# Advanced Mathematics

Nguyễn Quản Bá Hồng[1]

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[1]Independent Researcher, Ben Tre City, Vietnam
e-mail: nguyenquanbahong@gmail.com

# Contents

# Chapter 1

# Wikipedia's

## 1.1 Wikipedia/Symmetrization Methods

"In mathematics the *symmetrization methods* are algorithms of transforming a set $A \subset \mathbb{R}^n$ to a ball $B \subset \mathbb{R}^n$ with equal volume $\mathrm{vol}(B) = \mathrm{vol}(A)$ & centered at the origin. $B$ is called the *symmetrized version* of $A$, usually denoted $A^\star$. These algorithms show up in solving the classical isoperimetric inequality problem, which asks: Given all 2D shapes of a given area, which of them has the minimal perimeter. The conjectured answer was the disk & Steiner in 1838 showed this to be true using the Steiner symmetrization method. From this many other isoperimetric problems sprung & other symmetrization algorithms. E.g., Rayleigh's conjecture is that the 1st eigenvalue of the Dirichlet problem is minimized for the ball (see Rayleigh–Faber–Krahn inequality for details). Another problem is that the Newtonian capacity of a set $A$ is minimized by $A^\star$ & this was proved by Polya & G. Szego (1951) using circular symmetrization." – Wikipedia/symmetrization methods

### 1.1.1 Symmetrization

"If $\Omega \subset \mathbb{R}^n$ is measurable, then it is denoted by $\Omega^\star$ the symmetrized version of $\Omega$, i.e., a ball $\Omega^\star \coloneqq B_r(0) \subset \mathbb{R}^n$ s.t. $\mathrm{vol}(\Omega^\star) = \mathrm{vol}(\Omega)$. We denote by $f^\star$ the symmetric decreasing rearrangement of nonnegative measurable function $f$ & define it as $f^\star(x) \coloneqq \int_0^\infty 1_{\{y:f(y)>t\}^\star}(x)\mathrm{d}t$, where $\{y : f(y) > t\}^\star$ is the symmetrized version of preimage set $\{y : f(y) > t\}$. The methods described below have been proved to transform $\Omega$ to $\Omega^\star$, i.e., given a sequence of symmetrization transformations $\{T_k\}$ there is $\lim_{k\to\infty} d_{\mathrm{Ha}}(\Omega^\star, T_k(K)) = 0$, where $d_{\mathrm{Ha}}$ is the Hausdorff distance (for discussion & proofs see [Burchard2009])." – Wikipedia/symmetrization methods/symmetrization

### 1.1.2 Steiner symmetrization

Steiner Symmetrization of set $\Omega$.

"Steiner symmetrization was introduced by Steiner (1838) to solve the isoperimetric theorem stated above. Let $H \subset \mathbb{R}^n$ be a hyperplane through the origin. Rotate space so that $H$ is the $x_n = 0$ ($x_n$ is $n$th coordinate in $\mathbb{R}^n$) hyperplane. For each $\mathbf{x} \in H$ let the perpendicular line through $\mathbf{x} \in H$ be $L_{\mathbf{x}} = \{\mathbf{x} + y\mathbf{e}_n : y \in \mathbb{R}\}$. Then by replacing each $\Omega \cap L_{\mathbf{x}}$ by a line centered at $H$ & with length $|\Omega \cap L_{\mathbf{x}}|$ we obtain the *Steiner symmetrized version*.

$$\mathrm{St}(\Omega) \coloneqq \left\{\mathbf{x} + y\mathbf{e}_n : \mathbf{x} + z\mathbf{e}_n \in \Omega \text{ for some } \mathbf{z} \ \& \ |y| \le \frac{1}{2}|\Omega \cap L_x|\right\}.$$

It is denoted by $\mathrm{St}(f)$ the *Steiner symmetrization* w.r.t. $x_n = 0$ hyperplane of nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}$ & for fixed $x_1, \ldots, x_{n-1}$ define it as $\mathrm{St} : f(x_1, \ldots, x_{n-1}, \cdot) \mapsto (f(x_1, \ldots, x_{n-1}))^\star$.

