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Research Article

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Pointwise Gradient Estimates in Multi-dimensional Slow Diffusion Equations with a Singular Quenching Term

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Abstract: We consider the high-dimensional equation $\partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = 0$, extending the mathematical treatment made in 1992 by B. Kawohl and R. Kersner for the one-dimensional case. Besides the existence of a very weak solution $u \in \mathcal{C}([0, T]; L^1_\delta(\Omega))$, with $u^{-\beta}\chi_{\{u>0\}} \in L^1((0, T) \times \Omega)$, $\delta(x) = d(x, \partial\Omega)$, we prove some pointwise gradient estimates for a certain range of the dimension N, $m \geq 1$ and $\beta \in (0, m)$, mainly when the absorption dominates over the diffusion ($1 \leq m < 2 + \beta$). In particular, a new kind of universal gradient estimate is proved when $m + \beta \leq 2$. Several qualitative properties (such as the finite time quenching phenomena and the finite speed of propagation) and the study of the Cauchy problem are also considered.

Keywords: Singular Absorption and Nonlinear Diffusion Equations, Pointwise Gradient Estimates, Quenching Phenomenon, Free Boundary

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Dedicated to Laurent Véron in occasion of his 70th birthday

1 Introduction and Main Results

1.1 Introduction

The main goal of this paper is to extend to the high-dimensional case the 1992 mathematical treatment made by B. Kawohl and R. Kersner [50] for a one-dimensional degenerate diffusion equation with a singular absorption term. More precisely, we will study nonnegative solutions of the possibly degenerate reaction-diffusion multi-dimensional problem

$$\begin{cases} \partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = 0 & \text{in } (0, \infty) \times \Omega, \\ u^m = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is an open regular bounded domain of $\mathbb{R}^N$ (for instance, with $\partial\Omega$ of class $C^{1,\alpha}$ for some $\alpha \in (0, 1]$), $N \geq 1$, $m \geq 1$ ($m > 1$ corresponds to a typical slow diffusion) and mainly $\beta \in (0, m)$ (some remarks will be made on the case $\beta \geq m$ at the end of this paper). The case of the whole space, $\Omega = \mathbb{R}^N$, will be treated

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separately. Here $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (t,x) where $u(t,x)>0$, i.e.,

$$\chi_{\{u>0\}}(t,x) := \begin{cases} 1 & \text{if } u(t,x)>0, \\ 0 & \text{if } u(t,x)=0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular (and the diffusion becomes degenerate if $m>1$) when $u=0$, and that, by this normalization, we have $u^{-\beta}\chi_{\{u>0\}}(t,x)=0$ if $u(t,x)=0$. Notice that the boundary condition implies an automatic permanent singularity on the boundary $\partial\Omega$, in contrast to other related problems in which the singularity is permanently excluded of the boundary,

$$\begin{cases} \partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = 0 & \text{in } (0,\infty)\times\Omega, \\ u^m = 1 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{P(1)}$$

Notice also that the change of unknown $v = 1-u^m$, with u solution of (P(1)), in the semilinear case ($m=1$), for instance, leads to the formulation

$$\begin{cases} \partial_t v - \Delta v = \dfrac{\chi_{\{v<1\}}}{(1-v)^\beta} & \text{in } (0,\infty)\times\Omega, \\ v = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ v(0,x) = 1-u_0(x) & \text{in } \Omega. \end{cases}$$

In this way, the study of the associated Cauchy problem

$$\begin{cases} \partial_t u - \Delta u^m + u^{-\beta}\chi_{\{u>0\}} = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{CP}$$

can be regarded from two different points of view according to the assumptions made on the asymptotic behavior of the initial datum when $|x|\to+\infty$. The case $u_0(x)\searrow 0$ as $|x|\to+\infty$ can be considered as a limit of problems of type (P), and the case in which $u_0(x)$ is growing with $|x|$ as $|x|\to+\infty$ corresponds to a limit of problems of type (P(1)) (see, e.g., [43]). Our main goal in this paper is to analyze problems of type (P) and (CP) when $u_0(x)\searrow 0$ as $|x|\to+\infty$.

The literature on this type of problems increased very quickly in the last decades. Problem (P) (and (P(1))) was regarded as the limit case of the regularized Langmuir–Hinshelwood model in chemical catalyst kinetics (see [3, 25, 34, 39] for the elliptic case and [7, 55] for the parabolic equation). Some regularized singular absorption terms also arise in some models in enzyme kinetics [8]. See also many other references in the survey [44].

As mentioned before, what makes equations like (P) especially interesting is the fact that the solutions may raise to a free boundary defined as $\partial\{(t,x);\ u(t,x)>0\}$. In some contexts, problem (P(1)) was denoted as a *quenching problem*. It was soon pointed out the appearance of a blow-up time for $\partial_t u$ at the first time $T_c>0$ in which $u(T_c,x)=0$ at some point $x\in\Omega$ (see, e.g., [47, 52, 55]). More recently, parabolic problems with a singular absorption term of this type have been investigated by many authors (see, e.g., [19, 21–23, 49, 52, 55, 62] and references therein). Concerning the associate semilinear Cauchy problem, we mention the papers [40, 42, 43] and their references. The case $\beta\ge m$ presents special difficulties when the free boundary $\partial\{(t,x); u(t,x)>0\}$ is a nonempty hypersurface. This set corresponds to the so-called set of *rupture points* in the study of thin films [63]. This case, $\beta\ge m$, also arises in the modeling of micro-electromechanical systems (MEMS), in which mainly $m=1$ and $\beta=2$ (see [43, 54]).

A great amount of the previous papers in the literature concern only with the one-dimensional case. To explain some historical progress in founding gradient estimates for such kind of problems, we start by mentioning that the existence of weak solutions to (P) was obtained firstly by Phillips [55] for the case $N\ge 1$, $m=1$ and $\beta\in(0,1)$. Later, Dávila and Montenegro [23] proved an existence result to equation (P) with $m=1$ and including also a possible source term $f(u)$ satisfying a sublinear condition, i.e., $f(u)\le C(1+u)$. They proved that the pointwise gradient estimate

$$|\nabla u(t,x)| \le C u^{\frac{1-\beta}{2}}(t,x) \quad \text{in } (0,\infty)\times\Omega \tag{1.1}$$

plays a crucial role in proving the existence of solutions of (P). Besides, a partial uniqueness result was obtained by the same authors for a class of solutions with initial data $u_0(x) \geq C \operatorname{dist}(x, \partial\Omega)^\mu$ for $\mu \in (1, \frac{2}{1+\beta})$ and some constant $C > 0$ (see also [19] for a uniqueness result in another class of solutions). The uniqueness of solutions fails for general bounded nonnegative initial data [62].

Concerning the qualitative properties satisfied by the solutions of (P), one of the more peculiar facts is that the solutions may vanish after a finite time, even starting with a positive initial data. This phenomenon occurs by the presence of the singular absorption $u^{-\beta}\chi_{\{u>0\}}$ and can be understood as a generalization of the *finite extinction property* which arises for not so singular absorption terms of the form u^q, $0 < q < 1$. Another motivation of the present paper is to complete the previous work [27] in which the finite speed of propagation and other qualitative properties were proved by means of some energy methods (see, e.g., [2, 37]) in the class of *local* weak solutions of the more general formulation

$$\frac{\partial \psi(v)}{\partial t} - \operatorname{div} \mathbf{A}(x, t, v, Dv) + B(x, t, v, Dv) + C(x, t, v) = f(x, t, v)$$

for a singular absorption term. In that paper [27], the existence of weak solutions was merely assumed (and not proved), so our goal is to give some answers in this complementary direction. We also point out that, more specifically, when $m = 1$, $\beta \in (0, 1)$ and we consider equation (P) with a sublinear source term $\lambda f(u)$, $\lambda > 0$, it was shown in [53] that there is a real number $\lambda_0 > 0$ and a time $t_0 > 0$ such that $u_\lambda(t_0, x) = 0$ a.e. in Ω for all $\lambda \in (0, \lambda_0)$; the author called this phenomenon *complete quenching* (see a more general statement in [27, 40]). Other qualitative properties were studied in [42].

The extension from semilinear to some one-dimensional quasilinear degenerate equations of the p-Laplacian type was considered in [20, 41]. In that one-dimensional case, the formulation was

$$\begin{cases} \partial_t u - \partial_x(|u_x|^{p-2}u_x) + u^{-\beta}\chi_{\{u>0\}} = 0 & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

with $p > 2$, $\beta \in (0, 1)$. To obtain the existence of solutions of (1.2), it was proved in [20] the gradient estimate

$$|u_x(t, x)| \leq C u^{\frac{1-\beta}{p}}(t, x) \quad \text{in } (0, \infty) \times \Omega. \tag{1.3}$$

We note that (1.3) is a generalization of (1.1) as $p > 2$. Furthermore, it was shown in [20] that any solution of equation (1.2) must vanish after a finite time. A complete quenching result for equation (1.2) with a source $\lambda f(u)$ was obtained by the same authors in [21]. The extension of the gradient estimates to the higher-dimensional case remains today as an open problem.

As mentioned before, the first result in the literature for the one-dimensional problem (P) with a slow diffusion ($m > 1$) was due to Kawohl and Kersner [50] in 1992. Once again, a suitable gradient estimate was the key of the proof of the correct treatment of the problem. They proved that

$$|(u^{\frac{m+\beta}{2}})_x| \leq C \tag{1.4}$$

in the regime in which the *absorption dominates the nonlinear diffusion*, which corresponds to

$$1 \leq m < 2 + \beta. \tag{1.5}$$

Notice that the exponent in estimate (1.4) may be written also as $\frac{1}{\gamma}$ with $\gamma := \frac{2}{m+\beta}$. As a matter of fact, in [50], it was also considered the opposite regime in which the *diffusion dominates over the absorption* ($m \geq 2 + \beta$), and it was shown that the correct value for the pointwise gradient estimate is a different value of the exponent γ (this time $\frac{1}{m-1}$). We will not be especially interested in such a case in this paper, but in any case, see more details in the second part of Lemma 2.

Our N-dimensional approach to derive a pointwise gradient estimate of type (1.4) will adapt the classical Bernstein method (see, e.g., [5, 13, 32, 59]) with some ideas introduced by Ph. Bénilan (see, e.g., [6, 10, 13]). In fact, for the special case $N = 1$, we will extend the results of [50] to unbounded initial data. Our proof requires two technical additional assumptions:

$$1 \leq m < 1 + \frac{1}{\sqrt{N-1}} \tag{1.6}$$

and

$$\beta \in ((m-1-\sqrt{\Delta_{m,N}})_+, m-1+\sqrt{\Delta_{m,N}}) \quad \text{with} \quad \Delta_{m,N} := 1-(N-1)(m-1)^2. \tag{1.7}$$

We think that such auxiliary assumptions arise merely as some limitations of our technique of proof. The question of how to avoid them (in the framework in which the *absorption dominates the nonlinear diffusion*, $1 < m < 2 + \beta$) remains an open problem for us. Nevertheless, thanks to our technique of proof, we will prove a new gradient information for the case

$$\beta + m \leq 2 \tag{1.8}$$

(which applies to the semilinear framework), which seems to be unadvertised in the previous literature, either the L^∞ norm of gradient of $u^{\frac{m+\beta}{2}}(t)$ is smaller than or equal to $\|\nabla u_0^{\frac{m+\beta}{2}}\|_{L^\infty(\Omega)}$, or if the above norm is strictly smaller than this bound, then it is smaller than a universal constant $C = C(m, \beta, N)$, independent of Ω, then it is always smaller than this constant for $t \in (0, +\infty)$. Moreover, we will give some concrete examples proving the optimality of estimate (1.4).

For the existence of solutions, we will use a monotone family of regularized problems, and we will pass to the limit thanks to the monotonicity of the approximation of the singular nonlinear term and the contractive properties of the semigroup associated to the (unperturbed) nonlinear diffusion over suitable functional spaces. The pointwise gradient estimates will be previously obtained for solutions of the regularized problems and then extended to the solutions of (P) and (CP) by passing to the limit in the regularizing parameters. In the case of assumption (1.8), we will pass to the limit in the gradient term ∇u^m by means of a generalization of the *almost everywhere gradient convergence* technique (introduced initially for p-Laplace type operators in [15]). Finally, we will consider several qualitative properties of solutions of (P) and (CP) implying the *finite speed of propagation*, the *uniform localization of the support*, and the *instantaneous shrinking of the support property*. The well-known results for solutions of the porous media equation with a strong absorption (see, e.g., [1, 32, 46, 59]) remain valid for solutions of problem (P). Here we will get some sharper estimates rather than dealing with local solutions as in [27]. Our special interest is to analyze the differences arising among the behavior of solutions of the porous media equation with a strong absorption and the solutions of the porous media equation with a singular absorption term $u^{-\beta}\chi_{\{u>0\}}$. In the case in which the singularity is permanently excluded of the boundary, such as for problem (P(1)), the behavior of the solution (its “profile”) at the first time $t = \tau_0$ in which there is a quenching point was studied in [38]. In our formulation (P), we know that there is a permanent singularity on the boundary $\partial\Omega$, and thus our interest is to describe the profile of the solutions near the boundary $\partial\Omega$. We will construct a large class of solutions showing that their profile near the boundary follows the gradient estimate proved in this paper. So such gradient estimates are sharp. Some commentaries on the case $\beta \geq m$ will be also given at the end of the paper.

1.2 Main Results

Let us first introduce the notion of weak solution that we use for the case of Ω bounded and bounded initial data.

Definition 1. Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. A nonnegative function $u(t, x)$ is called a *weak solution* of (P) if

$$u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^\infty((0,\infty)\times\Omega), \quad u^{-\beta}\chi_{\{u>0\}} \in L^1((0,T)\times\Omega), \quad u^m \in L^2(0,T; H_0^1(\Omega))$$

for any $T > 0$ and u satisfies (P) in the sense of distributions $\mathcal{D}'((0,\infty)\times\Omega)$, i.e.,

$$\int_0^\infty \int_\Omega (-u\varphi_t + \nabla u^m \cdot \nabla\varphi + u^{-\beta}\chi_{\{u>0\}}\varphi)\,dx\,dt = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty((0,\infty)\times\Omega).$$

Any weak solution is also a *very weak solution* to equation (P) (see, e.g., [4, 50, 59]). Since the reaction term $u^{-\beta}\chi_{\{u>0\}}$ is required to be in $L^1((0,\infty)\times\Omega)$, a natural weaker notion of solution will be used sometimes in the paper for the class of nonnegative initial data which are merely in $L^1(\Omega)$.

Definition 2. Let $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, and $T > 0$. A nonnegative function $u \in \mathcal{C}([0,T]; L^1(\Omega))$ is called an L^1-*mild solution* of (P) if $u^{-\beta}\chi_{\{u>0\}} \in L^1((0,T)\times\Omega)$ and u coincides with the unique L^1-mild solution of the problem

$$\begin{cases} \partial_t u - \Delta u^m = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $f := -u^{-\beta}\chi_{\{u>0\}}$.

As a matter of fact, a weaker notion of solutions can be obtained when introducing the *distance to the boundary* as a weight, $u_0 \in L^1_\delta(\Omega) = \{v \in L^1_{\mathrm{loc}}(\Omega); \int_\Omega v(x)\delta(x)\,dx < \infty\}$, where $\delta(x) = d(x,\partial\Omega)$.

