Mathematical Analysis & Numerical Analysis
Giải Tích Toán Học & Giải Tích Số

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Ngày 6 tháng 11 năm 2024

Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
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1. *Mathematical Analysis – Giải Tích Toán Học*.
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Mục lục

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Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

1 Wikipedia

Resources – Tài nguyên.

1. [LL01]. Elliott Lieb, Michael Loss. *Analysis.*
2. [Rud76]. Walter Rudin. *Principles Principles of Mathematical Analysis.*
3. [Rud73; Rud87]. Walter Rudin. *Real & Complex Analysis.*
4. [Tao22a]. Terence Tao. *Analysis I.*
5. [Tao22b]. Terence Tao. *Analysis II.*

"Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones." – [LL01, Chap. 1, p. 1]

1.1 Wikipedia/A Mathematician's Apology

"*A Mathematician's Apology* [Har40; Har92; Har22] is a 1940 essay by British mathematician G. H. HARDY which defends the pursuit of mathematics for its own sake. Central to HARDY's "apology" – in the sense of a formal justification or defence (as in PLATO's *Apology of Socrates*) – is an argument that mathematics has value independent of its applications. HARDY located this value in what he called the beauty of mathematics & gave some examples of & criteria for mathematical beauty. The book also includes a brief autobiography which gives insight into the mind of a working mathematician.

1.1.1 Background

HARDY wished to justify his life's work in mathematics for 2 reasons. 1stly, having survived a heart attack & being at the age of 62, HARDY knew that he was approaching old age & that his mathematical creativity & skills were declining. By devoting time to writing the Apology, HARDY was admitting that his own time as a creative mathematician was finished. In his foreword to the 1967 edition of the book, C. P. SNOW describes the Apology as "a passionate lament for creative powers that used to be & that will never come again". In HARDY's words, "Exposition, criticism, appreciation, is work for 2nd-rate mind. [...] It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, & not to talk about what he or other mathematicians have done."

2ndly, at the start of World War II, HARDY, a committed pacifist, wanted to justify his belief that mathematics should be pursued for its own sake rather than for the sake of its applications. He began writing on this subject when he was invited to contribute an article to *Eureka*, the journal of The Archimedeans (the Cambridge University student mathematical society). 1 of the topics the editor suggested was "something about mathematics & the war", & the result was the article "Mathematics in war-time". HARDY later incorporated his article into *A Mathematician's Apology*.

HARDY wanted to write a book in which he would explain his mathematical philosophy to the next generation of mathematicians. He hoped that in this book he could inspire future generations about the importance of mathematics without appealing to its applied uses.

HARDY initially submitted *A Mathematician's Apology* to Cambridge University Press with the intention of personally paying for its printing, but the Press decided to fund publication with an initial run of 4000 copies. For the 1940 1st edition, HARDY sent postcards to the publisher requesting that presentation copies be sent to his sister GERTRUDE EMILY HARDY (1878–1963), C. D. BROAD, JOHN EDENSOR LITTLEWOOD, Sir ARTHUR EDDINGTON, C. P. Snow, the cricketer JOHN LOMAS (to whom G. H. HARDY dedicated the book), and others.

1.1.2 Summary

1 of the main themes of the book is the beauty that mathematics possesses, which HARDY compares to painting & poetry. For HARDY, the most beautiful mathematics was that which had no practical applications (pure mathematics) &, in particular number theory, HARDY's own field. HARDY contends that if useful knowledge is defined as knowledge which is likely to contribute to material comfort without respect to mere intellectual satisfaction, then most of higher mathematics is useless. He justifies the pursuit of pure mathematics with the argument that its very "uselessness" means that it cannot be misused to cause harm. On the other hand, HARDY denigrates much of the applied mathematics as either being "trivial", "ugly", or "dull" & contrasts it with "real mathematics", which is how he describes pure mathematics.

HARDY comments about a phrase attributed to CARL FRIEDRICH GAUSS: "Mathematics is the queen of the sciences & number theory is the queen of mathematics." One may believe that it is the relative sparseness of number theory in applied mathematics that led GAUSS to the above statement; however, HARDY points out that this is certainly not the case. If an application of number theory were to be found, then certainly no one would try to dethrone the "queen of mathematics" by it. What GAUSS meant, according to HARDY, is that the underlying concepts that constitute number theory are deeper & more elegant compared to those of any other branch of mathematics.

Another theme is that mathematics is a "young man's game". HARDY believed that anyone with a talent for mathematics should develop & use that talent while they are young, before their ability to create original mathematics starts to decline in middle age. This view reflects HARDY's increasing depression at the waning of his own mathematical skill. For HARDY, real mathematics was essentially a creative activity, rather than an explanatory or expository one.

1.1.3 Critiques

HARDY's opinions were heavily influenced by the academia culture of the universities Cambridge & Oxford between World War I & World War II.

Some of HARDY's examples seem unfortunate in retrospect. E.g., he writes, "No one has yet discovered many warlike purpose to be served by the theory of numbers or relativity, & it seems unlikely that anyone will do so for many years." Since then number theory was used to crack German Enigma codes, & much later figured prominently in public-key cryptography; furthermore, the inter-convertability of mass & energy predicted by special relativity forms the physical basis for nuclear weapons.

Applicability itself is not the reason that HARDY considered applied mathematics inferior to pure mathematics; it is the simplicity & vulgarity that belong to applied mathematics that led him to describe it as he did. He considered that Rolle's theorem, e.g., cannot be compared to the elegance & preeminence of the mathematics produced by Évasriste Galois & other pure mathematicians, although it is of some importance for calculus." – Wikipedia/A Mathematician's Apology

1.2 Wikipedia/curvature

A migrating wild-type *Dictyostelium discoideum* cell whose boundary is colored by curvature. Scale bar: 5μm. "In mathematics, *curvature* is any of several strongly related concepts in geometry that intuitively measure the amount by which a curve deviates from being a straight line or by which a surface deviates from being a plane. If a curve or surface is contained in a larger space, curvature can be defined *extrinsically* relative to the ambient space. Curvature of Riemannian manifolds of dimension at least 2 can be defined *intrinsically* without reference to a larger space.

For curves, the canonical example is that of a circle, which has a curvature equal to $\frac{1}{R}$. Smaller circles bend more sharply, & hence have higher curvature. The curvature *at a point* of a differentiable curve is the curvature of its osculating circle – i.e., the circle that best approximates the curve near this point. The curvature of a straight line is zero. In contrast to the tangent, which is a vector quantity, the curvature at a point is typically a scalar quantity, i.e., it is expressed by a single real number.

For surfaces (&, more generally for higher-dimensional manifolds), that are embedded in a Euclidean space, the concept of curvature is more complex, as it depends on the choice of a direction on the surface or manifold. This leads to the concepts of *maximal curvature*, minimal curvature, & mean curvature." – Wikipedia/curvature

1.2.1 History

In *Tractatus de configurationibus qualitatum et motuum*, the 14th-century philosopher & mathematician Nicole Oresme introduces the concept of curvature as a measure of departure from straightness; for circles he has the curvature as being inversely proportional to the radius; & he attempts to extend this idea to other curves as a continuously varying magnitude.

The curvature of a differentiable curve was originally defined through osculating circles. In this setting, Augustin-Louis Cauchy showed that the center of curvature is the intersection point of 2 infinitely close normal lines to the curve.

1.2.2 Plane curves

Intuitively, the curvature describes for any part of a curve how much the curve direction changes over a small distance traveled (e.g., angle in rad/m), so it is a measure of the instantaneous rate of change of *direction* of a point that moves on the curve: the larger the curvature, the larger this rate of change. In other words, the curvature measures how fast the unit tangent vector to the curve at point p rotates when point p moves at unit speed along the curve. In fact, it can be proved that this instantaneous rate of change is exactly the curvature. More precisely, suppose that the point is moving on the curve at a constant speed of 1 unit, i.e., the position of the point $P(s)$ is a function of the parameter s, which may be thought as the time or as the arc length from a given origin. Let $\mathbf{T}(s)$ be a unit tangent vector of the curve at $P(s)$, which is also the derivative of $P(s)$ w.r.t. s. Then, the derivative of $\mathbf{T}(s)$ w.r.t. s is a vector that is normal to the curve & whose length is the curvature.

To be meaningful, the definition of the curvature & its different characterizations require that the curve is continuously differentiable near P, for having a tangent that varies continuously; it requires also that the curve is twice differentiable at P, for insuring the existence of the involved limits, & the derivative of $\mathbf{T}(s)$.

The characterization of the curvature in terms of the derivative of the unit tangent vector is probably less intuitive than the definition in terms of the osculating circle, but formulas for computing the curvature are easier to deduce. Therefore, & also because of its use in kinematics, this characterization is often given as a definition of the curvature.

1. **Osculating circle.** Historically, the curvature of a differentiable curve was defined through the osculating circle, which is the circle that best approximates the curve at a point. More precisely, given a point P on a curve, every other point Q of the curve defines a circle (or sometimes a line) passing through Q & tangent to the curve at P. The osculating circle is the limit, if it exists, of this circle when Q tends to P. Then the *center* & the *radius of curvature* of the curve at P are the center & the radius of the osculating circle. The curvature is the reciprocal of radius of curvature. I.e., the curvature is $\kappa = \frac{1}{R}$, where R is the radius of curvature (the whole circle has this curvature, it can be read as turn π over the length $2\pi R$). This definition is difficult to manipulate & to express in formulas. Therefore, other equivalent definitions have been introduced.

2. **In terms of arc-length parametrization.**

3. **In terms of general parametrization.**

4. **Graph of a function.** The graph of a function $y = f(x)$, is a special case of a parameterized curve, of the form $x = t, y = f(t)$. As the 1st & 2nd derivatives of x are 1 & 0, previous formulas simplify to $\kappa = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$ for the curvature, & to $k = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$, for the signed curvature.

   In the general case of a curve, the sign of the signed curvature is somewhat arbitrary, as it depends on the orientation of the curve. In the case of the graph of a function, there is a natural orientation by increasing values of x. This makes significant the sign of the signed curvature.

   The sign of the signed curvature is the same as the sign of the 2nd derivative of f. If it is positive then the graph has an upward concavity, &, if it is negative the graph has a downward concavity. If it is zero, then one has an inflection point or an undulation point.

When the slope of the graph (i.e., the derivative of the function) is small, the signed curvature is well approximated by the 2nd derivative. More precisely, using big O notation, one has $k(x) = y''(1 + O(y'^2))$. It is common in physics & engineering to approximate the curvature with the 2nd derivative, e.g., in beam theory or for deriving the wave equation of a string under tension, & other applications where small slopes are involved. This often allows systems that are otherwise nonlinear to be treated approximately as linear.

5. **Polar coordinates.** If a curve is defined in polar coordinates by the radius expressed as a function of the polar angle, i.e. r is a function of θ, then its curvature is $\kappa(\theta) = \dfrac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{\frac{3}{2}}}$ where the prime refers to differentiation w.r.t. θ. This results from the formula for general parameterizations, by considering the parametrization $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$.

6. **Implicit curve.** For a curve defiend by an implicit equation $F(x, y) = 0$ with partial derivatives denoted $F_x, F_y, F_{xx}, F_{xy}, F_{yy}$, the curvature is given by $\kappa = \dfrac{|F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}|}{(F_x^2 + F_y^2)^{\frac{3}{2}}}$. The signed curvature is not defined, as it depends on an orientation of the curve that is not provided by the implicit equation. Note that changing F into $-F$ would not change the curve defined by $F(x, y) = 0$, but it would change the sign of the numerator if the absolute value were omitted in the preceding formula.

   A point of the curve where $F_x = F_y = 0$ is a singular point, which means that the curve is not differentiable at this point, & thus that the curvature is not defined (most often, the point is either a crossing point or a cusp). The above formula for the curvature can be derived from the expression of the curvature of the graph of a function by using the implicit function theorem & the fact that, on such a curve, one has $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

7. **Examples.** It can be useful to verify on simple examples that the different formulas given in the preceding sections give the same result.

   (a) **Circle.** A common parametrization of a circle of radius r is $\boldsymbol{\gamma}(t) = (r\cos t, r\sin t)$. The formula for the curvature gives $k(t) = \dfrac{r^2\sin^2 t + r^2\cos^2 t}{(r^2\sin^2 t + r^2\cos^2 t)^{\frac{3}{2}}} = \frac{1}{r}$. It follows, as expected, that the radius of curvature is the radius of the circle, & that the center of curvature is the center of the circle. The circle is a rare case where the arc-length parametrization is easy to compute, as it is $\boldsymbol{\gamma}(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right)$. It is an arc-length paramtrization, since the norm of $\boldsymbol{\gamma}'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right)$ is $= 1$. This parametrization gives the same value for the curvature, as it amounts to division by r^3 in both the numerator & the denominator in the preceding formula.

   The same circle can also be defined by the implicit equation $F(x, y) = 0$ with $F(x, y) = x^2 + y^2 - r^2$. Then, the formula for the curvature in this case gives $\kappa = \dfrac{|F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}|}{(F_x^2 + F_y^2)^{\frac{3}{2}}} = \dfrac{8y^2 + 8x^2}{(4x^2 + 4y^2)^{\frac{3}{2}}} = \dfrac{8r^2}{(4r^2)^{\frac{3}{2}}} = \dfrac{1}{r}$.

   (b) **Parabola.** Consider the parabola $y = ax^2 + bx + c$. It is the graph of a function, with derivative $2ax + b$, & the 2nd derivative $2a$. So, the signed curvature is $k(x) = \frac{2a}{(1+(2ax+b)^2)^{\frac{3}{2}}}$. It has the sign of a for all values of x. I.e., if $a > 0$, the concavity is upward directed everywhere; if $a < 0$, the concavity is downward directed; for $a = 0$, the curvature is zero everywhere, confirming that the parabola degenerates into a line in this case.

   The (unsigned) curvature is maximal for $x = -\frac{b}{2a}$, i.e., at the stationary point (zero derivative) of the function, which is the vertex of the parabola.

   Consider the parametrization $\boldsymbol{\gamma}(t) = (t, at^2 + bt + c) = (x, y)$. The 1st derivative of x is 1, & the 2nd derivative is 0. Substituting into the formula for general parametrization gives exactly the same result as above, with x replaced by t. If we use primes for derivatives w.r.t. the parameter t.

   The same parabola can also be defined by the implicit equation $F(x, y) = 0$ with $F(x, y) = ax^2 + bx + c - y$. As $F_y = -1$, $F_{yy} = F_{xy} = 0$, one obtains exactly the same value for the (unsigned) curvature. However, the signed curvature is meaningless here, as $-F(x, y) = 0$ is a valid implicit equation for the same parabola, which gives the opposite sign for the curvature.

8. **Frenet–Serret formulas for plane curves.** The vectors $\mathbf{T}, \mathbf{N}$ at 2 points on a plane curve, a translated version of the 2nd frame, & $\delta\mathbf{T}$ the change in $\mathbf{T}$. Here δs is the distance between the points. In the limit $\frac{d\mathbf{T}}{ds}$ will be in the direction $\mathbf{N}$. The curvature describes the rate of rotation of the frame. The expression of the curvature in terms of arc-length parametrization is essentially the 1st Frenet–Serret formula $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$, where the primes refer to the derivatives w.r.t. the arc length s, & $\mathbf{N}(s)$ is the normal unit vector in the direction of $\mathbf{T}'(S)$. As planar curves have zero torsion, the 2nd Frenet–Serret formula provides the relation $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} = -\kappa\frac{d\gamma}{ds}$. For a general parametrization by a parameter t, one needs expressions involving derivatives w.r.t. t. As these are obtained by multiplying by $\frac{ds}{dt}$ the derivatives w.r.t. s, one has, for any proper parametrization $\mathbf{N}'(t) = \kappa(t)\boldsymbol{\gamma}'(T)$.

9. **Curvature comb.** A *curvature comb* can be used to represent graphically the curvature of every point on a curve. If $t \mapsto x(t)$ is a parameterized curve its comb is defined as the parametrized curve $t \mapsto x(t) + d\kappa(t)n(t)$ where κ, n are the curvature & normal vector & d is a scaling factor (to be chosen as to enhance the graphical representation).

1.2.3 Space curves

1.2.4 Surfaces

1.2.5 Curvature of space

1.2.6 Generalizations

1.3 Wikipedia/cusp (singularity)

"In mathematics, a *cusp*, sometimes called *spinode* in old texts, is a point on a curve where a moving point must reverse direction. A typical example is given in the figure. A cusp is thus a type of singular point of a curve. For a plane curve defined by an analytic, parametric equation $x = f(t), y = g(t)$, a cusp is a point where both derivatives of f, g are zero, & the directional derivative, in the direction of the tangent, changes sign (the direction of the tangent is the direction of the slope $\lim \frac{g'(t)}{f'(t)}$). Cusps are *local singularities* in the sense that they involve only 1 value of the parameter t, in contrast to self-interaction points that involve more than 1 value. In some contexts, the condition on the directional derivative may be omitted, although, in this case, the singularity may look like a regular point.

For a curve defined by an implicit equation $F(x, y) = 0$, which is smooth, cusps are points where the terms of lowest degree of the Taylor expansion of F are a power of a linear polynomial; however, not all singular points that have this property are cusps. The theory of Puiseux series implies that, if F is an analytic function (e.g., a polynomial), a linear change of coordinates allow the curve to be parametrized, in a neighborhood of the cusps, as $x = at^m, y = S(t)$ where $a \in \mathbb{R}$, m is a positive even integer, & $S(t)$ is a power series of order $k > m$ (degree of the nonzero term of the lowest degree). The number m is sometimes called the *order* or the *multiplicity* of the cusp, & is equal to the degree of the nonzero part of lowest degree of F. In some contexts, the definition of a cusp is restricted to the case of cusps of order 2 – i.e., the case where $m = 2$.

The definitions for plane curves & implicitly-defined curves have been generalized by René Thom & Vladimir Arnold to curves defined by differentiable functions: a curve has a cusp at a point if there is a diffeomorphism of a neighborhood of the point in the ambient space, which maps the curve onto 1 of the above-defined cusps.

1.3.1 Classification in differential geometry

Consider a smooth real-valued function of 2 variables, say $f(x, y)$ where $x, y \in \mathbb{R}$. So f is a function from the plane to the line. The space of all such smooth functions is acted upon by the group of diffeomorphisms of the plane & the diffeomorphisms of the line, i.e., diffeomorphic changes of coordinate in both the source & the target. This action splits the whole function space up into equivalence classes, i.e., orbits of the group action.

1 such family of equivalence classes is denoted by $A_k^\pm$, where $k \in \mathbb{N}$. A function f is said to be of type $A_k^\pm$ if it lies in the orbit of $x^2 \pm y^{k+1}$, i.e., there exists a diffeomorphic change of coordinate in source & target which takes f into 1 of these forms. These simple forms $x^2 \pm y^{k+1}$ are said to give normal forms for the type $A_k^\pm$-singularities. Notice that the A_{2n}^+ are the same as the A_{2n}^- since the diffeomorphic change of coordinate $(x, y) \to (x, -y)$ in the source takes $x^2 + y^{k+1}$ to $x^2 - y^{2n+1}$. So we can drop the $\pm$ from $A_{2n}^\pm$ notation. The cusps are then given by the zero-level-sets of the representatives of the A_{2n} equivalence classes, where $n \in \mathbb{N}^\star$.

1.3.2 Examples

A cusp in the semicubical parabola $y^2 = x^3$.

- An *ordinary cusp* is given by $x^2 - y^3 = 0$, i.e., the zero-level-set of a type A_2-singularity. Let $f(x, y)$ be a smooth function of x & y & assume, for simplicity, that $f(0, 0) = 0$. Then a type A_2-singularity of f at $(0, 0)$ can be characterized by:
  1. Having a degenerate quadratic part, i.e., the quadratic terms in the Taylor series of f form a perfect square, say $L(x, y)^2$, where $L(x, y)$ is linear in x, y, &
  2. $L(x, y)$ does not divide the cubic terms in the Taylor series of $f(x, y)$.
- A *rhamphoid cusp* (from Greek 'beak-like') denoted originally a cusp s.t. both branches are on the same side of the tangent, such as for the curve of equation $x^2 - x^4 - y^5 = 0$. As such a singularity is in the same differential class as the cusp of equation $x^2 - y^5 = 0$, which is a singularity of type A_4, the term has been extended to all such singularities. These cusps are non-generic as caustics & wave fronts. The rhamphoid cusp & the ordinary cusp are non-diffeomorphic. A parametric form is $x = t^2, y = ax^4 + x^5$.

For a type A_4-singularity we need f to have a degenerate quadratic part (this gives type $A_{\ge 2}$), that L does divide the cubic terms (this gives type $A_{\ge 3}$), another divisibility condition (giving type $A_{\ge 4}$), & a final non-divisibility condition (giving type exactly A_4).

To see where these extra divisibility conditions come from, assume that f has a degenerate quadratic part L^2 & that L divides the cubic terms. It follows that the 3d order Taylor series of f is given by $L^2 \pm LQ$, where Q is quadratic in x, y. We can complete the square to show that $L^2 \pm LQ = \left(L \pm \frac{Q}{2}\right)^2 - \frac{Q^2}{4}$. We can now make a diffeomorphic change of variable (in this case we simply substitute polynomials with linearly independent linear parts) so that $\left(L \pm \frac{Q}{2}\right)^2 - \frac{Q^2}{4} \to x_1^2 + P_1$ where P_1 is quartic (order 4) in x_1, y_1. The divisibilty condition for type $A_{\ge 4}$ is that x_1 divides P_1. If x_1 does not divide P_1 then we have type exactly A_3 (the zero-level-set here is a tacnode). If x_1 divides P_1 we complete the square on $x_1^2 + P_1$ & change coordinates so that we have $x_2^2 + P_2$ where P_2 is quintic (order 5) in x_2, y_2. If x_2 does not divide P_2 then we have exactly type A_4, i.e., the zero-level-set will be a rhamphoid cusp.

1.3.3 Applications

An ordinary cusp occurring as the caustic of light rays in the bottom of a teacup. Cusps appear naturally when projecting into a plane a smooth curve in 3D Euclidean space. In general, such a projection is a curve whose singularities are self-crossing points & ordinary cusps. Self-crossing points appear when 2 different points of the curves have the same projection. Ordinary cusps appear when the tangent to the curve is parallel to the direction of projection (i.e., when the tangent projects on a single point). More complicated singularities occur when several phenomena occur simultaneously. E.g., rhamphoid cusps occur for inflection points (& for undulation points) for which the tangent is parallel to the direction of projection.

In many cases, & typically in computer vision & computer graphics, the curve that is projected is the curve of the critical points of the restriction to a (smooth) spatial object of the projection. A cusp appears thus as a singularity of the contour of the image of the object (vision) or of its shadow (computer graphics). Caustics & wave fronts are other examples of curves having cusps that are visible in the real world." – Wikipedia/cusp (singularity)

1.4 Wikipedia/functional analysis

"1 of the possible modes of vibration of an idealized circular drum head. These modes are eigenfunctions of a linear operator on a function space, a common construction in functional analysis. Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g., inner product, norm, or topology) & the linear funcitons defined on these spaces & suitably respecting these structures. The historical roots of functional analysis lie in the study of spaces of functions & the formulation of properties of transformations of functions e.g. Fourier transform as transformations defining, e.g., continuous or unitary operators between function spaces. This point of view turned out to be particularly useful for the study of differential equation & integral equation.

The usage of the word *functional* as a noun goes back to the calculus of variations, implying a function whose argument is a function. The term was 1st used in HADAMARD's 1910 book on that subject. However, the general concept of a functional had previously been introduced in 1887 by the Italian mathematician & physicist VITO VOLTERRA. The theory of nonlinear functionals was continued by students of HADAMARD, in particular FRÉCHET & LÉVY. HADAMARD also found the modern school of linear functional analysis further developed by RIESZ & the group of Polish mathematicians around STEFAN BANACH.

In modern introductory texts on functional analysis, the subject is seen as the study of vector spaces endowed with a topology, in particular infinite-dimensional spaces. In contrast, linear algebra deals mostly with finite-dimensional spaces, & does not use topology. An important part of functional analysis is the extension of the theories of measure, integration, & probability to infinite-dimensional spaces, also known as *infinite dimensional analysis*.

1.4.1 Normed vector spaces

The basic & historically 1st class of spaces studied in functional analysis are complete normed vector spaces over the $\mathbb{R}$ or $\mathbb{C}$, called Banach spaces. An important example is a Hilbert space, where the norm arises from an inner product. These spaces are of fundamental importance in many areas, including the mathematical formulation of quantum mechanics, machine learning ML, PDEs, & Fourier analysis.

More generally, functional analysis includes the study of Fréchet spaces & other topological vector spaes not endowed with a norm.

An important object of study in functional analysis are the continuous linear operators defined on Banach & Hilbert spaces. These lead naturally to the definition of $C^\star$-algebras & other operator algebras.

- Hilbert spaces. Hilbert spaces can be completely classified: there is a unique Hilbert space upto isomorphism for every cardinality of the orthonormal basis. Finite-dimensional Hilbert spaces are fully understood in linear algebra, & infinite-dimensional separable Hilbert spaces are isomorphic to $l^2(\aleph_0)$. Separability being important for applications, functional analysis of Hilbert spaces consequently mostly deals with this space. 1 of the open problems in functional analysis is to prove that every bounded linear operator on a Hilbert space has a proper invariant subspace. Many special cases of this invariant subspace problem have already been proven.

- Banach spaces. General Banach spaces are more complicated than Hilbert spaces, & cannot be classified in such a simple manner as those. In particular, many Banach spaces lack a notion analogous to an orthonormal basis. Examples of Banach spaces are L^p-spaces for $p \in [1, \infty)$. Given also a measure μ on set X, then $L^p(X)$, sometimes also denoted $L^p(X, \mu)$ or $L^p(\mu)$, has as its vectors equivalence classes $[f]$ of measurable functions whose absolute value's pth power has finite integral, i.e., functions f for which one has $\int_X |f(x)|^p \, \mathrm{d}\mu(x) < \infty$. If μ is the counting measure, then the integral may be replaced by a sum, i.e., require $\sum_{x\in X} |f(x)|^p < \infty$. Then it is not necessary to deal with equivalence classes, & the space is denoted $l^p(X)$, written more simply l^p in the case when $X = \mathbb{N}$.

  In Banach spaces, a large part of the study involves the dual space: the space of all continuous linear maps from the space into its underlying field, so-called functionals. A Banach space can be canonically identified with a subspace of its bidual, which is the dual of its dual space. The corresponding map is an isometry but in general not onto. A general Banach space & its bidual need not even be isometrically isomorphic in any way, contrary to the finite-dimensional situation. This s explained in the dual space article. Also, the notion of derivative can be extended to arbitrary functions between Banach spaces, see, e.g., Wikipedia/Fréchet derivative.

1.4.2 Linear functional analysis

1.4.3 Major & foundational results

There are 4 major theorems which are sometimes called the 4 pillars of functional analysis:

1. Hahn–Banach theorem
2. opening mapping theorem
3. closed graph theorem
4. uniform boundedness principle, also known as the Banach–Steinhaus theorem.

Important results of functional analysis include:

1. **Uniform bounded principle.** Main article: Wikipedia/Banach–Steinhause theorem. The uniform boundedness principle or Banach–Steinhaus theorem is 1 of the fundamental results in functional analysis. Together with the Hahn–Banach theorem & the opening mapping theorem, it is considered 1 of the cornerstones of the field. In its basic form, it asserts that for a family of continuous linear operators (& thus bounded operators) whose domain is a Banach space, pointwise boundedness is equivalent to uniform boundedness in operator form. The theorem was 1st published in 1927 by Stefan Banach & Hugo Steinhaus but it was also proven independently by Hans Hahn.

   Theorem 1 (Uniform Boundedness Principle)**.** *Let X be a Banach space & Y be a normed vector space. Suppose that F is a collection of continuous linear operators from $X \to Y$. If $\forall x \in X$, one has $\sup_{T\in F} \|T(x)\|_Y < \infty$, then $\sup_{T\in F} \|T\|_{B(X,Y)} < \infty$.*

2. **Spectral theorem.** Main article: Wikipedia/spectral theorem. There are many theorems known as the spectral theorem, but one in particular has many applications in functional analysis.

   Theorem 2 (Spectral theorem)**.** *Let A be a bounded self-adjoint operator on a Hilbert space H. Then there is a measure space (X, Σ, μ) & a real-valued essentially bounded measurable function f on X & a unitary operator $U : H \to L^2_\mu(X)$ s.t. $U^\star TU = A$ where T is the multiplication operator: $[T\varphi](x) = f(x)\varphi(x)$, & $\|T\| = \|f\|_\infty$.*

   This is the beginning of the vast research area of functional analysis called operator theory, see also spectral measure. There is also an analogous spectral theorem for bounded normal operators on Hilbert spaces. The only difference in the conclusion is that now f may be complex-valued.

3. **Hahn–Banach theorem.** Main article: Wikipedia/Hahn–Banach theorem. The Hahn–Banach theorem is a central tool in functional analysis. It allows the extension of bounded linear functionals defined on a subspace of some vector space to the whole space, & it also shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of the dual space "interesting".

   Theorem 3 (Hahn–Banach theorem)**.** *If $p : V \to \mathbb{R}$ is a sublinear function, & $\varphi : U \to \mathbb{R}$ is a linear functional on a linear subspace $U \subseteq V$ which is dominated by p on U, i.e., $\varphi(x) \le p(x)$, $\forall x \in U$, then there exists a linear extension $\psi : V \to \mathbb{R}$ of φ to the whole space V which is dominated by p on V, i.e., there exists a linear functional ψ s.t. $\psi(x) = \varphi(x)$, $\forall x \in U$, $\psi(x) \le p(x)$, $\forall x \in V$.*

4. **Open mapping theorem.** Main article: Wikipedia/open mapping theorem (functional analysis). The open mapping theorem, also known as the Banach–Schauder theorem (named after Stefan Banach & Juliusz Schauder), is a fundamental result which states that if a continuous linear operator between Banach spaces is surjective then it is an open map. More precisely,

   Theorem 4 (Open mapping theorem)**.** *If X, Y are Banach spaces & $A : X \to Y$ i a surjective continuous linear operator, then A is an open map (i.e., if U is an open set in X, then $A(U)$ is open in Y).*

   The proof uses the Baire category theorem, & completeness of both X, Y is essential to the theorem. The statement of the theorem is no longer true if either space is just assumed to be a normed space, but is true if X, Y are taken to be Fréchet spaces.

5. **Closed graph theorem.** Main article: Wikipedia/closed graph theorem

   Theorem 5 (Closed graph theorem)**.** *If X is a topological space & Y is a compact Hausdorff space, then the graph of a linear map $T : X \to Y$ is closed iff T is continuous.*

6. Other topics, see Wikipedia/list of functional analysis topics.

1.4.4 Foundations of mathematics considerations

Most spaces considered in functional analysis have infinite dimension. To show the existence of a vector space basis for such spaces may require Zorn's lemma. However, a somewhat different concept, the Schauder basis, is usually more relevant in functional analysis. Many theorems require the Hahn–Banach theorem, usually proved using the axiom of choice, although the strictly weaker Boolean prime ideal theorem suffices. The Baire category theorem, needed to prove many important theorems, also requires a form of axiom of choice.

1.4.5 Point of view

Functional analysis includes the following tendencies:

- *Abstract analysis.* An approach to analysis based on topological groups, topological rings, & topological vector spaces.
- *Geometry of Banach spaces* contains many topics. One is combinatorial approach connected with Jean Bourgain, another is a characterization of Banach spaces in which various forms of the law of large numbers hold.
- *Noncommutative geometry.* Developed by Alain Connes, partly building on earlier notions, e.g. George Mackey's approach to ergodic theory.
- *Connection with quantum mechanics.* Either narrowly defined as in mathematical physics, or broadly interpreted by, e.g., Israel Gelrand, to include most types of representation theory." – Wikipedia/functional analysis

1.5 Wikipedia/Helmholtz decomposition

"In physics & mathematics, the *Helmholtz decomposition theorem* or the *fundamental theorem of vector calculus* states that certain differentiable vector fields can be resolved into the sum of an irrotational (curl-free) vector field & a solenoidal (divergence-free) vector field. In physics, often only the decomposition of sufficiently smooth, rapidly decaying vector fields in 3D is discussed. It is named after Hermann von Helmholtz.

Definition 1. *For a vector field* $\mathbf{F} \in C^1(V, \mathbb{R}^n)$ *defined on a domain* $V \subseteq \mathbb{R}^n$*, a* Helmholtz decomposition *is a pair of vector fields* $\mathbf{G} \in C^1(V, \mathbb{R}^n), \mathbf{R} \in C^1(V, \mathbb{R}^n)$ *s.t.* $\mathbf{F}(\mathbf{r}) = \mathbf{G}(\mathbf{r}) + \mathbf{R}(\mathbf{r})$, $\mathbf{G}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, $\nabla \cdot \mathbf{R}(\mathbf{r}) = 0$. *Here,* $\Phi \in C^2(V, \mathbb{R})$ *is a scalar potential,* $\nabla\Phi$ *is its gradient, &* $\nabla \cdot \mathbf{R}$ *is the divergence of the vector field* $\mathbf{R}$*. The irrotational vector field* $\mathbf{G}$ *is called a* gradient field *&* $\mathbb{R}$ *is called a* solenoidal field *or* rotation field*. This decomposition does not exist for all vector fields & is not unique.*

1.5.1 History

The Helmholtz decomposition in 3D was 1st described in 1849 by George Gabriel Stokes for a theory of diffraction. Hermann von Helmholtz published his paper on some hydrodynamic basic equations in 1858, which was part of his research on the Helmholtz's theorems describing the motion of fluid in the vicinity of vortex lines. Their derivation required the vector fields to decay sufficiently fast at ∞. Later, this condition could be relaxed, & the Helmholtz decomposition could be extended to higher dimensions. For Riemannian manifolds, the Helmholtz-Hodge decomposition using differential geometry & tensor calculus was derived.

The decomposition has become an important tool for many problems in theoretical physics, but has also found applications in animation, computer vision as well as robotics

1.5.2 3D space

Many physics textbooks restrict the Helmholtz decomposition to 3D space & limit its application to vector fields that decay sufficiently fast at ∞ or to bump functions that are defined on a bounded domain. Then, a vector potential A can be defined, s.t. the rotation field is given by $\mathbf{R} = \nabla \times \mathbf{A}$, using the curl of a vector field.

Let $\mathbf{F}$ be a vector field on a bounded domain $V \subseteq \mathbb{R}^3$, which is twice continuously differentiable inside V, & let S be the surface that encloses the domain V. Then $\mathbf{F}$ can be decomposed into a curl-free component & a divergence-free component as $\boxed{\mathbf{F} = -\nabla\Phi + \nabla \times \mathbf{A}}$. [...]

1.5.3 Generalization to higher dimensions

1.5.4 Differential forms

The Hodge decomposition is closely related to the Helmholtz decomposition, generalizing from vector fields on $\mathbb{R}^3$ to differential forms on a Riemannian manifold M. Most formulations of the Hodge decomposition require M to be compact. Since this is not true of $\mathbb{R}^3$, the Hodge decomposition theorem is not strictly a generalization of the Helmholtz theorem. However, the compactness restriction in the usual formulation of the Hodge decomposition can be replaced by suitable decay assumptions at infinity on the differential forms involved, giving a proper generalization of the Helmholtz theorem.

1.5.5 Extensions to fields not decaying at infinity

1.5.6 Uniqueness of the solution

1.5.7 Applications

- **Electrodynamics.** The Helmholtz theorem is of particular interest in electrodynamics, since it can be used to write Maxwell's equations in the potential image & solve them more easily. The Helmholtz decomposition can be used to prove that, given electric current density & charge density, the electric field & the magnetic flux density can be determined. They are unique if the densities vanish at infinity & one assumes the same for the potentials.

- **Fluid dynamics.** In fluid dynamics, the Helmholtz projection plays an important role, especially for the solvability theory of NSEs. If the Helmholtz projection is applied to the linearized incompressible NSEs, the Stokes equation is obtained. This depends only on the velocity of the particles in the flow, but no longer on the static pressure, allowing the equation to be reduced to 1 unknown. However, both equations, the Stokes & linearization equations, are equivalent. The operator $P\Delta$ is called the Stokes operator.

- **Dynamical systems theory.** In the theory of dynamical systems, Helmholtz decomposition can be used to determine "quasipotentials" as well as to compute Lyapunov functions in some cases.

- **Medical Imaging.** In magnetic resonance elastography, a variant of MR imaging where mechanical waves are used to probe the viscoelasticity of organs, the Helmholtz decomposition is sometimes used to separate the measured displacement fields into its shear component (divergence-free) & its compression component (curl-free). In this way, the complex shear modulus can be calculated without contributions from compression waves.

- **Computer animation & robotics.** The Helmholtz decomposition is also used in the field of computer engineering. This includes robotics, image reconstruction but also computer animation, where the decomposition is used for realistic visualization of fluids or vector fields." – Wikipedia/Helmholtz decomposition

1.6 Wikipedia/integral operator

"An *integral operator* is an operator that involves integration. Special instances are:

- The operator of integration itself, denoted by the integral symbol
- Integral linear operators, which are linear operators induced by bilinear forms involving integrals
- Integral transforms, which are maps between 2 function spaces, which involve integrals." – Wikipedia/integral operator

1.7 Wikipedia/integral transform

Resources – Tài nguyên.

1. Đặng Đình Áng. *Biến Đổi Tích Phân*.

"In mathematics, an *integral transform* is a type of transform that maps a function from its original function space into another function space via integration, where some of the properties of the original function might be more easily characterized & manipulated than in the original function space. The transformed function can generally be mapped back to the original function space using the *inverse transform*.

1.7.1 General form

An integral transform is any transform T of the form $(Tf)(u)=\int_{t_1}^{t_2} f(t)K(t,u)\,\mathrm{d}t$. The input of this transform is a function f, & the output is another function Tf. An integral transform is a particular kind of mathematical operator. There are numerous useful integral transforms. Each is specified by a choice of the function K of 2 variables, called the *kernel* or *nucleus* (hạt nhân) of the transform.

Some kernels have an associated *inverse kernel* $K^{-1}(u,t)$ which (roughly speaking) yields an inverse transform: $f(t)=\int_{t_1}^{t_2}(Tf)(u)K^{-1}(u,t)\,\mathrm{d}u$. A *symmetric kernel* is one that is unchanged when the 2 variables are permuted; it is a kernel function K s.t. $K(t,u)=K(u,t)$. In the theory of integral equations, symmetric kernels corresponds to self-adjoint operators.

1.7.2 Motivation

There are many classes of problems that are difficult to solve – or at least quite unwieldy algebraically – in their original representations. An integral transform "maps" an equation from its original "domain" into another domain, in which manipulating & solving the equation may be much easier than in the original domain. The solution can then be mapped back to the original domain with the inverse of the integral transform.

There are many applications of probability that rely on integral transforms, such as "pricing kernel" or stochastic discount factor, or the smoothing of data recovered from robust statistics, see kernel (statistics).

1.7.3 History

The precursor of the transforms were the Fourier series to express functions in finite intervals. Later the Fourier transform was developed to remove the requirement of finite intervals.

Using the Fourier series, just about any practical function of time (e.g., the voltage across the terminals of an electronic device) can be represented as a sum of sines & cosines, each suitably scaled (multiplied by a constant factor), shifted (advanced or retarded in time) & "squeezed" or "stretched" (increasing or decreasing the frequency). The sines & cosines in the Fourier series are an example of an orthonormal basis.

1.7.4 Usage example

As an example of an application of integral transforms, consider the Laplace transform. This is a technique that maps differential or integro-differential equations in the "time" domain into polynomial equations in what is termed the "complex frequency" domain. (Complex frequency is similar to actual, physical frequency but rather more general. Specifically, the imaginary component ω of the complex frequency $s = -\sigma + i\omega$ corresponds to the usual concept of frequency, viz., the rate at which a sinusoid cycles, whereas the real component σ of the complex frequency corresponds to the degree of "damping", i.e., an exponential decrease of the amplitude.) The equation cast in terms of complex frequency is readily solved in the complex frequency domain (roots of the polynomial equations in the complex frequency domain correspond to eigenvalues in the time domain), leading to a "solution" formulated in the frequency domain. Employing the inverse transform, i.e., the inverse procedure of the original Laplace transform, one obtains a time-domain solution. In this example, polynomials in the complex frequency domain (typically occurring in the denominator) correspond to power series in the time domain, while axial shifts in the complex frequency domain correspond to damping by decaying exponentials in the time doamin.

The Laplace transform finds wide application in physics & particularly in electrical engineering, where the characteristic equations that describe the behavior of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially scaled & time-shifted damped sinusoids in the time domain. Other integral transforms find special applicability within other scientific & mathematical disciplines.

Another usage example is the kernel in the path integral $\psi(t,x) \coloneqq \int_{\mathbb{R}} \psi(t',x')K(t,x;t',x')\,\mathrm{d}x'$. This states that the total amplitude $\psi(t,x)$ to arrive at (t,x) is the sum (the integral) over all possible values x' of the total amplitude $\psi(t',x')$ to arrive at the point (t',x') multiplied by the amplitude to go from x' to x, i.e. $K(t,x;t',x')$. It is often referred to as the propagator for a given system. This (physics) kernel is the kernel of the integral transform. However, for each quantum system, there is a different kernel.

1.7.5 Table of transforms

1. Abel transform
2. Associated Legendre transform
3. Fourier transform
4. Fourier sine transform
5. Fourier cosine transform
6. Hankel transform
7. Hartley transform
8. Hermite transform
9. Hilbert transform
10. Jacobi transform
11. Laguerre transform
12. Laplace transform
13. Legendre transform
14. Mellin transform
15. 2-sided Laplace transform
16. Poisson kernel
17. Radon transform
18. Weierstrass transform
19. X-ray transform

In the limits of integration for the inverse transform, c is a constant which depends on the nature of the transform function. E.g., for the 1- & 2-sided Laplace transform, c must be greater than the largest real part of the zeros of the transform function. Note that there are alternative notations & conventions for the Fourier transform.

1.7.6 Different domains

Here integral transforms are defined for functions on $\mathbb{R}$, but they can be defined more generally for functions on a group.

- If instead one uses functions on the circle (periodic functions), integration kernels are then biperiodic functions; convolution by functions on the circle yields circular convolution.
- If one uses functions on the cyclic group of order n (C_n or $\mathbb{Z}/n\mathbb{Z}$), one obtains $n \times n$ matrices as integration kernels; convolution corresponds to circulant matrices.

1.7.7 General theory

Although the properties of integral transforms vary widely, they have some properties in common. E.g., every integral transform is a linear operator, since the integral is a linear operator, & in fact if the kernel is allowed to be a generalized function then all linear operators are integral transforms (a properly formulated version of this statement is the Schwartz kernel theorem).

The general theory of such integral equations is known as Fredholm theory. In this theory, the kernel is understood to be a compact operator acting on a Banach space of functions. Depending on the situation, the kernel is then variously referred to as the Fredholm operator, the nuclear operator or the Fredholm kernel." – Wikipedia/integral transform

1.8 Wikipedia/Lax equivalence theorem

"In numerical analysis, the *Lax equivalence theorem* is a fundamental theorem in the analysis of FDMs for the numerical solution of PDEs. It states that for a consistent FDM for a well-posed linear IVP, the method is convergent iff it is stable.