#### 1.1.2.1 Properties

It preserves convexity: if $\Omega$ is convex, then $\mathrm{St}(\Omega)$ is also convex. It is linear: $\mathrm{St}(\mathbf{x} + \lambda\Omega) = \mathrm{St}(\mathbf{x}) + \lambda\,\mathrm{St}(\Omega)$. Super-additive: $\mathrm{St}(K) + \mathrm{St}(U) \subset \mathrm{St}(K + U)$." – Wikipedia/symmetrization methods/Steiner symmetrization

### 1.1.3 Circular symmetrization

Fig. Circular symmetrization of set $\Omega$.

"A popular method for symmetrization in the plane is *Polya's circular symmetrization*. After, its generalization will be described to higher dimensions. Let $\Omega \subset \mathbb{C}$ be a domain; then its circular symmetrization $\mathrm{Circ}(\Omega)$ with regard to the positive real axis is defined as follows: Let $\Omega_t \coloneqq \{\theta \in [0, 2\pi] : te^{i\theta} \in \Omega\}$, i.e., contain the arcs of radius $t$ contained in $\Omega$. So it is defined

- If $\Omega_t$ is the full circle, then $\operatorname{Circ}(\Omega) \cap \{|z| = t\} \coloneqq \{|z| = t\}$.
- If the length is $\mathrm{m}(\Omega_t) = \alpha$, then $\operatorname{Circ}(\Omega) \cap \{|z| = t\} \coloneqq \left\{te^{i\theta} : |\theta| < \frac{\alpha}{2}\right\}$.
- $0, \infty \in \operatorname{Circ}(\Omega)$ if $0, \infty \in \Omega$.

In higher dimensions $\Omega \subset \mathbb{R}^n$, its spherical symmetrization $\operatorname{Sp}^n(\Omega)$ w.r.t. the positive axis of $x_1$ is defined as follows: Let $\Omega_r \coloneqq \{\mathbf{x} \in \mathbb{S}^{n-1} : r\mathbf{x} \in \Omega\}$, i.e., contain the caps of radius $r$ contained in $\Omega$. Also, for the 1st coordinate let $\operatorname{angle}(x_1) \coloneqq \theta$ if $x_1 = r\cos\theta$. So as above

- If $\Omega_r$ is the full cap, then $\operatorname{Sp}^n(\Omega) \cap \{|z| = r\} \coloneqq \{|z| = t\}$.
- If the surface area is $\mathrm{m}_{\mathrm{s}}(\Omega_t) = \alpha$, then $\operatorname{Sp}^n(\Omega) \cap \{|z| = r\} \coloneqq \{x : |x| = r \;\&\; 0 \le \operatorname{angle}(x_1) \le \theta_\alpha\} \eqqcolon C(\theta_\alpha)$ where $\theta_\alpha$ is picked so that its surface area is $\mathrm{m}_s(C(\theta_\alpha)) = \alpha$. In words, $C(\theta_\alpha)$ is a cap symmetric around the positive axis $x_1$ with the same area as the intersection $\Omega \cap \{|z| = r\}$.
- $0, \infty \in \operatorname{Sp}^n(\Omega)$ iff $0, \infty \in \Omega$." – Wikipedia/symmetrization methods/circular symmetrization

### 1.1.4 Polarization

Fig: Polarization of set $\Omega$.

"Let $\Omega \subset \mathbb{R}^n$ be a domain & $H^{n-1} \subset \mathbb{R}^n$ be a hyperplane through the origin. Denote the reflection across that plane to the positive halfspace $\mathbb{H}^+$ as $\sigma_H$ or just $\sigma$ when it is clear from the context. Also, the reflected $\Omega$ across hyperplane $H$ is defined as $\sigma\Omega$. Then, the polarized $\Omega$ is denoted as $\Omega^\alpha$ & defined as follows

- If $\mathbf{x} \in \Omega \cap \mathbb{H}^+$, then $\mathbf{x} \in \Omega^\alpha$.
- If $\mathbf{x} \in \Omega \cap \sigma(\Omega) \cap \mathbb{H}^-$, then $\mathbf{x} \in \Omega^\sigma$.
- If $\mathbf{x} \in (\Omega \backslash \sigma(\Omega)) \cap \mathbb{H}^-$, then $\sigma\mathbf{x} \in \Omega^\sigma$.

