Advanced Mathematics

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September 7, 2022

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Chapter 1

Wikipedia's

1.1 Wikipedia/Symmetrization Methods

"In mathematics the *symmetrization methods* are algorithms of transforming a set $A \subset \mathbb{R}^n$ to a ball $B \subset \mathbb{R}^n$ with equal volume $\mathrm{vol}(B) = \mathrm{vol}(A)$ & centered at the origin. B is called the *symmetrized version* of A, usually denoted $A^\star$. These algorithms show up in solving the classical isoperimetric inequality problem, which asks: Given all 2D shapes of a given area, which of them has the minimal perimeter. The conjectured answer was the disk & Steiner in 1838 showed this to be true using the Steiner symmetrization method. From this many other isoperimetric problems sprung & other symmetrization algorithms. E.g., Rayleigh's conjecture is that the 1st eigenvalue of the Dirichlet problem is minimized for the ball (see Rayleigh–Faber–Krahn inequality for details). Another problem is that the Newtonian capacity of a set A is minimized by $A^\star$ & this was proved by Polya & G. Szego (1951) using circular symmetrization." – Wikipedia/symmetrization methods

1.1.1 Symmetrization

"If $\Omega \subset \mathbb{R}^n$ is measurable, then it is denoted by $\Omega^\star$ the symmetrized version of Ω, i.e., a ball $\Omega^\star \coloneqq B_r(0) \subset \mathbb{R}^n$ s.t. $\mathrm{vol}(\Omega^\star) = \mathrm{vol}(\Omega)$. We denote by $f^\star$ the symmetric decreasing rearrangement of nonnegative measurable function f & define it as $f^\star(x) \coloneqq \int_0^\infty 1_{\{y:f(y)>t\}^\star}(x)\mathrm{d}t$, where $\{y : f(y) > t\}^\star$ is the symmetrized version of preimage set $\{y : f(y) > t\}$. The methods described below have been proved to transform Ω to $\Omega^\star$, i.e., given a sequence of symmetrization transformations $\{T_k\}$ there is $\lim_{k\to\infty} d_{\mathrm{Ha}}(\Omega^\star, T_k(K)) = 0$, where d_{Ha} is the Hausdorff distance (for discussion & proofs see [Burchard2009])." – Wikipedia/symmetrization methods/symmetrization

1.1.2 Steiner symmetrization

Steiner Symmetrization of set Ω.

"Steiner symmetrization was introduced by Steiner (1838) to solve the isoperimetric theorem stated above. Let $H \subset \mathbb{R}^n$ be a hyperplane through the origin. Rotate space so that H is the $x_n = 0$ (x_n is nth coordinate in $\mathbb{R}^n$) hyperplane. For each $\mathbf{x} \in H$ let the perpendicular line through $\mathbf{x} \in H$ be $L_\mathbf{x} = \{\mathbf{x} + y\mathbf{e}_n : y \in \mathbb{R}\}$. Then by replacing each $\Omega \cap L_\mathbf{x}$ by a line centered at H & with length $|\Omega \cap L_\mathbf{x}|$ we obtain the *Steiner symmetrized version.*

$$\mathrm{St}(\Omega) \coloneqq \left\{\mathbf{x} + y\mathbf{e}_n : \mathbf{x} + z\mathbf{e}_n \in \Omega \text{ for some } \mathbf{z} \;\&\; |y| \le \frac{1}{2}|\Omega \cap L_x|\right\}.$$

It is denoted by $\mathrm{St}(f)$ the *Steiner symmetrization* w.r.t. $x_n = 0$ hyperplane of nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}$ & for fixed $x_1, \ldots, x_{n-1}$ define it as $\mathrm{St} : f(x_1, \ldots, x_{n-1}, \cdot) \mapsto (f(x_1, \ldots, x_{n-1}))^\star$.

1.1.2.1 Properties

It preserves convexity: if Ω is convex, then $\mathrm{St}(\Omega)$ is also convex. It is linear: $\mathrm{St}(\mathbf{x} + \lambda\Omega) = \mathrm{St}(\mathbf{x}) + \lambda\,\mathrm{St}(\Omega)$. Super-additive: $\mathrm{St}(K) + \mathrm{St}(U) \subset \mathrm{St}(K + U)$." – Wikipedia/symmetrization methods/Steiner symmetrization

1.1.3 Circular symmetrization

Fig. Circular symmetrization of set Ω.

"A popular method for symmetrization in the plane is *Polya's circular symmetrization.* After, its generalization will be described to higher dimensions. Let $\Omega \subset \mathbb{C}$ be a domain; then its circular symmetrization $\mathrm{Circ}(\Omega)$ with regard to the positive real axis is defined as follows: Let $\Omega_t \coloneqq \{\theta \in [0, 2\pi] : te^{i\theta} \in \Omega\}$, i.e., contain the arcs of radius t contained in Ω. So it is defined

- If Ω_t is the full circle, then $\mathrm{Circ}(\Omega)\cap\{|z|=t\}\coloneqq\{|z|=t\}$.
- If the length is $\mathrm{m}(\Omega_t)=\alpha$, then $\mathrm{Circ}(\Omega)\cap\{|z|=t\}\coloneqq\left\{te^{i\theta}:|\theta|<\frac{\alpha}{2}\right\}$.
- $0,\infty\in\mathrm{Circ}(\Omega)$ if $0,\infty\in\Omega$.