The importance of the theorem is that while the convergence of the solution of the FDM to the solution of the PDE is what is desired, it is ordinarily difficult to establish because the numerical method is defined by a recurrence relation while the differential equation involves a differentiable function. However, consistency – the requirement that the FDM approximates the correct PDE – is straightforward to verify, & stability is typically much easier to show than convergence (& would be needed in any event to show that round-off error will not destroy the computation). Hence convergence is usually shown via the Lax equivalence theorem.

Stability in this context means that a matrix norm of the matrix used in the iteration is at most unity, called (practical) Lax–Richtmyer stability. Often a von Neumann stability analysis is substituted for convenience, although von Neumann stability only implies Lax–Richtmyer stability in certain cases.

This theorem is due to PETER LAX, sometimes called the *Lax–Richtmyer theorem*, after PETER LAX & ROBERT D. RICHTMYER." – Wikipedia/Lax equivalence theorem

1.9 Wikipedia/level set

Points at constant slices of $x_2 = f(x_1)$. Lines at constant slices of $x_3 = f(x_1, x_2)$. Planes at constant slices of $x_4 = f(x_1, x_2, x_3)$. $(n-1)$-dimensional level sets for functions of the form $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i = a_1 x_1 + \cdots + a_n x_n$ where $a_1, \ldots, a_n$ are constants, in $(n+1)$-dimensional Euclidean space, for $n = 1, 2, 3$. "In mathematics, a *level set* of a real-valued function f of n real variables is a set where the function takes on a given constant value c, i.e., $L_c(f) = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) = c\}$. When the number of independent variables is 2, a level set is called a *level curve*, also known as *contour line* or *isoline*; so a level curve is the set of all real-valued solutions of an equation in 2 variables x_1, x_2. When $n = 3$, a level set is called a *level surface* (or *isosurface*); so a level surface is the set of all real-valued roots of an equation in 3 variables x_1, x_2, x_3. For higher values of n, the level set is a *level hypersurface*, the set of all real-valued roots of an equation in $n > 3$ variables. A level set is a special case of a fiber. Contour curves at constant slices of $x_3 = f(x_1, x_2)$. Curved surfaces at constant slices of $x_4 = f(x_1, x_2, x_3)$. $(n-1)$-dimensional level sets of nonlinear functions $f(x_1, \ldots, x_n)$ in $(n+1)$-dimensional Euclidean space, for $n = 1, 2, 3$.

1.9.1 Alternative names

Level sets show up in many applications, often under different names. E.g., an implicit curve is a level curve, which is considered independently of its neighbor curves, emphasizing that such a curve is defined by an implicit equation. Analogously, a level surface is sometimes called an implicit surface or an isosurface.

The name isocontour is also used, which means a contour of equal height. In various application areas, isocontours have received specific names, which indicate often the nature of the values of the considered function, e.g. isobar, isotherm, isogon, isochrone, isoquant & indifference curve.

Example 1. *Consider the 2D Euclidean distance $d(x, y) = \sqrt{x^2 + y^2}$. A level set $L_r(d)$ of this function consists of those points that lie at a distance of r from the origin, that make a circle. E.g., $(3, 4) \in L_5(d)$, because $d(3, 4) = 5$. Geometrically, this means that the point $(3, 4)$ lies on the circle of radius 5 centered at the origin. More generally, a sphere in a metric space (M, m) with radius r centered at $x \in M$ can be defined as the level set $L_r(y \mapsto m(x, y))$.*

Example 2. Intersections of a co-ordinate function's level surfaces with a trefoil knot. *The plot of Himmelblau's function. Each curve shown is a level curve of the function, & they are spaced logarithmically: if a curve represents L_x, the curve directly "within" represents $L_{x/10}$, & the curve directly "outside" represents L_{10x}.* Log-spaced level curve plot of Himmelblau's function.

1.9.2 Level sets vs. gradient

Theorem 6. *If the function f is differentiable, the gradient of f at a point is either zero, or perpendicular to the level set of f at that point.*

To understand what this means, imagine that 2 hikers are at the same location on a mountain. 1 of them is bold, & decides to go in the direction where the slope is steepest. The other one is more cautious & does not want to either climb or descend, choosing a path which stays at the same height. In our analogy, the above theorem asys that the 2 hikers will depart in directions perpendicular to each other.

A consequence of this theorem (& its proof) is that f is differentiable, a level set is a hypersurface & a manifold outside the critical points of f. At a critical point, a level set may be reduced to a point (e.g., at a local extremum of f) or may have a singularity such as a self-intersection point or a cusp.

1.9.3 Sublevel & superlevel sets

A set of the form $L_c^-(f) \coloneqq \{(x_1,\ldots,x_n)|f(x_1,\ldots,x_n)\le c\}$ is called a *sublevel set* of f (or, alternatively, a *lower level set* or *trench* of f). A *strict sublevel* set of f is $\{(x_1,\ldots,x_n)|f(x_1,\ldots,x_n)<c\}$. Similarly $L_c^+(f) \coloneqq \{(x_1,\ldots,x_n)|f(x_1,\ldots,x_n)\ge c\}$ is called a *superlevel set* of f (or, alternatively, an *upper level set* of f). & a *strict superlevel set* of f is $\{(x_1,\ldots,x_n)|f(x_1,\ldots,x_n)>c\}$. Sublevel sets are important in minimization theory. By Weierstrass's theorem, the boundness of some nonempty sublevel set & the lower-semicontinuity of the function implies that a function attains its minimum. The convexity of all the sublevel sets characterizes quasiconvex functions." – Wikipedia/level set

1.10 Wikipedia/level-set method

"The *level-set method* (LSM) is a conceptual framework for using level sets as a tool for numerical analysis of surfaces & shapes. LSM can perform numerical computations involving curves & surfaces on a fixed Cartesian grid without having to parameterize these objects. LSM makes it easier to perform computations on shapes with sharp corners & shapes that change topology (e.g. by splitting in 2 or developing holes). These characteristics make LSM effective for modeling objects that vary in time, such as an airbag inflating or a drop of oil floating in water.

1.10.1 Overview

A bounded region with a well-behaved boundary. Below it, the red surface is the graph of a level set function φ determining this shape, & the flat blue region represents the X-Y plane. The boundary of the shape is then the zero-level set of φ, while the shape itself is the set of points in the plane for which φ is positive (interior of the shape) or zero (at the boundary).

In the top row, the shape's topology changes as it is split in 2. It is challenging to describe this transformation numerically by parameterizing the boundary of the shape & following its evolution. An algorithm can be used to detect the moment the shape splits in 2 & then construct parameterizations for the 2 newly obtained curves. On the bottom row, however, the plane at which the level set function is sampled is translated upwards, on which the shape's change in topology is described. It is less challenging to work with a shape's change in topology is described. It is less challenging to work with a shape through its level-set function rather than with itself directly, in which a method would need to consider all the possible deformations the shape might undergo.

Thus, in 2D, the level-set method amounts to representing a closed curve Γ (e.g. the shape boundary) using an auxiliary function φ, called the *level-set function*. The curve Γ is represented as the zero-level set of φ by $\Gamma \coloneqq \{(x,y)|\varphi(x,y)=0\}$, & the level-set method manipulates Γ *implicitly* through the function φ. This function φ is assumed to take positive values inside the region delimited by the curve Γ & negative values outside.

1.10.2 The level-set equation

If the curve Γ moves in the normal direction with a speed v, then by chain rule & implicit differentiation, it can be determined that the level-set function φ satisfies the *level-set equation* $\boxed{\partial_t\varphi = v|\nabla\varphi|}$. Here, $|\cdot|$ is the Euclidean norm (denoted customarily by single bars in PDEs), & t is time. This is a PDE, in particular a Hamilton–Jacobi equation, & can be solved numerically, e.g., by using finite differences on a Cartesian grid.

However, the numerical solution of the level set equation may require advanced techniques. Simple FDMs fails quickly. Upwinding methods e.g. Godunov method are considered better; however, the level set method does not guarantee preservative of the volume & shape of the set level in an advection field that maintains shape & size, e.g., a uniform or rotational velocity field. Instead, the shape of the level set may become distorted, & the level set may disappear over a few time steps. Therefore, high-order FDMs, e.g. high-order essentially non-oscillatory (ENO) schemes, are often required, & even then, the feasibility of long-term simulations is questionable. More advanced methods have been developed ito overcome this; e.g., combinations of the leveling method with tracking marker particles suggested by the velocity field.

1.10.3 Example

Consider a unit circle in $\mathbb{R}^2$, shrinking it on itself at a constant rate, i.e., each point on the boundary of the circle moves along its inwards pointing normally at some fixed speed. The circle will shrink & eventually collapse down to a point. If an initial distance field is constructed (i.e., a function whose value is the signed Euclidean distance to the boundary, positive interior, negative exterior) on the initial circle, the normalized gradient of this field will be the circle normal.

If the field has a constant value subtracted from it in time, the zero level (which was the initial boundary) of the new fields will also be circular & will similarly collapse to a point. This is due to this being effectively the temporal integration of the Eikonal equation with a fixed front velocity.

1.10.4 Applications

- In mathematical modeling of combustion, LSM is used to describe the instantaneous flame surface, known as G equation.
- Level-set data structures have been developed to facilitate the use of the level-set method in computer applications.
- CFD
- Trajectory planning
- Optimization
- Image processing
- Computational biophysics
- Discrete complex dynamics (visualization of the parameter plane & the dynamic plane)

1.10.5 History

The level-set method was developed in 1979 by Alain Dervieux, & subsequently popularized by Stanley Osher & James Sethian. It has since become popular in many disciplines, e.g., image processing, computer graphics, computational geometry, optimization, CFDs, & computational biology." – Wikipedia/level-set method

1.11 Wikipedia/minimal surface

A helicoid minimal surface formed by a soap film on a helical frame. "In mathematics, a *minimal surface* is a surface that locally minimizes its area. This is equivalent to having zero mean curvature.

The term "minimal surface" is used because these surfaces originally arose as surfaces that minimized total surface area subject to some constraint. Physical models of area-minimizing minimal surfaces can be made by dipping a wire frame into a soap solution, forming a soap film, which is a minimal surface whose boundary is the wire frame. However, the term is used for more general surfaces that may self-intersect or do not have constraints. For a given constraint there may also exist several minimal surfaces with different areas (e.g., see minimal surface of revolution): the standard definitions only relate to a local optimum, not a global optimum.

1.11.1 Definitions

Saddle tower minimal surface. While any small change of the surface increases its area, there exist other surfaces with the same boundary with a smaller total area. Minimal surface curvature planes. On a minimal surface, the curvature along the principal curvature planes are equal & opposite at every point. This makes the mean curvature zero. Minimal surfaces can be defined in several equivalent ways in $\mathbb{R}^3$. The fact that they are equivalent serves to demonstrate how minimal surface theory lies at the crossroads of several mathematical disciplines, especially differential geometry, calculus of variations, potential theory, complex analysis, & mathematical physics.

Definition 2 (Local least area definition)**.** *A surface $M \subset \mathbb{R}^3$ is minimal iff every point $p \in M$ has a neighborhood, bounded by a simple closed curve, which has the least area among all surfaces having the same boundary.*

This property is local: there might exist regions in a minimal surface, together with other surfaces of smaller area which have the same boundary. This property establishes a connection with soap films; a soap film deformed to have a wire frame as boundary will minimize area.

Definition 3 (Variational definition)**.** *A surface $M \subset \mathbb{R}^3$ is minimal iff it is a critical point of the area functional for all compactly supported variations.*

This definition makes minimal surfaces a 2D analogue to geodesics, which are analogously defined as critical points of the length functional.

Definition 4 (Mean curvature definition)**.** *A surface $M \subset \mathbb{R}^3$ is minimal iff its mean curvature is equal to zero at all points.*

A direct implication of this definition is that every point on the surface is a saddle point with equal & opposite principal curvatures. By the Young–Laplace equation, the mean curvature of a soap film is proportional to the difference in pressure between the sides. If the soap film does not enclose a region, then this will make its mean curvature zero. By contrast, a spherical soap bubble encloses a region which has a different pressure from the exterior region, & as such does not have zero mean curvature.

Definition 5 (Differential equation definition). *A surface $M \subset \mathbb{R}^3$ is minimal iff it can be locally expressed as the graph of a solution of* $(1+u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1+u_y)^2u_{xx} = 0$.

The PDE in this definition was originally found in 1762 by LAGRANGE, & JEAN BAPTISTE MEUSNIER discovered in 1776 that it implied a vanishing mean curvature.

Definition 6 (Energy definition). *A conformal immersion $X : M \to \mathbb{R}^3$ is minimal iff it is a critical point of the Dirichlet energy for all compactly supported variations, or equivalently if any point $p \in M$ has a neighborhood with least energy relative to its boundary.*

This definition ties minimal surfaces to harmonic functions & potential theory.

Definition 7 (Harmonic definition). *If $X = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ is an isometric immersion of a Riemann surface into 3-space, then X is said to be* minimal *whenever x_i is a harmonic function on M for each i.*

A direct implication of this definition & the maximum principle for harmonic functions is that there are no compact complete minimal surfaces in $\mathbb{R}^3$.

Definition 8 (Gauss map definition). *A surface $M \to \mathbb{R}^3$ is minimal iff its stereographically projected Gauss map $g : M \to \mathbb{C} \cup \{\infty\}$ is meromorphic w.r.t. the underlying Riemann surface structure, & M is not a piece of a sphere.*

This definition uses that the mean curvature is half of the trace of the shape operator, which is linked to the derivative of the Gauss map. If the projected Gauss map obeys the Cauchy–Riemann equations then either the trace vanishes or every point of M is umbilic, in which case it is a piece of a sphere.

The local least area & variational definitions allow extending minimal surfaces to other Riemann manifolds than $\mathbb{R}^3$.

1.11.2 History

Minimal surface theory originates with LAGRANGE who in 1762 considered the variational problem of finding the surface $z = z(x,y)$ of least area stretched across a given closed contour. He derived the Euler–Lagrange equation for the solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{z_x}{\sqrt{1+z_x^2+z_y^2}} + \frac{\mathrm{d}}{\mathrm{d}y}\frac{z_y}{\sqrt{1+z_x^2+z_y^2}} = 0. \tag{1}$$

He did not succeed in finding any solution beyond the plane. In 1776 JEAN BAPTISTE MARIE MEUSNIER discovered that the helicoid & catenoid satisfy the equation & that the differential expression corresponds to twice the mean curvature of the surface, concluding that surfaces with zero mean curvature are area-minimizing.

By expanding Lagrange's equation to $(1+z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1+z_y^2)z_{xx} = 0$, GASPARD MONGE & LEGENDRE in 1795 derived representation formulas for the solution surfaces. While these were successfully used by HEINRICH SCHERK in 1830 to derive his surfaces, they were generally regarded as practically unusable. CATALAN proved in 1842/43 that the helicoid is the only ruled minimal surface.

Progress had been fairly slow until the middle of the century when the Björling problem was solved using complex methods. The "1st golden age" of minimal surfaces began. HERMANN SCHWARZ found the solution of the Plateau problem for a regular quadrilateral in 1865 & for a general quadrilateral in 1867 (allowing the construction of his periodic surface families) using complex methods. WEIERSTRASS & ENNEPER developed more useful representation formulas, firmly linking minimal surfaces to complex analysis & harmonic functions. Other important contributions came from BELTRAMI, BONNET, DARBOUX, LIE, RIEMANN, SERRET, & WEINGARTEN.

Between 1925 & 1950 minimal surface theory revived, now mainly aimed at nonparametric minimal surfaces. The complete solution of the Plateau problem by JESSE DOUGLAS & TIBOR RADÓ was a major milestone. Bernstein's problem & ROBERT OSSERMAN's work on complete minimal surfaces of finite total curvature were also important.

Another revival began in the 1980s. 1 cause was the discovery in 1982 by CELSO COSTA of Costa's minimal surface that disproved the conjecture that the plane, the catenoid, & the helicoid are the only complete embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type. This not only stimulated new work on using the old parametric methods, but also demonstrated the importance of computer graphics to visualize the studied surfaces & numerical methods to solve the "period problem" (when using the conjugate surface method to determine surface patches that can be assembled into a larger symmetric surface, certain parameters need to be numerically matched to produce an embedded surface). Another cause was the verification by H. KARCHER that the triply periodic minimal surfaces originally described empirically by ALAN SCHOEN in 1970 actually exist. This had led to a rich menagerie of surface families & methods of deriving new surfaces from old, e.g. by adding handles or distorting them.

Currently the theory of minimal surfaces has diversified to minimal submanifolds in other ambient geometries, becoming relevant to mathematical physics (e.g., positive mass conjecture, Penrose conjecture) & 3-manifold geometry (e.g., Smith conjecture, Poincaré conjecture, Thurston Geometrization Conjecture).

1.11.3 Examples

Classical examples of minimal surfaces include:

- the plane, which is a trivial case
- catenoids: minimal surfaces made by rotating a catenary once around its directrix
- helicoids: A surface swept out by a line rotating with uniform velocity around an axis perpendicular to the line & simultaneously moving along the axis with uniform velocity

Surfaces from the 19th century golden age include:

- Schwarz minimal surfaces: triply periodic surfaces that fill $\mathbb{R}^3$
- Riemann's minimal surface: A posthumously described periodic surface
- Enneper surface
- Henneberg surface: the 1st non-orientable minimal surface
- Bour's minimal surface
- Neovius surface: a triply periodic surface

Modern surfaces include:

- Gyroid: 1 of SCHOEN's surfaces from 1970, a triply periodic surface of particular interest for liquid crystal structure
- Saddle tower family: generalizations of SCHERK's 2nd surface
- COSTA's minimal surface: Famous conjecture disproof. Described in 1982 by CELSO COSTA & later visualized by JIM HOFFMAN. JIM HOFFMAN, DAVID HOFFMAN, & WILLIAM MEEKS III then extended the definition to produce a family of surfaces with different rotational symmetries.
- Chen–Gackstatter surface family, adding handles to the Enneper surface.

1.11.4 Generalizations & links to other fields

Minimal surfaces can be defined in other manifolds than $\mathbb{R}^3$, such as hyperbolic space, higher-dimensional spaces or Riemannian manifolds.

The definition of minimal surfaces can be generalized/extended to cover constant-mean-curvature surfaces: surfaces with a constant mean curvature, which need not equal zero.

The curvature lines of an isothermal surface form an isothermal net.

In discrete differential geometry discrete minimal surfaces are studied: simplicial complexes of triangles that minimize their area under small perturbations of their vertex positions. Such discretizations are often used to approximate minimal surfaces numerically, even if no closed form expressions are known.

Brownian motion on a minimal surface leads to probabilistic proofs of several theorems on minimal surfaces.

Minimal surfaces have become an area of intense scientific study, especially in the areas of molecular engineering & materials science, due to their anticipated applications in self-assembly of complex materials. The endoplasmic reticulum, an important structure in cell biology, is proposed to be under evolutionary pressure to conform a nontrivial minimal surface.

In the fields of general relativity & Lorentzian geometry, certain extensions & modifications of the notion of minimal surface, known as apparent horizons, are significant. In contrast to the event horizon, they represent a curvature-based approach to understanding black hole boundaries.

Structures with minimal surfaces can be used as tents. Circus tent approximates a minimal surface.

Minimal surfaces are part of the generative design toolbox used by modern designers. In architecture there has been much interest in tensile structures, which are closely related to minimal surfaces. Notable examples can be seen in the work of FREI OTTO, SHIGERU BAN, & Zaha Hadid. The design of the Munich Olympic Stadium by FREI OTTO was inspired by soap surfaces. Another notable example, also by FREI OTTO, is the German Pavilion at Expo 67 in Montreal, Canada.

In the art world, minimal surfaces have been extensively explored in the sculpture of ROBERT ENGMAN (1927–2018), ROBERT LONGHURST (1949–), CHARLES O. PERRY (1929–2011), among others." – Wikipedia/minimal surface

1.12 Wikipedia/nonlocal operator

"In mathematics, a *nonlocal operator* is a mapping which maps functions on a topological space to functions, in such a way that the value of the output function at a given point cannot be determined solely from the values of the input function in any neighborhood of any point. An example of a nonlocal operator is the Fourier transform.

1.12.1 Formal definition

Let X be a topological space, Y a set, $F(X)$ a function space containing functions with domain X, & $G(Y)$ a function space containing functions with domain Y. 2 functions $u, v \in F(X)$ are called equivalent at $x \in X$ if there exists a neighborhood N of x s.t. $u(x') = v(x')$, $\forall x' \in N$. An operator $A : F(X) \to G(Y)$ is said to be local if $\forall y \in Y$, there exists an $x \in X$ s.t. $Au(y) = Av(y)$, $\forall u, v \in F(x)$ which are equivalent at x. A nonlocal operator is an operator which is not local. For a local operator it is possible (in principle) to compute the value $Au(y)$ using only knowledge of the values of u in an arbitrarily small neighborhood of a point x. For a nonlocal operator this is not possible.

1.12.2 Examples

Differential operators are examples of local operators. A large class of (linear) nonlocal operators is given by the integral transforms, such as the Fourier transform & the Laplace transform. For an integral transform of the form $(Au)(y) = \int_X u(x)K(x,y)\,\mathrm{d}x$, where K is some kernel function, it is necessary to know the values of u a.e. on the support of $K(\cdot, y)$ in order to compute the value of Au at y. An example of a singular integral operator is the fractional Laplacian $(-\Delta)^s f(x) = c_{d,s} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$. The prefactor $c_{d,s} \coloneqq \frac{4^s\Gamma(\frac{d}{2}+s)}{\pi^{\frac{d}{2}}|\Gamma(-s)|}$ involves the Gamma function & serves as a normalizing factor. The fractional Laplacian plays a role in, e.g., the study of nonlocal minimal surfaces.

1.12.3 Applications

Some examples of applications of nonlocal operators are:

- Time series analysis using Fourier transformations
- Analysis of dynamical systems using Laplace transformations
- Image denoising using non-local means
- Modeling Gaussian blur or motion blur in images using convolution with a blurring kernel or point spread function." – Wikipedia/nonlocal operator

1.13 Wikipedia/numerical diffusion

"*Numerical diffusion* is a difficulty with computer simulations of continua (e.g., fluids) wherein the simulated medium exhibits a higher diffusivity than the true medium. This phenomenon can be particularly egregious (= extremely bad) when the system should not be diffusive at all, e.g. an ideal fluid acquiring some spurious viscosity in a numerical model.

1.13.1 Explanation

In Eulerian simulations, time & space are divided into a discrete grid & the continuous differential equations of motion (e.g., NSEs) are discretization into finite-difference equations. The discrete equations are in general more diffusive than the original differential equations, so that the simulated system behaves differently than the intended physical system. The amount & character of the difference depends on the system being simulated & the type of discretization that is used. Most fluid dynamics or magnetohydrodynamic simulations seek to reduce numerical diffusion to the minimum possible, to achieve high fidelity – but under certain circumstances diffusion is added deliberately into the system to avoid singularities. E.g., shock waves in fluids & current sheets in plasmas are infinitely thin in some approximations; this can cause difficulty for numerical codes. A simple way to avoid the difficulty is to add diffusion that smooths out the shock or current sheet. Higher order numerical methods (including spectral methods) tend to have less numerical diffusion than low order methods.

1.13.2 Example

As an example of numerical diffusion, consider an Eulerian simulation using an explicit time-advance of a drop of green dye diffusing through water. If the water is flowing diagonally through the simulation grid, then it is impossible to move the dye in the exact direction of the flow: at each time step the simulation can at best transfer some dye in each of the vertical & horizontal directions. After a few time steps, the dye will have spread out through the grid due to this sideways transfer. This numerical effect takes the form of an extra high diffusion rate.

When numerical diffusion applies to the components of the momentum vector, it is called *numerical viscosity*, when it applies to a magnetic field, it is called numerical resistivity.

Phasefield Simulation of an airbubble within a phase of water. Consider a Phasefield-problem with a high pressure loaded air bubble (blue) within a phase of water. Since there are no chemical or thermodynamical reactions during expansion of air in water there is no possibility to come up with another (i.e., non read or blue) phase during the simulation. These inaccuracies between single phrases are based on numerical diffusion & can be decreased by mesh refining." – Wikipedia/numerical diffusion

1.14 Wikipedia/numerical error

Time series of the Tent map for the parameter $m = 2$ which shows numerical error: "the plot of time series (plot of x variable w.r.t. number of iterations) stops fluctuating & no values are observed after $n = 50$". Parameter $m = 2$, initial point is random. "In software engineering & mathematics, *numerical error* is the error in the numerical computations.

1.14.1 Types

It can be the combined effect of 2 kinds of error in a calculation.

1. the 1st is caused by the finite precision of computations involving floating-point or integer values

2. the 2nd usually called *truncation error* is the difference between the exact mathematical solution & the approximate solution obtained when simplifications are made to the mathematical equations to make them more amenable to calculation. The term truncation comes from the fact that either these simplifications usually involve the truncation of an infinite series expansion so as to make the computation possible & practical, or because the least significant bits of an arithmetic operation are thrown away.

1.14.2 Measure

Floating-point numerical error is often measured in ULP (unit in the last place)." – Wikipedia/numerical error

1.15 Wikipedia/numerical stability

"In the mathematical subfield of numerical analysis, *numerical stability* is a generally desirable property of numerical algorithms. The precise definition of stability depends on the context. One is numerical linear algebra & the other is algorithms for solving ODEs & PDEs by discrete approximation.

In numerical linear algebra, the principal concern is instabilities caused by proximity to singularities of various kinds, e.g. very small or nearly colliding eigenvalues. On the other hand, in numerical algorithms for differential equations the concern is the growth of round-off errors &/or small fluctuations in initial data which might cause a large deviation of final answer from the exact solution.

Some numerical algorithms may damp out the small fluctuations (errors) in the input data; others might magnify such errors. Calculations that can be proven not to magnify approximation errors are called *numerically stable*. 1 of the common tasks of numerical analysis is to try to select algorithms which are *robust* – i.e., do not produce a widely different result for very small change in the input data.

An opposite phenomenon is *instabillity*. Typically, an algorithm involves an approximative method, & in some cases one could prove that the algorithm would approach the right solution in some limit (when using actual real numbers, not floating point numbers). Even in this case, there is no guarantee that it would converge to the correct solution, because the floating-point round-off or truncation errors can be magnified, instead of damped, causing the deviation from the exact solution to grow exponentially.

1.15.1 Stability in numerical linear algebra

Diagram showing the *forward error* Δy & the *backward error* Δx, & their relation to the exact solution map f & the numerical solution $f^\star$. There are different ways to formalize the concept of stability. The following definitions of forward, backward, & mixed stability are often used in numerical linear algebra.

Consider the problem to be solved by the numerical algorithm as a function f mapping the data x to the solution y. The result of the algorithm, say $y^\star$, will usually deviate from the "true" solution y. The main causes of error are round-off error & truncation error. The *forward error* of the algorithm is the difference between the result & the solution; in this case, $\Delta y = y^\star - y$. The *backward error* is the smallest Δx s.t. $f(x + \Delta x) = y^\star$, i.e., the backward error tells us what problem the algorithm actually solved. The forward & backward error are related by the condition number: the forward error is at most as big in magnitude as the condition number multiplied by the magnitude of the backward error.

In many cases, it is more natural to consider the relative error $\frac{|\Delta x|}{|x|}$ instead of the absolute error Δx.

The algorithm is said to be *backward stable* if the backward error is small for all inputs x. Of course, "small" is a relative term & its definition will depend on the context. Often, we want the error to be of the same order as, or perhaps only a few orders of magnitude bigger than the unit round-off.

Mixed stability combines the concepts of forward error & backward error. The usual definition of numerical stability uses a more general concept, called *mixed stability*, which combines the forward error & the backward error. An algorithm is stable in this sense if it solves a nearby problem approximately, i.e., if there exists a Δx s.t. both Δx is small & $f(x + \Delta x) - y^\star$ is small. Hence, a backward stable algorithm is always stable.

An algorithm is *forward stable* if its forward error divided by the condition number of the problem is small, i.e., an algorithm is forward stable if it has a forward error of magnitude similar to some backward stable algorithm.

1.15.2 Stability in numerical differential equations

The above definitions are particularly relevant in situations where truncation errors are not important. In other contexts, e.g. when solving differential equations, a different definition of numerical stability is used.

In numerical ODEs, various concepts of numerical stability exist, e.g., A-stability. They are related to some concept of stability in the dynamatical systems sense, often Lyapunov stability. It is important to use a stable method when solving a stiff equation.

Yet another definition is used in numerical PDEs. An algorithm for solving a linear evolutionary PDE is stable if the total variation of the numerical solution at a fixed time remains bounded as the step size goes to 0. The Lax equivalence theorem states that an algorithm converges if it is consistent & stable (in this sense). Stability is sometimes achieved by including numerical diffusion. Numerical diffusion is a mathematical term which ensures that roundoff & other errors in the calculation get spread out & do not add up to cause the calculation to "blow up". Von Neumann stability analysis is a commonly used procedure for the stability analysis of finite difference schemes as applied to linear PDEs. These results do not hold for nonlinear PDEs, where a general, consistent definition of stability is complicated by many properties absent in linear equations.

1.15.3 Example

Computing $\sqrt{2}$ is a well-posed problem. Many algorithms solve this problem by starting with an initial approximation x_0 to $\sqrt{2}$, e.g., $x_0 = 1.4$, & then computing improved guesses x_1, x_2, etc. 1 such method is the famous Babylonian method, given by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Another method, called "method X", is given by $x_{n+1} = (x_n^2 - 2)^2 + x_n$. Observe that the Babylonian method converges quickly regardless of the initial guess, whereas Method X converges extremely slowly with initial guess $x_0 = 1.4$ & diverges for initial guess $x_0 = 1.42$. Hence, the Babylonian method is numerically stable, while Method X is numerically unstable.

Numerical stability is affected by the number of the significant digits the machine keeps. If a machine is used that keeps only the 4 most significant decimal digits, a good example on loss of significance can be given by the 2 equivalent functions $f(x) \coloneqq x(\sqrt{x+1} - \sqrt{x}) = g(x) \coloneqq \frac{x}{\sqrt{x+1}+\sqrt{x}}$. It is clear that loss of significance (caused here by catastrophic cancellation from subtracting approximations to the nearby numbers $\sqrt{501}, \sqrt{500}$, despite the subtraction being computed exactly) has a huge effect on the results, even though both functions are equivalent." – Wikipedia/numerical stability

1.16 Wikipedia/operator (mathematics)

"In mathematics, an *operator* is generally a mapping or function that acts on elements of a space to produce elements of another space (possibly & sometimes required to be the same space). There is no general definition of an *operator*, but the term is often used in place of *function* where the domain is a set of functions or other structured objects. Also, the domain of an operator is often difficult to characterize explicitly (e.g. in the case of an integral operator), & may be extended so as to act on related objects (an operator that acts on functions may act also on differential equations whose solutions are functions that satisfy the equation). See Wikipedia/operator (physics).

The most basic operators are linear maps, which act on vector spaes. Linear operators refer to linear maps whose domain & range are the same space, e.g. from $\mathbb{R}^d \to \mathbb{R}^d$. Such operators often preserve properties, e.g. continuity. E.g., differentiation & indefinite integration are linear operators; operators that are built from them are called differential operators, integral operators or integro-differential operators.

Operator is also used for denoting the symbol of a mathematical operation. This is related with the meaning of "operator" in computer programming, see Wikipedia/operator (computer programming).

1.16.1 Linear operators

Main article: Wikipedia/linear operator. The most common kind of operators encountered are *linear operators*. Let U, V be vector spaces over some field K. A mapping $A : U \to V$ is linear if $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$, $\forall \mathbf{x}, \mathbf{y} \in U$, $\forall \alpha, \beta \in K$, i.e., a linear operator preserves vector space operations, in the sense that it does not matter whether you apply the linear operator before or after the operations of addition & scalar multiplication. In more technical words, linear operators are morphisms between vector spaces. In the finite-dimensional case linear operators can be represented by matrices in the following way. Let K be a field, U, V be finite-dimensional vector spaces over K. Select a basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in U & $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in V. Then let $\mathbf{x} = x^i\mathbf{u}_i$ be an arbitrary vector in U (assuming Einstein convention), & $A : U \to V$ be a linear operator. Then $A\mathbf{x} = x^i A\mathbf{u}_i = x^i(A\mathbf{u}_i)^j\mathbf{v}_j$. Then $a_i^j \equiv (A\mathbf{u}_i)^j$, with all $a_i^j \in K$, is the matrix form of the operator A in the fixed basis $\{\mathbf{u}_i\}_{i=1}^n$. The tensor a_i^j does not depend on the choice of $\mathbf{x}$, & $A\mathbf{x} = \mathbf{y}$ if $a_i^j x^i = y^j$. Thus in fixed bases n-by-m matrices are in bijective correspondence to linear operators from $U \to V$.

The important concepts directly related to operators between finite-dimensional vector spaces are the ones of rank, determinant, inverse operator, & eigenspace.

Linear operators also play a great role in the infinite-dimensional case. The concepts of rank & determinant cannot be extended to infinite-dimensional matrices. This is why very different techniques are employed when studying linear operators (& operators in general) in the infinite-dimensional case. The study of linear operators in the infinite-dimensional case is known as functional analysis (so-called because various classes of functions form interesting examples of infinite-dimensional vector spaces).

The space of sequences of real numbers, or more generally sequences of vectors in any vector space, themselves form an infinite-dimensional vector space. The most important cases are sequences of real or complex numbers, & these spaces, together with linear subspaces, are known as sequence spaces. Operators on these spaces are known as sequence transformations.

Bounded linear operators over a Banach space form a Banach algebra in respect to the standard operator norm. The theory of Banach algebras develops a very general concept of spectra that elegantly generalizes the theory of eigenspaces.

1.16.2 Bounded operators

Main articles: Wikipedia/bounded operator, Wikipedia/operator norm, & Wikipedia/Banach algebra. Let U, V be 2 vector spaces over the same ordered field, e.g., $\mathbb{R}$, & they are equipped with norms. Then a linear operator from $U \to V$ is called *bounded* if there exists $c > 0$ s.t. $\|A\mathbf{x}\|_V \le c\|\mathbf{x}\|_U$, $\forall \mathbf{x} \in U$. Bounded operators form a vector space. On this vector space we can introduce a norm that is compatible with the norms of U, V: $\|A\| := \inf\{c : \|A\mathbf{x}\|_V \le c\|\mathbf{x}\|_U\}$. In case of operators from U to itself it can be shown that $\|AB\| \le \|A\|\|B\|$. Any unital normed algebra with this property is called a Banach algebra. It is possible to generalize spectral theory to such algebra. $C^\star$-algebras, which are Banach algebras with some additional structure, play an important role in quantum mechanics.

1.16.3 Examples

1. Analysis (calculus). Main articles: Wikipedia/differential operator & Wikipedia/integral operator. From the point of view of functional analysis, calculus is the study of 2 linear operators: the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$, & the Volterra operator $\int_0^t$.
2. Fundamental analysis operators on scalar & vector fields. Main articles: Wikipedia/vector calculus, Wikipedia/vector field, Wikipedia/scalar field, Wikipedia/gradient, Wikipedia/divergence, Wikipedia/curl. 3 operators are key to vector calculus:
   - Grad gradient (with operator symbol ∇) assigns a vector at every point in a scalar field that points in the direction of greatest rate of change of that field & whose norm measures the absolute value of that greatest rate of change.
   - Div divergence (with operator symbol $\nabla\cdot$) is a vector operator that measures a vector field's divergence from or convergence towards a given point.
   - Curl (with operator symbol $\nabla\times$) is a vector operator that measures a vector field's curling (winding around, rotating around) trend about a given point.

   As an extension of vector calculus operators in physics, engineering & tensor spaces, grad, div, & curl operators also are often associated with tensor calculus as well as vector calculus.
3. Geometry. Main articles: Wikipedia/general linear group, Wikipedia/isometry. In geometry, additional structures on vector spaces are sometimes studied. Operators that map such vector spaces to themselves bijectively are very useful in these studies, they naturally form groups by composition. E.g., bijective operators preserving the structure of a vector space are precisely the invertible linear operators. They form the general linear group under composition. However, they *do not* form a vector space under operator addition; since, e.g., both the identity & -identity are invertible (bijective), but their sum 0 is not. Operators preserving the Euclean metric on such a space form the isometry group, & those that fix the origin form a subgroup known as the orthogonal group. Operators in the orthogonal group that also preserve the orientation of vector tuples form the special orthogonal group, or the group of rotations.
4. Probability theory. Main article: Wikipedia/probability theory. Operators are also involved in probability theory, e.g. expectation, variance, & covariance, which are used to name both number statistics & the operators which produce them. Indeed, every covariance is basically a dot product: Every variance is a dot product of a vector with itself, & thus is a quadratic norm; every standard deviation is a norm (square root of the quadratic norm); the corresponding cosine to this dot product is the Pearson correlation coefficient; expected value is basically an integral operator (used to measure weighted shapes in the space).
5. Fourier series & Fourier transform. Main articles: Wikipedia/Fourier series, Wikipedia/Fourier transform. The Fourier transform is useful in applied mathematics, particularly physics & signal processing. It is another integral operator; it is useful mainly because it converts a function on 1 (temporal) domain to a function on another (frequency) domain, in a way effectively invertible. No information is lost, as there is an inverse transform operator. In the simple case of periodic functions, this result is based on the theorem that any continuous periodic function can be represented as the sum of a series of sine waves & cosine waves: $f(t) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\omega nt + b_n \sin\omega nt$. The tuple $(a_0, a_1, b_1, a_2, b_2, \ldots)$ is in fact an element of an infinite-dimensional vector space l^2, & thus Fourier series is a linear operator. When dealing with general function $\mathbb{R} \to \mathcal{C}$, the transform takes on an integral form $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega)e^{i\omega t}\,\mathrm{d}\omega$.s
6. Laplace transform. Main article: Wikipedia/Laplace transform. The *Laplace transform* is another integral operator & is involved in simplifying the process of solving differential equations. Given $f = f(s)$, it is defined by $F(s) = \mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$." – Wikipedia/operator (mathematics)

1.17 Wikipedia/operator algebra

"In functional analysis, a branch of mathematics, an *operator algebra* is an algebra of continuous linear operators on a topological vector space, with the multiplication given by the composition of mappings.

The results obtained in the study of operator algebras are often phrased in algebraic terms, while the techniques used are often highly analytic. Although the study of operators algebras is usually classified as a branch of functional analysis, it has direct applications to representation theory, differential geometry, quantum statistical mechanics, quantum information, & quantum field theory.

1.17.1 Overview

Operator algebras can be used to study arbitrary sets of operators with little algebraic relation *simultaneously*. From this point of view, operator algebras can be regarded as a generalization of spectral theory of a single operator. In general, operator algebras are non-commutative rings.

An operator algebra is typically required to be closed in a specified operator topology inside the whole algebra of continuous linear operators. In particular, it is a set of operators with both algebraic & topological closure properties. In some disciplines such properties are axiomatized & algebras with certain topological structure become the subject of the research.

Though algebras of operators are studied in various contexts (e.g., algebras of pseudo-differential operators acting on spaces of distributions), the term *operator algebra* is usually used in reference to algebras of bounded operators on a Banach space or, even more specially in reference to algebras of operators on a separable Hilbert space, endowed with the operator norm topology.

In the case of operators on a Hilbert space, the Hermitian adjoint map on operators gives a natural involution, which provides an additional algebraic structure that can be imposed on the algebra. In this context, the best studied examples are self-adjoint operator algebras, meaning that they are closed under taking adjoints. These include $C^\star$ algebras, von Neumann algebras, & AM*-algebras. $C^\star$-algebras can be easily characterize abstractly by a condition relating the norm, involution, & multiplication. Such abstractly defined $C^\star$-algebra can be identified to a certain closed subalgebra of the algebra of the continuous linear operators on a suitable Hilbert space. A similar result holds for von Neumann algebras.

Commutative self-adjoint operator algebras can be regarded as the algebra of complex-valued continuous functions on a locally compact space, or that of measurable functions on a standard measurable space. Thus, general operator algebras are often regarded as a noncommutative generalizations of these algebras, or the structure of the *base space* on which the functions are defined. This point of view is elaborated as the philosophy of noncommutative geometry, which tries to study various non-classical &/or pathological objects by noncommutative operator algebras.

Examples of operator algebras that are not self-adjoint include:

- nest algebras.
- many commutative subspace lattice algebras,
- many limit algebras." – Wikipedia/operator algebra

1.18 Wikipedia/physics-informed neural networks PINNs

"*Physics-informed neural networks* (PINNs), also referred as as *Theory-Trained Neural Networks* (TTNs), are a type of universal function approximators that can be embed the knowledge of any physical laws that govern a given data-set in the learning process, & can be described by PDEs. They overcome the low data availability of some biological & engineering systems that makes most state-of-the-art machine learning techniques lack robustness, rendering them ineffective in these scenarios. The prior knowledge of general physical laws acts in the training of neural networks (NNs) as a regularization agent that limits the space of admissible solutions, increasing the correctness of the function approximation. This way, embedding this prior information into a neural network results in enhancing the information content of the available data, facilitating the learning algorithm to capture the right solution & to generalize well even with a low amount of training examples. Physics-informed neural networks for solving NSEs.

1.18.1 Function approximation

Most of the physical laws that govern the dynamics of a system can be described by PDEs. E.g., the NSEs are a set of PDEs derived from the conservation laws (i.e., conservation of mass, momentum, & energy) that govern fluid mechanics. The solution of the NSEs with appropriate initial & boundary conditions allows the quantification of flow dynamics in a precisely defined geometry. However, these equations cannot be solved exactly & therefore numerical methods must be used (e.g. FDs, FEs, & FVs). In this setting, these governing equations must be solved while accounting for prior assumptions, linearization, & adequate time & space discretization.

Recently, solving the governing PDEs of physical phenomena using deep learning has emerged as a new field of scientific machine learning (SciML), leveraging the universal approximation theorem & high expressivity of neural networks. In general, deep neural networks could approximate any high-dimensional function given that sufficient training data are supplied. However, such networks do not consider the physical characteristics underlying the problem, & the level of approximation accuracy provided by them is still heavily dependent on careful specifications of the problem geometry as well as the initial & boundary conditions. Without this preliminary information, the solution is not unique & may lose physical correctness. On the other hand, physics-informed neural networks (PINNs) leverage governing physical equations in neural network training. Namely, PINNs are designed to be trained to satisfy the given training data as well as the imposed governing equations. In this fashion, a neural network can be guided with training data that do not necessarily need to be large & complete. Potentially, an accurate solution of PDEs can be found without knowing the boundary conditions. Therefore, with some knowledge about the physical characteristics of the problem & some form of training data (even sparse & incomplete), PINN may be used for finding an optimal solution with high fidelity.

PINNs allow for addressing a wide range of problems in computational science & represent a pioneering technology leading to the development of new classes of numerical solvers for PDEs. PINNs can be thought of as a meshfree alternative to traditional approaches (e.g., CFD for fluid dynamics), & new data-driven approaches for model inversion & system identification. Notably, the trained PINN network can be used for predicting the values on simulation grids of different resolutions without the need to be retrained. In addition, they allow for exploiting automatic differentiation (AD) to compute the required derivatives in the PDEs, a new class of differentiation techniques widely used to derive neural networks assessed to be superior to numerical differentiation or symbolic differentiation.

