Staggered and well-balanced discretization of shallow water equations

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Overview

Time-dependent fluid domain

- The fluid domain at time t:

$$\Omega_t := \Big\{(x,z) \in \mathbb{R}^{d+1}; -H_0 + b(t,x) < z < \zeta(t,x)\Big\}.$$

- 3D Navier-Stokes equations in $\mathbb{R}_+ \times \Omega_t$:

3D-NS

$$\begin{cases} \nabla \cdot u = 0, \\ \partial_t u + \nabla \cdot (u \otimes u) = -\dfrac{1}{\rho_0}\nabla P - g\mathbf{e}_z + \dfrac{1}{\rho_0}\nabla \cdot \overline{\mathbf{T}}, \end{cases} \qquad \text{(NS)}$$

where u: fluid velocity, ρ_0: constant fluid/mass density, P: pressure, $\overline{\mathbf{T}}$: matrix of stress terms.

Assumptions for fluids and flows

(H1) The fluid is homogeneous: $\rho \equiv \rho_0$, and inviscid: $\overline{\mathbf{T}} = \mathbf{0}_d$.

(H2) The fluid is incompressible: $\partial_t (\rho (t, x(t), z(t))) = 0$.

(H3) The flow is irrotational: $\mathrm{curl} u = 0$.

(H4) The surface and the bottom can be parametrized as graphs above the still water level: $\exists T_0 \in \mathbb{R}_+^\star$, s.t. $\zeta, b : [0, T_0) \times \mathbb{R}^d$ describe surface and bottom, respectively.

(H5) (Impermeability condition) The fluid particles do not cross the bottom.

Assumptions for fluids and flows (cont.)

(H6) (Kinetic boundary condition) The fluid particles do not cross the surface, i.e., no surging waves are allowed.

(H7) There is no surface tension and the external pressure is constant: $P = P_{\mathrm{atm}}$ on surface.

(H8) (Finite energy) The fluid is at rest at infinity.

(H9) ($\star$) The water depth is always bounded from below by a nonnegative constant.

$$\exists H_{\mathsf{min}} > 0 \text{ s.t. } h(t,x) := H_0 + \zeta(t,x) - b(t,x) \geq H_{\mathsf{min}}.$$

(H10) The bottom is time-independent: $\partial_t b = 0$.

Homogeneous free surface Euler equation

Write $u = (v, w)$, v: horizontal component, z: vertical component.

Homogeneous free surface Euler equation.

$$\begin{cases} \partial_t u + (u \cdot \nabla_{x,z})\, u = -\dfrac{1}{\rho_0} \nabla_{x,z} P - g\mathbf{e}_z, & \text{in } \Omega_t, \\ \nabla_{x,z} \cdot u = 0, & \text{in } \Omega_t, \\ \text{curl } u = 0, & \text{in } \Omega_t, \\ \partial_t \zeta = w - v \cdot \nabla \zeta, & \text{on } \Gamma_{\mathrm{top}}, \\ w = v \cdot \nabla b, & \text{on } \Gamma_{\mathrm{bot}}, \\ P = P_{\mathrm{atm}}, & \text{on } \Gamma_{\mathrm{top}}. \end{cases} \tag{hfsE}$$

Zakharov/Craig-Sulem formulation

Irrotationality $\Rightarrow \exists \Phi$ s.t. $u = \nabla_{x,z}\Phi$ in Ω_t, where Φ: velocity potential and its trace: $\psi := \Phi|_{z=\zeta}$.

Water waves system

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta, b]\,\psi = 0, & \text{in } \Omega_t, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \dfrac{\left(\mathcal{G}[\zeta, b]\,\psi + \nabla\zeta\cdot\nabla\psi\right)^2}{2\left(1+|\nabla\zeta|^2\right)} = 0, & \text{in } \Omega_t. \end{cases} \tag{ww}$$

where

$$\mathcal{G}[\zeta, b]\,\psi = -\nabla\cdot\left(\int_{-H_0+b(x)}^{\zeta(t,x)} \Phi(t,x,z)\,dz\right), \text{ in } [0, T_0)\times\mathbb{R}^d.$$

Derivation of shallow water equations

Layer-averaged horizontal approximations:

$$\begin{cases} \partial_t h + \nabla \cdot (h\bar{v}) = 0, \\ \partial_t (h\bar{v}) + h(\bar{v}\cdot\nabla)\bar{v} + \bar{v}\nabla\cdot(h\bar{v}) + gh\nabla(h+b) = 0. \end{cases} \quad (\mathrm{SW}_1)$$

Surface approximations:

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0, \\ \partial_t (hu) + h(u\cdot\nabla)u + u\nabla\cdot u + gh\nabla(h+b) = 0. \end{cases} \quad (\mathrm{SW}_2)$$

1DSW and Admissible meshes of $\mathbb{R}$

1D Shallow Water in $[0, T_0) \times \mathbb{R}$.

