Mathematical Analysis & Numerical Analysis
Giải Tích Toán Học & Giải Tích Số

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Ngày 17 tháng 10 năm 2024

Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
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1. *Mathematical Analysis – Giải Tích Toán Học.*
   PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.pdf.
   TEX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.tex.

Mục lục

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Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

1 Basic

Resources – Tài nguyên.

1. [LL01]. Elliott Lieb, Michael Loss. *Analysis*.
2. [Rud76]. Walter Rudin. *Principles Principles of Mathematical Analysis*.
3. [Rud73; Rud87]. Walter Rudin. *Real & Complex Analysis*.
4. [Tao22a]. Terence Tao. *Analysis I*.
5. [Tao22b]. Terence Tao. *Analysis II*.

"Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones." – [LL01, Chap. 1, p. 1]

2 C_0 Semigroup – Nửa Nhóm C_0

Resources – Tài nguyên.

1. [AK16]. Cung Thế Anh, Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng*.

"In mathematical analysis, a C_0*-semigroup*, also known as a *strongly continuous 1-parameter semigroup*, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant ODEs, strongly continuous semigroups provide solutions of linear constant coefficient ODEs in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations & PDEs. Formally, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+, +)$ on some Banach space X that is continuous in the strong toperator topology." -Wikipedia/C_0-semigroup

3 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

1. [Car16]. Manfredo P. do Carmo. *Differential Geometry of Curves & Surfaces*.
2. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries*.
3. [Küh15]. Wolfgang Kühnel. *Differential Geometry*.
4. [Wal15]. Shawn W. Walker. *The Shapes of Things*.

   "Differential geometry is the detailed study of the *shape* of a surface (manifold), including *local* & *global* properties. A plane in $\mathbb{R}^3$ is a very simple surface & does not require many tools to characterize. An "arbitrarily" shaped surface, e.g., hood of a car, has many distinguished geometric features (e.g., highly curved regions, regions of near flatness, etc.). Characterizing these features quantitatively & qualitatively requires the tools of differential geometry. Geometric details are important in many physical & biological processes, e.g., surface tension, biomembranes.

   The framework of differential geometry is built by 1st defining a local map (i.e., surface parameterization) which defines the surface. Then, a calculus framework is built up on the surface analogous to the standard "Euclidean calculus". Other approaches are also possible, e.g., those with implicit surfaces defined by level sets & distance functions. But parameterizations, though arbitrary, are quite useful in a variety of settings ⇒ stick mostly with those. Emphasize: The geometry of a surface does not depend on a particular parameterization. Otherwise, we will emphasize the distinction between `object 1` & `object 2`.

   We will use this "abuse" of notation when there is no possibility of ambiguity.

   Open set. The concept of open set is critical in multivariate calculus to properly define differentiability. The notation for referencing boundaries of sets, as well as the closure of sets, is practical for referencing geometric details of solid objects & their surfaces.

   Compactness. Compact support is useful for ignoring boundary effects. This concept is needed to keep the "action of a function" away from the boundary of a set, or to localize the function in a region of interest. 1 reason is to avoid potential difficulties with differentiating a function at its boundary of definition. Or, more commonly, we wish to ignore a quantity depending on the value of a function at a boundary point, e.g., $\int_{\partial S} f = 0$ if f has compact support in S.

Topological mapping/homeomorphism. A bijective, continuous mapping $\mathbf{\Phi}$ whose inverse $\mathbf{\Phi}^{-1}$ is also continuous is called a *topological mapping* or *homeomorphism*. Point sets that can be topologically mapped onto each other are said to be *homeomorphic*. Sets that are homeomorphic have the "same topology", i.e., their connectedness is the same; they have the same kinds of "holes". See [Wal15, Sect. 2.3.1] for what can happen when a mapping is not a homeomorphism.

Rigid motion mapping. A mapping $\mathbf{\Phi}$ is called a *rigid motion* if any pair of points $\mathbf{a}, \mathbf{b}$ are the same distance apart as the corresponding pair $\mathbf{\Phi}(\mathbf{a}), \mathbf{\Phi}(\mathbf{b})$.

