Some Topics in Mathematical Optimization

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Contents

Foreword

A collection of & some personal notes on Mathematical Optimization, especially the 3 major topics: Optimal Control, Shape Optimization, & Topology Optimization.

Keywords. Optimal control; Shape optimization; Topology optimization.

Chapter 1

Optimal Control

1.1 Introduction

"The mathematical optimization of process governed by PDEs has seen considerable progress in the past decade. Ever faster computational facilities & newly developed numerical techniques have opened the door to important practical applications in fields e.g. fluid flow, microelectronics[1], crystal[2] growth, vascular[3] surgery[4], & cardiac[5] medicine, to name just a few. As a consequence, the communities of numerical analysts & optimizers have taken a growing interest in applying their methods to optimal control problems involving PDEs ..." [...] "... the comprehensive text by J.-L. Lions Lions, 1971 covers much of the theory of linear equations & convex cost functionals." – Tröltzsch, 2010, Preface to the German edition, p. xiii

Tröltzsch, 2010 focuses on basic concepts & notions e.g.:

- Existence theory for linear & semilinear PDEs
- Existence of optimal controls
- Necessary optimality conditions & adjoint equations
- 2nd-order sufficient optimality conditions
- Foundation of numerical methods

Question 1.1. *What is optimal control?*

"The mathematical theory of optimal control has in the past few decades rapidly developed into an important & separate field of applied mathematics. 1 area of application of this theory lies in aviation[6] & space technology: aspects of optimization come into play whenever the motion of an aircraft or a space vessel[7] (which can be modeled by ODEs) has to follow a trajectory[8] that is "optimal" in a sense to be specified." – Tröltzsch, 2010, Sect. 1.1: *What is optimal control?*, p. 1

All the essential features of an *optimal control problem*:

- a *cost functional* to be minimized,
- an IVP for an ODE in order to determine the *state* y,
- a *control function* u, &
- various constraints that have to be obeyed.

[1] **microelectronics** [n] [uncountable] the design, production & use of very small electronic circuits.

[2] **crystal** [n] **1.** [countable] a small piece of a substance with many even sides, that is formed naturally when the substance becomes solid; in chemistry, a **crystal** is any solid that has its atoms, ions or molecules arranged in an ordered, symmetrical way; **2.** [uncountable] a clear mineral, e.g. quartz, used in making decorative objects.

[3] **vascular** [a] [usually before noun] (*medical*) connected with or containing veins.

[4] **surgery** [n] **1.** [uncountable, countable] medical treatment of injuries or diseases that involves cutting open a person's body, sewing up wounds, etc.; **2.** [countable] (*British English*) a place where a doctor sees patients; **3.** [countable] (*British English*) a time during which a doctor, an MP or another professional person is available to see people.

[5] **cardiac** [a] [only before noun] (*medical*) connected with the heart or heart disease; if somebody has a **cardiac arrest**, their heart suddenly stops temporarily or permanently.

[6] **aviation** [n] [uncountable] the activity of designing, building & flying aircraft.

[7] **vessel** [n] **1.** a tube that carries blood through the body of a person or an animal, or liquid through the parts of a plant; **2.** (*formal*) a large ship or boat; **3.** (*formal*) a container used for holding liquids, e.g. a bowl or cup.

[8] **trajectory** [n] (plural **trajectories**) (*specialist*) **1.** the curved part of something that has been fired, hit or thrown into the air; **2.** the way in which a person, an event or a process develops over a period of time, often leading to a particular result.

"The control u may be freely chosen within the given constraints, while the state is uniquely determined by the differential equation & the initial conditions. We have to choose u in such a way that the cost function is minimized. Such controls are called *optimal*." [...] "The optimal control of ODEs is of interest not only for aviation & space technology. In fact, it is also important in fields e.g. robotics[9], movement sequences in sports, & the control of chemical processes & power plants, to name just a few of the various applications. In many cases, however, the processes to be optimized can no longer be adequately modeled by ODEs; instead, PDEs have to be employed for their description. E.g., heat conduction[10], diffusion[11], electromagnetic[12] waves, fluid flows, freezing processes, & many other physical phenomenon[13] can be modeled by PDEs.

In these fields, there are numerous interesting problems in which a given cost functional has to be minimized subject to a differential equation & certain constraints being satisfied. The difference from the above problem "merely" consists of the fact that a PDE has to be dealt with in place of an ordinary one." – Tröltzsch, 2010, pp. 2–3

Tröltzsch, 2010 discusses, "through examples in the form of mathematically simplified case studies, the optimal control of heating processes, 2-phase problems, & fluid flows". Tröltzsch, 2010 focuses "on linear & semilinear elliptic & parabolic PDEs, since a satisfactory regularity theory is available for the solutions to such equations. This is not the case for hyperbolic equations. Also, the treatment of quasilinear PDEs is considerably more difficult, & the theory of their optimal control is still an open field in many respects." [...] "... the Hilbert space setting suffices as a functional analytic framework in the case of linear-quadratic theory." – Tröltzsch, 2010, p. 3

Quick notes

Primal-dual active set strategies, whose the exposition now leads to the systems of linear equations to be solved.

[9] **robotics** [n] [uncountable] the science of designing & operating robots.
[10] **conduction** [n] [uncountable] (*physics*) the process by which heat or electricity passes along or through a material.
[11] **diffusion** [n] [uncountable] **1.** the spreading of something more widely; **2.** the mixing of substances by the natural movement of their particles; **3.** the spreading of elements of culture from 1 region or group to another.
[12] **electromagnetic** [a] (*physics*) in which the electrical & magnetic properties of something are related.
[13] **phenomenon** [n] (plural **phenomena** a fact or an event in nature or society, especially one that is not fully understood.)

Chapter 2

Shape Optimization

Chapter 3

Topology Optimization

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