Report

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1 Stationary Navier-Stokes equations

In order to optimize the shape design of air ducts in combustion engines, we consider a shape optimization problem subject to a stationary incompressible viscous Navier-Stokes equations in 3D with appropriate physical boundary conditions in the duct geometry. An inflow profile is given at the inlet, a no-slip boundary condition is imposed on the wall, and a do-nothing boundary condition on the outlet. To find optimal shapes, we choose a mixed cost functional to achieve the flow uniformity at the outlet and minimize the dissipated power of our fluid dynamics device in a well balanced way.

To model the flow in the considered shape, we use the following initial boundary value problem for the stationary incompressible viscous Navier–Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f}_{\text{in}}, & \text{on } \Gamma_{\text{in}}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_{\text{wall}}, \\ -\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{out}}. \end{cases} \tag{NS}$$

Here, $\Omega \subset \mathbb{R}^N$ an open connected set, $\mathbf{u} : \Omega \to \mathbb{R}^N$ and $p : \Omega \to \mathbb{R}$ denote the *velocity vector field* and the *kinematic pressure*, respectively. We assume that the *kinematic viscosity* $\nu > 0$ and the density of the external volume force $\mathbf{f} : \Omega \to \mathbb{R}^N$, the inflow profile $\mathbf{f}_{\text{in}} : \Gamma_{\text{in}} \to \mathbb{R}^N$ are given.

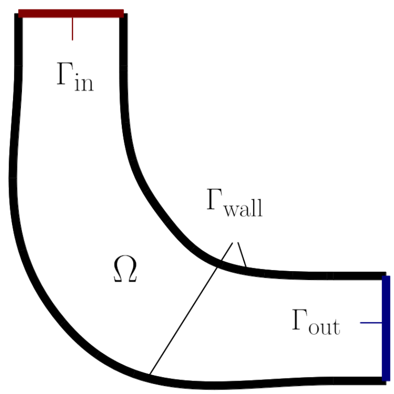

Figure 1: Simple sketch of a curved duct geometry.

1.1 Cost functionals

The uniformity of the flow leaving the outlet is an important design criterion of automotive air ducts to enhance the efficiency of distributing the air flow Othmer, 2008. To achieve this criterion, we minimize a cost functional capturing the distance between the normal component of the velocity and a given desired value of that quantity in the outlet:

$$J_1(\mathbf{u},\Omega) \coloneqq \frac{1}{2}\int_{\Gamma_{\text{out}}} (\mathbf{u}\cdot\mathbf{n} - \overline{u})^2 \,\mathrm{d}\Gamma, \text{ where } \overline{u} \coloneqq -\frac{1}{H_{N-1}(\Gamma_{\text{out}})}\int_{\Gamma_{\text{in}}} \mathbf{f}_{\text{in}}\cdot\mathbf{n}\mathrm{d}\Gamma, \tag{J_1}$$

where $H_{N-1}(\Gamma_{\text{out}})$ is the $(N-1)$-dimensional *Hausdorff measure* of the outflow Γ_{out}.

Another criterion engineers also want to minimize is the power dissipated by air ducts (and any fluid dynamics devices in general) Othmer, 2008. This dissipated power can be computed as the net inward flux of energy through the boundary of the considered tube:

$$J_2(\mathbf{u},p,\Omega) \coloneqq -\int_{\Gamma} \left(p + \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}\cdot\mathbf{n}\mathrm{d}\Gamma. \tag{J_2}$$

Concerning the regularity theory of Navier-Stokes equations with mixed boundary conditions (see, e.g., Maz'ya and Rossmann, 2007; Maz'ya and Rossmann, 2009), the spatial components of its solutions $(\mathbf{u},p)$ usually belong to the space $W^{1,2}(\Omega;\mathbb{R}^N)\times L^2(\Omega)$. Thus, the trace of p on the boundary Γ in the definition of $J_2(\mathbf{u},p,\Omega)$ is not well-defined, and we consider the following approximation of $J_2(\mathbf{u},p,\Omega)$, on account of the boundary condition on Γ_{wall}, instead:

$$\begin{aligned} J_2^\delta(\mathbf{u},p,\Omega) &\coloneqq -\frac{H_{N-1}(\Gamma_{\text{in}})}{\mathrm{m}_N(\Gamma_{\text{in}}^\delta)}\int_{\Gamma_{\text{in}}^\delta}\left(p+\frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}\cdot\mathbf{n}\mathrm{d}\mathbf{x} - \frac{H_{N-1}(\Gamma_{\text{out}})}{\mathrm{m}_N(\Gamma_{\text{out}}^\delta)}\int_{\Gamma_{\text{out}}^\delta}\left(p+\frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}\cdot\mathbf{n}\mathrm{d}\mathbf{x} \\ &= \int_\Omega k_\delta(\mathbf{x})\left(p+\frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}\cdot\mathbf{n}\mathrm{d}\mathbf{x}, \end{aligned} \tag{J_2^δ}$$

