Finite Volume Method in 1D

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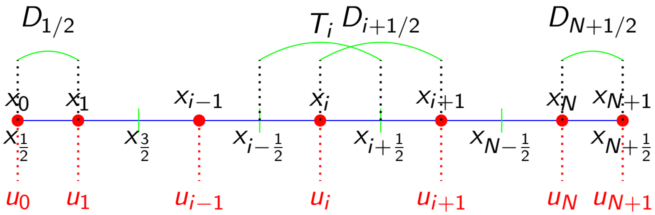
Introduction

The domain of the computation will be $\Omega =]0;1[$. Let the function $f \in L^2(\Omega)$, we will look for an appproximation of the following problem

$$\begin{cases} -u_{xx}(x) & = f(x) \text{ in } \Omega \\ u(0) & = 0, \\ u(1) & = 0. \end{cases} \tag{2.1}$$

by a cell-centered finite volume scheme



Let us choose $N+1$ points $\{x_{i+\frac{1}{2}}\}_{i\in\overline{0,N}}$ in $[0;1]$ such that

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = 1.$$

We set

$$T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],\ |T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \quad \forall i \in \overline{1,N}$$

$$x_0 = 0,\quad x_{N+1} = 1,\qquad x_i \in T_i\ \forall i \in \overline{1,N}$$

$$h = \max_{i\in\overline{1,N}}\{|T_i|\}$$

We call $(T_i)_{i\in\overline{1,N}}$ control volume and $(x_i)_{i\in\overline{0,N+1}}$ are control point.

Integrating the first equation in (2.1) over control volume T_i, there hold

$$\frac{1}{|T_i|}\int_{T_i} -u_{xx}dx = \frac{1}{|T_i|}\int_{T_i} f(x)dx \tag{2.2}$$

Applying the Green's formula, we obtain

$$\frac{-1}{|T_i|}\int_{T_i} -u_{xx}dx = \frac{-u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}})}{|T_i|}$$

and we put

$$f_i = \frac{1}{|T_i|}\int_{T_i} f(x)dx \quad \text{mean-value of } f \text{ over } T_i$$

Thus

$$\frac{-u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}})}{|T_i|} = f_i \tag{2.3}$$

♦ How to approximate the term $u_x(x_{i+\frac{1}{2}})$

Using the Taylor series expansion

$$
\begin{aligned}
u(x_{i+1}) =& u(x_{i+\frac{1}{2}}) + u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_{i+\frac{1}{2}}) \\
&+ \frac{u_{xx}(x_{i+\frac{1}{2}})}{2!}(x_{i+1} - x_{i+\frac{1}{2}})^2 + 0(h^3)
\end{aligned}
$$

$$
\begin{aligned}
u(x_i) =& u(x_{i+\frac{1}{2}}) + u_x(x_{i+\frac{1}{2}})(x_i - x_{i+\frac{1}{2}}) \\
&+ \frac{u_{xx}(x_{i+\frac{1}{2}})}{2!}(x_i - x_{i+\frac{1}{2}})^2 + 0(h^3)
\end{aligned}
$$

Thus

$$
\begin{aligned}
u(x_{i+1}) - u(x_i) = & (x_{i+1} - x_i)u_x(x_{i+\frac{1}{2}}) \\
&+ ((x_{i+1} - x_{i+\frac{1}{2}})^2 - (x_i - x_{i+\frac{1}{2}})^2)\frac{u_{xx}(x_{i+\frac{1}{2}})}{2!} + 0(h^3)
\end{aligned}
$$

Thus

$$u(x_{i+1}) - u(x_i) = (x_{i+1} - x_i)u_x(x_{i+\frac{1}{2}})$$
$$+ ((x_{i+1} - x_{i+\frac{1}{2}})^2 - (x_i - x_{i+\frac{1}{2}})^2)\frac{u_{xx}(x_{i+\frac{1}{2}})}{2!} + 0(h^3)$$

We have two cases:

Case 1: $x_{i+\frac{1}{2}}$ is the midpoint of segment $[x_i, x_{i+1}]$ then

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + 0(h^2)$$

Case 2: Otherwise,

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + 0(h)$$

From two cases, we get the approximation of the term $u_x(x_{i+\frac{1}{2}})$

$$u_x(x_{i+\frac{1}{2}}) = \frac{u_{i+1} - u_i}{|D_{i+1/2}|} \qquad \forall i \in \overline{0, N}$$

