Homework Assignment
Differential Geometry

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November 7, 2017

Exercise 5 (p.168, [DG_Carmo_1]) Consider the parametrized surface (Enneper's surface)

$$x(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right) \tag{1}$$

and show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0 \tag{2}$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0 \tag{3}$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2} \tag{4}$$

(d) The lines of curvature are the coordinate curves.

(e) The asymptotic curves are $u + v = \text{const}$, $u - v = \text{const}$

Solution

(a) We have

$$x_u(u,v) = \left(1 - u^2 + v^2, 2vu, 2u\right) \tag{5}$$
$$x_v(u,v) = \left(2uv, 1 - v^2 + u^2, -2v\right) \tag{6}$$

Using these, we can compute the coefficient of the first fundamental form

$$\begin{aligned}
E &= \langle x_u, x_u \rangle && (7)\\
&= (1 - u^2 + v^2)^2 + 4v^2u^2 + 4u^2 && (8)\\
&= (u^4 - 2u^2v^2 - 2u^2 + v^4 + 2v^2 + 1) + 4v^2u^2 + 4u^2 && (9)\\
&= (u^4 + 2u^2v^2 + 2u^2 + v^4 + 2v^2 + 1) && (10)\\
&= (1 + u^2 + v^2)^2 && (11)\\
F &= \langle x_u, x_v \rangle && (12)\\
&= (1 - u^2 + v^2)2uv + 2vu(1 - v^2 + u^2) - 4uv && (13)\\
&= 2uv - 2u^3v + 2uv^3 + 2vu - 2v^3u + 2vu^3 - 4uv && (14)\\
&= 0 && (15)\\
G &= \langle x_v, x_v \rangle && (16)\\
&= 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2 && (17)\\
&= (u^4 - 2u^2v^2 + 2u^2 + v^4 - 2v^2 + 1) + 4u^2v^2 + 4v^2 && (18)\\
&= (u^4 + 2u^2v^2 + 2u^2 + v^4 + 2v^2 + 1) && (19)\\
&= (1 + u^2 + v^2)^2 && (20)
\end{aligned}$$

(b) The unit normal vector at $x(u,v)$ is define as

$$\begin{aligned}
\nu(u,v) &= N(x(u,v)) && (21)\\
&= \frac{x_u \times x_v}{\|x_u \times x_v\|} && (22)\\
&= \frac{\left(-2u^3 - 2uv^2 - 2u, 2v^3 + 2vu^2 + 2v, -u^4 - 2u^2v^2 - v^4 + 1\right)}{\sqrt{(-2u^3 - 2uv^2 - 2u)^2 + (2v^3 + 2vu^2 + 2v)^2 + (-u^4 - 2u^2v^2 - v^4 + 1)^2}} && (23)\\
&= \frac{\left(-2u^3 - 2uv^2 - 2u, 2v^3 + 2vu^2 + 2v, -u^4 - 2u^2v^2 - v^4 + 1\right)}{(1 + u^2 + v^2)^2} && (24)
\end{aligned}$$

The second order partial derivative of x can be computed as

$$\begin{aligned}
x_{u,u}(u,v) &= (-2u, 2v, 2) && (25)\\
x_{u,v}(u,v) &= (2v, 2u, 0) && (26)\\
x_{v,v}(u,v) &= (2u, -2v, -2) = -x_{u,u}(u,v) && (27)
\end{aligned}$$

Hence, the coefficients of the second fundamental form are

$$\begin{aligned}
e &= \langle \nu, x_{u,u} \rangle && (28)\\
&= \frac{-2u(-2u^3 - 2uv^2 - 2u) + 2v(2v^3 + 2vu^2 + 2v) + 2(-u^4 - 2u^2v^2 - v^4 + 1)}{(1 + u^2 + v^2)^2} && (29)\\
&= 2\frac{2u^4 + 2u^2v^2 + 2u^2 + 2v^4 + 2v^2u^2 + 2v^2 - u^4 - 2u^2v^2 - v^4 + 1}{(1 + u^2 + v^2)^2} && (30)\\
&= 2\frac{u^4 + 2u^2v^2 + 2u^2 + v^4 + 2v^2 + 1}{(1 + u^2 + v^2)^2} && (31)\\
&= 2\frac{(1 + u^2 + v^2)^2}{(1 + u^2 + v^2)^2} && (32)\\
&= 2 && (33)\\
f &= \langle \nu, x_{u,v} \rangle && (34)\\
&= \frac{2u(-2u^3 - 2uv^2 - 2u) + 2v(2v^3 + 2vu^2 + 2v)}{(1 + u^2 + v^2)^2} && (35)\\
&= \frac{(-4u^4 - 4u^2v^2 - 4u^2) + (4v^4 + 4v^2u^2 + 4v^2)}{(1 + u^2 + v^2)^2} && (36)\\
&= \frac{0}{(1 + u^2 + v^2)^2} && (37)\\
&= 0 && (38)\\
g &= \langle \nu, x_{v,v} \rangle && (39)\\
&= -\langle \nu, x_{u,u} \rangle && (40)\\
&= -2 && (41)
\end{aligned}$$

(c)

The Gaussian curvature can be computed as

$$K = k_1 k_2 = \left| \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \right| \tag{42}$$

$$= \frac{\left| \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right|}{\left| \begin{bmatrix} E & F \\ F & G \end{bmatrix} \right|} \tag{43}$$

$$= \frac{eg - f^2}{EG - F^2} \tag{44}$$

The mean curvature can be computed as

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2} \operatorname{trace} \left(\begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \right) \tag{45}$$

$$= \frac{1}{2} \operatorname{trace} \left(\frac{1}{EG - F^2} \begin{bmatrix} Ge - Ff & Ef - Fe \\ Gf - Fg & Eg - Ff \end{bmatrix} \right) \tag{46}$$

$$= \frac{1}{2} \frac{Ge - 2Ff + Eg}{EG - F^2} \tag{47}$$

$$\tag{48}$$

Hence, we have

$$\begin{cases} k_1 k_2 & = K \\ k_1 + k_2 & = 2H \end{cases} \tag{49}$$

$$\Leftrightarrow \begin{cases} k_1(2H - k_1) & = K \\ k_2 & = 2H - k_1 \end{cases} \tag{50}$$

$$\Leftrightarrow \begin{cases} -k_1^2 + 2Hk_1 & = K \\ k_2 & = 2H - k_1 \end{cases} \tag{51}$$

$$\Leftrightarrow \begin{cases} k_1^2 - 2Hk_1 + K & = 0 \\ k_2 & = 2H - k_1 \end{cases} \tag{52}$$

$$\Leftrightarrow \begin{cases} k_1 & = H - \sqrt{H^2 - K} \quad \text{or} \quad k_1 = H + \sqrt{H^2 - K} \\ k_2 & = 2H - k_1 \end{cases} \tag{53}$$

