Ho Chi Minh City University of Science
Faculty of Mathematics and Computer Science

Bachelor Thesis

Multi-dimensional degenerate diffusion equation with a very strong absorption and a source terms

Author:
Nguyen Quan Ba Hong

Supervisor:
Dr. Dao Nguyen Anh
Dr. Bui Le Trong Thanh

A bachelor thesis submitted in fulfillment of the requirements for the degree of Bachelor of Science

in the

Department of Mathematical Analysis

May 7, 2019

Hong, Nguyen:

Multi-dimensional degenerate diffusion equation with a very strong absorption and a source terms.

Bachelor thesis, Ho Chi Minh City University of Science, 2018.

Declaration of Authorship

I, Nguyen Quan Ba Hong, declare that this bachelor thesis titled, "Multi-dimensional degenerate diffusion equation with a very strong absorption and a source terms" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this bachelor thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this bachelor thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the bachelor thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"*Pure mathematics is, in its way, the poetry of logical ideas*"

Albert Einstein

HO CHI MINH CITY UNIVERSITY OF SCIENCE

Abstract

Faculty of Mathematics and Computer Science

Department of Mathematical Analysis

Bachelor of Science

Multi-dimensional degenerate diffusion equation with a very strong absorption and a source terms

by NGUYEN QUAN BA HONG

We prove a local existence of weak solutions of semilinear parabolic equations with a strong singular absorption term and a general source. Some gradient estimates which act crucial roles in achieving existence and regularity results, are also provided.

Mathematics Subject Classification (2010): 35K55, 35K67, 35K65.

Keywords: quenching type parabolic equations, gradient estimates, strong absorption.

Acknowledgements

First and foremost, I have to thank my research supervisors, Dr. Dao Nguyen Anh, Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, and Dr. Bui Le Trong Thanh, Department of Mathematical Analysis, Faculty of Mathematics and Computer Science, Ho Chi Minh City, Vietnam. Without their assistance and dedicated involvement in every step throughout the process, this bachelor thesis would have never been accomplished. I would like to thank you very much for your support and understanding over these past four years.

I wish to express my sincere thanks to Prof. Dang Duc Trong, Assoc. Prof. Huynh Quang Vu, old and new Principals of the Faculty of Mathematics and Computer Science, for providing me with all then necessary facilities for the research.

I am also grateful to Dr. Ong Thanh Hai, in the field of Numerical Analysis, Department of Mathematical Analysis, for the continuous encouragement.

I place on record, my sincere thanks to Tra Quoc Khanh, Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, and Tran Nhat Luan, Institute of Computational Science and Technology, Ho Chi Minh, Vietnam,. I am extremely thankful and indebted to him for sharing expertise, and sincere and valuable guidance and encouragement extended to me.

I take this opportunity to express gratitude to all of the Department faculty members for their help and support. I also thank my parents for their immeasurable encouragement, support and attention.

I also place on record, my sense of gratitude to one and all, who directly or indirectly, have let their hands in this venture.

Ho Chi Minh City, July 2018.
Nguyen Quan Ba Hong

Contents

Dedicated to my beloved parents and my respected teacher, Dao Nguyen Anh, *who helped me through one of the most difficult turning points in my life, and guided conscientiously me to the world of PDEs.*

Preliminaries

1 Lipschitz Continuity and Lebesgue Integrability

Definition 0.1 (Lipschitz continuity). *Given two metric spaces (X, d_X) and (Y, d_Y), where d_X, d_Y denote the metrics on the set X, Y, respectively, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that*

$$d_Y\left(f\left(x_1\right), f\left(x_2\right)\right) \leq K d_X\left(x_1, x_2\right), \text{ for all } x_1, x_2 \in X.$$

Any such K is referred to as a Lipschitz constant for the function f. The smallest constant is sometimes called the (best) Lipschitz constant.

If $K = 1$ the function is called a short map, and if $0 \leq K < 1$ and f maps a metric space to itself, the function is called a contraction.

Definition 0.2 (Local Lipschitz continuous function). *A function is called locally Lipschitz continuity if for every $x \in X$ there exists a neighborhood U of x such that f restricted to U is Lipschitz continuous. Equivalently, if X is a locally compact metric space, then f is locally Lipschitz if and only if it is Lipschitz continuous on every compact subset of X.*

In spaces that are not locally compact, this is a necessary but not a sufficient condition.

Moreover, the notion Hölder continuity generalizes the notion of Lipschitz continuity.

Definition 0.3 (Hölder continuity). *A function f defined on X is said to be Hölder continuous or to satisfy a Hölder condition of order $\alpha > 0$ on X if there exists a constant $M > 0$ such that*

$$d_Y\left(f\left(x_1\right), f\left(x_2\right)\right) \leq M d_X(x_1, x_2)^{\alpha}, \text{ for all } x_1, x_2 \in X.$$

Sometimes a Hölder condition of order α is also called a uniform Lipschitz condition of order $\alpha > 0$.

Definition 0.4 (Bilipschitz continuity). *If there exists a $K \geq 1$ with*

$$\frac{1}{K} d_X\left(x_1, x_2\right) \leq d_Y\left(f\left(x_1\right), f\left(x_2\right)\right) \leq K d_X\left(x_1, x_2\right), \text{ for all } x_1, x_2 \in X.$$

then f is called bilipschitz (also written bi-Lipschitz).

A bilipschitz mapping is injective, and is in fact a homeomorphism onto its image. A bilipschitz function is the same thing as an injective Lipschitz function whose inverse function is also Lipschitz.

Here are some basic properties of Lipschitz continuity.

a) An everywhere differentiable function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, with $K := \sup |f'(x)|$ if and only if it has bounded first derivative. In particular, any continuously differentiable function is locally Lipschitz, as continuous functions are locally bounded so its gradient is locally bounded.

b) A Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous and therefore is differentiable almost everywhere, that is, differentiable at every point outside a set of Lebesgue measure zero. Its derivaitive is essentially bounded in magnitude by the Lipschitz constant, and for $a > b$, the difference $f(b) - f(a)$ is equal to the integral of the derivative f' on the interval $[a, b]$.

c) If $f : I \to \mathbb{R}$ is absolutely continuous and thus differentiable almost everywhere, and satisfies $|f'(x)| \leq K$ for a.e. $x \in I$, then f is Lipschitz continuous with Lipschitz constant at most K.

d) More generally, Rademacher's theorem extends the differentiability result to Lipschitz mappings between Euclidean spaces: a Lipschitz map $f : U \to \mathbb{R}^m$, where U is an open set in $\mathbb{R}^n$, is almost everywhere differentiable.

   Moreover, if K is the best Lipschitz constant of f, then $\|Df(x)\| \leq K$, whenever the total derivative Df exists.

e) For a differentiable Lipschitz map $f : U \to \mathbb{R}^m$ the inequality $\|Df\|_{\infty,U} \leq K$ holds for the best Lipschitz constant of f, and it turns out to be an equality if the domain U is convex.

f) Suppose that $\{f_n\}_{n\in\mathbb{Z}^+}$ is a sequence of Lipschitz continuous mappings between two metric spaces, and that all f_n's have Lipschitz constant bounded by some K. If f_n converges to a mapping uniformly, then f is also Lipschitz, with Lipschitz constant bounded by the same K.

   In particular, this implies that the set of real-valued functions on a compact metric space with a particular bound for the Lipschitz constant is a closed and convex subset of the Banach space of continuous functions. Nevertheless, this result does not hold for sequences in which the functions may have unbounded Lipschitz constants. In fact, the space of all Lipschitz functions on a compact metric space is a subalgebra of the Banach space of continuous functions, and thus dense in it, an elementary consequence of the Stone-Weiersstrass theorem (or as a consequence of Weierstrass approximation theorem, because every polynomial is locally Lipschitz continuous).

g) Every Lipschitz continuous map is uniformly continuous, and hence a fortiori continuous. More generally, a set of functions with bounded Lipschitz constant forms an equicontinuous set. The Arzelà-Ascoli theorem implies if $\{f_n\}_{n\in\mathbb{Z}^+}$ is a uniformly bounded sequence of functions with bounded Lipschitz constant, then it has a convergent subsequence. The limit function of this subsequence is also Lipschitz, with the same bound for the Lipschitz constant.

In particular, the set of all real-valued Lipschitz functions on a compact metric space X having Lipschitz constant $\leq K$ is a locally compact convex subset of the Banach space $\mathcal{C}(X)$.

h) For a family of Lipschitz continuous function f_α with common constant, the functions $\sup_\alpha f_\alpha$ and $\inf_\alpha f_\alpha$ are Lipschitz continuous as well, with the same Lipschitz constant, provided it assumes a finite value at least at a point.

i) If U is a subset of the metric space M and $f : U \to \mathbb{R}$ is a Lipschitz continuous function, there always exist Lipschitz continuous maps $M \to \mathbb{R}$ which extend f and have the same Lipschitz constant as f (see Kirsbraun theorem). An extension is provided by

$$\widetilde{f}(x) := \inf_{u \in U} \{f(u) + Kd(x, u)\},$$

where K is a Lipschitz constant for f on U.

A locally integrable function, or locally summable function, is a function which is integrable on every compact subset of its domain of definition. The importance of such functions lies in the fact that their function space is similar to L^p spaces, but its members are not required to satisfy any growth restriction on their behavior at infinity, i.e., locally integrable functions can grow arbitrarily fast at infinity, but are still manageable in a way similar to ordinary integrable functions.

Definition 0.5 (Local integrable functions - standard version)**.** *Let Ω be an open set in the Euclidean space $\mathbb{R}^N$ and $f : \Omega \to \mathbb{C}$ be a Lebesgue measurable function. If*

$$\int_K |f|\, dx < +\infty, \;\; \forall K \subset \Omega, \;\; K \text{ compact},$$

i.e., its Lebesgue integral is finite on all compact subsets K of Ω, then f is called locally integrable. The set of all locally integrable functions on Ω is denoted by $L^1_{\mathrm{loc}}(\Omega)$,

$$L^1_{\mathrm{loc}}(\Omega) := \left\{f : \Omega \to \mathbb{C} \text{ measurable}\, ; \, f|_K \in L^1(K), \;\; \forall K \subset \Omega, \;\; K \text{compact}\right\}.$$

The classical definition of local integrability involves only measure theoretic and topological concepts and can be carried over abstract to complex-valued functions on a topological measure space (X, Σ, μ). Nevertheless, due to the fact that the most common application of such functions is to distribution theory on Euclidean spaces, this important case is enough for our purpose. Here is another version of the notion of local integrable functions.

Definition 0.6 (Local integrable functions - alternative version). *Let Ω be an open set in the Euclidean space $\mathbb{R}^N$. Then a function $f : \Omega \to \mathbb{C}$ such that*

$$\int_\Omega |f\varphi|\, dx < +\infty, \text{ for all test fucntion } \varphi \in \mathcal{C}_c^\infty(\Omega),$$

is called locally integrable, and the set of such functions is denoted by $L^1_{\text{loc}}(\Omega)$. Here $\mathcal{C}_c^\infty(\Omega)$ denotes the set of all infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}$ with compact support contained in Ω.

This definition has its roots in the approach to measure and integration theory based on the concept of continuous linear functional on a topological vector space, developed by Nicolas Bourbaki and his school. This distribution-theoretic definition is equivalent the standard one, i.e., Definition 0.5 $\Leftrightarrow$ Definition 0.6.

Lemma 0.1 (Equivalence of definitions of local integrability). *A function $f : \Omega \to \mathbb{C}$ is locally integrable according to Definition 0.5 if and only if it is locally integrable in the sense of Definition 0.6, i.e.,*

$$\left(\int_K |f|\, dx < +\infty, \ \forall K \subset \Omega, \ K \text{ compact}\right) \Leftrightarrow \left(\int_\Omega |f\varphi|\, dx < +\infty, \ \forall \varphi \in \mathcal{C}_c^\infty(\Omega)\right).$$

Proof of Lemma 0.1. ($\Rightarrow$). Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ be a test function, φ is bounded by its supremum norm $\|\varphi\|_\infty$, measurable, and has a compact support. Hence,

$$\int_\Omega |f\varphi|\, dx = \int_{\text{Supp}(\varphi)} |f|\,|\varphi|\, dx \le \|\varphi\|_\infty \int_{\text{Supp}(\varphi)} |f|\, dx < +\infty.$$

($\Leftarrow$). Let K be an arbitrary compact subset of the open set $\Omega \subset \mathbb{R}^N$. The first step is to construct a test function $\varphi_K \in \mathcal{C}_c^\infty(\Omega)$ which majorizes the indicator function (or, characteristic function) χ_K of K.

The usual set distance between K and the boundary $\partial\Omega$ is strictly greater than zero, i.e., $\Delta := d(K, \partial\Omega) > 0$, hence it is possible to choose a real number δ such that $\delta \in \left(0, \frac{\Delta}{2}\right)$ (if $\partial\Omega = \emptyset$, take $\Delta = +\infty$). Let K_δ and $K_{2\delta}$ denote the closed δ-neighborhood and 2δ-neighborhood of K, respectively. They are likewise compact and satisfy

$$K \subset K_\delta \subset K_{2\delta} \subset \Omega, \text{ and } d(K_\delta, \partial\Omega) = \Delta - \delta > \delta > 0.$$

Now use convolution to define the function $\varphi_K : \Omega \to \mathbb{R}$ by

$$\varphi_K(x) := \chi_{K_\delta} \star \rho_\delta(x) = \int_{\mathbb{R}^N} \chi_{K_\delta}(y)\, \varphi_\delta(x - y)\, dy,$$

where ρ_δ is a mollifier constructed by using the standard positive symmetric one, see, e.g., Brezis, 2011. One has that φ_K is nonnegative, infinitely differentiable, and its support is contained in $K_{2\delta}$, in particular it is a test function. Since $\varphi_K(x) = 1$ for all $x \in K$, we have that $\chi_K \le \varphi_K$.

Let f be a locally integrable function according to Definition 0.6. Then

$$\int_K |f|\, dx = \int_\Omega |f|\, \chi_K dx \leq \int_\Omega |f|\, \varphi_K dx \leq +\infty.$$

Since this holds for every compact subset K of Ω, f is locally integrable in the sense of Definition 0.5. This completes our proof. □

Definition 0.7 (Local p-integrable functions - standard version). *Let Ω be an open set in the Euclidean space $\mathbb{R}^N$ and $f : \Omega \to \mathbb{C}$ be a Lebesgue measurable function. If, for a given $p \in [1, +\infty]$, f satisfies*

$$\int_K |f|^p dx < +\infty, \ \ \forall K \subset \Omega, \ \ K \text{ compact},$$

i.e., it belongs to $L^p(K)$ for all compact subsets K of Ω, then f is called locally p-integrable or also p-locally integrable. The set of all such functions is denoted by $L^p_{\text{loc}}(\Omega)$,

$$L^p_{\text{loc}}(\Omega) := \{f : \Omega \to C \text{ measurable}; \ f|_K \in L^p(K), \ \ \forall K \subset \Omega, \ \ K \text{ compact}\}.$$

Here is an alternative definition which is analogous to Definition 0.6.

Definition 0.8 (Local p-integrable functions - alternative version). *Let Ω be an open set in the Euclidean space $\mathbb{R}^N$. Then a function $f : \Omega \to \mathbb{C}$ such that*

$$\int_\Omega |f\varphi|^p dx < +\infty, \ \text{for all } \varphi \in \mathcal{C}^\infty_c(\Omega),$$

is called locally p-integrable, and the set of such functions is denoted by $L^1_{\text{loc}}(\Omega)$.

Lemma 0.2 (Equivalence of definitions of local p-integrability). *A function $f : \Omega \to \mathbb{C}$ is locally integrable according to Definition 0.7 if and only if it is locally integrable according to Definition 0.8, i.e.,*

$$\left(\int_K |f|^p dx < +\infty, \ \ \forall K \subset \Omega, \ \ K \text{ compact}\right) \Leftrightarrow \left(\int_\Omega |f\varphi|^p dx < +\infty, \ \ \forall \varphi \in \mathcal{C}^\infty_c(\Omega)\right).$$

Despite their apparent higher generality, locally p-integrable functions form a subset of locally integrable functions for every p such that $p \in (1, +\infty]$.

2 Divergence Theorem and Green's Identities

In vector calculus, the *divergence theorem*, also known as *Gauss's theorem* or *Ostrogradsky's theorem*, is a result that relates the flow (i.e., flux) of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that: *The sum of all sources (with sinks regarded as negative sources) gives the net flux out of a region.*

Theorem 0.1 (Divergence theorem). *Suppose Ω is a subset of $\mathbb{R}^N$, which is compact and has a piecewise smooth boundary $\partial\Omega$. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of Ω, then one has*

$$\int_\Omega \nabla \cdot \mathbf{F} d\Omega = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} d\sigma. \tag{0.1}$$

The left-hand side of (0.1) is a volume integral over the volume $\Omega \subset \mathbb{R}^N$, the right-hand side is the surface integral over the boundary $\partial\Omega$ of the volume Ω. The closed manifold $\partial\Omega$ is quite generally the boundary of Ω oriented by outward-pointing normals, and $\mathbf{n}$ is the outward pointing unit normal field of the boundary $\partial\Omega$. In terms of the intuitive description above, the left-hand side of (0.1) represents the total of the sources in the volume Ω, and the right-hand side represents the total flow across its boundary $\partial\Omega$.

Applying divergence theorem to the vector field $\mathbf{F} = \psi\nabla\varphi$, we derive the following corollaries, called Green's identities.

Corollary 0.1 (Green's first identity). *Let φ and ψ be scalar functions defined on some region $\Omega \subset \mathbb{R}^N$, and suppose that φ is twice continuously differentiable, and ψ is once continuously differentiable. Then*

$$\int_\Omega (\psi\Delta\varphi + \nabla\psi \cdot \nabla\varphi)\, d\Omega = \oint_{\partial\Omega} \psi\nabla\varphi \cdot \mathbf{n} d\sigma,$$

where Δ is the Laplace operator, $\mathbf{n}$ is the outward pointing unit normal of surface element, $d\sigma$ is the oriented surface element.

Choosing $\mathbf{F} = \psi\varepsilon\nabla\varphi - \varphi\varepsilon\nabla\psi$, one obtains

Corollary 0.2 (Green's second identity). *If φ and ψ are both twice continuously differentiable on $\Omega \subset \mathbb{R}^3$, and ε is continuously differentiable, the following identity holds*

$$\int_\Omega (\psi\nabla \cdot (\varepsilon\nabla\varphi) - \varphi\nabla \cdot (\varepsilon\nabla\psi))\, d\Omega = \oint_{\partial\Omega} \varepsilon \left(\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\psi}{\partial\mathbf{n}}\right) d\sigma.$$

For the special case of $\varepsilon = 1$ all across $\Omega \subset \mathbb{R}^3$, then

$$\int_\Omega (\psi\Delta\varphi - \varphi\Delta\psi)\, d\Omega = \oint_{\partial\Omega} \left(\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\psi}{\partial\mathbf{n}}\right) d\sigma.$$

In the equations above, $\frac{\partial\varphi}{\partial\mathbf{n}}$ is the directional derivative of φ in the direction of the outward pointing normal $\mathbf{n}$ to the surface element $d\sigma$,

$$\frac{\partial\varphi}{\partial\mathbf{n}} = \nabla\varphi\cdot\mathbf{n}.$$

Particularly, this demonstrates that the Laplacian is self-adjoint in the L^2 inner product for functions vanishing on the boundary.

3 L^p Spaces

Next, we recall some famous results of measurable functions and integrable functions. Given a measure space $(\Omega, \mathcal{M}, \mu)$, the following results hold.

Theorem 0.2 (Monotone convergence theorem, Beppo Levi)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

a) $f_1 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ *a.e. on* Ω,

b) $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $\|f_n - f\|_1 \to 0$.

Theorem 0.3 (Dominated convergence theorem, Lebesgue)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

a) $f_n(x) \to f(x)$ *a.e. on* Ω,

b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Lemma 0.3 (Fatou's lemma)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

a) for all n, $f_n \geq 0$ a.e.

b) $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$, we set $f(x) := \liminf_{n\to+\infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

In later chapters, we mainly use the basic case in which $\Omega \subseteq \mathbb{R}^N$, $\mathcal{M}$ consists of the Lebesgue measurable sets, and μ is the Lebesgue measure on $\mathbb{R}^N$.

Notation 0.1. *We denote by $\mathcal{C}_c\left(\mathbb{R}^N\right)$ the space of all continuous functions on $\mathbb{R}^N$ with compact support, i.e.,*

$$\mathcal{C}_c\left(\mathbb{R}^N\right) := \left\{f \in C\left(\mathbb{R}^N\right); f(x) = 0,\ \forall x \in \mathbb{R}^N \backslash K,\ \text{where } K \text{ is compact}\right\}.$$

Theorem 0.4 (Density)**.** *The space $\mathcal{C}_c\left(\mathbb{R}^N\right)$ is dense in $L^1\left(\mathbb{R}^N\right)$, i.e.,*

$$\forall f \in L^1\left(\mathbb{R}^N\right),\ \forall \varepsilon > 0,\ \exists f_1 \in C_c\left(\mathbb{R}^N\right)\ \text{s.t. } \|f - f_1\|_1 \leq \varepsilon.$$

Definition 0.9 (L^p spaces)**.** *Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set*

$$L^p(\Omega) := \left\{f : \Omega \to \mathbb{R}; f \text{ is measurable and } |f|^p \in L^1(\Omega)\right\},$$

with

$$\|f\|_{L^p} = \|f\|_{L^p} := \left(\int_\Omega |f(x)|^p d\mu\right)^{\frac{1}{p}}.$$

Definition 0.10 (L^∞ space)**.** *We set*

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{R}; f \text{ is measurable and } \exists C \text{ s.t. } |f(x)| \leq C \text{ a.e. on } \Omega\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty := \inf\{C; |f(x)| \leq C \text{ a.e. on } \Omega\}.$$

Notation 0.2 (Conjugate exponent)**.** *Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 0.5 (Hölder inequality)**.** *Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

Theorem 0.6 (Extended Hölder inequality)**.** *Assume that $f_1, \ldots, f_k$ are functions such that*

$$f_i \in L^{p_i},\ 1 \leq i \leq k,\ \text{with } \frac{1}{p} = \sum_{i=1}^k \frac{1}{p_i} \leq 1.$$

Then the product $f = f_1 f_2 \ldots f_k$ belongs to L^p and

$$\|f\|_p \leq \prod_{i=1}^k \|f_i\|_{p_i}.$$

In particular, if $f \in L^p \cap L^q$ *with* $1 \le p \le q \le \infty$, *then* $f \in L^r$ *for all* r, $p \le r \le q$, *and the following "interpolation inequality" holds*

$$\|f\|_r \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}, \text{ where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \ 0 \le \alpha \le 1.$$

Definition 0.11 (Mollifiers)**.** *A sequence of mollifiers* $(\rho_n)_{n \ge 1}$ *is any sequence of functions on* $\mathbb{R}^N$ *such that*

$$\rho_n \in \mathcal{C}_c^{\infty}\left(\mathbb{R}^N\right), \ \operatorname{Supp}(\rho_n) \subset \overline{B\left(0, \frac{1}{n}\right)}, \ \int \rho_n = 1, \ \rho_n \ge 0 \text{ on } \mathbb{R}^N.$$

Theorem 0.7 (Density)**.** *The space* $\mathcal{C}_c\left(\mathbb{R}^N\right)$ *is dense in* $L^p\left(\mathbb{R}^N\right)$ *for any* p, $1 \le p < \infty$.

