On Approximating Solution of One-Dimensional Elliptic Problems by Using Finite Volume Method

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Friday 7th April, 2017

Abstract

This assignment aims at solving one-dimensional elliptic problems with Dirichlet boundary conditions numerically by using finite volume method on admissible meshes.

We also implement one-dimensional elliptic problems with Dirichlet-Neumann boundary condition by MATLAB scripts, which is based on our teacher's MATLAB scripts, at the end of this context.

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Dedicated to our beloved teachers, ***Le Anh Ha***
and ***Phung Thanh Tam****, who teach us*
Finite Volume Method course.

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1 A Finite Volume Method for the Dirichlet Problem

1.1 Formulation of Finite Volume Scheme

We now consider the finite volume method for the academic Dirichlet problem, namely a second order differential operator without time dependent terms and with homogeneous Dirichlet boundary conditions.

Let f be a given function from $(0,1)$ to $\mathbb{R}$, consider the following differential equation

$$-u''(x) = f(x), x \in (0,1) \tag{1.1}$$
$$u(0) = 0 \tag{1.2}$$
$$u(1) = 0 \tag{1.3}$$

If $f \in C([0,1],\mathbb{R})$, there exists a unique solution $u \in C^2([0,1],\mathbb{R})$ to Problem (1.1)-(1.3). In the sequel, this exact solution will be denoted by u. Note that the equation $-u'' = f$ can be written in the conservative form $div(\mathbf{F}) = f$ with $\mathbf{F} = -u'$.

In order to compute a numerical approximation to the solution of this equation, let us define a mesh, denoted by $\mathcal{T}$, of the interval $(0,1)$ consisting of N cells (or *control volume*), denoted by K_i, $i = 1,\dots,N$, and N points of $(0,1)$, denoted by x_i, $\dots, i = 1,\dots,N$, satisfying the following assumptions.

Definition 1.1 (Admissible one-dimensional mesh). An admissible mesh of $(0,1)$, denoted by $\mathcal{T}$, is given by a family $(K_i)_{i=1,\dots,N}$, $N \in \mathbb{N}^*$, such that $K_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, and a family $(x_i)_{i=0,\dots,N+1}$ such that

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < \tag{1.4}$$
$$< x_{i+\frac{1}{2}} < \cdots < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1 \tag{1.5}$$

One sets

$$h_i = m(K_i) = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \;\; i = 1,\dots,N \tag{1.6}$$

and therefore

$$\sum_{i=1}^{N} h_i = 1 \tag{1.7}$$

and

$$h_i^- = x_i - x_{i-\frac{1}{2}}, \;\; i = 1,\dots,N \tag{1.8}$$
$$h_i^+ = x_{i+\frac{1}{2}} - x_i, \;\; i = 1,\dots,N \tag{1.9}$$
$$h_{i+\frac{1}{2}} = x_{i+1} - x_i, \;\; i = 0,\dots,N \tag{1.10}$$
$$size(\mathcal{T}) = h = \max\{h_i, \;\; i = 1,\dots,N\} \tag{1.11}$$

The discrete unknowns are denoted by u_i, $i = 1,\dots,N$, and are expected to be some approximation of u in the cell K_i (the discrete unknown u_i can be viewed as an approximation of the mean value of u over K_i, i.e.

$$u_i = \frac{1}{m(K_i)} \int_{K_i} u(x)\,dx = \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{K_i} u(x)\,dx = \frac{1}{h_i} \int_{K_i} u(x)\,dx \quad (1.12)$$

or of the value of $u(x_i)$, or of other values of u in the control volume K_i, ...). The first equation (1.1) is integrated over each cell K_i and yields

$$-u'\left(x_{i+\frac{1}{2}}\right) + u'\left(x_{i-\frac{1}{2}}\right) = \int_{K_i} f(x)\,dx, \quad i = 1, \ldots, N \quad (1.13)$$

A reasonable choice for the approximation of $-u'\left(x_{i+\frac{1}{2}}\right)$ (at least, for $i = 1, \ldots, N-1$) seems to be the differential quotient

$$F_{i+\frac{1}{2}} = -\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \quad (1.14)$$

This approximation is consistent in the sense that, if $u \in C^2([0,1], \mathbb{R})$, then

$$F^*_{i+\frac{1}{2}} = -\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \quad (1.15)$$

$$= -u'\left(x_{i+\frac{1}{2}}\right) + O(h) \quad (1.16)$$

where $|O(h)| \le Ch$ with $C \in \mathbb{R}_+$ only depending on u.

Remark 1.2. We can investigate (1.15)-(1.16) further as the following. Using the Taylor series expansion for $u(x_{i+1})$ and $u(x_i)$ yields

$$u(x_{i+1}) = u\left(x_{i+\frac{1}{2}}\right) + \left(x_{i+1} - x_{i+\frac{1}{2}}\right) u'\left(x_{i+\frac{1}{2}}\right) \quad (1.17)$$

$$+ \frac{1}{2}\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2 u''\left(x_{i+\frac{1}{2}}\right) + O\left(h^3\right) \quad (1.18)$$

$$u(x_i) = u\left(x_{i+\frac{1}{2}}\right) + \left(x_i - x_{i+\frac{1}{2}}\right) u'\left(x_{i+\frac{1}{2}}\right) \quad (1.19)$$

$$+ \frac{1}{2}\left(x_i - x_{i+\frac{1}{2}}\right)^2 u''\left(x_{i+\frac{1}{2}}\right) + O\left(h^3\right) \quad (1.20)$$

Thus,

$$u(x_{i+1}) - u(x_i) = (x_{i+1} - x_i)\, u'\left(x_{i+\frac{1}{2}}\right) \quad (1.21)$$

$$+ \frac{1}{2}\left(\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2 - \left(x_i - x_{i+\frac{1}{2}}\right)^2\right) u''\left(x_{i+\frac{1}{2}}\right) + O\left(h^3\right) \quad (1.22)$$

We have two cases.

1. **Case 1.** $x_{i+\frac{1}{2}}$ *is the midpoint of segment* $[x_i, x_{i+1}]$, then

$$u'\left(x_{i+\frac{1}{2}}\right) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + O\left(h^2\right) \quad (1.23)$$

2. **Case 2.** *Otherwise,*

$$u'\left(x_{i+\frac{1}{2}}\right) = \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{x_{i+1} - x_i} + O\left(h\right) \tag{1.24}$$

Assume that x_i is the center of K_i, i.e., $x_i = \frac{1}{2}x_{i+\frac{1}{2}} + \frac{1}{2}x_{i-\frac{1}{2}}$. Let $\widetilde{u}_i$ denote the mean value over K_i of the exact solution u to Problem (1.1)-(1.3), i.e.

$$\widetilde{u}_i = \frac{1}{m\left(K_i\right)}\int_{K_i} u\left(x\right)dx = \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}\int_{K_i} u\left(x\right)dx = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u\left(x\right)dx \tag{1.25}$$

One may then remark that

$$\left|\widetilde{u}_i - u\left(x_i\right)\right| \leq Ch_i^2 \tag{1.26}$$

with some C only depending on u.

Proof of (1.26) Indeed,

$$\left|\widetilde{u}_i - u\left(x_i\right)\right| = \left|\frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u\left(x\right)dx - \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u\left(x_i\right)dx\right| \tag{1.27}$$

$$= \left|\frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(u\left(x\right) - u\left(x_i\right)\right)dx\right| \tag{1.28}$$

$$\leq \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left|u\left(x\right) - u\left(x_i\right)\right|dx \tag{1.29}$$

We now deal with the quantity $u\left(x\right) - u\left(x_i\right)$ with $x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. To this end, since $x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$, we can represent x as

$$x = \left(1-t\right)x_{i-\frac{1}{2}} + tx_{i+\frac{1}{2}} \text{ for } t \in [0,1] \tag{1.30}$$

Then, we have $dx = \left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right)dt = h_i dt$.

Using Taylor series expansion for $u\left(x\right)$ and $u\left(x_i\right)$ yields

$$u\left(x\right) = u\left(\left(1-t\right)x_{i-\frac{1}{2}} + tx_{i+\frac{1}{2}}\right) \tag{1.31}$$

$$= u\left(x_{i-\frac{1}{2}}\right) + t\left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right)u'\left(x_{i-\frac{1}{2}}\right) + O\left(h_i^2\right) \tag{1.32}$$

$$= u\left(x_{i-\frac{1}{2}}\right) + th_i u'\left(x_{i-\frac{1}{2}}\right) + O\left(h_i^2\right) \tag{1.33}$$

$$u\left(x_i\right) = u\left(\frac{1}{2}x_{i-\frac{1}{2}} + \frac{1}{2}x_{i+\frac{1}{2}}\right) \tag{1.34}$$

$$= u\left(x_{i-\frac{1}{2}}\right) + \frac{1}{2}\left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right)u'\left(x_{i-\frac{1}{2}}\right) + O\left(h_i^2\right) \tag{1.35}$$

$$= u\left(x_{i-\frac{1}{2}}\right) + \frac{1}{2}h_i u'\left(x_{i-\frac{1}{2}}\right) + O\left(h_i^2\right) \tag{1.36}$$

Substituting (1.33) and (1.36) into (1.29) yields

$$\frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}|u(x)-u(x_i)|\,dx \tag{1.37}$$

$$=\frac{1}{h_i}\int_0^1\left|\left(t-\frac{1}{2}\right)h_iu'\left(x_{i-\frac{1}{2}}\right)+O\left(h_i^2\right)\right|h_idt \tag{1.38}$$

$$=\left(\frac{1}{2}h_iu'\left(x_{i-\frac{1}{2}}\right)t^2-\frac{1}{2}h_iu'\left(x_{i-\frac{1}{2}}\right)t\right)\Bigg|_0^1+O\left(h_i^2\right) \tag{1.39}$$

$$=O\left(h_i^2\right) \tag{1.40}$$

i.e., there exists a constant C only depending on u such that $|\widetilde{u}_i-u(x_i)|\leq Ch_i^2$, as stated. □

It follows easily that

$$\frac{\widetilde{u}_{i+1}-\widetilde{u}_i}{h_{i+\frac{1}{2}}}=u'\left(x_{i+\frac{1}{2}}\right)+O(h),\quad i=1,\ldots,N-1 \tag{1.41}$$

Proof 1 of (1.41). Indeed, applying the appendix *"An Easy Change of Variables"* with

$$a=x_{i+\frac{1}{2}},b=x_{i+\frac{3}{2}},c=x_{i-\frac{1}{2}},d=x_{i+\frac{1}{2}},f=u \tag{1.42}$$

$$S(x)=\frac{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}{x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}}x+\frac{x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}-x_{i+\frac{1}{2}}^2}{x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}} \tag{1.43}$$

$$=\frac{h_i}{h_{i+1}}x+\frac{x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}-x_{i+\frac{1}{2}}^2}{h_{i+1}},\quad\forall x\in\left[x_{i+\frac{1}{2}},x_{i+\frac{3}{2}}\right] \tag{1.44}$$

$$S^{-1}(y)=\frac{x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}y+\frac{x_{i+\frac{1}{2}}x_{i+\frac{1}{2}}-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}} \tag{1.45}$$

$$=\frac{h_{i+1}}{h_i}y+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i},\quad\forall y\in\left[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right] \tag{1.46}$$

yields

$$\widetilde{u}_{i+1}=\frac{1}{h_{i+1}}\int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}}u(x)\,dx \tag{1.47}$$

$$=\frac{1}{h_{i+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}u\left(\frac{h_{i+1}}{h_i}x+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i}\right)\frac{h_{i+1}}{h_i}dx \tag{1.48}$$

$$=\frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}u\left(\frac{h_{i+1}}{h_i}x+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i}\right)dx \tag{1.49}$$

Using Taylor series expansion as before, we obtain

$$u(x)=u\left((1-t)\,x_{i-\frac{1}{2}}+tx_{i+\frac{1}{2}}\right),\ \text{for } t\in[0,1] \tag{1.50}$$

$$=u\left(x_{i+\frac{1}{2}}\right)+(t-1)\,h_iu'\left(x_{i+\frac{1}{2}}\right)+O\left(h_i^2\right) \tag{1.51}$$

and

$$u\left(\frac{h_{i+1}}{h_i}x+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i}\right) \tag{1.52}$$

$$=u\left(\frac{h_{i+1}}{h_i}\left((1-t)\,x_{i-\frac{1}{2}}+tx_{i+\frac{1}{2}}\right)+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i}\right),\ \text{for } t\in[0,1] \tag{1.53}$$

$$=u\left(x_{i+\frac{1}{2}}\right)+h_{i+1}u'\left(x_{i+\frac{1}{2}}\right)t+O\left(h_{i+1}^2\right) \tag{1.54}$$

where we have used

$$\frac{h_{i+1}}{h_i}\left((1-t)\,x_{i-\frac{1}{2}}+tx_{i+\frac{1}{2}}\right)+\frac{x_{i+\frac{1}{2}}^2-x_{i+\frac{3}{2}}x_{i-\frac{1}{2}}}{h_i}-x_{i+\frac{1}{2}} \tag{1.55}$$

$$=\frac{h_{i+1}}{h_i}\left((1-t)\,x_{i-\frac{1}{2}}+tx_{i+\frac{1}{2}}\right)-\frac{\left(x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}\right)x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}} \tag{1.56}$$

$$=\frac{h_{i+1}}{h_i}\left(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}\right)t \tag{1.57}$$

$$=h_{i+1}t \tag{1.58}$$

Using (1.50)-(1.54) yields

$$\frac{\widetilde{u}_{i+1}-\widetilde{u}_i}{h_{i+\frac{1}{2}}} \tag{1.59}$$

$$=\frac{\int_0^1\left(h_{i+1}u'\left(x_{i+\frac{1}{2}}\right)t-(t-1)\,h_iu'\left(x_{i+\frac{1}{2}}\right)+O\left(h_{i+1}^2\right)+O\left(h_i^2\right)\right)h_idt}{h_ih_{i+\frac{1}{2}}} \tag{1.60}$$

$$=\frac{\int_0^1\left((h_{i+1}-h_i)\,u'\left(x_{i+\frac{1}{2}}\right)t+h_iu'\left(x_{i+\frac{1}{2}}\right)+O\left(h_{i+1}^2+h_i^2\right)\right)dt}{h_{i+\frac{1}{2}}} \tag{1.61}$$

$$=\frac{\left((h_{i+1}-h_i)\,u'\left(x_{i+\frac{1}{2}}\right)\frac{t^2}{2}+h_iu'\left(x_{i+\frac{1}{2}}\right)t\right)\Big|_0^1+O\left(h_{i+1}^2+h_i^2\right)}{h_{i+\frac{1}{2}}} \tag{1.62}$$

$$=\frac{h_{i+1}+h_i}{2h_{i+\frac{1}{2}}}u'\left(x_{i+\frac{1}{2}}\right)+\frac{1}{h_{i+\frac{1}{2}}}O\left(h_{i+1}^2+h_i^2\right) \tag{1.63}$$

$$=u'\left(x_{i+\frac{1}{2}}\right)+O\left(h\right),\quad i=1,\ldots,N-1 \tag{1.64}$$

where the last equality is deduced from

$$h_{i+1}+h_i=x_{i+\frac{3}{2}}-x_{i+\frac{1}{2}}+x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}} \tag{1.65}$$

$$=x_{i+\frac{3}{2}}+x_{i+\frac{1}{2}}-\left(x_{i+\frac{1}{2}}+x_{i-\frac{1}{2}}\right) \tag{1.66}$$

$$=2x_{i+1}-2x_i \tag{1.67}$$

$$=2h_{i+\frac{1}{2}},\quad i=1,\ldots,N-1 \tag{1.68}$$

and

$$\frac{h_{i+1}^2 + h_i^2}{h_{i+\frac{1}{2}}} = \frac{2\left(h_{i+1}^2 + h_i^2\right)}{h_{i+1} + h_i} \text{ by (1.68)} \tag{1.69}$$

$$\leq \frac{2(h_{i+1} + h_i)^2}{h_{i+1} + h_i} \tag{1.70}$$

$$= 2\left(h_{i+1} + h_i\right) \tag{1.71}$$

$$= O\left(h\right), \quad i = 1, \ldots, N-1 \text{ by (1.11)} \tag{1.72}$$

So (1.41) is proved. □

Proof 2 of (1.41). Instead of using a change of variables, we can approach more directly by looking at the common point $x_{i+\frac{1}{2}}$ of K_i and K_{i+1} as follows.

Using Taylor series expansion for $\widetilde{u}_i$ and $\widetilde{u}_{i+1}$ yields

$$\widetilde{u}_i = \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u\left(x\right) dx \tag{1.73}$$

$$= \frac{1}{h_i} \int_0^1 u\left(\left(1-t\right) x_{i-\frac{1}{2}} + t x_{i+\frac{1}{2}}\right) h_i dt \tag{1.74}$$

$$= \int_0^1 \left(u\left(x_{i+\frac{1}{2}}\right) + \left(t-1\right) h_i u'\left(x_{i+\frac{1}{2}}\right) + O\left(h_i^2\right)\right) dt \tag{1.75}$$

$$\widetilde{u}_{i+1} = \frac{1}{h_{i+1}} \int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}} u\left(x\right) dx \tag{1.76}$$

$$= \frac{1}{h_{i+1}} \int_0^1 u\left(\left(1-t\right) x_{i+\frac{1}{2}} + t x_{i+\frac{3}{2}}\right) h_{i+1} dt \tag{1.77}$$

$$= \int_0^1 \left(u\left(x_{i+\frac{1}{2}}\right) + t h_{i+1} u'\left(x_{i+\frac{1}{2}}\right) + O\left(h_{i+1}^2\right)\right) dt \tag{1.78}$$

then

$$\frac{\widetilde{u}_{i+1} - \widetilde{u}_i}{h_{i+\frac{1}{2}}} \tag{1.79}$$

$$= \frac{\int_0^1 \left(\left(h_{i+1} - h_i\right) u'\left(x_{i+\frac{1}{2}}\right) t + h_i u'\left(x_{i+\frac{1}{2}}\right) + O\left(h_i^2\right) + O\left(h_{i+1}^2\right)\right) dt}{h_{i+\frac{1}{2}}} \tag{1.80}$$

$$= \frac{\left(\left(h_{i+1} - h_i\right) u'\left(x_{i+\frac{1}{2}}\right) \frac{t^2}{2} + h_i u'\left(x_{i+\frac{1}{2}}\right) t\right)\Big|_0^1 + O\left(h_i^2 + h_{i+1}^2\right)}{h_{i+\frac{1}{2}}} \tag{1.81}$$

$$= \frac{\frac{1}{2}\left(h_{i+1} - h_i\right) u'\left(x_{i+\frac{1}{2}}\right) + h_i u'\left(x_{i+\frac{1}{2}}\right) + O\left(h_i^2 + h_{i+1}^2\right)}{h_{i+\frac{1}{2}}} \tag{1.82}$$

$$= \frac{h_{i+1} + h_i}{2h_{i+\frac{1}{2}}} u'\left(x_{i+\frac{1}{2}}\right) + \frac{1}{h_{i+\frac{1}{2}}} O\left(h_i^2 + h_{i+1}^2\right) \tag{1.83}$$

$$= u'\left(x_{i+\frac{1}{2}}\right) + O\left(h\right), \quad i = 1, \ldots, N-1 \tag{1.84}$$

where the last equality is handled as in the first proof. □

Hence the approximation of the flux is also consistent if the discrete unknowns u_i, $i = 1, \ldots, N$, are viewed as approximations of the mean value of u in the control volumes.

The Dirichlet boundary conditions are taken into account by using the values imposed at the boundaries to compute the fluxes on these boundaries. Taking these boundary conditions into consideration and setting

$$f_i = \frac{1}{h_i} \int_{K_i} f(x)\, dx, \quad i = 1, \ldots, N \tag{1.85}$$

(in actual computation, an approximation of f_i by numerical integration can be used), the finite volume scheme for problem (1.1)-(1.3) writes

$$F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = h_i f_i, \quad i = 1, \ldots, N \tag{1.86}$$

$$F_{i+\frac{1}{2}} = -\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}, \quad i = 1, \ldots, N-1 \tag{1.87}$$

$$F_{\frac{1}{2}} = -\frac{u_1}{h_{\frac{1}{2}}} \tag{1.88}$$

$$F_{N+\frac{1}{2}} = \frac{u_N}{h_{N+\frac{1}{2}}} \tag{1.89}$$

Note that (1.87)-(1.89) may also be written

$$F_{i+\frac{1}{2}} = -\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}, \quad i = 0, \ldots, N \tag{1.90}$$

setting

$$u_0 = u_{N+1} = 0 \tag{1.91}$$

The numerical scheme (1.86)-(1.89) may be written under the following matrix form

$$AU = b \tag{1.92}$$

where $U = (u_1, \ldots, u_N)^t, b = (b_1, \ldots, b_N)^t$ with (1.91) and with A and b defined by

$$(AU)_i = \frac{1}{h_i} \left(-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} \right), \quad i = 1, \ldots, N \tag{1.93}$$

$$b_i = \frac{1}{h_i} \int_{K_i} f(x)\, dx, \quad i = 1, \ldots, N \tag{1.94}$$

More explicitly, the matrix A is given by

$$A = \begin{bmatrix} \frac{1}{h_1}\left(\frac{1}{h_{\frac{1}{2}}} + \frac{1}{h_{\frac{3}{2}}}\right) & -\frac{1}{h_1 h_{\frac{3}{2}}} & & & \\ -\frac{1}{h_{\frac{3}{2}} h_2} & \frac{1}{h_2}\left(\frac{1}{h_{\frac{3}{2}}} + \frac{1}{h_{\frac{5}{2}}}\right) & -\frac{1}{h_2 h_{\frac{5}{2}}} & & \\ & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h_{N-\frac{1}{2}} h_N} & \frac{1}{h_N}\left(\frac{1}{h_{N-\frac{1}{2}}} + \frac{1}{h_{N+\frac{1}{2}}}\right) \end{bmatrix} \tag{1.95}$$

Put, for all $i = 1, \ldots, N$,

$$\alpha_i = -\frac{1}{h_{i-\frac{1}{2}} h_i} \tag{1.96}$$

$$\beta_i = \frac{1}{h_i}\left(\frac{1}{h_{i-\frac{1}{2}}} + \frac{1}{h_{i+\frac{1}{2}}}\right) \tag{1.97}$$

$$\gamma_i = -\frac{1}{h_i h_{i+\frac{1}{2}}} \tag{1.98}$$

we get

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = b_i, \quad i = 1, \ldots, N \tag{1.99}$$

Combining with the boundary conditions, we get the scheme for the cell-centered finite volume method

$$\begin{cases} \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = b_i, i = 1, \ldots, N \\ u_0 = u_{N+1} = 0 \end{cases} \tag{1.100}$$

With these notations, (1.95) can be rewritten as

$$A = \begin{bmatrix} \beta_1 & \gamma_1 & & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \alpha_3 & \beta_3 & \gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{N-1} & \beta_{N-1} & \gamma_N \\ & & & & \alpha_N & \beta_N \end{bmatrix} \tag{1.101}$$

The matrix A is tridiagonal and symmetric positive definite.

Remark 1.3 There are other finite volume schemes for problem (1.1)-(1.3).

1. For instance, it is possible, in Definition 2.1, to take $x_1 \geq 0, x_N \leq 1$ and, for the definition of the scheme (that is (1.94)-(1.89)), to write (1.94) only for $i = 2, \ldots, N-1$ and to replace (1.88) and (1.89) by $u_1 = u_N = 0$ (note that (1.87) does not change). For this so-called "modified finite volume" scheme, it is also possible to obtain an error estimate as for the scheme (1.94)-(1.89) (see Remark 2.3). Note that, with this scheme, the union of all control volumes for which the "conservation law" is written is slightly different from $[0,1]$ (namely $\left[x_{\frac{3}{2}}, x_{N-\frac{1}{2}}\right] \neq [0,1]$).

2. Another possibility is to take (primary) unknowns associated to the boundaries of the control volumes. We shall not consider this case here.

1.2 Comparison with a Finite Difference Scheme

With the same notations as in Section 1.1, consider that u_i is now an approximation of $u(x_i)$. It is interesting to notice that the expression

$$\frac{1}{h_i}\left(-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}}\right) \tag{1.102}$$

is not a consistent approximation of $-u''(x_i)$ in the finite difference sense, that is the error made by replacing the derivative by a difference quotient does not tend to 0 as h tends to 0. Indeed, let $\overline{U} = (u(x_1), \ldots, u(x_N))^t$; with the notations of (1.92)-(1.94), the truncation error may be defined as

$$r = A\overline{U} - b \tag{1.103}$$

with $r = (r_1, \ldots, r_N)^t$. Note that for f regular enough, which is assumed in the sequel, $b_i = f(x_i) + O(h)$. More explicitly, since $x_i \in \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$, there exists a constant t_0 such that $x_i = (1-t_0)\, x_{i-\frac{1}{2}} + t_0 x_{i+\frac{1}{2}}$. Then, as usual, we have

$$\begin{aligned}
b_i &= \frac{1}{h_i}\int_{K_i} f(x)\, dx &(1.104)\\
&= \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)\, dx &(1.105)\\
&= \frac{1}{h_i}\int_0^1 f\left((1-t)\, x_{i-\frac{1}{2}} + t x_{i+\frac{1}{2}}\right) h_i dt &(1.106)\\
&= \int_0^1 \left(f(x_i) + (t-t_0)\, h_i f'(x_i) + O\left(h_i^2\right)\right) dt &(1.107)\\
&= f(x_i) + \left(\frac{1}{2} - t_0\right) h_i f'(x_i) + O\left(h_i^2\right) &(1.108)
\end{aligned}$$

i.e., $b_i = f(x_i) + O(h)$. In addition, if $t_0 = \frac{1}{2}$, i.e., x_i is center of K_i, we have $b_i = f(x_i) + O\left(h^2\right)$.

An estimate of r is obtained by using Taylor's expansion

$$u(x_{i+1}) = u(x_i) + h_{i+\frac{1}{2}} u'(x_i) + \frac{1}{2} h_{i+\frac{1}{2}}^2 u''(x_i) + \frac{1}{6} h_{i+\frac{1}{2}}^3 u'''(\xi_i) \tag{1.109}$$

$$u(x_{i-1}) = u(x_i) - h_{i-\frac{1}{2}} u'(x_i) + \frac{1}{2} h_{i-\frac{1}{2}}^2 u''(x_i) - \frac{1}{6} h_{i-\frac{1}{2}}^3 u'''(\eta_i) \tag{1.110}$$

for some $\xi_i \in (x_i, x_{i+1})$ and $\eta_i \in (x_{i-1}, x_i)$, which yields

$$\begin{aligned}
r_i &= \left(A\overline{U}\right)_i - b_i &(1.111)\\
&= \frac{1}{h_i}\left(-\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} + \frac{u(x_i) - u(x_{i-1})}{h_{i-\frac{1}{2}}}\right) - f(x_i) + O(h) &(1.112)\\
&= \frac{1}{h_i}\left(\begin{array}{c} -u'(x_i) - \frac{1}{2} h_{i+\frac{1}{2}} u''(x_i) + O\left(h_{i+\frac{1}{2}}^2\right) \\ +u'(x_i) - \frac{1}{2} h_{i-\frac{1}{2}} u''(x_i) + O\left(h_{i-\frac{1}{2}}^2\right) \end{array}\right) - f(x_i) + O(h) &(1.113)\\
&= -\frac{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}}{2h_i} u''(x_i) + \frac{1}{h_i} O\left(h_{i-\frac{1}{2}}^2 + h_{i+\frac{1}{2}}^2\right) + u''(x_i) + O(h) &(1.114)
\end{aligned}$$

for $i = 1, \ldots, N$, which does not, in general tend to 0 as h tends to 0 (except in particular case, for instance, $\mathcal{T}$ is a uniform admissible mesh and x_i is the center of K_i, then $h_{i+\frac{1}{2}} = h_j, i = 0, \ldots, N; j = 1, \ldots, N$ and (1.114) becomes $r_i = O(h)$) as may be seen on the simple following example.

Example 1.4. Let $f \equiv 1$ and consider a mesh of $(0,1)$, in the sense of Definition 2.1, satisfying $h_i = h$ for even i, $h_i = \frac{h}{2}$ for odd i and $x_i = \frac{1}{2}x_{i-\frac{1}{2}} + \frac{1}{2}x_{i+\frac{1}{2}}$ for $i = 1, \ldots, N$. An easy computation shows that the truncation error r is

$$r_i = \begin{cases} -\frac{1}{4}, \text{ for even } i \\ \frac{1}{2}, \text{ for odd } i \end{cases} \tag{1.115}$$

Proof of (1.115). With $f = 1$, we have $u(x) = \frac{1}{2}x(1-x)$, $\forall x \in (0,1)$. It is easily to find out

$$x_{2k} = \frac{3}{2}kh - \frac{h}{2} \tag{1.116}$$

$$x_{2k+\frac{1}{2}} = \frac{3}{2}kh \tag{1.117}$$

$$x_{2k+1} = \frac{3}{2}kh + \frac{h}{4} \tag{1.118}$$

$$x_{2k+1+\frac{1}{2}} = \frac{3}{2}kh + \frac{h}{2} \tag{1.119}$$

Therefore,

$$r_{2k} = \frac{1}{h_{2k}}\left(-\frac{u(x_{2k+1}) - u(x_{2k})}{x_{2k+1} - x_{2k}} + \frac{u(x_{2k}) - u(x_{2k-1})}{x_{2k} - x_{2k-1}}\right) \tag{1.120}$$

$$- \frac{1}{h_{2k}} \int_{K_{2k}} f(x)\,dx \tag{1.121}$$

$$= \frac{1}{h}\left(\begin{array}{c} -\dfrac{u\left(\frac{3}{2}kh + \frac{h}{4}\right) - u\left(\frac{3}{2}kh - \frac{h}{2}\right)}{\frac{3}{2}kh + \frac{h}{4} - \left(\frac{3}{2}kh - \frac{h}{2}\right)} \\ +\dfrac{u\left(\frac{3}{2}kh - \frac{h}{2}\right) - u\left(\frac{3}{2}kh - \frac{5h}{4}\right)}{\frac{3}{2}kh - \frac{h}{2} - \left(\frac{3}{2}kh - \frac{5h}{4}\right)} \end{array}\right) - 1 \tag{1.122}$$

$$= \frac{1}{h}\left(\begin{array}{c} -\dfrac{u\left(\frac{3}{2}kh + \frac{h}{4}\right) - u\left(\frac{3}{2}kh - \frac{h}{2}\right)}{\frac{3h}{4}} \\ +\dfrac{u\left(\frac{3}{2}kh - \frac{h}{2}\right) - u\left(\frac{3}{2}kh - \frac{5h}{4}\right)}{\frac{3h}{4}} \end{array}\right) - 1 \tag{1.123}$$

$$= -\frac{1}{4} \tag{1.124}$$

and

$$r_{2k+1} = \frac{1}{h_{2k+1}}\left(-\frac{u(x_{2k+2}) - u(x_{2k+1})}{h_{2k+1+\frac{1}{2}}} + \frac{u(x_{2k+1}) - u(x_{2k})}{h_{2k+\frac{1}{2}}}\right) \tag{1.125}$$

$$- \frac{1}{h_{2k+1}} \int_{K_{2k+1}} f(x)\,dx \tag{1.126}$$

$$= \frac{2}{h}\left(\begin{array}{c} -\dfrac{u\left(\frac{3}{2}kh + h\right) - u\left(\frac{3}{2}kh + \frac{h}{4}\right)}{x_{2k+2} - x_{2k+1}} \\ +\dfrac{u\left(\frac{3}{2}kh + \frac{h}{4}\right) - u\left(\frac{3}{2}kh - \frac{h}{2}\right)}{x_{2k+1} - x_{2k}} \end{array}\right) - 1 \tag{1.127}$$

$$= \frac{2}{h}\left(\begin{array}{c} -\dfrac{u\left(\frac{3}{2}kh + h\right) - u\left(\frac{3}{2}kh + \frac{h}{4}\right)}{\frac{3}{2}kh + h - \left(\frac{3}{2}kh + \frac{h}{4}\right)} \\ +\dfrac{u\left(\frac{3}{2}kh + \frac{h}{4}\right) - u\left(\frac{3}{2}kh - \frac{h}{2}\right)}{\frac{3}{2}kh + \frac{h}{4} - \left(\frac{3}{2}kh - \frac{h}{2}\right)} \end{array}\right) - 1 \tag{1.128}$$

$$= \frac{2}{h}\left(\begin{array}{l} -\dfrac{u\left(\frac{3}{2}kh+h\right)-u\left(\frac{3}{2}kh+\frac{h}{4}\right)}{\frac{3h}{4}} \\ +\dfrac{u\left(\frac{3}{2}kh+\frac{h}{4}\right)-u\left(\frac{3}{2}kh-\frac{h}{2}\right)}{\frac{3h}{4}} \end{array}\right)-1 \tag{1.129}$$

$$= \frac{1}{2} \tag{1.130}$$

where (1.124) and (1.130) can be computed easily by hand, or by using the following short MATLAB code.

```
function u = u(x)
u = 1/2*x*(1-x);
```

and

```
% Example 2.3.
syms h k;
r_even = 1/h*(-(u(3/2*k*h+h/4)-u(3/2*k*h-h/2))/(3/4*h)+ ...
    (u(3/2*k*h-h/2)-u(3/2*k*h-5/4*h))/(3/4*h))-1;
simplify(r_even)
r_odd= 2/h*(-(u(3/2*k*h+h)-u(3/2*k*h+h/4))/(3/4*h)+ ...
    (u(3/2*k*h+h/4)-u(3/2*k*h-h/2))/(3/4*h))-1;
simplify(r_odd)
```

Hence, $\sup\{|r_i|, i=1,\ldots,N\} \not\to 0$ as $h \to 0$.

