

Advanced Mathematics

Nguyễn Quân Bá Hồng¹

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¹Independent Researcher, Ben Tre City, Vietnam
e-mail: nguyenquanbahong@gmail.com

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Chapter 1

Wikipedia's

1.1 Wikipedia/Symmetrization Methods

“In mathematics the *symmetrization methods* are algorithms of transforming a set $A \subset \mathbb{R}^n$ to a ball $B \subset \mathbb{R}^n$ with equal volume $\text{vol}(B) = \text{vol}(A)$ & centered at the origin. B is called the *symmetrized version* of A , usually denoted A^* . These algorithms show up in solving the classical *isoperimetric inequality* problem, which asks: Given all 2D shapes of a given area, which of them has the minimal *perimeter*. The conjectured answer was the disk & *Steiner* in 1838 showed this to be true using the Steiner symmetrization method. From this many other isoperimetric problems sprung & other symmetrization algorithms. E.g., Rayleigh's conjecture is that the 1st *eigenvalue* of the *Dirichlet problem* is minimized for the ball (see *Rayleigh–Faber–Krahn inequality* for details). Another problem is that the Newtonian *capacity of a set* A is minimized by A^* & this was proved by Polya & G. Szego (1951) using circular symmetrization.” – [Wikipedia/symmetrization methods](#)

1.1.1 Symmetrization

“If $\Omega \subset \mathbb{R}^n$ is measurable, then it is denoted by Ω^* the symmetrized version of Ω , i.e., a ball $\Omega^* := B_r(0) \subset \mathbb{R}^n$ s.t. $\text{vol}(\Omega^*) = \text{vol}(\Omega)$. We denote by f^* the *symmetric decreasing rearrangement* of nonnegative measurable function f & define it as $f^*(x) := \int_0^\infty 1_{\{y: f(y) > t\}^*}(x) dt$, where $\{y : f(y) > t\}^*$ is the symmetrized version of preimage set $\{y : f(y) > t\}$. The methods described below have been proved to transform Ω to Ω^* , i.e., given a sequence of symmetrization transformations $\{T_k\}$ there is $\lim_{k \rightarrow \infty} d_{\text{Ha}}(\Omega^*, T_k(K)) = 0$, where d_{Ha} is the *Hausdorff distance* (for discussion & proofs see [Burchard2009]).” – [Wikipedia/symmetrization methods/symmetrization](#)

1.1.2 Steiner symmetrization

Steiner Symmetrization of set Ω .

“Steiner symmetrization was introduced by Steiner (1838) to solve the isoperimetric theorem stated above. Let $H \subset \mathbb{R}^n$ be a *hyperplane* through the origin. Rotate space so that H is the $x_n = 0$ (x_n is n th coordinate in \mathbb{R}^n) hyperplane. For each $\mathbf{x} \in H$ let the perpendicular line through $\mathbf{x} \in H$ be $L_{\mathbf{x}} = \{\mathbf{x} + y\mathbf{e}_n : y \in \mathbb{R}\}$. Then by replacing each $\Omega \cap L_{\mathbf{x}}$ by a line centered at H & with length $|\Omega \cap L_{\mathbf{x}}|$ we obtain the *Steiner symmetrized version*.

$$\text{St}(\Omega) := \left\{ \mathbf{x} + y\mathbf{e}_n : \mathbf{x} + z\mathbf{e}_n \in \Omega \text{ for some } \mathbf{z} \text{ \& } |y| \leq \frac{1}{2}|\Omega \cap L_{\mathbf{x}}| \right\}.$$

It is denoted by $\text{St}(f)$ the *Steiner symmetrization* w.r.t. $x_n = 0$ hyperplane of nonnegative measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ & for fixed x_1, \dots, x_{n-1} define it as $\text{St} : f(x_1, \dots, x_{n-1}, \cdot) \mapsto (f(x_1, \dots, x_{n-1}))^*$.

1.1.2.1 Properties

It preserves convexity: if Ω is convex, then $\text{St}(\Omega)$ is also convex. It is linear: $\text{St}(\mathbf{x} + \lambda\Omega) = \text{St}(\mathbf{x}) + \lambda\text{St}(\Omega)$. Super-additive: $\text{St}(K) + \text{St}(U) \subset \text{St}(K + U)$.” – [Wikipedia/symmetrization methods/Steiner symmetrization](#)

1.1.3 Circular symmetrization

Fig. Circular symmetrization of set Ω .

“A popular method for symmetrization in the plane is *Polya's circular symmetrization*. After, its generalization will be described to higher dimensions. Let $\Omega \subset \mathbb{C}$ be a domain; then its circular symmetrization $\text{Circ}(\Omega)$ with regard to the positive real axis is defined as follows: Let $\Omega_t := \{\theta \in [0, 2\pi] : te^{i\theta} \in \Omega\}$, i.e., contain the arcs of radius t contained in Ω . So it is defined

- If Ω_t is the full circle, then $\text{Circ}(\Omega) \cap \{|z| = t\} := \{|z| = t\}$.
- If the length is $m(\Omega_t) = \alpha$, then $\text{Circ}(\Omega) \cap \{|z| = t\} := \{te^{i\theta} : |\theta| < \frac{\alpha}{2}\}$.
- $0, \infty \in \text{Circ}(\Omega)$ if $0, \infty \in \Omega$.

In higher dimensions $\Omega \subset \mathbb{R}^n$, its spherical symmetrization $\text{Sp}^n(\Omega)$ w.r.t. the positive axis of x_1 is defined as follows: Let $\Omega_r := \{\mathbf{x} \in \mathbb{S}^{n-1} : r\mathbf{x} \in \Omega\}$, i.e., contain the caps of radius r contained in Ω . Also, for the 1st coordinate let $\text{angle}(x_1) := \theta$ if $x_1 = r \cos \theta$. So as above

- If Ω_r is the full cap, then $\text{Sp}^n(\Omega) \cap \{|z| = r\} := \{|z| = r\}$.
- If the surface area is $m_s(\Omega_t) = \alpha$, then $\text{Sp}^n(\Omega) \cap \{|z| = r\} := \{x : |x| = r \text{ \& } 0 \leq \text{angle}(x_1) \leq \theta_\alpha\} =: C(\theta_\alpha)$ where θ_α is picked so that its surface area is $m_s(C(\theta_\alpha)) = \alpha$. In words, $C(\theta_\alpha)$ is a cap symmetric around the positive axis x_1 with the same area as the intersection $\Omega \cap \{|z| = r\}$.
- $0, \infty \in \text{Sp}^n(\Omega)$ iff $0, \infty \in \Omega$.” – [Wikipedia/symmetrization methods/circular symmetrization](#)

1.1.4 Polarization

Fig: Polarization of set Ω .