Notation 0.3. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set.*

a) $\mathcal{C}(\Omega)$ *is the space of continuous functions on* Ω.

b) $\mathcal{C}^k(\Omega)$ *is the space of functions* k *times continuously differentiable on* Ω *(*$k \ge 1$ *is an integer).*

c) $\mathcal{C}^{\infty}(\Omega) := \bigcap_{k \in Z^+} \mathcal{C}^k(\Omega)$.

d) $\mathcal{C}_c(\Omega)$ *is the space of continuous functions on* Ω *with compact support in* Ω, *i.e., which vanish outside some compact set* $K \subset \Omega$.

e) $\mathcal{C}_c^k(\Omega) := \mathcal{C}^k(\Omega) \cap \mathcal{C}_c(\Omega)$.

f) $\mathcal{C}_c^{\infty}(\Omega) := \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}_c(\Omega)$.

Theorem 0.8 (Density)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. Then* $\mathcal{C}_c^{\infty}(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $1 \le p < \infty$.

Theorem 0.9. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $u \in L^1_{\mathrm{loc}}(\Omega)$ *be such that*

$$\int uf = 0, \ \forall f \in \mathcal{C}_c^{\infty}(\Omega).$$

Then $u = 0$ *a.e. on* Ω.

We recall that the Ascoli-Arzelà theorem gives us a criterion to decide whether a family of functions in $\mathcal{C}(K)$ has compact closure in $\mathcal{C}(K)$, the space of continuous functions over a compact metric space K with values in $\mathbb{R}$.

Theorem 0.10 (Ascoli-Arzelà)**.** *Let* K *be a compact metric space and let* $\mathcal{H}$ *be a bounded subset of* $\mathcal{C}(K)$. *Assume that* $\mathcal{H}$ *is uniformly equicontinuous, i.e.,*

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon, \ \forall f \in \mathcal{H}.$$

Then the closure of $\mathcal{H}$ *in* $\mathcal{C}(K)$ *is compact.*

Notation 0.4 (Shift of function)**.** *We set* $(\tau_h f)(x) := f(x+h)$, $x \in \mathbb{R}^N$, $h \in \mathbb{R}^N$.

The following theorem and its corollary are L^p-versions of the Ascoli-Arzelà theorem.

Theorem 0.11 (Kolmogorov-M. Riesz-Fréchet)**.** *Let* $\mathcal{F}$ *be a bounded set in* $L^p(\mathbb{R}^N)$ *with* $1 \leq p < \infty$. *Assume that*

$$\lim_{|h|\to 0} \|\tau_h f - f\|_p = 0 \text{ uniformly for } f \in \mathcal{F}, \tag{0.2}$$

i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\|\tau_h f - f\|_p < \varepsilon$, $\forall f \in \mathcal{F}$, $\forall h \in \mathbb{R}^N$ *with* $|h| < \delta$.

Then the closure of $\mathcal{F}|_\Omega$ *in* $L^p(\Omega)$ *is compact for any measurable set* $\Omega \subset \mathbb{R}^N$ *with finite measure.*

Corollary 0.3. *Let* $\mathcal{F}$ *be a bounded set in* $L^p(\mathbb{R}^N)$ *with* $1 \leq p < \infty$. *Assume* (0.2) *and also*

$$\forall \varepsilon > 0, \ \ \exists \Omega \subset \mathbb{R}^N, \ \text{bounded, measurable s.t. } \|f\|_{L^p(\mathbb{R}^n \setminus \Omega)} < \varepsilon, \ \ \forall f \in \mathcal{F}.$$

Then $\mathcal{F}$ *has compact closure in* $L^p(\mathbb{R}^N)$.

4 Sobolev Spaces in N Dimensions

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 0.12. *The Sobolev space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) ; \exists g_1, \ldots, g_N \in L^p(\Omega) \ \text{ s.t. } \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \ \ \forall \varphi \in C_c^\infty(\Omega), \ \ \forall i = 1, \ldots N \right\}.$$

We set $H^1(\Omega) := W^{1,2}(\Omega)$.

For $u \in W^{1,2}(\Omega)$, we define $\partial_i u = \frac{\partial u}{\partial x_i} := g_i$, and write

$$\nabla u = \text{grad} u := (\partial_1 u, \ldots, \partial_N u).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_p + \sum_{i=1}^N \|\partial_i u\|_p,$$

or sometimes with the equivalent norm $\left(\|u\|_p^p + \sum_{i=1}^N \|\partial_i u\|_p^p \right)^{\frac{1}{p}}$ (if $1 \leq p < \infty$).

Definition 0.13. *Let $m \geq 2$ be an integer and let p be a real number with $1 \leq p \leq \infty$. We define by induction*

$$W^{m,p}(\Omega) := \left\{ u \in W^{m-1,p}(\Omega) ; \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \ \ \forall i = 1, \dots, N \right\}.$$

Alternatively, these sets could also be introduced as

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) ; \forall \alpha, \ \ |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \ \ s.t. \ \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi, \ \ \forall \varphi \in C_c^\infty(\Omega) \right\},$$

where we use the standard multi-index notation $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_i \geq 0$ an integer,

$$|\alpha| := \sum_{i=1}^{N} \alpha_i \ \text{and} \ D^\alpha \varphi := \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

We set $D^\alpha u = g_\alpha$. The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p$$

The space $H^m(\Omega) := W^{m,2}(\Omega)$ equipped with the scalar product

$$(u, v)_{H^m} := \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}$$

is a Hilbert space.

Definition 0.14. *Let $1 \leq p < \infty$; $W_0^{1,p}(\Omega)$ denotes the closure of $\mathcal{C}_c^1(\Omega)$ in $W^{1,p}(\Omega)$. Set $H_0^1(\Omega) := W_0^{1,2}(\Omega)$.*

Using a sequence of mollifiers, one has that $\mathcal{C}_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$.

Chapter 1

Some Related Results

In this chapter, we introduce the history of the study of quenching phenomenon through some related works chronically.

In Aronson, 1969, D. G. Aronson considered a homogeneous gas flowing through a homogeneous porous medium and supposed that the equation of state has the form $\gamma = \gamma_0 p^{\alpha}$, where γ is the density of the gas at any point, p is the pressure, and γ_0 is a constant. If the flow is isothermic then $\alpha = 1$, while if it is adiabatic then $\alpha \in (0, 1)$. By conservation of mass and Darcy's law, Aronson obtained the following equation

$$\partial_t u = \Delta u^m, \tag{1.1}$$

where u is essentially the density of the gas and Δ is the Laplace operator in the space variable x and $m = 1 + \alpha^{-1}$. Thus, the case $m = 2$ corresponds with the isothermic flow and the case $m > 2$ tallies with the adiabatic flow. Apart from constants, ∇u^{m-1} is the velocity and $u\nabla u^{m-1}$ is the flux vector. Thus, in particular, u^{m-1} is essentially pressure which, by Darcy's law, is also the velocity potential. For $m > 1$, (1.1) is a nonlinear equation which is parabolic for $u > 0$, but which degenerates when $u = 0$. The most arresting manifestation of the degeneracy of (1.1) is the finite speed of propagation of disturbances. If at some instant of time a solution of (1.1) has compact support, then it will continue to have that property for all later times. In general, the transition from a region where $u > 0$ to one where $u = 0$ is not smooth and it is decisive to introduce notions of some generalized solutions. The objective of this paper is to study the regularity properties of a class of generalized solutions of the Cauchy problem for (1.1), i.e.,

$$\begin{cases} \partial_t u = \Delta u^m, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

Equation (1.1) is a special case of the equation $\partial_t u = \partial_x^2 \varphi(x, u)$ studied by Oleinik, Kalashnikov and Yui-Lin'. In one-dimensional (1-D) setting, if u_0^m is Lipschitz continuous, then the corresponding Cauchy problem

$$\begin{cases} \partial_t u = \partial_x^2 (u^m), & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{1.3}$$

has a unique weak solution. Moreover, this solution satisfies the equation in the classical sense in a neighborhood of every point $(x,t) \in \mathbb{R} \times (0,\infty)$ at which the solution is positive. Aronson also demonstrated that, with respect to x, the velocity potential u^{m-1} is Lipschitz continuous, the flux $\partial_x (u^m)$ is continuous and the density u is Hölder continuous with exponent $\min\left\{1, (m-1)^{-1}\right\}$. This exponent is best possible as shown by an example of an explicit solution due to Pattle in Pattle's paper (1959). Moreover, for $1 < m < 2$, $\partial_x u$ actually exists and is continuous. To obtain the regularity results, Aronson considered instead of (2.3) the equation satisfied by the pressure $v = u^{m-1}$, i.e.,

$$\partial_t v = mv\partial_x^2 v + \frac{m}{m-1}(\partial_x v)^2, \tag{1.4}$$

and its corresponding Cauchy problem

$$\begin{cases} \partial_t v = mv\partial_x^2 v + \dfrac{m}{m-1}(\partial_x v)^2, & \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = v_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{1.5}$$

and derive an a priori estimate for the Lipschitz norm of v as a function of x.

In the subsequent work, Aronson, 1970, Aronson additionally asserted that in (2.3), if u_0 is smooth and everywhere positive, this problem can be solved by essentially standard techniques for nonlinear parabolic equations. On the other hand, if u_0 has compact support in $\mathbb{R}^n$, then any solution of the Cauchy problem must also have compact support in $\mathbb{R}^n$ for each $t > 0$. Aronson also studied a particular example of a flow which shows what kind of singularities can occur when the initial data is not strictly positive. By considering (1.5) with the initial data $v_0(x) = \cos^2 x$, he proved that real analytic initial data is insufficient to ensure the existence of a classical solution of (1.5) for all positive times. Moreover, it is impossible in general to estimate the second and higher order derivatives of the solution of (1.5) in terms of the bounds for derivatives of the initial data. Similar results also hold when the initial data has compact support and for the first boundary value problem.

In Oleinik's paper (1958), Oleinik, Kalashnikov and Yui-Lin' have shown that if u_0^m is Lipschitz continuous then the corresponding Cauchy problem (2.3) possesses a unique weak solution u, and u satisfies the equation in the classical sense in the neighborhood of every point of the semi-infinite strip $S := \mathbb{R} \times (0,\infty)$ at which u is positive. They also showed that if u_0 has compact support then u has compact support as a function of x for all $t > 0$.

In the paper of Kalashnikov (1967), Kalashnikov showed that if u_0 has compact support and $m \geq 2$ then, no matter how smooth u_0 is, there always exist points of discontinuity of $\partial_x u$ in $\mathbb{R} \times (\tau,\infty)$ for every $\tau > 0$.

In Kruzhkov's paper (1969), Kruzhkov has used the results of Aronson, 1969 together with a general result on nonlinear parabolic equations to prove that the weak solution u of (2.3) is also Hölder continuous as a function of t.

Chapter 2

1-D Degenerate Diffusion Equation with Very Strong Absorption and Source

1 Introduction

In one-dimensional case, consider the following Cauchy problem which is denoted by (P),

$$\partial_t u - \partial_x^2 (u^m) + u^{-\beta} \chi_{\{u>0\}} = f(u, x, t), \text{ in } \mathbb{R} \times (0, T), \tag{2.1}$$

$$u(x, 0) = u_0(x), \text{ for } x \in \mathbb{R}, \tag{2.2}$$

where $m \geq 1$ and $p < m$ are positive real numbers, and where $0 \leq u_0 \in L^\infty(\mathbb{R})$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the sets of points (x, t) where $u(x, t) > 0$, i.e., $\chi_{\{u>0\}}$ is equal to one if $u(x, t) > 0$ and it vanishes elsewhere. In this chapter, the source term $f : [0, \infty) \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ will be assumed a nonnegative function satisfying the following hypothesis

$$(H_f) \quad \begin{cases} f \in \mathcal{C}^1([0, \infty) \times \mathbb{R} \times [0, \infty)), & \\ f(0, x, t) = 0, & \text{for all } (x, t) \in \mathbb{R} \times (0, \infty), \\ f(u, x, t) \leq h(u), & \text{for all } (x, t) \in \mathbb{R} \times (0, \infty), \end{cases}$$

where h is a locally Lipschitz function on $[0, \infty)$, $h(0) = 0$. We denote by $D_u f$, $D_x f$, and $D_t f$ the partial derivatives of f with respect to the first variable, the space variable x, and the time variable t, respectively.

To establish a local existence of solutions for (2.1), it is necessary to introduce a notion of weak solution, as follows.

Definition 2.1. *Let $u_0 \in L^\infty(\mathbb{R})$. A nonnegative function $u(x, t)$ is called a weak solution of equation (2.1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0, T))$, and $u \in L^p\left(0, T; W^{1,2}_{\mathrm{loc}}(\mathbb{R})\right) \cap L^\infty_{\mathrm{loc}}(\mathbb{R} \times (0, T)) \cap$*

$C\left([0,T);L^1_{\text{loc}}(\mathbb{R})\right)$ *satisfies equation* (2.1) *in the sense of distribution* $\mathcal{D}'(\mathbb{R}\times(0,T))$, *i.e.*,

$$\iint_{\mathbb{R}\times(0,T)}\left(-u\partial_t\varphi + mu^{m-1}\partial_x u\partial_x\varphi + u^{-\beta}\chi_{\{u>0\}}\varphi - f(u,x,t)\varphi\right)dxdt = 0,$$

for all $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}\times(0,T))$.

Remark 2.1. *This definition requires that the integral* $\iint_{\mathbb{R}\times(0,T)}u^{-\beta}\chi_{\{u>0\}}\varphi dxdt$ *exists, i.e.,* $u^{-\beta}\chi_{\{u>0\}}\in L^1_{\text{loc}}(\mathbb{R}\times(0,T))$, *because all other terms in the above equality are finite. This do not (and cannot) exclude the existence of some physically interesting solutions to* (2.1) *for which this integral fails to exist, see Kawohl and Kersner, 1992, Remark 1.*

2 Existence of Solutions

Our local existence result is as follows.

Theorem 2.1 (Local existence)**.** *Suppose that* $0<\beta<m$, $u_0\in L^\infty(\mathbb{R})$, $u_0\neq 0$, f *satisfy* (H_f). *Then, there exists a finite time* $T>0$ *such that* (2.1) *has a maximal weak solution* u *in* $\mathbb{R}\times(0,T)$, *i.e., for any weak solution* v *in* $\mathbb{R}\times(0,T)$, *we have*

$$v\le u,\ \text{ in } \mathbb{R}\times(0,T).$$

Moreover, there is a positive constant C *depending only on* m, β, f, τ, T, *and* $\|u_0\|_\infty$ *such that*

$$|\partial_x u(x,t)| \le \begin{cases} Cu^{1-\frac{m+\beta}{2}}(x,t), & \text{if } m<2+\beta,\\ Cu^{2-m}(x,t), & \text{if } m\ge 2+\beta,\end{cases} \tag{2.3}$$

for a.e. $(x,t)\in\mathbb{R}\times(\tau,T)$.

Besides, if, in addition, $\partial_x\left(u_0^{1/\gamma}\right)\in L^\infty(\mathbb{R})$, *where*

$$\gamma := \begin{cases} \dfrac{2}{m+\beta}, & \text{if } m<2+\beta,\\ \dfrac{1}{m-1}, & \text{if } m\ge 2+\beta,\end{cases} \tag{2.4}$$

then there is a positive constant C *depending only on* m, β, f, T, $\|u_0\|_\infty$, *and* $\left\|\partial_x\left(u_0^{1/\gamma}\right)\right\|_\infty$ *such that the gradient estimate* (2.3) *holds for a.e.* $(x,t)\in\mathbb{R}\times[0,T)$.

Remark 2.2. *The result of Theorem 2.1 implies that* u *is continuous up to the boundary. Furthermore,* u *is continuous up to* $t=0$ *provided that* $\partial_x\left(u_0^{1/\gamma}\right)\in L^\infty(\mathbb{R})$.

If $f(u,x,t)=f(u)$, then Theorem 2.1 still holds for a locally Lipschitz function f on $[0,\infty)$, instead of the requirement $f\in\mathcal{C}^2([0,\infty))$ in the previous works (see, e.g., Dávila and Montenegro, 2005, Galaktionov and Vázquez, 1995). For example, the existence result stated can take into account the function $f(u) = (u-s)^+ u$, which is a locally Lipschitz function on $[0,\infty)$ for any $s>0$.

3 First Strategy for the Approximate Solutions

3.1 Gradient Estimates for the Approximate Solutions

To prove Theorem 2.1, we may assume, without loss of generality, that $u_0(x)$ is sufficiently smooth, e.g., $u_0 \in C^\infty(\mathbb{R})$. Otherwise, it suffices to use standard approximation arguments. The objective is to regularize the singularity generated by the absorption term $u^{-\beta}\chi_{\{u>0\}}$.

For any $\varepsilon \in (0,1)$ and $\eta \in (0,\varepsilon]$, we consider the following regularized problem

$$(P_{\varepsilon,\eta}) \quad \begin{cases} \partial_t u_\varepsilon - \partial_x^2 (u_\varepsilon^m) + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t), & \text{in } \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (0,T), \\ u_\varepsilon(x,0) = u_0(x) + \eta, & \text{for } x \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ u_\varepsilon\left(\pm\frac{1}{\varepsilon}, t\right) = u_0\left(\pm\frac{1}{\varepsilon}\right) + \eta, & \text{for } t \in (0,T), \end{cases}$$

where $g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s)$, $\psi_\varepsilon(s) = \psi\left(\frac{s}{\varepsilon}\right)$, and $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \psi \le 1$ is a non-decreasing function such that[1]

$$\psi(s) = \begin{cases} 0, & \text{if } s \le 1, \\ 1, & \text{if } s \ge 2. \end{cases}$$

The problem $(P_{\varepsilon,\eta})$ can be considered as a regularization of (P). We claim that the solution $u_{\varepsilon,\eta}$ of the regularized problem $(P_{\varepsilon,\eta})$ converges to a solution of (P) as $\varepsilon \downarrow 0$. Here is the first result in this section, which indicates existence and uniqueness of a solution of $(P_{\varepsilon,\eta})$.

Lemma 2.1 (Existence, uniqueness, and gradient estimates of $u_{\varepsilon,\eta}$)**.** *Let $u_0 \in C^\infty(\mathbb{R})$, $u_0 \neq 0$. The first boundary value problem $(P_{\varepsilon,\eta})$ has a unique classical solution, denoted by $u_{\varepsilon,\eta}$, in $(-1/\varepsilon, 1/\varepsilon) \times (0,T)$. In addition, for every fixed positive ε,*

$$0 < \eta \le u_{\varepsilon,\eta} \le \|u_0\|_\infty + \eta < \|u_0\|_\infty + 1, \ \text{in } \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (0,T).$$

a) Moreover, there is a constant $C>0$ only depending on m, β, f, τ, T, and $\|u_0\|_\infty$ such that

$$\left|\partial_x \left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(x,t)\right)\right| \le C, \quad \text{for a.e. } (x,t) \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (\tau, T),$$

where γ is given by (2.4).

[1] The constant $C>0$ in all gradient estimates presented in this thesis also depends on this cut-off function.

b) If $\partial_x\left(u_0^{1/\gamma}\right) \in L^\infty(\mathbb{R})$, *then there is a constant* $C > 0$ *depending only on* m, β, f, T, $\|u_0\|_\infty$, *and* $\left\|\partial_x\left(u_0^{1/\gamma}\right)\right\|_\infty$, *where* γ *is given by* (2.4), *such that*

$$\left|\partial_x\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(x,t)\right)\right| \le C, \quad \textit{for a.e. } (x,t) \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times [0,T).$$

Lemma 2.2 (Monotonicity lemma). *If* $0 < \eta_1 < \eta_2 < \varepsilon$ *then* $u_{\varepsilon,\eta_1}(x,t) < u_{\varepsilon,\eta_2}(x,t)$ *in* $(-1/\varepsilon, 1/\varepsilon) \times (0,T)$.

Thanks to Lemma 2.2, we now obtain a monotone bounded sequence $\{u_{\varepsilon,\eta}(x,t)\}_{\eta\in(0,\varepsilon)}$, whose limit

$$u_\varepsilon(x,t) := \lim_{\eta\to 0^+} u_{\varepsilon,\eta}(x,t), \text{ in } Q_{\varepsilon,T}.$$

A classical argument allows us to pass to the limit as $\eta \to 0$ in order to obtain $u_{\varepsilon,\eta} \to u_\varepsilon$ (resp., $\nabla u_{\varepsilon,\eta} \to \nabla u_\varepsilon$) so that u_ε is a unique classical solution of the following problem

$$(P_\varepsilon)\begin{cases} \partial_t u - \partial_x^2 u^m + g_\varepsilon(u) = f(u,x,t), & \text{in } \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (0,T), \\ u(x,0) = u_0(x), & \text{for } x \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ u\left(\pm\frac{1}{\varepsilon}, t\right) = u_0\left(\pm\frac{1}{\varepsilon}\right), & \text{for } t \in (0,T), \end{cases}$$

Then the gradient estimates for $u_{\varepsilon,\eta}$ presented in the next section also hold for u_ε. Next, we will pass $\varepsilon \to 0^+$ to obtain an existence of solution of (P).

3.2 Local Existence

In this subsection, we give a proof of Theorem 2.1.

Proof of Theorem 2.1. Let u_ε be the unique solution of (P_ε) in $(-1/\varepsilon, 1/\varepsilon) \times (0,T)$. Then, we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded monotone sequence.

Indeed, we have

$$g_{\varepsilon_1}(s) \ge g_{\varepsilon_2}(s), \text{ for any } 0 < \varepsilon_1 < \varepsilon_2.$$

This implies that u_{ε_1} is a sub-solution of the equation satisfied by u_{ε_2}. Therefore, the comparison principle yields

$$u_{\varepsilon_1} \le u_{\varepsilon_2}, \text{ in } \left(-\frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_2}\right) \times (0,T), \quad \forall 0 < \varepsilon_1 < \varepsilon_2,$$

so the conclusion follows. Consequently, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0^+$.

Obviously, one has

$$\begin{aligned} f\left(u_\varepsilon, x, t\right) &\leq h\left(u_\varepsilon\left(x,t\right)\right) \leq \operatorname{Lip}\left(h, \Gamma\left(t\right)\right) u_\varepsilon\left(x,t\right) \leq \operatorname{Lip}\left(h, \Gamma\left(t\right)\right) \Gamma\left(t\right) \\ &\leq \operatorname{Lip}\left(h, \Theta\left(\Gamma, T\right)\right) \Theta\left(\Gamma, T\right), \text{ in } \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times \left(0, T\right). \end{aligned}$$

since $u_\varepsilon\left(x,t\right)$ is bounded by $\Gamma\left(t\right)$.

We now show that u is a weak solution of (2.1)-(2.2) (see Definition 2.1). This will be done in several steps, partly following ideas in Phillips, 1987.

Step 1. *Claim:* $u^{-\beta}\chi_{\{u>0\}} \in L^1_{\text{loc}}\left(\mathbb{R} \times (0,T)\right)$.

