

# Elementary Mathematics

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# Chapter 1

## Wikipedia's

### 1.1 Wikipedia/How to Solve It

“*How to Solve It* (1945) is a small volume by mathematician [George Pólya](#) describing methods of [problem solving](#).” – [Wikipedia/how to solve it](#)

#### 1.1.1 4 principles

“*How to Solve It* suggests the following steps when solving a [mathematical problem](#):

1. 1st, you have to *understand the problem*.
2. After understanding, *make a plan*.
3. *Carry out the plan*.
4. *Look back* on your work. How could it be better?

If this technique fails, Pólya advises: “If you can’t solve a problem, then there is an easier problem you can solve: find it.” Or: “If you cannot solve the proposed problem, try to solve 1st some related problem. Could you imagine a more accessible related problem?” – [Wikipedia/how to solve it/4 principles](#)

##### 1.1.1.1 1st principle: Understand the problem

“Understanding the problem” is often neglected as being obvious & is not even mentioned in many mathematics classes. Yet students are often stymied in their efforts to solve it, simply because they don’t understand it fully, or even in part. In order to remedy this oversight, Pólya taught teachers how to prompt each student with appropriate questions, depending on the situation, such as:

- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture of a diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?
- Do you understand all the words used in stating the problem?
- Do you need to ask a question to get the answer?

The teacher is to select the question with the appropriate level of difficulty for each student to ascertain if each student understands at their own level, moving up or down the list to prompt each student, until each one can respond with something constructive.” – [Wikipedia/how to solve it/4 principles/1st principle: understand the problem](#)

### 1.1.1.2 2nd principle: Devise a plan

“Pólya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. You will find choosing a strategy increasingly easy. A partial list of strategies is included:

- Guess & check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation

Also suggested:

- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Use a model
- Work backward
- Use a formula
- Be creative
- Applying these rules to devise a plan takes your own skill & judgment.

Pólya lays a big emphasis on the teachers’ behavior. A teacher should support students with devising their own plan with a question method that goes from the most general questions to more particular questions, with the goal that the last step to having a plan is made by the student. He maintains that just showing students a plan, no matter how good it is, does not help them.” – [Wikipedia/how to solve it/4 principles/2nd principle: devise a plan](#)

### 1.1.1.3 3rd principle: Carry out the plan

“This step is usually easier than devising the plan. In general, all you need is care & patience, given that you have the necessary skills. Persist with the plan that you have chosen. If it continues not to work, discard it & choose another. Don’t be misled; this is how mathematics is done, even by professionals.” – [Wikipedia/how to solve it/4 principles/3rd principle: carry out the plan](#)

### 1.1.1.4 4th principle: Review/extend

“Pólya mentions that much can be gained by taking the time to reflect & look back at what you have done, what worked & what did not, & with thinking about other problems where this could be useful. Doing this will enable you to predict what strategy to use to solve future problems, if these relate to the original problem.” – [Wikipedia/how to solve it/4 principles/4th principle: review/extend](#)

## 1.1.2 Heuristics

“The book contains a dictionary-style set of **heuristics**, many of which have to do with generating a more accessible problem. E.g.:

**Heuristic** | **Informal Description** | **Formal analogue**

- **Analogy** | Can you find a problem analogous to your problem & solve that? | **map**
- **Auxiliary Elements** | Can you add some new element to your problem to get closer to a solution? | **Extension**

- **Generalization** | Can you find a problem more general than your problem? | **Generalization**
- **Induction** | Can you solve your problem by deriving a generalization from some examples? | **Induction**
- Variation of the Problem | Can you vary or change your problem to create a new problem (or set of problems) whose solution(s) will help you solve your original problem? | **Search**
- Auxiliary Problem | Can you find a subproblem or side problem whose solution will help you solve your problem? | **Subgoal**
- Here is a problem related to yours & solved before | Can you find a problem related to yours that has already been solved & use that to solve your problem? | **Pattern recognition, Pattern matching, Reduction**
- **Specialization** | Can you find a problem more specialized? | **Specialization**
- **Decomposing & Recombining** | Can you decompose the problem & “recombine its elements in some new manner”? | **Divide & conquer**
- **Working backward** | Can you start with the goal & work backwards to something you already know? | **Backward chaining**
- Draw a Figure | Can you draw a picture of the problem? | **Diagrammatic Reasoning**

” – [Wikipedia/how to solve it/heuristics](#)

### 1.1.3 Influence

- “The book has been translated into several languages & has sold over a million copies, & has been continuously in print since its 1st publication.
- **Marvin Minsky** said in his paper *Steps Toward Artificial Intelligence* that “everyone should know the work of George Pólya on how to solve problems.”
- Pólya’s book has had a large influence on mathematics textbooks as evidenced by the bibliographies for **mathematics education**.
- Russian inventor **Genrich Altshuller** developed an elaborate set of methods for problem solving known as **TRIZ**, which in many aspects reproduces or parallels Pólya’s work.
- **How to Solve it by Computer** is a computer science book by R. G. Dromey. It was inspired by Pólya’s work.” – [Wikipedia/how to solve it/influence](#)

## Chapter 2

# Polya, 2014. How to Solve It: A New Aspect of Mathematical Methods

### From the Preface to the 1st Printing

“A great discovery solves a great problem but there is a grain<sup>1</sup> of discovery in the solution of any problem. Your problem may be modest<sup>2</sup>; but it challenges your curiosity<sup>3</sup> & brings into play your inventive<sup>4</sup> faculties<sup>5</sup>, & if you solve it by your own means, you may experience the tension<sup>6</sup> & enjoy the triumph<sup>7</sup> of discovery. Such experiences at a susceptible<sup>8</sup> age may create a taste for mental work & leave their imprint<sup>9</sup> on mind & character for a lifetime<sup>10</sup>.