Therefore, the scheme obtained from (1.94)-(1.89) is not consistent in the finite difference sense, even though it is consistent in the finite volume sense, that is, the numerical approximation of the fluxes is conservative and the truncation error on the fluxes tends to 0 as h tends to 0.

If, for instance, x_i is the center of K_i, for $i=1,\ldots,N$, it is well known that for problem (1.1)-(1.3), the consistent finite difference scheme would be, omitting boundary conditions,

$$\frac{4}{2h_i+h_{i-1}+h_{i+1}}\left(-\frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}}+\frac{u_i-u_{i-1}}{h_{i-\frac{1}{2}}}\right)=f\left(x_i\right), \quad i=2,\ldots,N-1 \tag{1.131}$$

Remark 1.5. Assume that x_i is, for $i=1,\ldots,N$, the center of K_i and that the discrete unknown u_i of the finite volume scheme is considered as an approximation of the mean value $\widetilde{u}_i$ of u over K_i. Note that

$$\widetilde{u}_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u\left(x\right)dx \tag{1.132}$$

$$= \int_0^1 u\left(\left(1-t\right)x_{i-\frac{1}{2}}+tx_{i+\frac{1}{2}}\right)dt \tag{1.133}$$

$$= \int_0^1 \left(u\left(x_i\right)+\left(t-\frac{1}{2}\right)h_iu'\left(x_i\right)+\frac{1}{2}\left(t-\frac{1}{2}\right)^2 h_i^2u''\left(x_i\right)+O\left(h_i^3\right)\right)dt \tag{1.134}$$

$$= u\left(x_i\right)+\frac{h_i^2}{24}u''\left(x_i\right)+O\left(h^3\right) \tag{1.135}$$

if $u \in C^3([0,1],\mathbb{R})$ instead of $u(x_i)$, then again, the finite volume scheme, considered once more as a finite difference scheme, is not consistent in the finite difference sense. Indeed, let $\widetilde{R} = A\widetilde{U} - b$, with $\widetilde{U} = (\widetilde{u}_1, \ldots, \widetilde{u}_N)^t$, and $\widetilde{R} = \left(\widetilde{R}_1, \ldots, \widetilde{R}_N\right)^t$, then, in general, $\widetilde{R}_i$ does not go to 0 as h goes to 0. In fact, it will be shown later that the finite volume scheme, when seen as a finite difference scheme, is consistent in the finite difference sense if u_i is considered as an approximation of $u(x_i) - \frac{h_i^2}{8}u''(x_i)$. This is the idea upon which the first proof of convergence by Forsyth and Sammon in 1988 is based.

In the case of Problem (1.1)-(1.3), both the finite volume and finite difference schemes are convergent. The finite difference scheme (1.131) is convergent since it is stable, in the sense that $\|X\|_\infty \le C\|AX\|_\infty$, for all $X \in \mathbb{R}^N$, where C is a constant and $\|X\|_\infty = \sup(|X_1|, \ldots, |X_N|)$, $X = (X_1, \ldots, X_N)^t$, and consistent in the usual finite difference sense. Since $A\left(\overline{U} - U\right) = R$, the stability property implies that $\left\|\overline{U} - U\right\|_\infty \le C\|R\|_\infty$ which goes to 0, as h goes to 0, by definition of the consistency in the finite difference sense. The convergence of the finite volume scheme (1.94)-(1.89) needs some more work and is described later.

1.3 Comparison with a Mixed Finite Element Method

The finite volume method has often be thought of as a kind of mixed finite element method. Nevertheless, we show here that, on the simple Dirichlet problem (1.1)-(1.3), the two methods yields two different schemes. For Problem (1.1)-(1.3), the discrete unknowns of the finite volume method are the values $u_i,\ i = 1, \ldots, N$. However, the finite volume method also introduces one discrete unknown at each of the control volume extremities, namely the numerical flux between the corresponding control volumes. Hence, the finite volume method for elliptic problems may appear closely related to the mixed finite element method. Recall that the mixed finite element method consists in introducing in Problem (1.1)-(1.3) the auxiliary variable $q = -u'$, which yields the following system

$$q + u' = 0 \tag{1.136}$$

$$q_x = f \tag{1.137}$$

assuming $f \in L^2((0,1))$, a variational formulation of this system is

$$q \in H^1((0,1)), u \in L^2((0,1)) \tag{1.138}$$

$$\int_0^1 q(x)\,p(x)\,dx = \int_0^1 u(x)\,p_x(x)\,dx, \quad \forall p \in H^1((0,1)) \tag{1.139}$$

$$\int_0^1 q_x(x)\,v(x)\,dx = \int_0^1 f(x)\,v(x)\,dx, \quad \forall v \in L^2((0,1)) \tag{1.140}$$

Considering an admissible mesh of $(0,1)$ (see Definition 2.1), the usual discretization of this variational formulation consists in taking the classical piecewise linear finite element functions for the approximation H of $H^1((0,1))$ and the piecewise constant finite element for the approximation L of $L^2((0,1))$. Then, the discrete unknowns are $\{u_i, i = 1, \ldots, N\}$ and $\left\{q_{i+\frac{1}{2}}, i = 0, \ldots, N\right\}$ (u_i is an approximation of u in K_i and $q_{i+\frac{1}{2}}$ is an approximation of $u'\left(x_{i+\frac{1}{2}}\right)$).

The discrete equations are obtained by performing a Galerkin expansion of u and q with respect to the natural basis function $\psi_l,\ \ l = 1, \ldots, N$ (spanning L), and $q_{i+\frac{1}{2}},\ \ i = 0, \ldots, N$ (spanning H) and by taking $p = \varphi_{i+\frac{1}{2}},\ \ i = 0, \ldots, N$ in (1.139) and $v = \psi_k,\ k = 1, \ldots, N$ in (1.140). Let $h_0 = h_{N+1} = 0$, $u_0 = u_{N+1} = 0$ and $q_{-\frac{1}{2}} = q_{N+\frac{3}{2}} = 0$; then the discrete system obtained by the mixed finite element method has $2N + 1$ unknowns. It writes

$$q_{i+\frac{1}{2}} \left(\frac{h_i + h_{i+1}}{3} \right) + q_{i-\frac{1}{2}} \left(\frac{h_i}{6} \right) + q_{i+\frac{3}{2}} \left(\frac{h_{i+1}}{6} \right) = u_i - u_{i+1} \tag{1.141}$$

for $i = 0, \ldots, N$.

$$q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}} = \int_{K_i} f(x)\, dx, \quad i = 1, \ldots, N \tag{1.142}$$

Note that the unknowns $q_{i+\frac{1}{2}}$ cannot be eliminated from the system. The resolution of this system of equations does not give the same values $\{u_i,\ \ i = 1, \ldots, N\}$ than those obtained by using the finite volume scheme (1.94)-(1.89). In fact it is easily seen that, in this case, the finite volume scheme can be obtained from the mixed finite element scheme by using the following numerical integration for the left handside of (1.139)

$$\int_{K_i} g(x)\, dx = \frac{g(x_i) + g(x_{i+1})}{2} h_i \tag{1.143}$$

This is also true for some two-dimensional elliptic problems and therefore the finite volume error estimates for these problems may be obtained via the mixed finite element theory.

2 Convergence Theorems and Error Estimates for the Dirichlet Problem

2.1 A Finite Volume Error Estimate in a Simple Case

We shall now prove the following error estimate, which will be generalized to more general elliptic problems and in higher space dimensions.

Theorem 2.1. *Let $f \in C([0,1], \mathbb{R})$ and let $u \in C^2([0,1], \mathbb{R})$ be the (unique) solution of Problem* (1.1)-(1.3)*. Let $\mathcal{T} = (K_i)_{i=1,\ldots,N}$ be an admissible mesh in the sense of Definition 2.1. Then, there exists a unique vector $U = (u_1, \ldots, u_N)^t \in \mathbb{R}^N$ solution to* (1.94)-(1.89) *and there exists $C \geq 0$, only depending on u, such that*

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+\frac{1}{2}}} \leq C^2 h^2 \tag{2.1}$$

and

$$|e_i| \leq Ch, \quad \forall i \in \{1, \ldots, N\} \tag{2.2}$$

with $e_0 = e_{N+1} = 0$ *and* $e_i = u(x_i) - u_i$, *for all* $i \in \{1, \dots, N\}$.

This theorem is in fact a consequence of Theorem 2.7, which gives an error estimate for the finite volume discretization of a more general operator. However, we now give the proof of the error estimate in this first simple case.

PROOF OF THEOREM 2.5. First remark that there exists a unique vector $U = (u_1, \dots, u_N)^t \in \mathbb{R}^N$ solution to (1.94)-(1.89). Indeed, multiplying (1.94) by u_i and summing for $i = 1, \dots, N$ gives

$$\begin{aligned}\sum_{i=1}^{N} u_i h_i f_i &= \sum_{i=1}^{N} \left(-\frac{u_i (u_{i+1} - u_i)}{h_{i+\frac{1}{2}}} + \frac{u_i (u_i - u_{i-1})}{h_{i-\frac{1}{2}}} \right) && (2.3)\\
&= \sum_{i=1}^{N} \frac{u_i^2 - u_i u_{i+1}}{h_{i+\frac{1}{2}}} + \sum_{i=1}^{N} \frac{u_i^2 - u_{i-1} u_i}{h_{i-\frac{1}{2}}} && (2.4)\\
&= \sum_{i=1}^{N} \frac{u_i^2 - u_i u_{i+1}}{h_{i+\frac{1}{2}}} + \sum_{i=0}^{N-1} \frac{u_{i+1}^2 - u_i u_{i+1}}{h_{i+\frac{1}{2}}} && (2.5)\\
&= \frac{u_1^2}{h_{\frac{1}{2}}} + \sum_{i=1}^{N-1} \frac{u_i^2 - u_i u_{i+1}}{h_{i+\frac{1}{2}}} + \sum_{i=1}^{N-1} \frac{u_{i+1}^2 - u_i u_{i+1}}{h_{i+\frac{1}{2}}} + \frac{u_N^2}{h_{N+\frac{1}{2}}} && (2.6)\\
&= \frac{u_1^2}{h_{\frac{1}{2}}} + \sum_{i=1}^{N-1} \frac{(u_{i+1} - u_i)^2}{h_{i+\frac{1}{2}}} + \frac{u_N^2}{h_{N+\frac{1}{2}}} && (2.7)\end{aligned}$$

i.e., we obtain

$$\frac{u_1^2}{h_{\frac{1}{2}}} + \sum_{i=1}^{N-1} \frac{(u_{i+1} - u_i)^2}{h_{i+\frac{1}{2}}} + \frac{u_N^2}{h_{N+\frac{1}{2}}} = \sum_{i=1}^{N} u_i h_i f_i \tag{2.8}$$

Therefore, if $f_i = 0$ for any $i \in \{1, \dots, N\}$, then the unique solution to (1.94) is obtained by taking $u_i = 0$, for any $i \in \{1, \dots, N\}$. This gives existence and uniqueness of $U = (u_1, \dots, u_N)^t \in \mathbb{R}^N$ solution to (1.94) (with (1.87)-(1.89)). More explicitly, assuming that u and $\widehat{u}$ are two solutions to (1.94)

$$-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} = h_i f_i, \quad i = 1, \dots, N \tag{2.9}$$

$$-\frac{\widehat{u}_{i+1} - \widehat{u}_i}{h_{i+\frac{1}{2}}} + \frac{\widehat{u}_i - \widehat{u}_{i-1}}{h_{i-\frac{1}{2}}} = h_i f_i, \quad i = 1, \dots, N \tag{2.10}$$

Subtracting each side of (2.9) to (2.10) yields that $u - \widehat{u}$ is the solution to (1.94) with $f = 0$

$$-\frac{(u_{i+1} - \widehat{u}_{i+1}) - (u_i - \widehat{u}_i)}{h_{i+\frac{1}{2}}} + \frac{(u_i - \widehat{u}_i) - (u_{i-1} - \widehat{u}_{i-1})}{h_{i-\frac{1}{2}}} = 0, \quad i = 1, \dots, N \tag{2.11}$$

Hence, we can apply (2.8) for $u - \widehat{u}$ and $f = 0$ to obtain

$$\frac{(u_1 - \widehat{u}_1)^2}{h_{\frac{1}{2}}} + \sum_{i=1}^{N-1} \frac{((u_{i+1} - \widehat{u}_{i+1}) - (u_i - \widehat{u}_i))^2}{h_{i+\frac{1}{2}}} + \frac{(u_N - \widehat{u}_N)^2}{h_{N+\frac{1}{2}}} = 0 \tag{2.12}$$

Since the left hand side of (2.12) is nonnegative, it gives

$$u_1 - \widehat{u}_1 = 0 \tag{2.13}$$
$$u_{i+1} - \widehat{u}_{i+1} = u_i - \widehat{u}_i, \quad i = 1, \ldots, N-1 \tag{2.14}$$
$$u_N - \widehat{u}_N = 0 \tag{2.15}$$

i.e., $u_i = \widehat{u}_i, \quad i = 1, \ldots, N$, as stated.

One now proves (2.1). Let

$$\overline{F}_{i+\frac{1}{2}} = -u'\left(x_{i+\frac{1}{2}}\right), \quad i = 0, \ldots, N \tag{2.16}$$

Integrating the equation $-u'' = f$ over K_i yields

$$\overline{F}_{i+\frac{1}{2}} - \overline{F}_{i-\frac{1}{2}} = h_i f_i, \quad i = 1, \ldots, N \tag{2.17}$$

By (1.94), the numerical fluxes $F_{i+\frac{1}{2}}$ satisfy

$$F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = h_i f_i, \quad i = 1, \ldots, N \tag{2.18}$$

Therefore, with $G_{i+\frac{1}{2}} = \overline{F}_{i+\frac{1}{2}} - F_{i+\frac{1}{2}}, \quad i = 0, \ldots, N$,

$$G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}} = 0, \quad i = 1, \ldots, N \tag{2.19}$$

Using the consistency of the fluxes (1.15)-(1.16), there exists $C > 0$, only depending on u, such that

$$F^*_{i+\frac{1}{2}} = \overline{F}_{i+\frac{1}{2}} + R_{i+\frac{1}{2}} \text{ and } \left|R_{i+\frac{1}{2}}\right| \leq Ch \tag{2.20}$$

Hence with $e_i = u(x_i) - u_i$, for $i = 1, \ldots, N$, and $e_0 = e_{N+1} = 0$, one has

$$G_{i+\frac{1}{2}} = \overline{F}_{i+\frac{1}{2}} - F_{i+\frac{1}{2}} \tag{2.21}$$
$$= -\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - R_{i+\frac{1}{2}} + \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \tag{2.22}$$
$$= -\frac{(u(x_{i+1}) - u_{i+1}) - (u(x_i) - u_i)}{h_{i+\frac{1}{2}}} - R_{i+\frac{1}{2}} \tag{2.23}$$
$$= -\frac{e_{i+1} - e_i}{h_{i+\frac{1}{2}}} - R_{i+\frac{1}{2}}, \quad i = 0, \ldots, N \tag{2.24}$$

so that $(e_i)_{i=0,\ldots,N+1}$ satisfies (by (2.19) and (2.24))

$$-\frac{e_{i+1} - e_i}{h_{i+\frac{1}{2}}} - R_{i+\frac{1}{2}} + \frac{e_i - e_{i-1}}{h_{i-\frac{1}{2}}} + R_{i-\frac{1}{2}} = 0, \quad \forall i \in \{1, \ldots, N\} \tag{2.25}$$

Multiplying (2.25) by e_i and summing over $i = 1, \ldots, N$ yields

$$-\sum_{i=1}^{N} \frac{(e_{i+1} - e_i)\, e_i}{h_{i+\frac{1}{2}}} + \sum_{i=1}^{N} \frac{(e_i - e_{i-1})\, e_i}{h_{i-\frac{1}{2}}} = -\sum_{i=1}^{N} R_{i-\frac{1}{2}} e_i + \sum_{i=1}^{N} R_{i+\frac{1}{2}} e_i \tag{2.26}$$

Noting that $e_0 = 0, e_{N+1} = 0$ and reordering by parts, this yields (with (2.20))

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+\frac{1}{2}}} \leq Ch \sum_{i=0}^{N} |e_{i+1} - e_i| \tag{2.27}$$

Indeed, the left hand side of (2.26) can be rewritten as

$$- \sum_{i=1}^{N} \frac{(e_{i+1} - e_i)\, e_i}{h_{i+\frac{1}{2}}} + \sum_{i=1}^{N} \frac{(e_i - e_{i-1})\, e_i}{h_{i-\frac{1}{2}}} \tag{2.28}$$

$$= - \sum_{i=1}^{N} \frac{(e_{i+1} - e_i)\, e_i}{h_{i+\frac{1}{2}}} + \sum_{i=0}^{N-1} \frac{(e_{i+1} - e_i)\, e_{i+1}}{h_{i+\frac{1}{2}}} \tag{2.29}$$

$$= - \sum_{i=0}^{N} \frac{(e_{i+1} - e_i)\, e_i}{h_{i+\frac{1}{2}}} + \sum_{i=0}^{N} \frac{(e_{i+1} - e_i)\, e_{i+1}}{h_{i+\frac{1}{2}}}, \text{ by } e_0 = e_{N+1} = 0 \tag{2.30}$$

$$= \sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+\frac{1}{2}}} \tag{2.31}$$

and the right hand side of (2.26) can be estimated as

$$- \sum_{i=1}^{N} R_{i-\frac{1}{2}} e_i + \sum_{i=1}^{N} R_{i+\frac{1}{2}} e_i \tag{2.32}$$

$$= - \sum_{i=0}^{N-1} R_{i+\frac{1}{2}} e_{i+1} + \sum_{i=1}^{N} R_{i+\frac{1}{2}} e_i \tag{2.33}$$

$$= - \sum_{i=0}^{N} R_{i+\frac{1}{2}} e_{i+1} + \sum_{i=0}^{N} R_{i+\frac{1}{2}} e_i, \text{ by } e_0 = e_{N+1} = 0 \tag{2.34}$$

$$= \sum_{i=0}^{N} R_{i+\frac{1}{2}} (e_i - e_{i+1}) \tag{2.35}$$

$$\leq Ch \sum_{i=0}^{N} |e_{i+1} - e_i| \tag{2.36}$$

The Cauchy-Schwarz inequality applied to the right hand side of (2.26) give

$$\sum_{i=0}^{N} |e_{i+1} - e_i| \leq \left(\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\sum_{i=0}^{N} h_{i+\frac{1}{2}} \right)^{\frac{1}{2}} \tag{2.37}$$

Since $\sum_{i=0}^{N} h_{i+\frac{1}{2}} = 1$ in (2.37) and from (2.27), one deduces (2.1).

Since, for all $i \in \{1, \ldots, N\}$, $e_i = \sum_{j=1}^{i} (e_j - e_{j-1})$, one can deduce, from (2.37) and (2.1) that (2.2) holds. □

Remark 2.2. The error estimate given in this section does not use the discrete maximum principle[1] (that is the fact that $f_i \geq 0$, for all $i = 1, \ldots, N$,

[1] See Section 3.4.

implies $u_i \geq 0$, for all $i = 1, \ldots, N$), which is used in the proof of error estimates by the finite difference techniques, but the coerciveness of the elliptic operator, as in the proof of error estimates by the finite element techniques.

Remark 2.3.

1. The above proof of convergence give an error estimate of order h. It is sometimes possible to obtain an error estimate of order h^2. Indeed, this is the case, at least if $u \in C^4([0,1], \mathbb{R})$, if x_i is the center of K_i for all $i = 1, \ldots, N$. One obtains, in this case, $|e_i| \leq Ch^2$, for all $i \in \{1, \ldots, N\}$, where C only depends on u.

2. It is also possible to obtain an error estimate for the modified finite volume scheme described in the first item of Remark 2.2. It is even possible to obtain an error estimate of order h^2 in the case $x_1 = 0, x_N = 1$ and assuming that $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, for all $i = 1, \ldots, N-1$. In fact, in this case, one obtains $\left|R_{i+\frac{1}{2}}\right| \leq C_1 h^2$, for all $i = 1, \ldots, N-1$ (see (1.135)). Then, the proof of Theorem 2.1 gives (2.1) with h^4 instead of h^2 which yields $|e_i| \leq C_2 h^2$, for all $i \in \{1, \ldots, N\}$ (where C_1 and C_2 are only depending on u). Note that this modified finite volume scheme is also consistent in the finite difference sense. Then, the finite difference techniques yield also an error estimate on $|e_i|$, but only of order h.

3. It could be tempting to try and find error estimates with respect to the mean value of the exact solution on the control volumes rather than with respect to its value at some point of the control volumes. This is not such a good idea: Indeed, if x_i is not the center of K_i (this will be the general case in several space dimensions), then one does not have (in general) $|\widetilde{e}_i| \leq C_3 h^2$ (for some C_3 only depending on u) with $\widetilde{e}_i = \widetilde{u}_i - u_i$ where $\widetilde{u}_i$ denotes the mean value of u over K_i.

Remark 2.4.

1. If the assumption $f \in C([0,1], \mathbb{R})$ is replaced by the assumption $f \in L^2((0,1))$ in Theorem 2.1, then $u \in H^2((0,1))$ instead of $C^2([0,1], \mathbb{R})$, but the estimates of Theorem 2.1 still hold. Then, the consistency of the fluxes must be obtained with a Taylor expansion with an integral remainder. This is feasible for C^2 functions, and since the remainder only depends on the H^2 norm, a density argument allows to conclude.

2. If the assumption $f \in C([0,1], \mathbb{R})$ is replaced by the assumption $f \in L^1((0,1))$ in Theorem 2.1, then $u \in C^2([0,1], \mathbb{R})$ no longer holds (neither does $u \in H^2((0,1))$), but the convergence still holds; indeed there exists $C(u,h)$, only depending on u and h, such that $C(u,h) \to 0$, as $h \to 0$, and $|e_i| \leq C(u,h)$, for all $i = 1, \ldots, N$. The proof is similar to the one above, except that the estimate (2.20) is replaced by $\left|R_{i+\frac{1}{2}}\right| \leq C_1(u,h)$, for all $i = 0, \ldots, N$, with some $C_1(u,h)$, only depending on u and h, such that $C(u,h) \to 0$, as $h \to 0$.

Remark 2.5. Estimate (2.1) can be interpreted as a "discrete H_0^1" estimate on the error. A theoretical result which underlies the L^∞ estimate (2.2) is the fact that if Ω is an open bounded subset of $\mathbb{R}$, then $H_0^1(\Omega)$ is imbedded in $L^\infty(\Omega)$. This is no longer true in higher dimension. In two space dimensions, for instance, a discrete version of the imbedding of H_0^1 in L^p allows to obtain $\|e\|_p \le Ch$, for all finite p, which in turn yields $\|e\|_\infty \le Ch \ln h$ for convenient meshes.

The important features needed for the above proof seem to be the consistency of the approximation of the fluxes and the conservativity of the scheme; this conservativity is natural the fact that the scheme is obtained by integrating the equation over each cell, and the approximation of the flux on any interface is obtained by taking into account the flux balance (continuity of the flux in the case of no course term on the interface).

The above proof generalizes to other elliptic problems, such as a convection-diffusion equation of the form $-u'' + au' + bu = f$, and to equations of the form $-(\lambda u')_x = f$ where $\lambda \in L^\infty$ may be discontinuous, and is such that there exist α and β in $\mathbb{R}_+^*$ such that $\alpha \le \gamma \le \beta$. These generalizations are studied in the next section. Other generalizations include similar problems in 2 (or 3) space dimensions, with meshes consisting of rectangles (parallepipeds), triangles (tetrahedra), or general meshes of Voronoï type, and the corresponding evolutive (parabolic) problems. These generalizations will be addressed in further chapters.

2.2 Convergence and Error Analysis

Proposition 2.6. *The quantity $u'\left(x_{i+\frac{1}{2}}\right)$ is approximated by the differential quotient $\dfrac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}$. If $u \in C^2([0,1], \mathbb{R})$, this approximation is consistent in the sense that there exists $C \in \mathbb{R}_+$ only depending on u such that*

$$\left|\varepsilon_{i+\frac{1}{2}}\right| = \left|u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}}\right| \le Ch \tag{2.38}$$

where

$$\varepsilon_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \tag{2.39}$$

*is called **consistency error**.*

Proof. Using Taylor series expansion, there exist $\eta_{i+\frac{1}{2}} \in \left(x_{i+\frac{1}{2}}, x_{i+1}\right)$ and $\theta_{i+\frac{1}{2}} \in \left(x_i, x_{i+\frac{1}{2}}\right)$ such that

$$\frac{u(x_{i+1}) - u\left(x_{i+\frac{1}{2}}\right)}{h_{i+\frac{1}{2}}} - \frac{x_{i+1} - x_{i+\frac{1}{2}}}{h_{i+\frac{1}{2}}} u'\left(x_{i+\frac{1}{2}}\right) = \frac{\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2}{2h_{i+\frac{1}{2}}} u''\left(\eta_{i+\frac{1}{2}}\right) \tag{2.40}$$

$$\frac{u\left(x_{i+\frac{1}{2}}\right) - u(x_i)}{h_{i+\frac{1}{2}}} - \frac{x_{i+\frac{1}{2}} - x_i}{h_{i+\frac{1}{2}}} u'\left(x_{i+\frac{1}{2}}\right) = -\frac{\left(x_{i+\frac{1}{2}} - x_i\right)^2}{2h_{i+\frac{1}{2}}} u''\left(\theta_{i+\frac{1}{2}}\right) \tag{2.41}$$

Taking sum of the two expression gives

$$\varepsilon_{i+\frac{1}{2}} = -\frac{\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2}{2h_{i+\frac{1}{2}}} u''\left(\eta_{i+\frac{1}{2}}\right) + \frac{\left(x_{i+\frac{1}{2}} - x_i\right)^2}{2h_{i+\frac{1}{2}}} u''\left(\theta_{i+\frac{1}{2}}\right) \tag{2.42}$$

The following inequality holds

$$\begin{aligned}
\left|\varepsilon_{i+\frac{1}{2}}\right| &\leq \frac{\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2}{2h_{i+\frac{1}{2}}} \left|u''\left(\eta_{i+\frac{1}{2}}\right)\right| + \frac{\left(x_{i+\frac{1}{2}} - x_i\right)^2}{2h_{i+\frac{1}{2}}} \left|u''\left(\theta_{i+\frac{1}{2}}\right)\right| && (2.43)\\
&\leq \frac{1}{2} \sup_{x\in(0,1)} \left|u''(x)\right| \left(\frac{\left(x_{i+1} - x_{i+\frac{1}{2}}\right)^2}{h_{i+\frac{1}{2}}} + \frac{\left(x_{i+\frac{1}{2}} - x_i\right)^2}{h_{i+\frac{1}{2}}}\right) && (2.44)\\
&\leq \frac{1}{2} \sup_{x\in(0,1)} \left|u''(x)\right| \frac{\left(x_{i+1} - x_i\right)^2}{h_{i+\frac{1}{2}}} && (2.45)\\
&= \frac{1}{2} \sup_{x\in(0,1)} \left|u''(x)\right| h_{i+\frac{1}{2}} && (2.46)\\
&\leq Ch && (2.47)
\end{aligned}$$

where the last inequality is deduced by $h_{i+\frac{1}{2}} \leq h_i + h_{i+1} \leq 2h$. □

Let us defined the discrete H^1-norm

$$\|u\|_{1,h}^2 = \sum_{i=0}^{N} \frac{\left(u_{i+1} - u_i\right)^2}{h_{i+\frac{1}{2}}} \tag{2.48}$$

then (2.1) can be rewritten briefly as

$$\|e\|_{1,h} \leq Ch \tag{2.49}$$

3 Properties of Scheme

3.1 Definition of Discrete Derivatives and Discrete Scalar Products

Definition 3.1 (Discrete divergence operator).

$$d : \mathbb{R}^{N+1} \to \mathbb{R}^N \tag{3.1}$$

$$V = \left\{v_{i+\frac{1}{2}}\right\}_{i=0}^{N} \mapsto d(V) = \left\{(dV)_i\right\}_{i=1}^{N} \tag{3.2}$$

where

$$(dV)_i = \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i} \tag{3.3}$$

Definition 3.2 (Scalar product). Given $U = \{u_i\}_{i=1}^N$ and $W = \{w_i\}_{i=1}^N$

$$(U, W)_K = \sum_{i=1}^{N} h_i u_i w_i \tag{3.4}$$

$$\|U\|_{0,K}^2 = (U,U)_K = \sum_{i=1}^{N} h_i u_i^2 \tag{3.5}$$

Definition 3.3 (Discrete gradient operator).

$$g : \mathbb{R}^{N+2} \to \mathbb{R}^{N+1} \tag{3.6}$$

$$U = \{u_i\}_{i=0}^{N+1} \mapsto g(U) = \left\{(gU)_{i+\frac{1}{2}}\right\}_{i=0}^{N} \tag{3.7}$$

where

$$(gU)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \tag{3.8}$$

Definition 3.4 (Scalar product). Given $V^1 = \left\{v^1_{i+\frac{1}{2}}\right\}_{i=0}^{N}$, $V^2 = \left\{v^2_{i+\frac{1}{2}}\right\}_{i=0}^{N}$ and $U = \{u_i\}_{i=0}^{N+1}$

$$\left(V^1, V^2\right)_h = \sum_{i=0}^{N} h_{i+\frac{1}{2}} v^1_{i+\frac{1}{2}} v^2_{i+\frac{1}{2}} \tag{3.9}$$

$$\|V\|_{0,h}^2 = (V,V)_h \tag{3.10}$$

$$= \sum_{i=0}^{N} h_{i+\frac{1}{2}} \left(v_{i+\frac{1}{2}}\right)^2 \tag{3.11}$$

$$\|U\|_{1,h}^2 = \|g(U)\|_{0,h}^2 \tag{3.12}$$

$$= (g(U), g(U))_h \tag{3.13}$$

Proposition 3.5. *Given $V = \left\{v_{i+\frac{1}{2}}\right\}_{i=0}^{N}$ and $W = \{w_i\}_{i=0}^{N+1}$. Then*

$$\left(d(V), \{w_i\}_{i=1}^{N}\right)_K = -(V, g(W))_h + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{3.14}$$

Proof. Using Definition 3.1 and Definition 3.2, the left hand side of (3.14) can be rewritten as

$$\left(d(V), \{w_i\}_{i=1}^{N}\right)_K = \left(\left\{\frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i}\right\}_{i=1}^{N}, \{w_i\}_{i=1}^{N}\right)_K \tag{3.15}$$

$$= \sum_{i=1}^{N} h_i \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i} w_i \tag{3.16}$$

$$= \sum_{i=1}^{N} w_i \left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right) \tag{3.17}$$

Using Definition 3.3 and Definition 3.4 yields

$$(V, g(W))_h = \left(V, \left\{(gw)_{i+\frac{1}{2}}\right\}_{i=0}^{N}\right)_h \tag{3.18}$$

$$= \left(\left\{v_{i+\frac{1}{2}}\right\}_{i=0}^{N}, \left\{\frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}}\right\}_{i=0}^{N}\right)_h \tag{3.19}$$

$$= \sum_{i=1}^{N} h_{i+\frac{1}{2}} v_{i+\frac{1}{2}} \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{3.20}$$

$$= \sum_{i=1}^{N} v_{i+\frac{1}{2}} \left(w_{i+1} - w_i\right) \tag{3.21}$$

Thus, the right hand side of (3.14) can be rewritten as

$$- \left(V, g\left(W\right)\right)_h + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{3.22}$$

$$= - \sum_{i=0}^{N} v_{i+\frac{1}{2}} \left(w_{i+1} - w_i\right) + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{3.23}$$

$$= -v_{\frac{1}{2}} w_1 + \sum_{i=1}^{N-1} v_{i+\frac{1}{2}} \left(w_i - w_{i+1}\right) + v_{N+\frac{1}{2}} w_N \tag{3.24}$$

$$= w_1 \left(v_{\frac{3}{2}} - v_{\frac{1}{2}}\right) + w_2 \left(v_{\frac{5}{2}} - v_{\frac{3}{2}}\right) + \cdots + w_N \left(v_{N+\frac{1}{2}} - v_{N-\frac{1}{2}}\right) \tag{3.25}$$

$$= \sum_{i=1}^{N} w_i \left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right) \tag{3.26}$$

Comparing (3.17) and (3.26) yields that (3.14) holds. □

Let $\{u_i\}_{i=0}^{N+1}$ satisfy (1.94), there holds

$$-d\left(g\left(U\right)\right) = b \tag{3.27}$$

where $b = \{b_i\}_{i=1}^{N}$ and b_i is the mean-value of f in K_i, i.e., (1.94).

