Discrete Mathematics – Toán Rời Rạc

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3.1 Wikipedia/discrete mathematics

"*Discrete mathematics* is study of mathematical structures that can be considered "discrete" (in a way analogous to discrete variables, having a bijection with $\mathbb{N}$) rather than "continuous" (analogously to continuous functions). Objects studied in discrete mathematics include integers, graphs, & statements in logic. By contrast, discrete mathematics excludes topics in "continuous mathematics" e.g. real numbers, calculus or Euclidean geometry. Discrete objects can often be enumerated by integers; more formally, discrete mathematics has been characterized as branch of mathematics dealing with countable sets (finite sets or sets with same cardinality as $\mathbb{N}$). However, there is no exact definition of term "discrete mathematics".

Set of objects studied in discrete mathematics can be finite or infinite. Term *finite mathematics* is sometimes applied to parts of field of discrete mathematics that deals with finite sets, particularly those areas relevant to business.

Graphs e.g. these are among objects studied by discrete mathematics, for their interesting mathematical properties, their usefulness as models of real-world problems, & their importance in developing computer algorithms.

Research in discrete mathematics increased in latter half of 20th century partly due to development of digital computers which operate in "discrete" steps & store data in "discrete" bits. Concepts & notations from discrete mathematics are useful in studying & describing objects & problems in branches of computer science, e.g. computer algorithms, programming languages, cryptography, automated theorem proving, & software development. Conversely, computer implementations are significant in applying ideas from discrete mathematics to real-world problems.

Although main objects of study in discrete mathematics are discrete objects, analytic methods from "continuous" mathematics are often employed as well.

In university curricula, discrete mathematics are discrete objects, analytic methods from "continuous" mathematics are often employed as well.

In university curricula, discrete mathematics appeared in 1980s, initially as a computer science support course; its contents were somewhat haphazard at time. Curriculum has thereafter developed in conjunction with efforts by ACM & MAA into a course that is basically intended to develop mathematical maturity in 1st-year students; therefore, it is nowadays a prerequisite

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for mathematics majors in some universities as well. Some high-school-level discrete mathematics textbooks have appeared as well. At this level, discrete mathematics is sometimes seen as a preparatory course, like precalculus in this respect.

Fulkerson Prize is awarded for outstanding papers in discrete mathematics.

3.1.1 Topics

1. Theoretical computer science. Complexity studies time taken by algorithms, e.g. this quick sort. Theoretical computer science includes areas of discrete mathematics relevant to computing. It draws heavily on graph theory & mathematical logic. Included within theoretical computer science is study of algorithms & data structures. Computability studies what can be computed in principle, & has close ties to logic, while complexity studies time, space, & other resources taken by computations. Automata theory & formal language theory are closely related to computability. Petri nets & process algebras are used to model computer systems, & methods from discrete mathematics are used in analyzing VLSI electronic circuits.

   Computational geometry applies computer algorithms to representations of geometrical objects. Computational geometry applies algorithms to geometrical problems & representations of geometrical objects, while computer image analysis applies them to representations of images. Theoretical computer science also includes study of various continuous computational topics.

2. Information theory. ASCII codes for word "Wikipedia", given here in binary, provide a way of representing word in information theory, as well as for information-processing algorithms. Information theory involves quantification of information. Closely related is coding theory which is used to design efficient & reliable data transmission & storage methods. Information theory also includes continuous topics e.g.: analog signals, analog coding, analog encryption.

3. Logic. Mathematical logic is study of principles of valid reasoning & inference, as well as of consistency, soundness, & completeness. E.g., in most systems of logic (but not in intuitionistic logic) Peirce's law $(((P \to Q) \to P) \to P)$ is a theorem. For classical logic, it can be easily verified with a truth table. Study of mathematical proof is particularly important in logic, & has accumulated to automated theorem proving & formal verification of software.

   Logical formulas are discrete structures, as are proofs, which form finite trees or, more generally, directed acylic graph structures (with each inference step combining 1 or more premise branches to give a single conclusion). Truth values of logical formulas usually form a finite set, generally restricted to 2 values: true & false, but logic can also be continuous-valued, e.g., fuzzy logic. Concepts e.g. infinite proof trees or infinite derivation trees have also been studied, e.g., infinitary logic.