1.18.2 Modeling & computation

A general nonlinear PDE can be:

$$u_t + N[u;\lambda] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T],$$

where $u(t,\mathbf{x})$ denotes the solution, $N[\cdot;\lambda]$: a nonlinear operator parametrized by λ, & $\Omega \subset \mathbb{R}^d$. This general form of governing equations summarizes a wide range of problems in mathematical physics, e.g. conservative laws, diffusion process, advection-diffusion systems, & kinetic equations. Given noisy measurements of a generic dynamic system described by the equation above, PINNs can be designed to solve 2 classes of problems:

- data-driven solution
- data-driven discovery of PDEs.

Data-driven solution of PDEs. The *data-driven solution of PDE* computes the hidden state $u(t,\mathbf{x})$ of the system given boundary data &/or measurements z, & fixed model parameters λ. Solve:

$$u_t + N[u] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T].$$

By defining the residual $f(t,\mathbf{x})$ as $f \coloneqq u_t + N[u] = 0$, & approximating $u(t,\mathbf{x})$ by a deep neural network. This network can be differentiated using automatic differentiation. The parameters of $u(t,\mathbf{x})$ & $f(t,\mathbf{x})$ can be then learned by minimizing the following loss function $L_{\rm tot} \coloneqq L_u + L_f$ where $L_u \coloneqq \|u - z\|_\Gamma$ is the error between the PINN $u(t,\mathbf{x})$ & the set of boundary conditions & measured data on the set of points Γ where the boundary conditions & data are defined, & $L_f \coloneqq \|f\|_\Gamma$ is the mean-squared error of the residual function. This 2nd term encourages the PINN to learn the structural information expressed by the PDE during the training process.

This approach has been used to yield computationally efficient physics-informed surrogate models with applications in the forecasting of physical processes, model predictive control, multi-physics & multi-scale modeling, & simulation. It has been shown to converge to the solution of the PDE.

Data-driven discovery of PDEs. Given noisy & incomplete measurements z of the state of the system, the *data-driven discovery of PDE* results in computing the unknown state $u(t,\mathbf{x})$ & learning model parameters λ that best describe the observed data & it reads as follows:

$$u_t + N[u;\lambda] = 0,\ \mathbf{x} \in \Omega,\ t \in [0,T]. \tag{2}$$

By defining $f(t,\mathbf{x}) \coloneqq u_t + N[u;\lambda] = 0$, & approximating $u(t,\mathbf{x})$ by a deep neural network, $f(t,\mathbf{x})$ results in a PINN. This network can be derived using automatic differentiation. The parameters of $u(t,\mathbf{x}), f(t,\mathbf{x})$, together with the parameter λ of the differential operator can be then learned by minimizing the following loss function $L_{\rm tot} \coloneqq L_u + L_f$ where $L_u \coloneqq \|u - z\|_\Gamma$, with u,z: state solutions & measurements at sparse location Γ, respectively & $L_f \coloneqq \|f\|_\Gamma$ residual function. This 2nd term requires the structured information represented by the PDEs to be satisfied in the training process.

This strategy allows for discovering dynamic models described by nonlinear PDEs assembling computationally efficient & fully differentiable surrogate models that may find application in predictive forecasting, control, & data assimilation.

1.18.3 Physics-informed neural networks for piecewise function approximation

PINN is unable to approximate PDEs that have strong nonlinearity or sharp gradients that commonly occur in practical fluid flow problems. Piecewise approximation has been an old practice in the field of numerical approximation. With the capability of approximating strong nonlinearity extremely light weight PINNs are used to solve PDEs in much larger discrete subdomains that increases accuracy substantially & decreases computational load as well. DPINN (Distributed physics-informed neural networks) & DPIELM (Distributed physics-informed extreme learning machines) are generalizable space-time domain discretization for better approximation. DPIELM is an extremely fast & lightweight approximator with competitive accuracy. Domain scaling on the top has a special effect. Another school of thought is discretization for parallel computation to leverage usage of available computational resources.

XPINNs is a generalized space-time domain decomposition approach for the physics-informed neural networks (PINNs) to solve nonlinear PDEs on arbitrary complex-geometry domains. The XPINNs further pushes the boundaries of both PINNs as well as Conservative PINNs (cPINNs), which is a spatial domain decomposition approach in the PINN framework tailored to conservation laws. Compared to PINN, the XPINN method has large representation & parallelization capacity due to the inherent property of deployment of multiple neural networks in the smaller subdomains. Unlike cPINN, XPINN can be extended to any type of PDEs. Moreover, the domain can be decomposed in any arbitrary way (in space & time), which is not possible in cPINN. Thus, XPINN offers both space & time parallelization, thereby reducing the training cost more effectively. The XPINN is particularly effective for the large-scale problems (involving large data set) as well as for the high-dimensional problems where single network based PINN is not adequate. The rigorous bounds on the errors resulting from the approximation of the nonlinear PDEs (incompressible NSEs) with PINNS & XPINNs are proved. However, DPINN debunks the use of residual (flux) matching at the domain interfaces as they hardly seem to improve the optimization.

1.18.4 Physics-informed neural networks & functional interpolation

X-TFC framework scheme for PDE solution learning. In the PINN framework, initial & boundary conditions are not analytically satisfied, thus they need to be included in the loss function of the network to be simultaneously learned with the differential equation (DE) unknown functions. Having competing objectives during the network's training can lead to unbalanced gradients while using gradient-based techniques, which causes PINNs to often struggle to accurately learn the underlying DE solution. This drawback is overcome by using functional interpolation techniques such as the Theory of Functional Connections (TFC)'s constrained expression, in the Deep-TFC framework, which reduces the solution search space of constrained problems to the subspace of neural network that analytically satisfies the constraints. A further improvement of PINN & functional interpolation approach is given by the Extreme Theory of Functional Connections (X-TFC) framework, where a single-layer Neural Network & the extreme learning machine training algorithm are employed. X-TFC allows to improve the accuracy & performance of regular PINNs, & its robustness & reliability are proved for stiff problems, optimal control, aerospace, & rarefied gas dynamics applications.

1.18.5 Physics-informed PointNet (PIPN) for multiple sets of irregular geometries

Regular PINNs are only able to obtain the solution of a forward or inverse problem on a single geometry. It means that for any new geometry (computational domain), one must retrain a PINN. This limitation of regular PINNs imposes high computational costs, specifically for a comprehensive investigation of geometric parameters in industrial designs. Physics-informed PointNet (PIPN) is fundamentally the result of a combination of PINN's loss function with PointNet. In fact, instead of using a simple fully connected neural network, PIPN uses Pointnet as the core of its neural network. PointNet has been primarily designed for deep learning of 3D object classification & segmentation by the research group of Leonidas J. Guibas. PointNet extracts geometric features of input computational domains in PIPN. Thus, PIPN is able to solve governing equations on multiple computational domains (rather than only a single domain) with irregular geometries, simultaneously. The effectiveness of PIPN has been shown for incompressible flow, heat transfer, & linear elasticity.

1.18.6 Physics-informed neural networks (PINNs) for inverse computations

Physics-informed neural networks (PINNs) have proven particularly effective in solving inverse problems within differential equations, demonstrating their applicability across science, engineering, & economics. They have shown useful for solving inverse problems in a variety of fields, including nano-optics, topology optimization/characterization, multiphase flow in porous media, & high-speed fluid flow. PINNs have demonstrated flexibility when dealing with noisy & uncertain observation datasets. They also demonstrated clear advantages in the inverse calculation of parameters for multi-fidelity datasets, meaning datasets with different quality, quantity, & types of observations. Uncertainties in calculations can be evaluated using ensemble-based or Bayesian-based calculations.

1.18.7 Physics-informed neural networks (PINNs) with backward stochastic differential equation

Deep backward stochastic differential equation method is a numerical method that combines deep learning with Backward stochastic differential equation (BSDE) to solve high-dimensional problems in financial mathematics. By leveraging the powerful function approximation capabilities of deep neural networks, deep BSDE addresses the computational challenges faced by traditional numerical methods like FDMs or Monte Carlo simulations, which struggle with the curse of dimensionality. Deep BSDE methods use neural networks to approximate solutions of high-dimensional PDEs, effectively reducing the computational burden. Additionally, integrating Physics-informed neural networks (PINNs) into the deep BSDE framework enhances its capability by embedding the underlying physical laws into the neural network architecture, ensuring solutions adhere to governing stochastic differential equations, resulting in more accurate & reliable solutions.

1.18.8 Physics-informed neural networks for biology

An extension for adaptation of PINNs are Biologically-informed neural networks (BINNs). BINNs introduce 2 key adaptations to the typical PINN framework:

(i) the mechanistic terms of the governing PDE are replaced by neural networks, &

(ii) the loss function L_{tot} is modified to include L_{constr}, a term used to incorporate domain-specific knowledge that helps enforce biological applicability.

For (i), this adaptation has the advantage of relaxing the need to specify the governing differential equation a priori, either explicitly or by using a library of candidate terms. Additionally, this approach circumvents the potential issue of misspecifying regularization terms in stricter theory-informed cases.

A natural example of BINNs can be found in cell dynamics, where the cell density $u(t, \mathbf{x})$ is governed by a reaction-diffusion equation with diffusion & growth functions $D(u), G(u)$, respectively:

$$u_t = \nabla \cdot (D(u)\nabla u) + G(u)u,\ \mathbf{x} \in \Omega,\ t \in [0, T].$$

In this case, a component of $L_{\rm constr}$ could be $\|D\|_\Gamma$ for $D < D_{\min}$, $D > D_{\max}$, which penalizes values of D that fall outside a biologically relevant diffusion range defined by $D_{\min} \le D \le D_{\max}$. Furthermore, the BINN architecture, when utilizing multiplayer-perceptrons (MLPs), would function as follows: an MLP is used to construct $u_{\rm MLP}(t, \mathbf{x})$ from model inputs $(t, \mathbf{x})$, serving as a surrogate model for the cell density $u(t, \mathbf{x})$. This surrogate is then fed into the 2 additional MLPs, $D_{\rm MLP}(u_{\rm MLP}), G_{\rm MLP}(u_{\rm MLP})$, which model the diffusion & growth functions. Automatic differentiation can then be applied to compute the necessary derivatives of $u_{\rm MLP}, D_{\rm MLP}, G_{\rm MLP}$ to form the governing reaction-diffusion equation.

Note that since $u_{\rm MLP}$ is a surrogate for the cell density, it may contain errors, particularly in regions where the PDE is not fully satisfied. Therefore, the reaction-diffusion equation may be solved numerically, e.g. using a method-of-lines approach.

1.18.9 Limitations

Translation & discontinuous behavior are hard to approximate using PINNs. They fail when solving differential equations with slight advective dominance & hence asymptotic behavior causes the method to fail. Such PDEs could be solved by scaling variables. This difficulty in training of PINNs in advection-dominated PDEs can be explained by the Kolmogorov n-width of the solution. They also fail to solve a system of dynamical systems & hence have not been a success in solving chaotic equations. 1 of the reasons behind the failure of regular PINNs is soft-constraining of Dirichlet & Neumann boundary conditions which pose a multi-objective optimization problem which requires manually weighing the loss terms to be able to optimize. More generally, posing the solution of a PDE as an optimization problem brings with it all the problems that are faced in the world of optimization, the major one being getting stuck in local optima." – Wikipedia/physics-informed neural networks PINNs

1.19 Wikipedia/relaxation (iterative method)

"In numerical mathematics, *relaxation methods* are iterative methods for solving systems of equations, including nonlinear systems.

Relaxation methods were developed for solving large sparse linear systems, which arose as finite-difference discretizations of differential equations. They are also used for the solution of linear equations for linear least-squares problems & also for system of linear inequalities, e.g. those arising in linear programming. They have also been developed for solving nonlinear systems of equations.

Relaxation methods are important especially in the solution of linear systems used to model elliptic PDEs, e.g., Laplace's equation & its generalization, Poisson's equation. These equations describe BVPs, in which the solution-function's values are specified on boundary of a domain; the problem is to compute a solution also on its interior. Relaxation methods are used to solve the linear equations resulting from a discretization of the differential equation, e.g. by finite differences.

Iterative relaxation of solutions is commonly dubbed smoothing because with certain equations, e.g., Laplace's equation, it resembles repeated application of a local smoothing filter to the solution vector. These are not to be confused with relaxation methods in mathematical optimization, which approximate a difficult problem by a simpler problem whose "relaxed" solution provides information about the solution of the original problem.

1.19.1 Model problem of potential theory

Main article: Wikipedia/discrete Poisson equation. When φ is a smooth real-valued function on $\mathbb{R}$, its 2nd derivative can be approximated by

$$\frac{d^2\varphi(x)}{dx^2} = \frac{\varphi(x-h) - 2\varphi(x) + \varphi(x+h)}{h^2} + O(h^2). \tag{3}$$

Using this in both dimensions for a function φ of 2 arguments at the point (x, y), & solving for $\varphi(x, y)$, results in:

$$\varphi(x, y) = \frac{1}{4}\left(\varphi(x+h, y) + \varphi(x, y+h) + \varphi(x-h, y) + \varphi(x, y-h) - h^2\Delta\varphi(x, y)\right) + O(h^4). \tag{4}$$

To approximate the solution of Poisson equation $\Delta\varphi = \nabla^2\varphi = f$ numerically on a 2D grid with grid spacing h, the relaxation method assigns the given values of function φ to the grid points near the boundary & arbitrary values to the interior grid points, & then repeatedly performs the assignment $\varphi := \varphi^\star$ on the interior points, where $\varphi^\star$ is defined by

$$\varphi^\star(x, y) = \frac{1}{4}\left(\varphi(x+h, y) + \varphi(x, y+h) + \varphi(x-h, y) + \varphi(x, y-h) - h^2 f(x, y)\right), \tag{5}$$

until convergence. The method is easily generalized to other number of dimensions.

1.19.2 Convergence & acceleration

While the method converges under general conditions, it typically makes slower progress than competing methods. Nonetheless, the study of relaxation methods remains a core part of linear algebra, because the transformations of relaxation theory provide excellent preconditioners for new methods. Indeed, the choice of preconditioners is often more important than the choice of iterative method.

Multigrid methods may be used to accelerate the methods. One can 1st compute an approximation on a coarser grid – usually the double spacing $2h$ – & use that solution with interpolated values for the other grid points as the initial assignment. This can then also be done recursively for the coarser computation.

In linear systems, the 2 main classes of relaxation methods are stationary iterative methods, & the more general Krylov subspace methods. The Jacobi method is a simple relaxation method. The Gauss–Seidel method is an improvement upon the Jacobi method. Successive over-relaxation can be applied to either of the Jacobi & Gauss–Seidel methods to speed convergence." – Wikipedia/relaxation (iterative method)

1.20 Wikipedia/spectral theory

"In mathematics, *spectral theory* is an inclusive term for theories extending the eigenvector & eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. It is a result of studies of linear algebra & the solutions of systems of linear equations & their generalizations. The theory is connected to that of analytic functions because the spectral properties of an operator are related to analytic functions of the spectral parameter.

1.20.1 Mathematical backgrounds

The name *spectral theory* was introduced by David Hilbert in his original formulation of Hilbert space theory, which was cast in terms of quadratic forms in infinitely many variables. The original spectral theorem was therefore conceived as a version of the theorem on principal axes of an ellipsoid, in an infinite-dimensional setting. The later discovery in quantum mechanics that spectral theory could explain features of atomic spectra was therefore fortuitous. Hilbert himself was surprised by the unexpected application of this theory, noting that "I developed my theory of infinitely many variables from purely mathematical interests, & even called it 'spectral analysis' without any presentiment that it would later find application to the actual spectrum of physics."

There have been 3 main ways to formulate spectral theory, each of which find use in different domains. After Hilbert's initial formulation, the later development of abstract Hilbert spaces & the spectral theory of single normal operators on them were well suited to the requirements of physics, exemplified by the work of von Neumann. The further theory built on this to address Banach algebras in general. This development leads to the Gelfand representation, which covers the commutative case, & further into non-commutative harmonic analysis.

The difference can be seen in making the connection with Fourier analysis. The Fourier transform on the real line is in 1 sense the spectral theory of differentiation as a differential operator. But for that to cover the phenomena one has already to deal with generalized eigenfunctions (e.g., by means of a rigged Hilbert space). On the other hand, it is simple to construct a group algebra, the spectrum of which captures the Fourier transform's basic properties, & this is carried out by means of Pontryagin duality.

One can also study the spectral properties of operators on Banach spaces. E.g., compact operators on Banach spaces have many spectral properties similar to that of matrices.

1.20.2 Physical background

The background in the physics of vibration has been explained in this way:

> Spectral theory is connected with the investigation of localized vibrations of a variety of different objects, from atoms & molecules in chemistry to obstacles in acoustic waveguides. These vibrations have frequencies, & the issue is to decide when such localized vibrations occur, & how to go about computing the frequencies. This is a very complicated problem since every object has not only a fundamental tone but also a complicated series of overtones, which vary radically from 1 body to another.

Such physical ideas have nothing to do with the mathematical theory on a technical level, but there are examples of indirect involvement (see e.g. Mark Kac's question *Can you hear the shape of a drum?*). Hilbert's adoption of the term "spectrum" has been attributed to an 1897 paper of Wilhelm Wirtinger on Hill differential equation (by Jean Dieudonné), & it was taken up by his students during the 1st decade of the 20th century, among them Erhard Schmidt & Frigyes Riesz. It was almost 20 years later, when quantum mechanics was formulated in terms of the Schrödinger equation, that the connection was made to atomic spectra; a connection with the mathematical physics of vibration had been suspected before, as remarked by Henry Poincaré, but rejected for simple quantitative reasons, absent an explanation of the Balmer series. The later discovery in quantum mechanics that spectral theory could explain features of atomic spectra was therefore fortuitous, rather than being an object of Hilbert's spectral theory.

1.20.3 A definition of spectrum

1.20.4 Spectral theory briefly

1.20.5 Resolution of the identity

1.20.6 Resolvent operator

1.20.7 Operator equations

1.20.8 Spectral theorem & Rayleigh quotient

" – Wikipedia/spectral theory

1.21 Wikipedia/upwind scheme

"In computational physics, the term advection scheme refers to a class of numerical discretization methods for solving hyperbolic PDEs. In the so-called upwind scheme *typically*, the so-called upstream variables are used to calculate the derivatives in a flow field. I.e., derivatives are estimated using a set of data points biased to be more "upwind" of the query point, w.r.t. the direction of the flow. Historically, the origin of upwind methods can be traced back to the work of Courant, Isaacson, & Rees who proposed the CIR method.

1.21.1 Model equation

To illustrate the method, consider 1D linear advection equation $\partial_t u + a\partial_x u = 0$ which describes a wave propagating along the x-axis with a velocity a. This equation is also a mathematical mdoel for 1D linear advection. Consider a typical grind point i in the domain. In a 1D domain, there are only 2 directions associated with point i – left (towards negative infinity) & right (towards positive infinity). If $a > 0$, the traveling wave solution of the 1D linear advection equation propagates towards the right, the left side is called the *upwind* side & the right side is the *downwind* side. Similarly, if $a < 0$ the traveling wave solution propagates towards the left, the left side is called *downwind* side & right side is the *upwind* side. If the finite difference scheme for the spatial derivative, $\partial_x u$ contains more points in the upwind side, the scheme is called an *upwind-biased* or simply an *upwind scheme*.

1.21.2 1st-order upwind scheme

A simulation of a 1st-order upwind scheme in which $a = \sin t$. The simplest upwind scheme possible is the 1st-order upwind scheme. It is given by

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{\Delta t} + a\dfrac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & \text{if } a > 0, \\ \dfrac{u_i^{n+1} - u_i^n}{\Delta t} + a\dfrac{u_{i+1}^n - u_i^n}{\Delta x} = 0 & \text{if } a < 0, \end{cases} \tag{6}$$

where n refers to the t dimension & i refers to the x dimension. By comparison, a central difference scheme in this scenario would like like

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{7}$$

regardless of sign$a$.

1. **Compact form.** Define $a^+ := \max\{a, 0\}$, $a^- := \min\{a, 0\}$ & $u_x^+ := \frac{u_i^n - u_{i-1}^n}{\Delta x}$, $u_x^- := \frac{u_{i+1}^n - u_i^n}{\Delta x}$ the 2 conditional equations (6) can be combined & written in a compact form as

$$u_i^{n+1} = u_i^n - \Delta t[a^+ u_x^- + a^- u_x^+], \tag{8}$$

   which is a general way of writing any upwind-type schemes.

2. **Stability.** The upwind scheme is stable if the following Courant–Friedrichs–Lewy condition (CFL) is satisfied. $c := \left|\frac{a\Delta t}{\Delta x}\right| \le 1$ & $0 \le a$. A Taylor series analysis of the upwind scheme will show that it is 1st-order accurate in space & time. Modified wavenumber analysis shows that the 1st-order upwind scheme introduces severe numerical diffusion/dissipation in the solution where large gradients exist due to necessity of high wavenumbers to represent sharp gradients. The effects of the Courant number c on the stability of the 1st-order upward numerical scheme.

1.21.3 2nd-order upwind scheme

The spatial accuracy of the 1st-order upwind scheme can be improved by including 3 data points instead of just 2, which offers a more accurate finite difference stencil for the approximation of spatial derivative. For the 2nd-order upwind scheme, u_x^- becomes the 3-point backward difference in (8) & is defined as $u_x^+ := \dfrac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}$ & u_x^+ is the 3-point forward difference, defined as $u_x^- := \dfrac{-u_{i+2}^n + 4u_{i+1}^n - 3u_i^n}{2\Delta x}$. This scheme is less diffusive compared to the 1st-order accurate scheme & is called *linear upwind differencing (LUD) scheme*." – Wikipedia/upwind scheme

2 C_0 Semigroup – Nửa Nhóm C_0

Resources – Tài nguyên.

1. [AK16]. Cung Thế Anh, Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng.*

"In mathematical analysis, a C_0-*semigroup*, also known as a *strongly continuous 1-parameter semigroup*, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant ODEs, strongly continuous semigroups provide solutions of linear constant coefficient ODEs in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations & PDEs. Formally, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+, +)$ on some Banach space X that is continuous in the strong toperator topology." -Wikipedia/C_0-semigroup

3 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

1. [Car16]. Manfredo P. do Carmo. *Differential Geometry of Curves & Surfaces.*
2. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries.*
3. [Küh15]. Wolfgang Kühnel. *Differential Geometry.*
4. [Wal15]. Shawn W. Walker. *The Shapes of Things.*

"Differential geometry is the detailed study of the *shape* of a surface (manifold), including *local* & *global* properties. A plane in $\mathbb{R}^3$ is a very simple surface & does not require many tools to characterize. An "arbitrarily" shaped surface, e.g., hood of a car, has many distinguished geometric features (e.g., highly curved regions, regions of near flatness, etc.). Characterizing these features quantitatively & qualitatively requires the tools of differential geometry. Geometric details are important in many physical & biological processes, e.g., surface tension, biomembranes.

The framework of differential geometry is built by 1st defining a local map (i.e., surface parameterization) which defines the surface. Then, a calculus framework is built up on the surface analogous to the standard "Euclidean calculus". Other approaches are also possible, e.g., those with implicit surfaces defined by level sets & distance functions. But parameterizations, though arbitrary, are quite useful in a variety of settings ⇒ stick mostly with those. Emphasize: The geometry of a surface does not depend on a particular parameterization. Otherwise, we will emphasize the distinction between `object 1` & `object 2`.

We will use this "abuse" of notation when there is no possibility of ambiguity.

Open set. The concept of open set is critical in multivariate calculus to properly define differentiability. The notation for referencing boundaries of sets, as well as the closure of sets, is practical for referencing geometric details of solid objects & their surfaces.

Compactness. Compact support is useful for ignoring boundary effects. This concept is needed to keep the "action of a function" away from the boundary of a set, or to localize the function in a region of interest. 1 reason is to avoid potential difficulties with differentiating a function at its boundary of definition. Or, more commonly, we wish to ignore a quantity depending on the value of a function at a boundary point, e.g., $\int_{\partial S} f = 0$ if f has compact support in S.

Topological mapping/homeomorphism. A bijective, continuous mapping $\mathbf{\Phi}$ whose inverse $\mathbf{\Phi}^{-1}$ is also continuous is called a *topological mapping* or *homeomorphism*. Point sets that can be topologically mapped onto each other are said to be *homeomorphic*. Sets that are homeomorphic have the "same topology", i.e., their connectedness is the same; they have the same kinds of "holes". See [Wal15, Sect. 2.3.1] for what can happen when a mapping is not a homeomorphism.

Rigid motion mapping. A mapping $\mathbf{\Phi}$ is called a *rigid motion* if any pair of points $\mathbf{a}, \mathbf{b}$ are the same distance apart as the corresponding pair $\mathbf{\Phi}(\mathbf{a}), \mathbf{\Phi}(\mathbf{b})$.

Orthogonal Transformations. Define the (affine) linear map $\mathbf{\Phi}$ (transformation)

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{9}$$

If A satisfies the properties $A^{-1} = A^\top$, $\det A = 1$ then $\mathbf{\Phi}$ represents a rigid motion. Basically, $\mathbf{\Phi}$ consists of a rotation represented by A followed by a translation represented by $\mathbf{b}$. A rigid motion can be used to transition from 1 Cartesian coordinate system to another. If $\mathbf{b} = \mathbf{0}$ & $A^{-1} = A^\top$, $\det A = 1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is a linear map known as a *direct orthogonal transformation*, which is nothing more than a rotation of the coordinate system with the origin as the center. If $A^{-1} = A^\top$, $\det A = 1$ is replaced by $A^{-1} = A^\top$, $\det A = -1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is called an *opposite orthogonal transformation*, which consists of a rotation about the origin & a reflection in a plane. Both $A^{-1} = A^\top$, $\det A = \pm 1$ are examples of *orthogonal matrices*.

Interpretation of transformations. Can interpret $\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in 2 different ways. Consider a point $P \in \mathbb{R}^3$ with coordinates $\mathbf{x}$:

- *Alias* (Euler perspective). Viewing (9) as a transformation of coordinates, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of the same point w.r.t. 2 different coordinate systems, equivalently, the point is referenced by 2 different "names" (sets of coordinates).

- *Alibi* (Lagrange perspective). Viewing (9) as a mapping of sets, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of 2 different points w.r.t. the same coordinate system, equivalently, the point at $\widetilde{\mathbf{x}}$ "was previously" at $\mathbf{x}$ before applying the map.

The concept of material point is directly related to the alibi viewpoint. One can think of a "particle" of material, i.e., *material point*, initially located at $\mathbf{x}$, that then moves to $\widetilde{\mathbf{x}}$ because of some physical process. The transformation (9) simply represents the kinematic outcome of that physical process, which is a standard concept in deformable continuum mechanics, especially nonlinear elasticity.

General transformations. In general, transformation may not be linear. The alias viewpoint yields a *curvilinear* coordinate system. The alibi viewpoint implies that the set S is *deformed* into the set $S' = \mathbf{\Phi}(S)$.

Parametric approach – what is a surface? A *surface* is a set of points in space that is "regular enough". A random scattering of points in space does not match our intuitive notion of what a surface is, i.e., it is not regular enough. The boundary of a sphere does match our notion of a surface, i.e., regular enough to be a surface because a sphere is "smooth". *Intuition*: Can think of creating a surface as deforming a flat rubber sheet into a curved sheet. Let $U \subset \mathbb{R}^2$ be a "flat" domain & let $\mathbf{X} : U \to \mathbb{R}^3$ be this deforming transformation, i.e., for each point $(s_1, s_2)^\top \in U$ there is a corresponding point $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ s.t. $\mathbf{x} = \mathbf{X}(s_1, s_2)$. Let $\Gamma = \mathbf{X}(U)$ denote the surface obtained from "deforming" U. Call $\mathbf{x} = \mathbf{X}(s_1, s_2)$ a *parametric representation* of the surface Γ, where s_1, s_2 are called the *parameters* of the representation. Refer to U as a *reference* domain.

Allowable parameterization/immersion. If use $\mathbf{x} = \mathbf{X}(s_1, s_2)$ to define surfaces, then we must place assumptions on $\mathbf{X}$ to guarantee that $\Gamma = \mathbf{X}(U)$ is a valid surface. At the bare minimum, $\mathbf{X}$ must be continuous to avoid "tearing" the rubber sheet. But if want to perform calculus on Γ, need more:

Assumption 1 (Regularity assumptions on $\mathbf{X}$)**.** *An allowable parameterization/immersion is a parameterization of the form* $\mathbf{x} = \mathbf{X}(s_1, s_2)$ *satisfying:*

(A1) The function $\mathbf{X}(s_1, s_2) \in C^\infty(U)$ *& each point* $\mathbf{x} = \mathbf{X}(s_1, s_2) \in \Gamma$ *corresponds to just 1 point* $(s_1, s_2) \in U$*, i.e.,* $\mathbf{X}$ *is injective.*

(A2) The Jacobian matrix $J = [\partial_{s_1}\mathbf{X}, \partial_{s_2}\mathbf{X}]$ *is of rank* 2 *on* U*, i.e., the columns of* J *are linearly independent.*

Regular surface. The fundamental property that makes a set of points in $\mathbb{R}^3$ a surface is that it *locally looks like a plane* at every point. If you "zoom into" a surface, it should look flat. Definition defining a surface in terms of a parameterization is inadequate. Want to define a set in $\mathbb{R}^3$ that is "intrinsically" 2D & is smooth enough so we can perform calculus on it, without regard to a specific parameterization.

Definition 9 (Regular surface)**.**

Remark 1 (Local chart)**.**

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

3.1 Calculus on Surfaces

Goal. Define & develop the fundamental tools of calculus on a regular surface. Start with the notion of differentiability of functions defined only on a surface. Define the concept of vector fields in a surface. Then proceed to develop the gradient & Laplacian operators w.r.t. a surface. These operators allow for alternative expressions of the summed & Gaussian curvatures. Derive integration by parts on surfaces, i.e., the domain of integration is a surface. Conclude with some useful identities & inequalities. Always take Γ: a regular surface, either with or without a boundary.

4 Functional Analysis – Giải Tích Hàm

Resources – Tài nguyên.

1. [Alt16]. Hans Wilhelm Alt. *Linear Functional Analysis.*
2. [Bre11]. Haïm Brezis. *Functional Analysis, Sobolev Spaces & PDEs.*
3. [Eva10]. Lawrence C. Evans. *PDEs.*
4. [Rud91]. Walter Rudin. *Functional Analysis.*

   "Functional analysis is the study of certain topological-algebraic structures & of the methods by which knowledge of these structures can be applied to analytic problems."

   The material of a theory should be fully adequate for almost all applications to concrete problems. & this is what ought to be stressed in such a course: The close interplay between the abstract & the concrete is not only the most useful aspect of the whole subject but also the most fascinating one.

Many problems that analysts study are not primarily concerned with a single object such as a function, a measure, or an operator, but they deal instead with large classes of such objects. Most of the interesting classes that occur in this way turn out to be vector spaces, either with real scalars or with complex ones. Since limit processes play a role in every analytic problem (explicitly or implicitly), it should be no surprise that these vector spaces are supplied with metrics, or at least with topologies, that bear some natural relation to the objects of which the spaces are made up. The simplest & most important way of doing this is to introduce a *norm*. The resulting structure is called a *normed vector space*, or a normed linear space, or simply a *normed space*.

"The theory of distributions frees differential calculus from certain difficulties that arise because nondifferentiable functions exist. This is done by extending it to a class of objects (called *distributions* or *generalized functions*) which is much larger than the class of differential functions to which calculus applies in its original form. Here are some features that any such extension ought to have in order to be useful; our setting is some open subset of $\mathbb{R}^d$:

(a) Every continuous function should be a distribution.

(b) Every distribution should have partial derivatives which are again distributions. For differentiable functions, the new motion of derivative should coincide with the old one. (Every distribution should therefore be infinitely differentiable C^∞.)

(c) The usual formal rules of calculus should hold.

(d) There should be a supply of convergence theorems that is adequate for handling the usual limit processes." – [Rud91, Chap. 6, pp. 149–150]

5. [Sim87]. Jacques Simon. *Compact sets in the space $L^p(0,T;B)$.*

6. [Sim22]. Jacques Simon. *Distributions.*

7. [TTV24]. Đinh Ngọc Thanh, Bùi Lê Trọng Thanh, Huỳnh Quang Vũ. *Bài Giảng Giải Tích Hàm.*

8. Yosida.

4.1 Discrete Functional Analysis – Giải Tích Hàm Rời Rạc

Giải Tích Hàm Rời Rạc cung cấp các tools & theorems để chứng minh các kết quả bên Numerical Analysis.
Resources – Tài nguyên.

1. [Gal+18, Sect. 5: Appendix: Discrete Functional Analysis]. Gallouët, Herbin, Latché, J.-C., Mallem, K. *Convergence of the marker-and-cell scheme for the incompressible NSEs on non-uniform grids.*

5 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

1. [ABT18]. Richard Aster, Brian Borchers, Clifford H. Thurber. *Parameter Estimation & Inverse Problems.*

2. [Kir21]. Andreas Kirsch. *An Introduction to The Mathematical Theory of Inverse Problems.*

3. [IJ15]. Kazufumi Ito, Bangti Jin. *Inverse Problems.*

6 Measure & Integration – Độ Đo & Tích Phân

Resources – Tài nguyên.

1. [EG15]. Lawrence C. Evans, Ronald F. Gariepy. *Measure Theory & Fine Properties of Functions.*

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

"The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$-dimensional volume 'in the other direction' by 1st computing the n-dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y, which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*" – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

7 Mean-Field Game Theory – Lý Thuyết Trò Chơi Trường Trung Bình

Community – Cộng đồng. Nicholetta Tchou (French), Đào Mạnh Khang (Vietnamese), Michael Hintermüller (Austrian), Steven-Marian Stengl (German).

7.1 Wikipedia/mean-field game theory

"*Mean-field game theory* is the study of strategic decision making by small interacting agents in very large populations. It lies at the intersection of game theory with stochastic analysis & control theory. The use of the term "mean field" is inspired by mean-field theory in physics, which considers the behavior of systems of large numbers of particles where individual particles have negligible impacts upon the system. In other words, each agent acts according to his minimization or maximization problem taking into account other agents' decisions & because their population is large we can assume the number of agents goes to infinity & a representative agent exists.

In traditional game theory, the subject of study is usually a game with 2 players & discrete time space, & extends the results to more complex situations by induction. However, for games in continuous time with continuous states (differential games or stochastic differential games) this strategy cannot be used because of the complexity that the dynamic interactions generate. On the other hand with MFGs we can handle large numbers of players through the mean representative agent & at the same time describe complex state dynamics.

This class of problems was considered in the economics literature by Boyan Jovanovic & Robert W. Rosenthal, in the engineering literature by Minyi Huang, Roland Malhame, & Peter E. Caines & independently & around the same time by mathematicians Jean-Michel Lasry & Pierre-Louis Lions.

In continuous time a mean-field game is typically composed of a Hamilton-Jacobi-Bellman equation that describes the optimal control problem of an individual & a Fokker–Planck equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions it can be proved that a class of mean-field games is the limit as $N \to \infty$ of an N-player Nash equilibrium.

A related concept to that of mean-field games is "mean-field-type control". In this case, a social planner controls the distribution of states & chooses a control strategy. The solution to a mean-field-type control problem can typically be expressed as a dual adjoint Hamilton-Jacobi-Bellman equation coupled with Kolmogorov equation. Mean-field-type game theory is the multi-agent generalization of the single-agent mean-field-type control.

7.1.1 General Form of a Mean-field Game

The system of equations

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x,m,Du) = 0, \\ \partial_t m - \nu\Delta m - \nabla\cdot(D_pH(x,m,Du)m) = 0, \\ m(0) = m_0, \\ u(T,x) = G(x,m(T)), \end{cases}$$

can be used to model a typical Mean-field game. The basic dynamics of this set of equations can be explained by an average agent's optimal control problem. In a mean-field game, an average agent can control their movement α to influence the population's overall location by

$$dX_t = \alpha_t dt + \sqrt{2\nu}dB_t,$$

where ν: a parameter, B_t: a standard Brownian motion. By controlling their movement, the agent aims to minimize their overall expected cost C throughout the time period $[0,T]$:

$$C = \mathbb{E}\left[\int_0^T L(X_s,\alpha_s,m(s))\,\mathrm{d}s + G(X_T,m(T))\right],$$

where $L(X_s,\alpha_s,m(s))$ is the running cost at time s & $G(X_T,m(T))$ is the terminal cost at time T. By this definition, at time t & position x, the value function $u(t,x)$ can be determined as

$$u(t,x) = \inf_\alpha \mathbb{E}\left[\int_t^T L(X_s,\alpha_s,m(s))\,\mathrm{d}s + G(X_T,m(T))\right].$$

Given the definition of the value function $u(t,x)$, it can be tracked by the Hamilton-Jacobi equation. The optimal action of the average players $\alpha^*(t,x)$ can be determined as $\alpha^*(t,x) = D_pH(x,m,Du)$. As all agents are relatively small & cannot single-handedly change the dynamics of the population, they will individually adapt the optimal control & the population would move in that way. This is similar to a Nash Equilibrium, in which all agents act in response to a specific set of others' strategies. The optimal control solution then leads to the Kolmogorov-Fokker-Planck equation $\partial_t m - \nu\Delta m - \nabla\cdot(D_pH(x,m,Du)m) = 0$.

7.1.2 Finite State Games

A prominent category of mean field is games with a finite number of states & a finite number of actions per player. For those games, the analog of the Hamilton-Jacobi-Bellman equation is the Bellman equation, & the discrete version of the Fokker-Planck equation is the Kolmogorov equation. Specifically, for discrete-time models, the players' strategy is the Kolmogorv equation's probability matrix. In continuous time models, players have the ability to control the transition rate matrix.

A discrete mean field game can be defined by a tuple $\mathcal{G} = (\mathcal{E}, \mathcal{A}, \{Q_a\}, \mathbf{m}_0, \{c_a\}, \beta)$ where $\mathcal{E}$ is the state space, $\mathcal{A}$ the action set, Q_a the transition rate matrices, $\mathbf{m}_0$ the initial state, $\{c_a\}$ the cost functions & $\beta \in \mathbb{R}$ a discount factor. Furthermore, a mixed strategy is a measurable function $\pi : \mathbb{R}^+ \times \mathbb{E} \to \mathcal{P}(\mathcal{A})$, that associates to each state $i \in \mathcal{E}$ & each time $t \geq 0$ a probability measure $\pi_i(t) \in \mathcal{P}(\mathcal{A})$ on the set of possible actions. Thus $\pi_{i,a}(t)$ is the probability that, at time t a player in state i takes action a, under strategy π. Additionally, rate matrices $\{Q_a(\mathbf{m}^\pi(t))\}_{a\in\mathcal{A}}$ define the evolution over the time of population distribution, where $\mathbf{m}^\pi(t) \in \mathcal{P}(\mathcal{E})$ is the population distribution at time t.

7.1.3 Linear-quadratic Gaussian game problem

From Caines (2009), a relatively simple model of large-scale games is the linear-quadratic Gaussian model. The individual agent's dynamics are modeled as a stochastic differential equation

$$dX_i = (a_iX_i + b_iu_i)dt + \sigma_i dW_i,\ i = 1, \ldots, N,$$

where X_i: the state of the ith agent, u_i: control of the ith agent, W_i: independent Wiener processes $\forall i = 1, \ldots, N$. The individual agent's cost is

$$J_i(u_i, \nu) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}[(X_i - \nu)^2 + ru_i^2]\,\mathrm{d}t\right],\ \nu = \Phi\left(\frac{1}{N}\sum_{k\neq i}^N X_k + \eta\right).$$

The coupling between agents occurs in the cost function.

7.1.4 General & Applied Use

The paradigm of Mean Field Games has become a major connection between distributed decision-making & stochastic modeling. Starting out tin the stochastic control literature, it is gaining rapid adoption across a range of applications, including:

1. **Financial market.** Carmona reviews applications in financial engineering & economics that can be cast & tackled within the framework of the MFG paradigm. Carmona argues that models in macroeconomics, contract theory, finance, ..., greatly benefit from the switch to continuous time from the more traditional discrete-time models. He considers only continuous time models in his review chapter, including systemic risk, price impact, optimal execution, models for bank runs, high-frequency trading, & cryptocurrencies.
2. **Crowd motions.** MFG assumes that individuals are smart players which try to optimize their strategy & path w.r.t. certain costs (equilibrium with rational expectations approach). MFG models are useful to describe the anticipation phenomenon: the forward part describes the crowd evolution while the backward gives the process of how the anticipations are built. Additionally, compared to multi-agent microscopic model computations, MFG only requires lower computational costs for the macroscopic simulations. Some researchers have turned to MFG in order to model the interaction between populations & study the decision-making process of intelligent agents, including aversion & congestion behavior between 2 groups of pedestrians, departure time choice of morning commuters, & decision-making processes for autonomous vehicle.
3. **Control & mitigation of Epidemics.** Since the epidemic has affected society & individuals significantly, MFG & mean-field controls (MFCs) provide a perspective to study & understand the underlying population dynamics, especially in the context of the Covid-19 pandemic response. MFG has been used to extend the SIR-type dynamics with spatial effects or allowing for individuals to choose their behaviors & control their contributions to the spread of the disease. MFC is applied to design the optimal strategy to control the virus spreading within a spatial domain, control individuals' decisions to limit their social interactions, & support the government's nonpharmaceutical interventions." – Wikipedia/mean-field game theory

8 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

1. [Bre11]. Haïm Brezis. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations.*
2. [Eva10]. Lawrence C. Evans. *Partial Differential Equations.*
3. [GT01]. David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of 2nd Order.*

8.1 Weak solution – Nghiệm yếu

Definition 10 (Weak solution – Nghiệm yếu). *"In mathematics, a* weak solution *(also called a* generalized solution*) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of distributions." – Wikipedia/weak solution*

"Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough." – Wikipedia/weak solution

Example 3 (1st-order wave equation). *The 1st-order wave equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}^2$ with $u = u(t,x)$ has the weak form $\int_{\mathbb{R}^2} u\partial_t\varphi + u\partial_x\varphi \,\mathrm{d}t\,\mathrm{d}x = 0$ has a solution $u(t,x) = |t-x|$ which may be checked by splitting the integrals over region $\{x \geq t\}$ & $\{x \leq t\}$ where u is smooth.*

"The notion of weak solution based on distribution is sometimes inadequate. In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. Hamilton-Jacobi equation, there is a very different definition of weak solution called *viscosity solution*." – Wikipedia/weak solution

8.1.1 General idea

When solving a differential equation in u, one can rewrite it using a test function φ s.t. whatever derivatives in u show up in the equation, they are "transferred" via integration by parts to φ, resulting in an equation without derivatives of u. This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x},\partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}),$$

where the multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$ varies over some finite set in $\mathbb{N}^d$ & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x},\partial)u(\mathbf{x} = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

where the differential operator $Q(\mathbf{x},\partial)$ is given by the formula

$$Q(\mathbf{x},\partial)\varphi(\mathbf{x}) = \sum(-1)^{|\alpha|}\partial^\alpha[a_\alpha(\mathbf{x})\varphi(\mathbf{x})],$$

which is the formal adjoint of $P(\mathbf{x},\partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$-times differentiable function u defined on the open set W s.t. $P(\mathbf{x},\partial)u(\mathbf{x}) = 0$, $\forall\mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$, $\forall\varphi \in C_c^\infty(W)$.

8.2 Viscosity solution – Nghiệm trơn/nhớt

Example 4 (Viscosity solution for Hamilton–Jacobi equation). *Hamilton–Jacobi equation.*

8.3 Very weak solution – Nghiệm rất yếu

Example 5 (Very weak solution of porous medium equation (PME) [Váz07]). .