Since $2H - (H + \sqrt{H^2 - K}) = H - \sqrt{H^2 - K}$, we can choose

$$\begin{cases} k_1 & = H + \sqrt{H^2 - K} \\ k_2 & = H - \sqrt{H^2 - K} \end{cases} \tag{54}$$

We have

$$
\begin{aligned}
H^2 - K =& \frac{1}{4}\frac{(Ge - 2Ff + Eg)^2}{(EG - F^2)^2} - \frac{eg - f^2}{EG - F^2} && (55)\\
=& \frac{1}{4}\frac{(Ge - 2Ff + Eg)^2}{(EG - F^2)^2} - \frac{(eg - f^2)(EG - F^2)}{(EG - F^2)^2} && (56)\\
=& \frac{1}{4}\frac{E^2g^2 - 4EFfg + 2EGeg + 4F^2f^2 - 4FGef + G^2e^2}{(EG - F^2)^2} && (57)\\
& - \frac{1}{4}4\frac{F^2f^2 - egF^2 - EGf^2 + EGeg}{(EG - F^2)^2} && (58)\\
=& \frac{1}{4}\frac{E^2g^2 - 4EFfg + 2EGeg + 4F^2f^2 - 4FGef + G^2e^2}{(EG - F^2)^2} && (59)\\
& + \frac{1}{4}\frac{-4F^2f^2 + 4egF^2 + 4EGf^2 - 4EGeg}{(EG - F^2)^2} && (60)\\
=& \frac{1}{4}\frac{E^2g^2 - 4EFfg - 2EGeg + 4EGf^2 + 4F^2eg - 4FGef + G^2e^2}{(EG - F^2)^2} && (61)\\
& && (62)
\end{aligned}
$$

Since $F = f = 0$,

$$
H = \frac{1}{2}\frac{Ge + Eg}{EG} \tag{63}
$$

$$
\begin{aligned}
H^2 - K &= \frac{1}{4}\frac{E^2g^2 - 2EGeg + G^2e^2}{(EG)^2} && (64)\\
&= \frac{1}{4}\frac{(Eg - Ge)^2}{(EG)^2} && (65)
\end{aligned}
$$

Then, we have

$$
\begin{cases}
k_1 &= H + \sqrt{H^2 - K} = \frac{1}{2}\frac{Ge + Eg}{EG} + \frac{1}{2}\left|\frac{Eg - Ge}{EG}\right. \\
k_2 &= H - \sqrt{H^2 - K} = \frac{1}{2}\frac{Ge + Eg}{EG} - \frac{1}{2}\left.\frac{Eg - Ge}{EG}\right|
\end{cases} \tag{66}
$$

We can choose

$$
\begin{cases}
k_1 &= \frac{1}{2}\frac{Ge + Eg}{EG} - \frac{1}{2}\frac{Eg - Ge}{EG} = \frac{Ge}{EG} = \frac{e}{E} = \frac{2}{(1 + u^2 + v^2)^2} \\
k_2 &= \frac{1}{2}\frac{Ge + Eg}{EG} + \frac{1}{2}\frac{Eg - Ge}{EG} = \frac{Eg}{EG} = \frac{g}{G} = \frac{-2}{(1 + u^2 + v^2)^2}
\end{cases} \tag{67}
$$

(d)

According to p.161, [DG_Carmo_1], a connected regular curve C in the coordinate neighborhood of x is a line of curvature if and only if for any parametrization $\alpha(t) = x(u(t), v(t))$, $t \in I$ of C, we have

$$dN(\alpha'(t)) = \lambda(t)\alpha'(t) \tag{68}$$

It follows that $u'(t)$ and $v'(t)$ satisfy the differential equation of the lines of curvature

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0 \tag{69}$$

In our case, (69) is

$$-2(1 + u^2 + v^2)u'v' - 2(1 + u^2 + v^2)u'v' = 0 \tag{70}$$

or

$$(1 + u^2 + v^2)u'v' = 0 \tag{71}$$

Since $1 + u^2 + v^2 > 0$, this implies that either $u' = 0$ or $v' = 0$, which means either $u(t)$ or $v(t)$ must be invariant as t change. In other words, C must be a coordinate curve.

Therefore, the lines of curvature are the coordinate curves.

(e)

According to p.160, [DG_Carmo_1], a connected regular curve C in the coordinate neighborhood of x is an asymptotic curve if and only if for any parametrization $\alpha(t) = x(u(t), v(t))$, $t \in I$ of C, we have $II(\alpha'(t)) = 0$, for all $t \in I$, that is, if and only if u' and v' satisfy the differential equation of the asymptotic curves

$$e(u')^2 + 2fu'v' + g(v')^2 = 0 \tag{72}$$

In our case, (72) is

$$2(u')^2 - 2(v')^2 = 0 \tag{73}$$

or

$$(u')^2 = (v')^2 \tag{74}$$

which means that

$$u' = v' \quad \text{or} \quad u' = -v' \tag{75}$$

Integrating both sides, we have

$$u = v + \text{const} \quad \text{or} \quad u = -v + \text{const} \tag{76}$$

This is equivalent to

$$u - v = \text{const} \quad \text{or} \quad u + v = \text{const} \tag{77}$$

Therefore, the asymptotic curves are $u + v = \text{const}$, $u - v = \text{const}$.

Exercise 6 (p.168, [DG_Carmo_1]) (A surface with $K = -1$, the Pseudosphere)

(a) Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the tractrix).

(b) Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the pseudosphere) is regular and find out a parametrization in a neighborhood of a regular point.

(c) Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

Solution

(a) Let r be some line in a plane, with out loss of generality, we can choose the Cartesian coordinates such that the z-axis coincides with r and xOz is the plane where r is in.

Let $C(x) = (x, f(x))$ be a parameterization of a curve C in the plane xOz. Suppose that $(a, f(a))$ is a point on C, the tangent line of C at $(a, f(a))$ (denoted by $T_{C,a}(x, z)$) is given by

$$z = f(a) + f'(a)(x - a) \tag{78}$$

where $f'(a) = \dfrac{df}{dx}(a)$.

Let $(0, b)$ be the intersection of $T_{C,a}$ and the z-axis, b is given by

$$\begin{align} b &= f(a) + f'(a)(0 - a) \tag{79} \\ &= f(a) - af'(a) \tag{80} \end{align}$$

The distance between $(a, f(a))$ and $(0, b)$ is

$$\begin{align} \sqrt{(a-0)^2 + (f(a) - b)^2} &= \sqrt{a^2 + (f(a) - f(a) + af'(a))^2} \tag{81} \\ &= \sqrt{a^2 + a^2 f'^2(a)} \tag{82} \\ & \tag{83} \end{align}$$

Then, if the distance between $(a, f(a))$ and $(0, b)$ is constantly equal to 1, we have the equation

$$\sqrt{a^2 + a^2 f'^2(a)} = 1 \tag{84}$$

or

$$f'^2(a) = \frac{1 - a^2}{a^2} \tag{85}$$

with the condition $0 < |a| \leq 1$.

