Elementary Mathematics

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Chapter 1

Wikipedia's

1.1 Wikipedia/How to Solve It

"*How to Solve It* (1945) is a small volume by mathematician George Pólya describing methods of problem solving." – Wikipedia/how to solve it

1.1.1 4 principles

"*How to Solve It* suggests the following steps when solving a mathematical problem:

1. 1st, you have to *understand the problem.*
2. After understanding, *make a plan.*
3. *Carry out the plan.*
4. *Look back* on your work. How could it be better?

If this technique fails, Pólya advises: "If you can't solve a problem, then there is an easier problem you can solve: find it." Or: "If you cannot solve the proposed problem, try to solve 1st some related problem. Could you imagine a more accessible related problem?"" – Wikipedia/how to solve it/4 principles

1.1.1.1 1st principle: Understand the problem

"Understanding the problem" is often neglected as being obvious & is not even mentioned in many mathematics classes. Yet students are often stymied in their efforts to solve it, simply because they don't understand it fully, or even in part. In order to remedy this oversight, Pólya taught teachers how to prompt each student with appropriate questions, depending on the situation, such as:

- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture of a diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?
- Do you understand all the words used in stating the problem?
- Do you need to ask a question to get the answer?

The teacher is to select the question with the appropriate level of difficulty for each student to ascertain if each student understands at their own level, moving up or down the list to prompt each student, until each one can respond with something constructive." – Wikipedia/how to solve it/4 principles/1st principle: understand the problem

1.1.1.2 2nd principle: Devise a plan

"Pólya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. You will find choosing a strategy increasingly easy. A partial list of strategies is included:

- Guess & check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation

Also suggested:

- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Use a model
- Work backward
- Use a formula
- Be creative
- Applying these rules to devise a plan takes your own skill & judgment.

Pólya lays a big emphasis on the teachers' behavior. A teacher should support students with devising their own plan with a question method that goes from the most general questions to more particular questions, with the goal that the last step to having a plan is made by the student. He maintains that just showing students a plan, no matter how good it is, does not help them." – Wikipedia/how to solve it/4 principles/2nd principle: devise a plan

1.1.1.3 3rd principle: Carry out the plan

"This step is usually easier than devising the plan. In general, all you need is care & patience, given that you have the necessary skills. Persist with the plan that you have chosen. If it continues not to work, discard it & choose another. Don't be misled; this is how mathematics is done, even by professionals." – Wikipedia/how to solve it/4 principles/3rd principle: carry out the plan

1.1.1.4 4th principle: Review/extend

"Pólya mentions that much can be gained by taking the time to reflect & look back at what you have done, what worked & what did not, & with thinking about other problems where this could be useful. Doing this will enable you to predict what strategy to use to solve future problems, if these relate to the original problem." – Wikipedia/how to solve it/4 principles/4th principle: review/extend

1.1.2 Heuristics

"The book contains a dictionary-style set of heuristics, many of which have to do with generating a more accessible problem. E.g.:

Heuristic | **Informal Description** | **Formal analogue**

- Analogy | Can you find a problem analogous to your problem & solve that? | map
- Auxiliary Elements | Can you add some new element to your problem to get closer to a solution? | Extension

- Generalization | Can you find a problem more general than your problem? | Generalization
- Induction | Can you solve your problem by deriving a generalization from some examples? | Induction
- Variation of the Problem | Can you vary or change your problem to create a new problem (or set of problems) whose solution(s) will help you solve your original problem? | Search
- Auxiliary Problem | Can you find a subproblem or side problem whose solution will help you solve your problem? | Subgoal
- Here is a problem related to yours & solved before | Can you find a problem related to yours that has already been solved & use that to solve your problem? | Pattern recognization, Pattern matching, Reduction
- Specialization | Can you find a problem more specialized? | Specialization
- Decomposing & Recombining | Can you decompose the problem & "recombine its elements in some new manner"? | Divide & conquer
- Working backward | Can you start with the goal & work backwards to something you already know? | Backward chaining
- Draw a Figure | Can you draw a picture of the problem? | Diagrammatic Reasoning

" – Wikipedia/how to solve it/heuristics

1.1.3 Influence

- "The book has been translated into several languages & has sold over a million copies, & has been continuously in print since its 1st publication.
- Marvin Minsky said in his paper *Steps Toward Artificial Intelligence* that "everyone should know the work of George Pólya on how to solve problems."
- Pólya's book has had a large influence on mathematics textbooks as evidenced by the bibliographies for mathematics education.
- Russian inventor Genrich Altshuller developed an elaborate set of methods for problem solving known as TRIZ, which in many aspects reproduces or parallels Pólya's work.
- How to Solve it by Computer is a computer science book by R. G. Dromey. It was inspired by Pólya's work." – Wikipedia/how to solve it/influence

Chapter 2

Polya, 2014. How to Solve It: A New Aspect of Mathematical Methods

From the Preface to the 1st Printing

"A great discovery solves a great problem but there is a grain[1] of discovery in the solution of any problem. Your problem may be modest[2]; but it challenges your curiosity[3] & brings into play your inventive[4] faculties[5], & if you solve it by your own means, you may experience the tension[6] & enjoy the triumph[7] of discovery. Such experiences at a susceptible[8] age may create a taste for mental work & leave their imprint[9] on mind & character for a lifetime[10].

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted[11] time with drilling his students in routine operations he kills their interest, hampers[12] their intellectual development, & misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate[13] to their knowledge, & helps them to solve their problems with stimulating[14] questions, he may give them a taste for, & some means of, independent thinking.

Also a student whose college curriculum[15] includes some mathematics has a singular[16] opportunity. This opportunity is

[1] **grain** [n] **1.** [uncountable, countable] the small hard seeds of food plants such as wheat, rice, etc.; a single seed of such a plant; **2.** [countable] **grain (of something)** a small piece of a particular substance; usually a hard substance; **3.** [countable, usually singular] **grain of something** a very small amount; **4.** [countable] an individual particle or crystal in metal, rock, etc., usually explained with a lens or microscope.

[2] **modest** [a] **1.** fairly limited or small in amount; **2.** not expensive, rich or impressive; **3.** (of people, especially women, or their clothes) not showing too much of the body; not intended to attract attention, especially in a sexual way; **4.** (*approving*) not talking much about your own abilities or possessions.

[3] **curiosity** [n] (plural **curiosities**) **1.** [uncountable, singular] a strong desire to know about something; **2.** [countable] **curiosity (of something)** an unusual & interesting thing.

[4] **inventive** [a] **1.** (especially of people) able to create or design new things or think of new ideas; **2.** (of ideas) new & interesting.

[5] **faculty** [n] (plural **faculties**) **1.** [countable] a physical or mental ability, especially one that people are born with; **2.** [countable] **faculty (of something)** a department or group of related departments in a college or university; **3.** [countable + singular or plural verb] all the teachers in a faculty of a college or university; **4.** [countable, uncountable] (*NAE*) all the teachers of a particular university or college.

