Elementary Mathematics

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Contents

Chapter 1

Wikipedia's

1.1 Wikipedia/How to Solve It

"*How to Solve It* (1945) is a small volume by mathematician George Pólya describing methods of problem solving." – Wikipedia/how to solve it

1.1.1 4 principles

"*How to Solve It* suggests the following steps when solving a mathematical problem:

1. 1st, you have to *understand the problem*.
2. After understanding, *make a plan*.
3. *Carry out the plan*.
4. *Look back* on your work. How could it be better?

If this technique fails, Pólya advises: "If you can't solve a problem, then there is an easier problem you can solve: find it." Or: "If you cannot solve the proposed problem, try to solve 1st some related problem. Could you imagine a more accessible related problem?"" – Wikipedia/how to solve it/4 principles

1.1.1.1 1st principle: Understand the problem

"Understanding the problem" is often neglected as being obvious & is not even mentioned in many mathematics classes. Yet students are often stymied in their efforts to solve it, simply because they don't understand it fully, or even in part. In order to remedy this oversight, Pólya taught teachers how to prompt each student with appropriate questions, depending on the situation, such as:

- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture of a diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?
- Do you understand all the words used in stating the problem?
- Do you need to ask a question to get the answer?

The teacher is to select the question with the appropriate level of difficulty for each student to ascertain if each student understands at their own level, moving up or down the list to prompt each student, until each one can respond with something constructive." – Wikipedia/how to solve it/4 principles/1st principle: understand the problem

1.1.1.2 2nd principle: Devise a plan

"Pólya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. You will find choosing a strategy increasingly easy. A partial list of strategies is included:

- Guess & check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation

Also suggested:

- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Use a model
- Work backward
- Use a formula
- Be creative
- Applying these rules to devise a plan takes your own skill & judgment.

Pólya lays a big emphasis on the teachers' behavior. A teacher should support students with devising their own plan with a question method that goes from the most general questions to more particular questions, with the goal that the last step to having a plan is made by the student. He maintains that just showing students a plan, no matter how good it is, does not help them." – Wikipedia/how to solve it/4 principles/2nd principle: devise a plan

1.1.1.3 3rd principle: Carry out the plan

"This step is usually easier than devising the plan. In general, all you need is care & patience, given that you have the necessary skills. Persist with the plan that you have chosen. If it continues not to work, discard it & choose another. Don't be misled; this is how mathematics is done, even by professionals." – Wikipedia/how to solve it/4 principles/3rd principle: carry out the plan

1.1.1.4 4th principle: Review/extend

"Pólya mentions that much can be gained by taking the time to reflect & look back at what you have done, what worked & what did not, & with thinking about other problems where this could be useful. Doing this will enable you to predict what strategy to use to solve future problems, if these relate to the original problem." – Wikipedia/how to solve it/4 principles/4th principle: review/extend

1.1.2 Heuristics

"The book contains a dictionary-style set of heuristics, many of which have to do with generating a more accessible problem. E.g.:

Heuristic | Informal Description | Formal analogue

- Analogy | Can you find a problem analogous to your problem & solve that? | map
- Auxiliary Elements | Can you add some new element to your problem to get closer to a solution? | Extension

- Generalization | Can you find a problem more general than your problem? | Generalization
- Induction | Can you solve your problem by deriving a generalization from some examples? | Induction
- Variation of the Problem | Can you vary or change your problem to create a new problem (or set of problems) whose solution(s) will help you solve your original problem? | Search
- Auxiliary Problem | Can you find a subproblem or side problem whose solution will help you solve your problem? | Subgoal
- Here is a problem related to yours & solved before | Can you find a problem related to yours that has already been solved & use that to solve your problem? | Pattern recognization, Pattern matching, Reduction
- Specialization | Can you find a problem more specialized? | Specialization
- Decomposing & Recombining | Can you decompose the problem & "recombine its elements in some new manner"? | Divide & conquer
- Working backward | Can you start with the goal & work backwards to something you already know? | Backward chaining
- Draw a Figure | Can you draw a picture of the problem? | Diagrammatic Reasoning

" – Wikipedia/how to solve it/heuristics

1.1.3 Influence

- "The book has been translated into several languages & has sold over a million copies, & has been continuously in print since its 1st publication.
- Marvin Minsky said in his paper *Steps Toward Artificial Intelligence* that "everyone should know the work of George Pólya on how to solve problems."
- Pólya's book has had a large influence on mathematics textbooks as evidenced by the bibliographies for mathematics education.
- Russian inventor Genrich Altshuller developed an elaborate set of methods for problem solving known as TRIZ, which in many aspects reproduces or parallels Pólya's work.
- How to Solve it by Computer is a computer science book by R. G. Dromey. It was inspired by Pólya's work." – Wikipedia/how to solve it/influence

Chapter 2

Andreescu and Dospinescu, 2010. Problems from the Book

"Math isn't the art of answering mathematical questions, it is the art of asking the right questions, the questions that give you insight, the ones that lead you in interesting directions, the ones that connect with lots of other interesting questions – the ones with beautiful answers." – G. Chaitin

Preface

"What can a new book of problems in elementary mathematics possibly contribute to the vast existing collection of journals, articles, & books? This was our main concern when we decided to write this book. The inevitability[1] of this question does not facilitate[2] the answer, because after 5 years of writing & rewriting we still had something to add. It could be a new problem, a comment we considered pertinent[3], or a solution that escaped our rationale[4] until this predictive[5] moment, when we were supposed to bring it under the scrutiny[6] of a specialist in the field.

A mere perusal[7] of this book should be efficient to identify its target audience: students & coaches preparing for mathematical Olympiads, national or international. It takes more effort to realize that these are not the only potential beneficiaries[8] of this work. While the book is rife[9] with problems collected from various mathematical competitions & journals, one cannot neglect the classical results of mathematics, which naturally exceed the level of time-constrained competitions. & no, $\boxed{\text{classical does not mean easy!}}$ These mathematical beauties are more than just proof that elementary mathematics can produce jewels. They serve as an invitation to mathematics beyond competitions, regarded by many to be the "true mathematics". In this context, the audience is more diverse than one might think.

Even so, as it will be easily discovered, many of the problems in this book are very difficult. Thus, the theoretical portions are short, while the emphasis is squarely placed on the problems. Certainly, more subtle results like quadratic reciprocity & existence of primitive roots are related to the basic results in linear algebra or mathematical analysis. Whenever their proofs are particularly useful, they are provided. We will assume of the reader a certain familiarity with classical theorems of elementary mathematics, which we will use freely. The selection of problems was made by weighing the need for routine exercises that engender[10] familiarity with the joy of the difficult problems in which we find the truly beautiful ideas. We stroke to select only those problems, easy & hard, that best illustrate the ideas we wanted to exhibit.

Allow us to discuss in brief the structure of the book. What will most likely surprise the reader when browsing just the table of contents is the absence of any chapters on geometry. This book was not intended to be an exhaustive treatment of elementary mathematics; if even such a book appears, it will be $\boxed{\text{a sad day for mathematics}}$. Rather, we tried to assemble[11]

[1] **inevitability** [n] [uncountable, countable] (plural **inevitabilities**) the fact that something cannot be avoided or prevented.

[2] **facilitate** [v] (*formal*) **facilitate something** to make an action or a process possible or easier.

[3] **pertinent** [a] appropriate to a particular solution, SYNONYM: **relevant**.

[4] **rationale** [n] the principles or reasons that explain a particular decision, course of action or belief.

[5] **predictive** [a] connected with the ability to show what will happen in the future.

[6] **scrutiny** [n] [uncountable] careful & thorough examination, SYNONYM: **inspection**.

[7] **perusal** [n] [uncountable, singular] (*formal or humorous*) the act of reading something; especially in a careful way.

[8] **beneficiary** [n] (plural **beneficiaries**) **beneficiary (of something)** a person who gains a result of something.

[9] **rife** [a] [not before noun] **1.** if something bad or unpleasant is **rife** in a place, it is very common there, SYNONYM: **widespread**; **2. rife (with something)** full of something bad or unpleasant.

[10] **engender** [v] **engender something** to make a feeling or situation exist.

[11] **assemble** [v] **1.** [intransitive, transitive] to come together as a group; to bring people or things together as a group; **2.** [transitive] to fit together all the separate parts of something; **3.** [transitive] **assemble something** (*computing*) to change instructions that a human can read in an assembly language program into a code that a computer can understand & act on.

problems that enchanted[12] us in order to give a sense of techniques & ideas that become leitmotifs[13] not just in problem solving but in all of mathematics.

Furthermore, there are excellent books on geometry, & it was not hard to realize that it would be beyond our ability to create something new to add to this area of study. Thus, we preferred to elaborate[14] more on 3 important fields of elementary mathematics: algebra, number theory, & combinatorics. Even after this narrowing of focus there are many topics that are simply left out, either in consideration of the available space or else because of the fine existing literature on the subject. This is, e.g., the fate[15] of functional equations, a field which can spawn[16] extremely difficult, intriguing[17] problems, but one which does not have obvious recurring[18] themes that tie everything together.

Hoping that you have not abandoned the book because of these omissions, which might be considered major by many who do not keep in mind the stated objectives, we continue by elaborating on the contents of the chapters. To start out, we ordered the chapters in ascending order of difficulty of the mathematical tools used. Thus, the exposition starts out lightly with some classical substitution techniques in algebra, emphasizing a large number of examples & applications. These are followed by a topic dear to us: the Cauchy–Schwarz inequality & its variations. A sizable chapter presents applications of the Lagrange interpolation formula, which is known by most only through rôte, straightforward applications. The interested reader will find some genuine[19] pearls in this chapter, which should be enough to change his or her opinion about this useful mathematical tool. 2 rather difficult chapters, in which mathematical analysis mixes with algebra, are given at the end of the book. 1 of them is quite original, showing how simple consideration of integral calculus can solve very difficult inequalities. The other discusses properties of equidistribution & dense numerical series. Too many books consider the Weyl equidistribution theorem to be “much too difficult” to include, & we cannot resist contradicting them by presenting an elementary proof. Furthermore, the reader will quickly realize that for elementary problems we have not shied away from presenting the so-called *non-elementary solutions* which use mathematical analysis or advanced algebra. It would be a crime to consider these 2 types of mathematics as 2 different entities, & it would be even worse to present laborious elementary solutions without admitting the possibility of generalization for problems that have conceptual & easy non-elementary solutions. In the end we devote a whole chapter to discussing criteria for polynomial irreducibility. We observe that some extremely efficient criteria (like those of Peron & Capelli) are virtually unknown, even though they are more efficient than the well-known *Eisenstein criterion*.

The section dedicated to number theory is the largest. Some introductory chapters related to prime numbers of the form $4k+3$ & to the other of an element are included to provide a better understanding of fundamental results which are used later in the book. A large chapter develops a tool which is as simple as it is useful: the exponent of a prime in the factorization of an integer. Some mathematical diamonds belonging to Paul Erdős & others appear within. & even though quadratic reciprocity is brought up in many books, we included an entire chapter on this topic because the problems available to us were too ingenious[20] to exclude. Next come some difficult chapters concerning arithmetic properties of polynomials, the geometry of numbers (in which we present some arithmetic application of the famous Minkowski's theorem), & the properties of algebraic numbers. A special chapter studies some applications of the extremely simple idea that a convergent series of integers is eventually stationary! The reader will have the chance to realize that $\boxed{\text{in mathematics even simple ideas have great impact}}$: consider, e.g., the fundamental idea that in the interval $(-1,1)$ the only integer is 0. But how many fantastic results concerning irrational numbers follow simply from that! Another chapter dear to us concerns the sum of digits, a subject that always yields unexpected & fascinating problems, but for which we could not find a unique approach.

Finally, some words about the combinatorics section. The reader will immediately observe that our presentation of this topic takes an algebraic slant[21], which was, in fact, our intention. In this way we tried to present some unexpected applications of complex numbers in combinatorics, & a whole chapter is dedicated to useful formal series. Another chapter

[12]**enchant** [v] **1. enchant somebody** (*formal*) to attract somebody strongly & make them feel very interested, excited, etc., SYNONYM: **delight**; **2. enchant somebody/something** to place somebody/something under a magic spell (= magic words that have special powers), SYNONYM: **bewitch**.

[13]**leitmotif** [n] (also **leitmotiv**) (*from German*) **1.** (music) a short tune in a piece of music that is often repeated & is connected with a particular person, thing or idea; **2.** an idea or a phrase that is repeated often in a book or work of art, or is typical of a particular person or group.

[14]**elaborate** [v] [usually before noun] very complicated & details; carefully prepared & organized; [v] **1.** [intransitive, transitive] to explain or describe something in a more detailed way; **2.** [transitive] **elaborate something** to develop a plan, an idea, etc. & make it complicated or detailed; **3.** [transitive] (*biology*) (of a natural process) to produce a substance or structure from its elements or simpler constituents.

[15]**fate** [n] **1.** [countable] **fate (of somebody/something)** the things, especially bad things, that will happen or have happened to somebody/something; **2.** [uncountable] the power that is believed to control everything that happens & that cannot be stopped or changed.

[16]**spawn** [v] **1.** [intransitive, transitive] (of fish & some other creatures) to lay eggs; **2.** [transitive] to cause something to develop or be produced in large numbers.

[17]**intriguing** [a] very interesting because of being unusual or not having an obvious answer.

[18]**recur** [v] [intransitive] to happen again or a number of times..

[19]**genuine** [a] **1.** real; exactly what it appears/they appear to be; **2.** honest; that can be trusted, SYNONYM: **sincere**.

[20]**ingenious** [a] **1.** (of an object, a plan, an idea, etc.) very suitable for a particular purpose & resulting from clever new ideas; **2.** (of a person) having a lot of clever new ideas & good at inventing things.

[21]**slant** [v] **1.** [intransitive, transitive] to slope or to make something slope in a particular direction or at a particular angle; **2.** [transitive] **slant something (+ adv./prep.)** (*sometimes disapproving*) to present information based on a particular way of thinking, especially in an unfair way; [n] **1.** a sloping position; **2. slant (on something/somebody)** a way of thinking about something, especially one that shows support for a particular opinion or point of view.

shows how useful linear algebra can be when solving problems on set combinatorics. Of course, we are traditional in presenting applications of Turan's theorem & of graph theory in general, & the pigeonhole principle could not be omitted. We faced difficulties here, because this topic is covered extensively in other books, though rarely in a satisfying way. For this reason, we tried to present lesser-known problems, because this topic is so dear to elementary mathematics lovers. At the end, we included a chapter on special applications of polynomials in number theory & combinatorics, emphasizing the Combinatorial Nullstellensatz, a recent & extremely useful theorem by Noga Alon.

We end our description with some remarks on the structure of the chapters. In general, the main theoretical results are stated, & if they are sufficiently profound or obscure, a proof is given. Following the theoretical part, we present between 10 & 15 examples, most from mathematical contests or from journals such as Kvant, Komal, & American Mathematical Monthly. Others are new problems or classical results. Each chapter ends with a series of problems, the majority of which stem from the theoretical results.

Finally, a change that will please some & scare others: the end-of-chapter problems do not have solutions! We had several reasons for this. The 1st & most practical consideration was minimizing the mass of the book. But the 2nd & more important factor was this: we consider solving problems to necessarily include the inevitably lengthy process of trial & research to which the inclusion of solutions provides perhaps too tempting of a shortcut. Keeping this in mind, the selection of the problems was made with the goal that the diligent[22] reader could solve about $\frac{1}{3}$ of them, make some progress in the $\frac{2}{3}$ & have at least the satisfaction of looking for a solution in the remainder." – Andreescu and Dospinescu, 2010, Preface, pp. vii–xi

[22]**diligent** [a] (*formal*) showing care & effort in your work or duties.

2.1 Some Useful Substitutions

2.1.1 Theory & examples

2.1.2 Practice problems

2.2 Always Cauchy–Schwarz ...

2.2.1 Theory & examples

2.2.2 Practice problems

2.3 Look at the Exponent

2.3.1 Theory & examples

2.3.2 Practice problems

2.4 Primes & Squares

2.4.1 Theory & examples

2.4.2 Practice problems

2.5 T2's Lemma

2.5.1 Theory & examples

2.5.2 Practice problems

2.6 Some Classical Problems in Extremal Graph Theory

2.6.1 Theory & examples

2.6.2 Practice problems

2.7 Complex Combinatorics

2.7.1 Theory & examples

2.7.2 Practice problems

2.8 Formal Series Revisited

2.8.1 Theory & examples

2.8.2 Practice problems

2.9 A Brief Introduction to Algebraic Number Theory

2.9.1 Theory & examples

2.9.2 Practice problems

2.10 Arithmetic Properties of Polynomials

2.10.1 Theory & examples

2.10.2 Practice problems

2.11 Lagrange Interpolation Formula

2.11.1 Theory & examples

2.11.2 Practice problems

Chapter 3

Polya, 2014. How to Solve It: A New Aspect of Mathematical Methods

From the Preface to the 1st Printing

"A great discovery solves a great problem but there is a grain[1] of discovery in the solution of any problem. Your problem may be modest[2]; but it challenges your curiosity[3] & brings into play your inventive[4] faculties[5], & if you solve it by your own means, you may experience the tension[6] & enjoy the triumph[7] of discovery. Such experiences at a susceptible[8] age may create a taste for mental work & leave their imprint[9] on mind & character for a lifetime[10].

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted[11] time with drilling his students in routine operations he kills their interest, hampers[12] their intellectual development, & misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate[13] to their knowledge, & helps them to solve their problems with stimulating[14] questions, he may give them a taste for, & some means of, independent thinking.

Also a student whose college curriculum[15] includes some mathematics has a singular[16] opportunity. This opportunity is

[1]**grain** [n] **1.** [uncountable, countable] the small hard seeds of food plants such as wheat, rice, etc.; a single seed of such a plant; **2.** [countable] **grain (of something)** a small piece of a particular substance; usually a hard substance; **3.** [countable, usually singular] **grain of something** a very small amount; **4.** [countable] an individual particle or crystal in metal, rock, etc., usually explained with a lens or microscope.

[2]**modest** [a] **1.** fairly limited or small in amount; **2.** not expensive, rich or impressive; **3.** (of people, especially women, or their clothes) not showing too much of the body; not intended to attract attention, especially in a sexual way; **4.** (*approving*) not talking much about your own abilities or possessions.

[3]**curiosity** [n] (plural **curiosities**) **1.** [uncountable, singular] a strong desire to know about something; **2.** [countable] **curiosity (of something)** an unusual & interesting thing.

[4]**inventive** [a] **1.** (especially of people) able to create or design new things or think of new ideas; **2.** (of ideas) new & interesting.

[5]**faculty** [n] (plural **faculties**) **1.** [countable] a physical or mental ability, especially one that people are born with; **2.** [countable] **faculty (of something)** a department or group of related departments in a college or university; **3.** [countable + singular or plural verb] all the teachers in a faculty of a college or university; **4.** [countable, uncountable] (*NAE*) all the teachers of a particular university or college.

[6]**tension** [n] **1.** [uncountable, countable, usually plural] a situation in which people do not trust each other, or feel unfriendly towards each other, & which may cause them to attack each other; **2.** [countable, uncountable] **tension (between A & B)** a situation in which the fact that there are different needs or interests causes difficulties; **3.** [uncountable] a feeling of anxiety & stress that makes it impossible to relax; **4.** [uncountable] the feeling of fear & excitement that is created by a writer or a film director; **5.** [uncountable] the state of being stretched tight; the extent to which something is stretched tight.