“Let $\Omega \subset \mathbb{R}^n$ be a domain & $H^{n-1} \subset \mathbb{R}^n$ be a hyperplane through the origin. Denote the reflection across that plane to the positive halfspace \mathbb{H}^+ as σ_H or just σ when it is clear from the context. Also, the reflected Ω across hyperplane H is defined as $\sigma\Omega$. Then, the polarized Ω is denoted as Ω^α & defined as follows

- If $\mathbf{x} \in \Omega \cap \mathbb{H}^+$, then $\mathbf{x} \in \Omega^\alpha$.
- If $\mathbf{x} \in \Omega \cap \sigma(\Omega) \cap \mathbb{H}^-$, then $\mathbf{x} \in \Omega^\sigma$.
- If $\mathbf{x} \in (\Omega \setminus \sigma(\Omega)) \cap \mathbb{H}^-$, then $\sigma\mathbf{x} \in \Omega^\sigma$.

In words, $(\Omega \setminus \sigma(\Omega)) \cap \mathbb{H}^-$ is simply reflected to the halfspace \mathbb{H}^+ . It turns out that this transformation can approximate the above ones (in the [Hausdorff distance](#)) (see [Brock & Solynin2000]).” – [Wikipedia/symmetrization methods/polarization](#)

1.2 Wikipedia/Interpolation Space

“In the field of [mathematical analysis](#), an *interpolation space* is a space which lies “in between” 2 other [Banach spaces](#). The main applications are in [Sobolev spaces](#), where spaces of functions that have a noninteger number of [derivatives](#) are interpolated from the spaces of functions with integer number of derivatives.” – [Wikipedia/interpolation space](#)

1.2.1 History

“The theory of interpolation of vector spaces began by an observation of [Józef Marcinkiewicz](#), later generalized & now known as the [Riesz–Thorin theorem](#). In simple terms, if a linear function is continuous on a certain [space](#) L^p & also on a certain space L^q , then it is also continuous on the space L^r , for any intermediate r between p & q . In other words, L^r is a space which is intermediate between L^p & L^q .

In the development of Sobolev spaces, it became clear that the trace spaces were not any of the usual function spaces (with integer number of derivatives), & [Jacques-Louis Lions](#) discovered that indeed these trace spaces were constituted of functions that have a noninteger degree of differentiability.

Many methods were designed to generate such spaces of functions, including the [Fourier transform](#), complex interpolation, real interpolation, as well as other tools (see e.g. [fractional derivative](#)).” – [Wikipedia/interpolation space/history](#)

1.2.2 The setting of interpolation

“A [Banach space](#) X is said to be *continuously embedded* in a Hausdorff [topological vector space](#) Z when X is a linear subspace of Z s.t. the inclusion map from X into Z is continuous. A *compatible couple* (X_0, X_1) of Banach spaces consists of 2 Banach spaces X_0 & X_1 that are continuously embedded in the same Hausdorff topological vector space Z . The embedding in a linear space Z allows to consider the 2 linear subspaces $X_0 \cap X_1$ & $X_0 + X_1 = \{z \in Z; z = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$. Interpolation does not depend only upon the isomorphic (nor isometric) equivalence classes of X_0 & X_1 . It depends in an essential way from the specific *relative position* that X_0 & X_1 occupy in a larger space Z . One can define norms on $X_0 \cap X_1$ & $X_0 + X_1$ by $\|x\|_{X_0 \cap X_1} := \max(\|x\|_{X_0}, \|x\|_{X_1})$, $\|x\|_{X_0 + X_1} := \inf \{\|x_0\|_{X_0} + \|x_1\|_{X_1}; x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$. Equipped with these norms, the intersection & the sum are Banach spaces. The following inclusions are all continuous: $X_0 \cap X_1 \subset X_0$,

$X_1 \subset X_0 + X_1$. Interpolation studies the family of spaces X that are *intermediate spaces* between X_0 & X_1 in the sense that $X_0 \cap X_1 \subset X \subset X_0 + X_1$, where the 2 inclusions maps are continuous.

An example of this situation is the pair $(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, where the 2 Banach spaces are continuously embedded in the space Z of measurable functions on the real line, equipped with the topology of convergence in measure. In this situation, the spaces $L^p(\mathbb{R})$, for $1 \leq p \leq \infty$ are intermediate between $L^1(\mathbb{R})$ & $L^\infty(\mathbb{R})$. More generally,

$$L^{p_0}(\mathbb{R}) \cap L^{p_1}(\mathbb{R}) \subset L^p(\mathbb{R}) \subset L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R}), \text{ when } 1 \leq p_0 \leq p \leq p_1 \leq \infty,$$

with continuous injections, so that, under the given condition, $L^p(\mathbb{R})$ is intermediate between $L^{p_0}(\mathbb{R})$ & $L^{p_1}(\mathbb{R})$.

Definition 1.1 (Interpolation pair). *Given 2 compatible couples (X_0, X_1) & (Y_0, Y_1) , an interpolation pair is a couple (X, Y) of Banach spaces with the 2 following properties:*

- The space X is intermediate between X_0 & X_1 , & Y is intermediate between Y_0 & Y_1 .
- If L is any linear operator from $X_0 + X_1$ to $Y_0 + Y_1$, which maps continuously X_0 to Y_0 & X_1 to Y_1 , then it also maps continuously X to Y .

The interpolation pair (X, Y) is said to be of *exponent* θ (with $0 < \theta < 1$) if there exists a constant C s.t. $\|L\|_{X,Y} \leq C \|L\|_{X_0,Y_0}^{1-\theta} \|L\|_{X_1,Y_1}^\theta$ for all operators L as above. The notation $\|L\|_{X,Y}$ is for the norm of L as a map from X to Y . If $C = 1$, we say that (X, Y) is an *exact interpolation pair of exponent* θ . – [Wikipedia/interpolation space/the setting of interpolation](#)

1.2.3 Complex interpolation

“If the scalars are **complex numbers**, properties of complex **analytic functions** are used to define an interpolation space. Given a compatible couple (X_0, X_1) of Banach spaces, the linear space $\mathcal{F}(X_0, X_1)$ consists of all functions $f : \mathbb{C} \rightarrow X_0 + X_1$, that are analytic on $S = \{z : 0 < \operatorname{Re}(z) < 1\}$, continuous on $\bar{S} = \{z : 0 \leq \operatorname{Re}(z) \leq 1\}$, & for which all the following subsets are bounded: $\{f(z) : z \in S\} \subset X_0 + X_1$, $\{f(it) : t \in \mathbb{R}\} \subset X_0$, $\{f(1+it) : t \in \mathbb{R}\} \subset X_1$. $\mathcal{F}(X_0, X_1)$ is a Banach space under the norm

$$\|f\|_{\mathcal{F}(X_0, X_1)} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{X_1} \right\}.$$

Definition 1.2. For $0 < \theta < 1$, the complex interpolation space $(X_0, X_1)_\theta$ is the linear subspace of $X_0 + X_1$ consisting of all values $f(\theta)$ when f varies in the preceding space of functions, $(X_0, X_1)_\theta = \{x \in X_0 + X_1 : x = f(\theta), f \in \mathcal{F}(X_0, X_1)\}$. The norm on the complex interpolation space $(X_0, X_1)_\theta$ is defined by $\|x\|_\theta = \inf \{\|f\|_{\mathcal{F}(X_0, X_1)} : f(\theta) = x, f \in \mathcal{F}(X_0, X_1)\}$.