Example 6 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term [DDN20]). *Given $f \in L^1_\delta(\Omega), \lambda \geq 0$, a function $u \in L^1_\delta(\Omega)$ is called a* very weak solution *of*

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega,\\ |u|^{m-1}u = 1 & \text{on } \Gamma, \end{cases}$$

if $|u|^{m-1}u \in L^1(\Omega)$ and

$$\int_\Omega u^m\Delta\varphi + \lambda u\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi\,\mathrm{d}\mathbf{x} - \int_\Gamma \partial_{\mathbf{n}}\varphi\,\mathrm{d}\mathbf{x}.$$

Example 7 (Very weak solution of NSEs [Tsa18]). .

8.4 Navier–Stokes Equations [NSEs]

Resources – Tài nguyên.

1. [Ami+01]. Youcef Amirat, Didier Bresch, Jérôme Lemoine, Jacques Simon. *Effect of rugosity on a flow governed by stationary NSEs.*
2. [APS13]. Chérif Amrouche, Patrick Penel, Nour Seloula. *Some remarks on the boundary conditions in the theory of NSEs.*
3. [AR11]. Chérif Amrouche, M. Ángeles Rodríguez-Bellido. *Stationary Stokes, Oseen, & NSEs with singular data.*
4. [AL09]. Rong An, Kai Tai Li. *Existence of weak solution to nonhomogeneous steady NSEs with mixed boundary conditions.*
5. [ALL09]. Rong An, Yuan Li, Kai Tai Li. *Regularity of solutions to stationary NSEs with mixed boundary conditions.*
6. [Bän01]. Eberhard Bänsch. *Finite element discretization of the NSEs with a free capillary surface.*
7. [BNP04]. Hamid Bellout, Jiří Neustupa, Patrick Penel. *On the NSE with boundary conditions based on vorticity.*
8. [Ben11]. Michal Beneš. *Mixed IBVP for 3D NSEs in polyhedral domains.*
9. [BK07]. Michal Beneš, Petr Kučera. *Non-steady NSEs with homogeneous mixed boundary conditions and arbitrarily large initial condition.*
10. [BK12]. Michal Beneš, Petr Kučera. *On the Navier–Stokes flows for heat-conducting fluids with mixed boundary conditions.*
11. [BK16]. Michal Beneš, Petr Kučera. *Solutions to the NSEs with mixed boundary conditions in 2D bounded domains.*
12. [BHV09]. Christine Bernardi, Frédéric Hecht, Rüdiger Verfürth. *A finite element discretization of 3D NSEs with mixed boundary conditions.*
13. [BCY15]. Christine Bernardi, Tomás Chacón Rebollo, Driss Yakoubi. *Finite element discretization of the Stokes & NSEs with boundary conditions on the pressure.*
14. [BCY15]. Christine Bernardi, Toni Sayah. *A posteriori error analysis of the time dependent NSEs with mixed boundary conditions.*
15. [Ber09]. Luigi C. Berselli. *Some criteria concerning the vorticity and the problem of global regularity for the 3D NSEs.*
16. [BBP14]. Mahdi Boukrouche, Imane Boussetouan, Laetitia Paoli. *Non-isothermal Navier-Stokes system with mixed boundary conditions and friction law: uniqueness and regularity properties.*
17. [BM14]. Malte Braack, Piotr Boguslaw Mucha. *Directional do-nothing condition for the NSEs.*
18. [Bri09]. R. V. Brizitskiĭ. *Investigation of a class of control problems for stationary NSEs with mixed boundary conditions.*
19. [Bro+10]. R. Brown, I. Mitrea, M. Mitrea, M. Wright. *Mixed BVPs for the Stokes system.*
20. [Buc+08]. Dorin Bucur, Eduard Feireisl, Šárka Nečasová, Joerg Wolf. *On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries.*
21. [BMR07]. M. Bulíček, J. Málek, K. R. Rajagopal. *Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity.*
22. [CC88]. J. Cahouet, J.-P. Chabard. *Some fast 3D finite element solvers for the generalized Stokes problem.*
23. [Cam+18]. Jessika Camaño, Ricardo Oyarzúa, Ricardo Ruiz-Baier, Giordano Tierra. *Error analysis of an augmented mixed method for the Navier-Stokes problem with mixed boundary conditions.*
24. [Cat61]. Lamberto Cattabriga. *Su un problema al contorno relativo al sistema di equazioni di Stokes.*
25. [COQ09]. Gui-Qiang Chen, Dan Osborne, Zhongmin Qian. *The NSEs with the kinematic and vorticity boundary conditions on non-flat boundaries.*
26. [DY17]. Yasmina Daikh, Driss Yakoubi. *Spectral discretization of the Navier-Stokes problem with mixed boundary conditions.*
27. [DW95]. Paul Deuring, Wolf von Wahl. *Strong solutions of the Navier-Stokes system in Lipschitz bounded domains.*
28. [EF01]. Carsten Ebmeyer, Jens Frehse. *Steady NSEs with mixed boundary value conditions in 3D Lipschitzian domains.*
29. [EP99]. M. Elghaoui, R. Pasquetti. *Mixed spectral-boundary element embedding algorithms for the NSEs in the vorticity-stream function formulation.*

30. [FNP08]. Mohamed Farhloul, Serge Nicaise, Luc Paquet. *A refined mixed FEM for the stationary NSEs with mixed boundary conditions.*

31. [Fef06]. Charles L. Fefferman. *Existence and smoothness of the NSE*: **The millennium prize problems**.

32. [Foi+01]. C. Foias, O. Manley, R. Rosa, R. Temam. *NSEs & Turbulence.*

33. [FMT87]. C. Foias, O. Manley, R. Temam. *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension.*

34. [FK64]. Hiroshi Fujita, Tosio Kato. *On the Navier-Stokes IVP. I.*

35. [Fur80]. A. V. Fursikov. *Some control problems and results related to the unique solvability of the mixed BVP for the NSEs and Euler 3D systems.*

36. [Fur81]. A. V. Fursikov. *Control problems and theorems concerning unique solvability of a mixed BVP for 3D NSEs and Euler equations.*

37. [Fur82]. A. V. Fursikov. *Properties of solutions of control problems that are connected with the Navier-Stokes system.*

38. [Gal00]. Giovanni P. Galdi. *An introduction to the Navier-Stokes IBVP.*

39. [Gal11]. Giovanni P. Galdi. *An introduction to the mathematical theory of NSEs.*

40. [GK12]. Gung-Min Gie, James P. Kelliher. *Boundary layer analysis of the NSEs with generalized Navier boundary conditions.*

41. [GST15]. Telma Guerra, Adélia Sequeira, Jorge Tiago. *Existence of optimal boundary control for NSEs with mixed boundary conditions.*

42. [HRT96]. John G. Heywood, Rolf Rannacher, Stefan Turek. *Artificial boundaries and flux and pressure conditions for the incompressible NSEs.*

43. [Hoa10]. Luan Thach Hoang. *Incompressible fluids in thin domains with Navier friction boundary conditions (I).*

44. [Hoa13]. Luan Thach Hoang. *Incompressible fluids in thin domains with Navier friction boundary conditions (II).*

45. [HS10]. Luan Thach Hoang, George R. Sell. *NSEs with Navier boundary conditions for an oceanic model.*

46. [HP19]. Yanren Hou, Shuaichao Pei. *On the weak solutions to steady NSEs with mixed boundary conditions.*

47. [IR01]. Dragoş Iftimie, Geneviève Raugel. *Some results on NSEs in thin 3D domains.*

48. [IRS07]. Dragoş Iftimie, Geneviève Raugel, George R. Sell. *NSEs in thin 3D domains with Navier boundary conditions.*

49. [IS11]. Dragoş Iftimie, Franck Sueur. *Viscous boundary layers for the NSEs with the Navier slip conditions.*

50. [IJL02]. Traian Iliescu, Volker John, William J. Layton. *Convergence of finite element approximations of large eddy motion.*

51. [IC01]. A. A. Illarionov, A. Yu. Chebotarev. *On the solvability of a mixed BVP for stationary NSEs.*

52. [JM00]. Willi Jäger, Andro Mikelić. *On the interface boundary condition of Beavers, Joseph, & Saffman.*

53. [JM01]. Willi Jäger, Andro Mikelić. *On the roughness-induced effective boundary conditions for an incompressible viscous flow.*

54. [Joh02]. Volker John. *Slip with friction and penetration with resistance boundary conditions for the NSEs—numerical tests and aspects of the implementation.*

55. [JLS04]. Volker John, W. Layton, N. Sahin. *Derivation and analysis of near wall models for channel and recirculating flows.*

56. [KKM19]. K. Kaladhar, E. Komuraiah, K. Madhusudhan Reddy. *Soret and Dufour effects on chemically reacting mixed convection flow in an annulus with Navier slip and convective boundary conditions.*

57. [Kel06]. James P. Kelliher. *NSEs with Navier boundary conditions for a bounded domain in the plane.*

58. [KC15]. Tujin Kim, Daomin Cao. *Some properties on the surfaces of vector fields and its application to the Stokes and Navier-Stokes problems with mixed boundary conditions.*

59. [Kim16]. Tujin Kim. *Erratum to "Some properties on the surfaces of vector fields and its application to the Stokes and Navier-Stokes problems with mixed boundary conditions" [Nonlinear Anal. 113 (2015) 94–114] [MR3281848]*

60. [KC16]. Tujin Kim, Daomin Cao. *The steady Navier-Stokes and Stokes systems with mixed boundary conditions including one-sided leaks and pressure.*

61. [KC17]. Tujin Kim, Daomin Cao. *Non-stationary NSEs with mixed boundary conditions.*

62. [KH18]. Tujin Kim, Feimin Huang. *The non-steady Navier-Stokes systems with mixed boundary conditions including friction conditions.*

63. [KPR15]. Mikhail V. Korobkov, Konstantin Pileckas, Remigio Russo. *Solution of Leray's problem for stationary NSEs in plane and axially symmetric spatial domains.*

64. [KN01]. S. Kračmar, J. Neustupa. *A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions.*

65. [Kuč98b]. Petr Kučera. *A structure of the set of critical points to the NSEs with mixed boundary conditions.*

66. [Kuč98a]. Petr Kučera. *Solutions of the stationary Navier-Stokes equations with mixed boundary conditions in bounded domain.*

67. [Kuč09]. Petr Kučera. *Basic properties of solution of the non-steady NSEs with mixed boundary conditions in a bounded domain.*

68. [Kuč10]. Petr Kučera. *The time-periodic solutions of the NSEs with mixed boundary conditions.*

69. [KS98]. Petr Kučera, Zdeněk Skalák. *Local solutions to the NSEs with mixed boundary conditions.*

70. [Lad69]. O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.*

71. [LSŻ18]. Irena Lasiecka, Katarzyna Szulc, Antoni Żochowski. *Boundary control of small solutions to fluid-structure interactions arising in coupling of elasticity with NSE under mixed boundary conditions.*

72. [Lem02]. Pierre Gilles Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem.*

73. [RR93]. C. le Roux, B. D. Reddy. *The steady NSEs with mixed boundary conditions: application to free boundary flows.*

74. [Lem16]. Pierre Gilles Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century.*

75. [Ler33]. Jean Lerray. *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique.*

76. [Ler34a]. Jean Lerray. *Essai sur les mouvements plans d'un liquide visqueaux que limitent des parois.*

77. [Ler34b]. Jean Lerray. *Sur le mouvement d'un liquide visqueux emplissant l'espace.*

78. [LA08]. Kai Tai Li, Rong An. *On the rotating NSEs with mixed boundary conditions.*

79. [Lia01]. Anastasios Liakos. *Discretization of the NSEs with slip boundary condition.*

80. [Lia04]. Anastasios Liakos. *Discretization of the Navier-Stokes equations with slip boundary condition. II.*

81. [LY08]. Dongjie Liu, Dehao Yu. *The coupling method of natural boundary element and mixed finite element for stationary NSE in unbounded domains.*

82. [LP85]. I. I. Lyashko, N. Z. Prokhur. *Construction of stable schemes for solving a mixed BVP for evolution generalization of 3D NSEs.*

83. [MR07]. Vladimir Maz'ya, J. Rossmann. *L_p estimates of solutions to mixed BVPs for the Stokes system in polyhedral domains.*

84. [MR09]. Vladimir Maz'ya, J. Rossmann. *Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains.*

85. [Mih68]. V. P. Mihaĭlov. *The mixed boundary value problem for the Navier-Stokes system of equations.*

86. [MM09a]. Marius Mitrea, Sylvie Monniaux. *On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds.*

87. [MM09b]. Marius Mitrea, Sylvie Monniaux. *The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains.*

88. [Mon06]. Sylvie Monniaux. *NSEs in arbitrary domains: the Fujita-Kato scheme.*

89. [Muk92a]. F. Kh. Mukminov. *On the decay of the solution of the first mixed problem for a linearized system of NSEs in a domain with a noncompact boundary.*

90. [Muk92b]. F. Kh. Mukminov. *On the rate of decay of the solution of a mixed problem for a system of NSEs in a domain with a noncompact boundary.*

91. [NR15]. Nguyen Phuong Anh, Jean-Pierre Raymond. *Boundary stabilization of the NSEs in the case of mixed boundary conditions.*

92. [NPR07]. S. Nicaise, L. Paquet, Rafilipojaona. *A refined mixed finite element method for stationary NSEs with mixed boundary conditions using Lagrange multipliers.*

93. [Ovs78]. V. G. Ovsienko. *A mixed boundary value problem for nonstationary NSEs on the exterior of a circular cylinder.*

94. [Pap+15]. E. M. Papoutsis-Kiachagias, N. Magoulas, J. Mueller, C. Othmer, K. C. Giannakoglou. *Noise reduction in car aerodynamics using a surrogate objective function and the continuous adjoint method with wall functions*

95. [PR17]. Phan Đức Duy, Sérgio S. Rodrigues. *Gevrey regularity for NSEs under Lions boundary conditions.*

96. [PS08]. P. I. Plotnikov, J. Sokolowski. *Stationary BVPs for compressible NSEs.*

97. [PS12]. Pavel Plotnikov, Jan Sokołowski. *Compressible NSEs: Theory and Shape Optimization.*

98. [Ros09]. Jürgen Rossmann. *Mixed BVPs for Stokes and Navier-Stokes systems in polyhedral domains.*

99. [SNN19]. Abdou Sène, Timack Ngom, Evrad M. D. Ngom. *Global stabilization of the NSEs around an unstable steady state with mixed boundary kinetic energy controller.*

100. [Ser15]. Gregory Seregin. *Lecture notes on regularity theory for NSEs.*

101. [Soh01a; Soh01b]. Hermann Sohr. *The NSEs: An Elementary Functional Analytic Approach.*

    Primary objective. To develop an elementary & self-contained approach to the mathematical theory of a viscous incompressible fluid in a domain $\Omega\subset\mathbb{R}^d$, described by NSEs. Formulate the theory for a completely general domain Ω.

102. [SL08]. Jian Su, Kai Tai Li. *A FEM for 3D stationary rotating NSEs in primitive variables with mixed boundary conditions.*

103. [Tao13]. Terence Tao. *Localisation and compactness properties of the Navier-Stokes global regularity problem.*

104. [Tar06]. Luc Tartar. *An introduction to Navier-Stokes equation and oceanography.*

105. [Tem77; Tem00]. Roger Temam. *NSES: Theory & Numerical Analysis.*

106. [Tem83; Tem95]. Roger Temam. *NSEs & nonlinear functional analysis.*

107. [Tsa18]. Tai-Peng Tsai. *Lectures on NSEs.*

108. [Ver87]. Rüdiger Verfürth. *Finite element approximation of incompressible NSEs with slip boundary condition.*

109. [Ver91]. Rüdiger Verfürth. *Finite element approximation of incompressible NSEs with slip boundary condition. II.*

110. [Yud63]. V. I. Yudovich. *Non-stationary flow of an ideal incompressible liquid.*

111. [Jud67]. V. I. Yudovich. *An example of the loss of stability and the generation of a secondary flow of a fluid in a closed container.*

8.5 Schrödinger equations

Resources – Tài nguyên.

1. [Wei83]. Michael I. Weinstein. *Nonlinear Schrödinger equations and sharp interpolation estimates.*

    - NQBH. Master 2 Seminar: *On the smallest constant for a Gagliardo–Nirenberg functional inequality.* [report][summary][slide]

    Abstract. Obtain a sharp sufficient condition for global existence for nonlinear Schrödinger equation $2i\phi_t+\Delta\phi+|\phi|^{2\sigma}\phi=0$, in $\mathbb{R}^+\times\mathbb{R}^N$ in case $\sigma=\frac{2}{N}$, in terms of an exact stationary solution (nonlinear ground state) of NLS, derived by solving a variational problem to obtain the "best constant" for classical interpolation estimates of Nirenberg & Gagliardo.

    - Sect. 1: Introduction. The "best constant" of an interpolation estimate among various norms often has an analytical or geometrical significance.

      Goal 1. *Present a relationship between the best constant for a classical interpolation inequality due to Nirenberg & Gagliardo, & a sharp criterion for existence of global solutions to nonlinear Schrödinger equation:*

      $$2i\partial_t\phi+\Delta\phi+|\phi|^{2\sigma}\phi=0 \text{ in } \mathbb{R}^+\times\mathbb{R}^N,\ \phi(0,\mathbf{x})=\phi_0(\mathbf{x}) \tag{nSch}$$

      in critical case $\sigma=\frac{2}{N}$.

    - Sect. 2: Solution of a Variational Problem.
    - Sect. 3: Global Existence for IVP in Critical Case $\sigma=\frac{2}{N}$.
    - Sect. 4: Blowing Up Solutions.
    - Sect. 5: Numerical Observations & Open Questions.

8.6 Water Waves Systems

Community – Cộng đồng. VINCENT DUCHENE, DAVID LANNES, MICHAEL I. WEINSTEIN.
Resources – Tài nguyên.

1. VINCENT DUCHÊNE. *Many Models for Water Waves.*

2. [Lan13]. DAVID LANNES. *The Water Waves Problem.*

8.7 Elliptic PDEs

Resources – Tài nguyên.

1. [Agm65; Agm10]. SHMUEL AGMON. *Lectures on Elliptic BVPs.*

2. [GT01]. DAVID GILBARG, NEIL S. TRUDINGER. *Elliptic PDEs of 2nd Order.*

3. [Gri80]. PIERRE GRISVARD. *BVPs in Nonsmooth Domains.*

4. [Gri85; Gri11]. PIERRE GRISVARD. *Elliptic Problems in Nonsmooth Domains.*

5. [HL11]. QING HAN, FANGHUA LIN. ELLIPTIC PDES.

6. [NP20]. NGUYỄN QUỐC HƯNG, NGUYỄN CÔNG PHÚC. *Pointwise gradient estimates for a class of singular quasilinear equations with measure data.*

   Keywords. Riesz's potential, Wolff's potential, pointwise gradient estimate, Reifenberg flat domain.

   Abstract. Local & global pointwise gradient estimates are obtained for solutions to quasilinear elliptic equation with measure data $-\nabla\cdot(A(\mathbf{x},\nabla u))=\mu$ in a bounded & possibly nonsmooth domain $\Omega\subset\mathbb{R}^n$ where $\nabla\cdot(A(\mathbf{x},\nabla u))$ is modeled after the p-Laplacian. Extend earlier known results to the singular case in which $\frac{3n-2}{2n-1}<p\le 2-\frac{1}{n}$.

   - Sect. 1: Introduction & main results. Consider quasilinear elliptic equation with measure data $-\nabla\cdot(A(\mathbf{x},\nabla u))=\mu$ in a bounded open subset Ω of $\mathbb{R}^n$, $n\ge 2$, μ: a finite signed measure in Ω, nonlinearity $A=(A_1,\ldots,A_n):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is vector-valued function.

     Goal 2. *Obtain pointwise estimates for gradients of solutions to $-\nabla\cdot(A(\mathbf{x},\nabla u))=\mu$ by means of nonlinear potentials of Wolff type.*

     Assume $A=A(\mathbf{x},\xi)$ satisfies growth, ellipticity, & continuity assumptions. Dini's condition $\int_0^1\omega(r)^{\gamma_0}\frac{dr}{r}<\infty$.
     A typical model for main PDE is given by p-Laplace equation with measure data $-\Delta_p u\coloneqq-\nabla\cdot(|\nabla u|^{p-2}\nabla u)=\mu$ in Ω, or its nondegenerate version ($s>0$): $-\nabla\cdot((|\nabla u|+s^2)^{\frac{p-2}{2}}\nabla u)=\mu$ in Ω.
   - Sect. 2: Sharp quantitative $C^{1,\sigma}$ regularity estimates.
   - Sect. 3: Interior pointwise gradient estimates.
   - Sect. 4: Global pointwise gradient estimates.

7. [LU68]. OLGA A. LADYZHENSKAYA, NINA N. URAL'TSEVA. *Linear & Quasilinear Elliptic Equations.*

8. [MR10]. VLADIMIR MAZ'YA, JÜRGEN ROSSMANN. *Elliptic Equations in Polyhedral Domains.*

9. [Neč67; Neč12]. JINDŘICH NEČAS. *Les méthodes directes en théorie des équations elliptiques – Direct Methods in The Theory of Elliptic Equations.*

8.8 Parabolic PDEs

Resources – Tài nguyên.

1. [Fri64]. AVNER FRIEDMAN. *PDEs of Parabolic Type.*

2. [HS18]. DAVID HOLCMAN, ZEEV SCHUSS. *Asymptotics of Elliptic & Parabolic PDEs – & Their Applications in Statistical Physics, Computational Neuroscience, & Biophysics.*

3. [KA03]. PETER KNABNER, LUTZ ANGERMANN. *Numerical Methods for Elliptic & Parabolic PDEs.*

4. [Kry08]. N. V. KRYLOV. *Lectures on Elliptic & Parabolic Equations in Sobolev Spaces.*

5. [LSU68]. O. A. LADYŽENSKAJA, V. A. SOLONNIKOV, N. N. URAL'CEVA. *Linear & Quasi-linear Equations of Parabolic Type.*

6. [Lun95]. ALESSANDRA LUNARDI. *Analytic Semigroup & Optimal Regularity in Parabolic Problems.*

8.9 Hyperbolic PDEs

Resources – Tài nguyên.

1. [Ali09]. Serge Alinhac. *Hyperbolic PDEs.*
2. [BS07]. Sylvie Benzoni-Gavage, Denis Serre. *Multidimensional Hyperbolic PDEs.*
3. [Ika00]. Mitsuru Ikawa. *Hyperbolic PDEs & Wave Phenomena.*
4. [LT00]. Irena Lasiecka, Roberto Triggiani. *Control Theory for PDEs: Continuous & Approximation Theories. II: Abstract Hyperbolic-Like Systems Over a Finite Time Horizon.*
5. [Lax87]. Peter D. Lax. *Hyperbolic Systems of Conversation Laws & The Mathematical Theory of Shock Waves.*
6. [Lax06] . Peter D. Lax. *Hyperbolic PDEs.*
7. [Rau12]. Jeffrey Rauch. *Hyperbolic PDEs & Geometric Optics.*
8. [Tar08]. Luc Tartar. *From Hyperbolic Systems to Kinetic Theory.*

8.10 Porous Medium Equations [PMEs]

Resources – Tài nguyên.

1. [AK89]. Andrew F. Acker, Bernhard Kawohl. *Remarks on quenching.* [95 citations]
2. [BFV18]. Matteo Bonforte, Alessio Figalli, Juan Luis Vázquez. *Sharp global estimates for local & nonlocal porous medium-type equations in bounded domains.* [55 citations]

   Keywords. nonlocal diffusion, nonlinear equations, bounded domains, a priori estimates, positivity, boundary behavior, regularity, Harnack inequalities.

   Abstract. Provide a quantitative study of nonnegative solutions to nonlinear diffusion equations of porous medium-type of the form $\partial_t u + \mathcal{L}u^m = 0$, $m > 1$, where the operator $\mathcal{L}$ belongs to a general class of linear operators, & eqn is posed in a bounded domain $\Omega \subset \mathbb{R}^N$. As possible operators: include 3 most common definitions of the fractional Laplacian in a bounded domain with zero Dirichlet conditions, & also a number of other nonlocal versions. In particular, $\mathcal{L}$ can be a fractional power of a uniformly elliptic operator with C^1 coefficients. Since nonlinearity is given by u^m with $m > 1$, eqn is degenerate parabolic.

   Basic well-posed theory for this class of equations was recently developed by Bonforte & Vázquez. Address regularity theory: decay & positivity, boundary behavior, Harnack inequalities, interior & boundary regularity, & asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although focusing on fractional models, results cover also local case when $\mathcal{L}$ is a uniformly elliptic operator, & provide new estimates even in this setting.

   A surprising aspect discovered: possible presence of nonmatching powers for long-time boundary behavior, i.e., when $\mathcal{L} = (-\Delta)^s$ is a spectral power of Dirichlet Laplacian inside a smooth domain, can prove that: (i) when $s > 2\left(1 - \frac{1}{m}\right)$, for large times all solutions behave as $\operatorname{dist}^{\frac{1}{m}}$ near the boundary; (ii) when $s \le 2\left(1 - \frac{1}{m}\right)$, different solutions may exhibit different boundary behavior. This unexpected phenomenon is a completely new feature of nonlocal nonlinear structure of this model, & not present in semilinear elliptic equation $\mathcal{L}u^m = u$.

   - Sect. 1: Introduction.

     Goal 3. *Address question of obtaining a priori estimates, positivity, boundary behavior, Harnack inequalities, & regularity for a suitable class of weak solutions of nonlinear nonlocal diffusion equations of form $\partial_t u + \mathcal{L}F(u) = 0$ in $Q_\infty = (0,\infty) \times \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $N \ge 2$ (results work also in 1D if fractional exponent $0 < s < \frac{1}{2}$. The interval $\frac{1}{2} \le s < 1$ requires some minor modifications preferred to avoid.), $\mathcal{L}$: a linear operator representing diffusion of local or nonlocal type, the prototype example being fractional Laplacian (class of admissible operators).*

     Although arguments hold for a rather general class of nonlinearities $F : \mathbb{R} \to \mathbb{R}$, for simplicity, focus on model case $F(u) \coloneqq u^m$ with $m > 1$.

     Use of nonlocal operators in diffusion equations reflects the need to model presence of long-distance effects not included in evolution driven by Laplace operator: well documented in literature. Physical motivation & relevance of nonlinear diffusion models with nonlocal operators. Because u usually represents a density, all data & solutions are supposed to be nonnegative. Since the problem is posed on a bounded domain, need boundary or external conditions assumed to be of Dirichlet type.

     Extensively studied when $\mathcal{L} = -\Delta, F(u) = u^m$, $m > 1$, eqn becomes classical PME. Here interested in treating nonlocal diffusion operators, in particular fractional Laplacian operators. Since working on a bounded domain, the concept of fractional Laplacian operator admits several nonequivalent versions, the best known being the restricted fractional Laplacian (RFL), the spectral fractional Laplacian (SFL), & the censored fractional Laplacian (CFL). RFL is usually known as the *standard fractional Laplacian*, or plainly fractional Laplacian, & the CFL is often called the *regional fractional Laplacian.*

     The case of SFL operator with $F(u) = u^m$, $m > 1$, was already studied in [Bonforte & Vázquez 2015; 2016]. In [Bonforte & Vázquez 2016] presented a rather abstract setting where they were able to treat not only usual fractional Laplacians but also a large number of variants listed. Rather general increasing nonlinearities F were allowed. Basic questions of existence & uniqueness of suitable solutions for this problem were solved in [Bonforte & Vázquez 2016] in the class of "weak dual solutions", an extension of the concept of solution introduced in [Bonforte & Vázquez 2015] having proved to be quite flexible & efficient. Derived a number of a priori estimates (absolute bounds & smoothing effects) in that generality.

Since these basic facts are settled, here focus on finer aspects of theory, mainly sharp boundary estimates & decay estimates. Such upper & lower bounds will be formulated in terms of 1st eigenfunction Φ_1 of $\mathcal{L}$, which under our assumptions will satisfy $\Phi_1 \asymp \mathrm{dist}(\cdot,\Gamma)^\gamma$ for a certain characteristics power $\gamma \in (0,1]$ depending on particular operator being considered. Typical values: $\gamma = s$ (SFL), $\gamma = 1$ (RFL), $\gamma = s - \frac{1}{2}$ for $s > \frac{1}{2}$ (CFL) $\Rightarrow$ get various kinds of local & global Harnack-type inequalities.

Some of the boundary estimates obtained for parabolic case are essentially elliptic in nature. Study of this issue for stationary problems is done in a companion paper [Bonforte et al. 2017b]. Advantage: many arguments are clearer, since parabolic problem is more complicated than elliptic one. Clarifying such differences is 1 of main contributions. Prove both interior & boundary regularity, & to find large-time asymptotic behavior of solutions.

Notation. Some notation of general use. Notation $a \asymp b$ whenever there exist universal constants $c_0, c_1 > 0$ s.t. $c_0 b \le a \le c_1 b$. $a \vee b = \max\{a,b\}, a \wedge b = \min\{a,b\}$. Always consider bounded domains Ω with boundary of class C^2. Use short form "solution" to mean "weak dual solution", unless differently stated.

Presentation of results on sharp boundary behavior. A basic principle: sharp boundary estimates depend not only on $\mathcal{L}$ but also on behavior of nonlinearity $F(u)$ near $u = 0$, i.e., on exponent $m > 1$. [...]

Asymptotic behavior & regularity.

- Sect. 2: General class of operators & their kernels. The interest of theory developed here lies both in the sharpness of results & in wide range of applicability. Mentioned most relevant examples appearing in literature. Theory applies to a general class of operators with definite assumptions. Properties having to be assumed on class of admissible operators, which some of them already appeared in [Bonforte & Vázquez 2016]. To further develop theory, need to introduce more hypotheses. [Bonforte & Vázquez 2016] only uses properties of Green function, here make some assumptions also on kernel of $\mathcal{L}$ whenever it exists. Assumptions on the kernel K of $\mathcal{L}$ are needed for positivity results, because need to distinguish between local & nonlocal cases. Perform study of kernel K.
- Sect. 3: Reminders about weak dual solutions.
- Sect. 4: Upper boundary estimates.
- Sect. 5: Lower bounds.
- Sect. 6: Summary of general decay & boundary results.
- Sect. 7: Asymptotic behavior.
- Sect. 8: Regularity results.
- Sect. 9: Numerical evidence.
- Sect. 10: Complements, extensions, & further examples.

3. Matteo Bonforte, Maria Pia Gualdani, Peio Ibarrondo. *Time-Fractional Porous Medium Type Equations: Sharp Time Decay & Regularization.*.

Keywords. PME, fast diffusion equation, nonlocal operators, Caputo fractional time derivative, subdiffusion comparison principle, regularity estimates, long time behavior.

- Sect 1. Intro. Several phenomena, from physics to biology to finance, exhibit events during which fractional behavior & memory effects become predominant; e.g., viscoelastic materials (whose response depends on their current & past states), certain geographical processes including movement of groundwater or transportation through porous media, neuronal & gene regulation networks, but also control theory & more recent modeling of financial market. Fractional calculus provides a reliable tools to describe memory effects. In 1967 M. Caputo, in the context of modeling heterogeneous elastic fields, introduced the *Caputo time derivative* of order α:

$$D_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\tau)-f(0)}{(t-\tau)^\alpha}\,\mathrm{d}\tau,\ \alpha \in (0,1). \tag{10}$$

Caputo modeled certain type of fluid diffusing in porous media using this novel nonlocal operator. In these geothermal studies, Darcy's Law is adapted to describe fluids that may carry solid particles obstructing the pores, thus diminishing their size & creating a pattern of mineralization. This phenomenon has recently been observed in various other types of porous materials, including building materials, & zeolite.

Systems where particles exhibit anomalous diffusion (sub- or super-diffusion behavior) often involve memory effects, e.g., diffusion in porous media, turbulent flows, & biological transport processes. These applications require mathematical models allowing particles to do macroscopical long jumps (Lévy flights), leading to the use of nonlocal operators in the spatial & time variables. Many different mathematical models describing anomalous diffusion in a porous medium in literature. Main PDE:

$$D_t^\alpha = -\mathcal{L}u^m,\ m > 0, \tag{CPME}$$

& includes a general class of densely defined operators $\mathcal{L}$ both of local & nonlocal type. Eqn (CPME) is a *density dependent diffusion* resulting in a characteristic scaling $\frac{x}{t^{\frac{\alpha}{2+d(m-1)}}}$, whenever diffusion operator is $\mathcal{L} = -\Delta$.

From a mathematical point of view, characteristic scaling of the wetting front variable in anomalous diffusion differs from 1 of classical Heat Eqn $\frac{x}{\sqrt{t}}$. Originally, Caputo derivative arose in linear setting to achieve a subdiffusive characteristic scaling of form $\frac{x}{t^{\frac{\alpha}{2}}}$ with $\alpha \in (0,1)$. Non-locality in time, or memory effect, represents a "waiting time" phenomenon typically derived within stochastic framework of Continuous Time Random Walk. (CPME) encompasses a wide variety of anomalous diffusion models combining local & nonlocal spatial operators, Caputo fractional time derivative, & m-power like nonlinearities for any $m > 0$. Provide a comprehensive qualitative & quantitative study of (CPME). Beside global well-posedness, main contributions are 3fold: (i) study of comparison principle & time monotonicity formula, (ii) L^p-L^∞ smoothing effects, (iii) optimal long time behavior. Surprising facts: *regularity effects* & *non-extinction in finite time* for all solutions of (CPME) when $m \in (0,1)$. Notably: memory effect slows down diffusion, minimizing relevance of nonlinearity parameter m in ranges $m \in (0,1)$ & $m > 1$. Diffusive nature of eqn still provides a regularization of solution, a feature previously unknown for nonlinear equations involving Caputo derivatives. All these results are new even for classical Laplacian $\mathcal{L} = -\Delta$ & $\alpha \in (0,1)$.

Within mathematical framework, prototype subdiffusive PDE is "Heat Eqn with memory":

$$D_t^\alpha u = \Delta u. \tag{11}$$

Vast literature. well-posedness & regularity in $\mathbb{R}^d$, optimal asymptotic decay for solutions of (11) for Cauchy problem in $\mathbb{R}^d$. For Dirichlet problem on bounded domains, long time decay estimates, also allowing for general operators with variable coefficients in space & time. Global well-posedness of (CPME) with $m = 1$ for singular solutions on bounded domains. From a nonlinear perspective, consider several fractional nonlinear models & obtain sharp decay estimates of L^q norms using fractional ODEs techniques. Porous Medium with fractional pressure, associated to Caffarelli–Vázquez model studied in its Caputo derivative version. Develop a theory of fractional gradient flows in Hilbert spaces, analogous of Brezis-Komura theory with fractional time derivative. This theory provides well-posedness for both linear & for nonlinear problems.

Memory effects complicate analysis quite a lot. Memory effects somehow destroy semigroup structure, essential in the De Giorgi-Nash-Moser theory. A nontrivial adaptation of Green function method, achieved by employing novel time monotonicity estimates, a feature coming as a surprise in context of Caputo setting.

To provide further insights about results, some numerology[1] is in order: asymptotic estimates correspond to known estimates in formal limit $m \to 1$, which does not happen in the –formal– limit $\alpha \to 1^-$. Neither exponents of smoothing effects nor the ones in long time behavior estimates, correspond to known ones when $\alpha = 1$.

Principal findings:

- ◦ **Comparison principle & time monotonicity.**
- ◦ **Smoothing effects & boundary estimates.**
- ◦ **Case of unbounded domains.**
- ◦ **Optimal long time behavior.**

- Discretized problem.
- Continuous problem.
- Smoothing effects.
- Sharp time decay of L^p-norms.
- Open questions.
- Appendix A: Fractional ODEs.

Abstract. Consider a class of porous medium type of equations with Caputo time derivative. Prototype problem: $D_t^\alpha u = -\mathcal{L}u^m$ posed on a bounded Euclidean domain $\Omega \subset \mathbb{R}N$ with zero Dirichlet boundary conditions. The operator $\mathcal{L}$ falls within a wide class of either local or nonlocal operators, & nonlinearity is allowed to be of degenerate or singular type, namely, $0 < m < 1$ & $m > 1$. Most general form of a variety of models used to describe anomalous[2] diffusion processes with memory effects, & finds application in various fields, e.g., visco-elastic materials, signal processing, biological systems, & geophysical science. Prove existence of unique solution & new L^p-L^∞ smoothing effects. The comparison principle, provided in the most general setting, serves as a crucial tool in the proof & provides a novel monotonicity formula. Consequently, establish: regularizing effects from diffusion are stronger than memory effects introduced by fractional time derivative. Solution attains boundary conditions pointwise. Prove: solution does not vanish in finite time if $0 < m < 1$, unlike case with classical time derivative. Provide a sharp rate of decay for any L^p-norm of solution for any $m > 0$. Memory effects weaken the spatial diffusion & mitigate the difference between slow & fast diffusion.

4. [DDN20]. Đào Nguyên Anh, Jesus Ildefonso Díaz, Nguyễn Quản Bá Hồng. *Pointwise gradient estimates in multi-dimensional slow diffusion equations with a singular quenching term.* [4 citations]

Keywords. singular absorption, nonlinear diffusion equations, pointwise gradient estimates, quenching phenomenon, free boundary.

[1]the use of numbers to try to tell somebody what will happen in the future. số học.

[2]different from what is normal or expected. dị thường.

Abstract. Consider high-dimensional equation $\partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}}$, extend [KK92] 1D case. Prove existence of a very weak solution (VWS) $u \in C([0,T]; L^1_\delta(\Omega))$ with $u^{-\beta}\chi_{\{u>0\}} \in L^1((0,T)\times\Omega)$, $\delta(\mathbf{x}) := d(\mathbf{x},\partial\Omega)$. Prove some pointwise gradient estimates for a certain range of the dimension N, $m \ge 1$, $\beta \in (0,m)$, mainly when the absorption dominates over diffusion $1 \le m < 2+\beta$. Prove a new kind of universal gradient estimate when $m+\beta \le 2$. Consider several qualitative properties (e.g. finite time quenching phenomena & finite speed of propagation) & study of Cauchy problem.

Goal 4. *Extend to high-dimensional case [KK92] for a 1D degenerate diffusion equation with a singular absorption term. Study nonnegative solutions of possibly degenerate reaction-diffusion multi-dimensional problem $\partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}}$ in $(0,\infty)\times\Omega$, $u^m = 0$ on $(0,\infty)\times\Gamma$, $u(0,\mathbf{x}) = u_0(\mathbf{x})$ in Ω.*

Ω: an open regular bounded domain of $\mathbb{R}^N$, e.g., with Γ of calss $C^{1,\alpha}$ for some $\alpha \in (0,1]$, $N \ge 1$, $m \ge 1$ ($m > 1$ corresponds to a typical slow diffusion) & mainly $\beta \in (0,m)$ with some remarks for case $\beta \ge m$. Treat separately the case of whole space $\Omega = \mathbb{R}^N$. The absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular (& the diffusion becomes degenerate if $m > 1$) when $u = 0$, & by this normalization, have $u(t,\mathbf{x}) = 0 \Rightarrow u^{-\beta}\chi_{\{u>0\}}(t,\mathbf{x}) = 0$. Boundary condition implies an automatic permanent singularity on Γ, in contrast to other related problems in which the singularity is permanently excluded of the boundary $u^m = 1$ on $(0,\infty)\times\Gamma$. Change of unknown $v := 1 - u^m$ in semilinear case $m = 1$ leads to the formulation $\partial_t v - \Delta v = \frac{\chi_{\{v<1\}}}{(1-v)^\beta}$ on $(0,\infty)\times\Omega$. Study the associated Cauchy problem in $(0,\infty)\times\mathbb{R}^N$ can be regarded from 2 different points of view according to the assumptions made on the asymptotic behavior of the initial datum when $|\mathbf{x}| \to \infty$.

Goal 5. *Analyze problem of type (P) & (CP) when $u_0(\mathbf{x}) \searrow 0$ as $|\mathbf{x}| \to \infty$.*

Motivation. Problem (P) was regarded as the limit case of regularized Langmuir–Hinshelwood model in chemical catalyst kinetics for elliptic- & parabolic equations.

Interesting point. Solutions may raise to a free boundary defined as $\partial\{(t,\mathbf{x}); u(t,\mathbf{x}) > 0\}$. (P1) denoted as a *quenching problem*. Appearance of a blow-up time for $\partial_t u$ at the 1st time $T_c > 0$ in which $u(T_c,\mathbf{x}) = 0$ at some point $\mathbf{x} \in \Omega$.

Case $\beta \ge m$ presents special difficulties when the free boundary $\partial\{(t,\mathbf{x}); u(t,\mathbf{x}) > 0\}$ is a nonempty hypersurface, this set corresponds to the set of *rupture points* in study of thin films. This case $\beta \ge m$ also arises in the modeling of micro-electromechanical systems (MEMS), in which mainly $m = 1, \beta = 2$.

5. [Kaw92]. Bernhard Kawohl. *Remarks on quenching, blow up, & dead cores.*
6. [Kaw96]. Bernhard Kawohl. *Remarks on quenching.*
7. [KK92]. Bernhard Kawohl, Robert Kersner. *On degenerate diffusion with very strong absorption.*
8. [Phi87]. Daniel Phillips. *Existence of solutions of quenching problems.* [100 citations]

   Prove existence of weak solutions to PME when $N \ge 1$, $m = 1$, $\beta \in (0,1)$.
9. [Váz07]. Juan Luis Vázquez. *The Porous Medium Equation.*

   Note. Có nhiều bộ ký hiệu xung đột nhau do tác giả chắp vá quyển sách từ nhiều bài báo, công trình khác nhau. Nên cẩn thận khi thống nhất bộ ký hiệu.

9 Sobolev Spaces – Không Gian Sobolev

Resources – Tài nguyên.

1. [AF03]. Robert A. Adams, John J. F. Fournier. *Sobolev Spaces.*
2. [Gag57]. Emilio Gagliardo. *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.*
3. Necǎs.
4. [Tar06]. Luc Tartar. *An Introduction to Sobolev Spaces & Interpolation Spaces.*

10 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

1. [LeV07]. Randall J. LeVeque. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems.*

11 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

1. [BS08]. Susanne C. Brenner, L. Ridgway Scott. *The Mathematical Theory of FEMs.*
2. [EG04]. Alexandre Ern, Jean-Luc Guermond. *Theory & Practice of Finite Elements.*
3. [GR86]. Vivette Girault, Pierre-Arnaud Raviart. *FEMs for NSEs.*
4. [Gun89]. Max D. Gunzburger. *FEMs for Viscous Incompressible Flows.*
5. [Joh16]. Volker John. *FEMs for Incompressible Flow Problems.*

I met Volker John, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagonrinsky, k-ϵ & their simulations.