[7]**triumph** [n] **1.** [countable, uncountable] a great success, achievement or victory; **2.** [uncountable] the state of having achieved a great success or victory; the feeling of happiness that you get from this; [v] [intransitive] to defeat somebody/something; to be successful.

[8]**susceptible** [a] **1.** [not usually before noun] **susceptible (to somebody/something)** very likely to be influenced, harmed or affected by somebody/something; **2. susceptible (of something)** (*formal*) allowing something; capable of something.

[9]**imprint** [v] [often passive] **1.** to have a great effect on something so that it cannot be forgotten, changed, etc.; **2.** to print or press a mark or design onto a surface; [n] **1. imprint (of something) (in/on something)** a mark made by pressing something onto a surface; **2.** [usually singular] **imprint (of something) (on somebody/something)** (*formal*) the lasting effect that a person or an experience has on a place or a situation; **3.** (*specialist*) the name of the publisher of a book, usually printed below the title on the 1st page; a brand name under which books are published.

[10]**lifetime** [n] the length of time that somebody lives or that something lasts.

[11]**allot** [v] to give time, money, tasks, etc. to somebody/something as a share of what is available, SYNONYM: **allocate**.

[12]**hamper** [v] [often passive] to prevent something from being achieved easily or happening normally; to prevent somebody from easily doing something, SYNONYM: **hinder, impede**.

[13]**proportionate** [a] increasing or decreasing in size, amount or degree according to changes in something else, SYNONYM: **proportional**.

[14]**stimulating** [a] **1.** full of interesting or exciting ideas; making people feel enthusiastic; **2.** making you feel more active & healthy.

[15]**curriculum** [n] (plural **curricula**) the subjects that are included in a course of study or taught in a school, college or university.

[16]**singular** [n] [singular] (*grammar*) a form of a noun or verb that refers to 1 person or thing; [a] **1.** (*grammar*) connected with or having the form of a noun or verb that refers to 1 person or thing; **2.** especially great or obvious, SYNONYM: **outstanding**; **3.** (*mathematics, physics*) connected with a singularity.

lost, of course, if he regards[17] mathematics as a subject in which he has to earn so & so much credit & which he should forget after the final examination as quickly as possible. The opportunity may be lost even if the student has some natural talent for mathematics because he, as everybody else, must discover his talents & tastes; he cannot know that he likes raspberry pie if he has never tasted raspberry pie. He may manage to find out, however, that a mathematics problem may be as much fun as a crossword puzzle[18], or that vigorous[19] mental work may be an exercise as desirable as a fast game of tennis. Having tasted the pleasure in mathematics he will not forget it easily & then there is a good chance that mathematics will become something for him: a hobby, or a tool of his profession, or a great ambition,

The author remembers the time when he was a student himself, a somewhat ambitious student, eager to understand a little mathematics & physics. He listened to lectures, read books, tried to take in the solutions & facts presented, but there was a question that disturbed[20] him again & again: "Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? & how could I invent or discover such things by myself?" Today the author is teaching mathematics in a university; he thinks or hopes that some of his more eager students ask similar questions & he tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem but also the motives & procedures of the solution, & trying to explain these motives & procedures to others, he was finally led to write the present book. He hopes that it will be useful to teachers who wish to develop their students' ability to solve problems, & to students who are keen on developing their own abilities.

Although the present book pays special attention to the requirements of students & teachers of mathematics, it should interest anybody concerned with the ways & means of invention & discovery. Such interest may be more widespread[21] than one would assume without reflection[22]. The space devoted by popular newspapers & magazines to crossword puzzles & other riddles[23] seems to show that people spend some time in solving unpractical[24] problems. Behind the desire to solve this or that problem that confers[25] no material advantage, there may be a deeper curiosity, a desire to understand the ways & means, the motives & procedures, of solution.

The following pages are written somewhat concisely[26], but as simply as possible, & are based on a long & serious study of methods of solution. This sort of study, called *heuristic*[27][28] by some writers, is not in fashion nowadays but has a long past &, perhaps, some future.

Studying the methods of solving problems, we perceive[29] another face of mathematics. Yes, mathematics has 2 faces; it is the rigorous[30] science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a

[17]**regard** [v] [often passive] to think about somebody/something in a particular way; **as regards somebody/something** [idiom] concerning or in connection with somebody/something; [n] **1.** [uncountable] attention to or thought & care for somebody/something; **2.** [uncountable] **regard (for somebody/something)** respect or admiration for somebody/something. If you **hold somebody in high regard**, you have a good opinion of them.; **3.** (**regards**) [plural] used to send good wishes to somebody at the end of a letter or email; **have regard to something** [idiom] (*law*) to remember & think carefully about something; **in/with regard to somebody/something** [idiom] concerning somebody/something; **in this/that regard** [idiom] concerning what has just been mentioned.

[18]**crossword** [n] (also **crossword puzzle**) a game in which you have to fit words across & downwards into spaces with numbers in a square diagram. You find the words by solving clues.

[19]**vigorous** [a] **1.** involving physical strength, effort or energy; **2.** done with determination, energy or enthusiasm; **3.** strong & healthy.

[20]**disturb** [v] **1. disturb something** to change the arrangement of something, or affect how something functions; **2. disturb somebody/something** to interrupt somebody & prevent them from continuing with what they are doing; **3. disturb somebody** to make somebody feel anxious or upset.

[21]**widespread** [a] existing or happening over a large area or among many people, SYNONYM: **extensive**.

[22]**reflection** [n] **1.** [countable] **reflection of something** an account or description of what somebody/something is like; a thing that is a result of something else; **2.** [uncountable] careful thought about something, especially your work or studies; **3.** [countable, usually plural] **reflections (on something)** written or spoken thoughts about a particular subject; **4.** [uncountable] **reflection (of something)** the action or process of sending back light, heat, sound, etc. from a surface; **5.** (also **reflexion**) [countable, uncountable] **reflection (of something)** (*mathematics*) an operation on a shape to produce its mirror image.

[23]**riddle** [n] **1.** a question that is difficult to understand, & that has a surprising answer, that you ask somebody as a game; **2.** a mysterious event or situation that you cannot explain, SYNONYM: **mystery**; [v] **riddle somebody/something (with something)** to make a lot of holes in; **be riddle with something** [idiom] to be full of something, especially something bad or unpleasant.

[24]**unpractical** [a] **1.** not sensible or realistic; **2.** (9of people) not good at doing things that involve using the hands; not good at planning or organizing things, OPPOSITE: **practical**.

[25]**confer** [v] **1.** [transitive] to give somebody a particular power, right or honor; **2.** [transitive] to give somebody/something a particular advantage; **3.** [intransitive] **confer (with somebody) (on/about something)** to discuss something with somebody, in order to exchange opinions or get advice.

[26]**concise** [a] giving only the information that is necessary & important, using few words.

[27]**heuristic** [a] (*formal*) **heuristic** teaching or education encourages you to learn by discovering things for yourself.

[28]**heuristics** [n] [uncountable] (*formal*) a method of solving problems by finding practical ways of dealing with them, learning from past experience.

[29]**perceive** [v] **1.** to notice or become aware of something, SYNONYM: **notice**; **2.** to be aware of or experience something using the senses; **3.** [often passive] to understand or think of somebody/something in a particular way; to believe that a particular thing is true, SYNONYM: **see**.

[30]**rigorous** [a] **1.** done carefully & with a lot of attention to detail, SYNONYM: **thorough**; **2.** demanding that particular rules or processes are strictly followed, SYNONYM: **strict**.

systematic[31], deductive[32] science; but mathematics in the making appears as an experimental[33], inductive[34] science. Both aspects[35] are as old as the science of mathematics itself. But the 2nd aspect is new in 1 respect[36]; mathematics "in statu nascendi," in the process of being invented, has never before presented in quite this manner to the student, or to the teacher himself, or to the general public.

The subject of heuristic has manifold[37] connections; mathematicians, logicians[38], psychologists, educationalists[39], even philosophers may claim various parts of it as belonging to their special domains. The author, well aware of the possibility of criticism[40] from opposite[41] quarters[42] & keenly[43] conscious[44] of his limitations[45], has 1 claim to make: he has some experience in solving problems & in teaching mathematics on various levels.

The subject is more fully dealt with in a more extensive book by the author which is on the way to completion. *Stanford University, Aug 1, 1944*" – Polya, 2014, pp. v–vii

From the Preface to the 7th Printing

"[...] The 2 volumes *Induction & Analogy in Mathematics* & *Patterns of Plausible Inference* which constitute my recent work *Mathematics & Plausible Reasoning* continue the line of thinking begun in *How to Solve It. Zurich, Aug 30, 1954*" – Polya, 2014, p. viii

Preface to the 2nd Edition

"The present 2nd edition adds, besides a few minor improvements, a new 4th part, "Problems, Hints, Solutions."

As this edition was being prepared for print, a study appeared (Educational Testing Service, Princeton, N.J.; *cf. Time*, Jun 18, 1956) which seems to have formulated[46] a few pertinent[47] observations – they are not new to the people in the know, but it was high time to formulate them for the general public–: "... mathematics has the dubious[48], honor of being the least

[31]**systematic** [a] **1.** done according to a system or plan, in a thorough, efficient or determined way; **2.** (of an error) happening in the same way all through a process or set of results; caused by the system that is used.

[32]**deductive** [a] [usually before noun] using knowledge about things that are generally true in order to understand particular situations or problems.

[33]**experimental** [a] **1.** [usually before noun] connected with scientific experiments; **2.** based on new ideas, forms or methods that are used to find out what effect they have.

[34]**inductive** [a] (*specialist*) using particular facts & examples to form general rules & principles.

[35]**aspect** [n] **1.** [countable] a particular feature of a situation, an idea or a process; a way in which something may be considered; **2.** [countable, usually singular] **aspect (of something)** (*specialist*) a particular surface or side of an object or a part of the body; the direction in which something faces; **3.** [uncountable, countable] (*grammar*) the form of a verb that shows, e.g., whether the action happens once or many times, is completed or is still continuing.

[36]**respect** [n] **1.** [countable] a particular aspect or detail of something; **2.** [uncountable, singular] polite behavior towards or reasonable treatment of somebody/something; **3.** [uncountable, singular] a feeling of admiration for somebody/something because of their good qualities or achievements; **in respect of something** [idiom] (*formal*) **1.** concerning; **2.** in payment for something; **with respect to something** [idiom] concerning.

[37]**manifold** [a] (*formal*) many; of many different types; [n] (*specialist*) a pipe or chamber with several openings, especially 1 for taking gases in & out of a car engine.

[38]**logician** [n] a person who studies or is skilled in logic.

[39]**educationalists** [n] (also **educationist**) a specialist in theories & methods of teaching.

[40]**criticism** [n] **1.** [uncountable, countable] the act of expressing disapproval of somebody/something & opinions about their faults or bad qualities; a statement showing disapproval; **2.** [uncountable] the work or activity of analyzing & making fair, careful judgments about somebody/something, especially books, music, etc.

[41]**opposite** [a] **1.** [usually before noun] as different as possible from something; involving 2 different extremes; **2.** [usually before noun] on the other side of something or facing something; [n] **1.** (**the opposite**) [singular] the situation, idea or activity that is as different from another situation, etc. as it is possible to be, SYNONYM: **the reverse**; **2.** (**opposites**) [plural] people, ideas or situations that are as different as possible from each other; **the exact opposite** [idiom] a person or thing that is as different as possible from somebody/something else; [prep] on the other side of a particular area from somebody/something, & usually facing them.

[42]**quarter** [n] **1.** (also **fourth** *especially in NAE*) [countable] 1 of 4 equal parts of something; **2.** [countable] a period of 3 months, used especially as a period fo which bills are paid or a company's income is calculated; **3.** [countable] a person or group of people, especially as a source of help, information or a reaction; **4.** [countable, usually singular] a district or part of a town; **5.** (**quarters**) [plural] rooms that are provided for soldiers, servants, etc. to live in; **at/from close quarters** [idiom] very near.

[43]**keenly** [adv] **1.** very strongly or deeply; **2.** by people with different opinions that they express strongly.

[44]**conscious** [a] **1.** [not before noun] aware of something; noticing something, OPPOSITE: **unconscious**; **2.** able to use your senses & mental powers to understand what is happening, OPPOSITE: **unconscious**; **3.** (of actions, feelings, etc.) deliberate or controlled, OPPOSITE: **unconscious**; **4.** being particularly interested in something.

[45]**limitation** [n] **1.** [countable, usually plural] a limit on what somebody/something can do or how good they/it can be; **2.** [countable] a rule, fact or condition that limits something, SYNONYM: **restraint**; **3.** [uncountable] **limitation (of something)** the act or process of limiting or controlling somebody/something, SYNONYM: **restriction**; **4.** (also **limitation period**) [countable] (*law*) a legal limit on the period of time within which court proceedings can be taken or for which a property right continues.

[46]**formulate** [v] **1. formulate something** to create or prepare something carefully, giving particular attention to the details; **2. formulate something** to express your ideas in carefully chosen words.

[47]**pertinent** [a] appropriate to a particular situation, SYNONYM: **relevant**.

[48]**dubious** [a] **1.** that you cannot be sure about; that is probably not good. **Dubious** is also when you are stating that something is the opposite of a particular good quality. **2.** [usually before noun] probably not honest, SYNONYM: **suspicious**; **3.** [not usually before noun] **dubious about**

popular subject in the curriculum ... Future teachers pass through the elementary schools learning to detest[49] mathematics ... They return to the elementary school to teach a new generation to detest it."

I hope that the present edition, designed for wider diffusion[50], will convince some of its readers that mathematics, besides being a necessary avenue[51] to engineering jobs & scientific knowledge, may be fun & may also open up a vista[52] of mental activity on the highest level. *Zurich, Jun 30, 1956*" – Polya, 2014, pp. ix–

"How to Solve It" list

1st. You have to *understand* the problem.

Understanding the Problem.

What is the unknown? What are the data? What is the condition?

It is possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

2nd. Find the connection between the data & the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Devising a Plan.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! & try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours & solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve 1st some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown & the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3rd. *Carry out* your plan.

Carrying out the Plan.

Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4th. *Examine* the solution obtained.

Looking back.

Can you *check the result?* Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

– Polya, 2014, How to solve it, pp. xvi–xvii

something feeling uncertain about something; not knowing whether something is good or bad, synonym: **doubtful**.

[49]**detest** [v] (not used in the progressive tenses) to hate somebody/something very much, synonym: **loathe**.

[50]**diffusion** [n] [uncountable] **1.** the spreading of something more widely; **2.** the mixing of substances by the natural movement of their particles; **3.** the spreading of elements of culture from 1 region or group to another.

[51]**avenue** [n] a way of approaching a problem or making progress towards something.

[52]**vista** [n] **1.** (*literary*) a beautiful view, e.g., of the countryside, a city, etc., synonym: **panorama**; **2.** (*formal*) a range of things that might happen in the future, synonym: **prospect**.

Foreword by John H. Conway

"*How to Solve It* is a wonderful book! This I realized when I 1st read right through it as a student many years ago, but it has taken me a long time to appreciate just *how* wonderful it is. Why is that? 1 part of the answer is that the book is unique. In all my years as a student & teacher, I have never seen another that lives up to George Polya's title by teaching you how to go about solving problems. A. H. Schoenfeld correctly described its importance in his 1987 article "Polya, Problem Solving, & Education" in *Mathematics Magazine*: "For mathematics education & the world of problem solving it marked a line of demarcation[53] between 2 eras[54], problem solving before & after Polya."

It is 1 of the most successful mathematics books ever written, having sold over a million copies & been translated into 17 languages since it 1st appeared in 1945. Polya later wrote 2 more books about the art of doing mathematics, *Mathematics & Plausible Reasoning* (1954) & *Mathematical Discovery* (2 volumes, 1962 & 1965).

The book's title makes it seem that it is directed only toward students, but in fact it is addressed just as much to their teachers. Indeed, as Polya remarks in his introduction, the 1st part of the book takes the teacher's viewpoint more often than the student's.

Everybody gains that way. The student who reads the book on his own will find that overhearing[55] Polya's comments to his non-existent[56] teacher can bring that desirable person into being, as an imaginary but very helpful figure leaning over one's shoulder. This is what happened to me, & naturally I made heavy use of the remarks I'd found most important when I myself started teaching a few years later.

But it was some time before I read the book again, & when I did, I suddenly realized that it was even more valuable than I'd thought! Many of Polya's remarks that hadn't helped me as a student now made me a better teacher of those whose problems had differed from mine. Polya had met many more students than I had, & had obviously thought very hard about how to best help all of them learn mathematics. Perhaps his most important point is that $\boxed{\text{learning must be active}}$. As he said in a lecture on teaching, "Mathematics, you see, is not a spectator[57] sport. To understand mathematics means to be able to do mathematics. & what does it mean [to be] doing mathematics? In the 1st place, it means to be able to solve mathematical problems."

It is often said that to teach any subject well, one has to understand it "at least as well as one's students do." It is a paradoxical[58] truth that to teach mathematics well, one must also know how to misunderstand it at least to the extent one's students do! If a teacher's statement can be parsed[59] in 2 or more ways, it goes without saying that some students will understand it 1 way & others another, with results that can vary from the hilarious[60] to the tragic[61]. J. E. Littlewood gives 2 amusing[62] examples of assumptions that can easily be made unconsciously & misleadingly[63]. 1st, he remarks that the description of the coordinate axes ("Ox & Oy as in 2 dimensions, Oz vertical") in Lamb's book *Mechanics* is incorrect for him, sine he always worked in an armchair[64] with his feet up! Then, after asking how his reader would present the picture of a closed curve lying all on 1 side of its tangent, he states that there are 4 main schools (to left or right of vertical tangent, or above or below horizontal one) & that by lecturing without a figure, presuming that the curve was to the right of its vertical tangent, he had unwittingly[65] made nonsense[66] for the other 3 schools.

I know of no better remedy[67] for such presumptions[68] than Polya's counsel[69]: before trying to solve a problem, the

[53] **demarcation** [n] [uncountable, countable] a line or limit that separates 2 things, such as types of work, groups of people or areas of land.

[54] **era** [n] **1.** a period of time, usually in history, that is different from other perods because of particular characteristics or events; **2.** (*earth sciences*) a major division of time that can itself be divided into periods.

[55] **overhear** [v] to hear, especially by accident, a conversation in which you are not involved.

[56] **non-existent** [a] not existing; not real.

[57] **spectator** [n] a person who is watching a performance or an event.

[58] **paradoxical** [a] **1.** (of a person, thing or situation) having 2 opposite features & therefore seem strange; **2.** (of a statement) containing 2 opposite ideas that make it seem impossible or unlikely, although it is probably true.

[59] **parse** [v] (*grammar*) **parse something** to divide a sentence into parts & describe the grammar of each word or part.

[60] **hilarious** [a] extremely funny.

[61] **tragic** [a] **1.** making you feel very sad, usually because somebody has died or suffered a lot; **2.** [usually before noun] connected with tragedy (= the style of literature).

[62] **amusing** [a] funny & giving pleasure.

[63] **misleading** [a] giving the wrong idea & making people believe something that is not true, SYNONYM: **deceptive**.

