Functional Equation

Nguyễn Quản Bá Hồng*

Ngày 19 tháng 12 năm 2022

Tóm tắt nội dung

Mục lục

*Independent Researcher, Ben Tre City, Vietnam
e-mail: `nguyenquanbahong@gmail.com`; website: `https://nqbh.github.io`.

1 Wikipedia/Functional Equation

"In mathematics, a *functional equation* is, in the broadest meaning, an equation in which 1 or several functions appear as unknowns. So, differential equations & integral equations are functional equations. However, a more restricted meaning is often used, where a *functional equation* is an equation that relates several rules of the same function. E.g., the logarithm functions are essentially characterized by the *logarithmic functional equation* $\log(xy) = \log x + \log y$.

In the domain of the unknown function is supposed to be the natural numbers, the function is generally viewed as a sequence, &, in this case, a functional equation (in the narrower meaning) is called a recurrence relation. Thus the term *functional equation* is used mainly for real functions & complex functions. Moreover a smoothness condition is often assumed for the solutions, since without such a condition, most functional equations have very irregular solutions. E.g., the gamma function is a function that satisfies the functional equation $f(x+1) = xf(x)$ & the initial value $f(1) = 1$. There are many functions that satisfy these conditions, but the gamma function is the unique one that is meromorphic in the whole complex plane, & logarithmically convex for x real & positive (Bohr–Mollerup theorem)." – Wikipedia/functional equation

1.1 Examples

• "Recurrence relations can be seen as functional equations in functions over the integers or natural numbers, in which the differences between terms' indexes can be seen as an application of the shift operator. E.g., the recurrence relation defining the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ & $F_1 = 1$. • $f(x+P) = f(x)$, which characterizes the periodic functions. • $f(x) = f(-x)$, which characterizes the even functions, & likewise $f(x) = -f(-x)$, which characterizes the odd functions. • $f(f(x)) = g(x)$, which characterizes the functional square root of the function g. • $f(x+y) = f(x)+f(y)$ (Cauchy's functional equation), satisfied by linear maps. The equation may, contigent on the axiom of choice, also have other pathological nonlinear solutions, whose existence can be proven with a Hamel basis for the real numbers. • $f(x+y) = f(x) + f(y)$, satisfied by all exponential functions. Like Cauchy's additive functional equation, this too may have pathological, discontinuous solutions. ..." – Wikipedia/functional equation/example

1.2 Solution

"1 method of solving elementary functional equations is substitution. Some solutions to functional equations have exploited surjectivity, injectivity, oddness, & evenness.

Some functional equations have been solved with the use of ansatzes, mathematical induction.

Some classes of functional equations can be solved by computer-assisted techniques.

In dynamic programming a variety of successive approximation methods are used to solve Bellman's functional equation, including methods based on fixed point iterations." – Wikipedia/functional equation/solution

1.3 Pythagoras Equation

Problem 1.1. *Solve the equation $x^2 + y^2 = z^2$ for $x, y, z \in \mathbf{Z}$.*

The general solution is given by $x = k(m^2 - n^2)$, $y = 2kmn$, $z = k(m^2 + n^2)$, where $k, m, n \in \mathbb{N}$.

Definition 1.1 (Functional equation)**.** *"An equation in which unknowns are functions is called a* functional equation.

We are asked to find all functions satisfying some given relation(s)." – Venkatachala, 2013, p. 2

2 Function Equation on $\mathbb{N}$

3 Function Equation on $\mathbb{R}$

Problem 3.1 (Venkatachala, 2013, pp. 2–3)**.** *Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = -f(x)$ & $f(xy) = x^2 f(y)$, $\forall x, y \in \mathbb{R}$.*

Solution. We have $-f(xy) = f(-xy) = f((-x)y) = (-x)^2 f(y) = x^2 f(y) = f(xy) \Rightarrow f(xy) = 0$, $\forall x, y \in \mathbb{R}$. Taking $y = 1$, it implies $f(x) = 0$, $\forall x \in \mathbb{R}$. Thus the set of equations given has only 1 solution: $f(x) = 0$, $\forall x \in \mathbb{R}$. □

Tài liệu

Venkatachala, B. J. (2013). *Functional Equations: A Problem Solving Approach*. 2nd. Prism Books, pp. iii+265.