Assume that C is completely on the right side of Oz, using (85), we can find an equation for f by pluging $x \in (0, 1]$ into a,

$$f'^2(x) = \frac{1 - x^2}{x^2} \tag{86}$$

With (86), we note that C can be interpreted as an union of 2 curves, given respectively by the parameterization $C_1(x) = (x, f_1(x))$ and $C_2(x) = (x, f_2(x))$ where

$$f_1'(x) = \frac{\sqrt{1-x^2}}{x} \tag{87}$$

$$f_2'(x) = -\frac{\sqrt{1-x^2}}{x} \tag{88}$$

We notice that f_1' and f_2' coincide at $x = 1$ and we will not be able to determine the curve C using only the above differential equations, since those equations only give us the shape of the curve but not the position.

Therefore, we must specify a condition to constrain the position of C in the plane, in this this case, we can choose the condition $C(1) = (1, f(1)) = (1, 0)$, or $f_1(1) = f_2(1) = 0$.

Consider the initial value problem

$$\begin{cases} f_1'(x) &= \dfrac{\sqrt{1-x^2}}{x} \\ f_1(1) &= 0 \end{cases} \tag{89}$$

Using the Newton–Leibniz formula, integrating both sides of (87) yields

$$\begin{aligned} f_1(x) &= \int_1^x \frac{\sqrt{1-\bar{x}^2}}{\bar{x}} d\bar{x} + f_1(1) && (90)\\ &= \left(\sqrt{1-x^2} - \ln\frac{1+\sqrt{1-x^2}}{x}\right) + 0 && (91)\\ &= \sqrt{1-x^2} - \ln\frac{1+\sqrt{1-x^2}}{x} && (92)\\ &= \sqrt{1-x^2} - \operatorname{arsech} x && (93) \end{aligned}$$

Similarly, integrating both sides of (88) yields

$$\begin{aligned} f_2(x) &= \ln\frac{1+\sqrt{1-x^2}}{x} - \sqrt{1-x^2} && (94)\\ &= \operatorname{arsech} x - \sqrt{1-x^2} && (95)\\ &= -f_1(x) && (96) \end{aligned}$$

where arsech is the inverse hyperbolic secant function given by

$$\operatorname{arsech} x = \ln\frac{1+\sqrt{1-x^2}}{x} \tag{97}$$

with the derivative

$$\operatorname{arsech}' x = \frac{d}{dx}\operatorname{arsech} x = \frac{-1}{x\sqrt{1-x^2}} \tag{98}$$

By (93), (96) and the initial condition $f_1(1) = f_2(1) = 0$, we can see that C_1 and C_2 is symmetric about the x-axis.

Thus, for many purpose, we can evaluate the properties of the C_1 part of the curve C and deduce the properties of the rest using the property of symmetry.

(b) Let C be the generating curve for the surface of revolution S_C obtained by rotating C around the z-axis.

Since C is the union of two curve C_1 and C_2, which are symmetric about the x-axis, we consider partitioning S_C into 3 parts S_{C_1}, S_{C_2} and S_{C_0} as follows

• **Part 1:** S_{C_1} is the surface of revolution obtained by rotating the curve C_1 minus $C_1(1) = (1, 0)$ around the z-axis

• **Part 2:** S_{C_2} is the surface of revolution obtained by rotating the curve C_2 minus $C_2(1) = (1, 0)$ around the z-axis

• **Part 3:** S_{C_0} is the surface of revolution obtained by rotating the curve $C_0 = C_1 \cap C_2$. Actually, C_0 is the point $(1, 0)$ on the xOz plane. In other words S_{C_0} is the circle on the plane xOy with radius 1 and center at O.

Firstly, we consider a local parameterization of S_{C_1}

$$X_1(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)) \tag{99}$$

on the open domain

$$U_1 = \left\{(u, v) \in \mathbb{R}^2,\ \ 0 < v < 1,\ \ 0 < u < 2\pi\right\} \tag{100}$$

and

$$\begin{align}
\varphi(v) &= v \tag{101}\\
\psi(v) &= f_1(v) \tag{102}\\
&= \sqrt{1 - v^2} - \ln \frac{1 + \sqrt{1 - v^2}}{v} \tag{103}\\
&= \sqrt{1 - v^2} - \operatorname{arsech} v \tag{104}
\end{align}$$

In other words, $(\varphi(v), \psi(v))$ is the parameterization of the curve C_1 as in **(a)** but minus $C_1(1) = (1, 0)$.

We notice that this parameterization does not fully parameterize S_{C_1}. In particular, it miss the curve $(\varphi(v), 0, \psi(v))$. But since we care about local properties of S_{C_1} (in this exercise), we will use another local parameterization of S_{C_1} with another domain for the remaining point on the mentioned curve.

In this case, the local parameterization

$$X_2(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)) \tag{105}$$

on the open domain

$$U_2 = \left\{(u, v) \in \mathbb{R}^2,\ \ 0 < v < 1,\ \ -\pi < u < \pi\right\} \tag{106}$$

will account for the points on S_{C_1} not parameterized using the domain U_1.

Now, we will check the regularity of S_{C_1}.

Let $p \in S_{C_1}$, we can see that p must be in either $X_1(U_1)$ or $X_2(U_2)$, both can be proved to be open since X is continuous on U. Since the expressions for X_1 and X_2 are similar, we can use the notation X for both when there are no confusion. We also denote U as U_1 or U_2 such that $p \in X(U)$.

To prove that S_{C_1} is a regular curve, we must prove the following 3 condition (p.52, [DG_Carmo_1])

• **Condition 1:** X is differentiable. This means that all 3 components of $X(u, v)$ have continuous partial derivatives of all orders in U.

In our case, $\varphi(v)\cos u$, $\varphi(v)\sin u$ and $\psi(v)$ have continuous partial derivatives of all orders in U. Therefore, this condition is satisfied.

• **Condition 2:** X is a homeomorphism. Since X is already continuous by condition 1, this means that X has an inverse $X^{-1} : X(U) \cap S_{C_1} \to U$ which is continuous.

