predictors in nonlinear dynamical systems has increased significantly. They can exhibit a wide range of dynamics, due to feedback, & are also tractable nonlinear maps.

Neural network models are considered as massively interconnected nonlinear adaptive filters. The emphasis is on dynamics, stability, & spatio-temporal behavior of recurrent architectures & algorithms for prediction. However, wherever possible the material has been presented starting from feedforward networks & building up to the recurrent case.

Objective: to offer an accessible self-contained research monograph which can also be used as a graduate text. The material presented in the book is of interest to a wide population of researchers working in engineering, computing, science, finance, & biosciences. So that the topics are self-contained, assume familiarity with basic concepts of analysis & linear algebra. The material presented in Chaps. 1–6 can serve as an advanced text for courses on neural adaptive systems. The book encompasses traditional & advanced learning algorithms & architectures for recurrent neural networks. Although we emphasize the problem of time series prediction, the results are applicable to a wide range of problems, including other signal processing configurations e.g. system identification, noise cancellation, & inverse system modeling. Harmonize concepts of learning algorithms, embedded systems, representation of memory, neural network architectures & causal-noncausal dealing with time. A special emphasis is given to stability of algorithms – a key issue in real-time applications of adaptive systems.

- **Introduction.** Artificial neural network (ANN) models have been extensively studied with the aim of achieving human-like performance, especially in the field of pattern recognition. These networks are composed of a number of nonlinear computational elements which operate in parallel & are arranged in a manner reminiscent of biological neural interconnections. ANNs are

known by many names e.g. connectionist models, parallel distributed processing models & neuromorphic systems (Lippmann 1987). The origin of connectionist ideas can be traced back to the Greek philosopher, ARISTOTLE, & his ideas of mental associations. He proposed some of the basic concepts e.g. memory is composed of simple elements connected to each other via a number of different mechanisms (Medler 1998).

While early work in ANNs used anthropomorphic arguments to introduce the methods & models used, today neural networks used in engineering are related to algorithms & computation & do not question how brains might work (Hunt et al. 1992). E.g., recurrent neural networks have been attractive to physicists due to their isomorphism to spin glass systems (Ermentrout 1998). The following properties of neural networks make them important in signal processing (Hunt et al. 1992): they are nonlinear systems; they enable parallel distributed processing; they can be implemented in VLSI technology; they provide learning, adaptation & data fusion of both qualitative (symbolic data from artificial intelligence) & quantitative (from engineering) data; they realize multivariable systems.

The area of neural networks is nowadays considered from 2 main perspectives. 1st perspective: cognitive science, which is an interdisciplinary study of the mind. 2nd perspective is connectionism, which is a theory of information processing (Medler 1998). The neural networks in this work are approached from an engineering perspective, i.e., to make networks efficient in terms of topology, learning algorithms, ability to approximate functions & capture dynamics of time-varying systems. From the perspective of connection patterns, neural networks can be grouped into 2 categories: feedforward networks, in which graphs have no loops, & recurrent networks, where loops occur because of feedback connections. Feedforward networks are static, i.e., a given input can produce only 1 set of outputs, & hence carry no memory. In contrast, recurrent network architectures enable the information to be temporally memorized in the networks (Kung & Hwang 1998). Based on training by example, with strong support of statistical & optimization theories (Cichocki & Unbehauen 1993; Zhang & Constantinides 1992), neural networks are becoming 1 of the most powerful & appealing nonlinear signal processors for a variety of signal processing applications. As such, neural networks expand signal processing horizons (Chen 1997; Haykin 1996b), & can be considered as massively interconnected nonlinear adaptive filters. Our emphasis will be on dynamics of recurrent architectures & algorithms for prediction.

- **Some Important Dates in History of Connectionism.** In early 1940s pioneers of the field, MCCULLOCH & PITTS, studied potential of interconnection of a model of a neuron. They proposed a computational model based on a simple neuron-like element (McCulloch & Pitts 1943). Others, like Hebb were concerned with the adaptation laws involved in neural systems. In 1949 DONALD HEBB devised a learning rule for adapting the connections within artificial neurons (Hebb 1949). A period of early activity extends up to the 1960s with work of Rosenblatt (1962) & Widrow & Hoff (1960). In 1958, Rosenblatt coined the name ‘perceptron’²³. Based upon perceptron (Rosenblatt 1958), he developed the theory of statistical separability. Next major development: new formulation of learning rules by WIDROW & HOFF in their Adaline (Widrow & Hoff 1960). In 1969, Minsky & Papert (1969) provided a rigorous analysis of perceptron. Work of Grossberg in 1976 was based on biological & psychological evidence. He proposed several new architectures of nonlinear dynamical systems (Grossberg 1974) & introduced adaptive resonance theory (ART), which is a real-time ANN that performs supervised & unsupervised learning of categories, pattern classification & prediction. In 1982 HOPFIELD pointed out that neural networks with certain symmetries are analogues to spin glasses.

A seminal book on ANNs is by Rumelhart et al. (1986). Fukushima explored competitive learning in his biologically inspired Cognitron & Neocognitron (Fukushima 1975; Widrow & Lehr 1990). In 1971 Werbos developed a backpropagation learning algorithm which he published in his doctoral thesis (Werbos 1974). Rumelhart et al. rediscovered this technique in 1986 (Rumelhart et al. 1986). Kohonen (1982), introduced *self-organized maps* for pattern recognition (Burr 1993).

- **Structure of Neural Networks.** In neural networks, computational models or nodes are connected through weights that are adapted during use to improve performance. Main idea: to achieve good performance via dense interconnection of simple computational elements. The simplest node provides a linear combination of N weights w_1, \dots, w_N , & N inputs x_1, \dots, x_N , & passes the result through a nonlinearity Φ .

Models of neural networks are specified by the net topology, node characteristics & training or learning rules. From perspective of connection patterns, neural networks can be grouped into 2 categories: feedforward networks, in which graphs have no loops, & recurrent networks, where loops occur because of feedback connections. Neural networks are specified by (Tsoi & Back 1997). Connections within a node $y = \Phi(\sum_i w_i x_i + w_0)$.

- * Node: typically a sigmoid function
- * Layer: a set of nodes at the same hierarchical level
- * Connection: constant weights or weights as a linear dynamical system, feedforward or recurrent
- * Architecture: an arrangement of interconnected neurons
- * Mode of operation: analogue or digital.

Massively interconnected neural nets provide a greater degree of robustness or fault tolerance than sequential machines. By robustness we mean that small perturbations in parameters will also result in small deviations of the values of the signals from their nominal values.

In our work, hence, the term *neuron* will refer to an operator which performs the mapping: Neuron : $\mathbb{R}^{N+1} \rightarrow \mathbb{R}$. The equation $y = \Phi(\sum_{i=1}^N w_i x_i + w_0)$ represents a mathematical description of a neuron. The input vector is given by $\mathbf{x} =$

²³ **perceptron** [n] (*computing*) an artificial network which is intended to copy the brain’s ability to recognize things & see the differences between things.

$[x_1, \dots, x_N, 1]^\top$, whereas $\mathbf{w} = [w_1, \dots, w_N, w_0]^\top$ is referred to as the weight vector of a neuron. The weight w_0 is the weight which corresponds to the bias input, which is typically set to unity. The function $\Phi : \mathbb{R} \rightarrow (0, 1)$ is monotone & continuous, most commonly of a sigmoid shape. A set of interconnected neurons is a neural network (NN). If there are N input elements to an NN & M output elements of an NN, then an NN defines a continuous mapping $\text{NN} : \mathbb{R}^N \rightarrow \mathbb{R}^M$.

- **Perspective.** Before 1920s, prediction was undertaken by simply extrapolating the time series through a global fit procedure. The beginning of modern time series prediction was in 1927 when Yule introduced the autoregressive model in order to predict the annual number of sunspots. For the next half century the models considered were linear, typically driven by white noise. In 1980s, state-space representation & machine learning, typically by neural networks, emerged as new potential models for prediction of highly complex, nonlinear, & nonstationary phenomena. This was the shift from rule-based models to data-driven methods (Gershenfeld & Weigend 1993).

Time series prediction has traditionally been performed by use of linear parametric autoregressive (AR), moving-average (MA) or autoregressive moving-average (ARMA) models (Box & Jenkins 1976; Ljung & Soderstrom 1983; Makhoul 1975), the parameters of which are estimated either in a block or a sequential manner with least mean square (LMS) or recursive least-squares (RLS) algorithms (Haykin 1994). An obvious problem is that these processors are linear & are not able to cope with certain nonstationary signals, & signals whose mathematical model is not linear. On the other hand, neural networks are powerful when applied to problems whose solutions require knowledge which is difficult to specify, but for which there is an abundance of examples (Dillon & Manikopoulos 1991; Gent & Sheppard 1992; Townshend 1991). As time series prediction is conventionally performed entirely by inference of future behavior from examples of past behavior, it is a suitable application for a neural network predictor. The neural network approach to time series prediction is non-parametric in the sense that it does not need to know any information regarding the process that generates the signal. E.g., the order & parameters of an AR or ARMA process are not needed in order to carry out the prediction. This task is carried out by a process of learning from examples presented to the network & changing network weights in response to the output error.

Li (1992) has shown that the recurrent neural network (RNN) with a sufficiently large number of neurons is a realization of the nonlinear ARMA (NARMA) process. RNNs performing NARMA prediction have traditionally been trained by the real-time recurrent learning (RTRL) algorithm (Williams & Zipser 1989a) which provides the training process of the RNN 'on the run'. However, for a complex physical process, some difficulties encountered by RNNs e.g. high degree of approximation involved in RTRL algorithm for a higher-order MA part of underlying NARMA process, high computational complexity of $O(N^4)$, with N being the number of neurons in RNN, insufficient degree of nonlinearity involved, & relatively low robustness, induced a search for some other, more suitable schemes for RNN-based predictors.

In addition, in time series prediction of nonlinear & nonstationary signals, there is a need to learn long-time temporal dependencies. This is rather difficult with conventional RNNs because of the problem of vanishing gradient (Bengio et al. 1994). A solution to that problem might be NARMA models & nonlinear autoregressive moving average models with exogenous inputs (NARMAX) (Siegelmann et al. 1997) realized by recurrent neural networks. However, the quality of performance is highly dependent on the order of AR & MA parts in NARMAX model.

Main reasons for using neural networks for prediction rather than classical time series analysis are (Wu 1995)

- * they are computationally at least as fast, if not faster, than most available statistical techniques
- * they are self-monitoring (i.e. they learn how to make accurate predictions)
- * they are as accurate if not more accurate than most of the available statistical techniques
- * they provide iterative forecasts
- * they are able to cope with nonlinearity & nonstationarity of input processes
- * they offer both parametric & nonparametric prediction.
- **Neural Networks for Prediction: Perspective.** Many signals are generated from an inherently nonlinear physical mechanism & have statistically non-stationary properties, a classic example of which is speech. Linear structure adaptive filters are suitable for the nonstationary characteristics of such signals, but they do not account for nonlinearity & associated higher-order statistics (Shynk 1989). Adaptive techniques which recognize the nonlinear nature of the signal should therefore outperform traditional linear adaptive filtering techniques (Haykin 1996a; Kay 1993). The classic approach to time series prediction is to undertake an analysis of the time series data, which includes modeling, identification of the model & model parameter estimation phases (Makhoul 1975). The design may be iterated by measuring the closeness of the model to the real data. This can be a long process, often involving the derivation, implementation & refinement of a number of models before one with appropriate characteristics is found.

In particular, the most difficult systems to predict are

- * those with non-stationary dynamics, where the underlying behavior varies with time, a typical example of which is speech production
- * those which deal with physical data which are subject to noise & experimentation error, e.g. biomedical signals
- * those which deal with short time series, providing few data points on which to conduct the analysis, e.g. heart rate signals, chaotic signals & meteorological signals.

In all these situations, traditional techniques are severely limited & alternative techniques must be found (Bengio 1995; Haykin & Li 1995; Li & Haykin 1993; Niranjana & Kadiramanathan 1991).

On the other hand, neural networks are powerful when applied to problems whose solutions require knowledge which is difficult to specify, but for which there is an abundance of examples (Dillon & Manikopoulos 1991; Gent & Sheppard 1992;

Townshend 1991). From a system theoretic point of view, neural networks can be considered as a conveniently parametrized class of nonlinear maps (Narendra 1996).

There has been a recent resurgence in the field of ANNs caused by new net topologies, VLSI computational algorithms & introduction of massive parallelism into neural networks. As such, they are both universal function approximators (Cybenko 1989; Hornik et al. 1989) & arbitrary pattern classifiers. From the Weierstrass theorem, polynomials, & many other approximation schemes, can approximate arbitrarily well a continuous function. Kolmogorov's theorem (a negative solution of Hilbert's 13th problem (Lorentz 1976)) states that any continuous function can be approximated using only linear summations & nonlinear but continuously increasing functions of only 1 variable. This makes neural networks suitable for universal approximation, & hence prediction. Although sometimes computationally demanding (Williams & Zipser 1995), neural networks have found their place in the area of nonlinear autoregressive moving average (NARMA) (Bailer-Jones et al. 1998; Connor et al. 1992; Lin et al. 1996) prediction applications. Comprehensive survey papers on the use & role of ANNs can be found in Widrow & Lehr (1990), Lippmann (1987), Medler (1998), Ermentrout (1998), Hunt et al. (1992) & Billings (1980).

Only recently, neural networks have been considered for prediction. A recent competition by the Santa Fe Institute for Studies in the Science of Complexity (1991–1993) (Weigend & Gershenfeld 1994) showed that neural networks can outperform conventional linear predictors in a number of applications (Waibel et al. 1989). In journals, there has been an ever increasing interest in applying neural networks. A most comprehensive issue on recurrent neural networks is the issue of the *IEEE Transactions of Neural Networks*, vol. 5, no. 2, Mar 1994. In the signal processing community, there has been a recent special issue 'Neural Networks for Signal Processing' of *IEEE Transactions on Signal Processing*, vol. 45, no. 11, Nov 1997, & also the issue 'Intelligent Signal Processig' of the *Proceedings of IEEE*, vol. 86, no. 11, Nov 1998, both dedicated to the use of neural networks in signal processing applications.

Frequency of appearance of articles on recurrent neural networks in common citation index databases. Number of journal & conference articles on recurrent neural networks in IEE/IEEE publications between 1988 & 1999. The data were gathered using IEL Online service, & these publications are mainly periodicals & conferences in electronics engineering. Frequency of appearance for BIDS/ATHENS database, between 1988 & 2000, which also includes non-engineering publications. There is a clear growing trend in frequency of appearance of articles on recurrent neural networks. Therefore, felt that there was a need for a research monograph that would cover a part of the area with up to data ideas & results.

- **Structure of Book.** Divide book into 12 chapters & 10 appendices.

- * Chap. 1: An introduction to connectionism & notion of neural networks for prediction
- * Chap. 2: Detail fundamentals of adaptive signal processing & learning theory
- * Chap. 3: an initial overview of network architectures for prediction.
- * Chap. 4: a detailed discussion of activation functions & new insights are provided by consideration of neural networks within framework of modular groups from number theory.
- * Chap. 5: build material in Chap. 3 & provide more comprehensive coverage of recurrent neural network architectures together with concepts from nonlinear system modeling.
- * Chap. 6: Consider neural networks as nonlinear adaptive filters whereby develop necessary learning strategies for recurrent neural networks.
- * Chap. 7: Consider stability issues for certain recurrent neural network architectures through exploitation of fixed point theory & derive bounds for global asymptotic stability.
- * Chap. 8: Introduce a posteriori adaptive learning algorithms & highlight synergy with data-reusing algorithms.
- * Chap. 9: Derive a new class of normalized algorithms for online training of recurrent neural networks.
- * Chap. 10: Address convergence of online learning algorithms for neural networks.
- * Chap. 11: Present experimental results for prediction of nonlinear & nonstationary signals with recurrent neural networks.
- * Chap. 12: Describe exploitation of inherent relationships between parameters within recurrent neural networks.
- * Appendices A–J provide background to main chapters & cover key concepts from linear algebra, approximation theory, complex sigmoid activation functions, a precedent learning algorithm for recurrent neural networks, terminology in neural networks, *a posteriori* techniques in science & engineering, contraction mapping theory, linear relaxation & stability, stability of general nonlinear systems & deseasonalizing of time series. A comprehensive bibliography.

- **Readership.** This book is targeted at graduate students & research engineers active in the areas of communications, neural networks, nonlinear control, signal processing & time series analysis. It will also be useful for engineers & scientists working in diverse application areas, e.g., AI, biomedicine, earth sciences, finance & physics.

- **Fundamentals.**

- **Perspective.** Adaptive systems are at the very core of modern digital signal processing. There are many reasons for this, foremost amongst these is that adaptive filtering, prediction or identification do not require explicit *a priori* statistical knowledge of the input data. Adaptive systems are employed in numerous areas e.g. biomedicine, communications, control, radar, sonar, & video processing (Haykin 1996a).

Chap Summary. Introduce fundamentals of adaptive systems. Emphasis is 1st placed upon various structures available for adaptive signal processing, & includes predictor structure which is focus of this book. Detail basic learning algorithms & concepts in context of linear & nonlinear structure filters & networks. Discuss issue of modularity.

- **Adaptive Systems.** Adaptability, in essence, is ability to react in sympathy with disturbances to environment. A system that exhibits adaptability is said to be *adaptive*. Biological systems are adaptive systems; animals, e.g., can adapt to changes in their environment through a learning process (Haykin 1999a). **Block diagram of an adaptive system:** A generic adaptive system employed in engineering. It consists of:
 - * a set of adjustable parameters (weights) within some filter structure
 - * an error calculation block (difference between desired response & output of filter structure)
 - * a control (learning) algorithm for adaptation of weights.

Type of learning represented is so-called *supervised learning*, since the learning is directed by the desired response of the system. Goal: to adjust iteratively free parameters (weights) of adaptive system so as to minimize a prescribed cost function in some predetermined sense.²⁴ The filter structure within adaptive system may be linear, e.g. a finite impulse response (FIR) or infinite impulse response (IIR) filter, or nonlinear, e.g. a Volterra filter or a neural network.

- * **Configurations of Adaptive Systems Used in Signal Processing.** 4 typical configurations of adaptive systems used in engineering: (Jenkins et al. 1996)
 - System identification configuration
 - Noise canceling configuration
 - Prediction configuration
 - Inverse system configuration

Use notions of an adaptive filter & adaptive system interchangeably here. For the system identification configuration, both the adaptive filter & the unknown system are fed with the same input signal $x(k)$. Error signal $e(k)$ is formed at output as $e(k) := d(k) - y(k)$, & parameters of adaptive system are adjusted using this error information. An attractive point of this configuration is that desired response signal $d(k)$, also known as a *teaching/training signal*, is readily available from unknown system (plant). Applications of this scheme are in acoustic & electrical echo cancellation, control, & regulation of real-time industrial & other processes (plants). Knowledge about system is stored in set of converged weights of adaptive system. If dynamics of plant are not time-varying, possible to identify parameters (weights) of the plant to an arbitrary accuracy.

If desire to form a system which inter-relates noise components in input & desired response signals, noise canceling configuration can be implemented. The only requirement: noise in primary input & reference noise are correlated. This configuration subtracts an estimate of noise from received signal. Applications of this configuration include noise cancellation in acoustic environments & estimation of total ECG from mixture of maternal & foetal ECG (Widrow & Stearns 1985).

In adaptive prediction configuration, desired signal is input signal advanced relative to input of adaptive filter. This configuration has numerous applications in various areas of engineering, science, & technology & most of material in this book is dedicated to prediction. Prediction may be considered as a basis for any adaptation process, since adaptive filter is trying to predict desired response.

Inverse system configuration has an adaptive system cascaded with unknown system. A typical application is adaptive channel equalization in telecommunications, whereby an adaptive system tries to compensate for possibly time-varying communication channel, so that transfer function from input to output approximates a pure delay.

In most adaptive signal processing applications, parametric methods are applied which require *a priori* knowledge (or postulation) of a specific model in form of differential or difference equations. Thus, necessary to determine appropriate model order for successful operation, which will underpin data length requirements. On the other hand, nonparametric methods employ general model forms of integral equations or functional expansions valid for a broad class of dynamic nonlinearities. Most widely used nonparametric methods are referred to as Volterra–Wiener approach & are based on functional expansions.

- * **Blind Adaptive Techniques.** Presence of an explicit desired response signal $d(k)$ in all structures shown in **Block diagram of a blind equalization structure** implies that conventional, supervised, adaptive signal processing techniques may be applied for purpose of learning. When no such signal is available, may still be possible to perform learning by exploiting so-called *blind*, or *unsupervised*, methods. These methods exploit certain *a priori* statistical knowledge of input data. For a single signal, this knowledge may be in form of its constant modulus property, or, for multiple signals, their mutual statistical independence (Haykin 2000). Structure of a blind equalizer is shown, notice desired response is generated from output of a zero-memory nonlinearity. This nonlinearity is implicitly being used to test higher-order (i.e. greater than 2nd-order) statistical properties of output of adaptive equalizer. When ideal convergence of adaptive filter is achieved, zero-memory nonlinearity has no effect upon signal $y(k)$ & therefore $y(k)$ has identical statistical properties to that of channel input $s(k)$.
- **Gradient-Based Learning Algorithms.** A brief introduction to notion of gradient-based learning. Aim: to update iteratively weight vector \mathbf{w} of an adaptive system so that a nonnegative error measure $\mathcal{J}(\cdot)$ is reduced at each time step k , $\mathcal{J}(\mathbf{w} + \Delta\mathbf{w}) < \mathcal{J}(\mathbf{w})$, where $\Delta\mathbf{w}$ represents change in \mathbf{w} from 1 iteration to the next. This will generally ensure that after training, an adaptive system has captured relevant properties of unknown system that we are trying to model. Using a Taylor series expansion to approximate error measure, obtain $\mathcal{J}(\mathbf{w}) + \Delta\mathbf{w}\partial_{\mathbf{w}}\mathcal{J}(\mathbf{w}) + O(\mathbf{w}^2) < \mathcal{J}(\mathbf{w})$. This way, with the assumption that the higher-order terms in LHS can be neglected, $\mathcal{J}(\mathbf{w} + \Delta\mathbf{w}) < \mathcal{J}(\mathbf{w})$ can be rewritten as $\Delta\mathbf{w}\partial_{\mathbf{w}}\mathcal{J}(\mathbf{w}) < 0$. From this,

²⁴Aim: to minimize some function of error e . If $E[e^2]$ is minimized, consider minimum mean squared error (MSE) adaptation, the *statistical expectation operator* $E[\cdot]$ is due to random nature of inputs to adaptive system.

an algorithm that would continuously reduce error measure on the run, should change the weights in opposite direction of gradient $\partial_{\mathbf{w}}\mathcal{J}(\mathbf{w})$, i.e., $\Delta\mathbf{w} = -\eta\partial_{\mathbf{w}}\mathcal{J}$, where η is a small positive scalar called the *learning rate, step size or adaptation parameter*.

Examining $\Delta\mathbf{w} = -\eta\partial_{\mathbf{w}}\mathcal{J}$, if gradient of error measure $\mathcal{J}(\mathbf{w})$ is steep, large changes will be made to weights, & conversely, if gradient of error measure $\mathcal{J}(\mathbf{w})$ is small, namely a flat error surface, a larger step size η may be used. Gradient descent algorithms cannot, however, provide a sense of importance or hierarchy to weights (Agarwal & Mammone 1994). E.g., value of weight w_1 in Fig. 2.4 is 10 times greater than w_2 & 1000 times greater than w_4 . Hence, component of output of filter within adaptive system due to w_1 will, on average, be larger than that due to other weights. For a conventional gradient algorithm, however, change in w_1 will not depend upon relative sizes of coefficients, but relative sizes of input data. This deficiency provides motivation for certain partial update gradient-based algorithms (Douglas 1997).

Important to notice: *gradient-descent-based algorithms inherently forget old data*, which leads to a problem called *vanishing gradient* & has particular importance for learning in filters with recursive structures.

- **A General Class of Learning Algorithms.** To introduce a general class of learning algorithms & explain in very crude terms relationships between them, follow approach from Guo & Ljung (1995). Start from *linear regression equation* $y(k) = \mathbf{x}^\top(k)\mathbf{w}(k) + \nu(k)$ where $y(k)$: output signal, $\mathbf{x}(k)$: a vector comprising input signals, $\nu(k)$: a disturbance or noise sequence, & $\mathbf{w}(k)$: an unknown time-varying vector of weights (parameters) of adaptive system. Variation of weights at time k is denoted by $\mathbf{n}(k)$, & weight change equation becomes $\mathbf{w}(k) = \mathbf{w}(k-1) + \mathbf{n}(k)$. Adaptive algorithms can track weights only approximately, hence for the following analysis use symbol $\hat{\mathbf{w}}$. A general expression for weight update in an adaptive algorithm:

$$\hat{\mathbf{w}}(k+1) = \hat{\mathbf{w}}(k) + \eta\Gamma(k)(y(k) - \mathbf{x}^\top(k)\hat{\mathbf{w}}(k)), \quad (262)$$

where $\Gamma(k)$: adaptation gain vector, & η : step size. To assess how far an adaptive algorithm is from optimal solution, introduce *weight error vector* $\check{\mathbf{w}}(k)$, & a sample input matrix $\Sigma(k)$ as $\check{\mathbf{w}}(k) := \mathbf{w}(k) - \hat{\mathbf{w}}(k)$, $\Sigma(k) := \Gamma(k)\mathbf{x}^\top(k)$. Yield *weight error equation*:

$$\check{\mathbf{w}}(k+1) = (I - \eta\Sigma(k))\check{\mathbf{w}}(k) - \eta\Gamma(k)\nu(k) + \mathbf{n}(k+1). \quad (263)$$

For different gains $\Gamma(k)$, 3 well-known algorithms can be obtained from (262). Notice: role of η in RLS & KF algorithm is different to that in LMS algorithm. For RLS & KF may put $\eta = 1$ & introduce a forgetting factor instead.

1. Least mean square (LMS) algorithm: $\Gamma(k) = \mathbf{x}(k)$.
2. Recursive least-squares (RLS) algorithm:

$$\Gamma(k) = P(k)\mathbf{x}(k), \quad (264)$$

$$P(k) = \frac{1}{1-\eta} \left[P(k-1) - \eta \frac{P(k-1)\mathbf{x}(k)\mathbf{x}^\top(k)P(k-1)}{1-\eta + \eta\mathbf{x}^\top(k)P(k-1)\mathbf{x}(k)} \right]. \quad (265)$$

3. Kalman filter (KF) algorithm (Guo & Ljung 1995; Kay 1993):

$$\Gamma(k) = \frac{P(k-1)\mathbf{x}(k)}{R + \eta\mathbf{x}^\top(k)P(k-1)\mathbf{x}(k)}, \quad (266)$$

$$P(k) = P(k-1) - \frac{\eta P(k-1)\mathbf{x}(k)\mathbf{x}^\top(k)P(k-1)}{R + \eta\mathbf{x}^\top(k)P(k-1)\mathbf{x}(k)} + \eta Q. \quad (267)$$

The KF algorithm is the optimal algorithm in this setting if elements of $\mathbf{n}(k)$ & $\nu(k)$ in (2.5) & (2.6) are Gaussian noises with a covariance matrix $Q > 0$ & a scalar value $R > 0$, resp. (Kay 1993). All of these adaptive algorithms can be referred to as sequential estimators, since they refine their estimate as each new sample arrives. On the other hand, block-based estimators require all measurements to be acquired before the estimate is formed.

Although the most important measure of quality of an adaptive algorithm is generally covariance matrix of weight tracking error $E[\check{\mathbf{w}}(k)\check{\mathbf{w}}^\top(k)]$, due to statistical dependence between $\mathbf{x}(k)$, $\nu(k)$, $\mathbf{n}(k)$, precise expressions for this covariance matrix are extremely difficult to obtain.

To undertake statistical analysis of an adaptive learning algorithm, classical approach: assume $\mathbf{x}(k)$, $\nu(k)$, $\mathbf{n}(k)$ are statistically independent. Another assumption: homogeneous part of (2.9) $\check{\mathbf{w}}(k+1) = (I - \eta\Sigma(k))\check{\mathbf{w}}(k)$ & its averaged version $E[\check{\mathbf{w}}(k+1)] = (I - \eta E[\Sigma(k)])E[\check{\mathbf{w}}(k)]$ are exponentially stable in stochastic & deterministic senses (Guo & Ljung 1995).

- * **Quasi-Newton Learning Algorithm.** Quasi-Newton learning algorithm utilizes 2nd-order derivative of objective function to adapt weights. If change in objective function between iterations in a learning algorithm is modeled with a Taylor series expansion, have

$$\Delta E(\mathbf{w}) = E(\mathbf{w} + \Delta\mathbf{w}) - E(\mathbf{w}) \approx (\nabla_{\mathbf{w}}E(\mathbf{w}))^\top \Delta\mathbf{w} + \frac{1}{2}\Delta\mathbf{w}^\top \mathbf{H}\Delta\mathbf{w}. \quad (268)$$

After setting differential w.r.t. $\Delta\mathbf{w}$ to 0, weight update equation becomes $\Delta\mathbf{w} = -\mathbf{H}^{-1}\nabla_{\mathbf{w}}E(\mathbf{w})$. The Hessian \mathbf{H} in this equation determines not only the direction but also step size of gradient descent.

Conclude: adaptive algorithms mainly differ in their form of adaptation gains. The gains can be roughly divided into 2 classes: gradient-based gains (e.g. LMS, quasi-Newton) & Riccati equation-based gains (e.g. KF & RLS).

- **A Step-by-Step Derivation of Least Mean Square (LMS) Algorithm.** Consider a set of input-output pairs of data described by a mapping function f : $d(k) = f(\mathbf{x}(k))$, $k = 1, \dots, N$. Function $f(\cdot)$ is assumed to be unknown. Using concept of adaptive systems explained, aim: to approximate unknown function $f(\cdot)$ by a function $F(\cdot, \mathbf{w})$ with adjustable parameters \mathbf{w} , in some prescribed sense. Function F is defined on a system with a known architecture or structure. Convenient to define an instantaneous performance index,

$$J(\mathbf{w}(k)) = [d(k) - F(\mathbf{x}(k), \mathbf{w}(k))]^2, \quad (269)$$

which represents an energy measure. In that case, function F is most often just inner product $F = \mathbf{x}^\top(k)\mathbf{w}(k)$ & corresponds to operation of a linear FIR filter structure. Goal: to find an optimization algorithm that minimizes cost function $J(\mathbf{w})$. Common choice of algorithm is motivated by method of steepest descent, & generates a sequence of weight vectors $\mathbf{w}(1), \mathbf{w}(2), \dots$ as $\mathbf{w}(k+1) = \mathbf{w}(k) - \eta \mathbf{g}(k)$, $k = 0, 1, 2, \dots$ (2.21), where $\mathbf{g}(k)$ is gradient vector of cost function $J(\mathbf{w})$ at point $\mathbf{w}(k)$: $\mathbf{g}(k) = \partial_{\mathbf{w}} J(\mathbf{w})|_{\mathbf{w}=\mathbf{w}(k)}$. Parameter η in $\mathbf{w}(k+1) = \mathbf{w}(k) - \eta \mathbf{g}(k)$ determines behavior of algorithm:

- * for η small, algorithm (2.21) converges towards global minimum of error performance surface;
- * if value of η approaches some critical value η_c , trajectory of convergence on error performance surface is either oscillatory or overdamped;
- * if value of η is $> \eta_c$, system is unstable & does not converge.

These observations can only be visualized in 2D, i.e. for only 2 parameter values $w_1(k), w_2(k)$, & can be found in Widrow & Stearns (1985). If approximation function F in gradient descent algorithm (2.21) is linear, call such an adaptive system a *linear adaptive system*. Otherwise, describe it as a nonlinear adaptive system. Neural networks belong to this latter class.

- * **Wiener Filter.** Suppose system shown in Fig. 2.1 is modeled as a linear FIR filter: Fig. 2.5: Structure of a finite impulse response filter, have $F(\mathbf{x}, \mathbf{w}) = \mathbf{x}^\top \mathbf{w}$, dropping k index for convenience. Consequently, instantaneous cost function $J(\mathbf{w}(k))$ is a quadratic function of weight vector. Wiener filter is based upon minimizing ensemble average of this instantaneous cost function, i.e.,

$$J_{\text{Wiener}}(\mathbf{w}(k)) = E[[d(k) - \mathbf{x}^\top(k)\mathbf{w}(k)]^2], \quad (270)$$

& assuming $d(k), x(k)$ are zero mean & jointly wide sense stationary. To find minimum of cost function, differentiate w.r.t. \mathbf{w} & obtain

$$\partial_{\mathbf{w}} J_{\text{Wiener}} = -2E[e(k)\mathbf{x}(k)], \quad (271)$$

where $e(k) = d(k) - \mathbf{x}^\top(k)\mathbf{w}(k)$.

At Wiener solution, this gradient equals null vector $\mathbf{0}$. Solving (271) for this condition yields Wiener solution, $\mathbf{w} = \mathbf{R}_{\mathbf{x}, \mathbf{x}}^{-1} \mathbf{r}_{\mathbf{x}, d}$, where $\mathbf{R}_{\mathbf{x}, \mathbf{x}} = E[\mathbf{x}(k)\mathbf{x}^\top(k)]$ is the autocorrelation matrix of zero mean input data $\mathbf{x}(k)$ & $\mathbf{r}_{\mathbf{x}, d} = E[\mathbf{x}(k)d(k)]$ is crosscorrelation between input vector & desired signal $d(k)$. Wiener formula has same general form as block least-squares (LS) solution, when exact statistics are replaced by temporal averages.

RLS algorithm, as in (2.12), with assumption that input & desired response signals are jointly ergodic, approximates Wiener solution & asymptotically matches Wiener solution. More details about derivation of Wiener filter can be found in Haykin (1996a, 1999a).

- * **Further Perspective on Least Mean Square (LMS) Algorithm.** To reduce computational complexity of Wiener solution, which is a block solution, can use method of steepest descent for a recursive, or sequential, computation of weight vector \mathbf{w} . Derive LMS algorithm for an adaptive FIR filter, structure of which is shown in Fig. 2.5. In view of a general adaptive system, this FIR filter becomes filter structure within Fig. 2.1. Output of this filter: $y(k) = \mathbf{x}^\top(k)\mathbf{w}(k)$. Widrow & Hoff (1960) utilized this structure for adaptive processing & proposed instantaneous values of autocorrelation & crosscorrelation matrices to calculate gradient term within steepest descent algorithm. Cost function they proposed was $J(k) = \frac{1}{2}e^2(k)$, which is again based upon instantaneous output error $e(k) = d(k) - y(k)$. In order to derive weight update equation, start from instantaneous gradient $\partial_{\mathbf{w}(k)} J(k) = e(k)\partial_{\mathbf{w}(k)} e(k)$. Following same procedure as for general gradient descent algorithm, obtain

$$\partial_{\mathbf{w}(k)} e(k) = -\mathbf{x}(k), \quad \partial_{\mathbf{w}(k)} J(k) = -e(k)\mathbf{x}(k). \quad (272)$$

Set of equations that describes LMS algorithm is given by

$$\begin{cases} y(k) = \sum_{i=1}^N x_i(k)w_i(k) = \mathbf{x}^\top(k)\mathbf{w}(k), \\ e(k) = d(k) - y(k), \\ \mathbf{w}(k+1) = \mathbf{w}(k) + \eta e(k)\mathbf{x}(k). \end{cases} \quad (273)$$

LMS algorithm is a very simple yet extremely popular algorithm for adaptive filtering. Also optimal in H^∞ sense which justifies its practical utility (Hassibi et al. 1996).

- **On Gradient Descent for Nonlinear Structures.** Adaptive filters & neural networks are formally equivalent, in fact, structures of neural networks are generalizations of linear filters (Maass & Sontag 2000; Nerrand et al. 1991). Depending on architecture of a neural network & whether it is used online or offline, 2 broad classes of learning algorithms are available:
 - * techniques that use a direct computation of gradient, which is typical for linear & nonlinear adaptive filters
 - * techniques that involve backpropagation, which is commonplace for most offline applications of neural networks.

Backpropagation is a computational procedure to obtain gradients necessary for adaptation of weights of a neural network contained within its hidden layers & is not radically different from a general gradient algorithm.

As interested in neural networks for real-time signal processing, will analyze online algorithms that involve direct gradient computation. In this sect, introduce a learning algorithm for a nonlinear FIR filter, whereas learning algorithms for online training of recurrent neural networks will be introduced later. Start from a simple nonlinear FIR filter, which consists of standard FIR filter cascaded with a memoryless nonlinearity Φ as shown in Fig. 2.6: **Structure of a nonlinear adaptive filter**. This structure can be seen as a single neuron with a dynamical FIR synapse. This FIR synapse provides memory to neuron. Output of this filter is given by $y(k) = \Phi(\mathbf{x}^\top(k)\mathbf{w}(k))$. Nonlinearity $\Phi(\cdot)$ after tap-delay line is typically a sigmoid. Using ideas from LMS algorithm, if cost function is given by $J(k) = \frac{1}{2}e^2(k)$, have

$$e(k) = d(k) - \Phi(\mathbf{x}^\top(k)\mathbf{w}(k)), \quad (274)$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \eta \nabla_{\mathbf{w}} J(k), \quad (275)$$

where $e(k)$ is the *instantaneous error* at output neuron, $d(k)$ is some *teaching (desired) signal*, $\mathbf{w}(k) = [w_1(k), \dots, w_N(k)]^\top$: *weight vector* & $\mathbf{x}(k) = [x_1(k), \dots, x_N(k)]^\top$: *input vector*.

Gradient $\nabla_{\mathbf{w}} J(k)$ can be calculated as

$$\partial_{\mathbf{w}(k)} J(k) = e(k) \partial_{\mathbf{w}(k)} e(k) = -e(k) \Phi'(\mathbf{x}^\top(k)\mathbf{w}(k)) \mathbf{x}(k), \quad (276)$$

where $\Phi'(\cdot)$ represents 1st derivative of nonlinearity $\Phi(\cdot)$ & weight update equation (275) can be rewritten as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \eta \Phi'(\mathbf{x}^\top(k)\mathbf{w}(k)) e(k) \mathbf{x}(k). \quad (277)$$

This is the *weight update equation* for a direct gradient algorithm for a nonlinear FIR filter.

Extension to a General Neural Network. When deriving a direct gradient algorithm for a general neural network, network architecture should be taken into account. For large networks for offline processing, classical backpropagation is the most convenient algorithm. However, for online learning, extensions of previous algorithm should be considered.

- **On Some Important Notions From Learning Theory.** Discuss in more detail interrelations between error, error function, & objective function in learning theory.
- * **Relationship Between Error & Error Function.** Error at output of an adaptive system is defined as difference between output value of network & target (desired output) value. E.g., *instantaneous error* $e(k)$ is defined as $e(k) := d(k) - y(k)$. Instantaneous error can be positive, negative or zero, & is therefore not a good candidate for criterion function for training adaptive systems. Here look for another function, called *error function*, i.e., a function of instantaneous error, but is suitable as a criterion function for learning. Error functions are also called *loss functions*. They are defined so that an increase in the error function corresponds to a reduction in quality of learning, & they are nonnegative. An error function can be defined as $E(N) = \sum_{i=0}^N e^2(i)$ or as an average value $\bar{E}(N) = \frac{1}{N+1} \sum_{i=0}^N e^2(i)$.
- * **Objective Function.** *Objective function* is a function that we want to minimize during training. It can be equal to an error function, but often it may include other terms to introduce constraints. E.g. in generalization, too large a network might lead to overfitting. Hence objective function can consist of 2 parts, one for error minimization & the other which is either a penalty for a large network or a penalty term for excessive increase in weights of adaptive system or some other chosen function (Tikhonov et al. 1998). An example of such an objective function for online learning is:

$$J(k) = \frac{1}{N} \sum_{i=1}^N (e^2(k-i+1) + G(\|\mathbf{w}(k-i+1)\|_2^2)), \quad (278)$$

where G is some linear or nonlinear function. Often use symbols E, J interchangeably to denote cost function.

- * **Types of Learning w.r.t. Training Set & Objective Function.** *Batch learning* is also known as *epochwise*, or *offline learning*, & is a common strategy for offline training. Idea: to adapt weights once whole training set has been presented to an adaptive system. It can be described by following steps.

1. Initialize weights
2. Repeat
 - Pass all training data through network
 - Sum errors after each particular pattern
 - Update weights based upon total error
 - Stop if some prescribed error performance is reached

Counterpart of batch learning is so-called *incremental learning*, *online*, or *pattern training*. The procedure for this type of learning is as follows.

1. Initialize weights
2. Repeat
 - Pass 1 pattern through network
 - Update weights based upon instantaneous error

- Stop if some prescribed error performance is reached

Choice of type of learning is very much dependent upon application. Quite often, for networks that need initialization, perform 1 type of learning in initialization procedure, which is by its nature an offline procedure, & then use some other learning strategy while network is running. Such is the case with recurrent neural networks for online signal processing (Mandic & Chambers 1999f).

- * **Deterministic, Stochastic, & Adaptive Learning.** *Deterministic learning* is an optimization technique based on an objective function which always produces same result, no matter how many times we recompute it. Deterministic learning is always offline.

Stochastic learning is useful when objective function is affected by noise & local minima. It can be employed within context of a gradient descent learning algorithm. Idea: learning rate gradually decreases during training & hence steps on error performance surface in beginning of training are large which speeds up training when far from optimal solution. Learning rate is small when approaching optimal solution, hence reducing misadjustment. This gradual reduction of learning rate can be achieved by e.g. annealing (Kirkpatrick et al. 1983; Rose 1998; Szu & Hartley 1987).

The idea behind concept of adaptive learning is to forget the past when it is no longer relevant & adapt to changes in environment. The terms ‘adaptive learning’ or ‘gear-shifting’ are sometimes used for gradient methods in which learning rate is changed during training.

- * **Constructive Learning.** Constructive learning deals with change of architecture or interconnections in network during training. Neural networks for which topology can change over time are called *ontogenic* neural networks (Fiesler & Beale 1997). 2 basic classes of constructive learning are network growing & network pruning. In network growing approach, learning begins with a network with no hidden units, & if error is too big, new hidden units are added to network, training resumes, & so on. Most used algorithm based upon network growing is so-called *cascade-correlation algorithm* (Hoehfeld & Fahlman 1992). Network pruning starts from a large network & if error in learning is smaller than allowed, network size is reduced until desired ratio between accuracy & network size is reached (Reed 1993; Sum et al. 1999).
- * **Transformation of Input Data, Learning, & Dimensionality.** A natural question is whether to linearly/nonlinearly transform data before feeding them to an adaptive processor. This is particularly important for neural networks, which are nonlinear processors. If consider each neuron as a basic component of a neural network, then can refer to a general neural network as a system with componentwise nonlinearities. To express this formally, consider a scalar function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ & systems of form

$$\mathbf{y}(k) = \boldsymbol{\sigma}(\mathbf{A}\mathbf{x}(k)), \quad (279)$$

where matrix \mathbf{A} is an $N \times N$ matrix & $\boldsymbol{\sigma}$ is applied componentwise $\boldsymbol{\sigma}(x_1(k), \dots, x_N(k)) = (\sigma(x_1(k)), \dots, \sigma(x_N(k)))$. Systems of this type arise in a wide variety of situations. For a linear σ , have a linear system. If range of σ is finite, state vector of (279) takes values from a finite set, & dynamical properties can be analyzed in time which is polynomial in number of possible states. Throughout this book, interested in functions σ & combination matrices \mathbf{A} which would guarantee a fixed point of this mapping. Neural networks are commonly of form (279). In such a context call σ *activation function*. Results of Siegelmann & Sontag (1995) show: saturated linear systems (piecewise linear) can represent Turing machines, which is achieved by encoding transition rules of Turing machine in matrix \mathbf{A} .

Curse of dimensionality. Curse of dimensionality (Bellman 1961) refers to exponential growth of computation needed for a specific task as a function of dimensionality of input space. In neural networks, a network quite often has to deal with many irrelevant inputs which, in turn, increase dimensionality of input space. In such a case, network uses much of its resources to represent & compute irrelevant information, which hampers processing of desired information. A remedy for this problem is preprocessing of input data, e.g. feature extraction, & to introduce some importance function to input samples. Curse of dimensionality is particularly prominent in unsupervised learning algorithms. Radial basis functions are also prone to this problem. Selection of a neural network model must therefore be suited for a particular task. Some a priori information about data & scaling of inputs can help to reduce severity of problem.

Transformations on input data. Activation functions used in neural networks are centered around a certain value in their output space. E.g., mean of logistic function is 0.5, whereas the tanh function is centered around zero. Therefore, in order to perform efficient prediction, should match range of input data, their mean & variance, with range of chosen activation function. There are several operations that we could perform on input data, e.g. following.

1. Normalization, which in this context means dividing each element of input vector $\mathbf{x}(k)$ by its squared norm, i.e. $x_i(k) \in \mathbf{x}(k) \rightarrow \frac{x_i(k)}{\|\mathbf{x}(k)\|_2^2}$.
2. Rescaling, which means transforming input data in manner that we multiply/divide them by a constant & also add/subtract a constant from data.²⁵
3. Standardization, which is borrowed from statistics, where, e.g., a random Gaussian vector is standardized if its mean is subtracted from it, & vector is then divided by its standard deviation. Resulting random variable is called a ‘standard normal’ random variable with zero mean & unity standard deviation. Some examples of data standardization:
 - Standardization to zero mean & unity standard deviation can be performed as

$$\text{mean} = \frac{\sum_i X_i}{N}, \quad \text{std} = \sqrt{\frac{\sum_i (X_i - \text{mean})^2}{N - 1}}. \quad (280)$$

Standardized quantity becomes $S_i = \frac{X_i - \text{mean}}{\text{std}}$.

²⁵In real life a typical rescaling is transforming temperature from Celcius into Fahrenheit scale.

· Standardize X to midrange 0 & range 2. This can be achieved by

$$\text{midrange} = \frac{1}{2}(\max_i X_i + \min_i X_i), \text{ range} = \max_i X_i - \min_i X_i, S_i = \frac{X_i - \text{midrange}}{\text{range}/2}. \quad (281)$$

4. Principal component analysis (PCA) represents data by a set of unit norm vectors called *normalized eigenvectors*. Eigenvectors are positioned along directions of greatest data variance. Eigenvectors are found from covariance matrix \mathbf{R} of input dataset. An eigenvalue λ_i , $i = 1, \dots, N$, is associated with each eigenvector. Every input data vector is then represented by a linear combination of eigenvectors.

As pointed out earlier, standardizing input variables has an effect on training, since steepest descent algorithms are sensitive to scaling due to change in weights being proportional to value of gradient & input data.

Nonlinear transformations of data. This method to transform data can help when dynamic range of data is too high. In the case, e.g., typically apply log function to input data. Log function is often applied in error & objective functions for same purposes.

◦ **Learning Strategies.** To construct an optimal neural approximating model, have to determine an appropriate training set containing all relevant information of process & define a suitable topology that matches complexity & performance requirements. Training set construction issue requires 4 entities to be considered (Alippi & Piuri 1996; Bengio 1995; Haykin & Li 1995; Shadafan & Niranjan 1993):

- * number of training data samples N_D
- * number of patterns N_P constituting a batch
- * number of batches N_B to be extracted from training set
- * number of times generic batch is presented to network during learning.

Assumption is that training set is sufficiently rich so that it contains all the relevant information necessary for learning.

Requirement coincides with hypothesis that training data have been generated by a fully exciting input signal, e.g. white noise, which is able to excite all process dynamics. White noise is a persistently exciting input signal & is used for driving component of moving average (MA), autoregressive (AR), & autoregressive moving average (ARMA) models.

◦ **General Framework for Training of Recurrent Networks by Gradient-Descent-Based Algorithms.** Summarize some of important concepts mentioned earlier.

* **Adaptive vs. Nonadaptive Training.** Training of a network makes use of 2 sequences, sequence of inputs & sequence of corresponding desired outputs. If network is 1st trained (with a training sequence of finite length) & subsequently used (with fixed weights obtained from training), this mode of operation is referred to as *non-adaptive* (Nerrand et al. 1994). Conversely, the term *adaptive* refers to mode of operation whereby network is trained permanently throughout its application (with a training sequence of infinite length). Therefore, adaptive network is suitable for input processes which exhibit statistically non-stationary behavior, a situation which is normal in the fields of adaptive control & signal processing (Bengio 1995; Haykin 1996a; Haykin & Li 1995; Khotanzad & Lu 1990; Narendra & Parthasarathy 1990; Nerrand et al. 1994).

* **Performance Criterion, Cost Function, Training Function.** Computation of coefficients during training aims at finding a system whose operation is optimal w.r.t. some performance criterion which may be either qualitative, e.g. (subjective) quality of speech reconstruction, or quantitative, e.g. maximizing signal to noise ratio for spatial filtering. Goal: to define a positive *training function* which is s.t. a decrease of this function through modifications of coefficients of network leads to an improvement of performance of system (Bengio 1995; Haykin & Li 1995; Nerrand et al. 1994; Qin et al. 1992). In the case of non-adaptive training, training function is defined as a function of all data of training set (in such a case, usually termed as a *cost function*). The minimum of cost function corresponds to optimal performance of system. Training is an optimization procedure, conventionally using gradient-based methods.

