Fluid Dynamics

Theoretical and Computational Approaches

Second Edition

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Preface

This book is written as a graduate text for those students in fluid dynamics who have already acquired sufficient knowledge of physical concepts and want to learn the theoretical aspects leading to constructive techniques and results. The treatment of the subject of fluid dynamics from a theoretical viewpoint is important because it forms the basis for further advances. The aim in this book is to provide a connected treatment of the subject of fluid dynamics with due emphasis on the physical interpretations of the derived results. To attain this aim, and to have sufficient generality of the derived results, the method of vector and tensor analysis has been used starting from a level which even the beginners should have no difficulty in understanding. It is my opinion that tensor analysis creates a way of thinking and the gain of both the formulational and computational aptitudes overshadows the initial difficulties one may encounter in becoming familiar with it. After the initial development, the main thrust is on tackling the problems in incompressible and compressible laminar and turbulent flows.

The impetus of the work reported in this book is derived from the availability of high speed and large memory digital computers which have brought into sharp focus the need to pose and solve complicated fluid dynamic problems. Those problems which, a decade ago, were thought to be of only academic interest due to their computational intractability, are now routinely considered with renewed vigor without sacrificing their mathematical completeness or generality. In the light of these observations, there is a need for a text which, without abandoning the classical results, rises quickly to a level that conforms with the present day demand of mathematical and computational sophistications. The basic philosophy followed in this book is that the foundational aspects of fluid dynamics must first be developed in a simple manner without sacrificing completeness. This step then paves the way for the conversion of the developed principles into analytical forms, and finally to derive useful physical and computational results from these forms.

It is a generally accepted notion that a single text cannot cover all the topics and, therefore, cannot fulfill all the needs and aims of a course in fluid dynamics. Nevertheless, the present text sufficiently covers those topics in a comprehensive manner which are, and will remain, of much importance in fluid dynamics. In this text great emphasis has been put on the fundamental techniques and results because they form the basis for computational and experimental results. At various places, computational techniques leading to computer alogrithms have also been introduced to enhance the knowledge of ideas underlying the basic concepts, and to emphasize the inherent nature of computational schemes in fluid dynamics. For the purpose of providing immediate help in vectors and tensors, and for other small mathematical details, a series of mathematical expositions (abbreviated as MEs) have been provided. These MEs are meant to supplement the main text and are not intended to replace the available mathematical texts on various subjects. Material from the MEs can be assigned before the coverage of any chapter so as to spend more class time on the main subject matter.

The arrangement of topics and subtopics in a work covering extensive material is usually open to criticism. Suggestions for improvement, comments, and errors from interested readers, including students, will be greatly appreciated.

The work presented in this book could not have been completed without the help and encouragment of many people. I am deeply indebted to Charles B. Cliett for his constant encouragement and to a number of graduate students, both past and present, who, by their inquiries and discussions, made the writing of this book an intellectually satisfying experience. In particular, I am grateful to Kyle Anderson, Bob Barnard, Murali Beddhu, David Bridges, Walid Chakroun, Mohammad Hosni, Hyun Kim, and George Koomullil. Most of the sketches were drawn by Murali Beddhu for which I am grateful. My grateful thanks also go to Earl

Jennings and Saif Warsi for extensive technical discussions and for offering their opinions on portions of the manuscript. In the printing stage, the staff of CRC Press has been very cooperative. In particular, I am grateful to Russ Hall, the Engineering Editor, and Rosi Larrondo of the Editorial Department, and to Andrea Demby of the Editing, Design, and Production Department at CRC Press for doing a magnificent job on a difficult manuscript. My grateful thanks go to Bobbie White and Susan Price for their diligent typing and to Linda Kidd for a score of typographical works.

Last, but not least, I am thankful to my wife, Amina, for her encouragement, understanding, and for relinquishing long hours to me.

This second edition of the book follows the same plan and scope of the first, but some important topics of practical interest have been added, either by inserting a new subsection or through an example problem. Some additional exercises have also been added. Significantly new material on nonlinear turbulence modeling applicable to both the incompressible and compressible flows has been added. As far as possible the typographical errors and my own omissions in the first edition have been corrected.

I am extremely grateful to Mr. Robert Stern, Executive Editor of Mathematics and Engineering, for suggesting that we write a second edition of the book, and for his help and encouragement in this regard. Professor Charles L. Merkle, of Pennsylvania State University, and Professor Stanley Middleman, of the University of California at San Diego, deserve my sincere gratitude for their comments and suggestions.

My cordial thanks go to the staff of the CRC Press with whom it is always a pleasure to work.

Z. U. A. Warsi

Aim and Scope of the Book

This book is intended for graduate students in both engineering and applied mathematics whose main aim is to grasp and assimilate a constructive framework of the subject of modern fluid dynamics. By the word "constructive," I mean a logical and deductive framework which eventually leads to useful physical and computational results. This approach is in contrast to a purely illustrative and intuitive one in which the student is able to understand some aspects of the physical behaviors of fluid motion but is not trained to tackle new problems due to a lack of knowledge in some useful tools. This book fills that gap. All essential aspects of fluid dynamics have been touched upon in a way so that, at the end, the student has a set of algorithmic tools at his/her disposal to tackle other problems. The theoretical tools needed here are not at all advanced, and those needed have been collected in the book as "Mathematical Expositions," (MEs). To embrace a wider audience, drawing students and readers from all branches of engineering and applied mathematics, it must again be emphasized that the present book, by its very nature, is theoretical but not mathematical.

For ease in handling the material contained in the book it is essential to make Chapters 1 and 2 as requisites in the development of a course content for graduate students. These first two chapters lay down the foundations of theoretical fluid dynamics in a form which is free from any coordinate system and essentially covers all the basic equations. Chapter 3 is devoted to the technique of writing the Navier-Stokes equations and the Euler's equations in general steady and nonsteady curvilinear coordinates. This chapter also discusses the essential aspects of vorticity and stream functions. Chapter 4 is devoted to the inviscid incompressible and compressible flows. The development in this chapter is essentially of an elementary nature since it is meant to be a prelude to viscous flows. Thus, this chapter must be considered as an elementary treatment of inviscid flow to supplement the treatment of the boundary layer theory. Chapter 5 treats the exact solutions of the Navier-Stokes equations, classical and modern boundary layer theory, and incompressible and compressible forms of the Navier-Stokes formulations. The compressible formulation is preceded by a discussion of the hyperbolic system of partial differential equations. The idea of characteristics is introduced through solutions of the Burger's and related equations and not by a very elaborate theory. In Chapter 5 both the thin layer and the parabolized Navier-Stokes equations have also been introduced. Chapter 6 covers the stability problem of the laminar flows and the development of both the time and mass-weighted averaged Navier-Stokes and the associated turbulence equations. Some engineering results from the two-dimensional turbulent boundary layers have been discussed as a prelude to the physical consequences of solving the turbulence model equations.

Each chapter has a set of exercises at the end which either extend the ideas developed in the main body of the chapter or introduce those ideas which were not covered. This way a student can start on some research while studying the pertinent material. As far as possible only "source" references have been quoted with some exceptions in Chapters 5 and 6. Most of the figures have been taken from the books and articles of various authors which have been duely referenced in the figure captions.

Important Nomenclature

a	Speed of sound
c_p	Pressure coefficient
c_f	Skin friction coefficient
C_p	Specific heat at constant pressure
C_v	Specific heat at constant volume
$\mathbf{D}$	Rate-of-strain tensor
e	Internal energy per unit mass
e_t	Total energy: $e_t = e + \frac{1}{2}\lvert\mathbf{u}\rvert^2$
h	Enthalpy per unit mass
H	Total enthalpy per unit mass, or shape parameter; $H = h + \frac{1}{2}\lvert\mathbf{u}\rvert^2$, or $H = \delta^*/\theta$
ℓ	Mean free path, or a length scale in turbulence
s	Entropy per unit mass, or arc length
$\mathbf{T}$	Complete stress tensor
$\mathbf{W}$	Vorticity tensor
Γ	Circulation of velocity,or a closed curve
ϵ	Sum of internal, kinetic, and potential energies per unit mass, or dissipation of turbulence energy
ϵ_d	Total dissipation of turbulence energy
λ	Form parameter
Λ	Pohlhausen form parameter
$\boldsymbol{\sigma}$	Deviatoric stress tensor
$\boldsymbol{\tau}$	Stress vector
Φ	Dissipation of energy due to viscosity (dissipation function)
$\boldsymbol{\omega}$	Vorticity vector

Suggested Coverage

As has been mentioned in the Preface, it is highly unlikely that a single text can achieve the complete aim of a graduate course. Nevertheless, the following breakdown may help a teacher/student to form a course from this book.

I. One Semester* Course in Viscous Incompressible Flow
   (a) Chapters 1 and 2; all sections.
   (b) Chapter 3; sections 3.1, 3.3, 3.5–3.10, may be stopped at orthogonal coordinates. Further section is optional; 3.11, 3.12.
   (c) Chapter 5; sections 5.1, 5.4–5.10, 5.12–5.15, 5.19–5.21 (optional), 5.23.
   MEs: 1, 2, 3, and 4 (optional), 5, 6, and 8 (for CFD courses), 9 (optional).

II. One Semester Course in Viscous Compressible Flow
   (a) Chapters 1 and 2; all sections
   (b) Chapter 3; sections 3.1, 3.2, 3.5–3.10.
   (c) Chapter 4; all sections. Material from References 8,9, and 10 of Chapter 4 should be selected to broaden certain topics.
   (d) Chapter 5; sections 5.22, 5.24, 5.25, 5.27, and 5.28 (optional)
   MEs: 1, 2, 3, 4, 6, 8.

III. One Semester Course in Basic Turbulence and Turbulence Modeling
   (a) Review of Chapter 3, particularly those equations which are needed again in Chapter 6.
   (b) Chapter 6; sections 6.1–6.9, and 6.10 (if emphasis is on isotropic turbulence), 6.12–6.14, 6.15–6.20, 6.21 (optional), 6.22–6.32. Material from References 4, 7, 11, 12, 16, 29, 79, and 80 of Chapter 6 should be selected to broaden certain topics.
   MEs: 1, 2, 3, 4 (optional), 5, 6, and 9.

* A semester is roughly considered to be of four months duration, meeting three times a week.

Dedicated to the loving memory of my parents

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CHAPTER ONE

Kinematics of Fluid Motion

1.1 INTRODUCTION TO CONTINUUM MOTION

The analysis of fluid motion assumes that the body of fluid under consideration forms a physical continuum. A physical continuum is a medium filled with a continuous matter such that every part of the medium, however small, is itself a continuum and entirely filled with the matter. Since matter is composed of molecules, the continuum hypothesis implies that a very small volume will contain a large number of molecules. For example, 1 cm^3 of air contains 2.687×10^{19} molecules under normal conditions (Avogadro hypothesis). Thus, in a cube 0.001 cm on a side, there are 2.687×10^{10} molecules — which is a large number. We are not interested in the properties of each molecule at some point P but rather in the average over a large number of molecules in the neighborhood of the point P. Mathematically, the association of averaged values of properties at a point P also gives rise to a continuum of points and numbers. In summary, the continuum hypothesis implies the postulate: "*Matter is continuously distributed throughout the region under consideration with a large number of molecules even in macroscopically small volumes.*"

Though the single postulate of continuum mechanics satisfactorily describes fluid motion, it is imperative to consider some statistical aspects of molecular motions. These considerations distinguish different continua by their physical properties. The most common fluids are either gases or liquids. In gases, the molecules are far apart having an average separation between the molecules of the order of 3.5×10^{-7} cm. The cohesive forces between the molecules are weak. The molecules randomly collide and exchange their momentum, heat, and other properties and thus give rise to viscosity, thermal conductivity, etc. These effects, though molecular in origin, are considered the physical properties of the continuum itself. In liquids, the separation between the molecules is much smaller and the *cohesive forces* between a molecule and its neighbors are quite strong. Again, the averaged molecular properties resulting from these cohesive forces are taken as the properties of the medium. While air and water are treated through the same continuum hypothesis, the effects of their motions are different due to the differences in their molecular properties, e.g., viscosity, thermal conductivity, etc.

1.2 FLUID PARTICLES

In considering the motion of fluids, it is helpful to keep an infinitesimal volume of fluid as a geometrical point in a mathematical continuum of numbers and call it a *fluid particle*. Each fluid particle at a given moment of time, can be associated with an ordered triple of numbers, e.g., (a, b, c), and its motion followed in time. The state properties at the position (a, b, c) are called the state properties of the fluid particle itself at $t = t_0$ — thus giving a unique identity to this particle. As this particle moves about, its state properties will be understood to be the same as the local state properties of the continuum.

1.3 INERTIAL COORDINATE FRAMES

It is sufficient to consider Euclidean space to describe the laws of fluid motion. An Euclidean space is a curvature-free space in which a set of rectangular Cartesian coordinates can always be introduced on a global scale. One can introduce any other system of coordinates in this space without altering the basic nature of the space itself. These ideas form a logical environment for understanding coordinate transformation in Newtonian mechanics.

In fluid-dynamics, the speeds encountered are far smaller than the speed of light so that the relativistic effects are negligibly small. Time is considered as an absolute entity irrespective of the state of motion, and Newton's laws of motion are assumed to be exact. The concept of an inertial frame of reference is very important. It is precisely defined by Newton's first law of motion. It is a coordinate frame with respect to which bodies, under the absence of external forces, move with zero acceleration. (For transformation between reference frames, refer to ME.9.)

1.4 MOTION OF A CONTINUUM

The continuum hypothesis postulates fluid particles or the *material points* distributed continuously. To describe any kind of motion, a reference coordinate system is needed. Since fluid motion does not require any relativistic considerations, we take time as an absolute quantity common to any reference frame at rest or in motion. For simplicity, choose a reference frame at rest and refer the motion of all particles with respect to it. This coordinate system can be a curvilinear coordinate system. In what follows we shall consider an inertial frame for the description of fluid motion, as shown in Figure 1.1.

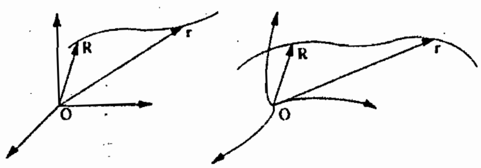

Fig. 1.1 Inertial Cartesian or curvilinear reference frame.

The material points at the fixed reference time t_0 or at any time t can now be located with respect to this inertial coordinate system by drawing position vectors from point o. Let **R** and **r** denote the position vectors of the fluid particles at times t_0 and t, respectively.* The vectors **R** and **r** can be denoted as an ordered triple of numbers in any of the following forms (refer to ME.1 for details):

$$\mathbf{R} = (X, Y, Z) = (X_i) = (X^i),$$

$$\mathbf{r} = (x, y, z) = (x_i) = (x^i)$$

where **R** is the reference configuration at t_0, the reference time.

In this text subscripted variables X_i, x_i denote the rectangular Cartesian coordinates, while X^i, x^i denote any general nonrectangular coordinates. Thus **r** referred to a rectangular Cartesian system is

$$\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = x_k\mathbf{i}_k \tag{1.1a}$$

where any two repeated indices imply summation, and $\mathbf{i}_k$ for $k = 1, 2, 3$ are the constant unit vectors along x_1, x_2, and x_3 axes, respectively. For general curvilinear coordinates (x^1, x^2, x^3) the position vector **r** is a function of these coordinates, i.e., $\mathbf{r} = \mathbf{r}(x^i)$, and it is not possible to write **r** linearly in terms of x^i as in Equation 1.1a. It must, however, be noted that the first differential $d\mathbf{r}$ is expressible linearly in dx^i for all coordinates as:

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial x^m}\, dx^m \text{ (sum on } m) \\ &= \mathbf{a}_m\, dx^m \end{aligned} \tag{1.1b}$$

* At some places we have also used the symbol **x** in lieu of **r**.

and $\mathbf{a}_m$ are called the *covariant base vectors*. If x^m are the rectangular Cartesian coordinates, i.e.:

$$x^1 = x_1, \qquad x^2 = x_2, \qquad x^3 = x_3$$

then from Equation 1.1a we find that:

$$\mathbf{a}_m = \mathbf{i}_m, \quad m = 1, 2, 3 \tag{1.1c}$$

where $\mathbf{i}_m$ are the constant unit vectors. (Refer to ME.1.)

In this text, we will use both the rectangular Cartesian and general curvilinear coordinates. From this discussion, the reader should have no difficulty in deciding about the implied coordinate system. Functions of the coordinates will be written as $F(\mathbf{R})$ or $F(\mathbf{r})$, etc.

Suppose at time $t = t_0$ we isolate a portion of the continuum and denote its bounding surface by S_0, as shown in Figure 1.2. A specific particle in the region enclosed by S_0 can be named $X_1 = a$, $X_2 = b$, $X_3 = c$, where (a, b, c) are the Cartesian coordinates of a particle at $t = t_0$. As this particle moves, its position at some other time $t > t_0$ can be located with respect to the reference system by (x_1, x_2, x_3), where:

$$x_i = \phi_i(a, b, c, t), \qquad i = 1, 2, 3 \tag{1.2}$$

The functions ϕ_i are to be determined by the laws of fluid motion. (The determination of ϕ_i is actually the "ultimate problem" of fluid dynamics, and the functions are not known *a priori*.) In the same manner any other particle labeled by its coordinates can be followed to a time $t > t_0$. Collectively, all these particles at time $t > t_0$ will be enclosed by a surface S_t.

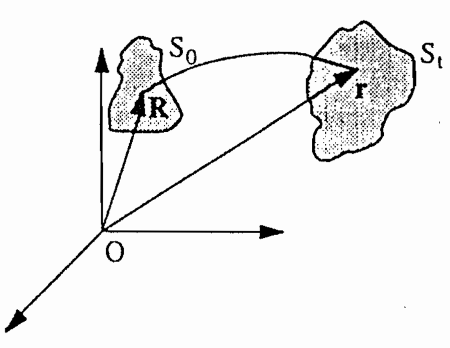

Fig. 1.2 Motion of an imaginary volume in a continuum.

This notion of mapping points at $t = t_0$ enclosed by an imaginary surface S_0 onto the points of the region enclosed by S_t at $t > t_0$ is of much help. This mapping should be one-to-one for distinct points to remain distinct. That is, two or more points enclosed by S_0 should not map onto a single point of the region enclosed by S_t.

Collectively, consider all the points at time $t = t_0$ and follow their motion in time $t > t_0$. The law of fluid motion should then be written as:

$$x_i = \phi_i(X_1, X_2, X_3, t), \qquad i = 1, 2, 3 \tag{1.3}$$

so that (X_1, X_2, X_3) can be any fluid particle at $t = t_0$. In vector form Equation 1.3 is written as:

$$\mathbf{r} = \boldsymbol{\phi}(\mathbf{R}, t) \tag{1.4}$$

where the vector-valued function ϕ must be such that at the time $t = t_0$, the position vectors $\mathbf{r}$ and $\mathbf{R}$ coincide. Thus:

$$\mathbf{r}(\mathbf{R}, t_0) = \phi(\mathbf{R}, t_0) = \mathbf{R} \tag{1.5}$$

The coordinates (X^i or X_i) are usually called the material or *Lagrangian coordinates*, and (x^i or x_i) are called the spatial or *Eulerian coordinates*. The following interpretation is in order:

1. If $\mathbf{R}$ is kept fixed and t varies, then Equation 1.4 represents the path or the trajectory followed by the chosen particle through time.
2. If t is kept fixed, then Equation 1.4 represents the transformation of coordinates occupied by the fluid particles at $t = t_0$ to the coordinates at time t. This transformation is generally nonlinear. Since the space remains Euclidean at $t = t_0$ and $t > t_0$, Equation 1.4 transforms an Euclidean space into itself. In essence, the map $\mathbf{F}$ defined as $\mathbf{F} : \mathbf{R} \rightarrow \phi(\mathbf{R}, t)$ is the fluid flow map.