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted<sup>11</sup> time with drilling his students in routine operations he kills their interest, hampers<sup>12</sup> their intellectual development, & misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate<sup>13</sup> to their knowledge, & helps them to solve their problems with stimulating<sup>14</sup> questions, he may give them a taste for, & some means of, independent thinking.

Also a student whose college curriculum<sup>15</sup> includes some mathematics has a singular<sup>16</sup> opportunity. This opportunity is

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<sup>1</sup>**grain** [n] **1.** [uncountable, countable] the small hard seeds of food plants such as wheat, rice, etc.; a single seed of such a plant; **2.** [countable] **grain (of something)** a small piece of a particular substance; usually a hard substance; **3.** [countable, usually singular] **grain of something** a very small amount; **4.** [countable] an individual particle or crystal in metal, rock, etc., usually explained with a lens or microscope.

<sup>2</sup>**modest** [a] **1.** fairly limited or small in amount; **2.** not expensive, rich or impressive; **3.** (of people, especially women, or their clothes) not showing too much of the body; not intended to attract attention, especially in a sexual way; **4.** (*approving*) not talking much about your own abilities or possessions.

<sup>3</sup>**curiosity** [n] (plural **curiosities**) **1.** [uncountable, singular] a strong desire to know about something; **2.** [countable] **curiosity (of something)** an unusual & interesting thing.

<sup>4</sup>**inventive** [a] **1.** (especially of people) able to create or design new things or think of new ideas; **2.** (of ideas) new & interesting.

<sup>5</sup>**faculty** [n] (plural **faculties**) **1.** [countable] a physical or mental ability, especially one that people are born with; **2.** [countable] **faculty (of something)** a department or group of related departments in a college or university; **3.** [countable + singular or plural verb] all the teachers in a faculty of a college or university; **4.** [countable, uncountable] (*NAE*) all the teachers of a particular university or college.

<sup>6</sup>**tension** [n] **1.** [uncountable, countable, usually plural] a situation in which people do not trust each other, or feel unfriendly towards each other, & which may cause them to attack each other; **2.** [countable, uncountable] **tension (between A & B)** a situation in which the fact that there are different needs or interests causes difficulties; **3.** [uncountable] a feeling of anxiety & stress that makes it impossible to relax; **4.** [uncountable] the feeling of fear & excitement that is created by a writer or a film director; **5.** [uncountable] the state of being stretched tight; the extent to which something is stretched tight.

<sup>7</sup>**triumph** [n] **1.** [countable, uncountable] a great success, achievement or victory; **2.** [uncountable] the state of having achieved a great success or victory; the feeling of happiness that you get from this; [v] [intransitive] to defeat somebody/something; to be successful.

<sup>8</sup>**susceptible** [a] **1.** [not usually before noun] **susceptible (to somebody/something)** very likely to be influenced, harmed or affected by somebody/something; **2.** **susceptible (of something)** (*formal*) allowing something; capable of something.

<sup>9</sup>**imprint** [v] [often passive] **1.** to have a great effect on something so that it cannot be forgotten, changed, etc.; **2.** to print or press a mark or design onto a surface; [n] **1.** **imprint (of something) (in/on something)** a mark made by pressing something onto a surface; **2.** [usually singular] **imprint (of something) (on somebody/something)** (*formal*) the lasting effect that a person or an experience has on a place or a situation; **3.** (*specialist*) the name of the publisher of a book, usually printed below the title on the 1st page; a brand name under which books are published.

<sup>10</sup>**lifetime** [n] the length of time that somebody lives or that something lasts.

<sup>11</sup>**allot** [v] to give time, money, tasks, etc. to somebody/something as a share of what is available, SYNONYM: **allocate**.

<sup>12</sup>**hamper** [v] [often passive] to prevent something from being achieved easily or happening normally; to prevent somebody from easily doing something, SYNONYM: **hinder, impede**.

<sup>13</sup>**proportionate** [a] increasing or decreasing in size, amount or degree according to changes in something else, SYNONYM: **proportional**.

<sup>14</sup>**stimulating** [a] **1.** full of interesting or exciting ideas; making people feel enthusiastic; **2.** making you feel more active & healthy.

<sup>15</sup>**curriculum** [n] (plural **curricula**) the subjects that are included in a course of study or taught in a school, college or university.