[64] **armchair** [n] a comfortable chair with sides on which you can rest your arms; [a] [only before noun] knowing about a subject through books, television, the Internet, etc., rather than by doing it for yourself.

[65] **unwittingly** [adv] without being aware of what you are doing or the situation that you are involved in, OPPOSITE: **wittingly**.

[66] **nonsense** [n] **1.** [uncountable, countable] ideas, statements or beliefs that you think are silly or not true, SYNONYM: **rubbish**; **2.** [uncountable] spoken or written words that have no meaning or make no sense; **3.** [uncountable] silly or unacceptable behavior; **make (a) nonsense of something** [idiom] to reduce the value of something by a lot; to make something seem silly.

[67] **remedy** [n] (plural **remedies**) **1.** a way of dealing with or improving an unpleasant or difficult situation, SYNONYM: **solution**; **2.** a treatment or medicine to cure a disease or to reduce pain that is not very serious; **3.** (*law*) a way of dealing with a problem, using the processes of the law, SYNONYM: **redress**; [v] **remedy something** to correct or improve something.

[68] **presumption** [n] [countable, uncountable] the act of supposing or accepting that something is true or exists, although it has not been proved; a belief that something is true or exists, SYNONYM: **assumption**. In legal contexts, **presumption** often means that something is being accepted as true until it is shown not to be true.

[69] **counsel** [n] [uncountable, countable] **1.** (*formal*) advice, especially given by older people or experts; a piece of advice; **2.** a lawyer or group of lawyers representing somebody in court; [v] (*formal*) **1. counsel somebody** to listen to & give support or professional advice to somebody who

student should demonstrate his or her understanding of its statement, preferably[70] to a real teacher, but in lieu[71] of that, to an imagined one. Experienced mathematicians know that often the hardest part of researching a problem is understanding precisely what that problem says. They often follow Polya's wise advice: "If you can't solve a problem, then there is an easier problem you can't solve: find it."

Readers who learn from this book will also want to learn about its author's life.[72]

George Polya was born György Pólya (he dropped the accents sometime later) on Dec 13, 1887, in Budapest, Hungary, to Jakab Pólya & his wife, the former Anna Deutsch. He was baptized into the Roman Catholic faith, to which Jakab, Anna, & their 3 previous children, Jenő, Ilona, & Flóra, had converted from Judaism[73] in the previous year. The 5th child, László, was born 4 years later.

Jakab had changed his surname from Pollák to the more Hungarian-sounding Pólya 5 years before György was born, believing that this might help him obtain a university post, which he eventually did, but only shortly before his untimely[74] death in 1897.

At the Dániel Berzsenyi Gymnasium[75], György studied Greek, Latin, & German, in addition to Hungarian. It is surprising to learn that there he was seemingly uninterested in mathematics, his work in geometry deemed merely "satisfactory" compared with his "outstanding" performance in literature, geography, & other subjects. His favorite subject, outside of literature, was biology.

He enrolled at the University of Budapest in 1905, initially studying law, which he soon dropped because he found it too boring. He then obtained the certification needed to teach Latin & Hungarian at a gymnasium, a certification that he never used but of which he remained proud. Eventually his professor, Bernát Alexander, advised him that to help his studies in philosophy, he should take some mathematics & physics courses. This was how he came to mathematics. Later, he joked that he "wasn't good enough for physics, & was too good for philosophy – mathematics is in between."

In Budapest he was taught physics by Eötvös & mathematics by Fejér & was awarded a doctorate after spending the academic year 1910–11 in Vienna, where he took some courses by Wirtinger & Mertens. He spent much of the next 2 years in Göttingen, where he met many more mathematicians – Klein, Caratheodory, Hilbert, Runge, Landau, Weyl, Hecke, Courant, & Toeplitz – & in 1914 visited Paris, where he became acquainted[76] with Picard & Hadamard & learned that Hurwitz had arranged an appointment for him in Zürich. He accepted this position, writing later: "I went to Zürich in order to be near Hurwitz, & we were in close touch for about 6 years, from my arrival in Zürich in 1914 to his passing [in 1919]. I was very much impressed by him & edited his works."

Of course, the 1st World War took place during this period. It initially had little effect on Polya, who had been declared unfit for service in the Hungarian army as the result of a soccer wound. But later when the army, more desperately[77] needing recruits[78], demanded that he return to fight for his country, his strong pacifist[79] views led him to refuse. As a consequence, he was unable to visit Hungary for many years, & in fact did not do so until 1967, 54 years after he left.

In the meantime, he had taken Swiss citizenship & married a Swiss girl, Stella Vera Weber, in 1918. Between 1918 & 1919, he published papers on a wide range of mathematical subjects, such as series, number theory, combinatorics, voting systems, astronomy, & probability. He was made an extraordinary professor at the Zürich ETH in 1920, & a few years later he & Gábor Szegő published their book *Aufgaben und Lehrsatze aus der Analysis* ("Problems & Theorems in Analysis"), described by G. L. Alexanderson & L. H. Lange in their obituary[80] of Polya as "a mathematical masterpiece[81] that assured[82]

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needs help; **2.** to advise a particular course of action; to advise somebody to do something.

[70]**preferable** [a] more attractive or more suitable; to be preferred to something.

[71]**lieu** [n] (*formal*) **in lieu (of something)** [idiom] instead of.

[72]The following biographical information is taken from that given by J. J. O'Connor & E. F. Robertson in the MacTutor History of Mathematics Archive (www-gap.dcs.st-and.ac.uk/~hisotry/).

[73]**Judaism** [n] [uncountable] the religion of the Jewish people, based mainly on the Bible & the Talmud (= a collection of ancient writings on Jewish law & traditions).

[74]**untimely** [a] (*formally*) **1.** [usually before noun] happening too soon or sooner than is normal or expected, SYNONYM: **premature**; **2.** happening at a time or in a situation that is not suitable, SYNONYM: **ill-timed**, OPPOSITE: **timely**.

[75]**gymnasium** (plural **gymnasiums, gymnasia**) (*formal*) a gym.

[76]**acquainted** [a] [not before noun] **1. acquainted with something** (*formal*) familiar with something, having read, seen or experienced it; **2.** not close friends with somebody, but having met a few times before.

[77]**desperate** [a] **1.** feeling or showing that you have little hope & are ready to do anything without worrying about danger to yourself or others; **2.** [usually before noun] (of an action) giving little hope of success; tried when everything else was failed; **3.** (of a situation) extremely serious or dangerous.

[78]**recruit** [v] **1.** [transitive, intransitive] to find new people to join a company, an organization, the armed forces, etc.; **2.** [transitive] to get people to help with or be involved in something; **3.** [transitive] **recruit something (from something)** to form a new army, team, etc. by persuading new people to join it; [n] **1.** a person who has recently joined the armed forces or the police; **2.** a person who joins an organization, a company, etc.

[79]**pacifist** [a] [usually before noun] holding or showing the belief that war & violence are always wrong; [n] a person who believes that war & violence are always wrong & refuses to fight in a war.

[80]**obituary** [n] (plural **obituaries**) an article about somebody's life & achievements, that is printed in a newspaper soon after they have died.

[81]**masterpiece** [n] (also **masterwork**) **1. masterpiece (of something)** a work of art such as a painting, film, book, etc. that is an excellent, or the best, example of the artist's work; **2. masterpiece of something** an extremely good example of something.

[82]**assure** [v] **1.** to tell somebody that something is definitely true or is definitely going to happen, especially when they have doubts about it; **2.** to make something certain to happen; to make somebody/something certain to get something; **3.** to make yourself certain about something.

their reputations[83]."

That book appeared in 1925, after Polya had obtained a Rockefeller Fellowship to work in England, where he collaborated with Hardy & Littlewood on what later become their book *Inequalities* (Cambridge University Press, 1936). He used a 2nd Rockefeller Fellowship to visit Princeton University in 1933, & while in the United States was invited by H. F. Blichfeldt to visit Stanford University, which he greatly enjoyed, & which ultimately became his home. Polya held a professorship at Stanford from 1943 until his retirement in 1953, & it was there, in 1978, that he taught his last course, in combinatorics; he died on Sep 7, 1985, at the age of 97.

Some readers will want to know about Polya's many contributions to mathematics. Most of them relate to analysis & are too technical to be understood by non-experts, but a few are worth mentioning.

In probability theory, Polya is responsible for the now-standard term "Central Limit Theorem" & for proving that the Fourier transform of a probability measure is a characteristic function & that a random walk on the integer lattice closes with probability 1 iff the dimension is at most 2.

In geometry, Polya independently re-enumerated the 17 plane crystallographic[84] groups (their 1st enumeration[85], by E. S. Fedorov, having been forgotten) & together with P. Niggli devised[86] a notation for them.

In combinatorics, Polya's Enumeration Theorem is now a standard way of counting configurations according to their symmetry. It has been described by R. C. Read as "a remarkable[87] theorem in a remarkable paper, & a landmark[88] in the history of combinatorial analysis."

How to Solve It was written in German during Polya's time in Zürich, which ended up in 1940, when the European situation forced him to leave for the United States. Despite the book's eventual success, 4 publishers rejected the English version before Princeton University Press brought it out in 1945. In their hands, *How to Solve It* rapidly became – & continues to be – 1 of the most successful mathematical books of all time." – Polya, 2014, Foreword, pp. xix–xxiv

Introduction

"The following consideration are grouped around the preceding list of questions & suggestions entitled[89] "How to Solve It." Any question or suggestion quoted from it will be printed in *italics*, & the whole list will be referred to simply as "the list" or as "our list."

The following pages will discuss the purpose of the list, illustrate its practical use by examples, & explain the underlying notions & mental operations. By way of preliminary explanation, this much may be said: If, using them properly, you address these questions & suggestions to yourself, they may help you to solve your problem. If, using them properly, you address the same questions & suggestions to 1 of your students, you may help him to solve his problem.

The book is divided into 4 parts.

The title of the 1st part is "In the Classroom." It contains 20 sections. Each section will be quoted by its number in heavy type as, e.g., "sect. **7.**" Sects. **1–5** discuss the "Purpose" of our list in general terms. Sects. **6–17** explain what are the "Main Divisions, Main Questions" of the list, & discuss a 1st practical example. Sects. **18–20** add "More Examples."

The title of the very short 2nd part is "How to Solve It." It is written in dialogue; a somewhat idealized teacher answers short questions of a somewhat idealized student.

The 3rd & most extensive part is a "Short Dictionary of Heuristic"; we shall refer to it as the "Dictionary." It contains 67 articles arranged alphabetically. E.g., the meaning of the term HEURISTIC (set in small capitals) is explained in an article with this title on p. 112. When the title of such an article is referred to within the text it will be set in small capitals. Certain paragraphs of a few articles are more technical; they are enclosed[90] in square brackets. Some articles are fairly closely connected with the 1st part to which they add further illustrations & more specific comments. Other articles go somewhat beyond the aim of the 1st part of which they explain the background. There is a key-article on MODERN HEURISTIC. It explains the connection of the main articles & the plan underlying the Dictionary; it contains also directions how to find information about particular items of the list. It must be emphasized that there is a common plan & a certain unity, because the articles of the Dictionary show the greatest outward variety. There are a few longer articles devoted to the systematic though condensed discussion of some general theme; others contain more specific comments; still others cross-references[91],

[83]**reputation** [n] the opinion that people have about what somebody/something is like, based on what has happened in the past.

[84]**crystallography** [n] [uncountable] the branch of science that deals with crystals.

[85]**enumeration** [n] [uncountable, countable] (*formal*) the act of naming things 1 by 1 in a list; a list of this sort.

[86]**devise** [v] **devise something** to plan or invent a procedure, system or method, especially one that is new or complicated, by using careful thought, SYNONYM: **think something up**.

[87]**remarkable** [a] unusual or surprising in a way that causes people to take notice.

[88]**landmark** [n] **1.** something, such as a large building, that you can see clearly from a distance & that will help you to know where you are; **2.** an event, a discovery or an invention that marks an important stage in something.

[89]**entitle** [v] **1.** [often passive] to give somebody the right to have or to do something; **2.** [usually passive] to give a title to a book, document, film, etc.

[90]**enclose** [v] **1.** [usually passive] to build a wall, fence, etc. around something; **2. enclose something** (especially of a wall, fence, etc.) to surround something; **3. enclose something (with something)** to put something in the same envelope or package as something else.

[91]**cross-reference** [v] **cross-reference something** to give cross references to another text or part of a text.

or historical data, or quotations, or aphorisms[92], or even jokes.

The Dictionary should not be read too quickly; its text is often condensed, & now & then somewhat subtle. The reader may refer to the Dictionary for information about particular points. If these points come from his experience with his own problems or his own students, the reading has a much better chance to be profitable[93].

The title of the 4th part is "Problems, Hints, Solutions." It proposes a few problems to the more ambitious reader. Each problem is followed (in proper distance) by a "hint" that may reveal a way to the result which is explained in the "solution."

We have mentioned repeatedly the "student" & the "teacher" & we shall refer to them again & again. It may be good to observe that the "student" may be a high school student, or a college student, or anyone else who is studying mathematics. Also the "teacher" may be a high school teacher, or a college instructor, or anyone interested in the technique of teaching mathematics. The author looks at the situation sometimes from the point of view of the student & sometimes from that of the teacher (the latter case is preponderant[94] in the 1st part). Yet most of the time (especially in the 3rd part) the point of view is that of a person who is neither teacher nor student but anxious to solve the problem before him." – Polya, 2014, Introduction, pp. xxv–xxvii

Part I. In The Classroom

Purpose

3.1 Helping the student

"1 of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion[95], & sound principles.

The student should acquire as much experience of independent work as possible. But if he is left alone with his problem without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much & not too little, so that the student shall have a *reasonable share of the work*.

If the student is not able to do much, the teacher should leave him at least some illusion[96] of independent work. In order to do so, the teacher should help the student discreetly[97], *unobtrusively*[98].

The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, & ask a question or indicate[99] a step that *could have occurred to the student himself*." – Polya, 2014, p. 1

3.2 Questions, recommendations, mental operations

"Trying to help the student effectively but unobtrusively & naturally, the teacher is led to ask the same questions & to indicate the same steps again & again. Thus, in countless problems, we have to ask the question: *What is the unknown?* We may vary the words, & ask the same thing in many different ways: What is required? What do you want to find? What are you supposed to seek? The aim of these questions is to focus the student's attention upon the unknown. Sometimes, we obtain the same effect more naturally with a suggestion: *Look at the unknown!* Question & suggestion aim at the same effect; they tend to provoke[100] the same mental operation.

[92]**aphorism** [n] (*formal*) a short phrase that says something true or wise.

[93]**profitable** [a] **1.** that makes or is likely to make money; **2.** that gives somebody an advantage or a useful result.

[94]**preponderant** [a] [usually before noun] (*formal*) larger in number or more important than other people or things in a group.

[95]**devotion** [n] [uncountable, singular] **1.** devotion (of somebody) (to somebody/something) great love, care & support for somebody/something; **2. devotion (to somebody/something)** the action of spending a lot of time or energy on something, SYNONYM: **dedication**; **3.** great religious feeling.

[96]**illusion** [n] **1.** [countable, uncountable] a false idea or belief; **2.** [countable] something that seems to exist but in fact does not, or seems to be something that it is not.

[97]**discreetly** [adv] in a careful way, in order to keep something secret or to avoid causing difficulty for somebody or making them feel embarrassed, SYNONYM: **tactfully**.

[98]**unobtrusively** [adv] (*formal, often approving*) in a way that does not attract unnecessary attention, OPPOSITE: **obtrusively**.

[99]**indicate** [v] **1.** to show that something is true or exists; **2.** to be a sign of something; to show that something is possible or likely, SYNONYM: **suggest**; **3. indicate something** to represent information without using words; **4.** to give information in writing; **5.** [usually passive] to suggest something as a necessary or recommend course of action; **6.** to mention something, especially in an indirect or brief way; **7. indicate something** (of an instrument for measuring things) to show a particular measurement.

[100]**provoke** [v] **1. provoke something** to cause a particular reaction or have a particular effect; **2.** to say or do something in order to produce a strong reaction from somebody, usually anger.

It seemed to the author that it might be worth while[101] to collect & to group questions & suggestions which are typically[102] helpful in discussing problems with students. The list we study contains questions & suggestions of this sort, carefully chosen & arranged; they are equally useful to the problem-solver who works by himself. If the reader is sufficiently acquainted with the list & can see, behind the suggestion, the action suggested, he may realize that the list enumerates, indirectly, *mental operations typically useful for the solution of problems.* These operations are listed in the order in which they are most likely to occur." – Polya, 2014, pp. 1–2

3.3 Generality

"Generality[103] is an important characteristic of the questions & suggestions contained in our list. Take the questions: *What is the unknown? What are the data? What is the condition?* These questions are generally applicable[104], we can ask them with good effect dealing with all sorts of problems. Their use is not restricted to any subject-matter[105]. Our problem may be algebraic or geometric, mathematical or nonmathematical, theoretical or practical, a serious problem or a mere puzzle; it makes no difference, the questions make sense & might help us to solve the problem.

There is a **restriction**[106], in fact, but it has nothing to do with the subject-matter. Certain questions & suggestions of the list are applicable to "problems to find" only, not to "problems to prove." If we have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS TO PROVE." – Polya, 2014, pp. 2–3

3.4 Common sense

"The questions & suggestions of our list are general, but, except for their generality, they are natural, simple, obvious, & proceed from plain common sense. Take the suggestion: *Look at the unknown! & try to think of a familiar problem having the same or a similar unknown.* This suggestion advises you to do what you would do anyhow[107], without any advice, if you were seriously concerned with your problem. Are you hungry? You wish to obtain food & you think of familiar ways of obtaining food. Have you a problem of geometric construction? You wish to construct a triangle & you think of familiar ways of constructing a triangle. Have you a problem of any kind? You wish to find a certain unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted from our list. & you are on the right track, too; the suggestion is a good one, it suggests to you a procedure which is very frequently successful.

All the questions & suggestions of our list are natural, simple, obvious, just plain common sense; but they state plain common sense in general terms. They suggest a certain conduct which comes naturally to any person who is seriously concerned with his problem & has some common sense. But the person who behaves the right way usually does not care to express his behavior in clear words &, possibly, he cannot express it so; our list tries to express it so." – Polya, 2014, p. 3

3.5 Teacher & student. Imitation & practice

"There are 2 aims which the teacher may have in view when addressing to his students a question or a suggestion of the list: 1st, to help the student to solve the problem at hand. 2nd, to develop the student's ability so that he may solve future problems by himself.

Experience shows that the questions & suggestions of our list, appropriately used, very frequently help the student. They have 2 common characteristics, common sense & generality. As they proceed from plain common sense they very often come naturally; they could have occurred to the student himself. As they are general, they help unobtrusively; they just indicate a general direction & leave plenty for the student to do.

But the 2 aims we mentioned before are closely connected; if the student succeeds in solving the problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that our questions are general, applicable in many

[101] **worth somebody's while (to do something/doing something)** [idiom] interesting or useful for somebody to do.

[102] **typically** [adv] **1.** used to say that something usually happens in the way that you are stating; **2.** in a way that shows the usual qualities or features of a particular type of person, thing or group.