• **Condition 3:** For each $q \in U$, the differential $dX_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

This condition is equivalent to the linear independence of the 2 vectors $X_u = \dfrac{\partial X}{\partial u}$ and $X_v = \dfrac{\partial X}{\partial v}$, where

$$\begin{aligned}
X_u &= (-\varphi(v)\sin u, \varphi(v)\cos u, 0) &(107)\\
&= (-v\sin u, v\cos u, 0) &(108)\\
X_v &= (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v)) &(109)\\
&= \left(\cos u, \sin u, -\frac{v}{\sqrt{1-v^2}} - \frac{-1}{v\sqrt{1-v^2}}\right) &(110)\\
&= \left(\cos u, \sin u, \frac{\sqrt{1-v^2}}{v}\right) &(111)
\end{aligned}$$

We can check the linear independence of those vectors by checking whether the cross product of these 2 vectors vanish

$$X_u \times X_v = \left(\sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, -v\right) \tag{112}$$

Since $0 < v < 1$ for all $q \in U$, we must have $X_u \times X_v \neq 0$ for all $q \in U$. In other words X_u and X_v are linearly independent for all $q \in U$.

(c) Let $p \in S_{C_1}$, we are going to compute the Gaussian curvature at p.

We have

$$X_u = (-v \sin u, v \cos u, 0) \tag{113}$$
$$X_v = \left(\cos u, \sin u, \frac{\sqrt{1 - v^2}}{v} \right) \tag{114}$$

Using these, we can compute the coefficient of the first fundamental form

$$E = \langle X_u, X_u \rangle \tag{115}$$
$$= v^2 \sin^2 u + v^2 \cos^2 uu + 0^2 \tag{116}$$
$$= v^2 \tag{117}$$
$$F = \langle X_u, X_v \rangle \tag{118}$$
$$= -v \sin u \cos u + v \cos u \sin u + 0 \tag{119}$$
$$= 0 \tag{120}$$
$$G = \langle X_v, X_v \rangle \tag{121}$$
$$= \cos^2 u + \sin^2 u + \frac{1 - v^2}{v^2} \tag{122}$$
$$= 1 + \frac{1}{v^2} - 1 \tag{123}$$
$$= \frac{1}{v^2} \tag{124}$$

The unit normal vector at $X(u, v)$ is computed as

$$\nu(u, v) = \frac{X_u \times X_v}{\|X_u \times X_v\|} \tag{125}$$
$$= \frac{\left(\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, -v\right)}{\sqrt{(\sqrt{1 - v^2} \cos u)^2 + (\sqrt{1 - v^2} \sin u)^2 + v^2}} \tag{126}$$
$$= \frac{\left(\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, -v\right)}{\sqrt{\sqrt{1 - v^2}^2 (\cos^2 u + \sin^2 u) + v^2}} \tag{127}$$
$$= \frac{\left(\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, -v\right)}{\sqrt{1 - v^2 + v^2}} \tag{128}$$
$$= \left(\sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, -v \right) \tag{129}$$
$$= X_u \times X_v \tag{130}$$

The second order partial derivative of x can be computed as

$$X_{u,u}(u, v) = (-v \cos u, -v \sin u, 0) \tag{131}$$
$$X_{u,v}(u, v) = (-\sin u, \cos u, 0) \tag{132}$$
$$X_{v,v}(u, v) = \left(0, 0, \frac{-1}{v^2 \sqrt{1 - v^2}} \right) \tag{133}$$

Hence, the coefficients of the second fundamental form are

$$
\begin{aligned}
e &= \langle \nu, X_{u,u} \rangle && (134)\\
&= -v\sqrt{1-v^2}\cos^2 u - v\sqrt{1-v^2}\sin^2 u + 0 && (135)\\
&= -v\sqrt{1-v^2} && (136)\\
f &= \langle \nu, X_{u,v} \rangle && (137)\\
&= -\sqrt{1-v^2}\cos u \sin u + \sqrt{1-v^2}\sin u \cos u + 0 && (138)\\
&= 0 && (139)\\
g &= \langle \nu, X_{v,v} \rangle && (140)\\
&= 0 + 0 + \frac{v}{v^2\sqrt{1-v^2}} && (141)\\
&= \frac{1}{v\sqrt{1-v^2}} && (142)
\end{aligned}
$$

The Gaussian curvature can be computed as

$$
\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} && (143)\\
&= \frac{-v\sqrt{1-v^2}\frac{1}{v\sqrt{1-v^2}} - 0^2}{v^2\frac{1}{v^2} - 0^2} && (144)\\
&= \frac{-1}{1} && (145)\\
&= -1 && (146)
\end{aligned}
$$

Exercise 7 (p.169, [DG_Carmo_1]) (Surfaces of Revolution with Constant Curvature.)

$x(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$ is given as a surface of revolution with constant Gaussian curvature K. To determine the functions φ and ψ, choose the parameter v in such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that v is the arc length of the generating curve $(\varphi(v), \psi(v))$. Show that

(a) φ satisfies $\varphi'' + K\varphi = 0$ and ψ is given by $\psi = \int \sqrt{1-(\varphi')^2}dv$; thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

(b) All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane xOy are given by

$$\varphi(v) = C\cos v, \ \ \psi(v) = \int_0^v \sqrt{1 - C^2\sin^2 v}dv \tag{147}$$

where C is a constant ($C = \varphi(0)$). Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane for the cases $C = 1$, $C > 1$, $C < 1$. Observe that $C = 1$ gives a sphere.

(c) All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:

1.

$$\varphi(v) = C\cosh v \tag{148}$$
$$\psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 v}dv \tag{149}$$

2.

$$\varphi(v) = C\sinh v \tag{150}$$
$$\psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 v}dv \tag{151}$$

3.

$$\varphi(v) = e^v \tag{152}$$
$$\psi(v) = \int_0^v \sqrt{1 - e^{2v}}dv \tag{153}$$

Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane.

(d) The surface of type 3 in **(c)** is the pseudosphere of **Exercise 6**.

