

Inequality

Nguyễn Quân Bá Hồng¹

August 25, 2022

¹Independent Researcher, Ben Tre City, Vietnam
e-mail: nguyenquanbahong@gmail.com; website: <https://nqbh.github.io>.

Contents

1	Wikipedia's	5
1.1	Wikipedia/Inequality (Mathematics)	5
2	Hardy, Littlewood, and Pólya, 1952. Inequality	6
2.1	Introduction	8
2.1.1	Finite, infinite, & integral inequalities	8
2.1.2	Notations	8
2.1.3	Positive inequalities	8
2.1.4	Homogeneous inequalities	8
2.1.5	The axiomatic basis of algebraic inequalities	8
2.1.6	Comparable functions	8
2.1.7	Selection of proofs	8
2.1.8	Selection of subjects	8
2.2	Elementary Mean Values	8
2.2.1	Ordinary means	8
2.2.2	Weighted means	8
2.2.3	Limiting cases of $\mathfrak{M}_r(a)$	8
2.2.4	Cauchy's inequality	8
2.2.5	The theorem of the arithmetic & geometric means	8
2.2.6	Other proofs of the theorem of the means	8
2.2.7	Hölder's inequality & its extensions	8
2.2.8	General properties of the means $\mathfrak{M}_r(a)$	8
2.2.9	The sums $\mathfrak{S}_r(a)$	8
2.2.10	Minkowski's inequality	8
2.2.11	A companion to Minkowski's inequality	8
2.2.12	Illustrations & applications of the fundamental inequalities	8
2.2.13	Inductive proofs of the fundamental inequalities	8
2.2.14	Elementary inequalities connected with Theorem 37	8
2.2.15	Elementary proof of Theorem 3	8
2.2.16	Tchebychef's inequality	8
2.2.17	Muirhead's theorem	8
2.2.18	Proof of Muirhead's theorem	8
2.2.19	An alternative theorem	8
2.2.20	Further theorems on symmetrical means	8
2.2.21	The elementary symmetric functions of n positive numbers	8
2.2.22	A note on definite forms	8
2.2.23	A theorem concerning strictly positive forms	8
2.2.24	Miscellaneous theorems & examples	8
2.3	Mean Values with an Arbitrary Function & the Theory of Convex Functions	8
2.3.1	Definitions	8
2.3.2	Equivalent means	8
2.3.3	A characteristic property of the means \mathfrak{M}_r	8
2.3.4	Comparability	8
2.3.5	Convex functions	8
2.3.6	Continuous convex functions	8
2.3.7	An alternative definition	8
2.3.8	Equality in the fundamental inequalities	8
2.3.9	Restatements & extensions of Theorem 85	8

2.3.10	Twice differentiable convex functions	8
2.3.11	Applications of the properties of twice differentiable convex functions	8
2.3.12	Convex functions of several variables	8
2.3.13	Generalizations of Hölder's inequality	8
2.3.14	Some theorems concerning monotonic functions	8
2.3.15	Sums with an arbitrary function: generalizations of Jensen's inequality	8
2.3.16	Generalizations of Minkowski's inequality	8
2.3.17	Comparison of sets	8
2.3.18	Further general properties of convex functions	8
2.3.19	Further properties of continuous convex functions	8
2.3.20	Discontinuous convex functions	8
2.3.21	Miscellaneous theorems & examples	8
2.4	Various Applications of the Calculus	8
2.4.1	Introduction	8
2.4.2	Applications of the mean value theorem	8
2.4.3	Further applications of elementary differential calculus	8
2.4.4	Maxima & minima of functions of 1 variable	8
2.4.5	Use of Taylor's series	8
2.4.6	Applications of the theory of maxima & minima of functions of several variables	8
2.4.7	Comparison of series & integrals	8
2.4.8	An inequality of W. H. Young	8
2.5	Infinite Series	8
2.5.1	Introduction	8
2.5.2	The means \mathfrak{M}_r	8
2.5.3	The generalization of Theorems 3 & 9	8
2.5.4	Hölder's inequality & its extensions	8
2.5.5	The means \mathfrak{M}_r (<i>cont.</i>)	8
2.5.6	The sums \mathfrak{S}_r	8
2.5.7	Minkowski's inequality	8
2.5.8	Tchebychef's inequality	8
2.5.9	A summary	8
2.5.10	Miscellaneous theorems & examples	8
2.6	Integrals	8
2.6.1	Preliminary remarks on Lebesgue integrals	8
2.6.2	Remarks on null sets & null functions	8
2.6.3	Further remarks concerning integration	8
2.6.4	Remarks on methods of proof	8
2.6.5	Further remarks on method: the inequality of Schwarz	8
2.6.6	Definition of the means $\mathfrak{M}_r(f)$ when $r \neq 0$	8
2.6.7	The geometric mean of a function	8
2.6.8	Further properties of the geometric mean	8
2.6.9	Hölder's inequality for integrals	8
2.6.10	General properties of the means $\mathfrak{M}_r(f)$	8
2.6.11	Convexity of $\log \mathfrak{M}_r$	8
2.6.12	Minkowski's inequality for integrals	8
2.6.13	Mean values depending on an arbitrary function	8
2.6.14	The definition of the Stieltjes integral	8
2.6.15	Special cases of the Stieltjes integral	8
2.6.16	Extensions of earlier theorems	8
2.6.17	The means $\mathfrak{M}_r(f; \phi)$	8
2.6.18	Distribution functions	8
2.6.19	Characterization of means values	8
2.6.20	Remarks on the characteristic properties	8
2.6.21	Completion of the proof of Theorem 215	8
2.6.22	Miscellaneous theorems & examples	8
2.7	Some Applications of the Calculus of Variations	8
2.7.1	Some general remarks	8
2.7.2	Object of the present chapter	8
2.7.3	Example of an inequality corresponding to an unattained extremum	8