[103] **generality** [n] (plural **generalities**) **1.** [uncountable] **generality (of something)** the quality of a theory or model that can be applied being across a wide range of cases & situations. The phrase **without loss of generality** means that a statement about 1 particular case can be easily applied to all other cases.; **2.** [countable, usually plural] a statement that makes general points rather than giving details or particular examples; **3.** (**the generality**) [singular + singular or plural verb] **generality (of somebody/something)** (*formal*) the greater part of a group of people or things, SYNONYM: **majority**.

[104] **applicable** [a] [not usually before noun] true about or appropriate to a particular situation, group of people, etc.

[105] **subject matter** [n] [uncountable] **subject matter (of something)** the ideas or information contained in a book, speech, painting, etc.

[106] **restriction** [n] **1.** [countable] a rule or law that limits what you can do or what can happen; **2.** [uncountable] the act of limiting or controlling somebody/something.

[107] **anyhow** [adv] **1.** (also **anyway**, also NAE, informal **anyways**) used when adding something to support an idea or argument; **2.** (also **anyway**, also NAE, informal **anyways**) despite something; even so; **3.** (also **anyway**, also NAE, informal **anyways**) used when changing the subject of a conversation, ending the conversation or returning to a subject; **4.** (also **anyway**, also NAE, informal **anyways**) used to correct or slightly change what you have said; **5.** in a careless way; not arranged in an order.

cases. If the same question is repeatedly helpful, the student will scarcely[108] fail to notice it & he will be induced[109] to ask the question by himself in a similar situation. Asking the question repeatedly, he may succeed once in eliciting[110] the right idea. By such a success, he discovers the right way of using the question, & then he has really assimilated[111] it.

The student may absorb[112] a few questions of our list so well that he is finally able to put to himself the right question in the right moment & to perform the corresponding mental operation naturally & vigorously[113]. Such a student has certainly derived the greatest possible profit from our list. What can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation[114] & practice. Trying to swim, you imitate what other people do with their hands & feet to keep their heads above water, &, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe & to imitate what other people do when solving problems &, finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill[115] some interest for problems into their minds & give them plenty of opportunity for imitation & practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions & suggestions of our list, he puts these questions & suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize[116] his ideas a little & he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions & suggestions, & doing so he will acquire something that is more important than the knowledge of any particular mathematical fact." – Polya, 2014, pp. 3–5

Main divisions, main questions

3.6 4 phases

"Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again & again. Our conception[117] of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions & suggestions of our list, we shall distinguish 4 phases of the work. 1st, we have to *understand* the problem; we have to see clearly what is required. 2nd, we have to see how the various items are connected, how the unknown is liked to the data, in order to obtain the idea of the solution, to make a *plan*. 3rd, we *carry out* our plan. 4th, we *look back* at the completed solution, we review & discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea & jumping all preparations blurts[118] out with the solution. Such lucky ideas, of course, are most desirable[119], but something very undesirable[120] & unfortunate may result if the student leaves out any of the 4 phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan*. Many mistakes can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may be lost if the student fails to reexamine & to *reconsider* the completed solution." – Polya, 2014, pp. 5–6

[108]**scarcely** [adv] only just; almost not.

[109]**induce** [v] **1. induce something** to cause something; **2. induce somebody to do something** to persuade or influence somebody to do something; **3. induce something** (*physics*) to produce an electric charge or current, or a magnetic state by induction; **4. induce something (from something)** to use particular facts & examples to form a general rule or principle.

[110]**elicit** [v] to get information or a reaction from somebody/something.

[111]**assimilate** [v] **1.** [intransitive, transitive] to become a part of a country or community rather than remaining in a separate group; to allow or cause people to do this; **2.** [transitive] **assimilate something** (of the body or any biological system) to absorb or take in a substance; **3. assimilate something** to think deeply about something & understand it fully, so that you can use it, SYNONYM: **absorb**; **4.** [transitive, often passive] **assimilate something (into/to something)** to accept an idea, information or activity; to make it fit into something.

[112]**absorb** [v] **1.** to take in a liquid, gas or other substance from the surface or space around; **2. absorb something** to take in & keep heat, light or other forms of energy, instead of reflecting it; **3.** [often passive] to take control of a smaller unit or group & make it part of something larger; **4.** to take something into the mind & learn or understand it, SYNONYM: **take something in**; **5. absorb something** to deal with or reduce the effects of changes or costs; **6. absorb something** to use up a large supply of something, especially money or time; **7. be absorbed in something** to be so interested in something that you pay no attention to anything else.

[113]**vigorously** [adv] **1.** with determination, energy or enthusiasm; **2.** in a way that involves physical strength, effort or energy.

[114]**imitation** [n] **1.** [countable] a copy of something, especially something expensive; **2.** [uncountable] the act of copying somebody/something.

[115]**instil** [v] (BE) (NAE **instill**) to graduate put an idea or attitude into somebody's mind; to make somebody feel, think or behave in a particular way over a period of time.

[116]**dramatize** [v] (BE also **dramatise**) **1.** to present a book or an event as a play or film; **2. dramatize something** to make something seem more exciting or important than it really is.

[117]**conception** [n] **1.** [countable, uncountable] an understanding or a belief of what something is or what something should be; **2.** [uncountable] the process of forming an idea or a plan; **3.** [uncountable, countable] the process of an egg being fertilized inside a woman's body so that he becomes pregnant.

[118]**blurt** [v] **blurt something (out)** | **blurt that** ... | **blurt what, how, etc.** ... | + **speech** to say something suddenly & without thinking carefully enough.

[119]**desirable** [a] that you would like to have or do; worth having or doing, OPPOSITE: **undesirable**.

[120]**undesirable** [a] not wanted or approved of; likely to cause trouble or problems, OPPOSITE: **desirable**.

3.7 Understanding the problem

"It is foolish[121] to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish & sad things often happen, in & out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult & not too easy, natural & interesting, & some time should be allowed for natural & interesting presentation.

1st of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement & the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principle parts of the problem attentively[122], repeatedly, & from various sides. If there is a figure connected with the problem he should *draw a figure* & point out on it the unknown & the data. If it is necessary to give names to these objects he should *introduce suitable notation*; devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory[123] stage provided that we do not expect a definitive[124] answer but just a provisional[125] answer, a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] "Understanding the problem" is subdivided into 2 stages: "Getting acquainted" & "Working for better understanding.")" – Polya, 2014, pp. 6–7

3.8 Example

"Let us illustrate some of the points explained in the foregoing[126] section. We take the following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, the width, & the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem of Pythagoras, & with some of its applications in plane geometry, but they may have very little systematic knowledge in solid geometry. The teacher may rely here upon the student's unsophisticated[127] familiarity[128] with spatial[129] relations.

The teacher can make the problem interesting by making it concrete[130]. The classroom is a rectangular parallelepiped whose dimensions could be measured, & can be estimated; the students have to find, to "measure indirectly," the diagonal of the classroom. The teacher points out the length, the width, & the height of the classroom, indicates the diagonal with a gesture, & enlivens[131] his figure, drawn on the blackboard, by referring repeatedly to the classroom.

The dialogue between the teacher & the students may start as follows:

"What is the unknown?"

"The length of the diagonal of a parallelepiped."

"What are the data?"

"The length, the width, & the height of the parallelepiped."

Introduce suitable notation. Which letter should denote the unknown?"

"x."

"Which letters would you choose for the length, the width, & the height?"

"a, b, c."

"What is the condition, linking a, b, c, & x?"

"x is the diagonal of the parallelepiped of which a, b, & c are the length, the width, & the height."

"Is it a reasonable problem? I mean, *is the condition sufficient to determine the unknown?*"

[121]**foolish** [a] **1.** not showing good sense or judgment, SYNONYM: **silly, stupid**; **2.** [not usually before noun] made to feel or look silly & embarrassed, SYNONYM: **silly, stupid**.

[122]**attentive** [a] **1.** reading, listening or watching carefully & with interest; **2.** helpful; making sure that people have what they need.

[123]**preparatory** [a] done in order to prepare for something.

[124]**definitive** [a] **1.** final; that cannot be changed; **2.** [usually before noun] considered to be the best of its kind & almost impossible to improve.

[125]**provisional** [a] made for the present time, possibly to be changed when more information is available or when a more permanent arrangement can be made.

[126]**foregoing** [a] [only before noun] **1.** used to refer to something that has just been mentioned, OPPOSITE: **following**; **2.** (**the foregoing**) [n, singular + singular or plural verb] what has just been mentioned.

[127]**unsophisticated** [a] **1.** not having or showing much experience of the world & social situations; **2.** simple & basic; not complicated, SYNONYM: **crude**, OPPOSITE: **sophisticated**.

[128]**familiarity** [n] **1.** [uncountable, singular] **familiarity with something** the state of knowing somebody/something well; the state of recognizing somebody/something; **2.** [uncountable] the fact of being well known to you.

[129]**spatial** [a] connected with space & the position, size, shape, etc. of things in it.

[130]**concrete** [a] **1.** made of concrete; **2.** based on facts or actions, not on ideas, guesses or intentions; **3.** a concrete object is one that you can see & feel, OPPOSITE: **abstract**; [n] [uncountable] building material that is made by mixing together cement, sand, small stones & water.

[131]**enliven** [v] (*formal*) **enliven something** to make something more interesting or more fun.

"Yes, it is. If we know a, b, c, we know the parallelepiped. If the parallelepiped is determined, the diagonal is determined."" – Polya, 2014, pp. 7–8

3.9 Devising a plan

"We have a plan when we know, or know at least in outline, which calculations, computations, or constructions we have to perform in order to obtain the unknown. The way from understanding the problem to conceiving[132] a plan may be long & tortuous[133]. In fact, the main achievement in the solution of a problem is to conceive the idea of a plan. This idea may emerge gradually. Or, after apparently[134] unsuccessful trials & a period of hesitation[135], it may occur suddenly[136], in a flash[137], as a "bright idea." The best that the teacher can do for the student is to procure[138] for him, by unobtrusive help, a bright idea. The questions & suggestions we are going to discuss tend to provoke such an idea.

In order to be able to see the student's position, the teacher should think of his own experience, of his difficulties & successes in solving problems.

We know, of course, that it is hard to have a good idea if we have little knowledge of the subject, & impossible to have it if we have no knowledge. Good ideas are based on past experience & formerly[139] acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent[140] facts; materials alone are not enough for constructing a house but we cannot construct a house without collecting the necessary materials. The materials necessary for solving a mathematical problem are certain relevant items of our formerly acquired mathematical knowledge, as formerly solved problems, or formerly proved theorems. Thus, it is often appropriate to start the work with the question: *Do you know a related problem?*

The difficulty is that there are usually too many problems which are somewhat related to our present problem, i.e., have some point in common with it. How can we choose the one, or the few, which are really useful? There is a suggestion that puts our finger on an essential common point: *Look at the unknown! & try to think of a familiar problem having the same or a similar unknown.*

If we succeed in recalling a formerly solved problem which is closely related to our present problem, we are lucky. We should try to deserve such luck; we may deserve it by exploiting[141] it. *Here is a problem related to yours & solved before. Could you use it?*

The foregoing[142] questions, well understood & seriously considered, very often help to start the right train of ideas; but they cannot help always, they cannot work magic. If they do not work, we must look around for some other appropriate point of contact, & explore the various aspects of our problem; we have to vary, to transform, to modify the problem. *Could you restate the problem?* Some of the questions of our list hint specific means to vary the problems, as generalization, specialization, use of analogy, dropping a part of the condition, & so on; the details are important but we cannot go into them now. Variation of the problem may lead to some appropriate auxiliary problem: *If you cannot solve the proposed problem try to solve 1st some related problem.*

Trying to apply various known problems or theorems, considering various modifications[143], experimenting with various auxiliary problems, we may stray so far from our original problem that we are in danger of losing it altogether. Yet there is a good question that may bring us back to it: *Did you use all the data? Did you use the whole condition?*" – Polya, 2014, pp. 8–9

[132]**conceive** [v] **1.** [transitive] to form an idea or plan in your mind; **2.** [transitive, intransitive] to think of something in a particular way; to imagine something; **3.** [intransitive, transitive] (of a woman) to become pregnant.

[133]**tortuous** [a] [usually before noun] (*formal*) **1.** (*usually disapproving*) not simple & direct; long, complicated & difficult to understand, SYNONYM: **convoluted**; **2.** (of a road, path, etc.) full of bends, SYNONYM: **winding**.

[134]**apparently** [adv] according to what you have heard or read; according to the way something appears.

[135]**hesitate** [v] **1.** [intransitive] **hesitate to do something** to be unwilling to do something, especially because you are not sure that it is right or appropriate; **2.** [intransitive] to be slow to speak or act because you feel uncertain or nervous.

[136]**suddenly** [adv] quickly & unexpectedly, OPPOSITE: **gradually**.

[137]**flash** [n] **1. flash (of something)** a sudden bright light that shines for a moment & then disappears; **2. flash of something** a particular feeling or idea that suddenly comes into your mind or shows in your face; [v] **1.** [intransitive, transitive] **flash (something)** to shine very brightly for a short time; to make something shine in this way; **2.** [intransitive, transitive] to appear on a television screen, computer screen, etc. for a short time; to make something do this.

[138]**procure** [v] **procure something (for somebody/something)** (*formal*) to obtain something, especially with effort.

[139]**formerly** [adv] in earlier times.

[140]**pertinent** [a] appropriate to a particular situation, SYNONYM: **relevant**.

[141]**exploit** [v] **1. exploit something** to use something well in order to gain as much from it as possible; **2.** to develop or use something for business or industry; **3. exploit somebody/something (for something)** (*disapproving*) to treat a person or situation as an opportunity to gain an advantage for yourself; **4. exploit somebody** (*disapproving*) to treat somebody unfairly by making them work & not giving them much in return.

[142]**foregoing** [a] [only before noun] **1.** used to refer to something that has just been mentioned, OPPOSITE: **following**; **2.** (**the foregoing**) [n, singular + singular or plural verb] what has just been mentioned.

[143]**modification** [n] [uncountable, countable] the act or process of changing something in order to improve it or make it more suitable; a change that is made, SYNONYM: **adaptation**.

3.10 Example

"We return to the example considered in Sect. 8. As we left it, the students just succeeded in understanding the problem & showed some mild[144] interest in it. They could now have some ideas of their own, some initiative[145]. If the teacher, having watched sharply[146], cannot detect[147] any sign of such initiative he has to resume[148] carefully his dialogue with the students. He must be prepared to repeat with some modification the questions which the student do not answer. He must be prepared to meet often with the disconcerning silence of the students (which will be indicated by dots ......).

"Do you know a related problem?"

......

"Look at the unknown! Do you know a problem having the same unknown?"

......

"Well, *what is the unknown?"*

"The diagonal of a parallelepiped."

"Do you know any *problem with the same unknown?"*

"No. We have not had any problem yet about the diagonal of a parallelepiped."

"Do you know any *problem with a similar unknown?"*

......

"You see, the diagonal is a segment, the segment of a straight line. Did you never solve a problem whose unknown was the length of a line?"

"Of course, we have solved such problems. E.g., to find a side of a right triangle."

"Good! *Here is a problem related to yours & solved before. Could you use it?"*

......

"You were lucky enough to remember a problem which is related to your present one & which you solved before. Would you like to use it? *Could you introduce some auxiliary element in order to make its use possible?"*

......

"Look here, the problem you remembered is about a triangle. Have you any triangle in your figure?"

Let us hope that the last hint was explicit enough to provoke the idea of the solution which is to introduce a right triangle, (emphasized in Fig. 1) of which the required diagonal is the hypotenuse[149]. Yet the teacher should be prepared for the case that even this fairly[150] explicit[151] hint is insufficient to shake the torpor[152] of the students; & so he should be prepared to use a whole gamut[153] of more & more explicit hints.

"Would you like to have a triangle in the figure?"

"What sort of triangle would you like to have in the figure?"

"You cannot find yet the diagonal; but you said that you could find the side of a triangle. Now, what will you do?"

"Could you find the diagonal, if it were a side of a triangle?"

When, eventually, with more or less help, the students succeed in introducing the decisive auxiliary element, the right triangle emphasized in Fig. 1, the teacher should convince himself that the students see sufficiently far ahead before encouraging them to go into actual calculations.

"I think that it was a good idea to draw that triangle. You have now a triangle; but have you the unknown?"

"The unknown is the hypotenuse of the triangle; we can calculate it by the theorem of Pythagoras."

"You can, if both legs are known; but are they?"

"1 leg is given, it is c. & the other, I think, is not difficult to find. Yes, the other leg is the hypotenuse of another right triangle."

"Very good! Now I see that you have a plan."" – Polya, 2014, pp. 10–12

[144] **mild** [a] (**milder, mildest**) **1.** not severe or strong; **2.** (of weather) not very cold, & therefore pleasant.

[145] **initiative** [n] **1.** [countable] a new plan for dealing with a particular problem or for achieving a particular purpose; **2.** [uncountable] the ability to decide & act on your own without waiting for somebody to tell you what to do; **3.** (**the initiative**) [singular] the power or opportunity to act before other people do.

[146] **sharply** [adv] **1.** suddenly & by a large amount; **2.** in a way that clearly shows the differences between 2 things; in a way that clearly emphasizes something; **3.** in a critical way; **4.** used to emphasize that something has a sharp point or edge.

[147] **detect** [v] to discover or notice something that is difficult to discover or notice.

[148] **resume** [v] (*formal*) [intransitive, transitive] (of an activity) to begin again after an interruption; to begin an activity again after an interruption.

[149] **hypotenuse** [n] (*geometry*) the side opposite the right angle of a right-angled triangle.

[150] **fairly** [adv] **1.** (before adjectives & adverbs) quite but not very; **2.** in a fair way; in a way that treats people equally & according to the rules or law.

[151] **explicit** [a] **1.** saying something clearly & exactly; **2.** showing or referring to sex in a very obvious or detailed way.

[152] **torpor** [n] [uncountable, singular] (*formal*) the state of not being active & having no energy or enthusiasm, SYNONYM: **lethargy**.

[153] **the gamut** [n] [singular] the complete range of a particular kind of thing.

3.11 Carrying out the plan

"To devise[154] a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, & 1 more thing: good luck. To carry out the plan is much easier; what we need is mainly patience[155].

The plan gives a general outline; we have to convince ourselves that the details fit into the outline, & so we have to examine the details one after the other, patiently[156], till everything is perfectly clear, & no obscure[157] corner remains in which an error could be hidden.

If the student has really conceived a plan, the teacher has now a relatively peaceful time. The main danger is that the student forgets his plan. This may easily happen if the student received his plan from outside, & accepted it on the authority of the teacher; but if he worked for it himself, even with some help, & conceived the final idea with satisfaction, he will not lose this idea easily. Yet the teacher must insist[158] that the student should *check each step.*

We may convince ourselves of the correctness of a step in our reasoning either "intuitively[159]" or "formally[160]." We may concentrate upon the point in question till we see it so clearly & distinctly that we have no doubt that the step is correct; or we may derive the point in question according to formal rules. (The difference between "insight" & "formal proof" is clear enough in many important cases; we may leave further discussion to philosophers.)