First proof of Step 1. The approximate solution u_ε satisfy the PDE in (P_ε) in the classical sense, i.e.,

$$\partial_t u_\varepsilon - \partial_x^2 u_\varepsilon^m + g_\varepsilon\left(u_\varepsilon\right) = f\left(u_\varepsilon, x, t\right), \text{ in } \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times \left(0, T\right).$$

Multiplying both sides of the last equation with a test function $\varphi \in C^{2,1}\left(R_\varepsilon\right)$ such that $\varphi\left(x_1, t\right) = \varphi\left(x_2, t\right) = 0$ in $[t_1, t_2]$, integrating these in R_ε, and then using integration by parts formula, where $R_\varepsilon = [x_1, x_2] \times [t_1, t_2] \subset (-1/\varepsilon, 1/\varepsilon) \times (0, T)$ is an arbitrary nonempty bounded rectangles, the following equation holds

$$\iint_{R_\varepsilon} \left(u_\varepsilon \partial_t \varphi + u_\varepsilon^m \partial_x^2 \varphi + f\left(u_\varepsilon, x, t\right)\varphi\right) dxdt - \int_{t_1}^{t_2} u_\varepsilon^m \partial_x \varphi\big|_{x_1}^{x_2} dt - \int_{x_1}^{x_2} u_\varepsilon \varphi\big|_{t_1}^{t_2} dx = \iint_{R_\varepsilon} g_\varepsilon\left(u_\varepsilon\right)\varphi dxdt.$$

We can pass to the limit in the left-hand side of this equation. So it follows from Fatou's lemma that $u^{-\beta}\chi_{\{u>0\}} \in L^1_{\text{loc}}\left(\mathbb{R} \times (0,T)\right)$, since

$$\iint_{R_\varepsilon} u^{-\beta}\chi_{\{u>0\}}\varphi dxdt \leq \varliminf_{\varepsilon \to 0^+} \iint_{R_\varepsilon} g_\varepsilon\left(u_\varepsilon\right)\varphi dxdt < \infty.$$

By the last equation and the boundedness of $f\left(u_\varepsilon, x, t\right)$, we can use a result of gradient convergence of Boccardo and Gallouët, 1989; Boccardo and Murat, 1992, in order to obtain

$$\partial_x u_\varepsilon \to \partial_x u \text{ as } \varepsilon \to 0^+, \text{ uniformly locally, for a.e. } (x,t) \in \mathbb{R} \times (0,T). \tag{2.5}$$

As a result, $\partial_x u$ fulfills the gradient estimates in Lemma 2.1 for a.e. $(x,t) \in \mathbb{R} \times (0,T)$, and

$$\partial_x u_\varepsilon \to \partial_x u \text{ as } \varepsilon \to 0^+, \text{ uniformly locally in } L^r\left(\mathbb{R} \times (0,T)\right), \ \forall r \in [1, \infty). \tag{2.6}$$

Step 2. *Claim: u solves (2.1) in the sense of distributions.*

For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\varphi$, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,T))$, to the PDE in (P_ε). Then, using integration by parts yields

$$\iint_{\mathrm{Supp}(\varphi)} \begin{pmatrix} -\Psi_\eta(u_\varepsilon)\partial_t\varphi + \frac{m}{\eta}u_\varepsilon^{m-1}\psi'\left(\frac{u_\varepsilon}{\eta}\right)(\partial_x u_\varepsilon)^2\varphi + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\varphi \\ +mu_\varepsilon^{m-1}\psi_\eta(u_\varepsilon)\partial_x u_\varepsilon\partial_x\varphi - f(u_\varepsilon,x,t)\psi_\eta(u_\varepsilon)\varphi \end{pmatrix} dxdt = 0,$$

with

$$\Psi_\eta(u_\varepsilon) = \int_0^{u_\varepsilon}\psi_\eta(s)\,ds.$$

Note that $\psi_\eta(\cdot)$ plays a role in avoiding the singularity of the term $u^{-\beta}\chi_{\{u>0\}}$ as u is near 0. Thus there is no problem of going to the limit as $\varepsilon \to 0$ in the indicated equation to obtain

$$\iint_{\mathrm{Supp}(\varphi)} \begin{pmatrix} -\Psi_\eta(u)\partial_t\varphi + \frac{m}{\eta}u^{m-1}\psi'\left(\frac{u}{\eta}\right)(\partial_x u)^2\varphi + u^{-\beta}\psi_\eta(u)\varphi \\ +mu^{m-1}\psi_\eta(u)\partial_x u\partial_x\varphi - f(u,x,t)\psi_\eta(u)\varphi \end{pmatrix} dxdt = 0.$$

Next, we go to the limit as $\eta \to 0$ in the last equation.

By (2.5), (2.6), and the local integrability of $u^{-\beta}\chi_{\{u>0\}}$ in $\mathbb{R}\times(0,T)$, it is not difficult to verify

$$\begin{cases} \lim\limits_{\eta\to 0^+}\iint_{\mathrm{Supp}(\varphi)}\Psi_\eta(u)\partial_t\varphi dxdt = \iint_{\mathrm{Supp}(\varphi)} u\partial_t\varphi dxdt, \\ \lim\limits_{\eta\to 0^+}\iint_{\mathrm{Supp}(\varphi)} mu^{m-1}\psi_\eta(u)\partial_x u\partial_x\varphi dxdt = \iint_{\mathrm{Supp}(\varphi)} mu^{m-1}\partial_x u\partial_x\varphi dxdt, \\ \lim\limits_{\eta\to 0^+}\iint_{\mathrm{Supp}(\varphi)} u^{-\beta}\psi_\eta(u)\varphi dxdt = \iint_{\mathrm{Supp}(\varphi)} u^{-\beta}\chi_{\{u>0\}}\varphi dxdt, \\ \lim\limits_{\eta\to 0^+}\iint_{\mathrm{Supp}(\varphi)} f(u,x,t)\psi_\eta(u)\varphi dxdt = \iint_{\mathrm{Supp}(\varphi)} f(u,x,t)\varphi dxdt. \end{cases}$$

Note that the assumption $f(0,x,t) = 0$ is used in the final limit.

To verify that u is a distributional solution of (2.1), it remains to show that the term involving ψ_η' vanishes in the limit, i.e.,

$$\lim_{\eta\to 0^+}\iint_{\mathrm{Supp}(\varphi)}\frac{m}{\eta}u^{m-1}\psi'\left(\frac{u}{\eta}\right)(\partial_x u)^2\varphi dxdt = 0.$$

Observing that

$$\iint_{\mathrm{Supp}(\varphi)}\frac{m}{\eta}u^{m-1}\psi'\left(\frac{u}{\eta}\right)(\partial_x u)^2\varphi dxdt \le \frac{m}{\eta}\|\psi'\|_\infty\int_{\mathrm{Supp}(\varphi)\cap\{\eta<u<2\eta\}}u^{m-1}(\partial_x u)^2\varphi dxdt,$$

we distinguish now two cases dictated by Theorem 2.1, or Lemma 2.1.

Case $m < 2 + \beta$**.** Lemma 2.1 and the gradient convergence result of Boccardo give us

$$u^{m-1}(\partial_x u)^2 \leq C u^{m-1} u^{2-m-\beta} = C u^{1-\beta},$$

where $C > 0$ is a constant only depending on m, β, f, T, $\|u_0\|_\infty$, and $\left\|\partial_x\left(u_0^{(m+\beta)/2}\right)\right\|_\infty$. Gathering these inequalities, one has

$$\begin{aligned}\iint_{\mathrm{Supp}(\varphi)} \frac{m}{\eta} u^{m-1} \psi'\left(\frac{u}{\eta}\right)(\partial_x u)^2 \varphi dx dt &\leq \frac{Cm}{\eta} \|\psi'\|_\infty \int_{\mathrm{Supp}(\varphi) \cap \{\eta < \mathrm{u} < 2\eta\}} u^{1-\beta} \varphi dx dt \\ &\leq Cm \|\psi'\|_\infty \iint_{\mathrm{Supp}(\varphi) \cap \{\eta < \mathrm{u} < 2\eta\}} u^{-\beta} \varphi dx dt \to 0 \text{ as } \eta \to 0^+,\end{aligned}$$

since $u^{-\beta}\chi_{\{u>0\}}$ is locally integrable on $\mathbb{R} \times (0, T)$.

Case $m \geq 2 + \beta$**.** The gradient estimate stated in Lemma 2.1 gives us

$$u^{m-1}(\partial_x u)^2 \leq C u \partial_x u,$$

where $C > 0$ is also a constant depending only on the terms stated in the first case.

Thus,

$$\begin{aligned}\iint_{\mathrm{Supp}(\varphi)} \frac{m}{\eta} u^{m-1} \psi'\left(\frac{u}{\eta}\right)(\partial_x u)^2 \varphi dx dt &\leq \frac{Cm}{\eta} \|\psi'\|_\infty \iint_{\mathrm{Supp}(\varphi) \cap \{\eta < \mathrm{u} < 2\eta\}} u \partial_x u \varphi dx dt \\ &\leq Cm \|\varphi\psi'\|_\infty \int_{\mathrm{Supp}(\varphi) \cap \{\eta < \mathrm{u} < 2\eta\}} |\partial_x u| \, dx dt.\end{aligned}$$

The last integral is a quantity tending to zero as $\eta \to 0^+$, because u_ε has locally finite lap-number, see Chen's paper (1990).

Sum up these cases, we deduce that

$$\iint_{\mathrm{Supp}(\varphi)} \left(-u \partial_t \varphi + m u^{m-1} \partial_x u \partial_x \varphi + u^{-\beta} \chi_{\{u>0\}} \varphi - f(u, x, t) \varphi\right) dx dt = 0.$$

In other words, u satisfies (2.1) in $\mathcal{D}'(\mathbb{R} \times (0, T))$. In conclusion, u is a weak solution of (2.1).

To complete the proof, it remains to show that u is the maximal solution of equation (2.1).

Proposition 2.1. *Let v be any weak solution of equation (2.1) on $\mathbb{R} \times (0, T)$. Then*

$$v(x, t) \leq u(x, t), \text{ for a.e. } (x, t) \in \mathbb{R} \times (0, T).$$

Proof of Proposition 2.1. In fact, we observe that

$$g_\varepsilon(v) \leq v^{-\beta} \chi_{\{v>0\}}, \text{ for all } \varepsilon > 0.$$

Thus,

$$\partial_t v - \partial_x^2 v^m + g_\varepsilon(v) \leq f(v, x, t), \text{ in } \mathcal{D}'(\mathbb{R} \times (0, T)),$$

which implies that v is a sub-solution of the PDE in (P_ε).

By the comparison principle, we get

$$v(x, t) \leq u_\varepsilon(x, t), \text{ for a.e. } (x, t) \in \mathbb{R} \times (0, T).$$

Letting $\varepsilon \to 0^+$ yields the desired result. □

This proposition also ends our proof. □

Remark 2.3. *If $f(u, x, t) = f(u)$, and f is a locally Lipschitz function on $[0, \infty)$. Then, the result of existence of a maximal solution still holds.*

3.3 Regularity is Optimal

Kawohl, 1992 presented two examples illustrating that the regularity is optimal when $f = 0$.

4 Second Strategy for the Approximate Solutions

Another approach which is deserved to mention is as follows. For any $\varepsilon \in (0, 1)$, we consider the problem $(P_{\varepsilon,\varepsilon})$, whose unique classical solution satisfies the following gradient estimates, which is a direct corollary of Lemma 2.1 by letting $\eta = \varepsilon$.

Corollary 2.1 (Existence, uniqueness, and gradient estimates of u_ε)**.** *Let $u_0 \in C^\infty(\mathbb{R})$, $u_0 \neq 0$. The first boundary value problem $(P_{\varepsilon,\varepsilon})$ has a unique classical solution, denoted by $u_{\varepsilon,\varepsilon}$, in $(-1/\varepsilon, 1/\varepsilon) \times (0, T)$. In addition, for every $\varepsilon > 0$,*

$$0 < \varepsilon \leq u_{\varepsilon,\varepsilon} \leq \|u_0\|_\infty + \varepsilon < \|u_0\|_\infty + 1, \ \textit{in} \ \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (0, T).$$

a) Moreover, there exists a constant $C > 0$ only depending on m, β, f, τ, T, and $\|u_0\|_\infty$ such that

$$\left|\partial_x \left(u_{\varepsilon,\varepsilon}^{\frac{1}{\gamma}}(x, t)\right)\right| \leq C, \ \textit{for a.e.} \ (x, t) \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (\tau, T),$$

where γ is given by (2.4).

b) If $[\partial_x\left(u_0^{1/\gamma}\right) \in L^\infty(\mathbb{R})$, *then there is a constant* $C > 0$ *depending only on* m, β, f, T, $\|u_0\|_\infty$, *and* $\left\|\partial_x\left(u_0^{1/\gamma}\right)\right\|_\infty$, *where* γ *is again given by* (2.4), *such that*

$$\left|\partial_x\left(u_{\varepsilon,\varepsilon}^{\frac{1}{\gamma}}(x,t)\right)\right| \leq C, \text{ for a.e. } (x,t) \in \left(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right) \times [0,T).$$

The following lemma, which is also a direct corollary of Lemma 2.2 by letting $\eta = \varepsilon$, indicates the monotonicity of the sequence $\{u_{\varepsilon,\varepsilon}(x,t)\}_{\varepsilon>0}$.

Corollary 2.2 (Monotonicity of u_ε's)**.** *If* $0 < \varepsilon_1 < \varepsilon_2$ *then* $u_{\varepsilon_1}(x,t) \leq u_{\varepsilon_2}(x,t)$ *in* $(-1/\varepsilon_2, 1/\varepsilon_2) \times (0,T)$.

Thanks to Corollary 2.2, we now obtain a monotone bounded sequence $\{u_{\varepsilon,\varepsilon}(x,t)\}_{\varepsilon>0}$, whose limit

$$u(x,t) := \lim_{\varepsilon\to 0^+} u_{\varepsilon,\varepsilon}(x,t), \text{ in } \mathbb{R}\times(0,T).$$

All the following steps are similar to the settings of the approximate solution $u_{\varepsilon,\eta}$ as in previous section.

Chapter 3

N-D Degenerate Diffusion Equation with Very Strong Absorption and Source

1 Introduction

In this chapter, we are interested in nonnegative solutions of the following problem in mutli-dimensions.

$$(P)\quad \begin{cases} \partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = f(u,x,t), & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ for some $N \in \mathbb{Z}^+$, $m \geq 1$, $0 < \beta < m$, $0 \leq u_0 \in L^\infty(\Omega)$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e.,

$$\chi_{\{u>0\}}(x,t) = \begin{cases} 1, & \text{if } u(x,t) > 0, \\ 0, & \text{if } u(x,t) \leq 0. \end{cases}$$

In addition, the constants m, β, and N are assumed to satisfy the following hypothesis, denoted by (H),

$$\left[\begin{array}{l} (H_1)\ \ N = 1 \text{ and } m \geq \beta + 2, \\ (H_2)\ \ N \in \mathbb{Z}^+,\ m = 1,\ \text{and } 0 < \beta < 1, \\ (H_3)\ \ N = 1 \text{ and } 1 < m < \beta + 2, \\ (H_4)\ \ N \geq 2,\ 1 < m < 1 + (N-1)^{-\frac{1}{2}},\ \text{and } \gamma_1(m,N) < 2/(m+\beta) < \gamma_2(m,N), \end{array}\right. \tag{3.2}$$

where $\gamma_1(m,N)$ and $\gamma_2(m,N)$ are the roots of the quadratic equation

$$\left[(N-1)(m-1)^2 + 4m(m-1)\right]\gamma^2 - 4(2m-1)\gamma + 4 = 0,$$

and given by

$$\gamma_1(m,N)=\frac{2\left(2m-1-\sqrt{1-(N-1)(m-1)^2}\right)}{(N-1)(m-1)^2+4m(m-1)},$$
$$\gamma_2(m,N)=\frac{2\left(2m-1+\sqrt{1-(N-1)(m-1)^2}\right)}{(N-1)(m-1)^2+4m(m-1)}.$$

Notice that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose tactically $u^{-\beta}\chi_{\{u>0\}}(x,t)=0$ whenever $u(x,t)=0$.

Through this thesis, the source term $f:[0,\infty)\times\overline{\Omega}\times[0,\infty)\to\mathbb{R}$ is assumed to be a nonnegative function satisfying the following hypothesis

$$(H_f)\quad\begin{cases} f\in\mathcal{C}^1\left([0,\infty)\times\overline{\Omega}\times[0,\infty)\right), & \\ f(0,x,t)=0, & \text{for all } (x,t)\in\Omega\times(0,\infty), \\ f(u,x,t)\le h(u), & \text{for all } (x,t)\in\Omega\times(0,\infty), \end{cases}$$

where h is a locally Lipschitz function on $[0,\infty)$, $h(0)=0$. We denote by D_uf and D_xf the partial derivatives of f with respect to the first variable and the space variable x, respectively.

In the sequel, we always consider nonnegative initial data $u_0\neq 0$. Before our discussion on the behaviors of solutions of equation (3.1), it is necessary to introduce a notion of weak solution, and establish first a local existence of solutions for equation (3.1).

Definition 3.1. *Let $u_0\in L^\infty(\Omega)$. A nonnegative function $u(x,t)$ is called a weak solution of equation (3.1) if $u^{-\beta}\chi_{\{u>0\}}\in L^1(\Omega\times(0,T))$, and $u\in L^p\left(0,T;W_0^{1,2}(\Omega)\right)\cap L^\infty(\Omega\times(0,T))\cap\mathcal{C}\left([0,T);L^1(\Omega)\right)$ satisfies equation (3.1) in the sense of distribution $\mathcal{D}'(\Omega\times(0,T))$, i.e.,*

$$\int_0^T\int_\Omega\left(-u\varphi_t+mu^{m-1}\nabla u\cdot\nabla\varphi+u^{-\beta}\chi_{\{u>0\}}\varphi-f(u,x,t)\varphi\right)dxdt=0,\tag{3.3}$$

for all $\varphi\in\mathcal{C}_c^\infty(\Omega\times(0,T))$.

2 Existence of Solutions

Our local existence result is as follows.

Theorem 3.1 (Local existence). *Suppose that (m,β,N) satisfies (H), $u_0\in L^\infty(\Omega)$, $u_0\neq 0$, f satisfy (H_f). Then, there exists a finite time $T>0$ such that (3.1) has a maximal weak solution u in $\Omega\times(0,T)$, i.e., for any weak solution v in $\Omega\times(0,T)$, we have*

$$v\le u,\ \ \text{in } \Omega\times(0,T),$$

Moreover, there is a positive constant C depending only on m, β, N, f, τ, T, and $\|u_0\|_\infty$ such that

$$\left|\nabla\left(u^{\frac{1}{\gamma}}(x,t)\right)\right| \leq C, \text{ for a.e. } (x,t) \in \Omega \times (\tau, T),$$

where

$$\gamma = \begin{cases} \dfrac{1}{m-1}, & \text{if } m, \beta, N \text{ satisfy } (H_1), \\ \dfrac{2}{m+\beta}, & \text{otherwise}. \end{cases} \tag{3.4}$$

Besides, if, in addition, $\nabla\left(u_0^{1/\gamma}\right) \in L^\infty(\Omega)$, where γ is also given by (3.4), then there is positive constant C depending only on m, β, N, f, T, $\|u_0\|_\infty$, and $\left\|\nabla\left(u_0^{1/\gamma}\right)\right\|_\infty$ such that

$$\left|\nabla\left(u^{\frac{1}{\gamma}}(x,t)\right)\right| \leq C, \text{ for a.e. } (x,t) \in \Omega \times [0, T).$$

Remark 3.1. *The result of Theorem 3.1 indicates that u is continuous up to the boundary. Furthermore, u is continuous up to $t = 0$ provided that $\nabla\left(u_0^{1/\gamma}\right) \in L^\infty(\Omega)$.*

If $f(u,x,t) = f(u)$, then Theorem 3.1 still holds for a locally Lipschitz function f on $[0,\infty)$, instead of the requirement $f \in \mathcal{C}^2([0,\infty))$ in the previous works (see, e.g., Dávila and Montenegro, 2005, Galaktionov and Vázquez, 1995). For instance, the existence result stated can allow for possibility the function $f(u) = (u-s)^+ u$, which is a locally Lipschitz function on $[0,\infty)$ for any $s > 0$.

3 Gradient Estimate for the Approximate Solution

We shall modify Bernstein's technique to obtain a variety of estimates on $|\nabla u|$. Roughly speaking, the gradient estimates that will be proved are of the type $\left|\nabla\left(u^{1/\gamma}\right)\right| \leq C$, i.e.,

$$|\nabla u(x,t)| \leq C u^{1-\frac{1}{\gamma}}(x,t) \text{ for a.e. } (x,t) \in \Omega \times (0,\infty).$$

It is known that such a gradient estimate plays a crucial role in proving the existence of a solutions. First we shall establish some gradient estimates for solutions of a regularized problem of (P).

For any $\varepsilon > 0$, let us set

$$g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s), \text{ with } \psi_\varepsilon(s) = \psi\left(\frac{s}{\varepsilon}\right), \tag{3.5}$$

where $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$ is a non-decreasing function such that

$$\psi(s) = \begin{cases} 0, & \text{if } s \leq 1, \\ 1, & \text{if } s \geq 2. \end{cases} \tag{3.6}$$

Now fix $\varepsilon \in (0,1)$, we consider the following regularized problem

$$(P_{\varepsilon,\eta}) \quad \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon^m + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t), & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon(x,t) = \eta, & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_0(x) + \eta, & \text{in } \Omega, \end{cases} \tag{3.7}$$

for any $0 < \eta < \varepsilon$.

Lemma 3.1 (Global Lipschitz continuity of g_ε)**.** *For any $\varepsilon > 0$, g_ε is a global Lipschitz function and the global Lipschitz constant of g_ε satisfies* $\mathrm{Lip}(g_\varepsilon) \le \beta + \varepsilon^{-1}\|\psi'\|_\infty$.

The problem $(P_{\varepsilon,\eta})$ can be contemplated as a regularization of (P). We will show that the solution[1] $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ tends to a solution of (P) when $\varepsilon \downarrow 0$. In passing to the limit, it is decisive to derive some gradient estimates for the solution $u_{\varepsilon,\eta}$. Before stating the gradient estimates presented below, we need some auxiliary results.

Lemma 3.2. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a mapping. Then the following equalities hold*

$$\begin{aligned} \nabla\left(|\nabla u|^2\right) &= 2D^2u \cdot \nabla u, \\ \Delta\left(|\nabla u|^2\right) &= 2\left|D^2u\right|^2 + 2\nabla u \cdot \nabla\Delta u, \end{aligned}$$

where we denote by D^2u the Hessian matrix of u, and $\left|D^2u\right|$ is its Frobenius norm.

The following lemma is a generalization of Bénilan's trick (see, e.g., Benachour, Iagar, and Laurençot, 2016), which is an efficient tool used to handle the term $|\nabla u|^2\Delta u$ appearing in many gradient estimates in $\mathbb{R}^N$ with $N \ge 2$.

Lemma 3.3 (Generalization of Bénilan's trick)**.** *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a mapping and $F : \mathbb{R} \to \mathbb{R}$ be a nonnegative function being of the class C^1. Then the following inequality holds*

$$F(u)\left|D^2u\right|^2 + F'(u)\left(\frac{1}{2}\nabla u \cdot \nabla\left(|\nabla u|^2\right) - |\nabla u|^2\Delta u\right) \ge -\frac{(N-1)(F')^2(u)|\nabla u|^4}{4F(u)},$$

at any point $x \in \mathbb{R}^N$ satisfying $F(u(x)) \ne 0$.

Proposition 3.1 (Gradient estimates for $u_{\varepsilon,\eta}$)**.** *Let $0 \le u_0 \in \mathcal{C}_c^\infty(\Omega)$, $u_0 \not\equiv 0$. Then, for any $0 < \eta < \varepsilon < 1$, there exists a unique classical solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ in $\Omega \times (0,T)$.*

a) In addition, for every $\tau > 0$, there is a positive constant $C > 0$ only depending on m, β, N, f, τ, T, and $\|u_0\|_\infty$ such that

$$\left|\nabla\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}\right)\right| \le C \text{ in } \Omega \times (\tau, T), \tag{3.8}$$

where γ is given by (3.4).