Besides ϕ being a one-to-one mapping of points, the function ϕ in Equation 1.4 must be spatially differentiable to some specified order. We will consider ϕ to be single valued and at least first order differentiable in the spatial variables. Let the inverse of Equation 1.4 be written as:

$$\mathbf{R} = \psi(\mathbf{r}, t) \tag{1.6}$$

For Equation 1.6 to exist, the Jacobian J of the map $\mathbf{F}$ defined below must not vanish at any point of the flow field:

$$J(\mathbf{R}, t) = \det\left(\frac{\partial x_i}{\partial X_j}\right) \neq 0 \tag{1.7a}$$

Another way of defining J is to introduce the gradient operator "Grad" in the reference configuration $\mathbf{R}$ and then a second order tensor or matrix $\mathbf{G}$ is obtained as:

$$\mathbf{G} = \text{Grad } \mathbf{r}$$

Thus:

$$J = \det(\mathbf{G})$$

In fully expanded form Equation 1.7a is

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\[2ex] \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\[2ex] \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} \neq 0 \tag{1.7b}$$

The condition $J \neq 0$ is known as the *smoothness condition* of fluid motion. Because of the continuity and differentiability of ϕ and ψ in Equations 1.4 and 1.6, we have:

$$\phi[\psi(\mathbf{r}, t), t] = \mathbf{r} \tag{1.8a}$$

$$\psi[\phi(\mathbf{R}, t), t] = \mathbf{R} \tag{1.8b}$$

$$J(\mathbf{R}, t_0) = 1 \tag{1.8c}$$

1.5 THE TIME DERIVATIVES

Suppose F to be either a scalar, vector, or tensor function of position and time representing some physical property of the flow. We can write it as $F(\mathbf{r}, t)$ or $F(\mathbf{R}, t)$ since the inverse of the transformation from $\mathbf{r}$ to $\mathbf{R}$ or $\mathbf{R}$ to $\mathbf{r}$ is assumed to exist. Physically, $F(\mathbf{R}, t)$ is the value of F experienced by a fluid particle at time t which at time t_0 was at $\mathbf{R}$, and $F(\mathbf{r}, t)$ is the value experienced by an observer at the fixed position $\mathbf{r}$ at time t. There are two possible time derivatives:

$$(i) \qquad \frac{\partial F}{\partial t} = \left.\frac{\partial F(\mathbf{r}, t)}{\partial t}\right|_{\mathbf{r}\,\text{fixed}} \tag{1.9a}$$

$$(i) \qquad \frac{dF}{dt} = \left.\frac{\partial F(\mathbf{R}, t)}{\partial t}\right|_{\mathbf{R}\,\text{fixed}} \tag{1.9b}$$

Here $\partial F/\partial t$ is the rate of change of F measured by an observer stationed at the fixed point $\mathbf{r}$ and is a local time variation of F. On the other hand, dF/dt is the rate of change of F measured by an observer moving with the fluid particle. The time derivative dF/dt is called the *material* or *substantive derivative*, or the *derivative following the motion*. The reason for these names is based on the fact that $F(\mathbf{R}, t)$ is purely a function of time and its derivative is a total derivative.

1.6 VELOCITY AND ACCELERATION

Any description of fluid motion using the Lagrangian coordinates $\mathbf{R} = (X_i)$ and time t is called a description in the *sense of Lagrange*. From an intuitive viewpoint, one should start describing the velocity and acceleration in the sense of Lagrange, as is done in the classical particle dynamics. It must be noted in this context that the $\mathbf{r}$ defined in Equation 1.4 represents the changing coordinates of a moving fluid particle. The velocity of a fluid particle at time t, which was at the position $\mathbf{R}$ at time $t = t_0$, is obtained by differentiating Equation 1.4, i.e.:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial\phi(\mathbf{R}, t)}{\partial t} = \mathbf{U}(\mathbf{R}, t) \tag{1.10}$$

and is called the *Lagrangian velocity*. Also:

$$d\mathbf{r} = \mathbf{U}(\mathbf{R}, t)\, dt$$

which on integration and using Equation 1.5 yields the particle trajectory:

$$\mathbf{r} = \mathbf{R} + \int_{t_0}^{t} \mathbf{U}(\mathbf{R}, t)\, dt \tag{1.11}$$

As is shown in Equation 1.10, $\mathbf{U}(\mathbf{R}, t)$ is a function of the material variables. Introducing the transformation (Equation 1.6) in $\mathbf{U}$, we get:

$$\mathbf{U}[\psi(\mathbf{r}, t), t] = \mathbf{u}(\mathbf{r}, t)$$

Thus:

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t) \tag{1.12}$$

is the absolute velocity of the fluid with respect to an inertial frame of reference. The dependence of $\mathbf{u}$ on $\mathbf{r}$ and t in Equation 1.12 shows that it is a function of the spatial or Eulerian coordinates and time, and is called the *Eulerian velocity*. It is important to note that in the Lagrangian description of fluid motion $\mathbf{r}$ is treated as a function of time but in the Eulerian description $\mathbf{r}$ is a fixed point in space. In component form Equation 1.12 can be written as follows:

1. Rectangular Cartesian coordinates

$$\frac{dx_m}{dt} = u_m(x_1, x_2, x_3, t), \qquad m = 1, 2, 3 \tag{1.13a}$$

where u_m are the Cartesian components of $\mathbf{u}$.

2. General coordinates

$$\frac{dx^i}{dt} = u^i(x^1, x^2, x^3, t), \qquad i = 1, 2, 3 \tag{1.13b}$$

where u^i are the contravariant components of $\mathbf{u}$. Refer to ME.1 for a thorough grounding in these topics.

Acceleration is the rate of change of velocity of a fluid particle. From Equation 1.10, the acceleration vector is given by:

$$\mathbf{A}(\mathbf{R}, t) = \frac{\partial \mathbf{U}(\mathbf{R}, t)}{\partial t} = \frac{d\mathbf{U}}{dt} \tag{1.14}$$

which is called the *Lagrangian acceleration*. Again making use of Equation 1.6 in Equation 1.14, we have:

$$\mathbf{A}[\boldsymbol{\psi}(\mathbf{r}, t), t] = \mathbf{a}(\mathbf{r}, t)$$

and

$$\mathbf{a}(\mathbf{r}, t) = \frac{d}{dt}\mathbf{u}(\mathbf{r}, t) \tag{1.15}$$

which is called the *Eulerian acceleration*. In Equation 1.15, the differential operator *d/dt* stands for the total time derivative.

Now realizing from Equation 1.4 that $\mathbf{r}$ is a function of time in the sense of Lagrange, hence $\mathbf{u}(\mathbf{r}, t)$ is both an explicit and implicit function of time. Using the chain rule of partial differentiation, we have:

$$\begin{aligned}\mathbf{a}(\mathbf{r}, t) &= \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{(\mathbf{r}\ \text{fixed})} + \frac{\partial \mathbf{u}}{\partial x_j}\frac{dx_j}{dt} \\ &= \frac{\partial \mathbf{u}}{\partial t} + u_j \frac{\partial \mathbf{u}}{\partial x_j}\end{aligned}$$

and after this operation the transformation from the Lagrangian to the Eulerian coordinates is complete. In the sense of Euler, **r** represents fixed points of space representing field points.

Introducing the "grad" operator in the Cartesian coordinates, Equation M1.8, i.e.:

$$\nabla = \text{grad} = \mathbf{i}_k \frac{\partial}{\partial x_k}$$

we obtain:

$$\mathbf{a}(\mathbf{r}, t) = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} \tag{1.16}$$

Introducing the tensor grad **u** (cf. Equations M1.10b and M1.22) we also have:

$$(\mathbf{u} \cdot \text{grad})\mathbf{u} = (\text{grad } \mathbf{u}) \cdot \mathbf{u}$$

and consequently:

$$\mathbf{a}(\mathbf{r}, t) = \frac{\partial \mathbf{u}}{\partial t} + (\text{grad } \mathbf{u}) \cdot \mathbf{u} \tag{1.17}$$

The substantive derivative operator in the Eulerian variables is then:

$$\frac{d}{dt} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \text{grad} \tag{1.18a}$$

$$= \frac{\partial}{\partial t} + [\text{grad}(\quad)] \cdot \mathbf{u} \tag{1.18b}$$

The part **u** · grad or [grad( )] · **u** is the convective part of D/Dt. Although (D/Dt) and d/dt are the same operators, the notation D/Dt introduced by Stokes cautions one to consider both the local and convective operations simultaneously for field points. Obviously, the substantive differential operator is

$$D = dt \frac{\partial}{\partial t} + (d\mathbf{r} \cdot \text{grad}) \tag{1.19a}$$

Thus:

$$D\mathbf{u} = dt \frac{\partial \mathbf{u}}{\partial t} + (d\mathbf{r} \cdot \text{grad})\mathbf{u} \tag{1.19b}$$

and:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{a}(\mathbf{r}, t) \tag{1.20}$$

Another method of obtaining the Eulerian acceleration vector is to consider the motion of a fluid particle at time t and $t + \Delta t$. At time t the particle is at **r**, while at $t + \Delta t$ it is at $\mathbf{r} + \Delta\mathbf{r}$, where it is important to note that $\Delta\mathbf{r} = \mathbf{u}\Delta t$. The new particle velocity is $\mathbf{u} + \Delta\mathbf{u}$, where:

$$\mathbf{u} + \Delta\mathbf{u} = \mathbf{u}(\mathbf{r} + \Delta\mathbf{r}, t + \Delta t)$$

Expanding by Taylor's theorem, we have:

$$\mathbf{u} + \Delta\mathbf{u} = \mathbf{u} + (\Delta\mathbf{r} \cdot \text{grad})\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}\Delta t + \cdots$$

$$= \mathbf{u} + (\mathbf{u}\Delta t \cdot \text{grad})\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}\Delta t + \cdots$$

Thus the principal part of the change in velocity is

$$\Delta\mathbf{u} = \left\{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u}\right\}\Delta t$$

In the limit as $\Delta t \rightarrow 0 : \Delta\mathbf{u} \rightarrow D\mathbf{u}$, so that:

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} \tag{1.21}$$

Example 1.1

Consider a unidirectional fluid flow which has been started from rest at time $t = 0$. Let a material or fluid particle be at a distance X_1 from the origin at $t = 0$, and it begins to move with a constant acceleration αX_1, where α is a constant having the dimension $1/(\text{time})^2$. Find the Lagrangian and Eulerian velocities, accelerations, and the particle position for $t > 0$.

Obviously:

$$A_1(X_1, t) = \frac{dU_1}{dt} = \alpha X_1$$

so that:

$$U_1(X_1, t) = \alpha X_1 t = \frac{dx_1}{dt}$$

On integration, we get:

$$x_1(X_1, t) = X_1 + \tfrac{1}{2}\alpha X_1 t^2$$

or:

$$X_1(x_1, t) = \frac{x_1}{1 + \frac{1}{2}\alpha t^2}$$

The solution is

$$A_1 = \alpha X_1, \qquad a_1 = \frac{\alpha x_1}{1 + \frac{1}{2}\alpha t^2}$$

$$U_1 = \alpha X_1 t, \qquad u_1 = \frac{\alpha x_1 t}{1 + \frac{1}{2}\alpha t^2}$$

$$X_1 = x_1/(1 + \tfrac{1}{2}\alpha t^2), \qquad x_1 = X_1(1 + \tfrac{1}{2}\alpha t^2)$$

1.7 STEADY AND NONSTEADY FLOW

It is apparent from the preceding analysis that fluid motion is always described with the aid of the variable t representing time. For example, the transformation of variables from the fixed system **R** to the transformed system **r** was achieved using the time variable t. Therefore, t is always implicit in the description of fluid motion.

When fluid motion is described purely in terms of the Eulerian variables **r** and t, then time can appear explicitly too, depending on the state of motion. This time dependency can also show up depending on how the coordinates **r** have been chosen in observing the fluid motion. For example, the motion of water produced by a steadily moving boat in a river is nonsteady when the coordinates **r** are attached with the bank of the river, but the same motion is steady if observations are made by attaching the coordinates with the moving boat. In the same problem, if the boat is moving unsteadily, or the flow in the river is nonsteady, then the explicit dependence on time cannot be neglected.

We, therefore, define a *steady flow* as the one in which there is no explicit dependence on time and the substantive derivative operator (Equation 1.18a) is only a convective operator given by:

$$\frac{D}{Dt} = \mathbf{u} \cdot \text{grad}$$

If time appears explicitly, then the flow is *unsteady* or *nonsteady*.

1.8 TRAJECTORIES OF FLUID PARTICLES AND STREAMLINES

The definition of fluid velocity vector, Equation 1.12, represents a system of first order ordinary differential equations. One needs to specify a set of initial conditions for a unique solution which are also available from Equation 1.5. Therefore, we have:

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t) \tag{1.22a}$$

$$\mathbf{r}(t_0) = \mathbf{R} \tag{1.22b}$$

A subsidiary system of equations from Equation 1.22a can be written by noting that for Cartesian coordinates $d\mathbf{r} = \mathbf{i}_k dx_k$, $\mathbf{u} = \mathbf{i}_k u_k$, so that:

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3} = dt \tag{1.22c}$$

while for curvilinear coordinates $d\mathbf{r} = \mathbf{a}_k dx^k$, $\mathbf{u} = \mathbf{a}_k u^k$, so that:

$$\frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} = dt \tag{1.22d}$$

where $\mathbf{a}_k$ are the covariant base vectors and u^k are the contravariant components of **u**.

A unique solution of Equation 1.22a or of its subsidiary counterpart is called a *trajectory* or *pathline* of a fluid particle. At every point of the pathline of a fluid particle, the velocity vector is necessarily tangential to the pathline.

Now realizing that $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$, therefore, for each fixed time, there are curves in a flow field on which $\mathbf{u}$ is tangent at every point. These curves are called the *streamlines*. Thus, a curve drawn in a flow field at a fixed time on which the velocity is tangent everywhere is called a streamline. The equations of a streamline at a fixed time t are obtained by the equation:

$$\mathbf{u} \times d\mathbf{r} = 0 \tag{1.23a}$$

which yields the equations:

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3} \tag{1.23b}$$

in the Cartesian coordinates, and:

$$\frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} \tag{1.23c}$$

in the curvilinear coordinates. (Refer to Equation M1.109 for the cross product of base vectors.) At any fixed time, the set of all streamlines constitutes a *flow pattern*. These flow patterns are different for different times.

From the definitions of the pathlines and streamlines, we conclude that the pathline of a fluid particle touches one of the streamlines of the set at any given moment of time. At any other time, this pathline will touch some other streamline of the changed flow pattern. However, when the flow is steady, the flow pattern is fixed and the pathlines and streamlines are the same. For a nonsteady velocity field $\mathbf{u}$, the pathlines and streamlines are same if $\mathbf{u} \times \partial\mathbf{u}/\partial t = 0$. Thus, it is possible to have steady streamlines for nonsteady flows under the condition $\mathbf{u} \times \partial\mathbf{u}/\partial t = 0$.

A closed curve drawn in any region of a field is said to be reducible if it can be shrunk to a point without going out of the region. Let streamlines be drawn from every point of this closed curve. The cylinder so generated is called a *stream tube*. A stream tube of infinitesimal cross section is called a *stream filament*.

1.9 MATERIAL VOLUME AND SURFACE

A very useful concept in fluid dynamics is that of a "closed system" or a "material volume". A *material volume* is an arbitrary collection of fluid of fixed identity and enclosed by a surface also formed of fluid particles. All points of the material volume, including the points of its boundary, move with the local continuum velocity. A material volume moves with the flow and deforms in shape as the flow progresses, with the stipulation that no mass ever fluxes in or out of the volume, viz., both the volume and its boundary are always composed of the same fluid particles. We shall denote a material volume by $\mathcal{V}(t)$ and its surface by $\mathcal{S}(t)$. Note that the use of material volume in fluid dynamics is in the form of a thought experiment in which one isolates a parcel of fluid out of the flow field and gives it a hypothetical surface. This helps formulate the conservation laws for fluid dynamics in a straightforward manner.

To obtain the necessary conditions for a bounding surface of a material volume, first note that the material volume is enclosed by a surface formed of fluid particles. This surface is called the *material bounding surface*. Since there cannot be a transfer of fluid across a material bounding surface, the fluid particles forming the inside surface of the material volume can never become the fluid particles forming the outside surface of the same material volume. Consequently, it qualifies as a material surface since it is always composed of the same material points.

Let $f(\mathbf{r}, t) = 0$ be the equation of a bounding surface $\mathcal{S}(t)$ enclosing the material volume $\mathcal{V}(t)$. Let $\mathbf{n}$ be the unit external normal to $\mathcal{S}(t)$; then:

$$\mathbf{n} = \text{grad}\, f / |\text{grad}\, f| \tag{1.24}$$

To obtain the necessary conditions for $\mathcal{S}(t)$ to be the bounding surface of a material volume, we follow Kelvin[1] who states that ". . . to express the fact that every particle of fluid remains on the same side of the surface, or that there is no flux across it, we must find the normal motion of the surface . . . ." Let v_n be the velocity of any point normal to $\mathcal{S}(t)$; then the equation of the surface at time $t + \delta t$ is

$$f(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = 0$$

where

$$\delta\mathbf{r} = \mathbf{n} v_n \delta t$$

Using Taylor's expansion, we have:

$$\frac{\partial f}{\partial t} + v_n(\mathbf{n} \cdot \text{grad}\, f) = 0$$

If the bounding surface is a material surface, then the velocity at any point normal to the surface must be equal to the normal continuum velocity, i.e., $v_n = \mathbf{u} \cdot \mathbf{n}$. Using Equation 1.24, we get:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \text{grad}\, f = 0 \tag{1.25}$$

Thus, for a bounding surface to be a material surface, the surface $f(\mathbf{r}, t)$ must satisfy Equation 1.25. Further, the set of equations (Equation 1.22a) must also have a unique solution in $\mathcal{V}(t)$. It has been shown in Example 3.7 that Equation 1.25 also holds for the kinematic condition on an impervious boundary.

