Mathematical Analysis & Numerical Analysis
Giải Tích Toán Học & Giải Tích Số

Nguyễn Quản Bá Hồng*

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Tóm tắt nội dung

This text is a part of the series *Some Topics in Advanced STEM & Beyond*:
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Latest version:

1. *Mathematical Analysis – Giải Tích Toán Học*.
PDF: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.pdf.
TEX: URL: https://github.com/NQBH/advanced_STEM_beyond/blob/main/analysis/NQBH_mathematical_analysis.tex.

Mục lục

*A Scientist & Creative Artist Wannabe. E-mail: nguyenquanbahong@gmail.com. Bến Tre City, Việt Nam.

Tôi được giải Nhì Giải tích Olympic Toán Sinh viên 2014 (VMC2014) khi còn học năm nhất Đại học & được giải Nhất Giải tích Olympic Toán Sinh viên 2015 (VMC2015) khi học năm 2 Đại học. Nhưng điều đó không có nghĩa là tôi giỏi Giải tích. Bằng chứng là 10 năm sau khi nhận các giải đó, tôi đang tự học lại Giải tích Toán học với hy vọng có 1 hay nhiều cách nhìn mới mẻ hơn & mang tính ứng dụng hơn cho các đề tài cá nhân của tôi.

1 Basic

Resources – Tài nguyên.

1. [LL01]. Elliott Lieb, Michael Loss. *Analysis.*
2. [Rud76]. Walter Rudin. *Principles Principles of Mathematical Analysis.*
3. [Rud73; Rud87]. Walter Rudin. *Real & Complex Analysis.*
4. [Tao22a]. Terence Tao. *Analysis I.*
5. [Tao22b]. Terence Tao. *Analysis II.*

"Analysis is the art of taking limits, & the constraint of having to deal with an integration theory that does not allow taking limits is much like having to do mathematics only with rational numbers & excluding the irrational ones." – [LL01, Chap. 1, p. 1]

2 C_0 Semigroup – Nửa Nhóm C_0

Resources – Tài nguyên.

1. [AK16]. Cung Thế Anh, Trần Đình Kế. *Nửa Nhóm Các Toán Tử Tuyến Tính & Ứng Dụng.*

"In mathematical analysis, a C_0*-semigroup*, also known as a *strongly continuous 1-parameter semigroup*, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant ODEs, strongly continuous semigroups provide solutions of linear constant coefficient ODEs in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations & PDEs. Formally, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+, +)$ on some Banach space X that is continuous in the strong toperator topology." -Wikipedia/C_0-semigroup

3 Differential Geometry – Hình Học Vi Phân

Resources – Tài nguyên.

1. [Car16]. Manfredo P. do Carmo. *Differential Geometry of Curves & Surfaces.*
2. [DZ01; DZ11]. Michael C. Delfour, Jean-Paul Zolésio. *Shapes & Geometries.*
3. [Küh15]. Wolfgang Kühnel. *Differential Geometry.*
4. [Wal15]. Shawn W. Walker. *The Shapes of Things.*

"Differential geometry is the detailed study of the *shape* of a surface (manifold), including *local* & *global* properties. A plane in $\mathbb{R}^3$ is a very simple surface & does not require many tools to characterize. An "arbitrarily" shaped surface, e.g., hood of a car, has many distinguished geometric features (e.g., highly curved regions, regions of near flatness, etc.). Characterizing these features quantitatively & qualitatively requires the tools of differential geometry. Geometric details are important in many physical & biological processes, e.g., surface tension, biomembranes.

The framework of differential geometry is built by 1st defining a local map (i.e., surface parameterization) which defines the surface. Then, a calculus framework is built up on the surface analogous to the standard "Euclidean calculus". Other approaches are also possible, e.g., those with implicit surfaces defined by level sets & distance functions. But parameterizations, though arbitrary, are quite useful in a variety of settings ⇒ stick mostly with those. Emphasize: The geometry of a surface does not depend on a particular parameterization. Otherwise, we will emphasize the distinction between `object 1` & `object 2`.

We will use this "abuse" of notation when there is no possibility of ambiguity.

Open set. The concept of open set is critical in multivariate calculus to properly define differentiability. The notation for referencing boundaries of sets, as well as the closure of sets, is practical for referencing geometric details of solid objects & their surfaces.

Compactness. Compact support is useful for ignoring boundary effects. This concept is needed to keep the "action of a function" away from the boundary of a set, or to localize the function in a region of interest. 1 reason is to avoid potential difficulties with differentiating a function at its boundary of definition. Or, more commonly, we wish to ignore a quantity depending on the value of a function at a boundary point, e.g., $\int_{\partial S} f = 0$ if f has compact support in S.

Topological mapping/homeomorphism. A bijective, continuous mapping $\mathbf{\Phi}$ whose inverse $\mathbf{\Phi}^{-1}$ is also continuous is called a *topological mapping* or *homeomorphism*. Point sets that can be topologically mapped onto each other are said to be *homeomorphic*. Sets that are homeomorphic have the "same topology", i.e., their connectedness is the same; they have the same kinds of "holes". See [Wal15, Sect. 2.3.1] for what can happen when a mapping is not a homeomorphism.

Rigid motion mapping. A mapping $\mathbf{\Phi}$ is called a *rigid motion* if any pair of points $\mathbf{a}, \mathbf{b}$ are the same distance apart as the corresponding pair $\mathbf{\Phi}(\mathbf{a}), \mathbf{\Phi}(\mathbf{b})$.

Orthogonal Transformations. Define the (affine) linear map $\mathbf{\Phi}$ (transformation)

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{1}$$

If A satisfies the properties $A^{-1} = A^\top$, $\det A = 1$ then $\mathbf{\Phi}$ represents a rigid motion. Basically, $\mathbf{\Phi}$ consists of a rotation represented by A followed by a translation represented by $\mathbf{b}$. A rigid motion can be used to transition from 1 Cartesian coordinate system to another. If $\mathbf{b} = \mathbf{0}$ & $A^{-1} = A^\top$, $\det A = 1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is a linear map known as a *direct orthogonal transformation*, which is nothing more than a rotation of the coordinate system with the origin as the center. If $A^{-1} = A^\top$, $\det A = 1$ is replaced by $A^{-1} = A^\top$, $\det A = -1$, then $\mathbf{\Phi}(\mathbf{x}) = A\mathbf{x}$ is called an *opposite orthogonal transformation*, which consists of a rotation about the origin & a reflection in a plane. Both $A^{-1} = A^\top$, $\det A = \pm 1$ are examples of *orthogonal matrices.*

Interpretation of transformations. Can interpret $\widetilde{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in 2 different ways. Consider a point $P \in \mathbb{R}^3$ with coordinates $\mathbf{x}$:

- *Alias* (Euler perspective). Viewing (1) as a transformation of coordinates, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of the same point w.r.t. 2 different coordinate systems, equivalently, the point is referenced by 2 different "names" (sets of coordinates).
- *Alibi* (Lagrange perspective). Viewing (1) as a mapping of sets, it appears that $\mathbf{x}, \widetilde{\mathbf{x}}$ are the coordinates of 2 different points w.r.t. the same coordinate system, equivalently, the point at $\widetilde{\mathbf{x}}$ "was previously" at $\mathbf{x}$ before applying the map.

The concept of material point is directly related to the alibi viewpoint. One can think of a "particle" of material, i.e., *material point*, initially located at $\mathbf{x}$, that then moves to $\widetilde{\mathbf{x}}$ because of some physical process. The transformation (1) simply represents the kinematic outcome of that physical process, which is a standard concept in deformable continuum mechanics, especially nonlinear elasticity.

General transformations. In general, transformation may not be linear. The alias viewpoint yields a *curvilinear* coordinate system. The alibi viewpoint implies that the set S is *deformed* into the set $S' = \mathbf{\Phi}(S)$.

Parametric approach – what is a surface? A *surface* is a set of points in space that is "regular enough". A random scattering of points in space does not match our intuitive notion of what a surface is, i.e., it is not regular enough. The boundary of a sphere does match our notion of a surface, i.e., regular enough to be a surface because a sphere is "smooth". *Intuition*: Can think of creating a surface as deforming a flat rubber sheet into a curved sheet. Let $U \subset \mathbb{R}^2$ be a "flat" domain & let $\mathbf{X} : U \to \mathbb{R}^3$ be this deforming transformation, i.e., for each point $(s_1, s_2)^\top \in U$ there is a corresponding point $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ s.t. $\mathbf{x} = \mathbf{X}(s_1, s_2)$. Let $\Gamma = \mathbf{X}(U)$ denote the surface obtained from "deforming" U. Call $\mathbf{x} = \mathbf{X}(s_1, s_2)$ a *parametric representation* of the surface Γ, where s_1, s_2 are called the *parameters* of the representation. Refer to U as a *reference* domain.