(e) The only surfaces of revolution with $K = 0$ are the right circular cylinder, the right circular cone, and the plane

Solution

(a) We have

$$x_u(u,v) = (-\varphi(v)\sin u, \varphi(v)\cos u, 0) \tag{154}$$
$$x_v(u,v) = (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v)) \tag{155}$$

Using these, we can compute the coefficient of the first fundamental form

$$E = \langle x_u, x_u\rangle \tag{156}$$
$$= \varphi^2(v)\sin^2 u + \varphi^2(v)\cos^2 u \tag{157}$$
$$= \varphi^2(v) \tag{158}$$
$$F = \langle x_u, x_v\rangle \tag{159}$$
$$= -\varphi(v)\varphi'(v)\sin u\cos u + \varphi(v)\varphi'(v)\cos u\sin u + 0 \tag{160}$$
$$= 0 \tag{161}$$
$$G = \langle x_v, x_v\rangle \tag{162}$$
$$= {\varphi'}^2(v)\sin^2 u + {\varphi'}^2(v)\cos^2 u + {\psi'}^2(v) \tag{163}$$
$$= {\varphi'}^2(v) + {\psi'}^2(v) \tag{164}$$
$$= 1 \tag{165}$$

The unit normal vector at $x(u,v)$ is define as

$$\nu(u,v) = N(x(u,v)) \tag{166}$$
$$= \frac{x_u \times x_v}{\|x_u \times x_v\|} \tag{167}$$
$$= \frac{(\varphi(v)\psi'(v)\cos u, \varphi(v)\psi'(v)\sin u, -\varphi(v)\varphi'(v))}{\sqrt{\varphi^2(v){\psi'}^2(v)\cos^2 u + \varphi^2(v){\psi'}^2(v)\sin^2 u + \varphi^2(v){\varphi'}^2(v)}} \tag{168}$$
$$= \frac{(\varphi(v)\psi'(v)\cos u, \varphi(v)\psi'(v)\sin u, -\varphi(v)\varphi'(v))}{\sqrt{\varphi^2(v)\left({\psi'}^2(v) + {\varphi'}^2(v)\right)}} \tag{169}$$
$$= \frac{(\varphi(v)\psi'(v)\cos u, \varphi(v)\psi'(v)\sin u, -\varphi(v)\varphi'(v))}{\sqrt{\varphi^2(v)\left({\psi'}^2(v) + {\varphi'}^2(v)\right)}} \tag{170}$$
$$= \frac{(\varphi(v)\psi'(v)\cos u, \varphi(v)\psi'(v)\sin u, -\varphi(v)\varphi'(v))}{\sqrt{\varphi^2(v)}} \tag{171}$$
$$= \frac{(\varphi(v)\psi'(v)\cos u, \varphi(v)\psi'(v)\sin u, -\varphi(v)\varphi'(v))}{|\varphi(v)|} \tag{172}$$
$$= \frac{\varphi(v)}{|\varphi(v)|}\,(\psi'(v)\cos u, \psi'(v)\sin u, -\varphi'(v)) \tag{173}$$
$$\tag{174}$$

The second order partial derivative of x can be computed as

$$x_{u,u}(u,v) = (-\varphi(v)\cos u, -\varphi(v)\sin u, 0) \tag{175}$$
$$x_{u,v}(u,v) = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0) \tag{176}$$
$$x_{v,v}(u,v) = (\varphi''(v)\cos u, \varphi''(v)\sin u, \psi''(v)) \tag{177}$$

Hence, the coefficients of the second fundamental form are

$$
\begin{align}
e &= \langle \nu, x_{u,u} \rangle \tag{178}\\
&= \frac{\varphi(v)}{|\varphi(v)|}\left[-\psi'(v)\varphi(v)\cos^2 u - \psi'(v)\varphi(v)\sin^2 u\right] \tag{179}\\
&= \frac{\varphi(v)}{|\varphi(v)|}\left[-\psi'(v)\varphi(v)\right] \tag{180}\\
&= -\frac{\varphi(v)^2\psi'(v)}{|\varphi(v)|} \tag{181}\\
&= -|\varphi(v)|\psi'(v) \tag{182}\\
f &= \langle \nu, x_{u,v} \rangle \tag{183}\\
&= \frac{\varphi(v)}{|\varphi(v)|}\left[-\psi'(v)\varphi'(v)\cos u \sin u + \psi'(v)\varphi'(v)\sin u \cos u\right] \tag{184}\\
&= \frac{\varphi(v)}{|\varphi(v)|}0 \tag{185}\\
&= 0 \tag{186}\\
g &= \langle \nu, x_{v,v} \rangle \tag{187}\\
&= \frac{\varphi(v)}{|\varphi(v)|}\left[\psi'(v)\varphi''(v)\cos^2 u + \psi'(v)\varphi''(v)\sin^2 u - \varphi'(v)\psi''(v)\right] \tag{188}\\
&= \frac{\varphi(v)}{|\varphi(v)|}\left[\psi'(v)\varphi''(v) - \varphi'(v)\psi''(v)\right] \tag{189}\\
& \tag{190}
\end{align}
$$

Similarly to **exercise 5**, the Gaussian curvature can be computed as

$$
\begin{align}
K &= \frac{eg - f^2}{EG - F^2} \tag{191}\\
&= \frac{(-|\varphi(v)|\psi'(v))\,\dfrac{\varphi(v)}{|\varphi(v)|}\left[\psi'(v)\varphi''(v) - \varphi'(v)\psi''(v)\right] - 0^2}{\varphi^2(v) - 0^2} \tag{192}\\
&= \frac{-\psi'(v)\varphi(v)\left[\psi'(v)\varphi''(v) - \varphi'(v)\psi''(v)\right]}{\varphi^2(v)} \tag{193}\\
&= \frac{-\psi'(v)\left[\psi'(v)\varphi''(v) - \varphi'(v)\psi''(v)\right]}{\varphi(v)} \tag{194}\\
&= \frac{-\psi'(v)\psi'(v)\varphi''(v) + \psi'(v)\varphi'(v)\psi''(v)}{\varphi(v)} \tag{195}\\
&= \frac{-\psi'^2(v)\varphi''(v) + \varphi'(v)\psi'(v)\psi''(v)}{\varphi(v)} \tag{196}
\end{align}
$$

Differentiating both sides of the equation $\varphi'^2 + \psi'^2 = 1$ with respect to v yields

$$
\begin{align}
\left(\varphi'(v)^2\right)' + \left(\psi'(v)^2\right)' &= (1)' \tag{197}\\
\Longleftrightarrow 2\varphi'(v)\varphi''(v) + 2\psi'(v)\psi''(v) &= 0 \tag{198}
\end{align}
$$

or

$$\psi'(v)\psi''(v) = -\varphi'(v)\varphi''(v) \tag{199}$$

Using this, we can rewrite (196) as

$$K = \frac{-\psi'^2(v)\varphi''(v) + \varphi'(v)\psi'(v)\psi''(v)}{\varphi(v)} \tag{200}$$

$$= \frac{-\psi'^2(v)\varphi''(v) - \varphi'(v)\varphi'(v)\varphi''(v)}{\varphi(v)} \tag{201}$$

$$= -\frac{\psi'^2(v)\varphi''(v) + \varphi'^2(v)\varphi''(v)}{\varphi(v)} \tag{202}$$