1.10 RELATION BETWEEN ELEMENTAL VOLUMES

At time $t = t_0$ we know that the coordinates of any point are given by $\mathbf{R}(X_i)$. We shall denote any arbitrary closed volume in the $\mathbf{R}$-space by $V(t_0)$ so that it is a volume "frozen" in time with surface $S(t_0)$. If in $V(t_0)$ we consider a rectangular parallelepiped of sides δX_1, δX_2, δX_3, then:

$$dV(t_0) = \delta X_1 \delta X_2 \delta X_3$$

and its diagonal is a directed vector element $\delta\mathbf{R}$. After a time $t > t_0$, this rectangular parallelepiped becomes a curvilinear parallelepiped of diagonal $\delta\mathbf{r}$. From Equation 1.4, keeping t fixed, we have $\delta\mathbf{r} = (\partial\mathbf{r}/\partial X_\alpha)\delta X_\alpha$, so that the edges of the parallelepiped are $(\partial\mathbf{r}/\partial X_1)\delta X_1$, $(\partial\mathbf{r}/\partial X_2)\delta X_2$, $(\partial\mathbf{r}/\partial X_3)\delta X_3$. The volume element at time t is then:

$$d\mathcal{V}(t) = \frac{\partial \mathbf{r}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial X_2} \times \frac{\partial \mathbf{r}}{\partial X_3}\right) dV(t_0)$$

or

$$d\mathcal{V}(t) = J\, dV(t_0) \tag{1.26}$$

where J has been defined in Equation 1.7.

1.11 THE KINEMATIC FORMULAE OF EULER AND REYNOLDS

A kinematic formula of much importance, known as *Euler's formula*, is obtained by taking the substantive derivative of the Jacobian J defined in Equation 1.7. In the determinant of J, let A_i be the cofactor of $\partial x_i/\partial X_\alpha$. Then:

$$J\delta_{ji} = \frac{\partial x_i}{\partial X_\alpha} A_{j\alpha} \tag{1.27}$$

where δ_{ji} is the Kronecker symbol. Writing either $i = j = 1$, or 2, or 3, gives the value of J. Thus for $i = j = 1$:

$$J = \frac{\partial x_1}{\partial X_\alpha} A_{1\alpha} \tag{1.28}$$

Differentiating Equation 1.28 with time t and using the definition of the velocity components (Equation 1.13a), we get on rearranging the terms:

$$\begin{aligned}
\frac{dJ}{dt} &= \frac{\partial u_k}{\partial X_\alpha} A_{k\alpha} \\
&= \frac{\partial u_k}{\partial x_j}\frac{\partial x_j}{\partial X_\alpha} A_{k\alpha} \\
&= J\delta_{kj}\frac{\partial u_k}{\partial x_j} = J\frac{\partial u_k}{\partial x_k}
\end{aligned}$$

Thus (noting that div $\mathbf{u}$ is the rate of fluid dilation) Euler's formula is

$$\frac{dJ}{dt} = J \operatorname{div} \mathbf{u} \tag{1.29}$$

The preceding result can also be obtained by starting from:

$$J = e_{pqr}\frac{\partial x_1}{\partial X_p}\frac{\partial x_2}{\partial X_q}\frac{\partial x_3}{\partial X_r}$$

where e_{pqr} is the permutation symbol; e.g., refer to Equation M1.5. Note that in general Eulerian coordinates x^i, Equation 1.29 is

$$\frac{dJ}{dt} = Ju,^i_i$$

where $u,^i_i$ is the formula for div $\mathbf{u}$ in curvilinear coordinates.

The other kinematic formula is due to Reynolds and is known as the *Reynolds' transport theorem*. Let $F(\mathbf{r}, t)$ be a physical property per unit volume. The amount of this property in a material volume $\mathcal{V}(t)$ is*

$$\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv \tag{1.30}$$

which is a well-defined function of time. The rate of change of Equation 1.30 as the volume moves with the flow is

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv \tag{1.31}$$

where $dv = d\mathcal{V}(t)$. To perform the time differentiation in Equation 1.31 we introduce the system of coordinates X_α, which amounts to replacing dv by dv_0 according to Equation 1.26, where $dv_0 = d\mathcal{V}(t_0)$. Thus, Equation 1.31 becomes:

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{V}(t_0)} F(\mathbf{R}, t)J\, dv_0 &= \int_{\mathcal{V}(t_0)} \frac{d}{dt}(JF)\, dv_0 \\ &= \int_{\mathcal{V}(t_0)} \left(J\frac{dF}{dt} + F\frac{dJ}{dt}\right) dv_0\end{aligned}$$

Reverting to the material volume and using Equation 1.29, we get:.

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv = \int_{\mathcal{V}(t)} \left(\frac{DF}{Dt} + F \operatorname{div} \mathbf{u}\right) dv \tag{1.32a}$$

Using the definition of D/Dt given in Equation 1.18, we also have:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv = \int_{\mathcal{V}(t)} \left[\frac{\partial F}{\partial t} + \operatorname{div}(F\mathbf{u})\right] dv \tag{1.32b}$$

We can also have a third form by using Gauss' divergence theorem (cf. Equation M2.2):

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv = \int_{\mathcal{V}(t)} \frac{\partial F}{\partial t}\, dv + \int_{\mathcal{S}(t)} F\mathbf{u}\cdot\mathbf{n}\, dS \tag{1.32c}$$

where $\mathbf{n}$ is the unit external normal to $\mathcal{S}(t)$.

The *Reynolds' transport theorem*, therefore, states that the rate of change of the total property F contained in a material volume is equal to the volume integral of the instantaneous changes of F while keeping $\mathcal{V}(t)$ fixed at a given time, plus the surface integral of the rate of spreading of F to the adjoining region due to the surface velocity $\mathbf{u}$.

Note that by using the Reynolds' transport theorem, the Euler's formula (Equation 1.29) can be obtained as a special case. Taking $\mathcal{V}(t)$ to be infinitesimally small, say $d\mathcal{V}(t)$, one obtains from Equation 1.32a:

$$\frac{d}{dt}(F d\mathcal{V}) = \left(\frac{DF}{Dt} + F \operatorname{div} \mathbf{u}\right) d\mathcal{V}$$

* In this book, we have used the pseudo symbols dv for volume element and dS for surface element, with an occasional sub- or superscript in all integrals.

Letting $F = 1$, the above equation becomes:

$$\frac{d}{dt}(d\mathcal{V}) = (\text{div } \mathbf{u})\, d\mathcal{V}$$

which is consistent with the definition of div **u** as the rate of change of volume per unit volume. Using Equation 1.26, we get:

$$\frac{dJ}{dt} = J \text{ div } \mathbf{u}$$

which is Equation 1.29. If J = 1 then the motion is called **isochoric.**

1.12 CONTROL VOLUME AND SURFACE

Any arbitrary shaped moving or fixed volume in a flow field with imaginary or physical boundaries is called a *control volume*. We shall denote a control volume by $V^*(t)$ and its surface by $S^*(t)$. In some cases, it is useful to imagine that at a *given instant* the boundaries of a material volume coincide with the surface and space enclosed by a given control volume instantaneously .

The Reynolds' transport theorem using a control volume can be stated as follows. Let $\mathbf{c}(\mathbf{r}, t)$ be the local absolute velocity of the surface $S^*(t)$, and $\mathbf{n}^*$ be the local unit normal vector on the surface. Here the most general case for a control volume (CV) has been considered. Specific cases are

1. $\mathbf{c} = 0$ or a constant for a CV at rest or moving uniformly, respectively.
2. $\mathbf{c} = \mathbf{c}(t)$ for a CV moving nonuniformly without change in shape.
3. $\mathbf{c} = \mathbf{c}(\mathbf{r}, t)$ for a CV arbitrarily changing in shape, and as a whole may be at rest or moving uniformly or nonuniformly.

Our aim is to find an expression for the time derivative:

$$\frac{d}{dt}\int_{V^*(t)} F(\mathbf{r}, t)\, dv^* \tag{1.33a}$$

For each volume element dv^*, the contribution due to the instantaneous changes in F is $(\partial F/\partial t)\, dv^*$, and thus the total of the instantaneous changes in $V^*(t)$ at a fixed time is

$$\int_{V^*(t)} \frac{\partial F}{\partial t}\, dv^* \tag{1.33b}$$

On the other hand, the rate of spreading of the quantity F to the adjoining region due to the movement of the element dS^* per unit time is $[(\mathbf{c} \cdot \mathbf{n}^*)\, dS^*]F$, and thus the total of this contribution for the whole surface is

$$\int_{S^*(t)} F\mathbf{c} \cdot \mathbf{n}^*\, dS^* \tag{1.33c}$$

The total of Equations 1.33b and c gives the value of Equation 1.33a, which is the Reynolds' transport theorem for a control volume**, and stated as

$$\frac{d}{dt}\int_{V^*(t)} F(\mathbf{r}, t)\, dv^* = \int_{V^*(t)} \frac{\partial F}{\partial t}\, dv^* + \int_{S^*(t)} F\mathbf{c} \cdot \mathbf{n}^*\, dS^* \tag{1.34}$$

Now to establish the relationship between the time rate of change of a physical quantity associated with a material volume to the changes in a control volume, imagine that at time t the material boundary is instantaneously coincident with the bounding surface of the control volume. Then Equation 1.32c becomes:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv = \int_{V^*(t)} \frac{\partial F}{\partial t}\, dv^* + \int_{S^*(t)} F\mathbf{u} \cdot \mathbf{n}^*\, dS^* \tag{1.35}$$

Eliminating $\int_{V^*(t)} (\partial F/\partial t)\, dv^*$ between Equations 1.34 and 1.35, we get:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} F(\mathbf{r}, t)\, dv = \frac{d}{dt}\int_{V^*(t)} F(\mathbf{r}, t)\, dv^* + \int_{S^*(t)} F(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^*\, dS^* \tag{1.36}$$

which is the required relation. Note that if $\mathbf{c} = \mathbf{u}$, then $V^*(t) = \mathcal{V}(t)$, while if $\mathbf{c} = 0$, then V^* is a fixed volume. It must be stressed that the operator d/dt on the first term of the right-hand side of Equation 1.36 means that the time differentiation is to be performed after the volume integral has been evaluated. If V^* is not dependent on time, i.e., the control surface is nondeformable, then the first term on the right-hand side of Equation 1.36 must be interpreted as the time rate of change of F within the control volume. It is then appropriate to use $\partial/\partial t$ in place of d/dt.

1.13 KINEMATICS OF DEFORMATION

The analysis of fluid motion depends on the concept that all the physical and kinematic quantities pertaining to the fluid, e.g., the velocity, pressure, temperature, etc. form a field of values. In a field, the quantities are distributed according to certain physical laws which are peculiar to that field. In this section, we are interested in the distribution of velocity in a fluid medium so as to uncover those kinematic quantities which play an important role in the structure of the field itself.

Consider two neighboring points P and Q. What mathematical expression is required to predict the velocity at Q if we are given the velocity and its first partial derivatives at P? Referring to Figure 1.3, let the coordinates of P be represented as $\mathbf{r}$ and those of Q as $\mathbf{r} + \delta\mathbf{r}$.

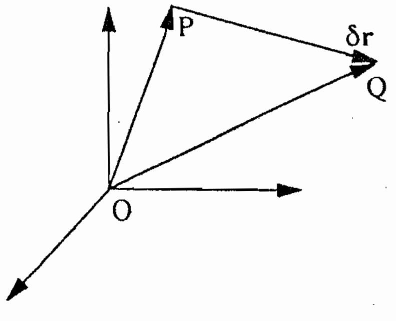

Fig. 1.3 Two closely spaced points, P, Q separated by the vector $\delta\mathbf{r}$.

** Following the same logic one can obtain the transport theorem for a material volume just by replacing $\mathbf{c}$ with $\mathbf{u}$ in Equation 1.33c.

The change in the velocity between points P and Q, or the relative velocity of Q with respect to P at a given time is obviously:

$$\mathbf{u}_Q - \mathbf{u}_P = \mathbf{u}(\mathbf{r} + \delta\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)$$

Applying Taylor's expansion and retaining only the first order terms in $\delta\mathbf{r}$, we have:

$$\delta\mathbf{u} = (\text{grad } \mathbf{u}) \cdot \delta\mathbf{r} \tag{1.37}$$

where as explained in ME.1, grad $\mathbf{u}$ is a second order tensor. The representation* of grad $\mathbf{u}$ in the Cartesian coordinates is

$$\nabla\mathbf{u} = \text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_k}\, \mathbf{i}_k \tag{1.38a}$$

while in the curvilinear coordinates, it is

$$\nabla\mathbf{u} = \text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^k}\, \mathbf{a}^k \tag{1.38b}$$

Since the tensor grad $\mathbf{u}$ is not a symmetric tensor, with the addition and subtraction of its transpose (grad $\mathbf{u})^{\mathrm{T}}$, we can write Equation 1.37 as:

$$\delta\mathbf{u} = (\mathbf{D} + \mathbf{W}) \cdot \delta\mathbf{r} \tag{1.39}$$

where

$$\mathbf{D} = \tfrac{1}{2}[\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^{\mathrm{T}}] \tag{1.40a}$$

$$\mathbf{W} = \tfrac{1}{2}[\text{grad } \mathbf{u} - (\text{grad } \mathbf{u})^{\mathrm{T}}] \tag{1.40b}$$

Note that $\mathbf{D}$ is symmetric while $\mathbf{W}$ is skew-symmetric.

The change in velocity between two neighboring points as described by Equation 1.37 is of profound interest because it also describes the deformation of a material line element $\delta\mathbf{r}$ as it moves with the flow. The purpose of Problem 1.7 in the Problems section is to show that:

$$\frac{D}{Dt}(\delta\mathbf{r}) = (\text{grad } \mathbf{u}) \cdot \delta\mathbf{r} = \frac{\partial \mathbf{u}}{\partial x_k}\, \mathbf{i}_k \cdot \delta\mathbf{r}$$

and then to obtain results both on the longitudinal and angular deformations.

We now use the Cartesian coordinates to explain the meaning of the terms appearing in Equation 1.39. For brevity, we write $\delta x_1 = q_1$, $\delta x_2 = q_2$, $\delta x_3 = q_3$; then Equation 1.39 in Cartesian coordinates becomes:

$$\delta u_i = D_{ij} q_j + W_{ij} q_j \tag{1.41}$$

where

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.42}$$

* Note from ME.1 that there is a duality in the representation of grad $\mathbf{u}$. In this book, we have adopted the convention of postmultiplication by the base vectors in the operator grad. Thus $\nabla\mathbf{u}$ is the transpose of the dyadic product of ∇ and $\mathbf{u}$.

and

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.43}$$

are the Cartesian components of **D** and W, respectively. With the aid of these expressions, the Cartesian components of the velocity at Q are represented in terms of those at P as:

$$(u_i)_Q = (u_i)_P + (D_{ij})_P q_j + (W_{ij})_P q_j \tag{1.44}$$

where $i = 1, 2, 3$ and:

$$\mathbf{q} = \delta\mathbf{r} = \mathbf{r}_Q - \mathbf{r}_P$$

The first term on the right-hand side of Equation 1.44 is the translational velocity at Q equal to the velocity at P. To interpret the next term, we write it without a P subscript, i.e.:

$$D_{ij}q_j = D_{i1}q_1 + D_{i2}q_2 + D_{i3}q_3 \tag{1.45}$$

and consider the following quadric having D_{ij} as coefficients:

$$\begin{aligned}\Phi &= \tfrac{1}{2}\mathbf{q}\cdot(\mathbf{D}\cdot\mathbf{q}) \\ &= \tfrac{1}{2}D_{mn}q_m q_n \\ &= \tfrac{1}{2}(q_1^2 D_{11} + q_2^2 D_{22} + q_3^2 D_{33}) + q_1 q_2 D_{12} + q_2 q_3 D_{23} + q_1 q_3 D_{13}\end{aligned} \tag{1.46}$$

where we have used the fact that D_{ij} is symmetric.

If now q_1, q_2, q_3 are treated as local current variables, then Φ = constant represents a surface and $\partial\Phi/\partial q_i$ $(i = 1, 2, 3)$ are proportional to the direction cosines of the normal to this surface. From Equation 1.45, we immediately note that:

$$\frac{\partial \Phi}{\partial q_i} = D_{ij}q_j$$

Thus, $D_{ij}q_j$ or $\mathbf{D}\cdot\mathbf{q}$, represents a velocity vector which is normal to the quadric Φ = constant. The problem becomes much simpler if one transforms from q_1, q_2, q_3 to $\bar{q}_1, \bar{q}_2, \bar{q}_3$ such that the quadratic form (Equation 1.46) does not contain any cross product terms $\bar{q}_i\bar{q}_j$ $(i \neq j)$. The coordinates $\bar{q}_i$ are then said to form the *principal axes* of the tensor **D**. On introducing the principal axes, the quadratic form becomes:

$$\Phi = \tfrac{1}{2}[\lambda_1(\bar{q}_1)^2 + \lambda_2(\bar{q}_2)^2 + \lambda_3(\bar{q}_3)^2] \tag{1.47a}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the symmetric tensor or matrix **D**, i.e., the roots of the equation:

$$|D_{ij} - \lambda\delta_{ij}| = 0$$

An important property of the eigenvalues is that:

$$\lambda_1 + \lambda_2 + \lambda_3 = D_{11} + D_{22} + D_{33} = \text{div } \mathbf{u} \tag{1.47b}$$

where

$$\overline{D}_{11} = \lambda_1, \quad \overline{D}_{22} = \lambda_2, \quad \overline{D}_{33} = \lambda_3$$

From Equation 1.47a, we conclude that the contribution to the velocity at Q due to $\overline{D} \cdot \mathbf{q}$, with reference to the principal axes, centered at P, is a vector with components:

$$\frac{\partial \Phi}{\partial \overline{q}_1} = \lambda_1 \overline{q}_1, \quad \frac{\partial \Phi}{\partial \overline{q}_2} = \lambda_2 \overline{q}_2, \quad \frac{\partial \Phi}{\partial \overline{q}_3} = \lambda_3 \overline{q}_3$$

Thus, any material line element parallel to the $\overline{q}_1$ axis is stretched at the rate $\lambda_1 = \overline{D}_{11}$. Similarly $\lambda_2 = \overline{D}_{22}$ and $\lambda_3 = \overline{D}_{33}$ are the rates of stretchings along the $\overline{q}_2$ and $\overline{q}_3$ axes, respectively. Thus, the tensor **D** is completely determined by the directions of the principal axes at any point and the three principal rates of strain $\lambda_1, \lambda_2, \lambda_3$. Because of this physical behavior, the contribution $D_{ij}q_j$ to the velocity at Q is that of deformation. The tensor **D** is called the *rate-of-strain tensor*. It is customary to define the deformation tensor as:

$$2\mathbf{D} = \text{def } \mathbf{u} = \text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T \tag{1.48}$$

With reference to the principal axes, the rate-of-strain quadric Equation 1.47 can be written as:

$$\frac{(\overline{q}_1)^2}{\mu_1^2} + \frac{(\overline{q}_2)^2}{\mu_2^2} + \frac{(\overline{q}_3)^2}{\mu_3^2} = 1$$

where $\mu_i^2 = 2\Phi/\lambda_i$ $(i = 1, 2, 3)$. Figure 1.4 demonstrates the shapes of the quadrics depending on the properties of λ_i (and thus of μ_i).

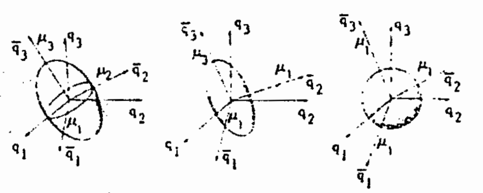

Fig. 1.4 Respective shapes of the rate-of-stain quadrics according to the eigenvalues of **D**.