2.7.4	1st proof of Theorem 254	8
2.7.5	2nd proof of Theorem 254	8
2.7.6	Further examples illustrative of variational methods	8
2.7.7	Further examples: Wirtinger's inequality	8
2.7.8	An example involving 2nd derivatives	8
2.7.9	A simpler theorem	8
2.7.10	Miscellaneous theorems & examples	8
2.8	Some Theorems Concerning Bilinear & Multilinear Forms	8
2.8.1	Introduction	8
2.8.2	An inequality for multilinear forms with positive variables & coefficients	8
2.8.3	A theorem of W. H. Young	8
2.8.4	Generalizations & analogues	8
2.8.5	Applications to Fourier series	8
2.8.6	The convexity theorem for positive multilinear forms	8
2.8.7	The convexity theorem for positive multilinear forms	8
2.8.8	General bilinear forms	8
2.8.9	Definition of a bounded bilinear form	8
2.8.10	Some properties of bounded forms in $[p, q]$	8
2.8.11	The Faltung of 2 forms in $[p, p']$	8
2.8.12	Some special theorems on forms in $[2, 2]$	8
2.8.13	Application to Hilbert's forms	8
2.8.14	The convexity theorem for bilinear forms with complex variables & coefficients	8
2.8.15	Further properties of a maximal set (x, y)	8
2.8.16	Proof of Theorem 295	8
2.8.17	Applications of the theorem of M. Riesz	8
2.8.18	Applications to Fourier series	8
2.8.19	Miscellaneous theorems & examples	8
2.9	Hilbert's Inequality & Its Analogues & Extensions	8
2.9.1	Hilbert's double series theorem	8
2.9.2	A general class of bilinear forms	8
2.9.3	The corresponding theorem for integrals	8
2.9.4	Extensions of Theorems 318 & 319	8
2.9.5	Best possible constants: proof of Theorem 317	8
2.9.6	Further remarks on Hilbert's theorems	8
2.9.7	Applications of Hilbert's theorems	8
2.9.8	Hardy's inequality	8
2.9.9	Further integral inequalities	8
2.9.10	Further theorems concerning series	8
2.9.11	Deduction of theorems on series from theorems on integrals	8
2.9.12	Carleman's inequality	8
2.9.13	Theorems with $0 < p < 1$	8
2.9.14	A theorem with 2 parameters p & q	8
2.9.15	Miscellaneous theorems & examples	8
2.10	Rearrangements	8
2.10.1	Rearrangements of finite sets of variables	8
2.10.2	A theorem concerning the rearrangements of 2 sets	8
2.10.3	A 2nd proof of Theorem 368	8
2.10.4	Restatement of Theorem 368	8
2.10.5	Theorems concerning the rearrangements of 3 sets	8
2.10.6	Reduction of Theorem 373 to a special case	8
2.10.7	Completion of the proof	8
2.10.8	Another proof of Theorem 371	8
2.10.9	Rearrangements of any number of sets	8
2.10.10	A further theorem on the rearrangement of any number of sets	8
2.10.11	Applications	8
2.10.12	The rearrangement of a function	8
2.10.13	On the rearrangement of 2 functions	8
2.10.14	On the rearrangement of 3 functions	8
2.10.15	Completion of the proof of Theorem 379	8