In higher dimensions $\Omega\subset\mathbb{R}^n$, its spherical symmetrization $\mathrm{Sp}^n(\Omega)$ w.r.t. the positive axis of x_1 is defined as follows: Let $\Omega_r\coloneqq\{\mathbf{x}\in\mathbb{S}^{n-1}:r\mathbf{x}\in\Omega\}$, i.e., contain the caps of radius r contained in Ω. Also, for the 1st coordinate let $\mathrm{angle}(x_1)\coloneqq\theta$ if $x_1=r\cos\theta$. So as above

- If Ω_r is the full cap, then $\mathrm{Sp}^n(\Omega)\cap\{|z|=r\}\coloneqq\{|z|=t\}$.
- If the surface area is $\mathrm{m_s}(\Omega_t)=\alpha$, then $\mathrm{Sp}^n(\Omega)\cap\{|z|=r\}\coloneqq\{x:|x|=r\ \&\ 0\le\mathrm{angle}(x_1)\le\theta_\alpha\}=:C(\theta_\alpha)$ where θ_α is picked so that its surface area is $\mathrm{m}_s(C(\theta_\alpha))=\alpha$. In words, $C(\theta_\alpha)$ is a cap symmetric around the positive axis x_1 with the same area as the intersection $\Omega\cap\{|z|=r\}$.
- $0,\infty\in\mathrm{Sp}^n(\Omega)$ iff $0,\infty\in\Omega$." – Wikipedia/symmetrization methods/circular symmetrization

1.1.4 Polarization

Fig: Polarization of set Ω.

"Let $\Omega\subset\mathbb{R}^n$ be a domain & $H^{n-1}\subset\mathbb{R}^n$ be a hyperplane through the origin. Denote the reflection across that plane to the positive halfspace $\mathbb{H}^+$ as σ_H or just σ when it is clear from the context. Also, the reflected Ω across hyperplane H is defined as $\sigma\Omega$. Then, the polarized Ω is denoted as Ω^α & defined as follows

- If $\mathbf{x}\in\Omega\cap\mathbb{H}^+$, then $\mathbf{x}\in\Omega^\alpha$.
- If $\mathbf{x}\in\Omega\cap\sigma(\Omega)\cap\mathbb{H}^-$, then $\mathbf{x}\in\Omega^\sigma$.
- If $\mathbf{x}\in(\Omega\backslash\sigma(\Omega))\cap\mathbb{H}^-$, then $\sigma\mathbf{x}\in\Omega^\sigma$.

In words, $(\Omega\backslash\sigma(\Omega))\cap\mathbb{H}^-$ is simply reflected to the halfspace $\mathbb{H}^+$. It turns out that this transformation can approximate the above ones (in the Hausdorff distance) (see [Brock & Solynin2000])." – Wikipedia/symmetrization methods/polarization

1.2 Wikipedia/Interpolation Space

"In the field of mathematical analysis, an *interpolation space* is a space which lies "in between" 2 other Banach spaces. The main applications are in Sobolev spaces, where spaces of functions that have a noninteger number of derivatives are interpolated from the spaces of functions with integer number of derivatives." – Wikipedia/interpolation space

1.2.1 History

"The theory of interpolation of vector spaces began by an observation of Józef Marcinkiewicz, later generalized & now known as the Riesz–Thorin theorem. In simple terms, if a linear function is continuous on a certain space L^p & also on a certain space L^q, then it is also continuous on the space L^r, for any intermediate r between p & q. In other words, L^r is a space which is intermediate between L^p & L^q.

In the development of Sobolev spaces, it became clear that the trace spaces were not any of the usual function spaces (with integer number of derivatives), & Jacques-Louis Lions discovered that indeed these trace spaces were constituted of functions that have a noninteger degree of differentiability.

Many methods were designed to generate such spaces of functions, including the Fourier transform, complex interpolation, real interpolation, as well as other tools (see e.g. fractional derivative)." – Wikipedia/interpolation space/history

1.2.2 The setting of interpolation

"A Banach space X is said to be *continuously embedded* in a Hausdorff topological vector space Z when X is a linear subspace of Z s.t. the inclusion map from X into Z is continuous. A *compatible couple* (X_0,X_1) of Banach spaces consists of 2 Banach spaces X_0 & X_1 that are continuously embedded in the same Hausdorff topological vector space Z. The embedding in a linear space Z allows to consider the 2 linear subspaces $X_0\cap X_1$ & $X_0+X_1=\{z\in Z;z=x_0+x_1,\,x_0\in X_0,\,x_1\in X_1\}$. Interpolation does not depend only upon the isomorphic (nor isometric) equivalence classes of X_0 & X_1. It depends in an essential way from the specific *relative position* that X_0 & X_1 occupy in a larger space Z. One can define norms on $X_0\cap X_1$ & X_0+X_1 by $\|x\|_{X_0\cap X_1}\coloneqq\max(\|x\|_{X_0},\|x\|_{X_1})$, $\|x\|_{X_0+X_1}\coloneqq\inf\{\|x_0\|_{X_0}+\|x_1\|_{X_1};x=x_0+x_1,\,x_0\in X_0,\,x_1\in X_1\}$. Equipped with these norms, the intersection & the sum are Banach spaces. The following inclusions are all continuous: $X_0\cap X_1\subset X_0$,

$X_1 \subset X_0 + X_1$. Interpolation studies the family of spaces X that are *intermediate spaces* between X_0 & X_1 in the sense that $X_0 \cap X_1 \subset X \subset X_0 + X_1$, where the 2 inclusions maps are continuous.