Allowable parameterization/immersion. If use $\mathbf{x} = \mathbf{X}(s_1, s_2)$ to define surfaces, then we must place assumptions on $\mathbf{X}$ to guarantee that $\Gamma = \mathbf{X}(U)$ is a valid surface. At the bare minimum, $\mathbf{X}$ must be continuous to avoid "tearing" the rubber sheet. But if want to perform calculus on Γ, need more:

Assumption 1 (Regularity assumptions on $\mathbf{X}$)**.** *An allowable parameterization/immersion is a parameterization of the form* $\mathbf{x} = \mathbf{X}(s_1, s_2)$ *satisfying:*

(A1) The function $\mathbf{X}(s_1, s_2) \in C^\infty(U)$ *& each point* $\mathbf{x} = \mathbf{X}(s_1, s_2) \in \Gamma$ *corresponds to just 1 point* $(s_1, s_2) \in U$*, i.e.,* $\mathbf{X}$ *is injective.*

(A2) The Jacobian matrix $J = [\partial_{s_1}\mathbf{X}, \partial_{s_2}\mathbf{X}]$ *is of rank* 2 *on* U*, i.e., the columns of* J *are linearly independent.*

Regular surface. The fundamental property that makes a set of points in $\mathbb{R}^3$ a surface is that it *locally looks like a plane* at every point. If you "zoom into" a surface, it should look flat. Definition defining a surface in terms of a parameterization is inadequate. Want to define a set in $\mathbb{R}^3$ that is "intrinsically" 2D & is smooth enough so we can perform calculus on it, without regard to a specific parameterization.

Definition 1 (Regular surface)**.**

Remark 1 (Local chart)**.**

1 trong những ứng dụng của Hình Học Vi Phân là *Shape Calculus & Tangential Calculus – Phép Tính Vi Tích Phân cho Tối Ưu Hình Dáng & Phép Tính Vi Tích Phân Trên Mặt Phẳng Tiếp Tuyến.*

3.1 Calculus on Surfaces

Goal. Define & develop the fundamental tools of calculus on a regular surface. Start with the notion of differentiability of functions defined only on a surface. Define the concept of vector fields in a surface. Then proceed to develop the gradient & Laplacian operators w.r.t. a surface. These operators allow for alternative expressions of the summed & Gaussian curvatures. Derive integration by parts on surfaces, i.e., the domain of integration is a surface. Conclude with some useful identities & inequalities. Always take Γ: a regular surface, either with or without a boundary.

4 Functional Analysis – Giải Tích Hàm

Resources – Tài nguyên.

1. [Rud91]. Walter Rudin. *Functional Analysis.*
2. Yosida.

5 Inverse Problems – Bài Toán Ngược

Resources – Tài nguyên.

1. [ABT18]. Richard Aster, Brian Borchers, Clifford H. Thurber. *Parameter Estimation & Inverse Problems.*
2. [Kir21]. Andreas Kirsch. *An Introduction to The Mathematical Theory of Inverse Problems.*
3. [IJ15]. Kazufumi Ito, Bangti Jin. *Inverse Problems.*

6 Measure & Integration – Độ Đo & Tích Phân

Resources – Tài nguyên.

1. [EG15]. Lawrence C. Evans, Ronald F. Gariepy. *Measure Theory & Fine Properties of Functions.*

The point of view of integration defined as a Riemann integral may be historically grounded & useful in many areas of mathematics but is far from being adequate for the requirements of modern analysis since Riemann integral can be defined only for a special class of functions & this class is not closed under the process of taking pointwise limits of sequence (not even monotonic sequences) of functions in this class.

"The useful & far-reaching idea of Lebesgue & others was to compute the $(n+1)$-dimensional volume 'in the other direction' by 1st computing the n-dimensional volume of the set where the function $> y$. This volume is a well-behaved, monotone nonincreasing function of y, which then can be integrated in the manner of Riemann. This method of integration not only works for a large class of functions (which is closed under taking pointwise limits), but it also greatly simplifies a problem that used to plague analysts: *Is it permissible to exchange limits & integration?*" – [LL01, Chap. 1, pp. 1–2]

Lebesgue integration theory is 1 of the great triumphs of 20th century mathematics & is the culmination of a long struggle to find the right perspective from which to view integration theory.

7 Mean-Field Game Theory – Lý Thuyết Trò Chơi Trường Trung Bình

Community – Cộng đồng. Nicholetta Tchou (French), Đào Mạnh Khang (Vietnamese), Michael Hintermüller (Austrian), Steven-Marian Stengl (German).

7.1 Wikipedia/mean-field game theory

"*Mean-field game theory* is the study of strategic decision making by small interacting agents in very large populations. It lies at the intersection of game theory with stochastic analysis & control theory. The use of the term "mean field" is inspired by mean-field theory in physics, which considers the behavior of systems of large numbers of particles where individual particles have negligible impacts upon the system. In other words, each agent acts according to his minimization or maximization problem taking into account other agents' decisions & because their population is large we can assume the number of agents goes to infinity & a representative agent exists.

In traditional game theory, the subject of study is usually a game with 2 players & discrete time space, & extends the results to more complex situations by induction. However, for games in continuous time with continuous states (differential games or stochastic differential games) this strategy cannot be used because of the complexity that the dynamic interactions generate. On the other hand with MFGs we can handle large numbers of players through the mean representative agent & at the same time describe complex state dynamics.

This class of problems was considered in the economics literature by Boyan Jovanovic & Robert W. Rosenthal, in the engineering literature by Minyi Huang, Roland Malhame, & Peter E. Caines & independently & around the same time by mathematicians Jean-Michel Lasry & Pierre-Louis Lions.

In continuous time a mean-field game is typically composed of a Hamilton-Jacobi-Bellman equation that describes the optimal control problem of an individual & a Fokker–Planck equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions it can be proved that a class of mean-field games is the limit as $N \to \infty$ of an N-player Nash equilibrium.

A related concept to that of mean-field games is "mean-field-type control". In this case, a social planner controls the distribution of states & chooses a control strategy. The solution to a mean-field-type control problem can typically be expressed as a dual adjoint Hamilton-Jacobi-Bellman equation coupled with Kolmogorov equation. Mean-field-type game theory is the multi-agent generalization of the single-agent mean-field-type control.

7.1.1 General Form of a Mean-field Game

The system of equations

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, m, Du) = 0, \\ \partial_t m - \nu\Delta m - \nabla \cdot (D_p H(x, m, Du)m) = 0, \\ m(0) = m_0, \\ u(T, x) = G(x, m(T)), \end{cases}$$

can be used to model a typical Mean-field game. The basic dynamics of this set of equations can be explained by an average agent's optimal control problem. In a mean-field game, an average agent can control their movement α to influence the population's overall location by

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t,$$

where ν: a parameter, B_t: a standard Brownian motion. By controlling their movement, the agent aims to minimize their overall expected cost C throughout the time period $[0, T]$:

$$C = \mathbb{E}\left[\int_0^T L(X_s, \alpha_s, m(s))\,\mathrm{d}s + G(X_T, m(T))\right],$$

where $L(X_s, \alpha_s, m(s))$ is the running cost at time s & $G(X_T, m(T))$ is the terminal cost at time T. By this definition, at time t & position x, the value function $u(t,x)$ can be determined as

$$u(t,x) = \inf_{\alpha} \mathbb{E}\left[\int_t^T L(X_s, \alpha_s, m(s))\,\mathrm{d}s + G(X_T, m(T))\right].$$

Given the definition of the value function $u(t,x)$, it can be tracked by the Hamilton-Jacobi equation. The optimal action of the average players $\alpha^*(t,x)$ can be determined as $\alpha^*(t,x) = D_pH(x,m,Du)$. As all agents are relatively small & cannot single-handedly change the dynamics of the population, they will individually adapt the optimal control & the population would move in that way. This is similar to a Nash Equilibrium, in which all agents act in response to a specific set of others' strategies. The optimal control solution then leads to the Kolmogorov-Fokker-Planck equation $\partial_t m - \nu\Delta m - \nabla\cdot(D_pH(x,m,Du)m) = 0$.

7.1.2 Finite State Games

A prominent category of mean field is games with a finite number of states & a finite number of actions per player. For those games, the analog of the Hamilton-Jacobi-Bellman equation is the Bellman equation, & the discrete version of the Fokker-Planck equation is the Kolmogorov equation. Specifically, for discrete-time models, the players' strategy is the Kolmogorv equation's probability matrix. In continuous time models, players have the ability to control the transition rate matrix.