In case of adaptive training, impossible, in most instances, to define a time-independent cost function whose minimization leads to a system that is optimal w.r.t. performance criterion. Therefore, training function is time dependent. Modification of coefficients is computed continually from gradient of training function. Latter involves data pertaining to a time window of finite length, which shifts in time (sliding window) & coefficients are updated at each sampling time.

* **Recursive vs. Nonrecursive Algorithms.** A nonrecursive algorithm employs a cost function (i.e. a training function defined on a fixed window), whereas a recursive algorithm makes use of a training function defined on a sliding window of data. An adaptive system must be trained by a recursive algorithm, whereas a non-adaptive system may be trained either by a nonrecursive or by a recursive algorithm (Nerrand et al. 1994).

* **Iterative vs. Noniterative Algorithms.** An iterative algorithm performs coefficient modifications several times from a set of data pertaining to a given data window, a non-iterative algorithm makes only 1 (Nerrand et al. 1994). E.g., conventional LMS algorithm (2.31) is thus a recursive, non-iterative algorithm operating on a sliding window.

* **Supervised vs. Unsupervised Algorithms.** A supervised learning algorithm performs learning by using a *teaching signal*, i.e. the desired output signal, while an unsupervised learning algorithm, as in blind signal processing, has no reference signal as a teaching input signal. An example of a supervised learning algorithm is the *delta rule*, while unsupervised learning algorithms are, e.g., the *reinforcement learning algorithm* & the *competitive rule* ('winner takes all') *algorithm*, whereby there is some sense of concurrency between elements of network structure (Bengio 1995; Haykin & Li 1995).

* **Pattern vs. Batch Learning.** Updating network weights by *pattern learning* means that weights of network are updated immediately after each pattern is fed in. Other approach is to take all data as a whole batch, & network is not updated until entire batch of data is processed. This approach is referred to as *batch learning* (Haykin & Li 1995; Qin et al. 1992). It can be shown (Qin et al. 1992) that while considering feedforward networks (FFN), after 1 training sweep through all data, pattern learning is a 1st-order approximation of batch learning w.r.t. learning rate η . Therefore, FFN pattern learning approximately implements FFN batch learning after 1 batch interval. After multiple sweeps through training data, difference between FFN pattern learning & FFN batch learning is of order²⁶ $O(\eta^2)$. Therefore, for small training rates, FFN pattern learning approximately implements FFN batch learning after multiple sweeps through training data. For recurrent networks, weight updating slopes for pattern learning & batch learning are different²⁷ (Qin et al. 1992). However, difference could also be controlled by learning rate η . Difference will converge to 0 as quickly as $\eta \rightarrow 0$ ²⁸ (Qin et al. 1992).

- **Modularity Within Neural Networks.** Hierarchical levels in neural network architectures are synapses, neurons, layers, & neural networks, & will be discussed in Chap. 5. Next step would be combinations of neural networks. In this case, consider modular neural networks. Modular neural networks are composed of a set of smaller subnetworks (modules), each performing a subtask of complete problem. To depict this problem, resource to case of linear adaptive filters described by a transfer function in z -domain $H(z)$ as

$$H(z) = \frac{\sum_{k=0}^M b(k)z^{-k}}{1 + \sum_{k=1}^N a(k)z^{-k}}. \quad (282)$$

Can rearrange this function either in a cascaded manner as

$$H(z) = A \prod_{k=1}^{\max\{M,N\}} \frac{1 - \beta_k z^{-1}}{1 - \alpha_k z^{-1}}, \quad (283)$$

or in a parallel manner as

$$H(z) = \sum_{k=1}^N \frac{A_k}{1 - \alpha_k z^{-1}}, \quad (284)$$

where for simplicity, have assumed 1st-order poles & zeros of $H(z)$. A cascade realization of a general system is shown in Fig. 2.7: A cascaded realization of a general system, whereas a parallel realization of a general system is shown in Fig. 2.8: A parallel realization of a general system. Can also combine neural networks in these 2 configurations. An example of cascaded neural network is the so-called *pipelined recurrent neural network*, whereas an example of a parallel realization of a neural network is associative Gaussian mixture model, or winner takes all network. Taking into account that neural networks are nonlinear systems, talk about nested modular architectures instead of cascaded architectures. Nested neural scheme can be written as

$$F(W, X) = \Phi \left(\sum_n w_n \Phi \left(\sum_i v_i \Phi \left(\cdots \Phi \left(\sum_j u_j X_j \right) \cdots \right) \right) \right), \quad (285)$$

where Φ is a sigmoidal function. It corresponds to a multilayer network of units that sum their inputs with ‘weights’ $W = \{w_n, v_i, u_j, \dots\}$ & then perform a sigmoidal transformation of this sum. Its motivation is: function

$$F(W, X) = \Phi \left(\sum_n w_n \Phi \left(\sum_j u_j X_j \right) \right) \quad (286)$$

can approximate arbitrarily well any continuous multivariate function (Funahashi 1989; Poggio & Girosi 1990).

Since use sigmoid ‘squashing’ activation functions, modular structures contribute to a general stability issue. Effects of a simple scheme of nested sigmoids are shown in Fig. 2.9: Effects of nesting sigmoid nonlinearities: 1st, 2nd, 3rd, & 4th pass. Pure nesting successively reduces range of output signal, bringing this composition of nonlinear functions to fixed point of employed nonlinearity for sufficiently many nested sigmoids.

Modular networks possess some advantages over over classical networks, since overall complex function is simplified & modules possibly do not have hidden units which speeds up training. Also, input data might be decomposable into subsets which can be fed to separate modules. Utilizing modular neural networks has not only computational advantages but also development advantages, improved efficiency, improved interpretability & easier hardware implementation. Also, there are strong suggestions from biology that modular structures are exploited in cognitive mechanisms (Fiesler & Beale 1997).

- **Summary.** Configurations of general adaptive systems have been provided, & prediction configuration has been introduced within this framework. Gradient-descent-based learning algorithms have then been developed for these configurations, with an emphasis on LMS algorithm. A thorough discussion of learning modes & learning parameters is given. Finally, modularity within neural networks has been addressed.

²⁶In fact, if data being processed exhibit highly stationary behavior, then average error calculated after FFN batch learning is very close to instantaneous error calculated after FFN pattern learning, e.g. speech data can be considered as being stationary within an observed frame. That forms the basis for use of various real-time & recursive learning algorithms, e.g. RTRL.

²⁷(Qin et al. 1992) showed for feedforward networks, updated weights for both pattern learning & batch learning adapt at same slope (derivative $d_\eta w$) w.r.t. learning rate η . For recurrent networks, this is not the case.

²⁸In which case, have a very slow learning process.

- Network Architectures for Prediction.

- **Perspective.** Architecture, or structure, of a predictor underpins its capacity to represent dynamic properties of a statistically nonstationary discrete time input signal & hence its ability to predict or forecast some future value \Rightarrow this chapter provides an overview of available structures for prediction of discrete time signals.
- **Introduction.** Basic building blocks of all discrete time predictors are adders, delayers, multipliers & for nonlinear case zero-memory nonlinearities. Manner in which these elements are interconnected describes architecture of a predictor. Foundations of linear predictors for statistically stationary signals are found in work of Yule (1927), Kolmogorov (1941), Wiener (1949). Later studies of Box & Jenkins (1970) & Makhoul (1975) were built upon these fundamentals. Such linear structures are very well established in digital signal processing & are classified either as finite impulse response (FIR) or infinite impulse response (IIR) digital filters (Oppenheim et al. 1999). FIR filters are generally realized without feedback, whereas IIR filters²⁹ utilize feedback to limit number of parameters necessary for their realization. Presence of feedback implies that consideration of stability underpins design of IIR filters. In statistical signal modeling, FIR filters are better known as moving average (MA) structures & IIR filters are named autoregressive (AR) or autoregressive moving average (ARMA) structures. Most straightforward version of nonlinear filter structures can easily be formulated by including a nonlinear operation in output stage of an FIR or an IIR filter. These represent simple examples of nonlinear autoregressive (NAR), nonlinear moving average (NMA) or nonlinear autoregressive moving average (NARMA) structures (Nerrand et al. 1993). Such filters have immediate application in prediction of discrete time random signals that arise from some nonlinear physical system, as for certain speech utterances. These filters, moreover, are strongly linked to single neuron neural networks.

Neuron, or node, is basic processing element within a neural network. Structure of a neuron is composed of multipliers, termed synaptic weights, or simply weights, which scale inputs, a linear combiner to form activation potential, & a certain zero-memory nonlinearity to model activation function. Different neural network architectures are formulated by combination of multiple neurons with various interconnections, hence term *connectionist modeling* (Rumelhart et al. 1986). Feedforward neural networks, as for FIR/MA/NMA filters, have no feedback within their structure. Recurrent neural networks, on the other hand, similarly to IIR/AR/NAR/NARMA filters, exploit feedback & hence have much more potential structural richness. Such feedback can either be local to neurons or global to network (Haykin 1999b; Tsoi & Back 1997). When inputs to a neural network are delayed versions of a discrete time random input signal correspondence between architectures of nonlinear filters & neural networks is evident.

From a biological perspective (Marmarelis 1989), *prototypical* neuron is composed of a cell body (soma), a tree-like element of fibres (dendrites) & a long fibre (axon) with sparse branches (collaterals). Axon is attached to soma at the *axon hillock*, &, together with its collaterals, ends at synaptic terminals (boutons), which are employed to pass information onto their neurons through *synaptic junctions*. Soma contains nucleus & is attached to trunk of dendritic tree from which it receives incoming information. Dendrites are conductors of input information to soma, i.e. input ports, & usually exhibit a high degree of arborisation.

Possible architectures for nonlinear filters or neural networks are manifold. State-space representation from system theory is established for linear systems (Kailath 1980; Kailath et al. 2000) & provides a mechanism for representation of structural variants. An insightful canonical form for neural networks is provided by Nerrand et al. (1993), by exploitation of state-space representation which facilitates a unified treatment of architectures of neural networks.³⁰

- **Overview.** An explanation of concept of prediction of a statistically stationary discrete time random signal. Building blocks for realization of linear & nonlinear predictors are then discussed. These same building blocks are also shown to be basic elements necessary for realization of a neuron. Emphasis is placed upon particular zero-memory nonlinearities used in output of nonlinear filters & activation functions of neurons.

An aim: to highlight correspondence between structures in nonlinear filtering & neural networks, so as to remove apparent boundaries between work of practitioners in control, signal processing, & neural engineering. Conventional linear filter models for discrete time random signals are introduced &, with aid of statistical modeling, motivate structures for linear predictors, their nonlinear counterparts are then developed.

A feedforward neural network is next introduced in which nonlinear elements are distributed throughout structure. To employ such a network as a predictor, shown: short-term memory is necessary, either at input or integrated within network. Recurrent networks follow naturally from feedforward neural networks by connecting output of network to its input. Implications of local & global feedback in neural networks are also discussed.

Role of state-space representation in architectures for neural networks is described & this leads to a canonical representation.

- **Prediction.** A real discrete time random signal $\{y(k)\}$, where k : *discrete time index*, is most commonly obtained by sampling some analogue measurement. Voice of an individual, e.g., is translated from pressure variation in air into a continuous time electrical signal by means of a microphone & then converted into a digital representation by an analogue-to-digital converter. Such discrete time random signals have statistics that are time-varying, but on a short-term basis, statistics may be assumed to be time invariant.

Principle of prediction of a discrete time signal is represented in Fig. 3.1: Basic concept of linear prediction & forms basis of linear predictive coding (LPC) which underlies many compression techniques. Value of signal $y(k)$ is predicted on basis of

²⁹FIR filters can be represented by IIR filters, however, in practice it is not possible to represent an arbitrary IIR filter with an FIR filter of finite length.

³⁰ARMA models also have a canonical (up to an invariant) representation.

a sum of p past values, i.e., $y(k-1), y(k-2), \dots, y(k-p)$, weighted, by coefficients a_i , $i = 1, \dots, p$, to form a prediction, $\hat{y}(k)$. Prediction error $e(k)$ thus become

$$e(k) = y(k) - \hat{y}(k) = y(k) - \sum_{i=1}^p a_i y(k-i). \quad (287)$$

Estimation of parameters a_i is based upon minimizing some function of error, most convenient form being mean square error $E[e^2(k)]$, where $E[\cdot]$ denotes *statistical expectation operator*, & $\{y(k)\}$ is assumed to be statistically wide sense stationary,³¹ with zero mean (Papoulis 1984). A fundamental advantage of mean square error criterion is so-called *orthogonality condition*, which implies

$$E[e(k), y(k-j)] = 0, \quad j = 1, 2, \dots, p, \quad (288)$$

is satisfied only when a_i , $i = 1, \dots, p$, take on their optimal values. As a consequence of (288) & linear structure of predictor, optimal weight parameters may be found from a set of linear equations, named the *Yule-Walker equations* (Box & Jenkins 1970), $\mathbf{R}_{yy}\mathbf{a} = \mathbf{r}_{yy}$ where $\mathbf{R}_{yy} = (r_{yy}(|i-j|))_{i,j=1}^p$, $r_{yy}(\tau) = E[y(k)y(k+\tau)]$ is value of autocorrelation function of $\{y(k)\}$ at lag τ . These equations may be equivalently written in matrix form as $\mathbf{R}_{yy}\mathbf{a} = \mathbf{r}_{yy}$ where $\mathbf{R}_{yy} \in \mathbb{R}^{p \times p}$: *autocorrelation matrix*, $\mathbf{a}, \mathbf{r}_{yy} \in \mathbb{R}^p$ are, resp., parameter vector of predictor & crosscorrelation vector. Toeplitz symmetric structure of \mathbf{R}_{yy} is exploited in Levinson-Durbin algorithm (Hayes 1997) to solve for optimal parameters in $O(p^2)$ operations. Quality of prediction is judged by minimum mean square error (MMSE), which is calculated from $E[e^2(k)]$ when weight parameters of predictor take on their optimal values. The MMSE is calculated from $r_{yy}(0) - \sum_{i=1}^p a_i r_{yy}(i)$.

Real measurements can only be assumed to be locally wide sense stationary & therefore, in practice, autocorrelation function values must be estimated from some finite length measurement in order to employ (3.3). A commonly used, but statistically biased & low variance (Kay 1993), autocorrelation estimator for application to a finite length N measurement, $\{y(0), y(1), \dots, y(N-1)\}$, is given by

$$\hat{r}_{yy}(\tau) = \frac{1}{N} \sum_{k=0}^{N-\tau-1} y(k)y(k+\tau), \quad \tau = 0, 1, \dots, p. \quad (289)$$

These estimates would then replace exact values in (3.3) from which weight parameters of predictor are calculated. This procedure, however, needs to be repeated for each new length N measurement, & underlies operation of a block-based predictor.

A 2nd approach to estimation of weight parameters $\mathbf{a}(k)$ of a predictor is sequential, adaptive or learning approach. Estimates of weight parameters are refined at each sample number k on basis of new sample $y(k)$ & prediction error $e(k)$. This yields an update equation of form $\hat{\mathbf{a}}(k+1) = \hat{\mathbf{a}}(k) + \eta f(e(k), \mathbf{y}(k))$, $k \geq 0$, where η is termed adaptation gain, $f(\cdot)$ is some function dependent upon particular learning algorithm, whereas $\hat{\mathbf{a}}(k), \mathbf{y}(k)$ are, resp., estimated weight vector & predictor input vector. Without additional prior knowledge, zero or random values are chosen for initial values of weight parameters in (3.6), i.e. $\hat{a}_i(0) = 0$, or n_i , $i = 1, \dots, p$, where n_i : a random variable drawn from a suitable distribution. Sequential approach to estimation of weight parameters is particularly suitable for operation of predictors in statistically nonstationary environments. Both block & sequential approach to estimation of weight parameters of predictors can be applied to linear & nonlinear structure predictors.

- **Building Blocks.** In Fig. 3.2: Building blocks of predictors: (a) delayer, (b) adder, (c) multiplier the basic building blocks of discrete time predictors are shown. A simple delayer has input $y(k)$ & output $y(k-1)$, note: sampling period is normalized to unity. From linear discrete time system theory, delay operation can also be conveniently represented in \mathcal{Z} -domain notation as the z^{-1} operator³² (Oppenheim et al. 1999). An adder, or sumer, simply produces an output which is the sum of all the components at its input. A multiplier, or scaler, used in a predictor generally has 2 inputs & yields an output which is product of 2 inputs. Manner in which delayers, adders, & multipliers are interconnected determines architecture of linear predictors. These architectures, or structures, are shown in block diagram form in the ensuing sections.

To realize nonlinear filters & neural networks, zero-memory nonlinearities are required. 3 zero-memory nonlinearities, as given in Haykin (1999b), with inputs $v(k)$ & outputs $\Phi(k)$ are described by following operations:

* Threshold:

$$\Phi(v(k)) = \begin{cases} 0 & v(k) < 0, \\ 1 & v(k) \geq 0, \end{cases} \quad (290)$$

* Piecewise-linear:

$$\Phi(v(k)) = \begin{cases} 0 & v(k) \leq -\frac{1}{2}, \\ v(k) & -\frac{1}{2} < v(k) < \frac{1}{2}, \\ 1 & v(k) \geq \frac{1}{2}, \end{cases} \quad (291)$$

³¹Wide sense stationarity implies that mean is constant, autocorrelation function is only a function of time lag & variance is finite.

³² z^{-1} operator is a delay operator s.t. $\mathcal{Z}(y(k-1)) = z^{-1}\mathcal{Z}(y(k))$.

* Logistic:

$$\Phi(v(k)) = \frac{1}{1 + e^{-\beta v(k)}}, \beta \geq 0. \quad (292)$$

The most commonly used nonlinearity is logistic function since it is continuously differentiable & hence facilitates analysis of operation of neural networks. This property is crucial in development of 1st- & 2nd-order learning algorithms. When $\beta \rightarrow \infty$, moreover, logistic function becomes unipolar threshold function. Logistic function is a strictly nondecreasing function which provides for a gradual transition from linear to nonlinear operation. Inclusion of such a zero-memory nonlinearity in output stage of structure of a linear predictor facilitates design of nonlinear predictors.

Threshold nonlinearity is well-established in neural network community as it was proposed in seminal work of McCulloch & Pitts (1943), however, it has a discontinuity at the origin. Piecewise-linear model, on the other hand, operates in a linear manner for $|v(k)| < \frac{1}{2}$ & otherwise saturates at zero or unity. Although easy to implement, neither of these zero-memory nonlinearities facilitates analysis of operation of nonlinear structures, because of badly behaved derivatives.

Neural networks are composed of basic processing units named neurons, or nodes, in analogy with biological elements present within human brain (Haykin 1999b). Basic building blocks of such artificial neurons are identical to those for nonlinear predictors. Block diagram of an artificial neuron³³ is shown in Fig. 3.3: Structure of a neuron for prediction. In context of prediction, inputs are assumed to be delayed versions of $y(k)$, i.e., $y(k-i)$, $i = 1, \dots, p$. There is also a constant bias input with unity value. These inputs are then passed through $(p+1)$ multipliers for scaling. In neural network parlance, this operation in scaling inputs corresponds to role of synapses in physiological neurons. A sumer then linearly combines (in fact this is an affine transformation) these scaled inputs to form an output $v(k)$ which is termed induced local field or activation potential of neuron. Save for presence of bias input, this output is identical to output of a linear predictor. This component of neuron, from a biological perspective, is termed synaptic part (Rao & Gupta 1993). Finally, $v(k)$ is passed through a zero-memory nonlinearity to form output $\hat{y}(k)$. This zero-memory nonlinearity is called (nonlinear) activation function of a neuron & can be referred to as somatic part (Rao & Gupta 1993). Such a neuron is a static mapping between its input & output (Hertz et al. 1991) & is very different from dynamic form of a biological neuron. Synergy between nonlinear predictors & neurons is therefore evident. Structural power of neural networks in prediction results, however, from interconnection of many such neurons to achieve overall predictor structure in order to distribute underlying nonlinearity.

- o **Linear Filters.** In digital signal processing & linear time series modeling, linear filters are well-established (Hayes 1997; Oppenheim et al. 1999) & have been exploited for structures of predictors. Essentially, there are 2 families of filters: those without feedback, for which their output depends only upon current & past input values; & those with feedback, for which their output depends both upon input values & past outputs. Such filters are best described by a constant coefficient difference equation, most general form of which is given by

$$y(k) = \sum_{i=1}^p a_i y(k-i) + \sum_{j=0}^q b_j e(k-j), \quad (293)$$

where $y(k)$: output, $e(k)$: input,³⁴ $a_i, i = 1, \dots, p$, are (AR) feedback coefficients & $b_j, j = 0, 1, \dots, q$, are (MA) feedforward coefficients. In causal systems, (293) is satisfied for $k \geq 0$ & initial conditions $y(i), i = -1, -2, \dots, -p$, are generally assumed to be zero. Block diagram for filter represented by (293) is shown in Fig. 3.4: Structure of an autoregressive moving average filter ARMA(p, q). Such a filter is termed an autoregressive moving average ARMA(p, q) filter, where p is order of autoregressive, or feedback, part of structure, & q : order of moving average, or feedforward, element of structure. Due to feedback present within this filter, impulse response, namely values of $y(k), k \geq 0$, when $e(k)$ is a discrete time impulse, is infinite in duration \Rightarrow such a filter is termed an infinite impulse response (IIR) filter within field of digital signal processing.

General form of (293) is simplified by removing feedback terms to yield

$$y(k) = \sum_{j=0}^q b_j e(k-j). \quad (294)$$

Such a filter is termed moving average MA(q) & has a finite impulse response, which is identical to parameters $b_j, j = 0, 1, \dots, q$. In digital signal processing \Rightarrow such a filter is named a finite impulse response (FIR) filter. Similarly, (294) is simplified to yield an autoregressive AR(p) filter

$$y(k) = \sum_{i=1}^p a_i y(k-i) + e(k), \quad (295)$$

which is also termed an IIR filter. Filter described by (3.12) is basis for modeling speech production process (Makhoul 1975). Presence of feedback within AR(p) & ARMA(p, q) filters implies that selection of $a_i, i = 1, \dots, p$, coefficients must be s.t. filters are BIBO stable, i.e. a bounded output will result from a bounded input (Oppenheim et al. 1999).³⁵ Most straightforward way to test stability is to exploit \mathcal{Z} -domain representation of transfer function of filter represented by (3.10):

$$H(z) = \frac{Y(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_q z^{-q}}{1 - a_1 z^{-1} - \dots - a_p z^{-p}} = \frac{N(z)}{D(z)}. \quad (296)$$

³³Term ‘artificial neuron’ will be replaced by ‘neuron’ in sequel.

³⁴Notice $e(k)$ is used as filter input, rather than $x(k)$, for consistency with later sections on prediction error filtering.

³⁵This type of stability is commonly denoted as BIBO stability in contrast to other types of stability, e.g. global asymptotic stability (GAS).

To guarantee stability, p roots of denominator polynomial of $H(z)$, i.e. values of z for which $D(z) = 0$, poles of transfer function, must lie within unit circle in z -plane, $|z| < 1$. In digital signal processing, cascade, lattice, parallel & wave filters have been proposed for realization of transfer function described by (3.13) (Oppenheim et al. 1999). For prediction applications, however, direct form, as in Fig. 3.4: Structure of an autoregressive moving average filter ARMA(p, q), & lattice structures are most commonly employed.

In signal modeling, rather than being deterministic, input $e(k)$ to filter in (3.10) is assumed to be an independent identically distributed (i.i.d.) discrete time random signal. This input is an integral part of a rational transfer function discrete time signal model. Filtering operations described by (3.10)–(3.12), together with such an i.i.d. input with prescribed finite variance σ_e^2 , represent resp., ARMA(p, q), MA(q), AR(p) signal models. Autocorrelation function of input $e(k)$ is given by $\sigma_e^2 \delta(k) \Rightarrow$ its power spectral density (PSD) is $P_e(f) = \sigma_e^2$ for all f . PSD of an ARMA model is therefore:

$$P_y(f) = |H(f)|^2 P_e(f) = \sigma_e^2 |H(f)|^2, \quad f \in \left(-\frac{1}{2}, \frac{1}{2}\right], \quad (297)$$

where f : normalized frequency. Quantity $|H(f)|^2$: magnitude squared frequency domain transfer function found from (3.13) by replacing $z = e^{j2\pi f}$. Role of filter is therefore to shape PSD of driving noise to match PSD of physical system. Such an ARMA model is well motivated by the Wold decomposition, which states: any stationary discrete time random signal can be split into sum of uncorrelated deterministic & random components. In fact, an ARMA(∞, ∞) model is sufficient to model any stationary discrete time random signal (Theiler et al. 1993).

- **Nonlinear Predictors.** If a measurement is assumed to be generated by an ARMA(p, q) model, optimal conditional mean predictor of discrete time random signal $\{y(k)\}$

$$\hat{y}(k) = E[y(k)|y(k-1), y(k-2), \dots, y(0)] \quad (298)$$

is given by

$$\hat{y}(k) = \sum_{i=1}^p a_i y(k-i) + \sum_{j=1}^q b_j \hat{e}(k-j), \quad (299)$$

where residuals $\hat{e}(k-j) = y(k-j) - \hat{y}(k-j)$, $j = 1, \dots, q$. Notice predictor described by (3.16) utilizes past value of actual measurement, $y(k-i)$, $i = 1, \dots, p$; whereas estimates of unobservable input signal $e(k-j)$, $j = 1, \dots, q$, are formed as difference between actual measurements & past predictions. Feedback present within (3.16), which is due to residuals $\hat{e}(k-j)$, results from presence of MA(q) part of model for $y(k)$ in (3.10). No information is available about $e(k) \Rightarrow$ it cannot form part of prediction. On this basis, simplest form of nonlinear autoregressive moving average NARMA(p, q) model takes form,

$$y(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) + \sum_{j=1}^q b_j e(k-j) \right) + e(k), \quad (300)$$

where $\Theta(\cdot)$ is an unknown differentiable zero memory nonlinear function. Notice $e(k)$ is not included within $\Theta(\cdot)$ as it is unobservable. Term NARMA(p, q) is adopted to define (3.17), since save for $e(k)$, output of an ARMA(p, q) model is simply passed through zero-memory nonlinearity $\Theta(\cdot)$.

Corresponding NARMA(p, q) predictor is given by

$$\hat{y}(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) + \sum_{j=1}^q b_j \hat{e}(k-j) \right), \quad (301)$$

where residuals $\hat{e}(k-j) = y(k-j) - \hat{y}(k-j)$, $j = 1, \dots, q$. Equivalently, simplest form of nonlinear autoregressive NAR(p) model is described by

$$y(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) \right) + e(k) \quad (302)$$

& its associated predictor is

$$\hat{y}(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) \right). \quad (303)$$

Associated structures for predictors described by (3.18) & (3.20) are shown in Fig. 3.5: Structure of NARMA(p, q) & NAR(p) predictors. Feedback is present within NARMA(p, q) predictor, whereas NAR(p) predictor is an entirely feedforward structure. Structures are simply those of linear filters described in Sect. 3.6 with incorporation of a zero-memory nonlinearity.

In control applications, most generally, NARMA(p, q) models also include so-called *exogeneous inputs* $u(k-s)$, $s = 1, \dots, r$, & following approach of (3.17) & (3.19) simplest example takes form

$$y(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) + \sum_{j=1}^q b_j e(k-j) + \sum_{s=1}^r c_s u(k-s) \right) + e(k), \quad (304)$$

& is termed a nonlinear autoregressive moving average with exogeneous inputs model, NARMAX(p, q, r), with associated predictor

$$\hat{y}(k) = \Theta \left(\sum_{i=1}^p a_i y(k-i) + \sum_{j=1}^q b_j e(k-j) + \sum_{s=1}^r c_s u(k-s) \right), \quad (305)$$

which again exploits feedback (Chen & Billings 1989; Siegelmann et al. 1997). This is most straightforward form of nonlinear predictor structure derived from linear filters.

- **Feedforward Neural Networks: Memory Aspects.** See also [NP23]: *A rigorous framework for the mean field limit of multilayer neural networks*. Nonlinearity present in predictors described by (3.18), (3.20), & (3.22) only appears at overall output, in same manner as in simple neuron depicted in Fig. 3.3. These predictors could therefore be referred to as single neuron structures. More generally, however, in neural networks, nonlinearity is distributed through certain layers, or stages, of processing.

In Fig. 3.6: **Multilayer feedforward neural network** a multiplayer feedforward neural network is shown. Measurement samples appear at input layer, & output prediction is given from output layer. To be consistent with problem of prediction of a single discrete time random signal, only a single output is assumed. In between, there exist so-called *hidden layers*. Notice outputs of each layer are only connected to inputs of adjacent layer. Nonlinearity inherent in network is due to overall action of all activation functions of neurons within structure.

In problem of prediction, nature of inputs to multilayer feedforward neural network must capture something about time evolution of underlying discrete time random signal. Simplest situation is for inputs to be time-delayed versions of signal, i.e. $y(k-i), i = 1, \dots, p$, & is commonly termed a tapped delay line or delay space embedding (Mozer 1993). Such a block of inputs provides network with a short-term memory of signal. At each time sample k , inputs of network only see effect of 1 sample of $y(k)$, & Mozer (1994) terms this a high-resolution memory. Overall predictor can then be represented as

$$\hat{y}(k) = \Phi(y(k-1), y(k-2), \dots, y(k-p)), \quad (306)$$

where Φ represents nonlinear mapping of neural network.

Other forms of memory for network include: samples with nonuniform delays, i.e. $y(k-i), i = \tau_1, \tau_2, \dots, \tau_p$; exponential, where each input to network, denoted $\tilde{y}_i(k), i = 1, \dots, p$, is calculated recursively from $\tilde{y}_i(k) = \mu_i \tilde{y}_i(k-1) + (1 - \mu_i) y_i(k)$, where $\mu_i \in [-1, 1]$: *exponential factor* which controls depth (Mozer 1993) or time spread of memory & $y_i(k) = y(k-i), i = 1, \dots, p$. A delay line memory is therefore termed high-resolution low-depth, while an exponential memory is low-resolution but high-depth. In continuous time, Principe et al. (1993) proposed Gamma memory, which provided a method to trade resolution for depth. A discrete time version of this memory is described by

$$\tilde{y}_{\mu,j}(k) = \mu \tilde{y}_{\mu,j}(k-1) + (1 - \mu) \tilde{y}_{\mu,j-1}(k-1), \quad (307)$$

where index j is included because necessary to evaluate (3.24) for $j = 0, 1, \dots, i$, where i : delay of particular input to network & $\tilde{y}_{\mu,-1}(k) = y(k+1), \forall k \geq 0$, & $\tilde{y}_{\mu,j}(0) = 0, \forall j \geq 0$. Form of equation is, moreover, a convex mixture. Choice of μ controls trade-off between depth & resolution; small μ provides low-depth & high-resolution memory, whereas high μ yields high-depth & low-resolution memory.

Restricting memory in a multilayer feedforward neural network to input layer may, however, lead to structures with an excessively large number of parameters. Wan (1993) therefore utilizes a time-delay network where memory is integrated within each layer of network. Fig. 3.7: **Structure of neuron of a time delay neural network** shows form of a neuron within a time-delay network, in which multipliers of basic neuron of Fig. 3.3 are replaced by FIR filters to capture dynamics of input signals. Networks formed from such neurons are functionally equivalent to networks with only memory at their input but generally have many fewer parameters, which is beneficial for learning algorithms.

Integration of memory into a multilayer feedforward network yields structure for nonlinear prediction \Rightarrow Clear: such networks belong to class of nonlinear filters.

- **Recurrent Neural Networks: Local & Global Feedback.** In Fig. 3.6, inputs to network are drawn from discrete time signal $y(k)$. Conceptually, straightforward to consider connecting delayed versions of output $\hat{y}(k)$ of network to its input. Such connections, however, introduce feedback into network & therefore stability of such networks must be considered, this is a particular focus of later parts of this book. Provision of feedback, with delay, introduces memory to network & so is appropriate for prediction.

Feedback within recurrent neural networks can be achieved in either a local or global manner. An example of a recurrent neural network is shown in Fig. 3.8: **Structure of a recurrent neural network with local & global feedback** with connections for both local & global feedback. Local feedback is achieved by introduction of feedback within hidden layer, whereas global feedback is produced by connection of network output to network input. Inter-neuron connections can also exist in hidden layer, but they are not shown in Fig. 3.8. Although explicit delays are not shown in feedback connections, they are assumed to be present within neurons in order that network is realizable. Operation of a recurrent network predictor that employs global feedback can now be represented by

$$\hat{y}(k) = \Phi(y(k-1), y(k-2), \dots, y(k-p), \hat{e}(k-1), \dots, \hat{e}(k-q)), \quad (308)$$

where again $\Phi(\cdot)$ represents nonlinear mapping of neural network &

$$\hat{e}(k-j) = y(k-j) - \hat{y}(k-j), \quad j = 1, \dots, q. \quad (309)$$

A taxonomy of recurrent neural networks architectures is presented by Tsoi & Back (1997). Choice of structure depends upon dynamics of signal, learning algorithm & ultimately prediction performance. There is, unfortunately, no hard & fast rule as to best structure to use for a particular problem (Personnaz & Dreyfus 1998).

- **State-Space Representation & Canonical Form.** Structures in this chapter have been developed on basis of difference equation representations. Simple nonlinear predictors can be formed by placing a zero-memory nonlinearity within output stage of a classical linear predictor. In this case, nonlinearity is restricted to output stage, as in a single layer neural network realization. On the other hand, if nonlinearity is distributed through many layers of weighted interconnections, concept of neural networks is fully exploited & more powerful nonlinear predictors may ensue. For purpose of prediction, memory stages may be introduced at input or within network. Most powerful approach is to introduce feedback & to unify feedback networks. Nerrand et al. (1994) proposed an insightful canonical state-space representation:

Any feedback network can be cast into a canonical form that consists of a feedforward (static) network:
 whose outputs are the outputs of the neurons that have desired values, & the values of the state variables,
 whose inputs are the inputs of the network & the values of the state variables, the latter being delayed by 1 time unit.

Note: in prediction of a single discrete-time random signal, network will have only 1 output neuron with a predicted value. For a dynamical system, e.g. a recurrent neural network for prediction, state represents *a set of quantities that summarizes all information about past behavior of system that is needed to uniquely describe its future behavior, except for purely external effects arising from applied input (excitation)* (Haykin 1999b).

Note: whereas always possible to rewrite a nonlinear input-output model in a state-space representation, an input-output model equivalent to a given state-space model might not exist &, if it does, it is surely of higher order. Under fairly general conditions of observability of a system, however, an equivalent input-output model does exist but it may be of high order. A state-space model is likely to have lower order & require a smaller number of past inputs &, hopefully, a smaller number of parameters. This has fundamental importance when only a limited number of data samples is available. Takens' theorem (Wan 1993) implies: for a wide class of deterministic systems, there exists a diffeomorphism (1-1 differential mapping) between a finite window of time series & underlying state of dynamic system which gives rise to time series. A neural network can therefore approximate this mapping to realize a predictor.

In Fig. 3.9: Canonical form of a recurrent neural network for prediction, general canonical form of a recurrent neural network is represented. If state is assumed to contain N variables, then a state vector is defined as $\mathbf{s}(k) = [s_1(k), \dots, s_N(k)]^T$, & a vector of p external inputs is given by $\mathbf{y}(k-1) = [y(k-1), y(k-2), \dots, y(k-p)]^T$. State evolution & output equations of recurrent network for prediction are given, resp., by

$$\mathbf{s}(k) = \varphi(\mathbf{s}(k-1), \mathbf{y}(k-1), \hat{y}(k-1)), \quad (310)$$

$$\hat{y}(k) = \psi(\mathbf{s}(k-1), \mathbf{y}(k-1), \hat{y}(k-1)), \quad (311)$$

where φ, Ψ represent general classes of nonlinearities. Particular choice of N minimal state variables is not unique, therefore several canonical forms³⁶ exist. A procedure for determination of N for an arbitrary recurrent neural network is described by Nerrand et al. (1994). NARMA & NAR predictors described by (3.18) & (3.20), however, follow naturally from canonical state-space representation because elements of state vector are calculated from inputs & outputs of network. Moreover, even if recurrent neural network contains local feedback & memory, still possible to convert network into above canonical form (Personnaz & Dreyfus 1998).

- **Summary.** Aim of this chapter: to show commonality between structures of nonlinear filters & neural networks. Basic building blocks for both structures have been shown to be adders, delayers, multipliers, & zero-memory nonlinearities, & manner in which these elements are interconnected defines particular structure. Theory of linear predictors, for stationary discrete time random signals, which are optimal in minimum mean square prediction error sense, has been shown to be well established. Structures of linear predictors have also been demonstrated to be established in signal processing & statistical modeling. Nonlinear predictors have then been developed on basis of defining dynamics of a discrete time random signal by a nonlinear model. In essence, in their simplest form these predictors have 2 stages: a weighted linear combination of inputs &/or past outputs, as for linear predictors, & a 2nd stage defined by a zero-memory nonlinearity.

Neuron, fundamental processing element in neural networks, has been introduced. Multilayer feedforward neural networks have been introduced in which nonlinearity is distributed throughout structure. To operate in a prediction mode, some local memory is required either at input or integral to network structure. Recurrent neural networks have then been formulated by connecting delayed versions of global output to input of a multilayer feedforward structure; or by introduction of local feedback within network. A canonical state-space form has been used to represent an arbitrary neural network.

• Activation Functions Used in Neural Networks.

- **Perspective.** Choice of nonlinear activation function has a key influence on complexity & performance of artificial neural networks, note: term *neural network* will be used interchangeably with term *artificial neural network*. Brief introduction to activation functions given in Chap. 3 is therefore extended. Although sigmoidal nonlinear activation functions are most common choice, there is no strong a priori justification why models based on such functions should be preferred to others.

³⁶These canonical forms stem from Jordan canonical forms of matrices & companion matrices. Notice: in fact $\hat{y}(k)$ is a state variable but shown separately to emphasize its role as predicted output.

⇒ introduce neural networks as universal approximators of functions & trajectories, based upon Kolmogorov universal approximation theorem, which is valid for both feedforward & recurrent neural networks. From these universal approximation properties, then demonstrate need for a sigmoidal activation function within a neuron. To reduce computational complexity, approximations to sigmoid function are further discussed. Use of nonlinear activation functions suitable for hardware realization of neural networks is also considered.

For rigor, extend analysis to complex activation functions & recognize that a suitable complex activation function is a Möbius transformation. In that context, a framework for rigorous analysis of some inherent properties of neural networks, e.g. fixed points, nesting & invertibility based upon theory of modular groups of Möbius transformations is provided.

All relevant defs, thms, & other mathematical terms are given in Appendices B–C.

- **Introduction.** A century ago, a set of 23 (originally) unsolved problems in mathematics was proposed by DAVID HILBERT (Hilbert 1901–1902). In his lecture ‘Mathematische Probleme’ at 2nd International Congress of Mathematics held in Paris in 1900, he presented 10 of them. These problems were designed to serve as examples for kinds of problems whose solutions would lead to further development of disciplines in mathematics. His 13th problem concerned solutions of polynomial equations. Although his original formulation dealt with properties of solution of 7th degree algebraic equation,³⁷ this problem can be restated as: *Prove that there are continuous functions of n variables, not representable by a superposition of continuous functions of $(n - 1)$ variables.* I.e., could a general algebraic equation of a high degree be expressed by sums & compositions of single-variable functions?³⁸ In 1957, KOLMOGOROV showed: conjecture of HILBERT was not correct (Kolmogorov 1957). Kolmogorov’s theorem is a general representation theorem stating: any real-valued continuous function f defined on an n -dimensional cube I^n , $n \geq 2$, can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} \Phi_q \left(\sum_{p=1}^n \psi_{pq}(x_p) \right), \quad (312)$$

where $\Phi_q(\cdot)$, $q = 1, \dots, 2n + 1$, & $\psi_{pq}(\cdot)$, $p = 1, \dots, n$, $q = 1, \dots, 2n + 1$, are typically nonlinear continuous functions of 1 variable.

For a neural network representation, this means: an activation function of a neuron has to be nonlinear to form a universal approximator. This also means: every continuous function of many variables can be represented by a 4-layered neural network with 2 hidden layers & an input & output layer, whose hidden units represent mappings Φ, ψ . However, this does not mean: a network with 2 hidden layers necessarily provides an accurate representation of function f . In fact, functions ψ_{pq} of KOLMOGOROV’s theorem are quite often highly nonsmooth, whereas for a neural network we want smooth nonlinear activation functions, as is required by gradient-descent learning algorithms (Poggio & Girosi 1990). Vitushkin (1954) showed: there are functions of > 1 variable which do not have a representation by superpositions of differentiable functions (Beiu 1998). Important questions about KOLMOGOROV’s representation are therefore existence, constructive proofs & bounds on size of a network needed for approximation.

KOLMOGOROV’s representation has been improved by several authors. Sprecher (1965) replaced functions ψ_{pq} in KOLMOGOROV REPRESENTATION by $\lambda^{pq}\psi_q$, where λ is a constant, & ψ_q are monotonic increasing functions which belong to class of Lipschitz functions. Lorentz (1976) showed: functions Φ_q can be replaced by only 1 function Φ . Hecht-Nielsen reformulated this result for MLPs so that they are able to approximate any function. In this case, functions ψ are nonlinear activation functions in hidden layers, whereas functions Φ are nonlinear activation functions in output layer. Functions Φ, ψ are found, however, to be generally highly nonsmooth. Further, in Katsuura & Sprecher (1994), function ψ is obtained through a graph that is limit point of an iterated composition of contraction mappings on their domain.

In applications of neural networks for universal approximation, existence proof for approximation by neural networks is provided by KOLMOGOROV’s theorem, which is neural network community was 1st recognized by Hecht-Nielsen (1987) & Lippmann (1987). 1st constructive proof of neural networks as universal approximators was given by Cybenko (1989). Most of analyzes rest on denseness property of nonlinear functions that approximate desired function in space in which desired function is defined. In CYBENKO’s results, e.g., if σ is a continuous discriminatory function,³⁹ then finite sums of form

$$g(\mathbf{x}) = \sum_{i=1}^N w_i \sigma(\mathbf{a}_i^\top \mathbf{x} + b_i), \quad (314)$$

where $w_i, b_i, i = 1, \dots, N$, are coefficients, are dense in space of continuous functions defined on $[0, 1]^n$. Following classical approach to approximation, this means: given any continuous function f defined on $[0, 1]^N$ & any $\varepsilon > 0$, there is a $g(\mathbf{x})$ given by (4.2) for which $|g(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$, $\forall \mathbf{x} \in [0, 1]^N$. CYBENKO then concludes: any bounded & measurable sigmoidal function is discriminatory (Cybenko 1989), & a 3-layer neural network with a sufficient number of neurons in its hidden layer can represent an arbitrary function (Beiu 1998; Cybenko 1989).

³⁷HILBERT conjectured: roots of equation $x^7 + ax^3 + bx^2 + cx + 1 = 0$ as functions of coefficients a, b, c are not representable by sums & superpositions of functions of 2 coefficients, or ‘Show impossibility of solving general 7th degree equation by functions of 2 variables.’

³⁸E.g., function xy is a composition of functions $g(\cdot) = \exp(\cdot)$, $h(\cdot) = \log \cdot$, therefore $xy = e^{\log x + \log y} = g(h(x) + h(y))$ (Gorban & Wunsch 1998).

³⁹ $\sigma(\cdot)$ is discriminatory if for a Borel measure μ on $[0, 1]^N$, $\int_{[0, 1]^N} \sigma(\mathbf{a}^\top \mathbf{x} + b) d\mu(\mathbf{x}) = 0$, $\forall \mathbf{a} \in \mathbb{R}^N, \forall b \in \mathbb{R}$, implies $\mu = 0$. The sigmoids CYBENKO considered had limits

$$\sigma(t) = \begin{cases} 0 & t \rightarrow -\infty, \\ 1 & t \rightarrow +\infty. \end{cases} \quad (313)$$

This justifies use of logistic function $\sigma(x) = \frac{1}{(1 + e^{-\beta x})}$ in neural network applications.

Funahashi (1989) extended this to include sigmoidal functions so that any continuous function is approximately realizable by 3-layer networks with bounded & monotonically increasing activation functions within hidden units. Hornik et al. (1989) showed: output function does not have to be continuous, & they also proved: a neural network can approximate simultaneously both a function & its derivative (Hornik et al. 1990). Hornik (1990) further showed: activation function has to be bounded & nonconstant (but not necessarily continuous), Kurkova (1992) revealed existence of an approximate representation of functions by superposition of nonlinear functions within constraints of neural networks. Leshno et al. (1993) relaxed condition for activation function to be ‘locally bounded piecewise continuous’ (i.e., iff activation function is not a polynomial). This result encompasses most of activation functions commonly used.

Funahashi & Nakamura (1993), in their article ‘Approximation of dynamical systems by continuous time recurrent neural networks’, proved: universal approximation theorem also holds for trajectories & patterns & for recurrent neural networks. Li (1992) also showed: recurrent neural networks are universal approximators. Some recent results, moreover, suggest: ‘smaller nets perform better’ (Elsken 1999), which recommends recurrent neural networks, since a small-scale RNN has dynamics that can be achieved only by a large scale feedforward neural network. Sprecher (1993) considered problem of dimensionality of neural networks & demonstrated: number of hidden layers is independent of number of input variables N . Barron (1993) described spaces of functions that can be approximated by relaxed algorithm of Jones using functions computed by single-hidden-layer networks or perceptrons. Attali & Pages (1997) provided an approach based upon Taylor series expansion. Maiorov & Pinkus have given lower bounds for neural network based approximation (Maiorov & Pinkus 1999). Approximation ability of neural networks has also been rigorously studied in Williamson & Helmke (1995).

Sigmoid neural units usually use a ‘bias’ or ‘threshold’ term in computing activation potential (combination function, net input $\text{net}(k) = \mathbf{x}^\top(k)\mathbf{w}(k)$) of neural unit. Bias term is a connection weight from a unit with a constant value as shown in Fig. 3.3. Bias unit is connected to every neuron in a neural network, weight of which can be trained just like any other weight in a neural network.

From geometric point of view, for an MLP with N output units, operation of network can be seen as defining an N -dimensional hypersurface in space spanned by inputs to network. Weights define position of this surface. Without a bias term, all hypersurfaces would pass through origin (Mandic & Chambers 2000c), which in turn means: universal approximation property of neural networks would not hold if bias was omitted.

A result by Hornik (1993) shows: a sufficient condition for universal approximation property without biases is that no derivative of activation function vanishes at origin, which implies that a fixed nonzero bias can be used instead of a trainable bias.

Why use activation functions? To introduce nonlinearity into a neural network, employ nonlinear activation (output) functions. Without nonlinearity, since a composition of linear functions is again a linear function, an MLP would not be functionally different from a linear filter & would not be able to perform nonlinear separation & trajectory learning for nonlinear & nonstationary signals.

Due to Kolmogorov theorem, almost any nonlinear function is a suitable candidate for an activation function of a neuron. However, for gradient-descent learning algorithms, this function ought to be differentiable. It also helps if function is bounded.⁴⁰ For output neuron, one should either use an activation function suited to distribution of desired (target) values, or preprocess inputs to achieve this goal. If, e.g., desired values are positive but have no known upper bound, an exponential nonlinear activation function can be used.

Important to identify classes of functions & processes that can be approximated by artificial neural networks. Similar problems occur in nonlinear circuit theory, where analogue nonlinear devices are used to synthesis desired transfer functions (gyrators, impedance converters), & in digital signal processing where digital filters are designed to approximate arbitrarily well any transfer function. Fuzzy sets are also universal approximators of functions & their derivatives (Kreinovich et al. 2000; Mitaim & Kosko 1996, 1997).

- **Overview.** 1st explain requirements of an activation function mathematically. Introduce different types of nonlinear activation functions & discuss their properties & realizability. Finally, a complex form of activation functions within framework of Möbius transformations will be introduced.
- **Neural Networks & Universal Approximation.** Learning an input–output relationship from examples using a neural network can be considered as problem of approximating an unknown function $f(x)$ from a set of data points (Girosi & Poggio 1989a). This is why analysis of neural networks for approximation is important for neural networks for prediction, & also system identification & trajectory tracking. Property of uniform approximation is also found in algebraic & trigonometric polynomials, e.g. in case of Weierstrass & Fourier representation, resp.

A neural activation function $\sigma(\cdot)$ is typically chosen to be a continuous & differentiable nonlinear function that belongs to class $S = \{\sigma_i | i = 1, \dots, n\}$ of sigmoid⁴¹ functions having following desirable properties⁴²

- * $\sigma_i \in S$ for $i = 1, \dots, n$
- * $\sigma_i(x_i)$ is a continuously differentiable function
- * $\sigma'_i(x_i) = \frac{d\sigma_i(x_i)}{dx_i} > 0, \forall x_i \in \mathbb{R}$
- * $\sigma_i(\mathbb{R}) = (a_i, b_i), a_i, b_i \in \mathbb{R}, a_i \neq b_i$

⁴⁰Function $f(x) = e^x$ is a suitable candidate for an activation function & is suitable for unbounded signals, is also continuously differentiable. However, to control dynamics, fixed points & invertibility of a neural network, desirable to have bounded, ‘squashing’ activation functions for neurons.

⁴¹Sigmoid means S-shaped.

⁴²Constraints we impose on sigmoidal functions are stricter than ones commonly employed.