$$= -\varphi''(v)\frac{\psi'^2(v) + \varphi'^2(v)}{\varphi(v)} \tag{203}$$

$$= -\frac{\varphi''(v)}{\varphi(v)} \tag{204}$$

Rewriting (204), we obtain

$$\varphi''(v) = -K\varphi(v) \tag{205}$$

or

$$\varphi''(v) + K\varphi(v) = 0 \tag{206}$$

As for ψ, using the equation $\varphi'^2 + \psi'^2 = 1$ again, we obtain

$$\psi'^2 = 1 - \varphi'^2 \tag{207}$$

We can choose

$$\psi' = \sqrt{1 - \varphi'^2} \tag{208}$$

Using the fundamental theorem of calculus,

$$\psi(v) = \int_a^v \sqrt{1 - \varphi'^2(\bar{v})}d\bar{v} \tag{209}$$

where v is in the interval $[a, b]$ such that $\psi(v)$ makes sense. In other words, v is such that

$$0 \leq \varphi'^2(v) \leq 1 \tag{210}$$

(b) In **(a)**, we already have the result (206): $\varphi'' + K\varphi = 0$, if $K = 1$, we have

$$\varphi'' + \varphi = 0 \tag{211}$$

This is a homogeneous second-order linear differential equation. The characteristic equation of (211) is

$$k^2 + 1 = 0 \tag{212}$$

The solution of this characteristic equation are

$$\begin{cases} k_1 & = i \\ k_2 & = -i \end{cases} \tag{213}$$

Hence, (211) has the solution

$$\varphi(v) = e^{0v}\left(C_1 \cos v + C_2 \sin v\right) \tag{214}$$
$$= C_1 \cos v + C_2 \sin v \tag{215}$$

where C_1 and C_2 are arbitrary constants.

Differentiating φ with respect to v, we have

$$\varphi'(v) = \left(C_1 \cos v + C_2 \sin v\right)' \tag{216}$$
$$= -C_1 \sin v + C_2 \cos v \tag{217}$$

Recall that the generating curve $(\varphi(v), \psi(v))$ is in the xOz plane and the surface of revolution $x(u, v)$ is generated by rotating $(\varphi(v), \psi(v))$ around the z-axis. In order for the surface $x(u, v)$ to intersect perpendicularly the plane xOy, the generating curve must also intersects perpendicularly with the x-axis.

Assume that the generating curve $(\varphi(v), \psi(v))$ (which is on the xOz plane) intersects the x-axis at the point $(C, 0)$ with $C = \varphi(0)$. We must have

$$(\varphi(0), \psi(0)) = (C, 0) \tag{218}$$

Furthermore, if $(\varphi(v), \psi(v))$ intersects the x-axis perpendicularly, which means that the tangent vector of $(\varphi(v), \psi(v))$ at the intersection $(C, 0)$ is perpendicular to the x-axis, we must have

$$(\varphi'(0), \psi'(0)) \cdot (1, 0) = \varphi'(0) = 0 \tag{219}$$

From (218), (219), (215) and (217), we have the system of equations

$$\begin{cases} \varphi(0) & = C \\ \varphi'(0) & = 0 \end{cases} \tag{220}$$

$$\iff \begin{cases} C_1 \cos 0 + C_2 \sin 0 & = C \\ -C_1 \sin 0 + C_2 \cos 0 & = 0 \end{cases} \tag{221}$$

$$\iff \begin{cases} C_1 & = C \\ C_2 & = 0 \end{cases} \tag{222}$$

Therefore,

$$\varphi(v) = C\cos v \tag{223}$$

and

$$\varphi'(v) = -C\sin v \tag{224}$$

As for $\psi'(v)$, using $\psi = \int\sqrt{1-\varphi'^2}dv$ in **(a)** and the fact that $\psi(0) = 0$ in (218), we can choose

$$\psi(v) = \int_0^v \sqrt{1 - C^2\sin^2\bar{v}}d\bar{v} \tag{225}$$

In order for (225) to makes sense,

$$0 \leq C^2\sin^2 v \leq 1 \tag{226}$$

or

$$\sin^2 v \leq \frac{1}{C^2} \tag{227}$$

or

$$-\frac{1}{|C|} \leq \sin v \leq \frac{1}{|C|} \tag{228}$$

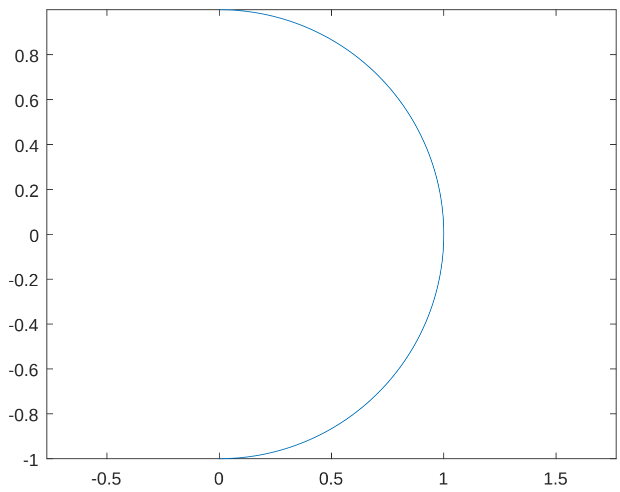

Figure 1: $C = 1$

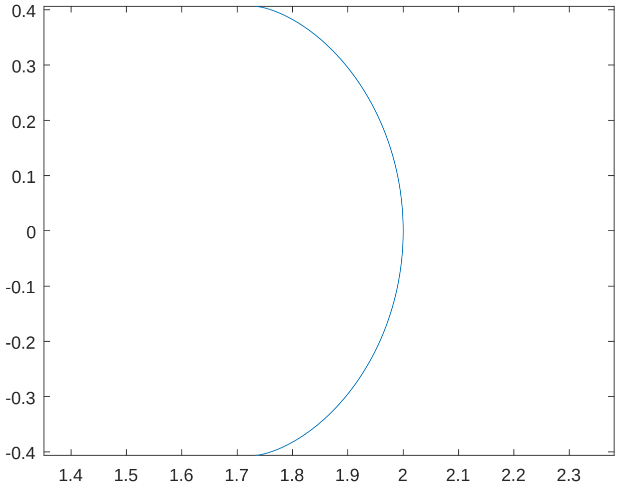

Figure 2: $C = 2$

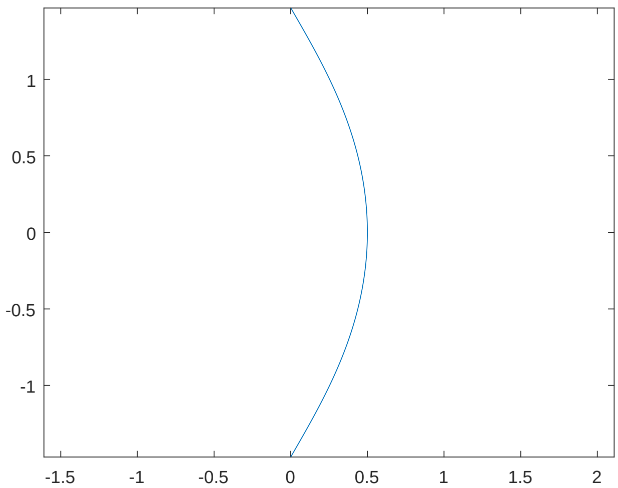

Figure 3: $C = \frac{1}{2}$

(c) Recall the definition of some hyperbolic cosine and Hyperbolic sine

$$\begin{cases} \cosh v &= \dfrac{e^v + e^{-v}}{2} \\ \sinh v &= \dfrac{e^v - e^{-v}}{2} \end{cases} \tag{229}$$

with the derivatives

$$\begin{cases} \cosh' v &= \dfrac{e^v - e^{-v}}{2} = \sinh v \\ \sinh' v &= \dfrac{e^v + e^{-v}}{2} = \cosh v \end{cases} \tag{230}$$