<sup>16</sup>**singular** [n] [singular] (*grammar*) a form of a noun or verb that refers to 1 person or thing; [a] **1.** (*grammar*) connected with or having the form of a noun or verb that refers to 1 person or thing; **2.** especially great or obvious, SYNONYM: **outstanding**; **3.** (*mathematics, physics*) connected with a singularity.

lost, of course, if he regards<sup>17</sup> mathematics as a subject in which he has to earn so & so much credit & which he should forget after the final examination as quickly as possible. The opportunity may be lost even if the student has some natural talent for mathematics because he, as everybody else, must discover his talents & tastes; he cannot know that he likes raspberry pie if he has never tasted raspberry pie. He may manage to find out, however, that a mathematics problem may be as much fun as a crossword puzzle<sup>18</sup>, or that vigorous<sup>19</sup> mental work may be an exercise as desirable as a fast game of tennis. Having tasted the pleasure in mathematics he will not forget it easily & then there is a good chance that mathematics will become something for him: a hobby, or a tool of his profession, or a great ambition,

The author remembers the time when he was a student himself, a somewhat ambitious student, eager to understand a little mathematics & physics. He listened to lectures, read books, tried to take in the solutions & facts presented, but there was a question that disturbed<sup>20</sup> him again & again: “Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? & how could I invent or discover such things by myself?” Today the author is teaching mathematics in a university; he thinks or hopes that some of his more eager students ask similar questions & he tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem but also the motives & procedures of the solution, & trying to explain these motives & procedures to others, he was finally led to write the present book. He hopes that it will be useful to teachers who wish to develop their students’ ability to solve problems, & to students who are keen on developing their own abilities.

Although the present book pays special attention to the requirements of students & teachers of mathematics, it should interest anybody concerned with the ways & means of invention & discovery. Such interest may be more widespread<sup>21</sup> than one would assume without reflection<sup>22</sup>. The space devoted by popular newspapers & magazines to crossword puzzles & other riddles<sup>23</sup> seems to show that people spend some time in solving unpractical<sup>24</sup> problems. Behind the desire to solve this or that problem that confers<sup>25</sup> no material advantage, there may be a deeper curiosity, a desire to understand the ways & means, the motives & procedures, of solution.

The following pages are written somewhat concisely<sup>26</sup>, but as simply as possible, & are based on a long & serious study of methods of solution. This sort of study, called *heuristic*<sup>27 28</sup> by some writers, is not in fashion nowadays but has a long past &, perhaps, some future.

Studying the methods of solving problems, we perceive<sup>29</sup> another face of mathematics. Yes, mathematics has 2 faces; it is the rigorous<sup>30</sup> science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a

<sup>17</sup>**regard** [v] [often passive] to think about somebody/something in a particular way; **as regards somebody/something** [idiom] concerning or in connection with somebody/something; [n] **1.** [uncountable] attention to or thought & care for somebody/something; **2.** [uncountable] **regard (for somebody/something)** respect or admiration for somebody/something. If you **hold somebody in high regard**, you have a good opinion of them.; **3.** (**regards**) [plural] used to send good wishes to somebody at the end of a letter or email; **have regard to something** [idiom] (*law*) to remember & think carefully about something; **in/with regard to somebody/something** [idiom] concerning somebody/something; **in this/that regard** [idiom] concerning what has just been mentioned.

<sup>18</sup>**crossword** [n] (also **crossword puzzle**) a game in which you have to fit words across & downwards into spaces with numbers in a square diagram. You find the words by solving clues.

<sup>19</sup>**vigorous** [a] **1.** involving physical strength, effort or energy; **2.** done with determination, energy or enthusiasm; **3.** strong & healthy.

<sup>20</sup>**disturb** [v] **1.** **disturb something** to change the arrangement of something, or affect how something functions; **2.** **disturb somebody/something** to interrupt somebody & prevent them from continuing with what they are doing; **3.** **disturb somebody** to make somebody feel anxious or upset.

<sup>21</sup>**widespread** [a] existing or happening over a large area or among many people, SYNONYM: **extensive**.

<sup>22</sup>**reflection** [n] **1.** [countable] **reflection of something** an account or description of what somebody/something is like; a thing that is a result of something else; **2.** [uncountable] careful thought about something, especially your work or studies; **3.** [countable, usually plural] **reflections (on something)** written or spoken thoughts about a particular subject; **4.** [uncountable] **reflection (of something)** the action or process of sending back light, heat, sound, etc. from a surface; **5.** (also **reflexion**) [countable, uncountable] **reflection (of something)** (*mathematics*) an operation on a shape to produce its mirror image.

<sup>23</sup>**riddle** [n] **1.** a question that is difficult to understand, & that has a surprising answer, that you ask somebody as a game; **2.** a mysterious event or situation that you cannot explain, SYNONYM: **mystery**; [v] **riddle somebody/something (with something)** to make a lot of holes in; **be riddle with something** [idiom] to be full of something, especially something bad or unpleasant.

<sup>24</sup>**unpractical** [a] **1.** not sensible or realistic; **2.** (9of people) not good at doing things that involve using the hands; not good at planning or organizing things, OPPOSITE: **practical**.

<sup>25</sup>**confer** [v] **1.** [transitive] to give somebody a particular power, right or honor; **2.** [transitive] to give somebody/something a particular advantage; **3.** [intransitive] **confer (with somebody) (on/about something)** to discuss something with somebody, in order to exchange opinions or get advice.

<sup>26</sup>**concise** [a] giving only the information that is necessary & important, using few words.

<sup>27</sup>**heuristic** [a] (*formal*) **heuristic** teaching or education encourages you to learn by discovering things for yourself.

<sup>28</sup>**heuristics** [n] [uncountable] (*formal*) a method of solving problems by finding practical ways of dealing with them, learning from past experience.