- * $\sigma'_i(x) \rightarrow 0$ as $x \rightarrow \pm\infty$
- * $\sigma'_i(x)$ takes a global maximal value $\max_{x \in \mathbb{R}} \sigma'_i(x)$ at a unique point $x = 0$
- * a sigmoidal function has only 1 inflection point, preferably at $x = 0$
- * from (iii), function σ_i is monotonically nondecreasing, i.e., if $x_1 < x_2$ for each $x_1, x_2 \in \mathbb{R} \Rightarrow \sigma_i(x_1) \leq \sigma_i(x_2)$
- * σ_i is uniformly Lipschitz, i.e. there exists a constant $L > 0$ s.t. $\|\sigma_i(x_1) - \sigma_i(x_2)\| \leq L\|x_1 - x_2\|$, $\forall x_1, x_2 \in \mathbb{R}$, i.e., $\frac{\sigma_i(x_1) - \sigma_i(x_2)}{x_1 - x_2} \leq L$, $\forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2$.

Briefly discuss some of above requirements. Property (ii) represents continuous differentiability of a sigmoid function, which is important for higher order learning algorithms, which require not only existence of Jacobian matrix, but also existence of a Hessian & matrices containing higher-order derivatives. This is also necessary if behavior of a neural network is to be described via Taylor series expansion about current point in state space of network. Property (iii) states: a sigmoid should have a positive 1st derivative, which in turns means: a gradient descent algorithm which is employed for training of a neural network should have gradient vectors pointing towards bottom of bowl shaped error performance surface, which is global minimum of surface. Property (vi) means: point around which 1st derivative is centered is origin. This is connected with property (vii) which means that 2nd derivative of activation function should change its sign at origin. Going back to error performance surface, this means: irrespective of whether current prediction error is positive or negative, gradient vector of network at that point should point downwards. Monotonicity, required by (vii) is useful for uniform convergence of algorithms & in search for fixed points of neural networks. Finally, Lipschitz condition is connected with boundedness of an activation function & degenerates into requirements of uniform convergence given by contraction mapping theorem for $L < 1$.

Surveys of neural transfer functions can be found in Duch & Jankowski (1999) & Cichocki & Unbehauen (1993). Examples of sigmoidal functions are

$$\sigma_1(x) = \frac{1}{1 + e^{-\beta x}}, \beta \in \mathbb{R}, \quad (315)$$

$$\sigma_2(x) = \tanh(\beta x) = \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta x} + e^{-\beta x}}, \beta \in \mathbb{R}, \quad (316)$$

$$\sigma_3(x) = \frac{2}{\pi} \arctan\left(\frac{\pi}{2}\beta x\right), \beta \in \mathbb{R}, \quad (317)$$

$$\sigma_4(x) = \frac{x^2}{1 + x^2} \operatorname{sgn} x, \quad (318)$$

where $\sigma(x) = \Phi(x)$ as in Chap. 3. For $\beta = 1$, these functions & their derivatives are given in Fig. 4.1. Function σ_1 , also known as *logistic function*,⁴³ is unipolar, whereas other 3 activation functions are bipolar. 2 frequently used sigmoid functions in neural networks are σ_1, σ_2 . Their derivatives are also simple to calculate, & are

$$\sigma'_1(x) = \beta \sigma_1(x)(1 - \sigma_1(x)), \quad (319)$$

$$\sigma'_2(x) = \beta \operatorname{sech}^2 x = \beta(1 - \sigma_2^2(x)). \quad (320)$$

Can easily modify activation functions to have different saturation values. For logistic function $\sigma_1(x)$, whose saturation values are $(0, 1)$, to obtain saturation values $(-1, 1)$, perform $\sigma_s(x) = \frac{2}{1 + e^{-\beta x}} - 1$. To modify input data to fall within range of an activation function, can normalize, standardize or rescale input data, using mean μ , standard deviation std & minimum & maximum range R_{\min}, R_{\max} .⁴⁴ Cybenko (1989) has shown: neural networks with a single hidden layer of neurons with sigmoidal functions are universal approximators & provided they have enough neurons, can approximate an arbitrary continuous function on a compact set with arbitrary precision. These results do not mean that sigmoidal functions always provide an optimal choice.⁴⁵ 2 functions determine way signals are processed by neurons.

* **Combination functions.** Each processing unit in a neural network performs some mathematical operation on values that are fed into it via synaptic connections (weights) from other units. Resulting value is called *activation potential* or ‘net input’. Any combination function is a net: $\mathbb{R}^N \rightarrow \mathbb{R}$ function, & its output is a scalar. Most frequently used combination functions are inner product (linear) combination functions (as in MLPs & RNNs) & Euclidean or Mahalanobis distance combination functions (a sin RBF networks).

* **Activation functions.** Neural networks for nonlinear processing of signals map their net input provided by a combination function onto the output of a neuron using a scalar function called a ‘nonlinear activation function’, ‘output function’ or

⁴³The logistic map $\dot{f} = rf\left(1 - \frac{f}{K}\right)$ (Strogatz 1994) is used to describe population dynamics, where f : growth of a population of organisms, r : growth rate & K : carrying capacity (population cannot grow unbounded). Fixed points of this map in phase space are 0 & K , hence population always approaches carrying capacity. Under these conditions, graph of $f(t)$ belongs to class of sigmoid functions.

⁴⁴To normalize input data to $\mu = 0$, std = 1, calculate $\mu = \frac{\sum_{i=1}^N x_i}{N}$, std = $\sqrt{\frac{\sum_{i=1}^N (x_i - \mu)^2}{N}}$, & perform standardization of input data as $\tilde{x}_i = \frac{x_i - \mu}{\text{std}}$. To translate data to midrange 0 & standardize to range R , perform

$$Z = \frac{\max_i \{x_i\} + \min_i \{x_i\}}{R}, \quad S_x = \max_i \{x_i\} - \min_i \{x_i\}, \quad x_i^n = \frac{x_i - Z}{S_x/R}. \quad (321)$$

⁴⁵Rational transfer functions (Leung & Haykin 1993) & Gaussian transfer functions also allow NNs to implement universal approximators.

sometimes even ‘activation function’. Entire functional mapping performed by a neuron (composition of a combination function & a nonlinear activation function) is sometimes called a ‘transfer’ function of a neuron $\sigma : \mathbb{R}^N \rightarrow \mathbb{R}$. Nonlinear activation functions with a bounded range are often called ‘squashing’ functions, e.g. commonly used tanh & logistic functions. If a unit does not transform its net input, it is said to have an ‘identity’ or ‘linear’ activation function.

Distance based combination functions (proximity functions) $D(\mathbf{x}, \mathbf{t}) \propto \|\mathbf{x} - \mathbf{t}\|$, are used to calculate how close \mathbf{x} is to a prototype vector \mathbf{t} . Also possible to use some combination of inner product & distance activation functions, e.g. in form $\alpha \mathbf{w}^\top \mathbf{x} + \beta \|\mathbf{x} - \mathbf{t}\|$ (Duch & Jankowski 1999). Many other functions can be used to calculate net input, as e.g. (Ridella et al. 1997).

$$A(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{i=1}^N w_i x_i + w_{N+1} \sum_{i=1}^N x_i^2. \quad (322)$$

- **Other Activation Functions.** By universal approximation theorems, there are many choices of nonlinear activation function \Rightarrow describe some commonly used application-motivated activation functions of a neuron.

Hard-limiter Heaviside (step) function was frequently used in 1st implementations of neural networks, due to its simplicity, given by

$$H(x) = \begin{cases} 0 & x \leq \theta, \\ 1 & x > \theta, \end{cases} \quad (323)$$

where θ is some threshold. A natural extension of step function is multistep function $H_{\text{MS}}(x; \boldsymbol{\theta}) = y_i$, $\theta_i \leq x \leq \theta_{i+1}$. A variant of this function resembles a staircase $\theta_1 < \theta_2 < \dots < \theta_N \Leftrightarrow y_1 < y_2 < \dots < y_N$, & is often called *staircase function*. Semilinear function is defined as

$$H_{\text{SL}}(x) = \begin{cases} 0 & x \leq \theta_1, \\ \frac{x - \theta_1}{\theta_2 - \theta_1} & \theta_1 < x \leq \theta_2, \\ 1 & x > \theta_2, \end{cases} \quad (324)$$

Functions (323) & (324) are depicted in Fig. 4.2: Step & semilinear activation function. Both mentioned functions have discontinuous derivatives, preventing use of gradient-based training procedures. Although they are, strictly speaking, S-shaped, do not use them for network networks for real-time processing, & this is why restricted ourselves to differentiable functions in our 9 requirements that a suitable activation function should satisfy. With development of neural network theory, these discontinuous functions were later generalized to logistic functions, leading to *graded response neurons*, which are suitable for gradient-based training. Indeed, logistic function

$$\sigma(x) = \frac{1}{1 + e^{-\beta x}} \quad (325)$$

degenerates into step function (323) as $\beta \rightarrow \infty$.

Many other activation functions have been designed for special purposes. E.g., a modified activation function which enables single layer perceptrons to solve some linearly inseparable problems has been proposed in Zhang & Sarhadi (1993) & takes form (4.9)

$$f(x) = \frac{1}{1 + e^{-(x^2 + \text{bias})}}. \quad (326)$$

Function (4.9) is differentiable \Rightarrow a network based upon this function can be trained using gradient descent methods. Square operation in exponential term of function enables individual neurons to perform limited nonlinear classification. This activation function has been employed for image segmentation (Zhang & Sarhadi 1993). There have been efforts to combine 2 or more forms of commonly used functions to obtain an improved activation function. E.g., a function defined by

$$f(x) = \lambda \sigma(x) + (1 - \lambda) H(x), \quad (327)$$

where $\sigma(x)$ is a sigmoid function, $H(x)$ is a hard-limiting function & $0 \leq \lambda \leq 1$, has been used in Jones (1990). Function (4.10) is a weighted sum of functions σ & H . Functions (4.9) & (4.10) are depicted in Fig. 4.3: Other activation functions.

Another possibility is to use a linear combination of sigmoid functions instead of a single sigmoid function as an activation function of a neuron. A sigmoid packet f is therefore defined as a linear combination of a set of sigmoid functions with different amplitudes h , slopes β , & biases b (Peng et al. 1998). This function is defined as

$$f(x) = \sum_{n=1}^N h_n \sigma_n = \sum_{n=1}^N \frac{h_n}{1 + e^{-\beta_n x + b_n}}. \quad (328)$$

During learning phase, all parameters (h, β, b) can be adjusted for adaptive shape-refining. Intuitively, a Gaussian-shaped activation function can be, e.g., approximated by a difference of 2 sigmoids, as shown in Fig. 4.4. Other options include spline neural networks⁴⁶ (Guarnieri et al. 1999; Vecchi et al. 1997) & wavelet based neural networks (Zhang et al. 1995), where structure of network is similar to RBF, except RBFs are replaced by orthonormal scaling functions that are not necessarily radial-symmetric.

⁴⁶Splines are piecewise polynomials (often cubic) that are smooth & can retain ‘squashing property’.

For neural systems operating on chaotic input signals, most commonly used activation function is a sinusoidal function. Another activation function that is often used in order to detect chaos in input signal is so-called *saturated-modulus function* given by (Dogaru et al. 1996; Nakagawa 1996)

$$\varphi(x) = \begin{cases} |x| & |x| \leq 1, \\ 1 & |x| > 1. \end{cases} \quad (329)$$

This activation function ensures chaotic behavior even for a very small number of neurons within network. This function corresponds to rectifying operation used in electronic instrumentation & is therefore called a *saturated modulus* or *saturated rectifier function*.

- **Implementation Driven Choice of Activation Functions.** When neurons of a neural network are realized in hardware, due to limitation of processing power & available precision, activation functions can be significantly different from their ideal forms (Al-Ruwaihi 1997; Yang et al. 1998). Implementations of nonlinear activation functions of neurons proposed by various authors are based on a look-up table, McLaurin polynomial approximation, piecewise linear approximation or stepwise linear approximation (Basaglia et al. 1995; Murtagh & Tsoi 1992). These approximations require more iterations of learning algorithm to converge as compared with standard sigmoids.

For neurons based upon look-up tables, samples of a chosen sigmoid are put into a ROM or RAM to store desired activation function. Alternatively, use simplified activation functions that approximate chosen activation function & are not demanding regarding processor time & memory. Thus, e.g., for logistic function, its derivative can be expressed as $\sigma'(x) = \sigma(x)(1 - \sigma(x))$, which is simple to calculate. Logistic function can be approximated using

$$f(x) = \begin{cases} 0 & x \leq -1, \\ 0.5 + x \left(1 - \frac{|x|}{2}\right) & -1 < x < 1, \\ 1 & x \geq 1. \end{cases} \quad (330)$$

Maximal absolute deviation between this function & logistic function is < 0.02 . Function (4.13) is compared with logistic function: Fig. 4.5: Logistic function & its approximation. There are several other approximations. To save on computational complexity, can approximate sigmoid functions with a series of straight lines, i.e. by a piecewise-linear functions. Another sigmoid was proposed by DAVID ELLIOTT (Elliott 1993)

$$\sigma(x) = \frac{x}{1 + |x|}, \quad \sigma'(x) = \frac{1}{(1 + |x|)^2} = (1 - |\sigma|)^2, \quad (331)$$

which is also easy to calculate. Function (4.14) & its derivative are shown in Fig. 4.6: Sigmoid function & its derivative.

In a digital VLSI implementation of an MLP, computation of activation function of each neuron is performed using a look-up table (LUT), i.e. a RAM or ROM memory which is addressed in some way (Piazza et al. 1993). An adaptive LUT based neuron is depicted in Fig. 4.7: LUT neuron.

Although sigmoidal functions are a typical choice for MLPs, several other functions have been considered. Recently, use of polynomial activation functions has been proposed (Chon & Cohen 1997; Piazza et al. 1992; Song & Manry 1993). Networks with polynomial neurons have been shown to be isomorphic to Volterra filters (Chon & Cohen 1997; Song & Manry 1993). However, calculating a polynomial activation $f(x) = a_0 + a_1(x) + \dots + a_M x^M$ for every neuron & every time instant is extremely computationally demanding & is unlikely to be acceptable for real-time applications. Since their calculation is much slower than simple arithmetic operations, other sigmoidal functions might be useful for hardware implementations of neural networks for online applications. An overview of such functions is given in Duch & Jankowski (1999).

- **MLP vs. RBF Networks.** MLP- & RBF-based neural networks are the 2 most commonly used types of feedforward networks. A fundamental difference between the 2 is way in which hidden units combine values at their inputs. MLP networks use inner products, whereas RBFs use Euclidean or Mahalanobis distance as a combination function. An RBF is given by

$$f(\mathbf{x}) = \sum_{i=1}^N c_i G(\mathbf{x}; \mathbf{t}_i), \quad (332)$$

where $G(\cdot)$ is a basis function, $c_i, i = 1, \dots, N$, are coefficients, $\mathbf{t}_i, i = 1, \dots, N$, are centers of radial bases, & \mathbf{x} : input vector. Both multilayer perceptrons & RBFs have good approximation properties & are related for normalized inputs. In fact, an MLP network can always simulate a Gaussian RBF network, whereas converse is true only for certain values of bias parameter (Poggio & Girosi 1990; Yee et al. 1999).

- **Complex Activation Functions.** Recent results suggest that despite existence of universal approximation property, approximation by real-valued neural networks might not be continuous (Kainen et al. 1999) for some standard types of neural networks, e.g. Heaviside perceptrons for Gaussian radial basis functions.⁴⁷ For many functions there is not a best approximation in \mathbb{R} . However, there is always a unique best approximation in \mathbb{C} .

⁴⁷Intuitively, since a measure of quality of an approximation is a distance function, e.g., \mathcal{L}_2 distance given by $\left(\int_a^b (f(x) - g(x))^2 dx\right)^{\frac{1}{2}}$, there might occur a case where we have to calculate an integral which is not possible to be calculated within field \mathbb{R} , but which is easy to calculate in field \mathbb{C} – recall function e^{-x^2} .

Many apparently real-valued mathematical problems can be better understood if they are embedded in complex plane. Every variable is then represented by a complex number $z = x + iy$, where $x, y \in \mathbb{R}, i = \sqrt{-1}$. Example problems cast into complex plane include analysis of transfer functions & polynomial equations. This has motivated researchers to generalize neural networks to complex plane (Clarke 1990). Concerning hardware realization, complex weights of neural network represent impedance as opposed to resistance in real-valued networks.

If again consider approximation, (4.17)

$$f(x) = \sum_{i=1}^N c_i \sigma(x - a_i), \quad (333)$$

where σ : a sigmoid function, different choices of σ will give different realizations of f . An extensive analysis of this problem is given in Helmke & Williamson (1995) & Williamson & Helmke (1995). Going back to elementary function approximation, if $\sigma(x) = x^{-1}$, then (4.17) represents a partial fraction expansion of a rational function f . Coefficients a_i, c_i are, resp., poles & residuals of (4.17). Notice, however, both a_i, c_i are allowed to be complex.⁴⁸

A complex sigmoid is naturally obtained by analytic continuation of a real-valued sigmoid onto complex plane. In order to extend a gradient-based learning algorithm for complex signals, employed activation function should be analytic. Using analytic continuation to extend an activation function to complex plane, however, has a consequence: by Liouville Theorem C.14, only bounded differentiable functions defined on entire complex plane are constant functions. For commonly used activation functions, however, singularities occur in a limited set.⁴⁹ For logistic function $\sigma(z) = \frac{1}{1+e^{-z}} = u + iv$, if z approaches any value in set $\{0 \pm i(2n+1)\pi, n \in \mathbb{Z}\}$, then $|\sigma(z)| \rightarrow \infty$. Similar conditions for \tanh are $\{0 \pm i\frac{2n+1}{2}\pi, n \in \mathbb{Z}\}$, whereas for e^{-z^2} , have singularities for $z = 0 + iy$ (Georgiou & Koutsougeras 1992).

Hence, a function obtained by an analytic continuation to complex plane is generally speaking not an appropriate activation function. Generalizing discussion for real activation functions, properties that a function $\sigma(z) = u(x, y) + iv(x, y)$ should satisfy so that it represents a suitable activation function in complex plane are (Georgiou & Koutsougeras 1992)

- * u, v are nonlinear in x, y
- * $\sigma(z)$ is bounded $\Rightarrow u, v$ are bounded
- * $\sigma(z)$ has bounded partial derivatives u_x, u_y, v_x, v_y , which satisfy Cauchy–Riemann conditions (Mathews & Howell 1997)
- * $\sigma(z)$ is not entire (not a constant).

Regarding fixed point iteration & global asymptotic stability of neural networks, which will be discussed in more detail in Chap. 7, complex-valued neural networks can generate dynamics $z \leftarrow \Phi(z)$. Functions of form $cz(1-z)$, e.g., give rise to Mandelbrot & Julia sets (Clarke 1990; Devaney 1999; Strogatz 1994). A single complex neuron with a feedback connection is thus capable of performing complicated discriminations & generation of nonlinear behavior.

To provide conditions on capability of complex neural networks to approximate nonlinear dynamics, a density theorem for complex MLPs with nonanalytic activation function & a hidden layer is proved in Arena et al. (1998b). The often cited denseness conditions are, as pointed out by Cotter (1990), special cases of Stone–Weierstrass theorem.

In context of learning algorithms, Leung & Haykin (1991) developed their complex backpropagation algorithm considering following activation function (4.19)

$$f(z) = \frac{1}{1 + e^{-z}} : \mathbb{C}^N \rightarrow \mathbb{C}, \quad (334)$$

whose magnitude is shown in Fig. 4.8: Complex extension of logistic function $\sigma(z) = \frac{1}{1+e^{-z}}$. This complex extension of logistic function has singularities due to complex exponential in denominator of (4.19). Safe to use (4.19) if inputs are scaled to range of complex logistic function where it is analytic. In Benvenuto & Piazza (1992), the activation function is proposed: (4.20)

$$\sigma(z) = \sigma(x) + i\sigma(y), \quad (335)$$

where $\sigma(z)$ is a 2D extension of a standard sigmoid. Magnitude of this function is shown in Fig. 4.9: Complex sigmoid $\sigma(z) = \sigma(z_r) + i\sigma(z_i)$. Function (4.20) is not analytic & bounded on \mathbb{C} . It is, however, discriminatory, & linear combinations of functions of this type are dense in \mathbb{C} (Arena et al. 1998a).

Another proposed complex sigmoid is (Benvenuto & Piazza 1992)

$$\sigma(z) = \frac{2c_1}{1 + e^{-2c_2 z}} - c_1, \quad (336)$$

where c_1, c_2 are suitable parameters. Derivative of this function is $\sigma'(z) = \frac{c_2}{2c_1}(c_1^2 - \sigma^2(z))$. Other work on complex back-propagation was proposed in Kim & Guest (1990).

A suitable complex activation function would have property: an excitation near 0 would remain close to 0, & large excitations would be mapped into bounded outputs. 1 such function is given by (Clarke 1990)

$$\sigma(z) = \frac{(\cos \theta + i \sin \theta)(z - s)}{1 - \alpha^* z}, \quad (337)$$

⁴⁸Going back to transfer function approximation in signal processing, functions of type (4.17) are able to approximate a Butterworth function of any degree if (4.17) is allowed to have complex coefficients (e.g. in case of an RLC realization). On the other hand, functions with real coefficients (an RC network) cannot approximate Butterworth function whose order is ≥ 2 .

⁴⁹Exponential function $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ maps set $\{z = a + (2k+1)i\pi, a \in \mathbb{R}, k \in \mathbb{Z}\}$ onto negative real axis, which determine singularities of complex sigmoids.

where θ : a rotation angle & α : a complex constant of magnitude < 1 . Operator $(\cdot)^*$: complex conjugation; sign of imaginary part of asterisked variable is changed. This function is a conformal mapping of unit disc in complex plane onto itself & is unique. Further, σ maps large complex numbers into $-\frac{1}{\alpha}$ & thus satisfies above criteria. 1 flaw in σ is a singularity at $z = \frac{1}{\alpha}$, but in view of Liouville's theorem this is unavoidable. Magnitude plot of this function is shown in Fig. 4.10: A complex activation function.

A simple function that satisfies all above properties is (Georgiou & Koutsougeras 1992)

$$f(z) = \frac{z}{c + \frac{|z|}{r}}, \quad (338)$$

where c, r : real positive constants. This function maps any point in complex plane onto open disc $\{z : |z| < r\}$ as shown in Fig. 4.11: Magnitude of function $f(z) = \frac{z}{c + \frac{|z|}{r}}$.

- **Complex Valued Neural Networks as Modular Groups of Compositions of Möbius Transformations.** Offer a different perspective upon some inherent properties of neural networks, e.g. fixed points, nesting & invertibility, by exposing representations of neural networks in framework of Möbius transformations. This framework includes consideration of complex weights & inputs to network together with complex sigmoidal activation functions.
 - * Möbius Transformation.
 - * Activation Functions & Möbius Transformations.
 - * Existence & Uniqueness of Fixed Points in a Complex Neural Network via Theory of Modular Groups.
- **Summary.** An overview of nonlinear activation functions used in neural networks has been provided. Started from problem of function approximation & trajectory learning & evaluated neural networks suitable for these problems. Properties of neural networks realized in hardware have also been addressed. For rigor, analysis has been extended to neural networks with complex activation functions, for which we have built a unified framework via modular groups of Möbius transformations. Existence & uniqueness conditions of fixed points & invertibility of such mappings have been derived. These results apply both for general input–output relationship in a neural network as well as for a single neuron. This analysis provides a strong mathematical background for further analysis of neural networks for adaptive filtering & prediction.
- **Recurrent Neural Networks Architectures.**
 - **Perspective.** Use of neural networks, in particular recurrent neural networks, in system identification, signal processing & forecasting is considered. Ability of neural networks to model nonlinear dynamical systems is demonstrated, & correspondence between neural networks & block-stochastic models is established. Provide further discussion of recurrent neural network architectures.
 - **Introduction.** There are numerous situations in which use of linear filters & models is limited. E.g., when trying to identify a saturation type nonlinearity, linear models will inevitably fail. This is also case when separating signals with overlapping spectral components.

Most real-world signals are generated, to a certain extent, by a nonlinear mechanism & therefore in many applications choice of a nonlinear model may be necessary to achieve an acceptable performance from an adaptive predictor. Communications channels, e.g., often need nonlinear equalizers to achieve acceptable performance. Choice of model has crucial importance⁵⁰ & practical applications have shown: nonlinear models can offer a better prediction performance than their linear counterparts. They also reveal rich dynamical behavior, e.g. limit cycles, bifurcations & fixed points, that cannot be captured by linear models (Gershenfeld & Weigend 1993).

By *system* we consider actual underlying physics⁵¹ that generate data, whereas by *model* we consider a mathematical description of system. Many variations of mathematical models can be postulated on basis of datasets collected from observations of a system, & their suitability assessed by various performance metrics. Since not possible to characterize nonlinear system by their impulse response, one has to resort to less general models, e.g. homomorphic filters, morphological filters & polynomial filters. Some of most frequently used polynomial filters are based upon Volterra series (Mathews 1991), a nonlinear analogue of linear impulse response, threshold autoregressive models (TAR) (Priestley 1991) & Hammerstein & Wiener models. The latter 2 represent structures that consist of a linear dynamical model & a static zero-memory nonlinearity. An overview of these models can be found in Haber & Unbehauen (1990). Notice: for nonlinear systems, ordering of modules within a modular structure⁵² plays an important role.

To illustrate some important features associated with nonlinear neurons, consider a squashing nonlinear activation function of a neuron, shown in Fig. 5.1: Effects of $y = \tanh v$ nonlinearity in a neuron model upon 2 example inputs. For 2 identical mixed sinusoidal inputs with different offsets, passed through this nonlinearity, output behavior varies from amplifying & slightly distorting input signal to attenuating & considerably nonlinearly distorting input signal. From viewpoint of system theory, neural networks represent nonlinear maps, mapping 1 metric space to another.

⁵⁰System identification, e.g., consists of choice of model, model parameter estimation & model validation.

⁵¹Technically, notions of *system* & *process* are equivalent (Pearson 1995; Sjöberg et al. 1995).

⁵²To depict this, for 2 modules performing nonlinear functions $H_1 = \sin x$, $H_2 = e^x$, have $H_1(H_2(x)) \neq H_2(H_1(x))$ since $\sin e^x \neq e^{\sin x}$. This is reason to use term *nesting* rather than cascading in modular neural networks.

Nonlinear system modeling has traditionally focused on Volterra–Wiener analysis. These models are nonparametric & computationally extremely demanding. Volterra series expansion is given by

$$y(k) = h_0 + \sum_{i=0}^N h_1(i)x(k-i) + \sum_{i=0}^N \sum_{j=0}^N h_2(i,j)x(k-i)x(k-j) + \dots \quad (339)$$

for representation of a causal system. A nonlinear system represented by a Volterra series is completely characterized by its Volterra kernels $h_i, i = 0, 1, 2, \dots$. The Volterra modeling of a nonlinear system requires a great deal of computation, & mostly 2nd- or 3rd-order Volterra systems are used in practice.

Since Volterra series expansion is a Taylor series expansion with memory, they both fail when describing a system with discontinuities, e.g. $y(k) = \text{Asgn}(x(k))$, where $\text{sgn}(\cdot)$ is signum function.

To overcome this difficulty, nonlinear parametric models of nonlinear systems, termed NARMAX, that are described by nonlinear difference equations, have been introduced (Billings 1980; Chon & Cohen 1997; Chon et al. 1999; Connor 1994). Unlike Volterra–Wiener representation, NARMAX representation of nonlinear systems offers compact representation.

NARMAX model describes a system by using a nonlinear functional dependence between lagged inputs, outputs &/or prediction errors. A polynomial expansion of transfer function of a NARMAX neural network does not comprise of delayed versions of input & output of order higher than those presented to network. Therefore, input of an insufficient order will result in undermodeling, which complies with Takens’ embedding theorem (Takens 1981).

Applications of neural networks in forecasting, signal processing & control require treatment of dynamics associated with input signal. Feedforward networks for processing of dynamical systems tend to capture dynamics by including past inputs in input vector. However, for dynamical modeling of complex systems, there is a need to involve feedback, i.e., to use recurrent neural networks. There are various configurations of recurrent neural networks, which are used by Jordan (1986) for control of robots, by Elman (1990) for problems in linguistics & by Williams & Zipser (1989a) for nonlinear adaptive filtering & pattern recognition. In Jordan’s network, past values of network outputs are fed back into hidden units, in Elman’s network, past values of outputs of hidden units are fed back into themselves, whereas in Williams–Zipser architecture, network is fully connected, having 1 hidden layer.

There are numerous modular & hybrid architectures, combining linear adaptive filters & neural networks. These include pipelined recurrent neural network & networks combining recurrent networks & FIR adaptive filters. Main idea: linear filter captures linear ‘portion’ of input process, whereas a neural network captures nonlinear dynamics associated with process.

- **Overview.** Introduce basic modes of modeling, e.g. *parametric, nonparametric, white box, black box & grey box* modeling. Afterwards, dynamical richness of neural models is addressed & feedforward & recurrent modeling for noisy time series are compared. Block-stochastic models are introduced & neural networks are shown to be able to represent these models. Chapter concludes with an overview of recurrent neural network architectures & recurrent neural networks for NARMAX modeling.
- **Basic Modes of Modeling.** Explain notions of parametric, nonparametric, black box, grey box, & white box modeling. These can be used to categorize neural network algorithms, e.g. direct gradient computation, a posteriori & normalized algorithms. Basic idea behind these approaches to modeling is *not to estimate what is already known* \Rightarrow One should utilize prior knowledge & knowledge about physics of system, when selecting neural network model prior to parameter estimation.
- * **Parametric vs. Nonparametric Modeling.** A review of nonlinear input–output modeling techniques is given in Pearson (1995). 3 classes of input–output models are *parametric, nonparametric, & semiparametric* models. Briefly address them:
 - *Parametric* modeling assumes a fixed structure for model. Model identification problem then simplifies to estimating a finite set of parameters of this fixed model. This estimation is based upon prediction of real input data, so as to best match input data dynamics. An example of this technique is broad class of ARIMA/NARMA models. For a given structure of model (e.g. NARMA) we recursively estimate parameters of chosen model.
 - *Nonparametric* modeling seeks a particular model structure from input data. Actual model is not known beforehand. An example taken from nonparametric regression is: look for a model in form of $y(k) = f(x(k))$ without knowing function $f(\cdot)$ (Pearson 1995).
 - *Semiparametric* modeling is combination of above. Part of model structure is completely specified & known beforehand, whereas other part of model is either not known or loosely specified.

Neural networks, especially recurrent neural networks, can be employed within estimators of all of above classes of models. Closely related to above concepts are white, grey, & black box modeling techniques.

- * **White, Grey, & Black Box Modeling.** To understand & analyze real-world physical phenomena, various mathematical models have been developed. Depending on some a priori knowledge about process, data & model, differentiate between 3 fairly general modes of modeling. Idea: to distinguish between 3 levels of prior knowledge, which have been ‘color-coded’. An overview of white, grey, & black box modeling techniques can be found in Aguirre (2000) & Sjöberg et al. (1995). Given data gathered from planet movements, then Kepler’s gravitational laws might well provide initial framework in building a mathematical model of process. This mode of modeling is referred to as *white box* modeling (Aguirre 2000), underlying its fairly deterministic nature. Static data are used to calculate parameters, & to do that underlying physical process has to be understood \Rightarrow possible to build a *white box* model entirely from physical insight & prior knowledge. However, underlying physics are generally not completely known, or are too complicated & often one has to resort to other types of modeling.

Exact form of input–output relationship that describes a real-world system is most commonly unknown, & therefore modeling is based upon a chosen set of known functions. In addition, if model is to approximate system with an arbitrary accuracy, set of chosen nonlinear continuous functions must be dense: case with polynomials. In this light, neural networks can be viewed as another mode of functional representations. *Black box* modeling therefore assumes no previous knowledge about system that produces data. However, chosen network structure belongs to architectures that are known to be flexible & have performed satisfactorily on similar problems. Aim: to find a function F that approximates process y based on previous observations of process y_{PAST} & input u as $y = F(u_{\text{PAST}}, u)$. This ‘black box’ establishes a functional dependence between input & output, which can be either linear or nonlinear. Downside: generally not possible to learn about true physical process that generates data, especially if a linear model is used. Once training process is complete, a neural network represents a *black box*, nonparametric process model. Knowledge about process is embedded in values of network parameters (i.e. synaptic weights).

A natural compromise between 2 previous models is so-called *grey box* modeling, obtained from *black box* modeling if some information about system is known a priori. This can be a probability density function, general statistics of process data, impulse response or attractor geometry. In Sjöberg et al. (1995), 2 subclasses of *grey box* models are considered: *physical* modeling, where a model structure is built upon understanding of underlying physics, as e.g. state-space model structure; & *semiphysical* modeling, where, based upon physical insight, certain nonlinear combinations of data structures are suggested, & then estimated by *black box* methodology.

- **NARMAX Models & Embedding Dimension.** For neural networks, number of input nodes specifies dimension of network input. In practice, true state of system is not observable & mathematical model of system that generates dynamics is not known. Question arises: is sequence of measurements $\{y(k)\}$ sufficient to reconstruct nonlinear system dynamics? Under some regularity conditions, Takens’ (1981) & Mane’s (1981) embedding theorems establish this connection. To ensure that dynamics of a nonlinear process estimated by a neural network are fully recovered, convenient to use Takens’ embedding theorem (Takens 1981), which states: to obtain a faithful reconstruction of system dynamics, *embedding dimension* d must satisfy $d \geq 2D + 1$, where D : dimension of system attractor. Takens’ embedding theorem (Takens 1981; Wan 1993) establishes a diffeomorphism between a finite window of time series $[y(k-1), y(k-2), \dots, y(k-N)]$ & underlying state of dynamic system which generates time series. This implies: a nonlinear regression $y(k) = g[y(k-1), y(k-2), \dots, y(k-N)]$ can model nonlinear time series. An important feature of delay-embedding theorem due to Takens (1981): it is physically implemented by delay lines.

There is a deep connection between time-lagged vectors & underlying dynamics. Delay vectors are not just a representation of a state of system, their length is key to recovering full dynamical structure of a nonlinear system. A general starting point would be to use a network for which input vector comprises delayed inputs & outputs, as shown in Fig. 5.2: **Nonlinear prediction configuration using a neural network model.** For network in Fig. 5.2, both input & output are passed through delay lines, hence indicating NARMAX character of this network. The switch in this figure indicates 2 possible modes of learning explained in Chap. 6.

- **How Dynamically Rich are Nonlinear Neural Models?** To make an initial step toward comparing neural & other nonlinear models, perform a Taylor series expansion of sigmoidal nonlinear activation function of a single neuron model as (Billings et al. 1992) (5.7)

$$\Phi(v(k)) = \frac{1}{1 + e^{-\beta v(k)}} = \frac{1}{2} + \frac{\beta}{4}v(k) - \frac{\beta^3}{48}v^3(k) + \frac{\beta^5}{480}v^5(k) - \frac{17\beta^7}{80640}v^7(k) + \dots \quad (340)$$

Depending on steepness β & activation potential $v(k)$, polynomial representation (5.7) of transfer function of a neuron exhibits a complex nonlinear behavior.

Consider a NARMAX recurrent perceptron with $p = 1, q = 1$ as shown in Fig. 5.3, which is a simple example of recurrent neural networks. Its mathematical description is given by

$$y(k) = \Phi(w_1x(k-1) + w_2y(k-1) + w_0). \quad (341)$$

Expanding this using (5.7):

$$y(k) = \frac{1}{2} + \frac{1}{4}[w_1x(k-1) + w_2y(k-1) + w_0] - \frac{1}{48}[w_1x(k-1) + w_2y(k-1) + w_0]^3 + \dots, \quad (342)$$

where $\beta = 1$. This expression illustrates dynamical richness of squashing activation functions. Associated dynamics, when represented in terms of polynomials are quite complex. Networks with more neurons & hidden layers will produce more complicated dynamics than those in (5.9). Following same approach, for a general recurrent neural network, obtain (Billings et al. 1992) (5.10)

$$y(k) = c_0 + c_1x(k-1) + c_2y(k-1) + c_3x^2(k-1) + c_4y^2(k-1) + c_5x(k-1)y(k-1) + c_6x^3(k-1) + c_7y^3(k-1) + c_8x^2(k-1)y(k-1) + \dots \quad (343)$$

(5.10) does not comprise delayed versions of input & output samples of order higher than those presented to network. If input vector were of an insufficient order, undermodeling would result, which complies with Takens’ embedding theorem \Rightarrow when modeling an unknown dynamical system or tracking unknown dynamics, important to concentrate on embedding dimension of network. Representation (5.10) also models an offset (mean value) c_0 of input signal.

- * **Feedforward vs. Recurrent Networks for Nonlinear Modeling.** Choice of which neural network to employ to represent a nonlinear physical process depends on dynamics & complexity of network that is best for representing problem in hand.

E.g., due to feedback, recurrent networks may suffer from instability & sensitivity to noise. Feedback networks, on the other hand, might not be powerful enough to capture dynamics of underlying nonlinear dynamical system. To illustrate this problem, resort to a simple IIR (ARMA) linear system described by following 1st-order difference equation (5.11)

$$z(k) = 0.5z(k-1) + 0.1x(k-1). \quad (344)$$

System (5.11) is stable, since pole of its transfer function is at 0.5, i.e., within unit circle in z -plane. However, in a noisy environment, output $z(k)$ is corrupted by noise $e(k)$, so that noisy output $y(k)$ of system (5.11) becomes $y(k) = z(k) + e(k)$, which will affect quality of estimation based on this model. This happens because noise terms accumulate during recursions⁵³ (5.11) as

$$y(k) = 0.5y(k-1) + 0.1x(k-1) + e(k) - 0.5e(k-1). \quad (345)$$

An equivalent FIR (MA) representation of same filter (5.11), using method of long division, gives

$$z(k) = 0.1x(k-1) + 0.05x(k-2) + 0.025x(k-3) + 0.0125x(k-4) + \dots \quad (346)$$

& representation of a noisy system now becomes (5.15)

$$y(k) = 0.1x(k-1) + 0.05x(k-2) + 0.025x(k-3) + 0.0125x(k-4) + \dots + e(k). \quad (347)$$

Clearly, noise in (5.15) is not correlated with previous outputs & estimates are unbiased.⁵⁴ Price to pay, however, is infinite length of exact representation of (5.11).

A similar principle applies to neural networks. In Chap. 6, address modes of learning in neural networks & discuss bias/variance dilemma for recurrent neural networks.

- **Wiener & Hammerstein Models & Dynamical Neural Networks.** Under relatively mild conditions,⁵⁵ output signal of a nonlinear model can be considered as a combination of outputs from some suitable submodels. Structure identification, model validation & parameter estimation based upon these submodels are more convenient than those of whole model. Block oriented stochastic models consist of static nonlinear & dynamical linear modules. Such models often occur in practice, examples of which are:

- * Hammerstein model, where a zero-memory nonlinearity is followed by a linear dynamical system characterized by its transfer function $H(z) = \frac{N(z)}{D(z)}$

- * Wiener model, where a linear dynamical system is followed by a zero-memory nonlinearity.

- * **Overview of Block-Stochastic Models.** Defs of certain stochastic models are given by

1. Wiener system

$$y(k) = g(H(z^{-1})u(k)), \quad (348)$$

where $u(k)$: input to system, $y(k)$: output, $H(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})}$: z -domain transfer function of linear component of system & $g(\cdot)$: a nonlinear function

2. Hammerstein system

$$y(k) = H(z^{-1})g(u(k)) \quad (349)$$

3. Uryson system, defined by

$$y(k) = \sum_{i=1}^M H_i(z^{-1})g_i(u(k)). \quad (350)$$

Theoretically, there are finite size neural systems with dynamic synapses which can represent all of above. Moreover, some modular neural architectures, e.g. PRNN (Haykin & Li 1995), are able to represent block-cascaded Wiener-Hammerstein systems described by (Mandic & Chambers 1999c)

$$y(k) = \Phi_N(H_N(z^{-1})\Phi_{N-1}(H_{N-1}(z^{-1})\dots\Phi_1(H_1(z^{-1})u(k))))), \quad (351)$$

$$y(k) = H_N(z^{-1})\Phi_N(H_{N-1}(z^{-1})\Phi_{N-1}\dots\Phi_1(H_1(z^{-1})u(k))) \quad (352)$$

under certain constraints relating size of networks & order of block-stochastic models. Due to its parallel nature, however, a general Uryson model is not guaranteed to be representable this way.

- * **Connection Between Block-Stochastic Models & Neural Networks.** Block diagrams of Wiener & Hammerstein systems are shown in Fig. 5.4: Nonlinear stochastic model used in control & signal processing. Nonlinear function from Fig. 5.4a can be generally assumed to be a polynomial⁵⁶ i.e., $v(k) = \sum_{i=0}^M \lambda_i u^i(k)$. Hammerstein model is a conventional parametric model, usually used to represent processes with nonlinearities involved with process inputs, as shown in Fig. 5.4a. Equation describing output of a SISO Hammerstein system corrupted with additive output noise $\eta(k)$ is

$$y(k) = \Phi[u(k-1)] + \sum_{i=2}^{\infty} h_i \Phi[u(k-i)] + \nu(k), \quad (353)$$

⁵³Notice: if noise $e(k)$ is zero mean & white it appears colored in (5.13), i.e., correlated with previous outputs, which leads to biased estimates.

⁵⁴Under usual assumption that external additive noise $e(k)$ is not correlated with input signal $x(k)$.

⁵⁵A finite degree polynomial steady-state characteristic.

⁵⁶By Weierstrass theorem, polynomials can approximate arbitrarily well any nonlinear function, including sigmoid functions.

where Φ is a nonlinear function which is continuous. Other requirements: linear dynamical subsystem is stable. This network is shown in Fig. 5.5: Discrete-time SISO Hammerstein model with observation noise.

Neural networks with locally distributed dynamics (LDNN) can be considered as locally recurrent networks with global feedforward features. An example of these networks is the *dynamical multilayer perceptron* (DMLP) which consists of dynamical neurons & is shown in Fig. 5.6: Dynamic perceptron. Model of this dynamic perceptron is described by

$$y(k) = \Phi(v(k)), \quad (354)$$

$$v(k) = \sum_{i=0}^{\deg N(z)} n_i(k)x(k-i) + 1 + \sum_{j=1}^{\deg D(z)} d_j(k)v(k-j), \quad (355)$$

$$x(k) = \sum_{l=1}^p w_l(k)u_l(k), \quad (356)$$

where n_i, d_i denote, resp., coefficients of polynomials in $N(z), D(z)$ & '1' is included for a possible bias input. From Fig. 5.6: transfer function between $y(k), x(k)$ represents a Wiener system. Hence, combinations of dynamical perceptrons (e.g. a recurrent neural network) are able to represent block-stochastic Wiener-Hammerstein models. Gradient-based learning rules can be developed for a recurrent neural network representing block-stochastic models. Both Wiener & Hammerstein models can exhibit a more general structure, as shown in Fig. 5.7: Generalized Hammerstein model, for Hammerstein model. Wiener & Hammerstein models can be combined to produce more complicated block-stochastic model. A representative of these models is Wiener-Hammerstein model, shown in Fig. 5.8: Wiener-Hammerstein model. This figure shows a Wiener stochastic model, followed by a linear dynamical system represented by its transfer function $H_2(z) = \frac{N_2(z)}{D_2(z)}$, hence building a Wiener-Hammerstein block-stochastic system. In practice, can build complicated block cascaded systems this way. Wiener & Hammerstein systems are frequently used to compensate each other (Kang et al. 1998). This includes finding an inverse of 1st module in combination. If these models are represented by neural networks, Chap. 4 provides a general framework for uniqueness, existence, & convergence of inverse neural models. Following example from Billings & Voon (1986) shows: Wiener model can be represented by a NARMA model, which, in turn can be modeled by a recurrent neural network.

Example 27. *Wiener model*

$$w(k) = 0.8w(k-1) + 0.4u(k-1), \quad (357)$$

$$y(k) = w(k) + w^3(k) + e(k), \quad (358)$$

was identified as [complicated (5.25)] which is a NARMA model, & hence can be realized by a recurrent neural network.

- **Recurrent Neural Network Architectures.** 2 straightforward ways to include recurrent connections in neural networks are *activation feedback* & *output feedback*, as shown, resp., in Fig. 5.9a: Activation feedback scheme & Fig. 5.9b: Output feedback scheme. These schemes are closely related to state space representation of neural networks. A comprehensive & insightful account of canonical forms & state space representation of general neural networks is given in Nerrand et al. (1993) & Dreyfus & Idan (1998). In Fig. 5.9: Recurrent neural network architectures, blocks labeled 'linear dynamical systems' comprise of delays & multipliers, hence providing linear combination of their input signals. Output of a neuron shown in Fig. 5.9a can be expressed as

$$v(k) = \sum_{i=0}^M w_{u,i}(k)u(k-i) + \sum_{j=1}^N w_{v,j}(k)v(k-j), \quad (359)$$

$$y(k) = \Phi(v(k)), \quad (360)$$

where $w_{u,i}, w_{v,j}$ correspond to weights associated with u, v , resp.

Transfer function of a neuron shown in Fig. 5.9b can be expressed as

$$v(k) = \sum_{i=0}^M w_{u,i}(k)u(k-i) + \sum_{j=1}^N w_{y,j}(k)y(k-j), \quad (361)$$

$$y(k) = \Phi(v(k)), \quad (362)$$

where $w_{y,j}$ correspond to weights associated with delayed outputs. A comprehensive account of types of synapses & short-term memories in dynamical neural networks is provided by Mozer (1993).

Networks mentioned so far exhibit a locally recurrent architecture, but when connected into a larger network, they have a feedforward structure. Hence they are referred to as locally recurrent-globally feedforward (LRGF) architectures. A general LRGF architecture is shown in Fig. 5.10: General LRGF architecture. This architecture allows for dynamic synapses both within input (represented by H_1, \dots, H_M) & output feedback (represented by H_{FB}), hence comprising some of aforementioned schemes.

Elman network is a recurrent network with a hidden layer, a simple example of which is shown in Fig. 5.11: An example of Elman recurrent neural network. This network consists of an MLP with an additional input which consists of delayed state

space variables of network. Even though it contains feedback connections, it is treated as a kind of MLP. Network shown in Fig. 5.12: **An example of Jordan recurrent neural network** is an example of Jordan network. It consists of a multilayer perceptron with 1 hidden layer & a feedback loop from output layer to an additional input called *context layer*. In context layer, there are self-recurrent loops. Both Jordan & Elman networks are structurally locally recurrent globally feedforward (LRGF), & are rather limited in including past information.

A network with a rich representation of past outputs, which will be extensively considered in this book, is a fully connected recurrent neural network, known as Williams–Zipser network (Williams & Zipser 1989a), shown in Fig. 5.13: **A fully connected recurrent neural network**. Give a detailed introduction to this architecture. This network consists of 3 layers: input layer, processing layer, & output layer. For each neuron $i = 1, \dots, N$, elements $u_j, j = 1, 2, \dots, p + N + 1$, of input vector to a neuron \mathbf{u} (5.31), are weighted, then summed to produce an internal activation function of a neuron v (5.30), which is finally fed through a nonlinear activation function Φ (5.28), to form output of i th neuron y_i (5.29). Function Φ is a monotonically increasing sigmoid function with slope β , as e.g. logistic function, $\Phi(v) = \frac{1}{1+e^{-\beta v}}$. At time instant k , for i th neuron, its weights form a $(p + N + 1) \times 1$ dimensional weight vector $\mathbf{w}_i^\top(k) = [w_{i,1}(k), \dots, w_{i,p+N+1}(k)]$, where p : number of external inputs, N : number of feedback connections & $(\cdot)^\top$ denotes vector transpose operation. 1 additional element of weight vector \mathbf{w} : bias input weight. Feedback consists of delayed output signals of RNN. Following equations fully describe RNN from Fig. 5.13: **A fully connected recurrent neural network**,

$$y_i(k) = \Phi(v_i(k)), \quad i = 1, \dots, N, \quad (363)$$

$$v_i(k) = \sum_{l=1}^{p+N+1} w_{i,l}(k) u_l(k), \quad (364)$$

$$\mathbf{u}_i^\top = [s(k-1), \dots, s(k-p), 1, y_1(k-1), y_2(k-1), \dots, y_N(k-1)], \quad (365)$$

where $(p + N + 1) \times 1$ dimensional vector \mathbf{u} comprises both external & feedback inputs to a neuron, as well as unity valued constant bias input.

- **Hybrid Neural Network Architectures.** These networks consist of a cascade of a neural network & a linear adaptive filter. If a neural network is considered as a complex adaptable nonlinearity, then hybrid neural networks resemble Wiener & Hammerstein stochastic models. An example of these networks is given in Khalaf & Nakayama (1999), for prediction of noisy time series. A neural subpredictor is cascaded with a linear FIR predictor, hence making a hybrid predictor. Block diagram of this type of neural network architecture is given in Fig. 5.14: **A hybrid neural predictor**. Neural network from Fig. 5.14 can be either a feedforward neural network or a recurrent neural network.