[1]The uniqueness of the solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ will be proved in the proof of Lemma 3.2.

b) Furthermore, if $\nabla\left((u_0(x)+\eta)^{1/\gamma}\right) \in L^\infty(\Omega)$ *for all* $\eta \in (0,\eta_0)$ *for some* $\eta_0 > 0$*, and*

$$U_0(x) := \sup_{\eta\in\left(0,\min\left\{\eta_0,\|u_0\|_\infty\right\}\right)} \left\|\nabla\left((u_0(x)+\eta)^{1/\gamma}\right)\right\|_\infty \in L^\infty(\Omega),$$

then there exists a positive constant $C > 0$ *merely depending on* m*,* β*,* N*,* f*,* T*,* $\|u_0\|_\infty$*, and* $\|U_0\|_\infty$ *such that*

$$\left|\nabla\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}\right)\right| \le C \text{ in } \Omega\times[0,T),$$

where γ *is also given by* (3.4)*.*

The original idea, in one-dimensional (1-D) settings, of Proposition 3.1 comes from Kawohl and Kersner, 1992. In their paper, they considered the function

$$\widetilde{g}_\varepsilon(u) := \frac{u}{\varepsilon + u^{1+\beta}}, \text{ for } \varepsilon > 0,$$

to regularize the given absorption term $u^{-\beta}\chi_{\{u>0\}}$.

Proposition 3.2 (Further gradient estimates for (H_2), Dao and Díaz, 2017; Dao, 2017)**.** *Let* (m,β,N) *satisfy* (H_2)*, and* $0 \le u_0 \in \mathcal{C}_c^\infty(\Omega)$*,* $u_0 \ne 0$*. There exists a classical unique solution* $u_{\varepsilon,\eta}$ *of* $(P_{\varepsilon,\eta})$ *in* $\Omega\times(0,T)$*.*

a) Moreover, there is a constant $C > 0$ *only depending on* β*,* T*,* f*,* $\|u_0\|_\infty$ *such that*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\left(\tau^{-1}+1\right), \text{ for any } (x,\tau)\in\Omega\times(0,T).$$

b) Furthermore, if $\nabla\left(u_0^{1/\gamma}\right) \in L^\infty(\Omega)$ *then we get*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau), \text{ for any } (x,\tau)\in\Omega\times(0,T),$$

with $C > 0$ *merely depends on* β*,* T*,* f*,* $\|u_0\|_\infty$*, and* $\left\|\nabla\left(u_0^{1/\gamma}\right)\right\|_\infty$*.*

Similarly, we can obtain further gradient estimates for (H_3), (H_4).

Remark 3.2. *If* $f(u,x,t)$ *is independent of the space variable* x*, then the term* $\Theta(D_x f,\cdot)$ *in the proofs of these further gradient estimates can be relaxed.*

If $f(u,x,t) = f(u)$, and f is merely a locally Lipschitz function on $[0,\infty)$, then we have the following result for (H_2).

Proposition 3.3 (Gradient estimates for (H_2), f local Lipschitz)**.** *Let* (m,β,N) *satisfy* (H_2)*. Suppose that* f *is a locally Lipschitz nonnegative function on* $[0,\infty)$*, and* $f(0) = 0$*. Then the* *equation* $(P_{\varepsilon,\eta})$ *has a unique solution in* $\Omega \times (0,T)$ *satisfying*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\left(1+\tau^{-1}\Theta^{1+\beta}(\Gamma,T)+\Theta^{1+\beta}(\Gamma,T)\operatorname{Lip}(f,\Theta(\Gamma,T))\right),$$

for $(x,\tau) \in \Omega \times (0,T)$*, where* $\operatorname{Lip}(f,r)$ *is the local Lipschitz constant of* f *on the closed interval* $[0,r]$*.*

Moreover, if $\nabla\left(u_0^{1/\gamma}\right) \in L^\infty(\Omega)$*, then we have*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\left(1+\Theta^{1+\beta}(\Gamma,T)\operatorname{Lip}(f,\Theta(\Gamma,T))\right).$$

Proof of Proposition 3.3. First of all we regularize f on $[0,\infty)$. To this end, we extend f by 0 in $(-\infty,0)$ (still denoted by f). Let f be the standard regularization of f on $\mathbb{R}$. Then, we consider the problem $(P_{\varepsilon,\eta})$ with the source $f_n(u_{\varepsilon,\eta})$ instead of $f(u_{\varepsilon,\eta})$. Thanks to Proposition 3.2 and 3.2, $(P_{\varepsilon,\eta})$ has a unique classical solution, denoted by $u_{\varepsilon,\eta,n}$, satisfying

$$|\nabla u_{\varepsilon,\eta,n}(x,\tau)|^2 \leq \frac{(1+\beta)^2}{2(1-\beta)}+\frac{c_0(1+\beta)}{2\tau(1-\beta)}\Theta^{1+\beta}(\Gamma,T)+\frac{1+\beta}{1-\beta}\Theta^{1+\beta}(\Gamma,T)\,\Theta\left({f_n}',\Theta(\Gamma,T)\right).$$

for any $(x,\tau) \in \Omega \times (0,T)$.

On the other hand, Rademacher's theorem (see Evans and Gariepy, 2015) implies that

$$\Theta\left({f_n}',\Theta(\Gamma,T)\right) \leq \operatorname{Lip}\left(f,\Theta(\Gamma,T)+\frac{1}{n}\right) \leq \operatorname{Lip}(f,2\Theta(\Gamma,T)).$$

By the last two inequalities, we observe that $|\nabla u_{\varepsilon,\eta,n}(x,\tau)|$ is bounded by a constant not depending on n. Then, the classical argument allows us to pass to the limit as $n \to \infty$ to get

$$u_{\varepsilon,\eta,n} \to u_{\varepsilon,\eta},\ \ \nabla u_{\varepsilon,\eta,n} \to \nabla u_{\varepsilon,\eta},\ \text{pointwise in } \Omega \times (0,T).$$

Thus, the first desired gradient estimate follows.

Similarly, we also obtain the second one. □

As a consequence of Proposition 3.2 (resp., proposition 3.3), we have the following regularity results.

Proposition 3.4 (Regularity)**.** *Let* (m,β,N) *satisfy* (H_2)*, and* $u_{\varepsilon,\eta}$ *be the solution of* $(P_{\varepsilon,\eta})$*. Then, we have*

$$|u_{\varepsilon,\eta}(x,t)-u_{\varepsilon,\eta}(y,s)| \leq C\left(|x-y|+|t-s|^{\frac{1}{3}}\right),\ \ \forall (x,t),(y,s) \in \Omega \times (\tau,T), \tag{3.9}$$

for any $\tau > 0$*, where* C *depends on* τ*,* T*,* f*, and* $\|u_0\|_\infty$*.*

Moreover, if $\nabla\left(u_0^{1/\gamma}\right) \in L^\infty(\Omega)$*, then* (**??**) *holds for any* (x,t)*,* $(y,s) \in \Omega \times (0,T)$*, and* C *depends on* T*,* f*,* $\|u_0\|_\infty$*, and* $\left\|\nabla\left(u_0^{1/\gamma}\right)\right\|_\infty$*.*

Proof of Proposition 3.4. We refer to the proof of Dao and Díaz, 2016, Proposition14, or Phillips, 1987. □

Obviously, all the above estimates are independent of ε, η. Thus, a classical argument allows us to pass to the limit as $\eta \to 0^+$ in order to obtain $u_{\varepsilon,\eta} \to u_\varepsilon$ (resp., $\nabla u_{\varepsilon,\eta} \to \nabla u_\varepsilon$) uniformly on $\Omega \times (0,T)$, where u_ε is the unique classical solution of the following equation

$$(P_\varepsilon) \quad \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon^m + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t), & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_0(x), & \text{in } \Omega. \end{cases}$$

Thus, the gradient estimates presented in Lemma 3.1 also hold for u_ε.

Next, we will send $\varepsilon \to 0^+$ to obtain an existence of solution of equation (3.1).

4 Local Existence

In this section, we provide a proof of Theorem 3.1.

Proof of Theorem 3.1. Let u_ε be the unique solution of equation (P_ε) in $\Omega \times (0,T)$. Then we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded monotone sequence.

Indeed, we have

$$g_{\varepsilon_1}(s) \geq g_{\varepsilon_2}(s), \text{ for any } 0 < \varepsilon_1 < \varepsilon_2.$$

This implies that u_{ε_1} is a sub-solution of the equation satisfied by u_{ε_2}. Therefore, the comparison principle yields

$$u_{\varepsilon_1} \leq u_{\varepsilon_2}, \text{ in } \Omega \times (0,T), \quad \forall \varepsilon_1 < \varepsilon_2,$$

so the conclusion follows. As a consequence, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0^+$.

Obviously, we have

$$\begin{aligned} f(u_\varepsilon, x, t) &\leq h(u_\varepsilon(x,t)) \leq \mathrm{Lip}(h, \Gamma(t))\, u_\varepsilon(x,t) \leq \mathrm{Lip}(h, \Gamma(t))\, \Gamma(t) \\ &\leq \mathrm{Lip}(h, \Theta(\Gamma, T))\, \Theta(\Gamma, T), \text{ in } \Omega \times (0,T), \end{aligned}$$

where $\Theta(g,r) := \max_{0 \leq s \leq r} |g(s)|$, since $u_\varepsilon(x,t)$ is bounded by $\Gamma(t)$.

Integrating the PDE in (P_ε) and then using Green's first identity yield

$$\int_\Omega u_\varepsilon(x,T)\, dx - \int_0^T \int_{\partial\Omega} m u_\varepsilon^{m-1} \nabla u_\varepsilon \cdot \mathbf{n} d\sigma dt + \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)\, dx dt$$

$$= \int_{\Omega} u_{\varepsilon}(x,0)\,dx + \int_0^T \int_{\Omega} f(u_{\varepsilon}, x, t)\,dxdt,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Since $\nabla u_{\varepsilon} \cdot \mathbf{n} \le 0$, we obtain

$$\int_0^T \int_{\Omega} g_{\varepsilon}(u_{\varepsilon})\,dxdt \le \int_{\Omega} u_0(x)\,dx + \int_0^T \int_{\Omega} f(u_{\varepsilon}, x, t)\,dxdt.$$

Combining the last inequality with the above Lipschitz-estimate yields

$$\int_0^T \int_{\Omega} g_{\varepsilon}(u_{\varepsilon})\,dxdt \le \int_{\Omega} u_0(x)\,dx + T\,|\Omega|\,\mathrm{Lip}\,(h, \Theta(\Gamma, T))\,\Theta(\Gamma, T)\,. \tag{3.10}$$

This implies that $\|g_{\varepsilon}(u_{\varepsilon})\|_{L^1(\Omega\times(0,T))}$ is bounded by a constant not depending on ε.

Thanks to Fatou's lemma, there is a function $\Upsilon \in L^1(\Omega \times (0,T))$ such that

$$\liminf_{\varepsilon\to 0^+} g_{\varepsilon}(u_{\varepsilon})(x,t) = \Upsilon, \text{ in } L^1(\Omega \times (0,T))\,. \tag{3.11}$$

Next, the monotonicity of $\{u_{\varepsilon}\}_{\varepsilon>0}$ implies

$$g_{\varepsilon}(u_{\varepsilon})(x,t) \ge g_{\varepsilon}(u_{\varepsilon})\,\chi_{\{u>0\}}(x,t)\,, \text{ for all } (x,t) \in \Omega \times (0,T)\,,$$

so

$$\liminf_{\varepsilon\to 0^+} g_{\varepsilon}(u_{\varepsilon})(x,t) = \Upsilon(x,t) \ge u^{-\beta}\chi_{\{u>0\}}(x,t)\,, \text{ for all } (x,t) \in \Omega \times (0,T)\,, \tag{3.12}$$

which implies that $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega \times (0,T)$.

Actually, we will prove

$$\Upsilon = u^{-\beta}\chi_{\{u>0\}}, \text{ in } L^1(\Omega \times (0,T))\,. \tag{3.13}$$

On the other hand, by (3.10) and the boundedness of $f(u_{\varepsilon}, x, t)$, we can use a result of gradient convergence of Boccardo and Gallouët, 1989, Boccardo and Murat, 1992 in order to obtain

$$\nabla u_{\varepsilon} \to \nabla u \text{ as } \varepsilon \to 0, \text{ for a.e. } (x,t) \in \Omega \times (0,T)\,. \tag{3.14}$$

As a consequence, ∇u fulfills the gradient estimates for $\nabla u_{\varepsilon,\eta}$ provided in Lemma 3.1 for a.e. $(x,t) \in \Omega \times (0,T)$, and

$$\nabla u_{\varepsilon} \to \nabla u \text{ as } \varepsilon \to 0, \text{ in } L^r(\Omega \times (0,T))\,, \ \forall r \in [1,\infty)\,. \tag{3.15}$$

Now, it suffices to demonstrate that u satisfies (P) in the sense of distribution.

For any $\eta > 0$ fixed, we multiply the test function $\psi_{\eta}(u_{\varepsilon})\,\varphi$, for any $\varphi \in \mathcal{C}_c^{\infty}(\Omega \times (0,T))$, to the PDE in (P_ε) satisfied by u_ε. Then, applying integration by parts for $\partial_t u_\varepsilon$ and Green's first identity for Δu_ε^m yields

$$\int_{\mathrm{Supp}(\varphi)} \begin{pmatrix} -\Psi_\eta(u_\varepsilon)\varphi_t + \frac{m}{\eta} u_\varepsilon^{m-1}\psi'\left(\frac{u_\varepsilon}{\eta}\right)|\nabla u_\varepsilon|^2\varphi + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\varphi \\ +m u_\varepsilon^{m-1}\psi_\eta(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\varphi - f(u_\varepsilon,x,t)\psi_\eta(u_\varepsilon)\varphi \end{pmatrix} dxdt = 0,$$

with

$$\Psi_\eta(u_\varepsilon) = \int_0^{u_\varepsilon} \psi_\eta(s)\,ds.$$

Note that $\psi_\eta(\cdot)$ plays a crucial role in avoiding the singularity generated by the absorption term $u^{-\beta}\chi_{\{u>0\}}$ as u is near 0. Thus, there is no problem of going to the limit $\varepsilon \to 0^+$ in the indicated equation in order to obtain

$$\int_{\mathrm{Supp}(\varphi)} \begin{pmatrix} -\Psi_\eta(u)\varphi_t + \frac{m}{\eta} u^{m-1}\psi'\left(\frac{u}{\eta}\right)|\nabla u|^2\varphi + u^{-\beta}\psi_\eta(u)\varphi \\ +m u^{m-1}\psi_\eta(u)\nabla u\cdot\nabla\varphi - f(u,x,t)\psi_\eta(u)\varphi \end{pmatrix} dxdt = 0. \tag{3.16}$$

Next, we pass to the limit as $\eta \to 0^+$ in the last equation.

By (3.14), (3.15), and the integrability of $u^{-\beta}\chi_{\{u>0\}}$ in $\Omega\times(0,T)$, it is not difficult to verify

$$\begin{cases} \lim\limits_{\eta\to 0^+}\int_{\mathrm{Supp}(\varphi)} \Psi_\eta(u)\varphi_t dxdt = \int_{\mathrm{Supp}(\varphi)} u\varphi_t dxdt, \\ \lim\limits_{\eta\to 0^+}\int_{\mathrm{Supp}(\varphi)} m u^{m-1}\psi_\eta(u)\nabla u\cdot\nabla\varphi dxdt = \int_{\mathrm{Supp}(\varphi)} m u^{m-1}\nabla u\cdot\nabla\varphi dxdt, \\ \lim\limits_{\eta\to 0^+}\int_{\mathrm{Supp}(\varphi)} u^{-\beta}\psi_\eta(u)\varphi dxdt = \int_{\mathrm{Supp}(\varphi)} u^{-\beta}\chi_{\{u>0\}}\varphi dxdt, \\ \lim\limits_{\eta\to 0^+}\int_{\mathrm{Supp}(\varphi)} f(u,x,t)\psi_\eta(u)\varphi dxdt = \int_{\mathrm{Supp}(\varphi)} f(u,x,t)\varphi dxdt. \end{cases} \tag{3.17}$$

Note that the assumption $f(0,x,t) = 0$ is used in the final limit.

To verify that u is a distributional solution of (P), it remains to show that the term involving ψ' vanishes in the limit, i.e.,

$$\lim_{\eta\to 0^+}\int_{\mathrm{Supp}(\varphi)} \frac{m}{\eta} u^{m-1}\psi'\left(\frac{u}{\eta}\right)|\nabla u|^2\varphi dxdt = 0. \tag{3.18}$$

Observing that

$$\int_{\mathrm{Supp}(\varphi)} \frac{m}{\eta} u^{m-1}\psi'\left(\frac{u}{\eta}\right)|\nabla u|^2\varphi dxdt \le \frac{m}{\eta}\|\psi'\|_\infty \int_{\mathrm{Supp}(\varphi)\cap\{\eta<\mathrm{u}<2\eta\}} u^{m-1}|\nabla u|^2\varphi dxdt,$$

we distinguish now two cases dictated by Theorem 3.1.

Case (m, β, N) **satisfies** (H_1)**.** This has been proved in the proof of Theorem 2.1[2].

Case (m, β, N) **satisfies** $(H) \setminus (H_1)$**.** Combining Lemma 3.1 and the gradient convergence result of Boccardo yields

$$u^{m-1}|\nabla u|^2 \leq C u^{m-1} u^{2-m-\beta} = C u^{1-\beta},$$

where $C > 0$ is a constant only depending on m, β, f, T, $\|u_0\|_\infty$, and $\left\|\nabla \left(u_0^{(m+\beta)/2}\right)\right\|_\infty$.

Gathering these inequalities, one has

$$\begin{aligned}\int_{\mathrm{Supp}(\varphi)} \frac{m}{\eta} u^{m-1} \psi'\left(\frac{u}{\eta}\right) |\nabla u|^2 \varphi dx dt &\leq \frac{Cm}{\eta} \|\psi'\|_\infty \int_{\mathrm{Supp}(\varphi)\cap\{\eta<\mathrm{u}<2\eta\}} u^{1-\beta} \varphi dx dt \\ &\leq Cm\|\varphi\psi'\|_\infty \int_{\mathrm{Supp}(\varphi)\cap\{\eta<\mathrm{u}<2\eta\}} u^{-\beta} dx dt \to 0 \text{ as } \eta \to 0^+,\end{aligned}$$

since $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$.

Combining (3.16)-(3.18), we deduce

$$\int_{\mathrm{Supp}(\varphi)} \left(-u\varphi_t + m u^{m-1} \nabla u \cdot \nabla \varphi + u^{-\beta} \chi_{\{u>0\}} \varphi - f(u,x,t)\varphi\right) dx dt = 0. \tag{3.19}$$

In other words, u solves (P) in $\mathcal{D}'(\Omega \times (0,T))$.

We now prove (3.13). From the PDE in (P_ε), one has

$$\int_{\mathrm{Supp}(\varphi)} \left(-u_\varepsilon \varphi_t + m u_\varepsilon^{m-1} \nabla u_\varepsilon \cdot \nabla \varphi + g_\varepsilon(u_\varepsilon)\varphi - f(u_\varepsilon, x, t)\varphi\right) dx dt = 0,$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$, $\varphi \geq 0$.

Then, sending $\varepsilon \to 0^+$ in the last equation gives us

$$\int_{\mathrm{Supp}(\varphi)} \left(-u\varphi_t + m u^{m-1} \nabla u \cdot \nabla \varphi - f(u,x,t)\varphi\right) dx dt + \lim_{\varepsilon \to 0^+} \int_{\mathrm{Supp}(\varphi)} g_\varepsilon(u_\varepsilon)\varphi dx dt = 0. \tag{3.20}$$

Compare (3.19) with (3.20), we get

$$\lim_{\varepsilon \to 0^+} \int_{\mathrm{Supp}(\varphi)} g_\varepsilon(u_\varepsilon)\varphi dx dt = \int_{\mathrm{Supp}(\varphi)} u^{-\beta} \chi_{\{u>0\}} \varphi dx dt. \tag{3.21}$$

According to (3.13), (3.21), and Fatou's lemma, we obtain

$$\int_{\mathrm{Supp}(\varphi)} u^{-\beta} \chi_{\{u>0\}} \varphi dx dt \geq \int_{\mathrm{Supp}(\varphi)} \Upsilon \varphi dx dt, \ \ \forall \varphi \in \mathcal{C}_c^\infty(\Omega \times (0,T)), \ \ \varphi \geq 0.$$

[2] Since this integral term is considered in the domain $\mathrm{Supp}(\varphi) \cap \{\eta < \mathrm{u} < 2\eta\}$, the difference between the unbounded domain $\mathbb{R} \times (0,T)$, which is considered in Chapter 2, and the bounded domain $\Omega \times (0,T)$ examined here is irrelevant.

Combining (3.12) and the last inequality yields (3.13).

The conclusion $u \in \mathcal{C}\left([0,T]; L^1(\Omega)\right)$ is well known (see the compactness result of Simon, 1987).

In conclusion, u is a weak solution of (P). To complete the proof, it remains to show that u is the maximal solution of (P).

Proposition 3.5 (Maximality). *Let v be any weak solution of (P) on $\Omega \times (0,t)$. Then, we have*

$$v(x,t) \leq u(x,t), \text{ for a.e. } (x,t) \in \Omega \times (0,T).$$

Proof of Proposition 3.5. In fact, one has

$$g_\varepsilon(v) \leq v^{-\beta}\chi_{\{v>0\}}, \text{ for all } \varepsilon > 0.$$

Thus,

$$\partial_t v - \Delta v^m + g_\varepsilon(v) \leq f(v,x,t), \text{ in } \mathcal{D}'(\Omega \times (0,T)),$$

which implies that v is a sub-solution of PDE in (P_ε).

