

Advanced Mathematics

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September 1, 2022

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Chapter 1

Wikipedia's

1.1 Wikipedia/Symmetrization Methods

“In mathematics the *symmetrization methods* are algorithms of transforming a set $A \subset \mathbb{R}^n$ to a ball $B \subset \mathbb{R}^n$ with equal volume $\text{vol}(B) = \text{vol}(A)$ & centered at the origin. B is called the *symmetrized version* of A , usually denoted A^* . These algorithms show up in solving the classical *isoperimetric inequality* problem, which asks: Given all 2D shapes of a given area, which of them has the minimal *perimeter*. The conjectured answer was the disk & *Steiner* in 1838 showed this to be true using the Steiner symmetrization method. From this many other isoperimetric problems sprung & other symmetrization algorithms. E.g., Rayleigh's conjecture is that the 1st *eigenvalue* of the *Dirichlet problem* is minimized for the ball (see *Rayleigh–Faber–Krahn inequality* for details). Another problem is that the Newtonian *capacity of a set* A is minimized by A^* & this was proved by Polya & G. Szego (1951) using circular symmetrization.” – [Wikipedia/symmetrization methods](#)

1.1.1 Symmetrization

“If $\Omega \subset \mathbb{R}^n$ is measurable, then it is denoted by Ω^* the symmetrized version of Ω , i.e., a ball $\Omega^* := B_r(0) \subset \mathbb{R}^n$ s.t. $\text{vol}(\Omega^*) = \text{vol}(\Omega)$. We denote by f^* the *symmetric decreasing rearrangement* of nonnegative measurable function f & define it as $f^*(x) := \int_0^\infty 1_{\{y: f(y) > t\}^*}(x) dt$, where $\{y : f(y) > t\}^*$ is the symmetrized version of preimage set $\{y : f(y) > t\}$. The methods described below have been proved to transform Ω to Ω^* , i.e., given a sequence of symmetrization transformations $\{T_k\}$ there is $\lim_{k \rightarrow \infty} d_{\text{Ha}}(\Omega^*, T_k(K)) = 0$, where d_{Ha} is the *Hausdorff distance* (for discussion & proofs see [Burchard2009]).” – [Wikipedia/symmetrization methods/symmetrization](#)

1.1.2 Steiner symmetrization

Steiner Symmetrization of set Ω .

“Steiner symmetrization was introduced by Steiner (1838) to solve the isoperimetric theorem stated above. Let $H \subset \mathbb{R}^n$ be a *hyperplane* through the origin. Rotate space so that H is the $x_n = 0$ (x_n is n th coordinate in \mathbb{R}^n) hyperplane. For each $\mathbf{x} \in H$ let the perpendicular line through $\mathbf{x} \in H$ be $L_{\mathbf{x}} = \{\mathbf{x} + y\mathbf{e}_n : y \in \mathbb{R}\}$. Then by replacing each $\Omega \cap L_{\mathbf{x}}$ by a line centered at H & with length $|\Omega \cap L_{\mathbf{x}}|$ we obtain the *Steiner symmetrized version*.

$$\text{St}(\Omega) := \left\{ \mathbf{x} + y\mathbf{e}_n : \mathbf{x} + z\mathbf{e}_n \in \Omega \text{ for some } \mathbf{z} \text{ \& } |y| \leq \frac{1}{2}|\Omega \cap L_{\mathbf{x}}| \right\}.$$

It is denoted by $\text{St}(f)$ the *Steiner symmetrization* w.r.t. $x_n = 0$ hyperplane of nonnegative measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ & for fixed x_1, \dots, x_{n-1} define it as $\text{St} : f(x_1, \dots, x_{n-1}, \cdot) \mapsto (f(x_1, \dots, x_{n-1}))^*$.