An example of this situation is the pair $(L^1(\mathbb{R}),\ L^\infty(\mathbb{R}))$, where the 2 Banach spaces are continuously embedded in the space Z of measurable functions on the real line, equipped with the topology of convergence in measure. In this situation, the spaces $L^p(\mathbb{R})$, for $1 \le p \le \infty$ are intermediate between $L^1(\mathbb{R})$ & $L^\infty(\mathbb{R})$. More generally,

$$L^{p_0}(\mathbb{R}) \cap L^{p_1}(\mathbb{R}) \subset L^p(\mathbb{R}) \subset L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R}),\ \text{when } 1 \le p_0 \le p \le p_1 \le \infty,$$

with continuous injections, so that, under the given condition, $L^p(\mathbb{R})$ is intermediate between $L^{p_0}(\mathbb{R})$ & $L^{p_1}(\mathbb{R})$.

Definition 1.1 (Interpolation pair)**.** *Given 2 compatible couples (X_0, X_1) & (Y_0, Y_1), an* interpolation pair *is a couple (X, Y) of Banach spaces with the 2 following properties:*

- *The space X is intermediate between X_0 & X_1, & Y is intermediate between Y_0 & Y_1.*
- *If L is any linear operator from $X_0 + X_1$ to $Y_0 + Y_1$, which maps continuously X_0 to Y_0 & X_1 to Y_1, then it also maps continuously X to Y.*

The interpolation pair (X, Y) is said to be of *exponent* θ (with $0 < \theta < 1$) if there exists a constant C s.t. $\|L\|_{X,Y} \le C\|L\|_{X_0,Y_0}^{1-\theta}\|L\|_{X_1,Y_1}^{\theta}$ for all operators L as above. The notation $\|L\|_{X,Y}$ is for the norm of L as a map from X to Y. If $C = 1$, we say that (X, Y) is an *exact interpolation pair of exponent* θ." – Wikipedia/interpolation space/the setting of interpolation

1.2.3 Complex interpolation

"If the scalars are complex numbers, properties of complex analytic functions are used to define an interpolation space. Given a compatible couple (X_0, X_1) of Banach spaces, the linear space $\mathcal{F}(X_0, X_1)$ consists of all functions $f : \mathbb{C} \to X_0 + X_1$, that are analytic on $S = \{z : 0 < \mathrm{Re}(z) < 1\}$, continuous on $\overline{S} = \{z : 0 \le \mathrm{Re}(z) \le 1\}$, & for which all the following subsets are bounded: $\{f(z) : z \in S\} \subset X_0 + X_1$, $\{f(it) : t \in \mathbb{R}\} \subset X_0$, $\{f(1 + it) : t \in \mathbb{R}\} \subset X_1$. $\mathcal{F}(X_0, X_1)$ is a Banach space under the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} \coloneqq \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{X_1}\right\}.$$

Definition 1.2. *For $0 < \theta < 1$, the* complex interpolation space *$(X_0, X_1)_\theta$ is the linear subspace of $X_0 + X_1$ consisting of all values $f(\theta)$ when f varies in the preceding space of functions, $(X_0, X_1)_\theta = \{x \in X_0 + X_1 : x = f(\theta),\ f \in \mathcal{F}(X_0, X_1)\}$. The norm on the complex interpolation space $(X_0, X_1)_\theta$ is defined by $\|x\|_\theta = \inf\left\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x,\ f \in \mathcal{F}(X_0, X_1)\right\}$.*

Equipped with this norm, the complex interpolation space $(X_0, X_1)_\theta$ is a Banach space.

Theorem 1.1. *Given 2 compatible couples of Banach spaces (X_0, X_1) & (Y_0, Y_1), the pair $((X_0, X_1)_\theta, (Y_0, Y_1)_\theta)$ is an exact interpolation pair of exponent θ, i.e., if $T : X_0 + X_1 \to Y_0 + Y_1$, is a linear operator bounded from X_j to Y_j, $j = 0, 1$, then T is bounded from $(X_0, X_1)_\theta$ to $(Y_0, Y_1)_\theta$ & $\|T\|_\theta \le \|T\|_0^{1-\theta}\|T\|_1^{\theta}$.*

The family of L^p spaces (consisting of complex valued functions) behaves well under complex interpolation. If (R, Σ, μ) is an arbitrary measure space, if $1 \le p_0, p_1 \le \infty$ & $0 < \theta < 1$, then

$$(L^{p_0}(R, \Sigma, \mu), L^{p_1}(R, \Sigma, \mu))_\theta = L^p(R, \Sigma, \mu),\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

with equality of norms. This fact is closely related to the Riesz–Thorin theorem." - Wikipedia/interpolation space/complex interpolation

1.2.4 Real interpolation

"There are 2 ways for introducing the *real interpolation method*. The 1st & most commonly used when actually identifying examples of interpolation spaces is the K-method. The 2nd method, the J-method, gives the same interpolation spaces as the K-method when the parameter θ is in $(0, 1)$. That the J- & K-methods agree is important for the study of duals of interpolation spaces: basically, the dual of an interpolation space constructed by the K-method appears to be a space constructed form the dual couple by the J-method." – Wikipedia/interpolation space/real interpolation

1.2.4.1 K-method

"The K-method of real interpolation can be used for Banach spaces over the field $\mathbb{R}$ of real numbers.