Substituding this approximation to the equation (2.3), we have

$$\frac{1}{|T_i|}\left[-\frac{u_{i+1} - u_i}{|D_{i+1/2}|} + \frac{u_i - u_{i-1}}{|D_{i-1/2}|}\right] = f_i \quad \forall i \in \overline{1, N} \tag{2.4}$$

Or

$$-\frac{u_{i-1}}{|D_{i-1/2}||T_i|} + \left[\frac{1}{|D_{i+1/2}||T_i|} + \frac{1}{|D_{i-1/2}||T_i|}\right] u_i - \frac{u_{i+1}}{|D_{i+1/2}||T_i|} = f_i \qquad \forall i \in \overline{1, N}$$

$$-\frac{u_{i-1}}{|D_{i-1/2}||T_i|} + \left[\frac{1}{|D_{i+1/2}||T_i|} + \frac{1}{|D_{i-1/2}||T_i|}\right] u_i - \frac{u_{i+1}}{|D_{i+1/2}||T_i|} = f_i \qquad \forall i \in \overline{1, N}$$

We set, for all $i \in \overline{1, N}$,

$$\alpha_i = \frac{-1}{|D_{i-1/2}||T_i|}$$

$$\beta_i = \frac{1}{|D_{i+1/2}||T_i|} + \frac{1}{|D_{i-1/2}||T_i|}$$

$$\gamma_i = \frac{-1}{|D_{i+1/2}||T_i|}$$

Thus, we get

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i \quad \forall i \in \overline{1, N}$$

Combining with the boundary conditions, we get the scheme for the cell-centered finite volume method

$$\begin{cases} \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i \quad \forall i \in \overline{1, N} \\ u_0 = u_{N+1} = 0 \end{cases}$$

Linear system for the scheme

$$\begin{cases} i = 1, \beta_1 u_1 + \gamma_1 u_2 & = f_1 \\ i = 2, \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_2 \\ i = 3, \qquad \alpha_3 u_2 + \beta_3 u_3 + \gamma_3 u_4 & = f_3 \\ \qquad \qquad \ldots & \\ i = N-1, \qquad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i = N, \qquad \alpha_N u_{N-1} \quad + \quad \beta_N u_N & = f_N \end{cases}$$

Matrix form $AU = F$, $\quad A \in \mathbb{R}^N \times \mathbb{R}^N, \quad U, F \in \mathbb{R}^N,$

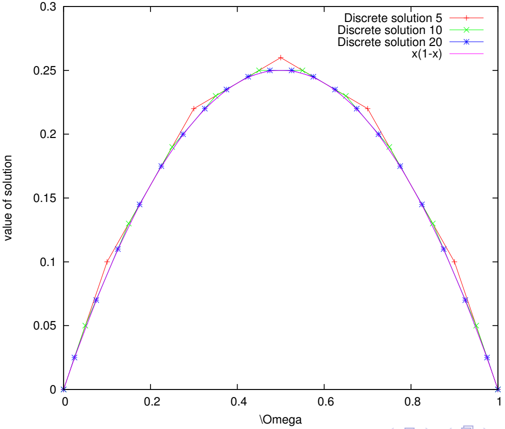
$$A = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\ 0 & 0 & 0 & 0 & \alpha_N & \beta_N \end{bmatrix}$$

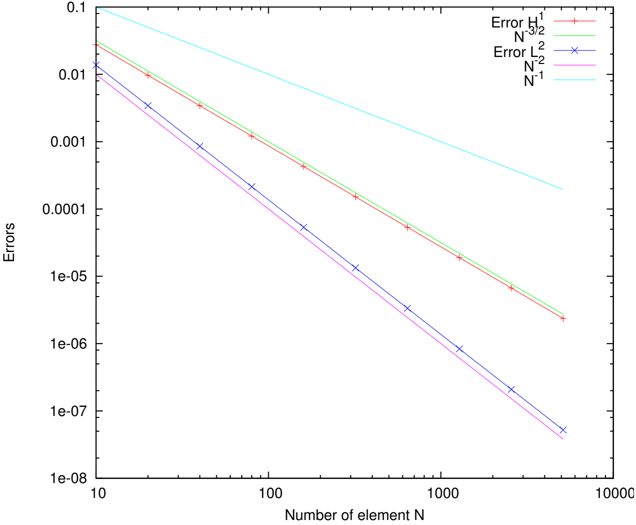
$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}$$