The main point is that the student should be honestly convinced of the correctness of each step. In certain cases, the teacher may emphasize the difference between "seeing" & "proving" *Can you see clearly that the step is correct?* But can you also *prove that the step is correct?*" – Polya, 2014, pp. 12–13

3.12 Example

"Let us resume our work at the point where we left it at the end of Sect. 10. The student, at last, has got the idea of the solution. He sees the right triangle of which the unknown x is the hypotenuse & the given height c is 1 of the legs; the other leg is the diagonal of a face. The student must, possibly, be urged to introduce suitable notation. He should choose y to denote that other leg, the diagonal of the face whose sides are a & b. Thus, he may see more clearly the idea of the solution which is to introduce an auxiliary problem whose unknown is y. Finally, working at 1 right triangle after the other, he may obtain (see Fig. 1) $x^2 = y^2 + c^2$, $y^2 = a^2 + b^2$, & hence, eliminating the auxiliary unknown y, $x^2 = a^2 + b^2 + c^2$, $x = \sqrt{a^2 + b^2 + c^2}$.

The teacher has no reason to interrupt the student if he carries out these details correctly except, possibly, to warn him that he should *check each step*. Thus, the teacher may ask:

"Can you *see clearly* that the triangle with sides x, y, c is a right triangle?"

To this question the student may answer honestly "Yes" but he could be much embarrassed if the teacher, not satisfied with the intuitive conviction[161] of the student, should go on asking:

"But can you *prove* that this triangle is a right triangle?"

Thus, the teacher should rather suppress[162] this question unless the class has had a good initiation[163] in solid geometry. Even in the latter case, there is some danger that the answer to an incidental[164] question may become the main difficulty for the majority of the students." – Polya, 2014, pp. 13–14

[154]**devise** [v] **devise something** to plan or invent a procedure, system or method, especially one that is new or complicated, by using careful thought, SYNONYM: **think something up**.

[155]**patience** [n] [uncountable] **1.** the ability to stay calm & accept delay, problems or suffering without complaining; **2.** the ability to spend a lot of time doing something difficult that needs a lot of attention & effort.

[156]**patient** [a] able to stay calm & accept delay, problems or suffering without complaining; showing this, OPPOSITE: **impatient**.

[157]**obscure** [v] to cover something; to make it difficult to see, hear or understand something.

[158]**insist** [v] **1.** [intransitive, transitive] to say firmly that something is true, especially when other people do not believe you; **2.** [intransitive, transitive] to demand that something happens or that somebody agrees to do something; **insist on/upon something** [idiom] to demand something & refuse to be persuaded to accept anything else.

[159]**intuitively** [adv] by using feelings rather than by considering facts.

[160]**formally** [adv] **1.** officially, OPPOSITE: **informally**; **2.** in the way that something appears or is presented; **3. formally trained/educated/taught** trained, etc. in a school, college or other institution.

[161]**conviction** [n] **1.** [countable, uncountable] the act of finding somebody guilty of a crime in court; the fact of having been found guilty; **2.** [countable, uncountable] a strong opinion or belief; **3.** [uncountable] the feeling of believing something strongly & of being sure about it.

[162]**suppress something** (of a government or ruler) to stop something by force, especially an activity or group that is believed to threaten authority; **2. suppress something** to prevent something from growing, developing or continuing; **3. suppress something** (*usually disapproving*) to prevent something from being published or made known; **4. suppress something** to prevent yourself from having or expressing a feeling or an emotion.

[163]**initiation** [n] [uncountable] **1.** the act of starting something; **2. initiation (into something)** the act of somebody becoming a member of a group, often with a special ceremony; the act of introducing somebody to an activity or a skill.

[164]**incidental** [a] **1. incidental (to something)** happening in connection with something else, but not as important as it, or not intended; **2. incidental to something** (*specialist*) happening as a natural result of something.

3.13 Looking back

"Even fairly good students, when they have obtained the solution of the problem & written down neatly[165] the argument, shut their books & look for something else. Doing so, they miss an important & instructive[166] phase of the work. By looking back at the completed solution, by reconsidering[167] & reexamining[168] the result & the path that led to it, they could consolidate[169] their knowledge & develop their ability to solve problems. A good teacher should understand & impress[170] on his students the view that no problem whatever is completely exhausted[171]. There remains always something to do; with sufficient study & penetration, we could improve any solution, &, in any case, we can always improve our understanding of the solution.

The student has now carried through his plan. He has written down the solution, checking each step. Thus, he should have good reasons to believe that his solution is correct. Nevertheless, errors are always possible, especially if the argument is long & involved. Hence, verifications[172] are desirable[173]. Especially, if there is some rapid & intuitive procedure to test either the result or the argument, it should not be overlooked[174]. *Can you check the result? Can you check the argument?*

In order to convince ourselves of the presence or of the quality of an object, we like to see & to touch it. & as we prefer perception[175] through 2 different senses, so we prefer conviction by 2 different proofs: *Can you derive the result differently?* We prefer, of course, a short & intuitive argument to a long & heavy one: *Can you see it at a glance?*

1 of the 1st & foremost[176] duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, & no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. The students will find looking back at the solution really interesting if they have made an honest effort, & have the consciousness[177] of having done well. Then they are eager[178] to see what else they could accomplish with that effort, & how they could do equally well another time. The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained. *Can you use the result, or the method for some other problem?*" – Polya, 2014, pp. 14–16

3.14 Example

"In Sect. 12, the students finally obtained the solution: If the 3 edges of a rectangular parallelogram, issued from the same corner, are a, b, c, the diagonal is $\sqrt{a^2 + b^2 + c^2}$.

Can you check the result? The teacher cannot expect a good answer to this question from inexperienced students. The students, however, should acquire fairly early the experience that problems "in letters" have a great advantage over purely numerical problems; if the problem is given "in letters" its result is accessible to several tests to which a problem "in numbers" is not susceptible at all. Our example, although fairly simple, is sufficient to show this. The teacher can ask several questions about the result which the students may readily[179] answer with "Yes"; but an answer "No" would show a serious flaw in the result.

Did you use all the data? Do all the data a, b, c appear in your formula for the diagonal?"

[165]**neatly** [a] (**neater, neatest**) **1.** in good order; carefully done or arranged; **2.** simple but clever; **3.** containing or made out of just 1 substance; not mixed with anything else.

[166]**instructive** [a] giving a lot of useful information.

[167]**reconsider** [v] [transitive, intransitive] to think about something again, especially because you might want to change a previous decision or opinion.

[168]**reexamine** [v] **reexamine something** to examine or think about something again, especially because you may need to change your opinion, SYNONYM: **reassess**.

[169]**consolidate** [v] **1. consolidate something** to make a position of power or success stronger so that it is more likely to continue; **2.** (*specialist*) to join things together into a single more effective whole; to join financial accounts or sums of money into a single overall account or sum.

[170]**impress** [v] [transitive, intransitive] if a person or thing impresses you, you feel admiration for them or it; **impress something on/upon somebody** [phrasal verb] to make somebody understand how important something is by emphasizing it; **impress something/itself on/upon somebody** [phrasal verb] to have a great effect on something, especially somebody's mind or imagination.

[171]**exhausted** [a] **1.** completely used or finished; **2.** very tired.

[172]**verification** [n] [uncountable] (*formal*) the act of showing or checking that something is true or accurate, SYNONYM: **confirmation**.

[173]**desirable** [a] that you would like to have or do; worth having or doing.

[174]**overlook** [v] **1. overlook something** to fail to see or notice something, SYNONYM: **miss**; **2. overlook something** if a building, etc. overlooks a place, you can see that place from the building; **3. overlook somebody (for something)** to not consider somebody for a job or position, even though they might be suitable.

[175]**perception** [n] **1.** [uncountable, countable] an idea, a belief or an image you have as a result of how you see or understand something; **2.** [uncountable] the way you notice things or the ability to notice things with the senses. In biology, **perception** refers to the process in the nervous system by which a living thing becomes aware of events & things outside itself.; **3.** [uncountable] the ability to understand the true nature of something, SYNONYM: **insight**.

[176]**foremost** [a] the most important or famous; in a position at the front; [adv] **1st & foremost** [idiom] more than anything else.

[177]**consciousness** [n] [uncountable] **1.** the state of being able to use your senses & mental powrs to understand what is happening; **2.** the state of being aware of something, SYNONYM: **awareness**; **3.** the ideas & opinions of a person or group.

[178]**eager** [a] very interested & excited by something that is going to happen or about something that you want to do, SYNONYM: **keen**.

[179]**readily** [adv] **1.** quickly & without difficulty, SYNONYM: **freely**; **2.** in a way that shows that you do not object to something, SYNONYM: **willingly**.

"Length, width, & height play the same role in our question; our problem is symmetric w.r.t. a, b, c. Is the expression you obtained for the diagonal symmetric in a, b, c? Does it remain unchanged when a, b, c are interchanged?

"Our problem is a problem of solid geometry: to find the diagonal of a parallelepiped with given dimensions a, b, c. Our problem is analogous to a problem of plane geometry: to find the diagonal of a rectangle with given dimensions a, b. Is the result of our 'solid' problem analogous to the result of the 'plane' problem?"

"If the height c decreases, & finally vanishes, the parallelepiped becomes a parallelogram. If you put $c = 0$ in your formula, do you obtain the correct formula for the diagonal of the rectangular parallelogram?"

"If the height c increases, the diagonal increases. Does your formula show this?"

"If all 3 measures a, b, c of the parallelepiped increase in the same proportion, the diagonal also increases in the same proportion, the diagonal also increases in the same proportion. If, in your formula, you substitute $12a, 12b, 12c$ for a, b, c respectively, the expression of the diagonal, owing to this substitution, should also be multiplied by 12. Is that so?"

"If a, b, c are measured in feet, your formula gives the diagonal measured in feet too; but if you change all measures into inches, the formula should remain correct. Is that so?"

(The 2 last questions are essentially equivalent; see TEST BY DIMENSION.)

These questions have several good effects. 1st, an intelligent student cannot help being impressed by the fact that the formula passes so many tests. He was convinced before that the formula is correct because he derived it carefully. But now he is more convinced, & his gain in confidence comes from a different source; it is due to a sort of "experimental evidence." Then, thanks to the foregoing questions, the details of the formula acquire new significance, & are linked up with various facts. The formula has therefore a better chance of being remembered, the knowledge of the student is consolidated. Finally, these questions can be easily transferred to similar problems. After some experience with similar problems, an intelligent student may perceive the underlying general ideas: use of all relevant data, variation of the data, symmetry, analogy. If he gets into the habit of directing his attention to such points, his ability to solve problems may definitely profit.

Can you check the argument? To recheck the argument step by step may be necessary in difficult & important cases. Usually, it is enough to pick out "touchy" points for rechecking. In our case, it may be advisable to discuss retrospectively the question which was less advisable to discuss as the solution was not yet attained: Can you *prove* that the triangle with sides x, y, c is a right triangle (See the end of Sect. 12.)

Can you use the result or the method for some other problem? With a little encouragement, & after 1 or 2 examples, the students easily find applications which consist essentially in giving some *concrete interpretation* to the abstract mathematical elements of the problem. The teacher himself used such a concrete interpretation as he took the room in which the discussion takes place for the parallelepiped of the problem. A dull student may propose, as application, to calculate the diagonal of the cafeteria instead of the diagonal of the classroom. If the students do not volunteer more imaginative remarks, the teacher himself may put a slightly different problem, e.g.: "Being given the length, the width, & the height of a rectangular parallelepiped, find the distance of the center from 1 of the corners."

The students may use the *result* of the problem they just solved, observing that the distance required is $\frac{1}{2}$ of the diagonal they just calculated. Or they may use the *method*, introducing suitable right triangles (the latter alternative is less obvious & somewhat more clumsy in the present case).

After this application, the teacher may discuss the configuration of the 4 diagonals of the parallelepiped, & the 6 pyramids of which the 6 faces are the bases, the center the common vertex, & the seidiagonals the lateral edges. When the geometric imagination of the students is sufficiently enlivened, the teacher should come back to his question: *Can you use the result, or the method, for some other problem?* Now there is a better chance that the students may find some more interesting concrete interpretation, e.g., the following:

"In the center of the flat rectangular top of building which is 21 yards long & 16 yards wide, a flagpole[180] is to be erected[181], 8 yards high. To support the pole, we need 4 equal cables. The tables should start from the same point, 2 yards under the top of the pole, & end at the 4 corners of the top of the building. How long is each cable?"

The students may use the *method* of the problem they solved in detail introducing a right triangle in a vertical plane, & another one in a horizontal plane. Or they may use the *result*, imagining a rectangular parallelepiped of which the diagonal, x, is 1 of the 4 cables & the edges are $a = 10.5$, $b = 8$, $c = 6$. By straightforward application of the formula, $x = 14.5$.

For more examples, see CAN YOU USE THE RESULT?" – Polya, 2014, pp. 16–19

3.15 Various approaches

"Let us still retain[182], for a while, the problem we considered in the foregoing Sects. 8, 10, 12, 14. The main work, the discovery of the plan, was described in Sect. 10. Let us observe that the teacher could have proceeded differently. Starting

[180]**flagpole** [n] (also **flagstaff**) a tall thin straight piece of wood or metal on which a flag is hung.

[181]**erect** [v] **1. erect something** to build something; **2. erect something** to create or establish something.

[182]**retain** [v] **1. retain somebody/something** to keep somebody/something; to continue to have something & not lose it or get rid of it; **2. retain something** to take in a substance & keep holding it; **3. retain something** to remember or continue to hold something; **4. retain somebody/something** (*law*) to employ a professional person such as a lawyer or doctor; to make regular payments to such a person in order to keep their services.

from the same point as in Sect. 10, he could have followed a somewhat different line, asking the following questions:

"Do you know any related problem?"

"Do you know an *analogous* problem?"

"You see, the proposed problem is a problem of solid geometry. Could you think of a simpler analogous problem of plane geometry?"

"You see, the proposed problem is about a figure in space, it is concerned with the diagonal of a rectangular parallelepiped. What might be an analogous problem about a figure in the plane? It should be concerned with – the diagonal–of–a rectangular–"

"Parallelogram."

The students, even if they are very slow & indifferent[183], & were not able to guess anything before, are obliged[184] finally to contribute at least a minute part of the idea. Besides, if the students are so slow, the teacher should not take up the present problem about the parallelepiped without having discussed before, in order to prepare the students, the analogous problem about the parallelogram. Then, he can go on now as follows:

"Here is a problem related to yours & solved before. Can you use it?"

"Should you introduce some auxiliary element in order to make its use possible?"

Eventually, the teacher may succeed in suggesting to the students the desirable idea. It consists in conceiving the diagonal of the given parallelepiped as the diagonal of a suitable parallelogram which must be introduced into the figure (as intersection of the parallelepiped with a plane passing through 2 opposite edges). The idea is essentially the same as before (Sect. 10) but the approach is different. In Sect. 10, the contact with the available knowledge of the students was established through the unknown; a formerly solved problem was recollected because its unknown was the same as that of the proposed problem. In the present section analogy provides the contact with the idea of the solution." – Polya, 2014, pp. 19–20

3.16 The teacher's method of questioning

"shown in the foregoing Sects. 8, 10, 12, 14, 15 is essentially this: Begin with a general question or suggestion of our list, &, if necessary, come down gradually to more specific & concrete questions or suggestions till you reach one which elicits[185] a response in the student's mind. If you have to help the student exploit his idea, start again, if possible, from a general question or suggestion contained in the list, & return again to some more special one if necessary; & so on.

Of course, our list is just a 1st list of this kind; it seems to be sufficient for the majority of simple cases, but there is no doubt that it could be perfected. It is important, however, that the suggestions from which we start should be simple, natural, & general, & that there list should be short.

The suggestions must be simple & natural because otherwise they cannot be *unobtrusive*.

The suggestions must be general, applicable not only to the present problem but to problems of all sorts, if they are to help develop the *ability* of the student & not just a special technique.

The list must be short in order that the questions may be often repeated, unartificially[186], & under varying circumstances; thus, there is a chance that they will be eventually assimilated[187] by the student & will contribute to the development of a *mental habit*.

It is necessary to come down gradually to specific suggestions, in order that the student may have as great a *share of the work* as possible.

This method of questioning is not a rigid one; fortunately so, because, in these matters, any rigid, mechanical, pedantical[188] procedure is necessarily bad. Our method admits a certain elasticity & variation, it admits various approaches (Sect. 15), it can be & should be so applied that questions asked by the teacher *could have occurred to the student himself*.

If a reader wishes to try the method here proposed in his class he should, of course, proceed with caution. He should study carefully the example introduced in Sect. 8, & the following examples in Sects. 18–20. He should prepare carefully the examples which he intends to discuss, considering also various approaches. He should start with a few trials & find out gradually how he can manage the method, how the students take it, & how much time it takes." – Polya, 2014, pp. 20–22

[183]**indifferent** [a] [not usually before noun] **indifferent (to somebody/something)** having or showing no interest in somebody/something.
[184]**oblige** [v] [transitive, usually passive] to make somebody do something, by law or because it is a rule or a duty.
[185]**elicit** [v] to get information or a reaction from somebody/something.
[186]**unartificially** [adv]
[187]**assimilate** [v] **1.** [intransitive, transitive] to become a part of a country or community rather than remaining in a separate group; to allow or cause people to do this; **2.** [transitive] **assimilate something** (of the body or any biological system) to absorb or take in a substance; **3.** [transitive] **assimilate something** to think deeply about something & understand it fully, so that you can use it, SYNONYM: **absorb**; **4.** [transitive, often passive] **assimilate something (into/to something)** to accept an idea, information or activity; to make it fit into something.
[188]**pedantic** [a] (*disapproving*) too worried about small details or rules.

3.17 Good questions & bad questions

"If the method of questioning formulated in the foregoing section is well understood it helps to judge, by comparison, the quality of certain suggestions which may be offered with the intention of helping the students.

Let us go back to the situation as it presented itself at the beginning of Sect. 10 when the question was asked: *Do you know a related problem?* Instead of this, with the best intention to help the students, the question may be offered: *Could you apply the theorem of Pythagoras?*

The intention may be the best, but the question is about the worst. We must realize in what situation it was offered; then we shall see that there is a long sequence of objections against that sort of "help."

1. If the student is near to the solution, he may understand the suggestion implied by the question; but if he is not, he quite possibly will not see at all the point at which the question is driving. Thus the question fails to help where help is most needed.

2. If the suggestion is understood, it gives the whole secret away, very little remains for the student to do.

3. The suggestion is of too special a nature. Even if the student can make use of it in solving the present problem, nothing is learned for future problems. The question is not instructive.

4. Even if he understands the suggestion, the student can scarcely[189] understand how the teacher came to the idea of putting such a question. & how could he, the student, find such a question by himself? It appears as an unnatural[190] surprise, as a rabbit pulled out of a hat; it is really not instructive.

None of these objections can be raised against the procedure described in Sect. 10, or against that in Sect. 15." – Polya, 2014, pp. 22–23

More examples

3.18 A problem of construction

"Inscribe[191] a square in a given triangle. 2 vertices of the square should be on the base of the triangle, the 2 other vertices of the square on the 2 other sides of the triangle, one on each.

"What is the unknown?"

"A square."

"What are the data?"

"A triangle is given, nothing else."

"What is the condition?"

"The 4 corners of the square should be on the perimeter of the triangle, 2 corners on the base, 1 corner on each of the other 2 sides."

"Is it possible to satisfy the condition?"

"I think so. I am not so sure."