Equipped with this norm, the complex interpolation space $(X_0, X_1)_\theta$ is a Banach space.

Theorem 1.1. *Given 2 compatible couples of Banach spaces (X_0, X_1) & (Y_0, Y_1) , the pair $((X_0, X_1)_\theta, (Y_0, Y_1)_\theta)$ is an exact interpolation pair of exponent θ , i.e., if $T : X_0 + X_1 \rightarrow Y_0 + Y_1$, is a linear operator bounded from X_j to Y_j , $j = 0, 1$, then T is bounded from $(X_0, X_1)_\theta$ to $(Y_0, Y_1)_\theta$ & $\|T\|_\theta \leq \|T\|_0^{1-\theta} \|T\|_1^\theta$.*

The family of L^p spaces (consisting of complex valued functions) behaves well under complex interpolation. If (R, Σ, μ) is an arbitrary **measure space**, if $1 \leq p_0, p_1 \leq \infty$ & $0 < \theta < 1$, then

$$(L^{p_0}(R, \Sigma, \mu), L^{p_1}(R, \Sigma, \mu))_\theta = L^p(R, \Sigma, \mu), \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

with equality of norms. This fact is closely related to the **Riesz–Thorin theorem**. – [Wikipedia/interpolation space/complex interpolation](#)

1.2.4 Real interpolation

“There are 2 ways for introducing the *real interpolation method*. The 1st & most commonly used when actually identifying examples of interpolation spaces is the K-method. The 2nd method, the J-method, gives the same interpolation spaces as the K-method when the parameter θ is in $(0, 1)$. That the J- & K-methods agree is important for the study of duals of interpolation spaces: basically, the dual of an interpolation space constructed by the K-method appears to be a space constructed from the dual couple by the J-method.” – [Wikipedia/interpolation space/real interpolation](#)

1.2.4.1 K-method

“The K-method of real interpolation can be used for Banach spaces over the field \mathbb{R} of **real numbers**.

Definition 1.3. Let (X_0, X_1) be a compatible couple of Banach spaces. For $t > 0$ & every $x \in X_0 + X_1$, let $K(x, t; X_0, X_1) = \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \}$. Changing the order of the 2 spaces results in: $K(x, t; X_0, X_1) = tK(x, t^{-1}; X_1, X_0)$. Let

$$\|x\|_{\theta, q; K} = \left(\int_0^\infty (t^{-\theta} K(x, t; X_0, X_1))^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 0 < \theta < 1, 1 \leq q < \infty,$$

$$\|x\|_{\theta, \infty; K} = \sup_{t > 0} t^{-\theta} K(x, t; X_0, X_1), \quad 0 \leq \theta \leq 1.$$

The K-method of real interpolation consists in taking $K_{\theta, q}(X_0, X_1)$ to be the linear subspace of $X_0 + X_1$ consisting of all x s.t. $\|x\|_{\theta, q; K} < \infty$.

Example 1.1. An important example is that of the couple $(L^1(\mathbb{R}, \Sigma, \mu), L^\infty(\mathbb{R}, \Sigma, \mu))$, where the functional $K(t, f; L^1, L^\infty)$ can be computed explicitly. The measure μ is supposed **σ -finite**. In this context, the best way of cutting the function $f \in L^1 + L^\infty$ as sum of 2 functions $f_0 \in L^1$ & $f_1 \in L^\infty$ is, for some $s > 0$ to be chosen as function of t , to let $f_1(x)$ be given for all $x \in \mathbb{R}$ by

$$f_1(x) = \begin{cases} f(x), & |f(x)| < s, \\ \frac{sf(x)}{|f(x)|}, & \text{otherwise.} \end{cases}$$

The optimal choice of s leads to the formula

$$K(f, t; L^1, L^\infty) = \int_0^1 f^*(u) du,$$

where f^* is the **decreasing rearrangement** of f .” – [Wikipedia/interpolation space/real interpolation/K-method](#)

1.2.4.2 J-method

“As with the K-method, the J-method can be used for real Banach spaces.

Definition 1.4. Let (X_0, X_1) be a compatible couple of Banach spaces. For $t > 0$ & for every vector $x \in X_0 \cap X_1$, let $J(x, t; X_0, X_1) = \max(\|x\|_{X_0}, t\|x\|_{X_1})$. A vector x in $X_0 + X_1$ belongs to the interpolation space $J_{\theta, q}(X_0, X_1)$ iff it can be written as $x = \int_0^\infty v(t) \frac{dt}{t}$, where $v(t)$ is measurable with values in $X_0 \cap X_1$ & s.t.

$$\Phi(v) = \left(\int_0^\infty (t^{-\theta} J(v(t), t; X_0, X_1))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

The norm of x in $J_{\theta, q}(X_0, X_1)$ is given by the formula

$$\|x\|_{\theta, q; J} := \inf_v \left\{ \Phi(v) : x = \int_0^\infty v(t) \frac{dt}{t} \right\}.$$

” – [Wikipedia/interpolation space/real interpolation/J-method](#)

1.2.4.3 Relations between the interpolation methods

“The 2 real interpolation methods are equivalent when $0 < \theta < 1$.

Theorem 1.2. Let (X_0, X_1) be a compatible couple of Banach spaces. If $0 < \theta < 1$ & $1 \leq q \leq \infty$, then $J_{\theta, q}(X_0, X_1) = K_{\theta, q}(X_0, X_1)$, with **equivalence of norms**.

The theorem covers degenerate cases that have not been excluded: e.g. if X_0 & X_1 form a direct sum, then the intersection & the J -spaces are the null space, & a simple computation shows that the K -spaces are also null.

When $0 < \theta < 1$, one can speak, up to an equivalent renorming, about the Banach space obtained by the real interpolation method with parameters θ & q . The notation for this real interpolation space is $(X_0, X_1)_{\theta, q}$. One has that

$$(X_0, X_1)_{\theta, q} = (X_1, X_0)_{1-\theta, q}, \quad 0 < \theta < 1, 1 \leq q \leq \infty.$$

For a given value of θ , the real interpolation spaces increase with q : if $0 < \theta < 1$ & $1 \leq q \leq r \leq \infty$, the following continuous inclusion holds true: $(X_0, X_1)_{\theta, q} \subset (X_0, X_1)_{\theta, r}$.