[6] **tension** [n] **1.** [uncountable, countable, usually plural] a situation in which people do not trust each other, or feel unfriendly towards each other, & which may cause them to attack each other; **2.** [countable, uncountable] **tension (between A & B)** a situation in which the fact that there are different needs or interests causes difficulties; **3.** [uncountable] a feeling of anxiety & stress that makes it impossible to relax; **4.** [uncountable] the feeling of fear & excitement that is created by a writer or a film director; **5.** [uncountable] the state of being stretched tight; the extent to which something is stretched tight.

[7] **triumph** [n] **1.** [countable, uncountable] a great success, achievement or victory; **2.** [uncountable] the state of having achieved a great success or victory; the feeling of happiness that you get from this; [v] [intransitive] to defeat somebody/something; to be successful.

[8] **susceptible** [a] **1.** [not usually before noun] **susceptible (to somebody/something)** very likely to be influenced, harmed or affected by somebody/something; **2. susceptible (of something)** (*formal*) allowing something; capable of something.

[9] **imprint** [v] [often passive] **1.** to have a great effect on something so that it cannot be forgotten, changed, etc.; **2.** to print or press a mark or design onto a surface; [n] **1. imprint (of something) (in/on something)** a mark made by pressing something onto a surface; **2.** [usually singular] **imprint (of something) (on somebody/something)** (*formal*) the lasting effect that a person or an experience has on a place or a situation; **3.** (*specialist*) the name of the publisher of a book, usually printed below the title on the 1st page; a brand name under which books are published.

[10] **lifetime** [n] the length of time that somebody lives or that something lasts.

[11] **allot** [v] to give time, money, tasks, etc. to somebody/something as a share of what is available, SYNONYM: **allocate**.

[12] **hamper** [v] [often passive] to prevent something from being achieved easily or happening normally; to prevent somebody from easily doing something, SYNONYM: **hinder, impede**.

[13] **proportionate** [a] increasing or decreasing in size, amount or degree according to changes in something else, SYNONYM: **proportional**.

[14] **stimulating** [a] **1.** full of interesting or exciting ideas; making people feel enthusiastic; **2.** making you feel more active & healthy.

[15] **curriculum** [n] (plural **curricula**) the subjects that are included in a course of study or taught in a school, college or university.

[16] **singular** [n] [singular] (*grammar*) a form of a noun or verb that refers to 1 person or thing; [a] **1.** (*grammar*) connected with or having the form of a noun or verb that refers to 1 person or thing; **2.** especially great or obvious, SYNONYM: **outstanding**; **3.** (*mathematics, physics*) connected with a singularity.

lost, of course, if he regards[17] mathematics as a subject in which he has to earn so & so much credit & which he should forget after the final examination as quickly as possible. The opportunity may be lost even if the student has some natural talent for mathematics because he, as everybody else, must discover his talents & tastes; he cannot know that he likes raspberry pie if he has never tasted raspberry pie. He may manage to find out, however, that a mathematics problem may be as much fun as a crossword puzzle[18], or that vigorous[19] mental work may be an exercise as desirable as a fast game of tennis. Having tasted the pleasure in mathematics he will not forget it easily & then there is a good chance that mathematics will become something for him: a hobby, or a tool of his profession, or a great ambition,

The author remembers the time when he was a student himself, a somewhat ambitious student, eager to understand a little mathematics & physics. He listened to lectures, read books, tried to take in the solutions & facts presented, but there was a question that disturbed[20] him again & again: "Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? & how could I invent or discover such things by myself?" Today the author is teaching mathematics in a university; he thinks or hopes that some of his more eager students ask similar questions & he tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem but also the motives & procedures of the solution, & trying to explain these motives & procedures to others, he was finally led to write the present book. He hopes that it will be useful to teachers who wish to develop their students' ability to solve problems, & to students who are keen on developing their own abilities.

Although the present book pays special attention to the requirements of students & teachers of mathematics, it should interest anybody concerned with the ways & means of invention & discovery. Such interest may be more widespread[21] than one would assume without reflection[22]. The space devoted by popular newspapers & magazines to crossword puzzles & other riddles[23] seems to show that people spend some time in solving unpractical[24] problems. Behind the desire to solve this or that problem that confers[25] no material advantage, there may be a deeper curiosity, a desire to understand the ways & means, the motives & procedures, of solution.

The following pages are written somewhat concisely[26], but as simply as possible, & are based on a long & serious study of methods of solution. This sort of study, called *heuristic*[27] [28] by some writers, is not in fashion nowadays but has a long past &, perhaps, some future.

Studying the methods of solving problems, we perceive[29] another face of mathematics. Yes, mathematics has 2 faces; it is the rigorous[30] science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a

[17] **regard** [v] [often passive] to think about somebody/something in a particular way; **as regards somebody/something** [idiom] concerning or in connection with somebody/something; [n] **1.** [uncountable] attention to or thought & care for somebody/something; **2.** [uncountable] **regard (for somebody/something)** respect or admiration for somebody/something. If you **hold somebody in high regard**, you have a good opinion of them.; **3.** (**regards**) [plural] used to send good wishes to somebody at the end of a letter or email; **have regard to something** [idiom] (*law*) to remember & think carefully about something; **in/with regard to somebody/something** [idiom] concerning somebody/something; **in this/that regard** [idiom] concerning what has just been mentioned.

[18] **crossword** [n] (also **crossword puzzle**) a game in which you have to fit words across & downwards into spaces with numbers in a square diagram. You find the words by solving clues.

[19] **vigorous** [a] **1.** involving physical strength, effort or energy; **2.** done with determination, energy or enthusiasm; **3.** strong & healthy.

[20] **disturb** [v] **1. disturb something** to change the arrangement of something, or affect how something functions; **2. disturb somebody/something** to interrupt somebody & prevent them from continuing with what they are doing; **3. disturb somebody** to make somebody feel anxious or upset.

[21] **widespread** [a] existing or happening over a large area or among many people, SYNONYM: **extensive**.

[22] **reflection** [n] **1.** [countable] **reflection of something** an account or description of what somebody/something is like; a thing that is a result of something else; **2.** [uncountable] careful thought about something, especially your work or studies; **3.** [countable, usually plural] **reflections (on something)** written or spoken thoughts about a particular subject; **4.** [uncountable] **reflection (of something)** the action or process of sending back light, heat, sound, etc. from a surface; **5.** (also **reflexion**) [countable, uncountable] **reflection (of something)** (*mathematics*) an operation on a shape to produce its mirror image.

[23] **riddle** [n] **1.** a question that is difficult to understand, & that has a surprising answer, that you ask somebody as a game; **2.** a mysterious event or situation that you cannot explain, SYNONYM: **mystery**; [v] **riddle somebody/something (with something)** to make a lot of holes in; **be riddle with something** [idiom] to be full of something, especially something bad or unpleasant.

[24] **unpractical** [a] **1.** not sensible or realistic; **2.** (9of people) not good at doing things that involve using the hands; not good at planning or organizing things, OPPOSITE: **practical**.