12 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

1. [BJ89]. Timothy J. Barth, Dennis C. Jespersen. *The Design and Application of Upwind Schemes on Unstructured Meshes.*
2. [DAM04]. M. Darwish, D. Asmar, F. Moukalled. *A comparative assessment within a multigrid environment of segregated pressure-based algorithms for fluid flow at all speeds.*
3. [DM96a]. M. Darwish, F. Moukalled. *A new approach for building bounded skew-upwind schemes.*
4. [DM96b]. M. Darwish, F. Moukalled. *The normalized weighting factor method: a novel technique for accelerating the convergence of high-resolution convective schemes.*
5. [DM06]. M. Darwish, F. Moukalled. *Convective Schemes for Capturing Interfaces of Free-Surface Flows on Unstructured Grids.*
6. [DMS01]. M. Darwish, F. Moukalled, B. Sekar. *A unified formulation of the segregated class of algorithms for multifluid flow at all speeds.*
7. [Dem15]. I. Demirdžić. *On the Discretization of the Diffusion Term in Finite-Volume Continuum Mechanics.*
8. [DM95]. I. Demirdžić, S. Muzaferija. *Numerical method for coupled fluid flow, heat transfer and stress analysis using unstructured moving meshes with cells of arbitrary topology,*
9. [DW14]. Fabian Denner, Berend G. M. van Wachem. *Compressive VOF method with skewness correction to capture sharp interfaces on arbitrary meshes.*
10. [EGH19]. Robert Eymard, Thierry Gallouët, Raphaèle Herbin. *Finite Volume Methods.*
11. [Gal+18]. T. Gallouët, R. Herbin, J.-C. Latché, K. Mallem. *Convergence of the marker-and-cell scheme for the incompressible NSEs on non-uniform grids.*
12. [Gal+17]. T. Gallouët, R. Herbin, D. Maltese, A.Novotny. *onvergence of the marker-and-cell scheme for the semi-stationary compressible Stokes problem.*
13. [GL88]. P. H. Gaskell, A. K. C. Lau. *Curvature-compensated convective transport: SMART, a new boundedness-preserving transport algorithm.*
14. [Har83]. Ami Harten. *High resolution schemes for hyperbolic conservation laws.*
15. [Har97]. Ami Harten. *High resolution schemes for hyperbolic conservation laws [MR0701178 (84g:65115)].*
16. [Iss86]. R. I. Issa. *Solution of the Implicit Discretized Fluid Flow Equations by Operator Splitting.*
17. [JJA86]. D. S. Jang, R. Jetli, S. Acharya. *Comparison of the PISO, SIMPLER, and SIMPLEC algorithms for the treatment of the pressure-velocity coupling in steady flow problems.*
18. [Jas96]. H. Jasak. *Error Analysis and Estimation for the Finite Volume Method with Applications for Fluid Flow.*
19. [JG00]. H. Jasak, A. D. Gosman. *Automatic Resolution Control for the Finite Volume Method, Part 1.*
20. [Leo79]. B. P. Leonard. *A stable and accurate convective modelling procedure based on quadratic upstream interpolation.*
21. [Leo87]. B. P. Leonard. *SHARP Simulation of Discontinuities in Highly Convective Steady Flow.*

22. [Leo88]. B. P. Leonard. *Universal Limiter for Transient Interpolation Modeling of the Advective Transport Equations: The ULTIMATE Conservative Difference Scheme.*

23. [LeV02]. Randall J. LeVeque. *FEMs for Hyperbolic Problems.*

24. [MM97]. S. R. Mathur, J. Y. Murthy. *A Pressure-Based Method for Unstructured Meshes.*

25. [MD00]. F. Moukalled, M. Darwish. *A unified formulation of the segregated class of algorithms for fluid flow at all speeds.*

26. [MD12]. F. Moukalled, M. Darwish. *Transient Schemes for Capturing Interfaces of Free-Surface Flows.*

27. [MMD16]. F. Moukalled, L. Mangani, M. Darwish. *The FVM in CFD.*

28. [Muz94]. S. Muzaferija. *Adaptive Finite Volume Method for Flow Predictions Using Unstructured Meshes and Multigrid Approach.*

29. [MG97]. S. Muzaferija, A. D. Gosman. *Finite Volume CFD Procedure and Adaptive Error Control Strategy for Grids of Arbitrary Topology.*

30. [Nic92]. R. A. Nicolaides. *Analysis and convergence of the MAC scheme. I. The linear problem.*

31. [NW96]. R. A. Nicolaides, X. Wu. *Analysis and convergence of the MAC scheme. II. NSEs.*

32. [OA02]. Carl Ollivier-Gooch, Michael Van Altena. *A High-Order-Accurate Unstructured Mesh Finite-Volume Scheme for the Advection–Diffusion Equation.*

33. [PBH04]. Sébastien Perron, Sylvain Boivin, Jean-Marc Hérard. *A FVM to solve the 3D NSEs on unstructured collocated meshes.*

34. [RC83]. C. M. Rhie, W. L. Chow. *Numerical Study of the Turbulent Flow Past an Airfoil with Trailing Edge Separation.*

35. [Swe84]. Peter K. Sweby. *High resolution schemes using flux limiters for hyperbolic conservation laws.*

36. [Swe85]. Peter K. Sweby. *High resolution TVD schemes using flux limiters.*

37. [VM07]. H. K. Versteeg, W. Malalasekera. *An Introduction to CFD: The FVM.*

38. [YL93]. Ruey-Hor Yen, Chen-Hua Liu. *Enhancement of the SIMPLE algorithm by an additional explicit corrector step.*

13 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

13.1 Wikipedia/Mathematician

Mathematician. Euclid (holding calipers), Greek mathematician, known as the "Father of Geometry"
Occupation.

- **Occupation type.** Academic

Description.

- **Competencies.** Mathematics, analytical skills & critical thinking skills.
- **Education required.** Doctoral degree, occasionally master's degree.
- **Fields of employment.**
  - universities,
  - private corporations,
  - financial industry,
  - government
- **Related jobs.** statistician, actuary.

A *mathematician* is someone who uses an extensive knowledge of mathematics in their work, typically to solve mathematical problems.

Mathematicians are concerned with numbers, data, quantity, structure, space, models, & change.

13.1.1 History

For broader coverage of this topic, see History of mathematics.

1 of the earliest known mathematicians was Thales of Miletus (c. 624–c.546 BC); he has been hailed as the 1st true mathematician & the 1st known individual to whom a mathematical discovery has been attributed.[Boyer (1991), *A History of Mathematics*, p. 43]

He is credited with the 1st use of deductive reasoning applied to geometry, by deriving 4 corollaries to Thales' Theorem.

The number of known mathematicians grew when Pythagoras of Samos (c. 582–c. 507 BC) established the Pythagorean School, whose doctrine it was that mathematics ruled the universe & whose motto was "All is number".[Boyer 1991, "*Ionia & the Pythagoreans*", p. 49]

It was the Pythagoreans who coined the term "mathematics", & with whom the study of mathematics for its own sake begins.

The 1st woman mathematician recorded by history was Hypatia of Alexandria (AD 350–415).

She succeeded her father as Librarian at the Great Library & wrote many works on applied mathematics.

Because of a political dispute, the Christian community in Alexandria punished her, presuming she was involved, by stripping her naked & scraping off her skin with clamshells (some say roofing tiles).["*Ecclesiastical History*, Bk VI: Chap. 15". Archived from the original on 2014-08-14. Retrieved 2014-11-19.]

Science & mathematics in the Islamic world during the Middle Ages followed various models & modes of funding varied based primarily on scholars.

It was extensive patronage & strong intellectual policies implemented by specific rulers that allowed scientific knowledge to develop in many areas.

Funding for translation of scientific texts in other languages was ongoing throughout the reign of certain caliphs,[Abattouy, M., Renn, J. & Weinig, P., 2001. *Transmission as Transformation: The Translation Movements in the Medieval East & West in a Comparative Perspective*. Science in Context, 14(1-2), 1-12.] & it turned out that certain scholars became experts in the works they translated & in turn received further support for continuing to develop certain sciences.

As these sciences received wider attention from the elite, more scholars were invited & funded to study particular sciences.

An example of a translator & mathematician who benefited from this type of support was al-Khawarizmi.

A notable feature of many scholars working under Muslim rule in medieval times is that they were often polymaths.

Examples include the work on optics, maths & astronomy of Ibn al-Haytham.

The Renaissance brought an increased emphasis on mathematics & science to Europe.

During this period of transition from a mainly feudal & ecclesiastical culture to a predominantly secular one, many notable mathematicians had other occupations: Luca Pacioli (founder of accounting); Niccolò Fontana Tartaglia (notable engineer & bookkeeper); Gerolamo Cardano (earliest founder of probability & binomial expansion); Robert Recorde (physician) & François Viètes (lawyer).

As time passed, many mathematicians gravitated towards universities.

An emphasis on free thinking & experimentation had begun in Britain's oldest universities beginning in the 17th century at Oxford with the scientists Robert Hooke & Robert Boyle, & at Cambridge where Isaac Newton was Lucasian Professor of Mathematics & Physics.

Moving into the 19th century, the objective of universities all across Europe evolved from teaching the "*regurgitation of knowledge*" to "*encourag[ing] productive thinking.*"[Röhrs, "*The Classical Idea of the University,*" Tradition & Reform of the University under an International Perspective p. 20]

In 1810, Humboldt convinced the King of Prussia to build a university in Berlin based on Friedrich Schleiermacher's liberal ideas; the goal was to demonstrate the process of the discovery of knowledge & to teach students to "*take account of fundamental laws of science in all their thinking.*"

Thus, seminars & laboratories started to evolve.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 5–6]

British universities of this period adopted some approaches familiar to the Italian & German universities, but as they already enjoyed substantial freedoms & autonomy the changes there had begun with the Age of Enlightenment, the same influences that inspired Humboldt.

The Universities of Oxford & Cambridge emphasized the importance of research, arguably more authentically implementing Humboldt's idea of a university than even German universities, which were subject to state authority.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 12]

Overall, science (including mathematics) became the focus of universities in the 19th & 20th centuries.

Students could conduct research in seminars or laboratories & began to produce doctoral theses with more scientific content.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 13]

According to Humboldt, the mission of the University of Berlin was to pursue scientific knowledge.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 16]

The German university system fostered professional, bureaucratically regulated scientific research performed in well-equipped laboratories, instead of the kind of research done by private & individual scholars in Great Britain & France.[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 17–18]

In fact, Rüegg asserts that the *German system is responsible for the development of the modern research university because it focused on the idea of "freedom of scientific research, teaching & study."*[Rüegg, "Themes", *A History of the University in Europe, Vol. III*, p. 31]

13.1.2 Required education

Mathematicians usually cover a breadth of topics within mathematics in their undergraduate education, & then proceed to specialize in topics of their own choice at the graduate level.

In some universities, a qualifying exam serves to test both the breadth & depth of a student's understanding of mathematics; the students, who pass, are permitted to work on a doctoral dissertation.

13.1.3 Activities

Emmy Noether, mathematical theorist & teacher.

Applied mathematics. Main article: Applied mathematics. Mathematicians involved with solving problems with applications in real life are called applied mathematicians.

Applied mathematicians are mathematical scientists who, with their specialized knowledge & professional methodology, approach many of the imposing problems presented in related scientific fields.

With professional focus on a wide variety of problems, theoretical systems, & localized constructs, applied mathematicians work regularly in the study & formulation of mathematical models.

Mathematicians & applied mathematicians are considered to be 2 of the STEM (science, technology, engineering, & mathematics) careers.

The discipline of applied mathematics concerns itself with mathematical methods that are typically used in science, engineering, business, & industry; thus, "applied mathematics" is a mathematical science with specialized knowledge.

The term "applied mathematics" also describes the professional specialty in which mathematicians work on problems, often concrete but sometimes abstract.

As professionals focused on problem solving, *applied mathematicians* look into the *formulation, study, & use of mathematical models* in science, engineering, business, & other areas of mathematical practice.

Abstract mathematics. Main article: Pure mathematics. Pure mathematics is mathematics that studies entirely abstract concepts.

From the 18th century onwards, this was a recognized category of mathematical activity, sometimes characterized as *speculative mathematics*,[See for example titles of works by Thomas Simpson from the mid-18th century: *Essays on Several Curious & Useful Subjects in Speculative & Mixed Mathematics, Miscellaneous Tracts on Some Curious & Very Interesting Subjects in Mechanics, Physical Astronomy & Speculative Mathematics. Chisholm*, Hugh, ed. (1911). "Simpson, Thomas". Encyclopædia Britannica. 25 (11th ed.). Cambridge University Press. p. 135.] & at variance with the trend towards meeting the needs of navigation, astronomy, physics, economics, engineering, & other applications.

Another insightful view put forth is that *pure mathematics is not necessarily applied mathematics*: it is possible to study abstract entities w.r.t. their intrinsic nature, & not be concerned with how they manifest in the real world.[Andy Magid, *Letter from the Editor, in Notices of the AMS*, Nov 2005, American Mathematical Society, p. 1173. [1] Archived 2016-03-03 at the Wayback Machine]

Even though the pure & applied viewpoints are distinct philosophical positions, in practice there is much overlap in the activity of pure & applied mathematicians.

To develop accurate models for describing the real world, many applied mathematicians draw on tools & techniques that are often considered to be "pure" mathematics.

On the other hand, many pure mathematicians draw on natural & social phenomena as inspiration for their abstract research.

Mathematics teaching. Many professional mathematicians also engage in the teaching of mathematics.

Duties may include:

- teaching university mathematics courses;
- supervising undergraduate & graduate research; and
- serving on academic committees.

Consulting. Many careers in mathematics outside of universities involve consulting.

E.g., actuaries assemble & analyze data to estimate the probability & likely cost of the occurrence of an event such as death, sickness, injury, disability, or loss of property.

Actuaries also address financial questions, including those involving the level of pension contributions required to produce a certain retirement income & the way in which a company should invest resources to maximize its return on investments in light of potential risk.

Using their broad knowledge, actuaries help design & price insurance policies, pension plans, & other financial strategies in a manner which will help ensure that the plans are maintained on a sound financial basis.

As another example, mathematical finance will derive & extend the mathematical or numerical models without necessarily establishing a link to *financial theory*, taking observed market prices as input.

Mathematical consistency is required, not compatibility with *economic theory*.

Thus, e.g., while a financial economist might study the structural reasons why a company may have a certain share price, a financial mathematician may take the share price as a given, & attempt to use stochastic calculus to obtain the corresponding value of derivatives of the stock (see: Valuation of options; Financial modeling).

13.1.4 Occupations

In 1938 in the United States, mathematicians were desired as teachers, calculating machine operators, mechanical engineers, accounting auditor bookkeepers, & actuary statisticians.

According to the Dictionary of Occupational Titles occupations in mathematics include the following.["020 OCCUPATIONS IN MATHEMATICS". *Dictionary Of Occupational Titles*. Archived from the original on 2012-11-02. Retrieved 2013-01-20.]

- Mathematician
- Operations-Research Analyst
- Mathematical Statistician
- Mathematical Technician
- Actuary
- Applied Statistician
- Weight Analyst

13.1.5 Quotations about mathematicians

The following are quotations about mathematicians, or by mathematicians.

> "*A mathematician is a device for turning coffee into theorems.*" - Attributed to both Alfréd Rényi[15] & Paul Erdös

> "*Die Mathematiker sind eine Art Franzosen; redet man mit ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.*"
> (Mathematicians are [like] a sort of Frenchmen; if you talk to them, they translate it into their own language, & then it is immediately something quite different.) - Johann Wolfgang von Goethe[16]

> "*Each generation has its few great mathematicians... & [the others'] research harms no one.*" - Alfred W. Adler (∼1930), "*Mathematics & Creativity*"[17]

> "In short, I never yet encountered the mere mathematician who could be trusted out of equal roots, or one who did not clandestinely hold it as a point of his faith that x squared + px was absolutely & unconditionally equal to q. Say to one of these gentlemen, by way of experiment, if you please, that you believe occasions may occur where x squared + px is not altogether equal to q, and, having made him understand what you mean, get out of his reach as speedily as convenient, for, beyond doubt, he will endeavor to knock you down." - Edgar Allan Poe, *The purloined letter*

> "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas." - G. H. Hardy, *A Mathematician's Apology*

> "*Some of you may have met mathematicians & wondered how they got that way.*" - Tom Lehrer

> "*It is impossible to be a mathematician without being a poet in soul.*" - Sofia Kovalevskaya

> "*There are 2 ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else - but persistent.*" - Raoul Bott

> "*Mathematics is the queen of the sciences & arithmetic the queen of mathematics.*" - Carl Friedrich Gauss [18]

13.1.6 Prizes in mathematics

There is no Nobel Prize in mathematics, though sometimes mathematicians have won the Nobel Prize in a different field, such as economics.

Prominent prizes in mathematics include the Abel Prize, the Chern Medal, the Fields Medal, the Gauss Prize, the Nemmers Prize, the Balzan Prize, the Crafoord Prize, the Shaw Prize, the Steele Prize, the Wolf Prize, the Schock Prize, & the Nevanlinna Prize.

The American Mathematical Society, Association for Women in Mathematics, & other mathematical societies offer several prizes aimed at increasing the representation of women & minorities in the future of mathematics.

13.1.7 Mathematical autobiographies

Several well known mathematicians have written autobiographies in part to explain to a general audience what it is about mathematics that has made them want to devote their lives to its study.

These provide some of the best glimpses into what it means to be a mathematician.

The following list contains some works that are not autobiographies, but rather essays on mathematics & mathematicians with strong autobiographical elements.

- *The Book of My Life* - Girolamo Cardano[19]
- A Mathematician's Apology - G.H. Hardy[20]
- A Mathematician's Miscellany (republished as Littlewood's miscellany) - J. E. Littlewood[Littlewood, J. E. (1990) [Originally *A Mathematician's Miscellany* published in 1953], Béla Bollobás (ed.), *Littlewood's miscellany*, Cambridge University Press, ISBN 0-521-33702 X]
- *I Am a Mathematician* - Norbert Wiener[Wiener, Norbert (1956), *I Am a Mathematician / The Later Life of a Prodigy*, The M.I.T. Press, ISBN 0-262-73007-3]
- *I Want to be a Mathematician* - Paul R. Halmos
- *Adventures of a Mathematician* - Stanislaw Ulam[Ulam, S. M. (1976), *Adventures of a Mathematician*, Charles Scribner's Sons, ISBN 0-684-14391-7]
- *Enigmas of Chance* - Mark Kac[Kac, Mark (1987), *Enigmas of Chance/An Autobiography*, University of California Press, ISBN 0-520-05986-7]
- *Random Curves* - Neal Koblitz
- *Love & Math* - Edward Frenkel
- *Mathematics Without Apologies* - Michael Harris[Harris, Michael (2015), *Mathematics without apologies/portrait of a problematic vocation*, Princeton University Press, ISBN 978-0-691-15423-7]

13.1.8 See also

- Lists of mathematicians
- Human computer
- Mathematical joke
- A Mathematician's Apology
- Men of Mathematics (book)
- Mental calculator □

13.2 Wikipedia/Henri Berestycki

Henri Berestycki (born Mar 25, 1951, in Paris)[1] is a French mathematician who obtained his PhD from Université Paris VI - Université Pierre et Marie Curie in 1975.

His Dissertation was titled *Contributions à l'étude des problèmes elliptiques non linéaires*, & his doctoral advisor was Haim Brezis.[2]

He was an L.E. Dickson Instructor in Mathematics at the University of Chicago from 1975–77, after which he returned to France to continue his research.

He has made many contributions in *nonlinear analysis*, ranging from *nonlinear elliptic equations, hamiltonian systems, spectral theory of elliptic operators*, & with applications to the description of *mathematical modelling of fluid mechanics & combustion.*

His current research interests include the mathematical modelling of financial markets, mathematical models in biology & especially in ecology, & modelling in social sciences (in particular, urban planning & criminology).

For these latter topics, he obtained an ERC Advanced grant in 2012.

He worked at the French National Center of Scientific Research (CNRS), then moved to an appointment as Professor at Univ. Paris XIII (1983–1985).

He became a Professor of Mathematics in 1988 at Université Pierre et Marie Curie, Paris VI (1988–2001 of "exceptional class" since 1993), & became Professor at Ecole normale supérieure, Paris (1994–1999), & part-time professor Ecole Polytechnique (1987–1999).

He is also a visiting Professor in the Department of Mathematics at the University of Chicago, & was also co-director of the Stevanovich Center of Financial Mathematics in Chicago.

He is currently the Directeur d'études (Research Professor) at École des hautes études en sciences sociales (EHESS), since 2001.

13.2.1 Services

- National Committee of French universities (1992–1995).
- Since 2002 director of Centre d'analyse et mathématique sociales (CAMS), CNRS -EHESS.
- Vice-president, EHESS (2004–2006).
- Member of the thesis prize committee of the universities of Paris (since 2006).

13.2.2 Awards

- Carrière Prize(1988)
- Prix Sophie Germain of the French Academy of Sciences (2004),
- Humboldt Prize in Germany (2004)
- French Legion of Honor in 2010.
- American Mathematical Society Fellowship (2012).[3]
- Foreign honorary member of the American Academy of Arts & Sciences, 2013.[4]

13.2.3 Articles

- Berestycki, Henri; Roquejoffre, Jean-Michel; Rossi, Luca; The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol.* 66 (2013), no. 4-5, 743–766.
- Barthélemy, Marc; Nadal, Jean-Pierre; Berestycki, Henri Disentangling collective trends from local dynamics. *Proc. Natl. Acad. Sci. USA* 107 (2010), no. 17, 7629–7634.
- Berestycki, Henri; Hamel, François; Nadirashvili, Nikolai Elliptic eigenvalue problems with large drift & applications to nonlinear propagation phenomena. *Comm. Math. Phys.* 253 (2005), no. 2, 451–480.
- Berestycki, Henri; Hamel, François Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* 55 (2002), no. 8, 949–1032.
- Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* 81 (1996), no. 2, 467–494.
- Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S. The principal eigenvalue & maximum principle for 2nd-order elliptic operators in general domains. *Comm. Pure Appl. Math.* 47 (1994), no. 1, 47–92.
- Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345; II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 347–375.
- Bahri, Abbas; Berestycki, Henri A perturbation method in critical point theory & applications. *Trans. Amer. Math. Soc.* 267 (1981), no. 1, 1–32. □

13.3 Wikipedia/Haïm Brezis

Haïm Brezis.

- **Born.** Jun 1m 1944 (age 76). Riom-ès-Montagnes, Cantal, France.
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.**
  - Brezis-Gallouet inequality
  - Bony-Brezis theorem
  - Brezis-Lieb lemma

Scientific career.

- **Fields.** Mathematics.
- **Institutions.** Pierre & Marie Curie University.
- **Doctoral advisor.**

- Gustave Choquet
- Jacques-Louis Lions

- **Doctoral students.**
  - Abbas Bahri
  - Henri Berestycki
  - Jean-Michel Coron
  - Jesús Ildefonso Díaz
  - Pierre-Louis Lions
  - Juan Luis Vázquez Suárez

Haïm Brezis (born Jun 1, 1944) is a French mathematician who works in functional analysis & partial differential equations.

13.3.1 Biography

Born in Riom-ès-Montagnes, Cantal, France.

Brezis is the son of a Romanian immigrant father, who came to France in the 1930s, & a Jewish mother who fled from the Netherlands.

His wife, Michal Govrin, a native Israeli, works as a novelist, poet, & theater director.[1]

Brezis received his Ph.D. from the University of Paris in 1972 under the supervision of Gustave Choquet.

He is currently a Professor at the Pierre & Marie Curie University & a Visiting Distinguished Professor at Rutgers University.

He is a member of the Academia Europaea (1988) & a foreign associate of the United States National Academy of Sciences (2003).

In 2012 he became a fellow of the American Mathematical Society.[2]

He holds honorary doctorates from several universities including National Technical University of Athens.[3]

Brezis is listed as an ISI highly cited researcher.[4]

13.3.2 Works

- *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (1973)
- *Analyse Fonctionnelle.* Théorie et Applications (1983)
- *Haïm Brezis. Un mathématicien juif.* Entretien Avec Jacques Vauthier. Collection Scientifiques & Croyants. Editions Beauchesne, 1999. ISBN 978-2-7010-1335-0, ISBN 2-7010-1335-6
- *Functional Analysis, Sobolev Spaces & Partial Differential Equations*, Springer; 1st Edition. edition (November 10, 2010), ISBN 978-0-387-70913-0, ISBN 0-387-70913-4

13.3.3 See also

- Bony-Brezis theorem
- Brezis-Gallouet inequality
- Brezis-Lieb lemma □

13.4 Lawrence Chris Evans

13.5 Wikipedia/Herbert Federer

Herbert Federer (Jul 23, 1920 – Apr 21, 2010)[“NAS Membership Directory: Federer, Herbert”. National Academy of Sciences. Retrieved Jun 15, 2010.] was an American mathematician.

He is 1 of the creators of geometric measure theory, at the meeting point of differential geometry & mathematical analysis.[Parks, H. (2012) *Remembering Herbert Federer (1920–2010)*, NAMS 59(5), 622–631.]

13.5.1 Career

Federer was born Jul 23, 1920, in Vienna, Austria.

After emigrating to the US in 1938, he studied mathematics & physics at the University of California, Berkeley, earning the Ph.D. as a student of Anthony Morse in 1944.

He then spent virtually his entire career as a member of the Brown University Mathematics Department, where he eventually retired with the title of Professor Emeritus.

Federer wrote more than 30 research papers in addition to his book *Geometric measure theory*.

The Mathematics Genealogy Project assigns him 9 Ph.D. students & well over a hundred subsequent descendants.

His most productive students include the late Frederick J. Almgren, Jr. (1933–1997) a professor at Princeton for 35 years, & his last student, Robert Hardt, now at Rice University.

Federer was a member of the National Academy of Sciences.

In 1987, he & his Brown colleague Wendell Fleming won the American Mathematical Society's Steele Prize "for their pioneering work in *Normal & Integral currents*."

13.5.2 Normal & integral currents

Federer's mathematical work separates thematically into the periods before & after his watershed 1960 paper *Normal & integral currents*, co-authored with Fleming.

That paper provided the *1st satisfactory general solution to Plateau's problem* - the problem of finding a $(k+1)$-dimensional least-area surface spanning a given k-dimensional boundary cycle in n-dimensional Euclidean space.

Their solution inaugurated a new & fruitful period of research on a large class of *geometric variational problems* - especially *minimal surfaces* - via what came to be known as *Geometric Measure Theory*.

13.5.3 Earlier work

During the 15 or so years prior to that paper, Federer worked at the technical interface of geometry & measure theory.

He focused particularly on surface area, rectifiability of sets, & the extent to which one could substitute rectifiability for smoothness in the analysis of surfaces.

His 1947 paper on the *rectifiable subsets of n-space* characterized purely unrectifiable sets by their "invisibility" under almost all projections.

A. S. Besicovitch had proven this for 1-dimensional sets in the plane, but Federer's generalization, valid for subsets of arbitrary dimension in any Euclidean space, was a major technical accomplishment, & later played a key role in *Normal & Integral Currents*.

In 1958, Federer wrote *Curvature Measures*, a paper that took some early steps toward understanding 2nd-order properties of surfaces lacking the differentiability properties typically assumed in order to discuss curvature.

He also developed & named what he called the coarea formula in that paper.

That formula has become a standard analytical tool.

13.5.4 Geometric measure theory

Federer is perhaps best known for his treatise *Geometric Measure Theory*, published in 1969.[Goffman, Casper (1971). "Review: Geometric measure theory, by Herbert Federer" (PDF). Bull. Amer. Math. Soc. 77 (1): 27–35. doi:10.1090/s0002-9904-1971-12603-4.]

Intended as both a text & a reference work, the book is unusually complete, general & authoritative: its nearly 600 pages cover a substantial amount of linear & multilinear algebra, give a profound treatment of measure theory, integration & differentiation, & then move on to rectifiability, theory of currents, & finally, variational applications.

Nevertheless, the book's unique style exhibits a rare & artistic economy that still inspires admiration, respect - & exasperation.

A more accessible introduction may be found in F. Morgan's book listed below.

13.5.5 See also

- Integral current
- Federer-Morse theorem

13.5.6 External links

- Federer's page a Brown □

13.6 Godfrey Harold Hardy

"Godfrey Harold Hardy FRS (Feb 7, 1877 – Dec 1, 1947) was an English mathematician, known for his achievements in number theory & mathematical analysis. In biology, he is known for the Hardy–Weinberg principle, a basic principle of population genetics.

G. H. Hardy is usually known by those outside the field of mathematics for his 1940 essay *A Mathematician's Apology*, often considered 1 of the best insights into the mind of a working mathematician written for the layperson[3].

Starting in 1914, Hardy was the mentor of the Indian mathematician Srinivasa Ramanujan, a relationship that has become celebrated. Hardy almost immediately recognized Ramanujan's extraordinary albeit untutored brilliance, & Hardy & Ramanujan became close collaborators. In an interview by Paul Erdős, when Hardy was asked what his greatest contribution to mathematics was, Hardy unhesitatingly replied that it was the discovery of Ramanujan. In a lecture on Ramanujan, Hardy said that "my association with him is the 1 romantic incident in my life".

13.6.1 Biography

G. H. Hardy was born on Feb 7, 1877, in Cranleigh, Surrey, England, into a teaching family. His father was Bursar & Art Master at Cranleigh School; his mother had been a senior mistress at Lincoln Training College for teachers. Both of his parents were mathematically inclined, though neither had a university education. He & his sister Gertrude "Gertie" Emily Hardy (1878–1963) were brought up by their educationally enlightened parents in a typical Victorian nursery attended by a nurse. At an early age, he argued with his nurse about the existence of Santa Clause & the efficacy of prayer. He read aloud to his sister books e.g., *Don Quixote*, *Gulliver's Travels*, & *Robinson Crusoe*.

Hardy's own natural affinity for mathematics was perceptible at an early age. When just 2 years old, he wrote numbers up to millions, & when taken to church he amused himself by factorizing the numbers of the hymns.

After schooling at Cranleigh, Hardy was awarded a scholarship to Winchester College for his mathematical work. In 1896, he entered Trinity College, Cambridge. He was 1st tutored under Robert Rumsey Webb, but found it unsatisfying, & briefly considered switching to history. He then was tutored by Augustus Love, who recommended him to read Camille Jordan's *Cours d'analyse*, which taught him for the 1st time "what mathematics really meant". After only 2 years of preparation under his coach, Robert Alfred Herman, Hardy was 4th in the Mathematics Tripos examination. Years later, he sought to abolish the Tripos system, as he felt that it was becoming more an end in itself than a means to an end. While at university, Hardy joined the Cambridge Apostles, an elite, intellectual secret society.

Hardy cited as his most important influence his independent study of *Cours d'analyse de l'École Polytechnique* by the French mathematician Camille Jordan, through which he became acquainted with the more precise mathematics tradition in continental Europe. In 1900 he passed part II of the Tripos, & in the same year he was elected to a Prize Fellowship at Trinity College. In 1903 he earned his M.A., which was the highest academic degree at English universities at that time. When his Prize Fellowship expired in 1906 he was appointed to the Trinity staff as a lecturer in mathematics, where teaching 6 hours per week left him time for research.

On Jan 16, 1913, Ramanujan wrote to Hardy, who Ramanujan had known from studying *Orders of Infinity* (1910). Hardy read the letter in the morning, suspected it was a crank or a prank, but thought it over & realized in the evening that it was likely genuine because "great mathematicians are commoner than thieves or humbugs of such incredible skills". He then invited Ramanujan to Cambridge & began "the 1 romantic incident in my life".

In the aftermath of the Bertrand Russell affair during World War I, in 1919 he left Cambridge to take the Savilian Chair of Geometry (& thus become a Fellow of New College) at Oxford. Hardy spent the academic year 1928–1929 at Princeton University in an academic exchange with Oswald Veblen, who spent the year at Oxford. Hardy gave the Josiah Willard Gibbs lecture for 1928. Hardy left Oxford & returned to Cambridge in 1931, becoming again a fellow of Trinity College & holding the Sadleirian Professorship until 1942. It is believed that he left Oxford for Cambridge to avoid the compulsory retirement at 65.

He was on the governing body of Abingdon School from 1922–1935.

In 1939, he suffered a coronary thrombosis, which prevented him from playing tennis, squash, etc. He also lost his creative powers in mathematics. He was constantly bored & distracted himself by writing a privately circulated memoir about the Bertrand Russell affair. In the early summer of 1947, he attempted suicide by barbiturate overdose. After that, he resolved to simply wait for death. He died suddenly 1 early morning while listening to his sister read out from a book of the history of Cambridge University cricket.

13.6.2 Work

Hardy is credited with reforming British mathematics by bringing rigor into it, which was previously a characteristic of French, Swiss, & German mathematics. British mathematicians had remained largely in the tradition of applied mathematics, in thrall to the reputation of Isaac Newton (see Cambridge Mathematical Tripos). Hardy was more in tune with the *cours d'analyse* methods dominant in France, & aggressively promoted his conception of pure mathematics, in particular against the hydrodynamics that was an important part of Cambridge mathematics.

Hardy preferred to work only 4 hours every day on mathematics, spending the rest of the day talking, playing cricket, & other gentlemanly activities.

From 1911, he collaborated with John Edensor Littlewood, in extensive work in mathematical analysis & analytic number theory. This (along with much else) led to quantitative progress on Waring's problem, as part of the Hardy–Littlewood circle method, as it became known. In prime number theory, they proved results & some notable conditional results. This was a major factor in the development of number theory as a system of conjectuers; e.g., the 1st Hardy–Littlewood conjecture & 2nd Hardy–Littlewood conjecture. Hardy's collaboration with Littlewood is among the most successful & famous collaborations

[3] 1. a person who does not have expert knowledge of a particular subject; 2. a person who is a member of a Church but is not a priest or member of the clergy.

in mathematical history. In a 1947 lecture, the Danish mathematician Harald Bohr reported a colleague as saying, "Nowadays, there are only 3 really great English mathematicians: Hardy, Littlewood, & Hardy–Littlewood."

Hardy is also known for formulating the Hardy–Weinberg principle, a basic principle of population principle, a basic principle of population genetics, independently from Wilhelm Weinberg in 1908. He played cricket with the geneticist Reginald Punnett, who introduced the problem to him in purely mathematical terms. Hardy, who had no interest in genetics & described the mathematical argument as "very simple", may never have realized how important the result became.

Hardy was elected an international honorary member of the American Academy of Arts & Sciences in 1921, an international member of the United States National Academy of Sciences in 1927, & an international member of the American Philosophical Society in 1939.

Hardy's collected papers have been published in 7 volumes by Oxford University Press.

Pure mathematics. Hardy preferred his work to be considered *pure mathematics*, perhaps because of his detestation of war & the military uses to which mathematics had been applied. He made several statements similar to that in his *Apology*:

> "I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."

However, aside from formulating the Hardy–Weinberg principle in population genetics, his famous work on integer partitions with his collaborator Ramanujan, known as the Hardy–Ramanujan asymptotic formula, has been widely applied in physics to find quantum partition functions of atomic nuclei (1st used by Niels Bohr) & to derive thermodynamic functions of non-interacting Bose–Einstein systems. Though Hardy wanted his maths to be "pure" & devoid[4] of any application, much of his work has found applications in other branches of science.

Moreover, Hardy deliberately pointed out in his *Apology* that mathematicians generally do not "glory in the uselessness of their work", but rather – because science can be used for evil ends as well as good – "mathematicians may be justified in rejoicing that there is 1 science at any rate, & that their own, whose very remoteness from ordinary human activities should keep it gentle & clean." Hardy also rejected as a "delusion" the belief that the difference between pure & applied mathematics had anything to do with their utility. Hardy regards as "pure" the kinds of mathematics that are independent of the physical world, but also considers some "applied" mathematicians, e.g. physicists Maxwell & Einstein, to be among the "real" mathematicians, whose work "has permanent aesthetic value" & "is eternal because the best of it may, like the best literature, continue to cause intense emotional satisfaction to thousands pf people after thousands of years." Although he admitted that what he called "real" mathematics may someday become useful, he asserted that, at the time in which the *Apology* was written, only the "dull & elementary parts" of either pure or applied mathematics could "work for good or ill".

13.6.3 Personality

Hardy was extremely shy as a child & was socially awkward, cold, & eccentric[5] throughout his life. During his school years, he was top of his class in most subjects, & won many prizes & awards but hated having to receive them in front of the entire school. He was uncomfortable being introduced to new people, & could not bear to look at his own reflection in a mirror. It is said that, when staying in hotels, he would cover all the mirrors with towels.

Socially, Hardy was associated with the Bloomsbury Group & the Cambridge Apostles; G. E. Moore, Bertrand Russell & J. M. Keynes were friends. Apart from close friendships, he had a few platonic relationships with young men who shared his sensibilities, & often his love of cricket. A mutual interest in cricket led him to befriend the young C. P. Snow. Hardy was a lifelong bachelor & in his final years he was cared for by his sister.

He was an avid cricket fan. Maynard Keynes observed that if Hardy had read the stock exchange for half an hour every day with as much interest & attention as he did the day's cricket scores, he would have become a rich man. He liked to speak of the best class of mathematician research as "the Hobbs class", & later, after Bradman appeared as an even greater batsman, "the Bradman class".

Around the age of 20, he decided that he did not believe in God, which proved a minor issue as attending the chapel was compulsory at Cambridge University. He wrote a letter to his parents explaining that, & from then on he refused to go into any college chapel, even for purely ritualistic duties.

He was at times politically involved, if not an activist. He took part in the Union of Democratic Control during World War I, & for Intellectual Liberty in the late 1930s. He admired America & Soviet Union roughly equally. He found both sides of the 2nd World War objectionable.

Paul Hoffman writes that "His concerns were wide-ranging, as evidenced by 6 New Year's resolutions he set in a postcard to a friend:

1. prove the Riemann hypothesis
2. make 211 not out in the 4th innings of the last Test Match at the Oval
3. find an argument for the nonexistence of God which shall convince the general public
4. be the 1st man at the top of Mount Everest

[4] *devoid of something* completely without something.
[5] considered by other people to be strange or unusual.

5. be proclaimed the 1st president of the U.S.S.R. of Great Britain & Germany

6. murder Mussolini.

13.6.4 Cultural references

HARDY is a key character, played by Jeremy Irons, in the 2015 film *The Man Who Knew Infinity*, based on the biography of RAMANUJAN with the same title. HARDY is a major character in DAVID LEAVITT's historical fiction novel *The Indian Clerk* (2007), which depicts his Cambridge years & his relationship with John Edensor Littlewood & RAMANUJAN. HARDY is a secondary character in *Uncle Petros & Goldbach's Conjecture* (1992), a mathematics novel by APOSTOLOS DOXIADIS. HARDY is also a character in the 2014 Indian film, *Ramanujan*, played by KEVIN MCGOWAN." – Wikipedia/G. H. Hardy

13.7 Peter Lax

13.8 Jacques-Louis Lions

- **Born.** May 3, 1928. Grasse, Alpes-Maritimes, France.
- **Died.** May 17, 2001 (aged 73).
- **Nationality.** French.
- **Alma mater.** University of Nancy.
- **Known for.** PDEs.
- **Awards.** Japan Prize (1991).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - École Polytechnique
  - Collège de France
- **Doctoral advisor.** Laurent Schwartz.
- **Doctoral students.**
  - Alain Bensoussan
  - Jean-Michel Bismut
  - Haïm Brezis
  - Erol Gelenbe
  - Roland Glowinski
  - Roger Temam

Jacques-Louis Lions ([1] 3 May 1928 - May 17, 2001) was a French mathematician who made contributions to the theory of partial differential equations & to stochastic control, among other areas.

He received the SIAM's John von Neumann Lecture prize in 1986 & numerous other distinctions.[2][3]

Lions is listed as an ISI highly cited researcher.[4]

13.8.1 Biography

After being part of the French Résistance in 1943 & 1944, J.-L. Lions entered the École Normale Supérieure in 1947.

He was a professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, & the École polytechnique.

In 1966 he sent an invitation to Gury Marchuk, the soviet mathematician to visit Paris.

This was hand delivered by General De Gaulle during his visit to Akademgorodok in June of that year.[5]

He joined the prestigious Collège de France as well as the French Academy of Sciences in 1973.

In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught & promoted the use of numerical simulations using finite elements integration.

Throughout his career, Lions insisted on the *use of mathematics in industry*, with a particular involvement in the French space program, as well as in domains such as energy & the environment.

This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992.

Lions was elected President of the International Mathematical Union in 1991 & also received the Japan Prize & the Harvey Prize that same year.[3]

In 1992, the University of Houston awarded him an honorary doctoral degree.

He was elected president of the French Academy of Sciences in 1996 & was also a Foreign Member of the Royal Society (ForMemRS)[6] & numerous other foreign academies.[2][3]

He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English & Russian, & major contributions to several collective works, including the 4000 pages of the monumental *Mathematical analysis & numerical methods for science & technology* (in collaboration with Robert Dautray), as well as the *Handbook of numerical analysis* in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994.[7]

Both father & son have received honorary doctorates from Heriot-Watt University in 1986 & 1995 respectively.[8]

13.8.2 Books

- with Enrico Magenes: *Problèmes aux limites non homogènes et applications.* 3 vols., 1968, 1970
- *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* 1968
- with L. Cesari: *Quelques méthodes de résolution des problèmes aux limites non linéaires.* 1969
- with Roger Dautray: *Mathematical analysis & numerical methods for science & technology.* 9 vols., 1984/5
- with Philippe Ciarlet: *Handbook of numerical analysis.* 7 vols.
- with Alain Bensoussan, Papanicolaou: *Asymptotic analysis of periodic structures.* North Holland 1978
- *Controlabilité exacte, perturbations et stabilisation de systèmes distribués*[9]
- with John E. Lagnese: *Modelling Analysis & Control of Thin Plates.* □

13.9 Andrew Joseph Majda

Resources – Tài nguyên.

1. [Eng+23]. Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, Vlad Vicol. *In memory of Andrew J. Majda.*

   "He was hard working until the end even though he suffered from serious health issues for quite some time."

   "He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, & observed & experimental data ... Andy Majda's modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis & numerical modeling, (iii) observed phenomena & experimental data, & (iv) qualitative and/or asymptotic modeling [Maj00]."

   "Andy's legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored."

   "The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed "Majda's model" for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived "Beale-Kato-Majda criterion," which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84]."

   "At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science."

13.10 Vladimir Mazya

13.11 Phan Thành Nam

"2008, sang Đan Mạch & làm TS tại ĐH Copenhagen dưới sự hướng dẫn của Prof. Jan Philip Solovej. Bảo vệ luận văn về ngành Vật Lý Toán năm 2011 với nhan đề *"Contributions to the Rigorous Study of the Structure of Atoms"*. 2011–2013: làm nghiên cứu viên sau tiến sĩ tại Đại học Cergy-Pontoise (Pháp) dưới sự hướng dẫn của Prof. Mathieu Lewin. 2013–2016, làm nghiên cứu viên sau tiến sĩ tại Viện khoa học & công nghệ Áo (IST) dưới sự hướng dẫn của Prof. Robert Seiringer. 2016: sang Cộng hòa Séc làm giáo sư trợ giảng tại Đại học Masaryk. 2017–now: GS tại Đại học Ludwig Maximilian Munich, CHLB Đức. 2020: đạt giải thưởng Hội Toán học Châu Âu EMS Prize.

> "Phan Thành Nam (1985–?) đã có những công trình đáng chú ý về toán học của hệ đa vật lượng tử (quantum many-body system) bao gồm hệ nguyên tử, phân tử cũng như các khí Bose & Fermi. Kết quả của anh anh liên quan đến sự cân bằng & tính chất động lực học của những hệ như vậy. Nhiều kết quả nổi tiếng trong lãnh vực này là do công của Nam. Chúng bao gồm những chặn tốt nhất cho ion hóa cực đại của các nguyên tử & những hằng số nổi tiếng của của bất đẳng thức Lieb–Thirring lừng danh. Hơn nữa, Nam & các cộng sự đã phát triển một cách tiếp cận tổng quát để thiết lập giới hạn trường trung bình (mean-field limit) của các hệ boson dựa trên định lý de Finetti lượng tử. Đó là thứ mà bây giờ trở tiêu chuẩn vàng trong lãnh vực này." – Prof. Jan Philip Solovej

Các lĩnh vực nghiên cứu của GS Phan Thành Nam là giải tích & vật lý toán, đặc biệt là cơ học lượng tử nhiều hạt, lý thuyết phổ, phép tính biến phân & phương trình đạo hàm riêng, giải tích số." – Wikipedia/Phan Thành Nam

13.12 Jindřich Nečas

13.13 Louis Nirenberg

Resources – Tài nguyên.