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = -gh\partial_x b. \end{cases} \tag{1DSW}$$

Definition (Admissible meshes of $\mathbb{R}$)

An admissible mesh $\mathcal{T}$ of $\mathbb{R}$ is given by an increasing sequence of reals $\left(x_{i+\frac{1}{2}}\right)_{i\in\mathbb{Z}}$ s.t. $\mathbb{R} = \bigcup_{i\in\mathbb{Z}} \overline{C_i}$ where $C_i := \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$. Then $\mathcal{T} := \{C_i; i \in \mathbb{Z}\}$. Set $\Delta x_i := \mathrm{length}\,(C_i)$, $x_i :=$ mid-point of C_i. Assume that

$$\Delta x := \mathrm{size}\,(\mathcal{T}) := \sup_{i\in\mathbb{Z}} \Delta x_i < \infty,$$

$$\exists \alpha_{\mathcal{T}} \in \mathbb{R}_+^\star \text{ s.t. } \alpha_{\mathcal{T}} \Delta x \le \inf_{i\in\mathbb{Z}} \Delta x_i.$$

DG-staggered upwind scheme

Given $\mathcal{T}$ an admissible mesh of $\mathbb{R}$ and $\Delta t \in \mathbb{R}_+^\star$ the time step. Set $t^n := n\Delta t$ for $n \in [T] := \{0, 1, \ldots, T\}$, with $T := \left\lfloor \frac{T_0}{\Delta t} \right\rfloor$.

- *Primal mesh*: $\left(x_{i+\frac{1}{2}}\right)_{i\in\mathbb{Z}}$, primal cells: C_i's, $x_{i+\frac{1}{2}}$'s: interfaces.
- *Dual mesh*: $(x_i)_{i\in\mathbb{Z}}$, duel cells: $C_{i+\frac{1}{2}} := (x_i, x_{i+1})$.

- Water height and bottom topography are discretized at the *dual mesh*, i.e., the center of the primal cells: h_i^n and b_i.
- Velocity is discretized at the *primal mesh*, i.e., the interfaces between primal cells: $u^n_{i+\frac{1}{2}}$.

DG-staggered upwind scheme

DG-staggered upwind scheme

$$
\begin{aligned}
h_i^{n+1} &:= h_i^n - \nu_i \left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} \right), \\
\bar{h}^{n+1}_{i+\frac{1}{2}} u^{n+1}_{i+\frac{1}{2}} &:= \bar{h}^n_{i+\frac{1}{2}} u^n_{i+\frac{1}{2}} - \nu_{i+\frac{1}{2}} \left(G^n_{i+1} - G^n_i \right) \\
&\quad - \frac{g}{2} \nu_{i+\frac{1}{2}} \left(\left(h^{n+1}_{i+1} \right)^2 - \left(h^n_{i+1} \right)^2 + 2 \bar{h}^{n+1}_{i+\frac{1}{2}} \left(b_{i+1} - b_i \right) \right).
\end{aligned}
\tag{DGsc}
$$

where $\nu_i := \frac{\Delta t}{\Delta x_i}$, $\nu_{i+\frac{1}{2}} := \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}}$ and

$$
F^n_{i+\frac{1}{2}} := h_i^n \left(u^n_{i+\frac{1}{2}} \right)_+ + h^n_{i+1} \left(u^n_{i+\frac{1}{2}} \right)_-, \quad \bar{h}^n_{i+\frac{1}{2}} := \frac{1}{2} \left(h_i^n + h^n_{i+1} \right),
$$

$$
G_i^n := \frac{1}{2} u^n_{i-\frac{1}{2}} \left(F^n_{i-\frac{1}{2}} + F^n_{i+\frac{1}{2}} \right)_+ + \frac{1}{2} u^n_{i+\frac{1}{2}} \left(F^n_{i-\frac{1}{2}} + F^n_{i+\frac{1}{2}} \right)_-.
$$

Properties of (DGsc)

Convention (Zero velocity of water in dry areas)

For all $i \in \mathbb{Z}$, $n \in [T]$, if $\bar{h}^n_{i+\frac{1}{2}} = 0$, then $u^n_{i+\frac{1}{2}} := 0$.