Another example of hybrid structures is so-called *pipelined recurrent neural network* (PRNN), introduced by Haykin & Li (1995) & shown in Fig. 5.15. It consists of a modular nested structure of small-scale fully connected recurrent neural networks & a cascaded FIR adaptive filter. In PRNN configuration, M modules, which are FCRNNs, are connected as shown in Fig. 5.15: **Pipelined recurrent neural network**. Cascaded linear filter is omitted. Description of this network follows approach from Mandic et al. (1998) & Baltersee & Chambers (1998). Uppermost module of PRNN, denoted by M , is simply an FCRNN, whereas in modules $(M-1, \dots, 1)$, only difference: feedback signal of output neuron within module m , denoted by $y_{m,1}, m = 1, \dots, M-1$, is replaced with appropriate output signal $y_{m+1,1}, m = 1, \dots, M-1$, from its left neighbor module $m+1$. $(p \times 1)$ -dimensional external signal vector $\mathbf{s}^\top(k) = [s(k), \dots, s(k-p+1)]$ is delayed by m time steps ($z^{-m}\mathbf{I}$) before feeding module m , where $z^{-m}, m = 1, \dots, M$, denotes m -step time delay operator & \mathbf{I} : $(p \times p)$ -dimensional identity matrix. Weight vectors \mathbf{w}_n of each neuron n , are embodied in an $(p + N + 1) \times N$ dimensional weight matrix $\mathbf{W}(k) = [\mathbf{w}_1(k), \dots, \mathbf{w}_N(k)]$, with N : number of neurons in each module. All modules operate using same weight matrix \mathbf{W} . Overall output signal of PRNN is $y_{\text{out}}(k) = y_{1,1}(k)$, i.e., output of 1st neuron of 1st module. A full mathematical description of PRNN is given in equations:

$$y_{i,n}(k) = \Phi(v_{i,n}(k)), \quad (366)$$

$$v_{i,n}(k) = \sum_{l=1}^{p+N+1} w_{n,l}(k) u_{i,l}(k), \quad (367)$$

$$\mathbf{u}_i^\top(k) = [s(k-i), \dots, s(k-i-p+1), 1, y_{i+1,1}(k), y_{i,2}(k-1), \dots, y_{i,N}(k-1)] \text{ for } 1 \leq i \leq M-1, \quad (368)$$

$$\mathbf{u}_M^\top(k) = [s(k-M), \dots, s(k-M-p+1), 1, y_{M,1}(k-1), y_{M,2}(k-1), \dots, y_{M,N}(k-1)] \text{ for } i = M. \quad (369)$$

At time step k for each module $i = 1, \dots, M$, 1-step forward prediction error $e_i(k)$ associated with a module is then defined as a difference between desired response of that module $s(k-i+1)$, which is actually next incoming sample of external input signal, & actual output of i th module $y_{i,1}(k)$ of PRNN, i.e., (5.36)

$$e_i(k) = s(k-i+1) - y_{i,1}(k), \quad i = 1, \dots, M. \quad (370)$$

Thus, overall cost function of PRNN becomes a weighted sum of all squared error signals $E(k) = \sum_{i=1}^M \lambda^{i-1} e_i^2(k)$, where $e_i(k)$ is defined in (5.36) & $\lambda \in (0, 1]$, is a forgetting factor.

Other architectures combining linear & nonlinear blocks include so-called ‘sandwich’ structure which was used for estimation of Hammerstein systems (Ibnkahla et al. 1998). Architecture used was a linear-nonlinear-linear combination.

- **Nonlinear ARMA Models & Recurrent Networks.** A general NARMA(p, q) recurrent network model can be expressed as (Chang and Hu 1997)

$$\hat{x}(k) = \Phi \left(\sum_{i=1}^p w_{1,i}(k)x(k-i) + w_{1,p+1}(k) + \sum_{j=p+2}^{p+q+1} w_{1,j}(k)\hat{e}(k+j-2-p-q) + \sum_{l=p+q+2}^{p+q+N} w_{1,l}(k)y_{l-p-q}(k-1) \right). \quad (371)$$

A realization of this model is shown in Fig. 5.16: **Alternative recurrent NARMA(p, q) network.** NARMA(p, q) scheme shown in Fig. 5.16 is a common Williams–Zipser type recurrent neural network, which consists of only 2 layers, output layer of output & hidden neurons y_1, \dots, y_N , & input layer of feedforward & feedback signals

$$x(k-1), \dots, x(k-p), +1, \hat{e}(k-1), \dots, \hat{e}(k-q), y_2(k-1), \dots, y_N(k-1). \quad (372)$$

Nonlinearity in this case is determined by both nonlinearity associated with output neuron of recurrent neural network & nonlinearities in hidden neurons.

Inputs to this network, given in (5.38), however, comprise prediction error terms (residuals) $\hat{e}(k-1), \dots, \hat{e}(k-q)$, which make learning in such networks difficult. Namely, well-known real-time recurrent learning (RTRL) algorithm (Haykin 1994; Williams & Zipser 1989a) was derived to minimize the instantaneous squared prediction error $\hat{e}(k)$, & hence cannot be applied directly to RNN realizations of NARMA(p, q) network since inputs to network comprise delayed prediction error terms $\{\hat{e}\} \Rightarrow$ desirable to find another equivalent representation of NARMA(p, q) network, which would be more suited for RTRL-based learning.

If, for sake of clarity, denote predicted values \hat{x} by y , i.e., to match notation common in RNNs with NARMA(p, q) theory, & have $y_1(k) = \hat{x}(k)$, & keep symbol x for exact values of input signal being predicted, NARMA network from (5.38), can be approximated further as (Connor 1994) [complicated (5.39)]. In that case, scheme shown in Fig. 5.16 should be redrawn, remaining topologically same, with y_1 replacing corresponding \hat{e} terms among inputs to network.

On the other hand, alternative expression for conditional mean predictor, depicted in Fig. 5.16 can be written as

$$\hat{x}(k) = \Phi \left(\sum_{i=1}^p w_{1,i}(k)x(k-i) + w_{1,p+1}(k) + \sum_{j=p+2}^{p+q+1} w_{1,j}(k)\hat{x}(k+j-2-p-q) + \sum_{l=p+q+2}^{p+q+N} w_{1,l}(k)y_{l-p-q}(k-1) \right) \quad (373)$$

or, bearing in mind (5.39), notation used earlier (Haykin & Li 1995; Mandic et al. 1998) for examples on prediction of speech, i.e., $x(k) = s(k)$, & $y_1(k) = \hat{s}(k)$,

$$\hat{s}(k) = \Phi \left(\sum_{i=1}^p w_{1,i}(k)s(k-i) + w_{1,p+1}(k) + \sum_{j=p+2}^{p+q+1} w_{1,j}(k)y_1(k+j-2-p-q) + \sum_{l=p+q+2}^{p+q+N} w_{1,l}(k)y_{l-p-q}(k-1) \right), \quad (374)$$

which is common RNN lookalike notation. This scheme offers a simpler solution to NARMA(p, q) problem, as compared to previous one, since only nonlinear function used is activation function of a neuron Φ , while set of signals being processed is same as in previous scheme. Furthermore, scheme given in (5.41) & depicted in Fig. 5.17: **Recurrent NARMA(p, q) implementation of prediction model** resembles basic ARMA structure.

Li (1992) has shown: recurrent network of (5.41) with a sufficiently large number of neurons & appropriate weights can be found by performing RTRL algorithm s.t. sum of squared prediction errors $E < \delta$ for an arbitrary $\delta > 0$. I.e., $\|\mathbf{s} - \hat{\mathbf{s}}\|_D < \delta$, where $\|\cdot\|_D$ denotes \mathcal{L}_2 norm w.r.t. training set D . Moreover, this scheme, shown also in Fig. 5.17, fits into well-known learning strategies, e.g. RTRL algorithm, which recommends this scheme for NARMA/NARMAX nonlinear prediction applications (Baldi & Atiya 1994; Draye et al. 1996; Kosmatopoulos et al. 1995; McDonnell & Waagen 1994; Nerrand et al. 1994; Wu & Niranjana 1994).

- **Summary.** A review of recurrent neural network architectures in fields of nonlinear dynamical modeling, system identification, control, signal processing, & forecasting has been provided. A relationship between neural network models & NARMA/NARMAX models, as well as Wiener & Hammerstein structures has been established. Particular attention has been devoted to fully connected recurrent neural network & its use in NARMA/NARMAX modeling has been highlighted.

● Neural Networks as Nonlinear Adaptive Filters.

- **Perspective.** Neural networks, in particular recurrent neural networks, are cast into framework of nonlinear adaptive filters. In this context, relation between recurrent neural networks & polynomial filters is 1st established. Learning strategies & algorithms are then developed for neural adaptive system identifiers & predictors. Finally, discuss issues concerning choice of a neural architecture w.r.t. bias & variance of prediction performance.
- **Introduction.** Representation of nonlinear systems in terms of NARMA/NARMAX models has been discussed at length in work of Billings & others (Billings 1980; Chen & Billings 1989; Connor 1994; Nerrand et al. 1994). Some cognitive aspects of neural nonlinear filters are provided in Maass & Sontag (2000). Pearson (1995), in his article on nonlinear input–output modeling, shows: block oriented nonlinear models are a subset of class of Volterra models. So, e.g., Hammerstein model, which consists of a static nonlinearity $f(\cdot)$ applied at output of a linear dynamical system described by its z -domain transfer function $H(z)$, can be represented⁵⁷ by Volterra series.

⁵⁷Under condition: function f is analytic & Volterra series can be thought of as a generalized Taylor series expansion, then coefficients of model (6.2) that do not vanish are $h_{i,j,\dots,z} \neq 0 \Leftrightarrow i = j = \dots = z$.

In previous chapter, shown: neural networks, be they feedforward or recurrent, cannot generate time delays of an order higher than dimension of input to network. Another important feature: capability to generate subharmonics in spectrum of output of a nonlinear neural filter (Pearson 1995). Key property for generating subharmonics in nonlinear systems is recursion, hence, recurrent neural networks are necessary for their generation. Notice: as pointed out in Pearson (1995), block-stochastic models are, generally speaking, not suitable for this application.

In Hakim et al. (1991), by using Weierstrass polynomial expansion theorem, relation between neural networks & Volterra series is established, which is then extended to a more general case & to continuous functions that cannot be expanded via a Taylor series expansion.⁵⁸ Both feedforward & recurrent networks are characterized by means of a Volterra series & vice versa.

Neural networks are often referred to as ‘adaptive neural networks’. Adaptive filters & neural networks are formally equivalent, & neural networks, employed as nonlinear adaptive filters, are generalizations of linear adaptive filters. However, in neural network applications, they have been used mostly in such a way that network is 1st trained on a particular training set & subsequently used. This approach is not an online adaptive approach, which is in contrast with linear adaptive filters, which undergo continual adaptation.

2 groups of learning techniques are used for training recurrent neural networks: a direct gradient computation technique (used in nonlinear adaptive filtering) & a recurrent backpropagation technique (commonly used in neural networks for offline applications). Real-time recurrent learning (RTRL) algorithm (Williams & Zipser 1989a) is a technique which uses direct gradient computation, & is used if the network coefficients change slowly with time. This technique is essentially an LMS learning algorithm for a nonlinear IIR filter. Notice: with same computation time, might be possible to unfold recurrent neural network into corresponding feedforward counterparts & hence to train it by backpropagation. Backpropagation through time (BPTT) algorithm is such a technique (Werbos 1990).

Some of benefits involved with neural networks as nonlinear adaptive filters are that no assumptions concerning Markov property, Gaussian distribution or additive measurement noise are necessary (Lo 1994). A neural filter would be a suitable choice even if mathematical models of input process & measurement noise are not known (black box modeling).

- **Overview.** Start with relationship between Volterra & bilinear filters & neural networks. Recurrent neural networks are then considered as nonlinear adaptive filters & neural architectures for this case are analyzed. Learning algorithms for online training of recurrent neural networks are developed inductively, starting from corresponding algorithms for linear adaptive IIR filters. Some issues concerning problem of vanishing gradient & bias/variance dilemma are finally addressed.
- **Neural Networks & Polynomial Filters.** Shown in Chap. 5: a small-scale neural network can represent high-order nonlinear systems, whereas a large number of terms are required for an equivalent Volterra series representation. E.g., as already shown, after performing a Taylor series expansion for output of a neural network depicted in Fig. 5.3, with input signals $u(k-1), u(k-2)$, obtain

$$y(k) = c_0 + c_1 u(k-1) + c_2 u(k-2) + c_3 u^2(k-1) + c_4 u^2(k-2) + c_5 u(k-1)u(k-2) + c_6 u^3(k-1) + c_7 u^3(k-2) + \dots, \quad (375)$$

which has form of a general Volterra series, given by (6.2)

$$y(k) = h_0 + \sum_{i=0}^N h_1(i)x(k-i) + \sum_{i=0}^N \sum_{j=0}^N h_2(i,j)x(k-i)x(k-j) + \dots \quad (376)$$

Representation by a neural network is therefore more compact. As pointed out in Schetzen (1981), Volterra series are not suitable for modeling saturation type nonlinear functions & systems with nonlinearities of a high order, since they require a very large number of terms for an acceptable representation. Order of Volterra series & complexity of kernels $h(\cdot)$ increase exponentially with order of delay in system (6.2). This problem restricts practical applications of Volterra series to small-scale systems.

Nonlinear system identification, on the other hand, has been traditionally based upon Kolmogorov approximation theorem (neural network existence theorem), which states: a neural network with a hidden layer can approximate an arbitrary nonlinear system. Kolmogorov’s theorem, however, is not that relevant in context of networks for learning (Giroi & Poggio 1989b). Problem: inner functions in Kolmogorov’s formula (4.1), although continuous, have to be highly nonsmooth. Following analysis from Chap. 5, straightforward: multilayered & recurrent neural networks have ability to approximate an arbitrary nonlinear system, whereas Volterra series fail even for simple saturation elements.

Another convenient form of nonlinear system: bilinear (truncated Volterra) system described by

$$y(k) = \sum_{j=1}^{N-1} c_j y(k-j) + \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} b_{i,j} y(k-j)x(k-i) + \sum_{i=0}^{N-1} a_i x(k-i). \quad (377)$$

Despite its simplicity, this is a powerful nonlinear model & a large class of nonlinear systems (including Volterra systems) can be approximated arbitrarily well using this model. Its functional dependence (6.3) shows: it belongs to a class of general recursive nonlinear models. A recurrent neural network that realizes a simple bilinear model is depicted in Fig. 6.1: **Recurrent neural network representation of bilinear model.** Multiplicative input nodes, denoted by \times , have to be introduced to represent bilinear model. Bias terms are omitted & chosen neuron is linear.

⁵⁸E.g. nonsmooth functions e.g. $|x|$.

Problem 13. Show: recurrent network shown in Fig. 6.1 realizes a bilinear model. Also show: this network can be described in terms of NARMAX models.

Solution. Functional description of recurrent network depicted in Fig. 6.1 is given by

$$y(k) = c_1 y(k-1) + b_{0,1} x(k) y(k-1) + b_{1,1} x(k-1) y(k-1) + a_0 x(k) + a_1 x(k-1), \quad (378)$$

which belongs to class of bilinear models (6.3). Functional description of network from Fig. 6.1 can also be expressed as $y(k) = F(y(k-1), x(k), x(k-1))$, which is a NARMA representation of model (6.4). \square

This example confirms duality between Volterra, bilinear, NARMA/NARMAX & recurrent neural models. To further establish connection between Volterra series & a neural network, express activation potential of nodes of network as

$$\text{net}_i(k) = \sum_{j=0}^M w_{i,j} x(k-j), \quad (379)$$

where $\text{net}_i(k)$: activation potential of i th hidden neuron, $w_{i,j}$: weights, $x(k-j)$: inputs to network. If nonlinear activation functions of neurons are expressed via an L th-order polynomial expansion⁵⁹ as

$$\Phi(\text{net}_i(k)) = \sum_{l=0}^L \xi_{il} \text{net}_i^l(k), \quad (380)$$

then neural model described in (6.6) & (6.7) can be related to Volterra model (6.2). Actual relationship is rather complicated, & Volterra kernels are expressed as sums of products of weights from input to hidden units, weights associated with output neuron, & coefficients ξ_{il} from (6.7). Chon et al. (1998) have used this kind of relationship to compare Volterra & neural approach when applied to processing of biomedical signals.

Hence, to avoid difficulty of excessive computation associated with Volterra series, an input-output relationship of a nonlinear predictor that computes output in terms of past inputs & outputs may be introduced as⁶⁰ (6.8)

$$\hat{y} = F(y(k-1), \dots, y(k-N), u(k-1), \dots, u(k-M)), \quad (381)$$

where $F(\cdot)$ is some nonlinear function. Function F may change for different input variables or for different regions of interest. A NARMAX model may therefore be a correct representation only in a region around some operating point. Leontaritis & Billings (1985) rigorously proved: a discrete time nonlinear time invariant system can always be represented by model (6.8) in vicinity of an equilibrium point provided that

- * response function of system is finite realizable, &
- * possible to linearize system around chosen equilibrium point.

Some of other frequently used models, e.g. bilinear polynomial filter, given by (6.3), are obviously cases of a simple NARMAX model.

- 6.5. Neural Networks & Nonlinear Adaptive Filters.
- 7. Stability Issues in RNN Architectures.
- 8. Data-Reusing Adaptive Learning Algorithms.
- 9. A Class of Normalized Algorithms for Online Training of Recurrent Neural Networks.
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- 11. Some Practical Considerations of Predictability & Learning Algorithms for Various Signals.
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- Appendix A: The \mathcal{O} Notation & Vector & Matrix Differentiation.
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⁵⁹Using Weierstrass theorem, this expansion can be arbitrarily accurate. However, in practice resort to a moderate order of this polynomial expansion.

⁶⁰This model is referred to as NARMAX model (nonlinear ARMAX), since it resembles linear model $\hat{y}(k) = a_0 + \sum_{j=1}^N a_j y(k-j) + \sum_{i=1}^M b_i u(k-i)$.

[209 citations]

- **Abstract.** Stability is a fundamental property of dynamical systems, yet to this date it has had little bearing on practice of RNNs. In this work, conduct a thorough investigation of stable recurrent models. Theoretically, prove stable RNNs are well approximated by feed-forward networks for purpose of both inference & training by gradient descent. Empirically (theo kinh nghiệm), demonstrate stable recurrent models often perform as well as their unstable counterparts on benchmark sequence tasks. Taken together, these findings shed light on effective power of recurrent networks & suggest much of sequence learning happens, or can be made to happen, in stable regime. Moreover, our results help to explain why in many cases practitioners succeed in replacing recurrent models by feed-forward models.
- **1. Introduction.** RNNs are a popular modeling choice for solving sequence learning problems arising in domains e.g. speech recognition & natural language processing. At outset, RNNs are nonlinear dynamical systems commonly trained to fit sequence data via some variant of gradient descent.

Stability is of fundamental importance in study of dynamical system. Surprisingly, however, stability has had little impact on practice of RNNs. Recurrent models trained in practice do not satisfy stability in an obvious manner, suggesting: perhaps training happens in a chaotic regime. Difficulty of training recurrent models has compelled practitioners to successfully replace recurrent models with non-recurrent, feed-forward architectures.

– Tính ổn định có tầm quan trọng cơ bản trong nghiên cứu hệ thống động. Tuy nhiên, điều đáng ngạc nhiên là tính ổn định có ít tác động đến việc thực hành RNN. Các mô hình hồi quy được đào tạo trong thực hành không đáp ứng được tính ổn định theo cách rõ ràng, cho thấy: có lẽ việc đào tạo diễn ra trong chế độ hỗn loạn. Khó khăn trong việc đào tạo các mô hình hồi quy đã buộc các học viên phải thay thế thành công các mô hình hồi quy bằng các kiến trúc không hồi quy, truyền thẳng.

This state of affairs raises important unresolved questions.

Is sequence modeling in practice inherently unstable? When & why are recurrent models really needed?

– Mô hình trình tự trong thực tế có thực sự không ổn định không? Khi nào & tại sao các mô hình tuần hoàn thực sự cần thiết?

In this work, shed light on both of these questions through a theoretical & empirical investigation of stability in recurrent models.

1st prove stable recurrent models can be approximated by feed-forward networks. In particular, not only are models equivalent for *inference*, they are also equivalent for *training* via gradient descent. While easy to contrive (chế tạo) nonlinear recurrent models that are some input sequence cannot be approximated by feed-forward models, our result implies such models are inevitably unstable. I.e., in particular they must have exploding gradients, which is in general an impediment to learnability via gradient descent.

– đầu tiên chứng minh các mô hình hồi quy ổn định có thể được xấp xỉ bằng các mạng truyền thẳng. Đặc biệt, các mô hình không chỉ tương đương với *suy luận*, mà còn tương đương với *đào tạo* thông qua giảm dần độ dốc. Trong khi các mô hình hồi quy phi tuyến tính dễ chế tạo (chế tạo) có một số chuỗi đầu vào không thể được xấp xỉ bằng các mô hình truyền thẳng, kết quả của chúng tôi ngụ ý rằng các mô hình như vậy chắc chắn không ổn định. Tức là, cụ thể chúng phải có độ dốc bùng nổ, nói chung là trở ngại đối với khả năng học thông qua giảm dần độ dốc.

2nd, across a variety of different sequence tasks, show how *recurrent models can often be made stable without loss in performance*. Also show models that are nominally unstable often operate in stable regime on data distribution. Combined with our 1st result, these observation helps to explain why an increasingly large body of empirical research succeeds in replacing recurrent models with feed-forward models in important applications, including translation [6,26], speech synthesis [25], & language modeling [5]. While stability does not always hold in practice to begin with, often possible to generate a high-performing stable model by *imposing stability* during training.

Our results also shed light on effective representational properties of recurrent networks trained in practice. In particular, stable models cannot have long-term memory. Therefore, when stable & unstable models achieve similar results, either task does not require long-term memory, or unstable model does not have it.

◦ **1.1. Contributions.** In this work, make following contributions.

1. Present a generic definition of stable recurrent models in terms of nonlinear dynamical systems & show how to ensure stability of several commonly used models. Previous work establishes stability for vanilla recurrent neural networks. Give new sufficient conditions for stability of long short-term memory (LSTM) networks. These sufficient conditions come with an efficient projection operator that can be used at training time to enforce stability.
2. Prove, under stability assumption, feed-forward networks can approximate recurrent networks for purposes of both inference & training by gradient descent. While simple in case of inference, training result lies on non-trivial stability properties of gradient descent.
3. Conduct extensive experimentation on a variety of sequence benchmarks, show stable models often have comparable performance with their unstable counterparts, & discuss when, if ever, there is an intrinsic performance price to using stable models.

- 2. **Stable Recurrent Models.** Define *stable recurrent models* & illustrate concept for various popular model classes. From a pragmatic perspective, stability roughly corresponds to criterion that gradients of training objective do not *explode* over time. Common recurrent models can operate in both stable & unstable regimes, depending on their parameters. To study stable variants of common architectures, give sufficient conditions to ensure stability & describe how to efficiently enforce these conditions during training.
- 2.1. **Defining Stable Recurrent Models.** A *recurrent model* is a nonlinear dynamical system given by a differentiable *state-transition map* $\phi_{\mathbf{w}} : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^n$, parameterized by $\mathbf{w} \in \mathbb{R}^m$. Hidden state $h_t \in \mathbb{R}^n$ evolves in discrete time steps according to update rule

$$h_t = \phi_{\mathbf{w}}(h_{t-1}, x_t),$$

where vector $x_t \in \mathbb{R}^d$ is an arbitrary input provided to system at time t . This general formulation allows us to unify many examples of interest. E.g., for a RNN, given weight matrices W, U , state evolves according to

$$h_t = \phi_{W,U}(h_{t-1}, x_t) = \tanh(W h_{t-1} + U x_t).$$

Recurrent models are typically trained using some variant of gradient descent. 1 natural – even if not strictly necessary – requirement for gradient descent to work: gradients of training objective do not explode over time. *Stable recurrent models* are precisely class of models where gradients cannot explode. They thus constitute a natural class of models where gradient descent can be expected to work. In general, define a stable recurrent model as follows.

Definition 10. A recurrent model $\phi_{\mathbf{w}}$ is stable if there exists some $\lambda < 1$ s.t., for any weights $\mathbf{w} \in \mathbb{R}^m$, states $\mathbf{h}, \mathbf{h}' \in \mathbb{R}^n$, & input $\mathbf{x} \in \mathbb{R}^d$,

$$\|\phi_{\mathbf{w}}(\mathbf{h}, \mathbf{x}) - \phi_{\mathbf{w}}(\mathbf{h}', \mathbf{x})\| \leq \lambda \|\mathbf{h} - \mathbf{h}'\|.$$

Equivalently, a recurrent model is stable if map $\phi_{\mathbf{w}}$ is λ -contractive in \mathbf{h} . If $\phi_{\mathbf{w}}$ is λ -stable, then $\|\nabla_{\mathbf{h}} \phi_{\mathbf{w}}(\mathbf{h}, \mathbf{x})\| < \lambda$, & for Lipschitz loss p , $\|\nabla_{\mathbf{w}} p\|$ is always bounded [21].

Stable models are particularly well-behaved & well-justified from a theoretical perspective. E.g., at present, only stable linear dynamical systems are known to be learnable via gradient descent [7]. In unstable models, gradients of objective can explode, & it is a delicate matter to even show: gradient descent converges to a stationary point. Following proposition offers 1 such example. Proof is provided in appendix.

Proposition 9. There exists an unstable system $\phi_{\mathbf{w}}$ where gradient descent does not converge to a stationary point, & $\|\nabla_{\mathbf{w}} p\| \rightarrow \infty$ as number of iterations $N \rightarrow \infty$.

- 2.2. **Examples of Stable Recurrent Models.** Provide sufficient conditions to ensure stability for several common recurrent models. These conditions offer a way to require learning happens in stable regime – after each iteration of gradient descent, one imposes corresponding stability condition via projection.

* **Linear dynamical systems & RNNs.** Given a Lipschitz, pointwise nonlinearity ρ & matrices $W \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{n \times d}$, state-transition map for a RNN is

$$h_t = \rho(W h_{t-1} + U x_t).$$

If ρ is identity, then system is a linear dynamical system, [10] show if ρ is L_ρ -Lipschitz, then model is stable provided $\|W\| < \frac{1}{L_\rho}$. Indeed, for any states \mathbf{h}, \mathbf{h}' , & any \mathbf{x}

$$\|\rho(W \mathbf{h} + U \mathbf{x}) - \rho(W \mathbf{h}' + U \mathbf{x})\| \leq L_\rho \|W \mathbf{h} + U \mathbf{x} - W \mathbf{h}' - U \mathbf{x}\| \leq L_\rho \|W\| \|\mathbf{h} - \mathbf{h}'\|.$$

In case of a linear dynamical system, model is stable provided $\|W\| < 1$. Similarly, for 1-Lipschitz \tanh -nonlinearity, stability obtains provided $\|W\| < 1$. In appendix, verify assumptions required by theorems given in next sect for this example. Imposing this condition during training corresponds to projecting onto spectral norm ball.

* **Long short-term memory networks.** Long Short-Term Memory (LSTM) networks are another commonly used class of sequence models [9]. State is a pair of vectors $\mathbf{s} = (c, h) \in \mathbb{R}^{2d}$, & model is parameterized by 8 matrices, $W_\square \in \mathbb{R}^{d \times d}, U_\square \in \mathbb{R}^{d \times n}$, for $\square \in \{i, f, o, z\}$. State-transition map ϕ_{LSTM} is given by

$$\begin{aligned} f_t &= \sigma(W_f \mathbf{h}_{t-1} + U_f \mathbf{x}_t), \\ i_t &= \sigma(W_i \mathbf{h}_{t-1} + U_i \mathbf{x}_t), \\ o_t &= \sigma(W_o \mathbf{h}_{t-1} + U_o \mathbf{x}_t), \\ z_t &= \tanh(W_z \mathbf{h}_{t-1} + U_z \mathbf{x}_t), \\ c_t &= i_t \circ z_t + f_t \circ c_{t-1}, \\ \mathbf{h}_t &= o_t \circ \tanh c_t, \end{aligned}$$

where \circ denotes elementwise multiplication, & σ : logistic function.

Provide conditions under which iterated system $\phi_{\text{LSTM}}^r = \phi_{\text{LSTM}} \circ \dots \circ \phi_{\text{LSTM}}$ is stable. Let $\|f\|_\infty = \sup_t \|f_t\|_\infty$. If weights W_f, U_f & inputs \mathbf{x}_t are bounded, then $\|f\|_\infty < 1$ since $|\sigma| < 1$ for any finite input. I.e., next state c_t must “forget” a nontrivial portion of c_{t-1} . Leverage this phenomenon to give sufficient conditions for ϕ_{LSTM} to be contractive in l_∞ norm, which in turn implies iterated system ϕ_{LSTM}^r is contractive in l_2 norm for $r = O(\log d)$. Let $\|W\|_\infty$ denote induced l_∞ matrix norm, which corresponds to maximum absolute row sum $\max_i \sum_j |W_{ij}|$.

Proposition 10. If $\|W_i\|_\infty, \|W_o\|_\infty < 1 - \|f\|_\infty, \|W_z\|_\infty \leq \frac{1}{4}(1 - \|f\|_\infty), \|W_f\|_\infty < (1 - \|f\|_\infty)^2$, & $r = O(\log d)$, then iterated system ϕ_{LSTM}^r is stable.

Proof is given in appendix. Conditions given in Prop. 2 are fairly restrictive. Somewhat surprisingly show in experiments models satisfying these stability conditions still achieve good performance on a number of tasks. Leave it as an open problem to find different parameter regimes where system is stable, as well as resolve whether original system ϕ_{LSTM} is stable. Imposing these conditions during training & corresponds to simple row-wise normalization of weight matrices & inputs.

- 3. Stable Recurrent Models Have Feed-forward Approximations. Prove stable recurrent models can be well-approximated by feed-forward networks for purposes of both inference & training by gradient descent. From a memory perspective, stable recurrent models are *equivalent* to feed-forward networks – both models use same amount of context to make predictions. This equivalence has important consequences for sequence modeling in practice. When a stable recurrent model achieves satisfactory performance on some tasks, a feedforward network can achieve similar performance. Consequently, if sequence learning in practice is inherently stable, then recurrent models may not be necessary. Conversely, if feed-forward models cannot match performance of recurrent models, then sequence learning in practice is in unstable regime.

– Chứng minh các mô hình hồi quy ổn định có thể được xấp xỉ tốt bằng các mạng truyền thẳng cho mục đích suy luận & đào tạo bằng phương pháp giảm dần độ dốc. Theo quan điểm về bộ nhớ, các mô hình hồi quy ổn định *tương đương* với các mạng truyền thẳng – cả hai mô hình đều sử dụng cùng một lượng ngữ cảnh để đưa ra dự đoán. Sự tương đương này có những hậu quả quan trọng đối với mô hình chuỗi trong thực tế. Khi một mô hình hồi quy ổn định đạt được hiệu suất thỏa đáng trên một số tác vụ, một mạng truyền thẳng có thể đạt được hiệu suất tương tự. Do đó, nếu việc học chuỗi trong thực tế vốn đã ổn định, thì các mô hình hồi quy có thể không cần thiết. Ngược lại, nếu các mô hình truyền thẳng không thể khớp với hiệu suất của các mô hình hồi quy, thì việc học chuỗi trong thực tế đang ở chế độ không ổn định.

- 3.1. Truncated recurrent models. For our purposes, salient distinction (sự phân biệt nổi bật) between a recurrent & feed-forward model is the latter has *finite-context*. Therefore, say a model is *feed-forward* if prediction made by model at step t is a function only of inputs x_{t-k}, \dots, x_t for some finite k .

While there are many choices for a feed-forward approximation, consider simplest one – truncation of system to some finite context k . I.e., feed-forward approximation moves over input sequence with a sliding window of length k producing an output every time sliding window advances by 1 step. Formally, for context length k chosen in advance, define *truncated model* via update rule

$$\mathbf{h}_t^k = \phi_{\mathbf{w}}(\mathbf{h}_{t-1}^k, \mathbf{x}_t), \mathbf{h}_{t-k}^k = \mathbf{0}.$$

Note: \mathbf{h}_t^k is a function only of previous k inputs $\mathbf{x}_{t-k}, \dots, \mathbf{x}_t$. While this definition is perhaps an abuse of term “feed-forward”, truncated model can be implemented as a standard autoregressive, depth- k feed-forward network, albeit with significant weight sharing.

Let f denote a prediction function that maps a state \mathbf{h}_t to outputs $f(\mathbf{h}_t) = y_t$. Let y_t^k denote predictions from truncated model. To simplify presentation, prediction function f is not parameterized. This is w.l.o.g. because always possible to fold parameters into system $\phi_{\mathbf{w}}$ itself. In sequel, study $\|y_t - y_t^k\|$ both during & after training.

- 3.2. Approximation during inference. Suppose train a full recurrent model $\phi_{\mathbf{w}}$ & obtain a prediction y_t . For an appropriate choice of context k , truncated model makes essentially same prediction y_t^k as full recurrent model. To show this result, 1st control difference between hidden states of both models.

Lemma 1. Assume $\phi_{\mathbf{w}}$ is λ -contractive in \mathbf{h} & L_x -Lipschitz in x . Assume input sequence $\|\mathbf{x}_t\| \leq B_x \forall t$. If truncation length $k \geq \log_{\frac{1}{\lambda}} \frac{L_x B_x}{(1-\lambda)\varepsilon}$, then difference in hidden states $\|\mathbf{h}_t - \mathbf{h}_t^k\| \leq \varepsilon$.

Lemma 1 effectively says stable models do not have long-term memory – distant inputs do not change states of system. If prediction function is Lipschitz, Lemma 1 immediately implies recurrent & truncated model make nearly identical predictions.

Proposition 11. If $\phi_{\mathbf{w}}$ is a L_x -Lipschitz & λ -contractive map, & f is L_f Lipschitz, & truncation length $k \geq \log_{\frac{1}{\lambda}} \frac{L_x B_x}{(1-\lambda)\varepsilon}$, then $\|y_t - y_t^k\| \leq \varepsilon$.

- 3.3. Approximation during training via gradient descent. Equipped with our inference result, return towards optimization. Show gradient descent for stable recurrent models at same point & track divergence in weights throughout course of gradient descent. Roughly, show if $k \approx O(\log \frac{N}{\varepsilon})$, then after N steps of gradient descent, difference in weights between recurrent & truncated models is at most ε . Even if gradients are similar for both models at same point, it is a priori possible: slight differences in gradients accumulate over time & lead to divergent weights where no meaningful comparison is possible. Building on similar techniques as [8], show: gradient descent itself is stable, & this type of divergence cannot occur.

Our gradient descent result requires 2 essential lemmas. 1st bounds difference in gradient between full & truncated model. 2nd establishes gradient map of both full & truncated models is Lipschitz. Defer proofs of both lemmas to appendix.

Let p_T denote loss function evaluated on recurrent model after T time steps, & define p_T^k similarly for truncated model. Assume there compact, convex domain $\Phi \subset \mathbb{R}^n$ so that map $\phi_{\mathbf{w}}$ is stable \forall choices of parameters $\mathbf{w} \in \Theta$.

Lemma 2. Assume p (& therefore p^k) is Lipschitz & smooth. Assume $\phi_{\mathbf{w}}$ is smooth, λ -contractive, & Lipschitz in x & \mathbf{w} . Assume inputs satisfy $\|\mathbf{x}_t\| \leq B_x$, then

$$\|\nabla_{\mathbf{w}} p_T - \nabla_{\mathbf{w}} p_T^k\| = \gamma k \lambda^k,$$

where $\gamma = O(B_x(1-\lambda)^{-2})$, suppressing dependence on Lipschitz & smoothness parameters.

Lemma 3. For any $w, w' \in \Theta$, suppose $\phi_{\mathbf{w}}$ is smooth, λ -contractive, \mathcal{E} Lipschitz in \mathbf{w} . If p is Lipschitz & smooth, then

$$\|\nabla_{\mathbf{w}} p_T(\mathbf{w}) - \nabla_{\mathbf{w}} p_T(\mathbf{w}')\| \leq \beta \|\mathbf{w} - \mathbf{w}'\|,$$

where $\beta = O((1 - \lambda)^{-3})$, suppressing dependence on Lipschitz & smoothness parameters.

Let w_{recurr}^i : weights of recurrent model on step i & define w_{trunc}^i similarly for truncated model. At initialization, $w_{\text{recurr}}^0 = w_{\text{trunc}}^0$. For k sufficiently large, Lemma 2 guarantees difference between gradient of recurrent & truncated models is negligible. Therefore, after a gradient update, $\|w_{\text{recurr}}^1 - w_{\text{trunc}}^1\|$ is small. Lemma 3 then guarantees: this small difference in weights does not lead to large differences in gradient on subsequent time step. For an appropriate choice of learning rate, formalizing this argument leads to following proposition.

Proposition 12. Under assumptions of Lemmas 2–3, for compact, convex Θ , after N steps of projected gradient descent with step size $\alpha_t = \frac{\alpha}{t}$, $\|w_{\text{recurr}}^N - w_{\text{trunc}}^N\| \leq \alpha \gamma k \lambda^k N^{\alpha\beta+1}$.

Decaying step size in our theorem is consistent with regime in which gradient descent is known to be stable for non-convex training objectives [8]. While decay is faster than many learning rates encountered in practice, classical results nonetheless show: with this learning rate gradient descent still converges to a stationary point; see p. 119 in [4] & references there. In appendix, give empirical evidence $O(\frac{1}{t})$ rate is necessary for our theorem & show examples of stable systems trained with constant or $O(\frac{1}{\sqrt{t}})$ rates that do not satisfy our bound.

Critically, bound in Prop. 4 goes to 0 as $k \rightarrow \infty$. In particular, if take $\alpha = 1$ & $k \geq \Omega \log \frac{\gamma N^\beta}{\epsilon}$, then after N steps of projected gradient descent, $\|w_{\text{recurr}}^N - w_{\text{trunc}}^N\| \leq \epsilon$. For this choice of k , obtain main theorem. Proof is left to appendix.

Theorem 2. Let p be Lipschitz & smooth. Assume $\phi_{\mathbf{w}}$ is smooth, λ -contractive, Lipschitz in x & \mathbf{w} . Assume inputs are bounded, \mathcal{E} prediction function f is L_f -Lipschitz. If $k \geq \Omega(\log \frac{\gamma N^\beta}{\epsilon})$, then after N steps of projected gradient descent with step size $\alpha_t = \frac{1}{t}$, $\|y_T - y_T^k\| \leq \epsilon$.

- 4. Experiments. In experiments, show stable recurrent models can achieve solid performance on several benchmark sequence tasks. Namely, show unstable recurrent models can often be made stable without a loss in performance. In some cases, there is a small gap between performance between unstable & stable models. Analyze whether this gap is indicative of a “price of stability” & show unstable methods involved are stable in a data-dependent sense.

- 4.1. Tasks. Consider 4 benchmark sequence problems – word-level language modeling, character-level language modeling, polyphonic music modeling, & slot-filling.

- * **Language modeling.** In language modeling, given a sequence of words or characters, model must predict next word or character. For character-level language modeling, train & evaluate models on Penn Treebank [15]. To increase coverage of our experiments, train & evaluate word-level language models on Wikitext-2 dataset, which is twice as large as Penn Treebank & features a larger vocabulary [17]. Performance is reported using bits-per-character for character-level models & perplexity for word-level models.

- * **Polyphonic music modeling.** In polyphonic music modeling, a piece is represented as a sequence of 88-bit binary codes corresponding to 88 keys on a piano, with a 1 indicating a key that is pressed at a given time. Given a sequence of codes, task: predict next code. Evaluate our models on JSB Chorales, a polyphonic music dataset consisting of 382 harmonized chorales by J.S. BACH [1]. Performance is measured using negative log-likelihood.

- * **Slot-filling.** In slot filling, model takes as input a query like “I want to Boston on Monday” & outputs a class label for each word in input, e.g. Boston maps to `Departure_City` & Monday maps to `Departure_Time`. Use Airline Travel Information Systems (ATIS) benchmark & report F1 score for each model [22].

- 4.2. Cf. Stable & Unstable Models. For each task, 1st train an unconstrained RNN & an unconstrained LSTM. All hyperparameters are chosen via grid-search to maximize performance of unconstrained model. For consistency with our theoretical results in Sect. 3 & stability conditions in Sect. 2.2, both models have a single recurrent layer & are trained using plain SGD. In each case, resulting model is unstable. However, then retrain best models using projected gradient descent to enforce stability *without retuning hyperparameters*. In RNN case, constrain $\|W\| < 1$. After each gradient update, project W onto spectral norm ball by computing SVD & thresholding singular values to lie in $[0, 1)$. In LSTM case, after each gradient update, normalize each row of weight matrices to satisfy sufficient conditions for stability given in Sect. 2.2.

Stable & unstable models achieve similar performance. Table 1: Comparison of stable & unstable models on a variety of sequence modeling tasks. For all tasks, stable & unstable RNNs achieve same performance. For polyphonic music & slot-filling, stable & unstable LSTMs achieve same results. On language modeling, there is a small gap between stable & unstable LSTMs. Discuss this in Sect. 4.3. Performance is evaluated on held-out test set. For negative log-likelihood (nll), bits per character (bpc), & perplexity, lower is better. For F1 score, higher is better. gives a comparison of performance between stable & unstable RNNs & LSTMs on each of different tasks. Each of reported metrics is computed on held-out test set. Also show a representative comparison of learning curves for word-level language modeling & polyphonic music modeling in Fig. 1: Stable & unstable variants of common recurrent architectures achieve similar performance across a range of different sequence tasks.

Across all tasks considered, stable & unstable RNNs have roughly same performance. Stable RNNs & LSTMs achieve results comparable to published baselines on slot-filling [18] & polyphonic music modeling [3]. On word & character level language modeling, both stable & unstable RNNs achieve comparable results to [3].

On language modeling tasks, however, there is a gap between stable & unstable LSTM models. Given restrictive conditions place on LSTM to ensure stability, surprising they work as well as they do. Weaker conditions ensuring stability of LSTM could reduce this gap. Also possible imposing stability comes at a cost in representational capacity required for some tasks.

- 4.3. What is “price of stability” in sequence modeling? Gap between stable & unstable LSTMs on language modeling raises question of whether there is an intrinsic performance cost for using stable models on some tasks. If measure stability in a data-dependent fashion, then unstable LSTM language models are stable, indicating this gap is illusory (ảo tưởng). However, in some cases with short sequences, instability can offer modeling benefits.
 - * **LSTM language models are stable in a “data-dependent” way.** Our notion of stability is conservative & requires stability to hold for every input & pair of hidden states. If instead consider a weak, data-dependent notion of stability, word & character-level LSTM models are stable (in iterated sense of Prop. 2). In particular, compute stability parameter only *using input sequences from data*. Furthermore, only evaluate stability on hidden states *reachable via gradient descent*. More precisely, to estimate λ , run gradient ascent to find worst-case hidden states \mathbf{h}, \mathbf{h}' to maximize $\frac{\|\phi_{\mathbf{w}}(\mathbf{h}, \mathbf{x}) - \phi_{\mathbf{w}}(\mathbf{h}', \mathbf{x})\|}{\|\mathbf{h} - \mathbf{h}'\|} \rightarrow \text{Appendix}$.
Data-dependent definition given above is a useful diagnostic – when sufficient stability conditions fail to hold, data-dependent condition addresses whether model is still operating in stable regime. Moreover, when input representation is fixed during training, our theoretical results go through without modification when using data-dependent definition. Using data-dependent measure, in Fig. 2: What is intrinsic “price of stability”? For language modeling, show unstable LSTMs are actually stable in weaker, data-dependent sense. On other hand, for polyphonic music modeling with short sequences, instability can improve model performance. (a) Data-dependent stability of character-level language models. Iterated-LSTM refers to iteration system $\phi_{\text{LSTM}}^r = \phi_{\text{LSTM}} \circ \dots \circ \phi_{\text{LSTM}}$, show: iterated character-level LSTM ϕ_{LSTM}^r is stable for $r \approx 80$ iterations. A similar result holds for word-level language model for $r \approx 100$. These findings are consistent with experiments in [14] which find LSTM trajectories converge after ≈ 70 steps *only when evaluated on sequences from data*. For language models, “price of stability” is therefore much smaller than gap in Table 1 suggests – even “unstable” models are operating in stable regime on data distribution.
 - * **Unstable systems can offer performance improvements for short-time horizons.** When sequences are short, training unstable models is less difficult because exploding gradients are less of an issue. In these case, unstable models can offer performance gains. To demonstrate this, train truncated unstable models on polyphonic music task for various values of truncation parameter k . In Fig. 2.(b): Unstable models can boost performance for short sequences., simultaneously plot performance of unstable model & stability parameter λ for converged model for each k . For short-sequences, final model is more unstable $\lambda \approx 3.5$ & achieves a better test-likelihood. For longer sequence lengths, λ decreases closer to stable regime $\lambda \approx 1.5$, & this improved test-likelihood performance disappears.
- 4.4. Unstable Models Operate in Stable Regime. In prev sect, showed: nominally unstable models often satisfy a data-dependent notion of stability (các mô hình không ổn định về mặt danh nghĩa thường thỏa mãn khái niệm về tính ổn định phụ thuộc vào dữ liệu). In this sect, offer further evidence unstable models are operating in stable regime. These results further help explain why stable & unstable models perform comparably in experiments.
 - * **Vanishing gradients.** Stable models necessarily have vanishing gradients, & indeed this ingredient is a key ingredient in proof of our training-time approximation result. For both word & character-level language models, find both *unstable RNNs & LSTMs also exhibit vanishing gradients*. In Fig. 3: Unstable word & character-level language models exhibit vanishing gradients. Plot norm of gradient w.r.t. inputs $\|\nabla_{x_t} p_{t+i}\|$, as distance between input & loss grows, averaged over entire training set. Gradient vanishes for moderate values of i for both RNNs & LSTMs, though decay is slower for LSTMs. (a) Distance i from input x_t to loss p_{t+i} . (b) Character-level language modeling., plot average gradient of loss at time $t + i$ w.r.t. input at time t , $\|\nabla_{x_t} p_{t+i}\|$ as t ranges over training set. For either language modeling task, LSTM & RNN suffer from limited sensitivity to distance inputs at initialization & throughout training. Gradients of LSTM vanish more slowly than those of RNN, but both models exhibit same qualitative behavior.
 - * **Truncating Unstable Models.** Results in Sect. 3 show stable models can be truncated without loss of performance. In practice, unstable models can also be truncated without performance loss. In Fig. 4: Effect of truncating unstable models. On both language & music modeling, RNNs & LSTMs exhibit diminishing returns for large values of truncation parameter k . In LSTMs, larger k doesn't affect performance, whereas for unstable RNNs, large k slightly decreases performance. (a) Word-level language modeling. (b) Polyphonic music modeling., show: performance of both LSTMs & RNNs for various values of truncation parameter k on word-level language modeling & polyphonic music modeling. Initially, increasing k increases performance because model can use more context to make predictions. However, in both cases, there is diminishing returns to larger values of truncation parameter k . LSTMs are unaffected by longer truncation lengths, whereas performance of RNNs slight degrades as k becomes very large, possibly due to training instability. In either case, diminishing returns to performance for large values of k means truncation & therefore feed-forward approximation is possible even for these unstable models.
 - * **Prop. 4 holds for unstable models.** In stable models, Prop. 4 in Sect. 3 ensures distance between weight matrices $\|\mathbf{w}_{\text{recurrent}} - \mathbf{w}_{\text{trunc}}\|$ grows slowly as training progresses, & this rate decreases as k becomes large. In Fig. 5: Qualitative version of Prop. 4 for unstable, word-level language models. Assume $k = 65$ well-captures full-recurrent model & plot $\|\mathbf{w}_{\text{recurrent}} - \mathbf{w}_{\text{trunc}}\| = \|W_k - W_{65}\|$ as training proceeds, where W denotes recurrent weights. As Prop. 4 suggests, this quantity grows slowly as training proceeds, & rate of growth decreases as k increases. (a) Unstable RNN language model. (b) Unstable LSTM language model., plot $\|W_k - W_{65}\|$ for $k \in \{5, 10, 15, 25, 35, 50, 64\}$ throughout training. As suggested by Prop. 4, after an initial rapid increase in distance, $\|W_k - W_{65}\|$ grows slowly. Moreover, there is a diminishing return to choosing larger values of truncation parameter k in terms of accuracy of approximation.
- 5. Are recurrent models truly necessary? Our experiments show recurrent models trained in practice operate in stable regime, & our theoretical results show stable recurrent models are approximable by feed-forward networks. As a consequence, conjecture

recurrent networks trained in practice are always approximable by feed-forward networks. Even with this conjecture, cannot yet conclude recurrent models as commonly conceived are unnecessary. 1st, our present proof techniques rely on truncated versions of recurrent models, & truncated recurrent architectures like LSTMs may provide useful inductive bias on some problems. Moreover, implementing truncated approximation as a feed-forward network increases number of weights by a factor of k over original recurrent model. Declaring recurrent models truly superfluous would require both finding more parsimonious feed-forward approximations & proving natural feed-forward models, e.g., fully connected networks or CNNs, can approximate stable recurrent models during training. This remains an important question for future work.