Proof of (3.27). Using Definition 3.1 and Definition 3.2 yields

$$-d\left(g\left(U\right)\right) = -d\left(\left\{(gU)_{i+\frac{1}{2}}\right\}_{i=0}^{N}\right) \tag{3.28}$$

$$= -d\left(\left\{\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}\right\}_{i=0}^{N}\right) \tag{3.29}$$

$$= -\left\{\frac{1}{h_i}\left(\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} - \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}}\right)\right\}_{i=1}^{N} \tag{3.30}$$

Hence, (3.27) is exactly (1.94)-(1.89). □

We consider any $W = \{w_i\}_{i=0}^{N+1}$, with $w_0 = w_{N+1} = 0$. Thanks to (3.14) and to the boundary condition on w, we get

$$\left(g\left(U\right), g\left(W\right)\right)_h = \left(\{w_i\}_{i=1}^{N}, b\right)_K \tag{3.31}$$

Proof of (3.31). The left hand side of (3.31) can be rewritten as

$$\left(g\left(U\right), g\left(W\right)\right)_h = \left(\left\{(gU)_{i+\frac{1}{2}}\right\}_{i=0}^{N}, \left\{(gW)_{i+\frac{1}{2}}\right\}_{i=0}^{N}\right)_h \tag{3.32}$$

$$= \left(\left\{ \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N}, \left\{ \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N} \right)_h \tag{3.33}$$

$$= \sum_{i=0}^{N} h_{i+\frac{1}{2}} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{3.34}$$

$$= \sum_{i=0}^{N} \frac{(u_{i+1} - u_i)(w_{i+1} - w_i)}{h_{i+\frac{1}{2}}} \tag{3.35}$$

$$= \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_{i+1} - \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i \tag{3.36}$$

$$= \sum_{i=1}^{N+1} \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} w_i - \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i \tag{3.37}$$

$$= \sum_{i=1}^{N} \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} w_i - \sum_{i=1}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i, \text{ by } w_0 = w_{N+1} = 0 \tag{3.38}$$

$$= \sum_{i=1}^{N} \left(\frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} - \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \right) w_i, \text{ by (1.94)} \tag{3.39}$$

$$= \sum_{i=1}^{N} h_i b_i w_i \tag{3.40}$$

The right hand side of (3.31) can be rewritten as

$$\left(\{w_i\}_{i=1}^{N}, b \right)_K = \left(\{w_i\}_{i=1}^{N}, \{b_i\}_{i=1}^{N} \right)_K \tag{3.41}$$

$$= \sum_{i=1}^{N} h_i w_i b_i \tag{3.42}$$

Comparing (3.40) and (3.42) yields (3.31). □

Thus the scheme may be written under the following discrete variational formulation.

Discrete Variational Formulation. *Find* $U = \{u_i\}_{i=0}^{N+1}$ *with* $u_0 = u_{N+1} = 0$, *such that for all* $W = \{w_i\}_{i=0}^{N+1}$, *there holds*

$$(g(U), g(W))_h = \left(\{w_i\}_{i=1}^{N}, b \right)_K \tag{3.43}$$

This is a discrete equivalent of the following continuous variational formulation.

Continuous Variational Formulation. *Find* $u \in H_0^1(\Omega)$ *such that for all* $w \in H_0^1(\Omega)$

$$(u', w')_{L^2(\Omega)} = (f, w)_{L^2(\Omega)} \tag{3.44}$$

The scheme is set as a system of $N+2$ equations and with $N+2$ unknowns $U = \{u_i\}_{i=0}^{N+1}$. In $\mathbb{R}^{N+2}$, existence and uniqueness are equivalent for square system. Let us prove uniqueness, for linear system, it is equivalent that if $b_i = 0$ for $i = 1, \ldots, N$ then $u_i = 0$ for $i = 1, \ldots, N$.

PROOF OF UNIQUENESS. If $b_i = 0$ for $i = 1, \ldots, N$ then $(b, w)_K = 0$ for all w. Since $u_0 = u_{N+1} = 0$, we can consider $w = u$. From (3.43), we get

$$(g(U), g(U))_h = \sum_{i=0}^{N} h_{i+\frac{1}{2}} (gU)_{i+\frac{1}{2}}^2 = 0 \tag{3.45}$$

Since $h_{i+\frac{1}{2}}$ is not vanish for $i = 0, \ldots, N$, (3.45) is equivalent to $(gU)_{i+\frac{1}{2}} = 0$ for $i = 0, \ldots, N$. According the definition of $(gU)_{i+\frac{1}{2}}$, we get $u_i = u_{i+1}$ for $i = 0, \ldots, N$, combining this with boundary condition, we get $u_i = 0$ for $i = 0, \ldots, N+1$. This completes our proof. □

3.2 The Case of Neumann Boundary Condition

We consider the equation with Neumann boundary condition

$$-u'' = f \text{ in } \Omega \tag{3.46}$$

$$u'(0) = u'(1) = 0 \tag{3.47}$$

Remark 3.6. The necessary condition over f to the solution of (3.46)-(3.47) to exist is

$$\int_\Omega f(x)\, dx = 0 \tag{3.48}$$

Remark 3.7. To determine unique solution to (3.43), we have

$$\int_\Omega u(x)\, dx = 0 \tag{3.49}$$

The equation $-u'' = f$ is discretized as Dirichlet boundary condition, we get

$$\frac{1}{h_i}\left(-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}}\right) = b_i, \quad i = 1, \ldots, N \tag{3.50}$$

Boundary conditions $u'(0) = u'(1) = 0$ are discretized by

$$(gU)_{\frac{1}{2}} = \frac{u_1 - u_0}{h_{\frac{1}{2}}} = 0 \tag{3.51}$$

$$(gU)_{N+\frac{1}{2}} = \frac{u_{N+1} - u_N}{h_{N+\frac{1}{2}}} = 0 \tag{3.52}$$

yields that

$$u_1 = u_0 \tag{3.53}$$

$$u_{N+1} = u_N \tag{3.54}$$

Moreover, (3.49) is discretized by

$$\sum_{i=1}^{N} h_i u_i = 0 \tag{3.55}$$

Thus, there are $N+3$ equations and but only $N+2$ unknowns. However, the set of equations (3.50) and (3.53)-(3.54) are not independent.

Summing (3.50) with index i runs over $i = 1, \ldots, N$ yields

$$\sum_{i=1}^{N} \left(-(gU)_{i+\frac{1}{2}} + (gU)_{i-\frac{1}{2}} \right) = \sum_{i=1}^{N} h_i b_i \tag{3.56}$$

i.e.,

$$-(gU)_{N+\frac{1}{2}} + (gU)_{\frac{1}{2}} = \sum_{i=1}^{N} h_i \frac{1}{h_i} \int_{T_i} f(x)\, dx \tag{3.57}$$

The left hand side of (3.57) is vanish because of (3.53)-(3.54), the right hand side is also vanish because of (3.48).

Thanks to (3.14) and to boundary condition (3.53)-(3.54), we have

$$(g(U), g(W))_h = \left(b, \{w_i\}_{i=1}^{N} \right)_K \tag{3.58}$$

for all $W = \{w_i\}_{i=0}^{N+1}$.

Proof of (3.58). Applying (3.14) for $V = g(U)$ yields

$$\begin{aligned}
(g(U), g(W))_h &= (gu)_{N+\frac{1}{2}} w_{N+1} - w_0 (gu)_{\frac{1}{2}} - \left(d(g(U)), \{w_i\}_{i=1}^{N} \right)_K && (3.59)\\
&= -\left(d(g(U)), \{w_i\}_{i=1}^{N} \right)_K, \text{ by (3.51)-(3.52)} && (3.60)\\
&= \left(b, \{w_i\}_{i=1}^{N} \right)_K && (3.61)
\end{aligned}$$

Thus, (3.58) holds. □

If b is vanish, then let $w = u$ in (3.58) to obtain $(g(U), g(U))_h = 0$. As in the proof of uniqueness in the previous subsection, we obtain $u_i = c$ for $i = 0, \ldots, N+1$ for some constant c. We use (3.55) to get $c = 0$. Thus, $u_i = 0$ for $i = 1, \ldots, N+1$.

Remark 3.8. Since $\sum\limits_{i=1}^{N} h_i f_i$ is not always vanish, We must make other $\widetilde{f_i}$ satisfy

$$\sum_{i=1}^{N} h_i \widetilde{f_i} = 0 \tag{3.62}$$

We can choose $\widetilde{f_i}$ given by

$$\widetilde{f_i} = f_i - \frac{\sum\limits_{i=1}^{N} h_i f_i}{\sum\limits_{i=1}^{N} h_i} \tag{3.63}$$

Proof of (3.62)-(3.63). It is straightforward to obtain

$$\sum_{i=1}^{N} h_i \widetilde{f}_i = \sum_{i=1}^{N} h_i \left(f_i - \frac{\sum\limits_{j=1}^{N} h_j f_j}{\sum\limits_{j=1}^{N} h_j} \right) \tag{3.64}$$

$$= \sum_{i=1}^{N} h_i f_i - \sum_{i=1}^{N} h_i \frac{\sum\limits_{j=1}^{N} h_j f_j}{\sum\limits_{j=1}^{N} h_j} \tag{3.65}$$

$$= \sum_{i=1}^{N} h_i f_i - \frac{\sum\limits_{i=1}^{N} h_i}{\sum\limits_{j=1}^{N} h_j} \sum_{j=1}^{N} h_j f_j \tag{3.66}$$

$$= 0 \tag{3.67}$$

Thus, (3.62) holds with $\widetilde{f}_i$ given by (3.63). □

3.3 The Case of Robin Boundary Condition

We consider the equation with Robin boundary condition

$$-u'' = f \text{ in } \Omega \tag{3.68}$$

$$u'(0) - \lambda_0 u(0) = u'(1) + \lambda_1 u(1) \tag{3.69}$$

we get the following discrete equation

$$\frac{1}{h_i} \left(-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} \right) = b_i, \quad i = 1, \dots, N \tag{3.70}$$

and

$$(gU)_{\frac{1}{2}} - \lambda_0 u_0 = 0 \tag{3.71}$$

$$(gU)_{N+\frac{1}{2}} + \lambda_1 u_{N+1} = 0 \tag{3.72}$$

3.4 Discrete Maximum Principle

Proposition 3.9. *Suppose that f is nonnegative on Ω and $u_0 = u_{N+1} = 0$. Then the discrete solution remains nonnegative on Ω, i.e., $u_i \geq 0$ for $i = 1, \dots, N$.*

Proof. We assume that for some $i \in \{1, \dots, N\}$, $u_i < 0$. Then there exists $i_1 \in \{1, \dots, N\}$ such that $u_{i_1} = \min\{u_i | i = 1, \dots, N\}$, thus $u_{i_1} < 0$. From discrete equation for $-u'' = f$, we get

$$\frac{1}{h_{i_1}} \left(\frac{u_{i_1} - u_{i_1+1}}{h_{i_1+\frac{1}{2}}} + \frac{u_{i_1} - u_{i_1-1}}{h_{i_1-\frac{1}{2}}} \right) = b_{i_1} \tag{3.73}$$

Since $u_{i_1} = \min\{u_i | i = 1, \dots, N\}$, we have $u_{i_1} - u_{i_1+1} \le 0$ and $u_{i_1} - u_{i_1-1} \le 0$, combining with (3.73) with $f_{i_1} \ge 0$, we have $u_{i_1-1} = u_{i_1} = u_{i_1+1}$. From that, we can prove that

$$u_i = u_{i_1}, \quad i = 0, \dots, N+1 \tag{3.74}$$

but $u_0 = 0$, which contradicts with $u_{i_1} < 0$. □

3.5 Estimation in Energy Norm

Definition 3.10. Since exact solution $u \in C^1\left(\overline{\Omega}\right)$, we can defined projection

$$\Pi : C^1\left(\overline{\Omega}\right) \to \mathbb{R}^{N+2} \tag{3.75}$$

$$u \mapsto \Pi u = \{\Pi u\}_{i=0}^{N+1} \tag{3.76}$$

where $(\Pi u)_i = u(x_i), \quad i = 0, \dots, N+1$.

Definition 3.11. Since $u' \in C^0\left(\overline{\Omega}\right)$, we can define projection

$$P : C^0\left(\overline{\Omega}\right) \to \mathbb{R}^{N+1} \tag{3.77}$$

$$u' \mapsto Pu' = \left\{(Pu')_{i+\frac{1}{2}}\right\}_{i=0}^{N} \tag{3.78}$$

where $(Pu')_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right), \quad i = 0, \dots, N$.

Lemma 3.12. *Let $W = \{w_i\}_{i=0}^{N+1}$ with $w_0 = w_{N+1}$ and if $U = \{u_i\}_{i=0}^{N+1}$ is the solution of finite volume method and u is the exact solution of the considered boundary value problem. We have*

$$(g(U), g(W))_h = (Pu', g(W))_h \tag{3.79}$$

Proof. The right hand side of (3.79) can be rewritten as

$$(Pu', g(W))_h = \left(\left\{(Pu')_{i+\frac{1}{2}}\right\}_{i=0}^{N}, \left\{(gW)_{i+\frac{1}{2}}\right\}_{i=0}^{N}\right)_h \tag{3.80}$$

$$= \left(\left\{u'\left(x_{i+\frac{1}{2}}\right)\right\}_{i=0}^{N}, \left\{\frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}}\right\}_{i=0}^{N}\right)_h \tag{3.81}$$

$$= \sum_{i=0}^{N} h_{i+\frac{1}{2}} u'\left(x_{i+\frac{1}{2}}\right) \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{3.82}$$

$$= \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right)(w_{i+1} - w_i) \tag{3.83}$$

$$= \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_{i+1} - \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i \tag{3.84}$$

$$= \sum_{i=1}^{N+1} u'\left(x_{i-\frac{1}{2}}\right) w_i - \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i \tag{3.85}$$

$$= \sum_{i=1}^{N} u'\left(x_{i-\frac{1}{2}}\right) w_i - \sum_{i=1}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i, \text{ by } w_0 = w_{N+1} = 0 \tag{3.86}$$

$$= \sum_{i=1}^{N} \left(u'\left(x_{i-\frac{1}{2}}\right) - u'\left(x_{i+\frac{1}{2}}\right)\right) w_i \tag{3.87}$$

$$= \sum_{i=1}^{N} h_i b_i w_i, \text{ by (1.13)} \tag{3.88}$$

$$= \left(\{w_i\}_{i=1}^{N}, b\right)_K \tag{3.89}$$

$$= (g(U), g(W))_h, \text{ by (3.31)} \tag{3.90}$$

Thus, (3.79) holds. □

Remark 3.13. When we handled Problem (1.1)-(1.3) with Dirichlet boundary condition, we considered $U = \{u_i\}_{i=1}^{N}$ since $u_0 = u_{N+1} = 0$ are known. But in Lemma 3.12, we consider the boundary value problem with more general boundary conditions, such as Neumann boundary condition, Robin boundary condition, which do not provide u_0 and u_{N+1}. Hence, we have to consider the solution U of our finite volume method with little extended form $U = \{u_i\}_{i=0}^{N+1}$.

We shall estimate the H_0^1 norm of $\|u - \Pi u\|_{1,h} = (g(u - \Pi u), g(u - \Pi u))_h^{\frac{1}{2}}$.

We set $w = U - \Pi u$, thanks to Lemma 3.12, since $w_0 = w_{N+1} = 0$, we can write

$$\|U - \Pi u\|_{1,h}^2 = (g(U - \Pi u), g(U - \Pi u))_h \tag{3.91}$$

$$= (g(U - \Pi u), g(W))_h \tag{3.92}$$

$$= (g(U), g(W))_h - (g(\Pi u), g(W))_h \tag{3.93}$$

$$= (Pu', g(W))_h - (g(\Pi u), g(W))_h \tag{3.94}$$

$$= (Pu' - g(\Pi u), g(W))_h \tag{3.95}$$

$$\leq \|Pu' - g(\Pi u)\|_{0,h} \|W\|_{1,h} \tag{3.96}$$

where the last inequality is deduced by *Cauchy Schwarz for scalar product*[2] (See [3]). More explicitly,

$$(Pu' - g(\Pi u), g(W))_h \tag{3.97}$$

$$\leq (Pu' - g(\Pi u), Pu' - g(\Pi u))_h^{\frac{1}{2}} (g(W), g(W))_h^{\frac{1}{2}} \tag{3.98}$$

$$= \|Pu' - g(\Pi u)\|_{0,h} \|W\|_{1,h}, \text{ by (3.10)-(3.13)} \tag{3.99}$$

Since $W = U - \Pi u$ then

$$\|U - \Pi u\|_{1,h} \leq \|Pu' - g(\Pi u)\|_{0,h} \tag{3.100}$$

[2]The CauchySchwarz inequality states that for all vectors u and v of an inner product space it is true that $|\langle u, v\rangle|^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle$ where $\langle \cdot, \cdot\rangle$ is the inner product. We here apply Cauchy Schwarz inequality for defined scalar product $(\cdot, \cdot)_h$.

Recall the consistency error defined by (2.39)

$$\varepsilon_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right) - \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h_{i+\frac{1}{2}}} \tag{3.101}$$

We can prove that[3]

$$h_{i+\frac{1}{2}}\varepsilon^2_{i+\frac{1}{2}} \leq \frac{2}{3}h^2 \int_{x_i}^{x_{i+1}} f^2\left(t\right)dt \tag{3.102}$$

Moreover, if $x_{i+\frac{1}{2}}$ is the midpoint of $[x_i, x_{i+1}]$ then [1] gives us the following estimate

$$h_{i+\frac{1}{2}}\varepsilon^2_{i+\frac{1}{2}} \leq \left(\frac{4}{15}\right)^2 h^4 \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.103}$$

Proof of (3.103). We compute the consistency error

$$\begin{aligned}
\varepsilon_{i+\frac{1}{2}} &= u'\left(x_{i+\frac{1}{2}}\right) - \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h_{i+\frac{1}{2}}} && (3.104)\\
&= \frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} u'\left(x_{i+\frac{1}{2}}\right)dt && (3.105)\\
&= \frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} \left(u'\left(x_{i+\frac{1}{2}}\right) - u'\left(t\right)\right)dt && (3.106)\\
&= -\frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} \left(\int_t^{x_{i+\frac{1}{2}}} f\left(s\right)ds\right)dt && (3.107)\\
&= -\frac{\int_{x_i}^{x_{i+\frac{1}{2}}}\left(\int_t^{x_{i+\frac{1}{2}}} f\left(s\right)ds\right)dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\left(\int_t^{x_{i+\frac{1}{2}}} f\left(s\right)ds\right)dt}{h_{i+\frac{1}{2}}} && (3.108)\\
&= \frac{-\int_{x_i}^{x_{i+\frac{1}{2}}}\left(\int_t^{x_{i+\frac{1}{2}}} f\left(s\right)ds\right)dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\left(\int_{x_{i+\frac{1}{2}}}^{t} f\left(s\right)ds\right)dt}{h_{i+\frac{1}{2}}} && (3.109)\\
&= \frac{-\int_{x_i}^{x_{i+\frac{1}{2}}}\left(\int_{x_i}^{s} f\left(s\right)dt\right)ds + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\left(\int_{s}^{x_{i+1}} f\left(s\right)dt\right)ds}{h_{i+\frac{1}{2}}} && (3.110)\\
&= \frac{-\int_{x_i}^{x_{i+\frac{1}{2}}}\left(s - x_i\right)f\left(s\right)ds + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\left(x_{i+1} - s\right)f\left(s\right)ds}{h_{i+\frac{1}{2}}} && (3.111)
\end{aligned}$$

Integrating by parts two integrals in (3.111) yields

$$\begin{aligned}
&h_{i+\frac{1}{2}}\varepsilon_{i+\frac{1}{2}} && (3.112)\\
&= \int_{x_i}^{x_{i+\frac{1}{2}}}\left(\frac{s^2}{2} - x_i s\right)f'\left(s\right)ds - \left(\frac{s^2}{2} - x_i s\right)f\left(s\right)\Big|_{x_i}^{x_{i+\frac{1}{2}}} && (3.113)\\
&+ \left(sx_{i+1} - \frac{s^2}{2}\right)f\left(s\right)\Big|_{x_{i+\frac{1}{2}}}^{x_{i+1}} - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\left(sx_{i+1} - \frac{s^2}{2}\right)f'\left(s\right)ds && (3.114)
\end{aligned}$$

[3] See Proof of Problem 4.10.

$$= \int_{x_i}^{x_{i+\frac{1}{2}}} \left(\frac{s^2}{2} - x_i s\right) f'(s)\, ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(s x_{i+1} - \frac{s^2}{2}\right) f'(s)\, ds \tag{3.115}$$

$$+ \frac{x_{i+1}^2}{2} f(x_{i+1}) - \frac{x_i^2}{2} f(x_i) + \underbrace{x_{i+\frac{1}{2}}}_{=\frac{1}{2}(x_i + x_{i+1})} (x_i - x_{i+1}) f\left(x_{i+\frac{1}{2}}\right) \tag{3.116}$$

$$= \int_{x_i}^{x_{i+\frac{1}{2}}} \left(\frac{s^2}{2} - x_i s\right) f'(s)\, ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(s x_{i+1} - \frac{s^2}{2}\right) f'(s)\, ds \tag{3.117}$$

$$+ \frac{x_{i+1}^2}{2} \left(f(x_{i+1}) - f\left(x_{i+\frac{1}{2}}\right)\right) + \frac{x_i^2}{2} \left(f\left(x_{i+\frac{1}{2}}\right) - f(x_i)\right) \tag{3.118}$$

$$= \int_{x_i}^{x_{i+\frac{1}{2}}} \left(\frac{s^2}{2} - x_i s\right) f'(s)\, ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(s x_{i+1} - \frac{s^2}{2}\right) f'(s)\, ds \tag{3.119}$$

$$+ \frac{x_{i+1}^2}{2} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f'(s)\, ds + \frac{x_i^2}{2} \int_{x_i}^{x_{i+\frac{1}{2}}} f'(s)\, ds \tag{3.120}$$

$$= \int_{x_i}^{x_{i+\frac{1}{2}}} \frac{(s - x_i)^2}{2} f'(s)\, ds + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{(x_{i+1} - s)^2}{2} f'(s)\, ds \tag{3.121}$$

$$\leq \left(\int_{x_i}^{x_{i+\frac{1}{2}}} \frac{(s - x_i)^4}{4} ds\right)^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} f'(s)^2 ds\right)^{\frac{1}{2}} \tag{3.122}$$

$$+ \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{(x_{i+1} - s)^4}{4} ds\right)^{\frac{1}{2}} \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f'(s)^2 ds\right)^{\frac{1}{2}} \tag{3.123}$$

Combining the obtained result with (9.6) yields

$$h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 \tag{3.124}$$

$$\leq \frac{1}{h_{i+\frac{1}{2}}} \begin{pmatrix} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} \frac{(s-x_i)^4}{4} ds\right)^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} f'(s)^2 ds\right)^{\frac{1}{2}} \\ + \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{(x_{i+1}-s)^4}{4} ds\right)^{\frac{1}{2}} \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f'(s)^2 ds\right)^{\frac{1}{2}} \end{pmatrix}^2 \tag{3.125}$$

$$\leq \frac{1}{h_{i+\frac{1}{2}}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} \frac{(s - x_i)^4}{4} ds + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{(x_{i+1} - s)^4}{4} ds\right) \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.126}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} \frac{(s - x_i)^4}{4} ds + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{(x_{i+1} - s)^4}{4} ds\right) \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.127}$$

$$= \frac{h_{i+\frac{1}{2}}^4}{320} \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.128}$$

$$\leq \frac{(h_i + h_{i+1})^4}{320} \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.129}$$

$$\leq \frac{h^4}{20} \int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.130}$$

Note that (3.130) is stronger than (3.103). □

Remark 3.14. If $\mathcal{T}$ is a uniform admissible mesh, (3.128) yields

$$h_{i+\frac{1}{2}}\varepsilon^2_{i+\frac{1}{2}} \leq \frac{h^4}{320}\int_{x_i}^{x_{i+1}} f'(s)^2 ds \tag{3.131}$$

Remark 3.15. We also obtain the following results about f. These results can be useful if you try to strengthen or generalize some results in this context.

1. Suppose that $f \in C^1[0,1]$. Applying mean value theorem for definite integral yields that there exists $\alpha_{i+\frac{1}{2}} \in (x_i, x_{i+1})$ such that

$$\overline{f}_{i+\frac{1}{2}} := f\left(\alpha_{i+\frac{1}{2}}\right) = \frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} f(t)\,dt \tag{3.132}$$

Hence, for all $x \in [x_i, x_{i+1}]$

$$\left|f(x) - f\left(\alpha_{i+\frac{1}{2}}\right)\right| \tag{3.133}$$

$$= \left|\frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} (f(x) - f(t))\,dt\right| \tag{3.134}$$

$$= \left|\frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} \left(\int_t^x f'(s)\,ds\right)dt\right| \tag{3.135}$$

$$= \frac{\left|\int_{x_i}^{x}\left(\int_t^x f'(s)\,ds\right)dt - \int_x^{x_{i+1}}\left(\int_x^t f'(s)\,ds\right)dt\right|}{h_{i+\frac{1}{2}}} \tag{3.136}$$

$$= \frac{\left|\int_{x_i}^{x}\left(\int_{x_i}^s f'(s)\,dt\right)ds - \int_x^{x_{i+1}}\left(\int_s^{x_{i+1}} f'(s)\,dt\right)ds\right|}{h_{i+\frac{1}{2}}} \tag{3.137}$$

$$= \frac{\left|\int_{x_i}^{x}(s-x_i) f'(s)\,ds - \int_x^{x_{i+1}}(x_{i+1}-s) f'(s)\,ds\right|}{h_{i+\frac{1}{2}}} \tag{3.138}$$

$$\leq \frac{\left|\int_{x_i}^{x}(s-x_i) f'(s)\,ds\right| + \left|\int_x^{x_{i+1}}(x_{i+1}-s) f'(s)\,ds\right|}{h_{i+\frac{1}{2}}} \tag{3.139}$$

$$\leq \frac{1}{h_{i+\frac{1}{2}}}\left(\begin{array}{c}\left(\int_{x_i}^{x}(s-x_i)^2 ds\right)^{\frac{1}{2}}\left(\int_{x_i}^{x} f'(s)^2 ds\right)^{\frac{1}{2}} \\ +\left(\int_x^{x_{i+1}}(x_{i+1}-s)^2 ds\right)^{\frac{1}{2}}\left(\int_x^{x_{i+1}} f'(s)^2 ds\right)^{\frac{1}{2}}\end{array}\right) \tag{3.140}$$

$$= \frac{1}{h_{i+\frac{1}{2}}}\left(\begin{array}{c}\frac{(x-x_i)^{\frac{3}{2}}}{\sqrt{3}}\left(\int_{x_i}^{x} f'(s)^2 ds\right)^{\frac{1}{2}} \\ +\frac{(x_{i+1}-x)^{\frac{3}{2}}}{\sqrt{3}}\left(\int_x^{x_{i+1}} f'(s)^2 ds\right)^{\frac{1}{2}}\end{array}\right) \tag{3.141}$$

$$\leq \frac{1}{h_{i+\frac{1}{2}}}\left(\frac{(x-x_i)^3 + (x_{i+1}-x)^3}{3}\right)^{\frac{1}{2}}\|f'\|_{L^2(x_i,x_{i+1})} \tag{3.142}$$

□

2. Using the obtained result in Step 1, we have

$$\left\|f - \overline{f}_{i+\frac{1}{2}}\right\|_{L^2(x_i,x_{i+1})} \tag{3.143}$$

$$= \left(\int_{x_i}^{x_{i+1}} \left(f(x) - \overline{f}_{i+\frac{1}{2}} \right)^2 dx \right)^{\frac{1}{2}} \tag{3.144}$$

$$\leq \left(\int_{x_i}^{x_{i+1}} \frac{(x - x_i)^3 + (x_{i+1} - x)^3}{3h_{i+\frac{1}{2}}^2} dx \right)^{\frac{1}{2}} \|f'\|_{L^2(x_i, x_{i+1})} \tag{3.145}$$

$$= \frac{h_{i+\frac{1}{2}}}{\sqrt{6}} \|f'\|_{L^2(x_i, x_{i+1})} \tag{3.146}$$

□

Return to our problem, we set

$$K_1 = \left\{ i : x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \right\} \tag{3.147}$$

$$K_2 = \left\{ i : x_{i+\frac{1}{2}} \neq \frac{x_i + x_{i+1}}{2} \right\} \tag{3.148}$$

Using (3.102) and (3.130) yields

$$\sum_{i=0}^{N} h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 = \sum_{i \in K_1} h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 + \sum_{i \in K_2} h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 \tag{3.149}$$

$$\leq \frac{h^4}{20} \sum_{i \in K_1} \|f'\|_{L^2(x_i, x_{i+1})}^2 + \frac{2}{3} h^2 \sum_{i \in K_2} \|f\|_{L^2(x_i, x_{i+1})}^2 \tag{3.150}$$

We have

$$\frac{h^4}{20} \sum_{i \in K_1} \|f'\|_{L^2(x_i, x_{i+1})}^2 \leq \frac{h^4}{20} \|f'\|_{L^2(0,1)}^2 \tag{3.151}$$

We suppose that $f \in H^1(\Omega) \cap C\left(\overline{\Omega}\right)$ then

$$\|f\|_{L^\infty(\Omega)}^2 \leq 2 \|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \tag{3.152}$$

Hence

$$\|f\|_{L^2(x_i, x_{i+1})}^2 = \int_{x_i}^{x_{i+1}} f^2(t)\, dt \tag{3.153}$$

$$\leq h_{i+\frac{1}{2}} \|f\|_{L^\infty(x_i, x_{i+1})}^2 \tag{3.154}$$

$$\leq 2h \left(2 \|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \right), \text{ by (3.152)} \tag{3.155}$$

So

$$\frac{2}{3} h^2 \sum_{i \in K_2} \|f\|_{L^2(x_i, x_{i+1})}^2 \leq \frac{2}{3} h^2 \sum_{i \in K_2} 2h \left(2 \|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \right) \tag{3.156}$$

$$= \frac{4}{3} h^3 |K_2| \left(2 \|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \right) \tag{3.157}$$

where $|K_2|$ is the cardinality of the set K_2.