1.1.2.1 Properties

It preserves convexity: if Ω is convex, then $\text{St}(\Omega)$ is also convex. It is linear: $\text{St}(\mathbf{x} + \lambda\Omega) = \text{St}(\mathbf{x}) + \lambda\text{St}(\Omega)$. Super-additive: $\text{St}(K) + \text{St}(U) \subset \text{St}(K + U)$.” – [Wikipedia/symmetrization methods/Steiner symmetrization](#)

1.1.3 Circular symmetrization

Fig. Circular symmetrization of set Ω .

“A popular method for symmetrization in the plane is *Polya's circular symmetrization*. After, its generalization will be described to higher dimensions. Let $\Omega \subset \mathbb{C}$ be a domain; then its circular symmetrization $\text{Circ}(\Omega)$ with regard to the positive real axis is defined as follows: Let $\Omega_t := \{\theta \in [0, 2\pi] : te^{i\theta} \in \Omega\}$, i.e., contain the arcs of radius t contained in Ω . So it is defined

- If Ω_t is the full circle, then $\text{Circ}(\Omega) \cap \{|z| = t\} := \{|z| = t\}$.
- If the length is $m(\Omega_t) = \alpha$, then $\text{Circ}(\Omega) \cap \{|z| = t\} := \{te^{i\theta} : |\theta| < \frac{\alpha}{2}\}$.
- $0, \infty \in \text{Circ}(\Omega)$ if $0, \infty \in \Omega$.

In higher dimensions $\Omega \subset \mathbb{R}^n$, its spherical symmetrization $\text{Sp}^n(\Omega)$ w.r.t. the positive axis of x_1 is defined as follows: Let $\Omega_r := \{\mathbf{x} \in \mathbb{S}^{n-1} : r\mathbf{x} \in \Omega\}$, i.e., contain the caps of radius r contained in Ω . Also, for the 1st coordinate let $\text{angle}(x_1) := \theta$ if $x_1 = r \cos \theta$. So as above

- If Ω_r is the full cap, then $\text{Sp}^n(\Omega) \cap \{|z| = r\} := \{|z| = r\}$.
- If the surface area is $m_s(\Omega_t) = \alpha$, then $\text{Sp}^n(\Omega) \cap \{|z| = r\} := \{x : |x| = r \text{ \& } 0 \leq \text{angle}(x_1) \leq \theta_\alpha\} =: C(\theta_\alpha)$ where θ_α is picked so that its surface area is $m_s(C(\theta_\alpha)) = \alpha$. In words, $C(\theta_\alpha)$ is a cap symmetric around the positive axis x_1 with the same area as the intersection $\Omega \cap \{|z| = r\}$.
- $0, \infty \in \text{Sp}^n(\Omega)$ iff $0, \infty \in \Omega$.” – [Wikipedia/symmetrization methods/circular symmetrization](#)

1.1.4 Polarization

Fig: Polarization of set Ω .

“Let $\Omega \subset \mathbb{R}^n$ be a domain & $H^{n-1} \subset \mathbb{R}^n$ be a hyperplane through the origin. Denote the reflection across that plane to the positive halfspace \mathbb{H}^+ as σ_H or just σ when it is clear from the context. Also, the reflected Ω across hyperplane H is defined as $\sigma\Omega$. Then, the polarized Ω is denoted as Ω^α & defined as follows

- If $\mathbf{x} \in \Omega \cap \mathbb{H}^+$, then $\mathbf{x} \in \Omega^\alpha$.
- If $\mathbf{x} \in \Omega \cap \sigma(\Omega) \cap \mathbb{H}^-$, then $\mathbf{x} \in \Omega^\sigma$.
- If $\mathbf{x} \in (\Omega \setminus \sigma(\Omega)) \cap \mathbb{H}^-$, then $\sigma\mathbf{x} \in \Omega^\sigma$.

In words, $(\Omega \setminus \sigma(\Omega)) \cap \mathbb{H}^-$ is simply reflected to the halfspace \mathbb{H}^+ . It turns out that this transformation can approximate the above ones (in the [Hausdorff distance](#)) (see [Brock & Solynin2000]).” – [Wikipedia/symmetrization methods/polarization](#)

Chapter 2

Terence Tao's

2.1 Tao, 2007. What Is Good Mathematics?

Abstract. “Some personal thoughts & opinions on what “good quality mathematics” is & whether one should try to define this term rigorously. As a case study, the story of Szemerédi’s theorem is presented.”