<sup>29</sup>**perceive** [v] **1.** to notice or become aware of something, SYNONYM: **notice**; **2.** to be aware of or experience something using the senses; **3.** [often passive] to understand or think of somebody/something in a particular way; to believe that a particular thing is true, SYNONYM: **see**.

<sup>30</sup>**rigorous** [a] **1.** done carefully & with a lot of attention to detail, SYNONYM: **thorough**; **2.** demanding that particular rules or processes are strictly followed, SYNONYM: **strict**.

systematic<sup>31</sup>, deductive<sup>32</sup> science; but mathematics in the making appears as an experimental<sup>33</sup>, inductive<sup>34</sup> science. Both aspects<sup>35</sup> are as old as the science of mathematics itself. But the 2nd aspect is new in 1 respect<sup>36</sup>; mathematics “in statu nascendi,” in the process of being invented, has never before presented in quite this manner to the student, or to the teacher himself, or to the general public.

The subject of heuristic has manifold<sup>37</sup> connections; mathematicians, logicians<sup>38</sup>, psychologists, educationalists<sup>39</sup>, even philosophers may claim various parts of it as belonging to their special domains. The author, well aware of the possibility of criticism<sup>40</sup> from opposite<sup>41</sup> quarters<sup>42</sup> & keenly<sup>43</sup> conscious<sup>44</sup> of his limitations<sup>45</sup>, has 1 claim to make: he has some experience in solving problems & in teaching mathematics on various levels.

The subject is more fully dealt with in a more extensive book by the author which is on the way to completion. *Stanford University, Aug 1, 1944*” – Polya, 2014, pp. v–vii

## From the Preface to the 7th Printing

“[...] The 2 volumes *Induction & Analogy in Mathematics & Patterns of Plausible Inference* which constitute my recent work *Mathematics & Plausible Reasoning* continue the line of thinking begun in *How to Solve It. Zurich, Aug 30, 1954*” – Polya, 2014, p. viii

## Preface to the 2nd Edition

“The present 2nd edition adds, besides a few minor improvements, a new 4th part, “Problems, Hints, Solutions.”

As this edition was being prepared for print, a study appeared (Educational Testing Service, Princeton, N.J.; cf. *Time*, Jun 18, 1956) which seems to have formulated<sup>46</sup> a few pertinent<sup>47</sup> observations – they are not new to the people in the know, but it was high time to formulate them for the general public–: “... mathematics has the dubious<sup>48</sup>, honor of being the least

<sup>31</sup>**systematic** [a] **1.** done according to a system or plan, in a thorough, efficient or determined way; **2.** (of an error) happening in the same way all through a process or set of results; caused by the system that is used.

<sup>32</sup>**deductive** [a] [usually before noun] using knowledge about things that are generally true in order to understand particular situations or problems.

<sup>33</sup>**experimental** [a] **1.** [usually before noun] connected with scientific experiments; **2.** based on new ideas, forms or methods that are used to find out what effect they have.

<sup>34</sup>**inductive** [a] (*specialist*) using particular facts & examples to form general rules & principles.

<sup>35</sup>**aspect** [n] **1.** [countable] a particular feature of a situation, an idea or a process; a way in which something may be considered; **2.** [countable, usually singular] **aspect (of something)** (*specialist*) a particular surface or side of an object or a part of the body; the direction in which something faces; **3.** [uncountable, countable] (*grammar*) the form of a verb that shows, e.g., whether the action happens once or many times, is completed or is still continuing.

<sup>36</sup>**respect** [n] **1.** [countable] a particular aspect or detail of something; **2.** [uncountable, singular] polite behavior towards or reasonable treatment of somebody/something; **3.** [uncountable, singular] a feeling of admiration for somebody/something because of their good qualities or achievements; **in respect of something** [idiom] (*formal*) **1.** concerning; **2.** in payment for something; **with respect to something** [idiom] concerning.

<sup>37</sup>**manifold** [a] (*formal*) many; of many different types; [n] (*specialist*) a pipe or chamber with several openings, especially 1 for taking gases in & out of a car engine.

<sup>38</sup>**logician** [n] a person who studies or is skilled in logic.

<sup>39</sup>**educationalists** [n] (also **educationist**) a specialist in theories & methods of teaching.

<sup>40</sup>**criticism** [n] **1.** [uncountable, countable] the act of expressing disapproval of somebody/something & opinions about their faults or bad qualities; a statement showing disapproval; **2.** [uncountable] the work or activity of analyzing & making fair, careful judgments about somebody/something, especially books, music, etc.

<sup>41</sup>**opposite** [a] **1.** [usually before noun] as different as possible from something; involving 2 different extremes; **2.** [usually before noun] on the other side of something or facing something; [n] **1.** (**the opposite**) [singular] the situation, idea or activity that is as different from another situation, etc. as it is possible to be, SYNONYM: **the reverse**; **2.** (**opposites**) [plural] people, ideas or situations that are as different as possible from each other; **the exact opposite** [idiom] a person or thing that is as different as possible from somebody/something else; [prep] on the other side of a particular area from somebody/something, & usually facing them.

