# Programming Problem: $n$th Roots & Trigonometry in Triangles
# Bài Tập Lập Trình: Căn Bậc $n$ & Lượng Giác trong Tam Giác

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# 1 Root

## 1.1 Square Root

## 1.2 Cube Root

## 1.3 $n$th Root

**Bài toán 1** (Root – Căn)**.**

# 2 Trigonometry in Right Triangles

"A *right triangle* (American English) or *right-angled triangle* (British English), or more formally an *orthogonal triangle*, formerly called a *rectangled triangle* is a triangle in which 1 angle is a right angle (i.e., a 90° angle), i.e., in which 2 sides are perpendicular. The relation between the sides & other angles of the right triangle is the basis for trigonometry."

"The side opposite to the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs* (or *catheti*, singular: cathetus)." – Wikipedia/right triangle

Given a right triangle $\Delta ABC$ with $\widehat{A} = 90^\circ$. Define $a \coloneqq BC$, $b \coloneqq CA$, $c \coloneqq AB$. Side $b$ is the side *adjacent to angle $C$* & *opposed to angle $B$*, while side $c$ may be identified as the side *adjacent to angle $B$* & opposed to (or *opposite*) angle $C$.

## 2.1 Pythagorean Triple

**Definition 1** (Pythagorean triple)**.** *If the lengths of all 3 sides of a right triangle are integers, the triangle is said to be a* Pythagorean triangle *& its side lengths are collectively known as a* Pythagorean triple*.*

"A *Pythagorean triple* consists of 3 positive integers $a, b, c$, such that $a^2 = b^2 + c^2$. Such a triple is commonly written $(b, c, a)$, & a well-known example is $(3, 4, 5)$. If $(b, c, a)$ is a Pythagorean triple, then so is $(kb, kc, ka)$ for any positive integer $k$. A *primitive Pythagorean triple* is one in which $a, b, c$ are coprime (i.e., they have no common divisor larger than 1), e.g., $(3, 4, 5)$ is a primitive Pythagorean triple whereas $(6, 8, 10)$ is not. A triangle whose sides form a Pythagorean triple is called a *Pythagorean triangle*, & is necessarily a right triangle.

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The name is derived from the Pythagorean theorem, stating that every right triangle has side lengths satisfying the formula $a^2 = b^2 + c^2$; thus, Pythagorean triples describe the 3 integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. E.g., the triangle with sides $(b, c, a) = (1, 1, \sqrt{2})$ is a right triangle, but $(1, 1, \sqrt{2})$ is not a Pythagorean triple because $\sqrt{2}$ is not an integer.[1] Moreover, 1 & $\sqrt{2}$ do not have an integer common multiple because $\sqrt{2}$ is irrational."

"When searching for integer solutions, the equation $b^2 + c^2 = a^2$ is a Diophantine equation. Thus Pythagorean triples are among the oldest known solutions of a nonlinear Diophantine equation." – Wikipedia/Pythagorean triple

**Problem 1** (Pythagorean triple)**.** *Write* Pascal, Python, C/C++ *programs to check if 3 integers $a, b, c$ input from the keyboard: (a) form a Pythagorean triangle or not. (b) form a primitive Pythagorean triple or not. If not, find & print out their primitive Pythagorean triple.*

**Problem 2** (List of primitive & non-primitive Pythagorean triples)**.** *Let $N$ be an integer input from the keyboard. Write* Pascal, Python, C/C++ *programs to print out all: (a) primitive Pythagorean triples of numbers up to $N$. (b) Pythagorean triples of numbers up to $N$.*

Sample: "There are 16 primitive Pythagorean triples of numbers up to 100:
$(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $(7, 24, 25)$, $(20, 21, 29)$, $(12, 35, 37)$, $(9, 40, 41)$, $(28, 45, 53)$, $(11, 60, 61)$, $(16, 63, 65)$, $(33, 56, 65)$, $(48, 55, 73)$, $(13, 84, 85)$, $(36, 77, 85)$, $(39, 80, 89)$, $(65, 72, 97)$.

Other small Pythagorean triples such as $(6, 8, 10)$ are not listed because they are not primitive; for instance $(6, 8, 10)$ is a multiple of $(3, 4, 5)$." [...] "Additionally, these are the remaining primitive Pythagorean triples of numbers up to 300:
$(20, 99, 101)$, $(60, 91, 109)$, $(15, 112, 113)$, $(44, 117, 125)$, $(88, 105, 137)$, $(17, 144, 145)$, $(24, 143, 145)$, $(51, 140, 149)$, $(85, 132, 157)$, $(119, 120, 169)$, $(52, 165, 173)$, $(19, 180, 181)$, $(57, 176, 185)$, $(104, 153, 185)$, $(95, 168, 193)$, $(28, 195, 197)$, $(84, 187, 205)$, $(133, 156, 205)$, $(21, 220, 221)$, $(140, 171, 221)$, $(60, 221, 229)$, $(105, 208, 233)$, $(120, 209, 241)$, $(32, 255, 257)$, $(23, 264, 265)$, $(96, 247, 265)$, $(69, 260, 269)$, $(115, 252, 277)$, $(160, 231, 281)$, $(161, 240, 289)$, $(68, 285, 293)$." – Wikipedia/Pythagorean triple/examples

"*Euclid's formula* is a fundamental formula for generating Pythagorean triples given an arbitrary pair of integers $m, n$ with $m > n > 0$. The formula states that the integers

$$\boxed{b = m^2 - n^2,\ c = 2mn,\ a = m^2 + n^2,\ \text{where } m, n \in \mathbb{N}^\star,\ m > n,} \tag{1}$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive iff $m, n$ are coprime & 1 of them is even. When both $m, n$ are odd, then $a, b, c$ will be even, & the triple will not be primitive; however, dividing $a, b, c$ by 2 will yield a primitive triple when $m, n$ are coprime.