Theorem 1.3. *Given $0 < \theta < 1$, $1 \leq q \leq \infty$ & 2 compatible couples (X_0, X_1) & (Y_0, Y_1) , the pair $((X_0, X_1)_{\theta, q}, (Y_0, Y_1)_{\theta, q})$ is an exact interpolation pair of exponent θ .*

A complex interpolation space is usually not isomorphic to 1 of the spaces given by the real interpolation method. However, there is a general relationship.

Theorem 1.4. *Let (X_0, X_1) be a compatible couple of Banach spaces. If $0 < \theta < 1$, then $(X_0, X_1)_{\theta, 1} \subset (X_0, X_1)_\theta \subset (X_0, X_1)_{\theta, \infty}$.*

Example 1.2. *When $X_0 = C([0, 1])$ & $X_1 = C^1([0, 1])$, the space of continuously differentiable functions on $[0, 1]$, the (θ, ∞) interpolation method, for $0 < \theta < 1$, gives the **Hölder space** $C^{0, \theta}$ of exponent θ . This is because the K -functional $K(f, t; X_0, X_1)$ of this couple is equivalent to*

$$\sup \left\{ |f(u)|, \frac{|f(u) - f(v)|}{1 + t^{-1}|u - v|} : u, v \in [0, 1] \right\}.$$

Only values $0 < t < 1$ are interesting here.

Real interpolation between L^p spaces gives the family of **Lorentz spaces**. Assuming $0 < \theta < 1$ & $1 \leq q \leq \infty$, one has

$$(L^1(\mathbb{R}, \Sigma, \mu), L^\infty(\mathbb{R}, \Sigma, \mu))_{\theta, q} = L^{p, q}(\mathbb{R}, \Sigma, \mu), \text{ where } \frac{1}{p} = 1 - \theta,$$

with equivalent norms. This follows from an **inequality of Hardy** & from the value given above of the K -functional for this compatible couple. When $q = p$, the Lorentz space $L^{p, p}$ is equal to L^p , up to renorming. When $q = \infty$, the Lorentz space $L^{p, \infty}$ is equal to **weak- L^p** . – [Wikipedia/interpolation space/real interpolation/relations between the interpolation methods](#)

1.2.5 The reiteration theorem

“An intermediate space X of the compatible couple (X_0, X_1) is said to be of *class* θ if $(X_0, X_1)_{\theta, 1} \subset X \subset (X_0, X_1)_{\theta, \infty}$, with continuous injections. Beside all real interpolation spaces $(X_0, X_1)_{\theta, q}$ with parameter θ & $1 \leq q \leq \infty$, the complex interpolation space $(X_0, X_1)_\theta$ is an intermediate space of class θ of the compatible couple (X_0, X_1) .

The reiteration theorems says, in essence, that interpolating with a parameter θ behaves, in some way, like forming a **convex combination** $a = (1 - \theta)x_0 + \theta x_1$: taking a further convex combination of 2 convex combinations gives another convex combination.

Theorem 1.5. *Let A_0, A_1 be intermediate spaces of the compatible couple (X_0, X_1) , of class θ_0 & θ_1 resp., with $0 < \theta_0 \neq \theta_1 < 1$. When $0 < \theta < 1$ & $1 \leq q \leq \infty$, one has $(A_0, A_1)_{\theta, q} = (X_0, X_1)_{\eta, q}$, $\eta = (1 - \theta)\theta_0 + \theta\theta_1$.*

It is notable that when interpolating with the real method between $A_0 = (X_0, X_1)_{\theta_0, q_0}$ & $A_1 = (X_0, X_1)_{\theta_1, q_1}$, only the values of θ_0 & θ_1 matter. Also, A_0 & A_1 can be complex interpolation spaces between X_0 & X_1 , with parameters θ_0 & θ_1 resp.

There is also a reiteration theorem for the complex method.

Theorem 1.6. *Let (X_0, X_1) be a compatible couple of complex Banach spaces, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1 . Let $A_0 = (X_0, X_1)_{\theta_0}$ & $A_1 = (X_0, X_1)_{\theta_1}$, where $0 \leq \theta_0 \leq \theta_1 \leq 1$. Assume further that $X_0 \cap X_1$ is dense in $A_0 \cap A_1$. Then, for every $0 \leq \theta \leq 1$,*

$$((X_0, X_1)_{\theta_0}, (X_0, X_1)_{\theta_1})_\theta = (X_0, X_1)_\eta, \quad \eta = (1 - \theta)\theta_0 + \theta\theta_1.$$

The density condition is always satisfied when $X_0 \subset X_1$ or $X_1 \subset X_0$. – [Wikipedia/interpolation space/the reiteration theorem](#)

1.2.6 Duality

“Let (X_0, X_1) be a compatible couple, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1 . In this case, the restriction map from the (continuous) **dual** X'_j of X_j , $j = 0, 1$, to the dual of $X_0 \cap X_1$ is 1-1. It follows that the pair of duals (X'_0, X'_1) is a compatible couple continuously embedded in the dual $(X_0 \cap X_1)'$.

For the complex interpolation method, the following duality result holds:

Theorem 1.7. *Let (X_0, X_1) be a compatible couple of complex Banach spaces, & assume that $X_0 \cap X_1$ is dense in X_0 & in X_1 . If X_0 & X_1 are **reflexive**, then the dual of the complex interpolation space is obtained by interpolating the duals, $((X_0, X_1)_\theta)' = (X'_0, X'_1)_\theta$, $0 < \theta < 1$.*

In general, the dual of the space $(X_0, X_1)_\theta$ is equal to $(X'_0, X'_1)^\theta$, a space defined by a variant of the complex method. The upper- θ & lower- θ methods do not coincide in general, but they do if at least 1 of X_0, X_1 is a reflexive space.