Definition 1.3. *Let (X_0, X_1) be a compatible couple of Banach spaces. For $t > 0$ & every $x \in X_0 + X_1$, let $K(x, t; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\ x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}$. Changing the order of the 2 spaces results in: $K(x, t; X_0, X_1) = tK(x, t^{-1}; X_1, X_0)$. Let*

$$\|x\|_{\theta,q;K} = \left(\int_0^\infty \left(t^{-\theta}K(x,t;X_0,X_1)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}},\ 0 < \theta < 1,\ 1 \le q < \infty,$$
$$\|x\|_{\theta,\infty;K} = \sup_{t>0} t^{-\theta}K(x,t;X_0,X_1),\ \ 0 \le \theta \le 1.$$

The K-method of real interpolation consists in taking $K_{\theta,q}(X_0, X_1)$ to be the linear subspace of $X_0 + X_1$ consisting of all x s.t. $\|x\|_{\theta,q;K} < \infty$.

Example 1.1. *An important example is that of the couple $(L^1(\mathbb{R}, \Sigma, \mu), L^\infty(\mathbb{R}, \Sigma, \mu))$, where the functional $K(t, f; L^1, L^\infty)$ can be computed explicitly. The measure μ is supposed σ-finite. In this context, the best way of cutting the function $f \in L^1 + L^\infty$ as sum of 2 functions $f_0 \in L^1$ & $f_1 \in L^\infty$ is, for some $s > 0$ to be chosen as function of t, to let $f_1(x)$ be given for all $x \in \mathbb{R}$ by*

$$f_1(x) = \begin{cases} f(x), & |f(x)| < s, \\ \dfrac{sf(x)}{|f(x)|}, & \textit{otherwise.} \end{cases}$$

The optimal choice of s leads to the formula

$$K(f,t;L^1,L^\infty) = \int_0^1 f^\star(u)\,\mathrm{d}u,$$

where $f^\star$ is the decreasing rearrangement of f." – Wikipedia/interpolation space/real interpolation/K-method

1.2.4.2 J-method

"As with the K-method, the J-method can be used for real Banach spaces.

Definition 1.4. *Let (X_0, X_1) be a compatible couple of Banach spaces. For $t > 0$ & for every vector $x \in X_0 \cap X_1$, let $J(x, t; X_0, X_1) = \max(\|x\|_{X_0}, t\|x\|_{X_1})$. A vector x in $X_0 + X_1$ belongs to the interpolation space $J_{\theta,q}(X_0, X_1)$ iff it can be written as $x = \int_0^\infty v(t)\frac{\mathrm{d}t}{t}$, where $v(t)$ is measurable with values in $X_0 \cap X_1$ & s.t.*

$$\Phi(v) = \left(\int_0^\infty \left(t^{-\theta}J(v(t),t;X_0,X_1)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty.$$

The norm of x in $J_{\theta,q}(X_0, X_1)$ is given by the formula

$$\|x\|_{\theta,q;J} \coloneqq \inf_v \left\{\Phi(v) : x = \int_0^\infty v(t)\frac{\mathrm{d}t}{t}\right\}.$$

" – Wikipedia/interpolation space/real interpolation/J-method

1.2.4.3 Relations between the interpolation methods

"The 2 real interpolation methods are equivalent when $0 < \theta < 1$.

Theorem 1.2. *Let (X_0, X_1) be a compatible couple of Banach spaces. If $0 < \theta < 1$ & $1 \le q \le \infty$, then $J_{\theta,q}(X_0, X_1) = K_{\theta,q}(X_0, X_1)$, with equivalence of norms.*

The theorem covers degenerate cases that have not been excluded: e.g. if X_0 & X_1 form a direct sum, then the intersection & the J-spaces are the null space, & a simple computation shows that the K-spaces are also null.

When $0 < \theta < 1$, one can speak, up to an equivalent renorming, about *the* Banach space obtained by the real interpolation method with parameters θ & q. The notation for this real interpolation space is $(X_0, X_1)_{\theta,q}$. One has that

$$(X_0,X_1)_{\theta,q} = (X_1,X_0)_{1-\theta,q},\ \ 0 < \theta < 1,\ 1 \le q \le \infty.$$

For a given value of θ, the real interpolation spaces increase with q: if $0 < \theta < 1$ & $1 \le q \le r \le \infty$, the following continuous inclusion holds true: $(X_0, X_1)_{\theta,q} \subset (X_0, X_1)_{\theta,r}$.

Theorem 1.3. *Given $0 < \theta < 1$, $1 \le q \le \infty$ & 2 compatible couples (X_0, X_1) & (Y_0, Y_1), the pair $((X_0, X_1)_{\theta,q}, (Y_0, Y_1)_{\theta,q})$ is an exact interpolation pair of exponent θ.*

A complex interpolation space is usually not isomorphic to 1 of the spaces given by the real interpolation method. However, there is a general relationship.

Theorem 1.4. *Let (X_0, X_1) be a compatible couple of Banach spaces. If $0 < \theta < 1$, then $(X_0, X_1)_{\theta,1} \subset (X_0, X_1)_\theta \subset (X_0, X_1)_{\theta,\infty}$.*

Example 1.2. *When $X_0 = C([0, 1])$ & $X_1 = C^1([0, 1])$, the space of continuously differentiable functions on $[0, 1]$, the (θ, ∞) interpolation method, for $0 < \theta < 1$, gives the Hölder space $C^{0,\theta}$ of exponent θ. This is because the K-functional $K(f, t; X_0, X_1)$ of this couple is equivalent to*

$$\sup\left\{|f(u)|, \frac{|f(u) - f(v)|}{1 + t^{-1}|u - v|} : u, v \in [0, 1]\right\}.$$

Only values $0 < t < 1$ are interesting here.

