A Survey on Navier–Stokes Equations

Collector: Nguyen Quan Ba Hong*

February 25, 2022

Abstract

A personal survey on Navier–Stokes equations (NSEs), especially its regularity and turbulence models.

Keywords. Navier–Stokes equations; turbulence models.

Contents

Quick notes

1. The 4 formulations appearing in the *Clay Millennium Prize* formulation Fefferman, 2006 of NSEs.

Disclaimer. This survey is used mainly for NQBH's personal purposes. Hence, several paragraphs are quoted without quotation marks. The main purpose here is to collect and to add personal remarks.

1 Incompressible NSEs

1.1 Various concepts of solutions to NSEs

To describe various formulations for NSEs, we must first define properly the concept of a solution to NSEs, including, e.g., *periodic solutions, finite energy solutions, H^1 solutions*, and *smooth solutions*, etc.

1.1.1 Smooth solutions of NSEs

> "Note that even within the category of smooth solutions, there is some choice in what decay hypotheses to place on the initial data and solution; for instance, one can require that the initial velocity $\mathbf{u}_0$ be Schwartz class, or merely smooth with finite energy. Intermediate between these two will be data which is smooth and in H^1."
> – Tao, 2013

Recall Tao, 2013, Def. 1.1:

Definition 1.1 (Smooth solutions to NSEs)**.** *A* smooth set of data *for NSEs up to time T is a triplet $(\mathbf{u}_0, \mathbf{f}, T)$, where $0 < T < \infty$ is a time, the initial velocity vector field $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and the forcing term $\mathbf{f} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are assumed to be smooth on $\mathbb{R}^3$ and $[0,T] \times \mathbb{R}^3$, respectively, (thus, $\mathbf{u}_0$ is infinitely differentiable in space, and $\mathbf{f}$ is infinitely differentiable in spacetime), and $\mathbf{u}_0$ is furthermore required to be divergence-free:*

$$\nabla \cdot \mathbf{u}_0 = 0, \ \textit{in } \mathbb{R}^3. \tag{1.1}$$

*Independent Researcher, Ben Tre City, Vietnam.
Email. `nguyenquanbahong@gmail.com`.

If $\mathbf{f} = \mathbf{0}$, we say that the data is homogeneous.

The total energy $E(\mathbf{u}_0, \mathbf{f}, T)$ *of a smooth set of data* $(\mathbf{u}_0, \mathbf{f}, T)$ *is defined by the quantity*

$$E(\mathbf{u}_0, \mathbf{f}, T) \coloneqq \frac{1}{2}\left(\|\mathbf{u}_0\|_{L^2_{\mathbf{x}}(\mathbb{R}^3)} + \|\mathbf{f}\|_{L^1_t L^2_{\mathbf{x}}([0,T]\times\mathbb{R}^3)}\right)^2, \tag{iiNS/E}$$

and $(\mathbf{u}_0, \mathbf{f}, T)$ *is said to have* finite energy *if* $E(\mathbf{u}_0, \mathbf{f}, T) < \infty$*. We define the* H^1 norm $\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T)$ *of the data to be the quantity*

$$\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T) \coloneqq \|\mathbf{u}_0\|_{H^1_x(\mathbb{R}^3)} + \|\mathbf{f}\|_{L^\infty_t H^1_x(\mathbb{R}^3)} < \infty, \tag{iiNS/$\mathcal{H}^1$}$$

and say that $(\mathbf{u}_0, \mathbf{f}, T)$ is H^1 *if* $\mathcal{H}^1(\mathbf{u}_0, \mathbf{f}, T) < \infty$*; note that the* H^1 *regularity is essentially 1 derivative higher than the energy regularity, which is at the level of* L^2*, and instead matches the regularity of the* initial enstrophy $\frac{1}{2}\int_{\mathbb{R}^3}\|\boldsymbol{\omega}_0(t,\mathbf{x})\|^2\,\mathrm{d}\mathbf{x}$*, where* $\omega_0 \coloneqq \nabla\times\mathbf{u}_0$ *is the* initial vorticity*. We say that a smooth set of data* $(\mathbf{u}_0, \mathbf{f}, T)$ *is* Schwartz *if, for all integers* $\alpha, m, k \ge 0$*, one has*

$$\sup_{\mathbf{x}\in\mathbb{R}^3}(1+\|\mathbf{x}\|)^k\|\nabla^\alpha_{\mathbf{x}}\mathbf{u}_0(\mathbf{x})\| < \infty \text{ and } \sup_{(t,\mathbf{x})\in[0,T]\times\mathbb{R}^3}(1+\|\mathbf{x}\|)^k\|\nabla^\alpha_{\mathbf{x}}\partial^m_t\mathbf{f}(\mathbf{x})\| < \infty. \tag{1.2}$$