– Các mô hình hồi quy có thực sự cần thiết không? Các thí nghiệm của chúng tôi cho thấy các mô hình hồi quy được đào tạo trong thực tế hoạt động ở chế độ ổn định, & kết quả lý thuyết của chúng tôi cho thấy các mô hình hồi quy ổn định có thể xấp xỉ được bằng các mạng truyền thẳng. Do đó, phỏng đoán *các mạng truyền thẳng được đào tạo trong thực tế luôn có thể xấp xỉ được bằng các mạng truyền thẳng.* Ngay cả với phỏng đoán này, vẫn chưa thể kết luận các mô hình hồi quy như thường được quan niệm là không cần thiết. Trước tiên, các kỹ thuật chứng minh hiện tại của chúng tôi dựa trên các phiên bản cắt cụt của các mô hình hồi quy, & các kiến trúc hồi quy cắt cụt như LSTM có thể cung cấp độ lệch quy nạp hữu ích cho một số vấn đề. Hơn nữa, việc triển khai phép xấp xỉ cắt cụt như một mạng truyền thẳng làm tăng số lượng trọng số lên gấp k so với mô hình hồi quy ban đầu. Việc tuyên bố các mô hình hồi quy thực sự là thừa sẽ yêu cầu cả việc tìm ra các phép xấp xỉ truyền thẳng tiết kiệm hơn & chứng minh các mô hình truyền thẳng tự nhiên, ví dụ như các mạng hoàn toàn kết nối hoặc CNN, có thể xấp xỉ các mô hình hồi quy ổn định trong quá trình đào tạo. Đây vẫn là một câu hỏi quan trọng cho công việc trong tương lai.

- 6. Related Work. Learning dynamical systems with gradient descent has been a recent topic of interest in ML community. [7] shows gradient descent can efficiently learn a class of stable, linear dynamical systems, [20] shows gradient descent learns a class of stable, nonlinear dynamical systems. Work by [23] gives a moment-based approach for learning some classes of stable nonlinear RNNs. Our work explores theoretical & empirical consequences of stability assumption made in these works. In particular, our empirical results show models trained in practice can be made closer to those currently being analyzed theoretically without large performance penalties.

For *linear* dynamical systems, [24] exploit connection between stability & truncation to learn a truncated approximation to full stable system. Their approximation result is same as our inference result for linear dynamical systems, & extend this result to nonlinear setting. Also analyze impact of truncation on training with gradient descent. Our training time analysis builds on stability analysis of gradient descent in [8], but increasingly uses it for an entirely different purpose. Results of this kind are completely new to our knowledge.

For RNNs, link between vanishing & exploding gradients & $\|W\|$ was identified in [21]. For 1-layer RNNs, [10] give sufficient conditions for stability in terms of norm $\|W\|$ & Lipschitz constant of nonlinearity. Our work additionally considers LSTMs & provides new sufficient conditions for stability. Moreover, study consequences of stability in terms of feed-forward approximation.

A number of recent works have sought to avoid vanishing & exploding gradients by ensuring system is an isometry, i.e. $\lambda = 1$. In RNN case, this amounts to constraining $\|W\| = 1$ [2,11,12,19,28]. [27] observes strictly requiring $\|W\| = 1$ reduces performance on several tasks, & instead proposes maintaining $\|W\| \in [1 - \epsilon, 1 + \epsilon]$. [29] maintains this “soft-isometry” constraint using a parameterization based on SVD that obviates (loại bỏ) need for projection step used in our stable-RNN experiments. [13] sidestep these issues & stabilizes training using a residual parameterization of model. At present, these unitary models have not yet seen widespread use, & our work shows much of sequence learning in practice, even with nominally unstable models, actually occurs in stable regime.

From an empirical perspective, [14] introduce a non-chaotic recurrent architecture & demonstrate it can perform as well more complex models like LSTMs. [3] conducts a detailed evaluation of recurrent & convolutional, feed-forward models outperform their recurrent counterparts. Their experiments are complimentary to ours; find recurrent models can often be replaced with stable recurrent models, which show are equivalent to feed-forward models.

- A. Proofs from Sect. 2.
 - A.2.1. RNNs.
 - A.2.2. LSTMs.
- B. Proofs from Sect. 3.
 - B.1. Proofs from Sect. 3.3.
 - * B.1.1. Gradient difference rule to truncation is negligible. Argue difference in gradient w.r.t. weights between recurrent & truncated models is $O(k\lambda^k)$. For sufficiently large k (independent of sequence length), impact of truncation is therefore negligible. Proof leverages “vanishing-gradient” phenomenon – long-term components of gradient of full recurrent model quickly vanish. Remaining challenge: show short-term components of gradient are similar for full & recurrent models.
 - * B.1.2. Stable recurrent models are smooth. Prove: gradient map $\nabla_{\mathbf{w}} p_T$ is Lipschitz. 1st, show on forward pass, difference between hidden states $\mathbf{h}_t(\mathbf{w})$, $\mathbf{h}'_t(\mathbf{w}')$ obtained by running model with weights \mathbf{w} , \mathbf{w}' , resp., is bounded in terms of $\|\mathbf{w} - \mathbf{w}'\|$. Using smoothness of ϕ , difference in gradients can be written in terms of $\|\mathbf{h}_t(\mathbf{w}) - \mathbf{h}'_t(\mathbf{w}')\|$, which in turn can be bounded in terms of $\|\mathbf{w} - \mathbf{w}'\|$. Repeatedly leverage this fact to conclude total difference in gradients must be similarly bounded. 1st show small differences in weights don’t significantly change trajectory of recurrent model.

Lemma 4. For some \mathbf{w}, \mathbf{w}' , suppose $\phi_{\mathbf{w}}, \phi_{\mathbf{w}'}$ are λ -contractive & $L_{\mathbf{w}}$ Lipschitz in \mathbf{w} . Let $\mathbf{h}_t(\mathbf{w}), \mathbf{h}_t(\mathbf{w}')$ be hidden state at time t obtain from running model with weights \mathbf{w}, \mathbf{w}' on common inputs $\{\mathbf{x}_t\}$. If $\mathbf{h}_0(\mathbf{w}) = \mathbf{h}_0(\mathbf{w}')$, then

$$\|\mathbf{h}_t(\mathbf{w}) - \mathbf{h}_t(\mathbf{w}')\| \leq \frac{L_{\mathbf{w}} \|\mathbf{w} - \mathbf{w}'\|}{1 - \lambda}.$$

Proof of Lem. 3 is similar in structure to Lem. 2 & follows from repeatedly using smoothness of ϕ & Lem. 5.

* B.1.3. Gradient descent analysis. Equipped with smoothness & truncation lemmas (Lems. 2–3), turn towards proving main gradient descent result.

• C. EXPERIMENTS.

- $O(\frac{1}{t})$ rate may be necessary. Key result underlying Thm. 1 is bound on parameter difference $\|\mathbf{w}_{\text{trunc}} - \mathbf{w}_{\text{recurr}}\|$ while running gradient descent obtained in Prop. 4. Show this bound has correct qualitative scaling using random instances & training randomly initialized, stable linear dynamical systems & tanh-RNNs. In Fig. 6, plot parameter error $\|\mathbf{w}_{\text{trunc}}^t - \mathbf{w}_{\text{recurr}}^t\|$ as training progresses for both models (averaged > 10 runs). Error scales comparably with bound given in Prop. 4. Also find for larger step-sizes like $\frac{\alpha}{\sqrt{t}}$ or constant α , bound fails to hold, suggesting $O(\frac{1}{t})$ condition is necessary.

Concretely, generate random problem instance by fixing a sequence length $T = 200$, sampling input data $x_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 4 \cdot I_{32})$, & sampling $y_T \sim \text{Unif}[-2, 2]$. Next, set $\lambda = 0.75$ & randomly initialize a stable linear dynamical system or RNN with tanh nonlinearity by sampling $U_{ij}, W_{ij} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 0.5)$ & thresholding singular values of W so $\|W\| \leq \lambda$. Use squared loss & prediction function $f(\mathbf{h}_t, \mathbf{x}_t) = C\mathbf{h}_t + D\mathbf{x}_t$, where $C, D \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_{32})$. Fix truncation length to $k = 35$, set learning rate to $\alpha_t = \frac{\alpha}{t}$ for $\alpha = 0.01$, & take $N = 200$ gradient steps. These parameters are chosen so that $\gamma k \lambda^k N^{\alpha\beta+1}$ bound from Prop. 4 does not become vacuous (trống rỗng) – by triangle inequality, always have $\|\mathbf{w}_{\text{trunc}} - \mathbf{w}_{\text{recurr}}\| \leq 2\lambda$.

- **Stable vs. unstable models.** Word & character level language modeling experiments are based on publically available code from [16]. Polyphonic music modeling code is based on code in [3], & slot-filling model is a reimplementation of [18]: Word-level language modeling code is based on https://github.com/pytorch/examples/tree/master/word_language_model, character-level code is based on <https://github.com/salesforce/awd-lstm-lm>, & polyphonic music modeling code is based on <https://github.com/locuslab/TCN>.

Since sufficient conditions for stability derived in Sect. 2.2. only apply for networks with a single layer, use a single layer RNN or LSTM \forall experiments. Further, our theoretical results are only applicable for vanilla SGD, & not adaptive gradient methods, so all models are trained with SGD. Table 2: Hyperparameters for all experiments contains a summary of all hyperparameters for each experiment.

All hyperparameters are shared between stable & unstable variants of both models. In RNN case, enforcing stability is conceptually simple, though computationally expensive. Since tanh is 1-Lipschitz, RNN is stable as long as $\|W\| < 1$. Therefore, after each gradient update, project W onto spectral norm ball by taking SVD & thresholding singular values to lie in $[0, 1)$. In LSTM case, enforcing stability is conceptually more difficult, but computationally simple. To ensure LSTM is stable, appeal to Prop. 2. Enforce following inequalities after each gradient update

1. Hidden-to-hidden forget gate matrix should satisfy $\|W_f\|_{\infty} < 0.128$, which is enforced by normalizing l_1 -norm of each row to have value at most 0.128.
2. Input vectors \mathbf{x}_t must satisfy $\|\mathbf{x}_t\|_{\infty} \leq B_{\mathbf{x}} = 0.75$, which is achieved by thresholding all values to lie in $[-0.75, 0.75]$.
3. Bias of forget gate b_f , must satisfy $\|b_f\|_{\infty} \leq 0.25$, which is again achieved by thresholding all values to lie in $[-0.25, 0.25]$.
4. Input-hidden forget gate matrix U_f should satisfy $\|U_f\|_{\infty} \leq 0.25$. This is enforced by normalizing l_1 -norm of each row to have value at most 0.25.
5. Given 1–4, forget gate can take value at most $f_{\infty} < 0.64$. Consequently, enforce $\|W_i\|_{\infty}, \|W_o\|_{\infty} \leq 0.36, \|W_z\| \leq 0.091, \|W_f\|_{\infty} < \min\{0.128, (1 - 0.64)^2\} = 0.128$.

After 1–5 are enforced, by Prop. 2, resulting (iterated)-LSTM is stable. Although above description is somewhat complicated, implementation boils down to normalizing rows of LSTM weight matrices, which can be done very efficiently in a few lines of PyTorch.

- **Data-dependent stability.** Unlike RNN, is an LSTM, not clear how to analytically compute stability parameter λ . Instead, rely on a heuristic method to estimate λ . Recall a model is stable if $\forall \mathbf{x}, \mathbf{h}, \mathbf{h}'$, have

$$S(\mathbf{h}, \mathbf{h}', \mathbf{x}) := \frac{\|\phi_{\mathbf{w}}(\mathbf{h}, \mathbf{x}) - \phi_{\mathbf{w}}(\mathbf{h}', \mathbf{x})\|}{\|\mathbf{h} - \mathbf{h}'\|} \leq \lambda < 1.$$

To estimate $\sup_{\mathbf{h}, \mathbf{h}', \mathbf{x}} S(\mathbf{h}, \mathbf{h}', \mathbf{x})$, do: 1st, take \mathbf{x} to be point in training set. In language modeling case, \mathbf{x} is 1 of learned word-vectors. Randomly sample & fix \mathbf{x} , & then perform gradient ascent on $S(\mathbf{h}, \mathbf{h}', \mathbf{x})$ to find worst-case \mathbf{h}, \mathbf{h}' . In our experiments, initialize $\mathbf{h}, \mathbf{h}' \sim \mathcal{N}(0, 0.1 \cdot I)$ & run gradient ascent with learning rate 0.9 for 1000 steps. This procedure is repeated 20 times, & estimate λ as maximum value of $S(\mathbf{h}, \mathbf{h}', \mathbf{x})$ encountered during any iteration from any of 20 random starting points.

4.4 ROBIN M. SCHMIDT. Recurrent Neural Networks (RNNs): A gentle Introduction & Overview

Abstract. State-of-the-art solutions in areas of “Language Modeling & Generating Text”, “Speech Recognition”, “Generating Image Descriptions” or “Video Tagging” have been using Recurrent Neural Networks as foundation for their approaches. Understanding underlying concepts is therefore of tremendous importance if want to keep up with recent or upcoming publications in those areas. In this work, give a short overview over some of most important concepts in realm of Recurrent Neural Networks which enables readers to easily understand fundamentals e.g. but not limited to “Backpropagation through Time” or “Long Short-Term Memory Units” as well as some of more recent advances like “Attention Mechanism” or “Pointer Networks”. Also give recommendations for further reading regarding more complex topics where necessary.

- **Introduction & Notation.** Recurrent Neural Networks (RNNs) are a type of neural network architecture which is mainly used to detect patterns in a sequence of data. Such data can be handwriting, genomes, text or numerical time series which are often produced in industry settings (e.g. stock markets or sensors) [7, 12]. However, they are also applicable to images if these get respectively decomposed into a series of patches & treated as a sequence [12]. On a higher level, RNNs find applications in *Lagrange Modeling & Generating Text*, *Speech Recognition*, *Generating Image Descriptions* or *Video Tagging*. What differentiates Recurrent Neural Networks from Feedforward Neural Networks also known as Multi-Layer Perceptrons (MLPs) is how information gets passed through network. While Feedforward Networks pass information through network without cycles, RNN has cycles & transmits information back into itself. This enables them to extend functionality of Feedforward Networks to also take into account previous inputs $\mathbf{X}_{0:t-1}$ & not only current input \mathbf{X}_t . This difference is visualized on a high level in Fig. 1: Visualization of differences between Feedforward NNs & Recurrent NNs. Note: option of having multiple hidden layers is aggregated to 1 Hidden Layer block \mathbf{H} . This block can obviously be extended to multiple hidden layers.

Can describe this process of passing information from previous iteration to hidden layer with mathematical notation proposed in [24]. For that, denote hidden state & input at time step t resp. as $\mathbf{H}_t \in \mathbb{R}^{n \times h}$, $\mathbf{X}_t \in \mathbb{R}^{n \times d}$ where n : number of samples, d : number of inputs of each sample, & h : number of hidden units. Further, use a weight matrix $\mathbf{W}_{xh} \in \mathbb{R}^{d \times h}$, hidden-state-to-hidden-state matrix $\mathbf{W}_{hh} \in \mathbb{R}^{h \times h}$ & a bias parameter $\mathbf{b}_h \in \mathbb{R}^{1 \times h}$. Lastly, all these informations get passed to a activation function ϕ which is usually a logistic sigmoid or tanh function to prepair gradients for usage in backpropagation. Putting all these notations together yields (1) as hidden variable & (2) as output variable.

$$\mathbf{H}_t = \phi_h(\mathbf{X}_t \mathbf{W}_{xh} + \mathbf{H}_{t-1} \mathbf{W}_{hh} + \mathbf{b}_h), \quad (382)$$

$$\mathbf{O}_t = \phi_o(\mathbf{H}_t \mathbf{W}_{ho} + \mathbf{b}_o). \quad (383)$$

Since \mathbf{H}_t recursively includes \mathbf{H}_{t-1} & this process occurs for every time step RNN includes traces of all hidden states that preceded \mathbf{H}_{t-1} as well as \mathbf{H}_{t-1} itself.

If compare that notation for RNNs with similar notation for Feedforward Neural Networks, can clearly see difference described earlier. In (3) can see computation for hidden variable while (4) shows output variable.

$$\mathbf{H} = \phi_h(\mathbf{X} \mathbf{W}_{xh} + \mathbf{b}_h), \quad (384)$$

$$\mathbf{O} = \phi_o(\mathbf{H} \mathbf{W}_{ho} + \mathbf{b}_o). \quad (385)$$

If you are familiar with training techniques for Feedforward Neural Networks e.g. backpropagation, 1 question: how to properly backpropagate error through a RNN. Here, a technique called Backpropagation Through Time (BPTT) is used which gets described in detail in next sect.

- **2. Backpropagation Through Time (BPTT) & Truncated BPTT.** Backpropagation Through Time (BPTT) is adaptation of backpropagation algorithm for RNNs [24]. In theory, this unfolds RNN to construct a traditional Feedforward Neural Network where can apply backpropagation. For that, use same notations for RNN as proposed before.

When forward pass input \mathbf{X}_t through network compute hidden state \mathbf{H}_t & output state \mathbf{O}_t 1 step at a time. Can then define a loss function $\mathcal{L}(\mathbf{O}, \mathbf{Y})$ to describe difference between all outputs \mathbf{O}_t & target values \mathbf{Y}_t as shown in (5). This basically sums up every loss term l_t of each update step so far. This loss term l_t can have different defs based on specific problem (e.g. Mean Squared Error, Hinge Loss, Cross Entropy Loss, etc.). (5)

$$\mathcal{L}(\mathbf{O}, \mathbf{Y}) = \sum_{t=1}^T l_t(\mathbf{O}_t, \mathbf{Y}_t). \quad (386)$$

Since have 3 weight matrices \mathbf{W}_{xh} , \mathbf{W}_{hh} , \mathbf{W}_{ho} need to compute partial derivative w.r.t. each of these weight matrices. With chain rule which is also used in normal backpropagation get to result for \mathbf{W}_{ho} shown in (6)

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}_{ho}} = \sum_{t=1}^T \frac{\partial l_t}{\partial \mathbf{O}_t} \cdot \frac{\partial \mathbf{O}_t}{\partial \phi_o} \cdot \frac{\partial \phi_o}{\partial \mathbf{W}_{ho}} = \sum_{t=1}^T \frac{\partial l_t}{\partial \mathbf{O}_t} \cdot \frac{\partial \mathbf{O}_t}{\partial \phi_o} \mathbf{H}_t. \quad (387)$$

For partial derivative w.r.t. \mathbf{W}_{hh} get

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}_{hh}} = \dots. \quad (388)$$

For partial derivative w.r.t. \mathbf{W}_{xh} get:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}_{xh}} = \dots. \quad (389)$$

Since each \mathbf{H}_t depends on previous time step, can substitute last part from above equations to get [(9)–(12)].

From here, can see: need to store powers of \mathbf{W}_{hh}^k as proceed through each loss term l_t of overall loss function \mathcal{L} which can become very large. For these large values this method becomes numerically unstable since eigenvalues < 1 & eigenvalues > 1 diverge [5]. 1 method of solving this problem is truncate sum at a computationally convenient size [24]. When you do this, you're using Truncated BPTT [22]. This basically establishes an upper bound for number of time steps gradient can flow back to [15]. One can think of this upper bound as a moving window of past time steps which RNN considers. Anything before cut-off time step doesn't get taken into account. Since BPTT basically unfolds RNN to create a new layer for each time step, can also think of this procedure as limiting number of hidden layers.

- 3. Problems of RNNs: Vanishing & Exploding Gradients. As in most neural networks, vanishing or exploding gradients is a key problem of RNNs [12]. In (9)–(10) can see $\frac{\partial \mathbf{H}_t}{\partial \mathbf{H}_k}$ which basically introduces matrix multiplication over (potentially very long) sequence, if there are small values (< 1) in matrix multiplication this causes gradient to decrease with each layer (or time step) & finally vanish [6]. This basically stops contribution of states that happened far earlier than current time step towards current time step [6]. Similarly, this can happen in opposite direction if have large values (> 1) during matrix multiplication causing an exploding gradient which is result values each weight too much & changes it heavily [6].

This problem motivated introduction of long short term memory units (LSTMs) to particularly handle vanishing gradient problem. This approach was able to outperform traditional RNNs on a variety of tasks [6]. In next sect, want to go deeper on proposed structure of LSTMs.

- 4. Long Short-Term Memory Units (LSTMs). Long Short-Term Memory Units (LSTMs) [9] were designed to properly handle vanishing gradient problem. Since they use a more constant error, they allow RNNs to learn over a lot more time steps (way over 1000) [12]. To achieve that, LSTMs store more information outside of traditional neural network flow in structures called *gated cells* [6, 12]. To make things work in an LSTM use an output gate \mathbf{O}_t to read entries of cell, an input gate \mathbf{I}_t to read data into cell & a forget gate \mathbf{F}_t to reset content of cell. Computations for these gates are shown in (13)–(15):

$$\mathbf{O}_t = \sigma(\mathbf{X}_t \mathbf{W}_{xo} + \mathbf{H}_{t-1} \mathbf{W}_{ho} + \mathbf{b}_o), \quad (390)$$

$$\mathbf{I}_t = \sigma(\mathbf{X}_t \mathbf{W}_{xi} + \mathbf{H}_{t-1} \mathbf{W}_{hi} + \mathbf{b}_i), \quad (391)$$

$$\mathbf{F}_t = \sigma(\mathbf{X}_t \mathbf{W}_{xf} + \mathbf{H}_{t-1} \mathbf{W}_{hf} + \mathbf{b}_f). \quad (392)$$

Shown equations use $\mathbf{W}_{xi}, \mathbf{W}_{xf}, \mathbf{W}_{xo} \in \mathbb{R}^{d \times h}$ & $\mathbf{W}_{hi}, \mathbf{W}_{hf}, \mathbf{W}_{ho} \in \mathbb{R}^{h \times h}$ as weight matrices while $\mathbf{b}_i, \mathbf{b}_f, \mathbf{b}_o \in \mathbb{R}^{1 \times h}$ are their respective biases. Further, they use sigmoid activation function σ to transform output $\in (0, 1)$ which each results in a vector with entries $\in (0, 1)$.

Next, need a candidate memory cell $\tilde{\mathbf{C}}_t \in \mathbb{R}^{n \times h}$ which has a similar computation as previously mentioned gates but instead uses a tanh activation function to have an output $\in (-1, 1)$. Further, it again has its own weights $\mathbf{W}_{xc} \in \mathbb{R}^{d \times h}$, $\mathbf{W}_{hc} \in \mathbb{R}^{h \times h}$ & biases $\mathbf{b}_c \in \mathbb{R}^{1 \times h}$. Respective computation is shown in (16):

$$\tilde{\mathbf{C}}_t = \tanh(\mathbf{X}_t \mathbf{W}_{xc} + \mathbf{H}_{t-1} \mathbf{W}_{hc} + \mathbf{b}_c). \quad (393)$$

To plug some things together, introduce old memory content $\mathbf{C}_{t-1} \in \mathbb{R}^{n \times h}$ which together with introduced gates controls how much of old memory content we want to preserve to get to new memory content \mathbf{C}_t . This is shown in (17)

$$\mathbf{C}_t = \mathbf{F}_t \odot \mathbf{C}_{t-1} + \mathbf{I}_t \odot \tilde{\mathbf{C}}_t, \quad (394)$$

where \odot denotes element-wise multiplication. Structure so far can be seen in Fig. 10 in Appendix A.

Last step: to introduce computation for hidden states $\mathbf{H}_t \in \mathbb{R}^{n \times h}$ into framework. This can be seen in (18):

$$\mathbf{H}_t = \mathbf{O}_t \odot \tanh \mathbf{C}_t. \quad (395)$$

With tanh function, ensure: each element of $\mathbf{H}_t \in (-1, 1)$. Full LSTM framework can be seen in Fig. 11: Computation of hidden state in an LSTM [24].

- 5. Deep Recurrent Neural Networks (DRNNs). Deep Recurrent Neural Networks (DRNNs) are in theory a really easy concept. To construct a deep RNN with L hidden layers, simply stack ordinary RNNs of any type on top of each other. Each hidden state $\mathbf{H}_t^{(l)} \in \mathbb{R}^{n \times h}$ is passed to next time step of current layer $\mathbf{H}_{t+1}^{(l)}$ as well as current time step of next layer $\mathbf{H}_t^{(l+1)}$. For 1st layer, compute hidden state as proposed in previous models shown in (19) while for subsequent layer use (20) where hidden state from previous layer is treated as input.

$$\mathbf{H}_t^{(1)} = \phi_1(\mathbf{X}_t, \mathbf{H}_{t-1}^{(1)}), \quad (396)$$

$$\mathbf{H}_t^{(l)} = \phi_l(\mathbf{H}_t^{(l-1)}, \mathbf{H}_{t-1}^{(l)}). \quad (397)$$

Output $\mathbf{O}_t \in \mathbb{R}^{n \times o}$ where o : number of outputs is then computed as shown in (21)

$$\mathbf{O}_t = \phi_o(\mathbf{H}_t^{(L)} \mathbf{W}_{ho} + \mathbf{b}_o) \quad (398)$$

where only use hidden state of layer L .

- 6. **Bidirectional Recurrent Neural Networks (BRNNs)**. Take an example of language modeling. Based on our current models, able to reliably predict next sequence element (i.e. next word) based on what we have seen so far. However, there scenarios where might want to fill in a gap in a sentence & part of sentence after gap conveys significant information. This information is necessary to take into account to perform well on this kind of task [24]. On a more generalized level, want to incorporate a look-ahead property for sequences.

To achieve this look-ahead property Bidirectional Recurrent Neural Networks (BRNNs) [14] got introduced which basically add another hidden layer which run sequence backwards starting from last element [24]. An architectural overview is visualized in Fig. 2: **Architecture of a bidirectional recurrent neural network**. Introduce a forward hidden state $\vec{\mathbf{H}}_t \in \mathbb{R}^{n \times h}$ & a backward hidden state $\overleftarrow{\mathbf{H}}_t \in \mathbb{R}^{n \times h}$. Their respective calculations are shown in (22)–(23): [technical details]

Keep in mind: 2 directions can have different number of hidden units.

- 7. **Encoder-Decoder Architecture & Sequence to Sequence (seq2seq)**. Encoder-Decoder architecture is a type of neural network architecture where network is 2fold. It consists of encoder network & a decoder network whose respective roles are to *encode* input into a state & *decode* state to an output. This state usually has shape of a vector or a tensor [24]. A visualization of this structure in shown in Fig. 3: **Encoder-Decoder Architecture Overview** alternated from [24].

Based on this Encoder-Decoder architecture a model called Sequence to Sequence (seq2seq) [16] got proposed for generating a sequence output based on a sequence input. This model uses RNNs for encoder as well as decoder where hidden state of encoder gets passed to hidden state of decoder. Common applications of model are Google Translate [16, 23], voice-enabled devices [13] or labeling video data [18]. It mainly focuses on mapping a fixed length input sequence of size n to an fixed length output sequence of size m where $n \neq m$ can be true but isn't a necessity.

A de-rello visualization of proposed architecture is shown in Fig. 4: **Visualization of Sequence to Sequence (seq2seq) Model**. Here, have a encoder which consists of a RNN accepting a single element of sequence \mathbf{X}_t where t is order of sequence element. These RNNs can be LSTMs or Gated Recurrent Units (GRUs) to further improve performance [16]. Further, hidden states \mathbf{H}_t are computed according to definition of hidden states in used RNN type (e.g. LSTM or GRU). Encoder Vector (context) is a representation of last hidden state of encoder network which aims to aggregate all information from all previous input elements. This functions as initial hidden state of decoder network of mode & enables decoder to make accurate predictions. Decoder network again is built of a RNN which predicts an output \mathbf{Y}_t at a time step t . Produced output is again a sequence where each \mathbf{Y}_t is a sequence element with order t . At each time step RNN accepts a hidden state from previous unit & itself produces an output as well as a new hidden state.

Encoder Vector (context) was shown to be a bottleneck for these type of models since it needed to contain all necessary information of a source sentence in a fixed-length vector which was particularly problematic for long sequences. There have been approaches to solve this problem by introducing Attention in e.g. [4] or [10]. In Sect. 8, take a closer look at proposed solutions.

- 8. **Attention Mechanism & Transformer**. Attention Mechanism for RNNs is partly motivated by human visual focus & peripheral perception [21]. It allows humans to focus on a certain region to achieve high resolution while adjacent objects are perceived with a rather low resolution. Based on these focus points & adjacent perception, can make inference about what we expect to perceive when shifting our focus point. Similarly, can transfer this method on our sequence of words where we are able to perform inference based on observed words. E.g., if perceive the word *eating* in sequence “She is eating a green apple” we assume to observe a food object in the near future [21].

Generally, Attention takes 2 sentences & transforms them into a matrix where each sequence element (i.e. a word) corresponds to a row or column. Based on this matrix layout, can fill in entries to identify relevant context or correlations between them. An example of this process can be seen in Fig. 5: **Example of an Alignment matrix of “L'accord sur la zone économique européen a été signé en août 1992” (French) & its English translation “The agreement on the European Economic Area was signed in August 1992”**: [4].

- 8.1. Def.
- 8.2. Different types of score functions.
- 8.3. Transformer.

- 9. **Pointer Networks (Ptr-Nets)**. Pointer Networks (Ptr-Nets) [19] adapts seq2seq model with attention to improve it by not fixing discrete categories (i.e. elements) of output dictionary *a priori*. Instead of yielding an output sequence generated from an input sequence, a pointer networks creates a succession of pointers to elements of input series [25]. In [19] show: using Pointer Networks they can solve combinatorial optimization problems e.g. computing planar convex hulls, Delaunay triangulations & symmetric planar Traveling Salesman Problem (TSP).

Generally, apply additive attention (from Table 1: Different score functions with their respective equations & usage alternated from [21]) between states & then normalize it by applying softmax function to model output conditional probability as seen in (29):

$$\mathbf{Y}_t = \text{softmax}(\text{score}(\mathbf{S}_t, \mathbf{H}_{t'})) = \text{softmax}(\mathbf{v}_a^\top \tanh \mathbf{W}_a[\mathbf{S}_t; \mathbf{H}_{t'}]). \quad (399)$$

Attention mechanism is simplified, as Ptr-Net does not blend encoder states into output with attention weights. In this way, output only responds to positions but not input content [21].

- 10. **Conclusion & Outlook.** In this work, gave an introduction into fundamentals for Recurrent Neural Networks (RNNs). This includes general framework for RNNs, Backpropagation through time, problems of traditional RNNs, LSTMs, Deep & Bidirectional RNNs as well as more recent advances e.g. Encoder-Decoder Architecture, seq2seq model, Attention, Transformer & Pointer Networks. Most topics are only covered conceptionally & don't go too deep into implementation specifications. To get a broader understanding of covered topics, recommend looking into some of cited original papers. Additionally, most recent publications use some of presented concepts so recommend taking a look at such papers.

1 recent publication which uses many of presented concepts is “Grandmaster level in StarCraft II using multi-agent reinforcement learning” by Vinyals et al. [20]. Here, they present their approach to train agents to play real-time strategy game Starcraft II with great success. If presented concepts were a little too theoretical for you, recommend reading that paper to see LSTMs, Transformer or Pointer Networks in a setting which can be deployed in a more practical environment.

5 Computer Music

5.1 EMMANOUIL BENETOS, SIMON DIXON, ZHIYAO DUAN, SEBASTIAN EWERT. **Automatic Music Transcription: An overview**

[371 citations]

- **Abstract.** Capability of transcribing music audio into music notation is a fascinating example of human intelligence. It involves perception (analyzing complex auditory scenes), cognition (recognizing musical objects), knowledge representation (forming musical structures), & inference (testing alternative hypotheses). Automatic music transcription (AMT), i.e., design of computational algorithms to convert acoustic music signals into some form of music notation, is a challenging task in signals into some form of music notation, is a challenging task in signal processing & AI. It comprises several subtasks, including multipitch estimation (MPE), onset & offset detection, instrument recognition, beat & rhythm tracking, interpolation of expressive timing & dynamics, & score typesetting.

Given number of subtasks it comprises & its wide application range, it is considered a fundamental problem in fields of music signal processing & music information retrieval [1,2]. Because of very nature of music signals, which often contain several sound sources (e.g., musical instruments & voice) that produce 1 or more concurrent sound events (e.g., notes & percussive sounds) that are meant to be highly correlated over both time & frequency, AMT is still considered a challenge & open problem in literature, particularly for music containing multiple instruments & many simultaneous notes (called *polyphonic music* in music signal processing literature) [2].

Typical data representation used in an AMT system are illustrated in FIG. 1: DATA REPRESENTED IN AN AMT SYSTEM: (A) INPUT WAVEFORM, (B) INTERNAL TIME-FREQUENCY REPRESENTATION, (C) OUTPUT PIANO-ROLL REPRESENTATION, (D) OUTPUT MUSIC SCORE, WITH NOTES A & D MARKED IN GRAY CIRCLES. EXAMPLE CORRESPONDS TO 1ST 6 S OF W.A. MOZART'S PIANO SONATA NO. 13, 3RD MOVEMENT. (IMAGES COURTESY OF MIDI ALIGNED PIANO SOUNDS DATABASE.). Usually, an AMT system takes an audio waveform as input, computes a time-frequency representation, & outputs a representation of pitches over time [also called a *piano-roll* representation] or a typeset music score.

In this article, provide a high-level overview of AMT, emphasizing intellectual merits & broader impacts of this topic & linking AMT to other problems found in wider field of digital signal processing. Give an overview of approaches to AMT, detailing methodology used in 2 main families of methods, based resp. on DL & nonnegative matrix factorization (NMF). Finally, provide an extensive discussion of open challenges for AMT. Regarding scope of article, emphasize approaches for transcribing polyphonic music produced by pitched instruments & voice. Outside scope of article are methods for transcribing nonpitched sounds, e.g. drums, for which a brief overview is given in “Percussion & Unpitched Sounds” (Bộ gõ & âm thanh không có cao độ) sect, as well as methods for transcribing specific sources within a polyphonic mixture, e.g. melody & bass lines.

- **Applications & impact.** A successful AMT system would enable a broad range of interactions between people & music, including music education (e.g., through systems for automatic instrument tutoring), music creation (e.g., dictating improvised musical ideas & automatic music accompaniment), music production (e.g., music content visualization & intelligent content-based editing), music search (e.g., indexing & recommendation of music by melody, bass, rhythm, or chord progression), & musicology (e.g., analyzing jazz improvisations & other nonnotated music). As such, AMT is an enabling technology with clear potential for both economic & societal impact.

AMT is closely related to other music signal processing tasks [3], e.g. audio source separation, which also involves estimation & inference of source signals from mixture observations. It is also useful for many high-level tasks in music information retrieval [4], e.g. structural segmentation, cover-song detection, & assessment of music similarity, since these tasks are much easier to address once musical notes are known. Thus, AMT provides main link between fields of music signal processing & symbolic music processing (i.e., processing of music notation & music language modeling). Integration of 2 aforementioned fields through AMT is discussed in sect “Further Extensions & Future Work”.

Given potential impact of AMT, problem has attracted commercial interest in addition to academic research. While it is outside scope of this article to provide a comprehensive list of commercial AMT software, commonly used applications include Melodyne <http://www.celemony.com/en/melodyne>, AudioScore <http://www.sibelius.com/products/audioscore/>, ScoreCloud <http://scorecloud.com/>, AnthemScore <https://www.lunaverus.com/>, & Transcribe! <https://www.seventhstring.com/xscribe/>. Worth noting: AMT papers in literature have refrained from making explicit comparisons with commercially

available music transcription software, possibly because of difference in scope & target application between commercial & academic tools.

AMT is an enabling technology with clear potential for both economic & societal impact.

- **Analogies to other fields.** AMT has close relations with other signal processing problems. W.r.t. field of speech processing, AMT is widely considered to be musical equivalent of automatic speech recognition (ASR), in sense: both tasks involve converting acoustic signals to symbolic sequences. Like cocktail party problem in speech, music usually involves multiple simultaneous voices, but, unlike speech, these voices are highly correlated in time & frequency (see challenges 2 & 3 in “Key Challenges” sect). In addition, both AMT & ASR systems benefit from language modeling components that are combined with acoustic components to produce plausible results. Thus, there are also clear links between AMT & wider field of natural language processing (NLP), with music having its own grammatical rules or statistical regularities, in a way similar to natural language [5]. Use of language models for AMT is detailed in sect “Further Extensions & Future Work”.

Within emerging field of sound scene analysis, there is a direct analogy between AMT & sound event detection (SED) [6], in particular with polyphonic SED, which involves detecting & classifying multiple overlapping events from audio. While everyday & natural sounds do not exhibit same degree of temporal regularity & intersource frequency dependence as found in music signals, there are close interactions between 2 problems in terms of methodologies used, as observed in literature [6].

Furthermore, AMT is related to image processing & computer vision, as musical objects, e.g. notes, can be recognized as 2D patterns in time-frequency representations. Compared with image processing & computer vision, where occlusion (sự tắc nghẽn) is a common issue, AMT systems are often affected by musical objects occupying same time-frequency regions (this is detailed in “Key Challenges” sect).

- **Key challenges.** Compared to other problems in music signal processing field or wider signal processing discipline, there are several factors that make AMT particularly challenging:
 1. Polyphonic music contains a mixture of multiple simultaneous sources (e.g., instruments & vocals) with different pitch, loudness, & timbre (sound quality), with each source producing 1 or more musical voices. Inferring musical attributes (e.g., pitch) from mixture signal is an extremely underdetermined problem.
 2. Overlapping sound events often exhibit harmonic relations with each other. For any consonant musical interval, fundamental frequencies form small integer ratios, so that their harmonics overlap in frequency, making separation of voices even more difficult. Taking a C major chord as an example, fundamental frequency ratio of its 3 notes C:E:G is 4:5:6, & percentages of harmonic positions that are overlapped by other notes are 46.7%, 33.3%, & 60% for C, E, & G, resp.
 3. Timing of musical voices is governed by regular metrical structure of music. In particular, musicians pay close attention to synchronization of onsets & offsets between different voices, which violates common assumption of statistical independence between sources, which otherwise facilitates separation.
 4. Annotation of ground-truth transcriptions for polyphonic music is very time consuming & requires high expertise. Lack of such annotations has limited use of powerful supervised-learning techniques to specific AMT subproblems, e.g. piano transcription, where annotation can be automated because of certain piano models that can automatically capture performance data. An approach to circumvent this problem was proposed in [7], but it requires professional music performers & thorough score pre- & postprocessing. Note: sheet music does not generally provide good ground-truth annotations for AMT; it is not time-aligned to audio signal, nor does it usually provide an accurate representation of a performance. Even when accurate transcriptions exist, it is not trivial to identify corresponding pairs of audio files & musical scores because of multitude of versions of any given musical work that are available from music distributors. At best, musical scores can be viewed as weak labels.

These key challenges are often not fully addressed in current AMT systems, leading to common issues in AMT outputs, e.g. octave errors, semitone errors, missed notes (in particular, in presence of dense chords), extra notes (often manifested as harmonic errors in presence of unseen timbres), merged or fragmented notes, incorrect onsets/offsets, or misassigned streams [1,2]. Remainder of this article will focus on ways to address previously mentioned challenges as well as on discussion of additional open problems for creation of robust AMT systems.

AMT provides main link between fields of music signal processing & symbolic music processing.

- **An overview of AMT methods.** In past 4 decades, many approaches have been developed for AMT for polyphonic music. While end goal of AMT: convert an acoustic music recording to some form of music notation, most approaches were designed to achieve a certain intermediate goal. Depending on level of abstraction & structures that need to be modeled for achieving such goals, AMT approaches can be generally organized into 4 categories: frame level, note level, stream level, & notation level.

Frame-level transcription, or MPE, is estimation of number & pitch of notes that are simultaneously present in each time frame (on order of 10 ms). This is usually performed independently in each frame, although contextual information is sometimes considered through filtering frame-level pitch estimates in a postprocessing stage. Fig. 2: Examples of (a) frame-level, (b) note-level, (c) stream-level transcriptions, produced by running methods proposed in [8–10], resp., of 1st phrase of J.S. BACH’s chorale *Ach Gott und Herr* from Bach10 dataset. All 3 levels are parametric descriptions of music performance. (a) shows an example of a frame-level transcription, where each black dot is a pitch estimate. Methods in this category do not form concept of musical notes & rarely model any high-level musical structures.

A large portion of existing AMT techniques operate at this level. Recent approaches include traditional signal processing methods [11,12], probabilistic modeling [8], Bayesian approaches [13], NMF [14–17], & neural networks (NNs) [18,19]. All of these methods have pros & cons, & research has not converged on a single approach. E.g., traditional signal processing methods are simple & fast & generalize better to different instruments, while deep NN methods generally achieve higher accuracy on specific instruments (e.g., piano). Bayesian approaches provide a comprehensive modeling of sound generation process, but models can be very complex & slow. Readers interested in a comparison of performance of different approaches are referred to Multiple Fundamental Frequency Estimation & Tracking task of annual Music Information Retrieval Evaluation eXchange (MIREX) <http://www.music-ir.org/mirex>. However, readers are reminded: evaluation results may be biased by limitations of datasets & evaluation metrics (see sects. “Key Challenges” & “Evaluation Metrics”).

Note-level transcription, or note tracking, is 1 level higher than MPE in terms of richness of structures of estimates. It not only estimates pitches in each time frame but also connects pitch estimates over time into notes. In AMT literature, a musical note is usually characterized by 3 elements: pitch, onset time, & offset time [1]. As note offsets can be ambiguous, they are sometimes neglected in evaluation of note-tracking approaches, &, as such, some note-tracking approaches only estimate pitch & onset times of notes. Fig. 2b shows an example of a note-level transcription, where each note is shown as a red circle (onset) followed by a black line (pitch contour). Many note-tracking approaches form notes by postprocessing MPE outputs (i.e., pitch estimates in individual frames). Techniques that have been used in this context include median filtering [12], hidden Markov models (HMMs) [20], & NNs [5]. This post-processing is frequently performed for each Musical Instrument Digital Interface (MIDI) pitch independently without considering interactions among simultaneous notes. This often leads to spurious (giả mạo) or missing notes that share harmonics with correctly estimated notes.

Some approaches have been proposed to consider note interactions through a spectral likelihood model [9] or a music language model [5, 18] (see “MLMs” sect). Another subset of approaches estimates notes directly from audio signal instead of building upon MPE outputs. Some approaches 1st detect onsets & then estimate pitches within each interonset interval [21], while others estimate pitch, onset, & sometimes offset in same framework [22–24].

From frame level to note level to stream level, transcription task becomes more complex as more musical structures & cues need to be modeled.

Stream-level transcription, also called *multipitch streaming (MPS)*, targets grouping of estimated pitches or notes into streams, where each stream typically corresponds to 1 instrument or musical voice & is closely related to instrument source separation. Fig. 2c shows an example of a stream-level transcription, where pitch streams of different instruments have different colors. Compared to note-level transcription, pitch contour of each stream is much longer than a single note & contains multiple discontinuities that are caused by silence, nonpitched sounds, & abrupt frequency changes. Therefore, techniques that are ordinarily used in note-level transcription are generally not sufficient for grouping pitches with a long & discontinuous contour. 1 important cue for MPS that is not explored in MPE & note tracking is timbre. Notes of same stream (source) generally show similar timbral characteristics compared to those in different streams. Therefore, stream-level transcription is also called *timbre tracking* or *instrument tracking* in literature. Existing works at this level are few, e.g. [10,16,25].

From frame level to note level to stream level, transcription task becomes more complex as more musical structures & cues need to be modeled. However, transcription outputs at these 3 levels are all parametric transcriptions, which are parametric descriptions of audio content. MIDI piano roll shown in Fig. 1c is a good example of such a transcription. It is, indeed, an abstraction of music audio, but it has not yet reached level of abstraction of music notation. Time is still measured in unit of seconds instead of beats; pitch is measured in MIDI numbers instead of spelled note names are compatible with key (e.g., C♯ vs. Db); & concepts of beat, bar, meter, key, harmony, & stream are lacking. Notation-level transcription aims to transcribe music audio into a human-readable musical score, e.g. staff notation widely used in Western classical music. Transcription at this level requires a deeper understanding of musical structures, including harmonic, rhythmic, & stream structures. Harmonic structures, e.g. keys & chords, influence note spelling of each MIDI pitch; rhythmic structures, e.g. beats & bars, help to quantize length of notes; & stream structures aid assignment of notes to different staves. There has been some work on estimation of musical structures from audio or MIDI representation of a performance. E.g., methods for pitch spelling [26], timing quantization [27], & voice separation [28] from performed MIDI files have been proposed. However, little work has been done in integrating these structures into a complete music notation transcription, especially for polyphonic music.

While there is a wide range of applicable methods, AMT has been dominated during last decade by 2 algorithmic families: NMF & NNs.

Several software packages, including Final, GarageBand, & MuseScore, provide functionality of converting a MIDI file into music notation, but results are typically not satisfying, & it is not clear what musical structures have been estimated & integrated during transcription process. Cogliati et al. [29] proposed a method to convert a MIDI performance into music notation, based on a systematic comparison of transcription performance with aforementioned software. In terms of audio-to-notation transcription, a proof-of-concept work using end-to-end NNs was proposed by CARVALHO & SMARAGDIS [30] to directly map music audio into music notation without explicitly modeling musical structures.

- **State of art.** While there is a wide range of applicable methods, AMT has been dominated during last decade by 2 algorithmic families: NMF & NNs. Both families have been used for a variety of tasks, from speech & image processing to recommender systems & NLP. Despite this wide applicability, both approaches offer a range of properties that make them particularly suitable for modeling music recordings at note level.
- **NMF for AMT.** Basic idea behind NMF & its variants: represent a given nonnegative time-frequency representation $\mathbf{V} \in \mathbb{R}_{\geq 0}^{M \times N}$, e.g., a magnitude spectrogram, as a product of 2 nonnegative matrices: a dictionary $\mathbf{D} \in \mathbb{R}_{\geq 0}^{M \times K}$ & an activation

matrix $\mathbf{A} \in \mathbb{R}_{\geq 0}^{K \times N}$ (see Fig. 3: An example of NMF, using same audio recording as in Fig. 1: (a) input spectrogram \mathbf{V} , (b) approximated spectrogram \mathbf{DA} , (c) dictionary \mathbf{D} (preextracted), (d) activation matrix \mathbf{A}). Computationally, goal: minimize a distance (or divergence) between \mathbf{V} & \mathbf{DA} w.r.t. \mathbf{D} & \mathbf{A} . As a straightforward approach to solving this minimization problem, multiplicative update rules have been central to success of NMF. E.g., generalized Kullback–Leibler divergence between \mathbf{V} & \mathbf{DA} is nonincreasing under following updates & guarantees nonnegativity of both \mathbf{D}, \mathbf{A} as long as both are initialized with positive real values [31]:

$$\mathbf{A} \leftarrow \mathbf{A} \odot \frac{\mathbf{D}^\top (\frac{\mathbf{V}}{\mathbf{DA}})}{\mathbf{D}^\top \mathbf{J}}, \quad \mathbf{D} \leftarrow \mathbf{D} \odot \frac{(\frac{\mathbf{V}}{\mathbf{DA}}) \mathbf{A}^\top}{\mathbf{JA}^\top},$$

where \odot operator denotes pointwise multiplication, $\mathbf{J} \in \mathbb{R}^{M \times N}$ denotes matrix of 1s, & division is pointwise. Intuitively, update rules can be derived by choosing a specific step size in a gradient (or, rather, coordinate) descent-based minimization of divergence [31].

In an AMT context, both unknown matrices have an intuitive interpretation. n th column of \mathbf{V} , i.e., spectrum at time point n , is modeled in NMF as a linear combination of K columns of \mathbf{D} , & corresponding K coefficients are given by n th column of \mathbf{A} . Given this point of view, each column of \mathbf{D} is generally referred to as a *spectral template* & usually represents expected spectral energy distribution associated with a specific note played on a specific instrument. For each template, corresponding row in \mathbf{A} is referred to as associated *activation* & encodes when & how intensely that note is played over time. Given nonnegativity constraints, NMF yields a purely constructive representation in sense: spectral energy modeled by 1 template cannot be canceled by another. This property is often seen as instrumental in identifying a parts-based & interpretable representation of input [31].

In Fig. 3, an NMF-based decomposition is illustrated. Magnitude spectrogram \mathbf{V} shown in Fig. 3a is modeled as a product of dictionary \mathbf{D} & activation matrix \mathbf{A} , shown in Fig. 3cd, resp. Product \mathbf{DA} is given in Fig. 3b. In this case, templates correspond to individual pitches, with clearly visible fundamental frequencies & harmonics. Additionally, comparing \mathbf{A} with piano-roll representation shown in Fig. 1c indicates correlation between NMF activations & underlying musical score.

While Fig. 3 illustrates principles behind NMF, it also indicates why AMT is difficult. Indeed, a regular NMF decomposition would rarely look as clean as Fig. 3. Compared to speech analysis, sound objects in music are highly correlated. E.g., even in a simple piece as in Fig. 1, most pairs of simultaneous notes are separated by musically consonant intervals, which acoustically means: many of their partials overlap [e.g., A & D notes around 4 s, marked with gray circles in Fig. 1d, share a high number of partials]. In this case, it can be difficult to disentangle how much energy belongs to which note. Task is further complicated by fact: spectrotemporal properties of notes vary considerably between different pitches, playing styles, dynamics, & recording conditions. Furthermore, stiffness property of strings affects travel speed of transverse waves based on their frequency. As a result, partials of instruments like piano are not found at perfect integer multiples of fundamental frequency. Because of this property, called *inharmonic*ity, positions of partials differ between individual pianos (Fig. 4: An illustration of inharmonicity: spectrum of a C#1 note played on a piano. Stiffness of strings causes partials to be shifted from perfect integer multiples of fundamental frequency (shown as vertical dotted lines). Here, 23rd partial is at position where 24th harmonic would be expected. Note: fundamental frequency of 34.65 Hz is missing, as piano soundboards typically do not resonate for modes with a frequency $< \approx 50$ Hz.).