2.1.1 The Many Aspects of Mathematical Quality

“We all agree that mathematicians should strive¹ to produce good mathematics. *But how does one define “good mathematics”, & should one even dare to try at all?* Let us 1st consider the former question. Almost immediately one realizes that there are many different types of mathematics which could be designated² “good”. E.g., “good mathematics” could refer (in no particular³ order) to

1. Good mathematical *problem solving* (e.g. a major⁴ breakthrough⁵ on an important mathematical problem);
2. Good mathematical *technique*⁶ (e.g. a masterful⁷ use of existing⁸ methods⁹ or the development¹⁰ of new tools¹¹);
3. Good mathematical *theory* (e.g. a conceptual¹² framework¹³ or choice of notation¹⁴ which systematically¹⁵ unifies¹⁶ &

¹**strive** [v] [intransitive] to try very hard to achieve something.

²**designate** [v] [often passive] **1.** to say officially that somebody/something has a particular character, name or purpose; to describe somebody/something in a particular way; **2.** to choose or name somebody/something for a particular job or position; **3.** (of a symbol) to identify or show something.

³**particular** [a] [only before noun] **1.** used to emphasize that you are referring to 1 individual person, thing or type of thing & not others, SYNONYM: **specific**; **2.** greater than usual; special; **in particular** [idiom] **1.** especially or particularly; **2.** special, SYNONYM: **specific**; **of particular note** [idiom] especially interesting; [n] **1.** [countable, usually plural] a fact or detail, especially one that is officially written down; **2.** (particulars) [plural] written information & details about a property, business, job, etc.

⁴**major** [a] **1.** [usually before noun] large, important or serious, OPPOSITE: **minor**; **2.** [only before noun] greater or more important; main, SYNONYM: **main**; [n] (*North American English*) **1.** the main subject or course of a student at college or university; **2.** a student studying a particular subject as the main part of their course.

⁵**breakthrough** [n] an important development or discovery that helps people to achieve or understand something.

⁶**technique** [n] **1.** [countable] a particular way of doing something that involves using a special skill or process; **2.** [uncountable, singular] a person’s skill or ability in a particular activity.

⁷**masterful** [a] **1.** (of a person, especially a man) able to control people or situations in a way that shows confidence as a leader; **2.** (also **masterly**) showing great skill or understanding.

⁸**existing** [a] [only before noun] found or used now or at the time being discussed.

⁹**method** [n] a particular way of doing something.

¹⁰**development** [n] **1.** [uncountable] the process of creating a new method, system, product or theory; **2.** [countable] a new or advanced method, system, product or theory; **3.** [uncountable] the process of making a country or area richer & more successful; **4.** [uncountable] the way in which a child or other living creature grows before & after birth.

¹¹**tool** [n] **1.** a thing that helps somebody to do a job or to achieve something; **2.** a piece of equipment held in the hand, that is used for making things or repairing things.

¹²**conceptual** [a] connected with or based on ideas.

¹³**framework** [n] **1.** a set of beliefs, ideas or principles that is based as the basis for examining or understanding something; **2.** a system of rules, laws or agreements that controls the way that something works in business, politics or society.

¹⁴**notation** [n] [uncountable, countable] **notation (for something)** a system of signs or symbols used to represent information, especially in mathematics, science & music.

¹⁵**systematically** [adv] **1.** in a way that follows a system; **2.** in the same way all through a process or set of results because of the system that is used.