The matrix A remains tridiagonal and symmetric positive definite

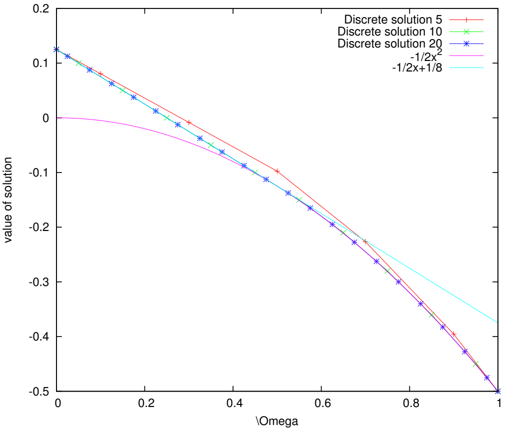
We set up with the following exact solution u and function f

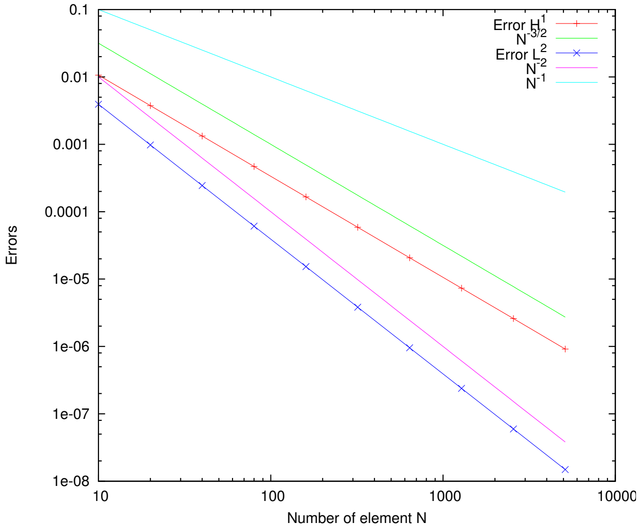
$$\begin{cases} u(x) & = x(1-x) \\ f(x) & = 2 \end{cases}$$




Error H1
N-3/2
Error L2
N-2
N-1
Errors
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
10
100
1000
10000
Number of element N

$$u(x) = \begin{cases} \frac{-1}{2}x^2 & \text{if } \frac{1}{2} \leq x \leq 1 \\ \frac{-1}{2}x + \frac{1}{8} & \text{if } 0 \leq x \leq \frac{1}{2} \end{cases}$$




Error H^1
$N^{-3/2}$
Error L^2
N^{-2}
N^{-1}
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
10
100
1000
10000
Errors
Number of element N

Existence and uniqueness of the solution

The discretized problem AU=F has a unique solution $U = (u_1, ..., u_N) \in \mathbb{R}^N$.

Proof

1. Existence:

It suffices to prove that A is invertible.

2. Uniqueness:

Let $U = (u_1, ..., u_N)$ and $\overline{U} = (\overline{u}_1, ..., \overline{u}_N)$ be two solutions of the discretized problem AU=F.

We have

$$\frac{u_i - u_{i-1}}{x_i - x_{i-1}} - \frac{u_{i+1} - u_i}{x_{i+1} - x_i} = |T_i| f_i$$

$$\frac{\overline{u}_i - \overline{u}_{i-1}}{x_i - x_{i-1}} - \frac{\overline{u}_{i+1} - \overline{u}_i}{x_{i+1} - x_i} = |T_i| f_i$$

Put $\Delta u_i = u_i - \overline{u}_i$. Then

$$\frac{\Delta u_i - \Delta u_{i-1}}{x_i - x_{i-1}} - \frac{\Delta u_{i+1} - \Delta u_i}{x_{i+1} - x_i} = 0$$

Multiplying two sides by Δu_i and taking sum over $i = 1, .., N$

$$\sum_{i=1}^{N} \frac{(\Delta u_i)^2 - \Delta u_{i-1}.\Delta u_i}{x_i - x_{i-1}} - \sum_{i=1}^{N} \frac{\Delta u_{i+1}.\Delta u_i - (\Delta u_i)^2}{x_{i+1} - x_i} = 0$$

Changing the index

$$\sum_{i=0}^{N-1} \frac{(\Delta u_{i+1})^2 - \Delta u_{i+1}.\Delta u_i}{x_{i+1} - x_i} - \sum_{i=1}^{N} \frac{\Delta u_{i+1}.\Delta u_i - (\Delta u_i)^2}{x_{i+1} - x_i} = 0$$

We get

$$\sum_{i=1}^{N-1} \frac{(\Delta u_{i+1} - \Delta u_i)^2}{x_{i+1} - x_i} + \frac{(\Delta u_1)^2}{x_1 - x_0} + \frac{(\Delta u_N)^2}{x_N - x_{N-1}} = 0$$