Example 1.2

Consider the two-dimensional flow field:

$$u_1 = ax_1 + 2bx_2, \qquad u_2 = 2cx_1 - ax_2, \qquad u_3 = 0$$

where a, b, c are constants. Find the Cartesian components and the eigenvalues of the rate-of-strain tensor **D**. Based on these data, find the principal axes of **D** and then represent it in terms of the unit vectors along the principal axes.

Here, based on the formula (Equation 1.42), we have:

$$D_{11} = a, \quad D_{12} = b + c, \quad D_{22} = -a, \quad D_{13} = D_{23} = D_{33} = 0$$

Thus, the eigenvalues are

$$\lambda_1 = K, \quad \lambda_2 = -K, \quad \lambda_3 = 0$$

where

$$K = [a^2 + (b + c)^2]^{1/2}$$

The directions of the principal axes referred to x_1, x_2 are given by the unit vectors (obtained by normalizing the eigenvectors):

$$\mathbf{u}_1 = [(K + a)\mathbf{i}_1 + (b + c)\mathbf{i}_2]/[2K(K + a)]^{1/2}$$

$$\mathbf{u}_2 = [(K - a)\mathbf{i}_1 - (b + c)\mathbf{i}_2]/[2K(K - a)]^{1/2}$$

$$\mathbf{u}_3 = -\mathbf{i}_3$$

Thus, the components of $\mathbf{D}$ referred to the principal axes are

$$\overline{D}_{11} = K, \quad \overline{D}_{22} = -K, \quad \overline{D}_{12} = 0$$

and

$$\mathbf{D} = D_{mn}\, \mathbf{i}_m\mathbf{i}_n = \overline{D}_{rs}\, \mathbf{u}_r\mathbf{u}_s$$

The remaining part of the tensor grad $\mathbf{u}$ is $\mathbf{W}$. It is a skew-symmetric tensor and, therefore, it has only three distinct components. To account for these components, we introduce a vector $\boldsymbol{\omega}$ defined as:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \text{curl } \mathbf{u} \tag{1.49}$$

and note by verification that the tensor $\mathbf{W}$ and the vector $\boldsymbol{\omega}$ are related through the equation:

$$2 \text{ div } \mathbf{W} = - \text{curl } \boldsymbol{\omega}$$

or much simply through the equation:

$$2\mathbf{W} \cdot \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$$

where $\mathbf{v}$ is an arbitrary nonzero vector.* Equating the coefficients of the components of $\mathbf{v}$, we obtain the Cartesian components of $\mathbf{W}$ as:

$$W_{ij} = -\frac{1}{2} e_{ijk}\omega_k \tag{1.50}$$

where e_{ijk} is the *permutation symbol* defined as:

$$e_{123} = e_{231} = e_{312} = 1$$

$$e_{132} = e_{321} = e_{213} = -1$$

otherwise zero.

In this connection, it is worthwhile to recapitulate the following formulae in the Cartesian coordinates:

$$(\text{curl } \mathbf{u})_i = \omega_i$$

$$= e_{ijk} \frac{\partial u_k}{\partial x_j} \tag{1.51a}$$

* Thus, $\mathbf{W} \cdot \boldsymbol{\omega} = 0$.

$$(\boldsymbol{\omega} \times \mathbf{r})_i = e_{ijk}\omega_j x_k = -e_{ijk}x_j\omega_k \tag{1.51b}$$

From Equation 1.50, the term $W_{ij}q_j$ appearing in Equation 1.44 is given by:

$$W_{ij}q_j = -\tfrac{1}{2}e_{ijk}q_j\omega_k$$

Since $\mathbf{q}$ is a vector of components q_1, q_2, q_3, we find according to Equation 1.51b that $W_{ij}q_j$ is the i-th component of the vector:

$$\tfrac{1}{2}\boldsymbol{\omega} \times \mathbf{q}$$

which represents a rigid body rotation of a fluid particle at the position $\mathbf{q}$ about an instantaneous axis with an angular velocity $(1/2)\boldsymbol{\omega}$. Thus, the contribution $W_{ij}q_j$ to the velocity at Q is that of a rigid body rotation, and the tensor $\mathbf{W}$ is called the *rotation tensor*.

In summary, we have established that the velocity at Q is composed of the following contributions at P:

1. A translation
2. A deformation
3. A rigid body rotation

1.14 KINEMATICS OF VORTICITY AND CIRCULATION

The vector $\boldsymbol{\omega}$ which is the curl of the velocity vector $\mathbf{u}$ and defined in Equation 1.49 is called the *angular velocity vector* or the *vorticity vector*. At every point in the flow, the fluid particles rotate about an instantaneous axis, and the vector $\boldsymbol{\omega}$ is represented by the direction of the instantaneous axis of rotation.

Vortex Line

A trajectory of the field of vorticity vectors is called a vortex line. In other words, a line in the flow whose tangents are directed along the local vorticity vectors forms a vortex line.

The fluid particles distributed along a vortex line rotate about the tangents to the vortex line at their respective positions, as is shown in Figure 1.5. A crude example of a vortex line is the thread passing through the holes of a number of beads, the beads being treated as fluid particles.

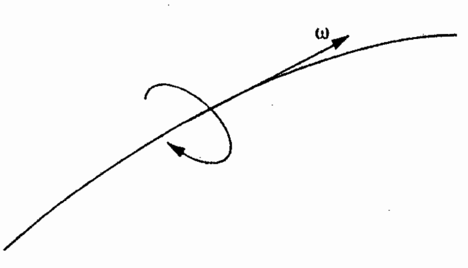

Fig. 1.5 A vortex line.

Vortex Tube

Any arbitrary closed curve drawn in the fluid is said to be reducible if it can be contracted to a point without passing out of the fluid region. If we draw vortex lines through each point of an arbitrary reducible curve, then we generate a curvilinear cylindrical surface. The part of fluid enclosed in this surface is called a *vortex tube*.

To analyze a vortex tube from a quantitative point of view, we consider a finite section of the tube *at a given time* as shown in Figure 1.6.

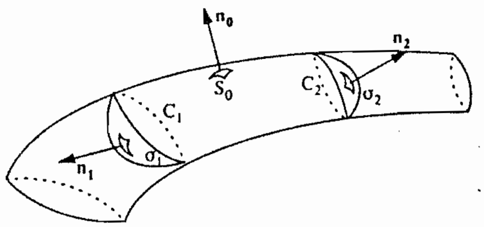

Fig. 1.6 Section of a vortex tube with capping surfaces.

The curves C_1 and C_2 are the section curves of the vortex tube whose surface is denoted as S_0. Treat the region enclosed by the vortex tube as a closed region by drawing capping surfaces σ_1 and σ_2, respectively, on the curves C_1 and C_2. The positive external normal to σ_1 and σ_2 are denoted as $\mathbf{n}_1$ and $\mathbf{n}_2$. The total surface σ is then given by the union:

$$\sigma = S_0 \cup \sigma_1 \cup \sigma_2$$

and V_0 the total volume enclosed by σ. Since the field is of vorticity $\boldsymbol{\omega}$, we apply Gauss' divergence theorem to this closed volume V_0 (refer to Equation M2.2) and have:

$$\begin{aligned}\int_{V_0} \operatorname{div} \boldsymbol{\omega}\, dv_0 &= \int_{\sigma} \boldsymbol{\omega} \cdot \mathbf{n}\, dS \\ &= \int_{S_0} \boldsymbol{\omega} \cdot \mathbf{n}_0\, dS + \int_{\sigma_1} \boldsymbol{\omega} \cdot \mathbf{n}_1\, dS + \int_{\sigma_2} \boldsymbol{\omega} \cdot \mathbf{n}_2\, dS \qquad (1.52)\end{aligned}$$

However, the vorticity field $\boldsymbol{\omega}$ is such that its divergence is always zero, i.e.:

$$\operatorname{div} \boldsymbol{\omega} = \operatorname{div} (\operatorname{curl} \mathbf{u}) = 0$$

and since $\boldsymbol{\omega} \cdot \mathbf{n}_0 = 0$ (S_0 being the vortex tube surface), Equation 1.52 simply gives:

$$\int_{\sigma_1} \boldsymbol{\omega} \cdot \mathbf{n}_1\, dS + \int_{\sigma_2} \boldsymbol{\omega} \cdot \mathbf{n}_2\, dS = 0$$

The terms in this equation represent the flux of vorticity through the arbitrarily chosen surfaces σ_1 and σ_2. To maintain the same definition of flux, we introduce $\mathbf{n}_1 = -\mathbf{n}_1'$ and then we have:

$$\int_{\sigma_1} \boldsymbol{\omega} \cdot \mathbf{n}_1'\, dS = \int_{\sigma_2} \boldsymbol{\omega} \cdot \mathbf{n}_2\, dS \qquad (1.53)$$

Since σ_1 and σ_2 are arbitrary, the result in Equation 1.53 shows that the flux of vorticity through any section of the tube remains unchanged. This constant flux through a section is called the *strength of the vortex tube*. We use Stokes' theorem, Equation M2.7, to write the strength of the vortex tube as a line integral. Thus:

$$\int_{\sigma_2} \boldsymbol{\omega} \cdot \mathbf{n}_2\, dS = \oint_{C_2} \mathbf{u} \cdot d\mathbf{r} \qquad (1.54)$$

The right-hand side of Equation 1.54 is the circulation of velocity around the contour C_2 on which the capping surface is σ_2. Thus, the strength of a vortex tube is equal to the circulation of velocity along the closed contour which lies on the vortex tube and encircles it once. Consequently, because of the constancy of strength, *the circulations about the curves C_1 and C_2 on a vortex tube at a given instant are the same. This result is known as the first theorem of Helmholtz.* It should be remarked here that the first theorem of Helmholtz is purely kinematic in nature and is generally applicable to all flow fields.

A very significant result can be obtained from Equation 1.53 for a vortex tube of very small sectional area, known as a *vortex filament*. The sections passing through the curves C_1 and C_2 are then essentially normal sections with uniform vorticity distributions on them. Denoting the normal section areas by σ_1 and σ_2 with vorticities ω_1 and ω_2, respectively, we have:

$$\omega_1\sigma_1 = \omega_2\sigma_2 \tag{1.55}$$

Consider the case of a vortex tube or filament in a fluid medium which, somewhere on its length, abruptly encounters or attains a zero vorticity distribution at a section. This possibility is clearly discounted either by Equation 1.53 or Equation 1.55.* This leads us to conclude that vortex tubes or filaments cannot abruptly begin or end in a fluid medium. The vortex tubes, therefore, form closed rings in fluid regions, or else extend to the boundaries of the fluid motion. For viscous fluid motions it will be shown in Chapter 3, Section 8, that because of the no-slip condition the normal component of $\boldsymbol{\omega}$ at the body surface is zero but the tangential component at the surface is not zero.

A *line vortex*, which is distinct from a *vortex line*, is a useful mathematical idealization of a vortex filament in which the vortex filament is assumed to converge onto its axis without changing its strength. In two dimensions, a line vortex is called a *point vortex*.

A *sheet vortex* (also known as a vortex sheet) is formed of contiguous line vortices of infinitesimal strength. Throughout the vortex sheet, there is a surface concentration of vorticity as a function of the position in the surface.

Rate of Change of Circulation

In Equation 1.54, we have expressed the strength of a vortex tube in terms of the circulation of velocity along a closed contour. Realizing that the vortex tubes are not rigid curvilinear cylinders fixed in space and time but move with the flow, it is of interest to know how the strength of a vortex tube changes as the tube moves with the flow. In essence, we have to find out how the circulation of velocity on a closed curve changes as the the curve moves with the flow. To analyze this aspect of the flow, we first consider a curve drawn from a point A to a point B in the fluid and determine the time rate of change of the quantity:

$$\int_A^B \mathbf{u} \cdot \delta\mathbf{r}$$

as it moves with the flow. Let the parametric representation of the curve joining A and B be $\mathbf{r}(s, t)$; s being the arc length with $0 \leq s \leq 1$. Then:

$$\begin{aligned}
\frac{d}{dt}\int_A^B \mathbf{u} \cdot \delta\mathbf{r} &= \frac{d}{dt}\int_0^1 \mathbf{u} \cdot \frac{d\mathbf{r}}{ds}\, ds \\
&= \int_0^1 \frac{d}{dt}\left(\mathbf{u} \cdot \frac{d\mathbf{r}}{ds}\right) ds \\
&= \int_A^B \frac{D\mathbf{u}}{Dt} \cdot \delta\mathbf{r} + \int_A^B \mathbf{u} \cdot \delta\mathbf{u} \\
&= \int_A^B \mathbf{a} \cdot \delta\mathbf{r} + \frac{1}{2}(u_B^2 - u_A^2)
\end{aligned} \tag{1.56}$$

* Another interpretation: Consider the case of a vortex tube in a fluid medium which, somewhere on its length, has narrowed down to such an extent that its section area becomes zero. This possibility is clearly discounted by Equation 1.55 for then the vorticity at the point of termination of the vortex must be infinitely large.

where $u = |\mathbf{u}|$. If now A and B coincide so as to form a simple closed curve $\mathscr{C}$ in the flow, then:

$$\frac{d}{dt}\oint_{\mathscr{C}} \mathbf{u} \cdot \delta\mathbf{r} = \oint_{\mathscr{C}} \mathbf{a} \cdot \delta\mathbf{r} \tag{1.57}$$

Thus, the rate of change of circulation of velocity or the strength of a vortex tube is equal to the circulation of acceleration along a contour enclosing the vortex tube.

If the acceleration can be expressed as the gradient of a scalar, i.e., $\mathbf{a} = -\text{grad } G$, then the circulation of velocity on a closed curve does not change as the curve moves with the flow. Such a flow is called circulation preserving.

Let $\mathscr{S}$ be the surface whose bounding curve is $\mathscr{C}$. On using the Stokes' theorem, M2.7, the right hand side of Equation 1.57 becomes

$$\int_{\mathscr{S}} (\text{curl } \mathbf{a}).\ \mathbf{n}\ dS$$

Using Equations 1.17, M1.37, and M1.43b, we also have

$$\frac{d}{dt}\oint_{\mathscr{C}} \mathbf{u} \cdot \delta\mathbf{r} = \int_{\mathscr{S}} \left[\frac{D\boldsymbol{\omega}}{Dt} - (\text{grad } \mathbf{u}) \cdot \boldsymbol{\omega} + \boldsymbol{\omega}(\text{div } \mathbf{u})\right] \cdot \mathbf{n} dS \tag{1.58}$$

REFERENCE

1. Kelvin, Lord, *Math. Phys. Pap.*, 1, 82.

PROBLEMS

1.1 Let a velocity field be given by the equations:

$$x_1 = \alpha X_1 F(t), \qquad x_2 = \beta X_2 G(t), \qquad x_3 = \gamma X_3 H(t)$$

where F, G, H are differentiable functions of time and α, β, γ are constants. As discussed in the text, X_1, X_2, X_3 are the coordinates at $t = t_0$.

(a) What are the values of F, G, H at $t = t_0$?

(b) Find the velocity and acceleration in both the Lagrangian and Eulerian coordinates.

(c) Find the Jacobian $J = \det(\partial x_i/\partial X_j)$ and establish Equation 1.29.

1.2 Given the velocity field:

$$u_1 = kx_2, \qquad u_2 = rx_1, \qquad u_3 = 0$$

where k and r are constants, find the components of $\mathbf{D}$ and $\mathbf{W}$. Find the eigenvalues of $\mathbf{D}$ and the principal unit vectors, in terms of $\mathbf{i}_1$ and $\mathbf{i}_2$, in the directions of the principal axes of $\mathbf{D}$. Express $\mathbf{D}$ in terms of these unit vectors.

1.3 Write the following quantities in the Cartesian component form, taking $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, as the unit vectors along x_1, x_2, x_3, respectively. Your answer should first be written by using the summation convention on repeated indices, and then in the expanded form to show all the components.

$$\text{grad } \phi, \quad \text{div } \mathbf{u}, \quad \text{curl } \mathbf{u}, \quad \frac{D}{Dt}, \quad \text{grad } \mathbf{u}, \quad (\text{grad } \mathbf{u})^T$$

1.4 With reference to rectangular Cartesian coordinates:

$$\mathbf{u} = u_m \mathbf{i}_m, \quad \text{grad } \mathbf{u} = \frac{\partial u_p}{\partial x_q} \mathbf{i}_p \mathbf{i}_q, \quad \text{grad} = \frac{\partial}{\partial x_r} \mathbf{i}_r$$

(a) Find the expanded forms of the quantities:

$$(\mathbf{u} \cdot \text{grad})\mathbf{u} \qquad \text{and} \qquad (\text{grad } \mathbf{u}) \cdot \mathbf{u}$$

and show that they are the same.

(b) Suppose the material volume collapses to a line segment $\alpha(t) \leq x < \beta(t)$. Show that

$$\mathbf{u} \cdot \mathbf{n}\Big|_{x=\alpha} = -\frac{d\alpha}{dt} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n}\Big|_{x=\beta} = \frac{d\beta}{dt}.$$

Next, particularize Equation 1.32c for this one-dimensional domain.

1.5 With reference to rectangular Cartesian coordinates: $\boldsymbol{\sigma} = \sigma_{pq} \mathbf{i}_p \mathbf{i}_q$, $\mathbf{D} = D_{mn} \mathbf{i}_m \mathbf{i}_n$.

(a) Write the fully expanded forms of the following quantities:

$$\boldsymbol{\sigma} \cdot \mathbf{u}, \qquad 2\mathbf{D} : \mathbf{D}, \qquad \text{div}[(\text{grad } \mathbf{u}) \cdot \mathbf{u}]$$

(For double dot product of tensors, refer to Equation M1.25.)

(b) Show that the magnitude of the vorticity vector $\boldsymbol{\omega}$ in Cartesian coordinates is

$$|\boldsymbol{\omega}|^2 = \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

1.6 Based on the definition of the Lagrangian coordinates $\mathbf{R}$, introduce the gradient operator:

$$\text{Grad} = \frac{\partial}{\partial X_a} \mathbf{i}_a$$

Deduce the following results:

$$\mathbf{I} = (\text{Grad } \mathbf{r}) \cdot (\text{grad } \mathbf{R}) \tag{i}$$

where $\mathbf{I}$ is the unit tensor:

$$\delta\mathbf{r} = (\text{Grad } \mathbf{r}) \cdot \delta\mathbf{R} \tag{ii}$$

$$\frac{D}{Dt}(\text{Grad } \mathbf{r}) = \text{Grad } \mathbf{u} \tag{iii}$$

1.7 Referring to the figure, let P_1P_2 be a spatial vector denoting the directed material fluid element between $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ at time t. In an infinitesimal time dt, the element undergoes both a displacement and a deformation, so that:

$$\overrightarrow{P_1'P_2'} = \delta\mathbf{r} + D(\delta\mathbf{r})$$

where D has been defined as Equation 1.19a.

(a) From the figure deduce that:

$$D(\delta\mathbf{r}) = \delta(D\mathbf{r})$$

(b) Using the result in (a) and the definition of velocity, show that:

$$\frac{D}{Dt}(\delta\mathbf{r}) = \delta\mathbf{u} = (\text{grad } \mathbf{u}) \cdot \delta\mathbf{r} \qquad \text{(i)}$$

(c) Without using the result in (a), establish the above result.
(Hint: Use results [i], [ii], and [iii] of Problem 1.6.)