In words, $(\Omega \backslash \sigma(\Omega)) \cap \mathbb{H}^-$ is simply reflected to the halfspace $\mathbb{H}^+$. It turns out that this transformation can approximate the above ones (in the Hausdorff distance) (see [Brock & Solynin2000])." – Wikipedia/symmetrization methods/polarization

## 1.2 Wikipedia/Interpolation Space

"In the field of mathematical analysis, an *interpolation space* is a space which lies "in between" 2 other Banach spaces. The main applications are in Sobolev spaces, where spaces of functions that have a noninteger number of derivatives are interpolated from the spaces of functions with integer number of derivatives." – Wikipedia/interpolation space

### 1.2.1 History

"The theory of interpolation of vector spaces began by an observation of Józef Marcinkiewicz, later generalized & now known as the Riesz–Thorin theorem. In simple terms, if a linear function is continuous on a certain space $L^p$ & also on a certain space $L^q$, then it is also continuous on the space $L^r$, for any intermediate $r$ between $p$ & $q$. In other words, $L^r$ is a space which is intermediate between $L^p$ & $L^q$.

In the development of Sobolev spaces, it became clear that the trace spaces were not any of the usual function spaces (with integer number of derivatives), & Jacques-Louis Lions discovered that indeed these trace spaces were constituted of functions that have a noninteger degree of differentiability.

Many methods were designed to generate such spaces of functions, including the Fourier transform, complex interpolation, real interpolation, as well as other tools (see e.g. fractional derivative)." – Wikipedia/interpolation space/history

### 1.2.2 The setting of interpolation

"A Banach space $X$ is said to be *continuously embedded* in a Hausdorff topological vector space $Z$ when $X$ is a linear subspace of $Z$ s.t. the inclusion map from $X$ into $Z$ is continuous. A *compatible couple* $(X_0, X_1)$ of Banach spaces consists of 2 Banach spaces $X_0$ & $X_1$ that are continuously embedded in the same Hausdorff topological vector space $Z$. The embedding in a linear space $Z$ allows to consider the 2 linear subspaces $X_0 \cap X_1$ & $X_0 + X_1 = \{z \in Z; z = x_0 + x_1,\, x_0 \in X_0,\, x_1 \in X_1\}$. Interpolation does not depend only upon the isomorphic (nor isometric) equivalence classes of $X_0$ & $X_1$. It depends in an essential way from the specific *relative position* that $X_0$ & $X_1$ occupy in a larger space $Z$. One can define norms on $X_0 \cap X_1$ & $X_0 + X_1$ by $\|x\|_{X_0 \cap X_1} \coloneqq \max(\|x\|_{X_0}, \|x\|_{X_1})$, $\|x\|_{X_0 + X_1} \coloneqq \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1}; x = x_0 + x_1,\, x_0 \in X_0,\, x_1 \in X_1\}$. Equipped with these norms, the intersection & the sum are Banach spaces. The following inclusions are all continuous: $X_0 \cap X_1 \subset X_0$,

$X_1 \subset X_0 + X_1$. Interpolation studies the family of spaces $X$ that are *intermediate spaces* between $X_0$ & $X_1$ in the sense that $X_0 \cap X_1 \subset X \subset X_0 + X_1$, where the 2 inclusions maps are continuous.

An example of this situation is the pair $(L^1(\mathbb{R}),\ L^\infty(\mathbb{R}))$, where the 2 Banach spaces are continuously embedded in the space $Z$ of measurable functions on the real line, equipped with the topology of convergence in measure. In this situation, the spaces $L^p(\mathbb{R})$, for $1 \le p \le \infty$ are intermediate between $L^1(\mathbb{R})$ & $L^\infty(\mathbb{R})$. More generally,

$$L^{p_0}(\mathbb{R}) \cap L^{p_1}(\mathbb{R}) \subset L^p(\mathbb{R}) \subset L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R}),\ \text{when } 1 \le p_0 \le p \le p_1 \le \infty,$$

with continuous injections, so that, under the given condition, $L^p(\mathbb{R})$ is intermediate between $L^{p_0}(\mathbb{R})$ & $L^{p_1}(\mathbb{R})$.