A discrete mean field game can be defined by a tuple $\mathcal{G} = (\mathcal{E}, \mathcal{A}, \{Q_a\}, \mathbf{m}_0, \{c_a\}, \beta)$ where $\mathcal{E}$ is the state space, $\mathcal{A}$ the action set, Q_a the transition rate matrices, $\mathbf{m}_0$ the initial state, $\{c_a\}$ the cost functions & $\beta \in \mathbb{R}$ a discount factor. Furthermore, a mixed strategy is a measurable function $\pi : \mathbb{R}^+ \times \mathbb{E} \to \mathcal{P}(\mathcal{A})$, that associates to each state $i \in \mathcal{E}$ & each time $t \geq 0$ a probability measure $\pi_i(t) \in \mathcal{P}(\mathcal{A})$ on the set of possible actions. Thus $\pi_{i,a}(t)$ is the probability that, at time t a player in state i takes action a, under strategy π. Additionally, rate matrices $\{Q_a(\mathbf{m}^\pi(t))\}_{a\in\mathcal{A}}$ define the evolution over the time of population distribution, where $\mathbf{m}^\pi(t) \in \mathcal{P}(\mathcal{E})$ is the population distribution at time t.

7.1.3 Linear-quadratic Gaussian game problem

From Caines (2009), a relatively simple model of large-scale games is the linear-quadratic Gaussian model. The individual agent's dynamics are modeled as a stochastic differential equation

$$dX_i = (a_iX_i + b_iu_i)dt + \sigma_i dW_i,\ i = 1,\ldots,N,$$

where X_i: the state of the ith agent, u_i: control of the ith agent, W_i: independent Wiener processes $\forall i = 1,\ldots,N$. The individual agent's cost is

$$J_i(u_i,\nu) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}[(X_i-\nu)^2 + ru_i^2]\,\mathrm{d}t\right],\ \nu = \Phi\left(\frac{1}{N}\sum_{k\neq i}^N X_k + \eta\right).$$

The coupling between agents occurs in the cost function.

7.1.4 General & Applied Use

The paradigm of Mean Field Games has become a major connection between distributed decision-making & stochastic modeling. Starting out tin the stochastic control literature, it is gaining rapid adoption across a range of applications, including:

1. **Financial market.** Carmona reviews applications in financial engineering & economics that can be cast & tackled within the framework of the MFG paradigm. Carmona argues that models in macroeconomics, contract theory, finance, ..., greatly benefit from the switch to continuous time from the more traditional discrete-time models. He considers only continuous time models in his review chapter, including systemic risk, price impact, optimal execution, models for bank runs, high-frequency trading, & cryptocurrencies.

2. **Crowd motions.** MFG assumes that individuals are smart players which try to optimize their strategy & path w.r.t. certain costs (equilibrium with rational expectations approach). MFG models are useful to describe the anticipation phenomenon: the forward part describes the crowd evolution while the backward gives the process of how the anticipations are built. Additionally, compared to multi-agent microscopic model computations, MFG only requires lower computational costs for the macroscopic simulations. Some researchers have turned to MFG in order to model the interaction between populations & study the decision-making process of intelligent agents, including aversion & congestion behavior between 2 groups of pedestrians, departure time choice of morning commuters, & decision-making processes for autonomous vehicle.

3. **Control & mitigation of Epidemics.** Since the epidemic has affected society & individuals significantly, MFG & mean-field controls (MFCs) provide a perspective to study & understand the underlying population dynamics, especially in the context of the Covid-19 pandemic response. MFG has been used to extend the SIR-type dynamics with spatial effects or allowing for individuals to choose their behaviors & control their contributions to the spread of the disease. MFC is applied to design the optimal strategy to control the virus spreading within a spatial domain, control individuals' decisions to limit their social interactions, & support the government's nonpharmaceutical interventions." – Wikipedia/mean-field game theory

8 Partial Differential Equations (PDEs) – Phương Trình Vi Phân Đạo Hàm Riêng

Resources – Tài nguyên.

1. [Bre11]. Haïm Brezis. *Functional Analysis, Sobolev Spaces, & Partial Differential Equations.*
2. [Eva10]. Lawrence C. Evans. *Partial Differential Equations.*
3. [GT01]. David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of 2nd Order.*

8.1 Weak solution – Nghiệm yếu

Definition 2 (Weak solution – Nghiệm yếu)**.** *"In mathematics, a* weak solution *(also called a* generalized solution*) to an ODE or PDE is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. 1 of the most important is based on the notion of distributions." – Wikipedia/weak solution*

"Avoiding the language of distributions, one starts with a differential equation & rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, & the solutions to it are called *weak solutions*). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; & the weak formulation allows one to find such solutions.

Weak solutions are important because many differential equations encountered in modeling real-world phenomena do not admit of sufficiently smooth solutions, & the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to 1st prove the existence of weak solutions & only alter show that those solutions are in fact smooth enough." – Wikipedia/weak solution

Example 1 (1st-order wave equation)**.** *The 1st-order wave equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}^2$ with $u = u(t,x)$ has the weak form $\int_{\mathbb{R}^2} u\partial_t\varphi + u\partial_x\varphi \,\mathrm{d}t\,\mathrm{d}x = 0$ has a solution $u(t,x) = |t-x|$ which may be checked by splitting the integrals over region $\{x \ge t\}$ & $\{x \le t\}$ where u is smooth.*

"The notion of weak solution based on distribution is sometimes inadequate. In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, & it is necessary to supplement it with *entropy conditions* or some other selection criterion. In fully nonlinear PDE e.g. Hamilton-Jacobi equation, there is a very different definition of weak solution called *viscosity solution*." – Wikipedia/weak solution

8.1.1 General idea

When solving a differential equation in u, one can rewrite it using a test function φ s.t. whatever derivatives in u show up in the equation, they are "transferred" via integration by parts to φ, resulting in an equation without derivatives of u. This new equation generalizes the original equation to include solutions which are not necessarily differentiable. The approach illustrated above works in great generality. Consider a linear differential operator in an open set $W \subset \mathbb{R}^d$:

$$P(\mathbf{x},\partial)u(\mathbf{x}) = \sum a_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}),$$

where the multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$ varies over some finite set in $\mathbb{N}^d$ & the coefficients a_α are smooth enough functions of $\mathbf{x} \in \mathbb{R}^d$. The differential equation $P(\mathbf{x},\partial)u(\mathbf{x} = 0$ can, after being multiplied by a smooth test function $\varphi \in C_c^\infty(W)$ & integrated by parts, be written as

$$\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

where the differential operator $Q(\mathbf{x},\partial)$ is given by the formula

$$Q(\mathbf{x},\partial)\varphi(\mathbf{x}) = \sum (-1)^{|\alpha|}\partial^\alpha[a_\alpha(\mathbf{x})\varphi(\mathbf{x})],$$

which is the formal adjoint of $P(\mathbf{x},\partial)$.

In summary, if the original (strong) problem was to find a $|\alpha|$-times differentiable function u defined on the open set W s.t. $P(\mathbf{x},\partial)u(\mathbf{x}) = 0$, $\forall\mathbf{x} \in W$ (a so-called *strong solution*), then an integrable function u would be said to be a *weak solution* if $\int_W u(\mathbf{x})Q(\mathbf{x},\partial)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$, $\forall\varphi \in C_c^\infty(W)$.

8.2 Viscosity solution – Nghiệm trơn/nhớt

Example 2 (Viscosity solution for Hamilton–Jacobi equation)**.** *Hamilton–Jacobi equation.*

8.3 Very weak solution – Nghiệm rất yếu

Example 3 (Very weak solution of porous medium equation (PME) [Váz07])**.** .

Example 4 (Very weak solution of multi-dimensional slow diffusion equations with a singular quenching term [DDN20])**.** *Given* $f \in L^1_\delta(\Omega), \lambda \geq 0$, *a function* $u \in L^1_\delta(\Omega)$ *is called a* very weak solution *of*

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \textit{in } \Omega, \\ |u|^{m-1}u = 1 & \textit{on } \Gamma, \end{cases}$$

if $|u|^{m-1}u \in L^1(\Omega)$ *and*

$$\int_\Omega u^m \Delta\varphi + \lambda u\varphi \,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi \,\mathrm{d}\mathbf{x} - \int_\Gamma \partial_{\mathbf{n}}\varphi \,\mathrm{d}\mathbf{x}.$$

Example 5 (Very weak solution of NSEs [Tsa18])**.** .

8.4 Navier–Stokes Equations

Resources – Tài nguyên.

1. [Lad69]. O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.*

2. [Soh01a; Soh01b]. Hermann Sohr. *The NSEs: An Elementary Functional Analytic Approach.*

   Primary objective. To develop an elementary & self-contained approach to the mathematical theory of a viscous incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$, described by NSEs. Formulate the theory for a completely general domain Ω.