It implies that

$$\Delta u_1 = \Delta u_N = \Delta u_{i+1} - \Delta u_i = 0, \quad i = 1, .., N-1$$

i.e.

$$\Delta u_i = 0, \quad i = 1, .., N$$

Consistency

$u_x(x_{i+1/2})$ is approximated by the differential quotient $\frac{u_{i+1}-u_i}{x_{i+1}-x_i}$.
If $u \in \mathbb{C}^2([0,1],\mathbb{R})$, this approximation is consistent in the sense that there exists $C \in \mathbb{R}_+$ only depending on u such that

$$|R_{i+1/2}| = |u_x(x_{i+1/2}) - \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}| \leq Ch$$

$R_{i+1/2}$ is called consistency error.

Proof

Using Taylor series expansion, there exist $\eta_{i+1/2} \in (x_{i+1/2}, x_{i+1})$ and $\theta_{i+1/2} \in (x_i, x_{i+1/2})$ such that

$$\frac{u_{x_{i+1}} - u_{x_{i+1/2}}}{x_{i+1} - x_i} - \frac{x_{i+1} - x_{i+1/2}}{x_{i+1} - x_i} u_x(x_{i+1/2}) = \frac{1}{2}\frac{(x_{i+1} - x_{i+1/2})^2}{x_{i+1} - x_i} u_{xx}(\eta_{i+1/2})$$

$$\frac{u_{x_{i+1/2}} - u_{x_i}}{x_{i+1} - x_i} - \frac{x_{i+1/2} - x_i}{x_{i+1} - x_i} u_x(x_{i+1/2}) = -\frac{1}{2}\frac{(x_{i+1/2} - x_i)^2}{x_{i+1} - x_i} u_{xx}(\theta_{i+1/2})$$

Consistency (cont.)

Taking sum of the two expressions gives

$$R_{i+1/2} = -\frac{1}{2}\frac{(x_{i+1}-x_{i+1/2})^2}{x_{i+1}-x_i}u_{xx}(\eta_{i+1/2}) + \frac{1}{2}\frac{(x_{i+1/2}-x_i)^2}{x_{i+1}-x_i}u_{xx}(\theta_{i+1/2})$$

The following inequality holds

$$\begin{aligned}
|R_{i+1/2}| &\leq \frac{1}{2}\frac{(x_{i+1}-x_{i+1/2})^2}{x_{i+1}-x_i}|u_{xx}(\eta_{i+1/2})| + \frac{1}{2}\frac{(x_{i+1/2}-x_i)^2}{x_{i+1}-x_i}|u_{xx}(\theta_{i+1/2})| \\
&\leq C\left(\frac{(x_{i+1}-x_{i+1/2})^2}{x_{i+1}-x_i} + \frac{(x_{i+1/2}-x_i)^2}{x_{i+1}-x_i}\right) \\
&\leq C\frac{(x_{i+1}-x_i)^2}{x_{i+1}-x_i} \\
&\leq Ch
\end{aligned}$$

Error estimate

Let $U = (u_1, ..., u_N) \in \mathbb{R}^N$ be the (unique) solution of the discrete problem AU=F. Then there exists $C > 0$, only depending on u, such that

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{x_{i+1} - x_i} \leq C^2 h^2 \tag{2.5}$$

and

$$|e_i| \leq Ch, \quad i = 1, .., N \tag{2.6}$$

with $e_0 = e_{N+1} = 0$ and $e_i = u(x_i) - u_i, \quad i = 1, .., N$.

Error estimate (cont.)

Proof

1. $\sum_{i=0}^{N} \frac{(e_{i+1}-e_i)^2}{x_{i+1}-x_i} \leq C^2 h^2$

Integrating equation $-u_{xx} = f$ over K_i yields

$$-u_x(x_{i+1/2}) + u_x(x_{i-1/2}) = |T_i| f_i$$

The approximate solution U satisfies

$$\frac{u_i - u_{i-1}}{x_i - x_{i-1}} - \frac{u_{i+1} - u_i}{x_{i+1} - x_i} = |T_i| f_i$$

Therefore

$$-u_x(x_{i+1/2}) + \frac{u_{i+1} - u_i}{x_{i+1} - x_i} + u_x(x_{i-1/2}) - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} = 0$$

Error estimate (cont.)

$$-u_x(x_{i+1/2}) + \frac{u_{i+1} - u_i}{x_{i+1} - x_i} = -R_{i+1/2} - \frac{e_{i+1} - e_i}{x_{i+1} - x_i}$$

$$u_x(x_{i-1/2}) - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} = R_{i-1/2} + \frac{e_i - e_{i-1}}{x_i - x_{i-1}}$$