Thus, e.g., the Schwartz property implies H^1*, which in turn implies finite energy. We also say that* $(\mathbf{u}_0, \mathbf{f}, T)$ *is* periodic *with some period* $L > 0$ *if one has* $\mathbf{u}_0(\mathbf{x} + L\mathbf{k}) = \mathbf{u}_0(\mathbf{x})$ *and* $\mathbf{f}(t, \mathbf{x} + L\mathbf{k}) = \mathbf{f}(t, \mathbf{x})$ *for all* $t \in [0,T]$*,* $\mathbf{x} \in \mathbb{R}^3$*, and* $\mathbf{k} \in \mathbb{Z}^3$*. Of course, periodicity is incompatible with the Schwartz,* H^1*, or finite energy properties, unless the data is zero. To emphasize the periodicity, we will sometimes write a periodic set of data* $(\mathbf{u}_0, \mathbf{f}, T)$ *as* $(\mathbf{u}_0, \mathbf{f}, T, L)$*.*

A smooth solution to the NSEs*, or a* smooth solution*, is a quintuplet* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$*, where* $(\mathbf{u}_0, \mathbf{f}, T)$ *is a smooth set of data, and the velocity vector field* $\mathbf{u} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ *and pressure field* $p : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ *are smooth functions on* $[0,T]\times\mathbb{R}^3$ *that obey the NSE:*[1]

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p + \mathbf{f}, \tag{1.3}$$

and the incompressibility property

$$\nabla\cdot\mathbf{u} = 0, \tag{1.4}$$

on all of $[0,T]\times\mathbb{R}^3$*,*[2] *and also the initial condition*

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),\ \forall\mathbf{x}\in\mathbb{R}^3. \tag{1.5}$$

We say that a smooth solution $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *has* finite energy *if the associated data* $(\mathbf{u}_0, \mathbf{f}, t)$ *has finite energy, and in addition one has*[3]

$$\|\mathbf{u}\|_{L^\infty_t L^2_{\mathbf{x}}([0,T]\times\mathbb{R}^3)} < \infty. \tag{1.6}$$

Similarly, we say that $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ is H^1 *if the associated data* $(\mathbf{u}_0, \mathbf{f}, T)$ *is* H^1*, and in addition one has*

$$\|\mathbf{u}\|_{L^\infty_t H^1_{\mathbf{x}}([0,T]\times\mathbb{R}^3)} + \|\mathbf{u}\|_{L^2_t H^2_{\mathbf{x}}([0,T]\times\mathbb{R}^3)} < \infty. \tag{1.7}$$

We say instead that a smooth solution $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *is* periodic *with period* $L > 0$ *if the associated data* $(\mathbf{u}_0, \mathbf{f}, T) = (\mathbf{u}_0, \mathbf{f}, T, L)$ *is periodic with period* L*, and if* $\mathbf{u}(t, \mathbf{x} + L\mathbf{k}) = \mathbf{u}(t, \mathbf{x})$ *for all* $t \in [0,T]$*,* $\mathbf{x} \in \mathbb{R}^3$*, and* $\mathbf{k} \in \mathbb{Z}^3$*. (Following Fefferman, 2006, however, we will not initially directly require any periodicity properties on the pressure.) As before, we will sometimes write a periodic solution* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *as* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T, L)$ *to emphasize the periodicity.*

Terence Tao [but I will never] sometimes abuse notation and refer to a solution $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ simply as $(\mathbf{u}, p)$ or even $\mathbf{u}$. Similarly, we will sometimes abbreviate a set of data $(\mathbf{u}_0, \mathbf{f}, T)$ as $(\mathbf{u}_0, \mathbf{f})$ or even $\mathbf{u}_0$ (in the homogeneous case $\mathbf{f} = \mathbf{0}$).

Remark 1.1. *In Fefferman, 2006, one considered*[4] *smooth finite energy solutions associated to Schwartz data, as well as periodic smooth solutions associated to periodic smooth data. In the latter case, one can of course normalize the period L to equal 1 by a simple scaling argument. In Tao, 2013, Terence Tao focused on the case when the data* $(\mathbf{u}_0, \mathbf{f}, T)$ *is large, although he did not study the asymptotic regime when* $T \to \infty$*.*

[1] NQBH: Why no viscosity ν? Any major differences in their mathematical analysis, especially the case $\nu = \nu(t, \mathbf{x}, \mathbf{u}, p)$ in turbulence models?

[2] NQBH: NSEs on the whole domain, hence useless for shape and topology optimizations, but useful for applying harmonic and Fourier analysis.

[3] Following Fefferman, 2006, Terence Tao omitted the *finite energy dissipation condition* $\nabla\mathbf{u} \in L^2_t L^2_{\mathbf{x}}([0,T]\times\mathbb{R}^3)$ often appearing in the literature, particular when discussing *Leray–Hopf weak solutions*. However, it turns out that this condition is actually automatic from (1.6) and smoothness; see Tao, 2013, Lem. 8.1. Similarly, from Tao, 2013, Corollary 11.1, the $L^2_t H^2_{\mathbf{x}}$ condition in (1.7) is redundant.