1. [Váz20]. JUAN LUIS VÁZQUEZ. *Remembering Louis Nirenberg & His Mathematics.*

Abstract

The article is dedicated to recalling the life & mathematics of Louis Nirenberg, a distinguished Canadian mathematician who recently died in New York, where he lived.

An emblematic figure of analysis & PDEs in the last century, he was awarded the Abel Prize in 2015.

From this watchtower at the Courant Institute in New York, he was for many years a global teacher & master.

He was a good friend of Spain.

> *1 of the wonders of mathematics is you go somewhere in the world & you meet other mathematicians, & it is like 1 big family.*
>
> *This large family is a wonderful joy.*[6]

13.13.1 Introduction

This article is dedicated to remembering the life & work of the prestigious Canadian mathematician Louis Nirenberg, born in Hamilton, Ontario, in 1925, who died in New York on Jan 26, 2020, at the age of 94.

Professor for much of his life at the mythical Courant Institute of New York University, he was considered 1 of the best mathematical analysts of the 20th century, a specialist in the analysis of PDEs.

When the news of his death was received, it was a very sad moment for many mathematicians, but it was also the opportunity of reviewing an exemplary life & underlining some of its landmarks.

His work unites diverse fields between what is considered Pure Mathematics & Applied Mathematics, & in particular he was cult figure in the discipline of PDEs, a key theory & tool in the mathematical formulation of many processes in science, in engineering, & in other branches of mathematics.

His work is a prodigy of sharpness & logical perfection, & at the same time its applications span today multiple scientific areas.

In recognition of his work, in 2015 he received the Abel Prize along with the another great mathematician, John Nash.

The Abel Prize is 1 of the greatest awards in Mathematics, comparable to the Nobel prizes in other sciences.

At that time, the Courant Institute, where he was for so many decades a renowned professor, published an article called *Beautiful Minds*[7] which is quite enjoyable reading.

He was a distinguished member of the AMS (American Mathematical Society).

Throughout his life, he received many other honors & awards, e.g. the AMS Bôcher Memorial Prize (1959), the Jeffery-Williams Prize (1987), the Steele Prize for Lifetime Achievement (1994 & 2014), the National Medal of Science (1995), the inaugural Crafoord Prize from the Royal Swedish Academy (1982), & the 1st Chern medal at the 2010 International Congress of Mathematicians, awarded by the International Mathematical Union & the Chern Foundation.

He was a plenary speaker at the International Congress of Mathematicians held in Stockholm in Aug 1962; the title of the conference was *"Some Aspects of Linear & Nonlinear PDE"*.

In 1969 he was elected Member of the U.S. National Academy of Sciences.

It was not honors that concerned him most, but rather his profession & the mathematical community that surrounded him.

In his long career at the Courant Institute he discovered many mathematical talents & collaborated in numerous relevant works with distinguished colleagues.

A wise man in science & life, he was 1 of the most influential & beloved mathematicians of the last century, & the current century too.

His teaching extended 1st to the international centers that he loved to visit, & then to the entire world.

Indeed, we live at this height of time in a world-wide scientific society whose close connection brings so many benefits to the pursuit of knowledge.

Many of his articles are among the most cited in the world.[8]

13.13.2 Starting

In order to start the tour of his mathematics, nothing better than to quote a few paragraphs from the mention of the Abel Prize Committee in 2015:[9]

Fig. Louis Nirenberg receiving the Abel Prize from King Harald V of Norway in the presence of John Nash (photo: Berit Roald/NTB scanpix).

Mathematical giants:

[6]From an interview with Louis Nirenberg appeared in *Notices of the AMS*, 2002, [43]

[7]Beautiful Minds: Courant's Nirenberg, Princeton's John Nash Win Abel Prize in Mathematics .

[8]Topic 35, PDEs, from the mathematical database Mathscinet, includes 3 articles by L. Nirenberg among the 10 most cited ever.

[9]John F. Nash, Jr. & Louis Nirenberg share the Abel Prize.

Nash & Nirenberg are 2 mathematical giants of the 20th century.

They are being recognized for their contributions to the field of PDEs, which are equations involving rates of change that originally arose to describe physical phenomena but, as they showed, are also helpful in analyzing abstract geometrical objects.

The Abel committee writes:

> *"Their breakthroughs have developed into versatile & robust techniques that have become essential tools for the study of nonlinear PDEs.*
>
> *Their impact can be felt in all branches of the theory."*

About Louis they say:

> *"Nirenberg has had 1 of the longest & most fêted careers in mathematics, having produced important results right up until his 70s.*
>
> *Unlike Nash, who wrote papers alone, Nirenberg preferred to work in collaboration with others, with more than 90% of his papers written jointly.*
>
> *Many results in the world of elliptic PDEs are named after him & his collaborators, e.g. the Gagliardo–Nirenberg inequalities, the John–Nirenberg inequality & the Kohn–Nirenberg theory of pseudo-differential operators."*

They conclude:

> *"Far from being confined to the solutions of the problems for which they were devised, the results proven by Nash & Nirenberg have become very useful tools & have found tremendous applications in further contexts."*

To be precise, Nirenberg made fundamental contributions to both linear & nonlinear PDEs, functional analysis, & their application in geometry & complex analysis.

Among the most famous contributions we will discuss are the Gagliardo-Nirenberg interpolation inequality, which is important in solving the elliptic PDEs that arise within many areas of mathematics; the formalization of the BMO spaces of bounded mean oscillation, & others that we will be seeing.

A work of utmost relevance was the work with Luis Caffarelli & Robert Kohn aimed at solving the big open problem of existence & smoothness of the solutions of the Navier-Stokes system of fluid mechanics.

This work was described by the AMS in 2002 as "1 of the best ever done."

The problem is on the Millennium Problems List (the list compiled by the Clay Foundation), & is 1 of the most appealing open problems of mathematical physics, raised nearly 2 centuries ago.

Fermat's Last Theorem & the Poincaré Conjecture have been defeated at the turn of the century, but the Navier-Stokes enigma (and in some sense its companion about the Euler's system) keep defying us.

We will deal with the issue in detail in Section 4.

The beginnings. From Canada to New York. Louis Nirenberg grew up in Montréal, where his fatehr was a Hebrew teacher.

After graduating[10] in 1945 at McGill University, Montréal, Louis found a summer job at the National Research Council of Canada, where he met the physicist Ernest Courant, the son of Richard Courant, a famous professor at New York University.

Ernest mentioned to Nirenberg that he was going to New York to see his father & Louis begged him for advice on a good place to apply for a master study in physics.

He returned with Richard Courant's invitation for Louis to go to New York University (NYU) for a master's degree in mathematics, after which he would be prepared for a physics program.

But once Louis began studying Mathematics at NYU, he never changed.

He defended his doctoral thesis under James Stoker in 1949, solving a problem in differential geometry.

The dice were cast.

We reach a crucial moment in Louis's life.

Breaking with the golden rule[11] according to which "a recent doctor should move to a different environment", Richard Courant kept his best students around him, including Louis Nirenberg, & he thus created the NYU Mathematical Institute, the famous Courant Institute, which has become a world benchmark for high mathematics, comparable only to the Princeton Institute for Advanced Study on the East Coast of the USA.

Louis was 1st a postdoc & then a permanent member of the faculty.

There he thrived & spent his life.

Equations & Geometry. The problem Stoker gave to Louis for his thesis, entitled *"The Determination of a Closed Convex Surface Having Given Line Elements"*, is called "the embedding problem" or "Weyl Problem".

It can be stated as follows: Given a 2D sphere with a Riemannian metric s.t. the Gaussian curvature is positive everywhere, the question is whether a surface can be constructed in 3D space so that the Riemannian distance function coincides with the distance inherited from the usual Euclidean distance in the 3D space (in other words, whether there is an isometric embedding as a convex surface in $\mathbb{R}^3$).

[10] With a degree in mathematics & physics, also in mathematics being bilingual counts.

[11] which is an essential part of the American professional practice.

The great German mathematician Hermann Weyl had taken a significant 1st step to solve the problem in 1916, & Nirenberg, as a student, completed Weyl's construction.

The work to do was to solve a system of nonlinear PDEs of the so-called "elliptic type".

It is the kind of equation & application that Louis Nirenberg has been working on ever since.

Progress has been slow but continued over time & is impressive at this moment.[12]

13.13.3 The power & beauty of inequalities

Focus on 1 of the most relevant topics in Louis Nirenberg's broad legacy, at the same time closest to our mathematical interest.

(Almost) every career in PDEs begin with the study of linear elliptic equations.

These form nowadays a well-established theory which combines Functional Analysis, Calculus of Variations, & explicit representations to produce solutions in suitable functional spaces.

For the classical equilibrium equations in the mechanics of continuous media, known as Laplace's & Poisson's equations, in symbols $\Delta u = f$, there is a classical "maximum principle" that provides the necessary estimates that guarantee the existence & uniqueness of solutions.

When combined with skillful tricks of the trade, it makes possible to obtain finer estimates, e.g. regularity & other properties.

Let us mention the estimates known under the names Harnack & Schauder, cf. [Eva10; GT01].

In this regard, Nirenberg is quoted as saying, either jokingly or seriously,

> *"I made a living off the maximum principle."*[13]

Many of the interesting problems that are proposed in Physics & other sciences & involve PDEs are **nonlinear**, e.g. the fluid equations or the curvature problems in geometry.

These nonlinear problems can seldom be solved by explicit formulas.

Because of that difficulty, the mathematical study of these problems has attracted increasing attention from the best mathematical minds of the past century, with remarkable success stories.

The usual approach goes as follows: the solution has to be obtained by some kind of approximation, & an essential technical point is usually to show that the proposed approximation procedure (or procedures) converge to a solution[14].

A complicated topology & functional analysis machinery has been developed over time & is available to test such convergence, provided certain estimates are fulfilled; their role is to allow for the approximation to be controlled.

See in this sense the book that many of us have studied as young people [Bre11].

Much of the work of an "EDP Analyst"[15] consists in finding estimates that control the passage to the limit that has to be applied, or to find a convenient fixed point theorem.

A common saying in our trade goes as follows: *Existence theorems come from a priori estimates & suitable functional analysis.*

Estimate, this is the key word in the world that Louis Nirenberg & his colleagues bequeathed us.

"Estimate" means the same thing as "inequality", & here Vázquez refers of course to a functional or numerical inequality.

It may look surprising to the reader, even weird, to find it so clearly stated: Inequalities, & not equalities (or identities), are the technical core of such a central theory of mathematics as PDEs.

However, this is precisely the mathematical revolution that was in the making when Louis was young.

Indeed, when he arrived at NYU, the most active & renowned researcher was probably Kurt Otto Friedrichs, who decisively influenced Nirenberg's future research career.

Friedrichs loved inequalities, as Louis put it:

> *"Friedrichs was a great lover of inequalities & that affected me very much.*
> *The point of view was that the inequalities are more interesting than the equalities."*

Carrying forward on that idea, Nirenberg has been unanimously recognized as a world master of inequalities".

Here is another saying by Louis:

> *"I love inequalities.*
> *So if somebody shows me a new inequality, I say: "Oh, that's beautiful, let me think about it," & I may have some ideas connected to it."*

For many years, mathematicians from all over the world came to the Courant Institute to seek his advice on issues involving inequalities.

And there we are.

We do not reject or despise the beauty of the exact solution if there is one, but functional inequalities are our firm support in an uncertain world that is yet to be discovered & described.

The key technical point of modern PDE theory is to establish the most needed & appropriate estimates in the strongest possible way.

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[12] Isometrically embedding low dimensional manifolds into higher dimensional Euclidean spaces is the contents of a famous paper by J. Nash in 1956.

[13] Curiously, it applies to Vázquez too.

Vázquez's most read article deals with the "Strong Maximum Principle", [74].

[14] Taken in some sense acceptable to physics, e.g., the solution in the weak sense or the solution in the distributional sense.

[15] *Analysis of PDEs* is an area of Mathematics in the US that perfectly describes our specialty which is neither pure nor applied, & does not need to declare itself in either direction.

Such a denomination is not much used in Spain & other countries; i.e., in Vázquez's opinion, the source of some persistent malfunctions.

Sobolev, Gagliardo & Nirenberg. There are many types of estimates the researcher needs in the study of nonlinear PDEs, but some have turned out to be much more relevant than others.

Vázquez will talk here about a type that has become particularly famous & useful.

They are often collectively called "Sobolev estimates" in honor of the great Russian mathematician Sergei Lvovich Sobolev because of his seminal work [68], 1938.

Briefly stated, they estimate the norms of functions belonging to the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, in terms of their (weak) derivatives of various orders.

In 1959 Emilio Gagliardo [35] & Louis Nirenberg [59] gave an independent & very simple proof of the following inequality:

Fig. The Talenti profile for different values of the parameters.

Theorem 7 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *Let $1 \le p < n$. There exists a constant $C > 0$ s.t. the following inequality*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \ p^* := \frac{np}{n-p},$$

holds true for all functions $u \in C_c^1(\mathbb{R}^n)$. The constant C depends only on p & n. The exponent p^* is called the Sobolev conjugate *of p. Du denotes the gradient vector.*

Gagliardo & Nirenberg included as their starting point the important case of exponent $p = 1$, left out by Sobolev.

The inequality implies the continuous inclusion of the Banach space called $W^{1,p}(\mathbb{R}^n)$ into $L^{p^*}(\mathbb{R}^n)$ (immersion theorem).

Versions for functions defined in bounded open sets $\mathbb{R}^n$ followed naturally.

This inequality soon attracted multiple applications & a wide array of variants & improvements.

Very interesting versions deal with functions defined on Riemannian manifolds.

Vázquez comments below 4 additional aspects that he finds appropriate for the curious reader.

(i) Thierry Aubin [3] & Giorgio Talenti [72] obtained in 1976 the *best constant* in this inequality, finding the functions that exhibit the *worst behavior*[16]

Indeed, when $1 < p < n$ the maximum quotient $\frac{\|u\|_{L^{p^*}(\mathbb{R}^n)}}{\|Du\|_{L^p(\mathbb{R}^n)}}$ is optimally realized by the function

$$U(\mathbf{x}) = \left(a + b|\mathbf{x}|^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}},$$

where $a, b > 0$ are arbitrary constants.[17]

It is the famous *Talenti profile*.

Note that $\frac{n-p}{p} = \frac{n}{p^*}$.

It happens that U is a probability density (integrable) if $\frac{n-p}{p-1} > n$, i.e., if $1 < p < p_c = \frac{2n}{n+1}$.

The U profile & its power appear recurrently in PDEs.

Thus, in nonlinear diffusion we find it as a power of the Barenblatt profile in fast diffusion, see Chap. 11 of [75], & the curiously critical exponent p_c also appears, but with consequences that go in the converse direction.

- Gagliardo & Nirenberg's work extends to the famous *Gagliardo-Nirenberg interpolation inequality*, a result in Sobolev's theory of spaces that estimates a certain norm of a function in terms of a product of norms of functions & derivatives thereof.

  We enter here a realm of higher complexity.[18]

  See details in [10].

(iii) In 1984 Luis Caffarelli, Bob Kohn & Louis Nirenberg needed inequalities of the previous type in functional Lebesgue spaces but with the novelty of including so-called *weights*, & this motivated the article [18], on the famous *CKN estimates originated* for spaces with power weights.

This was the beginning of an extensive literature.

A very striking effect arose in those studies: unlike GNS inequalities, there exists a phenomenon of symmetry breaking in the CKN inequalities, i.e., minimizers of such inequalities need not be symmetric functions, even when posed in the whole space or in balls.

The exact range of parameters for the symmetry breaking was found by J. Dolbeault, M. J. Esteban & M. Loss in [29].

(iv) In 2004 D. Cordero-Erausquin, B. Nazareth & C. Villani [24] used mass transport methods to obtain sharp versions of the Sobolev-Gagliardo-Nirenberg inequalities.

Mass transport is 1 of the most powerful new instruments used in PDE research.

This topic is related to the isoperimetric inequalities of ancient fame.[19] that now live moments of fruitful coincidence with Sobolev theory.

The survey [15] talks about this relationship.

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[16] This is an apparent grammatical contradiction that gives rise to beautiful functions.

[17] Consider the simple case $a = b = 1$, $p = 2$ in dimension $n = 4$.
The function looks a bit like Gaussian but it is not at all.

[18] Vázquez will avoid further details on these inequalities that can be found in the cited references.

[19] See Wikipedia/Isoperimetric Inequality.

The world of estimates that we have outlined has came to be an enormous space presided over by quite distinguished names, like H. Poincaré, J. Nash, G. H. Hardy, C. Morrey, J. Moser, N. Trudinger & other remarkable figures.

Hardy-Littlewood-Pólya's book [41] had a great influence on generations of analysts.

A commendable book on the importance of inequalities in Physics is the 2nd volume of Elliott Lieb's selected works, [53].

As a representative example chosen from among the numerous recent works, Vázquez mentions the arcile by M. del Pino & Jean Dolbeault [25].

It establishes a new optimal version of the Euclidean Gagliardo-Nirenberg inequalities.

This allows the authors to obtain the convergence rates to the equilibrium profiles of some nonlinear diffusion equations, e.g. those of the "porous media" type, 1 of the leitmotifs of Vázquez's research.

The authors completed the study & application with 2 new articles in 2003.

New functional inequalities based on entropy, maximum principles, & symmetrization processes allowed a group of Vázquez to find convergence rates for very fast diffusion equations in [7], thus solving in 2009 a much studied open problem.

It was almost 3 years of work by a team of 5 people.

Plus the work of previous authors.

Finally, there is a great deal of activity in the world of Sobolev spaces of fractional order (also called *Slobodeckii spaces*), & the associated fractional diffusions, cf. [21, 27].

It is a topic in full swing, a part of Vázquez's current mathematical efforts.[20]

New Spaces. John-Nirenberg space. Go back for a moment to the origins.

The limiting case of the Gagliardo-Nirenberg-Sobolev inequality happens for $p = n$.

Thanks to new inequalities due to C. Morrey, we know that for $p > n$ the resulting functions are Hölder continuous functions, [Eva10].

But the $p = n$ case was bizarre & it was left to Fritz John & Louis Nirenberg to solve the puzzle in 1961 by introducing the new BMO space of functions of *bounded mean oscillation*, see [44].

Actually, BMO is not a function space but rather a space of function classes modulo constants.

For this space there is the appropriate inequality.

Theorem 8 (John-Nirenberg)**.** *If $u \in W^{1,n}(\mathbb{R}^n)$ then u belongs to BMO and*

$$\|u\|_{BMO} \leq C\|Du\|_{L^n(\mathbb{R}^n)},$$

for a constant $C > 0$ depending only on n.[21]

The BMO spaces are once a very popular new object in functional & harmonic analysis, they replace L^∞ when it turns out so.

They were characterized by Charles Fefferman in [32].

The BMO spaces are slightly larger than L^∞.

The possible inequality (and functional immersion) of John-Nirenberg type using L^∞ instead of BMO as image space may seem reasonable but it is false.[22]

We ought to be very careful then with the critical cases, that Louis treated with utmost attention.

The John-Nirenberg spaces are used in analysis, in PDEs, in stochastic processes, & in multiple applications.

The reader may use the references [49] & [8] for some updates to recent work.

13.13.4 Navier-Stokes Equations

The Navier-Stokes system of equations describes the dynamics of an incompressible viscous fluid.

It was proposed in the 19th century to correct Euler's equations of ideal fluids, & adapt them to the more realistic viscous real world, [4].

The system reads **(1)**

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \nu\Delta\mathbf{u} + \dfrac{1}{\rho}\mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

where $\mathbf{u}$ is the *velocity vector*, p is the *pressure*, both variable, while ρ (the *density*) & ν (the *viscosity*) can be taken as positive constants.

It has had a spectacular success in practical science & engineering, but its essential mathematical aspects (existence, uniqueness, & regularity) have offered a stubborn resistance in the physical case of 3 space dimensions (3 or > 3 for the mathematician).

Nirenberg on the blackboard (photo: Courant Institute, NYU).

Fundamental works to cast the theory in a modern functional framework are due to Jean Leray [50, 51], who already in 1934 speaks of weak derivatives in spaces of integrable functions.

Using the new methods of functional analysis, authors soon obtained estimates that proved to be good enough to establish the existence & uniqueness of Leray solutions in 2 space dimensions, $n = 2$.

[20] There is a wide representation of Spanish mathematicians active in these subjects with remarkable results that would be well worth a review.

[21] The curious reader will wonder which function optimizes the constant. So?

[22] Find an elementary counterexample.

Furthermore, for regular initial data the solution is classical.

But the advance stopped sharply in higher dimensions, $n \geq 3$.

Vázquez gives the word to Charles Fefferman, of Princeton University, in his description of the open problem as the Clay Foundation Millennium Problem.

It is about proving or refuting the following Conjecture:

> *(A) Existence & smoothness of Navier-Stokes solutions on $\mathbb{R}^3$.*
> *Take viscosity $\nu > 0$ & $n = 3$.*
> *Let $\mathbf{u}_0(\mathbf{x})$ be any smooth, divergence-free vector field satisfying the regularity & decay conditions (*specified*).*
> *Take external force $f(t, \mathbf{x})$ to be identically zero.*
> *Then there exists smooth functions $p(t, \mathbf{x})$, $u_i(t, \mathbf{x})$ on $[0, \infty) \times \mathbb{R}^3$ that satisfy the Navier-Stokes system with initial conditions in the whole space.*[23]

The most significant advance in this field is in Vázquez's opinion the article [17] in which L. Caffarelli, R. Kohn & L. Nirenberg attached the problem of regularity & showed that if a solution with classical data develops singularities in a finite time, the set of such singularities must be in any case quite small in size.

More specifically, "the 1D measure, in the Hausdorff sense, of the set of possible singularities (located in space-time) is zero."

This implies that if the singular set is not empty, it cannot contain any line or filament.

In 1998 F. H. Lin [54] gave an interesting new proof of this result.

Vázquez is talking about 1 of the milestones of the authors' career; it happened during the stay of a young Luis Caffarelli at the Courant Institute at Louis's invitation, & was published in 1982.

The topic Fluids is completely different from the previous sections, but the functional estimates in Sobolev spaces play an essential role, along with the machinery of geometric measure theory.

The possible presence of these singularities was conjectured by Leray as a possible explanation for the phenomenon of *turbulence*.

According to this hypothesis, even for regular data, solutions in 3 or more dimensions can develop singularities in finite time in the form of points where the so-called *vorticity* becomes infinite.

In the elapsed time, it has not been possible to prove or refute Conjecture (A).

Many efforts have been invested & Vázquez believes that will bear fruit 1 day.

An account of the state of affairs in the Euler & NSEs around 2008 is due to P. Constantin [23].

At the present moment Vázquez is entertained by a number of trials & false proofs (some of them quite well published).

There are excellent general texts on Navier-Stokes, e.g. [36] & [73].

2 very recent texts are [66] & [67].

13.13.5 Elliptic Equations & the Calculus of Variations

For reasons of selection & space, Vázquez will be quite brief on a subject in which Louis made so many contributions.

Vázquez mentions 1st of all the article [11] by Haim Brezis & Louis Nirenberg, which figures among the most widely read among the works of both authors.

It deals with the existence of solutions of semilinear elliptic equations with critical exponent (once again!)

$$\Delta u + f(\mathbf{x}, u) + u^{\frac{n+2}{n-2}} = 0.$$

2 further articles that had great impact are work in collaboration with Shmuel Agmon & Avron Douglis [1], year 1959, & [2], year 1964.

They are near-the-boundary estimates for solutions of elliptic equations that satisfy general boundary conditions.

Behavior near the boundary of nonlinear or degenerate PDE solutions, or in domains with nonsmooth boundaries, is a really delicate issue.

Indeed, it is a topic of permanent interest in our community, in theory & also because of its practical interest[24].

The article [6]with Henri Berestycki & S. R. S. Varadhan links the study of the 1st eigenvalue with the maximum principle, a subject that Louis enjoyed so much.

In this context Vázquez finds the famous article on the method of the "moving planes" of 1991 [5] in collaboration with Henri Berestycki, which Vázquez consider a gem.

In the Calculus of Variations, Vázquez quotes the article [11] with Haim Brezis, about the difference between local minimizers in the spaces H^1 & C^1. See also [12].

A topic of great interest for Louis was the study of geometric properties e.g. symmetry.

The articles [37, 38] with Vasilis Gidas & Wei-Ming Ni deal with the radial symmetry of certain positive solutions of nonlinear elliptic equations that is imposed by the equation & the shape of the domain.

13.13.6 Other contributions

Vázquez collects here brief comments on important results obtained by Louis & his collaborators on various topics that would deserve a more extensive treatment.

[23] See full details of the presentation in https://www.claymath.org/sites/default/files/navierstokes.pdf.

[24] Think about the behavior of fluids in domains with corners.

Operator theory. Nirenberg & Joseph J. Kohn[25] introduced of a *pseudo-differential operator* that helped generate a huge amount of later work in the brilliant school of harmonic analysis.

In a 1965 article, [48], they dealt with pseudo-differential operators with a complete & algebraic view.

The operators in question act on the space of tempered distributions at $\mathbb{R}^n$, & are estimated in terms of Fourier transform norms.

The importance of these results is that they take into account all the "lower order terms", difficult to deal with in previous articles.

See also the volume [61] edited by Louis.

Free boundary problems. This is 1 of the favorite topics of this reviewer.

In 1977 Louis published with David Kinderlehrer the article [45] on the regularity of free boundary problems for elliptic equations, at the beginning of an era that was to witness great progress.

To put it clearly, let us assume that u is a solution to the problem

$$\Delta u \leq f, \; u \geq 0, \; (\Delta u - f)u = 0,$$

defined in a domain $D \subset \mathbb{R}^n$.

Boundary data are also given at the fixed boundary ∂D.

These data are intended to determine not only u but also the positivity domain $\Omega = \{\mathbf{x} \in D; u(\mathbf{x}) > 0\}$, or still better the boundary of Ω that lies within D, called the *free boundary*:

$$\Gamma(u) = \partial\Omega \cap D.$$

This is properly called an *obstacle problem*.

To get a physical idea, we can imagine a membrane in space $\mathbb{R}^3$ of height $z = U(x, y)$ that is subject to boundary conditions $U = h \geq 0$ in ∂D & must lie above a stable (obstacle) of height $U_{\text{obst}}(x, y) = 0$.

Fig. Free boundaries & obstacles (pictures: X. Ros-Oton).

Often, we want to consider a nontrivial obstacle φ, usually a concave function as in the figure.

This leads to an interesting equivalent formulation.

If we put $u = U + \varphi$, we arrive at the problem

$$\Delta u \leq g, \; u \geq \varphi, \; (\Delta u - g)(U - \varphi) = 0,$$

with *driving term* $g = f + \Delta\varphi$, & then we usually take $g = 0$.

In this formulation, u is constrained to stay above the obstacle $u_{\text{obst}}(\mathbf{x}) = var\phi$.

In any case, in the "free part", $\{\mathbf{x} \in \mathbb{R}^n; U(\mathbf{x}) > 0\} = \{\mathbf{x} \in \mathbb{R}^n; u(\mathbf{x}) > \varphi\}$, an elastic equation $\Delta U = f$ is satisfied, but a priori we do not know where that part could be located.

It is therefore a problem that combines PDEs & Geometry (again!).

This problem was known to have a unique *solution pair*, (u, Γ).

The attentive reader will have observed that once Γ is known, & with it Ω, the PDE problem to find u is rather elementary.

Therefore, the difficulty lies in principle in the geometry.

However, the solution to the puzzle was rather found in nonlinear analysis, [47], which also produces efficient numerical methods.

We then encounter a big theoretical problem: determining how regular is the set Γ, that we have found by abstract methods, & also determining how regular is u near Γ.

Even the simplest question: "is Γ a surface?" has to be answered.

D. Kinderlehrer & L. Nirenberg gave local conditions on f & assumed a certain initial regularity of u to conclude that then Γ is a very regular, even analytical, hyper-surface.

The study of free boundaries extends to problems evolving in time, e.g. the very famous Stefan problem discussed by Louis in [46].

The 1980s were years of great progress in the mathematical understanding of free boundaries, with reference books e.g. [28, 34].

This is a field of very intense activity, both theoretical & applied, in which Vázquez has worked with great delight for decades.

A required reference for in-depth study of the regularity of the free boundaries is the book [20] by L. Caffarelli & S. Salsa, see also A. Petrosyan et al. [64].

A study of tumor growth modeling, seen as a free boundary problem, was done by B. Perthame et al. in [63], it is just an example from a vast literature.

Geometric Equations. The article [55] with Charles Loewner in 1974 deals with PDEs that are invariant under conformal or projective transformations.

The reader will recall in this context the current relevance of PDEs linked to problems of Riemannian geometry, e.g. the Yamabe problem.

Vázquez refers to the lengthy overview [52] due to Yan Yan Li, Louis's doctoral student that has been for many years professor at Rutgers.

[25] J. J. Kohn is a brilliant Princeton analyst, not to be confused with R. Kohn from Courant.
J. J. Kohn speaks perfect Spanish with an Ecuadorian accent.

Complex geometry. The topic interested Louis a lot in his beginnings.

A Newlander & L. Nirenberg wrote in 1965 an article published in Annals of Mathematics [56] on analytical coordinates in quasi-complex manifolds.

The Newlander-Nirenberg Theorem states that any integrable quasi-complex structure is induced by a complex structure.

Integrability is expressed through a list of differential conditions.

Vázquez puts an end here to the mathematical journey, unfortunately unfair in many aspects due to the brevity of space & his ignorance in so many subjects.

Vázquez hopes that the extensive cited literature will serve as an indication of the profound influence of Louis Nirenberg & his world on the mathematicians & mathematics that have followed him.

For the curious reader, there are excellent articles dealing with the work & life of Louis Nirenberg: a congress in his honor on the occasion of the 75th anniversary was organized by Alice Chang et al. & is collected in [22].

He was interviewed by Allyn Jackson for the *AMS Notices* in 2002, [43], & Simon Donaldson, Fields Medal, wrote about him in the same journal in 2011, [30].

Yan-Yan Li's [52] 2010 article focuses on the analysis of geometric problems.

On the occasion of the Abel Prize, Xavier Cabré wrote a review in Catalan in [14] & Tristan Rivière reviews his work in PDEs in [65].

A mathematical description of the influence of his ideas appeared in 2016 in [69] with contributions of a number of experts: X. Cabré (symmetries of solutions), A. Chang (Gauss curvature problem), G. Seregin (Navier-Stokes problem), E. Carlen & A. Figalli (stability of the GNS inequality), M. T. Wang & S. T. Yau (Weyl problem & general relativity).

Finally, the book [42] presents the laureates of the Abel Prize in the period 2013–2017.

In it Robert V. Kohn devotes to L. Nirenberg the article *"A few of Louis Nirenberg's many contributions to the theory of PDEs"*.

By the way, there is a beautiful quotation from Abel as motto for the book:

> "Au reste il me paraît que si l'on veut faire des progrès dans les mathématiques il faut étudier les maîtres et non pas les écoliers."[26]

Update. the article *"A personal tribute to Louis Nirenberg"*, posted by Prof. Joel Spruck in the Arxiv repository in May 2021, [70].

As a person who met Louis Nirenberg in 1972 & became a Courant Instructor, his detailed report on a selection of Louis's works is a very commendable reading.

He concentrates on the work inspired by geometric problems beginning around 1974, especially the method of moving planes, & implicit fully nonlinear elliptic equations, & makes comments on Louis' personality.

Fig. Nirenberg in Barcelona in 2017 (photo: Jordi Play).

13.13.7 The quiet wise man & Spain

Vázquez's 1st memory of Louis Nirenberg sets them in Lisbon in the spring of 1982.[27]

Louis was already famous & Vázquez was a novice in the art.

In Lisbon Vázquez listened to 1 of his talks, which brought together the depth of the mathematics, the simplicity of the exposition & a grace to add some comment as timely as it was nice, characteristic features of Louis that delighted the public.

In the fall of that same year Vázquez set foot in the US, headed for the University of Minnesota,[28] to work on free boundary problems with Don Aronson & with Luis Caffarelli, who was back from his visit to Courant Institute.

Then Vázquez saw, through the group of great professor Vázquez had access to, that mathematical research offered a much better way of life.

Among that group of friends Vázquez counts Haim Brezis & Luis Caffarelli who have been Vázquez's masters, Louis Nirenberg, Constantine Dafermos, Donald Aronson, Mike Crandall, Hans Weinberger,... Vázquez will never cease from thanking them for that vision.

A few years later, Vázquez had the honor of participating in the organization of a summer course at the UIMP[29] which included Louis as lecturer along with Don G. Aronson (Minnesota), Philippe Bénilan (Besançon), Luis A. Caffarelli (IAS Princeton) & Constantine Dafermos (Brown Univ.).

These courses were inspired by Luis Caffarelli, close collaborator & friend of Louis, with the support of the Rector of the UIMP, Prof. Ernest Lluch,[30] & somehow they transmitted a certain spirit of mathematics that was being done around the Courant Institute.

The course had a remarkable consequence.

A young mathematician from Barcelona, Xavier Cabré, a student in the course, went to the Courant Institute with Louis Nirenberg & thus began an international mathematical career, like the ones that so many young people crave today.

His thesis, directed by Louis, dealt with "Estimates for Solutions of Elliptic & Parabolic Equations" (NYU, 1994).

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[26] In English: *"Finally, it appears to me that if one wants to make progress in mathematics, one should study the masters, not the students."* Taken from the book.

[27] At the International Symposium in Homage to Prof. J. Sebatião e Silva.

[28] This American university was very popular with young Spanish graduates & doctors for the excellence of its studies in Mathematics & Economics.

[29] Menéndez Pelayo International University, the course took place in 1987 at the Palacio de la Magdalena in Santander.

[30] Scholar of indelible memory, great protector of science & great conversationalist, he died tragically for being a good person at a very turbulent time.

Following his stay in New York, he published with Luis Caffarelli the beautiful book [16] on the so-called *completely nonlinear elliptic equations*.

Xavier Cabré is now an ICREA Professor at the UPC in Barcelona.

Louis Nirenberg visited Spain several times, specially Barcelona, & had many Spanish friends & admirers.

Although Vázquez did not become a collaborator of Louis, Vázquez had the opportunity of seeing him & talking to him on several occasions.

Vázquez highlights a stay at the Courant Institute in the winter of 1996 where Vázquez could appreciate the day-to-day life of the "quiet wise man", or a congress in Argentina in 2009 when Louis was already very senior but loved life as the 1st day.

The last event in which Vázquez saw him took place at Columbia University, New York, in May of last year (2019), in a congress in honor of Luis Caffarelli.

He went to some talks in his wheelchair at 94 years old, and, with his proverbial good humor he told them that it was a bit difficult for him to follow the lectures!

Impressed by his personality, the young mathematician David Fernández & Vázquez wrote a portrait of him in 2 entries in the blog *"The Republic of Mathematics"* that they edit in "Investigación y Ciencia" (Spanish partner of "Sciencific American").

They called the essays "Louis Nirenberg, the quiet wise man" (I) & (II).[31]

He was a teacher & master of science as those described by George Steiner in [71], where the relationship between teacher & pupil, master & disciple, is what matters.

Louis had 46 doctoral students, many of them well-known mathematicians.[32]

It was not his style to write long textbooks, he was the author of [60] & the recently published [62].

We will miss the teacher, master & senior friend who always looked gentle & kind, who loved Italy (*il bel paese*), culture, good food & talking about movies & friends, & with whom mathematics was easy & exciting.

Nirenberg lived in New York since 1949, in the Upper West Side, he was a perfect New Yorker & at the same time a citizen of the wide world.

He worked until the end of his life, frequently visiting "his" Institute.

Lucky soul, how Vázquez envies him, now & here the "elders" seem expendable for public utility.

Vázquez is proud to bear his name Louis = Luis, like Luis Caffarelli or Jacques Louis Lions or Luigi Ambrosio.

He is already a great name in mathematics & it is an honor that carries the burden of working as Louis Nirenberg, only for the best & always in a good mood, & that is not easy.

Rest in eternal peace, beloved Master.

In the Elysian fields you will have time to think about new functional inequalities, the beautiful functions that optimize them, & their surprising fruits.

In our own small way, we also follow them, as in [26]. □

13.14 Stanley Osher

13.15 Laurent Schwartz

Laurent Schwartz.

- **Born.** Mar 5, 1915. Paris, France.
- **Died.** Jul 4, 2002 (aged 87). Paris, France.
- **Nationality.** French.
- **Alma mater.** Ecole Normale Supérieure.
- **Known for.**
  - Theory of Distributions
  - Schwartz kernel theorem
  - Schwartz space
  - Schwartz-Bruhat function
  - Radonifying operator
  - Cylinder set measure
- **Awards.** Fields Medal (1950).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**

[31] https://www.investigacionyciencia.es/blogs/matematicas/75/posts.

[32] The 1st was Walter Littman (in 1956), whom Vázquez treated so much in Minnesota.

- University of Strashbourg
- University of Nancy
- University of Grenoble
- École Polytechnique
- Université de Paris VII

- **Doctoral advisor.** Georges Valiron.
- **Doctoral students.**
  - Maurice Audin
  - Georges Glaeser
  - Alexander Grothendieck
  - Jacques-Louis Lions
  - Bernard Malgrange
  - André Martineau
  - Bernard Maurey
  - Leopoldo Nachbin
  - Henri Hogbe Nlend
  - Gilles Pisier
  - François Treves
- **Influenced.** Per Enflo.

Laurent-Moïse Schwartz (Mar 5, 1915 - Jul 4, 2002) was a French mathematician.

He pioneered the theory of distributions, which gives a well-defined meaning to objects such as the Dirac delta function.

He was awarded the Fields Medal in 1950 for his work on the theory of distributions.

For several years he taught at the École polytechnique.

13.15.1 Biography

Family Laurent Schwartz came from a Jewish family of Alsatian origin, with a strong scientific background: his father was a well-known surgeon, his uncle Robert Debré (who contributed to the creation of UNICEF) was a famous pediatrician, & his great-uncle-in-law, Jacques Hadamard, was a famous mathematician.

During his training at Lycée Louis-le-Grand to enter the École Normale Supérieure, he fell in love with Marie-Hélène Lévy, daughter of the probabilist Paul Lévy who was then teaching at the École polytechnique.

Later they would have 2 children, Marc-André & Claudine.

Marie-Hélène was gifted in mathematics as well, as she contributed to the geometry of singular analytic spaces & taught at the University of Lille.

Angelo Guerraggio describes "Mathematics, politics & butterflies" as his "3 great loves".[1]

Education According to his teachers, Schwartz was an exceptional student.

He was particularly gifted in Latin, Greek & mathematics.

1 of his teachers told his parents: "*Beware, some will say your son has a gift for languages, but he is only interested in the scientific & mathematical aspect of languages: he should become a mathematician.*"

In 1934, he was admitted at the École Normale Supérieure, & in 1937 he obtained the agrégation (with rank 2).

World War II As a man of Trotskyist affinities & Jewish descent, life was difficult for Schwartz during World War II.

He had to hide & change his identity to avoid being deported after Nazi Germany overran France.

He worked for the University of Strasbourg (which had been relocated in Clermont-Ferrand because of the war) under the name of Laurent-Marie Sélimartin, while Marie-Hélène used the name Lengé instead of Lévy.

Unlike other mathematicians at Clermont-Ferrand such as Feldbau, the couple managed to escape the Nazis.

Later career Schwartz taught mainly at École Polytechnique, from 1958 to 1980.

At the end of the war, he spent one year in Grenoble (1944), then in 1945 joined the University of Nancy on the advice of Jean Delsarte & Jean Dieudonné, where he spent 7 years.

He was both an influential researcher & teacher, with students such as Bernard Malgrange, Jacques-Louis Lions, François Bruhat & Alexander Grothendieck.

He joined the science faculty of the University of Paris in 1952.

In 1958 he became a teacher at the École polytechnique after having at 1st refused this position.

From 1961 to 1963 the École polytechnique suspended his right to teach, because of his having signed the Manifesto of the 121 about the Algerian war, a gesture not appreciated by Polytechnique's military administration.

However, Schwartz had a lasting influence on mathematics at the École polytechnique, having reorganized both teaching & research there.

In 1965 he established the Centre de mathématiques Laurent-Schwartz (CMLS) as its 1st director.

In 1973 he was elected corresponding member of the French Academy of Sciences, & was promoted to full membership in 1975.

13.15.2 Mathematical legacy

In 1950 at the International Congress of Mathematicians, Schwartz was a plenary speaker[Schwartz, Laurent (1950). "*Théorie des noyaux*" (PDF). In: Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., Aug 30–Sep 6, 1950. vol. 1. pp. 220–230.] & was awarded the Fields Medal for his work on distributions.

He was the 1st French mathematician to receive the Fields medal.

Because of his sympathy for Trotskyism, Schwartz encountered serious problems trying to enter the United States to receive the medal; however, he was ultimately successful.

The theory of distributions clarified the (then) mysteries of the Dirac delta function & Heaviside step function.

It helps to extend the theory of Fourier transforms & is now of critical importance to the theory of partial differential equations.

13.15.3 Popular science

Throughout his life, Schwartz actively worked to promote science & bring it closer to the general audience.

Schwartz said:

> *"What are mathematics helpful for? Mathematics are helpful for physics.*
> *Physics helps us make fridges.*
> *Fridges are made to contain spiny lobsters, & spiny lobsters help mathematicians who eat them & have hence better abilities to do mathematics, which are helpful for physics, which helps us make fridges which..."[3]*

13.15.4 Entomology

Clanis schwartzi Paratype MHNT.

His mother, who was passionate about natural science, passed on her taste for entomology to Laurent.

His personal collection of 20,000 Lepidoptera specimens, collected during his various travels was bequeathed to the Muséum national d'histoire naturelle), the Science Museum of Lyon, the Museum of Toulouse & the Museo de Historia Natural Alcide d'Orbigny in Cochabamba (Bolivia).

Several species discovered by Schwartz bear his name.

13.15.5 Personal ideology

Apart from his scientific work, Schwartz was a well-known outspoken intellectual.

As a young socialist influenced by Leon Trotsky, Schwartz opposed the totalitarianism of the Soviet Union, particularly under Joseph Stalin.

Schwartz ultimately rejected Trotskyism for democratic socialism.

On his religious views, Schwartz called himself an atheist.[4]

13.15.6 Books

Research articles

- *Œuvres scientifiques. I.*

  With a general introduction to the works of Schwartz by Claude Viterbo & an appreciation of Schwartz by Bernard Malgrange.

  With 1 DVD.

  Documents Mathématiques (Paris), 9. Société Mathématique de France, Paris, 2011. x+523 pp. ISBN 978-2-85629-317-1 the 1st half of his works in analysis & partial differential equations.

After a preface by Claude Viterbo, which includes a few photos, one will find a note by Schwartz himself about his works, followed by a few original documents (letters, course notes), a presentation by Bernard Malgrange of the theory of distributions for which Schwartz received the Fields Medal in 1950, & a selection of articles covering the period 1944–1954.