"You do not seem to find the problem too easy. *If you cannot solve the proposed problem, try to solve 1st some related problem.* Could you satisfy a *part of the condition?"*

"You see, the condition is concerned with all the vertices of the square. How many vertices are there?"

"4."

"A part of the condition would be concerned with < 4 vertices. *Keep only a part of the condition, drop the other part.* What part of the condition is easy to satisfy?"

"It is easy to draw a square with 2 vertices on the perimeter of the triangle – or even one with 3 vertices on the perimeter!"

"Draw a figure!"

The student draws Fig. 2 (Polya, 2014, p. 24)

"You kept only a part of the condition, & you *dropped the other part. How far is the unknown now determined?"*

"The square is not determined if it has only 3 vertices on the perimeter of the triangle."

"Good! *Draw a figure."*

The student draws Fig. 3 (Polya, 2014, p. 24)

"The square, as you said, is not determined by the *part of the condition you kept. How can it vary?"*

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[189] **scarcely** [adv] only just; almost not, SYNONYM: **hardly**.

[190] **unnatural** [a] **1.** different from what is normal or expected, or from what is generally accepted as being right, OPPOSITE: **natural, normal**; **2.** different from anything in nature, OPPOSITE: **natural**.

[191] **inscribe** [v] **1.** [often passive] to write or cut words, your name, etc. onto something; **2.** [often passive] **inscribe something + adv./prep.** to make something present in, on, etc. something.

. . . . . .

"3 corners of your square are on the perimeter of the triangle but the 4th corner is not yet there where it should be. Your square, as you said, is undetermined, it can vary; the same is true of its 4th corner. *How can it vary?"*

. . . . . .

"Try it experimentally, if you wish. Draw more squares with 3 corners on the perimeter in the same way as the 2 squares already in the figure. Draw small squares & large squares. What seems to be the locus[192] of the 4th corner? *How can it vary?"*

The teacher brought the student very near to the idea of the solution. If the student is able to guess that the locus of the 4th corner is a straight line, he has got it." – Polya, 2014, pp. 23–25

3.19 A problem to prove

"2 angles are in different planes but each side of one is parallel to the corresponding side of the other, & has also the same direction. Prove that such angles are equal.

What we have to prove is a fundamental theorem of solid geometry. The problem may be proposed to students who are familiar with plane geometry & acquainted with those few facts of solid geometry which prepare the present theorem in Euclid's Elements. (The theorem that we have stated & are going to prove is the proposition 10 of Book XI of Euclid.) Not only questions & suggestions quoted from our list are printed in italics but also others which correspond to them as "problems to prove" correspond to "problems to find." (The correspondence is worked out systematically in PROBLEMS TO FIND, PROBLEMS TO PROVE 5, 6.)

"What is the hypothesis?"

"2 angles are in different planes. Each side of one is parallel to the corresponding side of the other, & has also the same direction.

"What is the conclusion?"

"The angles are equal."

"Draw a figure. Introduce suitable notation."

The student draws the lines of Fig. 4 & chooses, helped more or less by the teacher, the letters as in Fig. 4.

"What is the hypothesis? Say it, please, using your notation."

"A, B, C are not in the same plane as A', B', C'. & $AB||A'B'$, $AC||A'C'$. Also AB has the same direction as $A'B'$, & AC the same as $A'C'$."

"What is the conclusion?"

"$\angle BAC = \angle B'A'C'$."

"Look at the conclusion! & try to think of a familiar theorem having the same or a similar conclusion."

"If 2 triangles are congruent[193], the corresponding angles are equal."

"Very good! Now *here is a theorem related to yours & proved before. Could you use it?"*

"I think so but I do not see yet quite how."

"Should you introduce some auxiliary element in order to make its use possible?"

. . . . . .

"Well, the theorem which you quoted so well is about triangles, about a pair of congruent triangles. Have you any triangles in your figure?"

"No. But I could introduce some. Let me join B to C, & B' to C'. Then there are 2 triangles, ΔABC, $\Delta A'B'C'$."

"Well done. But what are these triangles good for?"

"To prove the conclusion, $\angle BAC = \angle B'A'C'$."

"Good! If you wish to prove this, what kind of triangles do you need?"

"Congruent triangles. Yes, of course, I may choose B, C, B', C' so that $AB = A'B', AC = A'C'$.

"Very good! Now, what do you wish to prove?"

"I wish to prove that the triangles are congruent, $\Delta ABC = \Delta A'B'C'$.

If I could prove this, the conclusion $\angle BAC = \angle B'A'C'$ would follow immediately."

"Fine! You have a new aim, you aim at a new conclusion. *Look at the conclusion! & try to think of a familiar theorem having the same or a similar conclusion."*

"2 triangles are congruent if—if the 3 sides of the one are equal respectively to the 3 sides of the other."

"Well done. You could have chosen a worse one. Now *here is a theorem related to yours & proved before. Could you use it?"*

"I could use it if I knew that $BC = B'C'$."

[192]**locus** [n] (plural **loci**) **1. locus (of something)** (*formal*) the exact place or position where something happens or which is thought to be the center of something; **2.** (*biology*) the position of a gene or mutation on a chromosome; **3. locus (of something)** the set of all points that share a particular property.

[193]**congruent** [a] **1.** (*geometry*) having the same size & shape; **2. congruent (with something)** (*formal*) in agreement with something; similar to something & not in conflict with it, SYNONYM: **compatible**.

"That is right! Thus, what is your aim?"
"To prove that $BC = B'C'$."
"Try to think of a familiar theorem having the same or a similar conclusion."
"Yes, I know a theorem finishing: '... then the 2 lines are equal.' But it does not fit in."
"Should you introduce some auxiliary element in order to make its use possible?"
......
"You see, how could you prove $BC = B'C'$ when there is no connection in the figure between BC & $B'C'$?"
......
"Did you use the hypothesis? What is the hypothesis?"
"We suppose that $AB||A'B', AC||A'C'$. Yes, of course, I must use that."
"Did you use the whole hypothesis? You say that $AB||A'B'$. Is that all that you know about these lines?"
"No; AB is also equal to $A'B'$, by construction. They are parallel & equal to each other. & so are AC & $A'C'$."
"2 parallel lines of equal length – it is an interesting configuration. *Have you seen it before?"*
"Of course! Yes! Parallelogram! Let me join A to A', B to B', & C to C'."
"The idea is not so bad. How many parallelograms have you now in your figure?"
"2. No, 3. No, 2. I mean, there are 2 of which you can prove immediately that they are parallelograms. There is a 3rd which seems to be a parallelogram; I hope I can prove that it is one. & then the proof will be finished!"

We could have gathered from this foregoing answers that the student is intelligent. But after this last remark of his, there is no doubt.

This student is able to guess a mathematical result & to distinguish clearly between proof & guess. He knows also that guesses can be more or less plausible. Really, he did profit something from his mathematics classes; he has some real experience in solving problems; he can conceive & exploit a good idea." – Polya, 2014, pp. 25–29

3.20 A rate problem

"*Water is flowing into a conical*[194] *vessel at the rate r. The vessel has the shape of a right circular cone, with horizontal base, the vertex pointing downwards; the radius of the base is a, the altitude of the cone b. Find the rate at which the surface is rising when the depth of the water is y. Finally, obtain the numerical value of the unknown supposing that $a = 4$ ft., $b = 3$ ft., $r = 2$ cu. ft. per minute, & $y = 1$ ft.*

The students are supposed to know the simplest rules of differentiation & the notion of "rate of change."

"What are the data?"

"The radius of the base of the cone $a = 4$ ft., the altitude of the cone $b = 3$ ft., the rate at which the water is flowing into the vessel $r = 2$ cu. ft. per minute, & the depth of the water at a certain moment, $y = 1$ ft."

"Correct. The statement of the problem seems to suggest that you should disregard, provisionally[195], the numerical values, work with the letters, express the unknown in terms of a, b, r, y & only finally, after having obtained the expression of the unknown in letters, substitute the numerical values. I would follow this suggestion. Now, *what is the unknown?"*

"The rate at which the surface is rising when the depth of the water is y."
"What is that? Could you say it in other terms?"
"The rate at which the depth of the water is increasing."
"What is that? *Could you restate it still differently?"*
"The rate of change of the depth of the water."
"That is right, the rate of change of y. But what is the rate of change? *Go back to the definition."*
"The derivative is the rate of change of a function."
"Correct. Now, is y a function? As we said before, we disregard the numerical value of y. Can you imagine that y changes?"
"Yes, y, the depth of the water, increases as the time goes by."
"Thus, y is a function of what?"
"Of the time t."
"Good. *Introduce suitable notation.* How would you press it in terms of a, b, r, y. By the way, 1 of these data is a 'rate.' Which one?"
"r is the rate at which water is flowing into the vessel."
"What is that? Could you say it in other terms?"
"r is the rate of change of the volume of the water in the vessel."
"What is that? *Could you restate it still differently?* How would you write it in *suitable notation?"*
"$r = \frac{dV}{dt}$."

[194]**conical** [a] shaped like a cone.
[195]**provisional** [a] made for the present time, possibly to be changed when more information is available or when a more permanent arrangement can be made.

"What is V?"

"The volume of the water in the vessel at the time t."

"Good. Thus, you have to express $\frac{dy}{dt}$ in terms of $a, b, \frac{dV}{dt}, y$. How will you do it?"

......

"If you cannot solve the proposed problem try to solve 1st some related problem. If you do not see yet the connection between $\frac{dy}{dt}$ & the data, try to bring in some simpler connection that could serve as a stepping stone."

......

"Do you not see that there are other connections? E.g., are y & V independent of each other?"

"No. When y increases, V must increase too."

"Thus, there is a connection. What is the connection?"

"Well, V is the volume of a cone of which the altitude is y. But I do not know yet the radius of the base."

"You may consider it, nevertheless. Call it something, say x."

"$V = \frac{\pi x^2 y}{3}$."

"Correct. Now, what about x? Is it independent of y?"

"No. When the depth of the water, y, increases the radius of the free surface, x, increases too."

"Thus, there is a connection. What is the connection?"

"Of course, similar triangles. $x : y = a : b$."

"1 more connection, you see. I would not miss profiting from it. Do not forget, you wished to know the connection between V & y."

"I have $x = \frac{ay}{b}$, $V = \frac{\pi a^2 y^3}{3b^2}$."

"Very good. This looks like a stepping stone, does it not? But you should not forget your goal. *What is the unknown?"*

"Well, $\frac{dy}{dt}$."

"You have to find a connection between $\frac{dy}{dt}, \frac{dV}{dt}$, & other quantities. & here you have one between y, V, & other quantities. What to do?"

"Differentiate! Of course! $\frac{dV}{dt} = \frac{\pi a^2 y^2}{b^2}\frac{dy}{dt}$. Here it is."

"Fine! & what about the numerical values?"

"If $a = 4$, $b = 3$, $\frac{dV}{dt} = r = 2$, $y = 1$, then $2 = \frac{\pi \cdot 16 \cdot 1}{9}\frac{dy}{dt}$."" – Polya, 2014, pp. 29–32

Part II. How to Solve It

3.21 A dialogue

3.21.1 Getting Acquainted

"*Where should I start?* Start from the statement of the problem.

What can I do? Visualize the problem as a whole as clearly & as vividly[196] as you can. Do not concern yourself with details for the moment.

What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind. The attention bestowed[197] on the problem may also stimulate[198] your memory & prepare for the recollection[199] of relevant points." – Polya, 2014, p. 33

3.21.2 Working for Better Understanding

"*Where should I start?* Start again from the statement of the problem. Start when this statement is so clear to you & so well impressed on your mind that you may lose sight of it for a while without fear of losing it altogether.

What can I do? Isolate[200] the principal parts of your problem. The hypothesis & the conclusion are the principal parts of a "problem to prove"; the unknown, the data, & the conditions are the principal parts of your problem, consider them 1

[196]**vivid** [a] **1.** (of memories, a description, etc.) producing very clear pictures in your mind, SYNONYM: **graphic**; **2.** (of light, colors, etc.) very bright.

[197]**bestow** [v] (*formal*) **bestow something (on/upon somebody)** to give something to somebody, especially to show how much they are respected.

[198]**stimulate** [v] **1. stimulate something** to make something develop or become more active, especially in a positive way; **2.** to make somebody interested & excited about something; **3. stimulate something** (*biology*) to make a part of the body function.

[199]**recollection** [n] (*formal*) **1.** [uncountable] the ability to remember something; the act of remembering something, SYNONYM: **memory**; **2.** [countable] a thing that you remember from the past, SYNONYM: **memory**.

[200]**isolate** [v] **1.** to separate somebody/something physically or socially from other people or things; **2.** to separate a single substance, cell, etc. from other so that you can study it; **3.** to separate a part of a situation, a problem or an idea so that you can see what it is & deal with it separately.

by 1, consider them in turn, consider them in various combinations, relating each detail to other details & each to the whole of the problem.

What can I gain by doing so? You should prepare & clarify details which are likely to play a role afterwards." – Polya, 2014, p. 33

3.21.3 Hunting for the Helpful Idea

"*Where should I start?* Start from the consideration of the principal parts of your problem. Start when these principal parts are distinctly arranged & clearly conceived, thanks to your previous work, & when your memory seems responsive.

What can I do? Consider your problem from various sides & seek contacts with your formerly acquired knowledge.

Consider your problem from various sides. Emphasize different parts, examine different details, examine the same details repeatedly but in different ways, combine the details differently, approach them from different sides. Try to see some new meaning in each detail, some new interpretation[201] of the whole.

Seek contacts with your formerly acquired knowledge. Try to think of what helped you in similar situations in the past. Try to recognize something familiar in what you examine, try to perceive something useful in what you recognize.

What could I perceive? A helpful idea, perhaps a decisive idea that shows you at a glance the way to the very end.

How can an idea be helpful? It shows you the whole of the way or a part of the way; it suggests to you more or less distinctly[202] how you can proceed. Ideas are more or less complete. You are lucky if you have any idea at all.

What can I do with an incomplete idea? You should consider it. If it looks advantageous you should consider it longer. If it looks reliable[203] you should ascertain[204] how far it leads you, & reconsider the situation. The situation has changed, thanks to your helpful idea. Consider the new situation from various sides & seek contacts with your formerly acquired knowledge.

What can I gain by doing so again? You may be lucky & have another idea. Perhaps your next idea will lead you to the solution right away. Perhaps you need a few more helpful ideas after the next. Perhaps you will be led astray[205] by some of your ideas. Nevertheless you should be grateful for all new ideas, also for the lesser ones, also for the hazy[206] ones, also for the supplementary ideas adding some precision[207] to a hazy one, or attempting the correction of a less fortunate one. Even if you do not have any appreciable[208] new ideas for a while you should be grateful if your conception of the problem becomes more complete or more coherent[209], more homogeneous[210] or better balanced[211]." – Polya, 2014, pp. 33–35

3.21.4 Carrying Out the Plan

"*Where should I start?* Start from the lucky idea that led you to the solution. Start when you feel sure of your grasp[212] of the main connection & you feel confident that you can supply the minor details that may be wanting.

What can I do? Make your grasp quite secure. Carry through in detail all the algebraic or geometric operations which you have recognized previously as feasible[213]. Convince yourself of the correctness of each step by formal reasoning, or by intuitive insight, or both ways if you can. If your problem is very complex you may distinguish "great" steps & "small" steps, each great step being composed of several small ones. Check 1st the great steps, & get down to the smaller ones afterwards.

What can I gain by doing so? A presentation of the solution each step of which is correct beyond doubt." – Polya, 2014, pp. 33–35

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[201] **interpretation** [n] **1.** [countable] the particular way in which something is understood or explained; **2.** [uncountable] **interpretation (of something)** the action of explaining the meaning of something. If something is **open to interpretation**, its meaning is not clear & can be understood in different ways.

[202] **distinctly** [adv] in a way that is clear & easily noticed; showing a clear difference.

[203] **reliable** [a] **1.** likely to be correct or true, OPPOSITE: **unreliable**; **2.** that can be trusted to do something well; that can be relied on, OPPOSITE: **unreliable**.

[204] **ascertain** [v] (*formal*) to find out the true or correct information about something.

[205] **astray** [adv] **1. go astray** [idiom] to become lost; to be stolen; **2. lead somebody astray** [idiom] to make somebody go in the wrong direction or do things that are wrong.

[206] **hazy** [a] (**hazier, haziest**) **1.** not clear because of haze; **2.** not clear because of a lack of memory, understanding or detail, SYNONYM: **vague**; **3.** (of a person) uncertain or confused about something.

[207] **precision** [n] [uncountable] the quality of being exact & accurate, SYNONYM: **accuracy**.

[208] **appreciable** [a] large or important enough to be noticed, SYNONYM: **considerable**.

[209] **coherent** [a] **1.** (of an argument, theory, statement or policy) logical & well organized; easy to understand & clear, OPPOSITE: **incoherent**; **2.** (of a person) able to talk & express yourself clearly; showing this, OPPOSITE: **incoherent**; **3.** made up of different parts tht fit or work well together; **4.** (*physics*) (of waves) in phase with each other, OPPOSITE: **incoherent**.

[210] **homogeneous** [a] **1.** (*formal*) consisting of things or people that are all the same or all of the same type, OPPOSITE: **heterogeneous**; **2.** (*chemistry*) used to describe a process involving substances in the same phase (solid, liquid or gas), OPPOSITE: **heterogeneous**.

[211] **balanced** [a] [usually before noun] (*approving*) **1.** having all parts in equal, correct or good amounts; **2.** giving careful thought to all opinions on a particular subject.

[212] **grasp** [v] **1.** to understand something completely; **2. grasp an opportunity** to take an opportunity without hesitating & use it; **3. grasp somebody/something** to take a firm hold of somebody/something, SYNONYM: **grip**; [n] [usually singular] **1.** a person's understanding of a subject; **2.** a firm hold of somebody/something or control over somebody/something; **3.** the ability to get or achieve something.

[213] **feasible** [a] that is possible & likely to be achieved, SYNONYM: **practicable**.

3.21.5 Looking Back

"*Where should I start?* From the solution, complete & correct in each detail.

What can I do? Consider the solution from various sides & seek contacts with your formerly acquired knowledge.

Consider the details of the solution & try to make them as simple as you can; survey more extensive parts of the solution & try to make them shorter; try to see the whole solution at a glance. Try to modify to their advantage smaller or larger parts of the solution, try to improve the whole solution, to make it intuitive, to fit it into your formerly acquired knowledge as naturally as possible. Scrutinize[214] the method that led you to the solution, try to see its point, & try to make use of it for other problems.

What can I gain by doing so? You may find a new & better solution, you may discover new & interesting facts. In any case, if you get into the habit of surveying & scrutinizing your solutions in this way, you will acquire some knowledge well ordered & ready to use, & you will develop your ability of solving problems." – Polya, 2014, p. 35

Part III. Short Dictionary of Heuristic

3.22 Analogy

"**Analogy**[215] is a sort of similarity[216]. Similar objects agree with each other in some respect, analogous objects *agree in certain relations* of their respective parts.

1. A rectangular parallelogram is analogous to a rectangular parallelepiped. In fact, the relations between the sides of the parallelogram are similar to those between the faces of the parallelepiped:

Each side of the parallelogram is parallel to just 1 other side, & is perpendicular to the remaining sides.

Each face of the parallelepiped is parallel to just 1 other face, & is perpendicular to the remaining faces.