Orthogonal Transformations. Define the (affine) linear map $\mathbf{\Phi}$ (transformation)

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{1}$$

If A satisfies the properties $A^{-1} = A^\top$, $\det A = 1$ then $\mathbf{\Phi}$ represents a rigid motion. Basically, $\mathbf{\Phi}$ consists of a rotation represented by A followed by a translation represented by $\mathbf{b}$. A rigid motion can be used to transition from 1 Cartesian coordinate system to another. If $\mathbf{b} = \mathbf{0}$ & $A^{-1} = A^\top$, $\det A = 1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is a linear map known as a *direct orthogonal transformation*, which is nothing more than a rotation of the coordinate system with the origin as the center. If $A^{-1} = A^\top$, $\det A = 1$ is replaced by $A^{-1} = A^\top$, $\det A = -1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is called an *opposite orthogonal transformation*, which consists of a rotation about the origin & a reflection in a plane. Both $A^{-1} = A^\top$, $\det A = \pm 1$ are examples of *orthogonal matrices*.

Interpretation of transformations. Can interpret $\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in 2 different ways. Consider a point $P \in \mathbb{R}^3$ with coordinates $\mathbf{x}$:

- *Alias* (Euler perspective). Viewing (1) as a transformation of coordinates, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of the same point w.r.t. 2 different coordinate systems, equivalently, the point is referenced by 2 different "names" (sets of coordinates).
- *Alibi* (Lagrange perspective). Viewing (1) as a mapping of sets, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of 2 different points w.r.t. the same coordinate system, equivalently, the point at $\widetilde{\mathbf{x}}$ "was previously" at $\mathbf{x}$ before applying the map.

The concept of material point is directly related to the alibi viewpoint. One can think of a "particle" of material, i.e., *material point*, initially located at $\mathbf{x}$, that then moves to $\widetilde{\mathbf{x}}$ because of some physical process. The transformation (1) simply represents the kinematic outcome of that physical process, which is a standard concept in deformable continuum mechanics, especially nonlinear elasticity.

General transformations. In general, transformation may not be linear. The alias viewpoint yields a *curvilinear* coordinate system. The alibi viewpoint implies that the set S is *deformed* into the set $S' = \mathbf{\Phi}(S)$.

Parametric approach – what is a surface? A *surface* is a set of points in space that is "regular enough". A random scattering of points in space does not match our intuitive notion of what a surface is, i.e., it is not regular enough. The boundary of a sphere does match our notion of a surface, i.e., regular enough to be a surface because a sphere is "smooth". *Intuition*: Can think of creating a surface as deforming a flat rubber sheet into a curved sheet. Let $U \subset \mathbb{R}^2$ be a "flat" domain & let $\mathbf{X} : U \to \mathbb{R}^3$ be this deforming transformation, i.e., for each point $(s_1, s_2)^\top \in U$ there is a corresponding point $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ s.t. $\mathbf{x} = \mathbf{X}(s_1, s_2)$. Let $\Gamma = \mathbf{X}(U)$ denote the surface obtained from "deforming" U. Call $\mathbf{x} = \mathbf{X}(s_1, s_2)$ a *parametric representation* of the surface Γ, where s_1, s_2 are called the *parameters* of the representation. Refer to U as a *reference* domain.

Allowable parameterization/immersion. If use $\mathbf{x} = \mathbf{X}(s_1, s_2)$ to define surfaces, then we must place assumptions on $\mathbf{X}$ to guarantee that $\Gamma = \mathbf{X}(U)$ is a valid surface. At the bare minimum, $\mathbf{X}$ must be continuous to avoid "tearing" the rubber sheet. But if want to perform calculus on Γ, need more:

Assumption 1 (Regularity assumptions on $\mathbf{X}$)**.** *An allowable parameterization/immersion is a parameterization of the form* $\mathbf{x} = \mathbf{X}(s_1, s_2)$ *satisfying:*

(A1) The function $\mathbf{X}(s_1, s_2) \in C^\infty(U)$ *& each point* $\mathbf{x} = \mathbf{X}(s_1, s_2) \in \Gamma$ *corresponds to just 1 point* $(s_1, s_2) \in U$*, i.e.,* $\mathbf{X}$ *is injective.*

(A2) The Jacobian matrix $J = [\partial_{s_1}\mathbf{X}, \partial_{s_2}\mathbf{X}]$ *is of rank* 2 *on* U*, i.e., the columns of* J *are linearly independent.*

Regular surface. The fundamental property that makes a set of points in $\mathbb{R}^3$ a surface is that it *locally looks like a plane* at every point. If you "zoom into" a surface, it should look flat. Definition defining a surface in terms of a parameterization is inadequate. Want to define a set in $\mathbb{R}^3$ that is "intrinsically" 2D & is smooth enough so we can perform calculus on it, without regard to a specific parameterization.