- *Œuvres scientifiques. II.*

  With an appreciation of Schwartz by Alain Guichardet.

  With 1 DVD.

  Documents Mathématiques (Paris), 10.

  Société Mathématique de France, Paris, 2011. x+507 pp. ISBN 978-2-85629-318-8

  the 2nd half of his works in analysis & partial differential equations.

  After a note by Alain Guichardet on Schwartz & his seminars, one will find a selection of articles covering the period 1954–1966.

- *Œuvres scientifiques. III.*

  With appreciations of Schwartz by Gilles Godefroy & Michel Émery.

  With 1 DVD.

  Documents Mathématiques (Paris), 11. Société Mathématique de France, Paris, 2011. x+619 pp. ISBN 978-2-85629-319-5

  his works on Banach space theory (1968–1987), introduced by Gilles Godefroy, & on probability theory (1970–1996), presented by Michel Émery, as well as some articles of a historical nature (1955–1994).

Technical books

- *Analyse hilbertienne.* Collection Méthodes. Hermann, Paris, 1979. ii+297 pp. ISBN 2-7056-5897-1
- *Application of distributions to the theory of elementary particles in quantum mechanics.* Gordon & Breach, New York, NY, 1968. 144pp. ISBN 9780677300900
- *Cours d'analyse. 1.* 2nd edition. Hermann, Paris, 1981. xxix+830 pp. ISBN 2-7056-5764-9
- *Cours d'analyse. 2.* 2nd edition. Hermann, Paris, 1981. xxiii+475+21+75 pp. ISBN 2-7056-5765-7

5 *Étude des sommes d'exponentielles. 2ième éd.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris 1959 151 pp.

- *Geometry & probability in Banach spaces.* Based on notes taken by Paul R. Chernoff. Lecture Notes in Mathematics, 852. Springer-Verlag, Berlin-New York, 1981. x+101 pp. ISBN 3-540-10691-X
- *Lectures on complex analytic manifolds.* With notes by M. S. Narasimhan. Reprint of the 1955 edition. Tata Institute of Fundamental Research Lectures on Mathematics & Physics, 4. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1986. iv+182 pp. ISBN 3-540-12877-8
- *Mathematics for the physical sciences.* Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966 358 pp.
- *Radon measures on arbitrary topological spaces & cylindrical measures.* Tata Institute of Fundamental Research Studies in Mathematics, No. 6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. xii+393 pp.
- *Semimartingales & their stochastic calculus on manifolds.* Edited & with a preface by Ian Iscoe. Collection de la Chaire Aisenstadt. Presses de l'Université de Montréal, Montreal, QC, 1984. 187 pp. ISBN 2-7606-0660-0
- *Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes.* Lecture Notes in Mathematics, 780. Springer, Berlin, 1980. xv+132 pp. ISBN 3-540-09749-X
- Les tenseurs. *Suivi de "Torseurs sur un espace affine" by Y. Bamberger & J.-P. Bourguignon.* 2nd edition. Hermann, Paris, 1981. i+203 pp. ISBN 2-7056-1376-5

6 *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris 1966 xiii+420 pp.

Seminar notes

- *Séminaire Schwartz in Paris 1953 bis 1961.* Online edition: [1]

Popular books

- *Pour sauver l'université.* Editions du Seuil, 1983. 122 pp. ISBN 2020065878
- *A mathematician grappling with his century.* Translated from the 1997 French original by Leila Schneps. Birkhäuser Verlag, Basel, 2001. viii+490 pp. ISBN 3-7643-6052-6

13.15.7 See also

- Schwartz distribution
- Schwartz kernel theorem
- Schwartz space
- Schwartz-Bruhat function
- Nicolas Bourbaki" – Wikipedia/Laurent Schwartz

13.16 Roger Temam

Roger Meyer Temam.

- **Born.** May 19, 1940 (age 80).
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.** Navier-Stokes equations.

Scientific career.

- **Fields.** Applied mathematics.
- **Institutions.**
  - Paris-Sud University (Orsay)
  - Indiana University
- **Doctoral advisor.** Jacques-Louis Lions.
- **Doctoral students.**
  - Etienne Pardoux
  - Denis Serre

Roger Meyer Temam (born May 19, 1940) is a French applied mathematician working in numerical analysis, nonlinear partial differential equations & fluid mechanics.

He graduated from the University of Paris - the Sorbonne in 1967, completing a doctorate (*thèse d'Etat*) under the direction of Jacques-Louis Lions.

He has published over 400 articles, as well as 12 (authored or co-authored) books.

13.16.1 Scientific work

The 1st work of Temam in his thesis dealt with the *fractional steps method*.

Thereafter, "he has continually explored & developed new directions & techniques":[2]

- calculus of variations, & the notion of *duality* (book #7), developing the mathematical framework for *discontinuous* (in displacement) *solutions*; a concept later used for his works on the *mathematical theory of plasticity* (book #5);
- mathematical formulation of the equilibrium of a plasma in a cavity, expressed as a nonlinear free boundary problem;[R. Temam, A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma, *Arch. Rational Mech. Anal.*, 60, 1975, 51–73.]
- Korteweg-de Vries equation;[R. Temam, Sur un problème non linéaire, *J. Math. Pures Appl.*, 48, 1969, 159–172.]
- Kuramoto-Sivashinsky equation;[5]
- Euler equations in a bounded domain;[R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.*, 20, 1975, 32–43.]

- infinite-dimensional dynamical systems theory.

  In particular, he studied the existence of the finite-dimensional global attractor for many dissipative equations of mathematical physics, including the incompressible Navier-Stokes equations.[P. Constantin, C. Foias, O. Manley & R. Temam, Determining modes & fractal dimension of turbulent flows, *J. Fluid Mech.*, 150, 1985, 427–440.][C. Foias, O.P. Manley & R. Temam, Physical estimates of the number of degrees of freedom in free convection, *Phys. Fluids*, 29, 1986, 3101–3103.]

  He was also the co-founder of the notion of *inertial manifolds*[C. Foias, G.R. Sell & R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Diff. Equ.*, 73, 1988, 309–353.] together with Ciprian Foias & George R. Sell & of exponential attractors[A. Eden, C. Foias, B. Nicolaenko & R. Temam, *Exponential attractors for dissipative evolution equations*, Collection Recherches en Mathématiques Appliquées, Masson, Paris, & John Wiley, England, 1994.] together with Alp Eden, Ciprian Foias & Basil Nicolaenko;[2]

- optimal control of the incompressible Navier-Stokes equations as a tool for the *control of turbulence*;[F. Abergel & R. Temam, On some control problems in fluid mechanics, *Theoret. Comput. Fluid Dynamics*, 1, 1990, 303–325.]

- boundary layer phenomena for incompressible flows.[12]

Temam's main activities concern the study of geophysical flows, the atmosphere & oceans.[2]

This started in the 1990s by collaboration with Jacques-Louis Lions & Shouhong Wang.[J.L. Lions, R. Temam & S. Wang, New formulations of the primitive equations of the atmosphere & applications, *Nonlinearity*, 5, 1992, 237–288.][J.L. Lions, R. Temam & S. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5, 1992, 1007–1053.][M. Coti Zelati, M. Frémond, R. Temam & J. Tribbia, Uniqueness, regularity & maximum principles for the equations of the atmosphere with humidity & saturation, *Physica D*, 264, 2013, 49-65, https://doi.org/10.1016/j.physd.2013.08.007][Y. Cao, M. Hamouda, R. Temam, J. Tribbia & X. Wang, The equations of the multi-phase humid atmosphere expressed as a quasi variational inequality, *Nonlinearity*, 31, 2018, 4692-4723, https://doi.org/10.1088/1361-6544/aad525.]

According to the Mathematical Genealogy Project database,[17][18] he holds the first position in the top 50 advisors.

More than 30 of his students are now full professors all over the world, & have themselves many descendants.[19]

13.16.2 Administrative activities

Temam became a professor at the Paris-Sud University at Orsay in 1968.

There, he co-founded the Laboratory of Numerical & Functional Analysis which he directed from 1972 to 1988.

He was also a *Maître de Conférences* at the Ecole Polytechnique in Paris from 1968 to 1986.[20]

In 1983, Temam co-founded the French Société de Mathématiques Appliquées et Industrielles (SMAI), analogous to the Society for Industrial & Applied Mathematics (SIAM), & served as its 1st president.[21]

He was also 1 of the founders of the International Congress on Industrial & Applied Mathematics (ICIAM) series & was the chair of the steering committee of the 1st ICIAM meeting held in Paris in 1987; & the chair of the standing committee of the 2nd ICIAM meeting held in Washington, D.C., in 1991.[22]

He was the Editor-in-Chief of the mathematical journal M2AN[23] from 1986 to 1997.

Temam has been the Director of the Institute for Scientific Computing & Applied Mathematics (ISCAM)[24] at Indiana University since 1986 (co-director with Ciprian Foias from 1986 to 1992).

He is also a College Professor (part-time till 2003) & he has been a Distinguished Professor since 2014.[25]

13.16.3 Books

1. (with G.-M. Gie, M. Hamouda & C.-Y. Jung): *Singular perturbations & boundary layers*, Springer-Verlag, New-York, 2018.

2. (with A. Miranville): *Mathematical Modelling in Continuum Mechanics*, Cambridge University Press, 2001. French Translation, Springer-Verlag France, 2002. Chinese Translation, Tsinghua University Press, 2004. 2nd English Edition 2005. Russian translation, Moskva Linom, 2013.

3. (with T. Dubois & F. Jauberteau): *Dynamic, multilevel methods & the numerical simulation of turbulence*; Cambridge University Press, 1999.

4. *Infinite Dimensional Dynamical Systems in Mechanics & Physics*, Springer-Verlag, New-York, Applied Mathematical Sciences Series, vol. 68, 1988. 2nd augmented edition, 1997. Reprinted in China by Beijing World Publishing Corp., 2000.

5. *Mathematical Problems in Plasticity*, Gauthier-Villars, Paris, 1983 (in French). English Transl., Gauthier-Villars, New-York, 1985. Russian Transl., Nauk, Moscow, 1991. "Republished by Dover books in Physics, 2018."

6. *Navier-Stokes Equations*, North-Holland Pub. Company, in English, 1977, 500 pages. Revised editions 1979, 1984 & 1985. Russian Translation, Mir, Moscow, 1981. "Republished in the AMS-Chelsea Series, AMS, Providence, 2001."

7. (with I. Ekeland): *Convex Analysis & Variational Problems.* Dunod, Paris, 1974, 350 pages (in French). English Translation, North-Holland, Amsterdam, 1976. Russian Translation, Mir, Moscow, 1979. "English version republished in the Series 'Classics in Applied Mathematics', SIAM, Philadelphia, 1999."

13.16.4 Awards & honors

- Fellow of the American Academy of Arts & Sciences (2015),[26] of the American Mathematical Society (2013),[27] of the American Association for the Advancement of Science (2011),[28] of the Society for Industrial & Applied Mathematics (2009).[29]
- Knight of the Legion of Honor, France, 2012.[30]
- Member of the French Academy of Sciences since 2007.[31]" – Wikipedia/Roger Temam

13.17 Karl Weierstrass

Karl Weierstrass/Karl Weierstraß.

- **Born.** Oct 31, 1815. Ostenfelde, Province of Westphalia, Kingdom of Prussia.
- **Died.** Feb 19, 1897 (aged 81). Berlin, Province of Brandenburg, Kingdom of Prussia.
- **Nationality.** German.
- **Alma mater.**
  - University of Bonn
  - Münster Academy
- **Known for.**
  - Weierstrass function
  - Weierstrass product inequality
  - (ε, δ)-definition of limit
  - Weierstrass-Erdmann condition
  - Weierstrass theorems
  - Bolzano-Weierstrass theorem
- **Awards.**
  - PhD (Hon): University of Königsberg (1854)
  - Copley Medal (1895)

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - Gewerbeinstitut
  - Friedrich Wilhelm University
- **Academic advisors.** Christoph Gudermann.
- **Doctoral students.**
  - Nikolai Bugaev
  - Georg Cantor
  - Georg Frobenius
  - Lazarus Fuchs
  - Wilhelm Killing
  - Leo Königsberger
  - Sofia Kovalevskaya
  - Mathias Lerch
  - Hans von Mangoldt
  - Eugen Netto
  - Adolf Piltz
  - Carl Runge
  - Arthur Schoenflies

- Friedrich Schottky
- Hermann Schwarz
- Ludwig Stickelberger
- Ernst Kötter

Karl Theodor Wilhelm Weierstrass (German: *Weierstraß*;[Duden. *Das Aussprachewörterbuch*. 7. Auflage. Bibliographisches Institut, Berlin 2015, ISBN 978-3-411-04067-4] Oct 31, 1815 - Feb 19, 1897) was a German mathematician often cited as the "*father of modern analysis*".

Despite leaving university without a degree, he studied mathematics & trained as a school teacher, eventually teaching mathematics, physics, botany & gymnastics.[Weierstrass, Karl Theodor Wilhelm. (2018). In Helicon (Ed.), *The Hutchinson unabridged encyclopedia with atlas & weather guide*. [Online]. Abington: Helicon. Available from: link [Accessed Jul 8, 2018].]

He later received an honorary doctorate & became professor of mathematics in Berlin.

Among many other contributions, Weierstrass formalized the definition of the continuity of a function, proved the intermediate value theorem & the Bolzano-Weierstrass theorem, & used the latter to study the properties of continuous functions on closed bounded intervals.

13.17.1 Biography

Weierstrass was born in Ostenfelde, part of Ennigerloh, Province of Westphalia.[O'Connor, J. J.; Robertson, E. F. (October 1998). "*Karl Theodor Wilhelm Weierstrass*". School of Mathematics & Statistics, University of St Andrews, Scotland. Retrieved Sep 7, 2014.]

Weierstrass was the son of Wilhelm Weierstrass, a government official, & Theodora Vonderforst.

His interest in mathematics began while he was a gymnasium student at the Theodorianum in Paderborn.

He was sent to the University of Bonn upon graduation to prepare for a government position.

Because his studies were to be in the fields of law, economics, & finance, he was immediately in conflict with his hopes to study mathematics.

He resolved the conflict by paying little heed to his planned course of study but continuing private study in mathematics.

The outcome was that he left the university without a degree.

He then studied mathematics at the Münster Academy (which was even then famous for mathematics) & his father was able to obtain a place for him in a teacher training school in Münster.

Later he was certified as a teacher in that city.

During this period of study, Weierstrass attended the lectures of Christoph Gudermann & became interested in elliptic functions.

In 1843 he taught in Deutsch Krone in West Prussia & since 1848 he taught at the Lyceum Hosianum in Braunsberg.

Besides mathematics he also taught physics, botany, & gymnastics.[3]

Weierstrass may have had an illegitimate child named Franz with the widow of his friend Carl Wilhelm Borchardt.[Biermann, Kurt-R.; Schubring, Gert (1996). "Einige Nachträge zur Biographie von Karl Weierstraß. (German) [Some postscripts to the biography of Karl Weierstrass]". *History of mathematics*. San Diego, CA: Academic Press. pp. 65–91.]

After 1850 Weierstrass suffered from a long period of illness, but was able to publish mathematical articles that brought him fame & distinction.

The University of Königsberg conferred an honorary doctor's degree on him on Mar 31, 1854.

In 1856 he took a chair at the *Gewerbeinstitut* in Berlin (an institute to educate technical workers which would later merge with the *Bauakademie* to form the Technical University of Berlin).

In 1864 he became professor at the Friedrich-Wilhelms-Universität Berlin, which later became the Humboldt Universität zu Berlin.

In 1870, at the age of 55, Weierstrass met Sofia Kovalevsky whom he tutored privately after failing to secure her admission to the University. They had a fruitful intellectual, but troubled personal, relationship that "far transcended the usual teacher-student relationship".

The misinterpretation of this relationship & Kovalevsky's early death in 1891 was said to have contributed to Weierstrass' later ill-health.

He was immobile for the last 3 years of his life, & died in Berlin from pneumonia.[*Dictionary of scientific biography*. Gillispie, Charles Coulston,, American Council of Learned Societies. New York. p. 223. ISBN 978-0-684-12926-6. OCLC 89822.]

13.17.2 Mathematical contributions

Soundness of calculus Weierstrass was interested in the soundness of calculus, & at the time there were somewhat ambiguous definitions of the foundations of calculus so that important theorems could not be proven with sufficient rigor.

Although Bolzano had developed a reasonably rigorous definition of a limit as early as 1817 (and possibly even earlier) his work remained unknown to most of the mathematical community until years later, & many mathematicians had only vague definitions of limits & continuity of functions.

The basic idea behind Delta-epsilon proofs is, arguably, 1st found in the works of Cauchy in the 1820s.

- Grabiner, Judith V. (March 1983), "Who Gave You the Epsilon? Cauchy & the Origins of Rigorous Calculus" (PDF), *The American Mathematical Monthly*, 90 (3): 185–194, doi:10.2307/2975545, JSTOR 2975545

- Cauchy, A.-L. (1823), "Septième Leçon – Valeurs de quelques expressions qui se présentent sous les formes indéterminées $\frac{\infty}{\infty}, \infty^0, \ldots$ Relation qui existe entre le rapport aux différences finies et la fonction dérivée", *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal*, Paris, archived from the original on 2009-05-04, retrieved 2009-05-01, p. 44.

Cauchy did not clearly distinguish between continuity & uniform continuity on an interval.

Notably, in his 1821 *Cours d'analyse*, Cauchy argued that the (pointwise) limit of (pointwise) continuous functions was itself (pointwise) continuous, a statement interpreted as being incorrect by many scholars.

The correct statement is rather that the uniform limit of continuous functions is continuous (also, the uniform limit of uniformly continuous functions is uniformly continuous).

This required the concept of uniform convergence, which was 1st observed by Weierstrass's advisor, Christoph Gudermann, in an 1838 paper, where Gudermann noted the phenomenon but did not define it or elaborate on it.

Weierstrass saw the importance of the concept, & both formalized it & applied it widely throughout the foundations of calculus.

The formal definition of continuity of a function, as formulated by Weierstrass, is as follows:

> $f(x)$ is continuous at $x = x_0$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every x in the domain of f, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
> In simple English, $f(x)$ is continuous at a point $x = x_0$ if for each x close enough to x_0, the function value $f(x)$ is very close to $f(x_0)$, where the "close enough" restriction typically depends on the desired closeness of $f(x_0)$ to $f(x)$.

Using this definition, he proved the Intermediate Value Theorem.

He also proved the Bolzano-Weierstrass theorem & used it to study the properties of continuous functions on closed & bounded intervals.

Calculus of variations Weierstrass also made advances in the field of calculus of variations.

Using the apparatus of analysis that he helped to develop, Weierstrass was able to give a complete reformulation of the theory that paved the way for the modern study of the calculus of variations.

Among several axioms, Weierstrass established a necessary condition for the existence of strong extrema of variational problems.

He also helped devise the Weierstrass-Erdmann condition, which gives sufficient conditions for an extremal to have a corner along a given extremum & allows one to find a minimizing curve for a given integral.

Other analytical theorems See also: List of things named after Karl Weierstrass.

- Stone-Weierstrass theorem
- Casorati-Weierstrass-Sokhotski theorem
- Weierstrass's elliptic functions
- Weierstrass function
- Weierstrass M-test
- Weierstrass preparation theorem
- Lindemann-Weierstrass theorem
- Weierstrass factorization theorem
- Enneper-Weierstrass parameterization

13.17.3 Students

- Edmund Husserl
- Sofia Kovalevskaya
- Gösta Mittag-Leffler
- Hermann Schwarz
- Carl Johannes Thomae
- Georg Cantor

13.17.4 Honors & awards

The lunar crater Weierstrass & the asteroid 14100 Weierstrass are named after him.

Also, there is the Weierstrass Institute for Applied Analysis & Stochastics in Berlin.

13.17.5 Selected works

- *Zur Theorie der Abelschen Funktionen* (1854)
- *Theorie der Abelschen Funktionen* (1856)
- *Abhandlungen-1*, Math. Werke. Bd. 1. Berlin, 1894
- *Abhandlungen-2*, Math. Werke. Bd. 2. Berlin, 1895
- *Abhandlungen-3*, Math. Werke. Bd. 3. Berlin, 1903
- *Vorl. ueber die Theorie der Abelschen Transcendenten*, Math. Werke. Bd. 4. Berlin, 1902
- *Vorl. ueber Variationsrechnung*, Math. Werke. Bd. 7. Leipzig, 1927

13.17.6 External links

- O'Connor, John J.; Robertson, Edmund F., "Karl Weierstrass", *MacTutor History of Mathematics archive*, University of St Andrews.
- Digitalized versions of Weierstrass's original publications are freely available online from the library of the *Berlin Brandenburgische Akademie der Wissenschaften.*
- Works by Karl Weierstrass at Project Gutenberg
- Works by or about Karl Weierstrass at Internet Archive" – Wikipedia/Karl Weierstrass

14 Miscellaneous

Tài liệu

[ABT18] Richard C. Aster, Brian Borchers, and Clifford H. Thurber. *Parameter estimation and inverse problems*. Third. Elsevier/Academic Press, Amsterdam, 2018, pp. xi+392. ISBN: 978-0-12-804651-7. DOI: 10.1016/C2015-0-02458-3. URL: https://doi.org/10.1016/C2015-0-02458-3.

[AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. ISBN: 0-12-044143-8.

[Agm10] Shmuel Agmon. *Lectures on elliptic boundary value problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr., Revised edition of the 1965 original. AMS Chelsea Publishing, Providence, RI, 2010, pp. x+216. ISBN: 978-0-8218-4910-1. DOI: 10.1090/chel/369. URL: https://doi.org/10.1090/chel/369.

[Agm65] Shmuel Agmon. *Lectures on elliptic boundary value problems*. Van Nostrand Mathematical Studies, No. 2. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965, pp. v+291.

[AK16] Cung Thế Anh and Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng*. Nhà Xuất Bản Đại Học Sư Phạm, 2016, p. 222.

[AK89] Andrew F. Acker and Bernhard Kawohl. "Remarks on quenching". In: *Nonlinear Anal.* 13.1 (1989), pp. 53–61. ISSN: 0362-546X. DOI: 10.1016/0362-546X(89)90034-5. URL: https://doi.org/10.1016/0362-546X(89)90034-5.

[AL09] Rong An and Kai Tai Li. "Existence of weak solution to nonhomogeneous steady Navier-Stokes equations with mixed boundary conditions". In: *Acta Math. Appl. Sin.* 32.4 (2009), pp. 664–672. ISSN: 0254-3079.

[Ali09] Serge Alinhac. *Hyperbolic partial differential equations*. Universitext. Springer, Dordrecht, 2009, pp. xii+150. ISBN: 978-0-387-87822-5. DOI: 10.1007/978-0-387-87823-2. URL: https://doi.org/10.1007/978-0-387-87823-2.

[ALL09] Rong An, Yuan Li, and Kai Tai Li. "Regularity of solutions to stationary Navier-Stokes equations with mixed boundary conditions". In: *Math. Appl. (Wuhan)* 22.1 (2009), pp. 83–89. ISSN: 1001-9847.

[Alt16] Hans Wilhelm Alt. *Linear functional analysis*. Universitext. An application-oriented introduction, Translated from the German edition by Robert Nürnberg. Springer-Verlag London, Ltd., London, 2016, pp. xii+435. ISBN: 978-1-4471-7279-6; 978-1-4471-7280-2. DOI: 10.1007/978-1-4471-7280-2. URL: https://doi.org/10.1007/978-1-4471-7280-2.

[Ami+01] Youcef Amirat, Didier Bresch, Jérôme Lemoine, and Jacques Simon. "Effect of rugosity on a flow governed by stationary Navier-Stokes equations". In: *Quart. Appl. Math.* 59.4 (2001), pp. 769–785. ISSN: 0033-569X. DOI: 10.1090/qam/1866556. URL: https://doi.org/10.1090/qam/1866556.

[APS13] Chérif Amrouche, Patrick Penel, and Nour Seloula. "Some remarks on the boundary conditions in the theory of Navier-Stokes equations". In: *Ann. Math. Blaise Pascal* 20.1 (2013), pp. 37–73. ISSN: 1259-1734. URL: http://ambp.cedram.org/item?id=AMBP_2013__20_1_37_0.

[AR11] Chérif Amrouche and M. Ángeles Rodríguez-Bellido. “Stationary Stokes, Oseen and Navier-Stokes equations with singular data”. In: *Arch. Ration. Mech. Anal.* 199.2 (2011), pp. 597–651. ISSN: 0003-9527. DOI: 10.1007/s00205-010-0340-8. URL: https://doi.org/10.1007/s00205-010-0340-8.

[Bän01] Eberhard Bänsch. “Finite element discretization of the Navier-Stokes equations with a free capillary surface”. In: *Numer. Math.* 88.2 (2001), pp. 203–235. ISSN: 0029-599X. DOI: 10.1007/PL00005443. URL: https://doi.org/10.1007/PL00005443.

[BBP14] Mahdi Boukrouche, Imane Boussetouan, and Laetitia Paoli. “Non-isothermal Navier-Stokes system with mixed boundary conditions and friction law: uniqueness and regularity properties”. In: *Nonlinear Anal.* 102 (2014), pp. 168–185. ISSN: 0362-546X. DOI: 10.1016/j.na.2014.02.009. URL: https://doi.org/10.1016/j.na.2014.02.009.

[BCY15] Christine Bernardi, Tomás Chacón Rebollo, and Driss Yakoubi. “Finite element discretization of the Stokes and Navier-Stokes equations with boundary conditions on the pressure”. In: *SIAM J. Numer. Anal.* 53.3 (2015), pp. 1256–1279. ISSN: 0036-1429. DOI: 10.1137/140972299. URL: https://doi.org/10.1137/140972299.

[Ben11] Michal Beneš. “Mixed initial-boundary value problem for the three-dimensional Navier-Stokes equations in polyhedral domains”. In: *Discrete Contin. Dyn. Syst.* Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. I (2011), pp. 135–144. ISSN: 1078-0947.

[Ber09] Luigi C. Berselli. “Some criteria concerning the vorticity and the problem of global regularity for the 3D Navier-Stokes equations”. In: *Ann. Univ. Ferrara Sez. VII Sci. Mat.* 55.2 (2009), pp. 209–224. ISSN: 0430-3202. DOI: 10.1007/s11565-009-0076-2. URL: https://doi.org/10.1007/s11565-009-0076-2.

[BFV18] Matteo Bonforte, Alessio Figalli, and Juan Luis Vázquez. “Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains”. In: *Anal. PDE* 11.4 (2018), pp. 945–982. ISSN: 2157-5045. DOI: 10.2140/apde.2018.11.945. URL: https://doi.org/10.2140/apde.2018.11.945.

[BHV09] Christine Bernardi, Frédéric Hecht, and Rüdiger Verfürth. “A finite element discretization of the three-dimensional Navier-Stokes equations with mixed boundary conditions”. In: *M2AN Math. Model. Numer. Anal.* 43.6 (2009), pp. 1185–1201. ISSN: 0764-583X. DOI: 10.1051/m2an/2009035. URL: https://doi.org/10.1051/m2an/2009035.

[BJ89] Timothy J. Barth and Dennis C. Jespersen. “The Design and Application of Upwind Schemes on Unstructured Meshes”. In: *27th Aerospace Sciences Meeting.* 1989, p. 12. DOI: 10.2514/6.1989-366. URL: https://arc.aiaa.org/doi/abs/10.2514/6.1989-366.

[BK07] Michal Beneš and Petr Kučera. “Non-steady Navier-Stokes equations with homogeneous mixed boundary conditions and arbitrarily large initial condition”. In: *Carpathian J. Math.* 23.1-2 (2007), pp. 32–40. ISSN: 1584-2851.

[BK12] Michal Beneš and Petr Kučera. “On the Navier-Stokes flows for heat-conducting fluids with mixed boundary conditions”. In: *J. Math. Anal. Appl.* 389.2 (2012), pp. 769–780. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2011.12.017. URL: https://doi.org/10.1016/j.jmaa.2011.12.017.

[BK16] Michal Beneš and Petr Kučera. “Solutions to the Navier-Stokes equations with mixed boundary conditions in two-dimensional bounded domains”. In: *Math. Nachr.* 289.2-3 (2016), pp. 194–212. ISSN: 0025-584X. DOI: 10.1002/mana.201400046. URL: https://doi.org/10.1002/mana.201400046.

[BM14] Malte Braack and Piotr Boguslaw Mucha. “Directional do-nothing condition for the Navier-Stokes equations”. In: *J. Comput. Math.* 32.5 (2014), pp. 507–521. ISSN: 0254-9409. DOI: 10.4208/jcm.1405-m4347. URL: https://doi.org/10.4208/jcm.1405-m4347.

[BMR07] M. Bulíček, J. Málek, and K. R. Rajagopal. “Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity”. In: *Indiana Univ. Math. J.* 56.1 (2007), pp. 51–85. ISSN: 0022-2518. DOI: 10.1512/iumj.2007.56.2997. URL: https://doi.org/10.1512/iumj.2007.56.2997.

[BNP04] Hamid Bellout, Jiří Neustupa, and Patrick Penel. “On the Navier-Stokes equation with boundary conditions based on vorticity”. In: *Math. Nachr.* 269/270 (2004), pp. 59–72. ISSN: 0025-584X. DOI: 10.1002/mana.200310165. URL: https://doi.org/10.1002/mana.200310165.

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599. ISBN: 978-0-387-70913-0.

[Bri09] R. V. Brizitskiĭ. “Investigation of a class of control problems for stationary Navier-Stokes equations with mixed boundary conditions”. In: *Sib. Zh. Ind. Mat.* 12.2 (2009), pp. 17–26. ISSN: 1560-7518. DOI: 10.1134/S1990478910030026. URL: https://doi.org/10.1134/S1990478910030026.

[Bro+10] R. Brown, I. Mitrea, M. Mitrea, and M. Wright. “Mixed boundary value problems for the Stokes system”. In: *Trans. Amer. Math. Soc.* 362.3 (2010), pp. 1211–1230. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-09-04774-6. URL: https://doi.org/10.1090/S0002-9947-09-04774-6.

[BS07] Sylvie Benzoni-Gavage and Denis Serre. *Multidimensional hyperbolic partial differential equations.* Oxford Mathematical Monographs. First-order systems and applications. The Clarendon Press, Oxford University Press, Oxford, 2007, pp. xxvi+508. ISBN: 978-0-19-921123-4; 0-19-921123-X.

[BS08] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods.* Third. Vol. 15. Texts in Applied Mathematics. Springer, New York, 2008, pp. xviii+397. ISBN: 978-0-387-75933-3. DOI: 10.1007/978-0-387-75934-0. URL: https://doi.org/10.1007/978-0-387-75934-0.

[Buc+08] Dorin Bucur, Eduard Feireisl, Šárka Nečasová, and Joerg Wolf. “On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries”. In: *J. Differential Equations* 244.11 (2008), pp. 2890–2908. ISSN: 0022-0396. DOI: 10.1016/j.jde.2008.02.040. URL: https://doi.org/10.1016/j.jde.2008.02.040.

[Cam+18] Jessika Camaño, Ricardo Oyarzúa, Ricardo Ruiz-Baier, and Giordano Tierra. “Error analysis of an augmented mixed method for the Navier-Stokes problem with mixed boundary conditions”. In: *IMA J. Numer. Anal.* 38.3 (2018), pp. 1452–1484. ISSN: 0272-4979. DOI: 10.1093/imanum/drx039. URL: https://doi.org/10.1093/imanum/drx039.

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition of [MR0394451]. Dover Publications, Inc., Mineola, NY, 2016, pp. xvi+510. ISBN: 978-0-486-80699-0; 0-486-80699-5.

[Cat61] Lamberto Cattabriga. “Su un problema al contorno relativo al sistema di equazioni di Stokes”. In: *Rend. Sem. Mat. Univ. Padova* 31 (1961), pp. 308–340. ISSN: 0041-8994. URL: http://www.numdam.org/item?id=RSMUP_1961__31__308_0.

[CC88] J. Cahouet and J.-P. Chabard. “Some fast 3D finite element solvers for the generalized Stokes problem”. In: *Internat. J. Numer. Methods Fluids* 8.8 (1988), pp. 869–895. ISSN: 0271-2091. DOI: 10.1002/fld.1650080802. URL: https://doi.org/10.1002/fld.1650080802.

[COQ09] Gui-Qiang Chen, Dan Osborne, and Zhongmin Qian. “The Navier-Stokes equations with the kinematic and vorticity boundary conditions on non-flat boundaries”. In: *Acta Math. Sci. Ser. B (Engl. Ed.)* 29.4 (2009), pp. 919–948. ISSN: 0252-9602. DOI: 10.1016/S0252-9602(09)60078-3. URL: https://doi.org/10.1016/S0252-9602(09)60078-3.

[DAM04] M. Darwish, D. Asmar, and F. Moukalled. “A comparative assessment within a multigrid environment of segregated pressure-based algorithms for fluid flow at all speeds”. In: *Numer. Heat Transfer B* 45.1 (2004), pp. 49–74. DOI: 10.1080/1040779049025487. URL: https://doi.org/10.1080/1040779049025487.

[DDN20] Nguyen Anh Dao, Jesus Ildefonso Díaz, and Quan Ba Hong Nguyen. “Pointwise gradient estimates in multi-dimensional slow diffusion equations with a singular quenching term”. In: *Adv. Nonlinear Stud.* 20.2 (2020), pp. 477–502. ISSN: 1536-1365. DOI: 10.1515/ans-2020-2076. URL: https://doi.org/10.1515/ans-2020-2076.

[Dem15] I. Demirdžić. “On the Discretization of the Diffusion Term in Finite-Volume Continuum Mechanics”. In: *Numer. Heat Transfer, Part B* 68.1 (2015), pp. 1–10. DOI: 10.1080/10407790.2014.985992. URL: https://doi.org/10.1080/10407790.2014.985992.

[DM06] M. Darwish and F. Moukalled. “Convective Schemes for Capturing Interfaces of Free-Surface Flows on Unstructured Grids”. In: *Numer. Heat Transfer B* 49.1 (2006), pp. 19–42. DOI: 10.1080/10407790500272137. URL: https://doi.org/10.1080/10407790500272137.

[DM95] I. Demirdžić and S. Muzaferija. “Numerical method for coupled fluid flow, heat transfer and stress analysis using unstructured moving meshes with cells of arbitrary topology”. In: *Comput. Methods Appl. Mech. Engrg.* 125.1–4 (1995), pp. 235–255. DOI: 10.1016/0045-7825(95)00800-G. URL: https://doi.org/10.1016/0045-7825(95)00800-G.

[DM96a] M. Darwish and F. Moukalled. “A new approach for building bounded skew-upwind schemes”. In: *Comput. Methods Appl. Mech. Engrg.* 129.3 (1996), pp. 221–233. ISSN: 0045-7825. DOI: 10.1016/0045-7825(95)00881-0. URL: https://doi.org/10.1016/0045-7825(95)00881-0.

[DM96b] M. Darwish and F. Moukalled. “The normalized weighting factor method: a novel technique for accelerating the convergence of high-resolution convective schemes”. In: *Numer. Heat Transfer B* 30.2 (1996), pp. 217–237. DOI: 10.1080/10407799608915080. URL: https://doi.org/10.1080/10407799608915080.

[DMS01] M. Darwish, F. Moukalled, and B. Sekar. “A unified formulation of the segregated class of algorithms for multifluid flow at all speeds”. In: *Numer. Heat Transfer B* 40.2 (2001), pp. 99–137. DOI: 10.1080/104077901750475887. URL: https://doi.org/10.1080/104077901750475887.

[DW14] Fabian Denner and Berend G. M. van Wachem. “Compressive VOF method with skewness correction to capture sharp interfaces on arbitrary meshes”. In: *J. Comput. Phys.* 279 (2014), pp. 127–144. ISSN: 0021-9991. DOI: 10.1016/j.jcp.2014.09.002. URL: https://doi.org/10.1016/j.jcp.2014.09.002.

[DW95] Paul Deuring and Wolf von Wahl. “Strong solutions of the Navier-Stokes system in Lipschitz bounded domains”. In: *Math. Nachr.* 171 (1995), pp. 111–148. ISSN: 0025-584X. DOI: 10.1002/mana.19951710108. URL: https://doi.org/10.1002/mana.19951710108.

[DY17] Yasmina Daikh and Driss Yakoubi. “Spectral discretization of the Navier-Stokes problem with mixed boundary conditions”. In: *Appl. Numer. Math.* 118 (2017), pp. 33–49. ISSN: 0168-9274. DOI: 10.1016/j.apnum.2017.02.002. URL: https://doi.org/10.1016/j.apnum.2017.02.002.

[DZ01] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*. Vol. 4. Advances in Design and Control. Analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, pp. xviii+482. ISBN: 0-89871-489-3.

[DZ11] M. C. Delfour and Jean-Paul Zolésio. *Shapes and geometries*. Second. Vol. 22. Advances in Design and Control. Metrics, analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, pp. xxiv+622. ISBN: 978-0-898719-36-9. DOI: 10.1137/1.9780898719826. URL: https://doi.org/10.1137/1.9780898719826.

[EF01] Carsten Ebmeyer and Jens Frehse. “Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains”. In: *Math. Ann.* 319.2 (2001), pp. 349–381. ISSN: 0025-5831. DOI: 10.1007/PL00004438. URL: https://doi.org/10.1007/PL00004438.

[EG04] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*. Vol. 159. Applied Mathematical Sciences. Springer-Verlag, New York, 2004, pp. xiv+524. ISBN: 0-387-20574-8. DOI: 10.1007/978-1-4757-4355-5. URL: https://doi.org/10.1007/978-1-4757-4355-5.

[EG15] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299. ISBN: 978-1-4822-4238-6.

[EGH19] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. “Finite Volume Methods”. In: *Handbook of Numerical Analysis, P.G. Ciarlet, J.L. Lions eds, 1997*. Vol. 7. 2019, pp. 713–1020. URL: https://hal.archives-ouvertes.fr/hal-02100732v2.

[Eng+23] Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, and Vlad Vicol. “In memory of Andrew J. Majda”. In: *Notices Amer. Math. Soc.* 70.10 (2023), pp. 1648–1666. ISSN: 0002-9920. DOI: 10.1090/noti2810. URL: https://doi.org/10.1090/noti2810.

[EP99] M. Elghaoui and R. Pasquetti. “Mixed spectral-boundary element embedding algorithms for the Navier-Stokes equations in the vorticity-stream function formulation”. In: *J. Comput. Phys.* 153.1 (1999), pp. 82–100. ISSN: 0021-9991. DOI: 10.1006/jcph.1999.6265. URL: https://doi.org/10.1006/jcph.1999.6265.

[Eva10] Lawrence C. Evans. *Partial Differential Equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. URL: https://doi.org/10.1090/gsm/019.

[Fef06] Charles L. Fefferman. “Existence and smoothness of the Navier-Stokes equation”. In: *The millennium prize problems*. Clay Math. Inst., Cambridge, MA, 2006, pp. 57–67.

[FK64] Hiroshi Fujita and Tosio Kato. “On the Navier-Stokes initial value problem. I”. In: *Arch. Rational Mech. Anal.* 16 (1964), pp. 269–315. ISSN: 0003-9527. DOI: 10.1007/BF00276188. URL: https://doi.org/10.1007/BF00276188.

[FMT87] C. Foias, O. Manley, and R. Temam. “Attractors for the Bénard problem: existence and physical bounds on their fractal dimension”. In: *Nonlinear Anal.* 11.8 (1987), pp. 939–967. ISSN: 0362-546X. DOI: 10.1016/0362-546X(87)90061-7. URL: https://doi.org/10.1016/0362-546X(87)90061-7.

[FNP08] Mohamed Farhloul, Serge Nicaise, and Luc Paquet. “A refined mixed finite-element method for the stationary Navier-Stokes equations with mixed boundary conditions”. In: *IMA J. Numer. Anal.* 28.1 (2008), pp. 25–45. ISSN: 0272-4979. DOI: 10.1093/imanum/drm005. URL: https://doi.org/10.1093/imanum/drm005.

[Foi+01] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*. Vol. 83. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2001, pp. xiv+347. ISBN: 0-521-36032-3. DOI: 10.1017/CBO9780511546754. URL: https://doi.org/10.1017/CBO9780511546754.

[Fri64] Avner Friedman. *Partial Differential Equations of Parabolic Type*. Robert E. Krieger Publishing Company, Malabar, Florida, 1964, pp. xiv+347.

[Fur80] A. V. Fursikov. “Some control problems and results related to the unique solvability of the mixed boundary value problem for the Navier-Stokes and Euler three-dimensional systems”. In: *Dokl. Akad. Nauk SSSR* 252.5 (1980), pp. 1066–1070. ISSN: 0002-3264.

[Fur81] A. V. Fursikov. “Control problems and theorems concerning unique solvability of a mixed boundary value problem for three-dimensional Navier-Stokes and Euler equations”. In: *Mat. Sb. (N.S.)* 115(157).2 (1981), pp. 281–306, 320. ISSN: 0368-8666.

[Fur82] A. V. Fursikov. “Properties of solutions of control problems that are connected with the Navier-Stokes system”. In: *Dokl. Akad. Nauk SSSR* 262.1 (1982), pp. 44–48. ISSN: 0002-3264.

[Gag57] Emilio Gagliardo. “Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili”. In: *Rend. Sem. Mat. Univ. Padova* 27 (1957), pp. 284–305. ISSN: 0041-8994. URL: http://www.numdam.org/item?id=RSMUP_1957__27__284_0.

[Gal+17] T. Gallouët, R. Herbin, D. Maltese, and A. Novotny. “Convergence of the marker-and-cell scheme for the semi-stationary compressible Stokes problem”. In: *Math. Comput. Simulation* 137 (2017), pp. 325–349. ISSN: 0378-4754. DOI: 10.1016/j.matcom.2016.10.003. URL: https://doi.org/10.1016/j.matcom.2016.10.003.

[Gal+18] T. Gallouët, R. Herbin, J.-C. Latché, and K. Mallem. “Convergence of the marker-and-cell scheme for the incompressible Navier-Stokes equations on non-uniform grids”. In: *Found. Comput. Math.* 18.1 (2018), pp. 249–289. ISSN: 1615-3375. DOI: 10.1007/s10208-016-9338-4. URL: https://doi.org/10.1007/s10208-016-9338-4.

[Gal00] Giovanni P. Galdi. “An introduction to the Navier-Stokes initial-boundary value problem”. In: *Fundamental directions in mathematical fluid mechanics*. Adv. Math. Fluid Mech. Birkhäuser, Basel, 2000, pp. 1–70.

[Gal11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Second. Springer Monographs in Mathematics. Steady-state problems. Springer, New York, 2011, pp. xiv+1018. ISBN: 978-0-387-09619-3. DOI: 10.1007/978-0-387-09620-9. URL: https://doi.org/10.1007/978-0-387-09620-9.

[GK12] Gung-Min Gie and James P. Kelliher. "Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions". In: *J. Differential Equations* 253.6 (2012), pp. 1862–1892. ISSN: 0022-0396. DOI: 10.1016/j.jde.2012.06.008. URL: https://doi.org/10.1016/j.jde.2012.06.008.

[GL88] P. H. Gaskell and A. K. C. Lau. "Curvature-compensated convective transport: SMART, a new boundedness-preserving transport algorithm". In: *Internat. J. Numer. Methods Fluids* 8.6 (1988), pp. 617–641. ISSN: 0271-2091. DOI: 10.1002/fld.1650080602. URL: https://doi.org/10.1002/fld.1650080602.