3. [Tem77; Tem00]. Roger Temam. *NSES: Theory & Numerical Analysis.*

4. [Tsa18]. Tai-Peng Tsai. *Lectures on NSEs.*

9 Sobolev Spaces – Không Gian Sobolev

Resources – Tài nguyên.

1. [AF03]. Robert A. Adams, John J. F. Fournier. *Sobolev Spaces.*

2. [Gag57]. Emilio Gagliardo. *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.*

3. Necǎs.

4. [Tar06]. Luc Tartar. *An Introduction to Sobolev Spaces & Interpolation Spaces.*

10 Finite Difference Methods FDMs – Phương Pháp Sai Phân Hữu Hạn

Resources – Tài nguyên.

1. [LeV07]. Randall J. LeVeque. *FDMs for ODE & PDEs: Steady-State & Time-Dependent Problems.*

11 Finite Element Methods FEMs – Phương Pháp Phần Tử Hữu Hạn

Resources – Tài nguyên.

1. [BS08]. Susanne C. Brenner, L. Ridgway Scott. *The Mathematical Theory of FEMs.*

2. [EG04]. Alexandre Ern, Jean-Luc Guermond. *Theory & Practice of Finite Elements.*

3. [GR86]. Vivette Girault, Pierre-Arnaud Raviart. *FEMs for NSEs.*

4. [Gun89]. Max D. Gunzburger. *FEMs for Viscous Incompressible Flows.*

5. [Joh16]. Volker John. *FEMs for Incompressible Flow Problems.*

I met Volker John, lead of Research Group 3 in WIAS in 2020 to discuss on turbulence models, e.g., Smagonrinsky, k-ϵ & their simulations.

12 Finite Volume Methods FVMs – Phương Pháp Thể Tích Hữu Hạn

Resources – Tài nguyên.

1. [EGH19]. Robert Eymard, Thierry Gallouët, Raphaèle Herbin. *Finite Volume Methods.*

2. [LeV02]. Randall J. LeVeque. *FEMs for Hyperbolic Problems.*

13 Mathematicians & Their Legacies – Các Nhà Toán Học & Các Di Sản

13.1 Wikipedia/Henri Berestycki

Henri Berestycki (born Mar 25, 1951, in Paris)[1] is a French mathematician who obtained his PhD from Université Paris VI - Université Pierre et Marie Curie in 1975.

His Dissertation was titled *Contributions à l'étude des problèmes elliptiques non linéaires*, and his doctoral advisor was Haim Brezis.[2]

He was an L.E. Dickson Instructor in Mathematics at the University of Chicago from 1975–77, after which he returned to France to continue his research.

He has made many contributions in *nonlinear analysis*, ranging from *nonlinear elliptic equations, hamiltonian systems, spectral theory of elliptic operators*, and with applications to the description of *mathematical modelling of fluid mechanics and combustion.*

His current research interests include the mathematical modelling of financial markets, mathematical models in biology and especially in ecology, and modelling in social sciences (in particular, urban planning and criminology).

For these latter topics, he obtained an ERC Advanced grant in 2012.

He worked at the French National Center of Scientific Research (CNRS), then moved to an appointment as Professor at Univ. Paris XIII (1983–1985).

He became a Professor of Mathematics in 1988 at Université Pierre et Marie Curie, Paris VI (1988–2001 of "exceptional class" since 1993), and became Professor at Ecole normale supérieure, Paris (1994–1999), and part-time professor Ecole Polytechnique (1987–1999).

He is also a visiting Professor in the Department of Mathematics at the University of Chicago, and was also co-director of the Stevanovich Center of Financial Mathematics in Chicago.

He is currently the Directeur d'études (Research Professor) at École des hautes études en sciences sociales (EHESS), since 2001.

13.1.1 Services

- National Committee of French universities (1992–1995).
- Since 2002 director of Centre d'analyse et mathématique sociales (CAMS), CNRS -EHESS.
- Vice-president, EHESS (2004–2006).
- Member of the thesis prize committee of the universities of Paris (since 2006).

13.1.2 Awards

- Carrière Prize(1988)
- Prix Sophie Germain of the French Academy of Sciences (2004),
- Humboldt Prize in Germany (2004)
- French Legion of Honor in 2010.
- American Mathematical Society Fellowship (2012).[3]
- Foreign honorary member of the American Academy of Arts and Sciences, 2013.[4]

13.1.3 Articles

- Berestycki, Henri; Roquejoffre, Jean-Michel; Rossi, Luca; The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol.* 66 (2013), no. 4-5, 743–766.
- Barthélemy, Marc; Nadal, Jean-Pierre; Berestycki, Henri Disentangling collective trends from local dynamics. *Proc. Natl. Acad. Sci. USA* 107 (2010), no. 17, 7629–7634.
- Berestycki, Henri; Hamel, François; Nadirashvili, Nikolai Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Comm. Math. Phys.* 253 (2005), no. 2, 451–480.

- Berestycki, Henri; Hamel, François Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* 55 (2002), no. 8, 949–1032.
- Berestycki, H.; Caffarelli, L. A.; Nirenberg, L. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* 81 (1996), no. 2, 467–494.
- Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S. The principal eigenvalue and maximum principle for 2nd-order elliptic operators in general domains. *Comm. Pure Appl. Math.* 47 (1994), no. 1, 47–92.
- Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345; II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 347–375.
- Bahri, Abbas; Berestycki, Henri A perturbation method in critical point theory and applications. *Trans. Amer. Math. Soc.* 267 (1981), no. 1, 1–32. □

13.2 Wikipedia/Haïm Brezis

Haïm Brezis.

- **Born.** Jun 1m 1944 (age 76). Riom-ès-Montagnes, Cantal, France.
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.**
  - Brezis-Gallouet inequality
  - Bony-Brezis theorem
  - Brezis-Lieb lemma

Scientific career.

- **Fields.** Mathematics.
- **Institutions.** Pierre and Marie Curie University.
- **Doctoral advisor.**
  - Gustave Choquet
  - Jacques-Louis Lions
- **Doctoral students.**
  - Abbas Bahri
  - Henri Berestycki
  - Jean-Michel Coron
  - Jesús Ildefonso Díaz
  - Pierre-Louis Lions
  - Juan Luis Vázquez Suárez

Haïm Brezis (born Jun 1, 1944) is a French mathematician who works in functional analysis and partial differential equations.

13.2.1 Biography

Born in Riom-ès-Montagnes, Cantal, France.

Brezis is the son of a Romanian immigrant father, who came to France in the 1930s, and a Jewish mother who fled from the Netherlands.

His wife, Michal Govrin, a native Israeli, works as a novelist, poet, and theater director.[1]

Brezis received his Ph.D. from the University of Paris in 1972 under the supervision of Gustave Choquet.

He is currently a Professor at the Pierre and Marie Curie University and a Visiting Distinguished Professor at Rutgers University.

He is a member of the Academia Europaea (1988) and a foreign associate of the United States National Academy of Sciences (2003).

In 2012 he became a fellow of the American Mathematical Society.[2]

He holds honorary doctorates from several universities including National Technical University of Athens.[3]

Brezis is listed as an ISI highly cited researcher.[4]

13.2.2 Works

- *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (1973)
- *Analyse Fonctionnelle.* Théorie et Applications (1983)
- *Haïm Brezis. Un mathématicien juif.* Entretien Avec Jacques Vauthier. Collection Scientifiques & Croyants. Editions Beauchesne, 1999. ISBN 978-2-7010-1335-0, ISBN 2-7010-1335-6
- *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer; 1st Edition. edition (November 10, 2010), ISBN 978-0-387-70913-0, ISBN 0-387-70913-4

13.2.3 See also

- Bony-Brezis theorem
- Brezis-Gallouet inequality
- Brezis-Lieb lemma □

13.3 Lawrence Chris Evans

13.4 Wikipedia/Herbert Federer

Herbert Federer (Jul 23, 1920 – Apr 21, 2010)["NAS Membership Directory: Federer, Herbert". National Academy of Sciences. Retrieved Jun 15, 2010.] was an American mathematician.

He is 1 of the creators of geometric measure theory, at the meeting point of differential geometry and mathematical analysis.[Parks, H. (2012) *Remembering Herbert Federer (1920–2010)*, NAMS 59(5), 622–631.]

13.4.1 Career

Federer was born Jul 23, 1920, in Vienna, Austria.

After emigrating to the US in 1938, he studied mathematics and physics at the University of California, Berkeley, earning the Ph.D. as a student of Anthony Morse in 1944.

He then spent virtually his entire career as a member of the Brown University Mathematics Department, where he eventually retired with the title of Professor Emeritus.

Federer wrote more than 30 research papers in addition to his book *Geometric measure theory*.

The Mathematics Genealogy Project assigns him 9 Ph.D. students and well over a hundred subsequent descendants.