2.10.16 An alternative proof	8
2.10.17 Applications	8
2.10.18 Another theorem concerning the rearrangement of a function in decreasing order	8
2.10.19 Proof of Theorem 384	8
2.10.20 Miscellaneous theorems & examples	8
2.11 Appendices	8
2.11.1 Appendix I: On strictly positive forms	8
2.11.2 Appendix II: Thorin's proof & extension of Theorem 295	8
2.11.3 Appendix III: On Hilbert's inequality	8
Bibliography	9

Chapter 1

Wikipedia's

1.1 **Wikipedia/Inequality (Mathematics)**

Chapter 2

Hardy, Littlewood, and Pólya, 1952. Inequality

“Oh! the little more, & how much it is! & the little less, & what worlds away!” – Robert Browning

Preface to 1st Edition

“This book was planned & begun in 1929. Our original intention was that it should be 1 of the *Cambridge Tracts*, but it soon became plain that a tract¹ would be much too short for our purpose.

Our subjects in writing the book are explained sufficiently in the introductory chapter, but we add a note here about history & bibliography². Historical & bibliographical questions are particularly troublesome³ in a subject like this, which has applications in every part of mathematics but has never been developed systematically.

It is often really difficult to trace the origin of a familiar inequality. It is quite likely to occur 1st as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; & no accessible statement of it may be quite complete. We have almost always found, even with the most famous inequalities, that we have a little new to add.

We have done our best to be accurate & have given all references we can, but we have never undertaken⁴ systematic bibliographical research. We follow the common practice, when a particular inequality is habitually⁵ associated with a particular mathematician’s name; we speak of the inequalities of Schwarz, Hölder, & Jensen, though all these inequalities can be traced further back; & we do not enumerate⁶ explicitly all the minor additions which are necessary for absolute completeness.

¹**tract** [n] **1.** (*biology*) a system of connected organs or tissues along which materials or messages pass; **2. tract (of something)** an area of land, especially a large one; **3.** a short piece of writing, especially on a religious, moral or political subject, that is intended to influence people’s ideas.

²**bibliography** [n] (plural **bibliographies**) the list of books, etc. that have been used by somebody writing an article, essay, etc.; a list of books or articles about a particular subject or by a particular author.

³**troublesome** [a] causing trouble, pain or difficulties.

⁴**undertake** [v] **1. undertake something** to make yourself responsible for something & start doing it; **2.** to agree or promise that you will do something.

⁵**habitual** [a] [only before noun] usual or typical of somebody/something.

⁶**enumerate** [v] (*formal*) **enumerate something** to name things on a list 1 by 1.

2.1 Introduction

- 2.1.1 Finite, infinite, & integral inequalities
- 2.1.2 Notations
- 2.1.3 Positive inequalities
- 2.1.4 Homogeneous inequalities
- 2.1.5 The axiomatic basis of algebraic inequalities
- 2.1.6 Comparable functions
- 2.1.7 Selection of proofs
- 2.1.8 Selection of subjects

2.2 Elementary Mean Values

- 2.2.1 Ordinary means
- 2.2.2 Weighted means
- 2.2.3 Limiting cases of $\mathfrak{M}_r(a)$
- 2.2.4 Cauchy's inequality
- 2.2.5 The theorem of the arithmetic & geometric means
- 2.2.6 Other proofs of the theorem of the means
- 2.2.7 Hölder's inequality & its extensions
- 2.2.8 General properties of the means $\mathfrak{M}_r(a)$
- 2.2.9 The sums $\mathfrak{G}_r(a)$
- 2.2.10 Minkowski's inequality
- 2.2.11 A companion to Minkowski's inequality
- 2.2.12 Illustrations & applications of the fundamental inequalities
- 2.2.13 Inductive proofs of the fundamental inequalities
- 2.2.14 Elementary inequalities connected with Theorem 37
- 2.2.15 Elementary proof of Theorem 3
- 2.2.16 Tchebychef's inequality
- 2.2.17 Muirhead's theorem
- 2.2.18 Proof of Muirhead's theorem
- 2.2.19 An alternative theorem
- 2.2.20 Further theorems on symmetrical means
- 2.2.21 The elementary symmetric functions of n positive numbers
- 2.2.22 A note on definite forms
- 2.2.23 A theorem concerning strictly positive forms
- 2.2.24 Miscellaneous theorems & examples

2.3 Mean Values with an Arbitrary Function & the Theory of Convex Functions

Bibliography

Hardy, G. H., J. E. Littlewood, and G. Pólya (1952). *Inequalities*. 2d ed. Cambridge, at the University Press, pp. xii+324.