<sup>42</sup>**quarter** [n] **1.** (also **fourth especially in NAE**) [countable] 1 of 4 equal parts of something; **2.** [countable] a period of 3 months, used especially as a period for which bills are paid or a company's income is calculated; **3.** [countable] a person or group of people, especially as a source of help, information or a reaction; **4.** [countable, usually singular] a district or part of a town; **5.** (**quarters**) [plural] rooms that are provided for soldiers, servants, etc. to live in; **at/from close quarters** [idiom] very near.

<sup>43</sup>**keenly** [adv] **1.** very strongly or deeply; **2.** by people with different opinions that they express strongly.

<sup>44</sup>**conscious** [a] **1.** [not before noun] aware of something; noticing something, OPPOSITE: **unconscious**; **2.** able to use your senses & mental powers to understand what is happening, OPPOSITE: **unconscious**; **3.** (of actions, feelings, etc.) deliberate or controlled, OPPOSITE: **unconscious**; **4.** being particularly interested in something.

<sup>45</sup>**limitation** [n] **1.** [countable, usually plural] a limit on what somebody/something can do or how good they/it can be; **2.** [countable] a rule, fact or condition that limits something, SYNONYM: **restraint**; **3.** [uncountable] **limitation (of something)** the act or process of limiting or controlling somebody/something, SYNONYM: **restriction**; **4.** (also **limitation period**) [countable] (*law*) a legal limit on the period of time within which court proceedings can be taken or for which a property right continues.

<sup>46</sup>**formulate** [v] **1.** **formulate something** to create or prepare something carefully, giving particular attention to the details; **2.** **formulate something** to express your ideas in carefully chosen words.

<sup>47</sup>**pertinent** [a] appropriate to a particular situation, SYNONYM: **relevant**.

<sup>48</sup>**dubious** [a] **1.** that you cannot be sure about; that is probably not good. **Dubious** is also when you are stating that something is the opposite of a particular good quality. **2.** [usually before noun] probably not honest, SYNONYM: **suspicious**; **3.** [not usually before noun] **dubious about**



popular subject in the curriculum ... Future teachers pass through the elementary schools learning to detest<sup>49</sup> mathematics ... They return to the elementary school to teach a new generation to detest it.”

I hope that the present edition, designed for wider diffusion<sup>50</sup>, will convince some of its readers that mathematics, besides being a necessary avenue<sup>51</sup> to engineering jobs & scientific knowledge, may be fun & may also open up a vista<sup>52</sup> of mental activity on the highest level. *Zurich, Jun 30, 1956*” – Polya, 2014, pp. ix–

## “How to Solve It” list

**1st.** You have to *understand* the problem.

UNDERSTANDING THE PROBLEM.

*What is the unknown? What are the data? What is the condition?*

It is possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

**2nd.** Find the connection between the data & the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

DEVISING A PLAN.

Have you seen it before? Or have you seen the same problem in a slightly different form?

*Do you know a related problem?* Do you know a theorem that could be useful?

*Look at the unknown!* & try to think of a familiar problem having the same or a similar unknown.

*Here is a problem related to yours & solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve 1st some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown & the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

**3rd.** *Carry out* your plan.

CARRYING OUT THE PLAN.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

**4th.** *Examine* the solution obtained.

LOOKING BACK.

Can you *check the result*? Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

– Polya, 2014, How to solve it, pp. xvi–xvii

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**something** feeling uncertain about something; not knowing whether something is good or bad, SYNONYM: **doubtful**.

<sup>49</sup>**detest** [v] (not used in the progressive tenses) to hate somebody/something very much, SYNONYM: **loathe**.

<sup>50</sup>**diffusion** [n] [uncountable] **1.** the spreading of something more widely; **2.** the mixing of substances by the natural movement of their particles;

**3.** the spreading of elements of culture from 1 region or group to another.

<sup>51</sup>**avenue** [n] a way of approaching a problem or making progress towards something.

<sup>52</sup>**vista** [n] **1.** (*literary*) a beautiful view, e.g., of the countryside, a city, etc., SYNONYM: **panorama**; **2.** (*formal*) a range of things that might happen in the future, SYNONYM: **prospect**.

## Foreword by John H. Conway

“*How to Solve It* is a wonderful book! This I realized when I 1st read right through it as a student many years ago, but it has taken me a long time to appreciate just *how* wonderful it is. Why is that? 1 part of the answer is that the book is unique. In all my years as a student & teacher, I have never seen another that lives up to George Polya’s title by teaching you how to go about solving problems. A. H. Schoenfeld correctly described its importance in his 1987 article “Polya, Problem Solving, & Education” in *Mathematics Magazine*: “For mathematics education & the world of problem solving it marked a line of demarcation<sup>53</sup> between 2 eras<sup>54</sup>, problem solving before & after Polya.”

It is 1 of the most successful mathematics books ever written, having sold over a million copies & been translated into 17 languages since it 1st appeared in 1945. Polya later wrote 2 more books about the art of doing mathematics, *Mathematics & Plausible Reasoning* (1954) & *Mathematical Discovery* (2 volumes, 1962 & 1965).

The book’s title makes it seem that it is directed only toward students, but in fact it is addressed just as much to their teachers. Indeed, as Polya remarks in his introduction, the 1st part of the book takes the teacher’s viewpoint more often than the student’s.