[25] **confer** [v] **1.** [transitive] to give somebody a particular power, right or honor; **2.** [transitive] to give somebody/something a particular advantage; **3.** [intransitive] **confer (with somebody) (on/about something)** to discuss something with somebody, in order to exchange opinions or get advice.

[26] **concise** [a] giving only the information that is necessary & important, using few words.

[27] **heuristic** [a] (*formal*) **heuristic** teaching or education encourages you to learn by discovering things for yourself.

[28] **heuristics** [n] [uncountable] (*formal*) a method of solving problems by finding practical ways of dealing with them, learning from past experience.

[29] **perceive** [v] **1.** to notice or become aware of something, SYNONYM: **notice**; **2.** to be aware of or experience something using the senses; **3.** [often passive] to understand or think of somebody/something in a particular way; to believe that a particular thing is true, SYNONYM: **see**.

[30] **rigorous** [a] **1.** done carefully & with a lot of attention to detail, SYNONYM: **thorough**; **2.** demanding that particular rules or processes are strictly followed, SYNONYM: **strict**.

systematic[31], deductive[32] science; but mathematics in the making appears as an experimental[33], inductive[34] science. Both aspects[35] are as old as the science of mathematics itself. But the 2nd aspect is new in 1 respect[36]; mathematics "in statu nascendi," in the process of being invented, has never before presented in quite this manner to the student, or to the teacher himself, or to the general public.

The subject of heuristic has manifold[37] connections; mathematicians, logicians[38], psychologists, educationalists[39], even philosophers may claim various parts of it as belonging to their special domains. The author, well aware of the possibility of criticism[40] from opposite[41] quarters[42] & keenly[43] conscious[44] of his limitations[45], has 1 claim to make: he has some experience in solving problems & in teaching mathematics on various levels.

The subject is more fully dealt with in a more extensive book by the author which is on the way to completion. *Stanford University, Aug 1, 1944*" – Polya, 2014, pp. v–vii

From the Preface to the 7th Printing

"[. . .] The 2 volumes *Induction & Analogy in Mathematics* & *Patterns of Plausible Inference* which constitute my recent work *Mathematics & Plausible Reasoning* continue the line of thinking begun in *How to Solve It. Zurich, Aug 30, 1954*" – Polya, 2014, p. viii

Preface to the 2nd Edition

"The present 2nd edition adds, besides a few minor improvements, a new 4th part, "Problems, Hints, Solutions."

As this edition was being prepared for print, a study appeared (Educational Testing Service, Princeton, N.J.; *cf. Time*, Jun 18, 1956) which seems to have formulated[46] a few pertinent[47] observations – they are not new to the people in the know, but it was high time to formulate them for the general public–: ". . . mathematics has the dubious[48], honor of being the least

[31]**systematic** [a] **1.** done according to a system or plan, in a thorough, efficient or determined way; **2.** (of an error) happening in the same way all through a process or set of results; caused by the system that is used.

[32]**deductive** [a] [usually before noun] using knowledge about things that are generally true in order to understand particular situations or problems.

[33]**experimental** [a] **1.** [usually before noun] connected with scientific experiments; **2.** based on new ideas, forms or methods that are used to find out what effect they have.

[34]**inductive** [a] (*specialist*) using particular facts & examples to form general rules & principles.

[35]**aspect** [n] **1.** [countable] a particular feature of a situation, an idea or a process; a way in which something may be considered; **2.** [countable, usually singular] **aspect (of something)** (*specialist*) a particular surface or side of an object or a part of the body; the direction in which something faces; **3.** [uncountable, countable] (*grammar*) the form of a verb that shows, e.g., whether the action happens once or many times, is completed or is still continuing.

[36]**respect** [n] **1.** [countable] a particular aspect or detail of something; **2.** [uncountable, singular] polite behavior towards or reasonable treatment of somebody/something; **3.** [uncountable, singular] a feeling of admiration for somebody/something because of their good qualities or achievements; **in respect of something** [idiom] (*formal*) **1.** concerning; **2.** in payment for something; **with respect to something** [idiom] concerning.

[37]**manifold** [a] (*formal*) many; of many different types; [n] (*specialist*) a pipe or chamber with several openings, especially 1 for taking gases in & out of a car engine.

[38]**logician** [n] a person who studies or is skilled in logic.

[39]**educationalists** [n] (also **educationist**) a specialist in theories & methods of teaching.

[40]**criticism** [n] **1.** [uncountable, countable] the act of expressing disapproval of somebody/something & opinions about their faults or bad qualities; a statement showing disapproval; **2.** [uncountable] the work or activity of analyzing & making fair, careful judgments about somebody/something, especially books, music, etc.

[41]**opposite** [a] **1.** [usually before noun] as different as possible from something; involving 2 different extremes; **2.** [usually before noun] on the other side of something or facing something; [n] **1.** (**the opposite**) [singular] the situation, idea or activity that is as different from another situation, etc. as it is possible to be, SYNONYM: **the reverse**; **2.** (**opposites**) [plural] people, ideas or situations that are as different as possible from each other; **the exact opposite** [idiom] a person or thing that is as different as possible from somebody/something else; [prep] on the other side of a particular area from somebody/something, & usually facing them.

[42]**quarter** [n] **1.** (also **fourth** *especially in NAE*) [countable] 1 of 4 equal parts of something; **2.** [countable] a period of 3 months, used especially as a period fo which bills are paid or a company's income is calculated; **3.** [countable] a person or group of people, especially as a source of help, information or a reaction; **4.** [countable, usually singular] a district or part of a town; **5.** (**quarters**) [plural] rooms that are provided for soldiers, servants, etc. to live in; **at/from close quarters** [idiom] very near.

[43]**keenly** [adv] **1.** very strongly or deeply; **2.** by people with different opinions that they express strongly.

[44]**conscious** [a] **1.** [not before noun] aware of something; noticing something, OPPOSITE: **unconscious**; **2.** able to use your senses & mental powers to understand what is happening, OPPOSITE: **unconscious**; **3.** (of actions, feelings, etc.) deliberate or controlled, OPPOSITE: **unconscious**; **4.** being particularly interested in something.

[45]**limitation** [n] **1.** [countable, usually plural] a limit on what somebody/something can do or how good they/it can be; **2.** [countable] a rule, fact or condition that limits something, SYNONYM: **restraint**; **3.** [uncountable] **limitation (of something)** the act or process of limiting or controlling somebody/something, SYNONYM: **restriction**; **4.** (also **limitation period**) [countable] (*law*) a legal limit on the period of time within which court proceedings can be taken or for which a property right continues.

[46]**formulate** [v] **1. formulate something** to create or prepare something carefully, giving particular attention to the details; **2. formulate something** to express your ideas in carefully chosen words.

[47]**pertinent** [a] appropriate to a particular situation, SYNONYM: **relevant**.

[48]**dubious** [a] **1.** that you cannot be sure about; that is probably not good. **Dubious** is also when you are stating that something is the opposite of a particular good quality. **2.** [usually before noun] probably not honest, SYNONYM: **suspicious**; **3.** [not usually before noun] **dubious about**

popular subject in the curriculum ... Future teachers pass through the elementary schools learning to detest[49] mathematics ... They return to the elementary school to teach a new generation to detest it."