¹⁶**unify** [v] **1.** **unify something** to join people or countries together so that they form a single unit; **2.** **unify something (into something)** to put things, especially ideas, together in a good or helpful way.

generalizes¹⁷ an existing¹⁸ body of results);

4. Good mathematical *insight*¹⁹ (e.g. a major conceptual simplification²⁰ or the realization²¹ of a unifying²² principle²³, analogy²⁴, or theme²⁵);
5. Good mathematical *discovery*²⁶ (e.g. the revelation²⁷ of an unexpected²⁸ & intriguing²⁹ new mathematical phenomenon³⁰, connection³¹, or counterexample³²);
6. Good mathematical *application*³³ (e.g. to important problems in physics, engineering, computer science, statistics, etc., or from 1 field of mathematics to another);
7. Good mathematical *exposition*³⁴ (e.g. a detailed³⁵ & informative³⁶ survey³⁷ on a timely³⁸ mathematical topic or a clear & well-motivated argument);
8. Good mathematical *pedagogy*³⁹ (e.g. a lecture⁴⁰ or writing style which enables others to learn & do mathematics more

¹⁷**generalize** [v] (*British English also generalise*) **1.** [intransitive] **generalize (from something)** to use a particular set of facts or ideas in order to form an opinion that is considered valid for a different situation; **2.** [intransitive] to make a general statement about something & not look at the details; **3.** [transitive, often passive] to apply a theory, idea, etc. to a wider group or situation than the original one.

¹⁸**existing** [a] [only before noun] found or used now or at the time being discussed.

¹⁹**insight** [n] **1.** [countable, uncountable] an understanding of a particular situation or thing; **2.** [uncountable] the ability to see & understand the truth about people or situations.

²⁰**simplification** [n] **1.** [uncountable] **simplification (of something)** the process of making something less complicated, or easier to do or understand; **2.** [countable] a change that makes a problem, statement, system, etc. less complicated or easier to understand or do.

²¹**realization** [n] (*British English also realisation*) **1.** [uncountable, singular] **realization (that)** ... the process of becoming aware of something, SYNONYM: **awareness**; **2.** [uncountable] **realization (of something)** the process of achieving a particular aim, etc., SYNONYM: **achievement**; **3.** [uncountable, countable] **realization (of something)** (*formal*) the act of producing something in an actual or physical form; the thing that is produced.

²²**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

²³**principle** [n] **1.** [countable] a law, rule or theory that something is based on; **2.** [singular] a general or scientific law that explains how something works or why something happens; **3.** [countable] a belief that is accepted as a reason for acting or thinking in a particular way; **4.** [countable, usually plural, uncountable] a moral rule or a strong belief that influences your actions; **in principle** [idiom] **1.** if something can be done in principle, there is no good reason why it should not be done although it has not yet been done & there may be some difficulties; **2.** in general but not in detail.

²⁴**analogy** [n] (plural **analogies**) [countable, uncountable] a comparison of 1 thing with another thing that has similar features, usually in order to explain it; a feature that is similar.

²⁵**theme** [n] the subject of a talk, piece of writing, exhibition, etc.; an idea that keeps returning in a piece of research or a work of art or literature.

²⁶**discovery** [n] (plural **discoveries**) **1.** [countable, uncountable] an act or the process of finding somebody/something, or learning about something that was not known about before; **2.** [countable] a thing, fact or person that is found or learned about for the 1st time.

²⁷**revelation** [n] **1.** [countable] a fact that people are made aware of, especially one that has been secret & is surprising, SYNONYM: **disclosure**; **2.** [uncountable] **revelation (of something)** the act of making people aware of something that has been secret, SYNONYM: **disclosure**; **3.** [countable, uncountable] something that is considered to be a sign or message from God.

²⁸**unexpected** [a] surprising; not expected.

²⁹**intriguing** [a] very interesting because of being unusual or not having an obvious answer.

³⁰**phenomenon** [n] (plural **phenomena**) a fact or an event in nature or society, especially one that is not fully understood.