Then

$$
\begin{aligned}
&\|U-\Pi u\|_{1,h} && (3.158)\\
&\leq \|Pu'-g(\Pi u)\|_{0,h} && (3.159)\\
&= \left\| \left\{ u'\left(x_{i+\frac{1}{2}}\right) \right\}_{i=0}^{N} - g\left(\{u(x_i)\}_{i=0}^{N+1} \right) \right\|_{0,h} && (3.160)\\
&= \left\| \left\{ u'\left(x_{i+\frac{1}{2}}\right) \right\}_{i=0}^{N} - g\left(\{u(x_i)\}_{i=0}^{N+1} \right) \right\|_{0,h} && (3.161)\\
&= \left\| \left\{ u'\left(x_{i+\frac{1}{2}}\right) \right\}_{i=0}^{N} - \left\{ \frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N+1} \right\|_{0,h} && (3.162)\\
&= \left\| \left\{ u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N} \right\|_{0,h} && (3.163)\\
&= \left\| \left\{ \varepsilon_{i+\frac{1}{2}} \right\}_{i=0}^{N} \right\|_{0,h} && (3.164)\\
&= \left(\sum_{i=0}^{h} h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 \right)^{\frac{1}{2}} && (3.165)\\
&\leq \left(\frac{h^4}{20} \sum_{i\in K_1} \|f'\|_{L^2(x_i,x_{i+1})}^2 + \frac{2}{3}h^2 \sum_{i\in K_2} \|f\|_{L^2(x_i,x_{i+1})}^2 \right)^{\frac{1}{2}} && (3.166)\\
&\leq \left(\frac{h^4}{20} \|f'\|_{L^2(\Omega)}^2 + \frac{4}{3}h^3 |K_2| \left(2\|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \right) \right)^{\frac{1}{2}} && (3.167)\\
&\qquad\qquad\qquad\qquad \text{by (3.151) and (3.157)} && (3.168)\\
&\leq \frac{h^2}{2\sqrt{5}} \|f'\|_{L^2(\Omega)} + \frac{2}{\sqrt{3}} h^{\frac{3}{2}} \sqrt{|K_2|} \left(2\|f\|_{L^2(\Omega)}^2 + \|f'\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} && (3.169)\\
&\leq \frac{h^2}{2\sqrt{5}} \|f'\|_{L^2(\Omega)} + \frac{2}{\sqrt{3}} \sqrt{|K_2|} h^{\frac{3}{2}} \left(\sqrt{2}\|f\|_{L^2(\Omega)} + \|f'\|_{L^2(\Omega)} \right) && (3.170)
\end{aligned}
$$

where (3.169) and (3.170) are deduced from the following obvious inequality

$$\left(a^2+b^2\right)^{\frac{1}{2}} \leq a+b, \quad \forall a \geq 0, \forall b \geq 0 \tag{3.171}$$

We have obtained the estimate

$$\|U-\Pi u\|_{1,h} \leq \frac{h^2}{2\sqrt{5}} \|f'\|_{L^2(\Omega)} + \frac{2}{\sqrt{3}} \sqrt{|K_2|} h^{\frac{3}{2}} \left(\sqrt{2}\|f\|_{L^2(\Omega)} + \|f'\|_{L^2(\Omega)} \right) \tag{3.172}$$

whose the leading term behaves like $O\left(h^{\frac{2}{2}}\right)$. If $|K_2|$ is bounded when h tends to 0, then convergence is at least of order $\frac{3}{2}$. $\square$

4 Theoretical Problems

Let $f \in L^2(0,1)$.

$$-u'' = f \text{ in } (0,1) \tag{4.1}$$

subject to a Neumann boundary condition

$$u'(0) = g_0 \tag{4.2}$$
$$-u'(1) = g_1 \tag{4.3}$$
$$\int_0^1 f(x)\,dx = g_0 + g_1 \tag{4.4}$$
$$\int_0^1 u(x)\,dx = 0 \tag{4.5}$$

Let $U = \{u_i\}_{i=0}^{N+1}$ with

$$(gu)_{\frac{1}{2}} = g_0 \tag{4.6}$$
$$(gu)_{N+\frac{1}{2}} = -g_1 \tag{4.7}$$

satisfy

$$-d(g(U)) = b \tag{4.8}$$

where $b = \{b_i\}_{i=1}^{N}$ and b_i is mean-value of f in K_i.

Problem 4.1. *Using that $w(x) - w(y) = \int_x^y w'(s)\,ds$, prove the following trace inequality for all $w \in H^1(0,1)$*

$$|w(x) - \overline{w}| \le \|w'\|_{L^2(0,1)} \tag{4.9}$$

and the following Poincare inequality for all $w \in H^1(0,1)$

$$\|w - \overline{w}\|_{L^2(0,1)} \le \|w'\|_{L^2(0,1)} \tag{4.10}$$

where we have used the notation

$$\overline{w} = \int_0^1 w(x)\,dx \tag{4.11}$$

Proof 1 of (4.9). We have

$$|w(x) - \overline{w}| = \left|\int_0^1 w(x)\,dt - \int_0^1 w(t)\,dt\right| \tag{4.12}$$
$$= \left|\int_0^1 (w(x) - w(t))\,dt\right| \tag{4.13}$$
$$= \left|\int_0^1 \left(\int_x^t w'(s)\,ds\right) dt\right| \tag{4.14}$$
$$\le \int_0^1 \left|\int_x^t w'(s)\,ds\right| dt \tag{4.15}$$

$$
\begin{aligned}
&\leq \int_0^1 \left| \int_x^t \left| w'(s) \right| ds \right| dt && (4.16)\\
&\leq \int_0^1 \left(\int_0^1 \left| w'(s) \right| ds \right) dt && (4.17)\\
&= \int_0^1 \left| w'(s) \right| ds && (4.18)\\
&= \| w' \|_{L^1(0,1)} && (4.19)\\
&\leq \| w' \|_{L^2(0,1)} && (4.20)
\end{aligned}
$$

where the last inequality is obtained by applying Cauchy Schwarz inequality. □

Proof 2 of (4.9).

$$
\begin{aligned}
\left| w(x) - \overline{w} \right| &= \left| \int_0^1 w(x)\, dt - \int_0^1 w(t)\, dt \right| && (4.21)\\
&= \left| \int_0^1 \left(w(x) - w(t) \right) dt \right| && (4.22)\\
&= \left| \int_0^1 \left(\int_x^t w'(s)\, ds \right) dt \right| && (4.23)\\
&= \left| - \int_0^x \left(\int_t^x w'(s)\, ds \right) dt + \int_x^1 \left(\int_x^t w'(s)\, ds \right) dt \right| && (4.24)\\
&= \left| - \int_0^x s w'(s)\, ds + \int_x^1 (1-s)\, w'(s)\, ds \right| && (4.25)\\
&\leq \left| \int_0^x s w'(s)\, ds \right| + \left| \int_x^1 (1-s)\, w'(s)\, ds \right| && (4.26)\\
&\leq \left(\int_0^x s^2 ds \right)^{\frac{1}{2}} \left(\int_0^x w'(s)^2 ds \right)^{\frac{1}{2}} && (4.27)\\
&+ \left(\int_x^1 (1-s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^x w'(s)^2 ds \right)^{\frac{1}{2}} && (4.28)\\
&= \sqrt{\frac{x^3}{3}} \left(\int_0^x w'(s)^2 ds \right)^{\frac{1}{2}} + \sqrt{\frac{(1-x)^3}{3}} \left(\int_x^1 w'(s)^2 ds \right)^{\frac{1}{2}} && (4.29)\\
&\leq \left(2 \left(\frac{x^3}{3} \int_0^x w'(s)^2 ds + \frac{(1-x)^3}{3} \int_x^1 w'(s)^2 ds \right) \right)^{\frac{1}{2}} && (4.30)\\
&\leq \left(2 \left(\frac{1}{3} \int_0^x w'(s)^2 ds + \frac{1}{3} \int_x^1 w'(s)^2 ds \right) \right)^{\frac{1}{2}}, \text{ by } x \in [0,1] && (4.31)\\
&\leq \sqrt{\frac{2}{3}} \| w' \|_{L^2(0,1)} && (4.32)\\
&\leq \| w' \|_{L^2(0,1)} && (4.33)
\end{aligned}
$$

where (4.30) is deduced from the familiar Cauchy-Schwarz inequality

$$
\sqrt{a} + \sqrt{b} \leq \sqrt{2\,(a+b)} \tag{4.34}
$$

with

$$a = \frac{x^3}{3} \int_0^x w'(s)^2 ds \tag{4.35}$$

$$b = \frac{(1-x)^3}{3} \int_x^1 w'(s)^2 ds \tag{4.36}$$

The inequality happens when $w' = 0$ in $[0,1]$, i.e., w is constant in $[0,1]$. □

Proof 3 of (4.9). Similar to Solution 2, we start at (4.26)

$$\begin{aligned}
|w(x) - \overline{w}| &\leq \left| \int_0^x s w'(s)\, ds \right| + \left| \int_x^1 (1-s)\, w'(s)\, ds \right| && (4.37)\\
&\leq \int_0^x |s w'(s)|\, ds + \int_x^1 |(1-s)\, w'(s)|\, ds && (4.38)\\
&\leq \int_0^x |w'(s)|\, ds + \int_x^1 |w'(s)|\, ds && (4.39)\\
&= \|w'\|_{L^1(0,1)} && (4.40)\\
&\leq \|w'\|_{L^2(0,1)} && (4.41)
\end{aligned}$$

□

Proof of (4.10). Using (4.9), we have

$$\begin{aligned}
\|w - \overline{w}\|_{L^2(0,1)} &= \left(\int_0^1 (w(x) - \overline{w})^2 dx \right)^{\frac{1}{2}} && (4.42)\\
&\leq \left(\int_0^1 \|w'\|_{L^2(0,1)}^2\, dx \right)^{\frac{1}{2}}, \text{ by (4.9)} && (4.43)\\
&\leq \|w'\|_{L^2(0,1)} && (4.44)
\end{aligned}$$

This completes the proof. □

Remark 4.2. We can strengthen (4.9) by improving some estimations in our previous proofs.

1. We note that we have obtained the following result in Proof 2 of (4.9).

$$\begin{aligned}
|w(x) - \overline{w}| &\leq \left(2 \left(\frac{x^3}{3} \int_0^x w'(s)^2 ds + \frac{(1-x)^3}{3} \int_x^1 w'(s)^2 ds \right) \right)^{\frac{1}{2}} && (4.45)\\
&\leq \sqrt{\frac{2}{3} \max\left\{ x^3, (1-x)^3 \right\}} \|w'\|_{L^2(0,1)}, \forall x \in [0,1] && (4.46)
\end{aligned}$$

2. Proof 4 of (4.9). By using (9.6), we also have the following estimation

$$\begin{aligned}
&|w(x) - \overline{w}| && (4.47)\\
&\leq \sqrt{\frac{x^3}{3}} \left(\int_0^x w'(s)^2 ds \right)^{\frac{1}{2}} + \sqrt{\frac{(1-x)^3}{3}} \left(\int_x^1 w'(s)^2 ds \right)^{\frac{1}{2}} && (4.48)
\end{aligned}$$

$$\leq \left(\frac{x^3}{3}+\frac{(1-x)^3}{3}\right)^{\frac{1}{2}}\left(\int_0^x w'(s)^2 ds+\int_x^1 w'(s)^2 ds\right)^{\frac{1}{2}} \tag{4.49}$$

$$= \left(x^2-x+\frac{1}{3}\right)^{\frac{1}{2}}\|w'\|_{L^2(0,1)}, \quad \forall x\in[0,1] \tag{4.50}$$

With these strengthened results, we can also strengthen (4.10) as follows.

1. Proof 2 of (4.10). Combining (4.32) and (4.42) yields

$$\|w-\overline{w}\|_{L^2(0,1)} = \left(\int_0^1 (w(x)-\overline{w})^2 dx\right)^{\frac{1}{2}} \tag{4.51}$$

$$\leq \left(\int_0^1 \frac{2}{3}\|w'\|_{L^2(0,1)}^2 dx\right)^{\frac{1}{2}} \tag{4.52}$$

$$= \sqrt{\frac{2}{3}}\|w'\|_{L^2(0,1)} \tag{4.53}$$

2. Proof 3 of (4.10). Combining (4.47)-(4.50) and (4.42) yields

$$\|w-\overline{w}\|_{L^2(0,1)} = \left(\int_0^1 (w(x)-\overline{w})^2 dx\right)^{\frac{1}{2}} \tag{4.54}$$

$$\leq \left(\int_0^1 \left(x^2-x+\frac{1}{3}\right)\|w'\|_{L^2(0,1)}^2 dx\right)^{\frac{1}{2}} \tag{4.55}$$

$$= \left(\int_0^1 \left(x^2-x+\frac{1}{3}\right) dx\right)^{\frac{1}{2}}\|w'\|_{L^2(0,1)} \tag{4.56}$$

$$= \frac{1}{\sqrt{6}}\|w'\|_{L^2(0,1)} \tag{4.57}$$

Can you obtain any better results for (4.9) and (4.10) than ours?

Problem 4.3. *Write the variational formulation of* (4.1) *and prove the following a priori estimate for a solution u of this variational formulation*

$$\|u'\|_{L^2(0,1)} \leq \|f\|_{L^2(0,1)} + (|g_0|+|g_1|) \tag{4.58}$$

Proof. Multiplying both side of (4.1) by a test function $v\in H^1(0,1)$ and then integrating both side of the obtained equality to obtain

$$-\int_0^1 u''v = \int_0^1 fv, \quad v\in H^1(0,1) \tag{4.59}$$

Integrating by parts the left hand side of (4.59) yields

$$-\int_0^1 u''v = -\left.u'v\right|_0^1 + \int_0^1 u'v' \tag{4.60}$$

$$= -u'(1)v(1)+u'(0)v(0)+\int_0^1 u'v' \tag{4.61}$$

$$= v(0)\, g_0 + v(1)\, g_1 + \int_0^1 u'v' \tag{4.62}$$

Then, we obtain the following variational formulation

$$\int_0^1 u'v' = \int_0^1 fv - v(0)\, g_0 - v(1)\, g_1, \quad v \in H^1(0,1) \tag{4.63}$$

For $f \in L^2(0,1)$, we have $u \in H^1(0,1)$. Hence, we can replace $v = u$ in (4.63) to obtain

$$\|u'\|^2_{L^2(0,1)} = \int_0^1 fu - u(0)\, g_0 - u(1)\, g_1 \tag{4.64}$$

Applying (4.9) for u and notice that $\overline{u} = 0$ (4.5), we obtain

$$|u(x)| \le \|u'\|_{L^2(0,1)}, \quad \forall x \in [0,1] \tag{4.65}$$

Similarly, applying (4.10) for u yields

$$\|u\|_{L^2(0,1)} \le \|u'\|_{L^2(0,1)} \tag{4.66}$$

Combining (4.64)-(4.66) yields

$$\begin{aligned} \|u'\|^2_{L^2(0,1)} &= \int_0^1 fu - u(0)\, g_0 - u(1)\, g_1 && (4.67)\\ &\le \|f\|_{L^2(0,1)}\|u\|_{L^2(0,1)} + |u(0)|\,|g_0| + |u(1)|\,|g_1| && (4.68)\\ &\le \|f\|_{L^2(0,1)}\|u'\|_{L^2(0,1)} + \|u'\|_{L^2(0,1)}\,|g_0| + \|u'\|_{L^2(0,1)}\,|g_1| && (4.69) \end{aligned}$$

If $\|u'\|_{L^2(0,1)} = 0$, (4.58) is obvious. If $\|u'\|_{L^2(0,1)} \ne 0$, dividing both side of (4.67)-(4.69) yields (4.58). □

Remark 4.4.

1. Using Remark 1.2, we can strengthen (4.58) as follows.

   (a) Using (4.46), (4.64) and (4.68) yields

   $$\|u'\|_{L^2(0,1)} \le \sqrt{\frac{2}{3}} \left(\|f\|_{L^2(0,1)} + |g_0| + |g_1| \right) \tag{4.70}$$

   (b) Using (4.50), (4.57) and (4.68) yields

   $$\|u'\|_{L^2(0,1)} \le \frac{1}{\sqrt{6}} \|f\|_{L^2(0,1)} + \frac{1}{\sqrt{3}} (|g_0| + |g_1|) \tag{4.71}$$

2. [4] We assume that $u \in C^2(0,1)$ during this remark. Again, we can strengthen (4.58) as follows.

   First of all, we fix an an arbitrary $\alpha \in [0,1]$. Due to (4.1), we have

   $$-\int_\alpha^x u''(t)\, dt = \int_\alpha^x f(t)\, dt, \quad \forall x \in [0,1] \tag{4.72}$$

[4] Nguyen An Thinh's remark.

i.e.,

$$u'(\alpha) - \int_{\alpha}^{x} f(t)\,dt = u'(x), \quad \forall x \in [0,1] \tag{4.73}$$

Hence

$$|u'(x)| \le \left| \int_{\alpha}^{x} f(t)\,dt \right| + |u'(\alpha)|, \quad \forall x \in [0,1] \tag{4.74}$$

Squaring and then integrating both sides of (4.74) yields

$$\|u'\|^2_{L^2(0,1)} \tag{4.75}$$

$$= \int_0^1 u'(x)^2 dx \tag{4.76}$$

$$\le \int_0^1 \left(\left| \int_{\alpha}^{x} f(t)\,dt \right| + |u'(\alpha)| \right)^2 dx \tag{4.77}$$

$$= \int_0^1 \left(\left| \int_{\alpha}^{x} f(t)\,dt \right|^2 + 2\,|u'(\alpha)| \left| \int_{\alpha}^{x} f(t)\,dt \right| + |u'(\alpha)|^2 \right) dx \tag{4.78}$$

$$= \int_0^1 \left| \int_{\alpha}^{x} f(t)\,dt \right|^2 dx \tag{4.79}$$

$$+ 2\,|u'(\alpha)| \int_0^1 \left| \int_{\alpha}^{x} f(t)\,dt \right| dx + |u'(\alpha)|^2 \tag{4.80}$$

$$\le \int_0^1 \left(\int_0^1 |f(t)|\,dt \right)^2 dx \tag{4.81}$$

$$+ 2\,|u'(\alpha)| \int_0^1 \left(\int_0^1 |f(t)|\,dt \right) dx + |u'(\alpha)|^2 \tag{4.82}$$

$$= \|f\|^2_{L^1(0,1)} + 2\,|u'(\alpha)|\,\|f\|_{L^1(0,1)} + |u'(\alpha)|^2 \tag{4.83}$$

$$= \left(\|f\|_{L^1(0,1)} + |u'(\alpha)| \right)^2 \tag{4.84}$$

Since α is taken arbitrarily and $u \in C^2(0,1)$, we have

$$\|u'\|_{L^2(0,1)} \le \|f\|_{L^1(0,1)} + \min_{x\in[0,1]} |u'(x)| \tag{4.85}$$

$$\le \|f\|_{L^2(0,1)} + \min_{x\in[0,1]} |u'(x)| \tag{4.86}$$

In particular, considering $\alpha = 0$ and $\alpha = 1$ yields

$$\|u'\|_{L^2(0,1)} \le \|f\|_{L^2(0,1)} + |g_0| \tag{4.87}$$

$$\|u'\|_{L^2(0,1)} \le \|f\|_{L^2(0,1)} + |g_1| \tag{4.88}$$

A question arises out of this remark naturally. Why can we obtain better results (4.86)-(4.88) than (4.58)? The answer to this question is quite easy. The main point is that we have use variational formulation (4.63) to deduce (4.58). This variational formulation weakened our problem before we estimate $\|u'\|_{L^2(0,1)}$. On the other hand, we deduce (4.86)-(4.88) from (4.1) without using variational formulation. This makes our estimate sharper.

Problem 4.5. *For $\{w_i\}_{i=0}^{N+1}$, we define*

$$\overline{w}_h = \sum_{i=1}^{N} h_i w_i \tag{4.89}$$

1. *For all $i, j = 0, 1, \ldots, N$, bound $|w_i - w_j|$ by a constant time $\|w\|_{1,h}$ and compute the constant.*
2. *Deduce the following **trace inequality** for all $i = 0, 1, \ldots, N+1$*

$$|w_i - \overline{w}_h| \leq \|w\|_{1,h} \tag{4.90}$$

3. *Deduce the following **discrete Poincare inequality***

$$\left\| \{w_i\}_{i=1}^{N} - \overline{w}_h \mathbf{1} \right\|_{0,K} \leq \|w\|_{1,h} \tag{4.91}$$

where $\mathbf{1} \in \mathbb{R}^N$ is the vector $(1, \ldots, 1)$.

Proof.

1. The case $i = j$ is trivial. If $i \neq j$, then, without lost of generality, we can assume that $i > j$. Then

$$|w_i - w_j| = \left| \sum_{k=j}^{i-1} (w_{k+1} - w_k) \right| \tag{4.92}$$

$$\leq \sum_{k=j}^{i-1} |w_{k+1} - w_k| \tag{4.93}$$

$$\leq \left(\sum_{k=j}^{i-1} h_{k+\frac{1}{2}} \right)^{\frac{1}{2}} \left(\sum_{k=j}^{i-1} \frac{1}{h_{k+\frac{1}{2}}} (w_{k+1} - w_k)^2 \right)^{\frac{1}{2}} \tag{4.94}$$

$$\leq \left(\sum_{k=j}^{i-1} h_{k+\frac{1}{2}} \right)^{\frac{1}{2}} \|w\|_{1,h} \tag{4.95}$$

$$= \sqrt{x_i - x_j} \|w\|_{1,h} \tag{4.96}$$

Similarly for $i < j$, we obtain

$$|w_i - w_j| \leq \sqrt{|x_i - x_j|} \|w\|_{1,h} \tag{4.97}$$

2. We have

$$|w_i - \overline{w}_h| = \left| w_i - \sum_{j=1}^{N} h_j w_j \right| \tag{4.98}$$

$$= \left| \sum_{j=1}^{N} h_j (w_i - w_j) \right|, \text{ by } \sum_{j=1}^{N} h_j = 1 \tag{4.99}$$

$$\leq \sum_{j=1}^{N} h_j \left| w_i - w_j \right| \tag{4.100}$$

$$\leq \sum_{j=1}^{N} h_j \sqrt{|x_i - x_j|} \|w\|_{1,h}, \text{ by (4.97)} \tag{4.101}$$

$$\leq \|w\|_{1,h} \sum_{j=1}^{N} h_j \tag{4.102}$$

$$= \|w\|_{1,h} \tag{4.103}$$

Note that we have proved the stronger estimate

$$|w_i - \overline{w}_h| \leq \left(\sum_{j=1}^{N} h_j \sqrt{|x_i - x_j|} \right) \|w\|_{1,h} \tag{4.104}$$

3. Using (4.104) yields

$$\left\| \{w_i\}_{i=1}^{N} - \overline{w}_h \mathbf{1} \right\|_{0,K} = \left\| \{w_i - \overline{w}_h\}_{i=1}^{N} \right\|_{0,K} \tag{4.105}$$

$$= \left(\sum_{i=1}^{N} h_i (w_i - \overline{w}_h)^2 \right)^{\frac{1}{2}} \tag{4.106}$$

$$\leq \left(\sum_{i=1}^{N} h_i \|w\|_{1,h}^2 \right)^{\frac{1}{2}}, \text{ by (4.90)} \tag{4.107}$$

$$= \|w\|_{1,h} \tag{4.108}$$

We finish our proof. □

Problem 4.6. *Given* $V = \left\{ v_{i+\frac{1}{2}} \right\}_{i=0}^{N}, W = \{w_i\}_{i=0}^{N+1}$. *Prove that*

$$(d(V), W)_K = -(V, g(W))_h + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{4.109}$$

PROOF. Using Definition 3.1 and Definition 3.2, the left hand side of (3.14) can be rewritten as

$$\left(d(V), \{w_i\}_{i=1}^{N} \right)_K = \left(\left\{ \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i} \right\}_{i=1}^{N}, \{w_i\}_{i=1}^{N} \right)_K \tag{4.110}$$

$$= \sum_{i=1}^{N} h_i \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i} w_i \tag{4.111}$$

$$= \sum_{i=1}^{N} w_i \left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} \right) \tag{4.112}$$

Using Definition 3.3 and Definition 3.4 yields

$$(V, g(W))_h = \left(V, \left\{ (gw)_{i+\frac{1}{2}} \right\}_{i=0}^{N} \right)_h \tag{4.113}$$

$$= \left(\left\{ v_{i+\frac{1}{2}} \right\}_{i=0}^{N}, \left\{ \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N} \right)_h \tag{4.114}$$

$$= \sum_{i=1}^{N} h_{i+\frac{1}{2}} v_{i+\frac{1}{2}} \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{4.115}$$

$$= \sum_{i=1}^{N} v_{i+\frac{1}{2}} \left(w_{i+1} - w_i \right) \tag{4.116}$$

Thus, the right hand side of (3.14) can be rewritten as

$$- \left(V, g\left(W \right) \right)_h + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{4.117}$$

$$= - \sum_{i=0}^{N} v_{i+\frac{1}{2}} \left(w_{i+1} - w_i \right) + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}} \tag{4.118}$$

$$= - v_{\frac{1}{2}} w_1 + \sum_{i=1}^{N-1} v_{i+\frac{1}{2}} \left(w_i - w_{i+1} \right) + v_{N+\frac{1}{2}} w_N \tag{4.119}$$

$$= w_1 \left(v_{\frac{3}{2}} - v_{\frac{1}{2}} \right) + w_2 \left(v_{\frac{5}{2}} - v_{\frac{3}{2}} \right) + \cdots + w_N \left(v_{N+\frac{1}{2}} - v_{N-\frac{1}{2}} \right) \tag{4.120}$$

$$= \sum_{i=1}^{N} w_i \left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} \right) \tag{4.121}$$

Comparing (4.112) and (4.121) yields that (3.14) holds. □

Problem 4.7. *For any* $\{w_i\}_{i=0}^{N+1}$, *and* $b = \{b_i\}_{i=1}^{N}$ *as before. Prove that*

$$\left(g\left(U \right), g\left(W \right) \right)_h = \left(b, \{w_i\}_{i=1}^{N} \right)_K - g_0 w_0 - g_1 w_{N+1} \tag{4.122}$$

Proof.

$$\left(g\left(U \right), g\left(W \right) \right)_h \tag{4.123}$$

$$= \left(\left\{ (gU)_{i+\frac{1}{2}} \right\}_{i=0}^{N}, \left\{ (gW)_{i+\frac{1}{2}} \right\}_{i=0}^{N} \right)_h \tag{4.124}$$

$$= \left(\left\{ \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N}, \left\{ \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \right\}_{i=0}^{N} \right)_h \tag{4.125}$$

$$= \sum_{i=0}^{N} h_{i+\frac{1}{2}} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{4.126}$$

$$= \sum_{i=0}^{N} \frac{\left(u_{i+1} - u_i \right) \left(w_{i+1} - w_i \right)}{h_{i+\frac{1}{2}}} \tag{4.127}$$

$$= \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_{i+1} - \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i \tag{4.128}$$

$$= \sum_{i=1}^{N+1} \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} w_i - \sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i \tag{4.129}$$

$$= \sum_{i=1}^{N} \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} w_i - \sum_{i=1}^{N} \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} w_i + \frac{u_{N+1} - u_N}{h_{N+\frac{1}{2}}} w_{N+1} - \frac{u_1 - u_0}{h_{\frac{1}{2}}} w_0 \tag{4.130}$$

$$= \sum_{i=1}^{N} \left(\frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} - \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \right) w_i + \frac{u_{N+1} - u_N}{h_{N+\frac{1}{2}}} w_{N+1} - \frac{u_1 - u_0}{h_{\frac{1}{2}}} w_0 \tag{4.131}$$

$$= \sum_{i=1}^{N} h_i b_i w_i + (gu)_{N+\frac{1}{2}} w_{N+1} - (gu)_{\frac{1}{2}} w_0 \tag{4.132}$$

$$= \left(b, \{w_i\}_{i=1}^{N}\right) - g_0 w_0 - g_1 w_{N+1} \tag{4.133}$$

Thus, (4.122) holds. □

Problem 4.8. *Using the previous Poincare and trace inequalities, deduce that the discrete solution $U = \{u_i\}_{i=0}^{N+1}$ satisfies the following a priori estimate*

$$\|U\|_{1,h} \leq \|b\|_{0,K} + (|g_0| + |g_1|) \tag{4.134}$$

PROOF. We recall that the condition (4.5) is discretized by (3.55), i.e.,

$$\overline{u}_h = \sum_{i=1}^{N} h_i u_i = 0 \tag{4.135}$$

Thus, applying (4.90) for U yields

$$|u_i| \leq \|U\|_{1,h}, \quad i = 0, \ldots, N+1 \tag{4.136}$$

Similarly, applying (4.91) for u yields

$$\|U\|_{0,K} \leq \|U\|_{1,h} \tag{4.137}$$

Then, we have

$$\begin{aligned}
\|U\|_{1,h}^2 &= \left(g\left(U\right), g\left(U\right)\right)_h && (4.138)\\
&= \left(b, \{u_i\}_{i=1}^{N}\right)_K - g_0 u_0 - g_1 u_{N+1}, \text{ by (4.122)} && (4.139)\\
&= \sum_{i=1}^{N} h_i u_i b_i - g_0 u_0 - g_1 u_{N+1} && (4.140)\\
&\leq \left(\sum_{i=1}^{N} h_i b_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} h_i u_i^2\right)^{\frac{1}{2}} + |g_0|\,|u_0| + |g_1|\,|u_{N+1}| && (4.141)\\
&= \|b\|_{0,K} \|U\|_{0,K} + |g_0|\,|u_0| + |g_1|\,|u_{N+1}| && (4.142)\\
&\leq \|b\|_{0,K} \|U\|_{1,h} + |g_0|\,\|U\|_{1,h} + |g_1|\,\|U\|_{1,h}, \text{ by (4.136),(4.137)} && (4.143)
\end{aligned}$$

If $\|U\|_{1,h} = 0$, (4.134) is obvious. If $\|U\|_{1,h} \neq 0$, then, dividing both side of (4.138)-(4.143) yields (4.134). □

Problem 4.9. *Since exact solution $u \in C^1\left(\overline{\Omega}\right)$. We can define projection*

$$\Pi : C^1\left(\overline{\Omega}\right) \to \mathbb{R}^{N+2} \tag{4.144}$$

$$u \mapsto (\Pi u)_i = u(x_i), \quad \forall i \in [0, N+1] \tag{4.145}$$

Since $u' \in C^0\left(\overline{\Omega}\right)$. We can define projection

$$P : C^0\left(\overline{\Omega}\right) \to \mathbb{R}^{N+1} \tag{4.146}$$

$$u' \mapsto (Pu')_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right), \quad \forall i \in [0, N] \tag{4.147}$$

Let $W = (w_i)_{i\in[0,N+1]}$, we prove that

$$(g(U), g(W))_h = (Pu', g(W))_h \tag{4.148}$$

Proof. The right hand side of (4.148) can be rewritten as

$$(Pu', g(W))_h \tag{4.149}$$

$$= \left(\left\{(Pu')_{i+\frac{1}{2}}\right\}_{i=0}^{N}, \left\{(gW)_{i+\frac{1}{2}}\right\}_{i=0}^{N}\right)_h \tag{4.150}$$

$$= \left(\left\{u'\left(x_{i+\frac{1}{2}}\right)\right\}_{i=0}^{N}, \left\{\frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}}\right\}_{i=0}^{N}\right)_h \tag{4.151}$$

$$= \sum_{i=0}^{N} h_{i+\frac{1}{2}} u'\left(x_{i+\frac{1}{2}}\right) \frac{w_{i+1} - w_i}{h_{i+\frac{1}{2}}} \tag{4.152}$$

$$= \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right)(w_{i+1} - w_i) \tag{4.153}$$

$$= \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_{i+1} - \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i \tag{4.154}$$

$$= \sum_{i=1}^{N+1} u'\left(x_{i-\frac{1}{2}}\right) w_i - \sum_{i=0}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i \tag{4.155}$$

$$= \sum_{i=1}^{N} u'\left(x_{i-\frac{1}{2}}\right) w_i - \sum_{i=1}^{N} u'\left(x_{i+\frac{1}{2}}\right) w_i, \text{ by } w_0 = w_{N+1} = 0 \tag{4.156}$$

$$= \sum_{i=1}^{N} \left(u'\left(x_{i-\frac{1}{2}}\right) - u'\left(x_{i+\frac{1}{2}}\right)\right) w_i \tag{4.157}$$

$$= \sum_{i=1}^{N} h_i b_i w_i, \text{ by (1.13)} \tag{4.158}$$

$$= \left(\{w_i\}_{i=1}^{N}, b\right)_K \tag{4.159}$$

$$= (g(U), g(W))_h, \text{ by (3.31)} \tag{4.160}$$

Thus, (4.148) holds. □

Problem 4.10. *We define*

$$\varepsilon_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1}) - u(x_i)}{\left|D_{i+\frac{1}{2}}\right|} \tag{4.161}$$

We prove that

$$\left|D_{i+\frac{1}{2}}\right| \varepsilon_{i+\frac{1}{2}}^{2} \leq \left(\frac{4}{3}\right)^{2} h^{2} \int_{D_{i+\frac{1}{2}}} f^{2}(t)\, dt \tag{4.162}$$

where $h = \max\{|T_i| : i \in [1, N]\}$.