For the real interpolation method, the duality holds provided that the parameter q is finite:

Theorem 1.8. *Let $0 < \theta < 1$, $1 \leq q < \infty$ & (X_0, X_1) a compatible couple of real Banach spaces. Assume that $X_0 \cap X_1$ is dense in X_0 & in X_1 . Then*

$$((X_0, X_1)_{\theta, q})' = (X'_0, X'_1)_{\theta, q'}, \text{ where } \frac{1}{q'} = 1 - \frac{1}{q}.$$

” – [Wikipedia/interpolation space/duality](#)

1.2.7 Discrete definitions

“Since the function $t \rightarrow K(x, t)$ varies regularly (it is increasing, but $\frac{1}{t}K(x, y)$ is decreasing), the definition of the $K_{\theta, q}$ -norm of a vector n , previously given by an integral, is equivalent to a definition given by a series. This series is obtained by breaking $(0, \infty)$ into pieces $(2^n, 2^{n+1})$ of equal mass for the measure $\frac{dt}{t}$,

$$\|x\|_{\theta, q; K} \simeq \left(\sum_{n \in \mathbb{Z}} (2^{-\theta n} K(x, 2^n; X_0, X_1))^q \right)^{\frac{1}{q}}.$$

In the special case where X_0 is continuously embedded in X_1 , one can omit the part of the series with negative indices n . In this case, each of the functions $x \rightarrow K(x, 2^n; X_0, X_1)$ defines an equivalent norm on X_1 .

The interpolation space $(X_0, X_1)_{\theta, q}$ is a “diagonal subspace” of an l^q -sum of a sequence of Banach spaces (each one being isomorphic to $X_0 + X_1$). Therefore, when q is finite, the dual of $(X_0, X_1)_{\theta, q}$ is a **quotient** of the l^p -sum of the duals, $\frac{1}{p} + \frac{1}{q} = 1$, which leads to the following formula for the discrete $J_{\theta, p}$ -norm of a functional x' in the dual of $(X_0, X_1)_{\theta, q}$:

$$\|x'\|_{\theta, p; J} \simeq \inf \left\{ \left(\sum_{n \in \mathbb{Z}} (2^{\theta n} \max(\|x'_n\|_{X'_0}, 2^{-n} \|x'_n\|_{X'_1}))^p \right)^{\frac{1}{p}} : x' = \sum_{n \in \mathbb{Z}} x'_n \right\}.$$

The usual formula for the discrete $J_{\theta, p}$ -norm is obtained by changing n to $-n$.

The discrete definition makes several questions easier to study, among which the already mentioned identification of the dual. Other such questions are compactness or weak-compactness of linear operators. Lions & Peetre have proved that:

Theorem 1.9. *If the linear operator T is **compact** from X_0 to a Banach space Y & bounded from X_1 to Y , then T is compact from $(X_0, X_1)_{\theta, q}$ to Y when $0 < \theta < 1$, $1 \leq q \leq \infty$.*

Davis, Figiel, Johnson, & Pełczyński have used interpolation in their proof of the following result:

Theorem 1.10. *A bounded linear operator between 2 Banach spaces is **weakly compact** iff it factors through a **reflexive space**.*

” – [Wikipedia/interpolation space/discrete definitions](#)

1.2.7.1 A general interpolation method

“The space l^q used for the discrete definition can be replaced by an arbitrary **sequence space** Y with **unconditional basis**, & the weights $a_n = 2^{-\theta n}$, $b_n = 2^{(1-\theta)n}$, that are used for the $K_{\theta, q}$ -norm, can be replaced by general weights $a_n, b_n > 0$, $\sum_{n=1}^{\infty} \min(a_n, b_n) < \infty$. The interpolation space $K(X_0, X_1, Y, \{a_n\}, \{b_n\})$ consists of the vectors x in $X_0 + X_1$ s.t.

$$\|x\|_{K(X_0, X_1)} = \sup_{m \geq 1} \left\| \sum_{n=1}^m a_n K\left(x, \frac{b_n}{a_n}; X_0, X_1\right) y_n \right\|_Y < \infty,$$

where $\{y_n\}$ is the unconditional basis of Y . This abstract method can be used, e.g., for the proof of the following result:

Theorem 1.11. *A Banach space with unconditional basis is isomorphic to a complemented subspace of a space with **symmetric basis**.*

” – [Wikipedia/interpolation space/discrete definitions/a general interpolation method](#)

1.2.8 Interpolation of Sobolev & Besov spaces

Several interpolation results are available for [Sobolev spaces](#) & [Besov spaces](#) on \mathbb{R}^n , H_p^s , $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $B_{p,q}^s$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. These spaces are spaces of [measurable functions](#) on \mathbb{R}^n when $s \geq 0$, & of [tempered distributions](#) on \mathbb{R}^n when $s < 0$. For the rest of the section, the following setting & notation will be used: $0 < \theta < 1$, $1 \leq p, p_0, p_1, q, q_0, q_1 \leq \infty$, $s, s_0, s_1 \in \mathbb{R}$, $s_\theta = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Complex interpolation works well on the class of Sobolev spaces H_p^s (the [Bessel potential spaces](#)) as well as Besov spaces: $(H_{p_0}^{s_0}, H_{p_1}^{s_1})_\theta = H_{p_\theta}^{s_\theta}$, $s_0 \neq s_1$, $1 < p_0, p_1 < \infty$. $(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_\theta = B_{p_\theta,q_\theta}^{s_\theta}$, $s_0 \neq s_1$.

Real interpolation between Sobolev spaces may give Besov spaces, except when $s_0 = s_1$, $(H_{p_0}^s, H_{p_1}^s)_{\theta,p_\theta} = H_{p_\theta}^s$. When $s_0 \neq s_1$ but $p_0 = p_1$, real interpolation between Sobolev spaces gives a Besov space: $(H_{p_0}^{s_0}, H_{p_1}^{s_1})_{\theta,q} = B_{p,q}^{s_\theta}$, $s_0 \neq s_1$. Also, $(B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} = B_{p,q}^{s_\theta}$, $s_0 \neq s_1$. $(B_{p,q_0}^s, B_{p,q_1}^s)_{\theta,q} = B_{p,q_\theta}^s$. $(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_{\theta,q_\theta} = B_{p_\theta,q_\theta}^{s_\theta}$, $s_0 \neq s_1$, $p_\theta = q_\theta$." – [Wikipedia/interpolation space/interpolation of Sobolev & Besov spaces](#)

Chapter 2

Terence Tao's

2.1 Tao, 2007. What Is Good Mathematics?

Abstract. “Some personal thoughts & opinions on what “good quality mathematics” is & whether one should try to define this term rigorously. As a case study, the story of Szemerédi’s theorem is presented.”

2.1.1 The Many Aspects of Mathematical Quality

“We all agree that mathematicians should strive¹ to produce good mathematics. *But how does one define “good mathematics”, & should one even dare to try at all?* Let us 1st consider the former question. Almost immediately one realizes that there are many different types of mathematics which could be designated² “good”. E.g., “good mathematics” could refer (in no particular³ order) to

1. Good mathematical *problem solving* (e.g. a major⁴ breakthrough⁵ on an important mathematical problem);
2. Good mathematical *technique*⁶ (e.g. a masterful⁷ use of existing⁸ methods⁹ or the development¹⁰ of new tools¹¹);
3. Good mathematical *theory* (e.g. a conceptual¹² framework¹³ or choice of notation¹⁴ which systematically¹⁵ unifies¹⁶ &

¹**strive** [v] [intransitive] to try very hard to achieve something.