By comparison principle, we get

$$v(x,t) \leq u_\varepsilon(x,t), \text{ for a.e. } (x,t) \in \Omega \times (0,T).$$

Sending $\varepsilon \to 0^+$ yields the result. □

This proposition also ends the proof of Theorem 3.1. □

Remark 3.3. *If $f(u,x,t) = f(u)$, and f is a locally Lipschitz function on $[0,\infty)$. Then, the result of existence of a maximal solution still holds.*

Appendix A

Proofs

1 Proofs of Stated Results in 1-D

Proof of Lemma 2.1. Define $Q_{\varepsilon,T} := \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times (0, T)$, $\phi_\eta \in \mathcal{C}^\infty(\mathbb{R})$ by

$$\phi_\eta(r) = \begin{cases} r, & \text{for } r \geq \eta, \\ \dfrac{\eta}{2}, & \text{for } r \leq 0, \\ \text{increasing on } [0, \eta], \end{cases}$$

and

$$\mathcal{L}(u_\varepsilon, g_\varepsilon, f) := \partial_t u_\varepsilon - \partial_x^2 (u_\varepsilon^m) + g_\varepsilon(u_\varepsilon) - f(u_\varepsilon, x, t).$$

Instead of $(P_{\varepsilon,\eta})$, consider the following first boundary value problem, denoted by $\left(P^\star_{\varepsilon,\eta}\right)$,

$$\begin{aligned} \mathcal{L}^\star(u_\varepsilon, \phi_\eta, g_\varepsilon, f) = 0, \text{ in } Q_{\varepsilon,T}, \\ u_\varepsilon(x, 0) = u_0(x) + \eta, \text{ for } x \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), \\ u_\varepsilon\left(\pm\frac{1}{\varepsilon}, t\right) = u_0\left(\pm\frac{1}{\varepsilon}\right) + \eta, \text{ for } t \in (0, T), \end{aligned}$$

where $\mathcal{L}^\star$ is defined by

$$\begin{aligned} \mathcal{L}^\star(u_\varepsilon, \phi_\eta, g_\varepsilon, f)(x, t) := \partial_t u_\varepsilon - m\phi_\eta^{m-1}(u_\varepsilon)\,\partial_x^2 u_\varepsilon - m(m-1)\,\phi_\eta^{m-2}(u_\varepsilon)\,(\partial_x u_\varepsilon)^2 \\ + g_\varepsilon(\phi_\eta(u_\varepsilon)) - f(\phi_\eta(u_\varepsilon), x, t). \end{aligned}$$

Due to known results (see Ladyˇ zenskaja and Ural' ceva, 1964) the problem $\left(P^\star_{\varepsilon,\eta}\right)$ has a unique classical solution, denoted by $u_{\varepsilon,\eta}$, with

$$|u_{\varepsilon,\eta}(x, t)| \leq 1 + \|u_0\|_\infty, \text{ for all } (x, t) \in Q_{\varepsilon,T}.$$

To prove that $u_{\varepsilon,\eta} \geq \eta$ in $Q_{\varepsilon,T}$, set

$$z_{\varepsilon,\eta}(x,t) := e^{-at}(u_{\varepsilon,\eta}(x,t) - \eta), \text{ for all } (x,t) \in Q_{\varepsilon,T},$$

or equivalently, $u_{\varepsilon,\eta} = e^{at} z_{\varepsilon,\eta} + \eta$, with $a > 0$ to be chosen later. One has

$$\partial_t u_{\varepsilon,\eta} = a e^{at} z_{\varepsilon,\eta} + e^{at}\partial_t z_{\varepsilon,\eta}, \;\; \partial_x u_{\varepsilon,\eta} = e^{at}\partial_x z_{\varepsilon,\eta}, \text{ and } \partial_x^2 u_{\varepsilon,\eta} = e^{at}\partial_x^2 z_{\varepsilon,\eta},$$

then the PDE for $z_{\varepsilon,\eta}$ is given by

$$\begin{aligned} a z_{\varepsilon,\eta} + \partial_t z_{\varepsilon,\eta} - m\phi_\eta^{m-1}\left(e^{at} z_{\varepsilon,\eta} + \eta\right)\partial_x^2 z_{\varepsilon,\eta} - m(m-1)\, e^{at}\phi_\eta^{m-2}\left(e^{at} z_{\varepsilon,\eta} + \eta\right)(\partial_x z_{\varepsilon,\eta})^2 \\ + e^{-at} g_\varepsilon\left(\phi_\eta\left(e^{at} z_{\varepsilon,\eta} + \eta\right)\right) = e^{-at} f\left(\phi_\eta\left(e^{at} z_{\varepsilon,\eta} + \eta\right), x, t\right), \text{ in } Q_{\varepsilon,T}. \end{aligned}$$

It is clear that

$$\begin{aligned} z_{\varepsilon,\eta}(x,0) &= u_{\varepsilon,\eta}(x,0) - \eta = u_0(x) \geq 0, \\ z_{\varepsilon,\eta}\left(\pm\frac{1}{\varepsilon}, t\right) &= e^{-at}\left(u_{\varepsilon,\eta}\left(\pm\frac{1}{\varepsilon}, t\right) - \eta\right) = e^{-at} u_0\left(\pm\frac{1}{\varepsilon}\right) \geq 0. \end{aligned}$$

Thus $z_{\varepsilon,\eta} \geq 0$ on the parabolic boundary of $Q_{\varepsilon,T}$. Suppose for the contrary that $z_{\varepsilon,\eta}$ attains negative values in $Q_{\varepsilon,T}$. Then it has to have a negative minimum at some (x_0, t_0) (since $\overline{Q_{\varepsilon,T}}$ is compact for all $\varepsilon > 0$ and $T > 0$), and there

$$\partial_t z_{\varepsilon,\eta}(x_0,t_0) \leq 0, \;\; \partial_x z_{\varepsilon,\eta}(x_0,t_0) = 0, \text{ and } \partial_x^2 z_{\varepsilon,\eta}(x_0,t_0) \geq 0,$$

which implies

$$\partial_t z_{\varepsilon,\eta}(x_0,t_0) - m\phi_\eta^{m-1}(u_\varepsilon)\,\partial_x^2 u_\varepsilon(x_0,t_0) \leq 0.$$

The PDE for $z_{\varepsilon,\eta}$ gives us at the point (x_0,t_0) the following equality

$$\begin{aligned} &\partial_t z_{\varepsilon,\eta}(x_0,t_0) - m\phi_\eta^{m-1}\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right)\partial_x^2 z_{\varepsilon,\eta}(x_0,t_0) \\ =& - a z_{\varepsilon,\eta}(x_0,t_0) - e^{-at_0}\left[f\left(\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right), x_0, t_0\right) + g_\varepsilon\left(\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right)\right)\right]. \end{aligned}$$

By the definition of ϕ_η and $z_{\varepsilon,\eta}$, we obtain

$$\begin{aligned} & e^{-at_0}\left[f\left(\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right), x_0, t_0\right) + g_\varepsilon\left(\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right)\right)\right] \\ \leq\; & e^{-at_0}\left[h\left(\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right)\right) + g_\varepsilon(\phi_\eta(\eta))\right] \\ \leq\; & e^{-at_0}\left[\mathrm{Lip}(h,\eta)\,\phi_\eta\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right) + g_\varepsilon(\eta)\right] \\ \leq\; & \eta e^{-at_0}\mathrm{Lip}(h,\eta). \end{aligned}$$

Thus, for $a > 0$ large enough, we have

$$\partial_t z_{\varepsilon,\eta}(x_0,t_0) - m\phi_\eta^{m-1}\left(e^{at_0} z_{\varepsilon,\eta}(x_0,t_0) + \eta\right)\partial_x^2 z_{\varepsilon,\eta}(x_0,t_0)$$

$$> -az_{\varepsilon,\eta}(x_0,t_0) - \eta e^{-at_0}\mathrm{Lip}(h,\eta) > 0,$$

which is a contradiction. Thus $u_{\varepsilon,\eta} \geq \eta$ in $Q_{\varepsilon,T}$. Similarly, we can easily obtain the remaining estimate $u_{\varepsilon,\eta} \leq \|u_0\|_\infty + \eta$ in $Q_{\varepsilon,T}$.

Finally, since $\mathcal{L}$ and $\mathcal{L}^\star$ coincide for every function $u_\varepsilon \geq \eta$, in particular,

$$\mathcal{L}^\star(u_{\varepsilon,\eta}, \phi_\eta, g_\varepsilon, f)(x,t) = \mathcal{L}(u_{\varepsilon,\eta}, g_\varepsilon, f)(x,t), \text{ for all } (x,t) \in Q_{\varepsilon,T}.$$

For the gradient estimates, see the Proof of Proposition 3.1.

The proof of Lemma 2.1 is complete. □

Proof of Lemma 2.2. We use the comparisons

$$u_{\varepsilon,\eta_1}(x,0) = u_0(x) + \eta_1 < u_0(x) + \eta_2 = u_{\varepsilon,\eta_2}(x,0),$$

initially and

$$u_{\varepsilon,\eta_1}\left(\pm\frac{1}{\varepsilon},t\right) = u_0\left(\pm\frac{1}{\varepsilon}\right) + \eta_1 < u_0\left(\pm\frac{1}{\varepsilon}\right) + \eta_2 = u_{\varepsilon,\eta_2}\left(\pm\frac{1}{\varepsilon},t\right),$$

on the boundary as well as the PDEs

$$\mathcal{L}(u_{\varepsilon,\eta_1}, g_\varepsilon, f) = 0 \text{ and } \mathcal{L}(u_{\varepsilon,\eta_2}, g_\varepsilon, f) = 0, \text{ in } Q_{\varepsilon,T}.$$

Our lemma follows from a standard comparison principle. □

2 Proofs of Stated Results in N-D

Proof of Lemma 3.1. The derivative of g_ε is given by

$$g_\varepsilon{}'(s) = -\frac{\beta}{s^{1+\beta}}\psi\left(\frac{s}{\varepsilon}\right) + \frac{1}{\varepsilon s^\beta}\psi'\left(\frac{s}{\varepsilon}\right),$$

for all $s \in \mathbb{R}$, and thus

$$\left|g_\varepsilon{}'(s)\right| \leq \sup_{s\geq 1}\frac{\beta}{s^{1+\beta}}\psi\left(\frac{s}{\varepsilon}\right) + \sup_{1\leq s\leq 2}\frac{1}{\varepsilon s^\beta}\psi'\left(\frac{s}{\varepsilon}\right) \leq \beta + \frac{1}{\varepsilon}\|\psi'\|_\infty,$$

for all $s \in \mathbb{R}$. Therefore, g_ε is global Lipschitz and its global Lipschitz constant satisfy the estimate $\mathrm{Lip}(\mathrm{g}_\varepsilon) \leq \beta + \varepsilon^{-1}\|\psi'\|_\infty$. □

Proof of Lemma 3.2. These equalities can be proved as follows,

$$\nabla\left(|\nabla u|^2\right) = \nabla\left(\sum_{i=1}^N (\partial_i u)^2\right)$$

$$
\begin{aligned}
&= \begin{bmatrix} \partial_1 \sum_{i=1}^{N} (\partial_i u)^2 \\ \vdots \\ \partial_N \sum_{i=1}^{N} (\partial_i u)^2 \end{bmatrix} = 2 \begin{bmatrix} \sum_{i=1}^{N} \partial_i u \partial_1 \partial_i u \\ \vdots \\ \sum_{i=1}^{N} \partial_i u \partial_N \partial_i u \end{bmatrix} \\
&= 2 \begin{bmatrix} \partial_1^2 u & \partial_1 \partial_2 u & \cdots & \partial_1 \partial_N u \\ \partial_2 \partial_1 u & \partial_2^2 u & \cdots & \partial_2 \partial_N u \\ \vdots & \ddots & \ddots & \vdots \\ \partial_N \partial_1 u & \partial_N \partial_2 u & \cdots & \partial_N^2 u \end{bmatrix} \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ \vdots \\ \partial_N u \end{bmatrix} \\
&= 2 D^2 u \cdot \nabla u,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta \left(|\nabla u|^2 \right) &= \operatorname{div} \left(\nabla \left(|\nabla u|^2 \right) \right) = \operatorname{div} \left(\nabla \left(\sum_{i=1}^{N} (\partial_i u)^2 \right) \right) \\
&= \operatorname{div} \left(\begin{bmatrix} \partial_1 \sum_{i=1}^{N} (\partial_i u)^2 & \cdots & \partial_N \sum_{i=1}^{N} (\partial_i u)^2 \end{bmatrix}^T \right) \\
&= 2 \operatorname{div} \left(\begin{bmatrix} \sum_{i=1}^{N} \partial_i u \partial_1 \partial_i u & \cdots & \sum_{i=1}^{N} \partial_i u \partial_N \partial_i u \end{bmatrix}^T \right) \\
&= 2 \sum_{j=1}^{N} \partial_j \sum_{i=1}^{N} \partial_i u \partial_i \partial_j u \\
&= 2 \sum_{j=1}^{N} \sum_{i=1}^{N} \left((\partial_i \partial_j u)^2 + \partial_i u \partial_i \partial_j^2 u \right) \\
&= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} (\partial_i \partial_j u)^2 + 2 \sum_{i=1}^{N} \partial_i u \sum_{j=1}^{N} \partial_i \partial_j^2 u \\
&= 2 \left| D^2 u \right|^2 + 2 \nabla u \cdot \nabla \Delta u.
\end{aligned}
$$

This completes our proof. □

Proof of Lemma 3.3. Put $w = |\nabla u|^2$, the left-hand side of the objective, denoted by $\mathcal{S}$, then can be rewritten as

$$
\mathcal{S} = F(u) \left| D^2 u \right|^2 + F'(u) \left(\frac{1}{2} \nabla u \cdot \nabla w - w \Delta u \right).
$$

Motivated by the application of Bénilan's trick in Benachour, Iagar, and Laurençot, 2016, we have

$$\mathcal{S} = F(u) \sum_{i=1}^{N} \sum_{j=1}^{N} (\partial_i \partial_j u)^2 + F'(u) \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \partial_i u \partial_j u \partial_i \partial_j u - w \sum_{i=1}^{N} \partial_i^2 u \right]$$

$$= F(u) \sum_{i=1}^{N} \left[\left(\partial_i^2 u\right)^2 + \frac{F'(u)}{F(u)} \left((\partial_i u)^2 - w \right) \partial_i^2 u \right] + F(u) \sum_{i \neq j} \left[(\partial_i \partial_j u)^2 + \frac{F'(u)}{F(u)} \partial_i u \partial_j u \partial_i \partial_j u \right].$$

We further estimate $\mathcal{S}$ as follows.

$$\begin{aligned}
\mathcal{S} =& F(u) \sum_{i=1}^{N} \left[\partial_i^2 u + \frac{F'(u)}{2F(u)} \left((\partial_i u)^2 - w \right) \right]^2 - \frac{F(u)}{4} \sum_{i=1}^{N} \left(\frac{F'}{F} \right)^2 (u) \left((\partial_i u)^2 - w \right)^2 \\
&+ F(u) \sum_{i \neq j} \left[\partial_i \partial_j u + \frac{F'(u)}{2F(u)} \partial_i u \partial_j u \right]^2 - \frac{F(u)}{4} \sum_{i \neq j} \left(\frac{F'}{F} \right)^2 (u) (\partial_i u)^2 (\partial_j u)^2 \\
\geq& - \frac{(F')^2}{4F}(u) \left[\sum_{i=1}^{N} \left((\partial_i u)^2 - w \right)^2 + \sum_{i \neq j} (\partial_i u)^2 (\partial_j u)^2 \right] \\
=& - \frac{(F')^2}{4F}(u) \left[\sum_{i=1}^{N} (\partial_i u)^4 - 2w \sum_{i=1}^{N} (\partial_i u)^2 + N w^2 + \sum_{i \neq j} (\partial_i u)^2 (\partial_j u)^2 \right] \\
=& - \frac{(N-1)(F')^2}{4F}(u) \, w^2,
\end{aligned}$$

where the last equality is obtained by using the following two identities

$$w = |\nabla u|^2 = \sum_{i=1}^{N} (\partial_i u)^2,$$

$$w^2 = \left(\sum_{i=1}^{N} (\partial_i u)^2 \right)^2 = \sum_{i=1}^{N} (\partial_i u)^4 + \sum_{i \neq j} (\partial_i u)^2 (\partial_j u)^2.$$

Therefore, we obtain the desired result. □

Proof of Proposition 3.1. a) Fix $\varepsilon \in (0, \|u_0\|_\infty)$, for any $\eta \in (0, \varepsilon]$, the existence and uniqueness of a classical solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ is well known (see, e.g., Theorem 5.2, Ladyˇ zenskaja and Ural' ceva, 1964, pp. 564-565). Let $\Gamma(t)$ be the flat solution of the ODE

$$\begin{cases} \partial_t \Gamma = h(\Gamma), \text{ in } [0, T'], \\ \Gamma(0) = 2\|u_0\|_\infty, \end{cases}$$

where h is the function introduced in (H_f), and T' is the maximal existence time of $\Gamma(t)$. Note that T' only depends on $\|u_0\|_\infty$, see Chapter 1, Coddington and Levinson, 1955.

It follows from the comparison principle that

$$\eta \leq u_{\varepsilon,\eta}(x,t) \leq \Gamma(t), \text{ for all } x \in \Omega, \ \ t \in \left[0, T'\right].$$

Given any $0 < \tau < T < \infty$, we set $h_{\varepsilon,\eta} = u_{\varepsilon,\eta}^{1/\gamma}$, or equivalently, $u_{\varepsilon,\eta} = h_{\varepsilon,\eta}^{\gamma}$ with $\gamma > 0$ chosen later. Computing the the terms in (3.1) in terms of m, γ, and $h_{\varepsilon,\eta}$ gives us

$$\begin{aligned}
\partial_t u_{\varepsilon,\eta} &= \gamma h_{\varepsilon,\eta}^{\gamma-1}\partial_t h_{\varepsilon,\eta}, \\
\nabla u_{\varepsilon,\eta}^m &= \nabla h_{\varepsilon,\eta}^{m\gamma} = m\gamma h_{\varepsilon,\eta}^{m\gamma-1}\nabla h_{\varepsilon,\eta}, \\
\Delta u_{\varepsilon,\eta}^m &= \operatorname{div}\left(\nabla u_{\varepsilon,\eta}^m\right) = \operatorname{div}\left(m\gamma h_{\varepsilon,\eta}^{m\gamma-1}\nabla h_{\varepsilon,\eta}\right) \\
&= m\gamma \nabla h_{\varepsilon,\eta}^{m\gamma-1}\cdot\nabla h_{\varepsilon,\eta} + m\gamma h_{\varepsilon,\eta}^{m\gamma-1}\Delta h_{\varepsilon,\eta} \\
&= m\gamma\left(m\gamma-1\right)h_{\varepsilon,\eta}^{m\gamma-2}|\nabla h_{\varepsilon,\eta}|^2 + m\gamma h_{\varepsilon,\eta}^{m\gamma-1}\Delta h_{\varepsilon,\eta}, \\
g_\varepsilon\left(u_{\varepsilon,\eta}\right) &= \frac{\psi_\varepsilon\left(h_{\varepsilon,\eta}^{\gamma}\right)}{h_{\varepsilon,\eta}^{\beta\gamma}}.
\end{aligned}$$

Plugging these into the equation of $(P_{\varepsilon,\eta})$ yields the following equation for $h_{\varepsilon,\eta}$:

$$\partial_t h_{\varepsilon,\eta} - m h_{\varepsilon,\eta}^{\gamma(m-1)}\Delta h_{\varepsilon,\eta} - m\left(m\gamma-1\right)h_{\varepsilon,\eta}^{\gamma(m-1)-1}|\nabla h_{\varepsilon,\eta}|^2 + \frac{\psi_\varepsilon\left(h_{\varepsilon,\eta}^{\gamma}\right)}{\gamma h_{\varepsilon,\eta}^{\gamma(\beta+1)-1}} = \frac{f\left(h_{\varepsilon,\eta}^{\gamma}, x, t\right)}{\gamma h_{\varepsilon,\eta}^{\gamma-1}}.$$

Following Aronson, 1969, Aronson, 1970, we consider the function $p : [0,1] \to \mathbb{R}$ defined by

$$p(y) = \frac{N_0}{3}y\left(4-y\right),$$

where $N_0 = \Theta^{\frac{1}{\gamma}}\left(\Gamma, T'\right)$, and $\Theta\left(\Gamma, T'\right) := \max_{0\leq t\leq T'}|\Gamma(t)|$.

For $y \in [0,1]$, one has

$$\begin{aligned}
p(y) &\in [0, N_0], \\
p'(y) &= \frac{2N_0}{3}\left(2-y\right) \in \left[\frac{2N_0}{3}, \frac{4N_0}{3}\right], \\
p''(y) &= -\frac{2N_0}{3}, \\
\left(\frac{p''}{p'}\right)'(y) &= -\frac{1}{\left(y-2\right)^2} \leq -\frac{1}{4}.
\end{aligned}$$

The function p is invertible and has the range $[0, N_0]$ on the unit interval. Its inverse function $p^{-1} : [0, N_0] \to [0,1]$ is given by

$$p^{-1}(y) = 2 - \left(4 - \frac{3y}{N_0}\right)^{\frac{1}{2}}, \text{ for } y \in [0, N_0].$$

Hence, we may define $v_{\varepsilon,\eta} := p^{-1}(h_{\varepsilon,\eta})$, or equivalently, $h_{\varepsilon,\eta} = p(v_{\varepsilon,\eta})$. Computing the terms in the equation for $h_{\varepsilon,\eta}$ in terms of m, γ, and $v_{\varepsilon,\eta}$ yields

$$\begin{aligned}
\partial_t h_{\varepsilon,\eta} &= p'(v_{\varepsilon,\eta})\,\partial_t v_{\varepsilon,\eta},\\
\nabla h_{\varepsilon,\eta} &= p'(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta},\\
\Delta h_{\varepsilon,\eta} &= \operatorname{div}(\nabla h_{\varepsilon,\eta}) = \operatorname{div}\left(p'(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta}\right)\\
&= p''(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 + p'(v_{\varepsilon,\eta})\,\Delta v_{\varepsilon,\eta}.
\end{aligned}$$

Plug these into the equation for $h_{\varepsilon,\eta}$, we derive the following equation for $v_{\varepsilon,\eta}$:

$$\begin{aligned}
&\partial_t v_{\varepsilon,\eta} - mp^{\gamma(m-1)}(v_{\varepsilon,\eta})\,\Delta v_{\varepsilon,\eta} - m(m\gamma-1)\left(p^{\gamma(m-1)-1}p'\right)(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2\\
&\quad - \frac{mp^{\gamma(m-1)}p''}{p'}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 + \frac{\psi_\varepsilon(p^\gamma)}{\gamma p' p^{\gamma(\beta+1)-1}}(v_{\varepsilon,\eta}) = \frac{f(p^\gamma(v_{\varepsilon,\eta}),x,t)}{\gamma(p'p^{\gamma-1})(v_{\varepsilon,\eta})}.
\end{aligned}$$

Differentiate both sides of this equation in the space variable x to obtain

$$\begin{aligned}
&\partial_t \nabla v_{\varepsilon,\eta} - mp^{\gamma(m-1)}(v_{\varepsilon,\eta})\,\nabla\Delta v_{\varepsilon,\eta} - m\gamma(m-1)\left(p^{\gamma(m-1)-1}p'\right)(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta}\Delta v_{\varepsilon,\eta}\\
&\quad - \left(m(m\gamma-1)\,p^{\gamma(m-1)-1}p' + \frac{mp^{\gamma(m-1)}p''}{p'}\right)'(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2\nabla v_{\varepsilon,\eta}\\
&\quad - \left(m(m\gamma-1)\,p^{\gamma(m-1)-1}p' + \frac{mp^{\gamma(m-1)}p''}{p'}\right)(v_{\varepsilon,\eta})\,\nabla\left(|\nabla v_{\varepsilon,\eta}|^2\right)\\
&\quad + \frac{1}{\gamma}\frac{\partial}{\partial v_{\varepsilon,\eta}}\left(\frac{\psi_\varepsilon(p^\gamma)}{p'p^{\gamma(\beta+1)-1}}\right)(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta} = \frac{1}{\gamma}\nabla\left(\frac{f(p^\gamma(v_{\varepsilon,\eta}),x,t)}{(p'p^{\gamma-1})(v_{\varepsilon,\eta})}\right).
\end{aligned} \tag{A.1}$$

Recall that (3.11) holds in $\Omega\times(0,\infty)$ and that we want to estimate $\left|\nabla u_{\varepsilon,\eta}^{1/\gamma}\right|$. Since

$$\left|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}\right| = |\nabla h_{\varepsilon,\eta}| = \left|p'(v_{\varepsilon,\eta})\right|\,|\nabla v_{\varepsilon,\eta}| \le \frac{4N_0}{3}\,|\nabla v_{\varepsilon,\eta}|,$$

it suffices to estimate $|\nabla v_{\varepsilon,\eta}|$.