where m_N denotes the N-dimensional *Lebesgue measure* and

$$k_\delta(\mathbf{x}) \coloneqq -\frac{H_{N-1}(\Gamma_{\text{in}})}{\mathrm{m}_N(\Gamma_{\text{in}}^\delta)}\chi_{\Gamma_{\text{in}}^\delta}(\mathbf{x}) - \frac{H_{N-1}(\Gamma_{\text{out}})}{\mathrm{m}_N(\Gamma_{\text{out}}^\delta)}\chi_{\Gamma_{\text{out}}^\delta}(\mathbf{x}),\ \forall \mathbf{x}\in\Omega.$$

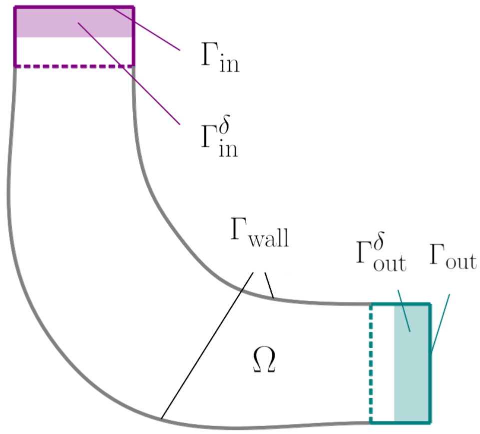

Figure 2: The duct geometry with δ-approximated inlet $\Gamma_{\text{in}}^{\delta}$ and δ-approximated outlet $\Gamma_{\text{out}}^{\delta}$.

Taking into account both objectives, we consider a mixed cost functional as a convex combination of the cost functionals above with a weighting parameter $\gamma \in [0,1]$:

$$\begin{aligned} J_{12}^{\delta,\gamma}(\mathbf{u},p,\Omega) &\coloneqq (1-\gamma)J_1(\mathbf{u},\Omega) + \gamma J_2^{\delta}(\mathbf{u},p,\Omega) \\ &= \frac{1-\gamma}{2}\int_{\Gamma_{\text{out}}} (\mathbf{u}\cdot\mathbf{n} - \overline{u})^2 \,\mathrm{d}\Gamma + \int_{\Omega} \gamma k_{\delta}(\mathbf{x}) \left(p + \frac{1}{2}|\mathbf{u}|^2\right) \mathbf{u}\cdot\mathbf{n}\mathrm{d}\mathbf{x}. \end{aligned} \tag{$J_{12}^{\delta,\gamma}$}$$

A typical PDE-constrained shape optimization problem (see, e.g., Delfour and J.-P. Zolésio, 2011; Sokołowski and Jean-Paul Zolésio, 1992) can be established by finding an admissible shape to minimize the mixed cost functional under the given Navier-Stokes equation: Optimize Ω over an appropriate class of admissible domains, denoted by $\mathcal{O}_{\text{ad}}$, such that the mixed cost functional $J_{12}^{\delta,\gamma}(\mathbf{u},p,\Omega)$ is minimized subject to (NS), i.e.:

$$\min_{\Omega\in\mathcal{O}_{\text{ad}}} J_{12}^{\delta,\gamma}(\mathbf{u},p,\Omega) \text{ such that } (\mathbf{u},p) \text{ solves (NS)}. \tag{sop}$$

2 Instationary Navier-Stokes equations

3 Smagorinsky turbulence models

4 k-ϵ turbulence models

Standard k-ϵ turbulence model:

$$\begin{cases} \partial_t\overline{\mathbf{u}} - \nabla\cdot((\nu+\nu_{\text{t}})\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} + \nabla\left(\overline{p} - \frac{2}{3}k\right) = \mathbf{f}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \partial_t k - \nabla\cdot\left(c_k\frac{k^2}{\epsilon}\nabla k\right) + \overline{\mathbf{u}}\cdot\nabla k = c_{\nu}\frac{k^2}{\epsilon}|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})|^2 - \epsilon, \\ \partial_t\epsilon - \nabla\cdot\left(c_{\varepsilon}\frac{k^2}{\epsilon}\nabla\epsilon\right) + \overline{\mathbf{u}}\cdot\nabla\epsilon = c_{\eta}k|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})|^2 - (c_{\varepsilon 2} - c_{\gamma})\frac{\epsilon^2}{k}, \end{cases} \tag{4.1}$$

where $\nu_{\text{t}} \coloneqq c_{\nu}\frac{k^2}{\epsilon}$.

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