- *Conservation of water volume*: $\mathfrak{h}^n = \mathfrak{h}^0$ for all $n \in [T]$, where $\mathfrak{h}^n$ is the 1D discrete water volume at $t = t^n$ defined by

$$\mathfrak{h}^n := \sum_{i \in \mathbb{Z}} \Delta x_i h^n_i, \ \ \forall n \in [T] .$$

- *Conservation of total mass*: $\mathcal{Z}^n = \mathcal{Z}^0$ for all $n \in [T]$, where $\mathcal{Z}^n$ is the 1D discrete total mass at $t = t^n$ defined by

$$\mathcal{Z}^n := \sum_{i \in \mathbb{Z}} \Delta x_i \left(h^n_i + b_i - H_0 \right), \ \ \forall n \in [T] .$$

Properties of (DGsc) (cont.)

- *Total momentum (in the flat bottom topography case $b = 0$)*: $\mathcal{M}^n = \mathcal{M}^0$ for all $n \in [T]$, where $\mathcal{M}^n$ is the 1D discrete total momentum at $t = t^n$ defined by

$$\mathcal{M}^n := \sum_{i \in \mathbb{Z}} \Delta x_{i+\frac{1}{2}} \bar{h}^n_{i+\frac{1}{2}} u^n_{i+\frac{1}{2}}, \quad \forall n \in [T],$$

- *Horizontal impulse (in the flat bottom topography case)*: In general, (DGsc) does not preserve the following 1D discrete horizontal impulse:

$$\mathcal{I}^n := \sum_{i \in \mathbb{Z}} \Delta x_{i+\frac{1}{2}} \left(\bar{h}^n_{i+\frac{1}{2}} + b_i - H_0 \right) u^n_{i+\frac{1}{2}}, \quad \forall n \in [T], \qquad \text{(dhi)}$$

Properties of (DGsc) (cont.)

- *Total energy*: In general, (DGsc) does not preserve the following 1D discrete total energy $\mathcal{H}^n_{\mathrm{SW},\alpha,\beta,\gamma,\delta}$ being equal to

$$\frac{1}{2}\sum_{i\in\mathbb{Z}}\left[\Delta x_{i+\alpha}\left(h^n_{i+\alpha}+b_i-H_0\right)^2+\Delta x_{i+\beta}h^n_{i+\gamma}\left(u^n_{i+\delta}\right)^2\right],$$

alternatively, $\overline{\mathcal{H}}^n_{\mathrm{SW},\alpha,\beta,\gamma,\delta}$ being equal to

$$\frac{1}{2}\sum_{i\in\mathbb{Z}}\left[\Delta x_{i+\alpha}\left(\bar{h}^n_{i+\frac{1}{2}}+b_i-H_0\right)^2+\Delta x_{i+\beta}\bar{h}^n_{i+\frac{1}{2}}\left(u^n_{i+\gamma}\right)^2\right],$$

Courant-Friedrichs-Lewy-like conditions

Courant-Friedrichs-Lewy-like (CFL-like) condition

$$\Delta t \leq \frac{\inf_{i\in\mathbb{Z}} \Delta x_i}{\sup\limits_{i\in\mathbb{Z}, n\in[T]} \left(\left(u^n_{i+\frac{1}{2}}\right)_+ - \left(u^n_{i-\frac{1}{2}}\right)_-\right)}, \tag{CFL3}$$

or stronger

$$\nu \leq \frac{\alpha_{\mathcal{T}}}{\sup\limits_{i\in\mathbb{Z}, n\in[T]} \left(\left(u^n_{i+\frac{1}{2}}\right)_+ - \left(u^n_{i-\frac{1}{2}}\right)_-\right)}, \tag{CFL4}$$

Properties of (DGsc) (cont.)

- *Nonnegativity conservation of water height*: Provided (CFL3) or (CFL4), (DGsc) is a monotone scheme. As a consequence, (DGsc) preserves the nonnegativity of water height:

$$(h_0 \geq 0 \text{ a.e. in } \mathbb{R}) \Rightarrow (h_i^n \geq 0, \;\; \forall i \in \mathbb{Z}, \;\; \forall n \in [T]) \, .$$

- *Positivity conservation of water height*: Provided *strict* (CFL3) or *strict* (CFL4), (DGsc) preserves the positivity of water height:

$$(h_0 > 0 \text{ a.e. in } \mathbb{R}) \Rightarrow (h_i^n > 0, \;\; \forall i \in \mathbb{Z}, \;\; \forall n \in [T]) \, .$$

Well-balanced property of (DGsc)

- *Fully wet case*: Provided strict (CFL3)/(CFL4) and $h_0 > 0$ a.e. in $\mathbb{R}$, (DGsc) preserves the following discrete still water steady state

$$\begin{cases} u^n_{i+\frac{1}{2}} = 0, & \forall i \in \mathbb{Z}, \\ h^n_i + b_i = C, & \forall i \in \mathbb{Z}, \end{cases} \Rightarrow \begin{cases} u^{n+1}_{i+\frac{1}{2}} = 0, & \forall i \in \mathbb{Z}, \\ h^{n+1}_i + b_i = C, & \forall i \in \mathbb{Z}. \end{cases} \tag{dsw}$$

- *Dry-wet case*: In general, (DGsc) does not preserve the discrete still water (dsw) in the case of fluid domain being dry and wet.