Definition 1 (Regular surface)**.**

Remark 1 (Local chart)**.**

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

3.1 Calculus on Surfaces

Goal. Define & develop the fundamental tools of calculus on a regular surface. Start with the notion of differentiability of functions defined only on a surface. Define the concept of vector fields in a surface. Then proceed to develop the gradient & Laplacian operators w.r.t. a surface. These operators allow for alternative expressions of the summed & Gaussian curvatures. Derive integration by parts on surfaces, i.e., the domain of integration is a surface. Conclude with some useful identities & inequalities. Always take Γ: a regular surface, either with or without a boundary.

4 Functional Analysis – Giải Tích Hàm

Resources – Tài nguyên.

1. [Rud91]. Walter Rudin. *Functional Analysis.*

2. Yosida.

5 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

1. [ABT18]. Richard Aster, Brian Borchers, Clifford H. Thurber. *Parameter Estimation & Inverse Problems.*

2. [Kir21]. Andreas Kirsch. *An Introduction to The Mathematical Theory of Inverse Problems.*

3. [IJ15]. Kazufumi Ito, Bangti Jin. *Inverse Problems.*

6 Measure & Integration – Độ Đo & Tích Phân

Resources – Tài nguyên.

1. [EG15]. Lawrence C. Evans, Ronald F. Gariepy. *Measure Theory & Fine Properties of Functions.*

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

"The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$-dimensional volume 'in the other direction' by 1st computing the n-dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y, which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*" – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

7 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

1. [Bre11]. Haïm Brezis. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations.*

2. [Eva10]. Lawrence C. Evans. *Partial Differential Equations.*

3. [GT01]. David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of 2nd Order.*

7.1 Weak solution – Nghiệm yếu

Definition 2 (Weak solution – Nghiệm yếu)**.** *"In mathematics, a* weak solution *(also called a* generalized solution*) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of distributions." – Wikipedia/weak solution*

"Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough." – Wikipedia/weak solution

Example 1 (1st-order wave equation)**.** *The 1st-order wave equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}^2$ with $u = u(t,x)$ has the weak form $\int_{\mathbb{R}^2} u\partial_t\varphi + u\partial_x\varphi \,\mathrm{d}t\,\mathrm{d}x = 0$ has a solution $u(t,x) = |t-x|$ which may be checked by splitting the integrals over region $\{x \ge t\}$ & $\{x \le t\}$ where u is smooth.*

"The notion of weak solution based on distribution is sometimes inadequate. In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. Hamilton-Jacobi equation, there is a very different definition of weak solution called *viscosity solution*." – Wikipedia/weak solution

7.1.1 General idea

When solving a differential equation in u, one can rewrite it using a test function φ s.t. whatever derivatives in u show up in the equation, they are "transferred" via integration by parts to φ, resulting in an equation without derivatives of u. This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x},\partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}),$$

where the multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$ varies over some finite set in $\mathbb{N}^d$ & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x},\partial)u(\mathbf{x} = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

where the differential operator $Q(\mathbf{x},\partial)$ is given by the formula

$$Q(\mathbf{x},\partial)\varphi(\mathbf{x}) = \sum (-1)^{|\alpha|}\partial^\alpha[a_\alpha(\mathbf{x})\varphi(\mathbf{x})],$$

which is the formal adjoint of $P(\mathbf{x},\partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$-times differentiable function u defined on the open set W s.t. $P(\mathbf{x},\partial)u(\mathbf{x}) = 0$, $\forall\mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$, $\forall\varphi \in C_c^\infty(W)$.

7.2 Viscosity solution – Nghiệm trơn/nhớt

Example 2 (Viscosity solution for Hamilton–Jacobi equation)**.** *Hamilton–Jacobi equation.*

7.3 Very weak solution – Nghiệm rất yếu

Example 3 (Very weak solution of porous medium equation (PME) [Váz07])**.** .

Example 4 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term [DDN20])**.** *Given $f \in L^1_\delta(\Omega), \lambda \ge 0$, a function $u \in L^1_\delta(\Omega)$ is called a* very weak solution *of*

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega,\\ |u|^{m-1}u = 1 & \text{on } \Gamma,\end{cases}$$

if $|u|^{m-1}u \in L^1(\Omega)$ and

$$\int_\Omega u^m\Delta\varphi + \lambda u\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi\,\mathrm{d}\mathbf{x} - \int_\Gamma \partial_{\mathbf{n}}\varphi\,\mathrm{d}\mathbf{x}.$$

Example 5 (Very weak solution of NSEs [Tsa18])**.** .