I hope that the present edition, designed for wider diffusion[50], will convince some of its readers that mathematics, besides being a necessary avenue[51] to engineering jobs & scientific knowledge, may be fun & may also open up a vista[52] of mental activity on the highest level. *Zurich, Jun 30, 1956*" – Polya, 2014, pp. ix–

"How to Solve It" list

1st. You have to *understand* the problem.

Understanding the Problem.

What is the unknown? What are the data? What is the condition?

It is possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

2nd. Find the connection between the data & the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Devising a Plan.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! & try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours & solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve 1st some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown & the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3rd. *Carry out* your plan.

Carrying out the Plan.

Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4th. *Examine* the solution obtained.

Looking back.

Can you *check the result?* Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

– Polya, 2014, How to solve it, pp. xvi–xvii

something feeling uncertain about something; not knowing whether something is good or bad, synonym: **doubtful**.

[49]**detest** [v] (not used in the progressive tenses) to hate somebody/something very much, synonym: **loathe**.

[50]**diffusion** [n] [uncountable] **1.** the spreading of something more widely; **2.** the mixing of substances by the natural movement of their particles; **3.** the spreading of elements of culture from 1 region or group to another.

[51]**avenue** [n] a way of approaching a problem or making progress towards something.

[52]**vista** [n] **1.** (*literary*) a beautiful view, e.g., of the countryside, a city, etc., synonym: **panorama**; **2.** (*formal*) a range of things that might happen in the future, synonym: **prospect**.

Foreword by John H. Conway

"*How to Solve It* is a wonderful book! This I realized when I 1st read right through it as a student many years ago, but it has taken me a long time to appreciate just *how* wonderful it is. Why is that? 1 part of the answer is that the book is unique. In all my years as a student & teacher, I have never seen another that lives up to George Polya's title by teaching you how to go about solving problems. A. H. Schoenfeld correctly described its importance in his 1987 article "Polya, Problem Solving, & Education" in *Mathematics Magazine*: "For mathematics education & the world of problem solving it marked a line of demarcation[53] between 2 eras[54], problem solving before & after Polya."

It is 1 of the most successful mathematics books ever written, having sold over a million copies & been translated into 17 languages since it 1st appeared in 1945. Polya later wrote 2 more books about the art of doing mathematics, *Mathematics & Plausible Reasoning* (1954) & *Mathematical Discovery* (2 volumes, 1962 & 1965).

The book's title makes it seem that it is directed only toward students, but in fact it is addressed just as much to their teachers. Indeed, as Polya remarks in his introduction, the 1st part of the book takes the teacher's viewpoint more often than the student's.

Everybody gains that way. The student who reads the book on his own will find that overhearing[55] Polya's comments to his non-existent[56] teacher can bring that desirable person into being, as an imaginary but very helpful figure leaning over one's shoulder. This is what happened to me, & naturally I made heavy use of the remarks I'd found most important when I myself started teaching a few years later.

But it was some time before I read the book again, & when I did, I suddenly realized that it was even more valuable than I'd thought! Many of Polya's remarks that hadn't helped me as a student now made me a better teacher of those whose problems had differed from mine. Polya had met many more students than I had, & had obviously thought very hard about how to best help all of them learn mathematics. Perhaps his most important point is that learning must be active. As he said in a lecture on teaching, "Mathematics, you see, is not a spectator[57] sport. To understand mathematics means to be able to do mathematics. & what does it mean [to be] doing mathematics? In the 1st place, it means to be able to solve mathematical problems."

It is often said that to teach any subject well, one has to understand it "at least as well as one's students do." It is a paradoxical[58] truth that to teach mathematics well, one must also know how to misunderstand it at least to the extent one's students do! If a teacher's statement can be parsed[59] in 2 or more ways, it goes without saying that some students will understand it 1 way & others another, with results that can vary from the hilarious[60] to the tragic[61]. J. E. Littlewood gives 2 amusing[62] examples of assumptions that can easily be made unconsciously & misleadingly[63]. 1st, he remarks that the description of the coordinate axes ("*Ox* & *Oy* as in 2 dimensions, *Oz* vertical") in Lamb's book *Mechanics* is incorrect for him, sine he always worked in an armchair[64] with his feet up! Then, after asking how his reader would present the picture of a closed curve lying all on 1 side of its tangent, he states that there are 4 main schools (to left or right of vertical tangent, or above or below horizontal one) & that by lecturing without a figure, presuming that the curve was to the right of its vertical tangent, he had unwittingly[65] made nonsense[66] for the other 3 schools.

I know of no better remedy[67] for such presumptions[68] than Polya's counsel[69]: before trying to solve a problem, the

[53] **demarcation** [n] [uncountable, countable] a line or limit that separates 2 things, such as types of work, groups of people or areas of land.

[54] **era** [n] **1.** a period of time, usually in history, that is different from other periods because of particular characteristics or events; **2.** (*earth sciences*) a major division of time that can itself be divided into periods.

[55] **overhear** [v] to hear, especially by accident, a conversation in which you are not involved.

[56] **non-existent** [a] not existing; not real.

[57] **spectator** [n] a person who is watching a performance or an event.

[58] **paradoxical** [a] **1.** (of a person, thing or situation) having 2 opposite features & therefore seem strange; **2.** (of a statement) containing 2 opposite ideas that make it seem impossible or unlikely, although it is probably true.

[59] **parse** [v] (*grammar*) **parse something** to divide a sentence into parts & describe the grammar of each word or part.

[60] **hilarious** [a] extremely funny.

[61] **tragic** [a] **1.** making you feel very sad, usually because somebody has died or suffered a lot; **2.** [usually before noun] connected with tragedy (= the style of literature).

[62] **amusing** [a] funny & giving pleasure.

[63] **misleading** [a] giving the wrong idea & making people believe something that is not true, SYNONYM: **deceptive**.

[64] **armchair** [n] a comfortable chair with sides on which you can rest your arms; [a] [only before noun] knowing about a subject through books, television, the Internet, etc., rather than by doing it for yourself.

[65] **unwittingly** [adv] without being aware of what you are doing or the situation that you are involved in, OPPOSITE: **wittingly**.

[66] **nonsense** [n] **1.** [uncountable, countable] ideas, statements or beliefs that you think are silly or not true, SYNONYM: **rubbish**; **2.** [uncountable] spoken or written words that have no meaning or make no sense; **3.** [uncountable] silly or unacceptable behavior; **make (a) nonsense of something** [idiom] to reduce the value of something by a lot; to make something seem silly.

[67] **remedy** [n] (plural **remedies**) **1.** a way of dealing with or improving an unpleasant or difficult situation, SYNONYM: **solution**; **2.** a treatment or medicine to cure a disease or to reduce pain that is not very serious; **3.** (*law*) a way of dealing with a problem, using the processes of the law, SYNONYM: **redress**; [v] **remedy something** to correct or improve something.

