A Survey on Navier–Stokes Equations

Nguyen Quan Ba Hong*

February 25, 2022

Abstract

A personal survey on Navier–Stokes equations (NSEs), especially its regularity and turbulence models.

Keywords. Navier–Stokes equations.

Contents

Quick notes

1. The 4 formulations appearing in the *Clay Millennium Prize* formulation Fefferman, 2006 of NSEs.

1 Incompressible NSEs

1.1 Various concepts of solutions to NSEs

To describe various formulations for NSEs, we must first define properly the concept of a solution to NSEs, including, e.g., *periodic solutions, finite energy solutions, H^1 solutions*, and *smooth solutions*, etc.

1.1.1 Smooth solutions of NSEs

> "Note that even within the category of smooth solutions, there is some choice in what decay hypotheses to place on the initial data and solution; for instance, one can require that the initial velocity $\mathbf{u}_0$ be Schwartz class, or merely smooth with finite energy. Intermediate between these two will be data which is smooth and in H^1."
> – Tao, 2013

Recall Tao, 2013, Def. 1.1:

Definition 1.1 (Smooth solutions to NSEs)**.** *A* smooth set of data *for NSEs up to time T is a triplet $(\mathbf{u}_0, \mathbf{f}, T)$, where $0 < T < \infty$ is a time, the initial velocity vector field $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and the forcing term $\mathbf{f} : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are assumed to be smooth on $\mathbb{R}^3$ and $[0, T] \times \mathbb{R}^3$, respectively, (thus, $\mathbf{u}_0$ is infinitely differentiable in space, and $\mathbf{f}$ is infinitely differentiable in spacetime), and $\mathbf{u}_0$ is furthermore required to be divergence-free:*

$$\nabla \cdot \mathbf{u}_0 = 0, \ \textit{in } \mathbb{R}^3. \tag{1.1}$$

If $\mathbf{f} = \mathbf{0}$, we say that the data is homogeneous.

The total energy *$E(\mathbf{u}_0, \mathbf{f}, T)$ of a smooth set of data $(\mathbf{u}_0, \mathbf{f}, T)$ is defined by the quantity*

$$E(\mathbf{u}_0, \mathbf{f}, T) \coloneqq \frac{1}{2} \left(\|\mathbf{u}_0\|_{L^2_{\mathbf{x}}(\mathbb{R}^3)} + \|\mathbf{f}\|_{L^1_t L^2_{\mathbf{x}}([0,T] \times \mathbb{R}^3)} \right)^2, \tag{iiNS/E}$$

*Independent Researcher, Ben Tre City, Vietnam.
Email. nguyenquanbahong@gmail.com.

and $(\mathbf{u}_0, \mathbf{f}, T)$ *is said to have* finite energy *if* $E(\mathbf{u}_0, \mathbf{f}, T) < \infty$. *We define the* H^1 norm $\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T)$ *of the data to be the quantity*

$$\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T) \coloneqq \|\mathbf{u}_0\|_{H_x^1(\mathbb{R}^3)} + \|\mathbf{f}\|_{L_t^\infty H_x^1(\mathbb{R}^3)} < \infty, \tag{iiNS/$\mathcal{H}^1$}$$

and say that $(\mathbf{u}_0, \mathbf{f}, T)$ is H^1 *if* $\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T) < \infty$; *note that the* H^1 *regularity is essentially 1 derivative higher than the energy regularity, which is at the level of* L^2, *and instead matches the regularity of the* initial enstrophy $\frac{1}{2}\int_{\mathbb{R}^3} \|\boldsymbol{\omega}_0(t, \mathbf{x})\|^2 \, \mathrm{d}\mathbf{x}$, *where* $\omega_0 \coloneqq \nabla \times \mathbf{u}_0$ *is the* initial vorticity. *We say that a smooth set of data* $(\mathbf{u}_0, \mathbf{f}, T)$ *is* Schwartz *if, for all integers* $\alpha, m, k \ge 0$, *one has*

$$\sup_{\mathbf{x}\in\mathbb{R}^3} (1 + \|\mathbf{x}\|)^k \|\nabla_{\mathbf{x}}^\alpha \mathbf{u}_0(\mathbf{x})\| < \infty \text{ and } \sup_{(t,\mathbf{x})\in[0,T]\times\mathbb{R}^3} (1 + \|\mathbf{x}\|)^k \|\nabla_{\mathbf{x}}^\alpha \partial_t^m \mathbf{f}(\mathbf{x})\| < \infty. \tag{1.2}$$

Thus, e.g., the Schwartz property implies H^1, *which in turn implies finite energy. We also say that* $(\mathbf{u}_0, \mathbf{f}, T)$ *is* periodic *with some period* $L > 0$ *if one has* $\mathbf{u}_0(\mathbf{x} + L\mathbf{k}) = \mathbf{u}_0(\mathbf{x})$ *and* $\mathbf{f}(t, \mathbf{x} + L\mathbf{k}) = \mathbf{f}(t, \mathbf{x})$ *for all* $t \in [0, T]$, $\mathbf{x} \in \mathbb{R}^3$, *and* $\mathbf{k} \in \mathbb{Z}^3$. *Of course, periodicity is incompatible with the Schwartz,* H^1, *or finite energy properties, unless the data is zero. To emphasize the periodicity, we will sometimes write a periodic set of data* $(\mathbf{u}_0, \mathbf{f}, T)$ *as* $(\mathbf{u}_0, \mathbf{f}, T, L)$.

A smooth solution to the NSEs, *or a* smooth solution, *is a quintuplet* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$, *where* $(\mathbf{u}_0, \mathbf{f}, T)$ *is a smooth set of data, and the velocity vector field* $\mathbf{u} : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ *and pressure field* $p : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ *are smooth functions on* $[0, T] \times \mathbb{R}^3$ *that obey the NSE:*[1]

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p + \mathbf{f}, \tag{1.3}$$

and the incompressibility property

$$\nabla \cdot \mathbf{u} = 0, \tag{1.4}$$

on all of $[0, T] \times \mathbb{R}^3$[2], *and also the initial condition*

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}^3. \tag{1.5}$$

We say that a smooth solution $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *has* finite energy *if the associated data* $(\mathbf{u}_0, \mathbf{f}, t)$ *has finite energy, and in addition one has*

$$\|\mathbf{u}\|_{L_t^\infty L_{\mathbf{x}}^2([0,T]\times\mathbb{R}^3)} < \infty. \tag{1.6}$$

2 Compressible NSEs

References

[TT's blog] Terence Tao's blog/Navier–Stokes equations.

- Terence Tao. *Localisation and compactness properties of the Navier-Stokes global regularity problem*. Aug 4, 2011.

[Wikipedia] Wikipedia.org

- Wikipedia/viscosity.

References

Fefferman, Charles L. (2006). "Existence and smoothness of the Navier-Stokes equation". In: *The millennium prize problems.* Clay Math. Inst., Cambridge, MA, pp. 57–67.

Tao, Terence (2013). "Localisation and compactness properties of the Navier-Stokes global regularity problem". In: *Anal. PDE* 6.1, pp. 25–107. ISSN: 2157-5045. DOI: `10.2140/apde.2013.6.25`. URL: `https://doi.org/10.2140/apde.2013.6.25`.

[1] NQBH: Why no viscosity ν? Any major differences in their mathematical analysis, especially the case $\nu = \nu(t, \mathbf{x}, \mathbf{u}, p)$ in turbulence models?

[2] NQBH: NSEs on the whole domain, hence useless for shape and topology optimizations, but useful for applying harmonic and Fourier analysis.