To address these challenges, basic NMF model has been extended by encouraging additional structure in dictionary & activations. E.g., an important principle: enforce sparsity in \mathbf{A} to obtain a solution dominated by activations that are few but substantial; success of sparsity paved way for a whole range of sparse-coding approaches, in which dictionary size K can considerably exceed input dimension M [32]. Other extensions focus on dictionary design. In case of supervised NMF, dictionary is precomputed & fixed using additionally available training material. E.g., given K recordings, each containing only a single note, dictionary shown in Fig. 3b was constructed by extracting 1 template from each recording. This way, templates are guaranteed to be free of interference from other notes & also to have a clear interpretation. As another example, Fig. 5: An illustration of harmonic NMF [15]. Each NMF template is represented as a linear combination of fixed narrow-band subtemplates. Resulting template is constrained to represent harmonic sounds by construction. illustrates an extension in which each NMF template is represented as a linear combination of fixed narrow-band subtemplates [15] which enforces a harmonic structure for all NMF templates. This way, a dictionary can be adapted to recording to be transcribed, while maintaining its clean, interpretable structure.

In shift-invariant dictionaries, a single template can be used to represent a range of different fundamental frequencies. In particular, using a logarithmic frequency axis, distances between individual partials of a harmonic sound are fixed, & thus shifting a template in frequency allows modeling of sounds of varying pitch. Sharing parameters between different pitches in this way has turned out to be effective for increasing model capacity (see, e.g., [16,17]). Furthermore, spectrotemporal dictionaries alleviate (làm giảm nhẹ) a specific weakness of NMF models. In NMF, difficult to express: notes often have a specific temporal evolution. E.g., beginning of a note (or attack phase) might have entirely different spectral properties than central part (decay phase). Such relationships are modeled in spectrotemporal dictionaries using a Markov process, which governs sequencing of templates across frames so that different subsets of templates can be used for attack & decay parts, resp. [16,23].

- NNs for AMT. As for many tasks relating to pattern recognition. NNs have, in recent years, had a considerable impact on problem of music transcription & on music signal processing in general. NNs are able to learn a nonlinear function (or a composition of functions) from input to output via an optimization algorithm, e.g. stochastic gradient descent [33]. Compared

to other fields, including image processing, progress on NNs for music transcription has been slower, & will discuss a few of underlying reasons.

1 of earliest approaches based on NNs was MAROLT’s Sonic system [21]. A central component in this approach was use of time-delay networks, which resemble convolutional networks in time direction [33] & were employed to analyze output of adaptive oscillators to track & group partials in output of a gammatone filterbank (bộ lọc gammatone). Although it was initially published in 2001, approach remains competitive & still appears in comparisons in more recent publications [23].

In context of more recent revival of NNs, a 1st successful system was presented by BÖCK & SCHEDL [34]. 1 of core ideas was to use 2 spectrograms as input to enable network to exploit both a high time accuracy (when estimating note onset position) & a high frequency resolution (when disentangling notes in lower frequency range). This input is processed using 1 (or more) long short-term memory (LSTM) layers [33]. Potential benefit of using LSTM layers is 2fold. 1st, spectral properties of a note evolve across input frames, & LSTM networks have capability to compactly model such sequences. 2nd, medium- & long-range dependencies between notes can potentially be captured. E.g., based on a popular chord sequence, after hearing C & G major chords followed by A minor, a likely successor is an F major chord. An investigation of whether such long-range dependencies are indeed modeled, however, was not within scope of this work.

Compared to other fields, including image processing, progress on NNs for music transcription has been slower.

Sigtia et al. [18] focused on long-range dependencies in music by combining an acoustic front end with a symbolic-level module resembling a language model, as used in speech processing. Using information obtained from MIDI files, a RNN was trained to predict active notes in next time frame, given those in past. This approach needed to learn & represent a very large joint probability distribution, i.e., a probability for every possible combination of active & inactive notes across time. Note: even in a single frame, there are 2^{88} possible combinations of notes on a piano. To render problem of modeling such an enormous probability space tractable, approach employed a specific NN architecture (Neural Autoregressive Distribution Estimator, also known as *NADE*), which represented a large joint probability as a long product of conditional probabilities, an approach quite similar to idea popularized recently by well-known WaveNet architecture. Despite use of a dedicated music language model, which was trained on relatively large MIDI-based data sets, only modest improvements over an HMM baseline could be observed, & thus question remains open regarding to which degree long-range dependencies were indeed captured.

To further disentangle influence of acoustic front end from language model on potential improvement in performance, Kelz et al. [19] focused on acoustic modeling, reporting on results of a larger-scale hyperparameter search & describing influence of individual system components. Trained using this careful & extensive procedure, resulting model outperformed existing models by a reasonable margin. I.e., while in speech processing, language models have led to a drastic improvement in performance, same effect is still to be demonstrated in an AMT system, a challenge discussed in more detail hereafter.

Development of NN-based AMT approaches continues. Current state-of-art method for general-purpose piano transcription was proposed by Google Brain [24]. Combining & extending ideas from existing methods, this approach combines 2 networks (Fig. 6: Google Brain’s Onset & Frames Network. Input is processed by an initial network detecting note onsets. Result is used as side information of a 2nd network focused on estimating note lengths. Bi LSTM: bidirectional LSTM layers; FC Sigmoid: fully connected sigmoid layer; Conv Stack: a series of convolutional layers [24].). One detects note onsets, & its output is used to inform a 2nd network, which focuses on perceiving note lengths. This can be interpreted from a probabilistic point of view. Note onsets are rare events compared to framewise note activity detections. Split into 2 network branches can thus be interpreted as splitting representation of a relatively complex joint probability distribution over onsets & frame activity into a probability over onsets & a probability over frame activities, conditioned on onset distribution. Since temporal dynamics of onsets & frame activities are quite different, this can lead to improved learning behavior for entire network when trained jointly.

- **A comparison of NMF & NN models.** Given popularity of NMF & NN-based methods for AMT, interesting to discuss their differences. In particular, neglecting nonnegativity constraints, NMF is a linear, generative model. Given that NMF-based methods are increasingly being replaced by NN-based ones, question arises in which way linearity could be a limitation for an AMT model.

To look into this, assume given an NMF dictionary with 2 spectral templates for each musical pitch. To represent an observed spectrum of a single pitch C4, can linearly combine 2 templates associated with C4. Set (or manifold) of valid spectra for C4 notes, however, is complex, & thus, in most cases, our linear interpolation will not correspond to a real-world recording of a C4. Could increase number of templates s.t. their interpolation could potentially get closer to a real C4 – but number of invalid spectra can represent increases much more quickly compared to number of valid spectra. Deep networks have shown considerable potential in recent years to implicitly represent such complex manifolds in a robust & comparatively efficient way [33]. An additional benefit over generative models like NMF: NNs can be trained in an end-to-end fashion, i.e., note detections can be a direct output of a network without need for additional postprocessing of model parameters (e.g. NMF activations).

Yet, despite these quite principled limitations, NMF-based methods remain competitive or even exceed results achieved using NNs. Currently, there are 2 main challenges for NN-based approaches. 1st, there are only a few, relatively small annotated data sets available, & these are often subject to severe biases [7]. Largest publicly available data set [11] contains several hours of piano music – but all recorded on only 7 different synthesizer-based & real pianos. While typical data augmentation strategies, e.g. pitch shifting or simulating different room acoustics, might mitigate some of effects, there is still a considerable risk that a network overfits acoustic properties of these specific instruments. For many types of instruments, even small data sets are not available. Other biases include musical style as well as distribution over central musical concepts, e.g. key, harmony, tempo, & rhythm.

A 2nd considerable challenge: adaptability to new acoustic conditions. Providing just a few examples of isolated notes of instrument to be transcribed, considerable improvements are observed in performance of NMF-based models. There is currently no corresponding, equally effective mechanism to retrain or adapt an NN-based AMT system on a few seconds of audio. Thus, error rate for nonadapted networks can be an order of magnitude higher than that of an adapted NMF system [23,24]. Overall, as both of these challenges cannot easily be overcome, NMF-based methods are likely to remain relevant in specific use cases.

In Fig. 7: Piano-roll representations of 1st 6 s of a recording of a Bach piece (BWV 582) for organ. Black corresponds to correctly detected pitches, red to false positives, & blue to false negatives. (a) Output of an NMF model trained on piano templates. (b) Output of piano-music-trained NN model of [24]., qualitatively illustrate some differences in behavior of systems based on supervised NMF & NNs. Both systems were specifically trained for transcribing piano recordings, & expose approaches to a recording of an organ. Like piano, organ is played with a keyboard, but its acoustic properties are quite different. Harmonics of organ are rich in energy & cover entire spectrum; energy of notes does not decay over time, & onsets are less pronounced. With this experiment, want to find out how gracefully system fail when they encounter a sound that is outside piano-sound manifold but still musically valid.

Comparing NMF output in Fig. 7a & NN output in Fig. 7b with ground truth, find: both methods detect additional notes (shown in red), mostly at octaves above & below correct fundamental. Given rich energy distribution, this behavior is expected. While use a simple baseline model for NMF – & thus some errors could be attributed to that choice – NN fails more gracefully. I.e., fewer octave errors & fewer spurious short note detections are observed. (Yet, in terms of recall, NMF-based approach identifies additional correct notes.)

As might be evident by surveying AMT literature, vast majority of approaches target only Western (or Eurogenetic) music.

Difficult to argue why acoustic model within network should be better prepared for such a situation. However, results suggest: network learned something additional: LSTM layers as used in network (cf. Fig. 6) seem to have learned how typical piano notes evolve in time, & thus most note lengths look reasonable & less spurious (ít giả tạo hơn). Similarly, bandwidth in which octave errors occur is narrower for NN, which could potentially indicate: network models likelihood of co-occurring notes or, i.e., a simple music language model (MLM). This leads us to our discussion of important remaining challenges in AMT.

- Further extensions & future work.

- **MLMs.** As outlined in “Analogies to Other Fields” sect, AMT is closely related to ASR. In same way that a typical ASR system consists of an acoustic component & a language component, an AMT system can model both acoustic sequences & underlying sequence of notes & other music cues over time. AMT systems have thus incorporated MLMs for modeling sequences of notes in a polyphonic context, with aim of improving transcription performance. Capabilities of DL methods toward modeling high-dimensional sequences have recently made polyphonic music sequence prediction possible. Boulanger-Lewandowski et al. [5] combined a restricted Boltzmann (RBM) with an RNN for polyphonic music prediction, which was used to postprocess acoustic output of an AMT system.

Sigtia et al. [18] also used aforementioned RNN-RBM as an MLM & combined acoustic & language predictions using a probabilistic graphical model. While these initial works showed promising results, there are several directions for future research in MLMs. These include creating unified acoustic & language models (as opposed to using MLMs as postprocessing steps) & modeling other musical cues, e.g. chords, key, & meter (as opposed to simply modeling note sequences).

- **Score-informed transcription.** If a known piece is performed, musical score provides a strong prior for transcription. In many cases, there are discrepancies between score & a given music performance, which may be due to a specific interpretation by a performer or to performance mistakes. For applications like music education, useful to identify such discrepancies, by incorporating musical score as additional prior information to simplify transcription process (score-informed music transcription [35]). Typically, systems for such transcription use a score-to-audio alignment method as a preprocessing step to align music score with input music audio prior to performing transcription, as in [35]. While specific instances of such systems have been developed for certain instruments (piano & violin), problem is still relatively unexplored, as is related & more challenging problem of lead-sheet-informed transcription & eventual integration of these methods toward development of automatic music tutoring systems.

- **Context-specific transcription.** While creation of a blind multi-instrument AMT system without specific knowledge of music style, instruments, & recording conditions is yet to be achieved, considerable progress has been reported on problem of context-specific transcription, where prior knowledge of sound of specific instrument model or manufacturer & recording environment is available. For context-specific piano transcription, multipitch detection accuracy can exceed 90% [22,23], making such systems appropriate for user-facing applications. Open work in this topic includes creation of context-specific AMT systems for multiple instruments.

- **Non-Western music.** As might be evident by surveying AMT literature, vast majority of approaches target only Western (or Eurogenetic) music. This allows several assumptions, regarding both instruments used & also way that music is represented & produced. Typical assumptions include octaves containing 12 equally spaced pitches; 2 modes, major & minor; & a standard tuning frequency of $A4 = 440$ Hz.

However, these assumptions do not hold true for other music styles from around world, where an octave is often divided into microtones (e.g., Arabic music theory is based on quartertones) or where there are modes not used in Western music (e.g., classical Indian music recognizes hundreds of modes, called *ragas*). Therefore, automatically transcribing non-Western music still remains an open problem with several challenges, including design of appropriate signal & music notation representations

while avoiding a so-called Western bias [36]. Another major issue is lack of annotated data sets for non-Western music, rendering application of data-intensive ML methods difficult.

- **Expressive pitch & timing.** Western notation conceptualizes music as sequences of unchanging pitches being maintained for regular durations & has little scope for representing expressive use of microtonality & microtiming or for detailed recording of timbre & dynamics. Research on automatic transcription has followed this narrow view, describing notes in terms of discrete pitches plus onset & offset times. E.g., no suitable notation exists for performed singing, most universal form of music making. Likewise, for other instruments without fixed pitch or with other expressive techniques, better representations are required. These richer representations can then be reduced to Western score notation, if required, by modeling musical knowledge & stylistic conventions.

Automatically transcribing non-Western music still remains an open problem with several challenges.

- **Percussion & unpitched sounds.** An active problem in music signal processing literature is that of detecting & classifying nonpitched sounds in music signals [1, Chap. 5]. In most cases, this is expressed as problem of drum transcription, since vast majority of contemporary music contains mixtures of pitched sounds & unpitched sounds produced by a drum set. Drum set components typically include bass drum, snare drum, hi-hat, cymbals, & toms. Problem in this case: detect & classify percussive sounds into 1 of aforementioned sound classes. Elements of drum transcription problem that make it particularly challenging are concurrent presence of several harmonic, inharmonic, & nonharmonic sounds in music signal, as well as requirement of an increased temporal resolution for drum transcription systems compared to typical multipitch detection systems. Approaches for pitched instrument transcription & drum transcription have largely been developed independently, & creation of a robust music transcription system that supports both pitched & unpitched sounds still remains an open problem.
- **Evaluation metrics.** Most AMT approaches are evaluated using set of metrics proposed for MIREX Multiple-F0 Estimation & Note Tracking public evaluation tasks <http://www.music-ir.org/mirex/>. 3 types of metrics are included: frame based, note based, & stream based, mirroring frame-level, note-level, & stream-level transcription categories presented in “Start of Art” sect. While above sets of metrics all have their merits, it could be argued: they do not correspond with human perception of music transcription accuracy, where, e.g., an extra note might be considered as a more severe error than a missed note, or where out-of-key note errors might be penalized more compared with in-key ones. Therefore, creation of perceptually relevant evaluation metrics for AMT & creation of evaluation metrics for notation-level transcription remain open problems.
- **Conclusions.** AMT has remained an active area of research in fields of music processing & music information retrieval for several decades, with several potential benefits in other areas & fields extending beyond music. As outlined in this article, there remain several hurdles (rào cản) to be overcome, i.e., on modeling music signals & on availability of data, as described in “Key Challenges” sect; w.r.t. limitations of state-of-art methodologies, as described in sect “A Comparison of NMF & NN-Models”; &, finally, on extensions beyond current area of existing tasks, as presented in “Further Extensions & Future Work” sect. Believe: addressing these challenges will lead toward creation of a complete music transcription system & unlock full potential of music signal processing technologies. Supplementary audio material related to this article can be found on companion website <http://c4dm.eecs.qmul.ac.uk/spm-amt-overview/>.

6 Computer Vision

6.1 XINYI WANG , JIANTENG PENG, SUFANG ZHANG, BIHUI CHEN, YI WANG, YANDONG GUO. A Survey Face Recognition. 2022

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- **Abstract.** Recent years witnessed breakthrough of face recognition with deep convolutional neural networks. Dozens of papers in field of FR are published every year. Some of them were applied in industrial community & played an important role in human life e.g. device unlock, mobile payment, & so on. This paper provides an introduction to face recognition, including its history, pipeline, algorithms based on conventional manually designed features or DL, mainstream training, evaluation datasets, & related applications. Have analyzed & compared state-of-art works as many as possible, & also carefully designed a set of experiments to find effect of backbone size & data distribution. This survey is a material of tutorial named *Practical Face Recognition Technology in Industrial World* in FG2023.
- **Keywords.** Face Recognition, DL, Industrial Application
- **1. Introduction.** As 1 of most important applications in field of AI & computer vision, face recognition (FR) has attracted wide attention of researchers. Almost every year, major CV conferences or journals, including CVPR, ICCV, PAMI, etc. publish dozens of papers in field of FR.

In this work, give an introduction to face recognition. Chap. 2 is about history of face recognition. Chap. 3 introduces pipeline in DL framework. Chap. 4 provides details of face recognition algorithms including loss functions, embedding techniques, face recognition with massive IDs, cross-domain, pipeline acceleration, closed-set training, mask face recognition, & privacy-preserving. In Chap. 5, carefully designed a set of experiments to find effect of backbone size & data distribution. Chap. 6 includes frequently-used training & test datasets & comparison results. Chap. 7 shows applications. Chap. 8 introduces competitions & open-source programs.

- **2. History.** Recent years witnessed breakthrough of face recognition (FR) with deep Convolutional Neural Networks. But before 2014, FR was processed in non-DL ways. In this sect, introduce conventional FR algorithms with manually designed features. Eigenface [1] is 1st method to solve problem of FR in computer vision. Eigenface reshaped a grayscale face image into a 1D vector as its feature. Then, a low-dimensional subspace can be found by Principal Component Analysis (PCA), where distributions of features from similar faces were closer. This subspace can be used to represent all face samples. For a query face image with an unknown ID, Eigenface adopted Euclidean distance to measure differences between its feature & all gallery face features with known categories in new subspace. Category of query image can be predicted by finding smallest distance to all features in gallery. Fisherface [2] pointed out: PCA used in Eigenface for dimensionality reduction maximizes variance of all samples in new subspace. Eigenface did not take sample category into account. Therefore, Fisherface considered face labels in training set & used Linear Discriminant Analysis (LDA) for dimensionality reduction, which maximized ratios of inter-class & intra-class variance. As a commonly used mathematical classification model, SVM was brought into field of FR. Since SVM & face feature extractors can be de-coupled, many SVM-based FR algorithms with different extractors have been proposed. Deniz et al. [3] used ICA[4] for feature extraction, & then used SVM to predict face ID. In order to speed up training of ICA+SVM, Kong et al. [5] designed fast Least Squares SVM to accelerate training process by modifying Least Squares SVM algorithm. Jianhong et al. [6] combined kernel PCA & Least Squares SVM to get a better result. As 1 of most successful image local texture extractors, Local Binary Pattern (LBP) [7] obtained characteristics of images through statistical histogram. Many scholars have proposed FR algorithms based on LBP feature extractor & its variants. Ahonen et al. [8] adopted LBP to extract features of all regions from 1 face image, & then concatenated those feature vectors for histogram statistics to obtain an embedding of whole image. Finally, they used Chi square sa distance metric to measure similarity of faces. Wolf et al. [9] proposed an improved region descriptor based on LBP, & got a better result with higher accuracy. Tan et al. [10] refined LBP & proposed Local Ternary Patterns (LTP), which had a higher tolerance of image noise.
- **3. Pipelines in DL framework.** As computing resources were increasing, mainstream face recognition methods applied DL to model this problem, which replaced aforementioned manually designed feature extraction methods. In this sect, give pipelines of ‘training’ a FR deep model & ‘inference’ to get ID of a face image (as shown in Fig. 1: Pipelines of training & inference in face recognition.). & will elaborate some representative methods related to them.

In general, pipeline of training FR model contains 2 steps: face images preprocessing (preprocessing) & model training (training). Getting FR result by trained model contains 3 parts: face images preprocessing (preprocessing), model inference to get face embedding (inference), & recognition by matching features between test image & images with known labels (recognition). In industry, face anti-spoofing will be added before inference in testing pipeline since FR systems are vulnerable to presentation attacks ranging from print, video replay, 3D mask, etc. Both preprocessing steps in training & inference should be same.

– Nhìn chung, quy trình đào tạo mô hình FR bao gồm 2 bước: tiền xử lý ảnh khuôn mặt (preprocessing) & đào tạo mô hình (training). Việc lấy kết quả FR bằng mô hình đã đào tạo bao gồm 3 phần: tiền xử lý ảnh khuôn mặt (preprocessing), suy luận mô hình để có được những khuôn mặt (inference), & nhận dạng bằng cách khớp các đặc điểm giữa ảnh thử nghiệm & ảnh có nhãn đã biết (recognition). Trong công nghiệp, chống giả mạo khuôn mặt sẽ được thêm vào trước khi suy luận trong quy trình thử nghiệm vì các hệ thống FR dễ bị tấn công trình bày từ in ấn, phát lại video, mặt nạ 3D, v.v. Cả hai bước tiền xử lý trong đào tạo & suy luận phải giống nhau.

◦ **3.1. Preprocessing.** In practice, faces for FR are located in a complicated image or a video. Therefore, need to apply face detection (with face landmark detection) to cut out specific face patch 1st, & use face alignment to transform this face patch in a good angle or position. After these 2 preprocessing, a face image patch is ready for further FR procedures. Noticed: face detection is necessary for FR preprocessing, while face alignment is not.

* **3.1.1. Face detection.** Face detection (or automatic face localization) is a long-standing problem in CV. Considering human face as an object, many object detection algorithms can be adopted to get a great face detection result, e.g. Faster-RCNN [11], SSD [12], YOLO (with different versions [13–15]).

However, some researchers treated human face as a special object & designed variable deep architectures for finding faces in wild with higher accuracy. Mtcnn [16] is 1 of most famous face detection methods in academy. Mtcnn adopted a cascaded structure with 3 stages of deep convolutional networks (P-net, R-net, O-net) that predict both face & landmark location in a coarse-to-fine manner. Non-maximum suppression (NMS) is employed to merge highly overlapped candidates, which are produced by P-net, R-net, & O-net. Faceness-Net [17] observed: facial attributes based supervision can effectively enhance capability of a face detection network in handling severe occlusions. As a result, Faceness-Net proposed a 2-stage network, where 1st stage applies several branches to generate response maps of different facial parts & 2nd stage refines candidate window using a multi-task CNN. At last, face attribute prediction & bounding box regression are jointly optimized.

As FPN (Feature Pyramid Network) is widely used in object detection, many researchers tried to bring in FPN in face detection & obtained success in detecting tiny faces. SRN (Selective Refinement Network) [18] adopted FPN to extract face features, & introduced 2-step classification & regression operations selectively into an anchor-based face detector to reduce false positives & improve location accuracy simultaneously. In particular, SRN consists of 2 modules: Selective 2-step Classification (STC) module & Selective 2-step Regression (STR) module. STC aims to filter out most simple negative anchors from low level detection layers to reduce search space for subsequent classifier, while STR is designed to coarsely adjust locations & sizes of anchors from high level detection layers to provide better initialization for subsequent regressor. Pipeline of SRN can be seen in Fig. 2: Pipeline of SRN.

RetinaFace [19] is another face detector using FPN to extract multi-level image features. However, different from 2-step detecting methods, e.g. SRN, RetinaFace presented a single-stage face detector. Furthermore, RetinaFace performed face detection in a multi-task way. In specific, RetinaFace predicted following 4 aspects at same time: (1) a face score, (2) a face box, (3) 5 facial landmarks, (4) dense 3D face vertices projected on image plane. Pipeline of RetinaFace can be seen in Fig. 3: Pipeline of RetinaFace.

CRFace [20] proposed a confidence ranker to refine face detection result in high resolution images or videos. HLA-Face [21] solved face detection problem in low light scenarios. As development of general 2D object detection, face detection has reached a high performance, even in real-world scenarios. As a result, innovative papers in face detection seldom show up in recent years.

- * 3.1.2. **Face anti-spoofing.** In industry, faces fed into recognition system need to be check whether they are real or not, in case someone uses printed images, videos or 3D mask to pass through system, especially in secure scenarios. As a result, while inference, face anti-spoofing (or presentation attack detection) is an important preprocessing step, which is located after face detection.

Major methods for face anti-spoofing usually input signals from RGB cameras. Both hand-crafted feature based (e.g., LBP [22], SIFT [23], SURF [24] & HoG [25]) & DL based methods have been proposed. CDCN [26] designed a novel convolutional operator called Central Difference Convolution to get fine-grained invariant information in diverse environments. Output feature map y can be formulated as

$$y(p_0) = \sum_{p_n \in R} w(p_n) \cdot x(p_0 + p_n)$$

where p_0 : current location on both input & output feature map, & p_n : location in R . In addition, CDCN used NAS to search backbone for face anti-spoofing task, which is supervised by depth loss instead of binary loss. DCDN [27] decoupled central gradient features into 2 cross directions (horizontal/vertical or diagonal) & achieved better performance with less computation cost when compared with CDCN. STDN [28] observed: there is little explanation of classifier's decision. As a result, STDN proposed an adversarial learning network to extract patterns differentiating a spoof (sự giả mạo) & live face. Pipeline of STDN can be seen in Fig. 4: Pipeline of STDN PatchNet [29] rephrased face anti-spoofing as a fine-grained material recognition problem. Specifically, PatchNet splitted categories finely based on capturing devices & presenting materials, & used patch-level inputs to learn discriminative features. In addition, asymmetric margin-based classification loss & self-supervises similarity loss were proposed to further improve generalization ability of spoof feature. SSAN [30] used a 2-stream structure to extract content & style features, & reassembled various content & style features to get a stylized feature space, which was used to distinguish a spoof & live face. In specific, adversarial learning was used to get a shared feature distribution for content information & a contrastive learning strategy was proposed to enhance liveness-related style information while suppress (kìm nén) domain-specific one.

Merely using RGB signals which belong to visible spectrum has a lot of limitations in face anti-spoofing, therefore an increasing numbers of methods tried to adopt multi-channel signals to achieve higher accuracy. FaceBagnet [31] used patch-level image to extract spoof-specific discriminative information & proposed a multi-modal features fusion strategy. CMFL [32] designed a cross-modal focal loss to modulate individual channel's loss contributions, thus captured complementary information among modalities.

- * 3.1.3. **Face alignment.** Face image patches from face detection are often different in shapes due to factors e.g. pose, perspective transformation & so on, which will lead to a decline on recognition performance. Under this condition, face alignment is an effective approach to alleviate (làm giảm nhẹ) this issue by transforming face patches into a similar angle & position [33,34]. Face image set with an identical ID will get a smaller intra-class difference after face alignment, further making training classifier more discriminative.

Common way to align face images is using a 2D transformation to calibrate facial landmarks to predefined frontal templates or mean face mode. This 2D transformation is normally processed by affine transformation [33]. Deepface [35] proposed a FR pipeline which employed a 3D alignment method in order to align faces undergoing out-of-plane rotations. Deepface used a generic 3D shape model & registered a 3D affine camera, which are used to warp 2D aligned crop to image plane of 3D shape. STN (Spatial Transform Networks) [36] introduced a learnable module which explicitly allowed spatial manipulation of image within network, & it can also be utilized in face alignment. Wu et al. [37] merged STN based face alignment & face feature extractor together, & put forward Recursive Spatial Transformer (ReST) for end-to-end FR training with alignment. 2D face alignment is faster than both 3D alignment & STN, & thus it is widely used in preprocessing step of FR. APA [38] proposed a more general 2D face alignment method. Instead of aligning all faces to near-frontal shape, APA adaptively learned multiple pose-specific templates, which preserved face appearance with less artifact & information loss.

As mainstream FR training sets are becoming larger gradually, some FR methods choose to omit face alignment step & train (or test) with face patches directly from face detection. Technically, face alignment is a way to increase intra-class compactness. & a well-trained FR model with more IDs can also obtain a compact feature distribution for each class. Therefore, face alignment may not be necessary nowadays.

- 3.2. **Training & testing a FR deep model.** After preprocessing, face images with their ground truth (sự thật cơ bản) IDs can be used to train a FR deep model. As progressing of computational hardware, any mainstream backbone is able to build a FR network. Different backbones with similar parameters amount have similar accuracy on FR. As a result, can use Resnet, ResNext, SEResnet, Inception net, Densenet, etc. to form your FR system. If need to design a FR system with limited

calculation resource, Mobilenet with its variation will be good options. In addition, NAS can be used to search a better network hyper-parameter. Network distilling technology is also welcomed while building industrial FR system.

Training algorithms will be elaborated in following sects. For testing, a face image after preprocessing can be inputted into a trained FR model to get its face embedding. Here, list some practical tricks which will be utilized in model training & testing steps. 1stly, training image augmentation is useful, especially for IDs with inadequate samples. Common augmentation ways, e.g. adding noise, blurring, modifying colors, are usually employed in FR training. However, randomly cropping should be avoided, since it will ruin face alignment results. 2ndly, flipping face images horizontally is a basic operation in both of training & testing FR model. In training, flipping faces can be considered as a data augmentation process. In testing, can put image I & its flipped mirror image I' into model, & get their embeddings f, f' . Then their mean feature $\frac{f+f'}{2}$ can be used as embedding of image I . This trick will further improve accuracy of FR results.

– Các thuật toán đào tạo sẽ được trình bày chi tiết trong các phần sau. Để thử nghiệm, một hình ảnh khuôn mặt sau khi xử lý trước có thể được đưa vào mô hình FR đã đào tạo để nhúng khuôn mặt của nó. Sau đây, hãy liệt kê một số thủ thuật thực tế sẽ được sử dụng trong các bước đào tạo & thử nghiệm mô hình. Đầu tiên, việc tăng cường hình ảnh đào tạo rất hữu ích, đặc biệt đối với các ID có mẫu không đủ. Các cách tăng cường phổ biến, ví dụ như thêm nhiễu, làm mờ, sửa đổi màu sắc, thường được sử dụng trong đào tạo FR. Tuy nhiên, nên tránh cắt ngẫu nhiên vì nó sẽ làm hỏng kết quả căn chỉnh khuôn mặt. Thứ hai, lật ảnh khuôn mặt theo chiều ngang là thao tác cơ bản trong cả đào tạo & thử nghiệm mô hình FR. Trong đào tạo, việc lật mặt có thể được coi là một quá trình tăng cường dữ liệu. Trong thử nghiệm, có thể đưa ảnh I & ảnh phản chiếu lật I' của nó vào mô hình, & lấy nhúng f, f' của chúng. Sau đó, đặc điểm trung bình $\frac{f+f'}{2}$ của chúng có thể được sử dụng làm nhúng của ảnh I . Thủ thuật này sẽ cải thiện thêm độ chính xác của kết quả FR.

- 3.3. Face recognition by comparing face embeddings. Last part of face inference pipeline is face recognition by comparing face embeddings. According to application, it can be divided into face verification & face identification. To apply FR, a face gallery needs to be built. 1st have a face ID set S , where each ID contains 1 (or several) face image(s). All face embeddings from gallery images will be extracted by a trained model & saved in a database.

Protocol of face verification (FV): giving a face image I & a specific ID in gallery, output a judgment whether face I belongs to ID. So FV is a 1:1 problem. Extract feature of image I & calculate its similarity $s(I, ID)$ with ID's embedding in database. $s(I, ID)$ is usually measured by cosine similarity. If $s(I, ID)$ is larger than a threshold μ , will output 'True' which represents image I belongs to ID, & vice versa.

However, face identification (FI) is a 1:N problem. Giving a face image I & a face ID set S , it outputs ID related to face image, or 'not recognized'. Similarly, extract feature of I & calculate its similarity $s(I, ID)$, $ID \in S$ with all IDs' embeddings in database. Then, find maximum of all similarity values. if $\max(s(I, ID))$ is larger than a threshold μ , output its related ID $\arg_{ID} \max(s(I, ID))$ as identification target; otherwise, output 'not recognized', which shows: person of image I is not in database. Phan et al. [39] enriched FI process by adopting an extra Earth Mover's Distance. 1st used cosine similarity to obtain a part of most similar faces of query image. Then a patch-wise Earth Mover's Distance was employed to re-rank these similarities to get final identification results. In Subsect. 4.5, introduce some methods to accelerate process of face identification.

- 4. Algorithms. In this sect, introduce FR algorithms in recent years. Based on different aspects in deep FR modeling, divided all FR methods into several categories: designing loss function, refining embedding, FR with massive IDs, FR on uncommon images, FR pipeline acceleration, & close-set training.

- 4.1. Loss Function.

* 4.1.1. Loss based on metric learning. Excerpt from softmax based classification, FR can be also regarded as extracting face features & performing feature matching. Therefore, training FR model is a process to learn a compact euclidean feature space, where distance directly correspond to a measure of face similarity. This is basic modeling of metric learning. In this sect, introduce some representative FR methods based on metric learning.

Inspired by pipeline of face verification, 1 intuitive loss designing is judging whether 2 faces in a pair have identical ID or not. Han et al. [40] used this designing in a cross-entropy way, & loss is: (2)

$$L_{ij} = -[y_{ij} \log p_{ij} + (1 - y_{ij}) \log(1 - p_{ij})]$$

where y_{ij} : binary GT of whether 2 compared face images i, j belong to same identity. p_{ij} : logits value.

Different from [40], contrastive loss [41] was proposed to directly compare features of 2 face images in a metric learning way. If 2 images belong to a same ID, their features in trained space should be closer to each other, & vice verses. As a result, contrastive loss is:

$$L_{ij} = \begin{cases} \frac{1}{2} \|f_i - f_j\|_2^2 & \text{if } y_{ij} = 1, \\ \frac{1}{2} \max(0, m - \|f_i - f_j\|_2)^2 & \text{if } y_{ij} = -1, \end{cases}$$

where i, j : 2 samples of a training pair, f_i, f_j : their features. $\|f_i - f_j\|_2^2$: their Euclidean distance. m : a margin for enlarging distance of sample pairs with different IDs (negative pairs). Furthermore, Euclidean distance can be replaced by cosine distance, & contrastive loss become following form:

$$L_{ij} = \frac{1}{2} (y_{ij} - \sigma(wd + b))^2, \quad d = \frac{f_i f_j}{\|f_i\|_2 \|f_j\|_2},$$

where d : cosine similarity between feature f_i, f_j , w, b : learnable scaling & shifting parameters, σ : sigmoid function. Different from contrastive loss, BioMetricNet [42] did not impose any specific metric on facial features. Instead, it shaped decision space by learning a latent representation in which matching (positive) & non-matching (negative) pairs are mapped onto clearly separated & well-behaved target distributions. BioMetricNet 1st extracted face features of matching & non-matching pairs, & mapped them into a new space in which a decision is made. Its loss used to measure statistics distribution of both matching & non-matching pairs in a complicated form.

FaceNet [34] proposed triplet loss to minimize distance between an anchor & a positive, both of which have same identity, & maximize distance between anchor & a negative of a different identity. Triplet loss was motivated in [43] in context of nearest-neighbor classification. Insight of triplet loss, is ensure: an image x^a (anchor) of a specific person is closer to all other images x^p (positive) of same person than it is to any image x^n (negative) of any other person. Thus loss is designed as: (5)

$$L = \sum_{x^a, x^p, x^n \in T} \max(\|f(x^a) - f(x^p)\|_2^2 - \|f(x^a) - f(x^n)\|_2^2 + \alpha, 0),$$

where T : set of all possible triplets in training set. $f(x)$: embedding of face x . α : a margin that is enforced between positive & negative pairs. In order to ensure fast convergence, crucial to select triplets that violate triplet constraint of:

$$\|f(x^a) - f(x^p)\|_2^2 + \alpha < \|f(x^a) - f(x^n)\|_2^2.$$

I.e., given x^a , want to select an x^p (hard positive) s.t. $\arg \max_{x^p} \|f(x^a) - f(x^p)\|_2^2$, & similarly x^n (hard negative) s.t. $\arg \min_{x^n} \|f(x^a) - f(x^n)\|_2^2$. As a result, accuracy of triplet loss model is highly sensitive to training triplet sample selection. Kang et al. [44] simplified both of contrastive & triplet loss & designed a new loss as

$$\begin{aligned} L &= \lambda_1 L_t + \lambda_2 L_p + \lambda_3 L_{\text{softmax}}, \\ L_t &= \sum_{(x^a, x^p, x^n) \in T} \max\left(0, 1 - \frac{\|f(x^a) - f(x^n)\|_2}{\|f(x^a) - f(x^p)\|_2 + m}\right), \\ L^p &= \sum_{(x^a, x^p, \dots) \in T} \|f(x^a) - f(x^p)\|_2^2, \end{aligned}$$

where $L_t, L_p, L_{\text{softmax}}$: modified triplet ratio loss, pairwise loss & regular softmax loss, resp. Factor m in L_t is a margin to enlarge difference between positive pair & negative pair, which has similar use with α in (5).

Contrastive loss measures difference of a image pair; triplet loss represented relations in a triplet: anchor, positive, & negative samples. Some researchers have developed more learning based FR losses by describing more samples.

In order to enhance discrimination of deeply learned features, center loss [45] simultaneously learned a center for deep features of each class & penalized distance between deep features & their corresponding class centers. Center loss can be added on softmax or other FR loss to refine feature distribution while training. Form of center loss: (8)

$$L = L_{\text{softmax}} + \lambda L_c, \quad L_c(i) = \frac{1}{2} \sum_{i=1}^m \|x_i - c_{y_i}\|_2^2,$$

where L_{softmax} is softmax loss on labeled training data. i is an image sample with its face label y_i , & x_i : its deep feature. m : number of training classes. c_{y_i} denotes y_i th class center of deep features. Formulation of center loss effectively characterizes intra-class variations.

* **4.1.2. Larger margin loss.** Face recognition is naturally treated as a classification problem, & thus using softmax as loss to train a FR model is an intuitive consideration. Larger margin loss (also is called as *angular margin based loss*) is derived from softmax. Larger margin loss is a major direction in FR, since it has largely improved performance of FR in academic & industry. Intuition: facial images with same identities are expected to be closer in representation space, while different identities expected to be far apart. As a result, larger margin losses encourage intra-class compactness & penalize similarity of different identities.

– **Mất biên độ lớn hơn.** Nhận dạng khuôn mặt được coi là một vấn đề phân loại, & do đó, sử dụng softmax làm mất mát để đào tạo mô hình FR là một cân nhắc trực quan. Mất mát biên độ lớn hơn (còn được gọi là *mất mát dựa trên biên độ góc*) được bắt nguồn từ softmax. Mất mát biên độ lớn hơn là một hướng chính trong FR, vì nó đã cải thiện đáng kể hiệu suất của FR trong học thuật & công nghiệp. Trực giác: hình ảnh khuôn mặt có cùng danh tính được mong đợi sẽ gần nhau hơn trong không gian biểu diễn, trong khi các danh tính khác nhau được mong đợi sẽ cách xa nhau. Do đó, mất mát biên độ lớn hơn khuyến khích tính chặt chẽ trong lớp & phạt sự giống nhau của các danh tính khác nhau.

1st, give formulation of softmax as follows:

$$L_i = -\log \frac{e^{W_{y_i} \cdot x_i + b_{y_i}}}{\sum_j e^{W_j \cdot x_j + b_j}},$$

where W : weighting vector of last fully connected layer, & W_j represents weight of class j ; x_i, y_i : face embedding of sample i & its ground truth ID. For class y_i , sample i is a positive sample; & for class $j \neq y_i$, sample i is a negative sample. These 2 definitions will be used in this whole paper. Consider positive & negative samples, softmax can be rewritten as

$$L_i = -\log \frac{e^{W_{y_i} \cdot x_i + b_{y_i}}}{e^{W_{y_i} \cdot x_i + b_{y_i}} + \sum_{j \neq y_i} e^{W_j \cdot x_j + b_j}}.$$

L-Softmax [46] 1st designed margin based loss by measuring features angles. 1st, it omits bias of each class b_j , & changes inner product of features & weights $W_j \cdot x_i$ to a new form $\|W_j\| \|x_i\| \cos \theta_j$, where θ_j : angle between x_i & weight W_j of class j . In order to enlarge margin of angels between each class, L-Softmax modifies $\cos \theta_{y_i}$ to $\psi \theta_{y_i}$ by narrowing down space between decision boundary & class centers. In specific, it has:

$$L_i = -\log \frac{e^{\|W_{y_i}\| \|x_i\| \psi(\theta_{y_i})}}{e^{\|W_{y_i}\| \|x_i\| \psi(\theta_{y_i})} + \sum_{j \neq y_i} e^{\|W_j\| \|x_i\| \cos \theta_j}},$$

$$\psi(\theta) = (-1)^k \cos m\theta - 2k,$$

where m is a fix parameter which is an integer; angle $\theta \in [0, \pi]$ has been divided in m intervals: $\left[\frac{k\pi}{m}, \frac{(k+1)\pi}{m}\right]$, $k \in \mathbb{Z} \cap [0, m-1]$. Therefore, it has $\psi(\theta) \leq \cos \theta$.

Similar with L-Softmax, Sphreface [47] proposed A-Softmax, which normalized each class weight W_j before calculating loss. Therefore, loss became:

$$L_i = -\log \frac{e^{\|x_i\| \psi(\theta_{y_i})}}{e^{\|x_i\| \psi(\theta_{y_i})} + \sum_{j \neq y_i} e^{\|x_i\| \cos \theta_j}}.$$

In training step of Sphreface, its actual loss is $(1-\alpha)*softmax + \alpha*A-softmax$, where α is a parameter in $[0, 1]$. During training, α is changing gradually from 0 to 1. This design is based on 2 motivations. 1stly, directly training A-Softmax loss will lead to hard convergence, since it brutally pushes features from different ID apart. 2ndly, training a softmax loss 1st will decrease angle θ_{y_i} between feature i & its related weight W_{y_i} , which causes part $\cos m\theta$ in A-softmax loss in a monotonous area. & easier to get a lower loss while gradient descending.

Besides weight of each class, NormFace [48] proposed: face embeddings need to be normalized as well. Also, NormFace scales up normalized face embeddings, which will alleviate data unbalancing problem of positive & negative samples for better convergence. However, NormFace doesn't use extra margin for positive samples as Sphreface. Therefore, NormFace loss is shown as

$$L_i = \log \frac{e^{s\tilde{W}_{y_i}\tilde{x}_i}}{\sum_j e^{s\tilde{W}_j\tilde{x}_j}} = -\log \frac{e^{s\cos \theta_{y_i}}}{\sum_j e^{s\cos \theta_j}},$$

where \tilde{W}, \tilde{x} : normalized class weights & face embeddings; s : a scaling up parameter & $s > 1$. To this end, both face embeddings & class weights are distributed on hyperphere manifold, due to normalization.

AM-Softmax [49] & CosFace [50] enlarged softmax based loss by minus a explicit margin m outside $\cos \theta$ as follows: p. 7/59

7 Geeksforgeeks

7.1 Introduction to Recurrent Neural Networks

“*Recurrent Neural Networks (RNNs)* were introduced in 1980s by researchers DAVID RUMELHART, GEOFFREY HINTON, & RONALD J. WILLIAMS. RNNs have laid foundation for advancements in processing sequential data, e.g. natural language & time-series analysis, & continue to influence AI research & applications today.

In this article, explore core principles of RNNs, understand how they function, & discuss why they are essential for tasks where previous inputs in a sequence influence further predictions.

7.1.1 What is RNNs?

In traditional **neural networks**, inputs & outputs are treated independently. However, tasks like predicting next word in a sentence require information from previous words to make accurate predictions. To address this limitation, RNNs were developed.

RNNs introduce a mechanism where *output from 1 step is fed back as input to next, allowing them to retain information (giữ lại thông tin) from previous inputs*. This design makes RNNs well-suited for tasks where context from earlier steps is essential, e.g. predicting next word in a sentence.

Defining feature of RNNs is their *hidden state* – also called *memory state* – which preserved essential information from previous inputs in sequence. By using same parameters across all steps, RNNs perform consistently across inputs, reducing parameter complexity compared to traditional neural networks. This capability makes RNNs highly effective for sequential tasks.

In simple terms, RNNs apply same network to each element in a sequence, RNNs preserve & pass on relevant information, enabling them to learn temporal dependencies that conventional neural networks cannot.

How RNN differs from Feedforward Neural Networks. **Feedforward Neural Networks (FNNs)** process data in 1 direction, from input to output, without retaining information from previous inputs. This makes them suitable for tasks with independent inputs, like image classification. However, FNNs struggle with sequential data since they lack memory.

RNNs solve this by incorporating loops that allow information from previous steps to be fed back into network. This feedback enables RNNs to remember prior inputs, making them ideal for tasks where context is important.

7.1.2 Key Components of RNNs

1. **Recurrent Neurons.** Fundamental processing unit in a RNN is a Recurrent Unit, which is not explicitly called a “Recurrent Neuron”. Recurrent units hold a hidden state that maintains information about previous inputs in a sequence. Recurrent units can “remember” information from prior steps by feeding back their hidden state, allowing them to capture dependencies across time.
2. **RNN Unfolding.** *RNN unfolding* or “*unrolling*” is process of expanding recurrent structure over time steps. During unfolding, each step of sequence is represented as a separate layer in a series, illustrating how information flows across each time step. This unrolling enables *backpropagation through time (BPTT)*, a learning process where errors are propagated across time steps to adjust network’s weights, enhancing RNN’s ability to learn dependencies within sequential data.

7.1.3 Types of RNNs

There are **4 types of RNNs** based on number of inputs & outputs in network:

1. **1-to-1 RNN.** *1-to-1 RNN* behaves as *Vanilla Neural Network*, is simplest type of neural network architecture. In this setup, there is a single input & a single output. Commonly used for straightforward classification tasks where input data points do not depend on previous elements.
2. **1-to-many RNN.** In a *1-to-many RNN*, network processes a single input to produce multiple outputs over time. This setup is beneficial when a single input element should generate a sequence of predictions.
E.g., for image captioning task, a single image as input, model predicts a sequence of words as a caption.
3. **Many-to-1 RNN.** *Many-to-1 RNN* receives a sequence of inputs & generates a single output. This type is useful when overall context of input sequence is needed to make 1 prediction.
In sentiment analysis (phân tích tình cảm), model receives a sequence of words (like a sentence) & produces a single output, which is sentiment of sentence (positive, negative, or neutral).
4. **Many-to-Many RNN.** *Many-to-Many RNN* type processes a sequence of inputs & generates a sequence of outputs. This configuration is ideal for tasks where input & output sequences need to align over time, often in a 1-to-1 or many-to-many mapping.
In language translation task, a sequence of words in 1 language is given as input, & a corresponding sequence is another language is generated as output.

7.1.4 Variants of RNNs

There are several variations of RNNs, each designed to address specific challenges or optimize for certain tasks:

1. **Vanilla RNN.** This simplest form of RNN consists of a single hidden layer, where weights are shared across time steps. Vanilla RNNs are suitable for learning short-term dependencies but are limited by vanishing gradient problem, which hampers long-sequence learning.
2. **Bidirectional RNNs.** **Bidirectional RNNs** process inputs in both forward & backward directions, capturing both past & future context for each time step. This architecture is ideal for tasks where entire sequence is available, e.g. named entity recognition & question answering.
3. **Long Short-Term Memory Networks (LSTMs).** **Long Short-Term Memory Networks (LSTMs)** introduced a memory mechanism to overcome vanishing gradient problem. Each LSTM cell has 3 gates:
 - *Input Gate:* Controls how much new information should be added to cell state.
 - *Forget Gate:* Decides what past information should be discarded.
 - *Output Gate:* Regulates what information should be output at current step. This selective memory enables LSTMs to handle long-term dependencies, making them ideal for tasks where earlier context is critical.
4. **Gated Recurrent Units (GRUs).** **Gated Recurrent Units (GRUs)** simplify LSTMs by combining input & forget gates into a single update gate & streamlining output mechanism. This design is computationally efficient, often performing similarly to LSTMs, & is useful in tasks where simplicity & faster training are beneficial.

7.1.5 RNN Architecture

RNNs share similarities in input & output structures with other DL architectures but differ significantly in how information flows from input to output. Unlike traditional deep neural networks, where each dense layer has distinct weight matrices, RNNs use shared weights across time steps, allowing them to remember information over sequences.

In RNNs, hidden state H_i is calculated for every input X_i to retain sequential dependencies. Computations follow these core formulas:

- **Hidden State Calculation:** $h = \sigma(U \cdot X + W \cdot h_{t-1} + B)$ where h represents current hidden state, U, W : weight matrices, B : bias.
- **Output Calculation:** $Y = O(V \cdot h + C)$. Output Y is calculated by applying O , an activation function, to weighted hidden state, where V, C represent weights & bias.
- **Overall Function:** $Y = f(X, h, W, U, V, B, C)$. This function defines entire RNN operation, where state matrix S holds each element s_i representing network's state at each time step i .
- **Key Parameters in RNNs:**
 - Weight Matrices: W, U, V
 - Bias Terms: B, C .

These parameters remain consistent across all time steps, enabling network to model sequential dependencies more efficiently, which is essential for tasks like language processing, time-series forecasting, & more.

7.1.6 How does RNN work?

In a RNN, each time step consists of units with a fixed activation function. Each unit contains an internal hidden state, which acts as memory by retaining information from previous time steps, thus allowing network to store past knowledge. Hidden state h_t is updated at each time step to reflect new input, adapting network's understanding of previous inputs.