(d) Consider a rectangular Cartesian system with P_1 as the origin. The vectors $\mathbf{i}_1\delta x_1$, $\mathbf{i}_2\delta x_2$, $\mathbf{i}_3\delta x_3$ are then directed along x_1, x_2, x_3 axes. Show by using the result in (b) that:

$$\frac{D}{Dt}(\mathbf{i}_k\delta x_k) = \frac{\partial \mathbf{u}}{\partial x_k}\delta x_k, \qquad k = 1, 2, 3.$$

where there is *no sum on k*, and from these equations find the expressions for D_{11}, D_{22}, D_{33}. Next consider:

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{i}_m\delta x_m \cdot \mathbf{i}_n\delta x_n), \qquad m \neq n$$

and again there is *no sum on repeated indices*. Open the differentiation and use the results in (d) to find the expressions for D_{12}, D_{13}, D_{23}. (Physically, the rate of change of a right angle initially formed by δx_m and δx_n ($m \neq n$) yields the shearing rate-of-strain component.)

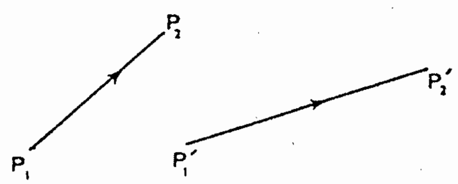

Fig. problem 1.7

1.8 Let $\delta\mathbf{r}$ be a material fluid element of length $\delta s = |\delta\mathbf{r}|$. Starting from:

$$\frac{D}{Dt}(\delta\mathbf{r} \cdot \delta\mathbf{r})$$

and using Equation i of Problem 1.7 and Equation M1.19, show that:

$$\frac{D}{Dt}(\delta s^2) = 2\delta\mathbf{r} \cdot \mathbf{D} \cdot \delta\mathbf{r}$$

1.9 Prove that:

$$\mathbf{D} : \mathbf{D} = \mathbf{D} : \text{grad } \mathbf{u}$$

and

$$\mathbf{W} : \text{grad } \mathbf{u} = \frac{1}{2}|\boldsymbol{\omega}|^2$$

1.10 Using the result in Problem 1.9, show that:

$$(\text{grad } \mathbf{u})^{\mathrm{T}} : (\text{grad } \mathbf{u}) = \mathbf{D} : \mathbf{D} - \tfrac{1}{2} |\boldsymbol{\omega}|^2$$

1.11 The quantity M_{T} defined as:

$$M_{\mathrm{T}} = |\boldsymbol{\omega}|/(2\mathbf{D} : \mathbf{D})^{1/2}$$

is called the Truesdell's vorticity number, (refer to Chapter 3 for details). Take the divergence of the acceleration vector **a** given in Equation 1.17 and show that it can be written in any one of the following forms:

$$\begin{aligned} \text{div } \mathbf{a} &= \frac{D\Delta}{Dt} + \mathbf{D} : \mathbf{D} - \tfrac{1}{2} |\boldsymbol{\omega}|^2 \\ &= \frac{D\Delta}{Dt} + (1 - M_{\mathrm{T}}^2)\mathbf{D} : \mathbf{D} \end{aligned}$$

where $\Delta = \text{div } \mathbf{u}$.

1.12 Referring to the figure, let X_0, Y_0, Z_0 be a set of fixed axes forming an inertial frame with $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ as the unit constant vectors along X_0, Y_0, Z_0, respectively. Let X, Y, Z be another coordinate system with unit vectors **i**, **j**, **k**, which beside being in acceleration with respect to the basic frame also rotates about an arbitrary axis in space. Let us denote the time operator with respect to the rest frame by d/dt and with respect to the moving frame by $\dot{d}/dt$. Thus, $\dot{d}/dt$ selectively treats **i**, **j**, **k** as constants.

(a) Show that:

$$\frac{d\mathbf{r}}{dt} = \frac{\dot{d}\mathbf{r}}{dt} + X\frac{d\mathbf{i}}{dt} + Y\frac{d\mathbf{j}}{dt} + Z\frac{d\mathbf{k}}{dt}$$

where **r** is the position vector referred to the moving axes.

(b) Using the properties:

$$\frac{1}{2}\frac{d}{dt}(\mathbf{i} \cdot \mathbf{i}) = \mathbf{i} \cdot \frac{d\mathbf{i}}{dt} = 0, \quad \text{etc.}$$

and the right-handed cross product equations for **i**, **j**, **k**, show that:

$$\frac{d\mathbf{i}}{dt} = a\mathbf{j} - b\mathbf{k}, \qquad \frac{d\mathbf{j}}{dt} = c\mathbf{k} - a\mathbf{i}, \qquad \frac{d\mathbf{k}}{dt} = b\mathbf{i} - c\mathbf{j}$$

where a, b, c are scalars.

(c) If $\boldsymbol{\Omega}$ is the angular velocity vector of the coordinate frame, then show that:

$$a = \Omega_z, \qquad b = \Omega_y, \qquad c = \Omega_x$$

where Ω_x, Ω_y, Ω_z are the Cartesian components of $\boldsymbol{\Omega}$. Using this result and the equations in (a) and (b), prove that:

$$\frac{d\mathbf{r}}{dt} = \frac{\dot{d}\mathbf{r}}{dt} + \boldsymbol{\Omega} \times \mathbf{r}$$

In general for vector functions of time:

$$\frac{d}{dt} = \frac{\dot{d}}{dt} + \mathbf{\Omega}\times$$

so that:

$$\mathbf{u}_r = \mathbf{v} + \mathbf{\Omega} \times \mathbf{r}; \qquad \mathbf{u}_r = \mathbf{u} - \dot{\mathbf{p}}$$

where $\mathbf{u}$ is the velocity of a fluid particle with respect to a fixed inertial frame and $\mathbf{v}$ is the velocity of the same particle with respect to the noninertial frame. Note that $\mathbf{u}_r$ is the absolute velocity of the fluid particle with respect to an inertial frame whose origin coincides instantaneously with that of the rotating frame.

(d) Use the time operator defined in (c) and show that:

$$\frac{d\mathbf{\Omega}}{dt} = \frac{\dot{d}\mathbf{\Omega}}{dt}$$

and

$$\frac{\dot{d}}{dt}(\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega} \times \mathbf{v} + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r}$$

(e) Using the results obtained in (a)–(d) show that, when reference is made to a Cartesian rotating frame, the acceleration is given by:

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \ddot{\mathbf{p}} + \frac{\dot{d}\mathbf{v}}{dt} + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

Note that: $(d\mathbf{\Omega}/dt) \times \mathbf{r}$ = acceleration due to the time variation of $\mathbf{\Omega}$.
$2\mathbf{\Omega} \times \mathbf{v}$ = Coriolis acceleration.
$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ = centripetal acceleration.

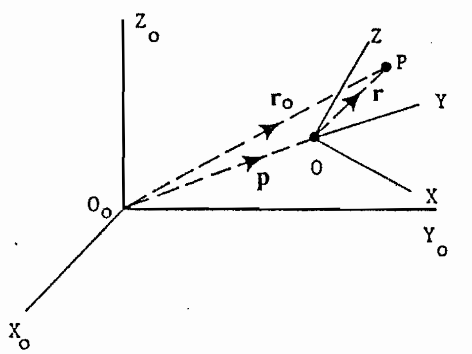

Fig. problem 1.12

1.13 In problem 1.12, if $\mathbf{\Omega}$ = constant vector, then show that the centripetal acceleration can be expressed as the gradient of a potential, i.e.:

$$\begin{aligned} \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) &= -\Omega^2 R \text{ grad } R \\ &= -\text{ grad}\left(\frac{\Omega^2 R^2}{2}\right) \end{aligned}$$

where R is the perpendicular distance of a point from the axis of rotation.

1.14 Using Equation 1.57, show that if **a** is derivable from a potential, then the strength of the vortex tube does not change as the vortex moves with the flow.

CHAPTER TWO

The Conservation Laws and the Kinetics of Flow

2.1 FLUID DENSITY AND THE CONSERVATION OF MASS

Let V_0 be a fixed volume containing a fluid of mass M. In an ordinary sense, the average density ρ of the fluid is the ratio M/V_0 which is a fixed number. However, this definition of fluid density is not of much use in general fluid dynamics, because we need a definition which, as a rule, is able to define density in a continuous manner both in space and time. In other words, the density should be defined as a *point function* $\rho(\mathbf{r}, t)$.

As a next step, we may take the volume V_0 to be vanishingly small and define:

$$\rho = \lim_{\substack{V_0 \to 0 \\ V_0 \neq 0}} \frac{M}{V_0} \tag{2.1}$$

Note that V_0 should never be exactly zero because then the value given by Equation 2.1 will be either zero or infinity depending upon whether the point lies outside a molecule or within it. From a continuum viewpoint Equation 2.1 defines ρ only at discrete points.

To circumvent these difficulties, we define the density $\rho(\mathbf{r}, t)$ as a point function such that its integral over an arbitrary volume V_0 gives the mass M contained in the volume. Thus:

$$M = \int_{V_0} \rho(\mathbf{r}, t)\, dv \tag{2.2}$$

defines the density $\rho(\mathbf{r}, t)$.

2.2 PRINCIPLE OF MASS CONSERVATION

The principle of conservation of mass can be stated either with reference to a moving material volume $\mathcal{V}(t)$ or with reference to a volume V_0 fixed in space. Both are stated as I and II following.

I. "The mass contained in a material volume $\mathcal{V}(t)$ does not change as the volume moves with the flow."

This principle comes directly from the definition of a material volume, discussed in Chapter 1. Since there can be no mass transfer to or from a material volume, the principle of mass conservation implies that:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho(\mathbf{r}, t)\, dv = 0 \tag{2.3}$$

where d/dt is the total time derivative.

To evaluate Equation 2.3, we use Equation 1.32a with $F = \rho$, so that:

$$\int_{\mathcal{V}(t)} \left(\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} \right) dv = 0 \tag{2.4}$$

The result in Equation 2.4 is valid for any choice of the volume $\mathcal{V}(t)$, therefore:

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0 \tag{2.5}$$

which is called the differential equation for the conservation of mass, or simply the *equation of continuity*. Using the definition of D/Dt, we can also write Equation 2.5 as:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{2.6}$$

II. "The rate of *decrease* of mass in a fixed volume V_0 is equal to the mass flux through its surface S_0."

This principle implies that:

$$-\frac{\partial}{\partial t}\int_{V_0} \rho \, dv = \int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS \tag{2.7}$$

According to Gauss' divergence theorem:

$$\int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS = \int_{V_0} \operatorname{div}(\rho \mathbf{u}) \, dv$$

hence Equation 2.7 becomes:

$$\int_{V_0} \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u})\right] dv = 0 \tag{2.8}$$

Since V_0 is an arbitrary fixed volume, we again get Equation 2.6.

Time Variation of ρP

Let P be a property per unit mass which may be a scalar, vector, or tensor function associated with the flow. Having defined the density ρ, we are now in a position to calculate the time rate of change of ρP as it moves with the flow. Let $\mathcal{V}(t)$ be an arbitrary material volume. Then:

$$\int_{\mathcal{V}(t)} \rho P \, dv$$

is the total property contained in $\mathcal{V}(t)$. Using Equations 1.32a and 2.5, we easily get:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho P \, dv = \int_{\mathcal{V}(t)} \rho \frac{DP}{Dt} \, dv \tag{2.9}$$

Particular Forms of the Continuity Equation

For steady flow, the equation of continuity (Equation 2.6) takes the form:

$$\operatorname{div}(\rho \mathbf{u}) = 0 \tag{2.10}$$

It is sometimes useful to state the continuity equation directly from Equation 2.7 for steady flows as:

$$\int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS = 0 \tag{2.11}$$

Fluid flows for which the density remains constant are termed *incompressible flows*. The differential form of the continuity equation then is simply:

$$\text{div } \mathbf{u} = 0 \tag{2.12}$$

Note that Equation 2.12 is applicable to both the steady and nonsteady incompressible flows.

2.3 MASS CONSERVATION USING A CONTROL VOLUME

Consider a control volume V^* of surface S^* which is moving with respect to a stationary inertial frame with a velocity $\mathbf{c}$. Since we have already developed the necessary equation for the time variation of a physical quantity F in a control volume in Chapter 1 (viz., Equation 1.36), we replace F by ρ in Equation 1.36 and use Equation 2.3. Thus:

$$\frac{d}{dt}\int_{V^*} \rho(\mathbf{r}, t)\, dv^* + \int_{S_m^*} \rho(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* = 0 \tag{2.13}$$

which is the mass conservation equation for a control volume. If the control volume is fixed, then denoting V^* and S^* by V_0 and S_0, respectively, we have:

$$\int_{V_0} \frac{\partial \rho}{\partial t}\, dv + \int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS = 0 \tag{2.14}$$

which is the same as Equation 2.7.

2.4 KINETICS OF FLUID FLOW

The two types of forces which act on fluid masses and are distributed throughout the fluid medium are

1. Body forces
2. Surface forces

The body forces are forces of an extensive character acting on the bulk portions of the fluid and arise due to some external cause. The external causes are (1) the force of gravity, (2) forces of electric and magnetic origin acting on a fluid carrying charged particles, (3) centrifugal forces due to the rotation of fluid masses about an axis, etc. The body force is proportional to the volume of fluid and therefore it is expressed as force per unit volume. We shall denote the body force per unit mass by the symbol $\mathbf{f}(\mathbf{r}, t)$ or simply $\mathbf{f}$, so that the body force per unit volume will be $\rho\mathbf{f}$.

The surface forces, on the other hand, are forces of an intensive or local nature. The surface forces arise due to mechanical interaction between contiguous portions of a fluid medium or between a portion of the fluid medium and a solid surface. To explain the phenomena from a continuum viewpoint*, consider two adjacent portions of a fluid medium separated by an

* In reality these interacting forces have a direct molecular origin which is easily explained if the fluid is a gas. At an imaginary surface in a gaseous medium there is a continual transfer of molecular momentum from one side to the other side of the surface and vice versa. The reaction to this molecular transport rate is the emergence of a force vector at each point of the surface. Refer to Section 2.16.

imaginary surface drawn in the fluid. At the separating surface there exists a direct mechanical contact between the fluid particles on the two sides of the surface, thus, giving rise to forces of action and reaction. If the fluid on one side is imagined to have been replaced by the force system which it has produced (as is usually done in solving problems in mechanics by using the free-body diagrams), then at each point of the imaginary surface there will be a force vector. The totality of these forces gives rise to a field of surface forces. Since the force per unit area of the surface (viz., the stress) is different both in magnitude and direction at different points, we shall exhibit it symbolically as $\boldsymbol{\tau}(\mathbf{r}, t; \mathbf{n})$ or simply as $\boldsymbol{\tau}(\mathbf{n})$, where $\mathbf{n}$ is the local outward drawn unit normal vector on the imaginary surface. The appearance of $\mathbf{n}$ in $\boldsymbol{\tau}$ is to emphasize the dependence of $\boldsymbol{\tau}$ on the orientation of the imaginary surface, as shown in Figure 2.1. Here it may be noted that through any point an infinite number of surfaces can be passed, each having a unique unit normal, thus, resulting in an infinite number of stress vectors associated with the same point. This ambiguity is removed through an automatic emergence of a stress tensor, which does not depend on the unit normals, (cf. Equation 2.25). The tensor which describes the state of stress is completely determined by passing three mutually perpendicular planes through the point as is discussed later in connection with Figure 2.2.

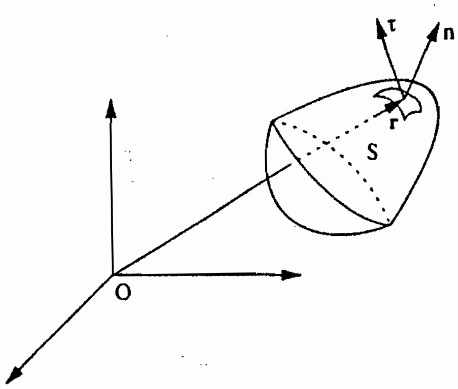

Fig. 2.1 Imaginary surface drawn in a fluid.

The aforementioned mechanical description for the existence of surface forces is fully contained in the stress principle of Cauchy which is stated as follows (refer, e.g., to Reference 1).

Stress Principle of Cauchy

The principle of the distribution of surface stresses is known as the stress principle of Cauchy which is stated as: "*Upon any imagined closed surface S there exists a distribution of stress vector $\boldsymbol{\tau}$ whose resultant and moment resultant are equipollent to the actual forces exerted by the material outside S upon that inside.*" This principle applies to non-polar fluids/materials which are those in which only the contact stress field $\boldsymbol{\tau}$ exists but the contact torque field is absent.

To obtain further results regarding the physical nature of $\boldsymbol{\tau}$, we have to first consider the law of conservation of momentum.

2.5 CONSERVATION OF LINEAR AND ANGULAR MOMENTUM

A. Conservation of Linear Momentum

The law of conservation of linear momentum states that *the rate of change of linear momentum of a material volume $\mathcal{V}(t)$ is equal to the resultant force on the volume.*

The conservation law for momentum is a restatement of Newton's second law of motion ($d(m\mathbf{u})/dt = \mathbf{F}$) applied to a closed system which by definition has a fixed mass. If $\mathbf{u}$ is the velocity vector at position $\mathbf{r}$ and time t and $\mathbf{f}$ is the body force per unit mass, then the law of conservation of linear momentum for the material volume $\mathcal{V}(t)$ is

$$\frac{d}{dt}\int_{V(t)} \rho\mathbf{u}\, dv = \int_{V(t)} \rho\mathbf{f}\, dv + \int_{\mathcal{S}} \boldsymbol{\tau}\, dS \tag{2.15}$$

Note that $\rho\mathbf{u}$ is the momentum per unit volume and $\int\boldsymbol{\tau}\, dS$ is the force exerted *by* the fluid on S. Using Equation 2.9 with $P = \mathbf{u}$ in Equation 2.15, we get:

$$\int_{V(t)} \rho \frac{D\mathbf{u}}{Dt}\, dv = \int_{V(t)} \rho\mathbf{f}\, dv + \int_{\mathcal{S}} \boldsymbol{\tau}\, dS \tag{2.16}$$

Equation 2.16 is directly applicable to a volume which is not a material volume, e.g., a fixed volume. (As a caution it must be stated that Equation 2.15 should never be used for a fixed volume since then the mass is not fixed.) Let V_0 be a fixed volume with a surface S_0; then Equation 2.16 becomes:

$$\int_{V_0} \rho \frac{D\mathbf{u}}{Dt}\, dv = \int_{V_0} \rho\mathbf{f}\, dv + \int_{S_0} \boldsymbol{\tau}\, dS \tag{2.17}$$

where the second term on the right represents the force exerted by the surrounding medium on the fluid in the volume V_0.