Definition 3. Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$, and $T > 0$. A nonnegative function $u \in \mathcal{C}([0,T]; L^1_\delta(\Omega))$ is called an L^1_δ-*mild solution* of (P) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(0,T; L^1_\delta(\Omega))$ and u coincides with the unique L^1_δ-mild solution of problem (1.9), with $f := -u^{-\beta}\chi_{\{u>0\}}$.

We recall that the notion of mild solution of the problem for the non-homogeneous problem (1.9) is well-defined thanks to the fact that the nonlinear diffusion operator $-\Delta u^m$ (with Dirichlet boundary conditions) is an m-accretive operator in $L^1(\Omega)$ with a dense domain (see, e.g., [10, 14, 59] and their references). The similar properties of this operator on the space $L^1_\delta(\Omega)$ will be shown in this paper as easy consequences of well-known results ([16, 17, 35, 57, 58] and [59, Section 6.6]). In fact, there are other equivalent formulations for very weak solutions obtained as an L^1_δ-mild solution of problem (1.9). One formulation which is especially useful for our purposes starts by introducing the auxiliary equivalent weight function $\zeta(x)$, $\zeta \in C^\infty(\Omega)\cap C^1(\overline{\Omega})$, $\zeta > 0$, given as the unique solution of the problem

$$\begin{cases} -\Delta\zeta = 1 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

It is well known that

$$\underline{C}\delta(x) \leq \zeta(x) \leq \overline{C}\delta(x) \quad \text{for any } x \in \Omega, \tag{1.11}$$

for some positive constants $\underline{C} < \overline{C}$ so that $L^1_\delta(\Omega) = L^1_\zeta(\Omega)$. Then it is easy to see that every L^1_δ-*mild solution* of (P) is a very weak solution of problem (1.9) in the sense that $u \in \mathcal{C}([0,T]; L^1_\delta(\Omega))$, $u \geq 0$, $u^m \in L^1((0,T)\times\Omega)$, $f = -u^{-\beta}\chi_{\{u>0\}} \in L^1(0,T; L^1_\delta(\Omega))$, and for any $t \in [0,T]$,

$$\int_\Omega u(t,x)\zeta(x)\,dx + \int_0^t\int_\Omega u^m(t,x)\,dx\,dt = \int_\Omega u_0(x)\zeta(x)\,dx + \int_0^t\int_\Omega f(t,x)\zeta(x)\,dx\,dt.$$

In what follows, our main interest is to deal with the case $N \geq 2$ and $m > 1$ since the two other cases ($N = 1$, $m \geq 1$ and $N \geq 1$, $m = 1$) were studied in [50] and [55], respectively. We also mention that some singular reaction terms were considered previously in the literature for the case of $m \in (0,1)$ (see, e.g., [18, 24]). Some of our results also hold for $m \in (0,1)$, but we will not pursue such a goal in this paper.

Our main result in this paper is the following one.

Theorem 1. *The following statements hold.*

(i) *Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$. Assume $m \geq 1$ and $\beta \in (0,m)$. Then problem* (P) *has a* maximal L^1_δ-*mild solution u. Moreover, if $u_0 \in L^1(\Omega)$, then u is also the* maximal L^1-*mild solution.*

(ii) *Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$, and assume* (1.5), (1.6) *and* (1.7). *Then*

$$\|\nabla u^{\frac{m+\beta}{2}}(t)\|_{L^\infty(\Omega)} \leq C\Big(\frac{1}{t^\omega} + 1\Big), \quad \textit{a.e. } t \in (0,+\infty),$$

for some positive constants $\omega = \omega(m,\beta,N)$ and $C = C(m,\beta,N,\Omega)$ if $m > 1$, $C = C(m,\beta,N,\|u_0\|_{L^1_\delta(\Omega)})$ if $m = 1$. Moreover, the maximal L^1-*mild solution is Hölder continuous on* $(0,T]\times\overline{\Omega}$.

(iii) *Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$ such that $\nabla u_0^{\frac{m+\beta}{2}} \in L^\infty(\Omega)$, and assume $m \geq 1$,* (1.5), (1.6), (1.7) *and* (1.8). *Then*

$$\|\nabla u^{\frac{m+\beta}{2}}(t)\|_{L^\infty(\Omega)} \leq \max\Big\{\|\nabla u_0^{\frac{m+\beta}{2}}\|_{L^\infty(\Omega)}, \frac{(m+\beta)\sqrt{2+\beta-m}}{\sqrt{2m(\Delta_{m,N} - (\beta+1-m)^2)}}\Big\}, \quad \textit{a.e. } t \in (0,+\infty).$$

We point out that, in the rest of the paper, we will denote by C different positive constants, possibly changing from line to line. Furthermore, any constant depending on some parameters will be emphasized by a parentheses indicating such a dependence; for instance, $C = C(m, \beta, N)$ will mean that C depends only on m, β, N.

Remark 1. Concerning the one-dimensional quasilinear case, $m > 1$, Theorem 1 extends the results of Kawohl and Kersner [50] to a class of more general initial data. Notice also that the gradient estimate given in part (iii) is new with respect to the paper [50] and also with respect to the literature on the semilinear problem. It can be useful for many different purposes (for instance, to control possible approximating algorithms when there are some additional perturbations in the right-hand side of the equation, and so on).

Remark 2. We emphasize that the gradient estimates prove (see Proposition 1 below) that in fact $u^{\frac{m+1}{2}}$ is Hölder continuous on $(0, \infty) \times \overline{\Omega}$ (and in fact also on $[0, \infty) \times \overline{\Omega}$ provided that $u_0^{\frac{m+1}{2}}$ is also Hölder continuous on $\overline{\Omega}$ and $\nabla u_0^{\frac{m+\beta}{2}} \in L^\infty(\Omega)$).

The existence of solutions to Cauchy problem (CP) can be obtained as a consequence of Theorem 1. Moreover, the above gradient estimates hold on $L^\infty(\mathbb{R}^N)$ for a.e. $t \in (0, T)$ (see Theorem 3 below).

This paper is organized as follows. In the next section, we will prove the pointwise gradient estimates of solutions of a regularized version of equation (P). Section 3 is devoted to prove Theorem 1 and its application to the study of the Cauchy problem (CP). Different qualitative properties will be considered in the final Section 4.

2 Technical Lemmas

In this section, we will adapt the classical Bernstein technique and some ideas of Ph. Bénilan and his collaborators to our framework in order to obtain a gradient estimate of the type $|\nabla u^{\frac{1}{\gamma}}| \le C$ with $\gamma := \frac{2}{m+\beta}$. Let $\psi \in \mathcal{C}^\infty(\mathbb{R} : [0, 1])$ be a non-decreasing real function such that

$$\psi(s) = \begin{cases} 0 & \text{if } s \le 1, \\ 1 & \text{if } s \ge 2. \end{cases}$$

For every $\varepsilon > 0$, we define $g_\varepsilon(s) := s^{-\beta}\psi_\varepsilon(s)$, where $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}$. It is straightforward to check that g_ε is a globally Lipschitz function for any $\varepsilon > 0$.

Now, for every $\varepsilon > 0$ and $\eta > 0$, we consider the regularized version of problem (P) given by

$$\begin{cases} \partial_t u - \Delta u^m + g_\varepsilon(u) = 0 & \text{in } (0, \infty) \times \Omega, \\ u = \eta & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) + \eta & \text{in } \Omega. \end{cases} \tag{$\mathrm{P}_{\varepsilon,\eta}$}$$

The main goal of this section is to get some pointwise estimates for $\nabla u_{\varepsilon,\eta}$ (with $u_{\varepsilon,\eta}$ the unique solution of ($\mathrm{P}_{\varepsilon,\eta}$)) which allow us to pass to the limit as $\eta, \varepsilon \downarrow 0$ to prove the gradient estimates indicated in Theorem 1.

We start by showing a general auxiliary result which is useful to handle expressions containing terms of the type $|\nabla u|^2 \Delta u$ arising in the study of gradient estimates in the multi-dimensional case. Our proof corresponds to a slight generalization of Bénilan's ideas (see, e.g., [6, 10] and the application made in [9]).

Lemma 1. *Let $u \in \mathcal{C}^2(\mathbb{R}^N : \mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R} : [0, \infty))$. Then the inequality*

$$g(u)|D^2 u|^2 + g'(u)\Big(\frac{1}{2}\nabla u \cdot \nabla(|\nabla u|^2) - |\nabla u|^2 \Delta u\Big) \ge -\frac{(N-1)g'(u)^2 |\nabla u|^4}{4g(u)}$$

holds over the set $\{x \in \mathbb{R}^N;\ g(u(x)) \ne 0\}$.

Proof. Set $w := |\nabla u|^2$, and denote by $\mathcal{S}(g, u)$ the left-hand side of the wanted inequality. Then $\mathcal{S}(g, u)$ can be rewritten as

$$\mathcal{S}(g, u) = g(u)|D^2 u|^2 + g'(u)\Big(\frac{1}{2}\nabla u \cdot \nabla w - w\Delta u\Big).$$

As in [9], we can adapt Bénilan's method presented in [10] in the following way:

$$
\begin{aligned}
\mathcal{S}(g,u) &= g(u)\sum_{i,j=1}^{N}(\partial_{ij}u)^2 + g'(u)\Big(\sum_{i,j=1}^{N}\partial_i u\partial_j u\partial_{ij}u - w\sum_{i=1}^{N}\partial_i^2 u\Big)\\
&= g(u)\sum_{i=1}^{N}\Big[(\partial_i^2 u)^2 + \frac{g'}{g}(u)((\partial_i u)^2 - w)\partial_i^2 u\Big] + g(u)\sum_{i\neq j}\Big[(\partial_{ij}u)^2 + \frac{g'}{g}(u)\partial_i u\partial_j u\partial_{ij}u\Big]\\
&= g(u)\sum_{i=1}^{N}\Big[\partial_i^2 u + \frac{g'}{2g}(u)((\partial_i u)^2 - w)\Big]^2 - \frac{g(u)}{4}\sum_{i=1}^{N}\Big(\frac{g'}{g}\Big)^2(u)((\partial_i u)^2 - w)^2\\
&\quad + g(u)\sum_{i\neq j}\Big(\partial_{ij}u + \frac{g'}{2g}(u)\partial_i u\partial_j u\Big)^2 - \frac{g(u)}{4}\sum_{i\neq j}\Big(\frac{g'}{g}\Big)^2(u)(\partial_i u)^2(\partial_j u)^2\\
&\geq -\frac{(g')^2}{4g}(u)\Big[\sum_{i=1}^{N}((\partial_i u)^2 - w)^2 + \sum_{i\neq j}(\partial_i u)^2(\partial_j u)^2\Big] = -\frac{(N-1)(g')^2}{4g}(u)w^2,
\end{aligned}
$$

which completes the proof. □

Given $u_0 \in \mathcal{C}_c^1(\Omega)$, $u_0 \geq 0$, $u_0 \neq 0$, $m \geq 1$ and $0 < \eta \leq \min\{\varepsilon, \|u_0\|_\infty\}$, the existence and uniqueness of a classical solution $u_{\varepsilon,\eta}$ of $(\mathrm{P}_{\varepsilon,\eta})$ is a well-known result (see, e.g., [51]). Moreover, the comparison principle applies, and thus

$$\eta \leq u_{\varepsilon,\eta}(t,x) \leq \|u_0\|_\infty + \eta \leq 2\|u_0\|_\infty \quad \text{in } (0,\infty)\times\Omega.$$

We will prove the gradient estimates in a separate way: first for the case $N \geq 2$ and then for $N = 1$.

Lemma 2. *Let $u_0 \in \mathcal{C}_c^1(\Omega)$ be nonnegative, $0 < \eta \leq \min\{\varepsilon, \|u_0\|_\infty\}$. Let $N \geq 2$ and $m \geq 1$ be such that $\Delta_{m,N} > 0$. Define $\gamma := \frac{2}{m+\beta}$, and assume* (1.7). *Then there is a positive constant $C = C(m,\beta,N)$ such that*

$$|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(t,x)|^2 \leq C(t^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta} + 1) \quad \text{in } (0,\infty)\times\Omega. \tag{2.1}$$

In addition, if one assumes (1.8) *and $\nabla u_0^{\frac{1}{\gamma}} \in L^\infty(\Omega)$, then*

$$|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(t,x)| \leq \max\Big\{\|\nabla u_0^{\frac{m+\beta}{2}}\|_{L^\infty(\Omega)}, \frac{(m+\beta)\sqrt{2+\beta-m}}{\sqrt{2m(\Delta_{m,N} - (\beta+1-m)^2)}}\Big\} \quad \text{in } (0,\infty)\times\Omega. \tag{2.2}$$

Proof. Let $h_{\varepsilon,\eta} := u_{\varepsilon,\eta}^{\frac{1}{\gamma}}$. Then $h_{\varepsilon,\eta}$ satisfies the following equation:

$$\partial_t h_{\varepsilon,\eta} - m h_{\varepsilon,\eta}^{\gamma(m-1)}\Delta h_{\varepsilon,\eta} - m(m\gamma-1)h_{\varepsilon,\eta}^{\gamma(m-1)-1}|\nabla h_{\varepsilon,\eta}|^2 + \gamma^{-1}\psi_\varepsilon(h_{\varepsilon,\eta}^\gamma)h_{\varepsilon,\eta}^{1-\gamma(1+\beta)} = 0. \tag{2.3}$$

Differentiating in (2.3) with respect to the variable x, we obtain

$$
\begin{aligned}
\partial_t\nabla h_{\varepsilon,\eta} - m h_{\varepsilon,\eta}^{\gamma(m-1)}\nabla\Delta h_{\varepsilon,\eta} &= m\gamma(m-1)h_{\varepsilon,\eta}^{\gamma(m-1)-1}\Delta h_{\varepsilon,\eta}\nabla h_{\varepsilon,\eta}\\
&\quad + m(m\gamma-1)(\gamma(m-1)-1)h_{\varepsilon,\eta}^{\gamma(m-1)-2}|\nabla h_{\varepsilon,\eta}|^2\nabla h_{\varepsilon,\eta}\\
&\quad + m(m\gamma-1)h_{\varepsilon,\eta}^{\gamma(m-1)-1}\nabla(|\nabla h_{\varepsilon,\eta}|^2) - \psi_\varepsilon'(h_{\varepsilon,\eta}^\gamma)h_{\varepsilon,\eta}^{-\beta\gamma}\nabla h_{\varepsilon,\eta}\\
&\quad - \gamma^{-1}(1-\gamma(1+\beta))\psi_\varepsilon(h_{\varepsilon,\eta}^\gamma)h_{\varepsilon,\eta}^{-\gamma(1+\beta)}\nabla h_{\varepsilon,\eta} \quad \text{in } (0,\infty)\times\Omega.
\end{aligned} \tag{2.4}
$$

For any $0 < \tau < T < \infty$, let $\zeta \in C^\infty(\mathbb{R} : [0,1])$ be a cut-off function such that

$$\zeta(t) = \begin{cases} 1 & \text{if } t \in [\tau, T],\\ 0 & \text{if } t \notin (\frac{\tau}{2}, T + \frac{\tau}{2}),\end{cases} \qquad \text{and} \qquad |\zeta'| \leq \frac{c_0}{\tau} \quad \text{for some positive constant } c_0.$$