[68] **presumption** [n] [countable, uncountable] the act of supposing or accepting that something is true or exists, although it has not been proved; a belief that something is true or exists, SYNONYM: **assumption**. In legal contexts, **presumption** often means that something is being accepted as true until it is shown not to be true.

[69] **counsel** [n] [uncountable, countable] **1.** (*formal*) advice, especially given by older people or experts; a piece of advice; **2.** a lawyer or group of lawyers representing somebody in court; [v] (*formal*) **1. counsel somebody** to listen to & give support or professional advice to somebody who

student should demonstrate his or her understanding of its statement, preferably[70] to a real teacher, but in lieu[71] of that, to an imagined one. Experienced mathematicians know that often the hardest part of researching a problem is understanding precisely what that problem says. They often follow Polya's wise advice: "If you can't solve a problem, then there is an easier problem you can't solve: find it."

Readers who learn from this book will also want to learn about its author's life.[72]

George Polya was born György Pólya (he dropped the accents sometime later) on Dec 13, 1887, in Budapest, Hungary, to Jakab Pólya & his wife, the former Anna Deutsch. He was baptized into the Roman Catholic faith, to which Jakab, Anna, & their 3 previous children, Jenő, Ilona, & Flóra, had converted from Judaism[73] in the previous year. The 5th child, László, was born 4 years later.

Jakab had changed his surname from Pollák to the more Hungarian-sounding Pólya 5 years before György was born, believing that this might help him obtain a university post, which he eventually did, but only shortly before his untimely[74] death in 1897.

At the Dániel Berzsenyi Gymnasium[75], György studied Greek, Latin, & German, in addition to Hungarian. It is surprising to learn that there he was seemingly uninterested in mathematics, his work in geometry deemed merely "satisfactory" compared with his "outstanding" performance in literature, geography, & other subjects. His favorite subject, outside of literature, was biology.

He enrolled at the University of Budapest in 1905, initially studying law, which he soon dropped because he found it too boring. He then obtained the certification needed to teach Latin & Hungarian at a gymnasium, a certification that he never used but of which he remained proud. Eventually his professor, Bernát Alexander, advised him that to help his studies in philosophy, he should take some mathematics & physics courses. This was how he came to mathematics. Later, he joked that he "wasn't good enough for physics, & was too good for philosophy – mathematics is in between."

In Budapest he was taught physics by Eötvös & mathematics by Fejér & was awarded a doctorate after spending the academic year 1910–11 in Vienna, where he took some courses by Wirtinger & Mertens. He spent much of the next 2 years in Göttingen, where he met many more mathematicians – Klein, Caratheodory, Hilbert, Runge, Landau, Weyl, Hecke, Courant, & Toeplitz – & in 1914 visited Paris, where he became acquainted[76] with Picard & Hadamard & learned that Hurwitz had arranged an appointment for him in Zürich. He accepted this position, writing later: "I went to Zürich in order to be near Hurwitz, & we were in close touch for about 6 years, from my arrival in Zürich in 1914 to his passing [in 1919]. I was very much impressed by him & edited his works."

Of course, the 1st World War took place during this period. It initially had little effect on Polya, who had been declared unfit for service in the Hungarian army as the result of a soccer wound. But later when the army, more desperately[77] needing recruits[78], demanded that he return to fight for his country, his strong pacifist[79] views led him to refuse. As a consequence, he was unable to visit Hungary for many years, & in fact did not do so until 1967, 54 years after he left.

In the meantime, he had taken Swiss citizenship & married a Swiss girl, Stella Vera Weber, in 1918. Between 1918 & 1919, he published papers on a wide range of mathematical subjects, such as series, number theory, combinatorics, voting systems, astronomy, & probability. He was made an extraordinary professor at the Zürich ETH in 1920, & a few years later he & Gábor Szegő published their book *Aufgaben und Lehrsatze aus der Analysis* ("Problems & Theorems in Analysis"), described by G. L. Alexanderson & L. H. Lange in their obituary[80] of Polya as "a mathematical masterpiece[81] that assured[82]

needs help; **2.** to advise a particular course of action; to advise somebody to do something.

[70] **preferable** [a] more attractive or more suitable; to be preferred to something.

[71] **lieu** [n] (*formal*) **in lieu (of something)** [idiom] instead of.

[72] The following biographical information is taken from that given by J. J. O'Connor & E. F. Robertson in the MacTutor History of Mathematics Archive (www-gap.dcs.st-and.ac.uk/~hisotry/).

[73] **Judaism** [n] [uncountable] the religion of the Jewish people, based mainly on the Bible & the Talmud (= a collection of ancient writings on Jewish law & traditions).

[74] **untimely** [a] (*formally*) **1.** [usually before noun] happening too soon or sooner than is normal or expected, SYNONYM: **premature**; **2.** happening at a time or in a situation that is not suitable, SYNONYM: **ill-timed**, OPPOSITE: **timely**.

[75] **gymnasium** (plural **gymnasiums, gymnasia**) (*formal*) a gym.

[76] **acquainted** [a] [not before noun] **1. acquainted with something** (*formal*) familiar with something, having read, seen or experienced it; **2.** not close friends with somebody, but having met a few times before.

[77] **desperate** [a] **1.** feeling or showing that you have little hope & are ready to do anything without worrying about danger to yourself or others; **2.** [usually before noun] (of an action) giving little hope of success; tried when everything else was failed; **3.** (of a situation) extremely serious or dangerous.

[78] **recruit** [v] **1.** [transitive, intransitive] to find new people to join a company, an organization, the armed forces, etc.; **2.** [transitive] to get people to help with or be involved in something; **3.** [transitive] **recruit something (from something)** to form a new army, team, etc. by persuading new people to join it; [n] **1.** a person who has recently joined the armed forces or the police; **2.** a person who joins an organization, a company, etc.

[79] **pacifist** [a] [usually before noun] holding or showing the belief that war & violence are always wrong; [n] a person who believes that war & violence are always wrong & refuses to fight in a war.

[80] **obituary** [n] (plural **obituaries**) an article about somebody's life & achievements, that is printed in a newspaper soon after they have died.

[81] **masterpiece** [n] (also **masterwork**) **1. masterpiece (of something)** a work of art such as a painting, film, book, etc. that is an excellent, or the best, example of the artist's work; **2. masterpiece of something** an extremely good example of something.

[82] **assure** [v] **1.** to tell somebody that something is definitely true or is definitely going to happen, especially when they have doubts about it; **2.** to make something certain to happen; to make somebody/something certain to get something; **3.** to make yourself certain about something.

their reputations[83]."

That book appeared in 1925, after Polya had obtained a Rockefeller Fellowship to work in England, where he collaborated with Hardy & Littlewood on what later become their book *Inequalities* (Cambridge University Press, 1936). He used a 2nd Rockefeller Fellowship to visit Princeton University in 1933, & while in the United States was invited by H. F. Blichfeldt to visit Stanford University, which he greatly enjoyed, & which ultimately became his home. Polya held a professorship at Stanford from 1943 until his retirement in 1953, & it was there, in 1978, that he taught his last course, in combinatorics; he died on Sep 7, 1985, at the age of 97.