Then

$$-\frac{e_{i+1} - e_i}{x_{i+1} - x_i} - R_{i+1/2} + \frac{e_i - e_{i-1}}{x_i - x_{i-1}} + R_{i-1/2} = 0, \quad i = 1, .., N$$

Multiplying by e_i and summing over $i = 1, .., N$ yields

$$-\sum_{i=1}^{N} \frac{(e_{i+1} - e_i)e_i}{x_{i+1} - x_i} + \sum_{i=1}^{N} \frac{(e_i - e_{i-1})e_i}{x_i - x_{i-1}} = \sum_{i=1}^{N} R_{i+1/2}e_i - \sum_{i=1}^{N} R_{i-1/2}e_i$$

Error estimate (cont.)

Changing the index gives

$$-\sum_{i=1}^{N}\frac{(e_{i+1}-e_i)e_i}{x_{i+1}-x_i}+\sum_{i=0}^{N-1}\frac{(e_{i+1}-e_i)e_{i+1}}{x_{i+1}-x_i}=\sum_{i=1}^{N}R_{i+1/2}e_i-\sum_{i=0}^{N-1}R_{i+1/2}e_{i+1}$$

Reordering and using $e_0 = e_N = 0$ yields

$$\sum_{i=0}^{N}\frac{(e_{i+1}-e_i)^2}{x_{i+1}-x_i}=\sum_{i=0}^{N}R_{i+1/2}(e_i-e_{i+1})$$

Using the consistency property, it implies

$$\sum_{i=0}^{N}\frac{(e_{i+1}-e_i)^2}{x_{i+1}-x_i}\leq Ch\sum_{i=0}^{N}|e_i-e_{i+1}| \tag{2.7}$$

Error estimate (cont.)

Applying Cauchy-Schwarz inequality

$$\sum_{i=1}^{N} |e_i - e_{i+1}| \leq \left(\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{x_{i+1} - x_i} \right)^{1/2} (\sum_{i=0}^{N} x_{i+1} - x_i)^{1/2}$$

e.g.

$$\sum_{i=1}^{N} |e_i - e_{i+1}| \leq \left(\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{x_{i+1} - x_i} \right)^{1/2}$$

From eq (2.7) it implies that

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{x_{i+1} - x_i} \leq Ch \left(\sum_{i=0}^{N} (\frac{(e_{i+1} - e_i)^2}{x_{i+1} - x_i} \right)^{1/2}$$

Error estimate (cont.)

Let us define the discrete H^1-norm

$$||u||_{1,h}^2 = \sum_{i=0}^{N} \frac{(u_{i+1} - u_i)^2}{x_{i+1} - x_i}$$

Then we can prove the error estimate

$$||e||_{1,h} \leq Ch$$

Error estimate (cont.)

2. $|e_i| \leq Ch, \quad i = 1, .., N$

$$|e_i| = |\sum_{j=1}^{i}(e_j - e_{j-1})|$$

$$|e_i| \leq \sum_{j=1}^{i}|e_j - e_{j-1}|$$

Using results from 1. we deduce that

$$|e_i| \leq \sum_{j=0}^{N}|e_j - e_{j-1}| \leq Ch$$

Definition: Discrete divergence operator

$$d : \mathbb{R}^{N+1} \to \mathbb{R}^N$$

$$\left\{ v_{i+\frac{1}{2}} \right\}_{i=0}^{N} \mapsto \{(dv)_i\}_{i=1}^{N}$$

where

$$(dv)_i = \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{|T_i|}, \quad T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

Definition: Scalar product
Given $\{u_i\}_{i=1}^{N}$, $\{w_i\}_{i=1}^{N}$

$$(u, w)_T = \sum_{i=1}^{N} u_i w_i |T_i|$$

$$\|u\|_{0,T}^2 = (u, u)_T$$

Definition: Discrete gradient operator

$$g : \mathbb{R}^{N+2} \to \mathbb{R}^{N+1}$$
$$\{u_i\}_{i=0}^{N+1} \mapsto \left\{(gu)_{i+\frac{1}{2}}\right\}_{i=0}^{N}$$

where

$$(gu)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{|D_{i+\frac{1}{2}}|}, \quad D_{i+\frac{1}{2}} = [x_i, x_{i+1}]$$