Consider now the function $v_{\varepsilon,\eta}(t,x) := \zeta(t)|\nabla h_{\varepsilon,\eta}(t,x)|^2$. Let $M := \max_{[0,\infty)\times\overline{\Omega}} v_{\varepsilon,\eta}$. It is enough to assume $M > 0$; otherwise, it is clear that $\nabla h_{\varepsilon,\eta} \equiv 0$, likewise $\nabla u_{\varepsilon,\eta} \equiv 0$. Therefore, there is a point

$$(t_0, x_0) \in \Big(\frac{\tau}{2}, T + \frac{\tau}{2}\Big)\times\Omega \quad \text{such that} \quad v_{\varepsilon,\eta}(t_0,x_0) = M$$

(since $v_{\varepsilon,\eta} = 0$ on $[0,\infty) \times \partial\Omega$). As a consequence, one has

$$\nabla(|\nabla h_{\varepsilon,\eta}|^2) = 0 \quad \text{and} \quad \partial_t v_{\varepsilon,\eta} - m h_{\varepsilon,\eta}^{\gamma(m-1)} \Delta v_{\varepsilon,\eta} \geq 0 \quad \text{at } (t_0, x_0). \tag{2.5}$$

This implies

$$\zeta' |\nabla h_{\varepsilon,\eta}|^2 + 2\zeta \nabla h_{\varepsilon,\eta} \cdot \partial_t \nabla h_{\varepsilon,\eta} \geq 2m\zeta h_{\varepsilon,\eta}^{\gamma(m-1)} (|D^2 h_{\varepsilon,\eta}|^2 + \nabla h_{\varepsilon,\eta} \cdot \nabla \Delta h_{\varepsilon,\eta}) \quad \text{at } (t_0, x_0),$$

or, equivalently,

$$\zeta \nabla h_{\varepsilon,\eta} \cdot (\partial_t \nabla h_{\varepsilon,\eta} - m h_{\varepsilon,\eta}^{\gamma(m-1)} \nabla \Delta h_{\varepsilon,\eta}) \geq -\frac{\zeta'}{2} |\nabla h_{\varepsilon,\eta}|^2 + m\zeta h_{\varepsilon,\eta}^{\gamma(m-1)} |D^2 h_{\varepsilon,\eta}|^2 \quad \text{at } (t_0, x_0).$$

Combining this with (2.4) and the former version of (2.5), we obtain

$$\begin{aligned} &m(m\gamma - 1)(1 - \gamma(m-1))\zeta h_{\varepsilon,\eta}^{\gamma(m-1)-2} |\nabla h_{\varepsilon,\eta}|^4 \\ &\quad \leq \frac{\zeta'}{2} |\nabla h_{\varepsilon,\eta}|^2 + m\gamma(m-1)\zeta h_{\varepsilon,\eta}^{\gamma(m-1)-1} \Delta h_{\varepsilon,\eta} |\nabla h_{\varepsilon,\eta}|^2 - m\zeta h_{\varepsilon,\eta}^{\gamma(m-1)} |D^2 h_{\varepsilon,\eta}|^2 \\ &\qquad - \zeta \psi_\varepsilon'(h_{\varepsilon,\eta}^\gamma) h_{\varepsilon,\eta}^{-\beta\gamma} |\nabla h_{\varepsilon,\eta}|^2 + (1 + \beta - \gamma^{-1})\zeta \psi_\varepsilon(h_{\varepsilon,\eta}^\gamma) h_{\varepsilon,\eta}^{-\gamma(1+\beta)} |\nabla h_{\varepsilon,\eta}|^2 \quad \text{at } (t_0, x_0). \end{aligned} \tag{2.6}$$

From (2.5), applying Lemma 1 to $g(s) = s^{\gamma(m-1)}$, we get

$$\begin{aligned} &h_{\varepsilon,\eta}^{\gamma(m-1)} |D^2 h_{\varepsilon,\eta}|^2 - \gamma(m-1) h_{\varepsilon,\eta}^{\gamma(m-1)-1} |\nabla h_{\varepsilon,\eta}|^2 \Delta h_{\varepsilon,\eta} \\ &\quad = h_{\varepsilon,\eta}^{\gamma(m-1)} |D^2 h_{\varepsilon,\eta}|^2 + \gamma(m-1) h_{\varepsilon,\eta}^{\gamma(m-1)-1} \Big(\frac{1}{2} \nabla h_{\varepsilon,\eta} \cdot \nabla(|\nabla h_{\varepsilon,\eta}|^2) - |\nabla h_{\varepsilon,\eta}|^2 \Delta h_{\varepsilon,\eta}\Big) \\ &\quad \geq -\frac{1}{4} \gamma^2 (N-1)(m-1)^2 h_{\varepsilon,\eta}^{\gamma(m-1)-2} |\nabla h_{\varepsilon,\eta}|^4 \quad \text{at } (t_0, x_0). \end{aligned}$$

A combination of this equality, (2.6) and $\nabla h_{\varepsilon,\eta}(t_0, x_0) \neq 0$ implies

$$\begin{aligned} &m\Big[(m\gamma - 1)(1 - \gamma(m-1)) - \frac{1}{4}\gamma^2 (N-1)(m-1)^2\Big] \zeta h_{\varepsilon,\eta}^{\gamma(m-1)-2} |\nabla h_{\varepsilon,\eta}|^2 \\ &\quad \leq \frac{\zeta'}{2} - \zeta \psi_\varepsilon'(h_{\varepsilon,\eta}^\gamma) h_{\varepsilon,\eta}^{-\beta\gamma} + (1 + \beta - \gamma^{-1}) \zeta \psi_\varepsilon(h_{\varepsilon,\eta}^\gamma) h_{\varepsilon,\eta}^{-\gamma(1+\beta)} \quad \text{at } (t_0, x_0). \end{aligned} \tag{2.7}$$

Denote

$$\mathcal{B} := m\Big[(m\gamma - 1)(1 - \gamma(m-1)) - \frac{1}{4}\gamma^2 (N-1)(m-1)^2\Big] = \frac{m[\Delta_{m,N} - (\beta + 1 - m)^2]}{(m+\beta)^2}.$$

Note that assumption (1.7) on β implies that $\mathcal{B} > 0$. Since $\psi_\varepsilon' \geq 0$, it is clear that the second term on the right-hand side of (2.7) is nonpositive. As a consequence, we get

$$\mathcal{B} v_{\varepsilon,\eta} = \mathcal{B}\zeta |\nabla h_{\varepsilon,\eta}|^2 \leq \frac{\zeta'}{2} h_{\varepsilon,\eta}^{2-\gamma(m-1)} + (1 + \beta - \gamma^{-1}) \zeta \psi_\varepsilon(h_{\varepsilon,\eta}^\gamma) h_{\varepsilon,\eta}^{2-\gamma(m+\beta)} \quad \text{at } (t_0, x_0).$$

Note that $2 - \gamma(m-1) = \frac{2(1+\beta)}{m+\beta} > 0$ and $1 + \beta - \gamma^{-1} = \frac{2+\beta-m}{2} > 0$ (since $\Delta_{m,N} > 0$ implies $m < 1 + \frac{1}{\sqrt{N-1}} \leq 2$ for all $N \geq 2$); the last inequality then implies

$$M \leq \frac{1}{2\mathcal{B}} \Big[\frac{c_0}{\tau} (2\|u_0\|_\infty)^{1+\beta} + 2 + \beta - m\Big].$$

Since $v_{\varepsilon,\eta}(t,x) \leq M$ in $(0,\infty) \times \Omega$, the last inequality implies, in particular, at $t = \tau$,

$$|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(\tau, x)|^2 \leq \frac{1}{2\mathcal{B}} (2^{1+\beta} c_0 \tau^{-1} \|u_0\|_\infty^{1+\beta} + 2 + \beta - m) \quad \text{for all } x \in \Omega.$$

The proof of the second statement is a small variation of the above case. For any $\tau > 0$, it suffices to make a slight modification by replacing the cut-off function $\zeta(t)$ by $\bar{\zeta}(t) \in C^\infty(\mathbb{R} : [0,1])$ defined by

$$\bar{\zeta}(t) = \begin{cases} 1 & \text{if } t \leq \tau, \\ 0 & \text{if } t \geq 2\tau, \end{cases} \qquad \text{and} \qquad \bar{\zeta}' \leq 0 \quad \text{in } \mathbb{R}.$$

Now if we define $\overline{v}_{\varepsilon,\eta} := \overline{\zeta}|\nabla h_{\varepsilon,\eta}|^2$ and assume that $\overline{v}_{\varepsilon,\eta}$ attains its maximum at $(0,\bar{x})$ for some $\bar{x} \in \Omega$, then we have

$$\begin{aligned}\overline{\zeta}(t)|\nabla h_{\varepsilon,\eta}(t,x)|^2 = \overline{v}_{\varepsilon,\eta}(t,x) \le \overline{v}_{\varepsilon,\eta}(0,\bar{x}) = |\nabla h_{\varepsilon,\eta}(0,\bar{x})|^2 &= \frac{1}{\gamma^2}(u_0(\bar{x})+\eta)^{2(\frac{1}{\gamma}-1)}|\nabla u_0(\bar{x})|^2\\ &\le \Big(\frac{u_0(\bar{x})}{u_0(\bar{x})+\eta}\Big)^{2(1-\frac{1}{\gamma})}\|\nabla u_0^{\frac{1}{\gamma}}\|_\infty \le \|\nabla u_0^{\frac{1}{\gamma}}\|_\infty,\end{aligned}$$

where we have used $\gamma \ge 1$ stemming from the additional assumption $\beta \le 2-m$. Thus

$$|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}| \le \|\nabla u_0^{\frac{1}{\gamma}}\|_\infty \quad \text{in } (0,\infty)\times\Omega.$$

Otherwise, $\overline{v}_{\varepsilon,\eta}$ must attain its maximum at some $(\bar{t}_0,\bar{x}_0) \in (0,2\tau)\times\Omega$ since

$$\overline{v}_{\varepsilon,\eta} = 0 \quad \text{on } \{(2\tau,\infty)\times\Omega\}\cup\{(0,\infty)\times\partial\Omega\}.$$

Then, repeating the proof of the first statement until (2.7), and from the fact that $\overline{\zeta}' \le 0$, we deduce

$$\mathcal{B}\overline{v}_{\varepsilon,\eta} = \mathcal{B}\overline{\zeta}|\nabla h_{\varepsilon,\eta}|^2 \le (1+\beta-\gamma^{-1})\overline{\zeta}\psi_\varepsilon(h_{\varepsilon,\eta}^\gamma) \quad \text{at } (\bar{t}_0,\bar{x}_0).$$

By the same argument, this leads us to

$$|\nabla u_{\varepsilon,\eta}^{\frac{1}{\gamma}}(t,x)| \le \Big(\frac{2+\beta-m}{2\mathcal{B}}\Big)^{\frac{1}{2}} \quad \text{in } (0,\infty)\times\Omega.$$

Then, combining both estimates, we arrive to the conclusion. □

Now we will consider the one-dimensional case to prove similar gradient estimates to the ones obtained in the above result. Moreover, we will get also a gradient estimate for the case in which the diffusion dominates over the absorption (similar to the one given in [48]).

Lemma 3. *Let $N=1$, $m\ge 1$, $\beta\in(0,m)$. Consider $u_0 \in \mathcal{C}_c^1(\Omega)$, $u_0 \ge 0$, $u_0 \not\equiv 0$ and $0<\eta\le\min\{\varepsilon,\|u_0\|_\infty\}$.*
(i) *If $m<\beta+2$, then there is a constant $C=C(m,\beta)$ such that*

$$|(u_{\varepsilon,\eta}^{\frac{1}{\gamma}})_x(t,x)|^2 \le C(t^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta}+1) \quad \text{in } (0,\infty)\times\Omega.$$

In addition, if we assume (1.8) and $(u_0^{\frac{1}{\gamma}})' \in L^\infty(\Omega)$, we get

$$|(u_{\varepsilon,\eta}^{\frac{1}{\gamma}})_x(t,x)| \le \max\Big\{\|(u_0^{\frac{1}{\gamma}})'\|_{L^\infty(\Omega)}, \frac{m+\beta}{\sqrt{2m(m-\beta)}}\Big\} \quad \text{in } [0,\infty)\times\Omega.$$

(ii) *If $m\ge\beta+2$, then there is a constant $C=C(m)$ such that*

$$|(u_{\varepsilon,\eta}^{m-1})_x(t,x)|^2 \le Ct^{-1}\|u_0\|_{L^\infty(\Omega)}^{m-1} \quad \text{in } (0,\infty)\times\Omega.$$

Proof. (i) Repeating the proof of Lemma 2 until (2.5), we get

$$\partial_x^2 h_{\varepsilon,\eta} = 0 \quad \text{and} \quad \partial_t v_{\varepsilon,\eta} - mh_{\varepsilon,\eta}^{\gamma(m-1)}\partial_x^2 v_{\varepsilon,\eta} \ge 0 \quad \text{at } (t_0,x_0).$$

Then

$$\zeta\partial_x h_{\varepsilon,\eta}(\partial_{tx}h_{\varepsilon,\eta} - mh_{\varepsilon,\eta}^{\gamma(m-1)}\partial_x^3 h_{\varepsilon,\eta}) \ge -\frac{\zeta}{2}(\partial_x h_{\varepsilon,\eta})^2 \quad \text{at } (t_0,x_0).$$

Combining this with the 1D-analogue of (2.3) and $\partial_x^2 h_{\varepsilon,\eta}(t_0,x_0)=0$, we obtain

$$\begin{aligned}&m(m\gamma-1)(1-\gamma(m-1))\zeta h_{\varepsilon,\eta}^{\gamma(m-1)-2}(\partial_x h_{\varepsilon,\eta})^2\\ &\quad\le \frac{\zeta'}{2} - \zeta\psi_\varepsilon'(h_{\varepsilon,\eta}^\gamma)h_{\varepsilon,\eta}^{-\beta\gamma} + (1+\beta-\gamma^{-1})\zeta\psi_\varepsilon(h_{\varepsilon,\eta}^\gamma)h_{\varepsilon,\eta}^{-\gamma(1+\beta)} \quad \text{at } (t_0,x_0).\end{aligned}$$

Using the same argument, we arrive at the desired estimate.