*Every* primitive triple arises (after the exchange of $b$ & $c$, if $b$ is even) from a *unique pair* of coprime numbers $m, n$, one of which is even. It follows that there are infinitely many primitive Pythagorean triples." [...] "Despite generating all primitive triples, Euclid's formula does not produce all triples, e.g., $(9, 12, 15)$ cannot be generated using integer $m, n$. This can be remedied by inserting an additional parameter $k$ to the formula. The following will generate all Pythagorean triples uniquely:

$$\boxed{b = k(m^2 - n^2),\ c = 2kmn,\ a = k(m^2 + n^2),\ \text{where } m, n, k \in \mathbb{N}^\star,\ m > n,\ \gcd(m, n) = 1,\ mn \,\vdots\, 2.} \tag{2}$$

These formulas generate Pythagorean triples can be verified by expanding $b^2 + c^2$ using elementary algebra & verifying that the result equals $c^2$. Since every Pythagorean triple can be divided through by some integer $k$ to obtain a primitive triple, every triple can be generated uniquely by using the formula with $m, n$ to generate its primitive counterpart & then multiplying through by $k$ as in the last equation (2).

Choosing $m, n$ from certain integer sequences gives interesting results, e.g., if $m, n$ are consecutive Pell numbers, $a, b$ will differ by 1. Many formulas for generating triples with particular properties have been developed since the time of Euclid." – Wikipedia/Pythagorean triple/generating a triple

**Problem 3.** *(a) Prove that $(a, b, c)$ given by either formulas (1) or (2) is a Pythagorean triple. (b) Compute* $\sin, \cos, \tan, \cot$ *of angles $B, C$ in terms of $m, n, k$.*

See Wikipedia/formulas for generating Pythagorean triples/proof of Euclid's formula for a (mathematically rigorous) proof. & Wikipedia/formulas for generating Pythagorean triples/interpretation of parameters in Euclid's formula.

*A variant of Euclid's formula for Pythagorean triples.* The following variant of Euclid's formula is sometimes more convenient, as being more symmetric in $m, n$ (same parity condition on $m, n$). Prove that

$$\boxed{b = mn,\ c = \frac{m^2 - n^2}{2},\ a = \frac{m^2 + n^2}{2},\ \text{where } m, n, k \in \mathbb{N}^\star,\ m > n,\ \gcd(m, n) = 1,\ mn \not\vdots\, 2.} \tag{3}$$

are 3 integers that form a Pythagorean triple, which is primitive iff $m, n$ are coprime. Conversely, every primitively Pythagorean triple arises (after the exchange of $b, c$, if $b$ is even) from a unique pair $m > n > 0$ of coprime odd integers. □

**Problem 4** (List of primitive & non-primitive Pythagorean triples)**.** *Let $n$ be an integer input from the keyboard. Use Euclid's formulas (1), (2), & (3) for generating Pythagorean triples, write* Pascal, Python, C/C++ *programs to print out all: (a) primitive Pythagorean triples of numbers up to $N$. (b) Pythagorean triples of numbers up to $N$.*

See also, Wikipedia/formulas for generating Pythagorean triples.

[1] $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, i.e., $\sqrt{2}$ is an irrational number (i.e., a real number which is not a rational number).

## 2.2 Principal Properties of Right Triangles

### 2.2.1 Sides – Cạnh

"The 3 sides of a right triangle are related by the Pythagorean theorem, which in modern algebraic notation can be written $b^2 + c^2 = a^2$, where $a$ is the length of the *hypotenuse* (side opposite the right angle), & $a, b$ are the lengths of the *legs* (remaining 2 sides). Pythagorean triples are integer values of $a, b, c$ satisfying this equation. This theorem was proven in antiquity, and is proposition I.47 in Euclid's *Elements*: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle."

### 2.2.2 Area – Diện Tích

"As with any triangle, the area is equal to one half the base multiplied by the corresponding height. In a right triangle, if 1 leg is taken as the base then the other is height, so the area of a right triangle is one half the product of the 2 legs. As a formula, the area $S$ is $S = \frac{1}{2}bc$, where $b, c$ are the legs of the triangle.

If the incircle is tangent to the hypotenuse $BC$ at point $P$, then denoting the semi-perimeter $\frac{a+b+c}{2}$ as $p$, we have $PB = p - b$, $PC = p - c$, & the area is given by $S = PB \cdot PC = (p-b)(p-c)$. This formula only applies to right triangles."

**Problem 5.** *Prove that the formula holds for any right triangle $\Delta ABC$ with the right angle $A$: $S = PB \cdot PC = (p-b)(p-c)$ where $p \coloneqq \frac{a+b+c}{2}$ is its semi-perimeter.*

### 2.2.3 Altitudes – Đường Cao

## 2.3 Solve Right Triangle

**Bài toán 2** (Solve right triangle – Giải tam giác vuông)**.**

# 3 Trigonometry in Triangles

Tổng quát hơn cho tam giác (không suy biến) bất kỳ (i.e., tam giác nhọn, vuông, tù).

## 3.1 Solve Triangle

**Bài toán 3** (Solve triangle – Giải tam giác)**.**