Definition: Scalar product

Given $\left\{v^1_{i+\frac{1}{2}}\right\}_{i=0}^{N}$, $\left\{v^2_{i+\frac{1}{2}}\right\}_{i=0}^{N}$

$$\left(v^1, v^2\right)_D = \sum_{i=1}^{N} v^1_{i+\frac{1}{2}} v^2_{i+\frac{1}{2}} |D_{i+\frac{1}{2}}|$$

$\|v\|^2_{0,D} = (v, v)_D$ and $|u|^2_{1,D} = \|g(u)\|^2_{0,D} = (g(u), g(u))_D$

Proposition

Given $\left\{v_{i+\frac{1}{2}}\right\}_{i=0}^{N}$, $\{w_i\}_{i=0}^{N+1}$. *Prove that*

$$(d(v), w)_T = -(v, g(w))_D + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \qquad ???? \qquad (3.1)$$

Let $\{u_i\}_{i\in[0,N+1]}$ satisfy (2.4), there hold

$$-d(g(u)) = f \tag{3.2}$$

where $f = \{f_i\}_{i=1}^{N}$ and f_i is mean-value of f in T_i.
We consider any $\{w_i\}_{i=0}^{N+1}$, with $w_0 = w_{N+1} = 0$. Thanks to (3.1) and to the boundary condition on w, we get

$$(g(u), g(w))_D = (w, f)_T \qquad ???? \tag{3.3}$$

Thus the scheme may be written under the discrete variational formulation: Find $(u_i)_{i\in[0,N+1]}$ with $u_0 = u_{N+1} = 0$, such that for all $(w_i)_{i\in[0,N+1]}$ with $w_0 = w_{N+1} = 0$, there holds

$$(g(u), g(w))_D = (w, f)_T \tag{3.4}$$

This is a discrete equivalent of continuous variational formulation: find $u \in H_0^1(\Omega)$ such that for all $w \in H_0^1(\Omega)$

$$(u', w')_{L^2(\Omega)} = (f, w)_{L^2(\Omega)}$$

The scheme is set as a system of $N+2$ equations and with $N+2$ unknowns $(u_i)_{i\in[0,N+1]}$. In $\mathbb{R}^{N+2}$, existence and uniqueness are equivalent for squrare system. Let us prove uniqueness, for linear sysstem, it is equivalent that if $f_i = 0$ for $i \in [1, n]$ then $u_i = 0$ for $i \in [0, N+1]$.
If f_i then $(f, w)_T = 0$ for all w. Since $u_0 = u_{N+1} = 0$, we can consider $w = u$. From (3.4), we get

$$(gu, gu)_D = \sum_{i=0}^{N} |D_{i+1/2}|(gu)^2_{i+1/2} = 0$$

Since $|D_{i+/12}|$ is not vanish, this is to equivalent $(gu)_{i+1/2} = 0$ for all $i \in [0, N]$. According the define of $(gu)_{i+1/2}$, we get $u_i = u_{i+1}$ for all $i \in [0, N]$, combining with boundary condition, we ge $u_i = 0$ for all $i \in [0, N+1]$

We consider the equation with Neumann boundary condition

$$\begin{cases} -u_{xx}(x) & = f(x) \text{ in } \Omega \\ u'(0) & = u'(1) = 0 \end{cases} \tag{3.5}$$

Remark

The necessary condition over f to the solution of (3.5) to exist is

$$\int_{\Omega} f(x)dx = 0 \tag{3.6}$$

Remark

To determine unique solution to (3.4), we have

$$\int_{\Omega} u(x)dx = 0 \tag{3.7}$$

The equation $-u_x x = f$ is discrtized same Dirichlet boundary condition, we get

$$\frac{1}{|T_i|}\left[-\frac{u_{i+1}-u_i}{|D_{i+1/2}|}+\frac{u_i-u_{i-1}}{|D_{i-1/2}|}\right]=f_i \quad \forall i \in \overline{1,N} \tag{3.8}$$

Boundary condtions $u'(0) = u'(1) = 0$ are discretized by $(gu)_{1/2} = (gu)_{N+1/2} = 0$, yields that

$$u_0 = u_1 \quad \text{and} \quad u_N = u_{N+1} \tag{3.9}$$

Moreover, (3.7) is discretized by

$$\sum_{i=1}^{N} |T_i| u_i = 0 \tag{3.10}$$

Thus, there are $N+3$ equations and but only $N+2$ unknowns. However, the set of equations (3.8) and (3.9) are not indepent. We have

$$
\begin{aligned}
\sum_{i=1}^{N}[-(gu)_{i+1/2}+(gu)_{i-1/2}] &= \sum_{i=1}^{N}|T_i|f_i \\
-(gu)_{N+1/2}+(gu)_{1/2} &= \sum_{i=1}^{N}|T_i|\frac{1}{|T_i|}\int_{T_i}f(x)dx \qquad (3.11)
\end{aligned}
$$

The left hand side of (3.11) is vanish because of (3.9), the right hand side is also vanish because of (3.6)

Thanks to (3.1) and to bondary condition (3.9), we have

$$(g(u), g(w))_D = (f, w)_T \tag{3.12}$$

If f is vanish and let us be $w = u$, $u_i = c$ for all $i \in [0, N+1]/$ We use (3.10), we get $c = 0$. Then $u_i = 0$ for all $i \in [0, N+1]$.

Remark

When we make numerical analysis, since $\sum_{i=1}^{N} |T_i| f_i$ is not always vanish. We must make orther $\tilde{f}_i$ satisfy $\sum_{i=1}^{N} |T_i| \tilde{f}_i = 0$

$$\tilde{f}_i = f_i - \frac{\sum_{i=1}^{N} |T_i| f_i}{\sum_{i=1}^{N} |T_i|}$$

We consider the equation with Robin boundary condition

$$\begin{cases} -u_{xx}(x) & = f(x) \text{ in } \Omega \\ u'(0) - \lambda_0 u(0) & = u'(1) + \lambda_1 u(1) = 0 \end{cases} \tag{3.13}$$

we get discrete equation following:

$$\frac{1}{|T_i|}\left[-\frac{u_{i+1}-u_i}{|D_{i+1/2}|} + \frac{u_i - u_{i-1}}{|D_{i-1/2}|}\right] = f_i \quad \forall i \in \overline{1, N} \tag{3.14}$$

and $(gu)_{1/2} - \lambda_0 u_0 = (gu)_{N+1/2} + \lambda_1 u_{N+1} = 0$.

How to prove the existence and uniqueness solution of the scheme???

We suppose that f is positive on Ω and $u_0 = u_{N+1} = 0$. We wish that we prove that the discrete solution remains positive on Ω, i.e $u_i \geq 0$ for all $i \in [1, N]$

.

Prove:

We assume that for give $i \in [1, N]$, $u_i < 0$. Then there exist $i_1 \in [1, N]$ such that $u_{i_1} = \min\{u_i : i \in [1, N]\}$, thus $u_{i_1} < 0$. From discrete equation for $-u_{xx} = f$, we get

$$\frac{1}{|T_{i_1}|}\left[\frac{u_{i_1} - u_{i_1+1}}{|D_{i_1+1/2}|} + \frac{u_{i_1} - u_{i_1-1}}{|D_{i_1-1/2}|}\right] = f_{i_1} \tag{3.15}$$

Since $u_{i_1} = \min\{u_i : i \in [1, N]\}$, thus $u_{i_1} - u_{i_1+1} \leq 0$ and $u_{i_1} - u_{i_1-1} \leq 0$, combining with (3.15) with $f_i \geq 0$, we have $u_{i1+1} = u_{i1} = u_{i1-1}$. From that, we can prove that

$$u_i = u_{i_1} \quad \text{for all} \quad i \in [0, N+1]$$

but while $u_0 = 0$ which is a contradiction

Since exact solution $u \in C^1(\bar{\Omega})$. We can define projection

$$\begin{aligned} \Pi : C^1(\bar{\Omega}) &\to \mathbb{R}^{N+2} \\ u &\mapsto (\Pi u)_i = u(x_i) \qquad \forall i \in [0, N+1] \end{aligned}$$

Since $u' \in C^0(\bar{\Omega})$. We can define projection

$$\begin{aligned} P : C^0(\bar{\Omega}) &\to \mathbb{R}^{N+1} \\ u' &\mapsto (Pu')_{i+1/2} = u'(x_{i+1/2}) \qquad \forall i \in [0, N] \end{aligned}$$

Lemma

Let $(w_i)_{i\in[0,N+1]}$ with $w_0 = w_{N+1} = 0$ and if u is the solution of finite volume method. we have

$$(gu, gw)_D = (Pu', gw)_D \qquad ???? \tag{4.1}$$

We shall estimate the H_0^1 norm of $u - \Pi u$ defined by

$$|u - \Pi u|_{1,D} = (g(u - \Pi u), g(u - \Pi u))_D^{1/2} \tag{4.2}$$

We set $w = u - \Pi u$, thanks to lema, since $w_0 = w_{N+1} = 0$, we can write

$$\begin{aligned} |u - \Pi u|_{1,D}^2 &= (g(u - \Pi u), g(u - \Pi u))_D \\ &= (g(u), g(w))_D - (g(\Pi u), g(w))_D \\ &= (Pu', g(w))_D - (g(\Pi u), g(w))_D \\ &= (Pu' - g(\Pi u), g(w))_D \\ &\leq \|Pu' - g(\Pi u)\|_{0,D} |w|_{1,D} \end{aligned}$$