(ii) Let now $\overline{\gamma} := \frac{1}{m-1}$, and define $h_{\varepsilon,\eta} := u_{\varepsilon,\eta}^{\frac{1}{\overline{\gamma}}}$. Then $h_{\varepsilon,\eta}$ satisfies

$$\partial_t h_{\varepsilon,\eta} - m h_{\varepsilon,\eta}\partial_x h_{\varepsilon,\eta} - \frac{m}{m-1}(\partial_x h_{\varepsilon,\eta})^2 + (m-1)\psi_\varepsilon(h_{\varepsilon,\eta}^{\overline{\gamma}})h_{\varepsilon,\eta}^{1-\overline{\gamma}(1+\beta)} = 0.$$

As in [5] (see also [32, 48]), we consider the auxiliary function $p(y) = \frac{N_0 y(4-y)}{3}$ for all $y \in [0,1]$, where $N_0 := (2\|u_0\|_\infty)^{m-1}$. Note that p is invertible and

$$p \in [0, N_0], \quad p' \in \Big[\frac{2N_0}{3}, \frac{4N_0}{3}\Big], \quad p'' = -\frac{2N_0}{3}, \quad \Big(\frac{p''}{p'}\Big)' \leq -\frac{1}{4} \quad \text{in } [0,1].$$

Its inverse function is given by $p^{-1}(z) = 2 - (4 - \frac{3z}{N_0})^{\frac{1}{2}}$ for all $z \in [0, N_0]$. Finally, define $v_{\varepsilon,\eta} := p^{-1} \circ h_{\varepsilon,\eta}$. We obtain the following equation, satisfied by $v_{\varepsilon,\eta}$:

$$\begin{aligned}\partial_t v_{\varepsilon,\eta} - mp(v_{\varepsilon,\eta})\partial_x^2 v_{\varepsilon,\eta} - \Big(\frac{m}{m-1}p' + mp(p')^{-1}p''\Big)(v_{\varepsilon,\eta})(\partial_x v_{\varepsilon,\eta})^2 \\ + (m-1)\psi_\varepsilon(p^{\overline{\gamma}})p^{1-\overline{\gamma}(1+\beta)}(p')^{-1}(v_{\varepsilon,\eta}) = 0 \quad \text{in } (0,\infty)\times\Omega.\end{aligned} \tag{2.8}$$

Differentiating in (2.8) with respect to the variable x, we obtain

$$\begin{aligned}\partial_{tx} v_{\varepsilon,\eta} - mp(v_{\varepsilon,\eta})\partial_x^3 v_{\varepsilon,\eta} &= mp'(v_{\varepsilon,\eta})\partial_x v_{\varepsilon,\eta}\partial_x^2 v_{\varepsilon,\eta} + \Big(\frac{m}{m-1}p' + mp(p')^{-1}p''\Big)'(v_{\varepsilon,\eta})(\partial_x v_{\varepsilon,\eta})^3 \\ &\quad + 2\Big(\frac{m}{m-1}p' + mp(p')^{-1}p''\Big)(v_{\varepsilon,\eta})\partial_x v_{\varepsilon,\eta}\partial_x^2 v_{\varepsilon,\eta} \\ &\quad - (m-1)(\psi_\varepsilon(p^{\overline{\gamma}})p^{1-\overline{\gamma}(1+\beta)}(p')^{-1})'(v_{\varepsilon,\eta})\partial_x v_{\varepsilon,\eta} \quad \text{in } (0,\infty)\times\Omega.\end{aligned} \tag{2.9}$$

Let us consider now the function $w_{\varepsilon,\eta} := \zeta(\partial_x v_{\varepsilon,\eta})^2$ and use the same argument as in the proof of Lemma 2. Then there is a point $(t_0, x_0) \in (\frac{\tau}{2}, T + \frac{\tau}{2}) \times \Omega$ where $w_{\varepsilon,\eta}$ attains its maximum, and thus

$$\partial_x^2 v_{\varepsilon,\eta} = 0 \quad \text{and} \quad \partial_t w_{\varepsilon,\eta} - mp(v_{\varepsilon,\eta})\partial_x^2 w_{\varepsilon,\eta} \geq 0 \quad \text{at } (t_0, x_0).$$

Then

$$\zeta\partial_x v_{\varepsilon,\eta}(\partial_{tx} v_{\varepsilon,\eta} - mp(v_{\varepsilon,\eta})\partial_x^3 v_{\varepsilon,\eta}) \geq -\frac{\zeta'}{2}(\partial_x v_{\varepsilon,\eta})^2 \quad \text{at } (t_0, x_0).$$

Combining this and (2.9), we get

$$\begin{aligned}&-m\Big(\frac{m}{m-1}p'' + p\Big(\frac{p''}{p'}\Big)'\Big)(v_{\varepsilon,\eta})\zeta(\partial_x v_{\varepsilon,\eta})^2 \\ &\quad\leq \frac{\zeta'}{2} - \zeta\psi_\varepsilon'(p^{\overline{\gamma}})p^{-\beta\overline{\gamma}}(v_{\varepsilon,\eta}) + (m-1)\zeta\psi_\varepsilon(p^{\overline{\gamma}})p^{1-\overline{\gamma}(1+\beta)}(p')^{-2}p''(v_{\varepsilon,\eta}) \\ &\qquad + (\beta + 2 - m)\zeta\psi_\varepsilon(p^{\overline{\gamma}})p^{-\overline{\gamma}(1+\beta)}(v_{\varepsilon,\eta}) \quad \text{at } (t_0, x_0).\end{aligned} \tag{2.10}$$

Note that the last three terms in the right-hand side of (2.10) are nonpositive, and

$$-m\Big(\frac{m}{m-1}p'' + p\Big(\frac{p''}{p'}\Big)'\Big)(v_{\varepsilon,\eta}) \geq \frac{2m^2 N_0}{3(m-1)} + \frac{m}{4}p(v_{\varepsilon,\eta}) \geq \frac{2m^2 N_0}{3(m-1)}.$$

Then (2.10) implies the following estimate:

$$\zeta(\partial_x v_{\varepsilon,\eta})^2(t_0, x_0) \leq \frac{3c_0(m-1)}{4m^2 N_0}\tau^{-1}.$$

By using the same arguments as in Lemma 2, the last inequality implies

$$\begin{aligned}(\partial_x h_{\varepsilon,\eta})^2(\tau, x) = (p')^2(v_{\varepsilon,\eta})(\partial_x v_{\varepsilon,\eta})^2(\tau, x) &\leq \Big(\frac{4N_0}{3}\Big)^2\frac{3c_0(m-1)}{4m^2 N_0}\tau^{-1} \\ &= \frac{2^{m+1}c_0(m-1)}{3m^2}\tau^{-1}\|u_0\|_\infty^{m-1} \quad \text{for all } x \in \Omega.\end{aligned}$$

The rest of the proof is straightforward. □

As in many other parabolic problems, the spatial gradient estimates given in Lemma 2 imply the global $\mathcal{C}^\alpha$-Hölder regularity of the solutions. Similar results hold for the one-dimensional case by using Lemma 3.

Proposition 1. *Assume the conditions of the first part of Lemma 2. Then, for any $\tau > 0$, the following estimates hold for all $(t,x),(s,y) \in [\tau,\infty)\times\Omega$:*

$$|u_{\varepsilon,\eta}^{\frac{m+1}{2}}(t,x) - u_{\varepsilon,\eta}^{\frac{m+1}{2}}(s,y)| \le C_1[C_2(|x-y| + |t-s|^{\frac{1}{3N}}) + C_3|t-s|^{\frac{1}{3}}],$$

$$C_1 = C(m,\beta,N)(\tau^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta} + 1)^{\frac{1}{2}}, \quad C_2 = \|u_0\|_{L^\infty(\Omega)}^{\frac{1-\beta}{2}}, \quad C_3 = |\Omega|^{\frac{1}{2}}\|u_0\|_{L^\infty(\Omega)}^{\frac{m-\beta}{2}},$$

if $\beta \le 1$, and

$$|u_{\varepsilon,\eta}^{\frac{m+1}{2}}(t,x) - u_{\varepsilon,\eta}^{\frac{m+1}{2}}(s,y)| \le \widehat{C_1}[\widehat{C_2}(|x-y| + |t-s|^{\frac{1}{3N}})^{\frac{m+1}{m+\beta}} + C_3|t-s|^{\frac{1}{3}}],$$

$$\widehat{C_1} = C(m,\beta,\|u_0\|_{L^\infty(\Omega)}), \quad \widehat{C_2} = 2(\tau^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta} + 1)^{\frac{m+1}{2(m+\beta)}},$$

if $\beta > 1$. Moreover, if $\beta + m \le 2$ and $\nabla u_0^{\frac{1}{\gamma}} \in L^\infty(\Omega)$, then

$$|u_{\varepsilon,\eta}^{\frac{m+1}{2}}(t,x) - u_{\varepsilon,\eta}^{\frac{m+1}{2}}(s,y)| \le K_1[(|x-y| + |t-s|^{\frac{1}{3N}}) + K_2|t-s|^{\frac{1}{3}}],$$

$$K_1 = 3\cdot 2^{\frac{1-\beta}{2}}\frac{m+1}{m+\beta}\|u_0\|_{L^\infty(\Omega)}^{\frac{1-\beta}{2}}\max\Big\{\|\nabla u_0^{\frac{1}{\gamma}}\|_{L^\infty(\Omega)}, \Big[\frac{(2+\beta-m)(m+\beta)^2}{2m(\Delta_{m,N} - (\beta+1-m)^2)}\Big]^{\frac{1}{2}}\Big\},$$

$$K_2 = C(m,\beta,N)|\Omega|^{\frac{1}{2}}(\tau^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta} + 1)^{\frac{1}{2}}\|u_0\|_{L^\infty(\Omega)}^{\frac{m-\beta}{2}},$$

for all $(t,x),(s,y) \in [0,\infty)\times\Omega$.

Proof. Let us first extend $u_{\varepsilon,\eta}$ by η outside Ω if needed and denote still by $u_{\varepsilon,\eta}$ to that extension. For arbitrary $t \ge s \ge \tau > 0$, by multiplying the equation by $\partial_t u_{\varepsilon,\eta}^m = m u_{\varepsilon,\eta}^{m-1}\partial_t u_{\varepsilon,\eta}$ and integrating by parts over $(s,t)\times\Omega$, we get

$$\int_s^t\int_\Omega m u_{\varepsilon,\eta}^{m-1}|\partial_t u_{\varepsilon,\eta}|^2\,dx\,d\sigma + \frac{1}{2}\frac{d}{dt}\int_s^t\int_\Omega |\nabla u_{\varepsilon,\eta}^m|^2\,dx\,d\sigma + \int_s^t\int_\Omega m u_{\varepsilon,\eta}^{m-1} g_\varepsilon(u_{\varepsilon,\eta})\partial_t u_{\varepsilon,\eta}\,dx\,d\sigma = 0.$$

Define $G_\varepsilon(r) := m\int_0^r s^{m-1}g_\varepsilon(s)\,ds$. Notice that

$$G_\varepsilon(r) \le m\int_0^r s^{m-\beta-1}\,ds = \frac{m}{m-\beta}r^{m-\beta} \quad \text{for all } r > 0.$$

Then the last equality implies

$$\int_s^t\int_\Omega m u_{\varepsilon,\eta}^{m-1}|\partial_t u_{\varepsilon,\eta}|^2\,dx\,d\sigma \le \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon,\eta}^m(s,x)|^2\,dx + \int_\Omega G_\varepsilon(u_{\varepsilon,\eta}(s,x))\,dx.$$

Let $z_{\varepsilon,\eta} := \frac{2\sqrt{m}u_{\varepsilon,\eta}^{\frac{m+1}{2}}}{m+1}$. Using (2.1), we get

$$\begin{aligned}\int_s^t\int_\Omega |\partial_t z_{\varepsilon,\eta}|^2\,dx\,d\sigma &\le C(m,\beta,N)(\tau^{-1}\|u_0\|_\infty^{1+\beta} + 1)\int_\Omega u_{\varepsilon,\eta}^{m-\beta}(s,x)\,dx \\ &\le C(m,\beta,N)|\Omega|(\tau^{-1}\|u_0\|_\infty^{1+\beta} + 1)\|u_0\|_\infty^{m-\beta} =: C_0.\end{aligned}$$

Given $x,y \in \Omega$, define $r := |x-y| + |t-s|^{\frac{1}{3N}}$. Then, for some $\bar{x} \in B_r(x)$,

$$\begin{aligned}|z_{\varepsilon,\eta}(t,\bar{x}) - z_{\varepsilon,\eta}(s,\bar{x})|^2 &\le (t-s)\int_s^t |\partial_t z_{\varepsilon,\eta}(\sigma,\bar{x})|^2\,d\sigma \\ &= \frac{t-s}{|B_r|}\int_s^t\int_{B_r(x)} |\partial_t z_{\varepsilon,\eta}(\sigma,z)|^2\,dz\,d\sigma \le \frac{C_0|t-s|}{\alpha_N r^N} \le \frac{C_0|t-s|^{\frac{2}{3}}}{\alpha_N},\end{aligned}$$

where $\alpha_N := |B_1| = \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$. From the triangle inequality, one has

$$|z_{\varepsilon,\eta}(t,x) - z_{\varepsilon,\eta}(s,y)| \le |z_{\varepsilon,\eta}(t,x) - z_{\varepsilon,\eta}(t,\bar{x})| + |z_{\varepsilon,\eta}(t,\bar{x}) - z_{\varepsilon,\eta}(s,\bar{x})| + |z_{\varepsilon,\eta}(s,\bar{x}) - z_{\varepsilon,\eta}(s,y)|.$$

Then, if $\beta \le 1$,

$$|z_{\varepsilon,\eta}(t,x) - z_{\varepsilon,\eta}(s,y)| \le \|\nabla z_{\varepsilon,\eta}(t)\|_\infty |x-\bar{x}| + \left(\frac{C_0}{\alpha_N}\right)^{\frac{1}{2}} |t-s|^{\frac{1}{3}} + \|\nabla z_{\varepsilon,\eta}(s)\|_\infty |\bar{x}-y|.$$

Combining this with the estimate

$$|\nabla z_{\varepsilon,\eta}(t,x)| = \sqrt{m} u_{\varepsilon,\eta}^{\frac{m-1}{2}}(t,x)|\nabla u_{\varepsilon,\eta}(t,x)| \le C(m,\beta,N) u_{\varepsilon,\eta}^{\frac{1-\beta}{2}}(t,x)(t^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta} + 1)^{\frac{1}{2}},$$

we get the first desired estimate.

If $\beta > 1$, then, since $z_{\varepsilon,\eta}(t,x) = C(m,\beta)(u_{\varepsilon,\eta}^{\frac{m+\beta}{2}})^\nu$ with $\nu = \frac{m+1}{m+\beta}$ and $\nu \in (0,1)$, using the Hölder continuity of the function $r \to r^\nu$, we get

$$\begin{aligned} |z_{\varepsilon,\eta}(t,x) - z_{\varepsilon,\eta}(t,\bar{x})| &\le C(m,\beta,\|u_0\|_{L^\infty(\Omega)}) |u_{\varepsilon,\eta}^{\frac{m+\beta}{2}}(t,x) - u_{\varepsilon,\eta}^{\frac{m+\beta}{2}}(t,\bar{x})|^\nu \\ &\le C(m,\beta,\|u_0\|_{L^\infty(\Omega)}) \|\nabla u_{\varepsilon,\eta}^{\frac{m+\beta}{2}}(t)\|_\infty^\nu |x-\bar{x}|^\nu, \end{aligned}$$

and we argue analogously with the term $|z_{\varepsilon,\eta}(s,\bar{x}) - z_{\varepsilon,\eta}(s,y)|$ to get the desired estimate.