PROOF 1[5]. We have

$$\varepsilon_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \tag{4.163}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} u'\left(x_{i+\frac{1}{2}}\right) dt \tag{4.164}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} \left(u'\left(x_{i+\frac{1}{2}}\right) - u'(t)\right) dt \tag{4.165}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt \tag{4.166}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt}{h_{i+\frac{1}{2}}} \tag{4.167}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{t} u''(s)\, ds\right) dt}{h_{i+\frac{1}{2}}} \tag{4.168}$$

Then

$$\left|\varepsilon_{i+\frac{1}{2}}\right| = \left|\frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{t} u''(s)\, ds\right) dt}{h_{i+\frac{1}{2}}}\right| \tag{4.169}$$

$$\leq \frac{\left|\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{t}^{x_{i+\frac{1}{2}}} u''(s)\, ds\right) dt\right| + \left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{t} u''(s)\, ds\right) dt\right|}{h_{i+\frac{1}{2}}} \tag{4.170}$$

$$\leq \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{t}^{x_{i+\frac{1}{2}}} |u''(s)|\, ds\right) dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{t} |u''(s)|\, ds\right) dt}{h_{i+\frac{1}{2}}} \tag{4.171}$$

$$\leq \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} |u''(s)|\, ds\right) dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |u''(s)|\, ds\right) dt}{h_{i+\frac{1}{2}}} \tag{4.172}$$

$$= \frac{\left(x_{i+\frac{1}{2}} - x_i\right) \int_{x_i}^{x_{i+\frac{1}{2}}} |u''(s)|\, ds + \left(x_{i+1} - x_{i+\frac{1}{2}}\right) \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} |u''(s)|\, ds}{h_{i+\frac{1}{2}}} \tag{4.173}$$

$$\leq \frac{h}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} |f(s)|\, ds \tag{4.174}$$

[5]Tran Nguyen Try's proof.

$$\leq \frac{h}{h_{i+\frac{1}{2}}} \left(\int_{x_i}^{x_{i+1}} |f(s)|^2 ds \right)^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+1}} 1^2 ds \right)^{\frac{1}{2}} \tag{4.175}$$

$$= \frac{h}{\sqrt{h_{i+\frac{1}{2}}}} \left(\int_{x_i}^{x_{i+1}} |f(s)|^2 ds \right)^{\frac{1}{2}} \tag{4.176}$$

Thus,

$$h_{i+\frac{1}{2}} \varepsilon_{i+\frac{1}{2}}^2 \leq h^2 \int_{x_i}^{x_{i+1}} |f(s)|^2 ds \tag{4.177}$$

This result is stronger than (4.162). □

Solution 2. The second proof is an improvement of the first one.

$$\varepsilon_{i+\frac{1}{2}} = u'\left(x_{i+\frac{1}{2}}\right) - \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} \tag{4.178}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} u'\left(x_{i+\frac{1}{2}}\right) dt \tag{4.179}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} \left(u'\left(x_{i+\frac{1}{2}}\right) - u'(t) \right) dt \tag{4.180}$$

$$= \frac{1}{h_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+1}} \left(\int_t^{x_{i+\frac{1}{2}}} u''(s)\, ds \right) dt \tag{4.181}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_t^{x_{i+\frac{1}{2}}} u''(s)\, ds \right) dt + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_t^{x_{i+\frac{1}{2}}} u''(s)\, ds \right) dt}{h_{i+\frac{1}{2}}} \tag{4.182}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_t^{x_{i+\frac{1}{2}}} u''(s)\, ds \right) dt - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_{x_{i+\frac{1}{2}}}^{t} u''(s)\, ds \right) dt}{h_{i+\frac{1}{2}}} \tag{4.183}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} \left(\int_{x_i}^{s} u''(s)\, dt \right) ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\int_s^{x_{i+1}} u''(s)\, dt \right) ds}{h_{i+\frac{1}{2}}} \tag{4.184}$$

$$= \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} (s - x_i)\, u''(s)\, ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - s)\, u''(s)\, ds}{h_{i+\frac{1}{2}}} \tag{4.185}$$

Hence

$$\left|\varepsilon_{i+\frac{1}{2}}\right| = \left| \frac{\int_{x_i}^{x_{i+\frac{1}{2}}} (s - x_i)\, u''(s)\, ds - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - s)\, u''(s)\, ds}{h_{i+\frac{1}{2}}} \right| \tag{4.186}$$

$$\leq \frac{\left| \int_{x_i}^{x_{i+\frac{1}{2}}} (s - x_i)\, u''(s)\, ds \right| + \left| \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - s)\, u''(s)\, ds \right|}{h_{i+\frac{1}{2}}} \tag{4.187}$$

$$\leq \frac{1}{h_{i+\frac{1}{2}}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} u''(s)^2 ds \right)^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} (s - x_i)^2 ds \right)^{\frac{1}{2}} \tag{4.188}$$

$$+\frac{1}{h_{i+\frac{1}{2}}}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} u''(s)^2 ds\right)^{\frac{1}{2}}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1}-s)^2 ds\right)^{\frac{1}{2}} \tag{4.189}$$

$$=\frac{\left(x_{i+\frac{1}{2}}-x_i\right)^{\frac{3}{2}}}{h_{i+\frac{1}{2}}\sqrt{3}}\left(\int_{x_i}^{x_{i+\frac{1}{2}}} f(s)^2 ds\right)^{\frac{1}{2}} \tag{4.190}$$

$$+\frac{\left(x_{i+1}-x_{i+\frac{1}{2}}\right)^{\frac{3}{2}}}{h_{i+\frac{1}{2}}\sqrt{3}}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f(s)^2 ds\right)^{\frac{1}{2}} \tag{4.191}$$

Using (9.6), we deduce

$$h_{i+\frac{1}{2}}\varepsilon_{i+\frac{1}{2}}^2 = h_{i+\frac{1}{2}}\left(\begin{array}{c}\frac{\left(x_{i+\frac{1}{2}}-x_i\right)^{\frac{3}{2}}}{h_{i+\frac{1}{2}}\sqrt{3}}\left(\int_{x_i}^{x_{i+\frac{1}{2}}} f(s)^2 ds\right)^{\frac{1}{2}} \\ +\frac{\left(x_{i+1}-x_{i+\frac{1}{2}}\right)^{\frac{3}{2}}}{h_{i+\frac{1}{2}}\sqrt{3}}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f(s)^2 ds\right)^{\frac{1}{2}}\end{array}\right)^2 \tag{4.192}$$

$$\leq \frac{1}{3h_{i+\frac{1}{2}}}\left(\left(x_{i+\frac{1}{2}}-x_i\right)^3+\left(x_{i+1}-x_{i+\frac{1}{2}}\right)^3\right)\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.193}$$

$$=\frac{\left(h_i^+\right)^2-h_i^+h_{i+1}^-+\left(h_{i+1}^-\right)^2}{3}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.194}$$

$$\leq \frac{\left(h_i^+\right)^2+\left(h_{i+1}^-\right)^2}{3}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.195}$$

$$\leq \frac{h_i^2+h_{i+1}^2}{3}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.196}$$

$$\leq \frac{2}{3}h^2\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.197}$$

We have obtain a stronger result than both (4.162) and (4.177). □

Remark 4.11.

1. If $x_{i+\frac{1}{2}}$ is the mid-point of $[x_i, x_{i+1}]$, we have

$$h_{i+\frac{1}{2}}\varepsilon_{i+\frac{1}{2}}^2 \leq \frac{\left(x_{i+\frac{1}{2}}-x_i\right)^3+\left(x_{i+1}-x_{i+\frac{1}{2}}\right)^3}{3h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.198}$$

$$=\frac{h_{i+\frac{1}{2}}^2}{12}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.199}$$

$$\leq \frac{\left(h_i+h_{i+1}\right)^2}{12}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.200}$$

$$\leq \frac{h^2}{3}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.201}$$

2. If $\mathcal{T}$ is a uniform admissible mesh, (4.199) yields

$$h_{i+\frac{1}{2}}\varepsilon_{i+\frac{1}{2}}^2 \leq \frac{h^2}{12}\int_{x_i}^{x_{i+1}} f(s)^2 ds \tag{4.202}$$

5 Practical Assignment

[6]Given a one-dimensional Poisson problem on $\Omega = (0,1)$

$$-u'' = f \text{ in } \Omega \tag{5.1}$$

Problem 5.1 (Dirichlet boundary condition).

1. *Solve equation* (5.1) *subject to homogeneous Dirichlet boundary condition*

$$u(0) = a \tag{5.2}$$
$$u(1) = b \tag{5.3}$$

by finite volume method on a regular grid and the control point is the mid point of each control volume

$$x_i = \frac{1}{2}x_{i-\frac{1}{2}} + \frac{1}{2}x_{i+\frac{1}{2}} \tag{5.4}$$

2. *Solve equation* (5.1) *with regular grid and control point is* $\frac{1}{3}$ *from the left of each control volume*

$$x_i = \frac{2}{3}x_{i-\frac{1}{2}} + \frac{1}{3}x_{i+\frac{1}{2}} \tag{5.5}$$

3. *How to approximate the mean-value of f over T_i and compare some ways approximation.*

4. *Solve equation* (5.1) *with singular grid (not uniform grid).*

Problem 5.2 (Neumann boundary condition). *Solve equation* (5.1) *subject to homogeneous Neumann boundary condition*

$$u'(0) = 0 \tag{5.6}$$
$$u'(1) = 0 \tag{5.7}$$
$$\int_0^1 f(x)\,dx = 0 \tag{5.8}$$
$$\int_0^1 u(x)\,dx = 0 \tag{5.9}$$

by finite volume method on a regular grid and singular grid with the control point be the mid point of each control volume

$$x_i = \frac{1}{2}x_{i-\frac{1}{2}} + \frac{1}{2}x_{i+\frac{1}{2}} \tag{5.10}$$

[6]Doan Tran Nguyen Tung's practical assignment.

6 MATLAB Scripts

6.1 Subscript f.m

```
function f1=f(x,k);
if k==1
    f1=1/2-x;
elseif k==2
    f1=400*pi^2*cos(20*pi*x);
elseif k==3
    f1=- 20000*x^3 + (957200*x^2)/33 - (1240655*x)/99 + 916835/594;
elseif k==4
    f1=-500*x^3*pi*sin(10*pi*x^5)^3*(sin(20*pi*x^5) + (75*pi*x^5)/2 ...
        + (125*x^5*pi*cos(20*pi*x^5))/2);
elseif k==5
    f1=4400*pi^2*sin(20*pi*x)^9*(11*sin(20*pi*x)^2 - 10);
elseif k==6
    f1=3*cos(x + 1/2)^3*sin(10*pi*(x + 1/2)^5)^5 ...
        - 6*cos(x + 1/2)*sin(x + 1/2)^2*sin(10*pi*(x + 1/2)^5)^5 ...
        + 12500*pi^2*cos(x + 1/2)^3*sin(10*pi*(x + 1/2)^5)^5*(x + 1/2)^8 ...
        - 50000*pi^2*cos(x + 1/2)^3*cos(10*pi*(x + 1/2)^5)^2*sin(10*pi* ...
        (x + 1/2)^5)^3*(x + 1/2)^8 - 1000*pi*cos(x + 1/2)^3* ...
        cos(10*pi*(x + 1/2)^5)*sin(10*pi*(x + 1/2)^5)^4*(x + 1/2)^3 ...
        + 1500*pi*cos(x + 1/2)^2*cos(10*pi*(x + 1/2)^5)*sin(x + 1/2)* ...
        sin(10*pi*(x + 1/2)^5)^4*(x + 1/2)^4;
end
```

6.2 u_exact.m

```
function uex=u_exact(x,k);

if k==1
    uex=(x^2*(2*x - 3))/12+1/24;
elseif k==2
    uex=sin(20*pi*x+pi/2);
elseif k==3
    uex=1000*(x-1/20)*(x-3/11)*(x-5/12)*(x-7/9)*(x-9/10);
elseif k==4
    uex=sin(10*pi*(x)^5)^5;
elseif k==5
    uex=sin(20*pi*(x))^11;
elseif k==6
    uex=(cos(x+1/2))^3*sin(10*pi*(x+1/2)^5)^5;
end
end
```

6.3 main_1d_laplace.m

```
% Solve 1D Laplace equation -uxx=f(x) in [a,b]
format long
```

```
clear all
% clc
close all

tic
ax=0;
bx=1;

cases=4;
BC=1;
grid=1;
cp=1;
integration=1; % Integration rule

fprintf('Case %d\n',cases);
if BC==1
    fprintf('Dirichlet Boundary Condition\n');
    bc='dirichlet';
elseif BC==2
    fprintf('Neumann Boundary Condition\n');
    bc='neumann';
elseif BC==3
    fprintf('Robin Boundary Condition\n');
    bc='robin';
end

if cases == 1
    N=4;% Number of control volume
    NO=N;
    M=6;% number of iteration when refine mesh
    if BC==1
        C0=1/24;
        C1=-1/24;
    elseif BC==2
        C0=0;
        C1=0;
        C=0;
    elseif BC==3
        L0=-1;
        L1=1;
        C0=1/24;
        C1=-1/24;
    end
    it=1;
elseif cases == 2
    N=4;% Number of control volume
    NO=N;
    M=12;% number of iteration when refine mesh
    if BC==1
        C0=1;
```

```
        C1=1;
    elseif BC==2
        C0=0;
        C1=0;
        C=0;
    elseif BC==3
        L0=-1;
        L1=-2;
        C0=1;
        C1=-2;
    end
    it=1;
elseif cases == 3
    N=4;% Number of control volume
    N0=N;
    M=6;% number of iteration when refine mesh
    if BC==1
        C0=-175/44;
        C1=2660/297;
    elseif BC==2
        C0=22415/198;
        C1=99175/594;
        C=5495/7128;
    elseif BC==3
        L0=-1;
        L1=0;
        C0=-175/44+22415/198;
        C1=99175/594;
    end
    it=1;
elseif cases == 4
    N=100;% Number of control volume
    N0=N;
    M=6;% number of iteration when refine mesh
    if BC==1
        C0=0;
        C1=0;
    elseif BC==2
        C0=0;
        C1=0;
        C=0.056334049139993;
    elseif BC==3
        L0=0;
        L1=3;
        C0=0;
        C1=0;
    end
    it=1;
elseif cases == 5
    N=100;% Number of control volume
```

```
    NO=N;
    M=30;% number of iteration when refine mesh
    if BC==1
        C0=0;
        C1=0;
    elseif BC==2
        C0=0;
        C1=0;
        C=0;
    elseif BC==3
        L0=-1;
        L1=1;
        C0=0;
        C1=0;
    end
    it=2;
elseif cases == 6
    N=100;% Number of control volume
    NO=N;
    M=6;% number of iteration when refine mesh
    if BC==1
        C0=sin((5*pi)/16)^5*cos(1/2)^3;
        C1=-sin(pi/16)^5*cos(3/2)^3;
    elseif BC==2
        C0=-(cos(1/2)^2*(24*sin((5*pi)/16)*sin(1/2) - ...
            125*pi*cos((5*pi)/16)*cos(1/2))*((2 - 2^(1/2))^(1/2)/4 - ...
            2^(1/2)/16 + 3/8))/8;
        C1=(3*cos(3/2)^2*(8*sin(pi/16)*sin(3/2) + ...
            3375*pi*cos(pi/16)*cos(3/2))*(2^(1/2)/16 - ...
            (2^(1/2) + 2)^(1/2)/4 + 3/8))/8;
        C=0.033404805943832;
    elseif BC==3
        L0=-1;
        L1=1;
        C0=sin((5*pi)/16)^5*cos(1/2)^3- ...
            (cos(1/2)^2*(24*sin((5*pi)/16)*sin(1/2) - ...
            125*pi*cos((5*pi)/16)*cos(1/2))*((2 - 2^(1/2))^(1/2)/4 - ...
            2^(1/2)/16 + 3/8))/8;
        C1=-sin(pi/16)^5*cos(3/2)^3+ ...
            (3*cos(3/2)^2*(8*sin(pi/16)*sin(3/2) + ...
            3375*pi*cos(pi/16)*cos(3/2))*(2^(1/2)/16 - ...
            (2^(1/2) + 2)^(1/2)/4 + 3/8))/8;
    end
    it=1;
end

norml2=zeros(M,1); % L2 norm
norme=zeros(M,1); % Energy norm

% grid=1;
```

```
if grid==1
    fprintf('Uniform grid\n');
    gr='uniform';
elseif grid==2
    fprintf('Singular grid\n');
    gr='singular';
end

% cp=2;
if cp==1
    fprintf('Each control point is the midpoint');
    fprintf('of the corresponding control volume\n');
    cps='midpoint_cp';
elseif cp==2
    fprintf('Each control point is 1/3 to the left');
    fprinft('of the corresponding control volume\n');
    cps='left_cp';
end

% integration=1; % Integration rule
if integration==1
    fprintf('Integration using Midpoint rule\n');
    inte='midpoint';
elseif integration==2
    fprintf('Integration using Trepozoidal rule\n');
    inte='trepozoidal';
elseif integration==3
    fprintf('Integration using Simpson rule\n');
    inte='simpson';
elseif integration==4
    fprintf('Integration using Boole rule\n');
    inte='boole';
end

ll=zeros(M,1);

for jj=1:M
    fprintf('N=%d\n',N);

    % Create the mesh point
    if grid==1
        dx=(bx-ax)/N;
        for i_iter=1:N+1
            x(i_iter)=ax+(i_iter-1)*dx;
        end
    elseif grid==2
        for i=1:N+1
            x(i) = ax + (1-cos(pi*i/2/(N+1)))*(bx-ax);
        end
    end
```

```
% create control point
x_cp=zeros(N+2,1);
for i_iter=1:N+2
    if(i_iter==1)
        x_cp(i_iter)=x(i_iter);
    else
        if(i_iter==N+2)
            x_cp(i_iter)=x(i_iter-1);
        else
            if cp==1
                x_cp(i_iter)=(x(i_iter-1)+x(i_iter))/2;
            elseif cp==2
                x_cp(i_iter)=(2*x(i_iter-1)+x(i_iter))/3;
            end
        end
    end
end

% Creare the Matrix A and the vector b
A=zeros(N,N);
b=zeros(N,1);
for i_iter=1:N
    a1=-1/((x(i_iter+1)-x(i_iter))*(x_cp(i_iter+1)-x_cp(i_iter)));
    c1=-1/((x(i_iter+1)-x(i_iter))*(x_cp(i_iter+2)-x_cp(i_iter+1)));
    b1=-a1-c1;

    if integration==1
        b(i_iter)=f((x(i_iter)+x(i_iter+1))/2.0,cases);
    elseif integration==2
        b(i_iter)=(f(x(i_iter),cases)+f(x(i_iter+1),cases))/2;
    elseif integration==3
        b(i_iter)=(f(x(i_iter),cases)+4*f((x(i_iter)+ ...
            x(i_iter+1))/2,cases)+f(x(i_iter+1),cases))/6;
    elseif integration==4
        b(i_iter)=(7*f(x(i_iter),cases)+32*f((x(i_iter)+ ...
            3*x(i_iter+1))/4,cases)+ ...
            12*f((x(i_iter)+x(i_iter+1))/2,cases)+ ...
            32*f((3*x(i_iter)+x(i_iter+1))/4,cases)+ ...
            7*f(x(i_iter+1),cases))/90;
    end

    if BC==1
        if(i_iter==1)
            A(i_iter,i_iter+1)=c1;
            A(i_iter,i_iter)=b1;
            b(i_iter)=b(i_iter)-a1*C0;
        else
            if(i_iter==N)
                A(i_iter,i_iter-1)=a1;
```

```
                A(i_iter,i_iter)=b1;
                b(i_iter)=b(i_iter)-c1*C1;
            else
                A(i_iter,i_iter-1)=a1;
                A(i_iter,i_iter+1)=c1;
                A(i_iter,i_iter)=b1;
            end
        end
    elseif BC==2
        if(i_iter==1)
            A(i_iter,i_iter+1)=c1;
            A(i_iter,i_iter)=-c1;
            b(i_iter)=b(i_iter)+a1*C0*(x_cp(2)-x_cp(1));
        else
            if(i_iter==N)
                A(i_iter,i_iter-1)=a1;
                A(i_iter,i_iter)=-a1;
                b(i_iter)=b(i_iter)-c1*C1*(x_cp(N+2)-x_cp(N+1));
            else
                A(i_iter,i_iter-1)=a1;
                A(i_iter,i_iter+1)=c1;
                A(i_iter,i_iter)=b1;
            end
        end
    elseif BC==3
        if(i_iter==1)
            A(i_iter,i_iter+1)=c1;
            A(i_iter,i_iter)=a1/(1+L0*(x_cp(2)-x_cp(1)))+b1;
            b(i_iter)=b(i_iter)+ ...
                a1*C0*(x_cp(2)-x_cp(1))/(1+L0*(x_cp(2)-x_cp(1)));
        else
            if(i_iter==N)
                A(i_iter,i_iter-1)=a1;
                A(i_iter,i_iter)=b1+c1/(1+L1*(x_cp(N+2)-x_cp(N+1)));
                b(i_iter)=b(i_iter)-c1*C1*(x_cp(N+2)- ...
                    x_cp(N+1))/(1+L1*(x_cp(N+2)-x_cp(N+1)));

            else
                A(i_iter,i_iter-1)=a1;
                A(i_iter,i_iter+1)=c1;
                A(i_iter,i_iter)=b1;
            end
        end
    end

end

if BC==1
    u=zeros(N,1);
    u=A\b;
```

```
    u_dis=zeros(N+2,1);
    u_dis(1)=C0;
    u_dis(N+2)=C1;
elseif BC==2
    K = 3;
    A(:,K)=[];
    A(K,:)=[];
    b(K)=[];
    u=A\b;
    u=[u(1:K-1) ; 0 ; u(K:length(u))];
    su=-C;
    for i=1:N
        su=su+u(i)*(x(i+1)-x(i));
    end
    u=u-su;
    u_dis=zeros(N+2,1);
    u_dis(1)=u(1)-C0*(x_cp(2)-x_cp(1));
    u_dis(N+2)=u(N)+C1*(x_cp(N+2)-x_cp(N+1));
elseif BC==3
    u=zeros(N,1);
    u=A\b;
    u_dis=zeros(N+2,1);
    u_dis(1)=(u(1)-C0*(x_cp(2)-x_cp(1)))/(1+ ...
        L0*(x_cp(2)-x_cp(1)));
    u_dis(N+2)=(u(N)+C1*(x_cp(N+2)-x_cp(N+1)))/(1+ ...
        L1*(x_cp(N+2)-x_cp(N+1)));
end

u_ex=zeros(N+2,1);
for i_iter=1:N+2
    u_ex(i_iter)=u_exact(x_cp(i_iter),cases);
end

for i_iter=1:N
    u_dis(i_iter+1)=u(i_iter);
end

fig=figure(jj);
set(fig, 'PaperSize', [5 6]);
plot(x_cp,u_dis,'red',x_cp,u_ex,'blue');
str=sprintf('Discrete and exact solutions, N = %d',N);
title(str);
legend('Discrete solution', 'Exact solution');
filename=sprintf('fig_%s_result_G%d_CP%d_I%d_N%d_M%d_C%d.pdf', ...
    bc,grid,cp,integration,N,M,cases);
orient(fig,'landscape');
fig.PaperPositionMode = 'auto';
%     saveas(fig,filename);
fprintf(filename); fprintf('\n');
```

```
    for i_iter=1:N
        norml2(jj)=norml2(jj)+(u_dis(i_iter+1)- ...
            u_ex(i_iter+1))^2*(x(i_iter+1)-x(i_iter));
    end
    norml2(jj)=sqrt(norml2(jj));
    fprintf('Error in L^2 norm: %df\n',norml2(jj));

    for i_iter=1:N+1
        norme(jj)=norme(jj)+((u_dis(i_iter+1)-u_ex(i_iter+1))- ...
            (u_dis(i_iter)-u_ex(i_iter)))^2/(x_cp(i_iter+1)-x_cp(i_iter));
    end
    norme(jj)=sqrt(norme(jj));
    fprintf('Error in energy norm: %df\n',norme(jj));

    ll(jj)=N;
    if it==1
        N=2*N;
    elseif it==2
        N=N+N0;
    end
    fprintf('\n');
end

fig=figure(jj+1);
set(fig, 'PaperSize', [5 6]);
if cases == 1
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)+2, 'black', log(ll), 2*log(ll)+2.5, ...
        'green', log(ll), log(ll)+2.5,'magenta');
elseif cases == 2
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)-7.5, 'black', log(ll), 2*log(ll)-6.1, ...
        'green', log(ll), log(ll)-11,'magenta');
elseif cases == 3
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)-6, 'black', log(ll), 2*log(ll)-6, ...
        'green', log(ll), log(ll)-5.5,'magenta');
elseif cases == 4
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)-9.5, 'black', log(ll), 2*log(ll)-6, ...
        'green', log(ll), log(ll)-11,'magenta');
elseif cases ==5
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)-9.5, 'black', log(ll), 2*log(ll)-6, ...
        'green', log(ll), log(ll)-11,'magenta');
elseif cases ==6
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll)-9.5, 'black', log(ll), 2*log(ll)-6, ...
        'green', log(ll), log(ll)-11,'magenta');
else
```

```
    plot(log(ll),-log(norml2),'r', log(ll), -log(norme),'blue', ...
        log(ll),1.5*log(ll), 'black', log(ll), 2*log(ll), ...
        'green', log(ll), log(ll),'magenta');
end

title('Error');
legend('L^2 Norm', 'H^1 norm', '3/2x', '2x','1x', ...
    'Location','northeastoutside');
filename=sprintf('fig_%s_error_G%d_CP%d_I%d_M%d_C%d.pdf', ...
    bc,grid,cp,integration,M,cases);
orient(fig,'landscape');
fig.PaperPositionMode = 'auto';
saveas(fig,filename);
fprintf(filename); fprintf('\n');
toc
```

7 Dirichlet Boundary Condition

In this section, we are going to test the Finite Volume Method for the one-dimensional Poisson equation subjected to Dirichlet boundary condition with several test cases using uniform grid and singular grid, varying positions of control points, integral approximation rules.

7.1 Test cases

Case 1.

$$u(x) = \frac{x^2\left(2x-3\right)}{12} + \frac{1}{24} \tag{7.1}$$

$$f(x) = \frac{1}{2} - x \tag{7.2}$$

$$u(0) = \frac{1}{24} \tag{7.3}$$

$$u(1) = -\frac{1}{24} \tag{7.4}$$

Case 2.

$$u(x) = (1000x-50)\left(x-\frac{3}{11}\right)\left(x-\frac{7}{9}\right)\left(x-\frac{5}{12}\right)\left(x-\frac{9}{10}\right) \tag{7.5}$$

$$f(x) = -20000x^3 + \frac{957200x^2}{33} - \frac{1240655x}{99} + \frac{916835}{594} \tag{7.6}$$

$$u(0) = -\frac{175}{44} \tag{7.7}$$

$$u(1) = \frac{2660}{297} \tag{7.8}$$

Case 3.

$$u(x) = \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^5\cos\left(x+\frac{1}{2}\right)^3 \tag{7.9}$$

$$f(x) = 3\cos\left(x+\frac{1}{2}\right)^3 \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^5 \tag{7.10}$$

$$+ 12500\pi^2\cos\left(x+\frac{1}{2}\right)^3 \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^5 \left(x+\frac{1}{2}\right)^8 \tag{7.11}$$

$$- 6\cos\left(x+\frac{1}{2}\right) \sin\left(x+\frac{1}{2}\right)^2 \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^5 \tag{7.12}$$

$$- 50000\pi^2\cos\left(x+\frac{1}{2}\right)^3 \cos\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^2 \times \tag{7.13}$$

$$\times \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^3 \left(x+\frac{1}{2}\right)^8 \tag{7.14}$$

$$- 1000\pi\cos\left(x+\frac{1}{2}\right)^3 \cos\left(10\pi\left(x+\frac{1}{2}\right)^5\right) \times \tag{7.15}$$

$$\times \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^4 \left(x+\frac{1}{2}\right)^3 \tag{7.16}$$

$$+ 1500\pi\cos\left(x+\frac{1}{2}\right)^2 \cos\left(10\pi\left(x+\frac{1}{2}\right)^5\right) \sin\left(x+\frac{1}{2}\right) \times \tag{7.17}$$

$$\times \sin\left(10\pi\left(x+\frac{1}{2}\right)^5\right)^4 \left(x+\frac{1}{2}\right)^4 \tag{7.18}$$

$$u(0) = \sin\left(\frac{5\pi}{16}\right)^5 \cos\left(\frac{1}{2}\right)^3 \tag{7.19}$$

$$u(1) = \sin\left(\frac{\pi}{16}\right)^5 \cos\left(\frac{3}{2}\right)^3 \tag{7.20}$$

7.2 Homogeneous Dirichlet boundary condition, regular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

7.2.1 Figures of Results

Case 1.