**Definition 1.1** (Interpolation pair). *Given 2 compatible couples $(X_0, X_1)$ & $(Y_0, Y_1)$, an* interpolation pair *is a couple $(X, Y)$ of Banach spaces with the 2 following properties:*

- *The space $X$ is intermediate between $X_0$ & $X_1$, & $Y$ is intermediate between $Y_0$ & $Y_1$.*
- *If $L$ is any linear operator from $X_0 + X_1$ to $Y_0 + Y_1$, which maps continuously $X_0$ to $Y_0$ & $X_1$ to $Y_1$, then it also maps continuously $X$ to $Y$.*

The interpolation pair $(X, Y)$ is said to be of *exponent* $\theta$ (with $0 < \theta < 1$) if there exists a constant $C$ s.t. $\|L\|_{X,Y} \le C\|L\|_{X_0,Y_0}^{1-\theta}\|L\|_{X_1,Y_1}^{\theta}$ for all operators $L$ as above. The notation $\|L\|_{X,Y}$ is for the norm of $L$ as a map from $X$ to $Y$. If $C = 1$, we say that $(X, Y)$ is an *exact interpolation pair of exponent* $\theta$." – Wikipedia/interpolation space/the setting of interpolation

### 1.2.3 Complex interpolation

### 1.2.4 Real interpolation

#### 1.2.4.1 K-method

##### 1.2.4.1.1 Example.

#### 1.2.4.2 J-method

##### 1.2.4.2.1 Example.

#### 1.2.4.3 Relations between the interpolation methods

##### 1.2.4.3.1 Example.

### 1.2.5 The reiteration theorem

### 1.2.6 Duality

### 1.2.7 Discrete definitions

#### 1.2.7.1 A general interpolation method

### 1.2.8 Interpolation of Sobolev & Besov spaces

# Chapter 2

# Terence Tao's

## 2.1 Tao, 2007. What Is Good Mathematics?

**Abstract.** "Some personal thoughts & opinions on what "good quality mathematics" is & whether one should try to define this term rigorously. As a case study, the story of Szemerédi's theorem is presented."

### 2.1.1 The Many Aspects of Mathematical Quality

"We all agree that mathematicians should strive[1] to produce good mathematics. *But how does one define "good mathematics", & should one even dare to try at all?* Let us 1st consider the former question. Almost immediately one realizes that there are many different types of mathematics which could be designated[2] "good". E.g., "good mathematics" could refer (in no particular[3] order) to

1. Good mathematical *problem solving* (e.g. a major[4] breakthrough[5] on an important mathematical problem);
2. Good mathematical *technique*[6] (e.g. a masterful[7] use of existing[8] methods[9] or the development[10] of new tools[11]);
3. Good mathematical *theory* (e.g. a conceptual[12] framework[13] or choice of notation[14] which systematically[15] unifies[16] &

[1]**strive** [v] [intransitive] to try very hard to achieve something.

[2]**designate** [v] [often passive] **1.** to say officially that somebody/something has a particular character, name or purpose; to describe somebody/something in a particular way; **2.** to choose or name somebody/something for a particular job or position; **3.** (of a symbol) to identify or show something.

[3]**particular** [a] [only before noun] **1.** used to emphasize that you are referring to 1 individual person, thing or type of thing & not others, SYNONYM: **specific**; **2.** greater than usual; special; **in particular** [idiom] **1.** especially or particularly; **2.** special, SYNONYM: **specific**; **of particular note** [idiom] especially interesting; [n] **1.** [countable, usually plural] a fact or detail, especially one that is officially written down; **2.** (**particulars**) [plural] written information & details about a property, business, job, etc.

[4]**major** [a] **1.** [usually before noun] large, important or serious, OPPOSITE: **minor**; **2.** [only before noun] greater or more important; main, SYNONYM: **main**; [n] (*North American English*) **1.** the main subject or course of a student at college or university; **2.** a student studying a particular subject as the main part of their course.

[5]**breakthrough** [n] an important development or discovery that helps people to achieve or understand something.

[6]**technique** [n] **1.** [countable] a particular way of doing something that involves using a special skill or process; **2.** [uncountable, singular] a person's skill or ability in a particular activity.

[7]**masterful** [a] **1.** (of a person, especially a man) able to control people or situations in a way that shows confidence as a leader; **2.** (also **masterly**) showing great skill or understanding.

[8]**existing** [a] [only before noun] found or used now or at the time being discussed.

[9]**method** [n] a particular way of doing something.

[10]**development** [n] **1.** [uncountable] the process of creating a new method, system, product or theory; **2.** [countable] a new or advanced method, system, product or theory; **3.** [uncountable] the process of making a country or area richer & more successful; **4.** [uncountable] the way in which a child or other living creature grows before & after birth.