The proof of the remaining statement can be obtained easily by using (2.2) instead of (2.1) in the last inequality. Note also that $\beta \le 2 - m < 1$. This completes our proof. □

Before ending this section, we point out that estimates (2.1) and (2.2) are independent of ε and η. Thus they play the role of some useful *a priori estimates* which will allow the passing to the limit as $\eta, \varepsilon \downarrow 0$, successively. So, for any $\varepsilon > 0$ fixed, since $g_\varepsilon(s)$ is a globally Lipschitz function, we can pass to the limit as $\eta \downarrow 0$ showing that $u_{\varepsilon,\eta} \to u_\varepsilon$ and that u_ε is the (unique) weak solution of the problem

$$\begin{cases} \partial_t u - \Delta u^m + g_\varepsilon(u) = 0 & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_{0,\varepsilon}(x) & \text{in } \Omega, \end{cases} \tag{P$_\varepsilon$}$$

where, more in general, we can assume that the initial datum is also depending on the parameter $\varepsilon > 0$, with $u_{0,\varepsilon} \in L^\infty(\Omega)$, $u_{0,\varepsilon} \ge 0$ (see details, e.g., in [4, 59]). Moreover, obviously, u_ε also satisfies the corresponding pointwise gradient estimates given in Lemma 2 and Lemma 3.

In the following section, we will justify that the limit $\varepsilon \downarrow 0$ allows us to prove the existence of solutions of equation (P) presented in Theorem 1.

3 Proof of Theorem 1 and Study of the Cauchy Problem

In order to complete the proof of Theorem 1, we will structure it in a series of steps.

Step 1: Monotone Convergence in $L^1(0,T;L^1_\delta(\Omega))$ for Bounded Initial Data

Let us first consider the case in which $u_0 = u_{0,\varepsilon} \in L^\infty(\Omega)$, $u_0 \ge 0$. The family of functions $(u_\varepsilon)_{\varepsilon>0}$, obtained at the end of the previous section, forms a bounded monotone sequence. Indeed, from the definition of g_ε, we see that $g_{\varepsilon_1}(s) \ge g_{\varepsilon_2}(s)$ for all $s \in \mathbb{R}$, for $0 < \varepsilon_1 < \varepsilon_2$. This implies that u_{ε_1} is a subsolution of the equation satisfied by u_{ε_2}, and then, since the comparison principle holds for problem (P$_\varepsilon$) (see, e.g., [4]), we get $u_{\varepsilon_1} \le u_{\varepsilon_2}$ in $(0,\infty)\times\Omega$ for $0 < \varepsilon_1 < \varepsilon_2$. Then there is a nonnegative function $u \in L^1(0,T;L^1_\delta(\Omega))$ such that $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$. From the $L^1_\delta(\Omega)$-contractivity proved in [59, Section 6.6] we know that, for all $T \in (0,\infty)$,

$$\int_\Omega u_\varepsilon(T,x)\zeta(x)\,dx + \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)\zeta(x)\,dx\,dt \le \int_\Omega u_0(x)\zeta(x)\,dx.$$

It follows from the last inequality and the dominated convergence theorem that there is a function Υ such that $\lim_{\varepsilon\downarrow 0} g_\varepsilon(u_\varepsilon) = \Upsilon$ in $L^1(0,T;L^1_\delta(\Omega))$. Moreover, the monotonicity of $(u_\varepsilon)_{\varepsilon>0}$ implies

$$g_\varepsilon(u_\varepsilon(t,x)) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(t,x) \quad \text{a.e. in } (0,\infty)\times\Omega,$$

so

$$\lim_{\varepsilon\downarrow 0} g_\varepsilon(u_\varepsilon(t,x)) = \Upsilon(t,x) \geq u^{-\beta}\chi_{\{u>0\}}(t,x) \quad \text{a.e. in } (0,\infty)\times\Omega. \tag{3.1}$$

Thus $\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(0,T;L^1_\delta(\Omega))} \leq \int_\Omega u_0(x)\zeta(x)\,dx$. As a matter of fact, we will prove later that

$$\Upsilon = u^{-\beta}\chi_{\{u>0\}} \quad \text{in } L^1(0,T;L^1_\delta(\Omega)). \tag{3.2}$$

Step 2: Passing to the Limit in $\mathcal{C}([0,T];L^1(\Omega))$ and $\mathcal{C}([0,T];L^1_\delta(\Omega))$ for Bounded Initial Data

Let us start by presenting some arguments which are valid to the case in which $u_0 \in L^1(\Omega)$, $u_0 \geq 0$. Since u_ε are limits of classical solutions, by applying [12, Section 3], we know that $(u_\varepsilon)_{\varepsilon>0}$ are generalized (and L^1-mild) solutions of the problems

$$\begin{cases} \partial_t u - \Delta u^m = f_\varepsilon & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x) = u_{0,\varepsilon}(x) & \text{in } \Omega, \end{cases} \tag{3.3}$$

with $f_\varepsilon \in L^1(0,T;L^1(\Omega))$ given by $f_\varepsilon(t,x) = -g_\varepsilon(u_\varepsilon(t,x))$.

From step 1, we know that $f_\varepsilon \to -\Upsilon$ in $L^1(0,T;L^1(\Omega))$ and $u_{0,\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon\downarrow 0$. Then, by [12, Theorem I], we know that $u_\varepsilon \to u$ in $\mathcal{C}([0,T];L^1(\Omega))$ with u the unique generalized (and L^1-mild) solution of the problem

$$\begin{cases} \partial_t u - \Delta u^m = -\Upsilon & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.4}$$

Let us now prove (3.2). Since u_ε is a weak solution of equation (P_ε), one has

$$\iint\limits_{\mathrm{Supp}(\varphi)} (-u_\varepsilon\varphi_t - u_\varepsilon^m\Delta\varphi + g_\varepsilon(u_\varepsilon)\varphi)\,dx\,dt = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty((0,T)\times\Omega),\ \varphi \geq 0.$$

Letting $\varepsilon\downarrow 0$ and since u is also a very weak solution of problem (3.4), we get

$$-\iint\limits_{\mathrm{Supp}(\varphi)} (u\varphi_t + u^m\Delta\varphi)\,dx\,dt + \lim_{\varepsilon\downarrow 0}\iint\limits_{\mathrm{Supp}(\varphi)} g_\varepsilon(u_\varepsilon)\varphi\,dx\,dt = 0.$$

Thus

$$\lim_{\varepsilon\downarrow 0}\iint\limits_{\mathrm{Supp}(\varphi)} g_\varepsilon(u_\varepsilon)\varphi\,dx\,dt = \iint\limits_{\mathrm{Supp}(\varphi)} u^{-\beta}\chi_{\{u>0\}}\varphi\,dx\,dt \quad \text{for all } \varphi \in C_c^\infty((0,T)\times\Omega),\ \varphi \geq 0. \tag{3.5}$$

Then $\Upsilon = u^{-\beta}\chi_{\{u>0\}}$ in $L^1(0,T;L^1(\Omega))$ follows from (3.1) and (3.5).

The same conclusion also holds for similar arguments for the more general case in which $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$. The only modification to be justified is the application of the continuous dependence result for mild solutions of (3.3). The main ingredient of the proof of [12, Theorem I] is that the abstract operator associated to problem (P_ε) is an m-T-accretive operator on the Banach space $X = L^1(\Omega)$, but the same conclusion arises once we prove the same properties on the space $X = L^1_\zeta(\Omega) = L^1_\delta(\Omega)$ (with ζ given by (1.10)). This is a more or less implicitly well-known property (see, e.g., [59, Section 6.6]), but since we are unable to find a more detailed proof, we will get here a short proof of this set of properties. Given $f \in L^1_\delta(\Omega)$ and $\lambda \geq 0$, we start by recalling the definition of very weak solution of the stationary problem

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega,\\ |u|^{m-1}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P(f, \lambda)}$$

Definition 4. Given $f \in L^1_\delta(\Omega)$ and $\lambda \geq 0$, a function $u \in L^1_\delta(\Omega)$ is called a *very weak solution* of $(\mathrm{P}(f,\lambda))$ if $|u|^{m-1}u \in L^1(\Omega)$ and for any $\psi \in W^{2,\infty}(\Omega) \cap W^{1,\infty}_0(\Omega)$,

$$\int_\Omega u^m(x)\Delta\psi(x)\,dx + \lambda \int_\Omega u(x)\psi(x)\,dx = \int_\Omega f(x)\psi(x)\,dx.$$

Lemma 4. *Let $X = L^1_\zeta(\Omega)$, $m > 0$, and define the operator $A\colon D(A) \to X$ given by*

$$Au = -\Delta(|u|^{m-1}u) =: f, \quad u \in D(A),$$

with

$$D(A) = \{u \in L^1_\zeta(\Omega);\ u \text{ is a very weak solution of } (\mathrm{P}(f,0)) \text{ for some } f \in L^1_\zeta(\Omega)\}.$$

Then A is an m-T-accretive operator on the Banach space X and $\overline{D(A)} = X$.

Proof. To show that A is a T-accretive operator on X, we have to show that, given $f, \widehat{f} \in L^1_\zeta(\Omega)$ and $\lambda > 0$, if $u, \widehat{u}$ are very weak solutions of $(\mathrm{P}(f,\lambda))$ and $(\mathrm{P}(\widehat{f},\lambda))$, respectively, then

$$\lambda\|[u-\widehat{u}]_+\|_{L^1_\zeta(\Omega)} \leq \|[f-\widehat{f}]_+\|_{L^1_\zeta(\Omega)}. \tag{3.6}$$

But by introducing $v = |u|^{m-1}u$, then $v \in L^1(\Omega)$ is a very weak solution of

$$\begin{cases} -\Delta v + \lambda|v|^{\frac{1}{m}-1}v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

(and similarly for $\widehat{v} = |\widehat{u}^{m-1}\widehat{u}|$). Assume for the moment that $f, \widehat{f} \geq 0$, and thus the positivity of $u, \widehat{u}$ was proved in [16] (see also [17]), and estimate (3.6) coincides exactly with [35, Theorem 2.5, estimate (19)] (notice that, although $L^1_\zeta(\Omega) = L^1_\delta(\Omega)$, thanks to (1.11), the norms $\|\cdot\|_{L^1_\zeta(\Omega)}$ and $\|\cdot\|_{L^1_\delta(\Omega)}$ are related by some constants; by replacing $\|\cdot\|_{L^1_\delta(\Omega)}$ with the norm $\|\cdot\|_{L^1_\zeta(\Omega)}$, then the constant C arising in [35, Theorem 2.5, estimate (19)] becomes exactly $C = 1$ as needed in (3.6)). By using the decomposition $f = f_+ - f_-$, estimate (3.6) holds for general $f, \widehat{f} \in L^1_\zeta(\Omega)$. An alternative proof can be obtained by applying the *local Kato inequality* given in [29, Theorem 4.4].

The proof of the m-accretivity of A (i.e., $R(A + \lambda I) = X$) was already proved in [16] (see also [17] and [35, Theorem 2.5]).

Moreover, given $f \in L^1_\zeta(\Omega)$, we consider $u_\alpha \in D(A)$ to be the unique solution of $\alpha A u_\alpha + u_\alpha = f$. Then, making $\alpha \downarrow 0$, we have (again by [35, Theorem 2.5]) that $u_\alpha \to f$ in $L^1_\zeta(\Omega)$, which proves that $\overline{D(A)} = X$. □

As a consequence of Lemma 4, we can apply the Crandall–Liggett theorem, and by the accretive operator theory, we know that $f_\varepsilon \to -\Upsilon$ in $L^1(0,T;L^1_\zeta(\Omega))$ and $u_{0,\varepsilon} \to u_0$ in $L^1_\zeta(\Omega)$ implies that $u_\varepsilon \to u$ in $\mathcal{C}([0,T];L^1_\zeta(\Omega))$ with u_ε and u the unique $L^1_\zeta(\Omega)$-mild solutions of problems (3.3) and (3.4), respectively, as $\varepsilon \downarrow 0$. Now the adaptation of the proof of [12, Theorem I] to show that $u_\varepsilon \to u$ in $\mathcal{C}([0,T];L^1_\zeta(\Omega))$ as generalized solutions is a trivial fact. This implies, as before, that $\Upsilon = u^{-\beta}\chi_{\{u>0\}}$ in $L^1(0,T;L^1_\delta(\Omega))$.

Remark 3. We point out that the uniqueness of a generalized (or L^1-mild) solution of problem (3.4) when $\Upsilon(t,x)$ is prescribed in $L^1(0,T;L^1_\zeta(\Omega))$ does not imply the uniqueness of the generalized (or L^1_ζ-mild) solution of the non-monotone problem (P). This question remains as an open problem; as in [62], the uniqueness of solutions fails even for general bounded nonnegative initial data. Some partial results are given in [28].

Step 3: Maximality of the Above Constructed Solution

Let us show that if v is a different solution of equation (P), then $v(t,x) \leq u(t,x)$ a.e. in $(0,\infty) \times \Omega$. Indeed, since $g_\varepsilon(v) \leq v^{-\beta}\chi_{\{v>0\}}$ for all $\varepsilon > 0$, then $\partial_t v - \Delta v^m + g_\varepsilon(v) \leq 0$ in $\mathcal{D}'((0,\infty) \times \Omega)$, which implies that v is a subsolution of problem (P_ε) (with the same initial datum). Since $g_\varepsilon(s)$ is a globally Lipschitz function, thanks to the L^1_ζ-contraction result (a consequence of the T-accretivity of A in $X = L^1_\zeta(\Omega)$, see also [4, 12]), we get $v(t,x) \leq u_\varepsilon(t,x)$ a.e. in $(0,\infty) \times \Omega$. Passing to the limit as $\varepsilon \downarrow 0$, we obtain the wanted inequality.

Step 4: Treatment of Unbounded Nonnegative Initial Data u_0

Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$, and let $u_{0,n}(x) = \inf\{u_0(x), n\}$. Then $u_{0,n} \in L^\infty(\Omega)$, $u_{0,n} \geq 0$ and $u_{0,n} \uparrow u_0$ in $L^1_\delta(\Omega)$ as $n \uparrow +\infty$. Then, as before, we can apply the comparison principle to deduce that, for any $\varepsilon > 0$, if $u_{\varepsilon,n}$ is the (unique) solution of problem (P_ε), then $u_{\varepsilon,n_1} \leq u_{\varepsilon,n_2}$ in $(0,\infty) \times \Omega$ if $n_1 \leq n_2$. Moreover, we have the uniform bound

$$0 \leq u_n(t,x) \leq U(t,x) \quad \text{a.e. in } (0,T) \times \Omega, \tag{3.7}$$

with $U \in \mathcal{C}([0,T]; L^1_\zeta(\Omega))$ the unique L^1_ζ-mild solution of the homogeneous problem

$$\begin{cases} \partial_t U - \Delta U^m = 0 & \text{in } (0,T) \times \Omega, \\ U = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Indeed, it suffices to use that, for any n and $\varepsilon > 0$, we have $-g_\varepsilon(u_{\varepsilon,n})(t,x) \leq 0$ in $(0,T) \times \Omega$ and to use the comparison principle for the unperturbed nonlinear diffusion problem. Then passing to the limit, as in step 2, we deduce that if u_n is the *maximal* L^1_ζ-mild solution of (P) associated to $u_{0,n} \in L^\infty(\Omega)$, then $u_{n_1} \leq u_{n_2}$ in $\mathcal{C}([0,T]; L^1_\zeta(\Omega))$ if $n_1 \leq n_2$. Moreover, $u_{n_1}^{-\beta} \geq u_{n_2}^{-\beta}$ on $\{(t,x) \in (0,\infty) \times \Omega,\, u_{n_1}(t,x) > 0\}$ if $n_1 \leq n_2$, and that, in fact, $\{u_{n_1} > 0\} \supset \{u_{n_2} > 0\}$. Then $\Upsilon_n := -u_n^{-\beta}\chi_{\{u_n > 0\}}$, is a monotone sequence of nonnegative functions in $L^1(0,T; L^1_\delta(\Omega))$ which converges to some Υ in $L^1(0,T; L^1_\delta(\Omega))$, and thus we can apply again the extension of the Benilan–Crandall–Sacks [12] argument to pass to the limit of L^1_ζ-mild solutions of problems of type (3.3), and thus we get $u_n \to u$ in $\mathcal{C}([0,T]; L^1_\zeta(\Omega))$ with u the unique $L^1_\zeta(\Omega)$-mild solution of problem (3.4) as $n \uparrow +\infty$. Arguing as in step 2, we get $\Upsilon = -u^{-\beta}\chi_{\{u>0\}}$, and thus $u^{-\beta}\chi_{\{u>0\}} \in L^1(0,T; L^1_\delta(\Omega))$. The proof of the maximality is again similar to the arguments of step 4.