4. Set theory. Set theory is branch of mathematics that studies sets, which are collections of objects, e.g. {blue, white, red} or (infinite) set of all prime numbers. Partially ordered sets & sets with other relations have applications in several areas.

   In discrete mathematics, countable sets (including finite sets) are main focus. Beginning of set theory as a branch of mathematics is usually marked by George Cantor's work distinguishing between different kinds of infinite set, motivated by study of trigonometric series, & further development of theory of infinite sets is outside scope of discrete mathematics. Indeed, contemporary work in descriptive set theory makes extensive use of traditional continuous mathematics.

5. Combinatorics. Combinatorics studies ways in which discrete structures can be combined or arranged. Enumerative combinatorics concentrates on counting number of certain combinatorial objects – e.g., 12fold way provides a unified framework for counting permutations, combinations, & partitions. Analytic combinatorics concerns enumeration (i.e., determining number) of combinatorial structures using tools from complex analysis & probability theory. In contrast with enumerative combinatorics which uses explicit combinatorial formulae & generating functions to describe results, analytic combinatorics aims at obtaining asymptotic formulae. Topological combinatorics concerns use of techniques from topology & algebraic topology/combinatorial topology in combinatorics. Design theory is a study of combinatorial designs, which are collections of subsets with certain intersection properties. Partition theory studies various enumeration & asymptotic problems related to integer partitions, & is closely related to q-series, special functions, & orthogonal polynomials. Originally a part of number theory & analysis, partition theory is now considered a part of combinatorics or an independent field. Order theory is study of partially ordered sets, both finite & infinite.

6. Graph theory. Graph theory has close links to group theory. This truncated tetrahedron graph is related to alternating group A_4. Graph theory, study of graphs & networks, is often considered part of combinatorics, but has grown large enough & distinct enough, with its own kind of problems, to be regarded as a subject in its own right. Graphs are 1 of prime objects of study in discrete mathematics. They are among most ubiquitous models of both natural & human-made structures. They can model many types of relations & process dynamics in physical, biological & social systems. In computer science, they can represent networks of communication, data organization, computational devices, flow of computation, etc. In mathematics, they are useful in geometry & certain parts of topology, e.g. knot theory. Algebraic graph theory has close links with group theory & topological graph theory has close links to topology. There are also continuous graphs; however, for most part, research in graph theory falls within domain of discrete mathematics.

7. Number theory. Ulam spiral of numbers, with black pixels showing prime numbers. This diagram hints at patterns in distribution of prime numbers. Number theory is concerned with properties of numbers in general, particularly integers. It has applications to cryptography & cryptanalysis, particularly with regard to modular arithmetic, diophantine equations, linear & quadratic congruences, prime numbers & primality testing. Other discrete aspects of number theory include geometry of numbers. In analytic number theory, techniques from continuous mathematics are also used. Topics that go beyond discrete objects include transcendental numbers, diophantine approximation, p-adic analysis & function fields.

8. Algebraic structures. Main article: Wikipedia/abstract algebra. Algebraic structures occur as both discrete examples & continuous examples. Discrete algebras include: Boolean algebra used in logic gates & programming; relational algebra used in databases; discrete & finite versions of groups, rings, & fields are important in algebraic coding theory; discrete semigroups & monoids appear in theory of formal languages.

9. Discrete analogues of continuous mathematics. There are many concepts & theories in continuous mathematics which have discrete versions, e.g. discrete calculus, discrete Fourier transforms, discrete geometry, discrete logarithms, discrete differential geometry, discrete exterior calculus, discrete Morse theory, discrete optimization, discrete probability theory, discrete probability distribution, difference equations, discrete dynamical systems, & discrete vector measures.