B. Conservation of Angular Momentum

The law of conservation of angular momentum states that the *rate of change of angular momentum of a material volume $V(t)$ is equal to the resultant moment on the volume.*

Let $\mathbf{r}$ be the position vector of a point in the flow with respect to the origin of an inertial coordinate system. Then $\rho(\mathbf{r} \times \mathbf{u})$ is the fluid angular momentum per unit volume. Similarly $\rho(\mathbf{r} \times \mathbf{f})$ is the moment of the body force per unit volume, and $\mathbf{r} \times \boldsymbol{\tau}$ is the moment of the stress vector per unit area of the surface. Using Equation 2.9 we have:

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \rho(\mathbf{r} \times \mathbf{u})\, dv &= \int_{V(t)} \rho \frac{D}{Dt}(\mathbf{r} \times \mathbf{u})\, dv \\ &= \int_{V(t)} \rho(\mathbf{r} \times \mathbf{f})\, dv + \int_{\mathcal{S}(t)} (\mathbf{r} \times \boldsymbol{\tau})\, dS \end{aligned}\tag{2.18}$$

For a fixed volume V_0, the law of angular momentum is given by:

$$\int_{V_0} \rho \frac{D}{Dt}(\mathbf{r} \times \mathbf{u})\, dv = \int_{V_0} \rho(\mathbf{r} \times \mathbf{f})\, dv + \int_{S_0} (\mathbf{r} \times \boldsymbol{\tau})\, dS \tag{2.19}$$

We shall return to both conservation laws after obtaining some important results on the stress vector $\boldsymbol{\tau}$.

Nature of the Stress Vector

To establish the nature of the stress vector $\boldsymbol{\tau}$, we consider Equation 2.17 applied to a very small volume V_0. Let ℓ be a linear dimension of the volume V_0, then:

$$V_0 \sim \ell^3, \qquad S_0 \sim \ell^2$$

Dividing each term of Equation 2.17 by ℓ^2, applying the mean value theorem of integral calculus, and taking the limit as the volume shrinks to a point (i.e., $\ell \to 0$) we get:

$$\lim_{\ell \to 0} \frac{1}{\ell^2}\int_{S_0} \boldsymbol{\tau}\, dS = 0 \tag{2.20}$$

Equation 2.20 states that the stress vectors at a point are in local equilibrium with each other, and is called the principle of *local stress equilibrium.**

We now apply Equation 2.20 to the surface of an infinitesimal tetrahedron whose vertex is at **r** with its three faces being parallel to the coordinate planes, as has been shown in Figure 2.2.

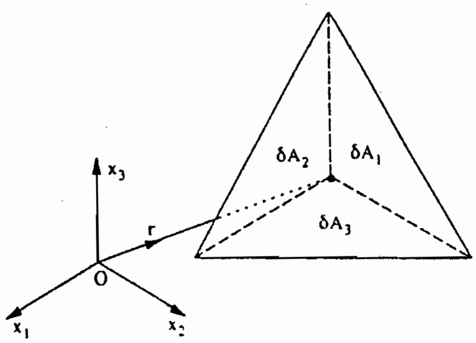

Fig. 2.2 Infinitesimal tetrahedron with vertex at **r** at time t.

Denoting the unit vectors along x_1, x_2, x_3 by $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, respectively, the outward normals to the areas δA_1, δA_2, δA_3 will be $-\mathbf{i}_1$, $-\mathbf{i}_2$, $-\mathbf{i}_3$, respectively. Let the area of the slanted face be denoted as σ and **n** its unit outward drawn normal. Then:

$$n_1 = \cos(\mathbf{n}, \mathbf{i}_1), \qquad n_2 = \cos(\mathbf{n}, \mathbf{i}_2), \qquad n_3 = \cos(\mathbf{n}, \mathbf{i}_3)$$

are the direction cosine of the normal **n**. The areas δA_1, δA_2, and δA_3 which are perpendicular to x_1, x_2, x_3 will then be $n_1\sigma$, $n_2\sigma$, $n_3\sigma$, respectively. Applying Equation 2.20 to the tetrahedron and using the integral mean value theorem, we get:

$$\lim_{\ell \to 0} \frac{1}{\ell^2} [\sigma \boldsymbol{\tau}(\boldsymbol{\lambda}, t; \mathbf{n}) + \sigma n_1 \boldsymbol{\tau}(\boldsymbol{\xi}, t; -\mathbf{i}_1) + \sigma n_2 \boldsymbol{\tau}(\boldsymbol{\eta}, t; -\mathbf{i}_2) + \sigma n_3 \boldsymbol{\tau}(\boldsymbol{\zeta}, t; -\mathbf{i}_3)] = 0 \tag{2.21}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are some points in the surface areas, σ, δA_1, δA_2, and δA_3, respectively. As the volume of the tetrahedron shrinks to zero, the area $\sigma = \ell^2 \to 0$ and the vectors, $\boldsymbol{\lambda}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ tend to the single value **r**, so that:

$$\boldsymbol{\tau}(\mathbf{r}, t; \mathbf{n}) = -n_k \boldsymbol{\tau}(\mathbf{r}, t; -\mathbf{i}_k)$$

or:

$$\boldsymbol{\tau}(\mathbf{n}) = -n_k \boldsymbol{\tau}(-\mathbf{i}_k) \tag{2.22}$$

To find $\boldsymbol{\tau}(-\mathbf{n})$ consider a thin fluid lamina in the form of a rectangular parallelepiped with two square faces as shown in Figure 2.3. Let ℓ be the length and $\epsilon\ell$ the thickness of the lamina, and **n** the normal on the square face.

Applying Equation 2.20 and the integral mean value theorem, we obtain:

$$\int_{S_0} \boldsymbol{\tau}\, dS = \ell^2[\boldsymbol{\tau}(\mathbf{n}) + \boldsymbol{\tau}(-\mathbf{n}) + \epsilon \mathbf{Q}]$$

* A violation of this principle would mean that there exists another process through which the "spared" stresses are transmitted to the rest of the continuum. Such a process is physically untenable.

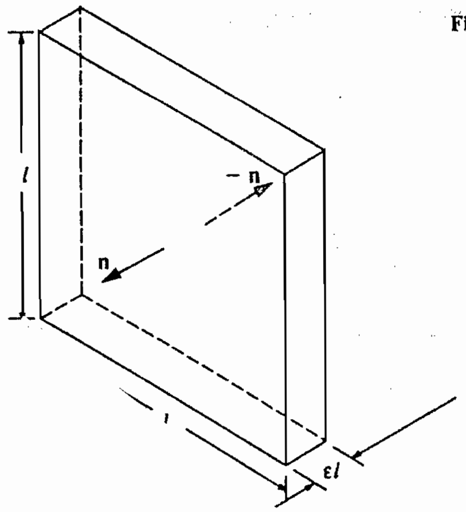

Fig. 2.3 Square-faced thin fluid lamina.

...om the faces each having area $\epsilon\ell^2$. Since ϵ is arbitrarily

$$= 0$$

$$= -\boldsymbol{\tau}(\mathbf{n}) \tag{2.23}$$

...quation 2.22 we get:

$$\cdot\, n_1\boldsymbol{\tau}(\mathbf{i}_1) + n_2\boldsymbol{\tau}(\mathbf{i}_2) + n_3\boldsymbol{\tau}(\mathbf{i}_3)$$

$$= n_k\boldsymbol{\tau}(\mathbf{i}_k) \tag{2.24}$$

...aving three components, we must introduce a two-index symbol to ...ector character and its dependence on k. Thus we write:

$$\tau_j(\mathbf{i}_k) = T_{kj}$$

or:

$$\boldsymbol{\tau}(\mathbf{i}_k) = T_{kj}\mathbf{i}_j$$

Consequently, Equation 2.24 becomes:

$$\boldsymbol{\tau}(\mathbf{n}) = n_k T_{kj}\mathbf{i}_j \tag{2.25}$$

The quantities T_{kj} are the Cartesian components of a second order tensor $\mathbf{T}$, and they do not depend on the components of $\mathbf{n}$. As is clear from above, the index k denotes the plane perpendicular to the k-direction and the index j denotes the direction of the component.

In component form Equation 2.25 is written as:

$$\tau_j(\mathbf{n}) = n_k T_{kj}, \qquad j = 1, 2, 3 \tag{2.26}$$

A formula which should be valid in *any* coordinate system is then:

$$\boldsymbol{\tau} = \mathbf{n} \cdot \mathbf{T} \tag{2.27a}$$

$$= \mathbf{T}^T \cdot \mathbf{n} \tag{2.27b}$$

where $\mathbf{T}^T$ is the transpose of the tensor $\mathbf{T}$. $\mathbf{T}$ is called the Cauchy stress tensor.
We now summarize the main properties of the stress vector $\boldsymbol{\tau}$:

1. The stress vectors are in local equilibrium (Equation 2.20).
2. The stress vector $\boldsymbol{\tau}$ is the force per unit area exerted by that part of the fluid into which $\mathbf{n}$ is directed.
3. The stress vector at a point is a *linear function* of the components of the normal to any surface passing through the point (Equation 2.26).

Symmetry of T

The stress tensor $\mathbf{T}$ introduced in Equation 2.27a is a second order tensor, and it has nine components when referred to a coordinate system in a three-dimensional space. For the purpose of studying an important property of the stress tensor $\mathbf{T}$, we consider the law of conservation of the angular momentum (Equation 2.19) for an arbitrary small volume $V_0 \sim \ell^3$, where ℓ is a linear dimension. Dividing each term of Equation 2.19 by ℓ^3, applying the integral mean value theorem, and then taking the limit as $\ell \to 0$, we find that:

$$\operatorname*{Lim}_{\ell \to 0} \frac{1}{\ell^3} \int_{S_0} (\mathbf{r} \times \boldsymbol{\tau})\, dS = 0 \tag{2.28}$$

The reason for this reduction is that the term on the left and the first term on the right of Equation 2.19 are each of the $0(\ell^4)$, and they vanish like ℓ as $\ell \to 0$. Using Equation 2.27b in Equation 2.28 and noting that:

$$\mathbf{r} \times \boldsymbol{\tau} = \mathbf{r} \times (\mathbf{T}^T \cdot \mathbf{n}) = (\mathbf{r} \times \mathbf{T}^T) \cdot \mathbf{n}$$

we apply Gauss' divergence theorem (Equation M2.2) and have:

$$\operatorname*{Lim}_{\ell \to 0} \frac{1}{\ell^3} \int_{V_0} \operatorname{div}(\mathbf{r} \times \mathbf{T}^T)\, dv = 0 \tag{2.29}$$

Using Cartesian coordinates, we can write Equation 2.29 as:

$$\operatorname*{Lim}_{\ell \to 0} \frac{1}{\ell^3} \int_{V_0} e_{ijk}\left(T_{jk} + x_j \frac{\partial T_{mk}}{\partial x_m}\right) dv = 0$$

Since the second term is $0(\ell^4/\ell^3)$, it vanishes. Thus, for the satisfaction of Equation 2.29 we must have:

$$e_{ijk} T_{jk} = 0, \qquad i = 1, 2, 3$$

Thus:

$$T_{jk} = T_{kj} \tag{2.30a}$$

or:

$$\mathbf{T}^{T} = \mathbf{T} \tag{2.30b}$$

i.e., the tensor **T** must be symmetric. From the preceding analysis we conclude that the law of conservation of angular momentum can be satisfied only when **T** is symmetric. The symmetry of **T** is also called Boltzmann's hypothesis. Noting the symmetry of **T** *from here onward* we shall write Equation 2.27 simply as:

$$\tau_j = T_{jk} n_k$$

or:

$$\boldsymbol{\tau} = \mathbf{T} \cdot \mathbf{n} \tag{2.31}$$

2.6 EQUATIONS OF LINEAR AND ANGULAR MOMENTUM

Having obtained the expression for $\boldsymbol{\tau}$ in terms of $\mathbf{T}^{T}$ (Equation 2.27b), we reconsider Equation 2.16 which becomes:

$$\int_{\mathcal{V}(t)} \rho \frac{D\mathbf{u}}{Dt}\, dv = \int_{\mathcal{V}(t)} \rho \mathbf{f}\, dv + \int_{\mathcal{S}(t)} \mathbf{T}^{T} \cdot n\, d\mathbf{S}$$

On using Gauss' theorm, Equation M2.2, we have:

$$\int_{\mathcal{V}(t)} \rho \frac{D\mathbf{u}}{Dt}\, dv = \int_{\mathcal{V}(t)} \rho \mathbf{f}\, dv + \int_{\mathcal{V}(t)} \operatorname{div} \mathbf{T}^{T}\, dv \tag{2.32}$$

which is the required integral form of the law of conservation of linear momentum. Equation 2.32 is, of course, true for a fixed volume V_0. Since Equation 2.32 is valid for any volume $\mathcal{V}(t)$, the differential form of the law of conservation of momentum becomes:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} + \operatorname{div} \mathbf{T}^{T}, \qquad (\mathbf{T}^{T} = \mathbf{T}) \tag{2.33}$$

which is the general equation for the motion of a continuum. In all the succeeding equations, we have used the symmetry property $\mathbf{T}^{T} = \mathbf{T}$. Referring to the conclusion following Equation 2.17, we conclude that div **T** is the force per unit volume exerted by the fluid surrounding the volume.

Another form of the equations can be obtained if we proceed from Equation 2.15 and use Equation 2.31, and Equation 1.32c with F replaced by $\rho\mathbf{u}$, thus having:

$$\int_{\mathcal{V}(t)} \frac{\partial}{\partial t} (\rho \mathbf{u})\, dv - \int_{\mathcal{S}(t)} \boldsymbol{\Pi} \cdot \mathbf{n}\, dS = \int_{\mathcal{V}(t)} \rho \mathbf{f}\, dv \tag{2.34}$$

where

$$\boldsymbol{\Pi} = \mathbf{T} - \rho \mathbf{u}\mathbf{u}$$

Equation 2.34 is also valid for a fixed volume V_0. Again using Gauss' theorem (Equation M2.3) in Equation 2.34, we have:

$$\int \left[\frac{\partial}{\partial t} (\rho \mathbf{u}) - \operatorname{div} \boldsymbol{\Pi} - \rho \mathbf{f} \right] dv = 0 \tag{2.35a}$$

where the above integral can be either on a material or a fixed volume. Since Equation 2.35a is valid for arbitrary volumes, the differential conservation law form of the equation of motion is

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) - \operatorname{div} \mathbf{\Pi} = \rho \mathbf{f} \tag{2.35b}$$

Following the same procedure or else cross multiplying each term of Equation 2.35b by the position vector **r** drawn from a fixed point, we get:

$$\frac{\partial}{\partial t}(\mathbf{r} \times \rho \mathbf{u}) - \operatorname{div}(\mathbf{r} \times \mathbf{\Pi}) = \mathbf{r} \times \rho \mathbf{f} \tag{2.35c}$$

where, because of the symmetry of $\mathbf{\Pi}$:

$$\mathbf{r} \times \operatorname{div} \mathbf{\Pi} = \operatorname{div}(\mathbf{r} \times \mathbf{\Pi})$$

Using the expression for $\mathbf{\Pi}$ given above, we have:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u}\mathbf{u}) = \rho \mathbf{f} + \operatorname{div} \mathbf{T} \tag{2.36a}$$

as the conservation law form of the linear momentum, and:

$$\frac{\partial}{\partial t}(\mathbf{r} \times \rho \mathbf{u}) + \operatorname{div}(\mathbf{r} \times \rho \mathbf{u}\mathbf{u}) = \mathbf{r} \times \rho \mathbf{f} + \operatorname{div}(\mathbf{r} \times \mathbf{T}) \tag{2.36b}$$

as the conservation law form of the angular momentum.

It is easy to check that by using the continuity equation (Equation 2.6) in Equation 2.33 one gets Equation 2.36a. It must, however, be clear that the original derivation of Equations 2.36a or 2.36b does not, in any way, depend on the satisfaction of the continuity equation.

2.7 MOMENTUM CONSERVATION USING A CONTROL VOLUME

The conservation of mass equation for a control volume has already been obtained earlier as Equation 2.13. To obtain the conservation of momentum equation for a control volume we replace F by ρu in Equation 1.36 and use Equations 2.15 and 2.31. Thus, the required equation is

$$\frac{d}{dt}\int_{V^*} \rho \mathbf{u} \, dv^* + \int_{S^*} \rho \mathbf{u}(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* = \int_{V^*} \rho \mathbf{f} \, dv^* + \int_{S^*} \mathbf{T} \cdot n^* \, dS^* \tag{2.37a}$$

In Equation 2.37a both **u** and **c** are the inertial velocities. If $\mathbf{c} = \mathbf{c}(t)$, then writing $\mathbf{u}_r = \mathbf{u} - \mathbf{c}$ in Equation 2.37a and using Equation 2.13 we get:

$$\frac{d}{dt}\int_{V^*} \rho \mathbf{u}_r \, dv^* + \int_{S^*} \rho \mathbf{u}_r \mathbf{u}_r \cdot \mathbf{n}^* \, dS^* = -\int_{V^*} \rho \frac{d\mathbf{c}}{dt} \, dv^* + \int_{V^*} \rho \mathbf{f} \, dv^* + \int_{S^*} \mathbf{T} \cdot n^* \, dS^* \tag{2.37b}$$

For a control volume which translates and also rotates, we introduce the velocity $\mathbf{u}_r$ which is measured with respect to the control volume and defined through:

$$\mathbf{u} = \dot{\mathbf{p}} + \mathbf{u}_r + \mathbf{\Omega} \times \mathbf{r}$$

where $\mathbf{p}$ is the position vector of the origin of the moving coordinates attached with the control volume and $\boldsymbol{\Omega}$ is the angular velocity of these coordiantes. Taking all the time derivatives to be measured with respect to the control volume, we have:

$$\frac{\partial}{dt}\int_{V^*} \rho \mathbf{u}_r\, dv^* + \int_{S^*} \rho \mathbf{u}_r(\mathbf{u}_r - \mathbf{c}) \cdot \mathbf{n}^*\, dS^* = -\int_{V^*} \rho \mathbf{A}\, dv^* + \int_{V^*} \rho \mathbf{f}\, dv^* + \int_{S^*} \mathbf{T} \cdot n^*\, dS^* \quad (2.38)$$

where A is the part of acceleration due to the noninertial frame. (Refer to Problem 3.8(b)). In Equation 2.38, $\mathbf{c}(\mathbf{r}, t)$ is the speed of deformation of the control volume surface. If the control surface is nondeforming, then $\mathbf{c}(\mathbf{r}, t) = 0$. For a nondeforming but rotating control volume:

$$\int_{V^*} \frac{\partial}{\partial t}(\rho \mathbf{u}_r) dv^* + \int_{S^*} \rho \mathbf{u}_r \mathbf{u}_r \cdot \mathbf{n}^*\, dS^* = -\int_{V^*} \rho \mathbf{A}\, dv^* + \int_{V^*} \rho \mathbf{f}\, dv^* + \int_{S^*} \mathbf{T} \cdot \mathbf{n}^*\, dS^*$$

2.8 CONSERVATION OF ENERGY

The third and the last law governing the motion of fluids is the law of conservation of energy. In this connection, we first note that the specific internal energy e is a consequence of the random motion of the molecules with respect to the averaged translatory motion of the molecules.[2] From a continuum viewpoint, we also have to consider kinetic energy of fluid masses due to the motion of fluid particles. Below we first develop an equation governing the change of kinetic energy fluid motion, which is independent of the law of conservation of energy.