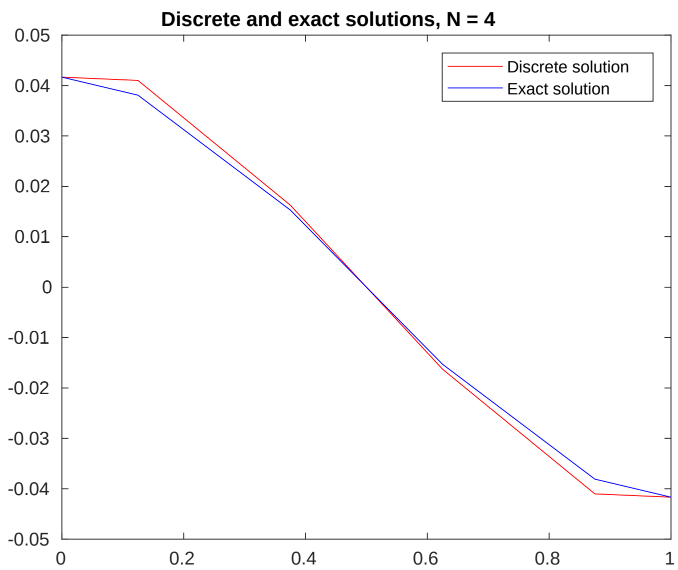

Figure 1: Dirichlet result, uniform mesh, midpoint, Case 1, $N = 4$.

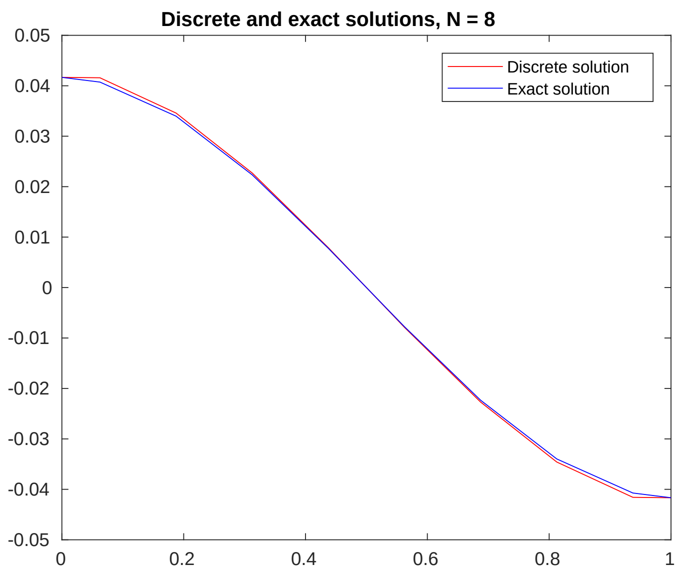

Figure 2: Dirichlet result, uniform mesh, midpoint rule, Case 1, $N = 8$.

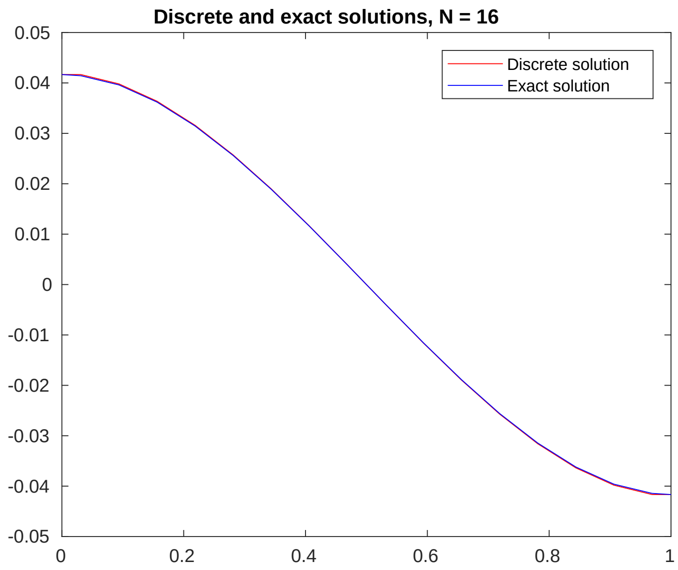

Figure 3: Dirichlet result, uniform mesh, midpoint rule, Case 1, $N = 16$.

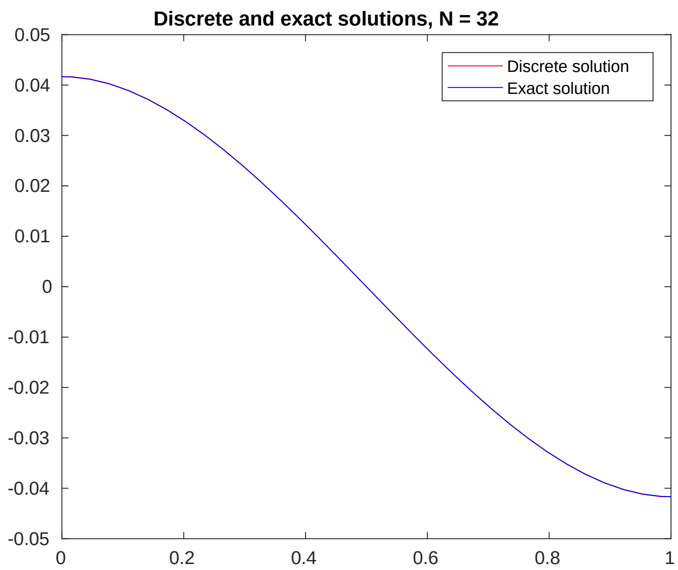

Figure 4: Dirichlet result, uniform mesh, midpoint rule, Case 1, $N = 32$.

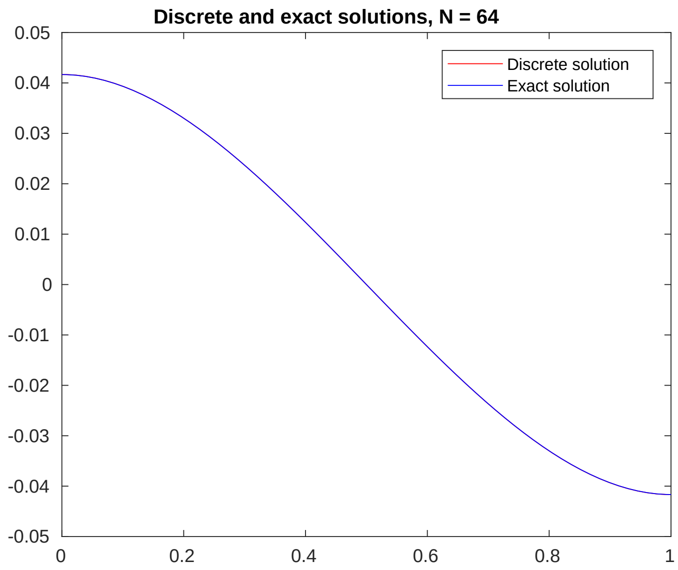

Figure 5: Dirichlet result, uniform mesh, midpoint rule, Case 1, $N = 64$.

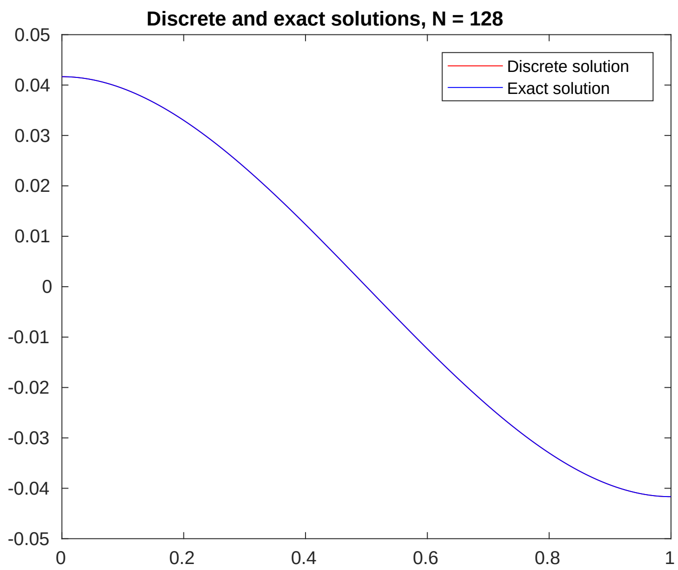

Figure 6: Dirichlet result, uniform mesh, midpoint rule, Case 1, $N = 128$.

Case 2.

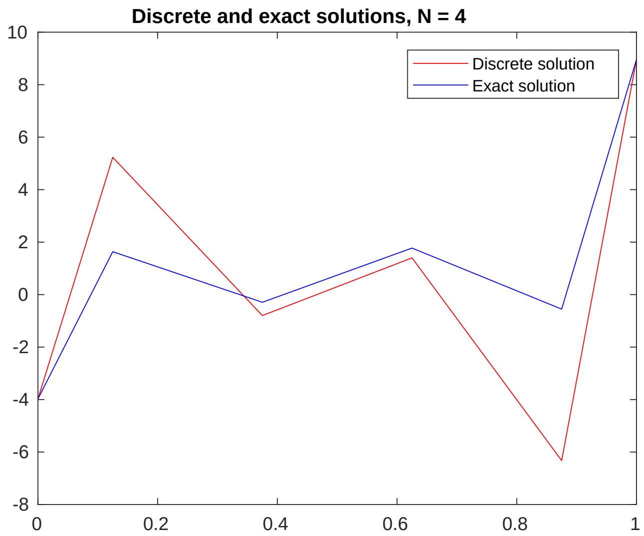

Figure 7: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 4$.

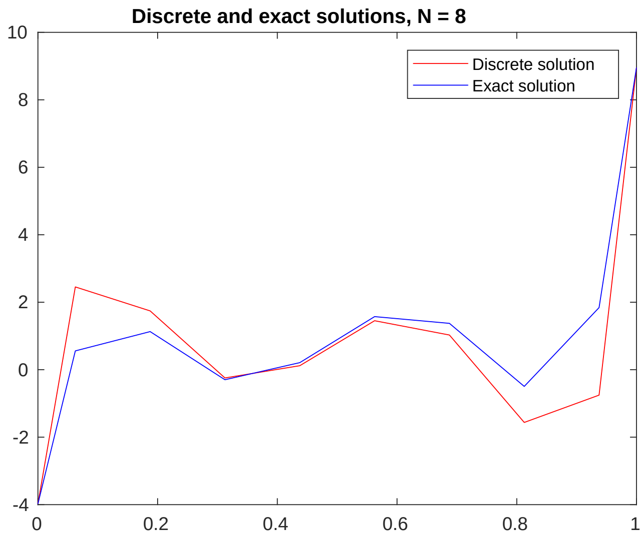

Figure 8: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 8$.

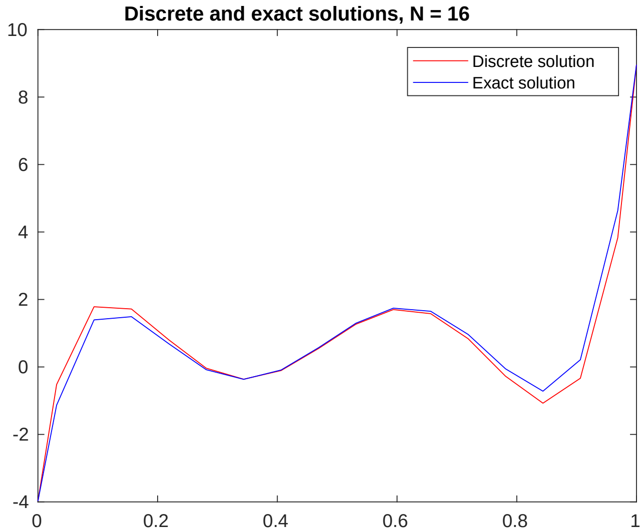

Figure 9: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 16$.

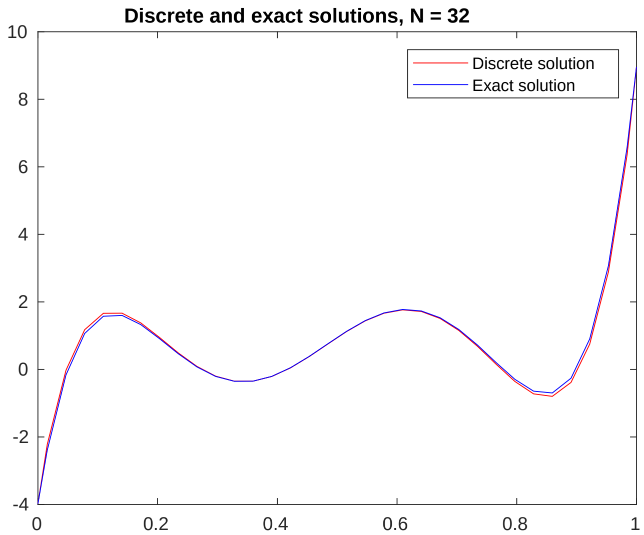

Figure 10: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 32$.

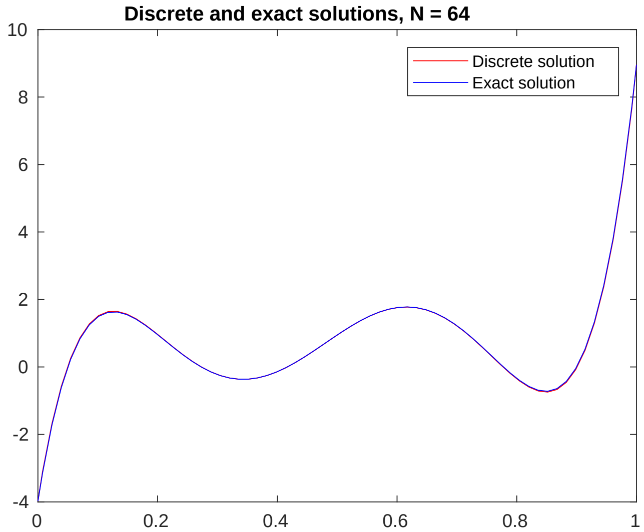

Figure 11: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 64$.

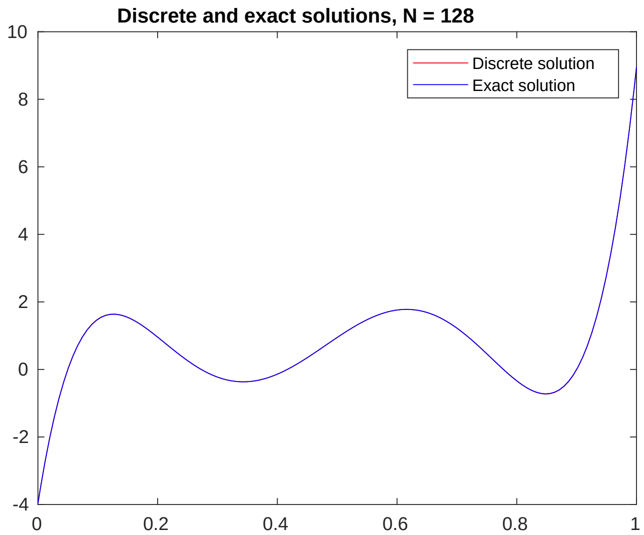

Figure 12: Dirichlet result, uniform mesh, midpoint rule, Case 2, $N = 128$.

Case 3.

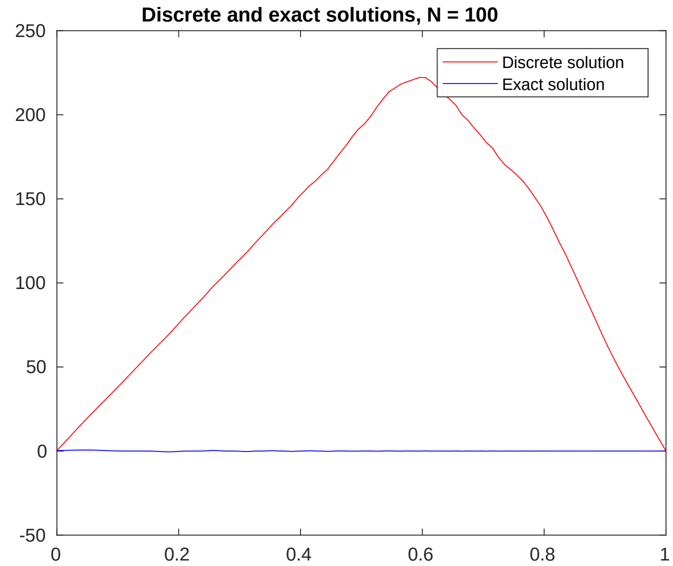

Figure 13: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 100$.

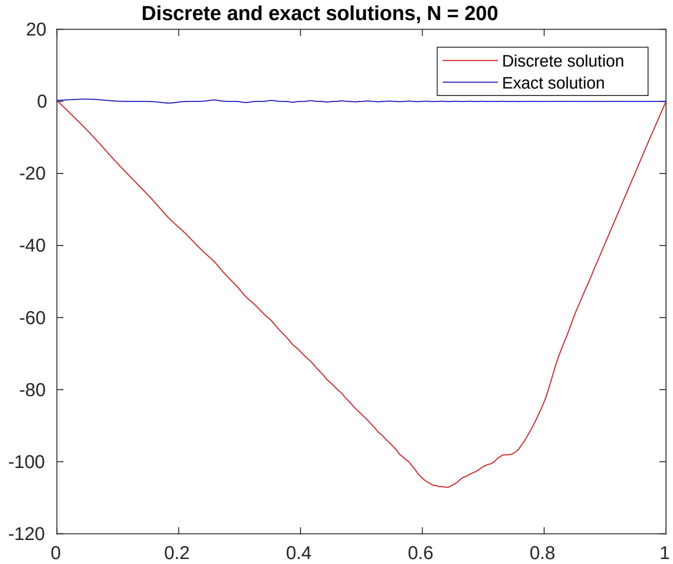

Figure 14: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 200$.

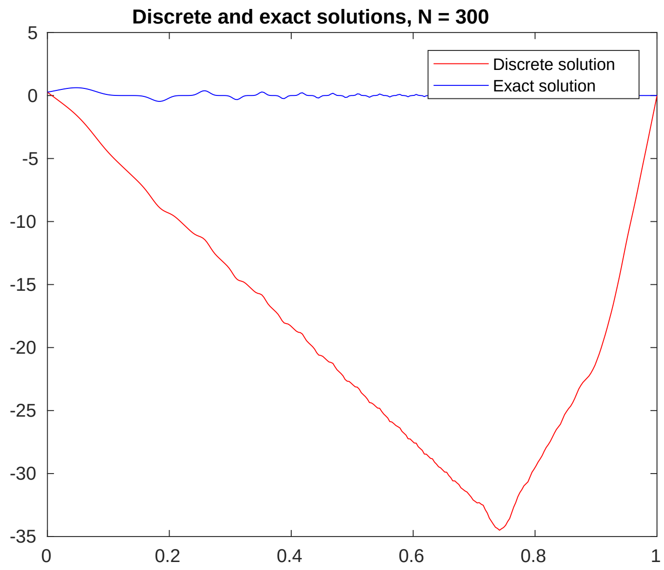

Figure 15: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 300$.

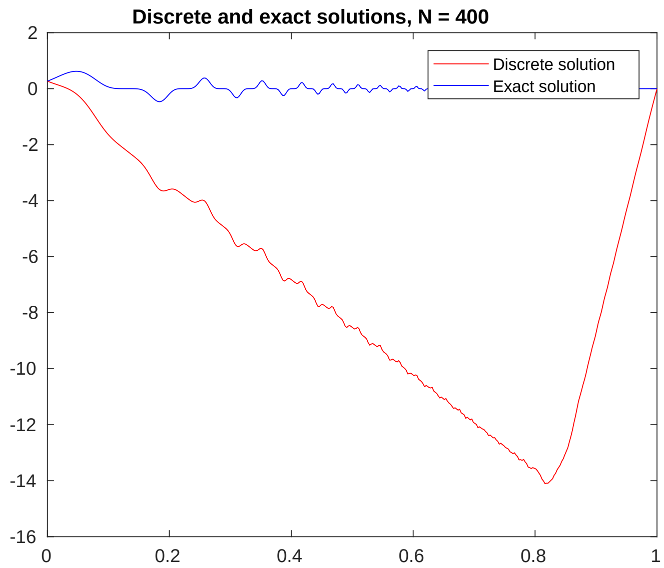

Figure 16: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 400$.

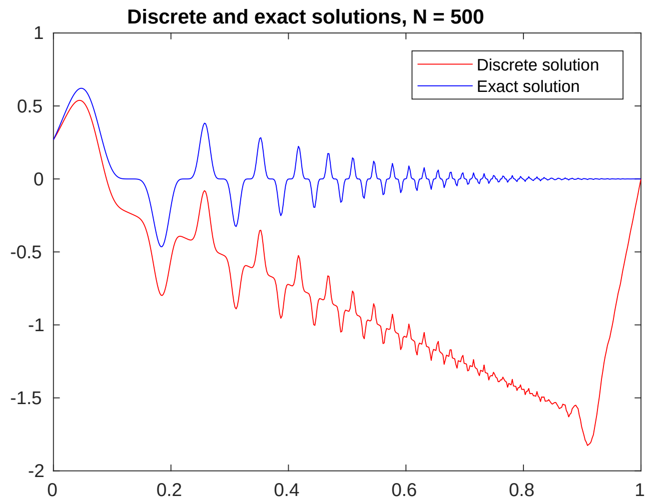

Figure 17: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 500$.

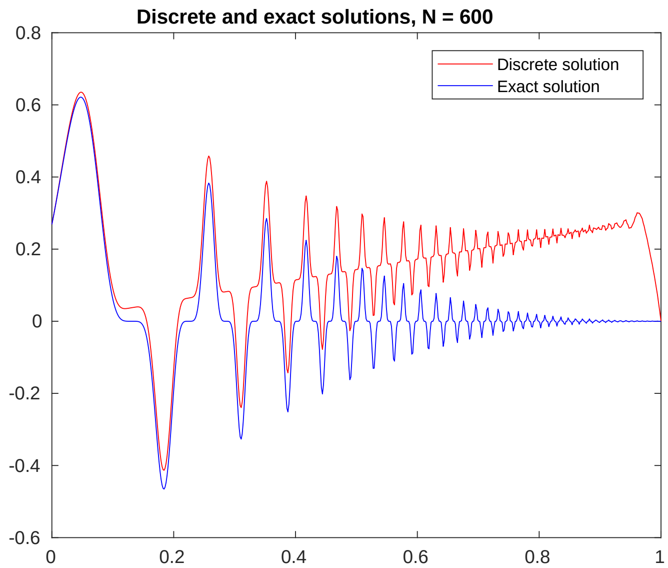

Figure 18: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 600$.

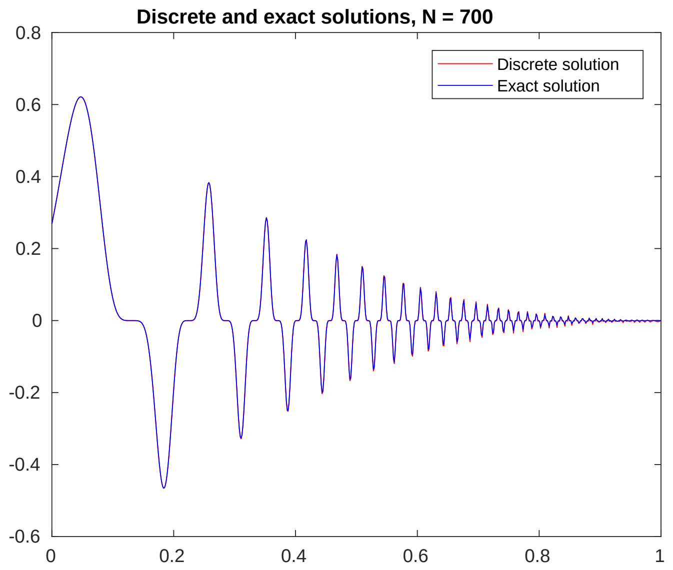

Figure 19: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 700$.

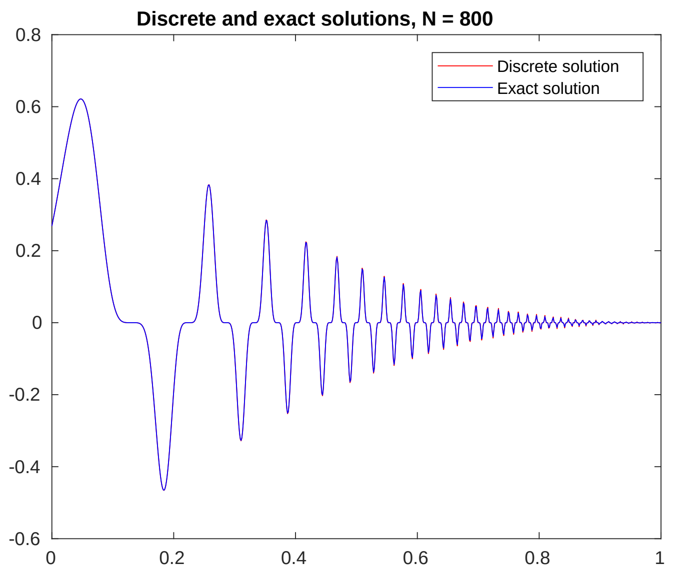

Figure 20: Dirichlet result, uniform mesh, midpoint rule, Case 3, $N = 800$.

7.2.2 Errors

Case 1.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	2.183660e-03	1.353165e-02
8	5.593964e-04	5.167483e-03
16	1.406791e-04	1.891105e-03
32	3.522145e-05	6.796587e-04
64	8.808591e-06	2.422258e-04
128	2.202349e-06	8.597891e-05

Table 1: Error table, Case 1.

Case 2.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	3.414525e+00	1.723527e-02
8	1.224066e+00	1.429127e+01
16	3.297560e-01	6.028506e+00
32	8.393195e-02	2.295482e+00
64	2.107644e-02	8.397813e-01
128	5.274953e-03	3.018207e-01

Table 2: Error table, Case 2.

Case 3.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	1.372029e+02	4.718806e+02
200	6.795356e+01	2.432018e+02
300	2.068556e+01	8.656798e+01
400	8.220398e+00	3.814539e+01
500	9.637916e-01	7.476626e+00
600	1.613099e-01	2.708167e+00
700	1.691419e-03	1.378369e+00
800	1.035571e-03	1.034955e+00

Table 3: Error table, Case 3.

7.2.3 Convergence Rate

Case 1.

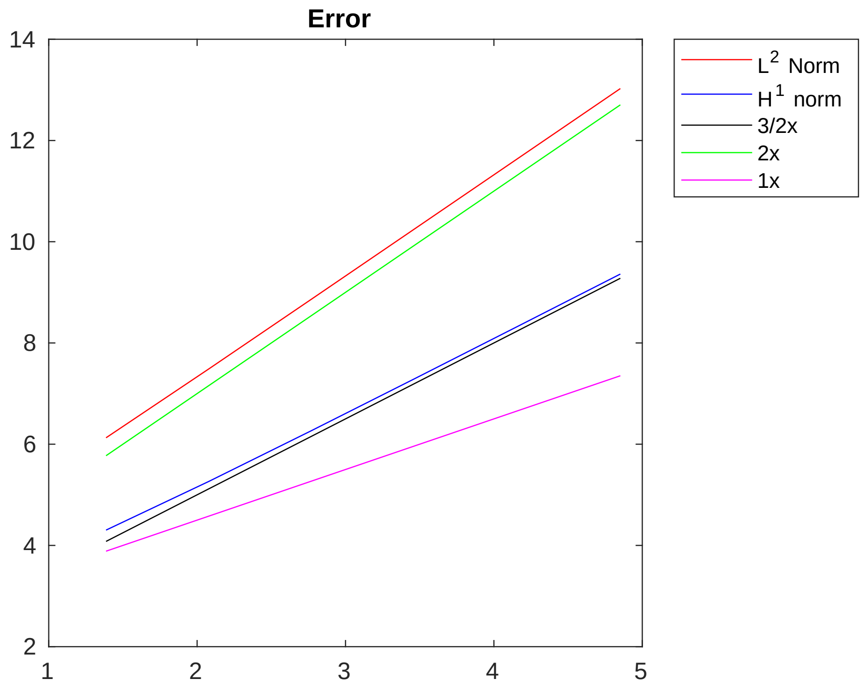

Figure 21: Dirichlet error, midpoint rule, uniform mesh, Case 1.

CASE 2.

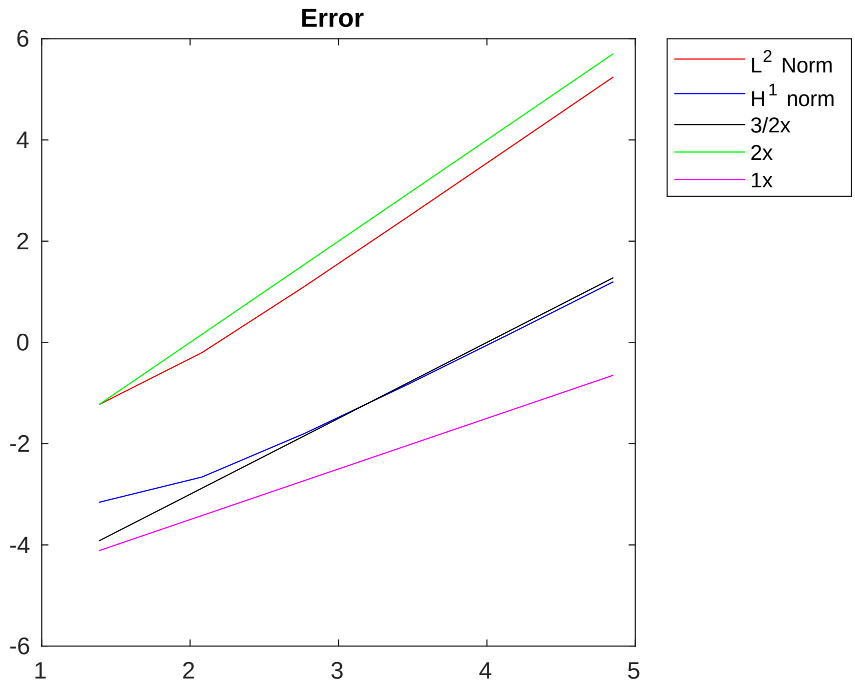

Figure 22: Dirichlet error, midpoint rule, uniform mesh, Case 3.

CASE 3.

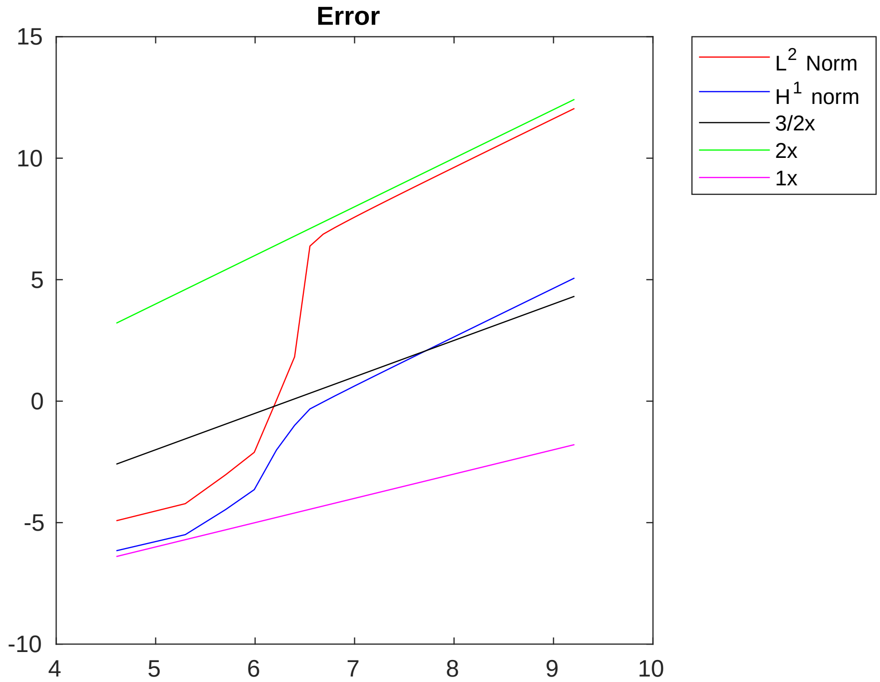

Figure 23: Dirichlet error, midpoint rule, uniform mesh, Case 3.