[11]**tool** [n] **1.** a thing that helps somebody to do a job or to achieve something; **2.** a piece of equipment held in the hand, that is used for making things or repairing things.

[12]**conceptual** [a] connected with or based on ideas.

[13]**framework** [n] **1.** a set of beliefs, ideas or principles that is based as the basis for examining or understanding something; **2.** a system of rules, laws or agreements that controls the way that something works in business, politics or society.

[14]**notation** [n] [uncountable, countable] **notation (for something)** a system of signs or symbols used to represent information, especially in mathematics, science & music.

[15]**systematically** [adv] **1.** in a way that follows a system; **2.** in the same way all through a process or set of results because of the system that is used.

[16]**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

generalizes[17] an existing[18] body of results);

4. Good mathematical *insight*[19] (e.g. a major conceptual simplification[20] or the realization[21] of a unifying[22] principle[23], analogy[24], or theme[25]);
5. Good mathematical *discovery*[26] (e.g. the revelation[27] of an unexpected[28] & intriguing[29] new mathematical phenomenon[30], connection[31], or counterexample[32]);
6. Good mathematical *application*[33] (e.g. to important problems in physics, engineering, computer science, statistics, etc., or from 1 field of mathematics to another);
7. Good mathematical *exposition*[34] (e.g. a detailed[35] & informative[36] survey[37] on a timely[38] mathematical topic or a clear & well-motivated argument);
8. Good mathematical *pedagogy*[39] (e.g. a lecture[40] or writing style which enables others to learn & do mathematics more

[17]**generalize** [v] (*British English also* **generalise**) **1.** [intransitive] **generalize (from something)** to use a particular set of facts or ideas in order to form an opinion that is considered valid for a different situation; **2.** [intransitive] to make a general statement about something & not look at the details; **3.** [transitive, often passive] to apply a theory, idea, etc. to a wider group or situation than the original one.

[18]**existing** [a] [only before noun] found or used now or at the time being discussed.

[19]**insight** [n] **1.** [countable, uncountable] an understanding of a particular situation or thing; **2.** [uncountable] the ability to see & understand the truth about people or situations.

[20]**simplification** [n] **1.** [uncountable] **simplification (of something)** the process of making something less complicated, or easier to do or understand; **2.** [countable] a change that makes a problem, statement, system, etc. less complicated or easier to understand or do.

[21]**realization** [n] (*British English also* **realisation**) **1.** [uncountable, singular] **realization (that)** ... the process of becoming aware of something, SYNONYM: **awareness**; **2.** [uncountable] **realization (of something)** the process of achieving a particular aim, etc., SYNONYM: **achievement**; **3.** [uncountable, countable] **realization (of something)** (*formal*) the act of producing something in an actual or physical form; the thing that is produced.

[22]**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

[23]**principle** [n] **1.** [countable] a law, rule or theory that something is based on; **2.** [singular] a general or scientific law that explains how something works or why something happens; **3.** [countable] a belief that is accepted as a reason for acting or thinking in a particular way; **4.** [countable, usually plural, uncountable] a moral rule or a strong belief that influences your actions; **in principle** [idiom] **1.** if something can be done in principle, there is no good reason why it should not be done although it has not yet been done & there may be some difficulties; **2.** in general but not in detail.

[24]**analogy** [n] (plural **analogies**) [countable, uncountable] a comparison of 1 thing with another thing that has similar features, usually in order to explain it; a feature that is similar.

[25]**theme** [n] the subject of a talk, piece of writing, exhibition, etc.; an idea that keeps returning in a piece of research or a work of art or literature.

[26]**discovery** [n] (plural **discoveries**) **1.** [countable, uncountable] an act or the process of finding somebody/something, or learning about something that was not known about before; **2.** [countable] a thing, fact or person that is found or learned about for the 1st time.

[27]**revelation** [n] **1.** [countable] a fact that people are made aware of, especially one that has been secret & is surprising, SYNONYM: **disclosure**; **2.** [uncountable] **revelation (of something)** the act of making people aware of something that has been secret, SYNONYM: **disclosure**; **3.** [countable, uncountable] something that is considered to be a sign or message from God.

[28]**unexpected** [a] surprising; not expected.

[29]**intriguing** [a] very interesting because of being unusual or not having an obvious answer.