Real interpolation between L^p spaces gives the family of Lorentz spaces. Assuming $0 < \theta < 1$ & $1 \le q \le \infty$, one has

$$(L^1(\mathbb{R}, \Sigma, \mu), L^\infty(\mathbb{R}, \Sigma, \mu))_{\theta,q} = L^{p,q}(\mathbb{R}, \Sigma, \mu), \text{ where } \frac{1}{p} = 1 - \theta,$$

with equivalent norms. This follows from an inequality of Hardy & from the value given above of the K-functional for this compatible couple. When $q = p$, the Lorentz space $L^{p,p}$ is equal to L^p, up to renorming. When $q = \infty$, the Lorentz space $L^{p,\infty}$ is equal to weak-L^p." – Wikipedia/interpolation space/real interpolation/relations between the interpolation methods

1.2.5 The reiteration theorem

"An intermediate space X of the compatible couple (X_0, X_1) is said to be of *class θ* if $(X_0, X_1)_{\theta,1} \subset X \subset (X_0, X_1)_{\theta,\infty}$, with continuous injections. Beside all real interpolation spaces $(X_0, X_1)_{\theta,q}$ with parameter θ & $1 \le q \le \infty$, the complex interpolation space $(X_0, X_1)_\theta$ is an intermediate space of class θ of the compatible couple (X_0, X_1).

The reiteration theorems says, in essence, that interpolating with a parameter θ behaves, in some way, like forming a convex combination $a = (1-\theta)x_0 + \theta x_1$: taking a further convex combination of 2 convex combinations gives another convex combination.

Theorem 1.5. *Let A_0, A_1 be intermediate spaces of the compatible couple (X_0, X_1), of class θ_0 & θ_1 resp., with $0 < \theta_0 \ne \theta_1 < 1$. When $0 < \theta < 1$ & $1 \le q \le \infty$, one has $(A_0, A_1)_{\theta,q} = (X_0, X_1)_{\eta,q}$, $\eta = (1-\theta)\theta_0 + \theta\theta_1$.*

It is notable that when interpolating with the real method between $A_0 = (X_0, X_1)_{\theta_0,q_0}$ & $A_1 = (X_0, X_1)_{\theta_1,q_1}$, only the values of θ_0 & θ_1 matter. Also, A_0 & A_1 can be complex interpolation spaces between X_0 & X_1, with parameters θ_0 & θ_1 resp.

There is also a reiteration theorem for the complex method.

Theorem 1.6. *Let (X_0, X_1) be a compatible couple of complex Banach spaces, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1. Let $A_0 = (X_0, X_1)_{\theta_0}$ & $A_1 = (X_0, X_1)_{\theta_1}$, where $0 \le \theta_0 \le \theta_1 \le 1$. Assume further that $X_0 \cap X_1$ is dense in $A_0 \cap A_1$. Then, for every $0 \le \theta \le 1$,*

$$((X_0, X_1)_{\theta_0}, (X_0, X_1)_{\theta_1})_\theta = (X_0, X_1)_\eta, \ \eta = (1-\theta)\theta_0 + \theta\theta_1.$$

The density condition is always satisfied when $X_0 \subset X_1$ or $X_1 \subset X_0$." – Wikipedia/interpolation space/the reinteration theorem

1.2.6 Duality

"Let (X_0, X_1) be a compatible couple, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1. In this case, the restriction map from the (continuous) dual X_j' of X_j, $j = 0, 1$, to the dual of $X_0 \cap X_1$ is 1-1. It follows that the pair of duals (X_0', X_1') is a compatible couple continuously embedded in the dual $(X_0 \cap X_1)'$.

For the complex interpolation method, the following duality result holds:

Theorem 1.7. *Let (X_0, X_1) be a compatible couple of complex Banach spaces, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1. If X_0 & X_1 are reflexive, then the dual of the complex interpolation space is obtained by interpolating the duals, $((X_0, X_1)_\theta)' = (X_0', X_1')_\theta$, $0 < \theta < 1$.*

In general, the dual of the space $(X_0, X_1)_\theta$ is equal to $(X_0', X_1')^\theta$, a space defined by a variant of the complex method. The upper-θ & lower-θ methods do not coincide in general, but they do if at least 1 of X_0, X_1 is a reflexive space.

For the real interpolation method, the duality holds provided that the parameter q is finite:

Theorem 1.8. *Let $0 < \theta < 1$, $1 \le q < \infty$ & (X_0, X_1) a compatible couple of real Banach spaces. Assume that $X_0 \cap X_1$ is dense in X_0 & in X_1. Then*

$$((X_0, X_1)_{\theta,q})' = (X_0', X_1')_{\theta,q'}, \text{ where } \frac{1}{q'} = 1 - \frac{1}{q}.$$

" – Wikipedia/interpolation space/duality

1.2.7 Discrete definitions

"Since the function $t \to K(x,t)$ varies regularly (it is increasing, but $\frac{1}{t}K(x,y)$ is decreasing), the definition of the $K_{\theta,q}$-norm of a vector n, previously given by an integral, is equivalent to a definition given by a series. This series is obtained by breaking $(0,\infty)$ into pieces $(2^n, 2^{n+1})$ of equal mass for the measure $\frac{\mathrm{d}t}{t}$,

$$\|x\|_{\theta,q;K} \simeq \left(\sum_{n\in\mathbb{Z}} \left(2^{-\theta n} K(x, 2^n; X_0, X_1)\right)^q \right)^{\frac{1}{q}}.$$

In the special case where X_0 is continuously embedded in X_1, one can omit the part of the series with negative indies n. In this case, each of the functions $x \to K(x, 2^n; X_0, X_1)$ defines an equivalent norm on X_1.