²**designate** [v] [often passive] **1.** to say officially that somebody/something has a particular character, name or purpose; to describe somebody/something in a particular way; **2.** to choose or name somebody/something for a particular job or position; **3.** (of a symbol) to identify or show something.

³**particular** [a] [only before noun] **1.** used to emphasize that you are referring to 1 individual person, thing or type of thing & not others, SYNONYM: **specific**; **2.** greater than usual; special; **in particular** [idiom] **1.** especially or particularly; **2.** special, SYNONYM: **specific**; **of particular note** [idiom] especially interesting; [n] **1.** [countable, usually plural] a fact or detail, especially one that is officially written down; **2.** (particulars) [plural] written information & details about a property, business, job, etc.

⁴**major** [a] **1.** [usually before noun] large, important or serious, OPPOSITE: **minor**; **2.** [only before noun] greater or more important; main, SYNONYM: **main**; [n] (*North American English*) **1.** the main subject or course of a student at college or university; **2.** a student studying a particular subject as the main part of their course.

⁵**breakthrough** [n] an important development or discovery that helps people to achieve or understand something.

⁶**technique** [n] **1.** [countable] a particular way of doing something that involves using a special skill or process; **2.** [uncountable, singular] a person’s skill or ability in a particular activity.

⁷**masterful** [a] **1.** (of a person, especially a man) able to control people or situations in a way that shows confidence as a leader; **2.** (also **masterly**) showing great skill or understanding.

⁸**existing** [a] [only before noun] found or used now or at the time being discussed.

⁹**method** [n] a particular way of doing something.

¹⁰**development** [n] **1.** [uncountable] the process of creating a new method, system, product or theory; **2.** [countable] a new or advanced method, system, product or theory; **3.** [uncountable] the process of making a country or area richer & more successful; **4.** [uncountable] the way in which a child or other living creature grows before & after birth.

¹¹**tool** [n] **1.** a thing that helps somebody to do a job or to achieve something; **2.** a piece of equipment held in the hand, that is used for making things or repairing things.

¹²**conceptual** [a] connected with or based on ideas.

¹³**framework** [n] **1.** a set of beliefs, ideas or principles that is based as the basis for examining or understanding something; **2.** a system of rules, laws or agreements that controls the way that something works in business, politics or society.

¹⁴**notation** [n] [uncountable, countable] **notation (for something)** a system of signs or symbols used to represent information, especially in mathematics, science & music.

¹⁵**systematically** [adv] **1.** in a way that follows a system; **2.** in the same way all through a process or set of results because of the system that is used.

¹⁶**unify** [v] **1.** **unify something** to join people or countries together so that they form a single unit; **2.** **unify something (into something)** to put things, especially ideas, together in a good or helpful way.

generalizes¹⁷ an existing¹⁸ body of results);

4. Good mathematical *insight*¹⁹ (e.g. a major conceptual simplification²⁰ or the realization²¹ of a unifying²² principle²³, analogy²⁴, or theme²⁵);
5. Good mathematical *discovery*²⁶ (e.g. the revelation²⁷ of an unexpected²⁸ & intriguing²⁹ new mathematical phenomenon³⁰, connection³¹, or counterexample³²);
6. Good mathematical *application*³³ (e.g. to important problems in physics, engineering, computer science, statistics, etc., or from 1 field of mathematics to another);
7. Good mathematical *exposition*³⁴ (e.g. a detailed³⁵ & informative³⁶ survey³⁷ on a timely³⁸ mathematical topic or a clear & well-motivated argument);
8. Good mathematical *pedagogy*³⁹ (e.g. a lecture⁴⁰ or writing style which enables others to learn & do mathematics more

¹⁷**generalize** [v] (*British English also generalise*) **1.** [intransitive] **generalize (from something)** to use a particular set of facts or ideas in order to form an opinion that is considered valid for a different situation; **2.** [intransitive] to make a general statement about something & not look at the details; **3.** [transitive, often passive] to apply a theory, idea, etc. to a wider group or situation than the original one.

¹⁸**existing** [a] [only before noun] found or used now or at the time being discussed.

¹⁹**insight** [n] **1.** [countable, uncountable] an understanding of a particular situation or thing; **2.** [uncountable] the ability to see & understand the truth about people or situations.

²⁰**simplification** [n] **1.** [uncountable] **simplification (of something)** the process of making something less complicated, or easier to do or understand; **2.** [countable] a change that makes a problem, statement, system, etc. less complicated or easier to understand or do.

²¹**realization** [n] (*British English also realisation*) **1.** [uncountable, singular] **realization (that)** ... the process of becoming aware of something, SYNONYM: **awareness**; **2.** [uncountable] **realization (of something)** the process of achieving a particular aim, etc., SYNONYM: **achievement**; **3.** [uncountable, countable] **realization (of something)** (*formal*) the act of producing something in an actual or physical form; the thing that is produced.

²²**unify** [v] **1. unify something** to join people or countries together so that they form a single unit; **2. unify something (into something)** to put things, especially ideas, together in a good or helpful way.

²³**principle** [n] **1.** [countable] a law, rule or theory that something is based on; **2.** [singular] a general or scientific law that explains how something works or why something happens; **3.** [countable] a belief that is accepted as a reason for acting or thinking in a particular way; **4.** [countable, usually plural, uncountable] a moral rule or a strong belief that influences your actions; **in principle** [idiom] **1.** if something can be done in principle, there is no good reason why it should not be done although it has not yet been done & there may be some difficulties; **2.** in general but not in detail.

²⁴**analogy** [n] (plural **analogies**) [countable, uncountable] a comparison of 1 thing with another thing that has similar features, usually in order to explain it; a feature that is similar.

²⁵**theme** [n] the subject of a talk, piece of writing, exhibition, etc.; an idea that keeps returning in a piece of research or a work of art or literature.

²⁶**discovery** [n] (plural **discoveries**) **1.** [countable, uncountable] an act or the process of finding somebody/something, or learning about something that was not known about before; **2.** [countable] a thing, fact or person that is found or learned about for the 1st time.

²⁷**revelation** [n] **1.** [countable] a fact that people are made aware of, especially one that has been secret & is surprising, SYNONYM: **disclosure**; **2.** [uncountable] **revelation (of something)** the act of making people aware of something that has been secret, SYNONYM: **disclosure**; **3.** [countable, uncountable] something that is considered to be a sign or message from God.

²⁸**unexpected** [a] surprising; not expected.

²⁹**intriguing** [a] very interesting because of being unusual or not having an obvious answer.