Step 5: Gradient Estimate for $u_0 \in L^1_\zeta(\Omega)$

Notice that, from (3.7) we get (after passing to the limit, as $n \uparrow +\infty$) $0 \leq u(t,x) \leq U(t,x)$ a.e. in $(0,T) \times \Omega$. On the other hand, by applying the smoothing effects shown in [60] (see also [58] for the semilinear case) and the explicit sharp estimate given in [59, (17.32)] (see a different proof via other rearrangement arguments in [26] combined with [35, Theorem 3.1]), we know that, for any $m \geq 1$,

$$\|U(t)\|_{L^\infty(\Omega)} \leq \frac{C(\Omega)}{t^\alpha} \|u_0\|^\sigma_{L^1_\zeta(\Omega)}, \tag{3.8}$$

with $\alpha = \frac{N}{N(m-1)+2}$ and $\sigma = \frac{2}{N(m-1)+2}$. In the special case of $m > 1$, we have a universal estimate for U (see, e.g. [59, Proposition 5.17]),

$$\|U(t)\|_{L^\infty(\Omega)} \leq C(m,N) R^{\frac{2}{m-1}} t^{-\frac{1}{m-1}}, \tag{3.9}$$

where R is the radius of a ball containing Ω.

Thus the same estimates, (3.8) for $m \geq 1$ and (3.9) for $m > 1$, also hold for u. Using Lemma 2, we get that, for any $t > 0$, a.e. $x \in \Omega$ and for any $\lambda \in (0,t)$, we have

$$|\nabla u_\varepsilon^{\frac{1}{\gamma}}(t,x)|^2 \leq C\Big(\frac{\|u(t-\lambda)\|^{1+\beta}_{L^\infty(\Omega)}}{t-\lambda} + 1\Big) \leq \begin{cases} C\Big(\dfrac{C(\Omega)^{1+\beta}\|u_0\|^{(1+\beta)\sigma}_{L^1(\Omega)}}{(t-\lambda)^{\alpha+1}} + 1\Big) & \text{if } m \geq 1, \\ C\Big(\dfrac{[C(m,N)R^{\frac{2}{m-1}}(t-\lambda)^{-\frac{1}{m-1}}]^{1+\beta}}{t-\lambda} + 1\Big) & \text{if } m > 1. \end{cases}$$

Passing to the limit, first as $\lambda \downarrow 0$ and then as $\varepsilon \downarrow 0$ (using the convergence of step 2 and weak-$\star$ convergence in $L^\infty(\Omega)$), we get the pointwise gradient estimate given in Theorem 1 (ii), with $\omega = \alpha + 1$ if $m \geq 1$ and $\omega = \frac{\beta+m}{m-1}$ if $m > 1$.

Now the proof of the fact that the *maximal* L^1-mild solution is Hölder continuous on $(0,T] \times \overline{\Omega}$ is a simple consequence of Proposition 1 and the above convergence arguments.

Step 6: Case $m + \beta < 2$, Gradient Convergence and Proof of Theorem 1 (iii)

In order to prove part (iii) of Theorem 1, we shall use another type of convergence arguments. As a matter of fact, we will prove a stronger result showing the gradient convergence as $\varepsilon \downarrow 0$,

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } (0,T)\times\Omega,$$

up to a subsequence. Indeed, from the equations satisfied by u_ε and $u_{\varepsilon'}$ for any $\varepsilon > \varepsilon' > 0$, we have

$$\partial_t(u_\varepsilon - u_{\varepsilon'}) - (\Delta u_\varepsilon^m - \Delta u_{\varepsilon'}^m) + g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}) = 0.$$

For any $\delta > 0$, let us define

$$T_\delta(s) = \begin{cases} s & \text{if } |s| < \delta, \\ \delta\,\mathrm{sign}(s) & \text{if } |s| \geq \delta, \end{cases} \qquad \text{and} \qquad S_\delta(r) = \int_0^r T_\delta(s)\,ds.$$

For any $0 < \tau < T < \infty$, by using $T_\delta(u_\varepsilon - u_{\varepsilon'})$ as a test function in (3.5) and integrating both sides of (3.5) on $(\tau, T)\times\Omega$, we obtain

$$\begin{aligned} \int_\Omega S_\delta(u_\varepsilon - u_{\varepsilon'})(T,x)\,dx &+ \int_\tau^T\int_\Omega (m u_\varepsilon^{m-1}\nabla u_\varepsilon - m u_{\varepsilon'}^{m-1}\nabla u_{\varepsilon'})\cdot\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,dt \\ &+ \int_\tau^T\int_\Omega (g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}))T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,dt = \int_\Omega S_\delta(u_\varepsilon - u_{\varepsilon'})(\tau,x)\,dx. \end{aligned}$$

It follows from the facts $S_\delta(r) \geq 0$ and $S_\delta(r) \leq \delta|r|$ for all $r \in \mathbb{R}$ that

$$\begin{aligned} &\int_\tau^T\int_\Omega m u_\varepsilon^{m-1}\nabla(u_\varepsilon - u_{\varepsilon'})\cdot\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,dt \\ &\quad + \int_\tau^T\int_\Omega m(u_\varepsilon^{m-1} - u_{\varepsilon'}^{m-1})\nabla u_{\varepsilon'}\cdot\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,dt \\ &\quad + \int_\tau^T\int_\Omega (g_\varepsilon(u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}))T_\delta(u_\varepsilon - u_{\varepsilon'})\,dx\,dt \leq \delta\int_\Omega |(u_\varepsilon - u_{\varepsilon'})(\tau,x)|\,dx. \end{aligned}$$

Since $|T_\delta(s)| \leq \delta$ for all $s \in \mathbb{R}$, we obtain, from the last inequality,

$$\iint_{\{|u_\varepsilon - u_{\varepsilon'}|<\delta\}} u_\varepsilon^{m-1}|\nabla(u_\varepsilon - u_{\varepsilon'})|^2\,dx\,dt \leq 4\delta\|u_0\|_{L^1(\Omega)} + \int_\tau^T\int_\Omega |(u_\varepsilon^{m-1} - u_{\varepsilon'}^{m-1})\nabla u_{\varepsilon'}\cdot\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})|\,dx\,dt.$$

Then, from (2.1) and the dominated convergence theorem, we get

$$\int_\tau^T\int_\Omega |(u_\varepsilon^{m-1} - u_{\varepsilon'}^{m-1})\nabla u_{\varepsilon'}\cdot\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})|\,dx\,dt \to 0 \quad \text{as } \varepsilon, \varepsilon' \downarrow 0$$

and

$$\iint_{\{|u_\varepsilon - u_{\varepsilon'}|<\delta\}} u_\varepsilon^{m-1}|\nabla(u_\varepsilon - u_{\varepsilon'})|^2\,dx\,dt \leq 4\delta\|u_0\|_{L^1(\Omega)} + o(\varepsilon,\varepsilon'),$$

where $o(\varepsilon,\varepsilon') \to 0$ as $\varepsilon, \varepsilon' \downarrow 0$. Moreover, it is clear that

$$\iint_{\{u_\varepsilon>\delta,\,|u_\varepsilon - u_{\varepsilon'}|<\delta\}} |\nabla(u_\varepsilon - u_{\varepsilon'})|^2\,dx\,dt \leq \delta^{1-m}\iint_{\{u_\varepsilon>\delta,\,|u_\varepsilon - u_{\varepsilon'}|<\delta\}} u_\varepsilon^{m-1}|\nabla(u_\varepsilon - u_{\varepsilon'})|^2\,dx\,dt.$$

It follows from the last inequality that

$$\iint_{\{u_\varepsilon>\delta,\,|u_\varepsilon-u_{\varepsilon'}|<\delta\}} |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt \le 4\delta^{2-m}\|u_0\|_{L^1(\Omega)}+\delta^{1-m}o(\varepsilon,\varepsilon').$$

Thanks to (2.1), we obtain

$$\iint_{\{u_\varepsilon\le\delta,\,|u_\varepsilon-u_{\varepsilon'}|<\delta\}} |\nabla u_\varepsilon|^2\,dx\,dt \le C \iint_{\{u_\varepsilon\le\delta,\,|u_\varepsilon-u_{\varepsilon'}|<\delta\}} u_\varepsilon^{2(1-\frac{1}{\gamma})}\,dx\,dt \le CT|\Omega|\delta^{2(1-\frac{1}{\gamma})},$$

where the constant $C>0$ is independent of ε,δ. Since $u_\varepsilon\ge u_{\varepsilon'}$ and by the same argument, we also obtain

$$\iint_{\{u_\varepsilon\le\delta,\,|u_\varepsilon-u_{\varepsilon'}|<\delta\}} |\nabla u_{\varepsilon'}|^2\,dx\,dt \le C\delta^{2(1-\frac{1}{\gamma})}.$$

Combining these, we get

$$\iint_{\{|u_\varepsilon-u_{\varepsilon'}|<\delta\}} |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt \lesssim \delta^{2-m}\|u_0\|_{L^1(\Omega)}+\delta^{1-m}o(\varepsilon,\varepsilon')+\delta^{2(1-\frac{1}{\gamma})}.$$

Here we used the notation $A\lesssim B$ in the sense that there is a constant $c>0$ such that $A\le cB$. Thanks to (2.1) and the fact that $u_\varepsilon\to u$, we obtain

$$\iint_{\{|u_\varepsilon-u_{\varepsilon'}|\ge\delta\}} |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt \le C_{\text{meas}}(\{|u_\varepsilon-u_{\varepsilon'}|\ge\delta\}) \le Co(\varepsilon,\varepsilon'),$$

with $C=C(m,\beta,N,\tau,T,\|u_0\|_\infty)$. It follows from that

$$\int_\tau^T\int_\Omega |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt \lesssim \delta^{2-m}\|u_0\|_{L^1(\Omega)}+(1+\delta^{1-m})o(\varepsilon,\varepsilon')+\delta^{2(1-\frac{1}{\gamma})}.$$

Hence

$$\limsup_{\varepsilon\downarrow 0}\int_\tau^T\int_\Omega |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt \le \delta^{2-m}\|u_0\|_{L^1(\Omega)}+\delta^{2(1-\frac{1}{\gamma})}.$$

The last inequality holds for any $\delta>0$, and since now $m+\beta<2$, we obtain

$$\limsup_{\varepsilon\downarrow 0}\int_\tau^T\int_\Omega |\nabla(u_\varepsilon-u_{\varepsilon'})|^2\,dx\,dt = 0.$$

Consequently, we have $\nabla u_\varepsilon\to\nabla u$ in $L^2((\tau,T)\times\Omega)$. Up to a subsequence, we deduce that $\nabla u_\varepsilon\to\nabla u$ a.e. in $(\tau,T)\times\Omega$. A diagonal argument implies that there is a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ (still denoted as $(u_\varepsilon)_{\varepsilon>0}$) such that $\nabla u_\varepsilon\to\nabla u$ a.e. in $(0,\infty)\times\Omega$. Hence u also satisfies the gradient estimates (2.1) and (2.2).

This puts an end to the proof of Theorem 1. □

Remark 4. An alternative proof of the regularity $u\in\mathcal{C}([0,\infty);L^1(\Omega))$ in part (iii) of Theorem 1, when $u_0\in L^\infty(\Omega)$, is the following: for any $1<p<2$, thanks to Lemma 2, we have that, for any finite time $T>0$,

$$\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt \le C\int_0^T\int_\Omega u^{p(1-\frac{1}{\gamma})}(t^{-1}\|u_0\|_{L^\infty(\Omega)}^{1+\beta}+1)^{\frac{p}{2}}\,dx\,dt \le C_1, \tag{3.10}$$

where $C_1>0$ only depends on $T,\Omega,\|u_0\|_{L^\infty(\Omega)}$ and the parameters involved. Since u is bounded on $(0,\infty)\times\Omega$, it follows from (3.10) that $\nabla u^m\in L^p((0,T),W_0^{1,p}(\Omega))$. This implies that

$$\partial_t u = \operatorname{div}(\nabla u^m) - u^{-\beta}\chi_{\{u>0\}} \in L^p((0,T),W^{-1,p}(\Omega))\cap L^1((0,T)\times\Omega),$$

where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$. Then, by a compactness embedding (see [56]), we obtain $u\in\mathcal{C}([0,T];L^1(\Omega))$.

The rest of this section is devoted to the associated Cauchy problem for initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The existence of solutions to Cauchy problem (CP) can be obtained as a consequence of Theorem 1. Here is a simplified statement.

Theorem 2. *Assume m, N, β as in Theorem 1. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$. Then problem* (CP) *has a weak solution $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty((0,\infty) \times \mathbb{R}^N)$ satisfying* (CP) *in the sense of distributions,*

$$\int_0^\infty \int_{\mathbb{R}^N} (-u\varphi_t - u^m \Delta\varphi + u^{-\beta}\chi_{\{u>0\}}\varphi)\,dx\,dt = 0 \quad \text{for all } \varphi \in \mathcal{D}((0,\infty) \times \mathbb{R}^N).$$

Moreover, the gradient estimates of Lemma 2 remain valid with $C = C(m, \beta, N, \|u_0\|_{L^1(\Omega)})$ for any $m \geq 1$.

Proof. We will start by constructing a sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of the regularized problem

$$\begin{cases} \partial_t u - \Delta u^m + g_\varepsilon(u) = 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

After that, we will prove that $u_\varepsilon \to u$, with u a weak solution of problem (CP).