The interpolation space $(X_0, X_1)_{\theta,q}$ is a "diagonal subspace" of an l^q-sum of a sequence of Banach spaces (each one being isomorphic to $X_0 + X_1$). Therefore, when q is finite, the dual of $(X_0, X_1)_{\theta,q}$ is a quotient of the l^p-sum of the duals, $\frac{1}{p} + \frac{1}{q} = 1$, which leads to the following formula for the discrete $J_{\theta,p}$-norm of a functional x' in the dual of $(X_0, X_1)_{\theta,q}$:

$$\|x'\|_{\theta,p;J} \simeq \inf \left\{ \left(\sum_{n\in\mathbb{Z}} \left(2^{\theta n} \max\left(\|x_n'\|_{X_0'}, 2^{-n}\|x_n'\|_{X_1'}\right)\right)^p \right)^{\frac{1}{p}} : x' = \sum_{n\in\mathbb{Z}} x_n' \right\}.$$

The usual formula for the discrete $J_{\theta,p}$-norm is obtained by changing n to $-n$.

The discrete definition makes several questions easier to study, among which the already mentioned identification of the dual. Other such questions are compactness or weak-compactness of linear operators. Lions & Peetre have proved that:

Theorem 1.9. *If the linear operator T is compact from X_0 to a Banach space Y & bounded from X_1 to Y, then T is compact from $(X_0, X_1)_{\theta,q}$ to Y when $0 < \theta < 1$, $1 \le q \le \infty$.*

Davis, Figiel, Johnson, & Pełczyński have used interpolation in their proof of the following result:

Theorem 1.10. *A bounded linear operator between 2 Banach spaces is weakly compact iff it factors through a reflexive space.*

" – Wikipedia/interpolation space/discrete definitions

1.2.7.1 A general interpolation method

"The space l^q used for the discrete definition can be replaced by an arbitrary sequence space Y with unconditional basis, & the weights $a_n = 2^{-\theta n}$, $b_n = 2^{(1-\theta)n}$, that are used for the $K_{\theta,q}$-norm, can be replaced by general weights $a_n, b_n > 0$, $\sum_{n=1}^\infty \min(a_n, b_n) < \infty$. The interpolation space $K(X_0, X_1, Y, \{a_n\}, \{b_n\})$ consists of the vectors x in $X_0 + X_1$ s.t.

$$\|x\|_{K(X_0,X_1)} = \sup_{m\ge1} \left\| \sum_{n=1}^m a_n K\left(x, \frac{b_n}{a_n}; X_0, X_1\right) y_n \right\|_Y < \infty,$$

where $\{y_n\}$ is the unconditional basis of Y. This abstract method can be used, e.g., for the proof of the following result:

Theorem 1.11. *A Banach space with unconditional basis is isomorphic to a complemented subspace of a space with symmetric basis.*

" – Wikipedia/interpolation space/discrete definitions/a general interpolation method

1.2.8 Interpolation of Sobolev & Besov spaces

Several interpolation results are available for Sobolev spaces & Besov spaces on $\mathbb{R}^n$, H_p^s, $s \in \mathbb{R}$, $1 \le p \le \infty$, $B_{p,q}^s$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$. These spaces are spaces of measurable functions on $\mathbb{R}^n$ when $s \ge 0$, & of tempered distributions on $\mathbb{R}^n$ when $s < 0$. For the rest of the section, the following setting & notation will be used: $0 < \theta < 1$, $1 \le p, p_0, p_1, q, q_0, q_1 \le \infty$, $s, s_0, s_1 \in \mathbb{R}$, $s_\theta = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Complex interpolation works well on the class of Sobolev spaces H_p^s (the Bessel potential spaces) as well as Besov spaces: $(H_{p_0}^{s_0}, H_{p_1}^{s_1})_\theta = H_{p_\theta}^{s_\theta}$, $s_0 \ne s_1$, $1 < p_0, p_1 < \infty$. $(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_\theta = B_{p_\theta,q_\theta}^{s_\theta}$, $s_0 \ne s_1$.

Real interpolation between Sobolev spaces may give Besov spaces, except when $s_0 = s_1$, $(H_{p_0}^s, H_{p_1}^s)_{\theta,p_\theta} = H_{p_\theta}^s$. When $s_0 \ne s_1$ but $p_0 = p_1$, real interpolation between Sobolev spaces gives a Besov space: $(H_p^{s_0}, H_p^{s_1})_{\theta,q} = B_{p,q}^{s_\theta}$, $s_0 \ne s_1$. Also, $(B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} = B_{p,q}^{s_\theta}$, $s_0 \ne s_1$. $(B_{p,q_0}^s, B_{p,q_1}^s)_{\theta,q} = B_{p,q_\theta}^s$. $(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_{\theta,q_\theta} = B_{p_\theta,q_\theta}^{s_\theta}$, $s_0 \ne s_1$, $p_\theta = q_\theta$." – Wikipedia/interpolation space/interpolation of Sobolev & Besov spaces

Chapter 2

Terence Tao's

2.1 Tao, 2007. What Is Good Mathematics?

Abstract. "Some personal thoughts & opinions on what "good quality mathematics" is & whether one should try to define this term rigorously. As a case study, the story of Szemerédi's theorem is presented."