The kinetic energy per unit volume of a fluid of density ρ is

$$\tfrac{1}{2}\rho(\mathbf{u} \cdot \mathbf{u}) = \tfrac{1}{2}\rho|\mathbf{u}|^2$$

Consider an arbitrary material volume $\mathcal{V}(t)$. The kinetic energy in the volume $\mathcal{V}(t)$ is

$$E = \int_{\mathcal{V}(t)} \tfrac{1}{2}\rho|\mathbf{u}|^2\, dv$$

To find the transport equation for E, we multiply each term of Equation 2.33 scalarly by $\mathbf{u}$ and integrate over the material volume $\mathcal{V}(t)$. Thus, using Equation 2.9, we get:

$$\frac{dE}{dt} = \int_{\mathcal{V}(t)} \rho(\mathbf{u} \cdot \mathbf{f})\, dv + \int_{\mathcal{V}(t)} (\operatorname{div} \mathbf{T}) \cdot \mathbf{u}\, dv \quad (2.39)$$

Consider now the term $\operatorname{div}(\mathbf{T} \cdot \mathbf{u})$, which on using Equation M1.44(b) and the symmetry of $\mathbf{T}$, can be written as:

$$\operatorname{div}(\mathbf{T} \cdot \mathbf{u}) = (\operatorname{div} \mathbf{T}) \cdot \mathbf{u} + \mathbf{T} : \operatorname{grad} \mathbf{u}$$

where $\mathbf{T} : \operatorname{grad} \mathbf{u}$ means the inner scalar multiplication of the tensors T and grad $\mathbf{u}$. (Refer to Equation M1.25.) Further since T is symmetric, we have:

$$\mathbf{T} : \operatorname{grad} \mathbf{u} = \mathbf{T} : \mathbf{D} \quad (2.40)$$

where $\mathbf{D}$ is the rate-of-strain tensor. Thus:

$$(\operatorname{div} \mathbf{T}) \cdot \mathbf{u} = \operatorname{div}(\mathbf{T} \cdot \mathbf{u}) - \mathbf{T} : \mathbf{D} \quad (2.41)$$

Using Equation 2.41 and the Gauss' divergence theorem, we get:

$$\frac{dE}{dt} = \int_{\mathcal{V}(t)} \rho(\mathbf{u} \cdot \mathbf{f})\, d\nu + \int_{\mathcal{S}} (\mathbf{T} \cdot \mathbf{u}) \cdot \mathbf{n}\, dS - \int_{\mathcal{V}(t)} \mathbf{T} : \mathbf{D}\, d\nu$$

However, according to Equation M1.19 and the symmetry of $\mathbf{T}$, we have:

$$\begin{aligned}(\mathbf{T} \cdot \mathbf{u}) \cdot \mathbf{n} &= \mathbf{u} \cdot (\mathbf{T} \cdot \mathbf{n}) \\ &= \mathbf{u} \cdot \boldsymbol{\tau}\end{aligned}$$

Thus:

$$\frac{dE}{dt} = \int_{\mathcal{V}(t)} \rho(\mathbf{u} \cdot \mathbf{f})\, d\nu + \int_{\mathcal{S}} \mathbf{u} \cdot \boldsymbol{\tau}\, dS - \int_{\mathcal{V}(t)} \mathbf{T} : \mathbf{D}\, d\nu \qquad (2.42)$$

To sum up, the rate of change of kinetic energy in a volume $\mathcal{V}$ of fluid *as it moves with the flow* is composed of three terms. The first two terms on the right-hand side account for the rate of work being done on the volume by the body and surface forces while the last term accounts for the rate at which work is being done in changing the volume and shape of fluid elements. Part of the work contained in the last term may be recoverable, but the remainder is the lost work which is destroyed or dissipated as heat due to the internal friction.

To find the differential equation, we consider Equation 2.39 along with the substitution of Equation 2.41 and the obvious identity:

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt} \int_{\mathcal{V}(t)} \tfrac{1}{2} \rho\, |\mathbf{u}|^2\, d\nu \\ &= \int_{\mathcal{V}(t)} \tfrac{1}{2} \rho \frac{D}{Dt} |\mathbf{u}|^2\, d\nu\end{aligned}$$

to have:

$$\rho \frac{D}{Dt} (\tfrac{1}{2} |\mathbf{u}|^2) = \rho(\mathbf{u} \cdot \mathbf{f}) + \operatorname{div}(\mathbf{T} \cdot \mathbf{u}) - \mathbf{T} : \mathbf{D} \qquad (2.43)$$

which is the differential equation of the kinetic energy in fluid motion.

The principle of conservation of energy is stated as: "*The rate of change of total energy of a material volume is equal to the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume.*"

Let e be the specific internal energy of the fluid. Then the internal energy in the material volume $\mathcal{V}(t)$ is

$$U = \int_{\mathcal{V}(t)} \rho e\, d\nu$$

Let $\mathbf{q}$ be the heat output rate and $\mathbf{n}$ the outward drawn normal on the surface $\mathcal{S}$ of $\mathcal{V}(t)$. Then $-\mathbf{q} \cdot \mathbf{n}\, dS$ is the heat conducted into the volume through the area element dS. The law of conservation of energy is then stated as:

$$\frac{d}{dt}(E + U) = \int_{\mathcal{V}(t)} \rho(\mathbf{u} \cdot \mathbf{f})\, d\nu + \int_{\mathcal{S}} (\mathbf{u} \cdot \boldsymbol{\tau})\, dS - \int_{\mathcal{S}} \mathbf{q} \cdot \mathbf{n}\, dS \qquad (2.44)$$

If the contribution of a distributed internal heat source is important to the energy balance, then the term $\int_V \rho r \, dv$ must be added on the right hand side of Equation 2.44, where r is the strength of the distributed internal heat source per unit mass. This term must then be carried in all the subsequent equations.

Using Equation 2.42 in Equation 2.44, we have:

$$\frac{dU}{dt} = \frac{d}{dt}\int_{\mathcal{V}(t)} \rho e \, dv$$
$$= \int_{\mathcal{V}(t)} \mathbf{T} : \mathbf{D} \, dv - \int_{\mathcal{S}} \mathbf{q} \cdot \mathbf{n} \, dS \tag{2.45}$$

Using Gauss' divergence theorem and Equation 2.9 in Equation 2.45, we get:

$$\int_{\mathcal{V}(t)} \rho \frac{De}{Dt} dv = \int_{\mathcal{V}(t)} \mathbf{T} : \mathbf{D} \, dv - \int_{\mathcal{V}(t)} \text{div}\, \mathbf{q} \, dv \tag{2.46}$$

Since $\mathcal{V}(t)$ is arbitrary, the differential equation for e is given as:

$$\rho \frac{De}{Dt} = \mathbf{T} : \mathbf{D} - \text{div}\, \mathbf{q} \tag{2.47}$$

In the same manner, using Gauss' divergence theorem and Equation 2.9 in Equation 2.44 and for brevity writing the total energy as:

$$e_t = e + \tfrac{1}{2}|\mathbf{u}|^2 \tag{2.48}$$

we get:

$$\rho \frac{De_t}{Dt} = \rho(\mathbf{f} \cdot \mathbf{u}) + \text{div}(\mathbf{T} \cdot \mathbf{u}) - \text{div}\, \mathbf{q} \tag{2.49}$$

as the equation of total energy.

Equations 2.47 and 2.49 are the basic equations for the rate of change of energy. Another useful form can be obtained by using the continuity equation (Equation 2.6) in Equation 2.49, which is

$$\frac{\partial}{\partial t}(\rho e_t) + \text{div}(\rho e_t \mathbf{u} - \mathbf{T} \cdot \mathbf{u} + \mathbf{q}) = \rho(\mathbf{f} \cdot \mathbf{u}) \tag{2.50a}$$

Further, if f is derivable from a potential χ, i.e.:

$$\mathbf{f} = -\text{grad}\, \chi$$

then the equation of energy is

$$\frac{\partial}{\partial t}(\rho e_t) + \text{div}(\rho e_t \mathbf{u} - \mathbf{T} \cdot \mathbf{u} + \mathbf{q}) = -\rho(\text{grad}\, \chi) \cdot \mathbf{u} \tag{2.50b}$$

If χ is *independent of time*, then using the continuity equation, Equation 2.50b can be written as:

$$\frac{\partial}{\partial t}\{\rho(e_t + \chi)\} + \operatorname{div}[\rho(e_t + \chi)\mathbf{u} - \mathbf{T}\cdot\mathbf{u} + \mathbf{q}] = 0 \tag{2.50c}$$

2.9 ENERGY CONSERVATION USING A CONTROL VOLUME

In Equation 1.36 replacing F by the total energy ρe_t per unit volume and using Equation 2.44, we get:

$$\frac{d}{dt}\int_{V^*} \rho e_t \, dv^* + \int_{S^*} \rho e_t(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^*$$

$$= \int_{V^*} \rho(\mathbf{u} \cdot \mathbf{f}) \, dv^* + \int_{S^*} (\mathbf{u} \cdot \boldsymbol{\tau}) \, dS^* - \int_{S^*} \mathbf{q} \cdot \mathbf{n}^* \, dS^* \tag{2.51}$$

which is the energy equation for a control volume.

2.10 GENERAL CONSERVATION PRINCIPLE

The integral forms of the conservation of mass, momentum, and energy as stated in Equations 2.3, 2.15, and 2.42, respectively, can all be joined together into a single statement. The general conservation law for classical fields in integral form for a material volume is stated as:

$$\int_{\mathcal{V}(t_2)} A \, dv - \int_{\mathcal{V}(t_1)} A \, dv + \int_{t_1}^{t_2} \int_{\mathcal{S}(t)} B \cdot \mathbf{n} \, dS \, dt = \int_{t_1}^{t_2} \int_{\mathcal{V}(t)} C \, dv \, dt \tag{2.52}$$

If all variables are assumed *continuous in time*, then the conservation law is

$$\frac{d}{dt}\int_{\mathcal{V}(t)} A \, dv + \int_{\mathcal{S}(t)} B \cdot \mathbf{n} \, dS = \int_{\mathcal{V}(t)} C \, dv \tag{2.53}$$

where $\mathbf{n}$ is the unit outward drawn normal to the surface $\mathcal{S}$ which encloses the material volume $\mathcal{V}(t)$.

The integral law (Equation 2.53) states that for a given material volume $\mathcal{V}(t)$ bounded by the surface $\mathcal{S}(t)$ the rate of change of what is contained in $\mathcal{V}(t)$ at time t plus the rate of flux out of $\mathcal{S}$ is equal to what is furnished to $\mathcal{V}(t)$. The quantities A, B, C are tensor quantities such that A and C have the same tensorial order. If $B \neq 0$, then it is a one order higher tensor than A.

Using the Reynolds transport theorem (Equation 1.32b) for the first term on the left of Equation 2.53, we obtain:

$$\int_{\mathcal{V}(t)} \left[\frac{\partial A}{\partial t} + \operatorname{div} f - C\right] dv = 0 \tag{2.54a}$$

where:

$$f = A\mathbf{u} + B \tag{2.54b}$$

Thus, Equation 2.54a is a general statement of the conservation law. Since it is valid for arbitrary volumes, the differential conservation law is

$$\frac{\partial A}{\partial t} + \operatorname{div} f = C \tag{2.55}$$

Equation 2.55 represents a model for the differential conservation law for classical fields.

Integral conservation form for a finite volume with arbitrarily moving boundaries can be directly obtained by using the formula for the change of time derivatives between a material volume and a control volume, i.e., Equation 1.36, in Equation 2.53. The required conservation law form for a finite volume in coordinate invariant form is

$$\frac{d}{dt}\int_{V^*(t)} A dv^* + \int_{S^*} F \cdot \mathbf{n}^* \, dS^* = \int_{V^*(t)} C dv^* \tag{2.56}$$

where:

$$F = A(\mathbf{u} - \mathbf{c}) + B$$

2.11 THE CLOSURE PROBLEM

The appearance of the stress tensor **T** in the governing equations of motion and energy renders the system of equations unsolvable unless **T** can be expressed in terms of the other dependent variables of the equations. The problem of specifying **T** in terms of other dependent variables is called the "*closure problem*". We proceed on the closure problem as follows.

First of all, since **T** is a second order symmetric tensor, it is also expressible as a 3 × 3 symmetric matrix in a 3-D space as:

$$\mathbf{T} = [T_{ij}]$$

where i stands for the row and j for the column. Further, because of symmetry:

$$T_{ij} = T_{ji}$$

The physical meaning of T_{ij} is that it is a stress (force per unit area) on a plane perpendicular to the i-axis in the direction of the j-axis. (Refer to the discussion following Equation 2.25.) Thus, T_{11}, T_{22}, T_{33} are the normal stresses, while T_{12}, T_{23}, T_{13} are the shear stresses at a point in the flow field. To analyze the problem more clearly, let us introduce a transformation from x_1, x_2, x_3 to $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ such that the tensor T referred to the new coordinates has no shearing components. The axes $\bar{x}_i$ are then called the principal axes. An important property of the principal axis transformation is that:

$$\bar{T}_{ii} = T_{ii} \quad \text{sum on } i \tag{2.57}$$

The transformed matrix **T** can now be written as the sum of two matrices as:[3]

$$\begin{pmatrix} \bar{T}_{11} & 0 & 0 \\ 0 & \bar{T}_{22} & 0 \\ 0 & 0 & \bar{T}_{33} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} \tag{2.58a}$$

where:

$$a = \tfrac{1}{3} T_{ii}, \qquad b = \bar{T}_{11} - a, \qquad c = \bar{T}_{22} - a, \qquad d = \bar{T}_{33} - a \tag{2.58b}$$

By virtue of Equation 2.57:

$$b + c + d = 0 \tag{2.58c}$$

The first matrix on the right of Equation 2.58a has all equal terms on its principal diagonal. Consequently, this matrix is isotropic and therefore has spherical symmetry. Physically it means that if a small sphere is drawn with **r** as center, then the force exerted per unit area across the surface is everywhere a. This uniform force for a fluid can only be a compression as is shown in Figure 2.4(a).

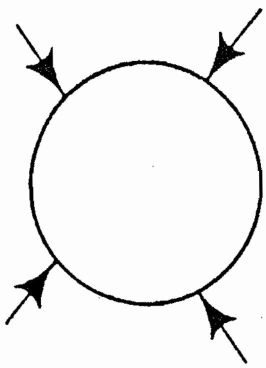

Fig. 2.4(a) Imaginary fluid sphere under uniform compression

The second term on the right of Equation 2.58a represents a departure of the stress from isotropy. In view of Equation 2.58c we conclude that a small fluid sphere is now under the influence of both tension and compression along the principal axes of such magnitudes that Equation 2.58c is satisfied. Under the influence of such forces the small sphere deforms to the shape of an ellipsoid in a very short moment of time, as is shown in Figure 2.4(b), thus ensuing a fluid motion.

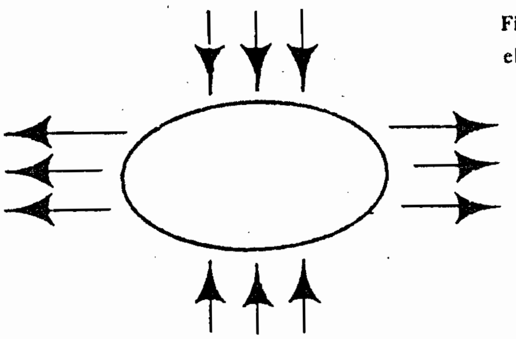

Fig. 2.4(b) Deformation of an imaginary fluid sphere to an ellipsoid in an infinitesimal time

From the preceding analysis we conclude that for a fluid at rest $b = c = d = 0$, which imply that:

$$\bar{T}_{11} = \bar{T}_{22} = \bar{T}_{33} = 1/3\ T_{ii}$$

Denoting the common value by $-p_s$, we find that:

$$T_{ii} = -3p_s$$

and which implies that for a fluid at rest:

$$\bar{T}_{ij} = -p_s\delta_{ij} \tag{2.59a}$$

If the principal axis-transformation of a tensor has components (Equation 2.59a), then the components of the original tensor referred to any rectangular Cartesian system have the same representation, i.e.:

$$T_{ij} = -p_s\delta_{ij} \tag{2.59b}$$

(Refer to ME.1, Section 9.) Therefore, if $\mathbf{n}$ is a local unit normal to any arbitrary surface drawn in a fluid at rest, then the force exerted across a unit area is $-p_s\mathbf{n}$.

The state of fluid at rest is a particular case of the state of fluid in motion. Following the decomposition in Equation 2.58a, we write the stress tensor of a fluid in motion as a sum of two parts. One part of $\mathbf{T}$ must be isotropic and the other nonisotropic or deviatoric. We therefore write:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\sigma} \tag{2.60}$$

where $-p\mathbf{I}$ is the isotropic part and $\boldsymbol{\sigma}$ the deviatoric part of $\mathbf{T}$. When rectangular Cartesian coordinates are used, the components of $\mathbf{T}$ are given by:

$$T_{ij} = -p\delta_{ij} + \sigma_{ij} \tag{2.61}$$

The scalar p is called the fluid pressure, or sometimes the static pressure, in a fluid in motion. For fluid at rest $\boldsymbol{\sigma} \equiv 0$ and $p = p_s$. A basic postulate in the dynamics of compressible fluids is the identification of pressure p with the thermodynamic pressure. From a macroscopic viewpoint, this postulate is convincing since for a gas at rest, the residual stress is only the thermodynamic pressure. From basic thermodynamics,[3] the pressure is

$$p = -\left(\frac{\partial e}{\partial v}\right)_s \tag{2.62}$$

where $v = 1/\rho$ is the specific volume. Further, the combined thermodynamic law for compressible fluids under the condition of reversibility is

$$T\,ds = de + p\,dv \tag{2.63}$$

where both in Equations 2.62 and 2.63 s is the entropy per unit mass.

For an incompressible flow ρ = constant and the pressure cannot be defined by Equation 2.62. Thus, the pressure in an incompressible flow is not a thermodynamic variable. From Equation 2.63 the combined thermodynamic law for an incompressible flow is

$$T\,ds = de \tag{2.64}$$

2.12 STOKES' LAW OF FRICTION

Having obtained the form (Equation 2.60) for the stress tensor $\mathbf{T}$, the next step is to obtain a constitutive equation for the deviatoric part $\boldsymbol{\sigma}$. The stress tensor $\boldsymbol{\sigma}$ is a consequence of the resistance to the rate of deformation of fluid elements and forms a field of force represented by $\boldsymbol{\sigma}$. The phenomenon of resistance to the rate of deformation is attributed to an inherent property of fluids, called the internal friction. (Refer to Section 2.4.) The property of internal friction is responsible for the appearance of shearing, compressive, and tensile forces in fluid motion.

The preceding observations suggest that $\boldsymbol{\sigma}$ must depend on the gradients of velocity, i.e., on grad $\mathbf{u}$ or on $\mathbf{D}$, the rate-of-strain tensor. There are many fluids with structural properties such that $\boldsymbol{\sigma}$ depends not only on $\mathbf{D}$ but also on their time derivatives. Such fluids are said to have a *memory*, because the effects of the past events in time have an influence on their response at the present time. Fluids that do not have memory are termed *Newtonian* or *Stokesian* in contrast to the *non-Newtonian*, which have memory. We shall be concerned mostly with the

Newtonian fluids and only touch upon a simple model of a non-Newtonian fluid in Section 2.18.

Let us assume that for a Newtonian fluid:

$$\boldsymbol{\sigma} = \boldsymbol{\phi}(\mathbf{u}, \text{grad } \mathbf{u}, \mathbf{D}) \tag{2.65}$$

with the stipulation that:

$$\boldsymbol{\sigma} = 0 \tag{2.66}$$

for rigid body-type of fluid motion.