Some readers will want to know about Polya's many contributions to mathematics. Most of them relate to analysis & are too technical to be understood by non-experts, but a few are worth mentioning.

In probability theory, Polya is responsible for the now-standard term "Central Limit Theorem" & for proving that the Fourier transform of a probability measure is a characteristic function & that a random walk on the integer lattice closes with probability 1 iff the dimension is at most 2.

In geometry, Polya independently re-enumerated the 17 plane crystallographic[84] groups (their 1st enumeration[85], by E. S. Fedorov, having been forgotten) & together with P. Niggli devised[86] a notation for them.

In combinatorics, Polya's Enumeration Theorem is now a standard way of counting configurations according to their symmetry. It has been described by R. C. Read as "a remarkable[87] theorem in a remarkable paper, & a landmark[88] in the history of combinatorial analysis."

How to Solve It was written in German during Polya's time in Zürich, which ended up in 1940, when the European situation forced him to leave for the United States. Despite the book's eventual success, 4 publishers rejected the English version before Princeton University Press brought it out in 1945. In their hands, *How to Solve It* rapidly became – & continues to be – 1 of the most successful mathematical books of all time." – Polya, 2014, Foreword, pp. xix–xxiv

Introduction

"The following consideration are grouped around the preceding list of questions & suggestions entitled[89] "How to Solve It." Any question or suggestion quoted from it will be printed in *italics*, & the whole list will be referred to simply as "the list" or as "our list."

The following pages will discuss the purpose of the list, illustrate its practical use by examples, & explain the underlying notions & mental operations. By way of preliminary explanation, this much may be said: If, using them properly, you address these questions & suggestions to yourself, they may help you to solve your problem. If, using them properly, you address the same questions & suggestions to 1 of your students, you may help him to solve his problem.

The book is divided into 4 parts.

The title of the 1st part is "In the Classroom." It contains 20 sections. Each section will be quoted by its number in heavy type as, e.g., "sect. **7.**" Sects. **1**–**5** discuss the "Purpose" of our list in general terms. Sects. **6**–**17** explain what are the "Main Divisions, Main Questions" of the list, & discuss a 1st practical example. Sects. **18**–**20** add "More Examples."

The title of the very short 2nd part is "How to Solve It." It is written in dialogue; a somewhat idealized teacher answers short questions of a somewhat idealized student.

The 3rd & most extensive part is a "Short Dictionary of Heuristic"; we shall refer to it as the "Dictionary." It contains 67 articles arranged alphabetically. E.g., the meaning of the term HEURISTIC (set in small capitals) is explained in an article with this title on p. 112. When the title of such an article is referred to within the text it will be set in small capitals. Certain paragraphs of a few articles are more technical; they are enclosed[90] in square brackets. Some articles are fairly closely connected with the 1st part to which they add further illustrations & more specific comments. Other articles go somewhat beyond the aim of the 1st part of which they explain the background. There is a key-article on MODERN HEURISTIC. It explains the connection of the main articles & the plan underlying the Dictionary; it contains also directions how to find information about particular items of the list. It must be emphasized that there is a common plan & a certain unity, because the articles of the Dictionary show the greatest outward variety. There are a few longer articles devoted to the systematic though condensed discussion of some general theme; others contain more specific comments; still others cross-references[91],

[83] **reputation** [n] the opinion that people have about what somebody/something is like, based on what has happened in the past.

[84] **crystallography** [n] [uncountable] the branch of science that deals with crystals.

[85] **enumeration** [n] [uncountable, countable] (*formal*) the act of naming things 1 by 1 in a list; a list of this sort.

[86] **devise** [v] **devise something** to plan or invent a procedure, system or method, especially one that is new or complicated, by using careful thought, SYNONYM: **think something up**.

[87] **remarkable** [a] unusual or surprising in a way that causes people to take notice.

[88] **landmark** [n] **1.** something, such as a large building, that you can see clearly from a distance & that will help you to know where you are; **2.** an event, a discovery or an invention that marks an important stage in something.

[89] **entitle** [v] **1.** [often passive] to give somebody the right to have or to do something; **2.** [usually passive] to give a title to a book, document, film, etc.

[90] **enclose** [v] **1.** [usually passive] to build a wall, fence, etc. around something; **2. enclose something** (especially of a wall, fence, etc.) to surround something; **3. enclose something (with something)** to put something in the same envelope or package as something else.

[91] **cross-reference** [v] **cross-reference something** to give cross references to another text or part of a text.

or historical data, or quotations, or aphorisms[92], or even jokes.

The Dictionary should not be read too quickly; its text is often condensed, & now & then somewhat subtle. The reader may refer to the Dictionary for information about particular points. If these points come from his experience with his own problems or his own students, the reading has a much better chance to be profitable[93].

The title of the 4th part is "Problems, Hints, Solutions." It proposes a few problems to the more ambitious reader. Each problem is followed (in proper distance) by a "hint" that may reveal a way to the result which is explained in the "solution."

We have mentioned repeatedly the "student" & the "teacher" & we shall refer to them again & again. It may be good to observe that the "student" may be a high school student, or a college student, or anyone else who is studying mathematics. Also the "teacher" may be a high school teacher, or a college instructor, or anyone interested in the technique of teaching mathematics. The author looks at the situation sometimes from the point of view of the student & sometimes from that of the teacher (the latter case is preponderant[94] in the 1st part). Yet most of the time (especially in the 3rd part) the point of view is that of a person who is neither teacher nor student but anxious to solve the problem before him." – Polya, 2014, Introduction, pp. xxv–xxvii

Part I. In The Classroom

Purpose

2.1 Helping the student

"1 of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion[95], & sound principles.

The student should acquire as much experience of independent work as possible. But if he is left alone with his problem without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much & not too little, so that the student shall have a *reasonable share of the work*.

If the student is not able to do much, the teacher should leave him at least some illusion[96] of independent work. In order to do so, the teacher should help the student discreetly[97], *unobtrusively*[98].

The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, & ask a question or indicate[99] a step that *could have occurred to the student himself*." – Polya, 2014, p. 1

2.2 Questions, recommendations, mental operations

"Trying to help the student effectively but unobtrusively & naturally, the teacher is led to ask the same questions & to indicate the same steps again & again. Thus, in countless problems, we have to ask the question: *What is the unknown?* We may vary the words, & ask the same thing in many different ways: What is required? What do you want to find? What are you supposed to seek? The aim of these questions is to focus the student's attention upon the unknown. Sometimes, we obtain the same effect more naturally with a suggestion: *Look at the unknown!* Question & suggestion aim at the same effect; they tend to provoke[100] the same mental operation.

[92] **aphorism** [n] (*formal*) a short phrase that says something true or wise.

[93] **profitable** [a] **1.** that makes or is likely to make money; **2.** that gives somebody an advantage or a useful result.

[94] **preponderant** [a] [usually before noun] (*formal*) larger in number or more important than other people or things in a group.