2.1.1 The Many Aspects of Mathematical Quality

"We all agree that mathematicians should strive[1] to produce good mathematics. *But how does one define "good mathematics", & should one even dare to try at all?* Let us 1st consider the former question. Almost immediately one realizes that there are many different types of mathematics which could be designated[2] "good". E.g., "good mathematics" could refer (in no particular[3] order) to

1. Good mathematical *problem solving* (e.g. a major[4] breakthrough[5] on an important mathematical problem);
2. Good mathematical *technique*[6] (e.g. a masterful[7] use of existing[8] methods[9] or the development[10] of new tools[11]);
3. Good mathematical *theory* (e.g. a conceptual[12] framework[13] or choice of notation[14] which systematically[15] unifies[16] &

[1]**strive** [v] [intransitive] to try very hard to achieve something.

[2]**designate** [v] [often passive] **1.** to say officially that somebody/something has a particular character, name or purpose; to describe somebody/something in a particular way; **2.** to choose or name somebody/something for a particular job or position; **3.** (of a symbol) to identify or show something.

[3]**particular** [a] [only before noun] **1.** used to emphasize that you are referring to 1 individual person, thing or type of thing & not others, SYNONYM: **specific**; **2.** greater than usual; special; **in particular** [idiom] **1.** especially or particularly; **2.** special, SYNONYM: **specific**; **of particular note** [idiom] especially interesting; [n] **1.** [countable, usually plural] a fact or detail, especially one that is officially written down; **2.** (**particulars**) [plural] written information & details about a property, business, job, etc.

[4]**major** [a] **1.** [usually before noun] large, important or serious, OPPOSITE: **minor**; **2.** [only before noun] greater or more important; main, SYNONYM: **main**; [n] (*North American English*) **1.** the main subject or course of a student at college or university; **2.** a student studying a particular subject as the main part of their course.

[5]**breakthrough** [n] an important development or discovery that helps people to achieve or understand something.

[6]**technique** [n] **1.** [countable] a particular way of doing something that involves using a special skill or process; **2.** [uncountable, singular] a person's skill or ability in a particular activity.

[7]**masterful** [a] **1.** (of a person, especially a man) able to control people or situations in a way that shows confidence as a leader; **2.** (also **masterly**) showing great skill or understanding.

[8]**existing** [a] [only before noun] found or used now or at the time being discussed.

[9]**method** [n] a particular way of doing something.

[10]**development** [n] **1.** [uncountable] the process of creating a new method, system, product or theory; **2.** [countable] a new or advanced method, system, product or theory; **3.** [uncountable] the process of making a country or area richer & more successful; **4.** [uncountable] the way in which a child or other living creature grows before & after birth.

[11]**tool** [n] **1.** a thing that helps somebody to do a job or to achieve something; **2.** a piece of equipment held in the hand, that is used for making things or repairing things.

[12]**conceptual** [a] connected with or based on ideas.

[13]**framework** [n] **1.** a set of beliefs, ideas or principles that is based as the basis for examining or understanding something; **2.** a system of rules, laws or agreements that controls the way that something works in business, politics or society.

[14]**notation** [n] [uncountable, countable] **notation (for something)** a system of signs or symbols used to represent information, especially in mathematics, science & music.

[15]**systematically** [adv] **1.** in a way that follows a system; **2.** in the same way all through a process or set of results because of the system that is used.

[16]**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

generalizes[17] an existing[18] body of results);

4. Good mathematical *insight*[19] (e.g. a major conceptual simplification[20] or the realization[21] of a unifying[22] principle[23], analogy[24], or theme[25]);
5. Good mathematical *discovery*[26] (e.g. the revelation[27] of an unexpected[28] & intriguing[29] new mathematical phenomenon[30], connection[31], or counterexample[32]);
6. Good mathematical *application*[33] (e.g. to important problems in physics, engineering, computer science, statistics, etc., or from 1 field of mathematics to another);
7. Good mathematical *exposition*[34] (e.g. a detailed[35] & informative[36] survey[37] on a timely[38] mathematical topic or a clear & well-motivated argument);
8. Good mathematical *pedagogy*[39] (e.g. a lecture[40] or writing style which enables others to learn & do mathematics more

[17]**generalize** [v] (*British English also* **generalise**) **1.** [intransitive] **generalize (from something)** to use a particular set of facts or ideas in order to form an opinion that is considered valid for a different situation; **2.** [intransitive] to make a general statement about something & not look at the details; **3.** [transitive, often passive] to apply a theory, idea, etc. to a wider group or situation than the original one.

[18]**existing** [a] [only before noun] found or used now or at the time being discussed.

[19]**insight** [n] **1.** [countable, uncountable] an understanding of a particular situation or thing; **2.** [uncountable] the ability to see & understand the truth about people or situations.

[20]**simplification** [n] **1.** [uncountable] **simplification (of something)** the process of making something less complicated, or easier to do or understand; **2.** [countable] a change that makes a problem, statement, system, etc. less complicated or easier to understand or do.

[21]**realization** [n] (*British English also* **realisation**) **1.** [uncountable, singular] **realization (that)** ... the process of becoming aware of something, SYNONYM: **awareness**; **2.** [uncountable] **realization (of something)** the process of achieving a particular aim, etc., SYNONYM: **achievement**; **3.** [uncountable, countable] **realization (of something)** (*formal*) the act of producing something in an actual or physical form; the thing that is produced.

[22]**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

[23]**principle** [n] **1.** [countable] a law, rule or theory that something is based on; **2.** [singular] a general or scientific law that explains how something works or why something happens; **3.** [countable] a belief that is accepted as a reason for acting or thinking in a particular way; **4.** [countable, usually plural, uncountable] a moral rule or a strong belief that influences your actions; **in principle** [idiom] **1.** if something can be done in principle, there is no good reason why it should not be done although it has not yet been done & there may be some difficulties; **2.** in general but not in detail.