Let us agree to call a side a "bounding element" of the parallelogram & a face a "bounding element" of the parallelepiped. Then, we may contract the 2 foregoing statements into one that applies equally to both figures:

Each bounding element is parallel to just 1 other bounding element & is perpendicular to the remaining bounding elements.

Thus, we have expressed certain relations which are common to the 2 systems of objects we compared, sides of the rectangle & faces of the rectangular parallelepiped. The analogy of these systems consists in this community of relations.

2. Analogy pervades[217] all our thinking, our everyday speech & our trivial conclusions as well as artistic[218] ways of expression & the highest scientific achievements. Analogy is used on very different levels. People often use vague, ambiguous[219], incomplete, or incompletely clarified[220] analogies, but analogy may reach the level of mathematical precision. All sorts of analogy may play a role in the discovery of the solution & so we should not neglect any sort.

3. We may consider ourselves lucky when, trying to solve a problem, we succeed in discovering a *simpler analogous problem*. In Sect. **15**, our original problem was concerned with the diagonal of a rectangular parallelepiped; the consideration of a simpler analogous problem, concerned with the diagonal of a rectangle, led us to the solution of the original problem. We are going to discuss 1 more case of the same sort. We have to solve the following problem:

Find the center of gravity of a homogeneous tetrahedron.

Without knowledge of the integral calculus, & with little knowledge of physics, this problem is not easy at all; it was a serious scientific problem in the days of Archimedes or Galileo. Thus, if we wish to solve it with a little preliminary knowledge as possible, we should look around for a simpler analogous problem. The corresponding problem in the plane occurs here naturally:

Find the center of gravity of a homogeneous triangle.

Now, we have 2 questions instead of 1. But 2 questions may be easier to answer than just 1 question – provided that the 2 questions are intelligently connected.

4. Laying aside, for the moment, our original problem concerning the tetrahedron, we concentrate upon the simpler analogous problem concerning the triangle. To solve this problem, we have to know something about centers of gravity. The following principle is plausible[221] & presents itself naturally.

[214]**scrutinize** [v] (BE also **scrutinise**) **scrutinize something** to look at or examine something carefully.

[215]**analogy** [n] (plural **analogies**) [countable, uncountable] a comparison of 1 thing with another thing that has similar features, usually in order to explain it; a feature that is similar.

[216]**similarity** [n] (plural **similarities**) **1.** [uncountable, singular] the state of being like somebody/something but not exactly the same, SYNONYM: **resemblance**, OPPOSITE: **difference, dissimilarity**; **2.** [countable] a feature that things or people have that makes them like each other, OPPOSITE: **difference, dissimilarity**.

[217]**pervade** [v] **pervade something** to spread through & be easy to notice in every part of something, SYNONYM: **permeate**.

[218]**artistic** [a] **1.** connected with art or artists; **2.** showing a natural skill in or enjoyment of art, especially being able to paint or draw well.

[219]**ambiguous** [a] **1.** that can be understood in more than 1 different way, SYNONYM: **equivocal**, OPPOSITE: **unambiguous**; **2.** not clearly stated or defined, SYNONYM: **equivocal**, OPPOSITE: **unambiguous**.

[220]**clarify** [v] to make something clearer or easier to understand.

[221]**plausible** [a] (of an excuse or explanation) reasonable & likely to be true, OPPOSITE: **implausible**.

If a system of masses S consists of parts, each of which has its center of gravity in the same plane, then this plane contains also the center of gravity of the whole system S.

This principle yields all that we need in the case of the triangle. 1st, it implies that the center of gravity of the triangle lies in the plane of the triangle. Then, we may consider the triangle as consisting of fibers (thin strips, "infinitely narrow" parallelograms) parallel to a certain side of the triangle (the side AB in Fig. 7). The center of gravity of each fiber (of any parallelogram) is, obviously, its midpoint, & all these midpoints lie on the line joining the vertex C opposite to the side AB to the midpoint M of AB (see Fig. 7).

Any plane passing through the mediam CM of the triangle contains the centers of gravity of all parallel fibers which constitute the triangle. Thus, we are led to the conclusion that the center of gravity of the whole triangle lies on the same median. Yet it must lie on the other 2 medians just as well, it must be the *common point of intersection of all 3 medians.*

It is desirable to verify now by pure geometry, independently of any mechanical assumption, that the 3 medians meet in the same point.

5. After the case of the triangle, the case of the tetrahedron is fairly easy. We have now solved a problem analogous to our proposed problem &, having solved it, we have a *model to follow*.

In solving the analogous problem which we use now as a model, we conceived ΔABC as consisting of fibers parallel to 1 of its sides, AB. Now, we conceive the tetrahedron $ABCD$ as consisting of fibers parallel to 1 of its edges, AB.

The midpoints of the fibers which constitute the triangle lie all on the same straight line, a median of the triangle, joining the midpoint M of the side AB to the opposite vertex C. The midpoints of the fibers which constitute the tetrahedron lie all in the same plane, joining the midpoint M of the edge AB to the opposite edge CD (see Fig. 8); we may call this plane MCD a *median plane* of the tetrahedron.

In the case of the triangle, we had 3 medians like MC, each of which has to contain the center of gravity of the triangle. Therefore, these 3 medians must meet in 1 point which is precisely the center of gravity. In the case of the tetrahedron we have 6 median planes like MCD, joining the midpoint of some edge to the opposite edge, each of which has to contain the center of gravity of the tetrahedron. Therefore, these 6 median planes must meet in 1 point which is precisely the center of gravity.

6. Thus, we have solved the problem of the center of gravity of the homogeneous tetrahedron. To complete our solution, it is desirable to verify now by pure geometry, independently of mechanical considerations, that the 6 median planes mentioned pass through the same point.

When we had solved the problem of the center of gravity of the homogeneous triangle, we found it desirable to verify, in order to complete our solution, that the 3 medians of the triangle pass through the same point. This problem is analogous to the foregoing but visibly simpler.

Again we may use, in solving the problem concerning the tetrahedron, the simpler analogous problem concerning the triangle (which we may suppose here as solved). In fact, consider the 3 median planes, passing through the 3 edges DA, DB, DC issued from the vertex D; each passes also through the midpoint of the opposite edge (the median plane through DC passes through M, see Fig. 8). Now, these 3 median planes intersect the plane of ΔABC in the 3 medians of this triangle. These 3 medians pass through the same point (this is the result of the simpler analogous problem) & this point, just as D, is a common point of the 3 median planes. The straight line, joining the 2 common points, is common to all 3 median planes.

We proved that those 3 among the 6 median planes which pass through the vertex D have a common straight line. The same must be true of those 3 median planes which pass through A; & also of the 3 median planes through B; & also of the 3 through C. Connecting these facts suitably, we may prove that the 6 median planes have a common point. (The 3 median planes passing through the sides of ΔABC determine a common point, & 3 lines of intersection which meet in the common point. Now, by what we have just proved, through each line of intersection 1 more median plane must pass.)

7. Both under **5** & under **6** we used a simpler analogous problem, concerning the triangle, to solve a problem about the tetrahedron. Yet the 2 cases are different in an important respect. Under **5**, we used the *method* of the simpler analogous problem whose solution we imitated point by point. Under **6**, we used the *result* of the simpler analogous problem, & we did not care how this result had been obtained. Sometimes, we may be able to use *both the method & the result* of the simpler analogous problem. Even our foregoing example shows that if we regard the considerations under **5** & **6** as different parts of the solution of the same problem.

Our example is typical. In solving a proposed problem, we can often use the solution of a simpler analogous problem; we may be able to use its method, or its result, or both. Of course, in more difficult cases, complications may arise which are not yet shown by our example. Especially, it can happen that the solution of the analogous problem cannot be immediately used for our original problem. Then, it may be worth while to reconsider the solution, to vary & to modify it till, after having tried various forms of the solution, we find eventually one that can be extended to our original problem.

8. It is desirable to foresee the result, or, at least, some features of the result, with some degree of plausibility. Such plausible forecasts are often based on analogy.

Thus, we may know that the center of gravity of a homogeneous triangle coincides with the center of gravity of its 3 vertices (i.e., of 3 material points with equal masses, placed in the vertices of the triangle). Knowing this, we may conjecture that the center of gravity of a homogeneous tetrahedron coincides with the center of gravity of its 4 vertices.

This conjecture is an "inference by analogy." Knowing that the triangle & the tetrahedron are alike in many respects, we conjecture that they are alike in 1 more respect. It would be foolish to regard the plausibility[222] of such conjectures as certainty[223], but it would be just as foolish, or even more foolish, to disregard such plausible conjectures.

Inference by analogy appears to be the most common kind of conclusion, & it is possibly the most essential kind. It yields more or less plausible conjectures which may or may not be confirmed by experience & stricter reasoning. The chemist, experimenting on animals in order to foresee the influence of his drugs on humans, draws conclusions by analogy. But so did a small boy I knew. His pet dog had to be taken to the veterinary[224], & he inquired:

"Who is the veterinary?"

"The animal doctor."

"Which animal is the animal doctor?"

9. An analogical conclusion from many parallel cases is stronger than one from fewer cases. Yet quality is still more important here than quantity. Clear-cut analogies weigh more heavily than vague similarities, systematically arranged instances count for more than random collections of cases.

In the foregoing (under **8**) we put forward a conjecture about the center of gravity of the tetrahedron. This conjecture was supported by analogy; the case of the tetrahedron is analogous to that of the triangle. We may strengthen the conjecture by examining 1 more analogous case, the case of a homogeneous rod (i.e., a straight line-segment of uniform density).

The analogy between: segment triangle tetrahedron has many aspects. A segment is contained in a straight line, a triangle in a plane, a tetrahedron in space. Straight line-segments are the simplest 1D bounded figures, triangles the simplest polygons, tetrahedrons the simplest polyhedrons.

The segment has 2 0D bounding elements (2 end-points) & its interior is 1D.

The triangle has 3 0D & 3 1D bounding elements (3 vertices, 3 sides) & its interior is 2D.

The tetrahedron has 4 0D, 6 1D, & 4 2D bounding elements (4 vertices, 6 edges, 4 faces), & its interior is 3D.

These numbers can be assembled into a table. The successive columns contain the numbers for the 0D, 1D, 2D, & 3D elements, the successive rows the numbers for the segment, triangle, & tetrahedron:

2	1		
3	3	1	
4	6	4	1.

Very little familiarity with the powers of a binomial is needed to recognize in these numbers a section of Pascal's triangle. We found a remarkable regularity in segment, triangle, & tetrahedron.

10. If we have experienced that the objects we compare are closely connected, "inferences by analogy," as the following, may have a certain weight with us.

The center of gravity of a homogeneous rod coincides with the center of gravity of its 2 end-points. The center of gravity of a homogeneous triangle coincides with the center of gravity of its 3 vertices. Should we not suspect that the center of gravity of a homogeneous tetrahedron coincides with the center of gravity of its 4 vertices?

Again, the center of gravity of a homogeneous rod divides the distance between its end-points in the proportion 1:1. The center of gravity of a triangle divides the distance between any vertex & the midpoint of the opposite side in the proportion 2:1. Should we not suspect that the center of gravity of a homogeneous tetrahedron divides the distance between any vertex & the center of gravity of the opposite face in the proportion 3:1?

It appears extremely unlikely that the conjectures suggested by these questions should be wrong, that such a beautiful regularity should be spoiled. The feeling that harmonious[225] simple order cannot be deceitful[226] guides the discoverer both in the mathematical & in the other sciences, & is expressed by the Latin saying: $\boxed{\textit{simplex sigillum veri}}$ (simplicity is the seal[227] of truth).

[The preceding suggests an extension to n dimensions. It appears unlikely that what is true in the 1st 3 dimensions, for $n = 1, 2, 3$, should cease to be true for higher values of n. This conjecture is an "inference by induction"; it illustrates that induction is naturally based on analogy. See INDUCTION & MATHEMATICAL INDUCTION.]

[**11.** We finish the present section by considering briefly the most important cases in which analogy attains the precision of mathematical ideas.

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[222]**plausibility** [n] [uncountable] **1.** the quality of being reasonable & likely to be true; **2.** (*disapproving*) the fact of being good at sounding honest & sincere, especially when trying to trick people.

[223]**certainty** [n] (plural **certainties**) **1.** [uncountable] the strong belief that something is true; **2.** [countable] something that you know is completely true or reliable; an event that is definitely going to happen; **3.** [uncountable] the quality of being definitely true or reliable.

[224]**veterinary** [a] [only before noun] connected with caring for the health of animals.

[225]**harmonious** [a] (of relationships between people) friendly, peaceful & without any disagreement.

[226]**deceitful** [a] (*formal*) behaving in a dishonest way by telling lies & making people believe things that are not true, SYNONYM: **dishonest**.

[227]**seal** [n] **1.** [singular] a thing that makes something definite; **2.** [countable] a device or substance used to join 2 things together or fill a crack so that air or liquid cannot get in or out; **3.** [countable] a sea animal that eats fish & lives around coasts. There are many types of seal, some of which are hunted for their fur; **set the seal on something** [idiom] (*formal*) to make something definite or complete; [v] **1.** [often passive] **seal something (with something)** to close a container tightly or fill a crack, especially so that air or liquid cannot get in or out; **2. seal something** to close an envelope, etc. by sticking the edges of the opening together; **3. seal something** to make something .

(I) 2 systems of mathematical objects, say S & S', are so connected that certain relations between the objects of S are governed by the same laws as those between the objects of S'.

This kind of analogy between S & S' is exemplified[228] by what we have discussed under 1; take as S the sides of a rectangle, as S' the faces of a rectangular parallelepiped.

(II) There is a 1-1 correspondence between the objects of the 2 systems S & S', preserving certain relations. I.e., if such a relation holds between the objects of 1 system, the same relation holds between the corresponding objects of the other system. Such a connection between 2 systems is a very precise sort of analogy; it is called *isomorphism* (or *holohedral isomorphism*).

(III) There is a 1-many correspondence between the objects of the 2 systems S & S' preserving certain relations. Such a connection (which is important in various branches of advanced mathematical study, especially in the Theory of Groups, & need not be discussed here in detail) is called *merohedral isomorphism* (or *homomorphism*; *homoiomorphism* would be, perhaps, a better term). Merohedral isomorphism may be considered as another very precise sort of analogy.]" – Polya, 2014, pp. 37–46

3.23 Auxiliary elements

"There is much more in our conception of the problem at the end of our work than was in it as we started working (PROGRESS & ACHIEVEMENT, 1). As our work progresses, we add new elements to those originally considered. An element that we introduce in the hope that it will further the solution is called an *auxiliary element*.

1. There are various kinds of auxiliary elements. Solving a geometric problem, we may introduce new lines into our figure, *auxiliary lines*. Solving an algebraic problem, we may introduce an *auxiliary unknown* (AUXILIARY PROBLEMS, 1). An *auxiliary theorem* is a theorem whose proof we undertake in the hope of promoting the solution of our original problem.

2. There are various reasons for introducing auxiliary elements. We are glad when we have succeeded in recollecting a *problem related to ours & solved before*. It is probable[229] that we can use such a problem but we do not know yet how to use it. E.g., the problem which we are trying to solve is a geometric problem, & the related problem which we have solved before & have now succeeded in recollecting is a problem about triangles. Yet there is no triangle in our figure; in order to make any use of the problem recollected we must have a triangle; therefore, we have to introduce one, by adding suitable auxiliary lines to our figure. In general, having recollected a formerly solved related problem & wishing to use it for our present one, we must often ask: *Should we introduce some auxiliary element in order to make its use possible?* (The example in Sect. **10** is typical.)

Going back to definitions, we have another opportunity to introduce auxiliary elements. E.g., explicating[230] the definition of a circle we should not only mention its center & its radius, but we should also introduce these geometric elements into our figure. Without introducing them, we could not make any concrete use of the definition; stating the definition without drawing something is mere lip-service.

Trying to use known results & going back to definitions are among the best reasons for introducing auxiliary elements; but they are not the only ones. We may add auxiliary elements to the conception of our problem in order to make it fuller, more suggestive, more familiar although we scarcely know yet explicitly how we shall be able to use the elements added. We may just feel that it is a "bright idea" to conceive the problem that way with such & such elements added.

We may have this or that reason for introducing an auxiliary element, but we should have some reason. We should not introduce auxiliary elements wantonly[231].

3. *Example.* Construct a triangle, being given 1 angle, the altitude[232] drawn from the vertex of the given angle, & the perimeter of the triangle.

We *introduce suitable notation.* Let α denote the given angle, h the given altitude drawn from the vertex A of α & p the given perimeter. We *draw a figure* in which we easily place α & h. *Have we used all the data?* No, our figure does not contain the given length p, equal to the perimeter of the triangle. Therefore we must introduce p. But how?

We may attempt to introduce p in various ways. The attempts exhibited in Figs. 9–10 appear clumsy[233]. If we try to make clear to ourselves why they appear so unsatisfactory, we may perceive that it is for lack of symmetry.

In fact, the triangle has 3 unknown sides a, b, c. We call a, as usual, the side opposite to A; we know that $a + b + c = p$. Now, the sides b & c play the same role; they are interchangeable; our problem is symmetric w.r.t. b & c. But b & c do

[228]**exemplify** [v] **1.** [usually passive] **be exemplified by/in something** to be a typical example of something; **2. exemplify something** to give an example in order to make something clearer, SYNONYM: **illustrative**.

[229]**probable** [a] likely to happen, exist or be true, OPPOSITE: **improbable**.

[230]**explicate** [v] (*formal*) **explicate something** to explain an idea or a work of literature in a lot of detail.

[231]**wantonly** [adv] **1.** (*formal*) in a way that causes harm or damage deliberately & for no acceptable reason; **2.** (*old-fashioned, disapproving*) in a way that is sexually immoral.

[232]**altitude** [n] **1.** [countable, usually singular] the height above sea level; **2.** [countable, usually plural, uncountable] a place that is high above sea level.

[233]**clumsy** [a] (**clumsier, clumsiest**) **1.** (of people & animals) moving or doing things in a way that is not smooth or steady or careful; **2.** (of actions & statements) done without skill or in a way that offends people; **3.** (of objects) difficult to move or use easily; not well designed; **4.** (of processes) too complicated to understand or use easily.

not play the same role in our Figs. 9–10; placing the length p we treated b & c differently; the Figs. 9–10 spoil the natural symmetry of the problem w.r.t. b & c. We should place p so that it has the same relation to b as to c.

This consideration may be helpful in suggesting to place the length p as in Fig. 11. We add to the side a of the triangle the segment CE of length b on 1 side & the segment BD of the length c on the other side so that p appears in Fig. 11 as the line ED of length $b+a+c=p$. If we have some little experience in solving problems of construction, we shall not fail to introduce into the figure, along with ED, the auxiliary lines AD & AE, each of which is the base of an isosceles triangle. In fact, it is not unreasonable to introduce elements into the problem which are particularly simple & familiar, as isosceles triangle.

We have been quite lucky in introducing our auxiliary lines. Examining the new figure we may discover that $\angle EAD$ has a simple relation to the given angle α. In fact, we find using the isosceles triangles ΔABD & ΔACE that $\angle DAE = \frac{\alpha}{2} + 90^\circ$. After this remark, it is natural to try the construction of ΔDAE. Trying this construction, we introduce an auxiliary problem which is much easier than the original problem.