[4] The viscosity parameter ν was not normalized in Fefferman, 2006 to 1, as Terence Tao did in Tao, 2013, but one can easily reduce to the $\nu = 1$ case by a simple rescaling.

NQBH: only true for the case $\nu = \text{const} > 0$, rescaling fails when ν varies, e.g., depends on $\mathbf{x}$ and/or $\mathbf{u}$, p.

Recall 2 standard *global regularity* conjectures for NSEs, using the formulation in Fefferman, 2006:

Conjecture 1.1 (Global regularity for homogeneous Schwartz data)**.** *Let* $(\mathbf{u}_0, 0, T)$ *be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution* $(\mathbf{u}, p, \mathbf{u}_0, 0, T)$ *with the indicated data.*

Conjecture 1.2 (Global regularity for homogeneous periodic data)**.** *Let* $(\mathbf{u}_0, 0, T)$ *be a smooth homogeneous periodic set of data. Then there exists a smooth periodic solution* $(\mathbf{u}, p, \mathbf{u}_0, 0, T)$ *with the indicated data.*

Extend these conjectures to the inhomogeneous case as follows:

Conjecture 1.3 (Global regularity for Schwartz data)**.** *Let* $(\mathbf{u}_0, \mathbf{f}, T)$ *be a Schwartz set of data. Then there exists a smooth finite energy solution* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *with the indicated data.*

Conjecture 1.4 (Global regularity for periodic data)**.** *Let* $(\mathbf{u}_0, \mathbf{f}, T)$ *be a smooth periodic set of data. Then there exists a smooth periodic solution* $(\mathbf{u}, p, \mathbf{u}_0, \mathbf{f}, T)$ *with the indicated data.*

As described in Fefferman, 2006, a positive answer to either Conjecture 1.3 or 1.4, or a negative answer to Conjecture 1.5 or 1.6, would qualify for the Clay Millennium Prize.

However, Conjecture 1.6 is not quite the "right" extension of Conjecture 1.4 to the inhomogeneous setting, and needs correcting slightly. This is because there is a technical quirk in the inhomogeneous periodic problem as formulated in Conjecture 1.6, due to the fact that p is not required to be periodic. This opens up a *Galilean invariance* in the problem which allows one to homogenize away the role of the forcing term. More precisely (established in Tao, 2013, Sect. 6):

Proposition 1.1 (Elimination of forcing term)**.** *Conjecture 1.6* $\Leftrightarrow$ *Conjecture 1.4.*

Terence Tao remarks that this is the only implication we know of that can deduce a global regularity result for the inhomogenous Navier–Stokes problem from a global regularity result for the homogeneous Navier–Stokes problem.

Tao, 2013, Prop. 1.7 exploits the technical loophole of nonperiodic pressure. The same loophole can also be used to easily demonstrate failure of uniqueness for the periodic Navier–Stokes problem (although this can also be done by the much simpler expedient of noting that one can adjust the pressure by an arbitrary constant without affecting the momentum equation). This suggests that in the nonhomogeneous case $\mathbf{f} \neq \mathbf{0}$, one needs an additional normalization to "fix" the periodic Navier–Stokes problem to avoid such loopholes. This can be done in the standard way, as follows: see Tao, 2013, p. 28…

1.2 OpenFOAM Solvers

1.2.1 OpenFOAM Solver `simpleFoam`

The `simpleFoam` solver employs the SIMPLE algorithm to solve:

$$\begin{cases} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbf{R} = -\nabla p + \mathbf{S}_{\mathbf{u}}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $\mathbf{R}$ is the stress tensor, and $\mathbf{S}_{\mathbf{u}}$ is the momentum source.

2 Compressible NSEs

2.1 OpenFOAM Solvers

References

[TT's blog] Terence Tao's blog/Navier–Stokes equations.

- Terence Tao. *Localisation and compactness properties of the Navier-Stokes global regularity problem*. Aug 4, 2011.
- Terence Tao. *254A, Notes 0: Physical derivation of the incompressible Euler and Navier-Stokes equations*. Sep 3, 2018.

[Wikipedia] Wikipedia.org

- Wikipedia/viscosity.

References

Fefferman, Charles L. (2006). “Existence and smoothness of the Navier-Stokes equation”. In: *The millennium prize problems*. Clay Math. Inst., Cambridge, MA, pp. 57–67.

Tao, Terence (2013). “Localisation and compactness properties of the Navier-Stokes global regularity problem”. In: *Anal. PDE* 6.1, pp. 25–107. ISSN: 2157-5045. DOI: 10.2140/apde.2013.6.25. URL: https://doi.org/10.2140/apde.2013.6.25.