The proof of the construction of $(u_\varepsilon)_{\varepsilon>0}$ is quite similar to the one given in the proof of Theorem 1. Thus we just sketch out the main idea. We start by considering the approximate problem over $(0,\infty) \times B_R$ for any $R > 0$, taking as initial data the function $u_0\chi_{B_R}$. By some classical results on the accretive operators theory (see, e.g., [4, 59]), we know that there is a unique weak solution $u_{\varepsilon,R}$ of the approximate problem in $(0,\infty) \times B_R$ and that (from the construction of the initial datum on B_R), for any ε, $R > 0$, we have the estimates

$$\begin{aligned} \|u_{\varepsilon,R}(t)\|_{L^1(B_R)} &\leq \|u_0\|_{L^1(\mathbb{R}^N)} && \text{for all } t > 0, \\ \|u_{\varepsilon,R}(t)\|_{L^\infty(B_R)} &\leq \|u_0\|_{L^\infty(\mathbb{R}^N)} && \text{for all } t > 0. \end{aligned}$$

Thanks to Lemma 2, we also know that

$$|\nabla u_{\varepsilon,R}^{\frac{1}{\gamma}}(t,x)|^2 \leq C(t^{-1}\|u_0\|_{L^\infty(\mathbb{R}^N)}^{1+\beta} + 1) \quad \text{in } (0,\infty) \times B_R.$$

Moreover, for any fixed $\varepsilon > 0$, it follows from the L^1-contraction property (for the unperturbed nonlinear diffusion problem) that the sequence $(u_{\varepsilon,R})_{R>0}$ is pointwise non-decreasing. Thus there exists a function, denoted by u_ε, such that $u_{\varepsilon,R} \uparrow u_\varepsilon$ as $R \to \infty$. Consequently, u_ε satisfies the corresponding estimates for the respective $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ norms. Moreover, since $g_\varepsilon(\cdot)$ is a globally Lipschitz function, the classical regularity result (see, e.g., [4, 59]) implies that

$$\nabla u_{\varepsilon,R}^m \to \nabla u_\varepsilon^m \quad \text{a.e. in } (0,\infty) \times \mathbb{R}^N,$$

up to a subsequence. Similarly to the proof of Theorem 1, we observe that $(u_\varepsilon)_{\varepsilon>0}$ is a non-decreasing sequence. Thus there exists a function u such that $u_\varepsilon \downarrow u$ in $(0,\infty) \times \mathbb{R}^N$, as $\varepsilon \downarrow 0$. Then we mimic the different steps in the proof of Theorem 1 to pass to the limit as $\varepsilon \downarrow 0$. We point out that the continuous dependence in $\mathcal{C}([0,T]; L^1(\mathbb{R}^N))$ is quite similar to the case of a bounded domain Ω since we do not need to approximate the nonlinear term $\psi(u) = u^m$. Then we get that u is a weak solution of equation (CP), and in fact, u is the maximal solution of problem (CP). □

Remark 5. In a similar way to the case of bounded domains, the accretivity in $L^1(\mathbb{R}^N)$ can be replaced by the accretivity in some weighted spaces $L^1_{\rho_\alpha}(\mathbb{R}^N)$ allowing to get the existence of solutions for the Cauchy problem for a more general class of initial data $u_0(x)$ growing with $|x|$ as $|x| \to +\infty$. That was started with the paper [11] and then developed and improved by several authors (see the exposition made in [59, Chapter 12]). The mentioned accretivity in $L^1_{\rho_\alpha}(\mathbb{R}^N)$ holds for any $m > 0$ and $N \geq 3$, for the weight given by

$$\rho_\alpha(x) = \frac{1}{(1+|x|^2)^\alpha},$$

with α given such that $0 < \alpha \leq \frac{N-2}{2}$. For other values of N and $\alpha > 0$, there is only existence of local-in-time solutions of the Cauchy problem [59]. This property could be used to get some generalizations of the results of [43] for the study of (CP) when $m > 1$, but we will not pursue this goal in this paper.

4 Qualitative Properties

We start by recalling that the existence of an L^1_δ-*mild solution* of (P(1)) (for more regular solutions, see, e.g., [59, Subsection 5.5.1]).

Definition 5. Let $u_0 \in L^1_\delta(\Omega)$, $u_0 \geq 0$, and $T > 0$. A nonnegative function $u \in \mathcal{C}([0,T]; L^1_\delta(\Omega))$ is called an L^1_δ-*mild solution* of (P(1)) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(0,T; L^1_\delta(\Omega))$ coincides with the unique L^1_δ-mild solution of the problem

$$\begin{cases} \partial_t u - \Delta u^m = f & \text{in } (0,T)\times\Omega, \\ u^m = 1 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $f := -u^{-\beta}\chi_{\{u>0\}}$.

The existence and uniqueness of an L^1_δ-*mild solution* of (4.1) for a given $f \in L^1(0,T; L^1_\delta(\Omega))$ is an easy modification of the results of [16, 57], [33, Theorem 1.10] (see also [61]) and step 2 of the above section. Indeed, given $f \in L^1_\delta(\Omega)$ and $\lambda \geq 0$, we start by recalling the definition of a very weak solution of the stationary problem

$$\begin{cases} -\Delta(|u|^{m-1}u) + \lambda u = f & \text{in } \Omega, \\ |u|^{m-1}u = 1 & \text{on } \partial\Omega. \end{cases} \tag{P(f, \lambda, 1)}$$

Definition 6. Given $f \in L^1_\delta(\Omega)$ and $\lambda \geq 0$, a function $u \in L^1_\delta(\Omega)$ is called a very weak solution of (P(f,λ)) if $|u|^{m-1}u \in L^1(\Omega)$ and, for any $\psi \in W^{2,\infty}(\Omega) \cap W^{1,\infty}_0(\Omega)$,

$$\int_\Omega u^m(x)\Delta\psi(x)\,dx + \int_\Omega \lambda u(x)\psi(x)\,dx = \int_\Omega f(x)\psi(x)\,dx - \int_{\partial\Omega} \frac{\partial\psi}{\partial n}(x)\,dx.$$

In a completely similar way to step 2 of the above section, we have the following result.

Lemma 5. *Let $X = L^1_\zeta(\Omega)$, $m > 0$, and define the operator $A\colon D(A) \to X$ given by*

$$Au = -\Delta(|u|^m u) := f, \quad u \in D(A),$$

with

$$D(A) = \{u \in L^1_\zeta(\Omega);\ u \text{ is a very weak solution of } (\mathrm{P}(f,0,1)) \text{ for some } f \in L^1_\zeta(\Omega)\}.$$

Then A is an m-T-accretive operator on the Banach space X and $\overline{D(A)} = X$.

Thus the Crandall–Liggett theorem can be applied to get the existence and uniqueness of $u \in \mathcal{C}([0,T]; L^1_\delta(\Omega))$, L^1_δ-*mild solution* of (4.1). Moreover, u is a very weak solution of (4.1) in the sense that $u \in \mathcal{C}([0,T]; L^1_\delta(\Omega))$, $u \geq 0$, $u^m \in L^1((0,T)\times\Omega)$, $f = u^{-\beta}\chi_{\{u>0\}} \in L^1(0,T; L^1_\delta(\Omega))$, and for any $t \in [0,T]$,

$$\int_\Omega u(t,x)\zeta(x)\,dx + \int_0^t\int_\Omega u^m(t,x)\,dx\,dt = \int_\Omega u_0(x)\zeta(x)\,dx + \int_0^t\int_\Omega f(t,x)\zeta(x)\,dx\,dt - \int_0^t\int_{\partial\Omega}\frac{\partial\psi}{\partial n}(x)\,dx.$$

The rest of arguments is completely similar to the case of problem (P).

Now let us present some explicit examples of solutions of (P(1)).

Lemma 6. *The following statements hold.*

(i) *Let $q \in (-\infty, 1)$, $x_0 \in \mathbb{R}^N$, and for $C > 0$, define the function*

$$v_{q,C}(x) = C|x - x_0|^{\frac{2}{1-q}}. \tag{4.2}$$

Then, for any $\lambda > 0$,

$$\mathcal{L}(v) := -\Delta v + \lambda v^q = \Big[\lambda C^2 - \frac{2(N(1-q)+2q)}{(1-q)^2}C\Big]|x - x_0|^{\frac{2q}{1-q}}.$$

In particular, if we define

$$K_{N,q,\lambda} = \left[\frac{\lambda(1-q)^2}{2(N(1-q))+2q}\right]^{\frac{1}{1+\beta/m}},$$

then $\mathcal{L}(v) \equiv 0$ *if* $C = K_{N,q,\lambda}$ *and* $\mathcal{L}(v) > 0$ *(resp.* $\mathcal{L}(v) < 0$*) if* $C < K_{N,q,\lambda}$ *(resp.* $C > K_{N,q,\lambda}$*).*

(ii) *If, for* $m > 0$ *and* $\beta \in (0, m)$*, we define*

$$u_{\beta,m,C}(x) = v_{q,C}^{\frac{1}{m}}(x) = C^{\frac{1}{m}}|x-x_0|^{\frac{2}{m+\beta}}, \quad \text{i.e., with } q = -\frac{\beta}{m},$$

then

$$-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} = \left[\lambda C^2 - \frac{2m(N(m+\beta)-2\beta)}{(m+\beta)^2}C\right]|x-x_0|^{-\frac{2\beta}{m+\beta}}. \tag{4.3}$$

(a) *Define*

$$K_{N,m,\beta,\lambda} = \left[\frac{\lambda(m+\beta)^2}{2m(N(m+\beta)-2\beta)}\right]^{\frac{m}{m+\beta}}, \tag{4.4}$$

then $K_{N,m,\beta,\lambda} > 0$ *and* $-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} = 0$ *in* $\mathbb{R}^N$ *if* $C = K_{N,q,\lambda}$*.*

(b) *If* $x_0 \in \overline{\Omega}$*, then* $-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \in L_\delta^1(\Omega)$ *and*

$$-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \gtrless 0 \quad \text{if} \quad C \lessgtr K_{N,q,\lambda}.$$

(iii) *If* $m > 0$ *and* $\beta \in [m, +\infty)$*, then equation* (4.3) *holds in* $\mathbb{R}^N$*. Moreover, the constant given by* (4.4) *is such that* $K_{N,m,\beta,\lambda} > 0$ *if and only if* $N \geq 2$*.*

(a) *If* $x_0 \in \partial\Omega$ *and* $\delta(x) = |x-x_0|$*, then* $-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \in L_\delta^1(\Omega)$ *and*

$$-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \gtrless 0 \quad \text{if} \quad C \lessgtr K_{N,q,\lambda}.$$

(b) *If* $x_0 \in \Omega$*, then* $-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \notin L_\delta^1(\Omega)$*.*

Proof. Part (i) was given in [25, Lemma 1.6]. Part (ii) results from (i) by a simple change of variable. Moreover, the fact that $-\Delta u_{\beta,m,C}^m + \lambda u_{\beta,m,C}^{-\beta} \in L_\delta^1(\Omega)$ holds because

$$-\frac{2\beta}{m+\beta} + 1 > -1 \tag{4.5}$$

for the case $x_0 \in \partial\Omega$, and since

$$-\frac{2\beta}{m+\beta} > -1 \tag{4.6}$$

(thanks to the condition $\beta \in (0, m)$) when $x_0 \in \Omega$. From definition (4.4), we see that if $\beta \in [m, +\infty)$, then the positivity of $K_{N,m,\beta,\lambda}$ fails only for $N = 1$. Moreover, inequality (4.5) still holds true, but we see that, for any interior point $x_0 \in \Omega$, the weight $\delta(x)$ is no help, and thus the singularity is not integrable (since condition (4.6) fails if $\beta \geq m$). □

Corollary 1. *Let* $\Omega = B_R(x_0)$*, and take* $u_0(x) = u_{\beta,m,C}(x)$ *with* $C = K_{N,m,\beta,\lambda}$ *and* $\lambda = 1$*. Let* $R > 0$ *be such that* $R^{\frac{2m}{m+\beta}} = 1$*. Then* $u(t,x) = u_{\beta,m,C}(x)$ *is the unique solution of* (P(1))*. Moreover,*

$$\|\nabla u^{\frac{m+\beta}{2}}(t)\|_{L^\infty(\Omega)} = C^* \quad \text{for some } C^* > 0,$$

and the exponent $\frac{m+\beta}{2}$ *cannot be replaced by any other greater exponent* α *such that* $\|\nabla u^\alpha(t)\|_{L^\infty(\Omega)} < +\infty$*.*

In order to prove some other qualitative properties (in the line of [30, 36, 45]), the following result is useful.

Lemma 7. *The following statements hold.*

(i) *Let* $q \in (-\infty, 1)$*,* $x_0 \in \mathbb{R}^N$*,* $t_0 \geq 0$*, and for* $C > 0$*, define the function* $v_{q,C}(x) = C|x-x_0|^{\frac{2}{1-q}}$*. Given* $t_0 \geq 0$*,* $\theta \geq 0$ *and* $\lambda > 0$*, let* $y_{q,\theta,\lambda}(t) = [\theta^{1-q} - \lambda(1-q)(t-t_0)]_+^{\frac{1}{1-q}}$ *for* $t \geq t_0$ *so that*

$$y_{q,\theta,\lambda}(t) = 0 \quad \text{for any } t \geq \frac{\theta^{1-q}}{\lambda(1-q)}.$$

Then, given $m \geq 1$*, if* $C \leq K_{N,q,\lambda}$*, the function*

$$U(t,x) = [v_{q,C}(x) + y_{q,\theta,\lambda}^m(t)]^{\frac{1}{m}} \tag{4.7}$$

satisfies $\partial_t U - \Delta U^m + \mu U^q \geq 0$ *on* $(t_0, +\infty) \times \mathbb{R}^N$*, with* $\mu = 2\lambda$*.*

(ii) *If, for $m \geq 1$ and $\beta \in (0, m)$, we define*

$$z_{m,\beta,\theta,\lambda}(t) = \Big[\theta^{\frac{m+\beta}{m}} - \lambda \frac{m+\beta}{m}(t - t_0)\Big]_+^{\frac{m}{m+\beta}} \quad \text{for } t \geq t_0,$$

and thus

$$W(t, x) = [u^m_{\beta,m,C}(x) + z^m_{m,\beta,\theta,\lambda}(t)]^{\frac{1}{m}},$$

then, if $\lambda = \frac{1}{2}$ and $C \geq K_{N,q,\lambda}$, we have $\partial_t W - \Delta W^m + W^{-\beta}\chi_{\{W>0\}} \leq 0$ on $(t_0, +\infty) \times \mathbb{R}^N$.

Proof. Notice that

$$\begin{cases} \dfrac{dy_{q,\theta,\lambda}}{dt} + \lambda y^q_{q,\theta,\lambda} = 0, \\ y_{q,\theta,\lambda}(t_0) = \theta. \end{cases}$$

Moreover, from the convexity of the function $s \to s^m$, we get

$$\partial_t U = U^{-\frac{m-1}{m}} y^{m-1}_{q,\theta,\lambda} \frac{dy_{q,\theta,\lambda}}{dt} \geq \frac{dy_{q,\theta,\lambda}}{dt},$$

and moreover, $-\Delta U^m = -\Delta v_{q,C}$. Notice also that $(a + b)^r \geq \frac{a^r+b^r}{2}$ for any $a, b \geq 0$ and $r > 0$. Then

$$\begin{aligned} \partial_t U - \Delta U^m + \mu U^q &\geq \frac{dy_{q,\theta,\lambda}}{dt} - \Delta v_{q,C} + 2\lambda[v_{q,C}(x) + y^m_{q,\theta,\lambda}(t)]^{\frac{q}{m}} \\ &\geq \Big(\frac{dy_{q,\theta,\lambda}}{dt} + \lambda y^q_{q,\theta,\lambda}\Big) - \Delta v_{q,C} + \lambda v^q \geq 0. \end{aligned}$$

The proof of (ii) is similar but uses now that $(a + b)^{-r} \leq \frac{a^{-r}+b^{-r}}{2}$ for any $a, b > 0$ and $r > 0$. □

Here are some applications of the above lemma.