Remark 7.1. We can see that in case 1 and case 2, the finite volume method has convergence rate of order 2 in L^2 norm and order $\frac{3}{2}$ in energy norm. But in case 3, finite volume method has convergence rate of order 2 in both L^2 and energy norm. Moreover, the convergence rate in case 3 only become consistent after the grid refinement reached a certain degree.

7.3 Homogeneous Dirichlet boundary condition, regular grid, each control point is $\frac{1}{3}$ from the left of corresponding control volume, integration using midpoint rule

7.3.1 Errors

Case 1.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	4.337732e-03	1.723527e-02
8	1.976011e-03	7.696433e-03
16	9.604458e-04	3.489791e-03
32	4.766545e-04	1.633552e-03
64	2.378772e-04	7.855739e-04
128	1.188822e-04	3.845069e-04

Table 4: Error table, Case 1.

Case 2.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	6.063755e+00	2.989757e+01
8	3.072003e+00	1.824323e+01
16	1.453481e+00	8.493182e+00
32	7.060401e-01	3.804418e+00
64	3.484227e-01	1.738566e+00
128	1.731597e-01	8.193746e-01

Table 5: Error table, Case 2.

Case 3.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	1.374387e+02	4.724692e+02
200	6.805998e+01	2.434741e+02
300	2.074894e+01	8.685629e+01
400	8.237658e+00	3.826618e+01
500	9.680948e-01	7.883261e+00
600	1.628207e-01	3.371928e+00
700	4.196379e-03	2.193966e+00
800	3.381889e-03	1.786729e+00

Table 6: Error table, Case 3.

7.3.2 Convergence rate

Case 1.

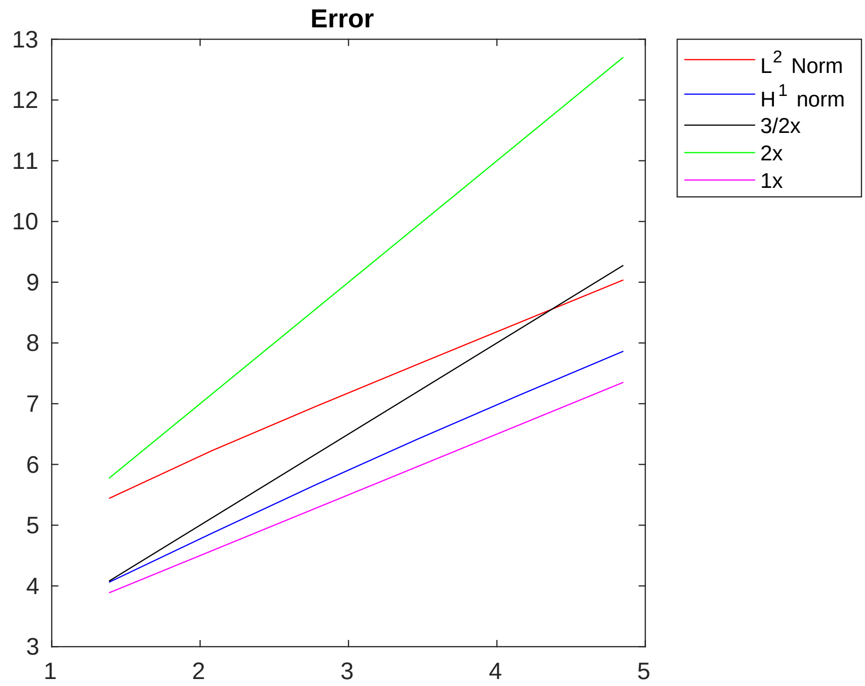

Figure 24: Dirichlet error left, uniform mesh, midpoint rule, Case 1.

Case 2.

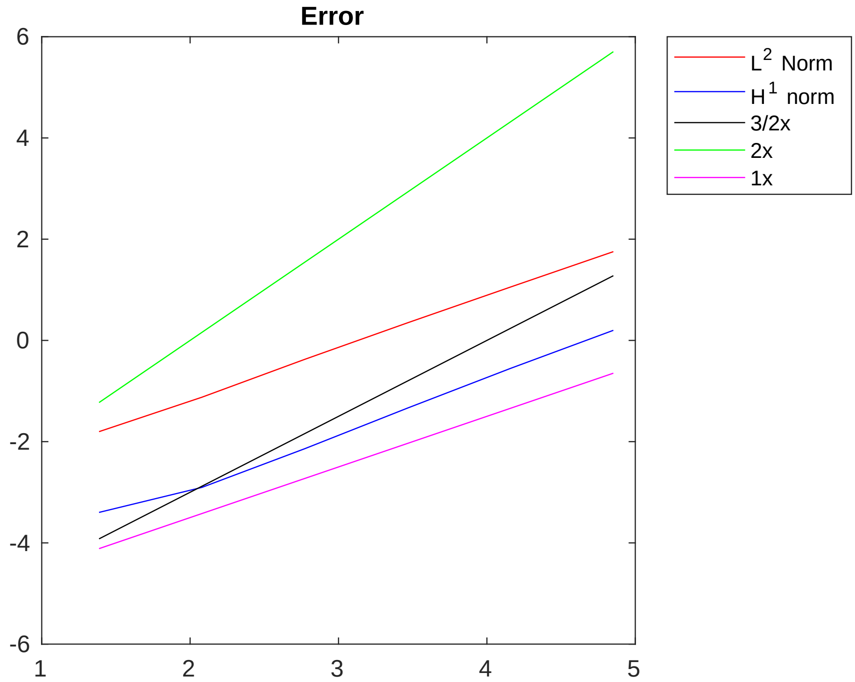

Figure 25: Dirichlet error left, uniform mesh, midpoint rule, Case 2.

Case 3.

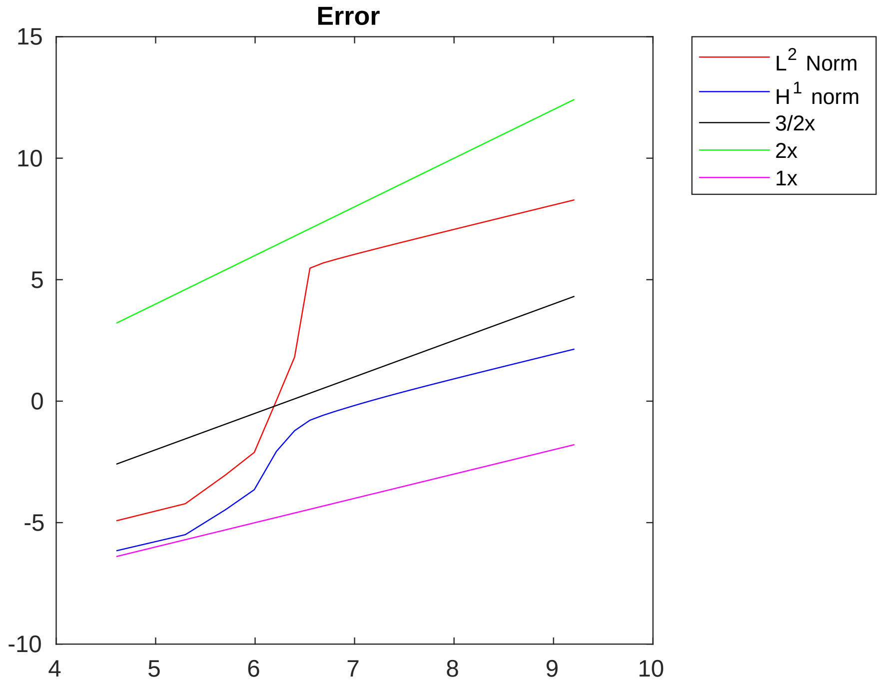

Figure 26: Dirichlet error left, uniform mesh, midpoint rule, Case 3.

Remark 7.2. Unlike the results in previous section, in all 3 cases, the finite volume method has convergence rate of order 1 in both L^p and energy norm. However, in case 3, similar to before, the convergence rate in case 3 only become consistent after the grid refinement reached a certain degree.

7.4 Approximation of the Mean Value of the Function f Over Control Volumes

7.4.1 Some Integral Approximation Rules

There are many ways to approximate integrals, both with equally spaced integration points and unequally spaced integration points. However, we are going to test only a few way of approximation with equally spaced integration points.

We need to approximate

$$\frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x) \tag{7.21}$$

with $T_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ being the control volume on which we need to approximate the mean value of f. For the sake of convenience, let $a = x_{i-\frac{1}{2}}$ and $b = x_{i+\frac{1}{2}}$.

Midpoint rule.

$$\frac{1}{b-a}\int_a^b f(x) = \frac{1}{b-a}(b-a)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\frac{(b-a)^3}{24}f^{(2)}(\xi) \tag{7.22}$$

$$= f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f^{(2)}(\xi) \tag{7.23}$$

For some ξ in $[a, b]$. □

Trapezoidal rule.

$$\frac{1}{b-a}\int_a^b f(x) = \frac{1}{b-a}\frac{b-a}{2}(f(a)+f(b)) - \frac{1}{b-a}\frac{1}{12}(b-a)^3 f^{(2)}(\xi) \tag{7.24}$$

$$= \frac{f(a)+f(b)}{2} - \frac{1}{12}(b-a)^2 f^{(2)}(\xi) \tag{7.25}$$

For some ξ in $[a, b]$. □

Simpson's rule. Let $h = \frac{b-a}{2}$, $t_1 = a$, $t_2 = a + h = \frac{b-a}{2}$, $t_3 = a + 2h = b$.

$$\frac{1}{b-a}\int_a^b f(x) = \frac{1}{2h}\frac{1}{3}\left(f(t_1) + 4f(t_2) + f(t_3)\right) + \frac{1}{2h}\frac{1}{90}h^5 f^{(4)}(\xi) \tag{7.26}$$

$$= \frac{1}{6h}\left(f(t_1) + 4f(t_2) + f(t_3)\right) + \frac{1}{180}h^4 f^{(4)}(\xi) \tag{7.27}$$

For some ξ in $[a, b]$. □

Boole's rule. Let $h = \frac{b-a}{4}$, $t_1 = a$, $t_2 = a + h$, $t_3 = a + 2h$, $t_4 = a + 3h$, $t_5 = a + 4h = b$.

$$\frac{1}{b-a}\int_a^b f(x) \tag{7.28}$$

$$= \frac{h}{90}\left(7f(t_1) + 32f(t_2) + 12f(t_3) + 32f(t_4) + 7f(t_5)\right) - \frac{8h^7}{945}f^{(6)}(\xi) \tag{7.29}$$

For some ξ in $[a, b]$. □

7.4.2 Effects in the Finite Volume Method

In this section, we will test case 3 with various integral approximation rules.

Errors.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	1.372029e+02	4.718806e+02
200	6.795356e+01	2.432018e+02
300	2.068556e+01	8.656798e+01
400	8.220398e+00	3.814539e+01
500	9.637916e-01	7.476626e+00
600	1.613099e-01	2.708167e+00
700	1.691419e-03	1.378369e+00
800	1.035571e-03	1.034955e+00

Table 7: Error table - Midpoint rule.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	3.773644e+01	2.412211e+02
200	8.391778e+01	3.030967e+02
300	2.038450e+01	8.486880e+01
400	8.220498e+00	3.799774e+01
500	9.637879e-01	7.212178e+00
600	1.612986e-01	2.251598e+00
700	1.234248e-03	8.617033e-01
800	5.799255e-04	6.097104e-01

Table 8: Error table - Trepozoidal rule.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	1.014161e+02	3.530643e+02
200	1.742818e+01	6.491145e+01
300	6.995754e+00	2.947754e+01
400	2.740099e+00	1.278054e+01
500	3.212656e-01	2.611736e+00
600	5.377526e-02	1.087361e+00
700	7.420314e-04	6.371791e-01
800	4.998059e-04	4.889503e-01

Table 9: Error table - Simpson's rule.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	2.543855e+01	9.312152e+01
200	1.936562e+00	1.267931e+01
300	5.423098e-01	3.614971e+00
400	1.670127e-01	1.894852e+00
500	3.154194e-02	1.063986e+00
600	1.971232e-03	8.160713e-01
700	6.508166e-04	5.954282e-01
800	4.865046e-04	4.672134e-01

Table 10: Error table - Boole's rule.

Remark 7.3. We can see that there are some variation in the accuracy when using different integration methods.

Convergence Rate.
Case 3, $M = 100$.

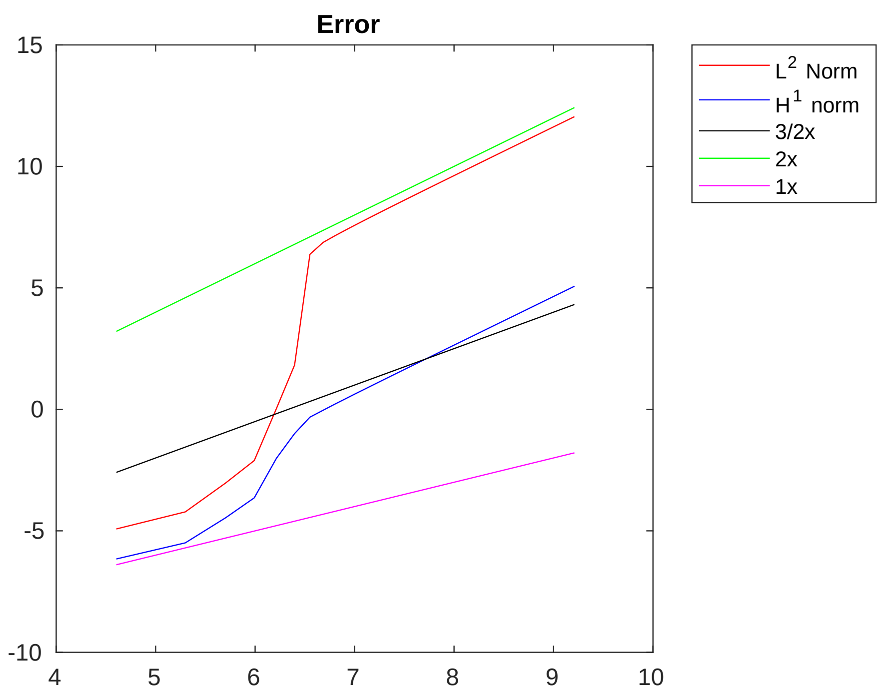

Figure 27: Dirichlet error left, uniform mesh, midpoint rule, Case 3.

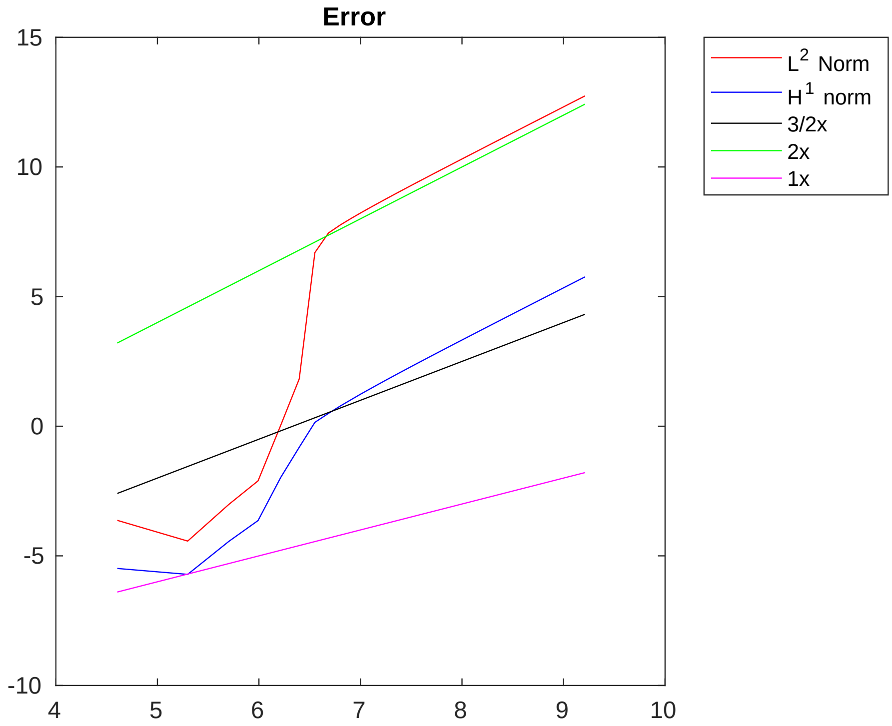

Figure 28: Dirichlet error left, uniform mesh, trapezoidal rule, Case 3.

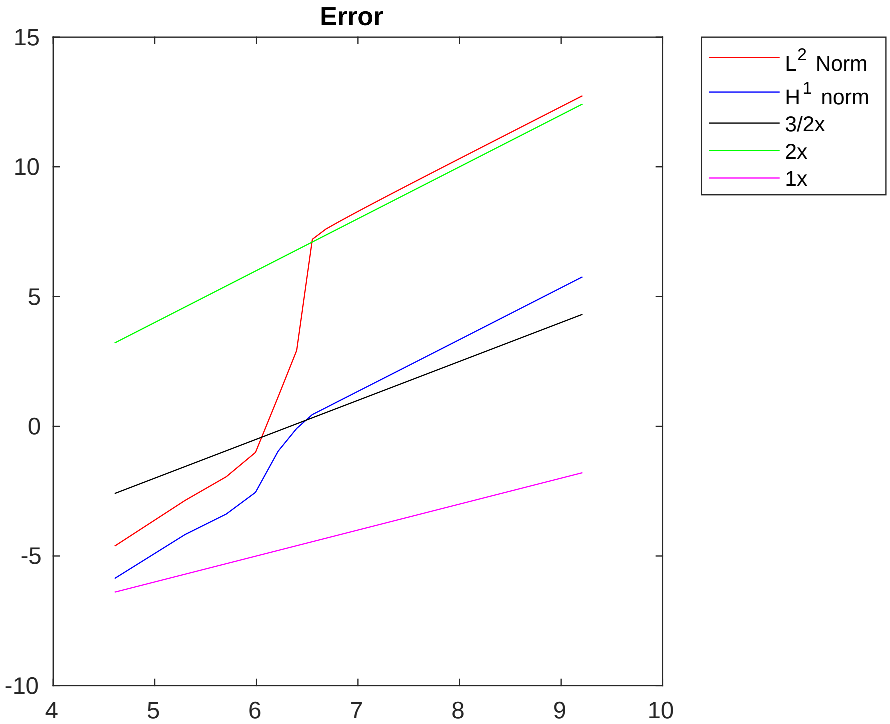

Figure 29: Dirichlet error left, uniform mesh, Simpson rule, Case 3.

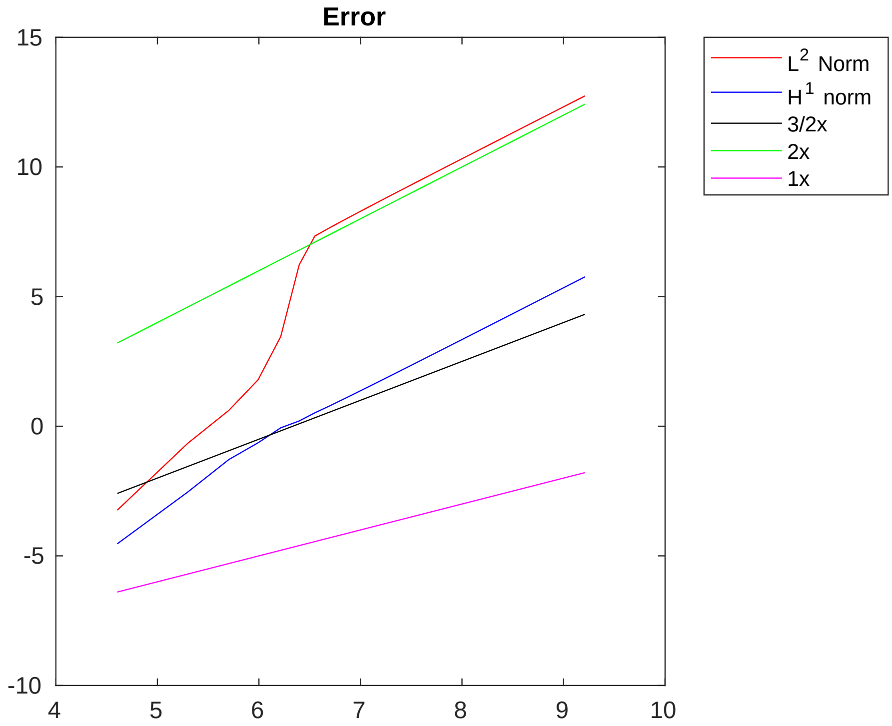

Figure 30: Dirichlet error left, uniform mesh, Boole rule, Case 3.

7.5 Homogeneous Dirichlet boundary condition, singular grid, each control point is the midpoint of corresponding control volume

In this section, we will consider the grid

$$x_i = 1 - \cos\left(\frac{\pi i}{2(N+1)}\right) \tag{7.30}$$

7.5.1 Errors

Case 1.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	2.572594e-03	1.408303e-02
8	8.312800e-04	6.052092e-03
16	2.366590e-04	2.399498e-03
32	6.309364e-05	9.021918e-04
64	1.628265e-05	3.291959e-04
128	4.135350e-06	1.182551e-04

Table 11: Error table, Case 1.

Case 2.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	5.147535e+00	2.212485e+01
8	2.426569e+00	1.848306e+01
16	7.617805e-01	9.789540e+00
32	2.084740e-01	4.495505e+00
64	5.417277e-02	2.011451e+00
128	1.378284e-02	9.183423e-01

Table 12: Error table, Case 2.

Case 3.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
100	2.382533e+02	7.816445e+02
200	1.021832e+02	3.524253e+02
300	6.461806e+01	2.324792e+02
400	3.011621e+01	1.225957e+02
500	2.319667e+01	9.540720e+01
600	1.878345e+00	1.109190e+01
700	2.656605e+00	1.539149e+01
800	2.128635e+00	1.401007e+01
900	2.328867e-01	3.282076e+00
1000	1.798795e-02	1.705853e+00
1100	1.554054e-03	1.344722e+00
1200	1.011087e-03	1.052726e+00

Table 13: Error table, Case 3.

7.5.2 Convergence rate

Case 1.

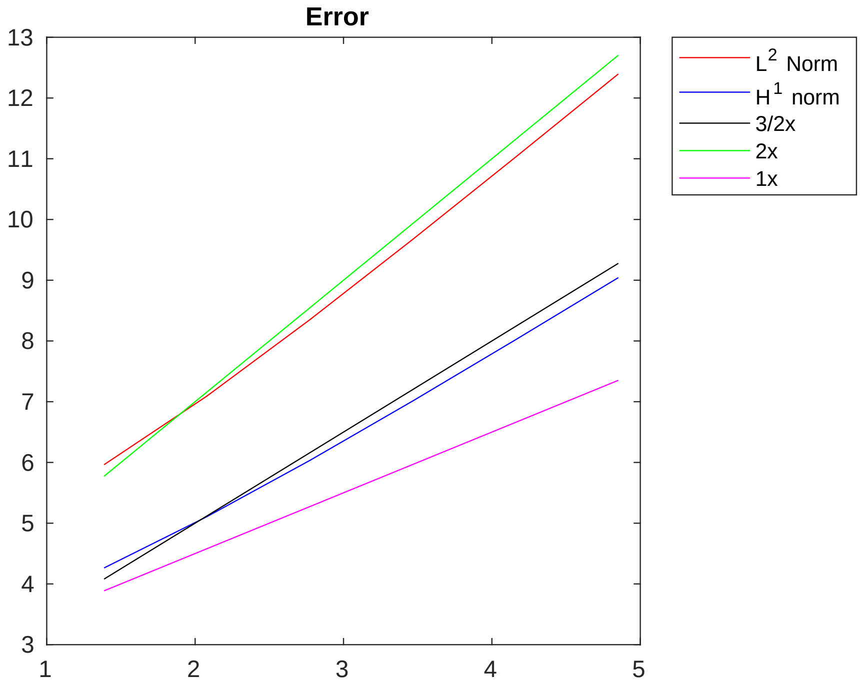

Figure 31: Dirichlet error, Case 1.

Case 2.

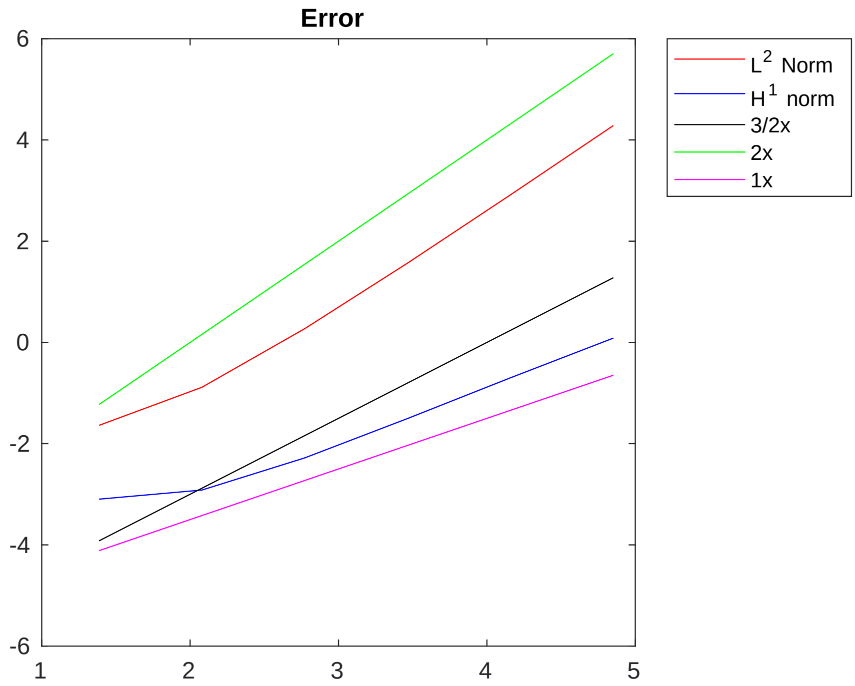

Figure 32: Dirichlet error, Case 2.

Case 3.

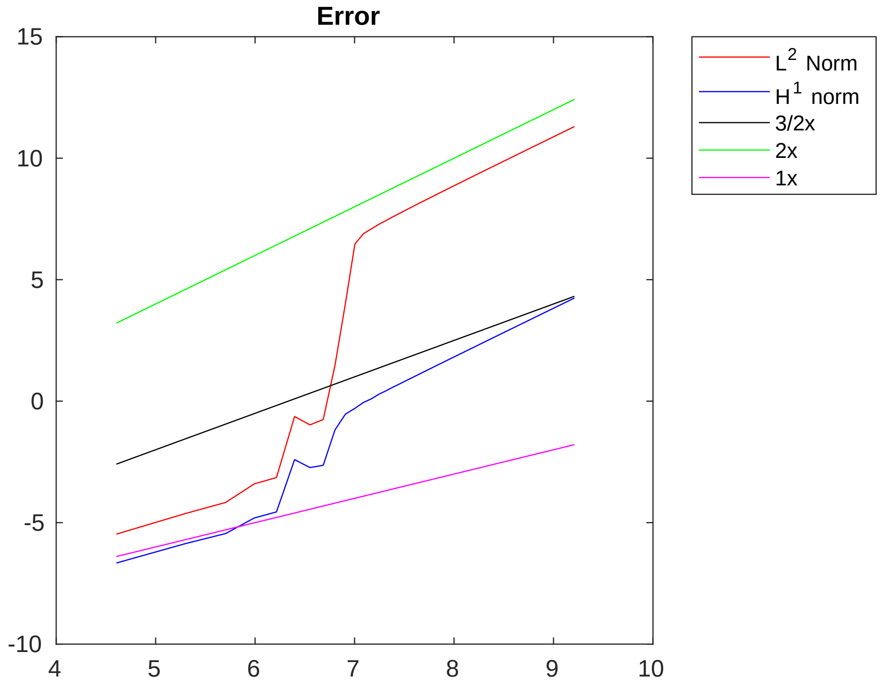

Figure 33: Dirichlet error, Case 3.

Remark 7.4.

1. In case 1, the method appears to have convergence rate of order 2 in L^2 norm and order $\frac{3}{2}$ in energy norm.

2. In case 2, the method appears to have convergence rate of order 2 in L^2 norm and order 1 in energy norm.

3. In case 3, the method appears to have convergence rate of order 2 in both L^2 energy norm and there are some "irregularities" in the convergence rate before the grid refinement reach a certain degree.

8 Neumann boundary condition

In this section, we are going to test the Finite Volume Method for the one-dimensional Poisson equation subjected to Neumann boundary condition with several test cases using uniform grid and singular grid.

8.1 Main Idea

We now review the problem we need to solve

$$-u''(x) = f(x), \text{ in } \Omega \tag{8.1}$$
$$u'(0) = 0 \tag{8.2}$$
$$u'(1) = 0 \tag{8.3}$$
$$\int_0^1 f(x)dx = 0 \tag{8.4}$$
$$\int_0^1 u(x)dx = 0 \tag{8.5}$$

We note that if we omit the condition (8.5), this problem will have infinitely many solutions of the form $u_1(x)+c$, with $u_1(x)$ being a solution of the problem. Let $U = [U_0, U_1, U_2, \dots, U_N, U_{N+1}]^t$ be the discrete solution of the problem.

When using derivative approximation with first order convergence rate, the boundary condition can be discretized as $U_0 = U_1$ and $U_N = U_{N+1}$. Then, we need to use the linear system to solve for $U = [U_1, U_2, \dots, U_N]^t$.

We also note that the discretized linear system has one independent variable. Therefore, we can choose to solve for one solution $\overline{U}$ such that $U_k = 0$ for some k and then use the condition $\int_0^1 u(x)dx = 0$ to find the constant C such that $U = \overline{U} + C$ satisfies the condition.

8.2 Test cases

Case 1.

$$u(x) = \frac{x^2(2x-3)}{12} + \frac{1}{24} \tag{8.6}$$
$$f(x) = \frac{1}{2} - x \tag{8.7}$$
$$\int_0^1 f(x) = 0 \tag{8.8}$$
$$u'(0) = 0 \tag{8.9}$$
$$u'(1) = 0 \tag{8.10}$$
$$\int_0^1 u(x) = 0 \tag{8.11}$$

Case 2.

$$u(x) = \sin\left(20\pi x + \frac{\pi}{2}\right) \tag{8.12}$$

$$f(x) = 400\pi^2 \cos(20\pi x) \tag{8.13}$$

$$\int_0^1 f(x) = 0 \tag{8.14}$$

$$u'(0) = 0 \tag{8.15}$$

$$u'(1) = 0 \tag{8.16}$$

$$\int_0^1 u(x) = 0 \tag{8.17}$$

8.3 Homogeneous Neumann boundary condition, regular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

8.3.1 Figures of results

CASE 1.

