Multidimensional Degenerate Diffusion Equation with Very Strong Absorption & Source Terms

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Overview

Gas Flow Models

In [Aronson, 1969], D. G. Aronson considered a homogeneous gas flowing through a homogeneous porous medium and obtained, by conservation of mass and Darcy's law,

$$\partial_t u = \Delta u^m, \tag{1}$$

where u: density of gas. For $m > 1$, (1) is a nonlinear equation which is parabolic for $u > 0$, but which degenerates when $u = 0$. The most interesting manifestation of the degeneracy of (1) is the finite speed of propagation of disturbances: If u has compact support at $t = t_0$, then u has compact support for all $t > t_0$.

Remarks on Quenching

In [Kawohl, 1996], Kawohl considered the parabolic problem

$$\partial_t u - \operatorname{div}\left(a\left(u, \nabla u\right) \nabla u\right) = -u^{-\beta}, \text{ in } \mathbb{R}^N \times \left(0, \infty\right). \tag{2}$$

and four special cases of (2):

$$\begin{aligned}
\partial_t u - \Delta u &= -u^{-\beta}, \\
\partial_t u - \partial_x\left(\varphi\left(\partial_x u\right)\right) &= -u^{-\beta}, \\
\partial_t u - \frac{\partial_x^2}{1 + \left(\partial_x u\right)^2} &= -\frac{1}{u}, \\
\partial_t u - \partial_x^2 u^m &= -u^{-\beta}.
\end{aligned}$$

Quenching phenomenon. When u reaches 0 in finite or infinite time, one says that u quenches in finite or infinite time.

1-D Degenerate Diffusion with very Strong Absorption

In [Kawohl & Kersner, 1992], they considered the Cauchy problem

$$\begin{aligned} \partial_t u - \partial_x^2 u^m + u^{-\beta}\chi_{u>0} &= 0, \text{ in } \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x), \end{aligned} \tag{3}$$

where $m \geq 1$, $0 < \beta < m$.

KAWOHL'S MISTAKES. Kawohl's notion of weak solution of (3) is inappropriate to pass through the limit. Kawohl and Kersner claimed that their method of proof will work in the general N-dimensional case. But this is wrong due to the appearance of a Laplace term in the gradient estimate when $N \geq 2$.

A General N-D Degenerate Diffusion with Source

We are interested in nonnegative solutions of the Cauchy problem

$$(P)\quad \begin{cases} \partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = f(u,x,t), & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$: a bounded domain, $m \geq 1$, $0 < \beta < m$, initial data $0 \leq u_0 \in L^\infty(\Omega)$, and the source term f satisfies

$$(H_f)\quad \begin{cases} f \in \mathcal{C}^1\left([0,\infty) \times \overline{\Omega} \times [0,\infty)\right), \\ f(0,x,t) = 0, & \text{for all } (x,t) \in \Omega \times (0,\infty), \\ f(u,x,t) \leq h(u), & \text{for all } (x,t) \in \Omega \times (0,\infty), \end{cases}$$

where h is a locally Lipschitz function on $[0,\infty)$, $h(0) = 0$.

Notion of Weak Solutions

Definition 1 (Weak Solutions).

Let $u_0 \in L^\infty(\Omega)$. A nonnegative function $u(x,t)$ is called a *weak solution* of (P) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$, and

$$u \in L^p\left(0,T; W_0^{1,2}(\Omega)\right) \cap L^\infty(\Omega \times (0,T)) \cap \mathcal{C}\left([0,T); L^1(\Omega)\right)$$

satisfies (P) in the sense of distribution $\mathcal{D}'(\Omega \times (0,T))$, i.e.,

$$\int \left(-u\varphi_t + mu^{m-1}\nabla u \cdot \nabla \varphi + u^{-\beta}\chi_{\{u>0\}}\varphi - f(u,x,t)\varphi\right) = 0,$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$.

Regularization Strategy

For any $\varepsilon > 0$, set $g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s)$, with $\psi_\varepsilon(s) = \psi\left(\frac{s}{\varepsilon}\right)$, where $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \psi \le 1$ is a non-decreasing function such that $\psi(s) = 0$ if $s \le 1$, and $\psi(s) = 1$ if $s \ge 2$.

1st Regularized Problems

$$(P_{\varepsilon,\eta}) \quad \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon^m + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t), & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon(x,t) = \eta, & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_0(x) + \eta, & \text{in } \Omega, \end{cases}$$

for any $0 < \eta < \varepsilon$.

Regularization Strategy (cont.)

As $\eta \to 0^+$, by classical arguments, $u_{\varepsilon,\eta} \to u_\varepsilon$ and $\nabla_{\varepsilon,\eta} \to \nabla u_\varepsilon$ uniformly in $\Omega \times (0, T)$, where u_ε is the unique classical solution of the regularized Cauchy problem

2nd Regularized Problems

$$(P_\varepsilon) \quad \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon^m + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon, x, t), & \text{in } \Omega \times (0, \infty), \\ u_\varepsilon(x, t) = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u_\varepsilon(x, 0) = u_0(x), & \text{in } \Omega. \end{cases}$$

Next, we will send $\varepsilon \to 0^+$ to obtain a local existence of a maximal weak solution of (P).