³¹**connection** [n] (*British English also, old-fashioned connexion*) **1.** [countable] something that connects 2 facts or ideas, SYNONYM: **link**; **2.** [countable] a relationship between people or groups of people, often for a particular purpose; **3.** [uncountable, countable] the action of connecting something to a supply of water, electricity, etc. or to a computer or telephone network; the fact of being connected in this way; **4.** [countable] a point, especially in an electrical system, where 2 parts connect; **5.** [countable, usually plural] a means of traveling to another place; **6.** [countable, usually plural] people that you know, who can help or advise you in your professional or social life; **in connection with somebody/something** [idiom] for reasons connected with somebody/something; **in this/that connection** [idiom] for reasons connected with something recently mentioned.

³²**counterexample** [n] **counterexample (to something)** an example that provides evidence against an idea or theory.

³³**application** [n] **1.** [uncountable, countable] the use of something such as an idea, method, rule, etc.; a use that something has; **2.** [countable] a formal (often written) request to an organization or authority for something, such as a job or permission to do something, or to join a group; **3.** [countable] a program or piece of software designed to do a particular job; **4.** [countable, uncountable] **application (of something) (to something)** the use of something to produce a particular physical effect; **5.** [countable, uncountable] **application (of something)** the action of putting or spreading something onto a surface or object.

³⁴**exposition** [n] [countable, uncountable] (*formal*) a full explanation of a theory, plan, etc.

³⁵**detailed** [a] giving many details; paying great attention to details.

³⁶**informative** [a] giving useful information.

³⁷**survey** [n] **1. survey (of somebody/something)** an investigation of the opinions, behavior, etc. of a particular group of people, which is usually done by asking them questions; **2.** an act of examining & recording the measurements, features, etc. of an area of land in order to make a map or plan of it; **3. survey (of something)** a general study, view or description of something; [v] **1. survey somebody/something** to investigate the opinions or behavior of a group of people by asking them a series of questions; **2. survey something** to study & give a general description of something; **3. survey something** to measure & record the features of an area of land, e.g. in order to make a map or in preparation for building; **4. survey something** to look carefully at the whole of something, especially in order to get a general impression of it, SYNONYM: **inspect**.

³⁸**timely** [a] happening at exactly the right time.

³⁹**pedagogy** [n] (plural **pedagogies**) [uncountable, countable] methods of teaching, especially as a subject of study or as a theory.

⁴⁰**lecture** [n] a talk that is given to a group of people to teach them about a particular subject, often as part of a university or college course; [v] [intransitive] **lecture (in/on something) (to somebody)** to give a talk or a series of talks to a group of people on a particular subject, especially as a way of teaching in a university or college.

- effectively, or contributions⁴¹ to mathematical education);
9. Good mathematical *vision*⁴² (e.g. a long-range⁴³ & fruitful program or set of conjectures⁴⁴);
 10. Good mathematical *taste* (e.g. a research goal which is inherently interesting & impacts important topics, themes, or questions);
 11. Good mathematical *public relations* (e.g. an effective showcasing of a mathematical achievement to non-mathematicians or from 1 field of mathematics to another);

⁴¹**contribution** [n] **1.** [usually singular] the part played by a person or thing in achieving, improving or causing something; **2.** a sum of money that is given to a person or an organization in order to help pay for something, SYNONYM: **donation**; **3.** **contribution (to something)** an item that forms part of a book, magazine, broadcast, discussion, etc.; **4.** a sum of money that you pay regularly to your employer or the government in order to pay for benefits such as health insurance or a pension.

⁴²**vision** [n] **1.** [uncountable] the ability to see; the area that you can see from a particular position; **2.** [countable] an idea or a picture in your imagination, especially of what the future will or could be like; **3.** [uncountable] the ability to think about or plan the future with great imagination & intelligence.

⁴³**long-range** [a] [only before noun] **1.** traveling a long distance; **2.** made for a period of time that will last a long way into the future.

⁴⁴**conjecture** [n] (*formal*) **1.** [countable] an opinion or idea that is not based on definite knowledge & is formed by guessing, SYNONYM: **guess**; **2.** [uncountable] the act of forming an opinion or idea that is not based on definite knowledge; [v] [intransitive, transitive] (*formal*) to form an opinion about something even though you do not have much information on it, SYNONYM: **guess**.

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