For any $\tau\in\left(0,\frac{T'}{3}\right)$, we suppose that $|\nabla v_{\varepsilon,\eta}(x,t)|$ is large at a point $x=x_0\in\Omega$ at some time $t=t_0\in\left(\tau,\frac{T'}{3}\right)$. Then let

$$\begin{aligned}
P &= \left(B_{\mathbb{R}^N}(x_0;2)\cap\Omega\right)\times\left(0,T'\right),\\
P_1 &= \left(B_{\mathbb{R}^N}(x_0;1)\cap\Omega\right)\times\left(\tau,\frac{T'}{3}\right),
\end{aligned}$$

be two open subsets containing (x_0, t_0). We consider a smooth cut-off function $\xi(x,t) \in C^\infty(\Omega \times (0,\infty))$, $0 \le \xi \le 1$ such that

$$\xi(x,t) = \begin{cases} 1, \text{ in } P_1, \\ 0, \text{ outside } \left(B_{\mathbb{R}^N}\left(x_0; \dfrac{3}{2}\right) \cap \Omega\right) \times \left(\dfrac{\tau}{2}, \dfrac{T'}{3} + \dfrac{\tau}{2}\right), \end{cases}$$

and $|\partial_t \xi| + |\nabla \xi| + |\Delta \xi| \le C_\xi$, for some constant $C_\xi > 0$ which is independent of ε.

Next, we consider the function $w_{\varepsilon,\eta} = \xi^2 |\nabla v_{\varepsilon,\eta}|^2$ in $\overline{P}$. The first partial derivative in the time variable, the gradient, and the Laplace of $w_{\varepsilon,\eta}$ are given by

$$\begin{aligned}
\partial_t w_{\varepsilon,\eta} &= 2\xi \partial_t \xi |\nabla v_{\varepsilon,\eta}|^2 + 2\xi^2 \nabla v_{\varepsilon,\eta} \cdot \partial_t \nabla v_{\varepsilon,\eta}, \\
\nabla w_{\varepsilon,\eta} &= 2\xi \nabla \xi |\nabla v_{\varepsilon,\eta}|^2 + \xi^2 \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right), \\
\Delta w_{\varepsilon,\eta} &= \operatorname{div}(\nabla w_{\varepsilon,\eta}) = \operatorname{div}\left(2\xi \nabla \xi |\nabla v_{\varepsilon,\eta}|^2 + \xi^2 \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right)\right) \\
&= 2\nabla \xi \cdot \nabla \left(\xi |\nabla v_{\varepsilon,\eta}|^2\right) + 2\xi |\nabla v_{\varepsilon,\eta}|^2 \Delta \xi + 2\xi \nabla \xi \cdot \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right) + \xi^2 \Delta \left(|\nabla v_{\varepsilon,\eta}|^2\right) \\
&= 2|\nabla \xi|^2 |\nabla v_{\varepsilon,\eta}|^2 + 2\xi \nabla \xi \cdot \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right) + 2\xi |\nabla v_{\varepsilon,\eta}|^2 \Delta \xi \\
&\quad + 2\xi \nabla \xi \cdot \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right) + \xi^2 \Delta \left(|\nabla v_{\varepsilon,\eta}|^2\right).
\end{aligned}$$

In a maximum point of $w_{\varepsilon,\eta}$, say (x_1, t_1), one has the equality $\nabla w_{\varepsilon,\eta} = 0$ and the inequality $\partial_t w_{\varepsilon,\eta} - m p^{\gamma(m-1)}(v_{\varepsilon,\eta}) \Delta w_{\varepsilon,\eta} \ge 0$. The former leads us to

$$2\nabla \xi(x_1,t_1) |\nabla v_{\varepsilon,\eta}(x_1,t_1)|^2 = -\xi(x_1,t_1) \nabla \left(|\nabla v_{\varepsilon,\eta}(x_1,t_1)|^2\right), \tag{A.2}$$

or, for short, $2\nabla \xi |\nabla v_{\varepsilon,\eta}|^2 = -\xi \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right)$ at (x_1, t_1), and the later yields

$$\begin{aligned}
0 &\le \partial_t w_{\varepsilon,\eta} - m p^{\gamma(m-1)}(v_{\varepsilon,\eta}) \Delta w_{\varepsilon,\eta} \\
&= 2\xi \partial_t \xi |\nabla v_{\varepsilon,\eta}|^2 + 2\xi^2 \nabla v_{\varepsilon,\eta} \cdot \partial_t \nabla v_{\varepsilon,\eta} \\
&\quad - m p^{\gamma(m-1)}(v_{\varepsilon,\eta}) \left[\begin{array}{l} 2|\nabla \xi|^2 |\nabla v_{\varepsilon,\eta}|^2 + 4\xi \nabla \xi \cdot \nabla \left(|\nabla v_{\varepsilon,\eta}|^2\right) + 2\xi \Delta \xi |\nabla v_{\varepsilon,\eta}|^2 \\ +2\xi^2 \left|D^2 v_{\varepsilon,\eta}\right|^2 + 2\xi^2 \nabla v_{\varepsilon,\eta} \cdot \nabla \Delta v_{\varepsilon,\eta} \end{array} \right] \\
&= 2\xi \partial_t \xi |\nabla v_{\varepsilon,\eta}|^2 + 2\xi^2 \nabla v_{\varepsilon,\eta} \cdot \partial_t \nabla v_{\varepsilon,\eta} \\
&\quad - 2m p^{\gamma(m-1)}(v_{\varepsilon,\eta}) \left[\begin{array}{l} |\nabla \xi|^2 |\nabla v_{\varepsilon,\eta}|^2 - 4|\nabla \xi|^2 |\nabla v_{\varepsilon,\eta}|^2 + \xi \Delta \xi |\nabla v_{\varepsilon,\eta}|^2 \\ +\xi^2 \left|D^2 v_{\varepsilon,\eta}\right|^2 + \xi^2 \nabla v_{\varepsilon,\eta} \cdot \nabla \Delta v_{\varepsilon,\eta} \end{array} \right],
\end{aligned}$$

at the point (x_1, t_1), where the last equality is obtain by using (A.2). This gives us the following inequality

$$\begin{aligned}
&\xi^2 \nabla v_{\varepsilon,\eta} \cdot \left[\partial_t \nabla v_{\varepsilon,\eta} - m p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \nabla \Delta v_{\varepsilon,\eta}\right] \\
&\geq \left[-\xi \partial_t \xi + m p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left(\xi \Delta \xi - 3|\nabla \xi|^2\right)\right] |\nabla v_{\varepsilon,\eta}|^2 + m \xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \left|D^2 v_{\varepsilon,\eta}\right|^2,
\end{aligned} \tag{A.3}$$

at the point (x_1, t_1).

Perform the scalar product of both sides of (3.11) with $\xi^2 \nabla v_{\varepsilon,\eta}$ and apply (A.2), (A.3), one has at (x_1, t_1),

$$\begin{aligned}
&\frac{1}{\gamma}\xi^2 \nabla v_{\varepsilon,\eta} \cdot \nabla\left(\frac{f\left(p^\gamma\left(v_{\varepsilon,\eta}\right), x, t\right)}{\left(p' p^{\gamma-1}\right)\left(v_{\varepsilon,\eta}\right)}\right) - \frac{1}{\gamma}\xi^2 \frac{\partial}{\partial v_{\varepsilon,\eta}}\left(\frac{\psi_\varepsilon\left(p^\gamma\right)}{p' p^{\gamma(\beta+1)-1}}\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2 \\
&= \xi^2 \nabla v_{\varepsilon,\eta} \cdot \left[\partial_t \nabla v_{\varepsilon,\eta} - m p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \nabla \Delta v_{\varepsilon,\eta}\right] \\
&\quad - m\gamma(m-1)\xi^2\left(p^{\gamma(m-1)-1} p'\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2 \Delta v_{\varepsilon,\eta} \\
&\quad - \xi^2\left(m(m\gamma-1) p^{\gamma(m-1)-1} p' + \frac{m p^{\gamma(m-1)} p''}{p'}\right)'\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^4 \\
&\quad - \xi^2\left(m(m\gamma-1) p^{\gamma(m-1)-1} p' + \frac{m p^{\gamma(m-1)} p''}{p'}\right)\left(v_{\varepsilon,\eta}\right) \nabla v_{\varepsilon,\eta} \cdot \nabla\left(|\nabla v_{\varepsilon,\eta}|^2\right) \\
&\geq \left[-\xi \partial_t \xi + m p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left(\xi \Delta \xi - 3|\nabla \xi|^2\right)\right] |\nabla v_{\varepsilon,\eta}|^2 + m \xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \left|D^2 v_{\varepsilon,\eta}\right|^2 \\
&\quad - m\gamma(m-1)\xi^2\left(p^{\gamma(m-1)-1} p'\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2 \Delta v_{\varepsilon,\eta} \\
&\quad - \xi^2\left(m(m\gamma-1) p^{\gamma(m-1)-1} p' + \frac{m p^{\gamma(m-1)} p''}{p'}\right)'\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^4 \\
&\quad + 2\xi\left(m(m\gamma-1) p^{\gamma(m-1)-1} p' + \frac{m p^{\gamma(m-1)} p''}{p'}\right)\left(v_{\varepsilon,\eta}\right) \nabla \xi \cdot \nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2.
\end{aligned}$$

This gives us at (x_1, t_1),

$$\begin{aligned}
&- m\xi^2\left[\begin{array}{c} ((2m-1)\gamma-1) p^{\gamma(m-1)-1} p'' + p^{\gamma(m-1)}\left(\frac{p''}{p'}\right)' \\ + (m\gamma-1)(\gamma(m-1)-1) p^{\gamma(m-1)-2}(p')^2 \end{array}\right]\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^4 \\
&\leq \left[\xi \partial_t \xi - m p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left(\xi \Delta \xi - 3|\nabla \xi|^2\right)\right] |\nabla v_{\varepsilon,\eta}|^2 \\
&\quad + m\gamma(m-1)\xi^2\left(p^{\gamma(m-1)-1} p'\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2 \Delta v_{\varepsilon,\eta} - m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left|D^2 v_{\varepsilon,\eta}\right|^2 \\
&\quad - 2m\xi\left[(m\gamma-1) p^{\gamma(m-1)-1} p' + \frac{p^{\gamma(m-1)} p''}{p'}\right]\left(v_{\varepsilon,\eta}\right) \nabla \xi \cdot \nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2 \\
&\quad + \frac{\xi^2}{\gamma}\left[\frac{\psi_\varepsilon\left(p^\gamma\right) p'' p^{1-\gamma(\beta+1)}}{(p')^2} - \gamma \psi_\varepsilon{}'\left(p^\gamma\right) p^{-\beta\gamma}\right]\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2
\end{aligned} \tag{A.4}$$

$$
\begin{aligned}
&+ \frac{\gamma(\beta+1)-1}{\gamma}\xi^2\psi_\varepsilon(p^\gamma)\,p^{-\gamma(\beta+1)}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 \\
&+ \xi^2 D_u f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,|\nabla v_{\varepsilon,\eta}|^2 + \frac{1}{\gamma}\xi^2\nabla v_{\varepsilon,\eta}\cdot D_x f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{1-\gamma}}{p'}(v_{\varepsilon,\eta}) \\
&- \frac{1}{\gamma}\xi^2 f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{1-\gamma}p''}{(p')^2}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 - \frac{\gamma-1}{\gamma}\xi^2 f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,p^{-\gamma}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2.
\end{aligned}
$$

where the terms involving f are obtained by the following calculations

$$
\begin{aligned}
&\nabla\left(\frac{f(p^\gamma(v_{\varepsilon,\eta}),x,t)}{p^{\gamma-1}p'(v_{\varepsilon,\eta})}\right) \\
&= \frac{\nabla f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,v(v_{\varepsilon,\eta}) - f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\left(p'p^{\gamma-1}\right)'(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta}}{\left(p^{\gamma-1}p'\right)^2(v_{\varepsilon,\eta})} \\
&= \frac{\gamma D_u f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,p^{\gamma-1}p'(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta} + D_x f(p^\gamma(v_{\varepsilon,\eta}),x,t)}{p^{\gamma-1}p'(v_{\varepsilon,\eta})} \\
&\quad - f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{1-\gamma}p''}{(p')^2}(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta} - (\gamma-1)\,f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,p^{-\gamma}(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta} \\
&= \gamma D_u f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\nabla v_{\varepsilon,\eta} + D_x f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{1-\gamma}}{p'}(v_{\varepsilon,\eta}) \\
&\quad - f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{1-\gamma}p''}{(p')^2}(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta} - (\gamma-1)\,f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,p^{-\gamma}(v_{\varepsilon,\eta})\,\nabla v_{\varepsilon,\eta}.
\end{aligned}
$$

Motivated by Lemma 3.3, we consider the following expression

$$
\begin{aligned}
\mathcal{S}(x,t) :=\ & m\xi^2 p^{\gamma(m-1)}(v_{\varepsilon,\eta})\,\left|D^2 v_{\varepsilon,\eta}\right|^2 \\
&+ m\gamma(m-1)\,\xi^2\left(p^{\gamma(m-1)-1}p'\right)(v_{\varepsilon,\eta})\left[\frac{1}{2}\nabla v_{\varepsilon,\eta}\cdot\nabla\left(|\nabla v_{\varepsilon,\eta}|^2\right) - |\nabla v_{\varepsilon,\eta}|^2\Delta v_{\varepsilon,\eta}\right].
\end{aligned}
$$

We now use Bénilan's trick (see, e.g., Benachour, Iagar, and Laurençot, 2016) to obtain

$$
\begin{aligned}
\mathcal{S} &= m\xi^2 p^{\gamma(m-1)}(v_{\varepsilon,\eta})\sum_{i=1}^{N}\sum_{j=1}^{N}(\partial_i\partial_j v_{\varepsilon,\eta})^2 \\
&\quad + m\gamma(m-1)\,\xi^2\left(p^{\gamma(m-1)-1}p'\right)(v_{\varepsilon,\eta})\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\partial_i v_{\varepsilon,\eta}\partial_j v_{\varepsilon,\eta}\partial_i\partial_j v_{\varepsilon,\eta} - |\nabla v_{\varepsilon,\eta}|^2\sum_{i=1}^{N}\partial_i^2 v_{\varepsilon,\eta}\right] \\
&= m\xi^2 p^{\gamma(m-1)}(v_{\varepsilon,\eta})\sum_{i=1}^{N}\left[\left(\partial_i^2 v_{\varepsilon,\eta}\right)^2 + \gamma(m-1)\,\frac{p'}{p}(v_{\varepsilon,\eta})\left((\partial_i v_{\varepsilon,\eta})^2 - |\nabla v_{\varepsilon,\eta}|^2\right)\partial_i^2 v_{\varepsilon,\eta}\right] \\
&\quad + m\xi^2 p^{\gamma(m-1)}(v_{\varepsilon,\eta})\sum_{i\neq j}\left[(\partial_i\partial_j v_{\varepsilon,\eta})^2 + \gamma(m-1)\,\frac{p'}{p}(v_{\varepsilon,\eta})\,\partial_i v_{\varepsilon,\eta}\partial_j v_{\varepsilon,\eta}\partial_i\partial_j v_{\varepsilon,\eta}\right].
\end{aligned}
$$

We further estimate $\mathcal{S}$ as follows,

$$
\begin{aligned}
\mathcal{S} = & \, m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \sum_{i=1}^{N}\left[\partial_i^2 v_{\varepsilon,\eta} + \frac{\gamma\left(m-1\right)}{2}\cdot\frac{p'}{p}\left(v_{\varepsilon,\eta}\right)\left(\left(\partial_i v_{\varepsilon,\eta}\right)^2 - |\nabla v_{\varepsilon,\eta}|^2\right)\right]^2 \\
& - m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \sum_{i=1}^{N}\frac{\gamma^2(m-1)^2}{4}\left(\frac{p'}{p}\right)^2\left(v_{\varepsilon,\eta}\right)\left(\left(\partial_i v_{\varepsilon,\eta}\right)^2 - |\nabla v_{\varepsilon,\eta}|^2\right)^2 \\
& + m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \sum_{i\neq j}\left[\partial_i\partial_j v_{\varepsilon,\eta} + \frac{\gamma\left(m-1\right)}{2}\cdot\frac{p'}{p}\left(v_{\varepsilon,\eta}\right)\partial_i v_{\varepsilon,\eta}\partial_j v_{\varepsilon,\eta}\right]^2 \\
& - m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) \sum_{i\neq j}\frac{\gamma^2(m-1)^2}{4}\left(\frac{p'}{p}\right)^2\left(v_{\varepsilon,\eta}\right)(\partial_i v_{\varepsilon,\eta})^2(\partial_j v_{\varepsilon,\eta})^2 \\
\geq & - \frac{m\gamma^2(m-1)^2}{4}\xi^2\left(p^{\gamma(m-1)-2}(p')^2\right)\left(v_{\varepsilon,\eta}\right)\left[\sum_{i=1}^{N}\left(\left(\partial_i v_{\varepsilon,\eta}\right)^2 - |\nabla v_{\varepsilon,\eta}|^2\right)^2 + \sum_{i\neq j}(\partial_i v_{\varepsilon,\eta})^2(\partial_j v_{\varepsilon,\eta})^2\right] \\
= & - \frac{m\gamma^2(m-1)^2}{4}\xi^2\left(p^{\gamma(m-1)-2}(p')^2\right)\left(v_{\varepsilon,\eta}\right)\left[\begin{array}{l}\sum_{i=1}^{N}(\partial_i v_{\varepsilon,\eta})^4 - 2|\nabla v_{\varepsilon,\eta}|^2\sum_{i=1}^{N}(\partial_i v_{\varepsilon,\eta})^2 \\ +N|\nabla v_{\varepsilon,\eta}|^4 + \sum_{i\neq j}(\partial_i v_{\varepsilon,\eta})^2(\partial_j v_{\varepsilon,\eta})^2\end{array}\right] \\
= & - \frac{m\gamma^2(m-1)^2\left(N-1\right)}{4}\xi^2\left(p^{\gamma(m-1)-2}(p')^2\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^4,
\end{aligned}
$$

where the last equality is obtained by using the following two identities

$$
\begin{aligned}
|\nabla v_{\varepsilon,\eta}|^2 &= \sum_{i=1}^{N}(\partial_i v_{\varepsilon,\eta})^2, \\
|\nabla v_{\varepsilon,\eta}|^4 &= \left(\sum_{i=1}^{N}(\partial_i v_{\varepsilon,\eta})^2\right)^2 = \sum_{i=1}^{N}(\partial_i v_{\varepsilon,\eta})^4 + \sum_{i\neq j}(\partial_i v_{\varepsilon,\eta})^2(\partial_j v_{\varepsilon,\eta})^2.
\end{aligned}
$$

Thus, the following inequality holds for all $(x,t)\in\Omega\times(0,T')$,

$$
\begin{aligned}
& m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left|D^2 v_{\varepsilon,\eta}\right|^2 - m\gamma\left(m-1\right)\xi^2\left(p^{\gamma(m-1)-1}p'\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2\Delta v_{\varepsilon,\eta} \\
& + \frac{m\gamma\left(m-1\right)}{2}\xi^2\left(p^{\gamma(m-1)-1}p'\right)\left(v_{\varepsilon,\eta}\right)\nabla v_{\varepsilon,\eta}\cdot\nabla\left(|\nabla v_{\varepsilon,\eta}|^2\right) \\
\geq & - \frac{m\gamma^2(m-1)^2\left(N-1\right)}{4}\xi^2\left(p^{\gamma(m-1)-2}(p')^2\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^4.
\end{aligned}
$$

In particular, combining the last inequality with (A.2) yields at the point (x_1,t_1),

$$
\begin{aligned}
& m\xi^2 p^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left|D^2 v_{\varepsilon,\eta}\right|^2 - m\gamma\left(m-1\right)\xi^2\left(p^{\gamma(m-1)-1}p'\right)\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2\Delta v_{\varepsilon,\eta} \\
& - m\gamma\left(m-1\right)\xi\left(p^{\gamma(m-1)-1}p'\right)\left(v_{\varepsilon,\eta}\right)\nabla\xi\cdot\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2
\end{aligned}
$$

$$\geq -\frac{m\gamma^2(m-1)^2(N-1)}{4}\xi^2\left(p^{\gamma(m-1)-2}(p')^2\right)(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^4.$$

Combine (A.4) with this, we obtain

$$\begin{aligned}
&-m\xi^2\left[\begin{array}{l}((2m-1)\gamma-1)\,p^{\gamma(m-1)-1}p''+p^{\gamma(m-1)}\left(\frac{p''}{p'}\right)'\\ +\left[\frac{\gamma^2(m-1)^2(N-1)}{4}+(m\gamma-1)\left(\gamma\left(m-1\right)-1\right)\right]p^{\gamma(m-1)-2}(p')^2\end{array}\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^4\\
\leq&\left[\xi\partial_t\xi-mp^{\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)\left(\xi\Delta\xi-3|\nabla\xi|^2\right)\right]|\nabla v_{\varepsilon,\eta}|^2\\
&-m\xi\left[\left(3m\gamma-\gamma-2\right)p^{\gamma(m-1)-1}p'+\frac{2p^{\gamma(m-1)}p''}{p'}\right](v_{\varepsilon,\eta})\,\nabla\xi\cdot\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2\\
&+\frac{\xi^2}{\gamma}\left[\frac{\psi_\varepsilon\left(p^\gamma\right)p''p^{1-\gamma(\beta+1)}}{(p')^2}-\gamma\psi_\varepsilon{}'\left(p^\gamma\right)p^{-\beta\gamma}\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2\\
&+\frac{\gamma\left(\beta+1\right)-1}{\gamma}\xi^2\psi_\varepsilon\left(p^\gamma\right)p^{-\gamma(\beta+1)}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2\\
&+\xi^2D_uf\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)|\nabla v_{\varepsilon,\eta}|^2+\frac{1}{\gamma}\xi^2\nabla v_{\varepsilon,\eta}\cdot D_xf\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)\frac{p^{1-\gamma}}{p'}\left(v_{\varepsilon,\eta}\right)\\
&-\frac{1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)\frac{p^{1-\gamma}p''}{(p')^2}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2-\frac{\gamma-1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)p^{-\gamma}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2.
\end{aligned}\tag{A.5}$$

We now prove the desired gradient estimate in the following cases depending on the values of m and N.