Passing to the limit in (DGsc) when $b = 0$

Some balances on kinetic energy, potential energy and topography energy holds. Then let both the mesh size and the time step tend to 0.

- *Consistency of* (DGsc) *with weak formulation of* (1DSW): Provided that some discrete BV estimates holds, the limit satisfies the weak formulation of (1DSW).
- *Consistency of* (DGsc) *with weak entropy inequality*: Provided that some discrete BV estimates holds, the limit satisfies the weak entropy inequality of (1DSW).

Description of OE-staggered schemes

$$
\begin{cases}
w_i^n := h_i^n + b_i, \\
b_{i+\frac{1}{2}}^n := \min\left(\max\left(b_i, b_{i+1}\right), \min\left(w_i^n, w_{i+1}^n\right)\right), \\
h_{i+\frac{1}{2}-}^n := \min\left(w_i^n - b_{i+\frac{1}{2}}^n, h_i^n\right), \\
h_{i+\frac{1}{2}+}^n := \min\left(w_{i+1}^n - b_{i+\frac{1}{2}}^n, h_{i+1}^n\right),
\end{cases}
$$

Offset equilibrium staggered schemes

$$
\begin{cases}
h_i^{n+1} := h_i^n - \nu_i\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right), \\
\overline{h}_{i+\frac{1}{2}}^{n+1} u_{i+\frac{1}{2}}^{n+1} := \overline{h}_{i+\frac{1}{2}}^n u_{i+\frac{1}{2}}^n - \nu_{i+\frac{1}{2}}\left(G_{i+1}^n - G_i^n\right) \\
\qquad - g\nu_{i+\frac{1}{2}} \overline{h}_{i+\frac{1}{2}}^{n+1}\left(h_{i+\frac{1}{2}+}^{n+1} - h_{i+\frac{1}{2}-}^{n+1}\right).
\end{cases}
\qquad \text{(OEsc)}
$$

Properties of (OEsc)

- In the flat bottom case $b = 0$ and case of fully wet domain, (OEsc) and (DGsc) coincide.
- (OEsc) preserves the 1D discrete water volume, total mass. For the flat bottom topography case, (OEsc) also preserves the 1D discrete total momentum. (OEsc) does not preserves the mentioned 1D discrete horizontal impulse and total energy.
- (OEsc) preserves the nonegativity (resp., positivity) of water height provided (CFL3) (resp., strict (CFL3)).
- For fully wet case, (OEsc) preserves the discrete still water (dsw).

Approximate dry and approximate wet

Let ε_0 be the machine epsilon, e.g., double precision for 64-bit, $\varepsilon_0 := 2^{-51} \approx 2.22 \times 10^{-16}$. Define for all $[0, T_0)$:

$$D^{\varepsilon_0,t} := \{x \in \mathbb{R}; h(t,x) \leq \varepsilon_0\}, \quad W^{\varepsilon_0,t} := \{x \in \mathbb{R}; h(t,x) > \varepsilon_0\},$$

Convention (Zero velocity of water in ε_0-dry areas)

For all $i \in \mathbb{Z}$, $n \in [T]$, if $\bar{h}^n_{i+\frac{1}{2}} \leq \varepsilon_0$, then $u^n_{i+\frac{1}{2}} := 0$.

