Programming Problem: nth Roots & Trigonometry in Triangles
Bài Tập Lập Trình: Căn Bậc n & Lượng Giác trong Tam Giác

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Mục lục

1 Root

1.1 Square Root

1.2 Cube Root

1.3 nth Root

Bài toán 1 (Root – Căn)**.**

2 Trigonometry in Right Triangles

"A *right triangle* (American English) or *right-angled triangle* (British English), or more formally an *orthogonal triangle*, formerly called a *rectangled triangle* is a triangle in which 1 angle is a right angle (i.e., a 90° angle), i.e., in which 2 sides are perpendicular. The relation between the sides & other angles of the right triangle is the basis for trigonometry."

"The side opposite to the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs* (or *catheti*, singular: cathetus)." – Wikipedia/right triangle

Given a right triangle ΔABC with $\widehat{A} = 90°$. Define $a \coloneqq BC$, $b \coloneqq CA$, $c \coloneqq AB$. Side b is the side *adjacent to angle C* & *opposed to angle B*, while side c may be identified as the side *adjacent to angle B* & opposed to (or *opposite*) angle C.

2.1 Pythagorean Triple

Definition 1 (Pythagorean triple)**.** *If the lengths of all 3 sides of a right triangle are integers, the triangle is said to be a* Pythagorean triangle *& its side lengths are collectively known as a* Pythagorean triple.

"A *Pythagorean triple* consists of 3 positive integers a, b, c, such that $a^2 = b^2 + c^2$. Such a triple is commonly written (b, c, a), & a well-known example is $(3, 4, 5)$. If (b, c, a) is a Pythagorean triple, then so is (kb, kc, ka) for any positive integer k. A *primitive Pythagorean triple* is one in which a, b, c are coprime (i.e., they have no common divisor larger than 1), e.g., $(3, 4, 5)$ is a primitive Pythagorean triple whereas $(6, 8, 10)$ is not. A triangle whose sides form a Pythagorean triple is called a *Pythagorean triangle*, & is necessarily a right triangle.

The name is derived from the Pythagorean theorem, stating that every right triangle has side lengths satisfying the formula $a^2 = b^2 + c^2$; thus, Pythagorean triples describe the 3 integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. E.g., the triangle with sides $(b, c, a) = (1, 1, \sqrt{2})$ is a right triangle, but $(1, 1, \sqrt{2})$

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is not a Pythagorean triple because $\sqrt{2}$ is not an integer.[1] Moreover, 1 & $\sqrt{2}$ do not have an integer common multiple because $\sqrt{2}$ is irrational."

"When searching for integer solutions, the equation $b^2 + c^2 = a^2$ is a Diophantine equation. Thus Pythagorean triples are among the oldest known solutions of a nonlinear Diophantine equation." – Wikipedia/Pythagorean triple

Problem 1 (Pythagorean triple)**.** *Write* Pascal, Python, C/C++ *programs to check if 3 integers a, b, c input from the keyboard: (a) form a Pythagorean triangle or not. (b) form a primitive Pythagorean triple or not. If not, find & print out their primitive Pythagorean triple.*

Problem 2 (List of primitive & non-primitive Pythagorean triples)**.** *Let N be an integer input from the keyboard. Write* Pascal, Python, C/C++ *programs to print out all: (a) primitive Pythagorean triples of numbers up to N. (b) Pythagorean triples of numbers up to N.*

Sample: "There are 16 primitive Pythagorean triples of numbers up to 100:
$(3,4,5)$, $(5,12,13)$, $(8,15,17)$, $(7,24,25)$, $(20,21,29)$, $(12,35,37)$, $(9,40,41)$, $(28,45,53)$, $(11,60,61)$, $(16,63,65)$, $(33,56,65)$, $(48,55,73)$, $(13,84,85)$, $(36,77,85)$, $(39,80,89)$, $(65,72,97)$.