Everybody gains that way. The student who reads the book on his own will find that overhearing<sup>55</sup> Polya’s comments to his non-existent<sup>56</sup> teacher can bring that desirable person into being, as an imaginary but very helpful figure leaning over one’s shoulder. This is what happened to me, & naturally I made heavy use of the remarks I’d found most important when I myself started teaching a few years later.

But it was some time before I read the book again, & when I did, I suddenly realized that it was even more valuable than I’d thought! Many of Polya’s remarks that hadn’t helped me as a student now made me a better teacher of those whose problems had differed from mine. Polya had met many more students than I had, & had obviously thought very hard about how to best help all of them learn mathematics. Perhaps his most important point is that learning must be active. As he said in a lecture on teaching, “Mathematics, you see, is not a spectator<sup>57</sup> sport. To understand mathematics means to be able to do mathematics. & what does it mean [to be] doing mathematics? In the 1st place, it means to be able to solve mathematical problems.”

It is often said that to teach any subject well, one has to understand it “at least as well as one’s students do.” It is a paradoxical<sup>58</sup> truth that to teach mathematics well, one must also know how to misunderstand it at least to the extent one’s students do! If a teacher’s statement can be parsed<sup>59</sup> in 2 or more ways, it goes without saying that some students will understand it 1 way & others another, with results that can vary from the hilarious<sup>60</sup> to the tragic<sup>61</sup>. J. E. Littlewood gives 2 amusing<sup>62</sup> examples of assumptions that can easily be made unconsciously & misleadingly<sup>63</sup>. 1st, he remarks that the description of the coordinate axes (“*Ox* & *Oy* as in 2 dimensions, *Oz* vertical”) in Lamb’s book *Mechanics* is incorrect for him, sine he always worked in an armchair<sup>64</sup> with his feet up! Then, after asking how his reader would present the picture of a closed curve lying all on 1 side of its tangent, he states that there are 4 main schools (to left or right of vertical tangent, or above or below horizontal one) & that by lecturing without a figure, presuming that the curve was to the right of its vertical tangent, he had unwittingly<sup>65</sup> made nonsense<sup>66</sup> for the other 3 schools.

I know of no better remedy<sup>67</sup> for such presumptions<sup>68</sup> than Polya’s counsel<sup>69</sup>: before trying to solve a problem, the

<sup>53</sup>**demarcation** [n] [uncountable, countable] a line or limit that separates 2 things, such as types of work, groups of people or areas of land.

<sup>54</sup>**era** [n] **1.** a period of time, usually in history, that is different from other periods because of particular characteristics or events; **2.** (*earth sciences*) a major division of time that can itself be divided into periods.

<sup>55</sup>**overhear** [v] to hear, especially by accident, a conversation in which you are not involved.

<sup>56</sup>**non-existent** [a] not existing; not real.

<sup>57</sup>**spectator** [n] a person who is watching a performance or an event.

<sup>58</sup>**paradoxical** [a] **1.** (of a person, thing or situation) having 2 opposite features & therefore seem strange; **2.** (of a statement) containing 2 opposite ideas that make it seem impossible or unlikely, although it is probably true.

<sup>59</sup>**parse** [v] (*grammar*) **parse something** to divide a sentence into parts & describe the grammar of each word or part.

<sup>60</sup>**hilarious** [a] extremely funny.

<sup>61</sup>**tragic** [a] **1.** making you feel very sad, usually because somebody has died or suffered a lot; **2.** [usually before noun] connected with tragedy (= the style of literature).

<sup>62</sup>**amusing** [a] funny & giving pleasure.

<sup>63</sup>**misleading** [a] giving the wrong idea & making people believe something that is not true, SYNONYM: **deceptive**.

<sup>64</sup>**armchair** [n] a comfortable chair with sides on which you can rest your arms; [a] [only before noun] knowing about a subject through books, television, the Internet, etc., rather than by doing it for yourself.

<sup>65</sup>**unwittingly** [adv] without being aware of what you are doing or the situation that you are involved in, OPPOSITE: **wittingly**.

<sup>66</sup>**nonsense** [n] **1.** [uncountable, countable] ideas, statements or beliefs that you think are silly or not true, SYNONYM: **rubbish**; **2.** [uncountable] spoken or written words that have no meaning or make no sense; **3.** [uncountable] silly or unacceptable behavior; **make (a) nonsense of something** [idiom] to reduce the value of something by a lot; to make something seem silly.

<sup>67</sup>**remedy** [n] (plural **remedies**) **1.** a way of dealing with or improving an unpleasant or difficult situation, SYNONYM: **solution**; **2.** a treatment or medicine to cure a disease or to reduce pain that is not very serious; **3.** (*law*) a way of dealing with a problem, using the processes of the law, SYNONYM: **redress**; [v] **remedy something** to correct or improve something.

<sup>68</sup>**presumption** [n] [countable, uncountable] the act of supposing or accepting that something is true or exists, although it has not been proved; a belief that something is true or exists, SYNONYM: **assumption**. In legal contexts, **presumption** often means that something is being accepted as true until it is shown not to be true.

<sup>69</sup>**counsel** [n] [uncountable, countable] **1.** (*formal*) advice, especially given by older people or experts; a piece of advice; **2.** a lawyer or group of lawyers representing somebody in court; [v] (*formal*) **1. counsel somebody** to listen to & give support or professional advice to somebody who

student should demonstrate his or her understanding of its statement, preferably<sup>70</sup> to a real teacher, but in lieu<sup>71</sup> of that, to an imagined one. Experienced mathematicians know that often the hardest part of researching a problem is understanding precisely what that problem says. They often follow Polya's wise advice: "If you can't solve a problem, then there is an easier problem you can't solve: find it."