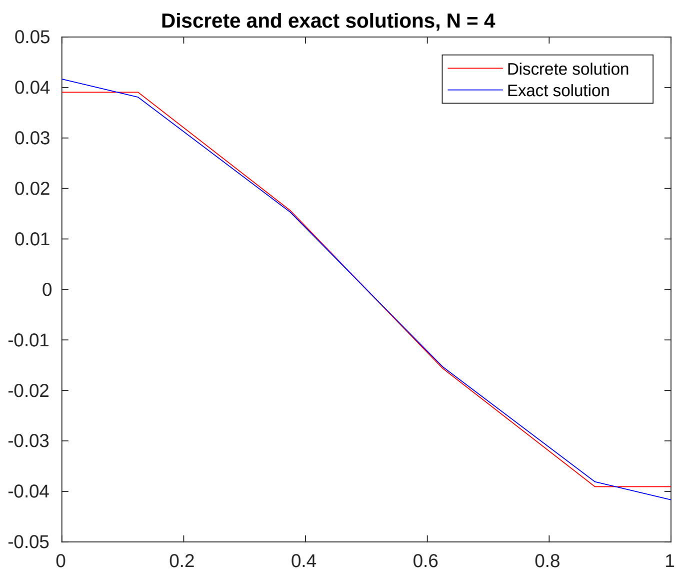

Figure 34: Neumann result, $N = 4$, Case 1.

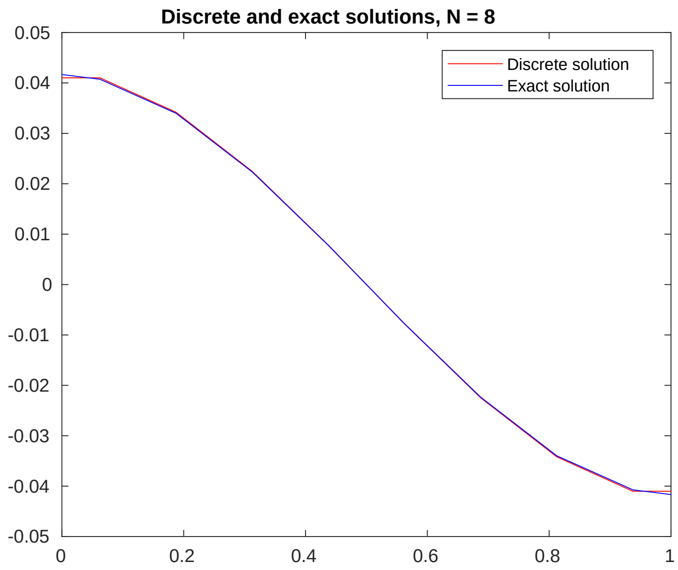

Figure 35: Neumann result, $N = 8$, Case 1.

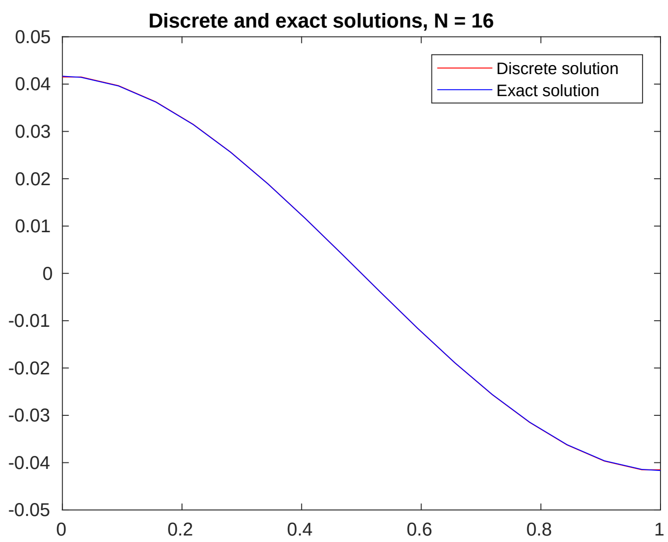

Figure 36: Neumann result, $N = 16$, Case 1.

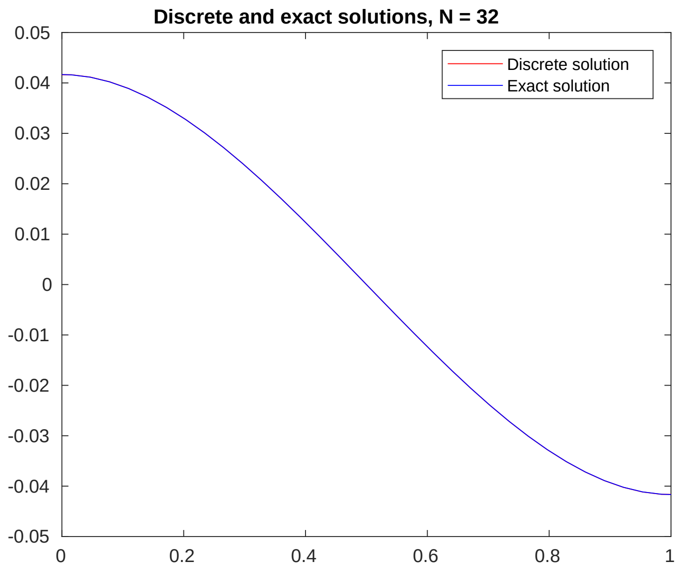

Figure 37: Neumann result, $N = 32$, Case 1.

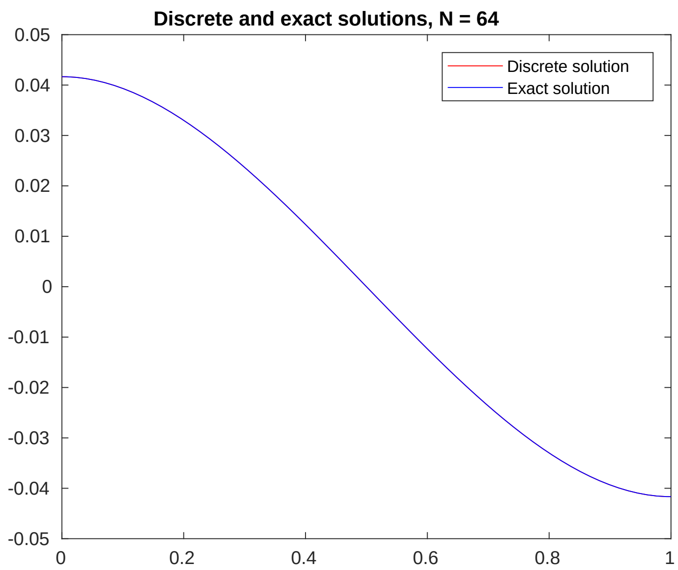

Figure 38: Neumann result, $N = 64$, Case 1.

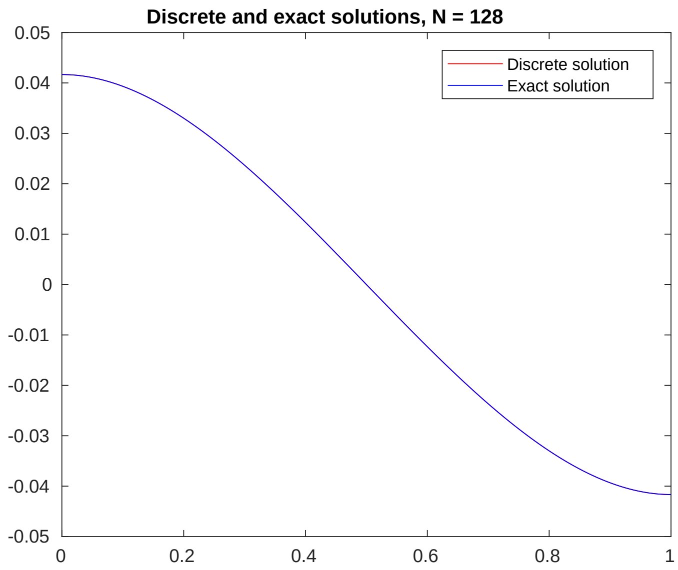

Figure 39: Neumann result, $N = 128$, Case 1.

Case 2.

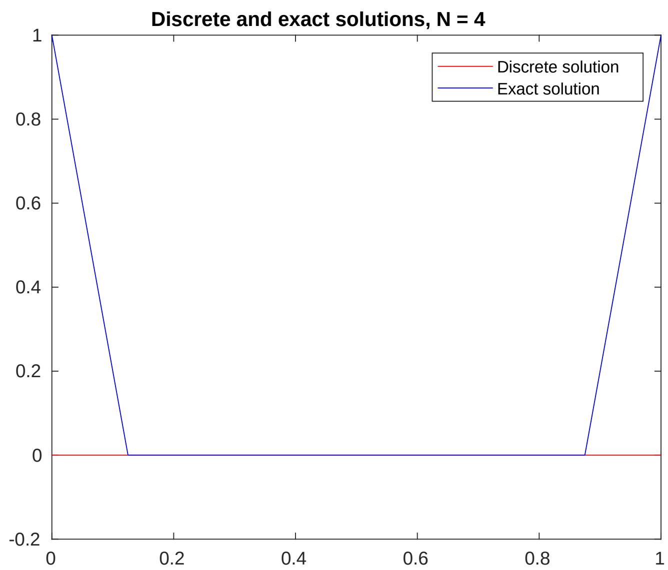

Figure 40: Neumann result, $N = 4$, Case 2.

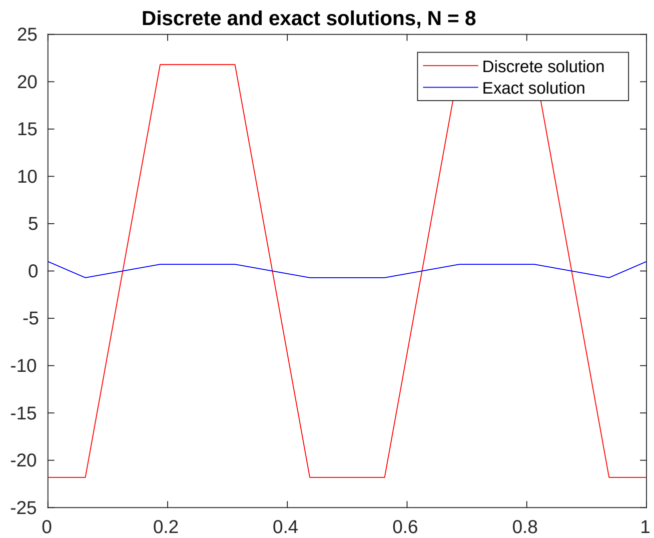

Figure 41: Neumann result, $N = 8$, Case 2.

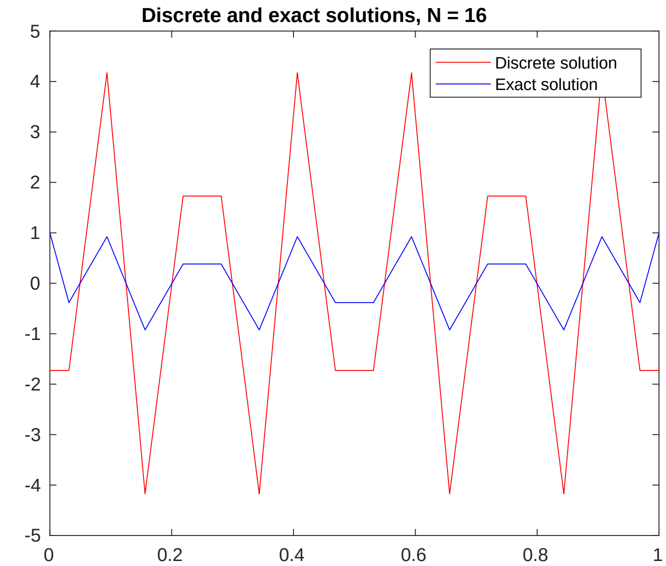

Figure 42: Neumann result, $N = 16$, Case 2.

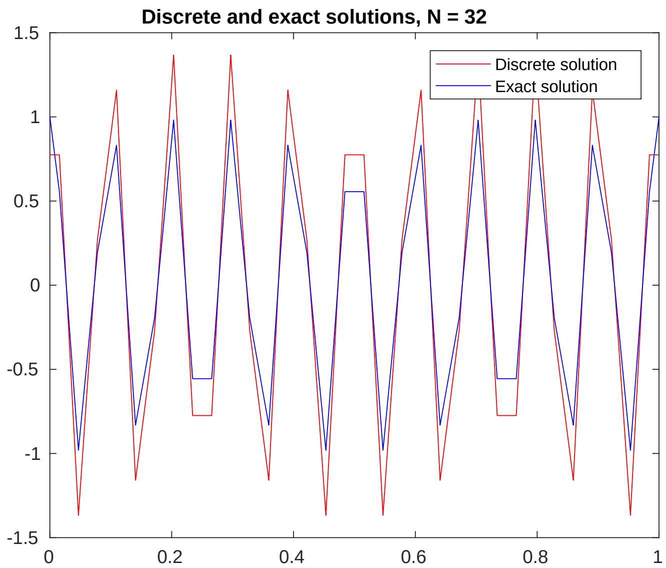

Figure 43: Neumann result, $N = 32$, Case 2.

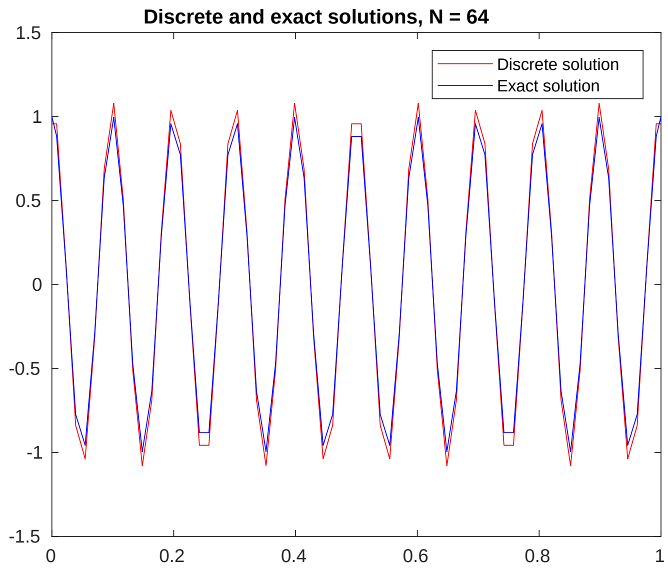

Figure 44: Neumann result, $N = 64$, Case 2.

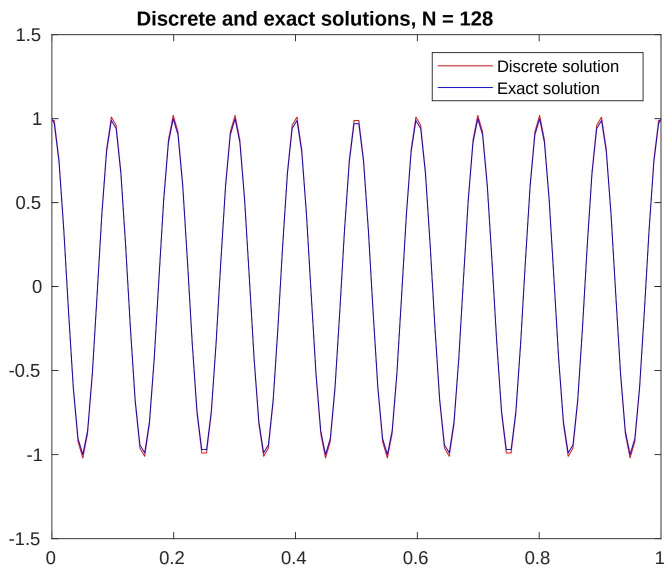

Figure 45: Neumann result G1 CP1 I1 N128 M9 C2

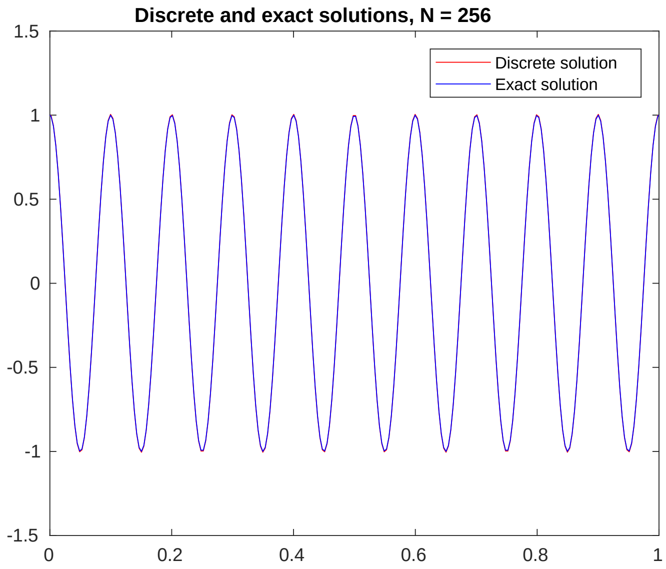

Figure 46: Neumann result, $N = 256$, Case 2.

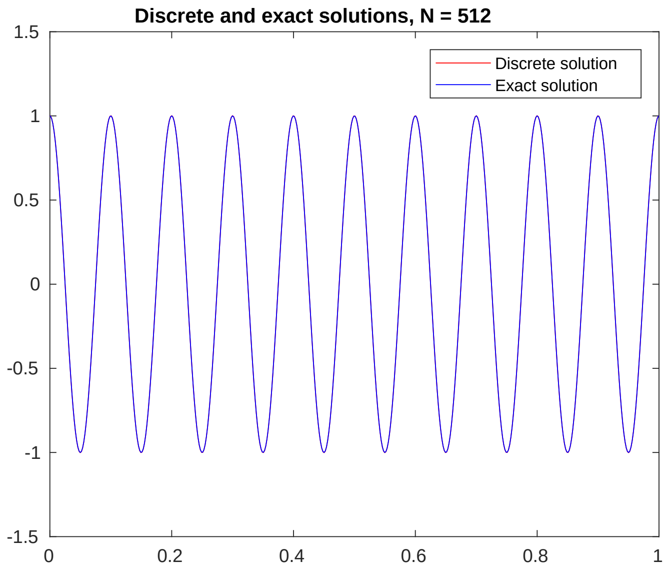

Figure 47: Neumann result, $N = 512$, Case 2.

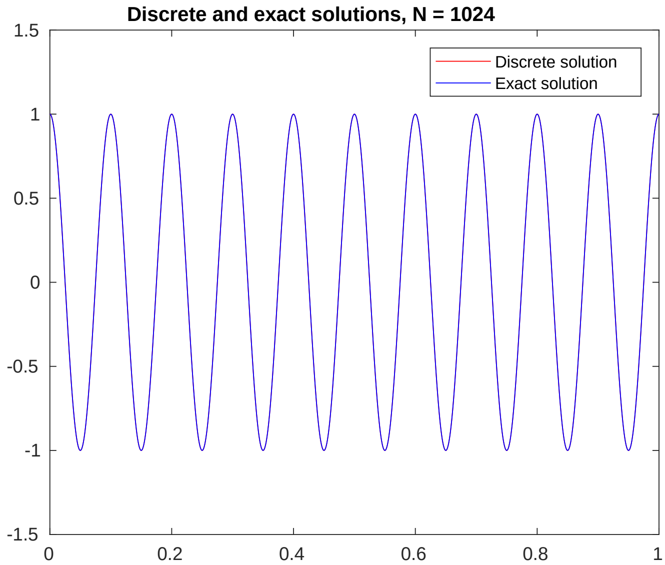

Figure 48: Neumann result, $N = 1024$, Case 2.

8.3.2 Errors

Case 1.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	7.278867e-04	1.449939e-02
8	1.864655e-04	5.329006e-03
16	4.689303e-05	1.918917e-03
32	1.174048e-05	6.845135e-04
64	2.936197e-06	2.430787e-04
128	7.341164e-07	8.612922e-05

Table 14: Error table

Case 2.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	3.446145e-13	4.000000e+00
8	2.110184e+01	2.389353e+02
16	2.486740e+00	7.434583e+01
32	2.787005e-01	1.565996e+01
64	5.963946e-02	4.064360e+00
128	1.437125e-02	1.122033e+00
256	3.560351e-03	3.281879e-01
512	8.880771e-04	1.017972e-01
1024	2.218939e-04	3.318688e-02

Table 15: Error table

8.3.3 Convergence rate

Case 1.

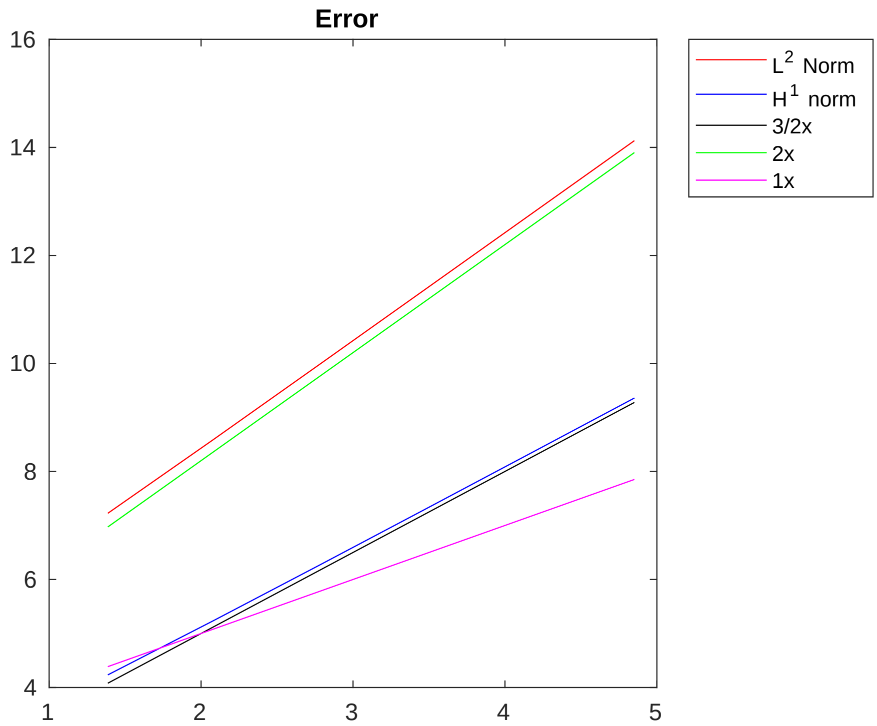

Figure 49: Neumann error, Case 1.

CASE 2.

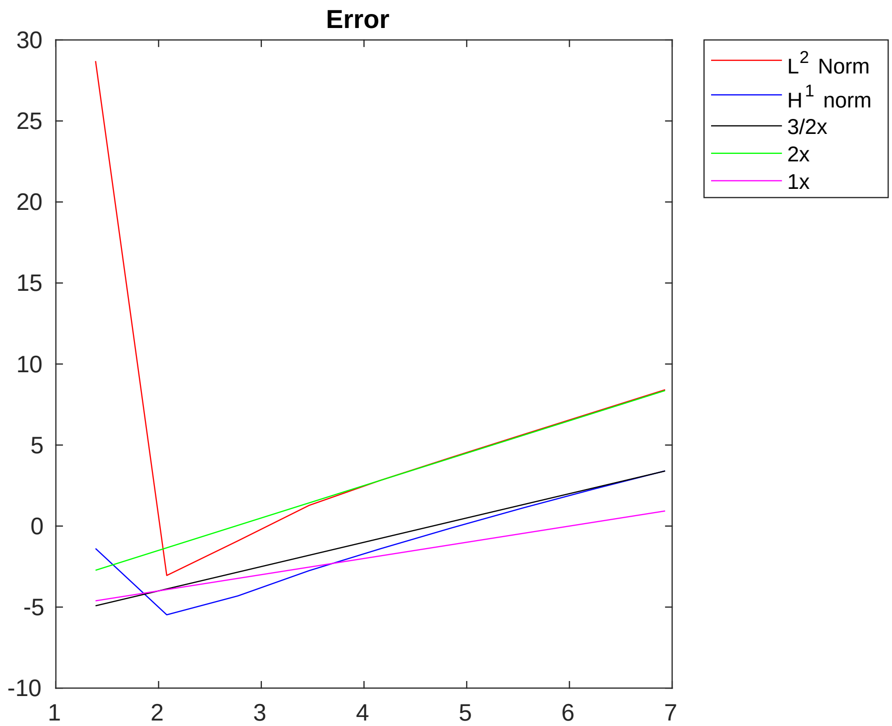

Figure 50: Neumann error, Case 2.

Remark 8.1. In both case 1 and 2, the method appears to have convergence rate of order 2 in L^2 norm and order $\frac{3}{2}$ in energy norm.

8.4 Homogeneous Neumann boundary condition, singular grid, each control point is the midpoint of corresponding control volume, integration using midpoint rule

Again, we will consider the grid

$$x_i = 1 - \cos\left(\frac{\pi i}{2(N+1)}\right) \tag{8.18}$$

in this section.

8.4.1 Errors

CASE 1.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	4.300279e-03	2.361702e-02
8	4.473365e-04	7.452580e-03
16	7.886546e-05	2.530245e-03
32	1.995312e-05	9.154423e-04
64	5.129577e-06	3.310069e-0
128	1.302358e-06	1.185516e-04

Table 16: Error table.

CASE 2.

N	$\|u_{discrete} - u_{exact}\|_{L^2}$	$\|u_{discrete} - u_{exact}\|_{H^1}$
4	2.518759e+02	8.229286e+02
8	2.924364e+01	2.378888e+02
16	9.888043e+01	3.764880e+02
32	5.525841e-01	2.551420e+01
64	1.099226e-01	6.825063e+00
128	2.584269e-02	1.840590e+00
256	6.388832e-03	5.196868e-01
512	1.595856e-03	1.549021e-01
1024	3.992680e-04	4.880113e-02

Table 17: Error table.

8.4.2 Convergence rate

Case 1.

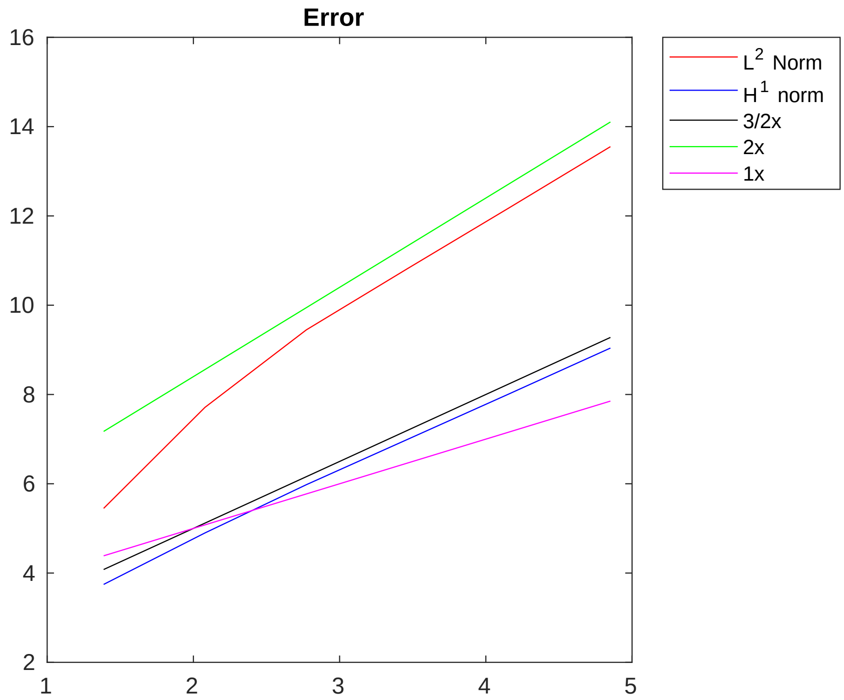

Figure 51: Neumann error, Case 1.

CASE 2.

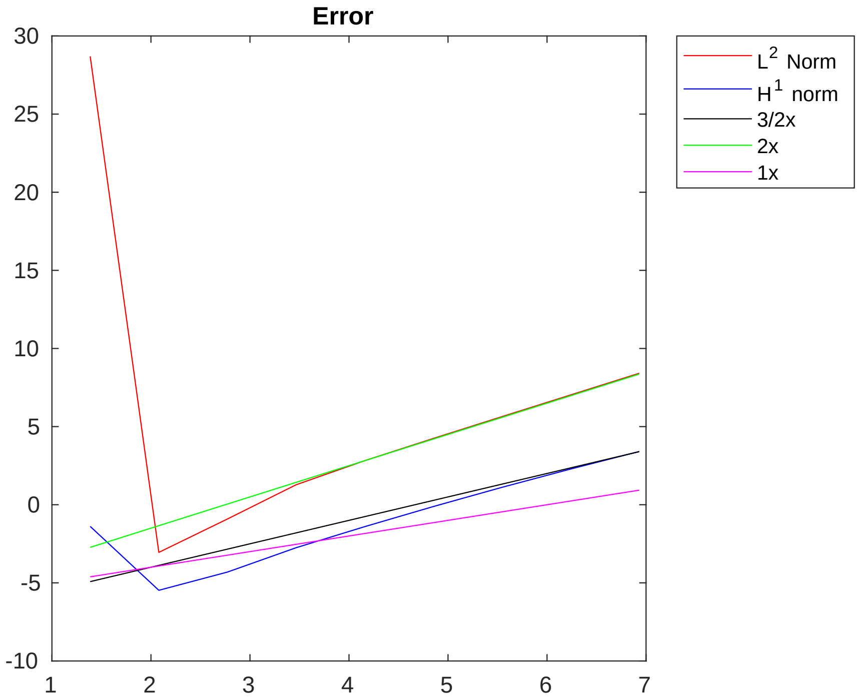

Figure 52: Neumann error, Case 2.

Remark 8.2. In both case 1 and 2, the method appears to have convergence rate of order 2 in L^2 norm and order $\frac{3}{2}$ in energy norm.

9 Appendices

9.1 An Easy Change of Variables

The purpose of this subsection is to transform an integral of the form $\int_a^b f(x)\,dx$ to an integral of the form $\int_c^d \widetilde{f}(x)\,dx$, where a, b, c, d are given real numbers.

To this end, we consider an arbitrary mapping S for which $S : [a, b] \mapsto [c, d]$. There are a lot of mappings S satisfying this requirement. For simplicity, we should choose the following linear mapping

$$y = S(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a}, \quad \forall x \in [a, b] \tag{9.1}$$

and the inverse mapping of S is given by

$$x = S^{-1}(y) = \frac{b-a}{d-c}y + \frac{ad-bc}{d-c}, \quad \forall y \in [c, d] \tag{9.2}$$

Then, we can use the change of variables $y = S(x)$ straightforward.

$$\int_a^b f(x)\,dx = \int_c^d f\left(S^{-1}(y)\right)\frac{b-a}{d-c}dy \tag{9.3}$$

Put

$$\widetilde{f}(y) = \frac{b-a}{d-c} f\left(S^{-1}(y)\right), \quad \forall y \in [c,d] \tag{9.4}$$

We can rewritten (9.3) as

$$\int_a^b f(x)\,dx = \int_c^d \widetilde{f}(x)\,dx \tag{9.5}$$

as required. □

9.2 Some Useful Inequalities

Proposition 9.1. *Let a, b, c, d be nonnegative real numbers. The following inequality holds*

$$\left(\sqrt{ab} + \sqrt{cd}\right)^2 \leq (a+c)(b+d) \tag{9.6}$$

PROOF. It is a direct consequence of Cauchy-Schwarz inequality. Indeed,

$$\begin{aligned}\left(\sqrt{ab} + \sqrt{cd}\right)^2 &= ab + cd + 2\sqrt{abcd} && (9.7)\\ &\leq ab + cd + bc + ad && (9.8)\\ &= (a+c)(b+d) && (9.9)\end{aligned}$$

The equality occurs if and only if $bc = ad$. □

THE END

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References

[1] Robert Eymard, Thierry Gallouët, Raphaèle Herbin, *Finite Volume Methods*, October 2006.

[2] Anh Ha Le, Huong Lan Tran, *Finite Volume Method in 1D*, University of Sciences, March 12, 2014.

[3] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality