My notes on $\star$ Sergiy Klymchuk, Counter-examples in Calculus

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Abstract

I take some notes when I learn the book [1].

Contents

1 Selected Counter-examples

Definition. A function $f(x)$ is said to be *increasing at the point* $x = a$ if in a certain neighborhood $(-\delta, \delta)$, where $\delta > 0$ the following is true: if $x < a$ then $f(x) < f(a)$ and if $x > a$ then $f(x) > f(a)$.

Problem 1. *If a function $f(x)$ is continuous and increasing at the point $x = a$ then there is a neighborhood $(x - \delta, x + \delta), \delta > 0$ where the function is also increasing.*

Counter-examples. The function

$$f(x) = \begin{cases} x + x^2 \sin \frac{2}{x}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases} \tag{1.1}$$

is increasing at the point $x = 0$ but it is not increasing in any neighborhood $(-\delta, \delta)$, $\delta > 0$. □

Problem 2. *If a function is not monotone then it doesn't have an inverse function.*

Counter-example.

$$f(x) = \begin{cases} x, \text{ if } x \in \mathbb{Q} \\ -x, \text{ if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} \tag{1.2}$$

□

Problem 3. *If a function is not monotone in (a, b) then its square cannot be monotone on (a, b).*

Counter-example.

$$f(x) = \begin{cases} x, \text{ if } x \in \mathbb{Q} \\ -x, \text{ if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} \tag{1.3}$$

defined on $(0, \infty)$. □

Problem 4. *If on the closed interval $[a, b]$ a function is*

1. *bounded.*
2. *takes its maximum and minimum values.*
3. *takes all its values between the maximum and minimum values.*
   then this function is continuous at some points or subintervals on $[a, b]$.

Counter-example.

$$f(x) = \begin{cases} 1, \text{ if } x = 0 \\ x, \text{ if } x \in \mathbb{Q} \backslash \{0, 1\} \\ -x, \text{ if } x \in \mathbb{R} \backslash \mathbb{Q} \\ 0, \text{ if } x = 1 \end{cases} \tag{1.4}$$

$\square$

Problem 5. *If a function is discontinuous at every point in its domain then the square and the absolute value of this function cannot be continuous.*

Counter-example.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} \tag{1.5}$$

$\square$

Problem 6. *A function cannot be continuous at only one point in its domain and discontinuous everywhere else.*

Counter-example.

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ -x, & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} \tag{1.6}$$

$\square$

Problem 7. *If a function $f(x)$ is differentiable on $(0, \infty)$ and $\lim\limits_{x \to \infty} f(x)$ exists then $\lim\limits_{x \to \infty} f'(x)$ also exists.*

Counter-example.

$$f(x) = \frac{\sin\left(x^2\right)}{x} \tag{1.7}$$

$\square$

Problem 8. *If a function $f(x)$ is differentiable and bounded on $(0, \infty)$ and $\lim\limits_{x \to \infty} f'(x)$ exists then $\lim\limits_{x \to \infty} f(x)$ also exists.*

Counter-example.

$$f(x) = \cos(\ln x) \tag{1.8}$$

$\square$

Problem 9. *If a function $f(x)$ is differentiable at the point $x = a$ then its derivative is continuous at $x = a$.*

Counter-example.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{1.9}$$

at the point $x = 0$. $\square$

Problem 10. *If the derivative of a function $f(x)$ is positive at the point $x = a$*

then there is a neighborhood about $x = a$ (no matter how small) where the function is increasing.

COUNTER-EXAMPLE.

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases} \tag{1.10}$$

□

Problem 11. *If a function $f(x)$ is continuous on (a, b) and has a local maximum at the point $c \in (a, b)$ then in a sufficiently small neighborhood of the point $x = c$ the function is increasing on the left and decreasing on the right from $x = c$.*

COUNTER-EXAMPLE.

$$f(x) = \begin{cases} 2 - x^2 \left(2 + \sin \frac{1}{x}\right), \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases} \tag{1.11}$$

□

Problem 12. *If a function $f(x)$ is differentiable at the point $x = a$ then there is a certain neighborhood of the point $x = a$ where the derivative of the function $f(x)$ is bounded.*

COUNTER-EXAMPLE.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases} \tag{1.12}$$

□

Problem 13. *If a function $f(x)$ at any neighborhood of the point $x = a$ has points where $f'(x)$ doesn't exist then $f'(a)$ doesn't exist.*

COUNTER-EXAMPLE.

$$f(x) = \begin{cases} x^2 \left|\cos \frac{\pi}{x}\right|, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases} \tag{1.13}$$

□

Problem 14. *A function cannot be differentiable only at one point in its domain and non-differentiable everywhere else in its domain.*

COUNTER-EXAMPLE.

$$f(x) = \begin{cases} x^2, \text{ if } x \in \mathbb{Q} \\ 0, \text{ if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} \tag{1.14}$$

□

Problem 15. *A continuous function cannot be non-differentiable at every point in its domain.*

COUNTER-EXAMPLE. The Weierstrass' function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos(3^n x) \tag{1.15}$$

$\square$

Problem 16. *If a function $f(x)$ is continuous and $\int\limits_a^\infty f(x)\,dx$ converges then*

$$\lim_{x\to\infty} f(x) = 0 \tag{1.16}$$

COUNTER-EXAMPLE. The Fresnel integral

$$\int\limits_a^\infty \sin x^2 dx \tag{1.17}$$

$\square$

Problem 17. *If a function $f(x)$ is continuous and non-negative and $\int\limits_a^\infty f(x)\,dx$ converges then $\lim\limits_{x\to\infty} f(x) = 0$.*

Problem 18. *If a function $f(x)$ is positive and unbounded for all real x then the integral $\int\limits_a^\infty f(x)\,dx$ diverges.*

Problem 19. *If a function $f(x)$ is continuous and not bounded for all real x then the integral $\int\limits_a^\infty f(x)\,dx$ diverges.*

COUNTER-EXAMPLE.

$$f(x) = x \sin x^4 \tag{1.18}$$

$\square$

Problem 20. *If a function $f(x)$ is continuous on $[1,\infty)$ and $\int\limits_1^\infty f(x)\,dx$ converges then $\int\limits_1^\infty |f(x)|\,dx$ also converges.*

COUNTER-EXAMPLE.

$$f(x) = \frac{\sin x}{x} \tag{1.19}$$

$\square$

Problem 21. *If the integral $\int\limits_a^\infty f(x)\,dx$ converges and a function $g(x)$ is bounded* *then the integral* $\int\limits_a^\infty f(x)g(x)\,dx$ *converges.*

Counter-example.

$$f(x) = \frac{\sin x}{x}, g(x) = \sin x \tag{1.20}$$

□

2 Selected examples

Problem 22. *A continuous curve that has a sharp corner at every point.*

Example. The Koch's snowflake. □

References

[1] Sergiy Klymchuk, *Counter-examples in Calculus*, November 2004.