Proposition 2. *Let $m \geq 1$, $\beta \in (0, m)$, and consider $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$.*

(i) *Complete quenching and formation of the free boundary: there is a finite time $\tau_0 > 0$ such that if u is the mild solution of* (P)*, then $u(t, x) = 0$ for all $t \in (\tau_0, \infty)$ and a.e. $x \in \Omega$.*

(ii) *Let $m \geq 1$, $\beta \in (0, m)$. Assume (for simplicity) $1 \geq u_0 \geq 0$. If u is the mild solution of* (P(1))*, then, for a.e. $x_0 \in \Omega$ such that $\delta(x_0) = d(x_0, \partial\Omega) \geq K_{N,q,\lambda}^{-\frac{2}{1-q}}$, there exists a $\tau_0 = \tau_0(x_0) \geq 0$ such that*

$$u(t, x_0) = 0 \quad \text{for all } t \in (\tau_0, \infty). \tag{4.8}$$

(iii) *Let $m \geq 1$, $\beta \in (0, m)$. If*

$$0 \leq u_0(x) \leq K_{N,q,\lambda}|x - x_0|^{\frac{2}{1-q}} \quad \text{a.e. on } B_{\delta(x_0)}(x_0) \cap \Omega \quad \text{and} \quad \delta(x_0) \geq \frac{1}{K_{N,q,\lambda}^{\frac{m+\beta}{2m}}},$$

then, if u is the mild solution of (P)*, we get*

$$0 \leq u(t, x) \leq K_{N,q,\lambda}|x - x_0|^{\frac{2}{1-q}} \quad \text{a.e. on } (0, +\infty) \times B_{\delta(x_0)}(x_0) \cap \Omega$$

and, in particular, $u(t, x_0) = 0$ for any $t > 0$.

(iv) *Let $m \geq 1$, $\beta \in (0, m)$, and assume*

$$u_0(x) \geq [C\delta^{\frac{2m}{m+\beta}}(x) + \theta^m]^{\frac{1}{m}}, \quad \delta(x) = d(x, \partial\Omega), \tag{4.9}$$

for some $C \geq K_{N,q,\lambda}$. Then, if u is the mild solution of (1.2) *and $\theta \leq 1$, we have*

$$u(t, x) \geq W(t, x) \quad \text{for any } x \in \Omega \text{ and any } t > 0.$$

In particular, if $\theta > 0$, then

$$u(t, x) > 0 \quad \text{for any } x \in \Omega \text{ and } t \in \Big[0, \frac{2m\theta^{\frac{m+\beta}{m}}}{m+\beta}\Big).$$

The conclusion holds for solutions of (P)*, for any $x \in \Omega$ and $t > 0$ if, in assumption* (4.9)*, we take $\theta = 0$.*

Proof. (i) Let $M = \|u_0\|_{L^\infty(\Omega)}$. Note that, since $u^{-\beta} \geq \mu u^\alpha$ for any $u \in (0, M]$ and any $q \in (0, 1)$ if $0 \leq \mu \leq M^{-(\alpha+\beta)}$, then $0 \leq u(t, x) \leq U_q(t, x)$ a.e. in $(0, T) \times \Omega$, with U_q the unique mild solution of the porous media homogeneous problem with a possible *strong absorption*

$$\begin{cases} \partial_t U - \Delta U^m + \lambda U^q = 0 & \text{in } (0, T) \times \Omega, \\ U^m = 0 & \text{on } (0, T) \times \partial\Omega, \\ U(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

since we know that $0 \leq u(t, x) \leq M$. Then, if U is given by (4.7), we get $0 \leq U_q(t, x) \leq U(t, x)$ on $(0, +\infty) \times \Omega$ if we take $t_0 = 0$ and $\theta \geq M$ (remember that $v_{q,C}(x) \geq 0$). Taking x_0 (in the definition of (4.2)) arbitrary in $\mathbb{R}^N$, we get the conclusion.

(ii) We argue as in (i), and thus $0 \leq u(t, x) \leq U_q(t, x)$ a.e. in $(0, T) \times \Omega$, but now with U_q the unique mild solution of the problem

$$\begin{cases} \partial_t U - \Delta U^m + \lambda U^q = 0 & \text{in } (0, T) \times \Omega, \\ U^m = 1 & \text{on } (0, T) \times \partial\Omega, \\ U(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

We use the function U given by (4.7) as supersolution, and we conclude that if we take $t_0 = 0$ and $\theta \geq M$ and $x_0 \in \Omega$ such that $\delta(x_0) = d(x_0, \partial\Omega) \geq K_{N,q,\lambda}^{-\frac{2}{1-q}}$, then (since $y_{q,\theta,\lambda}(t) \geq 0$)

$$U_q^m(t, x) \leq 1 \leq C\delta^{\frac{2}{1-q}}(x_0) \leq v_{q,C}(x) \leq U^m(t, x) \quad \text{for } x \in \partial B_{\delta(x_0)}(x_0),$$

and thus $0 \leq U_q(t, x) \leq U(t, x)$ on $(0, +\infty) \times B_{\delta(x_0)}(x_0)$ if we take $t_0 = 0$ and $\theta \geq \|u_0\|_{L^\infty(B_{\delta(x_0)}(x_0))}$, which proves (4.8).

The proof of (iii) is similar to the proof of (ii) but even simpler than before since now $u = 0$ on the boundary and the supersolution is nonnegative.

The comparison of solutions u of (1.2) (respectively (P)) with the subsolution $W(t, x)$ uses some properties of the function $\delta(x) = d(x, \partial\Omega)$ and follows the same arguments as [23] (see also [28] and [1, Theorem 2.3]) thanks to the assumption $\beta \leq m$. □

Remark 6. Conclusion (iv) of Proposition 2 is very useful in order to prove the uniqueness of the very weak solution of (P) (see, e.g., [23, 28]).

A sharper estimate on the complete quenching time can be obtained without passing by the porous media homogeneous problem with a possible *strong absorption*.

Proposition 3. *Assume the same conditions of Theorem 1, part* (i). *Then every weak solution of equation* (P) *must vanish after a finite time, i.e., there is a finite time $\tau_0 > 0$ such that $u(t, x) = 0$ for all $t \in (\tau_0, \infty)$ and a.e. $x \in \Omega$.*

Proof. By Theorem 1, it suffices to show that the maximal solution u constructed in the above section vanishes after a finite time $\tau_0 > 0$. Thanks to the smoothing effect, we can assume without loss of generality that the initial datum is a nonnegative bounded function $u_0 \in L^\infty(\Omega)$. We shall use some energy methods in the spirit of [2] and [20, Theorem 3]. For any $q \geq \beta + 2$, we can use u^{q-1} as a test function to equation (P), and we obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q(t, x)\,dx + \frac{4m(q-1)}{(m+q-1)^2}\int_\Omega |\nabla u^{\frac{m+q-1}{2}}(t, x)|^2\,dx + \int_\Omega u^{q-\beta-1}(t, x)\,dx = 0.$$

Define $v := u^{\frac{m+q-1}{2}}$. By applying the Sobolev embedding to v, one obtains

$$\|v(t)\|_{L^{2^*}(\Omega)} \leq C(N)\|\nabla v(t)\|_{L^2(\Omega)}, \tag{4.10}$$

with

$$2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ l \text{ for } l \in (1, \infty) & \text{if } N = 1, 2. \end{cases}$$

As we shall see, it is enough to consider the case of $N \geq 3$ since the cases of $N = 1, 2$ can be obtained by easy modifications. Observe that (4.10) is equivalent to

$$\|u(t)\|_{L^{q_\star}(\Omega)}^{\frac{q_\star(N-2)}{N}} \leq C(N) \int_\Omega |\nabla u^{\frac{m+q-1}{2}}(t,x)|^2 \, dx,$$

with $q_\star := \frac{(m+q-1)N}{N-2}$. Note that $q_\star > q$. From the interpolation inequality

$$\|u(t)\|_{L^q(\Omega)} \leq \|u(t)\|_{L^{q_\star}(\Omega)}^{\theta} \|u(t)\|_{L^{q-\beta-1}(\Omega)}^{1-\theta},$$

with $\frac{1}{q} = \frac{\theta}{q_\star} + \frac{1-\theta}{q-\beta-1}$, by a combination of the above inequalities, we deduce

$$\|u(t)\|_{L^q(\Omega)}^{\frac{q_\star(N-2)}{N}} \leq C\|\nabla u^{\frac{m+q-1}{2}}\|_{L^2(\Omega)}^{2\theta} \|u(t)\|_{L^{q-\beta-1}(\Omega)}^{\frac{(1-\theta)q_\star(N-2)}{N}} \leq CA^\theta A^{\frac{(1-\theta)q_\star(N-2)}{(q-\beta-1)N}} = CA^{\theta+\frac{(1-\theta)q_\star(N-2)}{(q-\beta-1)N}},$$

where

$$A := \int_\Omega |\nabla u^{\frac{m+q-1}{2}}(t,x)|^2 \, dx + \int_\Omega u^{q-\beta-1}(t,x) \, dx.$$

This implies

$$\|u(t)\|_{L^q(\Omega)} \leq C(N,m,q) A^{\frac{\theta}{q_\star}\frac{N}{N-2}+\frac{1-\theta}{q-1-\beta}} \leq CA^{\frac{1}{q}+\frac{2\theta}{(N-2)q_\star}}.$$

Then

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q(t,x)\,dx + C(m,q)A \leq 0.$$

In particular, we obtain that $y(t) := \|u(t)\|_{L^q(\Omega)}^q$ satisfies the ordinary differential inequality

$$y'(t) + Cy^\sigma(t) \leq 0, \tag{4.11}$$

with $\sigma := \big(1 + \frac{2q\theta}{(N-2)q_\star}\big)^{-1} \in (0,1)$. Then, as in [2], we deduce that there is a time $\tau_0 > 0$ such that $y(\tau_0) = 0$ and then $y(t) = 0$ for any $t > \tau_0$ since $y(t)$ is a nonnegative function. Thus $u(t,x) = 0$ in $(\tau_0, \infty) \times \Omega$. Indeed, if, on the contrary, we assume that $y(t) > 0$ for every $t > 0$, then, by solving (4.11), we get $y^{1-\sigma}(t) + Ct \leq y^{1-\sigma}(0)$, and since this inequality holds for any $t > 0$, we arrive at a contradiction for t large enough. This ends the proof. □

Remark 7. We note that the above arguments are independent of the size of Ω. Thus one can easily verify that the quenching result also holds for the case $\Omega = \mathbb{R}^N$ as pointed out in the introduction. Moreover, the formation of the free boundary given in Proposition 2 can be also adapted to solutions of the Cauchy problem.

Remark 8. Although several energy methods were developed in the literature (see, e.g., [2, 25] and their references), the main new aspect was the application to the case of singular absorption terms. The method applies to the class of *local* weak solutions of the more general formulation

$$\frac{\partial \psi(v)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,v,Dv) + B(x,t,v,Dv) + C(x,t,v) = f(x,t,v), \tag{4.12}$$

in which the absorption term can be singular and then including equation (P) as a special case. More precisely the assumptions made in [27] were the following: under the general structural assumptions

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C|\mathbf{q}|, \quad C|\mathbf{q}|^2 \leq \mathbf{A}(x,t,r,\mathbf{q}) \cdot \mathbf{q}, \quad C|r|^{\theta+1} \leq G(r) \leq C^*|r|^{\theta+1},$$

where $G(r) = \psi(r)r - \int_0^r \psi(\tau)\,d\tau$, and $C|r|^\alpha \leq C(x,t,r)r$, $f(x,t,r)r \leq \lambda|r|^{q+1} + g(x,t)r$, with $p > 1$, $q \in \mathbb{R}$ and the main assumptions $\theta \in (0,1)$ and $\alpha \in (0, \min\{1, 2\theta\})$. Notice that, by defining $v = u^m$ (and thus $u = v^{\frac{1}{m}}$), problem (P) can be formulated as

$$\begin{cases} \partial_t v^{\frac{1}{m}} - \Delta v + v^{-\frac{\beta}{m}}\chi_{\{v>0\}} = 0 & \text{in } (0,\infty)\times\Omega, \\ v = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ v(0,x) = u_0^{\frac{1}{m}}(x) & \text{in } \Omega. \end{cases}$$

Thus it corresponds to equation (4.12) with

$$\mathbf{A}(x,t,v,Dv)=Dv,\quad B(x,t,v,Dv)=0,\quad f(x,t,u)=0,\quad C(x,t,r)=v^{-\frac{\beta}{m}}\chi_{\{v>0\}}\quad\text{and}\quad \psi(v)=v^{\frac{1}{m}}.$$

Then the corresponding exponents are $\theta=\frac{1}{m}$, $\alpha=\frac{m-\beta}{m}$, and the energy method apply presented in [27] applies to the cases

$$\beta\in\begin{cases}(0,m) & \text{if } m\in[1,2],\\ (m-2,m) & \text{if } m>2.\end{cases}$$

Theorem 1 of [27] shows the finite speed of propagation and, more exactly, a stronger property which usually is as called "stable (or uniform) localization property" (see also [2, Chapter 3]). A sufficient condition for the existence of *local waiting time* (or what we can call perhaps more properly the *non-dilation of the initial support*), i.e. the free boundary cannot invade the subset where the initial datum is nonzero, was given in [27, Theorem 3]. Finally, the local quenching property (i.e. the formation of a *region where* $u=0$ even for strictly positive initial data, sometimes called also the *instantaneous shrinking of the support property*, see [2] and its references) was shown in [27, Theorem 4].

Remark 9. Let us recall that, in the case of the semilinear formulation of Lemma 5 with $\beta\geq 1$, it is known that there is a finite time blow up τ_0 of the time derivative $\partial_t u$ in the interior points $x_0\in\Omega$ where the solution quenches ($u(\tau_0,x_0)=0$) and that weak solutions cease to exist for $t>\tau_0$ (see, e.g., the exposition made in [38, 43, 47, 50, 52]). Nevertheless, it is possible to show that, in the case in which the singularity is automatically present on the boundary of Ω from the initial time $t=0$, the existence of a very weak solution can be obtained at least until the time in which the solution also quenches in some interior point $x_0\in\Omega$. The main reason of this fact is that the weight $\delta(x)=d(x,\partial\Omega)$ used in the definition of very weak solution, when asking that $u^{-\beta}\chi_{\{u>0\}}\in L^1(0,T;L^1_\delta(\Omega))$, allows to compensate the singularity arising in the boundary (but obviously it is ineffective for singularities arising in the interior of the domain Ω). In fact, the above compensation of the boundary singularity, when $\beta\geq m$, with the weight $\delta(x)$, was already pointed out in parts (iii) (a) and (b) of Lemma 6. A global example which requires some additional assumptions and holds for a modified equation is

$$\partial_t u-\Delta u^m+\lambda\delta^\nu(x)u^{-\beta}\chi_{\{u>0\}}=0\quad\text{in } (0,\infty)\times\Omega$$

for some suitable values of $\lambda>0$ and $\nu>1$. This corresponds to an easy adaptation to the framework of the slow diffusion with a singular term some of the results announced in [31] and [58, Section 7] concerning the associate semilinear problems.

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