If $K = -1$, using the result in **(a)**, we have the equation

$$\varphi'' - \varphi = 0 \tag{231}$$

This is a homogeneous second-order linear differential equation. The characteristic equation of (231) is

$$k^2 - 1 = 0 \tag{232}$$

The solutions of this characteristic equation are

$$\begin{cases} k_1 &= 1 \\ k_2 &= -1 \end{cases} \tag{233}$$

Hence, (231) has the solution

$$\varphi(v) = C_1 e^v + C_2 e^{-v} \tag{234}$$

where C_1 and C_2 are arbitrary constants.

Differentiating φ with respect to v, we have

$$\varphi'(v) = \left(C_1 e^v + C_2 e^{-v}\right)' \tag{235}$$

$$= C_1 e^v - C_2 e^{-v} \tag{236}$$

Note that we have to specify the conditions for C_1, C_2 in order to determine the surfaces of revolution.

• **Case 1:** $C_1 = C_2 = \dfrac{C}{2}$

$$\varphi(v) = \frac{C}{2}(e^v + e^{-v}) \tag{237}$$

$$= C\frac{e^v + e^{-v}}{2} \tag{238}$$

$$= C\cosh v \tag{239}$$

$$\varphi'(v) = C\sinh v \tag{240}$$

$$\psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 \bar{v}}\, d\bar{v} \tag{241}$$

In order for (225) to makes sense,

$$0 \leq C^2 \sinh^2 v \leq 1 \tag{242}$$

or

$$\sinh^2 v \leq \frac{1}{C^2} \tag{243}$$

or

$$-\frac{1}{|C|} \leq \sinh v \leq \frac{1}{|C|} \tag{244}$$

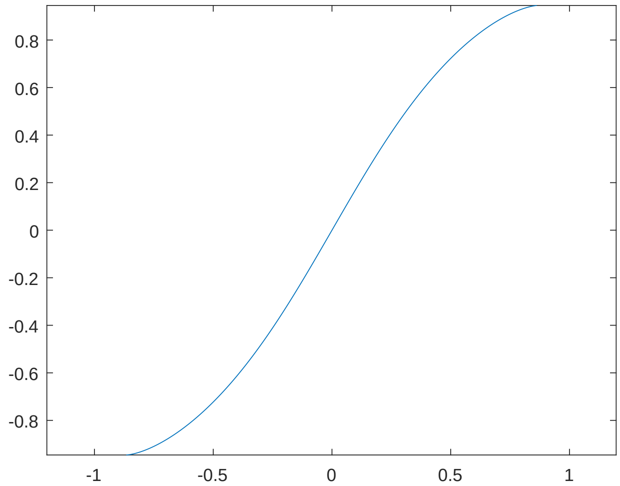

Figure 4: Case 1, $C = 1$

- **Case 2:** $C_1 = \dfrac{C}{2}$ and $C_2 = -C_1 = \dfrac{C}{2}$

$$\begin{align}
\varphi(v) &= \frac{C}{2}(e^v - e^{-v}) \tag{245}\\
&= C\frac{e^v - e^{-v}}{2} \tag{246}\\
&= C \sinh v \tag{247}\\
\varphi'(v) &= C \cosh v \tag{248}\\
\psi(v) &= \int_0^v \sqrt{1 - C^2 \cosh^2 \bar{v}}\, d\bar{v} \tag{249}
\end{align}$$

In order for (225) to makes sense,

$$0 \leq C^2 \cosh^2 v \leq 1 \tag{250}$$

or

$$\cosh^2 v \leq \frac{1}{C^2} \tag{251}$$

or

$$-\frac{1}{|C|} \leq \cosh v \leq \frac{1}{|C|} \tag{252}$$

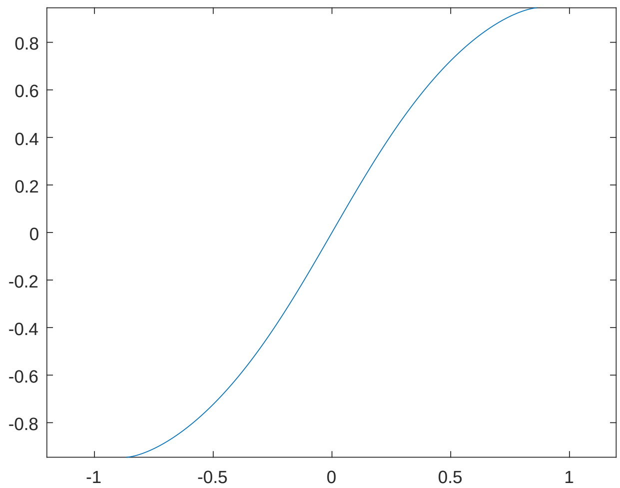

Figure 5: Case 2, $C = 0.5$

- **Case 3:** $C_1 = 1$ and $C_2 = 0$

$$\varphi(v) = e^v \tag{253}$$
$$\varphi'(v) = e^v \tag{254}$$
$$\psi(v) = \int_0^v \sqrt{1 - e^2 \bar{v}} d\bar{v} \tag{255}$$

In order for (225) to makes sense,

$$0 \leq e^{2v} \leq 1 \tag{256}$$

or

$$e^v \leq 1 \tag{257}$$

or

$$v \leq 0 \tag{258}$$

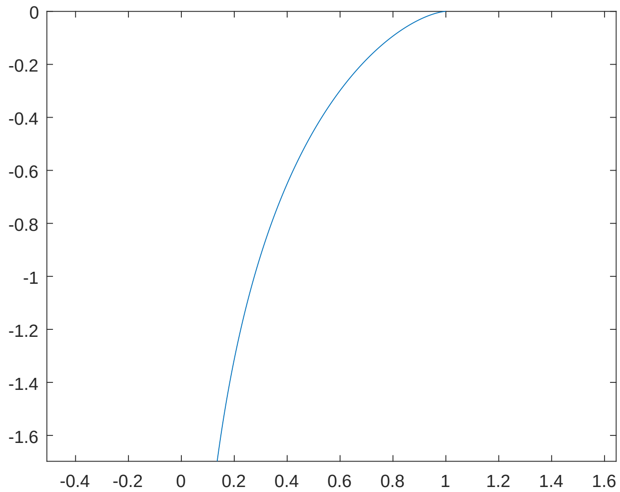

Figure 6: Case 3

(d) To check whether the surface of type 3 in **(c)** is the pseudosphere in excercise 6, we can check whether if the generating curve $(\varphi(v), \psi(v))$ in the plane xOz is the tractrix as defined in excercise 6.