Case I: $m = 1$**.** It can be proved that the choice $\gamma = \frac{2}{1+\beta}$ fulfills the gradient estimate (3.8) when m, β, and N satisfy (H_2) as follows. Multiplying (A.5) by $p^2\left(v_{\varepsilon,\eta}\right)$ gives us

$$\begin{aligned}
&-\xi^2\left[\left(\gamma-1\right)pp''+p^2\left(\frac{p''}{p'}\right)'-\left(\gamma-1\right)\left(p'\right)^2\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^4\\
\leq&\left[\xi\partial_t\xi-\left(\xi\Delta\xi-3|\nabla\xi|^2\right)\right]p^2\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2-2\xi\left[\left(\gamma-1\right)pp'+\frac{p^2p''}{p'}\right](v_{\varepsilon,\eta})\,\nabla\xi\cdot\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2\\
&+\frac{\xi^2}{\gamma}\left[\frac{\psi_\varepsilon\left(p^\gamma\right)p''p^{3-\gamma(\beta+1)}}{(p')^2}-\gamma\psi_\varepsilon{}'\left(p^\gamma\right)p^{2-\beta\gamma}\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2\\
&+\frac{\gamma\left(\beta+1\right)-1}{\gamma}\xi^2\psi_\varepsilon\left(p^\gamma\right)p^{2-\gamma(\beta+1)}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2\\
&+\xi^2D_uf\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)p^2\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2+\frac{1}{\gamma}\xi^2\nabla v_{\varepsilon,\eta}\cdot D_xf\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)\frac{p^{3-\gamma}}{p'}\left(v_{\varepsilon,\eta}\right)\\
&-\frac{1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)\frac{p^{3-\gamma}p''}{(p')^2}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2-\frac{\gamma-1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right),x,t\right)p^{2-\gamma}\left(v_{\varepsilon,\eta}\right)|\nabla v_{\varepsilon,\eta}|^2.
\end{aligned}\tag{A.6}$$

The coefficient of $\xi^2|\nabla v_{\varepsilon,\eta}|^4$ on the left-hand side of the last inequality is positive if $\gamma > 1$. Assume $\gamma > 1$, this coefficient is then bounded below by a positive constant since

$$
\begin{aligned}
&-\left[(\gamma-1)\,pp''+p^2\left(\frac{p''}{p'}\right)'-(\gamma-1)\left(p'\right)^2\right](v_{\varepsilon,\eta})\\
&\geq \frac{2N_0\,(\gamma-1)}{3}p\,(v_{\varepsilon,\eta})+\frac{1}{4}p^2\,(v_{\varepsilon,\eta})+\frac{4N_0^2\,(\gamma-1)}{9}\geq\frac{4N_0^2\,(\gamma-1)}{9}.
\end{aligned}
$$

All of the coefficients of $|\nabla v_{\varepsilon,\eta}|^2$ and $\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2$ on the right-hand side of (A.6) which do not involve ψ_ε and f are bounded. It suffices to choose $\gamma > 1$ such that the remaining terms of (A.6) (i.e., the ones involving ψ_ε and f) are bounded uniformly with respect to ε or have the negative signs.

The first two terms involving ψ_ε have the negative signs, so we can drop them. Note that $\gamma\,(\beta+1)-1 > 0$ (since we assume $\gamma > 1$), the remaining term involving ψ_ε has the positive sign. Its coefficient is bounded if $2-\gamma\,(\beta+1)\geq 0$, i.e., $\gamma\leq\frac{2}{1+\beta}$. Moreover, the last term in (A.6) has the negative sign and thus can be dropped. The coefficients of the remaining terms involving f in (A.6) are bounded if $\gamma\leq 3$. After all, since $\frac{2}{1+\beta}\in(1,2)$, the choice $\gamma:=\frac{2}{1+\beta}$, as indicated by (3.4), accomplishes (3.8) when (H_2) holds.

Case II: $m > 1$**.** Assume

$$
\frac{\gamma^2(m-1)^2\,(N-1)}{4}+(m\gamma-1)\,(\gamma\,(m-1)-1)\leq 0,
$$

we consider the left-hand side of this assumption as a quadratic polynomial in the variable γ, i.e.,

$$
\left[\frac{(N-1)\,(m-1)^2}{4}+m\,(m-1)\right]\gamma^2-(2m-1)\,\gamma+1\leq 0.
$$

The discriminant of this quadratic polynomial is given by $\Delta_\gamma = 1-(N-1)\,(m-1)^2$. We consider the following sub-cases depending on the value of N.

Case II.1: $N\geq 2$**.** Assume that $\Delta_\gamma > 0$, i.e., $1 < m < 1+(N-1)^{-\frac{1}{2}}$, the above assumption is then equivalent to $\gamma_1\,(m,N) < \gamma < \gamma_2\,(m,N)$, where

$$
\begin{aligned}
\gamma_1\,(m,N)&:=\frac{2\left(2m-1-\sqrt{1-(N-1)\,(m-1)^2}\right)}{(N-1)\,(m-1)^2+4m\,(m-1)},\\
\gamma_2\,(m,N)&:=\frac{2\left(2m-1+\sqrt{1-(N-1)\,(m-1)^2}\right)}{(N-1)\,(m-1)^2+4m\,(m-1)},
\end{aligned}
$$

are two distinct roots of the given quadratic polynomial. Assume $\gamma \in (\gamma_1(m,N), \gamma_2(m,N))$, multiplying (A.5) by $p^{2-\gamma(m-1)}(v_{\varepsilon,\eta})$ yields

$$
\begin{aligned}
&-m\xi^2\left[\begin{array}{c} ((2m-1)\gamma-1)\,pp''+p^2\left(\frac{p''}{p'}\right)' \\ +\left[\frac{\gamma^2(m-1)^2(N-1)}{4}+(m\gamma-1)(\gamma(m-1)-1)\right](p')^2 \end{array}\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^4 \\
\leq &\left[\xi\partial_t\xi p^{2-\gamma(m-1)}(v_{\varepsilon,\eta})-mp^2(v_{\varepsilon,\eta})\left(\xi\Delta\xi-3|\nabla\xi|^2\right)\right]|\nabla v_{\varepsilon,\eta}|^2 \\
&-m\xi\left[(3m\gamma-\gamma-2)\,pp'+\frac{2p^2p''}{p'}\right](v_{\varepsilon,\eta})\,\nabla\xi\cdot\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2 \\
&+\frac{\xi^2}{\gamma}\left[\psi_\varepsilon(p^\gamma)\,\frac{p''p^{3-\gamma(m+\beta)}}{(p')^2}-\gamma\psi_\varepsilon{}'(p^\gamma)\,p^{2-\gamma(m+\beta-1)}\right](v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 \qquad\qquad (A.7)\\
&+\frac{\gamma(\beta+1)-1}{\gamma}\xi^2\psi_\varepsilon(p^\gamma)\,p^{2-\gamma(m+\beta)}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2 \\
&+\xi^2p^{2-\gamma(m-1)}(v_{\varepsilon,\eta})\,D_uf(p^\gamma(v_{\varepsilon,\eta}),x,t)\,|\nabla v_{\varepsilon,\eta}|^2+\frac{1}{\gamma}\xi^2\nabla v_{\varepsilon,\eta}\cdot D_xf(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p^{3-m\gamma}}{p'}(v_{\varepsilon,\eta}) \\
&-\frac{1}{\gamma}\xi^2f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,\frac{p''p^{3-m\gamma}}{(p')^2}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2-\frac{\gamma-1}{\gamma}\xi^2f(p^\gamma(v_{\varepsilon,\eta}),x,t)\,p^{2-m\gamma}(v_{\varepsilon,\eta})\,|\nabla v_{\varepsilon,\eta}|^2.
\end{aligned}
$$

The coefficient of $\xi^2|\nabla v_{\varepsilon,\eta}|^4$ on the left-hand side of (A.7) is positive if we assume, in addition, that $(2m-1)\gamma-1\geq 0$, i.e., $\gamma\geq\frac{1}{2m-1}$. Assume $\gamma\geq\frac{1}{2m-1}$, this coefficient is then bounded below by a positive constant since

$$
\begin{aligned}
&-\left[\frac{\gamma^2(m-1)^2(N-1)}{4}+(m\gamma-1)(\gamma(m-1)-1)\right](p')^2(v_{\varepsilon,\eta}) \\
\geq &-\frac{4N_0^2}{9}\left[\frac{\gamma^2(m-1)^2(N-1)}{4}+(m\gamma-1)(\gamma(m-1)-1)\right].
\end{aligned}
$$

All of the coefficients of $|\nabla v_{\varepsilon,\eta}|^2$ and $\nabla v_{\varepsilon,\eta}|\nabla v_{\varepsilon,\eta}|^2$ on the right-hand side of (A.7) which do not involve ψ_ε and f are bounded if $2-\gamma(m-1)\geq 0$, i.e., $\gamma\leq\frac{2}{m-1}$. It suffices to choose $\gamma\in\left[\frac{1}{2m-1},\frac{2}{m-1}\right]\cap(\gamma_1(m,N),\gamma_2(m,N))$ such that the remaining terms of (A.7) (i.e., the ones involving ψ_ε and f) are bounded uniformly with respect to ε or have the negative signs.

As in case I), the first two terms involving ψ_ε have the negative signs, so we can drop them. The coefficient of the remaining term involving ψ_ε: $\gamma^{-1}(\gamma(\beta+1)-1)\,\xi^2\psi_\varepsilon(p^\gamma)\,p^{2-\gamma(m+\beta)}(v_{\varepsilon,\eta})$, has the negative sign if $\gamma(\beta+1)-1\leq 0$, i.e., $\gamma\leq\frac{1}{1+\beta}$. In addition, it has the positive sign and is bounded uniformly with respect to ε if $\gamma(\beta+1)-1>0$ and $2-\gamma(m+\beta)\geq 0$, i.e., $\gamma\in\left(\frac{1}{1+\beta},\frac{2}{m+\beta}\right]$. Moreover, the four terms involving f in (A.7) are bounded if

$$\min\{2-\gamma(m-1),3-m\gamma,2-m\gamma\}\geq 0,$$

i.e., $\gamma \leq \frac{2}{m}$. After all, we need to choose

$$\gamma \in \left[\frac{1}{2m-1}, \frac{2}{m}\right] \cap \left(\gamma_1\left(m, N\right), \gamma_2\left(m, N\right)\right) \cap \left(\frac{1}{1+\beta}, \frac{2}{m+\beta}\right],$$

in order that (A.7) gives us the boundedness of $|\nabla v_{\varepsilon,\eta}|$. We have

$$\begin{aligned} \frac{2}{m+\beta} - \frac{1}{1+\beta} &= \frac{\beta+2-m}{(m+\beta)(1+\beta)} > 0, \\ \frac{2}{m+\beta} - \frac{1}{2m-1} &= \frac{3m-\beta-2}{(m+\beta)(2m-1)} > 0, \end{aligned}$$

since $N \geq 2$, $m < 1 + (N-1)^{-\frac{1}{2}} < 1 + (2-1)^{-\frac{1}{2}} = 2$, and $\beta < m$. Thus, if we assume $\frac{2}{m+\beta} \in \left(\gamma_1\left(m, N\right), \gamma_2\left(m, N\right)\right)$, the choice $\gamma := \frac{2}{m+\beta}$, as indicated by (3.4), achieves (3.8) when (H_4) is active.

Case II.2: $N = 1$. In one-dimensional setting, we have $\Delta_\gamma = 1$, $\gamma_1\left(m, 1\right) = \frac{1}{m}$, and $\gamma_2\left(m, 1\right) = \frac{1}{m-1}$. We now recover the result in Kawohl and Kersner, 1992, Theorem 2, p. 473, with a modification of the additional source term f, as follows.

Case II.2.1: $m < \beta + 2$. Assume that $\frac{1}{m} < \gamma < \frac{1}{m-1}$. Multiplying (A.6) by $p^{2-\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)$ yields

$$\begin{aligned} &- m\xi^2 \left[\begin{array}{l} \left(\left(2m-1\right)\gamma-1\right)pp'' + p^2\left(\frac{p''}{p'}\right)' \\ + \left(m\gamma-1\right)\left(\gamma\left(m-1\right)-1\right)\left(p'\right)^2 \end{array} \right]\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^4 \\ \leq &\left[\xi\partial_t\xi p^{2-\gamma(m-1)}\left(v_{\varepsilon,\eta}\right) - mp^2\left(v_{\varepsilon,\eta}\right)\left(\xi\partial_x^2\xi - 3(\partial_x\xi)^2\right)\right]\left(\partial_x v_{\varepsilon,\eta}\right)^2 \\ &- m\xi\left[\left(3m\gamma-\gamma-2\right)pp' + \frac{2p^2p''}{p'}\right]\left(v_{\varepsilon,\eta}\right)\partial_x\xi(\partial_x v_{\varepsilon,\eta})^3 \\ &+ \frac{\xi^2}{\gamma}\left[\psi_\varepsilon\left(p^\gamma\right)\frac{p''p^{3-\gamma(m+\beta)}}{\left(p'\right)^2} - \gamma{\psi_\varepsilon}'\left(p^\gamma\right)p^{2-\gamma(m+\beta-1)}\right]\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2 \\ &+ \frac{\gamma\left(\beta+1\right)-1}{\gamma}\xi^2\psi_\varepsilon\left(p^\gamma\right)p^{2-\gamma(m+\beta)}\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2 \\ &+ \xi^2p^{2-\gamma(m-1)}\left(v_{\varepsilon,\eta}\right)D_uf\left(p^\gamma\left(v_{\varepsilon,\eta}\right), x, t\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2 + \frac{1}{\gamma}\xi^2\partial_x v_{\varepsilon,\eta}D_xf\left(p^\gamma\left(v_{\varepsilon,\eta}\right), x, t\right)\frac{p^{3-m\gamma}}{p'}\left(v_{\varepsilon,\eta}\right) \\ &- \frac{1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right), x, t\right)\frac{p^{3-m\gamma}p''}{\left(p'\right)^2}\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2 - \frac{\gamma-1}{\gamma}\xi^2f\left(p^\gamma\left(v_{\varepsilon,\eta}\right), x, t\right)p^{2-m\gamma}\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2. \end{aligned} \tag{A.8}$$

The coefficient of $\xi^2(\partial_x v_{\varepsilon,\eta})^4$ on the left-hand side of (A.8) is bounded below by a positive constant since

$$\begin{aligned} &- \left(\left(2m-1\right)\gamma-1\right)pp''\left(v_{\varepsilon,\eta}\right) - p^2\left(\frac{p''}{p'}\right)'\left(v_{\varepsilon,\eta}\right) - \left(m\gamma-1\right)\left(\gamma\left(m-1\right)-1\right)\left(p'\right)^2\left(v_{\varepsilon,\eta}\right) \\ \geq &\frac{2N_0}{3}\left(\left(2m-1\right)\gamma-1\right)p\left(v_{\varepsilon,\eta}\right) + \frac{1}{4}p^2\left(v_{\varepsilon,\eta}\right) - \left(m\gamma-1\right)\left(\gamma\left(m-1\right)-1\right)\left(p'\right)^2\left(v_{\varepsilon,\eta}\right) \end{aligned}$$

$$\geq \frac{4N_0^2}{9}\left(m\gamma-1\right)\left(1-\gamma\left(m-1\right)\right),$$

where we have used the inequality $\gamma > \frac{1}{2m-1}$ deduced from $\gamma \in \left(\frac{1}{m}, \frac{1}{m-1}\right)$ and

$$\frac{1}{m}-\frac{1}{2m-1}=\frac{m-1}{m\left(2m-1\right)}>0.$$

All of the coefficients of $\left(\partial_x v_{\varepsilon,\eta}\right)^2$ and $\left(\partial_x v_{\varepsilon,\eta}\right)^3$ on the right-hand side of (A.8) which do not involve ψ_ε and f are bounded. It suffices to choose $\gamma \in \left(\frac{1}{m}, \frac{1}{m-1}\right)$ such that the remaining terms of (A.8) are bounded uniformly with respect to ε or have the negative signs.

As in the previous cases, the first two terms involving ψ_ε have the negative signs and thus can be dropped. The coefficient of the remaining term involving ψ_ε has the positive sign and is bounded uniformly with respect to ε if $\gamma\left(\beta+1\right)-1>0$ and $2-\gamma\left(m+\beta\right)\geq 0$, i.e., $\gamma \in \left(\frac{1}{1+\beta}, \frac{2}{m+\beta}\right]$. Moreover, the four terms involving f in (A.8) are bounded if $\gamma \leq \frac{2}{m}$. After all, it is decisive to choose

$$\gamma \in \left(\frac{1}{m}, \frac{1}{m-1}\right) \cap \left(\frac{1}{1+\beta}, \frac{2}{m+\beta}\right] \cap \left(0, \frac{2}{m}\right],$$

so that (A.8) implies the boundedness of $\left|\partial_x v_{\varepsilon,\eta}\right|$. Since

$$\begin{aligned}\frac{2}{m+\beta}-\frac{1}{m}&=\frac{m-\beta}{m\left(m+\beta\right)}>0,\\ \frac{1}{m-1}-\frac{2}{m+\beta}&=\frac{\beta+2-m}{\left(m-1\right)\left(m+\beta\right)}>0,\end{aligned}$$

we obtain $\frac{2}{m+\beta} \in \left(\frac{1}{m}, \frac{1}{m-1}\right)$. Therefore, the choice $\gamma := \frac{2}{m+\beta}$, as indicated in (3.4), accomplishes (3.8) when (H_3) is active.

Case II.2.2: $m \geq \beta+2$. We choose γ as the larger root of the above quadratic polynomial, i.e., $\gamma = \frac{1}{m-1}$. Then (A.6) becomes

$$\begin{aligned}&-m\xi^2\left[\frac{m}{m-1}p''+p\left(\frac{p''}{p'}\right)'\right]\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^4\\ &\leq \left[\xi\partial_t\xi-mp\left(v_{\varepsilon,\eta}\right)\left(\xi\partial_x^2\xi-3(\partial_x\xi)^2\right)\right]\left(\partial_x v_{\varepsilon,\eta}\right)^2-m\xi\partial_x\xi\left(\frac{m+1}{m-1}p'+\frac{2pp''}{p'}\right)\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^3\\ &\quad+\xi^2\left[\left(m-1\right)\psi_\varepsilon\left(p^{\frac{1}{m-1}}\right)\frac{p''p^{\frac{m-\beta-2}{m-1}}}{\left(p'\right)^2}-\psi_\varepsilon{}'\left(p^{\frac{1}{m-1}}\right)p^{-\frac{\beta}{m-1}}\right]\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2\\ &\quad+\left(\beta+2-m\right)\xi^2\psi_\varepsilon\left(p^{\frac{1}{m-1}}\right)p^{-\frac{\beta+1}{m-1}}\left(v_{\varepsilon,\eta}\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2 \qquad\qquad \text{(A.9)}\\ &\quad+\xi^2D_uf\left(p^{\frac{1}{m-1}}\left(v_{\varepsilon,\eta}\right),x,t\right)\left(\partial_x v_{\varepsilon,\eta}\right)^2+\left(m-1\right)\xi^2\partial_x v_{\varepsilon,\eta}D_xf\left(p^{\frac{1}{m-1}}\left(v_{\varepsilon,\eta}\right),x,t\right)\frac{p^{\frac{m-2}{m-1}}}{p'}\left(v_{\varepsilon,\eta}\right)\end{aligned}$$

$$- (m-1)\,\xi^2 f\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta}),x,t\right)\frac{p'' p^{\frac{m-2}{m-1}}}{(p')^2}(v_{\varepsilon,\eta})(\partial_x v_{\varepsilon,\eta})^2$$
$$+ (m-2)\,\xi^2 f\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta}),x,t\right)p^{-\frac{1}{m-1}}(v_{\varepsilon,\eta})(\partial_x v_{\varepsilon,\eta})^2.$$

The coefficient of $\xi^2(\partial_x v_{\varepsilon,\eta})^4$ on the left-hand side of (A.9) is bounded below by a positive constant. All of the coefficients of $(\partial_x v_{\varepsilon,\eta})^2$ and $(\partial_x v_{\varepsilon,\eta})^3$ on the right-hand side of (A.9) not involving ψ_ε and f are bounded. We also notice that all of three terms involving ψ_ε have the negative signs, where the negative sign of the last one is deduced from the assumption $m \geq \beta+2$, and thus can be dropped. Thanks to the smoothness assumption on f given in (H_f) and $m > 2$, the first two terms involving f in (A.9) are bounded uniformly with respect to ε. Due to the Lipchitz-type dominance assumption on f proposed in (H_f), the coefficients of the remaining terms involving f in (A.9) is bounded. Indeed,

$$\begin{aligned}
- (m-1)\,\xi^2 f\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta}),x,t\right)&\frac{p'' p^{\frac{m-2}{m-1}}}{(p')^2}(v_{\varepsilon,\eta})\\
&= \frac{2(m-1)N_0}{3}\xi^2 f\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta}),x,t\right)\frac{p^{\frac{m-2}{m-1}}}{(p')^2}(v_{\varepsilon,\eta})\\
&\leq \frac{2N_0(m-1)}{3}\xi^2 h\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta})\right)\frac{p^{\frac{m-2}{m-1}}}{(p')^2}(v_{\varepsilon,\eta})\\
&= \frac{2N_0(m-1)}{3}\xi^2 \mathrm{Lip}\left(h,\Theta\left(\Gamma,T'\right)\right)\frac{p}{(p')^2}(v_{\varepsilon,\eta}),
\end{aligned}$$

and

$$\begin{aligned}
(m-2)\,\xi^2 f\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta}),x,t\right)p^{-\frac{1}{m-1}}(v_{\varepsilon,\eta}) &\leq (m-2)\,\xi^2 h\left(p^{\frac{1}{m-1}}(v_{\varepsilon,\eta})\right)p^{-\frac{1}{m-1}}(v_{\varepsilon,\eta})\\
&\leq (m-2)\,\xi^2 \mathrm{Lip}\left(h,\Theta\left(\Gamma,T'\right)\right),
\end{aligned}$$

where $\mathrm{Lip}(h,r)$ is the local Lipschitz constant of h on the closed interval $[0,r]$. Therefore, the choice $\gamma = \frac{1}{m-1}$, as illustrated in (3.4), attain (3.8) when (H_1) is active.

Therefore, a) follows by choosing $T = \frac{T'}{3}$.

b) To demonstrate the second claim of Proposition 3.1, it suffices to choose $\xi(x,t) = \xi(x)$ so that it is independent of t. Indeed, given arbitrary $\eta \in (0, \min\{\eta_0, \|u_0\|_\infty\})$, the assumption $U_0 \in L^\infty(\Omega)$ gives us

$$\begin{aligned}
\|U_0\|_\infty \geq \nabla\left((u_0(x)+\eta)^{\frac{1}{\gamma}}\right) &= \nabla\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(x,0)\right) = \nabla\left(h_{\varepsilon,\eta}(x,0)\right)\\
&= \nabla\left(p\left(v_{\varepsilon,\eta}(x,0)\right)\right) = p'\left(v_{\varepsilon,\eta}(x,0)\right)\nabla v_{\varepsilon,\eta}(x,0),\ \text{a.e. in } \Omega,
\end{aligned}$$

and thus

$$\nabla v_{\varepsilon,\eta}(x,0) \le \frac{\|U_0\|_\infty}{p'(v_{\varepsilon,\eta}(x,0))} \le \frac{3\|U_0\|_\infty}{2N_0}, \text{ a.e. in } \Omega,$$

or equivalently, $\nabla v_{\varepsilon,\eta}(\cdot,0) \in L^\infty(\Omega)$ and $\|\nabla v_{\varepsilon,\eta}(\cdot,0)\|_\infty \le \frac{3\|U_0\|_\infty}{2N_0}$.

We again suppose that $|\nabla v_{\varepsilon,\eta}(x,t)|$ is large at a point $x = x_0 \in \Omega$ at some time $t = t \in \left[0, \frac{T'}{3}\right)$. Then let

$$Q = B_{\mathbb{R}^N(x_0;2)} \cap \Omega, \;\; Q_1 = B_{\mathbb{R}^N(x_0;1)} \cap \Omega,$$

be two open subsets containing x_0. We consider a smooth cut-off function $\xi(x) \in C^\infty(\Omega)$[1], $0 \le \xi \le 1$ such that

$$\xi(x) = \begin{cases} 1, \text{ in } Q_1, \\ 0, \text{ out side } B_{\mathbb{R}^N}\left(x_0; \dfrac{3}{2}\right) \cap \Omega, \end{cases}$$

and $|\nabla\xi| + |\Delta\xi| \le C_\xi$ for some $C_\xi > 0$ being independent of ε.