³⁰**phenomenon** [n] (plural **phenomena**) a fact or an event in nature or society, especially one that is not fully understood.

³¹**connection** [n] (*British English also, old-fashioned connexion*) **1.** [countable] something that connects 2 facts or ideas, SYNONYM: **link**; **2.** [countable] a relationship between people or groups of people, often for a particular purpose; **3.** [uncountable, countable] the action of connecting something to a supply of water, electricity, etc. or to a computer or telephone network; the fact of being connected in this way; **4.** [countable] a point, especially in an electrical system, where 2 parts connect; **5.** [countable, usually plural] a means of traveling to another place; **6.** [countable, usually plural] people that you know, who can help or advise you in your professional or social life; **in connection with somebody/something** [idiom] for reasons connected with somebody/something; **in this/that connection** [idiom] for reasons connected with something recently mentioned.

³²**counterexample** [n] **counterexample (to something)** an example that provides evidence against an idea or theory.

³³**application** [n] **1.** [uncountable, countable] the use of something such as an idea, method, rule, etc.; a use that something has; **2.** [countable] a formal (often written) request to an organization or authority for something, such as a job or permission to do something, or to join a group; **3.** [countable] a program or piece of software designed to do a particular job; **4.** [countable, uncountable] **application (of something) (to something)** the use of something to produce a particular physical effect; **5.** [countable, uncountable] **application (of something)** the action of putting or spreading something onto a surface or object.

³⁴**exposition** [n] [countable, uncountable] (*formal*) a full explanation of a theory, plan, etc.

³⁵**detailed** [a] giving many details; paying great attention to details.

³⁶**informative** [a] giving useful information.

³⁷**survey** [n] **1. survey (of somebody/something)** an investigation of the opinions, behavior, etc. of a particular group of people, which is usually done by asking them questions; **2.** an act of examining & recording the measurements, features, etc. of an area of land in order to make a map or plan of it; **3. survey (of something)** a general study, view or description of something; [v] **1. survey somebody/something** to investigate the opinions or behavior of a group of people by asking them a series of questions; **2. survey something** to study & give a general description of something; **3. survey something** to measure & record the features of an area of land, e.g. in order to make a map or in preparation for building; **4. survey something** to look carefully at the whole of something, especially in order to get a general impression of it, SYNONYM: **inspect**.

³⁸**timely** [a] happening at exactly the right time.

³⁹**pedagogy** [n] (plural **pedagogies**) [uncountable, countable] methods of teaching, especially as a subject of study or as a theory.

⁴⁰**lecture** [n] a talk that is given to a group of people to teach them about a particular subject, often as part of a university or college course; [v] [intransitive] **lecture (in/on something) (to somebody)** to give a talk or a series of talks to a group of people on a particular subject, especially as a way of teaching in a university or college.

- effectively, or contributions⁴¹ to mathematical education);
9. Good mathematical *vision*⁴² (e.g. a long-range⁴³ & fruitful program or set of conjectures⁴⁴);
 10. Good mathematical *taste* (e.g. a research goal which is inherently interesting & impacts important topics, themes, or questions);
 11. Good mathematical *public relations* (e.g. an effective showcasing of a mathematical achievement to non-mathematicians or from 1 field of mathematics to another);
 12. Good *meta-mathematics* (e.g. advances in the foundations, philosophy, history, scholarship, or practice of mathematics);
 13. *Rigorous* mathematics (with all details correctly & carefully given in full);
 14. *Beautiful* mathematics (e.g. the amazing identities of Ramanujan, results which are easy (& pretty) to state but not to prove);
 15. *Elegant* mathematics (e.g. Paul Erdős' concept of "proofs from *The Book*", achieving a difficult result with a minimum of effort);
 16. *Creative* mathematics (e.g. a radically new & original technique, viewpoint, or species of result);
 17. *Useful* mathematics (e.g. a lemma or method which will be used repeatedly in future work on the subject);
 18. *Strong* mathematics (e.g. a sharp result that matches the known counterexamples or a result which deduces an unexpectedly strong conclusion from a seemingly weak hypothesis);
 19. *Deep* mathematics (e.g. a result which is manifestly nontrivial, e.g. by capturing a subtle phenomenon beyond the reach of more elementary tools);
 20. *Intuitive* mathematics (e.g. an argument which is natural & easily visualizable);
 21. *Definitive* mathematics (e.g. a classification of all objects of a certain type, the final word on a mathematical topic);
 22. etc., etc.⁴⁵

As the above list demonstrates, the concept of mathematical quality is a high-dimensional one & lacks an obvious canonical total ordering.⁴⁶ I believe this is because mathematics is itself complex & high-dimensional & evolves in unexpected & adaptive ways; each of the above qualities represents a different way in which we as a community improve our understanding & usage of the subject. There does not appear to be universal agreement as to the relative importance or weight to each of the above qualities. This is partly due to tactical considerations: a field of mathematics at a given stage of development may be more receptive to 1 approach to mathematics than another. It is also partly due to cultural considerations: any given field or school of mathematics tends to attract like-minded mathematicians who prefer similar approaches to a subject. It also reflects the diversity of mathematical ability; different mathematicians tend to excel in different mathematical styles & are thus well suited for different types of mathematical challenges. (See also [Gowers2000] for some related discussion.)

I believe that this diverse & multifaceted nature of "good mathematics" is very healthy for mathematics as a whole, as it allows us to pursue many different approaches to the subject & exploit many different types of mathematical talent towards our common goal of greater mathematical progress & understanding. While each 1 of the above attributes is generally accepted to be a desirable trait to have in mathematics, it can become detrimental⁴⁷ to a field to pursue only 1 or 2 of them at the expense of all the others. Consider e.g. the following hypothetical (& somewhat exaggerated) scenarios:

⁴¹**contribution** [n] **1.** [usually singular] the part played by a person or thing in achieving, improving or causing something; **2.** a sum of money that is given to a person or an organization in order to help pay for something, SYNONYM: **donation**; **3.** **contribution (to something)** an item that forms part of a book, magazine, broadcast, discussion, etc.; **4.** a sum of money that you pay regularly to your employer or the government in order to pay for benefits such as health insurance or a pension.

⁴²**vision** [n] **1.** [uncountable] the ability to see; the area that you can see from a particular position; **2.** [countable] an idea or a picture in your imagination, especially of what the future will or could be like; **3.** [uncountable] the ability to think about or plan the future with great imagination & intelligence.

⁴³**long-range** [a] [only before noun] **1.** traveling a long distance; **2.** made for a period of time that will last a long way into the future.

⁴⁴**conjecture** [n] (*formal*) **1.** [countable] an opinion or idea that is not based on definite knowledge & is formed by guessing, SYNONYM: **guess**; **2.** [uncountable] the act of forming an opinion or idea that is not based on definite knowledge; [v] [intransitive, transitive] (*formal*) to form an opinion about something even though you do not have much information on it, SYNONYM: **guess**.