   - Calculus of finite differences, discrete analysis. In discrete calculus & calculus of finite differences, a function defined on an interval of integers is usually called a sequence. A sequence could be a finite sequence from a data source or an infinite sequence from a discrete dynamical system. Such a discrete function could be defined explicitly by a list (if its domain is finite), or by a formula for its general term, or it could be given implicitly by a recurrence relation or difference equation. Difference equations are similar to differential equations, but replace differentiation by taking difference between adjacent terms; they can be used to approximate differential equations or (more often) studied in their own right. Many questions & methods concerning differential equations have counterparts for difference equations. E.g., where there are integral transforms in harmonic analysis for studying continuous functions for analogue signals, there are discrete transforms for discrete functions or digital signals. As well as discrete metric spaces, there are more general discrete topological spaces, finite metric spaces, finite topological spaces.

     Time scale calculus is a unification of theory of difference equations with that of differential equations, which has applications to fields requiring simultaneous modeling of discrete & continuous data. Another way of modeling such a situation is notion of hybrid dynamical systems.

   - Discrete geometry. Discrete geometry & combinatorial geometry are about combinatorial properties of *discrete collections* of geometrical objects. A long-standing topic in discrete geometry is tiling of plane.

     In algebraic geometry, concept of a curve can be extended to discrete geometries by taking spectra of polynomials rings over finite fields to be models of affine spaces over that field, & letting subvarieties or spectra of other rings provide curves that lie in that space. Although space in which curves appear has a finite number of points, curves are not so much sets of points as analogues of curves in continuous settings. E.g., every point of form $V(x-c) \subset \mathrm{Spec} K[x] = \mathbb{A}^1$ for K a field can be studied either as $\mathrm{Spec} K[x]/(x-c) \cong \mathrm{Spec} K$, a point, or as spectrum $\mathrm{Spec} K[x]_{(x-c)}$ of local ring at $(x-c)$, a point together with a neighborhood around it. Algebraic varieties also have a well-defined notion of tangent space called Zariski tangent space, making many features of calculus applicable even in finite settings.

   - Discrete modeling. In applied mathematics, discrete modeling is discrete analogue of continuous modeling. In discrete modeling, discrete formulae are fit to data. A common method in this form of modeling is to use recurrence relation. Discretization concerns process of transferring continuous models & equations into discrete counterparts, often for purposes of making calculations easier by using approximations. Numerical analysis provides an important example.

3.1.2 Challenges

Much research in graph theory was motivated by attempts to prove: all maps can be colored using only 4 colors so that no areas of same color share an edge. Kenneth Appel & Wolfgang Haken proved this in 1976.

History of discrete mathematics has involved a number of challenging problems which have focused attention within areas of field. In graph theory, much research was motivated by attempts to prove 4 color theorem, 1st stated in 1852, but not proved until 1976 (by Kenneth Appel & Wolfgang Haken, using substantial computer assistance).

In logic, 2nd problem on David Hilbert's list of open problems presented in 1900 was to prove: axioms of arithmetic are consistent. Gödel's 2nd incompleteness theorem, proved in 1931, showed: this was not possible – at least not within arithmetic itself. Hilbert's 10th problem was to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution. In 1970, Yuri Matiyasevich proved: this could not be done.

Need to break German codes in World War II led to advances in cryptography & theoretical computer science, with 1st programmable digital electronic computer being developed at England's Bletchley Park with guidance of Alan Turing & his seminal work, *On Computable Numbers*. Cold War meant: cryptography remained important, with fundamental advances e.g. public-key cryptography being developed in following decades. Telecommunication industry has also motivated advances in discrete mathematics, particularly in graph theory & information theory. Formal verification of statements in logic has been necessary for software development of safety-critical systems, & advances in automated theorem proving have been driven by this need.

Computational geometry has been an important part of computer graphics incorporated into modern video games & computer-aiddd design tools.

Several fields of discrete mathematics, particularly theoretical computer science, graph theory, & combinatorics, are important in addressing challenging bioinformatics problems associated with understanding tree of life.

Currently, 1 of most famous open problems in theoretical science is P = NP problem, which involves relationship between complexity classes P & NP. Clay Mathematics Institute has offered a $1 million USD prize for 1st correct proof, along with prizes for 6 other mathematical problems." – Wikipedia/discrete mathematics