[95] **devotion** [n] [uncountable, singular] **1.** devotion (of somebody) (to somebody/something) great love, care & support for somebody/something; **2. devotion (to somebody/something)** the action of spending a lot of time or energy on something, SYNONYM: **dedication**; **3.** great religious feeling.

[96] **illusion** [n] **1.** [countable, uncountable] a false idea or belief; **2.** [countable] something that seems to exist but in fact does not, or seems to be something that it is not.

[97] **discreetly** [adv] in a careful way, in order to keep something secret or to avoid causing difficulty for somebody or making them feel embarrassed, SYNONYM: **tactfully**.

[98] **unobtrusively** [adv] (*formal, often approving*) in a way that does not attract unnecessary attention, OPPOSITE: **obtrusively**.

[99] **indicate** [v] **1.** to show that something is true or exists; **2.** to be a sign of something; to show that something is possible or likely, SYNONYM: **suggest**; **3. indicate something** to represent information without using words; **4.** to give information in writing; **5.** [usually passive] to suggest something as a necessary or recommend course of action; **6.** to mention something, especially in an indirect or brief way; **7. indicate something** (of an instrument for measuring things) to show a particular measurement.

[100] **provoke** [v] **1. provoke something** to cause a particular reaction or have a particular effect; **2.** to say or do something in order to produce a strong reaction from somebody, usually anger.

It seemed to the author that it might be worth while[101] to collect & to group questions & suggestions which are typically[102] helpful in discussing problems with students. The list we study contains questions & suggestions of this sort, carefully chosen & arranged; they are equally useful to the problem-solver who works by himself. If the reader is sufficiently acquainted with the list & can see, behind the suggestion, the action suggested, he may realize that the list enumerates, indirectly, *mental operations typically useful for the solution of problems.* These operations are listed in the order in which they are most likely to occur." – Polya, 2014, pp. 1–2

2.3 Generality

"Generality[103] is an important characteristic of the questions & suggestions contained in our list. Take the questions: *What is the unknown? What are the data? What is the condition?* These questions are generally applicable[104], we can ask them with good effect dealing with all sorts of problems. Their use is not restricted to any subject-matter[105]. Our problem may be algebraic or geometric, mathematical or nonmathematical, theoretical or practical, a serious problem or a mere puzzle; it makes no difference, the questions make sense & might help us to solve the problem.

There is a **restriction**[106], in fact, but it has nothing to do with the subject-matter. Certain questions & suggestions of the list are applicable to "problems to find" only, not to "problems to prove." If we have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS TO PROVE." – Polya, 2014, pp. 2–3

2.4 Common sense

"The questions & suggestions of our list are general, but, except for their generality, they are natural, simple, obvious, & proceed from plain common sense. Take the suggestion: *Look at the unknown! & try to think of a familiar problem having the same or a similar unknown.* This suggestion advises you to do what you would do anyhow[107], without any advice, if you were seriously concerned with your problem. Are you hungry? You wish to obtain food & you think of familiar ways of obtaining food. Have you a problem of geometric construction? You wish to construct a triangle & you think of familiar ways of constructing a triangle. Have you a problem of any kind? You wish to find a certain unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted from our list. & you are on the right track, too; the suggestion is a good one, it suggests to you a procedure which is very frequently successful.

All the questions & suggestions of our list are natural, simple, obvious, just plain common sense; but they state plain common sense in general terms. They suggest a certain conduct which comes naturally to any person who is seriously concerned with his problem & has some common sense. But the person who behaves the right way usually does not care to express his behavior in clear words &, possibly, he cannot express it so; our list tries to express it so." – Polya, 2014, p. 3

2.5 Teacher & student. Imitation & practice

"There are 2 aims which the teacher may have in view when addressing to his students a question or a suggestion of the list: 1st, to help the student to solve the problem at hand. 2nd, to develop the student's ability so that he may solve future problems by himself.

Experience shows that the questions & suggestions of our list, appropriately used, very frequently help the student. They have 2 common characteristics, common sense & generality. As they proceed from plain common sense they very often come naturally; they could have occurred to the student himself. As they are general, they help unobtrusively; they just indicate a general direction & leave plenty for the student to do.

But the 2 aims we mentioned before are closely connected; if the student succeeds in solving the problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that our questions are general, applicable in many

[101] **worth somebody's while (to do something/doing something)** [idiom] interesting or useful for somebody to do.

[102] **typically** [adv] **1.** used to say that something usually happens in the way that you are stating; **2.** in a way that shows the usual qualities or features of a particular type of person, thing or group.

[103] **generality** [n] (plural **generalities**) **1.** [uncountable] **generality (of something)** the quality of a theory or model that can be applied being across a wide range of cases & situations. The phrase **without loss of generality** means that a statement about 1 particular case can be easily applied to all other cases.; **2.** [countable, usually plural] a statement that makes general points rather than giving details or particular examples; **3.** (**the generality**) [singular + singular or plural verb] **generality (of somebody/something)** (*formal*) the greater part of a group of people or things, SYNONYM: **majority**.

[104] **applicable** [a] [not usually before noun] true about or appropriate to a particular situation, group of people, etc.

[105] **subject matter** [n] [uncountable] **subject matter (of something)** the ideas or information contained in a book, speech, painting, etc.

[106] **restriction** [n] **1.** [countable] a rule or law that limits what you can do or what can happen; **2.** [uncountable] the act of limiting or controlling somebody/something.

[107] **anyhow** [adv] **1.** (also **anyway**, also NAE, informal **anyways**) used when adding something to support an idea or argument; **2.** (also **anyway**, also NAE, informal **anyways**) despite something; even so; **3.** (also **anyway**, also NAE, informal **anyways**) used when changing the subject of a conversation, ending the conversation or returning to a subject; **4.** (also **anyway**, also NAE, informal **anyways**) used to correct or slightly change what you have said; **5.** in a careless way; not arranged in an order.

cases. If the same question is repeatedly helpful, the student will scarcely[108] fail to notice it & he will be induced[109] to ask the question by himself in a similar situation. Asking the question repeatedly, he may succeed once in eliciting[110] the right idea. By such a success, he discovers the right way of using the question, & then he has really assimilated[111] it.

The student may absorb[112] a few questions of our list so well that he is finally able to put to himself the right question in the right moment & to perform the corresponding mental operation naturally & vigorously[113]. Such a student has certainly derived the greatest possible profit from our list. What can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation[114] & practice. Trying to swim, you imitate what other people do with their hands & feet to keep their heads above water, &, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe & to imitate what other people do when solving problems &, finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill[115] some interest for problems into their minds & give them plenty of opportunity for imitation & practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions & suggestions of our list, he puts these questions & suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize[116] his ideas a little & he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions & suggestions, & doing so he will acquire something that is more important than the knowledge of any particular mathematical fact." – Polya, 2014, pp. 3–5

Main divisions, main questions

2.6 4 phases

"Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again & again. Our conception[117] of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions & suggestions of our list, we shall distinguish 4 phases of the work. 1st, we have to *understand* the problem; we have to see clearly what is required. 2nd, we have to see how the various items are connected, how the unknown is liked to the data, in order to obtain the idea of the solution, to make a *plan*. 3rd, we *carry out* our plan. 4th, we *look back* at the completed solution, we review & discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea & jumping all preparations blurts[118] out with the solution. Such lucky ideas, of course, are most desirable[119], but something very undesirable[120] & unfortunate may result if the student leaves out any of the 4 phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan*. Many mistakes can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may be lost if the student fails to reexamine & to *reconsider* the completed solution." – Polya, 2014, pp. 5–6

[108] **scarcely** [adv] only just; almost not.