Demand of Gradient Estimates

By comparison principle, we obtain the monotonicity of the sequence u_ε's.

Lemma (Monotone lemma)

The sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded monotone.

Thus, $u_\varepsilon \to u$ pointwise as $\varepsilon \to 0^+$.
Consider the variational formulation of u_ε with test functions $\psi_\eta(u_\varepsilon)\varphi$, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$, and pass to limit $\varepsilon \to 0^+$: u is a distribution solution of (P) if

$$\lim_{\eta\to0^+}\int_{\mathrm{Supp}(\varphi)} \frac{m}{\eta}u^{m-1}\psi'\left(\frac{u}{\eta}\right)|\nabla u|^2\varphi dxdt = 0. \qquad (4)$$

Thus, we need some kind of gradient estimates in order that (4) holds.

Gradient Estimates

Lemma (Gradient estimates for $u_{\varepsilon,\eta}$)

Let $0 \leq u_0 \in \mathcal{C}_c^{\infty}(\Omega)$, $u_0 \neq 0$. *Then, for any* $0 < \eta < \varepsilon < 1$, *there exists a unique classical solution* $u_{\varepsilon,\eta}$ *of* $(P_{\varepsilon,\eta})$ *in* $\Omega \times (0, T)$.
a) In addition, for every $\tau > 0$, *there is a positive constant* $C > 0$ *only depending on* m, β, N, f, τ, T, *and* $\|u_0\|_\infty$ *such that*

$$\left|\nabla\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}\right)\right| \leq C \text{ in } \Omega \times (\tau, T),$$

$$\gamma = \begin{cases} \dfrac{1}{m-1}, & \text{if } m, \beta, N \text{ satisfy } (H_1), \\ \dfrac{2}{m+\beta}, & \text{otherwise}. \end{cases} \tag{5}$$

Gradient Estimates (cont.)

Lemma (Gradient estimates (cont.))

b) Furthermore, if $\nabla\left(\left(u_0(x)+\eta\right)^{1/\gamma}\right)\in L^\infty(\Omega)$ *for all* $\eta\in(0,\eta_0)$ *for some* $\eta_0>0$*, and*

$$U_0(x) := \sup_{\eta\in\left(0,\min\left\{\eta_0,\|u_0\|_\infty\right\}\right)} \left\|\nabla\left(\left(u_0(x)+\eta\right)^{1/\gamma}\right)\right\|_\infty \in L^\infty(\Omega),$$

then there exists a positive constant $C>0$ *merely depending on* m, β, N, f, T, $\|u_0\|_\infty$, *and* $\|U_0\|_\infty$ *such that*

$$\left|\nabla\left(u_{\varepsilon,\eta}^{\frac{1}{\gamma}}\right)\right| \le C \text{ in } \Omega\times[0,T),$$

where γ *is also given by* (5).

Conditions of (m, β, N).

In order that our gradient estimate works, the triple (m, β, N) is assumed to satisfy the following hypothesis (H),

$$\left[\begin{array}{l} (H_1)\ \ N = 1 \text{ and } m \geq \beta + 2, \\ (H_2)\ \ N \in \mathbb{Z}^+,\ m = 1,\ \text{and } 0 < \beta < 1, \\ (H_3)\ \ N = 1 \text{ and } 1 < m < \beta + 2, \\ (H_4)\ \ N \geq 2,\ 1 < m < 1 + (N-1)^{-\frac{1}{2}},\ \gamma_1 < \dfrac{2}{m+\beta} < \gamma_2, \end{array}\right.$$

where $\gamma_1(m, N)$ and $\gamma_2(m, N)$ are the roots of the quadratic equation

$$\left[(N-1)(m-1)^2 + 4m(m-1)\right]\gamma^2 - 4(2m-1)\gamma + 4 = 0.$$

Local Existence of Maximal Weak Solution

Theorem (Local existence)

Suppose that (m,β,N) *satisfies* (H), $u_0 \in L^\infty(\Omega)$, $u_0 \neq 0$, f *satisfy* (H_f). *Then, there exists a finite time* $T>0$ *such that* (P) *has a maximal weak solution* u *in* $\Omega\times(0,T)$, *i.e., for any weak solution* v *in* $\Omega\times(0,T)$, *we have*

$$v \leq u, \text{ in } \Omega\times(0,T),$$

Moreover, the gradient estimates for $u_{\varepsilon,\eta}$ *also holds for* u.

Some Special Cases

The local existence results and gradient estimates presented in my bachelor thesis includes some published works.

1. $N = 1$, $f = 0$ (see (H_1) and (H_3)): [Kawohl & Kersner, 1992].
2. $m = 1$ (see (H_2)): [Dao, 2017] and some other works of my main supervisor.
3. $m = 1$, $f = 0$ (see (H_2)): [Phillips, 1987].

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Thank for your attention

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