Smallest constant for a Gagliardo-Nirenberg functional inequality

Supervisor: Prof. Florian Méhats

NGUYEN Quan Ba Hong

Université de Rennes 1, France

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Variational problem

GOAL: *Determine the best constant $C_{\sigma,N}$ for the interpolation estimate*

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leq C_{\sigma,N}^{2\sigma+2} \|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}, \tag{1}$$

for $0 < \sigma < \frac{2}{N-2}$, $N \geq 2$.

To compute $C_{\sigma,N}$, it suffices to minimize the functional

$$J^{\sigma,N}(f) := \frac{\|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}. \tag{2}$$

Main result

Theorem 1

For $0 < \sigma < \frac{2}{N-2}$, $\alpha := \inf\limits_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$ *is attained at a function* ψ *with the following properties:*

1) ψ *is positive and a functional of* $|x|$ *alone.*
2) $\psi \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.
3) ψ *is a solution of equation*

$$\frac{\sigma N}{2}\Delta\psi - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0, \tag{3}$$

   of minimal L^2 *norm (the ground state).*

In addition, $\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}$.

Compactness lemma

Lemma 2 (Compactness lemma)

For $0 < \sigma < \frac{2}{N-2}$, *the embedding*

$$H^1_{\mathrm{radial}}\left(\mathbb{R}^N\right) \hookrightarrow L^{2\sigma+2}\left(\mathbb{R}^N\right)$$

is compact.

Strauss's estimate: If $N \geq 2$ and $u \in H^1_{\mathrm{radial}}\left(\mathbb{R}^N\right)$, then

$$|u(x)| \leq \frac{C}{|x|^{\frac{N-1}{2}}} \|u\|_{H^1}.$$

Proof of the main theorem:

- $J^{\sigma,N}$ is invariant under the scaling $u^{\lambda,\mu}(x) := \mu u(\lambda x)$:

$$J^{\sigma,N}\left(u^{\lambda,\mu}\right) = J^{\sigma,N}(u), \ \forall \mu \in \mathbb{R}, \ \lambda \in \mathbb{R}.$$

- $\exists \{u_n\}_n \subset H^1\left(\mathbb{R}^N\right) \cap L^{2\sigma+2}\left(\mathbb{R}^N\right)$ s.t.

$$\alpha = \inf J^{\sigma,N}(u) = \lim_{n\to\infty} J^{\sigma,N}(u_n) < \infty.$$

- We can assume $u_n > 0$ and u_n radially symmetric.

Symmetrization

The *symmetric-decreasing rearrangement* f^* of a function f satisfies:

1) f^* is radially symmetric.
2) For $f \in L^p\left(\mathbb{R}^N\right)$, $\|f\|_p = \|f^*\|_p$, $\forall 1 \leq p \leq \infty$.
3) $\|\nabla f\|_2 \geq \|\nabla f^*\|_2$.

For each u_n,

$$\begin{cases} \|u_n^*\|_2 = \|u_n\|_2, \ \|u_n^*\|_{2\sigma+2} = \|u_n\|_{2\sigma+2}, \\ \|\nabla u_n^*\|_2 \leq \|\nabla u_n\|_2, \end{cases}$$

and thus

$$J^{\sigma,N}\left(u_n^*\right) \leq J^{\sigma,N}\left(u_n\right).$$

- With suitable μ_n, λ_n, we obtain a sequence $\psi_n(x) := u^{\lambda_n,\mu_n}(x)$ satisfying
  a) $\psi_n \geq 0$ and radially symmetric,
  b) $\psi_n \in H^1\left(\mathbb{R}^N\right)$,
  c) $\|\psi_n\|_2 = \|\nabla\psi_n\|_2 = 1$,
  d) $J^{\sigma,N}(\psi_n) \downarrow \alpha$ as $n \to \infty$.
- Since $(\psi_n)_n$ is bounded in $H^1\left(\mathbb{R}^N\right)$, some subsequence has a weak H^1 limit ψ^*.

- By Compactness lemma, we can take ψ_n strongly convergent to ψ^* in $L^{2\sigma+2}\left(\mathbb{R}^N\right)$ for $0 < \sigma < \frac{2}{N-2}$.
- Prove $\|\psi^*\|_2 = \|\nabla\psi^*\|_2 = 1$ and $\psi_n \to \psi^*$ in H^1.
- The minimizing function ψ^* satisfies the Euler-Lagrange equation

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J^{\sigma,N}\left(\psi^* + \varepsilon\eta\right) = 0, \quad \forall\eta \in C_0^\infty\left(\mathbb{R}^N\right).$$

- ψ^* satisfies

$$\frac{\sigma N}{2}\Delta\psi^* - \left(1 + \frac{\sigma}{2}\left(2 - N\right)\right)\psi^* + \alpha\left(\sigma + 1\right)\left(\psi^*\right)^{2\sigma+1} = 0 \text{ in } \mathcal{D}'.$$

- Let $\psi = [\alpha(\sigma+1)]^{\frac{1}{2\sigma}}\psi^*$,

$$\frac{\sigma N}{2}\Delta\psi - \left(1+\frac{\sigma}{2}(2-N)\right)\psi + \psi^{2\sigma+1} = 0 \text{ in } \mathcal{D}'.$$

- Regularize ψ by bootstrap argument: $\psi \in C^\infty\left(\mathbb{R}^N\right)$.
- The infimum of $J^{\sigma,N}$ is given by

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}.$$

Minimal L^2 norm

Multiply

$$\frac{\sigma N}{2}\Delta\varphi - \left(1 + \frac{\sigma}{2}(2-N)\right)\varphi + \varphi^{2\sigma+1} = 0$$

- by φ and integrate over $\mathbb{R}^N$, obtain

$$\frac{\sigma N}{2}\|\nabla\varphi\|_2^2 + \left(1 + \frac{\sigma}{2}(2-N)\right)\|\varphi\|_2^2 = \|\varphi\|_{2\sigma+2}^{2\sigma+2},$$

- by $x \cdot \nabla\varphi$, obtain

$$\frac{\sigma N}{2}\left(\frac{N}{2} - 1\right)\|\nabla\varphi\|_2^2 + \frac{N}{2}\left(1 + \frac{\sigma}{2}(2-N)\right)\|\varphi\|_2^2 = \frac{N}{2\sigma+2}\|\varphi\|_{2\sigma+2}^{2\sigma+2}.$$

- Solving $\|\nabla\varphi\|_2^2$ and $\|\varphi\|_{2\sigma+2}^{2\sigma+2}$ in terms of $\|\varphi\|_2^2$:

$$\|\nabla\varphi\|_2^2 = \|\varphi\|_2^2, \;\; \|\varphi\|_{2\sigma+2}^{2\sigma+2} = (\sigma+1)\,\|\varphi\|_2^2.$$

  Thus

$$J^{\sigma,N}(\varphi) = \frac{\|\varphi\|_2^{2\sigma}}{\sigma+1},$$

  in particular,

$$J^{\sigma,N}(\psi) = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}.$$

- $J^{\sigma,N}(\varphi) \geq J^{\sigma,N}(\psi)$ then implies $\|\varphi\|_2 \geq \|\psi\|_2$.

Corollary 3

Let $0 < \sigma < \frac{2}{N-2}$. *Then, the following equation*

$$\Delta u - u + u^{2\sigma+1} = 0$$

has a positive, radial solution of class $H^1\left(\mathbb{R}^N\right)$.

Corollary 4

The smallest constant for which the interpolation estimate (1) *holds is given by*

$$C_{\sigma,N} = \left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}.$$