His most productive students include the late Frederick J. Almgren, Jr. (1933–1997) a professor at Princeton for 35 years, and his last student, Robert Hardt, now at Rice University.

Federer was a member of the National Academy of Sciences.

In 1987, he and his Brown colleague Wendell Fleming won the American Mathematical Society's Steele Prize "for their pioneering work in *Normal & Integral currents*."

13.4.2 Normal & integral currents

Federer's mathematical work separates thematically into the periods before and after his watershed 1960 paper *Normal and integral currents*, co-authored with Fleming.

That paper provided the *1st satisfactory general solution to Plateau's problem* - the problem of finding a $(k+1)$-dimensional least-area surface spanning a given k-dimensional boundary cycle in n-dimensional Euclidean space.

Their solution inaugurated a new and fruitful period of research on a large class of *geometric variational problems* - especially *minimal surfaces* - via what came to be known as *Geometric Measure Theory*.

13.4.3 Earlier work

During the 15 or so years prior to that paper, Federer worked at the technical interface of geometry and measure theory.

He focused particularly on surface area, rectifiability of sets, and the extent to which one could substitute rectifiability for smoothness in the analysis of surfaces.

His 1947 paper on the *rectifiable subsets of n-space* characterized purely unrectifiable sets by their "invisibility" under almost all projections.

A. S. Besicovitch had proven this for 1-dimensional sets in the plane, but Federer's generalization, valid for subsets of arbitrary dimension in any Euclidean space, was a major technical accomplishment, and later played a key role in *Normal and Integral Currents*.

In 1958, Federer wrote *Curvature Measures*, a paper that took some early steps toward understanding 2nd-order properties of surfaces lacking the differentiability properties typically assumed in order to discuss curvature.

He also developed and named what he called the coarea formula in that paper.

That formula has become a standard analytical tool.

13.4.4 Geometric measure theory

Federer is perhaps best known for his treatise *Geometric Measure Theory*, published in 1969.[Goffman, Casper (1971). "Review: Geometric measure theory, by Herbert Federer" (PDF). Bull. Amer. Math. Soc. 77 (1): 27–35. doi:10.1090/s0002-9904-1971-12603-4.]

Intended as both a text and a reference work, the book is unusually complete, general and authoritative: its nearly 600 pages cover a substantial amount of linear and multilinear algebra, give a profound treatment of measure theory, integration and differentiation, and then move on to rectifiability, theory of currents, and finally, variational applications.

Nevertheless, the book's unique style exhibits a rare and artistic economy that still inspires admiration, respect - and exasperation.

A more accessible introduction may be found in F. Morgan's book listed below.

13.4.5 See also

- Integral current
- Federer-Morse theorem

13.4.6 External links

- Federer's page a Brown □

13.5 Peter Lax

13.6 Jacques-Louis Lions

- **Born.** May 3, 1928. Grasse, Alpes-Maritimes, France.
- **Died.** May 17, 2001 (aged 73).
- **Nationality.** French.
- **Alma mater.** University of Nancy.
- **Known for.** PDEs.
- **Awards.** Japan Prize (1991).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - École Polytechnique
  - Collège de France
- **Doctoral advisor.** Laurent Schwartz.
- **Doctoral students.**
  - Alain Bensoussan
  - Jean-Michel Bismut
  - Haïm Brezis
  - Erol Gelenbe
  - Roland Glowinski
  - Roger Temam

Jacques-Louis Lions ([1] 3 May 1928 - May 17, 2001) was a French mathematician who made contributions to the theory of partial differential equations and to stochastic control, among other areas.

He received the SIAM's John von Neumann Lecture prize in 1986 and numerous other distinctions.[2][3]

Lions is listed as an ISI highly cited researcher.[4]

13.6.1 Biography

After being part of the French Résistance in 1943 and 1944, J.-L. Lions entered the École Normale Supérieure in 1947.

He was a professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, and the École polytechnique.

In 1966 he sent an invitation to Gury Marchuk, the soviet mathematician to visit Paris.

This was hand delivered by General De Gaulle during his visit to Akademgorodok in June of that year.[5]

He joined the prestigious Collège de France as well as the French Academy of Sciences in 1973.

In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught and promoted the use of numerical simulations using finite elements integration.

Throughout his career, Lions insisted on the *use of mathematics in industry*, with a particular involvement in the French space program, as well as in domains such as energy and the environment.

This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992.

Lions was elected President of the International Mathematical Union in 1991 and also received the Japan Prize and the Harvey Prize that same year.[3]

In 1992, the University of Houston awarded him an honorary doctoral degree.

He was elected president of the French Academy of Sciences in 1996 and was also a Foreign Member of the Royal Society (ForMemRS)[6] and numerous other foreign academies.[2][3]

He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English and Russian, and major contributions to several collective works, including the 4000 pages of the monumental *Mathematical analysis and numerical methods for science and technology* (in collaboration with Robert Dautray), as well as the *Handbook of numerical analysis* in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994.[7]

Both father and son have received honorary doctorates from Heriot-Watt University in 1986 and 1995 respectively.[8]

13.6.2 Books

- with Enrico Magenes: *Problèmes aux limites non homogènes et applications.* 3 vols., 1968, 1970
- *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* 1968
- with L. Cesari: *Quelques méthodes de résolution des problèmes aux limites non linéaires.* 1969
- with Roger Dautray: *Mathematical analysis and numerical methods for science and technology.* 9 vols., 1984/5
- with Philippe Ciarlet: *Handbook of numerical analysis.* 7 vols.
- with Alain Bensoussan, Papanicolaou: *Asymptotic analysis of periodic structures.* North Holland 1978
- *Controlabilité exacte, perturbations et stabilisation de systèmes distribués*[9]
- with John E. Lagnese: *Modelling Analysis and Control of Thin Plates.* □

13.7 Andrew Joseph Majda

Resources – Tài nguyên.

1. [Eng+23]. Bjorn Engquist, Panagiotis Souganidis, Samuel N. Stechmann, Vlad Vicol. *In memory of Andrew J. Majda.*

   "He was hard working until the end even though he suffered from serious health issues for quite some time."

   "He advocated a philosophy for applied mathematics research that involves the interaction of math theory, asymptotic modeling, numerical modeling, and observed and experimental data ... Andy Majda's modus operandi of modern applied mathematics, as a symbiotic relationship between (i) rigorous mathematical theory, (ii) numerical analysis and numerical modeling, (iii) observed phenomena and experimental data, and (iv) qualitative and/or asymptotic modeling [Maj00]."

   "Andy's legacy lives on in the mathematical science he created, but also in the many students & postdocs he so enthusiastically taught & mentored."

   "The period at UCLA was followed by 5 years at Berkeley, 1979–1984. During this productive time, he developed "Majda's model" for combustion in reactive flows, & together with Tosio Kato & Tom Beale derived "Beale-Kato-Majda criterion," which characterizes a putative incompressible Euler singularity in terms of the accumulation of vorticity [BKM84]."

   "At Courant, Andy shifted his research efforts to cross-disciplinary research in modern applied mathematics with climate–atmosphere–ocean science."

13.8 Vladimir Mazya

13.9 Jindřich Nečas

13.10 Louis Nirenberg

[Váz20]

13.11 Stanley Osher

13.12 Wikipedia/Laurent Schwartz

Laurent Schwartz.

- **Born.** Mar 5, 1915. Paris, France.
- **Died.** Jul 4, 2002 (aged 87). Paris, France.
- **Nationality.** French.
- **Alma mater.** Ecole Normale Supérieure.
- **Known for.**
  - Theory of Distributions
  - Schwartz kernel theorem
  - Schwartz space
  - Schwartz-Bruhat function
  - Radonifying operator
  - Cylinder set measure
- **Awards.** Fields Medal (1950).

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - University of Strashbourg
  - University of Nancy
  - University of Grenoble
  - École Polytechnique
  - Université de Paris VII
- **Doctoral advisor.** Georges Valiron.
- **Doctoral students.**
  - Maurice Audin
  - Georges Glaeser
  - Alexander Grothendieck
  - Jacques-Louis Lions
  - Bernard Malgrange
  - André Martineau
  - Bernard Maurey
  - Leopoldo Nachbin
  - Henri Hogbe Nlend
  - Gilles Pisier
  - François Treves
- **Influenced.** Per Enflo.

Laurent-Moïse Schwartz (Mar 5, 1915 - Jul 4, 2002) was a French mathematician.

He pioneered the theory of distributions, which gives a well-defined meaning to objects such as the Dirac delta function.

He was awarded the Fields Medal in 1950 for his work on the theory of distributions.

For several years he taught at the École polytechnique.

13.12.1 Biography

Family Laurent Schwartz came from a Jewish family of Alsatian origin, with a strong scientific background: his father was a well-known surgeon, his uncle Robert Debré (who contributed to the creation of UNICEF) was a famous pediatrician, and his great-uncle-in-law, Jacques Hadamard, was a famous mathematician.

During his training at Lycée Louis-le-Grand to enter the École Normale Supérieure, he fell in love with Marie-Hélène Lévy, daughter of the probabilist Paul Lévy who was then teaching at the École polytechnique.

Later they would have 2 children, Marc-André and Claudine.

Marie-Hélène was gifted in mathematics as well, as she contributed to the geometry of singular analytic spaces and taught at the University of Lille.

Angelo Guerraggio describes "Mathematics, politics and butterflies" as his "3 great loves".[1]

Education According to his teachers, Schwartz was an exceptional student.

He was particularly gifted in Latin, Greek and mathematics.

1 of his teachers told his parents: "*Beware, some will say your son has a gift for languages, but he is only interested in the scientific and mathematical aspect of languages: he should become a mathematician.*"

In 1934, he was admitted at the École Normale Supérieure, and in 1937 he obtained the agrégation (with rank 2).

World War II As a man of Trotskyist affinities and Jewish descent, life was difficult for Schwartz during World War II.

He had to hide and change his identity to avoid being deported after Nazi Germany overran France.

He worked for the University of Strasbourg (which had been relocated in Clermont-Ferrand because of the war) under the name of Laurent-Marie Sélimartin, while Marie-Hélène used the name Lengé instead of Lévy.

Unlike other mathematicians at Clermont-Ferrand such as Feldbau, the couple managed to escape the Nazis.

Later career Schwartz taught mainly at École Polytechnique, from 1958 to 1980.

At the end of the war, he spent one year in Grenoble (1944), then in 1945 joined the University of Nancy on the advice of Jean Delsarte and Jean Dieudonné, where he spent 7 years.

He was both an influential researcher and teacher, with students such as Bernard Malgrange, Jacques-Louis Lions, François Bruhat and Alexander Grothendieck.

He joined the science faculty of the University of Paris in 1952.

In 1958 he became a teacher at the École polytechnique after having at 1st refused this position.

From 1961 to 1963 the École polytechnique suspended his right to teach, because of his having signed the Manifesto of the 121 about the Algerian war, a gesture not appreciated by Polytechnique's military administration.

However, Schwartz had a lasting influence on mathematics at the École polytechnique, having reorganized both teaching and research there.

In 1965 he established the Centre de mathématiques Laurent-Schwartz (CMLS) as its 1st director.

In 1973 he was elected corresponding member of the French Academy of Sciences, and was promoted to full membership in 1975.

13.12.2 Mathematical legacy

In 1950 at the International Congress of Mathematicians, Schwartz was a plenary speaker[Schwartz, Laurent (1950). "*Théorie des noyaux*" (PDF). In: Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., Aug 30–Sep 6, 1950. vol. 1. pp. 220–230.] and was awarded the Fields Medal for his work on distributions.

He was the 1st French mathematician to receive the Fields medal.

Because of his sympathy for Trotskyism, Schwartz encountered serious problems trying to enter the United States to receive the medal; however, he was ultimately successful.

The theory of distributions clarified the (then) mysteries of the Dirac delta function and Heaviside step function.

It helps to extend the theory of Fourier transforms and is now of critical importance to the theory of partial differential equations.

13.12.3 Popular science

Throughout his life, Schwartz actively worked to promote science and bring it closer to the general audience.

Schwartz said:

> *"What are mathematics helpful for? Mathematics are helpful for physics.*
> *Physics helps us make fridges.*
> *Fridges are made to contain spiny lobsters, and spiny lobsters help mathematicians who eat them and have hence better abilities to do mathematics, which are helpful for physics, which helps us make fridges which…"[3]*

13.12.4 Entomology

Clanis schwartzi Paratype MHNT.

His mother, who was passionate about natural science, passed on her taste for entomology to Laurent.

His personal collection of 20,000 Lepidoptera specimens, collected during his various travels was bequeathed to the Muséum national d'histoire naturelle), the Science Museum of Lyon, the Museum of Toulouse and the Museo de Historia Natural Alcide d'Orbigny in Cochabamba (Bolivia).

Several species discovered by Schwartz bear his name.

13.12.5 Personal ideology

Apart from his scientific work, Schwartz was a well-known outspoken intellectual.

As a young socialist influenced by Leon Trotsky, Schwartz opposed the totalitarianism of the Soviet Union, particularly under Joseph Stalin.

Schwartz ultimately rejected Trotskyism for democratic socialism.

On his religious views, Schwartz called himself an atheist.[4]

13.12.6 Books

Research articles

- *Œuvres scientifiques. I.*

  With a general introduction to the works of Schwartz by Claude Viterbo and an appreciation of Schwartz by Bernard Malgrange.

  With 1 DVD.

  Documents Mathématiques (Paris), 9. Société Mathématique de France, Paris, 2011. x+523 pp. ISBN 978-2-85629-317-1

  > the 1st half of his works in analysis and partial differential equations.
  > After a preface by Claude Viterbo, which includes a few photos, one will find a note by Schwartz himself about his works, followed by a few original documents (letters, course notes), a presentation by Bernard Malgrange of the theory of distributions for which Schwartz received the Fields Medal in 1950, and a selection of articles covering the period 1944–1954.

- *Œuvres scientifiques. II.*

  With an appreciation of Schwartz by Alain Guichardet.

  With 1 DVD.

  Documents Mathématiques (Paris), 10.

  Société Mathématique de France, Paris, 2011. x+507 pp. ISBN 978-2-85629-318-8

  > the 2nd half of his works in analysis and partial differential equations.
  > After a note by Alain Guichardet on Schwartz and his seminars, one will find a selection of articles covering the period 1954–1966.

- *Œuvres scientifiques. III.*

  With appreciations of Schwartz by Gilles Godefroy and Michel Émery.

  With 1 DVD.

  Documents Mathématiques (Paris), 11. Société Mathématique de France, Paris, 2011. x+619 pp. ISBN 978-2-85629-319-5

  > his works on Banach space theory (1968–1987), introduced by Gilles Godefroy, and on probability theory (1970–1996), presented by Michel Émery, as well as some articles of a historical nature (1955–1994).

Technical books

- *Analyse hilbertienne.* Collection Méthodes. Hermann, Paris, 1979. ii+297 pp. ISBN 2-7056-5897-1
- *Application of distributions to the theory of elementary particles in quantum mechanics.* Gordon and Breach, New York, NY, 1968. 144pp. ISBN 9780677300900
- *Cours d'analyse. 1.* 2nd edition. Hermann, Paris, 1981. xxix+830 pp. ISBN 2-7056-5764-9
- *Cours d'analyse. 2.* 2nd edition. Hermann, Paris, 1981. xxiii+475+21+75 pp. ISBN 2-7056-5765-7

5 *Étude des sommes d'exponentielles. 2ième éd.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris 1959 151 pp.

- *Geometry and probability in Banach spaces.* Based on notes taken by Paul R. Chernoff. Lecture Notes in Mathematics, 852. Springer-Verlag, Berlin-New York, 1981. x+101 pp. ISBN 3-540-10691-X

- *Lectures on complex analytic manifolds.* With notes by M. S. Narasimhan. Reprint of the 1955 edition. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 4. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1986. iv+182 pp. ISBN 3-540-12877-8

- *Mathematics for the physical sciences.* Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966 358 pp.

- *Radon measures on arbitrary topological spaces and cylindrical measures.* Tata Institute of Fundamental Research Studies in Mathematics, No. 6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. xii+393 pp.

- *Semimartingales and their stochastic calculus on manifolds.* Edited and with a preface by Ian Iscoe. Collection de la Chaire Aisenstadt. Presses de l'Université de Montréal, Montreal, QC, 1984. 187 pp. ISBN 2-7606-0660-0

- *Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes.* Lecture Notes in Mathematics, 780. Springer, Berlin, 1980. xv+132 pp. ISBN 3-540-09749-X

- Les tenseurs. *Suivi de "Torseurs sur un espace affine" by Y. Bamberger and J.-P. Bourguignon.* 2nd edition. Hermann, Paris, 1981. i+203 pp. ISBN 2-7056-1376-5

6 *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris 1966 xiii+420 pp.

Seminar notes

- *Séminaire Schwartz in Paris 1953 bis 1961.* Online edition: [1]

Popular books

- *Pour sauver l'université.* Editions du Seuil, 1983. 122 pp. ISBN 2020065878

- *A mathematician grappling with his century.* Translated from the 1997 French original by Leila Schneps. Birkhäuser Verlag, Basel, 2001. viii+490 pp. ISBN 3-7643-6052-6

13.12.7 See also

- Schwartz distribution
- Schwartz kernel theorem
- Schwartz space
- Schwartz-Bruhat function
- Nicolas Bourbaki □

13.13 Wikipedia/Roger Temam

Roger Meyer Temam.

- **Born.** May 19, 1940 (age 80).
- **Nationality.** French.
- **Alma mater.** University of Paris.
- **Known for.** Navier-Stokes equations.

Scientific career.

- **Fields.** Applied mathematics.
- **Institutions.**
  - Paris-Sud University (Orsay)
  - Indiana University
- **Doctoral advisor.** Jacques-Louis Lions.

- **Doctoral students.**
  - Etienne Pardoux
  - Denis Serre

Roger Meyer Temam (born May 19, 1940) is a French applied mathematician working in numerical analysis, nonlinear partial differential equations and fluid mechanics.

He graduated from the University of Paris - the Sorbonne in 1967, completing a doctorate (*thèse d'Etat*) under the direction of Jacques-Louis Lions.

He has published over 400 articles, as well as 12 (authored or co-authored) books.

13.13.1 Scientific work

The 1st work of Temam in his thesis dealt with the *fractional steps method*.

Thereafter, "he has continually explored and developed new directions and techniques":[2]

- calculus of variations, and the notion of *duality* (book #7), developing the mathematical framework for *discontinuous* (in displacement) *solutions*; a concept later used for his works on the *mathematical theory of plasticity* (book #5);
- mathematical formulation of the equilibrium of a plasma in a cavity, expressed as a nonlinear free boundary problem;[R. Temam, A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma, *Arch. Rational Mech. Anal.*, 60, 1975, 51–73.]
- Korteweg-de Vries equation;[R. Temam, Sur un problème non linéaire, *J. Math. Pures Appl.*, 48, 1969, 159–172.]
- Kuramoto-Sivashinsky equation;[5]
- Euler equations in a bounded domain;[R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.*, 20, 1975, 32–43.]
- infinite-dimensional dynamical systems theory.

  In particular, he studied the existence of the finite-dimensional global attractor for many dissipative equations of mathematical physics, including the incompressible Navier-Stokes equations.[P. Constantin, C. Foias, O. Manley and R. Temam, Determining modes and fractal dimension of turbulent flows, *J. Fluid Mech.*, 150, 1985, 427–440.][C. Foias, O.P. Manley and R. Temam, Physical estimates of the number of degrees of freedom in free convection, *Phys. Fluids*, 29, 1986, 3101–3103.]

  He was also the co-founder of the notion of *inertial manifolds*[C. Foias, G.R. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Diff. Equ.*, 73, 1988, 309–353.] together with Ciprian Foias and George R. Sell and of exponential attractors[A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations*, Collection Recherches en Mathématiques Appliquées, Masson, Paris, and John Wiley, England, 1994.] together with Alp Eden, Ciprian Foias and Basil Nicolaenko;[2]
- optimal control of the incompressible Navier-Stokes equations as a tool for the *control of turbulence*;[F. Abergel and R. Temam, On some control problems in fluid mechanics, *Theoret. Comput. Fluid Dynamics*, 1, 1990, 303–325.]
- boundary layer phenomena for incompressible flows.[12]

Temam's main activities concern the study of geophysical flows, the atmosphere and oceans.[2]

This started in the 1990s by collaboration with Jacques-Louis Lions and Shouhong Wang.[J.L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of the atmosphere and applications, *Nonlinearity*, 5, 1992, 237–288.][J.L. Lions, R. Temam and S. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5, 1992, 1007–1053.][M. Coti Zelati, M. Frémond, R. Temam and J. Tribbia, Uniqueness, regularity and maximum principles for the equations of the atmosphere with humidity and saturation, *Physica D*, 264, 2013, 49-65, https://doi.org/10.1016/j.physd.2013.08.007][Y. Cao, M. Hamouda, R. Temam, J. Tribbia and X. Wang, The equations of the multi-phase humid atmosphere expressed as a quasi variational inequality, *Nonlinearity*, 31, 2018, 4692-4723, https://doi.org/10.1088/1361-6544/aad525.]

According to the Mathematical Genealogy Project database,[17][18] he holds the first position in the top 50 advisors.

More than 30 of his students are now full professors all over the world, and have themselves many descendants.[19]

13.13.2 Administrative activities

Temam became a professor at the Paris-Sud University at Orsay in 1968.

There, he co-founded the Laboratory of Numerical and Functional Analysis which he directed from 1972 to 1988.

He was also a *Maître de Conférences* at the Ecole Polytechnique in Paris from 1968 to 1986.[20]

In 1983, Temam co-founded the French Société de Mathématiques Appliquées et Industrielles (SMAI), analogous to the Society for Industrial and Applied Mathematics (SIAM), and served as its 1st president.[21]

He was also 1 of the founders of the International Congress on Industrial and Applied Mathematics (ICIAM) series and was the chair of the steering committee of the 1st ICIAM meeting held in Paris in 1987; and the chair of the standing committee of the 2nd ICIAM meeting held in Washington, D.C., in 1991.[22]

He was the Editor-in-Chief of the mathematical journal M2AN[23] from 1986 to 1997.

Temam has been the Director of the Institute for Scientific Computing & Applied Mathematics (ISCAM)[24] at Indiana University since 1986 (co-director with Ciprian Foias from 1986 to 1992).

He is also a College Professor (part-time till 2003) and he has been a Distinguished Professor since 2014.[25]

13.13.3 Books

1. (with G.-M. Gie, M. Hamouda and C.-Y. Jung): *Singular perturbations and boundary layers*, Springer-Verlag, New-York, 2018.
2. (with A. Miranville): *Mathematical Modelling in Continuum Mechanics*, Cambridge University Press, 2001. French Translation, Springer-Verlag France, 2002. Chinese Translation, Tsinghua University Press, 2004. 2nd English Edition 2005. Russian translation, Moskva Linom, 2013.
3. (with T. Dubois and F. Jauberteau): *Dynamic, multilevel methods and the numerical simulation of turbulence*; Cambridge University Press, 1999.
4. *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New-York, Applied Mathematical Sciences Series, vol. 68, 1988. 2nd augmented edition, 1997. Reprinted in China by Beijing World Publishing Corp., 2000.
5. *Mathematical Problems in Plasticity*, Gauthier-Villars, Paris, 1983 (in French). English Transl., Gauthier-Villars, New-York, 1985. Russian Transl., Nauk, Moscow, 1991. "Republished by Dover books in Physics, 2018."
6. *Navier-Stokes Equations*, North-Holland Pub. Company, in English, 1977, 500 pages. Revised editions 1979, 1984 and 1985. Russian Translation, Mir, Moscow, 1981. "Republished in the AMS-Chelsea Series, AMS, Providence, 2001."
7. (with I. Ekeland): *Convex Analysis and Variational Problems*. Dunod, Paris, 1974, 350 pages (in French). English Translation, North-Holland, Amsterdam, 1976. Russian Translation, Mir, Moscow, 1979. "English version republished in the Series 'Classics in Applied Mathematics', SIAM, Philadelphia, 1999."

13.13.4 Awards & honors

- Fellow of the American Academy of Arts and Sciences (2015),[26] of the American Mathematical Society (2013),[27] of the American Association for the Advancement of Science (2011),[28] of the Society for Industrial and Applied Mathematics (2009).[29]
- Knight of the Legion of Honor, France, 2012.[30]
- Member of the French Academy of Sciences since 2007.[31] □

13.14 Wikipedia/Karl Weierstrass

Karl Weierstrass/Karl Weierstraß.

- **Born.** Oct 31, 1815. Ostenfelde, Province of Westphalia, Kingdom of Prussia.
- **Died.** Feb 19, 1897 (aged 81). Berlin, Province of Brandenburg, Kingdom of Prussia.
- **Nationality.** German.
- **Alma mater.**
  - University of Bonn
  - Münster Academy
- **Known for.**
  - Weierstrass function
  - Weierstrass product inequality
  - (ε, δ)-definition of limit
  - Weierstrass-Erdmann condition
  - Weierstrass theorems
  - Bolzano-Weierstrass theorem
- **Awards.**
  - PhD (Hon): University of Königsberg (1854)
  - Copley Medal (1895)

Scientific career.

- **Fields.** Mathematics.
- **Institutions.**
  - Gewerbeinstitut
  - Friedrich Wilhelm University
- **Academic advisors.** Christoph Gudermann.
- **Doctoral students.**
  - Nikolai Bugaev
  - Georg Cantor
  - Georg Frobenius
  - Lazarus Fuchs
  - Wilhelm Killing
  - Leo Königsberger
  - Sofia Kovalevskaya
  - Mathias Lerch
  - Hans von Mangoldt
  - Eugen Netto
  - Adolf Piltz
  - Carl Runge
  - Arthur Schoenflies
  - Friedrich Schottky
  - Hermann Schwarz
  - Ludwig Stickelberger
  - Ernst Kötter

Karl Theodor Wilhelm Weierstrass (German: *Weierstraß*;[Duden. *Das Aussprachewörterbuch*. 7. Auflage. Bibliographisches Institut, Berlin 2015, ISBN 978-3-411-04067-4] Oct 31, 1815 - Feb 19, 1897) was a German mathematician often cited as the "*father of modern analysis*".

Despite leaving university without a degree, he studied mathematics and trained as a school teacher, eventually teaching mathematics, physics, botany and gymnastics.[Weierstrass, Karl Theodor Wilhelm. (2018). In Helicon (Ed.), *The Hutchinson unabridged encyclopedia with atlas and weather guide*. [Online]. Abington: Helicon. Available from: link [Accessed Jul 8, 2018].]

He later received an honorary doctorate and became professor of mathematics in Berlin.

Among many other contributions, Weierstrass formalized the definition of the continuity of a function, proved the intermediate value theorem and the Bolzano-Weierstrass theorem, and used the latter to study the properties of continuous functions on closed bounded intervals.

13.14.1 Biography

Weierstrass was born in Ostenfelde, part of Ennigerloh, Province of Westphalia.[O'Connor, J. J.; Robertson, E. F. (October 1998). "*Karl Theodor Wilhelm Weierstrass*". School of Mathematics and Statistics, University of St Andrews, Scotland. Retrieved Sep 7, 2014.]

Weierstrass was the son of Wilhelm Weierstrass, a government official, and Theodora Vonderforst.

His interest in mathematics began while he was a gymnasium student at the Theodorianum in Paderborn.

He was sent to the University of Bonn upon graduation to prepare for a government position.

Because his studies were to be in the fields of law, economics, and finance, he was immediately in conflict with his hopes to study mathematics.

He resolved the conflict by paying little heed to his planned course of study but continuing private study in mathematics.

The outcome was that he left the university without a degree.

He then studied mathematics at the Münster Academy (which was even then famous for mathematics) and his father was able to obtain a place for him in a teacher training school in Münster.

Later he was certified as a teacher in that city.

During this period of study, Weierstrass attended the lectures of Christoph Gudermann and became interested in elliptic functions.

In 1843 he taught in Deutsch Krone in West Prussia and since 1848 he taught at the Lyceum Hosianum in Braunsberg.

Besides mathematics he also taught physics, botany, and gymnastics.[3]

Weierstrass may have had an illegitimate child named Franz with the widow of his friend Carl Wilhelm Borchardt.[Biermann, Kurt-R.; Schubring, Gert (1996). "Einige Nachträge zur Biographie von Karl Weierstraß. (German) [Some postscripts to the biography of Karl Weierstrass]". *History of mathematics*. San Diego, CA: Academic Press. pp. 65–91.]

After 1850 Weierstrass suffered from a long period of illness, but was able to publish mathematical articles that brought him fame and distinction.

The University of Königsberg conferred an honorary doctor's degree on him on Mar 31, 1854.

In 1856 he took a chair at the *Gewerbeinstitut* in Berlin (an institute to educate technical workers which would later merge with the *Bauakademie* to form the Technical University of Berlin).

In 1864 he became professor at the Friedrich-Wilhelms-Universität Berlin, which later became the Humboldt Universität zu Berlin.

In 1870, at the age of 55, Weierstrass met Sofia Kovalevsky whom he tutored privately after failing to secure her admission to the University. They had a fruitful intellectual, but troubled personal, relationship that "far transcended the usual teacher-student relationship".

The misinterpretation of this relationship and Kovalevsky's early death in 1891 was said to have contributed to Weierstrass' later ill-health.

He was immobile for the last 3 years of his life, and died in Berlin from pneumonia.[*Dictionary of scientific biography*. Gillispie, Charles Coulston,, American Council of Learned Societies. New York. p. 223. ISBN 978-0-684-12926-6. OCLC 89822.]

13.14.2 Mathematical contributions

Soundness of calculus Weierstrass was interested in the soundness of calculus, and at the time there were somewhat ambiguous definitions of the foundations of calculus so that important theorems could not be proven with sufficient rigor.

Although Bolzano had developed a reasonably rigorous definition of a limit as early as 1817 (and possibly even earlier) his work remained unknown to most of the mathematical community until years later, and many mathematicians had only vague definitions of limits and continuity of functions.

The basic idea behind Delta-epsilon proofs is, arguably, 1st found in the works of Cauchy in the 1820s.

- Grabiner, Judith V. (March 1983), "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" (PDF), *The American Mathematical Monthly*, 90 (3): 185–194, doi:10.2307/2975545, JSTOR 2975545

- Cauchy, A.-L. (1823), "Septième Leçon – Valeurs de quelques expressions qui se présentent sous les formes indéterminées $\frac{\infty}{\infty}, \infty^0, \ldots$ Relation qui existe entre le rapport aux différences finies et la fonction dérivée", *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal*, Paris, archived from the original on 2009-05-04, retrieved 2009-05-01, p. 44.

Cauchy did not clearly distinguish between continuity and uniform continuity on an interval.

Notably, in his 1821 *Cours d'analyse*, Cauchy argued that the (pointwise) limit of (pointwise) continuous functions was itself (pointwise) continuous, a statement interpreted as being incorrect by many scholars.

The correct statement is rather that the uniform limit of continuous functions is continuous (also, the uniform limit of uniformly continuous functions is uniformly continuous).

This required the concept of uniform convergence, which was 1st observed by Weierstrass's advisor, Christoph Gudermann, in an 1838 paper, where Gudermann noted the phenomenon but did not define it or elaborate on it.

Weierstrass saw the importance of the concept, and both formalized it and applied it widely throughout the foundations of calculus.

The formal definition of continuity of a function, as formulated by Weierstrass, is as follows:

> $f(x)$ is continuous at $x = x_0$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every x in the domain of f, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
> In simple English, $f(x)$ is continuous at a point $x = x_0$ if for each x close enough to x_0, the function value $f(x)$ is very close to $f(x_0)$, where the "close enough" restriction typically depends on the desired closeness of $f(x_0)$ to $f(x)$.

Using this definition, he proved the Intermediate Value Theorem.

He also proved the Bolzano-Weierstrass theorem and used it to study the properties of continuous functions on closed and bounded intervals.

Calculus of variations Weierstrass also made advances in the field of calculus of variations.

Using the apparatus of analysis that he helped to develop, Weierstrass was able to give a complete reformulation of the theory that paved the way for the modern study of the calculus of variations.

Among several axioms, Weierstrass established a necessary condition for the existence of strong extrema of variational problems.

He also helped devise the Weierstrass-Erdmann condition, which gives sufficient conditions for an extremal to have a corner along a given extremum and allows one to find a minimizing curve for a given integral.

Other analytical theorems See also: List of things named after Karl Weierstrass.

- Stone-Weierstrass theorem
- Casorati-Weierstrass-Sokhotski theorem
- Weierstrass's elliptic functions
- Weierstrass function
- Weierstrass M-test
- Weierstrass preparation theorem
- Lindemann-Weierstrass theorem
- Weierstrass factorization theorem
- Enneper-Weierstrass parameterization

13.14.3 Students

- Edmund Husserl
- Sofia Kovalevskaya
- Gösta Mittag-Leffler
- Hermann Schwarz
- Carl Johannes Thomae
- Georg Cantor

13.14.4 Honors & awards

The lunar crater Weierstrass and the asteroid 14100 Weierstrass are named after him.

Also, there is the Weierstrass Institute for Applied Analysis and Stochastics in Berlin.

13.14.5 Selected works

- *Zur Theorie der Abelschen Funktionen* (1854)
- *Theorie der Abelschen Funktionen* (1856)
- *Abhandlungen-1*, Math. Werke. Bd. 1. Berlin, 1894
- *Abhandlungen-2*, Math. Werke. Bd. 2. Berlin, 1895
- *Abhandlungen-3*, Math. Werke. Bd. 3. Berlin, 1903
- *Vorl. ueber die Theorie der Abelschen Transcendenten*, Math. Werke. Bd. 4. Berlin, 1902
- *Vorl. ueber Variationsrechnung*, Math. Werke. Bd. 7. Leipzig, 1927

13.14.6 External links

- O'Connor, John J.; Robertson, Edmund F., "Karl Weierstrass", *MacTutor History of Mathematics archive*, University of St Andrews.
- Digitalized versions of Weierstrass's original publications are freely available online from the library of the *Berlin Brandenburgische Akademie der Wissenschaften*.
- Works by Karl Weierstrass at Project Gutenberg
- Works by or about Karl Weierstrass at Internet Archive □

14 Miscellaneous

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