We then consider the function $w_{\varepsilon,\eta}(x,t) = \xi^2(x)\,|\nabla v_{\varepsilon,\eta}(x,t)|^2$ in $\overline{Q} \times \left[0, \frac{T'}{3}\right)$. Then $w_{\varepsilon,\eta}$ attains its maximum at either the initial data, i.e., at some point $(x_1, 0)$ with $x_1 \in B_{\mathbb{R}^N}\left(x_0; \frac{3}{2}\right)$, or at some interior point $(x_2, t_2) \in B_{\mathbb{R}^N}\left(x_0; \frac{3}{2}\right) \times \left(0, \frac{T'}{3}\right)$.

In the former case, the assumption posed on the initial data u_0 yields

$$\max_{(x,t)\in\overline{Q}\times\left[0,\frac{T'}{3}\right)} w_{\varepsilon,\eta}(x,t) = w_{\varepsilon,\eta}(x_1,0) = \xi^2(x_1)\,|\nabla v_{\varepsilon,\eta}(x_1,0)|^2 \le \|\nabla v_{\varepsilon,\eta}(\cdot,0)\|_\infty^2 \le \frac{9\,\|U_0\|_\infty^2}{4N_0^2},$$

which implies the desired gradient estimate.

In the later case, we repeat the proof of the first statement to establish the desired gradient estimate. This completes our proof. □

Proof of Proposition 3.2. We utilize the settings suggested in the proof of Proposition 3.1 until the equation for $h_{\varepsilon,\eta}$ is established. By Proposition 3.1, we set $\gamma = \frac{2}{1+\beta}$. Recall that the equation for $h_{\varepsilon,\eta}$ in this case is given by

$$\partial_t h_{\varepsilon,\eta} - \Delta h_{\varepsilon,\eta} - (\gamma - 1)\frac{|\nabla h_{\varepsilon,\eta}|^2}{h_{\varepsilon,\eta}} + \frac{\psi_\varepsilon\left(h_{\varepsilon,\eta}^\gamma\right)}{\gamma h_{\varepsilon,\eta}} = \frac{f\left(h_{\varepsilon,\eta}^\gamma, x, t\right)}{\gamma h_{\varepsilon,\eta}^{\gamma-1}}.$$

[1] Compare this with the proof of the statement a).

For any $\tau \in \left(0, \frac{T'}{3}\right)$, let us consider a cut-off function $\xi(t) \in C^\infty(0,\infty)$, $0 \le \xi(t) \le 1$, such that

$$\xi(t) = \begin{cases} 1, & \text{on } \left[\tau, \dfrac{T'}{3}\right], \\ 0, & \text{out side } \left(\dfrac{\tau}{2}, \dfrac{T'}{3} + \dfrac{\tau}{2}\right), \end{cases}$$

and $|\xi'| \le \dfrac{c_0}{\tau}$ for some constant $c_0 > 0$.

Next, we set $w_{\varepsilon,\eta}(x,t) := \xi(t)\,|\nabla h_{\varepsilon,\eta}(x,t)|^2$.

If $\max_{\Omega\times[0,T']} w_{\varepsilon,\eta} = 0$, then $\nabla u_{\varepsilon,\eta} = 0$ in $\Omega \times \left[\tau, \frac{T'}{3}\right]$, so the desired gradient estimate is trivial.

If this is not the case, there is a point $(x_0,t_0) \in \Omega \times \left(0, \frac{2T'}{3}\right)$ such that $\max_{\Omega\times[0,T']} w_{\varepsilon,\eta} = w_{\varepsilon,\eta}(x_0,t_0) > 0$. Thus, we have at (x_0,t_0),

$$\partial_t w_{\varepsilon,\eta} = \nabla w_{\varepsilon,\eta} = 0, \quad \Delta w_{\varepsilon,\eta} \le 0.$$

One has

$$\begin{aligned} \partial_t w_{\varepsilon,\eta} &= \xi' |\nabla h_{\varepsilon,\eta}|^2 + 2\xi \nabla h_{\varepsilon,\eta} \cdot \partial_t \nabla h_{\varepsilon,\eta}, \\ \nabla w_{\varepsilon,\eta} &= \xi \nabla \left(|\nabla h_{\varepsilon,\eta}|^2\right), \\ \Delta w_{\varepsilon,\eta} &= \xi \Delta \left(|\nabla h_{\varepsilon,\eta}|^2\right) = 2\xi \left|D^2 h_{\varepsilon,\eta}\right|^2 + 2\xi \nabla h_{\varepsilon,\eta} \cdot \nabla \Delta h_{\varepsilon,\eta}, \end{aligned}$$

where we have used Lemma 3.2.

Thus, one has, at (x_0,t_0), $\nabla\left(|\nabla h_{\varepsilon,\eta}|^2\right) = 0$, and

$$0 \le \partial_t w_{\varepsilon,\eta} - \Delta w_{\varepsilon,\eta} = \xi' |\nabla h_{\varepsilon,\eta}|^2 + 2\xi \nabla h_{\varepsilon,\eta} \cdot \partial_t \nabla h_{\varepsilon,\eta} - 2\xi \left|D^2 h_{\varepsilon,\eta}\right|^2 - 2\xi \nabla h_{\varepsilon,\eta} \cdot \nabla \Delta h_{\varepsilon,\eta},$$

Now, consider at the point (x_0,t_0), the latter is equivalent to

$$\xi' |\nabla h_{\varepsilon,\eta}|^2 + 2\xi \nabla h_{\varepsilon,\eta} \cdot \nabla\left(\partial_t h_{\varepsilon,\eta} - \Delta h_{\varepsilon,\eta}\right) \ge 2\xi \left|D^2 h_{\varepsilon,\eta}\right|^2 \ge 0,$$

Combining the last inequality with the PDE satisfied by $h_{\varepsilon,\eta}$ yields

$$\xi' |\nabla h_{\varepsilon,\eta}|^2 + 2\xi \nabla h_{\varepsilon,\eta} \cdot \nabla \left((\gamma - 1) \frac{|\nabla h_{\varepsilon,\eta}|^2}{h_{\varepsilon,\eta}} - \frac{\psi_\varepsilon\left(h_{\varepsilon,\eta}^\gamma\right)}{\gamma h_{\varepsilon,\eta}} + \frac{f\left(h_{\varepsilon,\eta}^\gamma, x_0, t_0\right)}{\gamma h_{\varepsilon,\eta}^{\gamma-1}} \right) \ge 0.$$

Expanding this and noticing that $\nabla\left(|\nabla h_{\varepsilon,\eta}(x_0,t_0)|^2\right) = 0$, we obtain

$$(\gamma-1)\, h_{\varepsilon,\eta}^{-2} |\nabla h_{\varepsilon,\eta}|^4 \le \frac{1}{2}\xi^{-1}\xi' |\nabla h_{\varepsilon,\eta}|^2 - \psi_\varepsilon\left(h_{\varepsilon,\eta}^\gamma\right) h_{\varepsilon,\eta}^{\gamma-2} |\nabla h_{\varepsilon,\eta}|^2 + \frac{1}{\gamma} h_{\varepsilon,\eta}^{-2} \psi_\varepsilon\left(h_{\varepsilon,\eta}^\gamma\right) |\nabla h_{\varepsilon,\eta}|^2$$

$$+ D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)|\nabla h_{\varepsilon,\eta}|^2 + \frac{1}{\gamma} h_{\varepsilon,\eta}^{1-\gamma} \nabla h_{\varepsilon,\eta} \cdot D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right) + \frac{1-\gamma}{\gamma} f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right) h_{\varepsilon,\eta}^{-\gamma} |\nabla h_{\varepsilon,\eta}|^2.$$

Multiply both sides of the last inequality with $h_{\varepsilon,\eta}^2$ and eliminate the negative terms in the right-hand side, one has

$$\begin{aligned}(\gamma-1)|\nabla h_{\varepsilon,\eta}|^4 \leq \frac{1}{2}\xi^{-1}\left|\xi'\right| h_{\varepsilon,\eta}^2 |\nabla h_{\varepsilon,\eta}|^2 + \frac{1}{\gamma}|\nabla h_{\varepsilon,\eta}|^2 + h_{\varepsilon,\eta}^2 \left|D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| |\nabla h_{\varepsilon,\eta}|^2 \\ + \frac{1}{\gamma} h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| |\nabla h_{\varepsilon,\eta}|.\end{aligned} \tag{A.10}$$

If $|\nabla h_{\varepsilon,\eta}(x_0,t_0)| \leq 1$, then $w_{\varepsilon,\eta}(x_0,t_0) \leq 1$. This leads us to $w_{\varepsilon,\eta}(x,\tau) \leq 1$, thereby proves

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq \frac{4}{(1+\beta)^2} u_{\varepsilon,\eta}^{1-\beta}(x,\tau).$$

Then, the desired gradient estimate follows immediately.

If this is not the case, we have $|\nabla h_{\varepsilon,\eta}(x_0,t_0)| > 1$. Multiplying the last term in (A.10) with $|\nabla h_{\varepsilon,\eta}(x_0,t_0)|$, and then simplifying the term $|\nabla h_{\varepsilon,\eta}|^2$ in both sides, we obtain

$$(\gamma-1)|\nabla h_{\varepsilon,\eta}|^2 \leq \frac{1}{2}\xi^{-1}\left|\xi'\right| h_{\varepsilon,\eta}^2 + \frac{1}{\gamma} + h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| + \frac{1}{\gamma} h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right|.$$

Multiplying both sides of the last inequality with $\xi(t_0)$ yields

$$(\gamma-1)\xi(t_0)|\nabla h_{\varepsilon,\eta}|^2 \leq \frac{1}{2}\left|\xi'\right| h_{\varepsilon,\eta}^2 + \frac{1}{\gamma}\xi(t_0)\left(1 + \gamma h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| + h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right|\right).$$

Remind that $w(x_0,t_0) = \xi(t_0)|\nabla h_{\varepsilon,\eta}(x_0,t_0)|^2$, $0 \leq \xi(t) \leq 1$, $|\xi'| \leq c_0\tau^{-1}$. Since

$$h_{\varepsilon,\eta}^{\gamma}(x,t) = u_{\varepsilon,\eta}(x,t) \in \left[\eta, \Theta\left(\Gamma, T'\right)\right], \text{ for all } (x,t) \in \Omega \times \left[0, T'\right],$$

and

$$w_{\varepsilon,\eta}(x_0,t_0) \geq \xi(t_0) w_{\varepsilon,\eta}(x_0,t_0) \geq w_{\varepsilon,\eta}(x,\tau) = |\nabla h_{\varepsilon,\eta}(x,\tau)|^2,$$

we deduce from the last estimate and

$$\begin{aligned}|\nabla h_{\varepsilon,\eta}(x,\tau)|^2 &\leq \frac{c_0}{2\tau(\gamma-1)} h_{\varepsilon,\eta}^2 + \frac{1}{\gamma-1}\xi(t_0) h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| \\ &\quad + \frac{\xi(t_0)}{\gamma(\gamma-1)}\left(1 + h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right|\right) \\ &\leq \frac{c_0(1+\beta)}{2\tau(1-\beta)} h_{\varepsilon,\eta}^2 + \frac{1+\beta}{1-\beta}\xi(t_0) h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right| \\ &\quad + \frac{(1+\beta)^2}{2(1-\beta)}\left(1 + h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma}, x_0, t_0\right)\right|\right)\end{aligned}$$

$$\le \frac{c_0(1+\beta)}{2\tau(1-\beta)}\Theta^{1+\beta}(\Gamma,T') + \frac{1+\beta}{1-\beta}\Theta^{1+\beta}(\Gamma,T')\,\Theta\left(D_u f(\cdot,x_0,t_0),\Theta(\Gamma,T')\right)$$
$$+\frac{(1+\beta)^2}{2(1-\beta)}\left[1+\Theta^{\frac{1+3\beta}{2}}(\Gamma,T')\,\Theta\left(D_x f(\cdot,x_0,t_0),\Theta(\Gamma,T')\right)\right]$$
$$\le \frac{c_0(1+\beta)}{2\tau(1-\beta)}\Theta^{1+\beta}(\Gamma,T') + \frac{1+\beta}{1-\beta}\Theta^{1+\beta}(\Gamma,T')\,\Theta\left(\max_{(x,t)\in\overline{\Omega}\times[0,2T'/3]} D_u f(\cdot,x,t),\Theta(\Gamma,T')\right)$$
$$+\frac{(1+\beta)^2}{2(1-\beta)}\left[1+\Theta^{\frac{1+3\beta}{2}}(\Gamma,T')\,\Theta\left(\max_{(x,t)\in\overline{\Omega}\times[0,2T'/3]} D_x f(\cdot,x,t),\Theta(\Gamma,T')\right)\right].$$

Thus, a) follows by choosing $T=\frac{T'}{3}$.

Now, we prove b). The proof of the second statement is similar to that of the first one. We just make a slight change by considering a cut-off function $\overline{\xi}\in\mathcal{C}^\infty(\mathbb{R})$ (instead of $\xi(t)$ above), such that $0\le\overline{\xi}\le 1$, $\overline{\xi}'\le 0$, and

$$\overline{\xi}(t)=\begin{cases}1, & \text{if } t\le \dfrac{T'}{3},\\[2mm] 0, & \text{if } t\ge \dfrac{2T'}{3}.\end{cases}$$

Then, the function $w_{\varepsilon,\eta}(x,t)$ considered above will attains its maximum at either the initial data or the interior points.

In the former case, there is a point $x_0\in\Omega$ such that

$$\max_{(x,t)\in\Omega\times[0,T']} w_{\varepsilon,\eta}(x,t)=w_{\varepsilon,\eta}(x_0,0)=\overline{\xi}(0)\left|\nabla h_{\varepsilon,\eta}(x_0,0)\right|^2\le\left|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(x_0,0)\right|^2$$
$$=\left|\nabla\left((u_0(x_0)+\eta)^{\frac{1}{\gamma}}\right)\right|^2\le U_0^2(x_0)\le\|U_0\|_\infty^2,$$

which implies

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2\le\frac{4}{(1+\beta)^2}\|U_0\|_\infty^2\,u_{\varepsilon,\eta}^{1-\beta}(x,\tau),\ \text{for any } x\in\Omega.$$

Thus, we get the desired gradient estimate immediately.

In the later case, there is a point $(x_0,t_0)\in\Omega\times\left(0,\frac{2T'}{3}\right)$ such that

$$\max_{(x,t)\in\Omega\times[0,T']} w_{\varepsilon,\eta}(x,t)=w_{\varepsilon,\eta}(x_0,t_0).$$

Then, we repeat the proof of the first statement for this case until (A.10) to get

$$(\gamma-1)\left|\nabla h_{\varepsilon,\eta}\right|^4\le\frac{\overline{\xi}'}{2\overline{\xi}}h_{\varepsilon,\eta}^2\left|\nabla h_{\varepsilon,\eta}\right|^2+\frac{1}{\gamma}\left|\nabla h_{\varepsilon,\eta}\right|^2+h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^\gamma,x_0,t_0\right)\right|\left|\nabla h_{\varepsilon,\eta}\right|^2$$

$$+\frac{1}{\gamma}h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma},x_0,t_0\right)\right|\left|\nabla h_{\varepsilon,\eta}\right|.$$

Since $\overline{\xi}'(t)\leq 0$, the last inequality gives us

$$(\gamma-1)\left|\nabla h_{\varepsilon,\eta}\right|^4\leq\frac{1}{\gamma}\left|\nabla h_{\varepsilon,\eta}\right|^2+h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma},x_0,t_0\right)\right|\left|\nabla h_{\varepsilon,\eta}\right|^2+\frac{1}{\gamma}h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma},x_0,t_0\right)\right|\left|\nabla h_{\varepsilon,\eta}\right|.$$

If $\left|\nabla h_{\varepsilon,\eta}(x_0,t_0)\right|\leq 1$, then

$$\left|\nabla u_{\varepsilon,\eta}(x,\tau)\right|^2\leq\frac{4}{(1+\beta)^2}u_{\varepsilon,\eta}^{1-\beta}(x,\tau),\ \text{for all} x\in\Omega,$$

as above.

If this is not the case, we have $\left|\nabla h_{\varepsilon,\eta}(x_0,t_0)\right|>1$. Repeating the remaining part of the proof of a), we obtain

$$\begin{aligned}\left|\nabla h_{\varepsilon,\eta}\right|^2&\leq\frac{1}{\gamma(\gamma-1)}+\frac{1}{\gamma-1}h_{\varepsilon,\eta}^2\left|D_u f\left(h_{\varepsilon,\eta}^{\gamma},x_0,t_0\right)\right|+\frac{1}{\gamma(\gamma-1)}h_{\varepsilon,\eta}^{3-\gamma}\left|D_x f\left(h_{\varepsilon,\eta}^{\gamma},x_0,t_0\right)\right|\\&\leq\frac{1+\beta}{1-\beta}\Theta^{1+\beta}\left(\Gamma,T'\right)\Theta\left(D_u f(\cdot,x_0,t_0),\Theta\left(\Gamma,T'\right)\right)\\&\quad+\frac{(1+\beta)^2}{2(1-\beta)}\left[1+\Theta^{\frac{1+3\beta}{2}}\left(\Gamma,T'\right)\Theta\left(D_x f(\cdot,x_0,t_0),\Theta\left(\Gamma,T'\right)\right)\right]\\&\leq\frac{1+\beta}{1-\beta}\Theta^{1+\beta}\left(\Gamma,T'\right)\Theta\left(\max_{(x,t)\in\overline{\Omega}\times[0,2T'/3]}D_u f(\cdot,x,t),\Theta\left(\Gamma,T'\right)\right)\\&\quad+\frac{(1+\beta)^2}{2(1-\beta)}\left[1+\Theta^{\frac{1+3\beta}{2}}\left(\Gamma,T'\right)\Theta\left(\max_{(x,t)\in\overline{\Omega}\times[0,2T'/3]}D_x f(\cdot,x,t),\Theta\left(\Gamma,T'\right)\right)\right].\end{aligned}$$

Finally, we obtain the following gradient estimate

$$\left|\nabla u_{\varepsilon,\eta}(x,\tau)\right|^2\leq Cu_{\varepsilon,\eta}^{1-\beta}(x,\tau),\ \text{for all }\ (x,\tau)\in\Omega\times[0,T),$$

where $T=\frac{T'}{3}$, and C is the positive constant given by

$$C:=\max\left\{\begin{array}{l}\frac{4}{(1+\beta)^2}\left\|U_0\right\|_\infty^2,\frac{1+\beta}{1-\beta}\Theta^{1+\beta}(\Gamma,T')\,\Theta\left(\max\limits_{(x,t)\in\overline{\Omega}\times[0,2T'/3]}D_u f(\cdot,x,t),\Theta(\Gamma,T')\right)\\+\frac{(1+\beta)^2}{2(1-\beta)}\left[1+\Theta^{\frac{1+3\beta}{2}}(\Gamma,T')\,\Theta\left(\max\limits_{(x,t)\in\overline{\Omega}\times[0,2T'/3]}D_x f(\cdot,x,t),\Theta(\Gamma,T')\right)\right]\end{array}\right\}.$$

This completes our proof. □

Bibliography

Aronson, D. G. (Mar. 1969). "Regularity properties of flows through porous media". In: *SIAM Journal on Applied Mathematics* 17.2, pp. 461–467. DOI: 10.1137/0117045. URL: http://www.jstor.org/stable/2099578.

— (1970). "Regularity properties of flows through porous media: A counterexample". In: *SIAM J. Appl. Math.* 19, pp. 299–307. ISSN: 0036-1399. DOI: 10.1137/0119027. URL: https://doi.org/10.1137/0119027.

Benachour, Said, Razvan Gabriel Iagar, and Philippe Laurençot (2016). "Large time behavior for the fast diffusion equation with critical absorption". In: *J. Differential Equations* 260.11, pp. 8000–8024. ISSN: 0022-0396. DOI: 10.1016/j.jde.2016.02.008. URL: https://doi.org/10.1016/j.jde.2016.02.008.

Boccardo, Lucio and Thierry Gallouët (1989). "Nonlinear elliptic and parabolic equations involving measure data". In: *J. Funct. Anal.* 87.1, pp. 149–169. ISSN: 0022-1236. DOI: 10.1016/0022-1236(89)90005-0. URL: https://doi.org/10.1016/0022-1236(89)90005-0.

Boccardo, Lucio and François Murat (1992). "Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations". In: *Nonlinear Anal.* 19.6, pp. 581–597. ISSN: 0362-546X. DOI: 10.1016/0362-546X(92)90023-8. URL: https://doi.org/10.1016/0362-546X(92)90023-8.

Brezis, Haim (2011). *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, pp. xiv+599. ISBN: 978-0-387-70913-0.

Coddington, Earl A. and Norman Levinson (1955). *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, pp. xii+429. DOI: 10.1063/1.3059875.

Dao, Anh Nguyen (2017). "Instantaneous shrinking of compact support of solutions of semi-linear parabolic equations with singular absorption". In: *An. Univ. Craiova Ser. Mat. Inform.* 44.1, pp. 156–168. ISSN: 1223-6934.

Dao, Nguyen Anh and Jesus Ildefonso Díaz (2016). "A gradient estimate to a degenerate parabolic equation with a singular absorption term: the global quenching phenomena". In: *J. Math. Anal. Appl.* 437.1, pp. 445–473. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2015.11.059. URL: https://doi.org/10.1016/j.jmaa.2015.11.059.

— (2017). "The extinction versus the blow-up: global and non-global existence of solutions of source types of degenerate parabolic equations with a singular absorption". In: *J. Differential Equations* 263.10, pp. 6764–6804. ISSN: 0022-0396. DOI: 10.1016/j.jde.2017.07.029. URL: https://doi.org/10.1016/j.jde.2017.07.029.

Dávila, Juan and Marcelo Montenegro (2005). "Existence and asymptotic behavior for a singular parabolic equation". In: *Trans. Amer. Math. Soc.* 357.5, pp. 1801–1828. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-04-03811-5. URL: https://doi.org/10.1090/S0002-9947-04-03811-5.

Evans, Lawrence C. and Ronald F. Gariepy (2015). *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, pp. xiv+299. ISBN: 978-1-4822-4238-6.

Galaktionov, Victor A. and Juan L. Vázquez (1995). "Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations". In: *Arch. Rational Mech. Anal.* 129.3, pp. 225–244. ISSN: 0003-9527. DOI: 10.1007/BF00383674. URL: https://doi.org/10.1007/BF00383674.

Kawohl, Bernhard (1992). "Remarks on quenching, blow up and dead cores". In: *Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989)*. Vol. 7. Progr. Nonlinear Differential Equations Appl. Birkhäuser Boston, Boston, MA, pp. 275–286. DOI: 10.1007/978-1-4612-0393-3_20.

Kawohl, Bernhard and Robert Kersner (1992). "On degenerate diffusion with very strong absorption". In: *Math. Methods Appl. Sci.* 15.7, pp. 469–477. ISSN: 0170-4214. DOI: 10.1002/mma.1670150703. URL: https://doi.org/10.1002/mma.1670150703.

Ladyˇ zenskaja, O. A. and N. N. Ural' ceva (1964). "On the Hölder continuity of the solutions and the derivatives of linear and quasi-linear equations of elliptic and parabolic types". In: *Trudy Mat. Inst. Steklov.* 73, pp. 172–220. ISSN: 0371-9685.

Phillips, Daniel (1987). "Existence of solutions of quenching problems". In: *Appl. Anal.* 24.4, pp. 253–264. ISSN: 0003-6811. DOI: 10.1080/00036818708839668. URL: https://doi.org/10.1080/00036818708839668.

Simon, Jacques (1987). "Compact sets in the space $L^p(0,T;B)$". In: *Ann. Mat. Pura Appl. (4)* 146, pp. 65–96. ISSN: 0003-4622. DOI: 10.1007/BF01762360. URL: https://doi.org/10.1007/BF01762360.