[GR86] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*. Vol. 5. Springer Series in Computational Mathematics. Theory and algorithms. Springer-Verlag, Berlin, 1986, pp. x+374. ISBN: 3-540-15796-4. DOI: 10.1007/978-3-642-61623-5. URL: https://doi.org/10.1007/978-3-642-61623-5.

[Gri11] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. Vol. 69. Classics in Applied Mathematics. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, pp. xx+410. ISBN: 978-1-611972-02-3. DOI: 10.1137/1.9781611972030.ch1. URL: https://doi.org/10.1137/1.9781611972030.ch1.

[Gri80] Pierre Grisvard. *Boundary Value Problems in Non-Smooth Domains*. Univ. of Maryland, Dept. of Math. Lecture Notes no. 19. 1980.

[Gri85] P. Grisvard. *Elliptic problems in nonsmooth domains*. Vol. 24. Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston, MA, 1985, pp. xiv+410. ISBN: 0-273-08647-2.

[GST15] Telma Guerra, Adélia Sequeira, and Jorge Tiago. "Existence of optimal boundary control for the Navier-Stokes equations with mixed boundary conditions". In: *Port. Math.* 72.2-3 (2015), pp. 267–283. ISSN: 0032-5155. DOI: 10.4171/PM/1968. URL: https://doi.org/10.4171/PM/1968.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001, pp. xiv+517. ISBN: 3-540-41160-7.

[Gun89] Max D. Gunzburger. *Finite element methods for viscous incompressible flows*. Computer Science and Scientific Computing. A guide to theory, practice, and algorithms. Academic Press, Inc., Boston, MA, 1989, pp. xviii+269. ISBN: 0-12-307350-2.

[Har22] G. H. Hardy. *A Annotated Mathematician's apology*. Annotations & complementary by Alan J. Cain, Lisbon 2019. Cambridge University Press, Cambridge, 2022, p. 196. ISBN: 0-521-42706-1. DOI: 10.1017/CBO9781139644112. URL: https://doi.org/10.1017/CBO9781139644112.

[Har40] G. H. Hardy. *A Mathematician's Apology*. Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1940, pp. vii+93.

[Har83] Ami Harten. "High resolution schemes for hyperbolic conservation laws". In: *J. Comput. Phys.* 49.3 (1983), pp. 357–393. ISSN: 0021-9991. DOI: 10.1016/0021-9991(83)90136-5. URL: https://doi.org/10.1016/0021-9991(83)90136-5.

[Har92] G. H. Hardy. *A Mathematician's Apology*. Canto. With a foreword by C. P. Snow, Reprint of the 1967 edition. Cambridge University Press, Cambridge, 1992, p. 153. ISBN: 0-521-42706-1. DOI: 10.1017/CBO9781139644112. URL: https://doi.org/10.1017/CBO9781139644112.

[Har97] Ami Harten. "High resolution schemes for hyperbolic conservation laws [MR0701178 (84g:65115)]". In: vol. 135. 2. With an introduction by Peter Lax, Commemoration of the 30th anniversary {of J. Comput. Phys.}. 1997, pp. 259–278. DOI: 10.1006/jcph.1997.5713. URL: https://doi.org/10.1006/jcph.1997.5713.

[HL11] Qing Han and Fanghua Lin. *Elliptic partial differential equations*. Second. Vol. 1. Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011, pp. x+147. ISBN: 978-0-8218-5313-9.

[Hoa10] Luan Thach Hoang. "Incompressible fluids in thin domains with Navier friction boundary conditions (I)". In: *J. Math. Fluid Mech.* 12.3 (2010), pp. 435–472. ISSN: 1422-6928. DOI: 10.1007/s00021-009-0297-2. URL: https://doi.org/10.1007/s00021-009-0297-2.

[Hoa13] Luan Thach Hoang. "Incompressible fluids in thin domains with Navier friction boundary conditions (II)". In: *J. Math. Fluid Mech.* 15.2 (2013), pp. 361–395. ISSN: 1422-6928. DOI: 10.1007/s00021-012-0123-0. URL: https://doi.org/10.1007/s00021-012-0123-0.

[HP19] Yanren Hou and Shuaichao Pei. "On the weak solutions to steady Navier-Stokes equations with mixed boundary conditions". In: *Math. Z.* 291.1-2 (2019), pp. 47–54. ISSN: 0025-5874. DOI: 10.1007/s00209-018-2072-7. URL: https://doi.org/10.1007/s00209-018-2072-7.

[HRT96] John G. Heywood, Rolf Rannacher, and Stefan Turek. "Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations". In: *Internat. J. Numer. Methods Fluids* 22.5 (1996), pp. 325–352. ISSN: 0271-2091. DOI: 10.1002/(SICI)1097-0363(19960315)22:5<325::AID-FLD307>3.0.CO;2-Y. URL: https://doi.org/10.1002/(SICI)1097-0363(19960315)22:5%3C325::AID-FLD307%3E3.0.CO;2-Y.

[HS10] Luan T. Hoang and George R. Sell. "Navier-Stokes equations with Navier boundary conditions for an oceanic model". In: *J. Dynam. Differential Equations* 22.3 (2010), pp. 563–616. ISSN: 1040-7294. DOI: 10.1007/s10884-010-9189-7. URL: https://doi.org/10.1007/s10884-010-9189-7.

[HS18] David Holcman and Zeev Schuss. *Asymptotics of elliptic and parabolic PDEs—and their applications in statistical physics, computational neuroscience, and biophysics*. Vol. 199. Applied Mathematical Sciences. Springer, Cham, 2018, pp. xxiii+444. ISBN: 978-3-319-76894-6; 978-3-319-76895-3. DOI: 10.1007/978-3-319-76895-3. URL: https://doi.org/10.1007/978-3-319-76895-3.

[IC01] A. A. Illarionov and A. Yu. Chebotarev. "On the solvability of a mixed boundary value problem for stationary Navier-Stokes equations". In: *Differ. Uravn.* 37.5 (2001), pp. 689–695, 719. ISSN: 0374-0641. DOI: 10.1023/A:1019276817715. URL: https://doi.org/10.1023/A:1019276817715.

[IJ15] Kazufumi Ito and Bangti Jin. *Inverse problems*. Vol. 22. Series on Applied Mathematics. Tikhonov theory and algorithms. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015, pp. x+318. ISBN: 978-981-4596-19-0.

[IJL02] Traian Iliescu, Volker John, and William J. Layton. "Convergence of finite element approximations of large eddy motion". In: *Numer. Methods Partial Differential Equations* 18.6 (2002), pp. 689–710. ISSN: 0749-159X. DOI: 10.1002/num.10027. URL: https://doi.org/10.1002/num.10027.

[Ika00] Mitsuru Ikawa. *Hyperbolic partial differential equations and wave phenomena*. Vol. 189. Translations of Mathematical Monographs. Translated from the 1997 Japanese original by Bohdan I. Kurpita, Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xxii+190. ISBN: 0-8218-1021-9. DOI: 10.1090/mmono/189. URL: https://doi.org/10.1090/mmono/189.

[IR01] Dragoş Iftimie and Geneviève Raugel. "Some results on the Navier-Stokes equations in thin 3D domains". In: vol. 169. 2. Special issue in celebration of Jack K. Hale's 70th birthday, Part 4 (Atlanta, GA/Lisbon, 1998). 2001, pp. 281–331. DOI: 10.1006/jdeq.2000.3900. URL: https://doi.org/10.1006/jdeq.2000.3900.

[IRS07] Dragoş Iftimie, Geneviève Raugel, and George R. Sell. "Navier-Stokes equations in thin 3D domains with Navier boundary conditions". In: *Indiana Univ. Math. J.* 56.3 (2007), pp. 1083–1156. ISSN: 0022-2518. DOI: 10.1512/iumj.2007.56.2834. URL: https://doi.org/10.1512/iumj.2007.56.2834.

[IS11] Dragoş Iftimie and Franck Sueur. "Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions". In: *Arch. Ration. Mech. Anal.* 199.1 (2011), pp. 145–175. ISSN: 0003-9527. DOI: 10.1007/s00205-010-0320-z. URL: https://doi.org/10.1007/s00205-010-0320-z.

[Iss86] R. I. Issa. "Solution of the Implicit Discretized Fluid Flow Equations by Operator Splitting". In: *J. Comput. Phys.* 62.1 (1986), pp. 40–65. DOI: 10.1016/0021-9991(86)90099-9. URL: https://doi.org/10.1016/0021-9991(86)90099-9.

[Jas96] H. Jasak. "Error Analysis and Estimation for the Finite Volume Method with Applications for Fluid Flow". PhD Thesis, University of London. PhD thesis. 1996, p. 396.

[JG00] H. Jasak and A. D. Gosman. "Automatic Resolution Control for the Finite Volume Method, Part 1". In: *Numer. Heat Transfer B* 38.3 (2000), pp. 237–256. DOI: 10.1080/10407790050192753. URL: https://doi.org/10.1080/10407790050192753.

[JJA86] D. S. Jang, R. Jetli, and S. Acharya. "Comparison of the PISO, SIMPLER, and SIMPLEC algorithms for the treatment of the pressure-velocity coupling in steady flow problems". In: *Numer. Heat Transfer* 10.3 (1986), pp. 209–228. DOI: 10.1080/10407788608913517. URL: https://doi.org/10.1080/10407788608913517.

[JLS04] V. John, W. Layton, and N. Sahin. "Derivation and analysis of near wall models for channel and recirculating flows". In: *Comput. Math. Appl.* 48.7-8 (2004), pp. 1135–1151. ISSN: 0898-1221. DOI: 10.1016/j.camwa.2004.10.011. URL: https://doi.org/10.1016/j.camwa.2004.10.011.

[JM00] Willi Jäger and Andro Mikelić. "On the interface boundary condition of Beavers, Joseph, and Saffman". In: *SIAM J. Appl. Math.* 60.4 (2000), pp. 1111–1127. ISSN: 0036-1399. DOI: 10.1137/S003613999833678X. URL: https://doi.org/10.1137/S003613999833678X.

[JM01] Willi Jäger and Andro Mikelić. "On the roughness-induced effective boundary conditions for an incompressible viscous flow". In: *J. Differential Equations* 170.1 (2001), pp. 96–122. ISSN: 0022-0396. DOI: 10.1006/jdeq.2000.3814. URL: https://doi.org/10.1006/jdeq.2000.3814.

[Joh02] Volker John. "Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equations—numerical tests and aspects of the implementation". In: *J. Comput. Appl. Math.* 147.2 (2002), pp. 287–300. ISSN: 0377-0427. DOI: 10.1016/S0377-0427(02)00437-5. URL: https://doi.org/10.1016/S0377-0427(02)00437-5.

[Joh16] Volker John. *Finite element methods for incompressible flow problems*. Vol. 51. Springer Series in Computational Mathematics. Springer, Cham, 2016, pp. xiii+812. ISBN: 978-3-319-45749-9; 978-3-319-45750-5. DOI: 10.1007/978-3-319-45750-5. URL: https://doi.org/10.1007/978-3-319-45750-5.

[Jud67] V. I. Judovič. "An example of the loss of stability and the generation of a secondary flow of a fluid in a closed container". In: *Mat. Sb. (N.S.)* 74(116) (1967), pp. 565–579. ISSN: 0368-8666.

[KA03] Peter Knabner and Lutz Angermann. *Numerical methods for elliptic and parabolic partial differential equations*. Vol. 44. Texts in Applied Mathematics. Springer-Verlag, New York, 2003, pp. xvi+424. ISBN: 0-387-95449-X.

[Kaw92] Bernhard Kawohl. "Remarks on quenching, blow up and dead cores". In: *Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989)*. Vol. 7. Progr. Nonlinear Differential Equations Appl. Birkhäuser Boston, Boston, MA, 1992, pp. 275–286.

[Kaw96] Bernd Kawohl. "Remarks on quenching". In: *Doc. Math.* 1 (1996), No. 09, 199–208. ISSN: 1431-0635.

[KC15] Tujin Kim and Daomin Cao. "Some properties on the surfaces of vector fields and its application to the Stokes and Navier-Stokes problems with mixed boundary conditions". In: *Nonlinear Anal.* 113 (2015), pp. 94–114. ISSN: 0362-546X. DOI: 10.1016/j.na.2014.09.017. URL: https://doi.org/10.1016/j.na.2014.09.017.

[KC16] Tujin Kim and Daomin Cao. "The steady Navier-Stokes and Stokes systems with mixed boundary conditions including one-sided leaks and pressure". In: *Methods Appl. Anal.* 23.4 (2016), pp. 329–363. ISSN: 1073-2772. DOI: 10.4310/MAA.2016.v23.n4.a3. URL: https://doi.org/10.4310/MAA.2016.v23.n4.a3.

[KC17] Tujin Kim and Daomin Cao. "Non-stationary Navier-Stokes equations with mixed boundary conditions". In: *J. Math. Sci. Univ. Tokyo* 24.2 (2017), pp. 159–194. ISSN: 1340-5705.

[Kel06] James P. Kelliher. "Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane". In: *SIAM J. Math. Anal.* 38.1 (2006), pp. 210–232. ISSN: 0036-1410. DOI: 10.1137/040612336. URL: https://doi.org/10.1137/040612336.

[KH18] Tujin Kim and Feimin Huang. "The non-steady Navier-Stokes systems with mixed boundary conditions including friction conditions". In: *Methods Appl. Anal.* 25.1 (2018), pp. 13–50. ISSN: 1073-2772. DOI: 10.4310/MAA.2018.v25.n1.a2. URL: https://doi.org/10.4310/MAA.2018.v25.n1.a2.

[Kim16] Tujin Kim. "Erratum to "Some properties on the surfaces of vector fields and its application to the Stokes and Navier-Stokes problems with mixed boundary conditions" [Nonlinear Anal. 113 (2015) 94–114] [MR3281848]". In: *Nonlinear Anal.* 135 (2016), pp. 249–250. ISSN: 0362-546X. DOI: 10.1016/j.na.2016.01.003. URL: https://doi.org/10.1016/j.na.2016.01.003.

[Kir21] Andreas Kirsch. *An introduction to the mathematical theory of inverse problems.* Third. Vol. 120. Applied Mathematical Sciences. Springer, Cham, 2021, p. 400. ISBN: 978-3-030-63343-1; 978-3-030-63342-4. DOI: 10.1007/978-3-030-63343-1. URL: https://doi.org/10.1007/978-3-030-63343-1.

[KK92] Bernhard Kawohl and Robert Kersner. "On degenerate diffusion with very strong absorption". In: *Math. Methods Appl. Sci.* 15.7 (1992), pp. 469–477. ISSN: 0170-4214. DOI: 10.1002/mma.1670150703. URL: https://doi.org/10.1002/mma.1670150703.

[KKM19] K. Kaladhar, E. Komuraiah, and K. Madhusudhan Reddy. "Soret and Dufour effects on chemically reacting mixed convection flow in an annulus with Navier slip and convective boundary conditions". In: *Appl. Math. Nonlinear Sci.* 4.2 (2019), pp. 475–488. DOI: 10.2478/AMNS.2019.2.00045. URL: https://doi.org/10.2478/AMNS.2019.2.00045.

[KN01] S. Kračmar and J. Neustupa. "A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions". In: *Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000).* Vol. 47. 6. 2001, pp. 4169–4180. DOI: 10.1016/S0362-546X(01)00534-X. URL: https://doi.org/10.1016/S0362-546X(01)00534-X.

[KPR15] Mikhail V. Korobkov, Konstantin Pileckas, and Remigio Russo. "Solution of Leray's problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains". In: *Ann. of Math. (2)* 181.2 (2015), pp. 769–807. ISSN: 0003-486X. DOI: 10.4007/annals.2015.181.2.7. URL: https://doi.org/10.4007/annals.2015.181.2.7.

[Kry08] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces.* Vol. 96. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xviii+357. ISBN: 978-0-8218-4684-1. DOI: 10.1090/gsm/096. URL: https://doi.org/10.1090/gsm/096.

[KS98] Petr Kučera and Zdeněk Skalák. "Local solutions to the Navier-Stokes equations with mixed boundary conditions". In: *Acta Appl. Math.* 54.3 (1998), pp. 275–288. ISSN: 0167-8019. DOI: 10.1023/A:1006185601807. URL: https://doi.org/10.1023/A:1006185601807.

[Kuč09] Petr Kučera. "Basic properties of solution of the non-steady Navier-Stokes equations with mixed boundary conditions in a bounded domain". In: *Ann. Univ. Ferrara Sez. VII Sci. Mat.* 55.2 (2009), pp. 289–308. ISSN: 0430-3202. DOI: 10.1007/s11565-009-0082-4. URL: https://doi.org/10.1007/s11565-009-0082-4.

[Kuč10] Petr Kučera. "The time-periodic solutions of the Navier-Stokes equations with mixed boundary conditions". In: *Discrete Contin. Dyn. Syst. Ser. S* 3.2 (2010), pp. 325–337. ISSN: 1937-1632. DOI: 10.3934/dcdss.2010.3.325. URL: https://doi.org/10.3934/dcdss.2010.3.325.

[Kuč98a] P. Kučera. "Solutions of the stationary Navier-Stokes equations with mixed boundary conditions in bounded domain". In: *Analysis, numerics and applications of differential and integral equations (Stuttgart, 1996).* Vol. 379. Pitman Res. Notes Math. Ser. Longman, Harlow, 1998, pp. 127–131.

[Kuč98b] Petr Kučera. "A structure of the set of critical points to the Navier-Stokes equations with mixed boundary conditions". In: *Navier-Stokes equations: theory and numerical methods (Varenna, 1997).* Vol. 388. Pitman Res. Notes Math. Ser. Longman, Harlow, 1998, pp. 201–205.

[Küh15] Wolfgang Kühnel. *Differential geometry.* Vol. 77. Student Mathematical Library. Curves—surfaces—manifolds, Third edition [of MR1882174], Translated from the 2013 German edition by Bruce Hunt, with corrections and additions by the author. American Mathematical Society, Providence, RI, 2015, pp. xii+402. ISBN: 978-1-4704-2320-9. DOI: 10.1090/stml/077. URL: https://doi.org/10.1090/stml/077.

[LA08] Kai Tai Li and Rong An. "On the rotating Navier-Stokes equations with mixed boundary conditions". In: *Acta Math. Sin. (Engl. Ser.)* 24.4 (2008), pp. 577–598. ISSN: 1439-8516. DOI: 10.1007/s10114-007-1032-7. URL: https://doi.org/10.1007/s10114-007-1032-7.

[Lad69] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969, pp. xviii+224.

[Lan13] David Lannes. *The water waves problem*. Vol. 188. Mathematical Surveys and Monographs. Mathematical analysis and asymptotics. American Mathematical Society, Providence, RI, 2013, pp. xx+321. ISBN: 978-0-8218-9470-5. DOI: 10.1090/surv/188. URL: https://doi.org/10.1090/surv/188.

[Lax06] Peter D. Lax. *Hyperbolic partial differential equations*. Vol. 14. Courant Lecture Notes in Mathematics. With an appendix by Cathleen S. Morawetz. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2006, pp. viii+217. ISBN: 978-0-8218-3576-0; 0-8218-3576-9. DOI: 10.1090/cln/014. URL: https://doi.org/10.1090/cln/014.

[Lax87] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1987, pp. v+48.

[Lem02] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. Vol. 431. Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2002, pp. xiv+395. ISBN: 1-58488-220-4. DOI: 10.1201/9781420035674. URL: https://doi.org/10.1201/9781420035674.

[Lem16] Pierre Gilles Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016, pp. xxii+718. ISBN: 978-1-4665-6621-7. DOI: 10.1201/b19556. URL: https://doi.org/10.1201/b19556.

[Leo79] B. P. Leonard. "A stable and accurate convective modelling procedure based on quadratic upstream interpolation". In: *Comput. Methods Appl. Mech. Engrg.* 19.1 (1979), pp. 59–98. DOI: 10.1016/0045-7825(79)90034-3. URL: https://doi.org/10.1016/0045-7825(79)90034-3.

[Leo87] B. P. Leonard. *SHARP Simulation of Discontinuities in Highly Convective Steady Flow*. NASA Technical Memorandum 100240, ICOMP-87-9. 1987.

[Leo88] B. P. Leonard. *Universal Limiter for Transient Interpolation Modeling of the Advective Transport Equations: The ULTIMATE Conservative Difference Scheme*. NASA Technical Memorandum 100916, ICOMP-88-11. 1988.

[Ler33] Jean Leray. *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*. NUMDAM, [place of publication not identified], 1933, p. 82. URL: http://www.numdam.org/item?id=THESE_1933__142__1_0.

[Ler34a] Jean Leray. "Essai sur les mouvements plans d'un liquide visqueaux que limitent des parois". In: *J. Math. Pures Appl.* 13 (1934), pp. 331–418.

[Ler34b] Jean Leray. "Sur le mouvement d'un liquide visqueux emplissant l'espace". In: *Acta Math.* 63.1 (1934), pp. 193–248. ISSN: 0001-5962. DOI: 10.1007/BF02547354. URL: https://doi.org/10.1007/BF02547354.

[LeV02] Randall J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002, pp. xx+558. ISBN: 0-521-81087-6; 0-521-00924-3. DOI: 10.1017/CBO9780511791253. URL: https://doi.org/10.1017/CBO9780511791253.

[LeV07] Randall J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations*. Steady-state and time-dependent problems. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007, pp. xvi+341. ISBN: 978-0-898716-29-0. DOI: 10.1137/1.9780898717839. URL: https://doi.org/10.1137/1.9780898717839.

[Lia01] Anastasios Liakos. "Discretization of the Navier-Stokes equations with slip boundary condition". In: *Numer. Methods Partial Differential Equations* 17.1 (2001), pp. 26–42. ISSN: 0749-159X. DOI: 10.1002/1098-2426(200101)17:1<26::AID-NUM2>3.0.CO;2-D. URL: https://doi.org/10.1002/1098-2426(200101)17:1%3C26::AID-NUM2%3E3.0.CO;2-D.

[Lia04] A. Liakos. "Discretization of the Navier-Stokes equations with slip boundary condition. II". In: *Comput. Math. Appl.* 48.7-8 (2004), pp. 1153–1166. ISSN: 0898-1221. DOI: 10.1016/j.camwa.2004.10.012. URL: https://doi.org/10.1016/j.camwa.2004.10.012.

[LL01] Elliott H. Lieb and Michael Loss. *Analysis*. Second. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xxii+346. ISBN: 0-8218-2783-9. DOI: 10.1090/gsm/014. URL: https://doi.org/10.1090/gsm/014.

[LP85] I. I. Lyashko and N. Z. Prokhur. "Construction of stable schemes for solving a mixed boundary value problem for evolution generalization of three-dimensional Navier-Stokes equations". In: *Dokl. Akad. Nauk Ukrain. SSR Ser. A* 12 (1985), pp. 14–16, 86. ISSN: 0201-8446.

[LSU68] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and Quasi-linear Equations of Parabolic Type*. Translations of Mathematical Monographs, Volume 23. American Mathematical Society, 1968, pp. xiii+648.

[LSŻ18] Irena Lasiecka, Katarzyna Szulc, and Antoni Żochowski. “Boundary control of small solutions to fluid-structure interactions arising in coupling of elasticity with Navier-Stokes equation under mixed boundary conditions”. In: *Nonlinear Anal. Real World Appl.* 44 (2018), pp. 54–85. ISSN: 1468-1218. DOI: 10.1016/j.nonrwa.2018.04.004. URL: https://doi.org/10.1016/j.nonrwa.2018.04.004.

[LT00] Irena Lasiecka and Roberto Triggiani. *Control theory for partial differential equations: continuous and approximation theories. II.* Vol. 75. Encyclopedia of Mathematics and its Applications. Abstract hyperbolic-like systems over a finite time horizon. Cambridge University Press, Cambridge, 2000, i–xxii and 645–1067 and I1–I4. ISBN: 0-521-58401-9. DOI: 10.1017/CBO9780511574801.002. URL: https://doi.org/10.1017/CBO9780511574801.002.

[LU68] Olga A. Ladyzhenskaya and Nina N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968, pp. xviii+495.

[Lun95] Alessandra Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Modern Birkhäuser Classics. [2013 reprint of the 1995 original] [MR1329547]. Birkhäuser/Springer Basel AG, Basel, 1995, pp. xviii+424. ISBN: 978-3-0348-0556-8; 978-3-0348-0557-5.

[LY08] Dongjie Liu and Dehao Yu. “The coupling method of natural boundary element and mixed finite element for stationary Navier-Stokes equation in unbounded domains”. In: *CMES Comput. Model. Eng. Sci.* 37.3 (2008), pp. 305–329. ISSN: 1526-1492.

[MD00] F. Moukalled and M. Darwish. “A unified formulation of the segregated class of algorithms for fluid flow at all speeds”. In: *Numer. Heat Transfer B* 37.1 (2000), pp. 103–139. DOI: 10.1080/104077900275576. URL: https://doi.org/10.1080/104077900275576.

[MD12] F. Moukalled and M. Darwish. “Transient Schemes for Capturing Interfaces of Free-Surface Flows”. In: *Numer. Heat Transfer B* 61.3 (2012), pp. 171–203. DOI: 10.1080/10407790.2012.666145. URL: https://doi.org/10.1080/10407790.2012.666145.

[MG97] S Muzaferija and A. D. Gosman. “Finite Volume CFD Procedure and Adaptive Error Control Strategy for Grids of Arbitrary Topology”. In: *J. Comput. Phys.* 138.2 (1997), pp. 766–787. DOI: 10.1006/jcph.1997.5853. URL: https://doi.org/10.1006/jcph.1997.5853.

[Mih68] V. P. Mihaĭlov. “The mixed boundary value problem for the Navier-Stokes system of equations”. In: *Sibirsk. Mat. Ž.* 9 (1968), pp. 1221–1237. ISSN: 0037-4474.

[MM09a] Marius Mitrea and Sylvie Monniaux. “On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds”. In: *Trans. Amer. Math. Soc.* 361.6 (2009), pp. 3125–3157. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-08-04827-7. URL: https://doi.org/10.1090/S0002-9947-08-04827-7.

[MM09b] Marius Mitrea and Sylvie Monniaux. “The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains”. In: *Differential Integral Equations* 22.3-4 (2009), pp. 339–356. ISSN: 0893-4983.

[MM97] S. R. Mathur and J. Y. Murthy. “A Pressure-Based Method for Unstructured Meshes”. In: *Numer. Heat Transfer B* 31.2 (1997), pp. 195–215. DOI: 10.1080/10407799708915105. URL: https://doi.org/10.1080/10407799708915105.

[MMD16] F. Moukalled, L. Mangani, and M. Darwish. *The Finite Volume Method in Computational Fluid Dynamics*. Vol. 113. Fluid Mechanics and its Applications. An advanced introduction with OpenFOAM® and Matlab®. Springer, Cham, 2016, pp. xxiii+791. ISBN: 978-3-319-16873-9; 978-3-319-16874-6. DOI: 10.1007/978-3-319-16874-6. URL: https://doi.org/10.1007/978-3-319-16874-6.

[Mon06] Sylvie Monniaux. “Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme”. In: *Math. Res. Lett.* 13.2-3 (2006), pp. 455–461. ISSN: 1073-2780. DOI: 10.4310/MRL.2006.v13.n3.a9. URL: https://doi.org/10.4310/MRL.2006.v13.n3.a9.

[MR07] V. Maz'ya and J. Rossmann. “L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains”. In: *Math. Nachr.* 280.7 (2007), pp. 751–793. ISSN: 0025-584X. DOI: 10.1002/mana.200610513. URL: https://doi.org/10.1002/mana.200610513.

[MR09] V. Maz'ya and J. Rossmann. “Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains”. In: *Arch. Ration. Mech. Anal.* 194.2 (2009), pp. 669–712. ISSN: 0003-9527. DOI: 10.1007/s00205-008-0171-z. URL: https://doi.org/10.1007/s00205-008-0171-z.

[MR10] Vladimir Maz'ya and Jürgen Rossmann. *Elliptic equations in polyhedral domains*. Vol. 162. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010, pp. viii+608. ISBN: 978-0-8218-4983-5. DOI: 10.1090/surv/162. URL: https://doi.org/10.1090/surv/162.

[Muk92a] F. Kh. Mukminov. “On the decay of the solution of the first mixed problem for a linearized system of Navier-Stokes equations in a domain with a noncompact boundary”. In: *Mat. Sb.* 183.10 (1992), pp. 123–144. ISSN: 0368-8666. DOI: 10.1070/SM1994v077n01ABEH003438. URL: https://doi.org/10.1070/SM1994v077n01ABEH003438.

[Muk92b] F. Kh. Mukminov. “On the rate of decay of the solution of a mixed problem for a system of Navier-Stokes equations in a domain with a noncompact boundary”. In: *Dokl. Akad. Nauk* 324.6 (1992), pp. 1149–1151. ISSN: 0869-5652.

[Muz94] S. Muzaferija. "Adaptive Finite Volume Method for Flow Predictions Using Unstructured Meshes and Multigrid Approach". PhD Thesis, University of London. PhD thesis. 1994, pp. x+237.

[Neč12] Jindřich Nečas. *Direct methods in the theory of elliptic equations.* Springer Monographs in Mathematics. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader. Springer, Heidelberg, 2012, pp. xvi+372. ISBN: 978-3-642-10454-1; 978-3-642-10455-8. DOI: 10.1007/978-3-642-10455-8. URL: https://doi.org/10.1007/978-3-642-10455-8.

[Neč67] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques.* Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967, p. 351.

[Nic92] R. A. Nicolaides. "Analysis and convergence of the MAC scheme. I. The linear problem". In: *SIAM J. Numer. Anal.* 29.6 (1992), pp. 1579–1591. ISSN: 0036-1429. DOI: 10.1137/0729091. URL: https://doi.org/10.1137/0729091.

[NP20] Quoc-Hung Nguyen and Nguyen Cong Phuc. "Pointwise gradient estimates for a class of singular quasilinear equations with measure data". In: *J. Funct. Anal.* 278.5 (2020), pp. 108391, 35. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2019.108391. URL: https://doi.org/10.1016/j.jfa.2019.108391.

[NPR07] S. Nicaise, L. Paquet, and Rafilipojaona. "A refined mixed finite element method for stationary Navier-Stokes equations with mixed boundary conditions using Lagrange multipliers". In: *Comput. Methods Appl. Math.* 7.1 (2007), pp. 83–100. ISSN: 1609-4840. DOI: 10.2478/cmam-2007-0005. URL: https://doi.org/10.2478/cmam-2007-0005.

[NR15] Phuong Anh Nguyen and Jean-Pierre Raymond. "Boundary stabilization of the Navier-Stokes equations in the case of mixed boundary conditions". In: *SIAM J. Control Optim.* 53.5 (2015), pp. 3006–3039. ISSN: 0363-0129. DOI: 10.1137/13091364X. URL: https://doi.org/10.1137/13091364X.

[NW96] R. A. Nicolaides and X. Wu. "Analysis and convergence of the MAC scheme. II. Navier-Stokes equations". In: *Math. Comp.* 65.213 (1996), pp. 29–44. ISSN: 0025-5718. DOI: 10.1090/S0025-5718-96-00665-5. URL: https://doi.org/10.1090/S0025-5718-96-00665-5.

[OA02] Carl Ollivier-Gooch and Michael Van Altena. "A High-Order-Accurate Unstructured Mesh Finite-Volume Scheme for the Advection–Diffusion Equation". In: *J. Comput. Phys.* 181.2 (2002), pp. 729–752. DOI: 10.1006/jcph.2002.7159. URL: https://doi.org/10.1006/jcph.2002.7159.

[Ovs78] V. G. Ovsienko. "A mixed boundary value problem for nonstationary Navier-Stokes equations on the exterior of a circular cylinder". In: *Vychisl. Prikl. Mat. (Kiev)* 35 (1978), pp. 127–130, 167. ISSN: 0321-4117.

[Pap+15] E. M. Papoutsis-Kiachagias, N. Magoulas, J. Mueller, C. Othmer, and K. C. Giannakoglou. "Noise reduction in car aerodynamics using a surrogate objective function and the continuous adjoint method with wall functions". In: *Comput. & Fluids* 122 (2015), pp. 223–232. ISSN: 0045-7930. DOI: 10.1016/j.compfluid.2015.09.002. URL: https://doi.org/10.1016/j.compfluid.2015.09.002.

[PBH04] Sébastien Perron, Sylvain Boivin, and Jean-Marc Hérard. "A finite volume method to solve the 3D Navier-Stokes equations on unstructured collocated meshes". In: *Comput. & Fluids* 33.10 (2004), pp. 1305–1333. ISSN: 0045-7930. DOI: 10.1016/j.compfluid.2003.10.006. URL: https://doi.org/10.1016/j.compfluid.2003.10.006.

[Phi87] Daniel Phillips. "Existence of solutions of quenching problems". In: *Appl. Anal.* 24.4 (1987), pp. 253–264. ISSN: 0003-6811. DOI: 10.1080/00036818708839668. URL: https://doi.org/10.1080/00036818708839668.

[PR17] Duy Phan and Sérgio S. Rodrigues. "Gevrey regularity for Navier-Stokes equations under Lions boundary conditions". In: *J. Funct. Anal.* 272.7 (2017), pp. 2865–2898. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2017.01.014. URL: https://doi.org/10.1016/j.jfa.2017.01.014.

[PS08] P. I. Plotnikov and J. Sokolowski. "Stationary boundary value problems for compressible Navier-Stokes equations". In: *Handbook of differential equations: stationary partial differential equations. Vol. VI.* Handb. Differ. Equ. Elsevier/North-Holland, Amsterdam, 2008, pp. 313–410. DOI: 10.1016/S1874-5733(08)80022-8. URL: https://doi.org/10.1016/S1874-5733(08)80022-8.

[PS12] Pavel Plotnikov and Jan Sokolowski. *Compressible Navier-Stokes Equations: Theory and Shape Optimization.* Monografie Matematyczne, Instytut Matematyczny PAN, Vol. 73, New Series. Birkhäuser, 2012, pp. xvi+457. ISBN: 978-3-0348-0765-4. DOI: 10.1007/978-3-0348-0367-0.

[Rau12] Jeffrey Rauch. *Hyperbolic partial differential equations and geometric optics.* Vol. 133. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xx+363. ISBN: 978-0-8218-7291-8. DOI: 10.1090/gsm/133. URL: https://doi.org/10.1090/gsm/133.

[RC83] C. M. Rhie and W. L. Chow. "Numerical Study of the Turbulent Flow Past an Airfoil with Trailing Edge Separation". In: *AIAA J.* (1983). DOI: 10.2514/3.8284. URL: https://doi.org/10.2514/3.8284.

[Ros09] Jürgen Rossmann. "Mixed boundary value problems for Stokes and Navier-Stokes systems in polyhedral domains". In: *Analysis, partial differential equations and applications.* Vol. 193. Oper. Theory Adv. Appl. Birkhäuser Verlag, Basel, 2009, pp. 269–280. DOI: 10.1007/978-3-7643-9898-9_19. URL: https://doi.org/10.1007/978-3-7643-9898-9_19.

[RR93] C. le Roux and B. D. Reddy. "The steady Navier-Stokes equations with mixed boundary conditions: application to free boundary flows". In: *Nonlinear Anal.* 20.9 (1993), pp. 1043–1068. ISSN: 0362-546X. DOI: 10.1016/0362-546X(93)90094-9. URL: https://doi.org/10.1016/0362-546X(93)90094-9.

[Rud73] Walter Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, pp. xiii+397.

[Rud76] Walter Rudin. *Principles of mathematical analysis*. Third. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, pp. x+342.

[Rud87] Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

[Rud91] Walter Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424. ISBN: 0-07-054236-8.

[Ser15] Gregory Seregin. *Lecture notes on regularity theory for the Navier-Stokes equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015, pp. x+258. ISBN: 978-981-4623-40-7.

[Sim22] Jacques Simon. *Distributions*. Mathematics and Statistics Series. Analysis for PDEs Set, 3. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2022, pp. xviii+377. ISBN: [9781786305251].

[Sim87] Jacques Simon. "Compact sets in the space $L^p(0,T;B)$". In: *Ann. Mat. Pura Appl. (4)* 146 (1987), pp. 65–96. ISSN: 0003-4622. DOI: 10.1007/BF01762360. URL: https://doi.org/10.1007/BF01762360.

[SL08] Jian Su and Kai Tai Li. "A finite element method for the three-dimensional stationary rotating Navier-Stokes equations in primitive variables with mixed boundary conditions". In: *Math. Numer. Sin.* 30.3 (2008), pp. 235–246. ISSN: 0254-7791.

[SNN19] Abdou Sène, Timack Ngom, and Evrad M. D. Ngom. "Global stabilization of the Navier-Stokes equations around an unstable steady state with mixed boundary kinetic energy controller". In: *J. Dyn. Control Syst.* 25.2 (2019), pp. 197–218. ISSN: 1079-2724. DOI: 10.1007/s10883-018-9406-y. URL: https://doi.org/10.1007/s10883-018-9406-y.

[Soh01a] Hermann Sohr. *The Navier-Stokes equations*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. An elementary functional analytic approach. Birkhäuser Verlag, Basel, 2001, pp. x+367. ISBN: 3-7643-6545-5. DOI: 10.1007/978-3-0348-8255-2. URL: https://doi.org/10.1007/978-3-0348-8255-2.

[Soh01b] Hermann Sohr. *The Navier-Stokes equations*. Modern Birkhäuser Classics. An elementary functional analytic approach, [2013 reprint of the 2001 original] [MR1928881]. Birkhäuser/Springer Basel AG, Basel, 2001, pp. x+367. ISBN: 978-3-0348-0550-6; 978-3-0348-0551-3.

[Swe84] Peter K. Sweby. "High resolution schemes using flux limiters for hyperbolic conservation laws". In: *SIAM J. Numer. Anal.* 21.5 (1984), pp. 995–1011. ISSN: 0036-1429. DOI: 10.1137/0721062. URL: https://doi.org/10.1137/0721062.

[Swe85] Peter K. Sweby. "High resolution TVD schemes using flux limiters". In: *Large-scale computations in fluid mechanics, Part 2 (La Jolla, Calif., 1983)*. Vol. 22. Lectures in Appl. Math. Amer. Math. Soc., Providence, RI, 1985, pp. 289–309.

[Tao13] Terence Tao. "Localisation and compactness properties of the Navier-Stokes global regularity problem". In: *Anal. PDE* 6.1 (2013), pp. 25–107. ISSN: 2157-5045. DOI: 10.2140/apde.2013.6.25. URL: https://doi.org/10.2140/apde.2013.6.25.

[Tao22a] Terence Tao. *Analysis I*. Vol. 37. Texts and Readings in Mathematics. Fourth edition [of 2195040]. Hindustan Book Agency, New Delhi, [2022] ©2022, pp. xvi+355. ISBN: 978-81-951961-9-7.

[Tao22b] Terence Tao. *Analysis II*. Vol. 38. Texts and Readings in Mathematics. Fourth edition [of 2195041]. Springer, Singapore; Hindustan Book Agency, New Delhi, [2022] ©2022, pp. xvii+195. ISBN: 978-9-81197-284-3. DOI: 10.1007/978-981-19-7284-3. URL: https://doi.org/10.1007/978-981-19-7284-3.

[Tar06] Luc Tartar. *An introduction to Navier-Stokes equation and oceanography*. Vol. 1. Lecture Notes of the Unione Matematica Italiana. Springer-Verlag, Berlin; UMI, Bologna, 2006, pp. xxviii+245. ISBN: 978-3-540-35743-8; 3-540-35743-2. DOI: 10.1007/3-540-36545-1. URL: https://doi.org/10.1007/3-540-36545-1.

[Tar08] Luc Tartar. *From hyperbolic systems to kinetic theory*. Vol. 6. Lecture Notes of the Unione Matematica Italiana. A personalized quest. Springer-Verlag, Berlin; UMI, Bologna, 2008, pp. xxviii+279. ISBN: 978-3-540-77561-4. DOI: 10.1007/978-3-540-77562-1. URL: https://doi.org/10.1007/978-3-540-77562-1.

[Tem00] Roger Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. AMS Chelsea Publishing, 2000, p. 408. ISBN: 0-8218-2737-5.

[Tem77] Roger Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, pp. x+500. ISBN: 0-7204-2840-8.

[Tem83] Roger Temam. *Navier-Stokes equations and nonlinear functional analysis*. Vol. 41. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1983, pp. xii+122. ISBN: 0-89871-183-5.

[Tem95] Roger Temam. *Navier-Stokes equations and nonlinear functional analysis*. Second. Vol. 66. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995, pp. xiv+141. ISBN: 0-89871-340-4. DOI: 10.1137/1.9781611970050. URL: https://doi.org/10.1137/1.9781611970050.

[Tsa18] Tai-Peng Tsai. *Lectures on Navier-Stokes equations*. Vol. 192. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2018, pp. xii+224. ISBN: 978-1-4704-3096-2.

[TTV24] Đinh Ngọc Thanh, Bùi Lê Trọng Thanh, and Huỳnh Quang Vũ. *Bài Giảng Giải Tích Hàm*. HCMUS, 2024, p. 114.

[Váz07] Juan Luis Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. Mathematical theory. The Clarendon Press, Oxford University Press, Oxford, 2007, pp. xxii+624. ISBN: 978-0-19-856903-9; 0-19-856903-3.

[Váz20] Juan Luis Vázquez. "Remembering Louis Nirenberg and his mathematics". In: *Gac. R. Soc. Mat. Esp.* 23.2 (2020), pp. 243–261. ISSN: 1138-8927.

[Ver87] Rüdiger Verfürth. "Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition". In: *Numer. Math.* 50.6 (1987), pp. 697–721. ISSN: 0029-599X. DOI: 10.1007/BF01398380. URL: https://doi.org/10.1007/BF01398380.

[Ver91] R. Verfürth. "Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition. II". In: *Numer. Math.* 59.6 (1991), pp. 615–636. ISSN: 0029-599X. DOI: 10.1007/BF01385799. URL: https://doi.org/10.1007/BF01385799.

[VM07] H. K. Versteeg and W. Malalasekera. *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Second edition. Pearson, Prentice Hall, 2007, pp. xiv+503. ISBN: 978-0131274983.

[Wal15] Shawn W. Walker. *The shapes of things*. Vol. 28. Advances in Design and Control. A practical guide to differential geometry and the shape derivative. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015, pp. ix+154. ISBN: 978-1-611973-95-2. DOI: 10.1137/1.9781611973969.ch1. URL: https://doi.org/10.1137/1.9781611973969.ch1.

[Wei83] Michael I. Weinstein. "Nonlinear Schrödinger equations and sharp interpolation estimates". In: *Comm. Math. Phys.* 87.4 (1982/83), pp. 567–576. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103922134.

[YL93] Ruey-Hor Yen and Chen-Hua Liu. "Enhancement of the SIMPLE algorithm by an additional explicit corrector step". In: *Numer. Heat Transfer B* 24.1 (1993), pp. 127–141. DOI: 10.1080/10407799308955885. URL: https://doi.org/10.1080/10407799308955885.

[Yud63] V. I. Yudovich. "Non-stationary flow of an ideal incompressible liquid". In: *USSR Computational Mathematics and Mathematical Physics* 3 (1963), pp. 1407–1456. DOI: 10.1016/0041-5553(63)90247-7. URL: https://doi.org/10.1016/0041-5553(63)90247-7.