7.4 Navier–Stokes Equations

Resources – Tài nguyên.

1. [Lad69]. O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.*
2. [Soh01a; Soh01b]. Hermann Sohr. *The NSEs: An Elementary Functional Analytic Approach.*

   Primary objective. To develop an elementary & self-contained approach to the mathematical theory of a viscous incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$, described by NSEs. Formulate the theory for a completely general domain Ω.
3. [Tem77; Tem00]. Roger Temam. *NSES: Theory & Numerical Analysis.*
4. [Tsa18]. Tai-Peng Tsai. *Lectures on NSEs.*

8 Sobolev Spaces – Không Gian Sobolev

Resources – Tài nguyên.

1. [AF03]. Robert A. Adams, John J. F. Fournier. *Sobolev Spaces.*
2. [Gag57]. Emilio Gagliardo. *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.*
3. Necǎs.
4. [Tar06]. Luc Tartar. *An Introduction to Sobolev Spaces & Interpolation Spaces.*

9 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

1. [LeV07]. Randall J. LeVeque. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems.*

10 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

1. [BS08]. Susanne C. Brenner, L. Ridgway Scott. *The Mathematical Theory of FEMs.*
2. [EG04]. Alexandre Ern, Jean-Luc Guermond. *Theory & Practice of Finite Elements.*
3. [GR86]. Vivette Girault, Pierre-Arnaud Raviart. *FEMs for NSEs.*
4. [Gun89]. Max D. Gunzburger. *FEMs for Viscous Incompressible Flows.*
5. [Joh16]. Volker John. *FEMs for Incompressible Flow Problems.*

I met Volker John, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagonrinsky, k-ϵ & their simulations.

11 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

1. [EGH19]. Robert Eymard, Thierry Gallouët, Raphaèle Herbin. *Finite Volume Methods.*
2. [LeV02]. Randall J. LeVeque. *FEMs for Hyperbolic Problems.*

12 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

12.1 Haïm Brezis

12.2 Lawrence Chris Evans

12.3 Peter Lax

12.4 Andrew Joseph Majda

Resources – Tài nguyên.

1. [Eng+23]. Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, Vlad Vicol. *In memory of Andrew J. Majda.*

   "He was hard working until the end even though he suffered from serious health issues for quite some time."

   "He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, and observed and experimental data ... Andy Majda's modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis and numerical modeling, (iii) observed phenomena and experimental data, and (iv) qualitative and/or asymptotic modeling [Maj00]."

   "Andy's legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored."

   "The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed "Majda's model" for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived "Beale-Kato-Majda criterion," which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84]."

   "At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science."

12.5 Vladimir Mazya

12.6 Jindřich Nečas

12.7 Louis Nirenberg

[Váz20]

12.8 Stanley Osher

13 Miscellaneous

Tài liệu

[ABT18] Richard C. Aster, Brian Borchers, and Clifford H. Thurber. *Parameter estimation and inverse problems*. Third. Elsevier/Academic Press, Amsterdam, 2018, pp. xi+392. ISBN: 978-0-12-804651-7. DOI: 10.1016/C2015-0-02458-3. URL: https://doi.org/10.1016/C2015-0-02458-3.

[AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. ISBN: 0-12-044143-8.

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[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition of [MR0394451]. Dover Publications, Inc., Mineola, NY, 2016, pp. xvi+510. ISBN: 978-0-486-80699-0; 0-486-80699-5.

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[DZ01] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*. Vol. 4. Advances in Design and Control. Analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, pp. xviii+482. ISBN: 0-89871-489-3.

[DZ11] M. C. Delfour and Jean-Paul Zolésio. *Shapes and geometries*. Second. Vol. 22. Advances in Design and Control. Metrics, analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, pp. xxiv+622. ISBN: 978-0-898719-36-9. DOI: 10.1137/1.9780898719826. URL: https://doi.org/10.1137/1.9780898719826.

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[Eva10] Lawrence C. Evans. *Partial Differential Equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. URL: https://doi.org/10.1090/gsm/019.

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[GR86] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*. Vol. 5. Springer Series in Computational Mathematics. Theory and algorithms. Springer-Verlag, Berlin, 1986, pp. x+374. ISBN: 3-540-15796-4. DOI: 10.1007/978-3-642-61623-5. URL: https://doi.org/10.1007/978-3-642-61623-5.

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