[109] **induce** [v] **1. induce something** to cause something; **2. induce somebody to do something** to persuade or influence somebody to do something; **3. induce something** (*physics*) to produce an electric charge or current, or a magnetic state by induction; **4. induce something (from something)** to use particular facts & examples to form a general rule or principle.

[110] **elicit** [v] to get information or a reaction from somebody/something.

[111] **assimilate** [v] **1.** [intransitive, transitive] to become a part of a country or community rather than remaining in a separate group; to allow or cause people to do this; **2.** [transitive] **assimilate something** (of the body or any biological system) to absorb or take in a substance; **3. assimilate something** to think deeply about something & understand it fully, so that you can use it, SYNONYM: **absorb**; **4.** [transitive, often passive] **assimilate something (into/to something)** to accept an idea, information or activity; to make it fit into something.

[112] **absorb** [v] **1.** to take in a liquid, gas or other substance from the surface or space around; **2. absorb something** to take in & keep heat, light or other forms of energy, instead of reflecting it; **3.** [often passive] to take control of a smaller unit or group & make it part of something larger; **4.** to take something into the mind & learn or understand it, SYNONYM: **take something in**; **5. absorb something** to deal with or reduce the effects of changes or costs; **6. absorb something** to use up a large supply of something, especially money or time; **7. be absorbed in something** to be so interested in something that you pay no attention to anything else.

[113] **vigorously** [adv] **1.** with determination, energy or enthusiasm; **2.** in a way that involves physical strength, effort or energy.

[114] **imitation** [n] **1.** [countable] a copy of something, especially something expensive; **2.** [uncountable] the act of copying somebody/something.

[115] **instil** [v] (BE) (NAE **instill**) to graduate put an idea or attitude into somebody's mind; to make somebody feel, think or behave in a particular way over a period of time.

[116] **dramatize** [v] (BE also **dramatise**) **1.** to present a book or an event as a play or film; **2. dramatize something** to make something seem more exciting or important than it really is.

[117] **conception** [n] **1.** [countable, uncountable] an understanding or a belief of what something is or what something should be; **2.** [uncountable] the process of forming an idea or a plan; **3.** [uncountable, countable] the process of an egg being fertilized inside a woman's body so that he becomes pregnant.

[118] **blurt** [v] **blurt something (out)** | **blurt that** ... | **blurt what, how, etc.** ... | + **speech** to say something suddenly & without thinking carefully enough.

[119] **desirable** [a] that you would like to have or do; worth having or doing, OPPOSITE: **undesirable**.

[120] **undesirable** [a] not wanted or approved of; likely to cause trouble or problems, OPPOSITE: **desirable**.

2.7 Understanding the problem

"It is foolish[121] to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish & sad things often happen, in & out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult & not too easy, natural & interesting, & some time should be allowed for natural & interesting presentation.

1st of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement & the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principle parts of the problem attentively[122], repeatedly, & from various sides. If there is a figure connected with the problem he should *draw a figure* & point out on it the unknown & the data. If it is necessary to give names to these objects he should *introduce suitable notation*; devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory[123] stage provided that we do not expect a definitive[124] answer but just a provisional[125] answer, a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] "Understanding the problem" is subdivided into 2 stages: "Getting acquainted" & "Working for better understanding.")" – Polya, 2014, pp. 6–7

2.8 Example

"Let us illustrate some of the points explained in the foregoing[126] section. We take the following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, the width, & the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem of Pythagoras, & with some of its applications in plane geometry, but they may have very little systematic knowledge in solid geometry. The teacher may rely here upon the student's unsophisticated[127] familiarity[128] with spatial[129] relations.

The teacher can make the problem interesting by making it concrete[130]. The classroom is a rectangular parallelepiped whose dimensions could be measured, & can be estimated; the students have to find, to "measure indirectly," the diagonal of the classroom. The teacher points out the length, the width, & the height of the classroom, indicates the diagonal with a gesture, & enlivens[131] his figure, drawn on the blackboard, by referring repeatedly to the classroom.

The dialogue between the teacher & the students may start as follows:

"What is the unknown?"

" – Polya, 2014, pp. 7–8

[121] **foolish** [a] **1.** not showing good sense or judgment, SYNONYM: **silly, stupid**; **2.** [not usually before noun] made to feel or look silly & embarrassed, SYNONYM: **silly, stupid**.

[122] **attentive** [a] **1.** reading, listening or watching carefully & with interest; **2.** helpful; making sure that people have what they need.

[123] **preparatory** [a] done in order to prepare for something.

[124] **definitive** [a] **1.** final; that cannot be changed; **2.** [usually before noun] considered to be the best of its kind & almost impossible to improve.

[125] **provisional** [a] made for the present time, possibly to be changed when more information is available or when a more permanent arrangement can be made.

[126] **foregoing** [a] [only before noun] **1.** used to refer to something that has just been mentioned, OPPOSITE: **following**; **2.** (**the foregoing**) [n, singular + singular or plural verb] what has just been mentioned.

[127] **unsophisticated** [a] **1.** not having or showing much experience of the world & social situations; **2.** simple & basic; not complicated, SYNONYM: **crude**, OPPOSITE: **sophisticated**.

[128] **familiarity** [n] **1.** [uncountable, singular] **familiarity with something** the state of knowing somebody/something well; the state of recognizing somebody/something; **2.** [uncountable] the fact of being well known to you.

[129] **spatial** [a] connected with space & the position, size, shape, etc. of things in it.

[130] **concrete** [a] **1.** made of concrete; **2.** based on facts or actions, not on ideas, guesses or intentions; **3.** a concrete object is one that you can see & feel, OPPOSITE: **abstract**; [n] [uncountable] building material that is made by mixing together cement, sand, small stones & water.

[131] **enliven** [v] (*formal*) **enliven something** to make something more interesting or more fun.

Analogy

Auxiliary elements

Auxiliary problem

Bolzano

Bright idea

Can you check the result?

Can you derive the result differently?

Can you use the result?

Carrying out

Condition

Contradictory

Corollary

Could you derive something useful from the data?

Could you restate the problem?

Decomposing & recombining

Definition

Descartes

Determination, hope, success

Diagnosis

Did you see all the data?

Do you know a related problem?

Draw a figure

Examine your guess

Figures

Generalization

Have you seen it before?

Here is a problem related to yours & solved before

Problems

Hints

Solutions

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