4. Teachers & authors of textbooks should not forget that the intelligent student & THE INTELLIGENT READER are not satisfied by verifying that the steps of a reasoning are correct but also want to know the motive & the purpose of the various steps. The introduction of an auxiliary element is a conspicuous[234] step. If a tricky auxiliary line appears abruptly[235] in the figure, without any motivation, & solves the problem surprisingly, intelligent students & readers are disappointed; they feel that they are cheated. Mathematics is interesting in so far as it occupies our reasoning & invention if the motive & purpose of the most conspicuous step remain incomprehensible[236]. To make such steps comprehensible[237] by suitable remarks (as in the foregoing, under **3**) or by carefully chosen questions & suggestions (as in Sects. **10, 18**–**20**) takes a lot of time & effort; but it may be worth while." – Polya, 2014, pp. 46–50

3.24 Auxiliary problem

"*Auxiliary problem* is a problem which we consider, not for its own sake, but because we hope that its consideration may help us to solve another problem, our original problem. The original problem is the end we wish to attain, the auxiliary problem a means by which we try to attain our end.

An insect tries to escape through the windowpane[238], tries the same again & again, & does not try the next window which is open & through which it came into the room. A man is able, or at least should be able, to act more intelligently. Human superiority consists in going around an obstacle that cannot be overcome directly, in devising a suitable auxiliary problem when the original problem appears insoluble[239]. To devise[240] an auxiliary problem is an important operation of the mind. To raise a clear-cut[241] new problem subservient[242] to another problem, to conceive[243] distinctly[244] as an end what is means to another end, is a refined achievement of the intelligence. It is an important task to learn (or to teach) how to handle auxiliary problems intelligently.

1. *Example.* Find x, satisfying the equation $x^4 - 13x^2 + 36 = 0$. If we observe that $x^4 = (x^2)^2$ we may see the advantage of introducing $y = x^2$. We obtain now a new problem: Find y, satisfying the equation $y^2 - 13y + 36 = 0$. The new problem is an auxiliary problem; we intend to use it as a means of solving our original problem. The unknown of our auxiliary problem, y, is appropriately called *auxiliary unknown.*

2. *Example.* Find the diagonal of a rectangular parallelepiped being given the lengths of 3 edges drawn from the same corner.

Trying to solve this problem (Sect. **8**) we may be led, by analogy (Sect. **15**), to another problem: Find the diagonal of a rectangular parallelogram being given the lengths of 2 sides drawn from the same vertex.

The new problem is an auxiliary problem; we consider it because we hope to derive some profit for the original problem from its consideration.

[234] **conspicuous** [a] easy to see or notice; likely to attract attention; **conspicuous by your absence** [idiom] not present in a situation or place, when it is obvious that you should be there.

[235] **abrupt** [a] **1.** sudden & unexpected, often in an unpleasant way; **2.** steep; not gradual or gentle; **3.** speaking or acting in a way that seems unfriendly & rude; not taking time to say more than is necessary.

[236] **incomprehensible** [a] **incomprehensible (to somebody)** impossible to understand, OPPOSITE: **comprehensible**.

[237] **comprehensible** [a] that can be understood by somebody, OPPOSITE: **incomprehensible**.

[238] **windowpane** [n] a piece of glass in a window.

[239] **insoluble** [a] **1. insoluble (in something)** (of a substance) that does not dissolve in a particular liquid, OPPOSITE: **soluble**; **2.** (of a problem or mystery) that cannot be solved or explained, OPPOSITE: **soluble**.

[240] **devise** [v] **devise something** to plan or invent a procedure, system or method, especially one that is new or complicated, by using careful thought, SYNONYM: **think something up**.

[241] **clear-cut** [a] definite & easy to see or identify.

[242] **subservient** [a] **1. subservient (to somebody/something)** (*disapproving*) too willing to obey other people; **2. subservient (to something)** (*formal*) less important than something else.

[243] **conceive** [v] **1.** [transitive] to form an idea or plan in your mind; **2.** [transitive, intransitive] to think of something in a particular way; to imagine something; **3.** [intransitive, transitive] (of a woman) to become pregnant.

[244] **distinctly** [adv] in a way that is clear & easily noticed; showing a clear difference.

3. *Profit.* The profit that we derive from the consideration of an auxiliary problem may be of various kinds. We may use the *result* of the auxiliary problem. Thus, in example 1, having found by solving the quadratic equation for y that y is equal to 4 or to 9, we infer that $x^2 = 4$ or $x^2 = 9$ & derive hence all possible values of x. In other cases, we may use the *method* of the auxiliary problem. Thus, in example 2, the auxiliary problem is a problem of plane geometry; it is analogous to, but simpler than, the original problem which is a problem of solid geometry. It is reasonable to introduce an auxiliary problem of this kind in the hope that it will be instructive[245], that it will give us opportunity to familiarize ourselves with certain methods, operations, or tools, which we may use afterwards for our original problem. In example 2, the choice of the auxiliary problem is rather lucky; examining it closely we find that we can use both its method & its result. (See Sect. **15**, & DID YOU USE ALL THE DATA?)

4. *Risk.* We take away from the original problem the time & the effort that we devote to the auxiliary problem. If our investigation of the auxiliary problem fails, the time & effort we devoted to it may be lost. Therefore, we should exercise our judgment in choosing an auxiliary problem. We may have various good reasons for our choice. The auxiliary problem may appear more accessible than the original problem; or it may appear instructive; or it may have some sort of aesthetic[246] appeal[247]. Sometimes the only advantage of the auxiliary problem is that it is new & offers unexplored[248] possibilities; we choose it because we are tired of the original problem all approaches to which seem to be exhausted.

5. *How to find one.* The discovery of the solution of the proposed problem often depends on the discovery of a suitable auxiliary problem. Unhappily[249] there is no infallible[250] method of discovering the solution. There are, however, questions & suggestions which are frequently helpful, as LOOK AT THE UNKNOWN. We are often led to useful auxiliary problems by VARIATION OF THE PROBLEM.

6. *Equivalent problems.* 2 problems are *equivalent* if the solution of each involves the solution of the other. Thus, in our example 1, the original problem & the auxiliary problem are equivalent.

Consider the following theorems: A. In any equilateral triangle, each angle is equal to 60°. B. In any equiangular triangle, each angle is equal to 60°.

These 2 theorems are not identical. They contain different notions; one is concerned with equality of the sides, the other with equality of the angles of a triangle. But each theorem follows from the other. Therefore, the problem to prove A is equivalent to the problem to prove B.

If we are required to prove A, there is a certain advantage in introducing, as an auxiliary problem, the problem to prove B. The theorem B is a little easier to prove than A &, what is more important, we may *foresee* that B is easier than A, we may judge so, we may find plausible from the outset that B is easier than A. In fact, the theorem B, concerned only with angles, is more "homogeneous" than the theorem A which is concerned with both angles & sides.

The passage from the original problem to the auxiliary problem is called *convertible*[251] reduction, or *bilateral*[252] reduction, or *equivalent* reduction if these 2 problems, the original & the auxiliary, are equivalent. Thus, the reduction of A to B (see above) is convertible & so is the reduction in example 1. Convertible reductions are, in a certain respect, more important & more desirable than other ways to introduce auxiliary problems, but auxiliary problems which are not equivalent to the original problem may also be very useful; take example 2.

7. *Chains of equivalent auxiliary problems* are frequent in mathematical reasoning. We are required to solve a problem A; we cannot see the solution, but we may find that A is equivalent to another problem B. Considering B we may run into a 3rd problem C equivalent to B. Proceeding in the same way, we reduce C to D, & so on, until we come upon a last problem L whose solution is known or immediate. Each problem being equivalent to the preceding, the last problem L must be equivalent to our original problem A. Thus we are able to infer the solution of the original problem A from the problem L which we attained as the last link in a chain of auxiliary problems.

Chains of problems of this kind were noticed by the Greek mathematicians as we may see from an important passage of PAPPUS. For an illustration, let us reconsider our example 1. Let us call (A) the condition imposed upon the unknown x: (A) $x^4 - 13x^2 + 36 = 0$. 1 way of solving the problem is to transform the proposed condition into another condition which we shall

[245]**instructive** [a] giving a lot of useful information.

[246]**aesthetic** [a] NAE also **esthetic** **1.** concerned with beauty & art & the understanding of beautiful things; **2.** beautiful to look at; [n] **1.** [countable] **aesthetic (of something)** a set of principles that express the aesthetic qualities & ideas of a particular artist or a particular group of artists, writers, etc.; **2.** (**aesthetics**) [uncountable] the branch of philosophy that studies the principles of beauty, especially in art.

[247]**appeal** [n] **1.** [countable, uncountable] a formal request to a court or to somebody in authority for a judgment or a decision to be changed; **2.** [uncountable] a quality that makes somebody/something attractive or interesting; **3.** [countable] **appeal (for something)** an urgent request for money, help or information; [v] **1.** [intransitive] to make a formal request to a court or to somebody in authority for a judgment or a decision to be changed. In North American English, the form **appeal (something) (to somebody/something)** is usually used, without a preposition.; **2.** [intransitive] **appeal to somebody** to attract or interest somebody; **3.** [intransitive] to make a serious & urgent request; **4.** [intransitive] **appeal to something** to try to persuade somebody to do something by suggesting that it is a fair, reasonable or honest thing to do.

[248]**unexplored** [a] **1.** (of a country or an area of land) that no one has investigated or put on a map; that has not been explored; **2.** (of an area of activity or thought) that has not yet been examined or discussed thoroughly.

[249]**unhappily** [adv] **1.** in an unhappy way; **2.** used to say that a particular situation or fact makes you sad or disappointed, SYNONYM: **unfortunately**, OPPOSITE: **happily**.

[250]**infallible** [a] **1.** never wrong; never making mistakes, OPPOSITE: **fallible**; **2.** that never fails; always doing what is supposed to do.

[251]**convertible** [a] **1.** that can be changed to a different form, use or character; **2.** (of a currency) that can be freely changed into another currency or gold.

[252]**bilateral** [a] **1.** involving 2 groups of people or 2 countries; **2.** (*medical*) involving both of 2 parts or sides of the body or brain.

call (B): $(2x^2)^2 - 2(2x^2)13 + 144 = 0$. Observe that the conditions (A) & (B) are different. They are only slightly different if you wish to say so, they are certainly equivalent as you may easily convince yourself, but they are definitely not identical. The passage from (A) to (B) is not only correct but has a clear-cut purpose, obvious to anybody who is familiar with the solution of quadratic equations. Working further in the same direction we transform the condition (B) into still another condition (C): $(2x^2)^2-2(2x^2)13+169 = 25$. Proceeding in the same way, we obtain (D) $(2x^2-13)^2 = 25$, (E) $2x^2-13 = \pm 5$, (F) $x^2 = \frac{13\pm 5}{2}$, (G) $x = \pm\sqrt{\frac{13\pm 5}{2}}$, (H) $x = 3$, or -3, or 2, or -2. Each reduction that we made was convertible. Thus, the last condition (H) is equivalent to the 1st condition (A) so that $\pm 3, \pm 2$ are all possible solutions of our original equation.

In the foregoing, we derived from an original condition (A) a sequence of conditions (B), (C), (D), ... each of which was equivalent to the foregoing. This point deserves the greatest care. Equivalent conditions are satisfied by the same objects. Therefore, if we pass from a proposed condition to a new condition equivalent to it, we have the same solutions. But if we pass from a proposed condition to a narrower one, we lose solutions, & if we pass to a wider one we admit improper, adventitious solutions which have nothing to do with the proposed problem. If, in a series of successive reductions, we pass to a narrower & then again to a wider condition we may lose track of the original problem completely. In order to avoid this danger, we must check carefully the nature of each newly introduced condition: Is it equivalent to the original condition? This question is still more important when we do not deal with a single equation as here but with a system of equations, or when the condition is not expressed by equations as, e.g., in problems of geometric construction.

(Compare PAPPUS, especially comments 2, 3, 4, 8. The description on p. 143, lines 4–21, is unnecessarily restricted; it describes a chain of "problems to find," each of which has a different unknown. The example considered here has just the opposite speciality[253]: all problems of the chain have the same unknown & differ only in the form of the condition. Of course, no such restriction is necessary.)

8. *Unilateral*[254] *reduction.* We have 2 problems, A & B, both unsolved. If we could solve A we could hence derive the full solution of B. But not conversely; if we could solve B, we would obtain, possibly, some information about A, but we would not know how to derive the full solution of A from that of B. In such a case, more is achieved by the solution of A than by the solution of B. Let us call A the *more ambitious*, & B the *less ambitious* of the 2 problems.

If, from a proposed problem, we pass either to a more ambitious or to a less ambitious auxiliary problem we call the step a *unilateral reduction.* There are 2 kinds of unilateral reduction, & both are, in some way or other, more risky than a bilateral or convertible reduction.

Our example 2 shows a unilateral reduction to a less ambitious problem. In fact, if we could solve the original problem, concerned with a parallelepiped whose length, width, & height are a, b, c respectively, we could move on to the auxiliary problem putting $c = 0$ & obtaining a parallelogram with length a & width b. For another example of a unilateral reduction to a less ambitious problem see SPECIALIZATION, 3, 4, 5. These examples show that, with some luck, we may be able to use a less ambitious auxiliary problem as a *stepping stone*, combining the solution of the auxiliary problem with some appropriate supplementary remark to obtain the solution of the original problem.

Unilateral reduction to a more ambitious problem may also be successful. (See GENERALIZATION, 2, & the reduction of the 1st to the 2nd problem considered in INDUCTION & MATHEMATICAL INDUCTION, 1, 2.) In fact, the more ambitious problem may be more accessible; this is the INVENTOR'S PARADOX." – Polya, 2014, pp. 50–57

3.25 Bolzano, Bernard

"BOLZANO, BERNARD (1781–1848), logician & mathematician, devoted an extensive part of his comprehensive presentation of logic, Wissenschaftslehre[255], to the subject of heuristic (vol. 3, pp. 293–575). He writes about this part of his work: "I do not think at all that I am able to present here any procedure of investigation that was not perceived long ago by all men of talent; & I do not promise at all that you can find here anything quite new of this kind. But I shall take pains to state in clear words the rules & ways of investigation which are followed by all able men, who in most cases are not even conscious of following them. Although I am free from the illusion that I shall fully succeed even in doing this, I still hope that the little that is presented here may please some people & have some application afterwards." – Polya, 2014, pp. 57–58

[253] **speciality** [n] (BE) (also **speciality** NAE, BE) (plural **specialities**) **1.** a particular area of medicine that somebody studies & becomes an expert in; **2.** a type of food or product that a restaurant or place is famous for because it is so good; **3.** (in compounds) + **noun** satisfying particular tastes or needs; **4.** an area of work or study that somebody gives most of their attention to & knows a lot about; something that somebody is good at.

[254] **unilateral** [a] **1.** [usually before noun] done by 1 member of a group or organization without the agreement of the other members; **2.** (*medical*) involving only 1 side of an organ or the body.

[255] **wissenschaftslehre**: theory of science.

3.26 Bright idea

"*Bright idea*, or "good idea," or "seeing the light," is a colloquial[256] expression describing a sudden advance toward the solution; see PROGRESS & ACHIEVEMENT, 6. the coming of a bright idea is an experience familiar to everybody but difficult to describe & so it may be interesting to notice that a very suggestive description of it has been incidentally given by an authority as old as Aristotle.

Most people will agree that conceiving a bright idea is an "act of sagacity[257]." Aristotle defines "sagacity" as follows: "Sagacity is a hitting by guess upon the essential connection in an inappreciable[258] time. As e.g., if you see a person talking with a rich man in a certain way, you may instantly guess that that person is trying to borrow money. Or observing that the bright side of the moon is always toward the sun, you may suddenly perceive why this is; namely, because the moon shines by the light of the sun."[259]

The 1st example is not bad but rather trivial; not much sagacity is needed to guess things of this sort about rich men & money, & the idea is not very bright. The 2nd example, however, is quite impressive if we make a little effort of imagination to see it in its proper setting.

We should realize that a contemporary[260] of Aristotle had to watch the sun & the stars if he wished to know the time since there were no wristwatches[261], & had to observe the phases of the moon if he planned traveling by night since there were no street lights. He was much better acquainted[262] with the sky than the modern city-dweller[263], & his natural intelligence was not dimmed[264] by undigested fragments of journalistic presentations of astronomical theories. He saw the full moon as a flat disc, similar to the disc of the sun but much less bright. He must have wondered at the incessant changes in the shape & position of the moon. He observed the moon occasionally also at daytime, about sunrise or sunset, & found out "that the bright side of the moon is always toward the sun" which was in itself a respectable achievement. & now he perceives that the varying aspects of the moon are like the various aspects of a ball which is illuminated from 1 side so that one half of it is shiny & the other half dark. He conceives the sun & the moon not as flat discs but as round bodies, one giving & the other receiving the light. He understands the essential connection, he rearranges his former conceptions instantly, "in an inappreciable time": there is a sudden leap of the imagination, a bright idea, a flash of genius." – Polya, 2014, pp. 58–

[256]**colloquial** [a] (of words & languages) used in conversation but not in formal speech or writing, SYNONYM: **informal**.
[257]**sagacity** [n] [uncountable] (*formal*) good judgment & understanding, SYNONYM: **wisdom**.
[258]**appreciable** [a] large or important enough to be noticed, SYNONYM: **considerable**.
[259]"The text is slightly rearranged. For a more exact translation see William Whewell, *The Philosophy of the Inductive Sciences* (1847), vol. II, p. 131."
[260]**contemporary** [a] **1.** belonging to the present time, SYNONYM: **modern**; **2.** (especially of people & society) belong to the same time as somebody/something else; [n] (plural **contemporaries**) a person or thing living or existing at the same time as somebody/something else, especially somebody who is about the same age as somebody else.
[261]**wristwatch** [n] a watch that you wear on your wrist.
[262]**acquainted** [a] [not before noun] **1. acquainted with something** (*formal*) familiar with something, having read, seen or experienced it; **2.** not close friends with somebody, but having met a few times before.
[263]**dweller** [n] (especially in compounds) a person who lives in the particular place that is mentioned.
[264]**dim** [a] (**dimmer, dimmest**) **1.** not bright; **2.** (of a situation) not giving any reason to have hope; not good; **3.** not remembered or imagined clearly, SYNONYM: **vague**; **4.** (of a room or other space) difficult to see in because there is not much light; **5.** (of an object or shape) difficult to see because there is not much light.

3.27 Can you check the result?

3.28 Can you derive the result differently?

3.29 Can you use the result?

3.30 Carrying out

3.31 Condition

3.32 Contradictory

3.33 Corollary

3.34 Could you derive something useful from the data?

3.35 Could you restate the problem?

3.36 Decomposing & recombining

3.37 Definition

3.38 Descartes

3.39 Determination, hope, success

3.40 Diagnosis

3.41 Did you see all the data?

3.42 Do you know a related problem?

3.43 Draw a figure

3.44 Examine your guess

3.45 Figures

3.46 Generalization

3.47 Have you seen it before?

3.48 Here is a problem related to yours & solved before

3.49 Heuristic

3.50 Heuristic reasoning

3.51 If you cannot solve the proposed problem

3.52 Induction & mathematical induction

3.53 Inventor's paradox

3.89 Problems

3.90 Hints

3.91 Solutions

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