[24]**analogy** [n] (plural **analogies**) [countable, uncountable] a comparison of 1 thing with another thing that has similar features, usually in order to explain it; a feature that is similar.

[25]**theme** [n] the subject of a talk, piece of writing, exhibition, etc.; an idea that keeps returning in a piece of research or a work of art or literature.

[26]**discovery** [n] (plural **discoveries**) **1.** [countable, uncountable] an act or the process of finding somebody/something, or learning about something that was not known about before; **2.** [countable] a thing, fact or person that is found or learned about for the 1st time.

[27]**revelation** [n] **1.** [countable] a fact that people are made aware of, especially one that has been secret & is surprising, SYNONYM: **disclosure**; **2.** [uncountable] **revelation (of something)** the act of making people aware of something that has been secret, SYNONYM: **disclosure**; **3.** [countable, uncountable] something that is considered to be a sign or message from God.

[28]**unexpected** [a] surprising; not expected.

[29]**intriguing** [a] very interesting because of being unusual or not having an obvious answer.

[30]**phenomenon** [n] (plural **phenomena**) a fact or an event in nature or society, especially one that is not fully understood.

[31]**connection** [n] (*British English also, old-fashioned* **connexion**) **1.** [countable] something that connects 2 facts or ideas, SYNONYM: **link**; **2.** [countable] a relationship between people or groups of people, often for a particular purpose; **3.** [uncountable, countable] the action of connecting something to a supply of water, electricity, etc. or to a computer or telephone network; the fact of being connected in this way; **4.** [countable] a point, especially in an electrical system, where 2 parts connect; **5.** [countable, usually plural] a means of traveling to another place; **6.** [countable, usually plural] people that you know, who can help or advise you in your professional or social life; **in connection with somebody/something** [idiom] for reasons connected with somebody/something; **in this/that connection** [idiom] for reasons connected with something recently mentioned.

[32]**counterexample** [n] **counterexample (to something)** an example that provides evidence against an idea or theory.

[33]**application** [n] **1.** [uncountable, countable] the use of something such as an idea, method, rule, etc.; a use that something has; **2.** [countable] a formal (often written) request to an organization or authority for something, such as a job or permission to do something, or to join a group; **3.** [countable] a program or piece of software designed to do a particular job; **4.** [countable, uncountable] **application (of something) (to something)** the use of something to produce a particular physical effect; **5.** [countable, uncountable] **application (of something)** the action of putting or spreading something onto a surface or object.

[34]**exposition** [n] [countable, uncountable] (*formal*) a full explanation of a theory, plan, etc.

[35]**detailed** [a] giving many details; paying great attention to details.

[36]**informative** [a] giving useful information.

[37]**survey** [n] **1. survey (of somebody/something)** an investigation of the opinions, behavior, etc. of a particular group of people, which is usually done by asking them questions; **2.** an act of examining & recording the measurements, features, etc. of an area of land in order to make a map or plan of it; **3. survey (of something)** a general study, view or description of something; [v] **1. survey somebody/something** to investigate the opinions or behavior of a group of people by asking them a series of questions; **2. survey something** to study & give a general description of something; **3. survey something** to measure & record the features of an area of land, e.g. in order to make a map or in preparation for building; **4. survey something** to look carefully at the whole of something, especially in order to get a general impression of it, SYNONYM: **inspect**.

[38]**timely** [a] happening at exactly the right time.

[39]**pedagogy** [n] (plural **pedagogies**) [uncountable, countable] methods of teaching, especially as a subject of study or as a theory.

[40]**lecture** [n] a talk that is given to a group of people to teach them about a particular subject, often as part of a university or college course; [v] [intransitive] **lecture (in/on something) (to somebody)** to give a talk or a series of talks to a group of people on a particular subject, especially as a way of teaching in a university or college.

effectively, or contributions[41] to mathematical education);

9. Good mathematical *vision*[42] (e.g. a long-range[43] & fruitful program or set of conjectures[44]);
10. Good mathematical *taste* (e.g. a research goal which is inherently interesting & impacts important topics, themes, or questions);
11. Good mathematical *public relations* (e.g. an effective showcasing of a mathematical achievement to non-mathematicians or from 1 field of mathematics to another);

[41]**contribution** [n] **1.** [usually singular] the part played by a person or thing in achieving, improving or causing something; **2.** a sum of money that is given to a person or an organization in order to help pay for something, SYNONYM: **donation**; **3. contribution (to something)** an item that forms part of a book, magazine, broadcast, discussion, etc.; **4.** a sum of money that you pay regularly to your employer or the government in order to pay for benefits such as health insurance or a pension.

[42]**vision** [n] **1.** [uncountable] the ability to see; the area that you can see from a particular position; **2.** [countable] an idea or a picture in your imagination, especially of what the future will or could be like; **3.** [uncountable] the ability to think about or plan the future with great imagination & intelligence.

[43]**long-range** [a] [only before noun] **1.** traveling a long distance; **2.** made for a period of time that will last a long way into the future.

[44]**conjecture** [n] (*formal*) **1.** [countable] an opinion or idea that is not based on definite knowledge & is formed by guessing, SYNONYM: **guess**; **2.** [uncountable] the act of forming an opinion or idea that is not based on definite knowledge; [v] [intransitive, transitive] (*formal*) to form an opinion about something even though you do not have much information on it, SYNONYM: **guess**.

Bibliography

Tao, Terence (2007). "What is good mathematics?" In: *Bull. Amer. Math. Soc. (N.S.)* 44.4, pp. 623–634. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-07-01168-8. URL: https://doi.org/10.1090/S0273-0979-07-01168-8.