We have

$$\varphi(v) = e^v \tag{259}$$

$$\psi(v) = \int_0^v \sqrt{1 - e^{2\bar{v}}} d\bar{v} \tag{260}$$

The tangent vector of the generating curve $(\varphi(v), \psi(v))$ is

$$(\varphi'(v), \psi'(v)) = \left(e^v, \sqrt{1 - e^{2v}}\right) \tag{261}$$

Pick a point $\left(e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w}\right)$ on the generating curve ($w \in (0, 1)$).

We are going to find the tangent line of the curve $(\varphi(v), \psi(v))$ at the point $\left(e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w}\right)$. We denote this tangent line as $T_w(x) = (x, ax + b)$ with a and b being unknown constants.

Since $T_w(x)$ go through $\left(e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w}\right)$, we have the equation

$$e^w a + b = \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} \tag{262}$$

We also know that the tangent vector at $\left(e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w}\right)$ is

$$(\varphi'(w), \psi'(w)) = \left(e^w, \sqrt{1 - e^{2w}}\right) \tag{263}$$

This implies that $T_w(x)$ must also go through the point

$$\left(e^w + e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} + \sqrt{1 - e^{2w}}\right) = \left(2e^w, \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} + \sqrt{1 - e^{2w}}\right) \tag{264}$$

Thus, we have an other equation

$$2e^w a + b = \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} \sqrt{1 - e^{2w}} \tag{265}$$

We have the linear system of equation

$$\begin{cases} e^w a + b & = \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} \\ 2e^w a + b & = \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} + \sqrt{1 - e^{2w}} \end{cases} \tag{266}$$

Solve this system for a and b, we obtain

$$\begin{cases} a & = \dfrac{\int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} - \left(\int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} + \sqrt{1 - e^2 w}\right)}{e^w - 2e^w} \\ b & = \dfrac{e^w \left(\int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} + \sqrt{1 - e^{2w}}\right) - 2e^w \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w}}{e^w - 2e^w} \end{cases} \tag{267}$$

$$\iff \begin{cases} a & = \dfrac{\sqrt{1 - e^2 w}}{e^w} \\ b & = \int_0^w \sqrt{1 - e^{2\bar{w}}} d\bar{w} - \sqrt{1 - e^{2w}} \end{cases} \tag{268}$$

Hence, we can find that the intersection of the tangent line of the generating curve at $\left(e^w, \int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w}\right)$ and the z-axis is

$$(0,b) = \left(0, \int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w} - \sqrt{1-e^{2w}}\right) \tag{269}$$

The distance between $\left(e^w, \int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w}\right)$ and $\left(0, \int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w} - \sqrt{1-e^{2w}}\right)$ is

$$\sqrt{(0-e^w)^2 + \left(\int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w} - \sqrt{1-e^{2w}} - \int_0^w \sqrt{1-e^{2\bar{w}}}d\bar{w}\right)^2} \tag{270}$$

$$=\sqrt{(e^w)^2 + \left(\sqrt{1-e^{2w}}\right)^2} \tag{271}$$

$$=\sqrt{e^{2w}+1-e^{2w}} \tag{272}$$

$$=1 \tag{273}$$

Therefore, the generating curve $(\varphi(v), \psi(v))$ is a tractrix and the surface of revolution it generates is a pseudosphere.

(e) If $K = 0$, using the result in **(a)**, we have the equation

$$\varphi'' = 0 \tag{274}$$

This is a homogeneous second-order linear differential equation. The characteristic equation of (274) is

$$k^2 = 0 \tag{275}$$

The solution of this characteristic equation is

$$\left\{k = 0\right. \tag{276}$$

Hence, (274) has the solution

$$\begin{aligned} \varphi(v) &= e^{0v}(C_1 + C_2 v) && (277) \\ &= C_1 + C_2 v && (278) \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

We also have

$$\begin{aligned} \varphi'(v) &= (C_1 + C_2 v)' && (279) \\ &= C_2 && (280) \end{aligned}$$

and

$$\begin{aligned} \psi(v) &= \int_0^v \sqrt{1 - \varphi'^2(\bar{v})} d\bar{v} && (281) \\ &= \int_0^v \sqrt{1 - C_2^2} d\bar{v} && (282) \\ &= v\sqrt{1 - C_2^2} && (283) \end{aligned}$$

where $-1 \le C_2 \le 1$

• **Case 1:** $C_2 = 1$ or $C_2 = -1$

In this case, the generating curve is

$$\begin{aligned} (\varphi(v), \psi(v)) &= \left(C_1 + C_2 v, v\sqrt{1 - C_2^2}\right) && (284) \\ &= (C_1 + C_2 v, 0) && (285) \end{aligned}$$

which is a line orthogonal to the z-axis.

Therefore, the surface of revolution in this case is a plane.

• **Case 2:** $C_2 = 0$

In this case, the generating curve is

$$\begin{aligned} (\varphi(v), \psi(v)) &= \left(C_1, v\sqrt{1 - 0^2}\right) && (286) \\ &= (C_1, v) && (287) \end{aligned}$$

which is a line orthogonal to the x-axis.

Therefore, the surface of revolution in this case is a right circular cylinder.

• **Case 3:** $C_1 \neq 0$ and $0 < |C_2| < 1$

In this case, the generating curve is

$$(\varphi(v), \psi(v)) = \left(C_1 + C_2 v, v\sqrt{1 - C_2^2}\right) \tag{288}$$

which is a line x-axis and it intersect the z-axis at $\left(0, -\frac{C_1\sqrt{1-C_2^2}}{C_2}\right)$.

Therefore, the surface of revolution in this case is a right circular cone.

Bibliography

[DG_Kunnel] Wolfgang Kunnel, *Differential Geometry: Curves, Surfaces, Manifolds*, 2nd edition, AMS, 2006.

[DG_Carmo_1] Manfredo P. Do Carmo, *Differential Geometry of Curves and Surfaces*, 1st edition, Prentice-Hall, 1976.