Updating Hidden State in RNNs. Current hidden state h_t depends on previous state h_{t-1} & current input x_t , & is calculated using following relations:

1. **State Update.** $h_t = f(h_{t-1}, x_t)$ where h_t : current state, h_{t-1} : previous state, x_t : input at current time step.
2. **Activation Function Application.** $h_t = \tanh(W_{hh} \cdot h_{t-1} + W_{xh} \cdot x_t)$ where W_{hh} : weight matrix for recurrent neuron, W_{xh} : weight matrix for input neuron.
3. **Output Calculation.** $y_t = W_{hy} \cdot h_t$ where y_t : output & W_{hy} : weight at output layer.

These parameters are updated using backpropagation. However, since RNN works on sequential data here use an updated backpropagation which is known as *backpropagation through time*.

Backpropagation through time (BPTT) in RNNs. In a Recurrent Neural Network (RNN), data flows sequentially, where each time step's output depends on previous time step. This ordered data structure necessitates applying backpropagation across all hidden states, or time steps, in sequence. This unique approach is called *Backpropagation Through Time (BPTT)*, essential for updating network parameters that rely on temporal dependencies.

BPTT Process in RNNs. SKIPPED MATH.

Training Process in RNNs. Training RNNs involves feeding input data through multiple time steps, capturing dependencies across these steps, & updating model through backpropagation.

Steps in RNN training include:

1. **Input at Each Time Step:** A single time step of input sequence is provided to network.
2. **Calculate Hidden State:** Using current input & prev hidden state, network calculates current hidden state h_t .
3. **State Transition:** Current hidden state h_t then becomes h_{t-1} for next time step.
4. **Sequential Processing:** This process continues across all time steps to accumulate information from prev states.
5. **Output Generation & Error Calculation:** Final hidden state is used to compute network's output, which is then compared to actual target output to generate an error.
6. **Backpropagation Through Time (BPTT):** This error is backpropagated through each time step to update weights & train RNN.

7.1.7 Implementing a Text Generator Using RNNs

Create a character-based text generator using RNN in TensorFlow & Keras. Implement an RNN that learns patterns from a text sequence to generate new text character-by-character.

- **Step 1: Import Necessary Libraries.** Start by importing essential libraries for data handling & building neural network.

```
import numpy as np
import tensorflow as tf
from tensorflow.keras.models import Sequential
from tensorflow.keras.layers import SimpleRNN, Dense
```

- **Step 2: Define Input Text & Prepare Character Set.** Define input text & identify unique characters in text, which encode for model.

```
text = "This is GeeksforGeeks a software training institute"
chars = sorted(list(set(text)))
char_to_index = {char: i for i, char in enumerate(chars)}
index_to_char = {i: char for i, char in enumerate(chars)}
```

- **Step 3: Create Sequences & Labels.** To train RNN, need sequences of fixed length `seq_length` & character following each sequence as label.

```
seq_length = 3
sequences = []
labels = []

for i in range(len(text) - seq_length):
    seq = text[i:i + seq_length]
    label = text[i + seq_length]
    sequences.append([char_to_index[char] for char in seq])
    labels.append(char_to_index[label])

X = np.array(sequences)
y = np.array(labels)
```

- **Step 4: Convert Sequences & Labels to 1-Hot Encoding.** For training, convert X, y into 1-hot encoded tensors.

```
X_one_hot = tf.one_hot(X, len(chars))
y_one_hot = tf.one_hot(y, len(chars))
```

- **Step 5: Build RNN Model.** Create a simple RNN model with a hidden layer of 50 units & a Dense output layer with **softmax activation**.

```
model = Sequential()
model.add(SimpleRNN(50, input_shape=(seq_length, len(chars)), activation='relu'))
model.add(Dense(len(chars), activation='softmax'))
```

- **Step 6: Compile & Train Model.** Compile model using **categorical_crossentropy** loss & train it for 100 epochs.

```
model.compile(optimizer='adam', loss='categorical_crossentropy', metrics=['accuracy'])
model.fit(X_one_hot, y_one_hot, epochs=100)
```

- **Step 7: Generate New Text Using Trained Model.** After training, use a starting sequence to generate new text character-by-character.

```
start_seq = "This is G"
generated_text = start_seq
```



```

for i in range(50):
    x = np.array([[char_to_index[char] for char in generated_text[-seq_length:]])
    x_one_hot = tf.one_hot(x, len(chars))
    prediction = model.predict(x_one_hot)
    next_index = np.argmax(prediction)
    next_char = index_to_char[next_index]
    generated_text += next_char

print("Generated Text:")
print(generated_text)

```

Output:

Generated Text: This is Geeks a software training instituteais is is is is

NQBH's output: ***WHY?***

Generated Text:
This is GeeksforGeeksforGeeksforGeeksforGeeksforGeeksforGee

Advantages of RNNs.

- *Sequential Memory*: RNNs retain information from previous inputs, making them ideal for time-series predictions where past data is crucial. This capability is often called *Long Short-Term Memory* (LSTM).
- *Enhanced Pixel Neighborhoods*: RNNs can be combined with convolutional layers to capture extended pixel neighborhoods, improving performance in image & video data processing.

Limitations of RNNs. While RNNs excel at handling sequential data, they face 2 main training challenges, i.e., vanishing gradient & exploding gradient problem:

1. *Vanishing Gradient*: During backpropagation, gradients diminish as they pass through each time step, leading to minimal weight updates. This limits RNN's ability to learn long-term dependencies, which is crucial for tasks like language translation.
2. *Exploding Gradient*: Sometimes, gradients grow uncontrollably, causing excessively large weight updates that destabilize training. Gradient clipping is a common technique to manage this issue.

These challenges can hinder performance (cản trở hiệu suất) of standard RNNs on complex, long-sequence tasks.

Applications of RNNs. RNNs are used in various applications where data is sequential or time-based:

- **Time-Series Prediction**: RNNs excel in forecasting tasks, e.g. stock market predictions & weather forecasting.
- **Natural Language Processing (NLP)**: RNNs are fundamental in NLP tasks like language modeling, sentiment analysis, & machine translation.
- *Speech Recognition*: RNNs capture temporal patterns in speech data, aiding in speech-to-text & other audio-related applications.
- *Image & Video Processing*: When combined with convolutional layers, RNNs help analyze video sequences, facial expressions, & gesture recognition.

FAQs on RNNs.

- Which type of problem can be solved by RNN? Modeling time-dependent & sequential data problems, like text generation, machine translation, & stock market prediction, is possible with RNNs. Nevertheless, will discover: gradient problem makes RNN difficult to train. Vanishing gradients issue affects RNNs.
- What are types of RNN? 1 to 1, 1 to many, many to 1, many to many.
- What are differences between RNN & CNN? Key distinctions between CNNs & RNNs: CNNs are frequently employed in solution of problems involving spatial data, like images. Text & video data that is temporally & sequentially organized is better analyzed by RNNs. RNNs & CNNs are not designed alike.

” – [Geeksforgeeks/introduction to recurrent neural networks](#)

8 Miscellaneous

8.1 Scholarpedia/recurrent neural networks

“A *recurrent neural network* (RNN) is any network whose **neurons** send feedback signals to each other. This concept includes a huge number of possibilities. A number of reviews already exist of some types of RNNs. These include:

- **Wikipedia/recurrent neural network**
- **Recurrent Neural Networks for Temporal Data Processing** edited by HUBERT CARDOT. The RNNs (Recurrent Neural Networks) are a general case of artificial neural networks where the connections are not feed-forward ones only. In RNNs, connections between units form directed cycles, providing an implicit internal memory. Those RNNs are adapted to problems dealing with signals evolving through time. Their internal memory gives them the ability to naturally take time into account. Valuable approximation results have been obtained for dynamical systems.
- **Jürgen Schmidhuber/Recurrent Neural Networks.**

Typically, these reviews consider RNNs that are artificial neural networks (aRNN) useful in technological applications. To complement these contributions, the present summary focuses on biological recurrent neural networks (bRNN) that are found in the **brain**. Since feedback is ubiquitous in the brain, this task, in full generality, could include most of the brain's **dynamics**. The current review divides bRNNs into those in which feedback signals occur in neurons within a single processing layer, which occurs in networks for such diverse functional roles as storing spatial patterns in short-term **memory**, winner-take-all decision making, contrast enhancement & normalization, hill climbing, **oscillations** of multiple types (**synchronous**, **traveling waves**, chaotic), storing temporal sequences of events in **working memory**, & serial learning of lists; & those in which feedback signals occur between multiple processing layers, e.g. occurs when bottom-up adaptive filters activate learned recognition categories & top-down learned expectations focus **attention** on expected patterns of critical features & thereby modulate both types of learning.

8.1.1 Types of Recurrent Neural Networks

There are at least 3 streams of bRNN research: binary, linear, & continuous-nonlinear (Grossberg, 1988):

1. **Binary.** Binary systems were inspired in part by neurophysiological observations showing that signals between many neurons are carried by all-or-none spikes. The binary stream was initiated by the classical MCCULLOCH & PITTS (1943) model of threshold **logic** system that describes how the activities, or short-term memory (STM) traces, x_i of the i th node in a network interact in discrete time according to the equation:

$$x_i(t+1) = \text{sgn} \left(\sum_j A_{ij} x_j(t) - B_j \right).$$

The McCulloch-Pitts model had an influence far beyond the field of neural networks through its influence on VON NEUMANN's development of the digital computer.

Caianiello (1961) used a binary STM equation that is influenced by activities at multiple times in the past:

$$x_i(T+\tau) = 1 \left[\sum_{j=1}^n \sum_{k=0}^{l(m)} A_{ij}^{(k)} x_j(t-k\tau) - B_i \right],$$

where $l(w) = 1$ if $w \geq 0$ & 0 if $w < 0$.

Rosenblatt (1962) used an STM equation that evolves in continuous time, whose activities can spontaneously decay, & which can generate binary signals above a nonzero threshold:

$$\frac{d}{dt} x_i = -A x_i + \sum_{j=1}^n \phi(B_j + x_j) C_{ij},$$

where $\phi(w)$ if $w \geq \theta$ & 0 if $w < \theta$. This equation was used in the classical Perceptron model.

Both Caianiello (1961) & Rosenblatt (1962) introduced equations to change the weights $A_{ij}^{(k)}$ in (2) & C_{ij} in (3) through learning. Such adaptive weights are often called *long-term memory* (LTM) traces. In both these models, interactions between STM & LTM were uncoupled in order to simplify the analysis. These LTM equations also had a digital aspect. the Caianiello (1961) LTM equations increased or decreased at constant rates until they hit finite upper or lower bounds. The Rosenblatt (1962) LTM equations were used to classify patterns into 2 distinct classes, as in the Perception Learning Theorem.

2. **Linear.** Widrow (1962) drew inspiration from the brain to introduce the gradient Adeline adaptive pattern recognition machine. Anderson (1968) initially described his intuitions about neural pattern recognition using a spatial cross-correlation function. Concepts from linear system theory were adapted to represent some aspects of neural dynamics, including solutions of simultaneous linear equations $Y = AX$ using matrix theory, & concepts about cross-correlation. Kohonen (1971) made a transition from linear algebra concepts e.g. the Moore-Penrose pseudoinverse to more biologically motivated studies that are summarized in his books (Kohonen, 1977, 1984). These ideas began with a mathematically familiar engineering framework before moving towards more biologically motivated nonlinear interactions.

3. **Continuous-Nonlinear.** Continuous-nonlinear network laws typically arose from an analysis of behavioral or neural data. Neurophysiological experiments on the lateral eye of the Limulus, or horseshoe crab, led to the award of a Nobel prize to H. K. HARTLINE. These data inspired the steady state **Ratcliff model** (Hartline & Ratcliff, 1957):

$$r_i = e_i - \sum_{j=1}^n k_{ij} [r_j - r_{ij}]^+, \quad (400)$$

where $[w]^+ := \max\{w, 0\}$. Equation (4) describes how cell activations e_i are transformed into smaller net responses r_i due to recurrent inhibitory threshold-linear signals $-k_{ij}[r_i - r_{ij}]^+$. The Hartline-Ratcliff model is thus a kind of continuous threshold-logic system. Ratcliff et al. (1963) extended this steady-state model to a dynamical model:

$$r_i(t) = e_i(t) - \sum_{j=1}^n k_{ij} \left[\frac{1}{\tau} \int_0^t e^{-\frac{t-s}{\tau}} r_j(s) ds - r_{ij} \right]^+, \quad (401)$$

which also behaves linearly above threshold. This model is a precursor of the Additive Model that is described below.

Another classical tradition arose from the analysis of how the excitable membrane of a single neuron can generate electrical spikes capable of rapidly & non-decrementally traversing the axon, or pathway, from 1 neuron's cell body to a neuron to which it is sending signals. This experimental & modeling work on the squid giant axon by Hodgkin & Huxley (1952) also led to the award of a Nobel prize. Since this work focused on individual neurons rather than neural networks, it will not be further discussed herein except to note that it provides a foundation for the Shunting Model described below.

Another source of continuous-nonlinear RNNs arose through a study of adaptive behavior in real time, which led to the derivation of neural networks that form the foundation of most current biological neural network research (Grossberg, 1967, 1968b, 1968c). These laws were discovered in 1957–58 when Grossberg, then a college Freshman, introduced the paradigm of using nonlinear systems of differential equations to model how brain mechanisms can control behavioral functions. The laws were derived from an analysis of how psychological data about human & animal learning can arise in an individual learner adapting autonomously in real time. Apart from the Rockefeller Institute student monograph Grossberg (1964), it took a decade to get them published.

4. **Additive STM equation.** The following equation is called the Additive Model because it adds the terms, possibly nonlinear, that determine the rate of change of neuronal activities, or potentials, x_i :

$$\frac{d}{dt}x_i = -A_i x_i + \sum_{j=1}^n f_j(x_j) B_{ji} z_{ji}^{(+)} - \sum_{j=1}^n g_j(x_j) C_{ji} z_{ji}^{(-)} + I_i. \quad (402)$$

Equation (6) includes a term for passive decay $-A_i x_i$, positive feedback $\sum_{j=1}^n f_j(x_j) B_{ji} z_{ji}^{(+)}$, negative feedback $-\sum_{j=1}^n g_j(x_j) C_{ji} z_{ji}^{(-)}$, & input I_i . Each feedback term includes an activity-dependent (possibly) nonlinear signal ($f_j(x_j), g_j(x_j)$); a connection, or path, strength (B_{ji}, C_{ji}), & an adaptive weight, or LTM trace ($z_{ji}^{(+)}, z_{ji}^{(-)}$). If the positive & negative feedback terms are lumped together & the connection strengths are lumped with the LTM traces, then the Additive Model may be written in the simpler form:

$$\frac{d}{dt}x_i = -A_i x_i + \sum_{j=1}^n f_j(x_j) z_{ji} + I_i. \quad (403)$$

Early applications of the Additive Model included computational analyses of **vision**, learning, recognition, **reinforcement learning**, & learning of temporal order in speech, **language**, & sensory-motor control (Grossberg, 1969b, 1969c, 1969d, 1970a, 1970b, 1971a, 1971b, 1972a, 1972b, 1974, 1975; Grossberg & Pepe, 1970, 1971). The Additive Model has continued to be a cornerstone of neural network research to the present time; e.g., in decision-making (Usher & McClelland, 2001). Physicists & engineers unfamiliar with the classical status of the Additive Model in neural networks called it the **Hopfield model** after the 1st application of this equation in Hopfield (1984). Grossberg (1988) summarizes historical factors that contributed to their unfamiliarity with the neural network literature. The Additive Model in (7) may be generalized in many ways, including the effects of delays & other factors. In the limit of infinitely many cells, an abstraction which does not exist in the brain, the discrete sum in (7) may be replaced by an integral (see **Neural fields**).

5. **Shunting STM equation.** Grossberg (1964, 1968b, 1969b) also derived an STM equation for neural networks that more closely model the shunting dynamics of individual neurons (Hodgkin, 1964). In such a shunting equation, each STM trace is bounded within an interval $[-D, B]$. Automatic gain control, instantiated by multiplicative shunting, or mass action, terms, interacts with balanced positive & negative signals & inputs to maintain the sensitivity of each STM trace within its interval (see **The Noise-Saturation Dilemma**):

$$\frac{d}{dt}x_i = -A_i x_i + (B - x_i) \left[\sum_{j=1}^n f_j(x_j) C_{ji} z_{ji}^{(+)} + I_i \right] - (D + x_i) \left[\sum_{j=1}^n g_j(x_j) E_{ji} z_{ji}^{(-)} + J_i \right]. \quad (404)$$

The Shunting Model is approximated by the Additive Model in cases where the inputs are sufficiently small that the resulting activities x_i do not come close to their saturation values $-D, B$.

The Wilson-Cowan model (Wilson & Cowan, 1972) also uses a combination of shunting & additive terms, as in (8). However, instead of using sums of sigmoid signals that are multiplied by shunting terms, as in RHS of (8), the Wilson-Cowan model uses a sigmoid of sums that is multiplied by a shunting term, as in the expression $(B - x_i)f_j(\sum_j C_{ji}x_jz_{ji}^{(+)} - x_jE_{ji}z_{ji}^{(-)} + I_i)$. This form can saturate activities when inputs or recurrent signals get large, unlike (8), as noted in Grossberg (1973).

6. **Generalized STM equation.** Equations (6) & (8) are special cases of an STM equation, introduced in Grossberg (1968c), which includes LTM & medium-term memory (MTM) terms that changes at a rate intermediate between the faster STM & the slower LTM. The laws for STM, MTM, & LTM are specialized to deal with different evolutionary pressures in neural models of different brain systems, including additional factors e.g. transmitter mobilization (Grossberg, 1969c, 1969b). This generalized STM equation is:

$$\frac{dx_i}{dt} = -Ax_i + (B - Cx_i) \left[\sum_{k=1}^n f_k(x_k)D_{ki}y_{ki}z_{ki} + I_i \right] - (E + Fx_i) \left[\sum_{k=1}^n g_k(x_k)G_{ki}Y_{ki}Z_{ki} + J_i \right]. \quad (405)$$

In the shunting model, the parameters $C \neq 0, F \neq 0$. The parameter $E = 0$ when there is “silent” **shunting inhibition**, whereas $E \neq 0$ describes the case of hyperpolarizing shunting inhibition. In the Additive Model, parameters $C = F = 0$. The excitatory interaction term $[\sum_{k=1}^n f_k(x_k)D_{ki}y_{ki}z_{ki} + I_i]$ describes an external input I_i plus the total excitatory feedback signal $[\sum_{k=1}^n f_k(x_k)D_{ki}y_{ki}z_{ki}]$ that is a sum of signals from other populations via their output signals $f_k(x_k)$. The term D_{ki} is a constant connection strength between cell populations v_k, v_i , whereas terms y_{ki} & z_{ki} describe MTM & LTM variables, resp. The inhibitory interaction term $[\sum_{k=1}^n g_k(x_k)G_{ki}Y_{ki}Z_{ki} + J_i]$ has a similar interpretation. Equation (9) assumes “fast inhibition”, i.e., inhibitory

7. MTM: Habituate Transmitter Gates & Depressing Synapses.
8. LTM: Gated steepest descent learning: Not Hebbian learning.

8.1.2 Processing & STM of Spatial Patterns

1. Transformation & short-term storage of distributed input patterns by neural networks.
2. The Noise-Saturation Dilemma.
3. A though experiment to solve the noise-saturation dilemma.
4. Automatic gain control by the off surround prevents saturation.
5. Contrast normalization & pattern processing by real-time probabilities.
6. Weber Law & shift property.
7. Physiological interpretation of shunting dynamics: The membrane equation of neurophysiology.
8. Recurrent competitive fields.
9. Winner-take-all, contrast enhancement, normalization, & quenching threshold.
10. Shunting dynamics in cortical models.
11. Decision-making in Competitive Systems: Liapunov methods.
12. Competition, decision, & consensus.
13. Adaptation level systems: Globally-consistent decision-making.
14. Cohen-Grossberg model, Liapunov function, & theorem.
15. Symmetry does not imply convergence: Synchronized oscillations.
16. Unifying horizontal, bottom-up, & top-down STM & LTM interactions.

8.1.3 Interactions of STM & LTM during Neuronal Learning

1. Unbiased spatial pattern learning by Generalized Additive RNNs.
2. Outstar learning theorem.
3. Sparse stable category learning theorem.
4. Adaptive bidirectional associative memory.
5. Adaptive resonance theory.

8.1.4 Working memory: processing & STM of temporal sequences

1. Relative activity codes temporal order in working memory.
2. Working memory design enables stable learning of list chunks.
3. LTM Invariance & Normalization rule are realized by specialized RCFs.
4. Primacy, recency, & bowed activation gradients.
5. Experimental support.
6. Stable chunk learning implies the Magical Numbers 4 & 7.
7. Equations for some Item-&-Order RNNs.

8.1.5 Serial Learning: From Command Cells to values, Decisions, & Plans

1. Avalanches.
2. Command cells & nonspecific arousal.
3. Self-organizing avalanches: Instar-outstar maps & serial learning of temporal order.
4. Context-Sensitive Self-Organizing Avalanches: What categories control temporal order?
5. Serial learning.

” – [Scholarpedia/recurrent neural networks](#)

9 Wikipedia’s

9.1 Wikipedia/diffusion model

“

” – [Wikipedia/diffusion model](#)

9.2 Wikipedia/large language model

“A *large language model* (*LLM*) is a type of ML model designed for [natural language processing](#) tasks e.g. language [generation](#). LLMs are [language models](#) with many parameters, & are trained with [self-supervised learning](#) on a vast amount of text.

The largest & most capable LLMs are [generative pretrained transformers](#) (GPTs). Modern models can be [fine-tuned](#) for specific tasks or guided by [prompt engineering](#). These models acquire [predictive power](#) regarding [syntax](#), [semantics](#), & [ontologies](#) inherent in human language corpora, but they also inherit inaccuracies & [biases](#) present in [data](#) they are trained in.

9.2.1 History

9.2.2 Dataset preprocessing

9.2.3 Training & architecture

9.2.4 Training cost

Qualifier “large” in “large language model” is inherently vague, as there is no definitive threshold for number of parameters required to qualify as “large”. As time goes on, what was previously considered “large” may evolve. [GPT-1](#) of 2018 is usually considered 1st LLM, even though it has only 0.117 billion parameters. Tendency towards larger models is visible in [list of LLMs](#).

Advances in software & hardware have reduced cost substantially since 2020, s.t. in 2023 training of a 12-billion-parameter LLM computational cost is 72300 [A100-GPU](#)-hours, while in 2020 cost of training a 1.5-billion-parameter LLM (which was 2 orders of magnitude smaller than state of art in 2020) was between \$80,000 & \$1,600,000. Since 2020, large sums were invested in increasingly large models. E.g., training of GPT-2 (i.e., a 1.5-billion-parameters model) in 2019 cost \$50,000, while training of PaLM (i.e. a 540-billion-parameters model) in 2022 cost \$8 million, & Megatron-Turing NLG 530B (in 2021) cost around \$11 million.

For Transformer-based LLM, training cost is much higher than inference cost. It costs 6 [FLOPs](#) per parameter to train on 1 token, whereas it costs 1–2 FLOPs per parameter to infer on 1 token.

9.2.5 Tool use

There are certain tasks that, in principle, cannot be solved by any LLM, at least not without use of external tools or additional software. An example of such a task is responding to user's input $354 * 139 =$, provided that LLM has not already encountered a continuation of this calculation in its training corpus. In such cases, LLM needs to resort to running program code that calculates result, which can then be included in its response. Another example is “What is time now? It is”, where a separate program interpreter would need to execute a code to get system time on computer, so that LLM can include it in its reply. This basic strategy can be sophisticated with multiple attempts of generated programs, & other sampling strategies.

Generally, in order to get an LLM to use tools, one must fine-tune it for tool-use. If number of tools is finite, then fine-tuning may be done just once. If number of tools can grow arbitrarily, as with online **API** services, then LLM can be fine-tuned to be able to read API documentation & call API correctly.

A simple form of tool use is **retrieval-augmented generation**: augmentation of an LLM with **document retrieval**. Given a query, a document retriever is called to retrieve most relevant documents. This is usually done by encoding query & documents into vectors, then finding documents with vectors (usually stored in a **vector database**) most similar to vector of query. LLM then generates an output based on both query & context included from retrieved documents.

9.2.6 Agency

9.2.7 Compression

9.2.8 Multimodality

9.2.9 Properties

9.2.10 Interpretation

9.2.11 Evaluation

9.2.12 Wider impact

” – [Wikipedia/large language model](#)

9.3 Wikipedia/recurrent neural network

“*Recurrent neural networks (RNNs)* are a class of **artificial neural network** commonly used for sequential data processing. Unlike **feedforward neural networks**, which process data in a single pass, RNNs process data across multiple time steps, making them well-adapted for modeling & processing text, speech, & **time series**.

– *Mạng nơ-ron hồi quy (RNN)* là 1 lớp mạng nơ-ron nhân tạo thường được sử dụng để xử lý dữ liệu tuần tự. Không giống như mạng nơ-ron truyền thẳng, xử lý dữ liệu trong 1 lần chạy, RNN xử lý dữ liệu qua nhiều bước thời gian, khiến chúng thích ứng tốt với việc mô hình hóa & xử lý văn bản, giọng nói, & chuỗi thời gian.

The building block of RNNs is the *recurrent unit*. This unit maintains a hidden state, essentially a form of memory, which is updated at each time step based on the current input & the previous hidden state. This feedback loop allows the network to learn from past inputs, & incorporate that knowledge into its current processing.

– Khối xây dựng của RNN là *đơn vị tái diễn*. Đơn vị này duy trì trạng thái ẩn, về cơ bản là 1 dạng bộ nhớ, được cập nhật tại mỗi bước thời gian dựa trên đầu vào hiện tại & trạng thái ẩn trước đó. Vòng phản hồi này cho phép mạng học hỏi từ các đầu vào trước đó, & kết hợp kiến thức đó vào quá trình xử lý hiện tại của nó.

Early RNNs suffered from the **vanishing gradient problem**, limiting their ability to learn long-range dependencies. This was solved by the **long short-term memory (LSTM)** variant in 1997, thus making it the standard architecture for RNN.

– Các RNN ban đầu gặp phải vấn đề độ dốc biến mất, hạn chế khả năng học các phụ thuộc tầm xa. Vấn đề này đã được giải quyết bằng biến thể bộ nhớ dài hạn ngắn hạn (LSTM) vào năm 1997, do đó trở thành kiến trúc tiêu chuẩn cho RNN.

RNNs have been applied to tasks e.g. unsegmented, connected **handwriting recognition**, **speech recognition**, **natural language processing**, & **neural machine translation**.

– RNN đã được áp dụng cho các nhiệm vụ như nhận dạng chữ viết tay không phân đoạn, có kết nối, nhận dạng giọng nói, xử lý ngôn ngữ tự nhiên và dịch máy thần kinh.

9.3.1 History

- Before modern. 1 origin of RNN was neuroscience. The word “recurrent” is used to describe loop-like structures in anatomy. In 1901, **CAJAL** observed “recurrent semicircles” in the **cerebellar cortex** formed by **parallel fiber**, **Purkinje cells**, & **granule cells**. In 1933, **LORENTE DE NÓ** discovered “recurrent, reciprocal connections” by **Golgi's method**, & proposed that excitatory loops explain certain aspects of the **vestibulo-ocular reflex**. During 1940s, multiple people proposed the existence of feedback in the brain, which was a contrast to the previous understanding of the neural system as a purely feedforward structure. **HEBB** considered “reverberating circuit” as an explanation for short-term memory. The McCulloch & Pitts paper (1943), which proposed the **McCulloch-Pitts neuron** model, considered networks that contains cycles. The current activity of such networks can be affected by activity indefinitely far in the past. They were both interested in closed loops as possible explanations for e.g. **epilepsy** & **causalgia**. **Recurrent inhibition** was proposed in 1946 as a negative feedback mechanism in motor control. Neural feedback loops were a common topic of discussion at the **Macy conferences**. Grossberg, Stephen (2013-02-22). “Recurrent Neural Networks”. Scholarpedia: An extensive review of recurrent neural network models in neuroscience.

A close-loop cross-coupled perceptron network. **FRANK ROSENBLATT** in 1960 published “close-loop cross-coupled perceptrons”, which are 3-layered **perceptron** networks whose middle layer contains recurrent connections that change by a **Hebbian learning** rule. Later, in *Principles of Neurodynamics* (1961), he described “closed-loop cross-coupled” & “back-coupled” perceptron networks, & made theoretical & experimental studies for Hebbian learning in these networks, & noted that a fully cross-coupled perceptron network is equivalent to an infinitely deep feedforward network.

Similar networks were published by **KAORU NAKANO** in 1971, **SHUN’ICHI AMARI** in 1972, & **WILLIAM A. LITTLE** in 1974, who was acknowledged by **HOPFIELD** in his 1982 paper.

Another origin of RNN was **statistical mechanics**. The **Ising model** was developed by **WILHELM LENZ** & **Ernest Ising** in the 1920s as a simple statistical mechanical model of magnets at equilibrium. **GLAUBER** in 1963 studied by Ising model evolving in time, as a process towards equilibrium (**Glauber dynamics**), adding in the component of time.

The **Sherrington-Kirkpatrick model** of spin glass, published in 1975, is the Hopfield network with random initialization. **SHERINGTON & KIRKPATRICK** found that it is highly likely for the energy function of the SK model to have many local minima. In the 1982 paper, **HOPFIELD** applied this recently developed theory to study the Hopfield network with binary activation functions. In a 1984 paper he extended this to continuous activation functions. It became a standard model for the study of neural networks through statistical mechanics.

- **Modern.** Modern RNN networks are mainly based on 2 architectures: LSTM & BRNN.

At the resurgence of neural networks in the 1980s, recurrent networks were studied again. They were sometimes called “iterated nets”. 2 early influential works were the **Jordan network** (1986) & the **Elman network** (1990), which applied RNN to study **cognitive psychology**. In 1993, a neural history compressor system solved a “Very Deep Learning” task that required > 1000 subsequent **layers** in an RNN unfolded in time.

Long short-term memory (LSTM) networks were invented by **HOCHREITER & SCHMIDHUBER** in 1995 & set accuracy records in multiple applications domains. It became the default choice for RNN architecture.

Bidirectional recurrent neural networks (BRNN) uses 2 RNN that processes the same input in opposite directions. These 2 are often combined, giving the bidirectional LSTM architecture.

Around 2006, bidirectional LSTM started to revolutionize **speech recognition**, outperforming traditional models in certain speech applications. They also improved large-vocabulary speech recognition & **text-to-speech** synthesis & was used in **Google voice search**, & dictation on **Android devices**. They broke records for improved **machine translation**, **language modeling**, & Multilingual Language Processing. Also, LSTM combined with **convolutional neural networks** (CNNs) improved **automatic image captioning**.

The idea of encoder-decoder sequence transduction had been developed in the early 2010s. The papers most commonly cited as the originators that produced **seq2seq** are 2 papers from 2014. A **seq2seq** architecture employs 2 RNN, typically LSTM, an “encoder” & a “decoder”, for sequence transduction, e.g. machine translation. They became state of the art in machine translation, & was instrumental in the development of **attention mechanism** & **Transformer**.

9.3.2 Configurations

Main article: **Layer (deep learning)**. An RNN-based model can be factored into 2 parts: configuration & architecture. Multiple RNN can be combined in data flow, & the data flow itself is the configuration. Each RNN itself may have any architecture, including LSTM, GRU, etc.

– Một mô hình dựa trên RNN có thể được chia thành hai phần: cấu hình & kiến trúc. Nhiều RNN có thể được kết hợp trong 1 luồng dữ liệu & luồng dữ liệu đó chính là cấu hình. Mỗi RNN có thể có bất kỳ kiến trúc nào, bao gồm LSTM, GRU, v.v.

- **Standard.** Compressed (left) & unfolded (right) basic recurrent neural network. RNNs come in many variants. Abstractly speaking, an RNN is a function f_θ of type $(x_t, h_t) \mapsto (y_t, h_{t+1})$, where x_t : input vector, h_t : hidden vector, y_t : output vector, θ : neural network parameters. In words, it is a neural network that maps an input x_t into an output y_t , with the hidden vector h_t playing the role of “memory”, a partial record of all previous input-output pairs. At each step, it transforms input to an output, & modifies its “memory” to help it to better perform future processing.

The illustration to the right may be misleading to many because practical neural network topologies are frequently organized in “layers” & the drawing gives that appearance. However, what appears to be **layers** are, in fact, different steps in time, “unfolded” to produce the appearance of layers.

- **Stacked RNN.** A *stacked RNN*, or *deep RNN*, is composed of multiple RNNs stacked one above the other. Abstractly, it is structured as follows

1. Layer 1 has hidden vector $h_{1,t}$, parameters θ_1 & maps $f_{\theta_1} : (x_{0,t}, h_{1,t}) \mapsto (x_{1,t}, h_{1,t+1})$.
2. Layer 2 has hidden vector $h_{2,t}$, parameters θ_2 , & maps $f_{\theta_2} : (x_{1,t}, h_{2,t}) \mapsto (x_{2,t}, h_{2,t+1})$.
3. ...
4. Layer n has hidden vector $h_{n,t}$, parameters θ_n , & maps $f_{\theta_n} : (x_{n-1,t}, h_{n,t}) \mapsto (x_{n,t}, h_{n,t+1})$.

Each layer operates as a stand-alone RNN, & each layer’s output sequence is used as the input sequence to the layer above. There is no conceptual limit to the depth of stacked RNN.

- **Bidirectional.** Main article: [Wikipedia/bidirectional recurrent neural networks](#). A *bidirectional RNN* (biRNN) is composed of 2 RNNs, one processing the input sequence in 1 direction, & another in the opposite direction. Abstractly, it is structured as follows:

- The forward RNN processes in 1 direction: $f_{\theta}(x_0, h_0) = (y_0, h_1), f_{\theta}(x_1, h_1) = (y_1, h_2), \dots$
- The backward RNN processes in the opposite direction: $f'_{\theta'}(x_N, h'_N) = (y'_N, h'_{N-1}), f'_{\theta'}(x_{N-1}, h'_{N-1}) = (y'_{N-1}, h'_{N-2}), \dots$

The 2 output sequences are then concatenated to give the total output: $((y_0, y'_0), (y_1, y'_1), \dots, (y_N, y'_N))$.

Bidirectional RNN allows the model to process a token both in the context of what came before it & what came after it. By stacking multiple bidirectional RNNs together, the model can process a token increasingly contextually. The **ELMo** model (2018) is a stacked bidirectional **LSTM** which takes character-level as inputs & produces word-level embeddings.

- **Encoder-decoder.** Main article: [seq2seq](#). A decoder without an encoder. 2 RNNs can be run front-to-back in an *encoder-decoder* configuration. The encoder RNN processes an input sequence into a sequence of hidden vectors, & the decoder RNN processes the sequence of hidden vectors to an output sequence, with an optional **attention mechanism**. This was used to construct state of the art **neural machine translators** during the 2014–2017 period. This was an instrumental step towards the development of **Transformers**. Encoder-decoder RNN without attention mechanism. Encoder-decoder RNN with attention mechanism.
- **PixelRNN.** An RNN may process data with more than 1D. PixelRNN processes 2D data, with many possible directions. E.g., the row-by-row direction processes $n \times n$ grid of vectors $x_{i,j}$ in the following order: $x_{1,1}, x_{1,2}, \dots, x_{1,n}, x_{2,1}, x_{2,2}, \dots, x_{2,n}, \dots, x_{n,n}$. The *diagonal BiLSTM* uses 2 LSTMs to process the same grid. One processes it from the top-left corner to the bottom-right, s.t. it processes $x_{i,j}$ depending on its hidden state & cell state on the top & the left side: $h_{i-1,j}, c_{i-1,j}$ & $h_{i,j-1}, c_{i,j-1}$. The other processes it from the top-right corner to the bottom-left.

9.3.3 Architectures

- **Fully recurrent.** A fully connected RNN with 4 neurons. *Fully recurrent neural networks* (FRNN) connect the outputs of all neurons to the inputs of all neurons. I.e., it is a **fully connected network**. This is the most general neural network topology, because all other topologies can be represented by setting some network topology, because all other topologies can be represented by setting some connection weights to 0 to simulate the lack of connections between those neurons.
- **Hopfield.** Main article: [Wikipedia/Hopfield network](#). The **Hopfield network** is an RNN in which all connections across layers are equally sized. It requires **stationary** inputs & is thus not a general RNN, as it does not process sequences of patterns. However, it guarantees that it will converge. If the connections are trained using **Hebbian learning**, then the Hopfield network can perform as **robust content-addressable memory**, resistant to connection alteration.
- **Elman networks & Jordan networks.** A simple Elman network where $\sigma_h = \tanh, \sigma_y = \text{Identity}$. An **Elman** network is a 3-layer network (arranged horizontally as x, y, z in the illustration) with the addition of a set of context units (u in the illustration). The middle (hidden) layer is connected to these context units fixed with a weight of 1. At each time step, the input is fed forward & a **learning rule** is applied. The fixed back-connections save a copy of the previous values of the hidden units in the context units (since they propagate over the connections before the learning rule is applied). Thus the network can maintain a sort of state, allowing it to perform tasks e.g. sequence-prediction that are beyond the power of a standard **multilayer perceptron**.

Jordan networks are similar to Elman networks. The contexts units are fed from the output layer instead of the hidden layer. The context units in a Jordan network are also called the *state layer*. They have a recurrent connection to themselves.

Elman & Jordan networks are also known as “Simple recurrent networks” (SRN).

Elman network:

$$h_t = \sigma_h(W_h x_t + U_h h_{t-1} + b_h), y_t = \sigma_y(W_y h_t + b_y). \quad (\text{Elman nw})$$

Jordan network:

$$h_t = \sigma_h(W_h x_t + U_h s_t + b_h), y_t = \sigma_y(W_y h_t + b_y), s_t = \sigma_s(W_{s,s} s_{t-1} + S_{s,y} y_{t-1} + b_s). \quad (\text{Jordan nw})$$

Variables & functions: x_t : input vector, h_t : hidden layer vector, s_t : “state” vector, y_t : output vector, W, U, b : parameter matrices & vector, σ : **activation functions**.

- **Long short-term memory.** Long short-term memory unit. Main article: [Wikipedia/long short-term memory](#). *Long short-term memory* (LSTM) is the most widely used RNN architecture. It was designed to solve the **vanishing gradient problem**. LSTM is normally augmented by recurrent gates called “forget gates”. LSTM prevents backpropagated errors from vanishing or exploding. Instead, errors can flow backward through unlimited numbers of virtual layers unfolded in space. I.e., LSTM can learn tasks that require memories of events that happened thousands or even millions of discrete time steps earlier. Problem-specific LSTM-like topologies can be evolved. LSTM works even given long delays between significant events & can handle signals that mix low & high-frequency components.

Many applications use stacks of LSTMs, for which it is called “deep LSTM”. LSTM can learn to recognize **context-sensitive languages** unlike previous models based on **hidden Markov models** (HMM) & similar concepts.

- **Gated recurrent unit.** Gated recurrent unit. Main article: [Wikipedia/gated recurrent unit](#). *Gated recurrent unit* (GRU), introduced in 2014, was designed as a simplification of LSTM. They are used in the full form & several further simplified variants. They have fewer parameters than LSTM, as they lack an output gate.

Their performance on polyphonic music modeling & speech signal modeling was found to be similar to that of long short-term memory. There does not appear to be particular performance difference between LSTM & GRU.

- **Bidirectional associative memory.** Main article: [Wikipedia/bidirectional associative memory](#). Introduced by BART KOSKO, a bidirectional associative memory (BAM) network is a variant of a Hopfield network that stores associative data as a vector. The bidirectionality comes from passing information through a matrix & its transpose. Typically, bipolar encoding is preferred to binary encoding of the associative pairs. Recently, stochastic BAM models using [Markov](#) stepping were optimized for increased network stability & relevance to real-world applications.

A BAM network has 2 layers, either of which can be driven as an input to recall an association & produce an output on the other layer.

- **Echo state.** Main article: [Wikipedia/echo state network](#). *Echo state network* (ESN) have a sparsely connected random hidden layer. The weights of output neurons are the only part of the network that can change (be trained). ESNs are good at reproducing certain [time series](#). A variant for [spiking neurons](#) is known as a [liquid state machine](#).

- **Recursive.** Main article: [Wikipedia/recursive neural network](#). A [recursive neural network](#) is created by applying the same set of weights [recursively](#) over a differentiable graph-like structure by traversing the structure in [topological order](#). Such networks are typically also trained by the reverse mode of [automatic differentiation](#). They can process [distributed representations](#) of structure, e.g. [logical terms](#). A special case of recursive neural networks is the RNN whose structure corresponds to a linear chain. Recursive neural networks have been applied to [natural language processing](#). The Recursive Neural Tensor Network uses a [tensor](#)-based composition function for all nodes in the tree.

- **Neural Turing machines.** Main articles: [Wikipedia/neural Turing machine](#) & [Wikipedia/differentiable neural computer](#). *Neural Turing machines* (NTMs) are a method of extending recurrent neural networks by coupling them to external [memory](#) resources with which they interact. The combined system is analogous to a [Turing machine](#) or [Von Neumann architecture](#) but is [differentiable](#) end-to-end, allowing it to be efficiently trained with [gradient descent](#).

Differentiable neural computers (DNCs) are an extension of Neural Turing machines, allowing for the usage of fuzzy amounts of each memory address & a record of chronology.

Neural network pushdown automata (NNPDA) are similar to NTMs, but tapes are replaced by analog stacks that are differentiable & trained. In this way, they are similar in complexity to recognizers of [context free grammars](#) (CFGs).

Recurrent neural networks are [Turing complete](#) & can run arbitrary programs to process arbitrary sequences of inputs.

9.3.4 Training

- **Teacher forcing.** Encoder-decoder RNN without attention mechanism. Teacher forcing is shown in red. An RNN can be trained into a conditionally [generative model](#) of sequences, aka *autoregression*.

Concretely, let us consider the problem of machine translation, i.e., given a sequence (x_1, x_2, \dots, x_n) of English words, the model is to produce a sequence (y_1, \dots, y_m) of French words. It is to be solved by a [seq2seq](#) model.

Now, during training, the encoder half of the model would 1st ingest (x_1, x_2, \dots, x_n) , then the decoder half would start generating a sequence $(\hat{y}_1, \hat{y}_2, \dots, \hat{y}_l)$. The problem is that if the model makes a mistake early on, say at \hat{y}_2 , then subsequent tokens are likely to also be mistakes. This makes it inefficient for the model to obtain a learning signal, since the model would mostly learn to shift \hat{y}_2 towards y_2 , but not the others.

Teacher forcing makes it so that the decoder uses the correct output sequence for generating the next entry in the sequence. So e.g., it would see (y_1, \dots, y_k) in order to generate \hat{y}_{k+1} .

- **Gradient descent.** Main articles: [Wikipedia/gradient descent](#) & [Wikipedia/vanishing gradient problem](#). Gradient descent is a 1st-order iterative optimization algorithm for finding the minimum of a function. In neural networks, it can be used to minimize the error term by changing each weight in proportion to the derivative of the error w.r.t. that weight, provided the nonlinear [activation functions](#) are [differentiable](#).

The standard method for training RNN by gradient descent is the “[backpropagation through time](#)” (BPTT) algorithm, which is a special case of the general algorithm of [backpropagation](#). A more computationally expensive online variant is called “Real-Time Recurrent Learning” or RTRL, which is an instance of [automatic differentiation](#) in the forward accumulation mode with stacked tangent vectors. Unlike BPTT, this algorithm is local in time but not local in space.

In this context, local in space means that a unit’s weight vector can be updated using only information stored in the connected units & the unit itself s.t. update complexity of a single unit is linear in the dimensionality of the weight vector. Local in time means that the updates take place continually (on-line) & depend only on the most recent time step rather than on multiple time steps within a given time horizon as in BPTT. Biological neural networks appear to be local w.r.t. both time & space.

For recursively computing the partial derivatives, RTRL has a time-complexity of $O(\text{number of hidden} \cdot \text{number of weights})$ per time step for computing the [Jacobian matrices](#), while BPTT only takes $O(\text{number of weights})$ per time step, at the cost

of storing all forward activations within the given time horizon. An online hybrid between BPTT & RTRL with intermediate complexity exists, along with variants for continuous time.

A major problem with gradient descent for standard RNN architectures is that **error gradients vanish** exponentially quickly with the size of the time lag between important events. LSTM combined with a BPTT/RTRL hybrid learning method attempts to overcome these problems. This problem is also solved in the independently recurrent neural network (IndRNN) by reducing the context of a neuron to its own past state & the cross-neuron information can then be explored in the following layers. Memories of different ranges including long-term memory can be learned without the gradient vanishing & exploding problem.

The on-line algorithm called causal recursive backpropagation (CRBP), implements & combines BPTT & RTRL paradigms for locally recurrent networks. It works with the most general locally recurrent networks. The CRBP algorithm can minimize the global error term. This fact improves the stability of the algorithm, providing a unifying view of gradient calculation techniques for recurrent networks with local feedback.

1 approach to gradient information computation in RNNs with arbitrary architectures is based on signal-flow graphs diagrammatic derivation. It uses the BPTT batch algorithm, based on LEE's theorem for network sensitivity calculations. It was proposed by WAN & BEAUFAYS, while its fast online version was proposed by CAMPOLUCCI, UNCINI, & PIAZZA.

- **Connectionist temporal classification.** The **connectionist temporal classification** (CTC) is a specialized loss function for training RNNs for sequence modeling problems where the timing is variable.
- **Global optimization methods.** Training the weights in a neural network can be modeled as a nonlinear **global optimization** problem. A target function can be formed to evaluate the fitness or error of a particular weight vector as follows: 1st, the weights in the network are set according to the weight vector. Next, the network is evaluated against the training sequence. Typically, the sum-squared difference between the predictions & the target values specified in the training sequence is used to represent the error of the current weight vector. Arbitrary global optimization techniques may then be used to minimize this target function.

The most common global optimization method for training RNNs is **genetic algorithms**, especially in unstructured networks.

Initially, the genetic algorithm is encoded with the neural network weights in a predefined manner where 1 gene in the **chromosome** represents 1 weight link. The whole network is represented as a single chromosome. The fitness function is evaluated as follows:

- Each weight encoded in the chromosome is assigned to the respective weight link of the network.
- The training set is presented to the network which propagates the input signals forward.
- The mean-squared error is returned to the fitness function.
- This function drives the genetic selection process.

Many chromosomes make up the population; therefore, many different neural networks are evolved until a stopping criterion is satisfied. A common stopping scheme is:

- When the neural network has learned a certain percentage of the training data or
- When the minimum value of the mean-squared-error is satisfied or
- When the maximum number of training generations has been reached.

The fitness function evaluates the stopping criterion as it receives the mean-squared error reciprocal from each network during training. Therefore, the goal of the genetic algorithm is to maximize the fitness function, reducing the mean-squared error.

Other global (&/or evolutionary) optimization techniques may be used to seek a good set of weights, e.g. **simulated annealing** or **particle swarm optimization**.

9.3.5 Other architectures

- **Independently RNN (IndRNN).** The independently recurrent neural network (IndRNN) addresses the gradient vanishing & exploding problems in the traditional fully connected RNN. Each neuron in 1 layer only receives its own past state as context information (instead of full connectivity to all other neurons in this layer) & thus neurons are independent of each other's history. The gradient backpropagation can be regulated to avoid gradient vanishing & exploding in order to keep long or short-term memory. The cross-neuron information is explored in the next layers. IndRNN can be robustly trained with non-saturated nonlinear functions e.g. ReLU. Deep networks can be trained using skip connections.
- **Neural history compressor.** The neural history compressor is an unsupervised stack of RNNs. At the input level, it learns to predict its next input from the previous inputs. Only unpredictable inputs of some RNN in the hierarchy become inputs to the next higher level RNN, which therefore recomputes its internal state only rarely. Each higher level RNN thus studies a compressed representation of the information in the RNN below. This is done s.t. the input sequence can be precisely reconstructed from the representation at the highest level.

The system effectively minimizes the description length or the negative **logarithm** of the probability of the data. Given a lot of learnable predictability in the incoming data sequence, the highest level RNN can use supervised learning to easily classify even deep sequences with long intervals between important events.

It is possible to distill the RNN hierarchy into 2 RNNs: the “conscious” chunker (higher level) & the “subconscious” automatizer (lower level). Once the chunker has learned to predict & compress inputs that are unpredictable by the automatizer, then the automatizer can be forced in the next learning phase to predict or imitate through additional unit the hidden units of the more slowly changing chunker. This makes it easy for the automatizer to learn appropriate, rarely changing memories across long intervals. In turn, this helps the automatizer to make many of its once unpredictable inputs predictable, s.t. the chunker can focus on the remaining unpredictable events.

A **generative model** partially overcame the **vanishing gradient problem** of **automatic differentiation** or **backpropagation** in neural networks in 1992. In 1993, such a system solved a “Very Deep Learning” task that required > 1000 subsequent layers in an RNN unfolded in time.

- **2nd order RNNs.** 2nd-order RNNs use higher order weights w_{ijk} instead of the standard w_{ij} weights, & states can be a product. This allows a direct mapping to a **finite-state machine** both in training, stability, & representation. Long short-term memory is an example of this but has no such formal mappings or proof of stability.