Define the sets of ε_0-dry and ε_0-wet indices in the discrete level:

$$I_D^{\varepsilon_0,n} := \{i \in \mathbb{Z}; 0 \leq h_i^n \leq \varepsilon_0\}, \quad I_W^{\varepsilon_0,n} := \{i \in \mathbb{Z}; h_i^n > \varepsilon_0\}.$$

Nonegativity conservation of water height implies

$$\mathbb{Z} = I_D^{\varepsilon_0,n} \cup I_W^{\varepsilon_0,n}, \quad \forall n \in [T],$$

Discrete lake at rest

Define the discrete lake at rest as

$$
\begin{cases}
\begin{cases}
u^n_{i+\frac{1}{2}} = 0, & \forall i \in \mathbb{Z}, \\
h^n_j + b_j = C_i, \quad \forall j \in I^n_{W,i}, & \forall i \in \mathbb{Z},
\end{cases} \\
\Rightarrow \begin{cases}
u^{n+1}_{i+\frac{1}{2}} = 0, & \forall i \in \mathbb{Z}, \\
h^{n+1}_j + b_j = C_i, \quad \forall j \in I^{n+1}_{W,i}, & \forall i \in \mathbb{Z},
\end{cases} \\
I^n_{W,i} = I^{n+1}_{W,i}, \quad I^n_{D,i} = I^{n+1}_{D,i}, \quad \forall i \in \mathbb{Z}.
\end{cases}
\tag{dlar}
$$

- In general, (OEsc) does not preserve (dlar) in the case of dry-wet fluid domain.
- Even the notion of ε_0-lake at rest is defined, (OEsc) still does not preserve it.

Discrete lake at rest equilibrium

Distinguish 4 cases: "wet-wet" front (WW) (interior interface), dry-wet front (DW), wet-dry front (WD), and "dry-dry" front (DD):

$$
\begin{cases}
\left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(w^n_i = w^n_{i+1}\right), & \forall i \in \mathbb{Z} \text{ s.t. } \left(i \in I^n_W\right) \wedge \left(i+1 \in I^n_W\right), \\
\left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(b_i \geq w^n_{i+1}\right), & \forall i \in \mathbb{Z} \text{ s.t. } \left(i \in I^n_D\right) \wedge \left(i+1 \in I^n_W\right), \\
\left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(w^n_i \leq b_{i+1}\right), & \forall i \in \mathbb{Z} \text{ s.t. } \left(i \in I^n_W\right) \wedge \left(i+1 \in I^n_D\right), \\
\qquad u^n_{i+\frac{1}{2}} := 0, & \forall i \in \mathbb{Z} \text{ s.t. } \left(i \in I^n_D\right) \wedge \left(i+1 \in I^n_D\right).
\end{cases}
\tag{DLAR}
$$

Approximate discrete lake at rest equilibrium

Distinguish 4 cases: "ε_0-wet-wet" front (ε_0-WW) (interior interface), ε_0-dry-wet front (ε_0-DW), ε_0-wet-dry front (ε_0-WD), and "ε_0-dry-dry" front (ε_0-DD):

$$\begin{cases} \left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(w^n_i = w^n_{i+1}\right), & \left(i \in I^{\varepsilon_0,n}_W\right) \wedge \left(i+1 \in I^{\varepsilon_0,n}_W\right), \\ \left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(b_i + \varepsilon_0 \geq w^n_{i+1}\right), & \left(i \in I^{\varepsilon_0,n}_D\right) \wedge \left(i+1 \in I^{\varepsilon_0,n}_W\right), \\ \left(u^n_{i+\frac{1}{2}} = 0\right) \wedge \left(w^n_i \leq b_{i+1} + \varepsilon_0\right), & \left(i \in I^{\varepsilon_0,n}_W\right) \wedge \left(i+1 \in I^{\varepsilon_0,n}_D\right), \\ \qquad u^n_{i+\frac{1}{2}} := 0, & \left(i \in I^{\varepsilon_0,n}_D\right) \wedge \left(i+1 \in I^{\varepsilon_0,n}_D\right). \end{cases} \tag{aDLAR}$$

(OEsc) resolves dry-wet transition issue successfully!

Theorem (Well-balanced property for dry-wet horizontal fluid domain $\mathbb{R}$)

Assume that the initial data satisfies $(h_0, u_0) \in L^\infty(\mathbb{R}; \mathbb{R}_+) \times L^\infty(\mathbb{R})$. Let $\mathcal{T}$ be an admissible mesh of $\mathbb{R}$ and $\Delta t \in \mathbb{R}_+^\star$ be the time step. Let $\left(h_i^n, u_{i+\frac{1}{2}}^n\right)_{i\in\mathbb{Z}, n\in[T]}$ be the discrete finite volume approximate solution generated by (OEsc). Assume that the CFL-like condition (CFL3) or (CFL4) holds. Let ε_0 be an arbitrary nonnegative real. Assume that the horizontal fluid domain $\mathbb{R}$ contains at least one non-degenerate dry area (not 1-point dry area type) but not fully dry, the offset equilibrium staggered scheme (OEsc) is ε_0-well-balanced for (1DSW) in the dry-wet horizontal fluid domain $\mathbb{R}$ in the discrete level.

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