Readers who learn from this book will also want to learn about its author's life.<sup>72</sup>

George Polya was born György Pólya (he dropped the accents sometime later) on Dec 13, 1887, in Budapest, Hungary, to Jakab Pólya & his wife, the former Anna Deutsch. He was baptized into the Roman Catholic faith, to which Jakab, Anna, & their 3 previous children, Jenő, Ilona, & Flóra, had converted from Judaism<sup>73</sup> in the previous year. The 5th child, László, was born 4 years later.

Jakab had changed his surname from Pollák to the more Hungarian-sounding Pólya 5 years before György was born, believing that this might help him obtain a university post, which he eventually did, but only shortly before his untimely<sup>74</sup> death in 1897.

At the Dániel Berzsenyi Gymnasium<sup>75</sup>, György studied Greek, Latin, & German, in addition to Hungarian. It is surprising to learn that there he was seemingly uninterested in mathematics, his work in geometry deemed merely "satisfactory" compared with his "outstanding" performance in literature, geography, & other subjects. His favorite subject, outside of literature, was biology.

He enrolled at the University of Budapest in 1905, initially studying law, which he soon dropped because he found it too boring. He then obtained the certification needed to teach Latin & Hungarian at a gymnasium, a certification that he never used but of which he remained proud. Eventually his professor, Bernát Alexander, advised him that to help his studies in philosophy, he should take some mathematics & physics courses. This was how he came to mathematics. Later, he joked that he "wasn't good enough for physics, & was too good for philosophy – mathematics is in between."

In Budapest he was taught physics by Eötvös & mathematics by Fejér & was awarded a doctorate after spending the academic year 1910–11 in Vienna, where he took some courses by Wirtinger & Mertens. He spent much of the next 2 years in Göttingen, where he met many more mathematicians – Klein, Caratheodory, Hilbert, Runge, Landau, Weyl, Hecke, Courant, & Toeplitz – & in 1914 visited Paris, where he became acquainted<sup>76</sup> with Picard & Hadamard & learned that Hurwitz had arranged an appointment for him in Zürich. He accepted this position, writing later: "I went to Zürich in order to be near Hurwitz, & we were in close touch for about 6 years, from my arrival in Zürich in 1914 to his passing [in 1919]. I was very much impressed by him & edited his works."

Of course, the 1st World War took place during this period. It initially had little effect on Polya, who had been declared unfit for service in the Hungarian army as the result of a soccer wound. But later when the army, more desperately<sup>77</sup> needing recruits<sup>78</sup>, demanded that he return to fight for his country, his strong pacifist<sup>79</sup> views led him to refuse. As a consequence, he was unable to visit Hungary for many years, & in fact did not do so until 1967, 54 years after he left.

In the meantime, he had taken Swiss citizenship & married a Swiss girl, Stella Vera Weber, in 1918. Between 1918 & 1919, he published papers on a wide range of mathematical subjects, such as series, number theory, combinatorics, voting systems, astronomy, & probability. He was made an extraordinary professor at the Zürich ETH in 1920, & a few years later he & Gábor Szegő published their book *Aufgaben und Lehrsätze aus der Analysis* ("Problems & Theorems in Analysis"), described by G. L. Alexanderson & L. H. Lange in their obituary<sup>80</sup> of Polya as "a mathematical masterpiece<sup>81</sup> that assured<sup>82</sup>

needs help; **2.** to advise a particular course of action; to advise somebody to do something.

<sup>70</sup>**preferable** [a] more attractive or more suitable; to be preferred to something.

<sup>71</sup>**lieu** [n] (*formal*) **in lieu (of something)** [idiom] instead of.

<sup>72</sup>The following biographical information is taken from that given by J. J. O'Connor & E. F. Robertson in the MacTutor History of Mathematics Archive ([www-gap.dcs.st-and.ac.uk/~history/](http://www-gap.dcs.st-and.ac.uk/~history/)).

<sup>73</sup>**Judaism** [n] [uncountable] the religion of the Jewish people, based mainly on the Bible & the Talmud (= a collection of ancient writings on Jewish law & traditions).

<sup>74</sup>**untimely** [a] (*formally*) **1.** [usually before noun] happening too soon or sooner than is normal or expected, SYNONYM: **premature**; **2.** happening at a time or in a situation that is not suitable, SYNONYM: **ill-timed**, OPPOSITE: **timely**.

<sup>75</sup>**gymnasium** (plural **gymnasiums**, **gymnasia**) (*formal*) a gym.

<sup>76</sup>**acquainted** [a] [not before noun] **1. acquainted with something** (*formal*) familiar with something, having read, seen or experienced it; **2.** not close friends with somebody, but having met a few times before.

<sup>77</sup>**desperate** [a] **1.** feeling or showing that you have little hope & are ready to do anything without worrying about danger to yourself or others; **2.** [usually before noun] (of an action) giving little hope of success; tried when everything else was failed; **3.** (of a situation) extremely serious or dangerous.