Other small Pythagorean triples such as $(6,8,10)$ are not listed because they are not primitive; for instance $(6,8,10)$ is a multiple of $(3,4,5)$." [...] "Additionally, these are the remaining primitive Pythagorean triples of numbers up to 300:
$(20,99,101)$, $(60,91,109)$, $(15,112,113)$, $(44,117,125)$, $(88,105,137)$, $(17,144,145)$, $(24,143,145)$, $(51,140,149)$, $(85,132,157)$, $(119,120,169)$, $(52,165,173)$, $(19,180,181)$, $(57,176,185)$, $(104,153,185)$, $(95,168,193)$, $(28,195,197)$, $(84,187,205)$, $(133,156,205)$, $(21,220,221)$, $(140,171,221)$, $(60,221,229)$, $(105,208,233)$, $(120,209,241)$, $(32,255,257)$, $(23,264,265)$, $(96,247,265)$, $(69,260,269)$, $(115,252,277)$, $(160,231,281)$, $(161,240,289)$, $(68,285,293)$." – Wikipedia/Pythagorean triple/examples

"*Euclid's formula* is a fundamental formula for generating Pythagorean triples given an arbitrary pair of integers m, n with $m > n > 0$. The formula states that the integers

$$\boxed{b = m^2 - n^2,\ c = 2mn,\ a = m^2 + n^2,\ \text{where } m, n \in \mathbb{N}^\star,\ m > n,} \tag{1}$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive iff m, n are coprime & 1 of them is even. When both m, n are odd, then a, b, c will be even, & the triple will not be primitive; however, dividing a, b, c by 2 will yield a primitive triple when m, n are coprime.

Every primitive triple arises (after the exchange of b & c, if b is even) from a *unique pair* of coprime numbers m, n, one of which is even. It follows that there are infinitely many primitive Pythagorean triples." [...] "Despite generating all primitive triples, Euclid's formula does not produce all triples, e.g., $(9,12,15)$ cannot be generated using integer m, n. This can be remedied by inserting an additional parameter k to the formula. The following will generate all Pythagorean triples uniquely:

$$\boxed{b = k(m^2 - n^2),\ c = 2kmn,\ a = k(m^2 + n^2),\ \text{where } m, n, k \in \mathbb{N}^\star,\ m > n,\ \gcd(m, n) = 1,\ mn \vdots 2.} \tag{2}$$

These formulas generate Pythagorean triples can be verified by expanding $b^2 + c^2$ using elementary algebra & verifying that the result equals c^2. Since every Pythagorean triple can be divided through by some integer k to obtain a primitive triple, every triple can be generated uniquely by using the formula with m, n to generate its primitive counterpart & then multiplying through by k as in the last equation (2).

Choosing m, n from certain integer sequences gives interesting results, e.g., if m, n are consecutive Pell numbers, a, b will differ by 1. Many formulas for generating triples with particular properties have been developed since the time of Euclid." – Wikipedia/Pythagorean triple/generating a triple

Problem 3. *(a) Prove that (a, b, c) given by either formulas (1) or (2) is a Pythagorean triple. (b) Compute* $\sin, \cos, \tan, \cot$ *of angles B, C in terms of m, n, k.*

See Wikipedia/formulas for generating Pythagorean triples/proof of Euclid's formula for a (mathematically rigorous) proof. & Wikipedia/formulas for generating Pythagorean triples/interpretation of parameters in Euclid's formula.

A variant of Euclid's formula for Pythagorean triples. The following variant of Euclid's formula is sometimes more convenient, as being more symmetric in m, n (same parity condition on m, n). Prove that

$$\boxed{b = mn,\ c = \frac{m^2 - n^2}{2},\ a = \frac{m^2 + n^2}{2},\ \text{where } m, n, k \in \mathbb{N}^\star,\ m > n,\ \gcd(m, n) = 1,\ mn \not\vdots 2.} \tag{3}$$

are 3 integers that form a Pythagorean triple, which is primitive iff m, n are coprime. Conversely, every primitively Pythagorean triple arises (after the exchange of b, c, if b is even) from a unique pair $m > n > 0$ of coprime odd integers. □

Problem 4 (List of primitive & non-primitive Pythagorean triples)**.** *Let n be an integer input from the keyboard. Use Euclid's formulas (1), (2), & (3) for generating Pythagorean triples, write* Pascal, Python, C/C++ *programs to print out all: (a) primitive Pythagorean triples of numbers up to N. (b) Pythagorean triples of numbers up to N.*

See also, Wikipedia/formulas for generating Pythagorean triples

[1] $\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$, i.e., $\sqrt{2}$ is an irrational number (i.e., a real number which is not a rational number).

2.2 Solve Right Triangle

Bài toán 2 (Solve right triangle – Giải tam giác vuông)**.**

3 Trigonometry in Triangles

Tổng quát hơn cho tam giác (không suy biến) bất kỳ (i.e., tam giác nhọn, vuông, tù).

3.1 Solve Triangle

Bài toán 3 (Solve triangle – Giải tam giác)**.**