Since $w = u - \Pi u m$ then

$$|u - \Pi u|_{1,D} \leq \|Pu' - g(\Pi u)\|_{0,D} \tag{4.3}$$

There hold

$$\|Pu' - g(\Pi u)\|_{0,D}^2 = \sum_{i=0}^{N} |D_{i+1/2}|\varepsilon_{i+1/2}^2 \tag{4.4}$$

where $\varepsilon_{i+1/2}$ is the diffrence $u'(x_{i+1/2})$ and finite diffrence $\frac{u(x_{i+1})-u(x_i)}{|D_{i+1/2}|}$:

$$\varepsilon_{i+1/2} = u'(x_{i+1/2}) - \frac{u(x_{i+1}) - u(x_i)}{|D_{i+1/2}|} \tag{4.5}$$

We prove that

$$|D_{i+1/2}|\varepsilon_{i+1/2}^2 \leq \left(\frac{2}{3}\right)^2 4h^2 \int_{D_{i+1/2}} f^2(t)dt \quad ???? \tag{4.6}$$

If $x_{i+1/2}$ is midpoint of $D_{i+/12}$ then

$$|D_{i+1/2}|\varepsilon_{i+1/2}^2 \leq \left(\frac{4}{15}\right)^2 h^4 \|f'\|_{L^2(D_{i+1/2})}^2 \quad ???? \tag{4.7}$$

We set:
$K_1 = \{i : x_{i+1/2} = \frac{x_i + x_{i+1}}{2}\}$ and $K_2 = \{i : x_{i+1/2} \neq \frac{x_i + x_{i+1}}{2}\}$
There holds

$$\sum_{i=0}^{N} |D_{i+1/2}|\varepsilon_{i+1/2}^2 = \sum_{i \in K1} |D_{i+1/2}|\varepsilon_{i+1/2}^2 + \sum_{i \in K2} |D_{i+1/2}|\varepsilon_{i+1/2}^2$$

$$\leq \left(\frac{4}{15}\right)^2 h^4 \sum_{i \in K_1} \|f'\|_{L^2(D_{i+1/2})}^2 + \left(\frac{2}{3}\right)^2 4h^2 \sum_{i \in K_2} \int_{D_{i+1/2}} f^2(t)dt \tag{4.8}$$

We have

$$\left(\frac{4}{15}\right)^2 h^4 \sum_{i \in K_1} \|f'\|_{L^2(D_{i+1/2})}^2 \leq \left(\frac{4}{15}\right)^2 h^4 \|f'\|_{L^2(\Omega)}^2$$

We suppose that $f \in H^1(\Omega)$ and f continuous on $\bar{\Omega}$ then

$$\|f\|^2_{L\infty(\Omega)} \leq (2\|f\|^2_{L^2(\Omega)} + \|f'\|^2_{L^2(\Omega)})$$

$$\int_{D_{i+1/2}} f^2(t)dt \leq 2h(2\|f\|^2_{L^2(\Omega)} + \|f'\|^2_{L^2(\Omega)})$$

So

$$\left(\frac{2}{3}\right)^2 4h^2 \sum_{i\in K_2} \int_{D_{i+1/2}} f^2(t)dt \leq \left(\frac{2}{3}\right)^2 8|K_2|h^3(2\|f\|^2_{L^2(\Omega)}+\|f'\|^2_{L^2(\Omega)})$$

Then

$$\begin{aligned}|u - \Pi u|_{1,D} \leq & \left(\frac{4}{15}\right) h^2\|f'\|_{L^\Omega} \\ & + \left(\frac{2}{3}\right) 2\sqrt{2}\sqrt{|K_2|}h^{3/2}(\sqrt{2}\|f\|_{L^2(\Omega)} + \|f'\|_{L^2(\Omega)})\end{aligned}$$

which leading term behaves like $O(h^{3/2})$. If K_2 is bounded when h tends to zero, then convergence is at least of 1.5 order

Lemma (Discrete Poincare inequality)

Let $(w_i)_{i\in[0,N+1]}$ *such that* $w_0 = 0$ *then* $\|w\|_{0,T} \leq |w|_{1,D}$

How to prove this Lemma????