<sup>78</sup>**recruit** [v] **1.** [transitive, intransitive] to find new people to join a company, an organization, the armed forces, etc.; **2.** [transitive] to get people to help with or be involved in something; **3.** [transitive] **recruit something (from something)** to form a new army, team, etc. by persuading new people to join it; [n] **1.** a person who has recently joined the armed forces or the police; **2.** a person who joins an organization, a company, etc.

<sup>79</sup>**pacifist** [a] [usually before noun] holding or showing the belief that war & violence are always wrong; [n] a person who believes that war & violence are always wrong & refuses to fight in a war.

<sup>80</sup>**obituary** [n] (plural **obituaries**) an article about somebody's life & achievements, that is printed in a newspaper soon after they have died.

<sup>81</sup>**masterpiece** [n] (also **masterwork**) **1. masterpiece (of something)** a work of art such as a painting, film, book, etc. that is an excellent, or the best, example of the artist's work; **2. masterpiece of something** an extremely good example of something.

<sup>82</sup>**assure** [v] **1.** to tell somebody that something is definitely true or is definitely going to happen, especially when they have doubts about it; **2.** to make something certain to happen; to make somebody/something certain to get something; **3.** to make yourself certain about something.

their reputations<sup>83</sup>.”

That book appeared in 1925, after Polya had obtained a Rockefeller Fellowship to work in England, where he collaborated with Hardy & Littlewood on what later become their book *Inequalities* (Cambridge University Press, 1936). He used a 2nd Rockefeller Fellowship to visit Princeton University in 1933, & while in the United States was invited by H. F. Blichfeldt to visit Stanford University, which he greatly enjoyed, & which ultimately became his home. Polya held a professorship at Stanford from 1943 until his retirement in 1953, & it was there, in 1978, that he taught his last course, in combinatorics; he died on Sep 7, 1985, at the age of 97.

Some readers will want to know about Polya’s many contributions to mathematics. Most of them relate to analysis & are too technical to be understood by non-experts, but a few are worth mentioning.

In probability theory, Polya is responsible for the now-standard term “Central Limit Theorem” & for proving that the Fourier transform of a probability measure is a characteristic function & that a random walk on the integer lattice closes with probability 1 iff the dimension is at most 2.

In geometry, Polya independently re-enumerated the 17 plane crystallographic<sup>84</sup> groups (their 1st enumeration<sup>85</sup>, by E. S. Fedorov, having been forgotten) & together with P. Niggli devised<sup>86</sup> a notation for them.

In combinatorics, Polya’s Enumeration Theorem is now a standard way of counting configurations according to their symmetry. It has been described by R. C. Read as “a remarkable<sup>87</sup> theorem in a remarkable paper, & a landmark<sup>88</sup> in the history of combinatorial analysis.”

*How to Solve It* was written in German during Polya’s time in Zürich, which ended up in 1940, when the European situation forced him to leave for the United States. Despite the book’s eventual success, 4 publishers rejected the English version before Princeton University Press brought it out in 1945. In their hands, *How to Solve It* rapidly became – & continues to be – 1 of the most successful mathematical books of all time.” – Polya, 2014, Foreword, pp. xix–xxiv

## Introduction

# Part I. In The Classroom

## Purpose

### 2.1 Helping the student

### 2.2 Questions, recommendations, mental operations

### 2.3 Generality

### 2.4 Common sense

### 2.5 Teacher & student. Imitation & practice

#### Main divisions, main questions

<sup>83</sup>**reputation** [n] the opinion that people have about what somebody/something is like, based on what has happened in the past.

<sup>84</sup>**crystallography** [n] [uncountable] the branch of science that deals with crystals.

<sup>85</sup>**enumeration** [n] [uncountable, countable] (*formal*) the act of naming things 1 by 1 in a list; a list of this sort.

<sup>86</sup>**devise** [v] **devise something** to plan or invent a procedure, system or method, especially one that is new or complicated, by using careful thought, SYNONYM: **think something up**.

<sup>87</sup>**remarkable** [a] unusual or surprising in a way that causes people to take notice.

<sup>88</sup>**landmark** [n] **1.** something, such as a large building, that you can see clearly from a distance & that will help you to know where you are; **2.** an event, a discovery or an invention that marks an important stage in something.

- 2.6 4 phases
- 2.7 Understanding the problem
- 2.8 Example
- 2.9 Devising a plan
- 2.10 Example
- 2.11 Carrying out the plan
- 2.12 Example
- 2.13 Looking back
- 2.14 Example
- 2.15 Various approaches
- 2.16 The teacher's method of questioning
- 2.17 Good questions & bad questions

More examples

- 2.18 A problem of construction
- 2.19 A problem to prove
- 2.20 A rate problem

## Part II. How to Solve It

A dialogue

## Part III. Short Dictionary of Heuristic

Analogy

Auxiliary elements

Auxiliary problem

Bolzano

Bright idea

Can you check the result?

Can you derive the result differently?

Can you use the result?

Carrying out

Condition

Contradictory

Corollary

Could you derive something useful from the data?

Could you restate the problem?

Decomposing & recombining

Definition

Descartes

Determination, hope, success

Diagnosis

Did you see all the data?

Do you know a related problem?

Draw a figure

Examine your guess

Figures

Generalization

Have you seen it before?

Here is a problem related to yours & solved before

## Problems

## Hints

## Solutions

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