- **Hierarchical recurrent neural network.** Hierarchical recurrent neural networks (HRNN) connect their neurons in various ways to decompose hierarchical behavior into useful subprograms. Such hierarchical structures of cognition are present in theories of memory presented by philosopher **Henri Bergson**, whose philosophical views have inspired hierarchical models.

Hierarchical recurrent neural networks are useful in **forecasting**, helping to predict disaggregated inflation components of the **consumer price index** (CPI). The HRNN model leverages information from higher levels in the CPI hierarchy to enhance lower-level predictions. Evaluation of a substantial dataset from the US CPI-U index demonstrates the superior performance of the HRNN model compared to various established **inflation** prediction methods.

- **Recurrent multiplayer perceptron network.** Generally, a recurrent multiplayer perceptron network (RMLP network) consists of cascaded subnetworks, each containing multiple layers of nodes. Each subnetwork is feed-forward except for the last layer, which can have feedback connections. Each of these subnets is connected only by feed-forward connections.

- **Multiple timescales model.** A multiple timescales recurrent neural network (MTRNN) is a neural-based computational model that can simulate the functional hierarchy of the brain through self-organization depending on the spatial connection between neurons & on distinct types of neuron activities, each with distinct time properties. With such varied neuronal activities, continuous sequences of any set of behaviors are segmented into reusable primitives, which in turn are flexibly integrated into diverse sequential behaviors. The biological approval of such a type of hierarchy was discussed in the **memory-prediction** theory of brain function by **HAWKINS** in his book *On Intelligence*. Such a hierarchy also agrees with theories of memory posited by philosopher **HENRI BERGSON**, which have been incorporated into an MTRNN model.

- **Memristive networks.** GREG SNIDER of **HP Labs** describes a system of cortical computing with memristive nanodevices. The **memristors** (memory resistors) are implemented by thin film materials in which the resistance is electrically tuned via the transport of ions or oxygen vacancies within the film. **DARPA’s SyNAPSE project** has funded IBM Research & HP Labs, in collaboration with the Boston University Department of Cognitive & Neural Systems (CNS), to develop neuromorphic architectures that may be based on memristive systems. Memristive networks are a particular type of **physical neural network** that have very similar properties to (Little-)Hopfield networks, as they have continuous dynamics, a limited memory capacity & natural relaxation via the minimization of a function which is asymptotic to the **Ising model**. In this sense, the dynamics of a memristive circuit have the advantage compared to a Resistor-Capacitor network to have a more interesting nonlinear behavior. From this point of view, engineering analog memristive networks account for a peculiar type of **neuromorphic engineering** in which the device behavior depends on the circuit wiring or topology. The evolution of these networks can be studied analytically using variations of the Caravelli-Traversa-Di **Ventra** equations.

- **Continuous-time.** A continuous-time recurrent neural network (CTRNN) uses a system of ODEs to model the effects on a neuron of the incoming inputs. They are typically analyzed by **dynamical systems theory**. Many RNN models in neuroscience are continuous-time.

For a neuron i in the network with activation y_i , the rate of change of activation is given by $\tau_i \dot{y}_i = -y_i + \sum_{j=1}^n w_{ji} \sigma(y_j - \Theta_j) + I_i(t)$ where τ_i : time constant of **postsynaptic** node, y_i : activation of postsynaptic node, \dot{y}_i : rate of change of activation of postsynaptic node, w_{ji} : weight of connection from pre to postsynaptic node, $\sigma(x)$: **sigmoid** of x e.g. $\sigma(x) = \frac{1}{1+e^{-x}}$, y_j : activation of presynaptic node, Φ_j : bias of presynaptic node, $I_i(t)$: input (if any) to node. CTRNNs have been applied to **evolutionary robotics** where they have been used to address vision, co-operation, & minimal cognitive behavior.

Note that, by the **Shannon sampling theorem**, discrete-time recurrent neural networks can be viewed as continuous-time recurrent neural networks where the differential equations have transformed into equivalent **difference equations**. This transformation can be thought of as occurring after the post-synaptic node activation functions $y_i(t)$ have been low-pass filtered but prior to sampling.

They are in fact **recursive neural networks** with a particular structure: that of a linear chain. Whereas recursive neural networks operate on any hierarchical structure, combining child representations into parent representations, recurrent neural networks operate on the linear progression of time, combining the previous time step & a hidden representation into the representation for the current time step.

From a time-series perspective, RNNs can appear as nonlinear versions of **finite impulse respons** & **infinite impulse response** filters & also as a **nonlinear autoregressive exogenous model** (NARX). RNN has infinite impulse response whereas **convolutional**

neural networks have **finite impulse** response. Both classes of networks exhibit temporal **dynamic behavior**. A finite impulse recurrent network is a **directed acyclic graph** that can be unrolled & replaced with a strictly feedforward neural network, while an infinite impulse recurrent network is a **directed cyclic graph** that cannot be unrolled.

The effect of memory-based learning for the recognition of sequences can also be implemented by a more biological-based model which uses the silencing mechanism exhibited in neurons with a relatively high frequency spiking activity.

Additional stored states & the storage under direct control by the network can be added to both **infinite-impulse** & **finite-impulse** networks. Another network or graph can also replace the storage if that incorporates time delays or has feedback loops. Such controlled states are referred to as gated states or gated memory & are part of **long short-term memory** networks (LSTMs) & **gated recurrent units**. This is also called Feedback Neural Network (FNN).

9.3.6 Libraries

Modern libraries provide runtime-optimized implementations of the above functionality or allow to speed up the slow loop by **just-in-time compilation**.

- **Apache Singa**
- **Caffe**: Created by the Berkeley Vision & Learning Center (BVLC). It supports both CPU & GPU. Developed in C++, & has Python & MATLAB wrappers.
- **Chainer**: Fully in Python, production support for CPU, GPU, distributed training.
- **Deeplearning4j**: Deep learning in Java & Scala on multi-GPU-enabled Spark.
- **Flux**: includes interfaces for RNNs, including GRUs & LSTMs, written in Julia.
- **Keras**: High-level API, providing a wrapper to many other deep learning libraries.
- **Microsoft Cognitive Toolkit**
- **MXNet**: an open-source deep learning framework used to train & deploy deep neural networks.
- **PyTorch**: Tensors & Dynamic neural networks in Python with GPU acceleration.
- **TensorFlow**: Apache 2.0-licensed Theano-like library with support for CPU, GPU & Google's proprietary **TPU**, mobile
- **Theano**: A deep-learning library for Python with an API largely compatible with the **NumPy** library.
- **Torch**: A scientific computing framework with support for machine learning algorithms, written in C & Lua.

9.3.7 Applications

Applications of recurrent neural networks include:

- **Machine translation**
- **Robot control**
- **Time series prediction**
- **Speech recognition**
- **Speech synthesis**
- **Brain-computer interfaces**
- Time series anomaly detection
- **Text-to-Video model**
- Rhythm learning
- Music composition
- Grammar learning
- **Handwriting recognition**
- Human action recognition
- Protein homology detection
- Predicting subcellular localization of proteins

- Several prediction tasks in the area of business process management
- Prediction in medical care pathways
- Predictions of fusion plasma disruptions in reactors (Fusion Recurrent Neural Network (FRNN) code)– [Wikipedia/recurrent neural network](#)

9.4 Wikipedia/supervised learning

“In supervised learning, training data is labeled with expected answers, while in **unsupervised learning**, model identifies patterns or structures in unlabeled data. In ML, *supervised learning (SL)* is a paradigm where a **statistical model** is trained using input objects (e.g., a vector of predictor variables) & desired output values (also known as a *supervisory signal*), which are often human-made labels. Training process builds a function that maps new data to expected output values. An optimal scenario will allow for algorithm to accurately determine output values for unseen instances. This requires learning algorithm to **generalize** from training data to unseen situations in a “reasonable” way (see **inductive bias**). This statistical quality of an algorithm is measured via a **generalization error**.

9.4.1 Steps to follow

To solve a given problem of supervised learning, following steps must be performed:

1. Determine type of training samples. Before doing anything else, user should decide what kind of data is to be used as a **training set**. In case of **handwriting analysis**, e.g., this might be a single handwritten character, an entire handwritten word, an entire sentence of handwriting, or a full paragraph of handwriting.
2. Gather a training set. Training set needs to be representative of real-world use of function. Thus, a set of input objects is gathered together with corresponding outputs, either from **human experts** or from measurements.
3. Determine input **feature** representation of learned function. Accuracy of learned function depends strongly on how input object is represented. Typically, input object is transformed into a **feature vector**, which contains a number of features that are descriptive of object. Number of features should not be too large, because of **curse of dimensionality**; but should contain enough information to accurately predict output.
4. Determine structure of learned function & corresponding learning algorithm. E.g., one may choose to use **support-vector machines** or **decision trees**.
5. Complete design. Run learning algorithm on gathered training set. Some supervised learning algorithms require use to determine certain **control parameters**. These parameters may be adjusted by optimizing performance on a subset (called a **validation set**) of training set, or via **cross-validation**.
6. Evaluate accuracy of learned function. After parameter adjustment & learning, performance of resulting function should be measured on a **test case** that is separate from training set.

9.4.2 Algorithm choice

A wide range of supervised learning algorithms are available, each with its strengths & weaknesses. There is no single learning algorithm that works best on all supervised learning problems (see **no free lunch theorem**).

There are 4 major issues to consider in supervised learning:

- Bias-variance tradeoff.
- Function complexity & amount of training data.
- Dimensionality of input space.
- Noise in output values.
- Other factors to consider.
- Algorithms.

9.4.3 How supervised learning algorithms work

- Empirical risk minimization.
- Structural risk minimization.

9.4.4 Generative training

9.4.5 Generalizations

9.4.6 Approaches & algorithms

9.4.7 Applications

9.4.8 General issues

” – [Wikipedia/supervised learning](#)

9.5 Wikipedia/torch (ML)

“Torch is an open-source ML library, a [scientific computing](#) framework, & a [scripting language](#) based on [Lua](#). It provides [LuaJIT](#) interfaces to DL algorithms implemented in C. It was created by [Idiap Research Institute](#) at [EPFL](#). Torch development moved in 2017 to [PyTorch](#), a port of library to Python.

9.5.1 torch

Core package of Torch is `torch`. It provides a flexible N -dimensional array or [Tensor](#), which supports basic routines for indexing, slicing, transposing, type-casting, resizing, sharing storage & cloning. This object is used by most other packages & thus forms core object of library. Tensor also supports mathematical operations like `max`, `min`, `sum`, statistical distributions like uniform, normal, & multinomial, & BLAS (Basic Linear Algebra Subprograms) operations like [dot product](#), [matrix-vector multiplication](#), [matrix-matrix multiplication](#) & [matrix product](#).

Following exemplifies using torch via its [REPL](#) interpreter:

```
> a = torch.randn(3, 4)

> =a
-0.2381 -0.3401 -1.7844 -0.2615
0.1411  1.6249  0.1708  0.8299
-1.0434  2.2291  1.0525  0.8465
[torch.DoubleTensor of dimension 3x4]

> a[1][2]
-0.34010116549482

> a:narrow(1,1,2)
-0.2381 -0.3401 -1.7844 -0.2615
0.1411  1.6249  0.1708  0.8299
[torch.DoubleTensor of dimension 2x4]

> a:index(1, torch.LongTensor{1,2})
-0.2381 -0.3401 -1.7844 -0.2615
0.1411  1.6249  0.1708  0.8299
[torch.DoubleTensor of dimension 2x4]

> a:min()
-1.7844365427828
```

`torch` package also simplifies [object-oriented programming](#) & [serialization](#) by providing various convenience functions which are used throughout its packages. `torch.class(classname, parentclass)` function can be used to create [object factories](#) ([classes](#)). When [constructor](#) is called, `torch` initializes & sets a [Lua table](#) with user-defined [metatable](#), which makes table an [object](#).

Objects created with torch factory can also be serialized, as long as they do not contain references to objects that cannot be serialized, e.g. Lua [coroutines](#), & Lua *userdata*. However, *userdata* can be serialized if it is wrapped by a table (or metatable) that provides `read()`, `write()` methods.

9.5.2 nn

`nn` package is used for building [neural networks](#). It is divided into modular objects that share a common `Module` interface. Modules have a `forward()`, `backward()` method that allow them to [feedforward](#) & [backpropagation](#), resp. Modules can be joined using module [composites](#), like `Sequential`, `Parallel`, `Concat` to create complex task-tailored graphs. Simpler modules like `Linear`, `Tanh`, `Max` make up basic component modules. This modular interface provides 1st-order [automatic gradient differentiation](#). What follows is an example use-case for building a [multilayer perceptron](#) using Modules:

```
> mlp = nn.Sequential()
> mlp:add(nn.Linear(10, 25)) -- 10 input, 25 hidden units
```



```
> mlp:add(nn.Tanh()) -- some hyperbolic tangent transfer function
> mlp:add(nn.Linear(25, 1)) -- 1 output
> =mlp:forward(torch.randn(10))
-0.1815
[torch.Tensor of dimension 1]
```

Loss functions are implemented as sub-classes of **Criterion**, which has a similar interface to **Module**. It also has **forward()**, **backward()** methods for computing loss & backpropagating gradients, resp. Criteria are helpful to train neural network on classical tasks. Common criteria are **mean squared error** criterion implemented in *MSECriterion* & **cross-entropy** criterion implemented in *ClassNLLCriterion*. What follows is an example of a Lua function that can be iteratively called to train an **mlp** Module on input Tensor **x**, target Tensor **y** with a scalar **learningRate**:

```
function gradUpdate(mlp, x, y, learningRate)
    local criterion = nn.ClassNLLCriterion()
    local pred = mlp:forward(x)
    local err = criterion:forward(pred, y);
    mlp:zeroGradParameters();
    local t = criterion:backward(pred, y);
    mlp:backward(x, t);
    mlp:updateParameters(learningRate);
end
```

It also has **StochasticGradient** class for training a neural network using **stochastic gradient descent**, although **optim** package provides much more options in this respect, like momentum & weight decay **regularization**.

9.5.3 Other packages

Many packages other than above official packages are used with Torch. These are listed in torch cheatsheet. These extra packages provide a wide range of utilities e.g. parallelism, asynchronous input/output, image processing, & so on. They can be installed with **LuaRocks**, Lua package manager which is also included with Torch distribution.

9.5.4 Applications

Torch is used by Facebook AI Research Group, **IBM**, **Yandex**, & **Idiap Research Institute**. Torch has been extended for use on **Android** & **iOS**. It has been used to build hardware implementations for data flows like those found in neural networks.

Facebook has released a set of extension modules as open source software.” – [Wikipedia/torch \(ML\)](#)

9.6 Wikipedia/types of artificial neural networks

There are many *types of artificial neural networks (ANN)*.

Artificial neural networks are **computational models** inspired by **biological neural networks**, & are used to **approximate** functions that are generally unknown. Particularly, they are inspired by behavior of **neurons** & electrical signals they convey between input (e.g. from eyes or nerve endings in hand), processing, & output from brain (e.g. reacting to light, touch, or heat). The way neurons semantically communicate is an area of ongoing research. Most artificial neural networks bear only some resemblance to their more complex biological counterparts, but are very effective at their intended tasks (e.g. classification or segmentation).

Some artificial neural networks are **adaptive systems** & are used e.g. to **model population** & environments, which constantly change.

Neural networks can be hardware- (neurons are represented by physical components) or **software-based** (computer models), & can use a variety of topologies & learning algorithms.

9.6.1 Feedforward

Main article: [Wikipedia/feedforward neural network](#). Feedforward neural network: 1st & simplest type. In this network information moves only from input layer directly through any hidden layers to output layer without cycles/loops. Feedforward networks can be constructed with various types of units, e.g. binary **McCulloch–Pitts neurons**, simplest of which is **perceptron**. Continuous neurons, frequently with sigmoidal **activation**, are used in context of **backpropagation**.

- Group method of data handling. Main article: [Wikipedia/group method of data handling](#). Group Method of Data Handling (GMDH) features fully automatic structural & parametric model optimization. Node activation functions are **Kolmogorov–Gabor polynomials** that permit additions & multiplications. It uses a deep multilayer perceptron with 8 layers. It is a **supervised learning** network that grows layer by layer, where each layer is trained by **regression analysis**. Useless items are detected using a **validation set**, & pruned through **regularization**. Size & depth of resulting network depends on task.
- Autoencoder. Main article: [Wikipedia/autoencoder](#). An autoencoder, autoassociator or Diabolo network: is similar to **multilayer perceptron** (MLP) – with an input layer, an output layer & 1 or more hidden layers connecting them. However, output layer has

same number of units as input layer. Its purpose is to reconstruct its own inputs (instead of emitting a target value. Therefore, autoencoders are **unsupervised learning** models. An autoencoder is used for **unsupervised learning of efficient coding**, typically for purpose of **dimensionality reduction** & for learning **generative models** of data.

- Probabilistic. Main article: **Wikipedia/probabilistic neural network**. A probabilistic neural network (PNN) is a 4-layer feedforward neural network. Layers are Input, hidden pattern/summation, & output. In PNN algorithm, parent probability distribution function (PDF) of each class is approximated by a **Parzen window** & a non-parametric function. Then, using PDF of each class, class probability of a new input is estimated & Bayes' rule is employed to allocate it to class with highest posterior probability. It was derived from **Bayesian network** & a statistical algorithm called **Kernel Fisher discriminant analysis**. It is used for classification & pattern recognition.
- Time delay. Main article: **Wikipedia/time delay neural network**. A time delay neural network (TDNN) is a feedforward architecture for sequential data that recognizes **features** independent of sequence position. In order to achieve time-shift invariance, delays are added to input so that multiple data points (points in time) are analyzed together.

It usually forms part of a larger pattern recognition system. It has been implemented using a perceptron network whose connection weights were trained with back propagation (supervised learning).

- Convolutional. Main article: **Wikipedia/convolutional neural network**. A convolutional neural network (CNN, or ConvNet or shift invariant or space invariant) is a class of deep network, composed of 1 or more **convolutional** layers with fully connected layers (matching those in typical ANNs) on top. It uses tied weights & **pooling layers**. In particular, max-pooling. It is often structured via Fukushima's convolutional architecture. They are variations of **multilayer perceptrons** that use minimal **preprocessing**. This architecture allows CNNs to take advantage of 2d structure of input data.

Its unit connectivity pattern is inspired by organization of **visual cortex**. Units respond to stimuli in a restricted region of space known as **receptive field**. Receptive fields partially overlap, overcovering entire **visual field**. Unit response can be approximated mathematically by a convolution operation.

CNNs are suitable for processing visual & other 2D data. They have shown superior results in both image & speech applications. They can be trained with standard backpropagation. CNNs are easier to train than other regular, deep, feed-forward neural networks & have many fewer parameters to estimate.

Capsule Neural Networks (CapsNet) add structures called *capsules* to a CNN & reuse output from several capsules to form more stable (w.r.t. various perturbations) representations.

Examples of applications in computer vision include **DeepDream** & **robot navigation**. They have wide applications in **image & video recognition**, **recommender systems**, & **natural language processing**.

- Deep stacking network. A deep stacking network (DSN) (deep convex network) is based on a hierarchy of blocks of simplified neural network modules. It was introduced in 2011 by Deng & Yu. It formulates learning as a **convex optimization problem** with a **closed-form solution**, emphasizing mechanism's similarity to **stacked generalization**. Each DSN block is a simple module that is easy to train by itself in a **supervised** fashion without backpropagation for entire blocks.

Each block consists of a simplified multi-layer perceptron (MLP) with a single hidden layer. Hidden layer **h** has logistic **sigmoidal units**, & output layer has linear units. Connections between these layers are represented by weight matrix U ; input-to-hidden-layer connections have weight matrix W . Target vectors **t** form columns of matrix T , & input data vectors **x** form columns of matrix X . Matrix of hidden units is $H = \sigma(W^T X)$. Modules are trained in order, so lower-layer weights W are known at each stage. Function performs element-wise **logistic sigmoid** operation. Each block estimates same final label **class** **y**, & its estimate is concatenated with original input X to form expanded input for next block. Thus, input to 1st block contains original data only, while downstream blocks' input adds output of preceding blocks. Then learning upper-layer weight matrix U given other weights in network can be formulated as a convex optimization problem $\min_{U^T} f = \|U^T H - T\|_F^2$, which has a closed-form solution.

Unlike other deep architectures, e.g. **DBNs**, goal: not to discover transformed **feature** representation. Structure of hierarchy of this kind of architecture makes parallel learning straightforward, as a batch-mode optimization problem. In purely **discriminative tasks**, DSNs outperform conventional DBNs.

- Tensor deep stacking networks. This architecture is a DSN extension. It offers 2 important improvements: it uses higher-order information from **covariance** statistics, & it transforms **non-convex problem** of a lower-layer to a convex sub-problem of an upper-layer. TDSNs use covariance statistics in a **bilinear mapping** from each of 2 distinct sets of hidden units in same layer to prediction, via a 3rd-order **tensor**.

While parallelization & scalability are not considered seriously in conventional DNNs, all learning for DSNs & TDSNs is done in batch mode, to allow parallelization. Parallelization allows scaling design to larger (deeper) architectures & data sets.

Basic architecture is suitable for diverse tasks e.g. **classification** & **regression**.

9.6.2 Regulatory feedback

Regulatory feedback networks started as a model to explain brain phenomena found during recognition including network-wide **bursting** & **difficulty with similarity** found universally in sensory recognition. A mechanism to perform optimization during

recognition is created using inhibitory feedback connections back to same inputs that activate them. This reduces requirements during learning & allows learning & updating to be easier while still being able to perform complex recognition.

A regulatory feedback network makes inferences using **negative feedback**. Feedback is used to find optimal activation of units. It is most similar to a **non-parametric method** but is different from **K-nearest neighbor** in that it mathematically emulates feedforward networks.

9.6.3 Radial basis function

Main article: **Wikipedia/radial basis function network**. Radial basis functions are functions that have a distance criterion w.r.t. a center. Radial basis functions have been applied as a replacement for sigmoidal hidden layer transfer characteristic in multi-layer perceptrons. RBF networks have 2 layers: In the 1st, input is mapped onto each RBF in 'hidden' layer. RBF chosen is usually a Gaussian. In regression problems output layer is a linear combination of hidden layer values representing mean predicted output. This interpretation of this output layer value is the same as a **regression model** in statistics. In classification problems output layer is typically a **sigmoid function** of a linear combination of hidden layer values, representing a posterior probability. Performance in both cases is often improved by **shrinkage** techniques, known as **ridge regression** in classical statistics. This corresponds to a prior belief in small parameter values (& therefore smooth output functions) in a **Bayesian** framework.

RBF networks have advantage of avoiding local minima in same way as multi-layer perceptrons. This is because only parameters that are adjusted in learning process are linear mapping from hidden layer to output layer. Linearity ensures: error surface is quadratic & therefore has a single easily found minimum. In regression problems this can be found in 1 matrix operation. In classification problems fixed nonlinearity introduced by sigmoid output function is most efficiently dealt with using **iteratively re-weighted least squares**.

RBF networks have disadvantage of requiring good coverage of input space by radial basis functions. RBF centers are determined with reference to distribution of input data, but without reference to prediction task. As a result, representational resources may be wasted on areas of input space that are irrelevant to task. A common solution is to associate each data point with its own center, although this can expand linear system to be solved in final layer & requires shrinkage techniques to avoid **overfitting**.

Associating each input datum with an RBF leads naturally to kernel methods e.g. **support vector machines** (SVM) & Gaussian processes (RBF is **kernel function**). All 3 approaches use a nonlinear kernel function to project input data into a space where learning problem can be solved using a linear model. Like Gaussian processes, & unlike SVMs, RBF networks are typically trained in a maximum likelihood framework by maximizing probability (minimizing error). SVMs avoid overfitting by maximizing instead a **margin**. SVMs outperform RBF networks in most classification applications. In regression applications they can be competitive when dimensionality of input space is relatively small.

- **How RBF networks work.** RBF neural networks are conceptually similar to **K-nearest neighbor** (k-NN) models. Basic idea: similar inputs produce similar outputs.

Assume: each case in a training set has 2 predictor variables x, y , & target variable has 2 categories, positive & negative. Given a new case with predictor values $x = 6, y = 5.1$, how is target variable computed?

Nearest neighbor classification performed for this example depends on how many neighboring points are considered. If 1-NN is used & closest point is negative, then new point should be classified as negative. Alternatively, if 9-NN classification is used & closest 9 points are considered, then effect of surrounding 8 positive points may outweigh closest 9th (negative) point.

An RBF network positions neurons in space described by predictor variables (x, y in this example). This space has as many dimensions as predictor variables. Euclidean distance is computed from new point to center of each neuron, & a radial basis function (RBF, also called a *kernel function*) is applied to distance to compute weight (influence) for each neuron. Radial basis function is so named because radius distance is argument to function. $\text{Weight} = \text{RBF}(\text{distance})$.

- **Radial basis function.** Value for new point is found by summing output values of RBF functions multiplied by weights computed for each neuron.

Radial basis function for a neuron has a center & a radius (also called a *spread*). Radius may be different for each neuron, &, in RBF networks generated by DTREG, radius may be different in each dimension.

With larger spread, neurons at a distance from a point have a greater influence.

- **Architecture.** RBF networks have 3 layers:

- * **Input layer.** 1 neuron appears in input layer for each predictor variable. In case of **categorical variables**, $N - 1$ neurons are used where N : number of categories. Input neurons standardizes value ranges by subtracting **median** & dividing by **interquartile** range. Input neurons then feed values to each of neurons in hidden layer.

- * **Hidden layer.** This layer has a variable number of neurons (determined by training process). Each neuron consists of a radial basis function centered on a point with as many dimensions as predictor variables. Spread (radius) of RBF function may be different for each dimension. Centers & spreads are determined by training. When presented with x vector of input values from input layer, a hidden neuron computes Euclidean distance of test case from neuron's center point & then applies RBF kernel function to this distance using spread values. Resulting value is passed to summation layer.

- * **Summation layer.** Value coming out of a neuron in hidden layer is multiplied by a weight associated with neuron & adds to weighted values of other neurons. This sum becomes output. For classification problems, 1 output is produced (with a separate set of weights & summation unit) for each target category. Value output for a category is probability that the case being evaluated has that category.

- **Training.** Following parameters are determined by training process:
 - * Number of neurons in hidden layer
 - * Coordinates of center of each hidden-layer RBF function
 - * Radius (spread) of each RBF function in each dimension
 - * Weights applied to RBF function outputs as they pass to summation layer

Various methods have been used to train RBF networks. 1 approach 1st uses **K-means clustering** to find cluster centers which are then used as centers for RBF functions. However, K-means clustering is computationally intensive & it often does not generate optimal number of centers. Another approach is to use a random subset of training points as centers.

DTREG uses a training algorithm that uses an evolutionary approach to determine optimal center points & spreads for each neuron. It determines when to stop adding neurons to network by monitoring estimated leave-1-out (LOO) error & terminating when LOO error begins to increase because of overfitting.

Computation of optimal weights between neurons in hidden layer & summation layer is done using ridge regression. An iterative procedure computes optimal regularization Lambda parameter that minimizes generalized cross-validation (GCV) error.

- **General regression neural network.** Main article: **Wikipedia/General regression neural network**. A GRNN is an associative memory neural network that is similar to **probabilistic neural network** but it is used for regression & approximation rather than classification.

9.6.4 Deep belief network

Main article: **Wikipedia/deep belief network**. A **restricted Boltzmann machine** (RBM) with fully connected visible & hidden units. Note there are no hidden-hidden or visible-visible connections. A deep belief network (DBN) is a probabilistic, **generative model** made up of multiple hidden layers. It can be considered a **composition** of simple learning modules.

A DBN can be used to generatively pre-train a deep neural network (DNN) by using learned DBN weights as initial DNN weights. Various discriminative algorithms can then tune these weights. This is particularly helpful when training data are limited, because poorly initialized weights can significantly hinder learning. These pre-trained weights end up in a region of weight space that is closer to optimal weights than random choices. This allows for both improved modeling & faster ultimate convergence.

9.6.5 Recurrent neural network

Main article: **Wikipedia/recurrent neural network**. Recurrent neural networks (RNN) propagate data forward, but also backwards, from later processing stages to earlier stages. RNN can be used as general sequence processors.

- **Fully recurrent.** This architecture was developed in 1980s. Its network creates a directed connection between every pair of units. Each has a time-varying, real-valued (more than just 0 or 1) activation (output). Each connection has a modifiable real-valued weight. Some of nodes are called *labeled nodes*, some output nodes, the rest hidden nodes.

For **supervised learning** in discrete time settings, training sequences of real-valued input vectors become sequences of activations input nodes, 1 input vector at a time. At each time step, each non-input unit computes its current activation as a nonlinear function of weighted sum of activations of all units from which it receives connections. System can explicitly activate (independent of incoming signals) some output units at certain time steps. E.g., if input sequence is a speech signal corresponding to a spoken digit, final target output at end of sequence may be a label classifying digit. For each sequence, its error is sum of deviations of all activations computed by network from corresponding target signals. For a training set of numerous sequences, total error is sum of errors of all individual sequences.

To minimize total error, **gradient descent** can be used to change each weight in proportion to its derivative w.r.t. error, provided nonlinear activation functions are differentiable. Standard method is called “**backpropagation through time**” or BPTT, a generalization of backpropagation for feedforward networks. A more computationally expensive online variant is called “**Real-Time Recurrent Learning**” or RTRL. Unlike BPTT this algorithm is *local in time but not local in space*. An online hybrid between BPTT & RTRL with intermediate complexity exists, with variants for continuous time. A major problem with gradient descent for standard RNN architectures: error gradients vanish exponentially quickly with size of time lag between important events. **Long short-term memory** architecture overcomes these problems.

In **reinforcement learning** settings, no teacher provides target signals. Instead a **fitness function** or **reward function** or **utility function** is occasionally used to evaluate performance, which influences its input stream through output units connected to actuators that affect environment. Variants of **evolutionary computation** are often used to optimize weight matrix.

- **Hopfield.** **Hopfield network** (like similar attractor-based networks) is of historic interest although it is not a general RNN, as it is not designed to process sequences of patterns. Instead it requires stationary inputs. It is an RNN in which all connections are symmetric. It guarantees that it will converge. If connections are trained using **Hebbian learning** Hopfield network can perform as robust **content-addressable memory**, resistant to connection alteration.
- **Boltzmann machine.** **Boltzmann machine** can be thought of as a noisy Hopfield network. It is 1 of 1st neural networks to demonstrate learning of **latent variables** (hidden units). Boltzmann ML was at 1st slow to simulate, but contrastive divergence algorithm speeds up training for Boltzmann machines & **Products of Experts**.

- Self-organizing map. Self-organizing map (SOM) uses **unsupervised learning**. A set of neurons learn to map points in an input space to coordinates in an output space. Input space can have different dimensions & topology from output space, & SOM attempts to preserve these.
- Learning vector quantization. **Learning vector quantization** (LVQ) can be interpreted as a neural network architecture. Prototypical representatives of classes parametrize, together with an appropriate distance measure, in a distance-based classification scheme.
- Simple recurrent. Simple recurrent networks have 3 layers, with addition of a set of “context units” in input layer. These units connect from hidden layer or output layer with a fixed weight of 1. At each time step, input is propagated in a standard feedforward fashion, & then a backpropagation-like learning rule is applied (not performing **gradient descent**). Fixed back connections leave a copy of previous values of hidden units in context units (since they propagate over connections before learning rule is applied).
- Reservoir computing. **Reservoir computing** is a computation framework that may be viewed as an extension of neural networks. Typically an input signal is fed into a fixed (random) **dynamical system** called a *reservoir* whose dynamics map input to a higher dimension. A *readout* mechanism is trained to map reservoir to desired output. Training is performed only at readout stage. **Liquid-state machine** are a type of reservoir computing.
- Echo state. **Echo state network** (ESN) employs a sparsely connected random hidden layer. Weights of output neurons are only part of network that are trained. ESN are good at reproducing certain **time series**.
- Long short-term memory. **Long short-term memory** (LSTM) avoids **vanishing gradient problem**. It works even when with long delays between inputs & can handle signals that mix low & high frequency components. LSTM RNN outperformed other RNN & other sequence learning methods e.g. **HMM** in applications e.g. language learning & connected handwriting recognition.
- Bi-directional. Main article: **Wikipedia/bidirectional recurrent neural network**. Bi-directional RNN, or BRNN, use a finite sequence to predict or label each element of a sequence based on both past & future context of element. This is done by adding outputs of 2 RNNs: one processing sequence from left to right, the other one from right to left. Combined outputs are predictions of teacher-given target signals. This technique proved to be especially useful when combined with LSTM.
- Hierarchical. **Hierarchical RNN** connects elements in various ways to decompose hierarchical behavior into useful subprograms.
- Stochastic. A distinct from conventional neural networks, **stochastic artificial neural network** used as an approximation to random functions.
- Genetic scale. A RNN (often a LSTM) where a series is decomposed into a number of scales where every scale informs primary length between 2 consecutive points. A 1st order scale consists of a normal RNN, a 2nd order consists of all points separated by 2 indices & so on. The N th order RNN connects 1st & last node. Outputs from all various scales are treated as a **Committee of Machines** & associated scores are used genetically for next iteration.

9.6.6 Modular

- Committee of machines.
- Associative.

9.6.7 Physical

A **physical neural network** includes electrically adjustable resistance material to simulate artificial synapses. Examples include **ADALINE memristor**-based neural network. An **optical neural network** is a physical implementation of an artificial neural network with **optical components**.

9.6.8 Dynamic

Unlike static neural networks, dynamic neural networks adapt their structure &/or parameters to input during inference showing time-dependent behavior, e.g. transient phenomena & delay effects. Dynamic neural networks in which parameters may change over time are related to fast weights architecture (1987), where 1 neural network outputs weights of another neural network.

- CASCADING. Cascade correlation is an architecture & **supervised learning** algorithm. Instead of just adjusting weights in a network of fixed topology, Cascade-Correlation begins with a minimal network, then automatically trains & adds new hidden units 1 by 1, creating a multilayer structure. Once a new hidden unit has been added to network, its input-side weights are frozen. This unit then becomes a permanent feature-detector in network, available for producing outputs or for creating other, more complex feature detectors. Cascade-Correlation architecture has several advantages: It learns quickly, determines its own size & topology, retains structures it has built even if training set changes & requires no **backpropagation**.
- Neuro-fuzzy. A **neuro-fuzzy** network is a **fuzzy inference system** in body of an artificial neural network. Depending on FIS type, several layers simulate processes involved in a fuzzy inference-like **fuzzification**, inference, aggregation, & **defuzzification**. Embedding an FIS in a general structure of an ANN has benefit of using available ANN training methods to find parameters of a fuzzy system.

- **Compositional pattern-producing.** **Compositional pattern-producing networks** (CPPNs) are a variation of artificial neural networks which differ in their set of **activation functions** & how they are applied. While typical artificial neural networks often contain only **sigmoid functions** (& sometimes **Gaussian functions**), CPPNs can include both types of functions & many others. Furthermore, unlike typical artificial neural networks, CPPNs are applied across entire space of possible inputs so that they can represent a complete image. Since they are compositions of functions, CPPNs in effect encode images at infinite resolution & can be sampled for a particular display at whatever resolution is optimal.

9.6.9 Memory networks

Memory networks incorporate **long-term memory**. Long-term memory can be read & written to, with goal of using it for prediction. These models have been applied in context of **question answering** (QA) where long-term memory effectively acts as a (dynamic) knowledge base & output is a textual response.

In **sparse distributed memory** or **hierarchical temporal memory**, patterns encoded by neural networks are used as addresses for **content-addressable memory**, with “neurons” essentially serving as address encoders & **decoders**. However, early controllers of such memories were not differentiable.

- **1-shot associative memory.** This type of network can add new patterns without re-training. It is done by creating a specific memory structure, which assigns each new pattern to an orthogonal plane using adjacently connected hierarchical arrays. Network offers real-time pattern recognition & high scalability; this requires parallel processing & is thus best suited for platforms e.g. **wireless sensor networks**, **grid computing**, & **GPGPUs**.

- **Hierarchical temporal memory.** **Hierarchical temporal memory** (HTM) models some of structural & **algorithmic** properties of **neocortex**. HTM is a **biomimetic** model based on **memory-prediction** theory. HTM is a method for discovering & inferring high-level causes of observed input patterns & sequences, thus building an increasingly complex model of world.

HTM combines existing ideas to mimic neocortex with a simple design that provides many capabilities. HTM combines & extends approaches used in **Bayesian networks**, spatial & temporal clustering algorithms, while using a tree-shaped hierarchy of nodes that is common in **neural networks**.

- **Holographic associative memory.** **Holographic Associative Memory** (HAM) is an analog, correlation-based, associative, stimulus-response system. Information is mapped onto phase orientation of complex numbers. Memory is effective for **associative memory** tasks, generalization & pattern recognition with changeable attention. Dynamic search localization is central to biological memory. In visual perception, humans focus on specific objects in a pattern. Humans can change focus from object to object without learning. HAM can mimic this ability by creating explicit representations for focus. It uses bi-modal representation of pattern & a hologram-like complex spherical weight state-space. HAMs are useful for optical realization because underlying hyper-spherical computations can be implemented with optical computation.

- **LSTM-related differentiable memory structures.** Apart from **long short-term memory** (LSTM), other approaches also added differentiable memory to recurrent functions. E.g.:

- Differentiable push & pop actions for alternative memory networks called *neural stack machines*
- Memory networks where control network’s external differentiable storage is in fast weights of another network
- LSTM forget gates
- Self-referential RNNs with special output units for addressing & rapidly manipulating RNN’s own weights in differentiable fashion (internal storage)
- Learning to transduce with unbounded memory

- **Neural Turing machines.** **Neural Turing machines** (NTM) couple LSTM networks to external memory resources, with which they can interact by attentional processes. Combined system is analogous to a **Turing machine** but is differentiable end-to-end, allowing it to be efficiently trained by **gradient descent**. Preliminary results demonstrate: neural Turing machines can infer simple algorithms e.g. copying, sorting & associative recall from input & output examples.

Differential neural computers (DNC) are an NTM extension. They out-performed Neural Turing machines, long short-term memory systems & memory networks on sequence-processing tasks.

- **Semantic hashing.** Approaches that represent previous experiences directly & **use a similar experience to form a local model** are often called **nearest neighbor** or **k-nearest neighbors** methods. Deep learning is useful in semantic hashing where a deep **graphical model** word-count vectors obtained from a large set of documents. Documents are mapped to memory addresses in such a way that semantically similar documents are located at nearby addresses. Documents similar to a query document can then be found by accessing all addresses that differ by only a few bits from address of query document. Unlike **sparse distributed memory** that operates on 1000-bit addresses, semantic hashing works on 32 or 64-bit addresses found in a conventional computer architecture.

- **Pointer networks.** Deep neural networks can be potentially improved by deepening & parameter reduction, while maintaining trainability. While training extremely deep (e.g., 1 million layers) neural networks might not be practical, **CPU-like** architectures e.g. pointer networks & neural random-access machines overcome this limitation by using external **random-access memory** & other components that typically belong to a **computer architecture** e.g. **registers**, **ALU**, & **pointers**. Such systems operate on

probability distribution vectors stored in memory cells & registers. Thus, model is fully differentiable & trains end-to-end. Key characteristic of these models: their depth, size of their short-term memory, & number of parameters can be altered independently.

9.6.10 Hybrids

- **Encoder-decoder networks.** Encoder-decoder frameworks are based on neural networks that map highly **structured** input to highly structured output. Approach arose in context of **machine translation**, where input & output are written sentences in 2 natural languages. In that work, an LSTM RNN or CNN was used as an encoder to summarize a source sentence, & summary was decoded using a conditional RNN **language model** to produce translation. These systems share building blocks: gated RNNs & CNNs & trained attention mechanisms.

9.6.11 Other types

- **Instantaneously trained.** **Instantaneously trained neural networks** (ITNN) were inspired by phenomenon of short-term learning that seems to occur instantaneously. In these networks weights of hidden & output layers are mapped directly from training vector data. Ordinarily, they work on binary data, but versions for continuous data that require small additional processing exist.
- **Spiking.** **Spiking neural networks** (SNN) explicitly consider timing of inputs. Network input & output are usually represented as a series of spikes (**delta function** or more complex shapes). SNN can process information in **time domain** (signals that vary over time). They are often implemented as recurrent networks. SNN are also a form of **pulse computer**.

Spiking neural networks with axonal conduction delays exhibit polychronization, & hence could have a very large memory capacity.

SNN & temporal correlations of neural assemblies in such networks – have been used to model figure/ground separation & region linking in visual system.

- **Spatial.** **Spatial neural networks** (SNNs) constitute a supercategory of tailored **neural networks (NNs)** for representing & predicting geographic phenomena. They generally improve both statistical accuracy & **reliability** of a-spatial/classic NNs whenever they handle **geo-spatial datasets**, & also of the other spatial (**statistical**) **models** (e.g. spatial regression models) whenever geo-spatial **Datasets**; variables depict **nonlinear relations**. Examples of SNNs are OSFA spatial neural networks, SVANNs & GWNNs.
- **Neocognitron.** **Neocognitron** is a hierarchical, multilayered network that was modeled after **visual cortex**. It uses multiple types of units, (originally 2, called **simple** & **complex** cells), as a cascading model for use in pattern recognition tasks. Local features are extracted by S-cells whose deformation is tolerated by C-cells. Local features in input are integrated gradually & classified at higher layers. Among various kinds of neocognitron are systems that can detect multiple patterns in same input by using back propagation to achieve **selective attention**. It has been used for **pattern recognition** tasks & inspired **convolutional neural networks**.
- **Compound hierarchical-deep models.** Compound hierarchical-deep models compose deep networks with non-parametric **Bayesian models**. **Features** can be learned using deep architectures e.g. **DBNs**, **deep Boltzmann machines** (DBM), deep auto encoders, convolutional variants, **ssRBMs**, deep coding networks, DBNs with sparse feature learning, **RNNs**, conditional DBNs, **denoising autoencoders**. This provides a better representation, allowing faster learning & more accurate classification with high-dimensional data. However, these architectures are poor at learning novel classes with few examples, because all network units are involved in representing input (a *distributed representation*) & must be adjusted together (high **degree of freedom**). Limiting degree of freedom reduces number of parameters to learn, facilitating learning of new classes from few examples. **Hierarchical Bayesian (HB) models** allow learning from few examples, e.g. for **computer vision**, statistics, & **cognitive science**.

Compound HD architectures aim to integrate characteristics of both HB & deep networks. Compound HDP-DBM architecture is a **hierarchical Dirichlet process** (HDP) as a hierarchical model, incorporating DBM architecture. It is a full **generative model**, generalized from abstract concepts flowing through model layers, which is able to synthesize new examples in novel classes that look “reasonably” natural. All levels are learned jointly by maximizing a joint **log-probability score**.

In a DBM with 3 hidden layers, probability of a visible input v is:

$$p(v, \psi) = \frac{1}{Z} \sum_h \exp \left(\sum_{ij} W_{ij}^{(1)} v_i h_j^1 + \sum_{jl} W_{jl}^{(2)} h_j^1 h_l^2 + \sum_{lm} W_{lm}^{(3)} h_l^2 h_m^3 \right), \quad (406)$$

where $h = \{h^{(1)}, h^{(2)}, h^{(3)}\}$: set of hidden units, $\psi = \{W^{(1)}, W^{(2)}, W^{(3)}\}$: model parameters, representing visible-hidden & hidden-hidden symmetric interaction terms.

A learned DBM model is an undirected model that defines **joint distribution** $P(v, h^1, h^2, h^3)$. 1 way to express what was been learned is **conditional model** $P(v, h^1, h^2 | h^3)$ & a **prior** term $P(h^3)$.

Here $P(v, h^1, h^2|h^3)$ represents a conditional DBM model, which can be viewed as a 2-layer DBM but with bias terms given by states of h^3 :

$$p(v, h^1, h^2|h^3) = \frac{1}{Z(\psi, h^3)} \sum_h \exp \left(\sum_{ij} W_{ij}^{(1)} v_i h_j^1 + \sum_{jl} W_{jl}^{(2)} h_j^1 h_l^2 + \sum_{lm} W_{lm}^{(3)} h_l^2 h_m^3 \right). \quad (407)$$

- **Deep predictive coding networks.** A deep predictive coding network (DPCN) is a **predictive** coding scheme that uses top-down information to empirically adjust priors needed for a bottom-up **inference** procedure by means of a deep, locally connected, **generative model**. This works by extracting sparse **features** from time-varying observations using a linear dynamical model. Then, a pooling strategy is used to learn invariant feature representations. These units compose to form a deep architecture & are trained by **greedy** layer-wise **unsupervised learning**. Layers constitute a kind of **Markov chain** s.t. states at any layer depend only on preceding & succeeding layers.

DPCNs predict representation of layer, by using a top-down approach using information in upper layer & temporal dependencies from previous states.

DPCNs can be extended to form a **convolutional network**.

- **Multilayer kernel machine.** Multilayer kernel machines (MKM) are a way of learning highly nonlinear functions by iterative application of weakly nonlinear kernels. They use **kernel principal component analysis** (KPCA), as a method for **unsupervised** greedy layer-wise pre-training step of deep learning.

Layer $l + 1$ learns representation of previous layer l , extracting n_l **principal component** (PC) of projection layer l output in feature domain induced by kernel. To reduce **dimensionality** of updated representation in each layer, a **supervised strategy** selects best informative features among features extracted by KPCA. Process is:

- rank n_l features according to their **mutual information** with class labels;
- for different values of K & $m_l \in \{1, \dots, n_l\}$, compute classification error rate of a **K-nearest neighbor** (K-NN) classifier using only m_l most informative features on a **validation set**
- value of m_l with which classifier has reached lowest error rate determines number of features to retain.

Some drawbacks accompany KPCA method for MKMs.

A more straightforward way to use kernel machines for deep learning was developed for spoken language understanding. Main idea: to use a kernel machine to approximate a shallow neural net with an infinite number of hidden units, then use a **deep stacking network** to splice output of kernel machine & raw input in building the next, higher level of kernel machine. Number of levels in deep convex network is a **hyper-parameter** of overall system, to be determined by **cross validation**.” – Wikipedia/types of artificial neural networks

Tài liệu

- [ABT18] Richard C. Aster, Brian Borchers, and Clifford H. Thurber. *Parameter estimation and inverse problems*. Third. Elsevier/Academic Press, Amsterdam, 2018, pp. xi+392. ISBN: 978-0-12-804651-7. DOI: [10.1016/C2015-0-02458-3](https://doi.org/10.1016/C2015-0-02458-3). URL: <https://doi.org/10.1016/C2015-0-02458-3>.
- [Bac24] Francis Bach. “Learning Theory from First Principles”. In: Adaptive Computation and Machine Learning series (2024), p. 496.
- [BHH06] Francis R. Bach, David Heckerman, and Eric Horvitz. “Considering cost asymmetry in learning classifiers”. In: *J. Mach. Learn. Res.* 7 (2006), pp. 1713–1741. ISSN: 1532-4435.
- [Cho21] François Chollet. *Deep Learning with Python*. 2nd edition. Manning, 2021, p. 478.
- [DFO24] Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong. *Mathematics for Machine Learning*. 1st edition. Cambridge University Press, 2024, pp. iii+411.
- [DG24] Shlomo Dubnov and Ross Greer. *Deep & Shallow: Machine Learning in Music & Audio*. 1st edition. Chapman & Hall/CRC Machine Learning & Pattern Recognition. CRC Press, 2024, p. 328.
- [Kut23] Gitta Kutyniok. “The mathematics of artificial intelligence”. In: *ICM—International Congress of Mathematicians. Vol. 7. Sections 15–20*. EMS Press, Berlin, [2023] ©2023, pp. 5118–5139.
- [MC01] Danilo P. Mandic and Jonathon A. Chambers. “Recurrent Neural Networks for Prediction: Learning Algorithms, Architectures and Stability”. In: Wiley Series in Adaptive and Learning Systems for Signal Processing, Communications, and Control (2001), p. 304.
- [NP23] Phan-Minh Nguyen and Huy Tuan Pham. “A rigorous framework for the mean field limit of multilayer neural networks”. In: *Math. Stat. Learn.* 6.3-4 (2023), pp. 201–357. ISSN: 2520-2316. DOI: [10.4171/msl/42](https://doi.org/10.4171/msl/42). URL: <https://doi.org/10.4171/msl/42>.
- [RHP21] Rishikesh Ranade, Chris Hill, and Jay Pathak. “DiscretizationNet: a machine-learning based solver for Navier-Stokes equations using finite volume discretization”. In: *Comput. Methods Appl. Mech. Engrg.* 378 (2021), Paper No. 113722, 20. ISSN: 0045-7825. DOI: [10.1016/j.cma.2021.113722](https://doi.org/10.1016/j.cma.2021.113722). URL: <https://doi.org/10.1016/j.cma.2021.113722>.

- [Ric+22] Michael M. Richter, Sheuli Paul, Veton Këpuska, and Marius Silaghi. *Signal Processing and Machine Learning with Applications*. Springer, 2022, p. 607.
- [RLM22] Sebastian Raschka, Yuxi (Hayden) Liu, and Vahid Mirjalili. *Machine Learning with PyTorch & Scikit-Learn: Develop Machine Learning & Deep Learning Models with Python*. 1st edition. Packt Publishing, 2022, p. 741.
- [Rud87] Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.