[30]**phenomenon** [n] (plural **phenomena**) a fact or an event in nature or society, especially one that is not fully understood.

[31]**connection** [n] (*British English also, old-fashioned* **connexion**) **1.** [countable] something that connects 2 facts or ideas, SYNONYM: **link**; **2.** [countable] a relationship between people or groups of people, often for a particular purpose; **3.** [uncountable, countable] the action of connecting something to a supply of water, electricity, etc. or to a computer or telephone network; the fact of being connected in this way; **4.** [countable] a point, especially in an electrical system, where 2 parts connect; **5.** [countable, usually plural] a means of traveling to another place; **6.** [countable, usually plural] people that you know, who can help or advise you in your professional or social life; **in connection with somebody/something** [idiom] for reasons connected with somebody/something; **in this/that connection** [idiom] for reasons connected with something recently mentioned.

[32]**counterexample** [n] **counterexample (to something)** an example that provides evidence against an idea or theory.

[33]**application** [n] **1.** [uncountable, countable] the use of something such as an idea, method, rule, etc.; a use that something has; **2.** [countable] a formal (often written) request to an organization or authority for something, such as a job or permission to do something, or to join a group; **3.** [countable] a program or piece of software designed to do a particular job; **4.** [countable, uncountable] **application (of something) (to something)** the use of something to produce a particular physical effect; **5.** [countable, uncountable] **application (of something)** the action of putting or spreading something onto a surface or object.

[34]**exposition** [n] [countable, uncountable] (*formal*) a full explanation of a theory, plan, etc.

[35]**detailed** [a] giving many details; paying great attention to details.

[36]**informative** [a] giving useful information.

[37]**survey** [n] **1. survey (of somebody/something)** an investigation of the opinions, behavior, etc. of a particular group of people, which is usually done by asking them questions; **2.** an act of examining & recording the measurements, features, etc. of an area of land in order to make a map or plan of it; **3. survey (of something)** a general study, view or description of something; [v] **1. survey somebody/something** to investigate the opinions or behavior of a group of people by asking them a series of questions; **2. survey something** to study & give a general description of something; **3. survey something** to measure & record the features of an area of land, e.g. in order to make a map or in preparation for building; **4. survey something** to look carefully at the whole of something, especially in order to get a general impression of it, SYNONYM: **inspect**.

[38]**timely** [a] happening at exactly the right time.

[39]**pedagogy** [n] (plural **pedagogies**) [uncountable, countable] methods of teaching, especially as a subject of study or as a theory.

[40]**lecture** [n] a talk that is given to a group of people to teach them about a particular subject, often as part of a university or college course; [v] [intransitive] **lecture (in/on something) (to somebody)** to give a talk or a series of talks to a group of people on a particular subject, especially as a way of teaching in a university or college.

effectively, or contributions[41] to mathematical education);

9. Good mathematical *vision*[42] (e.g. a long-range[43] & fruitful program or set of conjectures[44]);
10. Good mathematical *taste* (e.g. a research goal which is inherently interesting & impacts important topics, themes, or questions);
11. Good mathematical *public relations* (e.g. an effective showcasing of a mathematical achievement to non-mathematicians or from 1 field of mathematics to another);

[41]**contribution** [n] **1.** [usually singular] the part played by a person or thing in achieving, improving or causing something; **2.** a sum of money that is given to a person or an organization in order to help pay for something, SYNONYM: **donation**; **3. contribution (to something)** an item that forms part of a book, magazine, broadcast, discussion, etc.; **4.** a sum of money that you pay regularly to your employer or the government in order to pay for benefits such as health insurance or a pension.

[42]**vision** [n] **1.** [uncountable] the ability to see; the area that you can see from a particular position; **2.** [countable] an idea or a picture in your imagination, especially of what the future will or could be like; **3.** [uncountable] the ability to think about or plan the future with great imagination & intelligence.

[43]**long-range** [a] [only before noun] **1.** traveling a long distance; **2.** made for a period of time that will last a long way into the future.

[44]**conjecture** [n] (*formal*) **1.** [countable] an opinion or idea that is not based on definite knowledge & is formed by guessing, SYNONYM: **guess**; **2.** [uncountable] the act of forming an opinion or idea that is not based on definite knowledge; [v] [intransitive, transitive] (*formal*) to form an opinion about something even though you do not have much information on it, SYNONYM: **guess**.

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