⁴⁵The above list is not meant to be exhaustive. In particular, it focuses primarily on the type of mathematics found in mathematical research papers as opposed to classrooms, textbooks, or papers in disciplines close to mathematics, such as the natural sciences.

⁴⁶In particular, it is worth pointing out that mathematical rigor, while highly important, is only 1 component of what determines quality piece of mathematics.

⁴⁷**detrimental** [a] (*formal*) harmful, SYNONYM: **damaging**.

- A field which becomes increasingly ornate & baroque, in which individual results are generalized & refined for their own sake, but the subject as a whole drifts aimlessly without any definite direction or sense of progress;
- A field which becomes filled with many astounding conjectures but with no hope of rigorous progress on any of them;
- A field which now consists primarily of using *ad hoc* methods to solve a collection of problems which have no unifying theme, connections, or purpose;
- A field which has become overly dry & theoretical, continually recasting & unifying previous results in increasingly technical forward frameworks but not generating any exciting new breakthroughs as a consequence; or
- A field which reverses classical results & continually presents shorter, simpler, & more elegant proofs of these results but which does not generate any truly original & new results beyond the classical literature.

In each of these cases, the field of mathematics exhibits much activity & progress in the short term but risks a decline of relevance & a failure to attract younger mathematicians to the subject in the longer term. Fortunately, it is hard for a field to stagnate in this manner when it is constantly being challenged & revitalized by its connections to other fields of mathematics (or to related sciences) & by exposure to (& respect for) multiple cultures of “good mathematics”. These self-correcting mechanisms help to keep mathematics balanced, unified, productive, & vibrant.

Let us turn now to the other question posed above, namely whether we should try to pin down a definition of “good mathematics” at all. In doing so, we run the risk of arrogance & hubris; in particular, we might fail to recognize exotic examples of genuine mathematical progress because they fall outside mainstream definitions⁴⁸ of “good mathematics”. On the other hand, there is a risk also in the opposite position – that all approaches to mathematics are equally suitable & deserving of equal resources⁴⁹ for any given mathematical field of study or that all contributions to mathematics are equally important; such positions may be admirable for their idealism, but they sap mathematics of its sense of direction & purpose & can also lead to a sub-optimal allocation of mathematical resources.⁵⁰ The true situation lies somewhere in between; for each area of mathematics, the existing body of results, folklore, intuition & experience (or lack thereof) will indicate which types of approaches are likely to be fruitful & thus deserve the majority of resources & which ones are more speculative & which might warrant inspection by only a handful of independently minded mathematicians, just to cover all bases. E.g., in mature & well-developed fields, it may make sense to pursue systematic programs & develop general theories in a rigorous manner, conservatively following tried-&-true methods & established intuition, whereas in newer & less settled fields, a greater emphasis might be placed on making & solving conjectures, experimenting with different approaches, & relying to some extent on non-rigorous heuristics & analogies. It thus makes sense from a tactical point of view to have at least a partial (but evolving) consensus within each field as to what qualities of mathematical progress one should prize the most so that one can develop & advance the field as effectively as possible at each stage of its development. E.g., 1 field may be in great need of solutions to pressing problems; another field may be crying out for a theoretical framework to organize the clutter of existing results or a grand program or series of conjectures to stimulate new results; other fields would greatly benefit from now, simpler, & more conceptual proofs of key theorems; yet more fields may require good publicity & lucid introductions to the subject in order to attract more activity & interest. Thus the determination of what would constitute good mathematics for a field can & should depend highly on the state of the field itself. It should also be a determination which is continually updated & debated, both within a field & by external observers to that field; as mentioned earlier, it is quite possible for a consensus on how a field should progress to lead to imbalances within that field if they are not detected & corrected in time.

It may seem from the above discussion that the problem of evaluating mathematical quality, while important, is a hopelessly complicated one, especially since many good mathematical achievements may score highly on some of the qualities listed above but not on others. Also, many of these qualities are subjective & difficult to measure precisely except with hindsight. However, there is the remarkable phenomenon⁵¹ that good mathematics in 1 of the above senses tends to beget more good mathematics in many of the other senses as well, leading to the tentative conjecture that perhaps there is, after all, a universal notion of good quality mathematics & all the specific metrics listed above represent different routes to uncover new mathematics or different stages or aspects of the evolution of a mathematical story.” – Tao, 2007, pp. 623–626

2.1.2 Case Study: Szemerédi’s theorem

2.1.3 Conclusion

“As we can see from the above case study, the very best examples of good mathematics do not merely fulfill 1 or more of the criteria of mathematical quality listed at the beginning of this article, but are more importantly part of a greater mathematical

⁴⁸A related difficulty is that, with the notable exception of mathematical rigor, most of the above qualities are somewhat subjective & contain some inherent imprecision or uncertainty. We thank Gil Kalai for emphasizing this point.

⁴⁹Examples of scarce resources include money, time, attention, talent, & pages in top journals.

⁵⁰Another solution to this problem is to exploit the fact that mathematical resources are also high-dimensional; e.g. one can award prizes for exposition, for creativity, etc., or have different journals devoted to different types of achievement. We thank Gil Kalai for this observation.

⁵¹This phenomenon is also somewhat related to the “unreasonable effectiveness of mathematics” observed by Wigner [Wigner1960]

story, which then unfurls to generate many further pieces of good mathematics of many different types. Indeed, one can view the history of entire fields of mathematics as being primarily generated by a handful of these great stories, their evolution through time, & their interaction with each other. I would thus conclude that good mathematics is not merely measured by 1 or more of the “local” qualities listed previously (though these are certainly important & worth pursuing & debating), but also depends on the more “global” question of how it fits in with other pieces of good mathematics, either by building upon earlier achievements or encouraging the the development of future breakthroughs. Of course, without the benefit of hindsight it is difficult to predict with certainty what types of mathematics will have such a property. There does however seem to be some undefinable sense that a certain piece of mathematics is “on to something”, that is a piece of a larger puzzle waiting to be explored further. & it seems to me that the pursuit of such intangible promises of future potential is at least as important an aspect of mathematical progress as the more concrete & obvious aspects of mathematical quality listed previously. Thus I believe that good mathematics is more than simply the process of solving problems, building theories, & making arguments shorter, stronger, clearer, more elegant, or more rigorous, though these are of course all admirable goals. While achieving all of these tasks (& debating which ones should have higher priority within any given field), we should also be aware of any possible larger context that one’s results could be placed in, as this may well lead to the greatest long-term benefit for the result, for the field, & for mathematics as a whole.” – Tao, 2007, pp. 632–633

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