In Equation 2.65, $\boldsymbol{\sigma}$ is a tensor-valued function of its arguments. Based on the following two simple observations, in both of which $\boldsymbol{\sigma} = 0$ because of rigid body-type motion (Equation 2.66) we can reduce the number of arguments in Equation 2.65 to only one:

1. If a fluid mass translates as a rigid body, then every point has the same constant velocity. Thus, $\mathbf{u}$ = constant everywhere in the fluid mass. Also, there is no relative motion between the fluid layers and consequently no deformation of the fluid elements. Since grad $\mathbf{u} = 0$, $\mathbf{D} = 0$, and $\boldsymbol{\sigma} = 0$, $\boldsymbol{\sigma}$ cannot depend explicitly on $\mathbf{u}$ but can depend only on grad $\mathbf{u}$ and $\mathbf{D}$.
2. If a fluid mass rotates as a rigid body with a constant angular velocity $\boldsymbol{\Omega}$ about an axis, then the velocity of a fluid particle at position vector $\mathbf{r}$ from the axis has velocity $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$. It can easily be verified by differentiation that for this velocity field grad $\mathbf{u} \neq 0$, but $\mathbf{D} \equiv 0$. Since in this case $\boldsymbol{\sigma} \equiv 0$, we conclude that $\boldsymbol{\sigma}$ does not depend explicitly on grad $\mathbf{u}$ but only on $\mathbf{D}$.

   These arguments suggest that in general:

$$\boldsymbol{\sigma} = \boldsymbol{\phi}(\mathbf{D}) \tag{2.67}$$

viz., $\boldsymbol{\sigma}$ is only a function of the rate-of-strain tensor $\mathbf{D}$, with $\boldsymbol{\phi}$ as a tensor-valued function.

Stokes[1] stated the following basic postulates regarding the intrinsic nature of the stress tensor $\boldsymbol{\sigma}$.

The Postulates of Stokes

These postulates are

1. $\boldsymbol{\sigma}$ is a continuous function of $\mathbf{D}$ and is independent of all other kinematic variables.
2. The *form* of $\boldsymbol{\sigma}$ as a function of $\mathbf{D}$ does not depend either on position in space or on any preferred direction.
3. $\boldsymbol{\sigma}$ is a Galilean invariant.
4. For inviscid fluids $\boldsymbol{\sigma} = 0$ and $\mathbf{T} = -p\mathbf{I}$.

The first hypothesis is, of course, the Equation 2.67. The second hypothesis may alternatively be interpreted to state that when $\boldsymbol{\sigma}$ is referred to rectangular Cartesian axes then the *form* of $\boldsymbol{\sigma}$ is independent of the position in space and the orientation of the axes. That is

$$\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^{-1} = \boldsymbol{\phi}(\mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^{-1})$$

for all orthogonal transformations $\mathbf{Q}$. (Refer to ME.9.) The third hypothesis states that when $\boldsymbol{\sigma}$ is referred to a Galilean frame of reference then its *form* is not different from the *form* in a stationary frame of reference. The fourth hypothesis pertains to a hypothetical fluid medium which does not produce any resistance to deformation for which $\mathbf{D} \neq 0$ but $\boldsymbol{\sigma} = 0$.

Stokesian Stress Tensor

Besides these noted postulates, Stokes further assumed that $\boldsymbol{\sigma}$ is a *linear function* of $\mathbf{D}$. Let us consider a system of rectangular Cartesian coordinates and according to Equation 2.67, write:

$$\sigma_{ij} = B_{ijk\ell} D_{k\ell} \tag{2.68}$$

where there is an implicit sum on k and ℓ. According to Stokes' second postulate the tensor $B_{ijk\ell}$ must be an isotropic tensor.* Taking the result of Equation iii in Example 2.1(b) we then have:

$$B_{ijk\ell} = A_2 \delta_{ij} \delta_{k\ell} + A_3 \delta_{ik} \delta_{j\ell} + A_4 \delta_{i\ell} \delta_{jk} \tag{2.69}$$

where A_2, A_3, A_4 are scalar invariants. Substituting Equation 2.69 in Equation 2.68 and using the symmetry of σ_{ij}, we get:

$$A_4 = A_3$$

Thus:

$$\begin{aligned} \sigma_{ij} &= A_2(\text{div } \mathbf{u})\delta_{ij} + A_3\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= A_2(\text{div } \mathbf{u})\delta_{ij} + 2A_3 D_{ij} \end{aligned}$$

Writing:

$$A_2 = \lambda, \qquad A_3 = \mu$$

we then have:

$$\sigma_{ij} = \lambda(\text{div } \mathbf{u})\delta_{ij} + 2\mu D_{ij} \tag{2.70}$$

and thus the complete stress components are

$$T_{ij} = -p\delta_{ij} + \lambda(\text{div } \mathbf{u})\delta_{ij} + 2\mu D_{ij} \tag{2.71}$$

The scalar invariants μ and λ depend on the thermodynamic state variables (e.g., T, e, etc.) and are called the first and second coefficients of viscosity, respectively. The tensor-invariant form of Equation 2.70 is then:

$$\boldsymbol{\sigma} = \lambda(\text{div } \mathbf{u})\mathbf{I} + 2\mu\mathbf{D} \tag{2.72}$$

and therefore the complete stress tensor is

$$\mathbf{T} = -p\mathbf{I} + \lambda(\text{div } \mathbf{u})\mathbf{I} + 2\mu\mathbf{D} \tag{2.73}$$

It is sometimes useful to define a quantity μ', called the bulk coefficient of viscosity, and defined as:

$$\mu' = \lambda + 2/3\,\mu$$

* In fluids having statistically isotropic molecular structures the stress $\boldsymbol{\sigma}$ generated in an element of fluid by a given $\mathbf{D}$ is independent of the orientation of the element.

In terms of μ', the constitutive equation (Equation 2.73) is

$$\mathbf{T} = [-p + (\mu' - \tfrac{2}{3}\mu)\text{div } \mathbf{u}]\mathbf{I} + 2\mu\mathbf{D} \tag{2.74}$$

For the case of *incompressible flow*, div $\mathbf{u} = 0$ and then the constitutive equation simplifies to:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} \tag{2.75}$$

Example 2.1

(a) Show that an *isotropic* tensor of the second order can always be represented in terms of one *invariant* α as:

$$A_{ij} = \alpha\delta_{ij} \tag{i}$$

in the Cartesian coordinates, and as:

$$A_{ij} = \alpha g_{ij} \tag{ii}$$

in general curvilinear coordinates. In Equation i A_{ij} are the Cartesian components, while in Equation ii A_{ij} are the covariant components of $\mathbf{A}$.

(b) Show that an *isotropic* tensor of the fourth order can always be represented in terms of three invariants α, β, γ as:

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \tag{iii}$$

in the Cartesian coordinates, and as:

$$A_{ijkl} = \alpha g_{ij}g_{kl} + \beta g_{ik}g_{jl} + \gamma g_{il}g_{jk} \tag{iv}$$

in general curvilinear coordinates. In Equation iii A_{ijkl} are the Cartesian components, while in Equation iv A_{ijkl} are the covariant components of $\mathbf{A}$.

The Cartesian components of a tensor in an Euclidean space which follow Equation M9.10 form a Cartesian tensor; refer to ME.9. An isotropic tensor is a Cartesian tensor which transforms into itself under a rotation of axes. This exercise enables us to establish the *forms* of second and fourth order isotropic tensors in terms of their invariants, with reference to both the Cartesian and the embedded curvilinear coordinates.

An invariant is a quantity which does not change its value at a point when referred to different coordinate systems. Invariants come into existence either due to the physical nature of quantities (e.g., temperature, absolute density, etc. at a point), or due to some mathematical constructions. As an example the invariant of a vector $\mathbf{u}$ is its magnitude $(\mathbf{u} \cdot \mathbf{u})^{1/2}$. Similarly, for two vectors $\mathbf{u}$ and $\mathbf{v}$, the invariants are

$$\mathbf{u} \cdot \mathbf{u}, \qquad \mathbf{v} \cdot \mathbf{v}, \qquad \mathbf{u} \cdot \mathbf{v}$$

On the other hand, a tensor of second order has three invariants which depend on its eigenvalues. If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues, then the three invariants are

$$\mathrm{I} = \lambda_1 + \lambda_2 + \lambda_3$$

$$\mathrm{II} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$\mathrm{III} = \lambda_1\lambda_2\lambda_3$$

For an isotropic second order tensor all the eigenvalues are real and equal, viz., $\lambda_1 = \lambda_2 = \lambda_3$.

To prove the results in Equations i and ii we consider a second order tensor **A** and select two arbitrary vectors **u** and **v**. We form the invariant:

$$(\mathbf{A} \cdot \mathbf{v}) \cdot \mathbf{u} \equiv A_{ij}u_iv_j \tag{v}$$

and impose the requirement that for **A** to be isotropic the quantity in Equation v must be equal to a bilinear function in the components of **u** and **v**. That is:

$$(\mathbf{A} \cdot \mathbf{v}) \cdot \mathbf{u} = \alpha(\mathbf{u} \cdot \mathbf{v})$$

or:

$$A_{ij}u_iv_j = \alpha u_m v_m$$

However:

$$v_m = \delta_{mr}v_r$$

so that:

$$\begin{aligned} A_{ij}u_iv_j &= \alpha\delta_{mr}u_mv_r \\ &= \alpha\delta_{ij}u_iv_j \end{aligned}$$

Thus:

$$A_{ij} = \alpha\delta_{ij}$$

which is the result in Equation i of Example 2.1. Note that this satisfies the above noted property on eigenvalues.

To prove the result in Equation ii of Example 2.1 we consider the equation:

$$A_{ij}u^iv^j = \alpha u^m v_m$$

However, according to Equation M1.78c:

$$v_m = g_{mr}v^r$$

so that:

$$\begin{aligned} A_{ij}u^iv^j &= \alpha g_{mr}u^mv^r \\ &= \alpha g_{ij}u^iv^j \end{aligned}$$

Thus:

$$A_{ij} = \alpha g_{ij}$$

where A_{ij} are the covariant components of **A** and g_{ij} are the covariant components of the metric tensor **I**.

To prove the results of the problem in Example 2.1(b) we consider a fourth order tensor and select four arbitrary vectors **u**, **v**, **w**, **p**. The invariant in Cartesian coordinates is

$$[((\mathbf{A} \cdot \mathbf{p}) \cdot \mathbf{w}) \cdot \mathbf{v}] \cdot \mathbf{u} \equiv A_{ijk\ell} u_i v_j w_k p_\ell \qquad \text{(vi)}$$

Again, because of the isotropy of A, we impose the requirement that the invariant in Equation vi must be quarticlinear in the components of the chosen vectors. Thus:

$$A_{ijk\ell} u_i v_j w_k p_\ell = \alpha(\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{p}) + \beta(\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{p}) + \gamma(\mathbf{u} \cdot \mathbf{p})(\mathbf{w} \cdot \mathbf{v})$$

where α, β, γ are the invariants. For Cartesian coordinates:

$$A_{ijk\ell} u_i v_j w_k p_\ell = \alpha u_m v_m w_n p_n + \beta u_m w_m v_n p_n + \gamma u_m p_m w_n v_n$$

In each term using (e.g., $v_m = \delta_{mr} v_r$, etc.) we get by manipulating the dummy indices:

$$A_{ijk\ell} = \alpha \delta_{ij} \delta_{k\ell} + \beta \delta_{ik} \delta_{j\ell} + \gamma \delta_{i\ell} \delta_{jk}$$

Following the same technique for curvilinear coordinates, but noting that the invariant has the form:

$$A_{ijk\ell} u^i v^j w^k p^\ell$$

and $\mathbf{u} \cdot \mathbf{v} = u^m \cdot v_m$, etc. we get:

$$A_{ijk\ell} = \alpha g_{ij} g_{k\ell} + \beta g_{ik} g_{j\ell} + \gamma g_{i\ell} g_{jk}$$

where $A_{ijk\ell}$ are the covariant components of A.

Example 2.2

Based on the available expression for the stress tensor T derive an expression for the stress vector $\boldsymbol{\tau}$.

In this connection it is useful first to verify the following tensor identities:

$$\mathbf{I} \cdot \mathbf{n} = \mathbf{n} \qquad \text{(i)}$$

$$2\mathbf{W} \cdot \mathbf{n} = \boldsymbol{\omega} \times \mathbf{n} \qquad \text{(ii)}$$

$$\mathbf{D} \cdot \mathbf{n} = \mathbf{W} \cdot \mathbf{n} + \mathbf{n}(\text{div } \mathbf{u}) + (\mathbf{n} \times \text{grad}) \times \mathbf{u} \qquad \text{(iii)}$$

Based on the identities in Equations i–iii it is easy to show that:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{T} \cdot \mathbf{n} \\ &= [-p + (\lambda + 2\mu)\text{div } \mathbf{u}]\mathbf{n} + \mu(\boldsymbol{\omega} \times \mathbf{n}) + 2\mu(\mathbf{n} \times \text{grad}) \times \mathbf{u} \end{aligned} \qquad \text{(iv)}$$

2.13 THE INTERPRETATION OF PRESSURE

It was mentioned earlier in Section 2.11 that the p appearing in the constitutive equation (Equations 2.60, 2.73, 2.74) is the thermodynamic pressure. Conditions in which it is the mean of the three normal stresses are discussed next.

Referring Equation 2.74 to rectangular Cartesian coordinates and adding the three normal stresses, we get:

$$T_{ii} = -3p + 3\mu'(\text{div } \mathbf{u})$$

Defining the mean pressure $\bar{p}$, which reduces to the fluid-static pressure p_s for a fluid at rest, as $\bar{p} = -1/3\, T_{ii} = -1/3(\mathbf{I} : \mathbf{T})$, we obtain:

$$p - \bar{p} = \mu' \text{ div } \mathbf{u} \tag{2.76}$$

Equation 2.76 shows that $p = \bar{p}$ when either $\mu' = 0$, or div $\mathbf{u} = 0$. The condition $\mu' = 0$ implies that:

$$3\lambda + 2\mu = 0 \tag{2.77}$$

which is called the Stokes relation and has been found to be strictly valid only for monatomic* gases. Now since p is a thermodynamic variable, it should depend only on the local instantaneous values of ρ and e and not on div $\mathbf{u}$. On the other hand, if $\bar{p} \neq p$, then a multiplication of both sides of Equation 2.76 by div $\mathbf{u}$ shows a nonzero energy dissipation due to μ', which will alter the nature of p as a thermodynamic variable.** Consequently, under normal conditions (i.e., div $\mathbf{u}$ not very large) it is safe to assume the validity of Equation 2.77 for all gases even though it may not be strictly true. Thus, both for the compressible and incompressible flows, the pressure is the negative of the mean of the three normal stresses.***

2.14 THE DISSIPATION FUNCTION

Having introduced the complete stress tensor for a Newtonian fluid, it is now possible to interpret the term $\mathbf{T} : \mathbf{D}$ appearing in the energy equations (Equations 2.43 and 2.47). Using Equation 2.60 we have:

$$\begin{aligned} \mathbf{T} : \mathbf{D} &= -p(\mathbf{I} : \mathbf{D}) + \boldsymbol{\sigma} : \mathbf{D} \\ &= -p \text{ div } \mathbf{u} + \boldsymbol{\sigma} : \mathbf{D} \end{aligned} \tag{2.78}$$

The term $\boldsymbol{\sigma} : \mathbf{D}$ represents the rate of work done by the viscous part of the stresses in the deformation process of a unit volume of fluid. Mathematically $\boldsymbol{\sigma} : \mathbf{D}$ is the scalar inner product of two symmetric tensors. (Refer to Equation M1.25.) Because of the symmetry of $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} : \mathbf{D} = \boldsymbol{\sigma} : \text{grad } \mathbf{u}$$

which is a scalar Φ. Using Equation 2.72, we then have:****

$$\Phi = \lambda(\text{div } \mathbf{u})^2 + \mu[\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T] : \text{grad } \mathbf{u} \tag{2.79}$$

Since the contribution of this term in the mechanical energy equation (Equation 2.43) is negative, the function Φ is called the *dissipation function*. That is, Φ is the rate of energy dissipation per unit volume of the fluid. It must also be noted that the contribution of Φ to the internal energy equation (Equation 2.47) is positive; therefore, the dissipated energy increases the internal energy and hence the temperature of the fluid.

For these arguments to be true, we have to find conditions under which Φ is always non-negative, i.e., $\Phi \geqslant 0$. To find these conditions, we first write Equation 2.79 with reference to a rectangular Cartesian system:

* For example, helium (He), argon (Ar), neon (Ne), etc.

** For a discussion here, refer to Reference 3, pp. 154–155.

*** Implying $\sigma_{ii} = 0$.

**** Also $\Phi = \lambda(\text{div } \mathbf{u})^2 + 2\mu\mathbf{D} : \mathbf{D}$

$$\Phi = \lambda(\text{div } \mathbf{u})^2 + \mu \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.80}$$

which, for viscous incompressible flow is simply:

$$\Phi = \mu \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.81}$$

We now write Φ in terms of the eigenvalues of $\mathbf{D}$. Let d_1, d_2, d_3 be the eigenvalues of $\mathbf{D}$; then:

$$d_1 + d_2 + d_3 = D_{11} + D_{22} + D_{33} \tag{2.82a}$$

$$d_1d_2 + d_1d_3 + d_2d_3 = \begin{vmatrix} D_{22} & D_{32} \\ D_{23} & D_{33} \end{vmatrix} + \begin{vmatrix} D_{11} & D_{21} \\ D_{12} & D_{22} \end{vmatrix} + \begin{vmatrix} D_{11} & D_{31} \\ D_{13} & D_{33} \end{vmatrix} \tag{2.82b}$$

$$d_1d_2d_3 = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{22} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix} \tag{2.82c}$$

Using Equation 2.82a, b it is easy to show that Φ in Equation 2.80 can be written as:

$$\Phi = \lambda(d_1 + d_2 + d_3)^2 + 2\mu(d_1^2 + d_2^2 + d_3^2)$$

On further rearrangement of terms:

$$3\Phi = (3\lambda + 2\mu)(d_1 + d_2 + d_3)^2 + 2\mu[(d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_1 - d_3)^2]$$

Thus, $\Phi \geqslant 0$ provided that:

$$3\lambda + 2\mu \geqslant 0 \quad \text{and} \quad \mu \geqslant 0 \tag{2.83}$$

The inequalities in Equation 2.83 provide the necessary conditions for $\Phi \geqslant 0$. For a real fluid the inequalities in Equation 2.83 must always be satisfied.

2.15 CONSTITUTIVE EQUATION FOR NON-NEWTONIAN FLUIDS

It was mentioned earlier that a non-Newtonian fluid has a memory which affects its response to applied stresses in a manner different from that of a Newtonian fluid. To take account of the history of the deformation of a fluid we have to introduce certain new entities which are summarized next.

Consider a fluid particle whose spatial positon at the current time t is $\mathbf{r}$. Suppose at an earlier time τ, where $-\infty < \tau < t$, the spatial position of the same fluid particle was $\boldsymbol{\xi}$. Since the particle is the same, it is logical to conclude that there must be a functional relationship between the two coordinates, viz.:

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r}, \tau)$$

Taking the current time t as a reference time, the function must be such that:

$$\mathbf{r} = \boldsymbol{\xi}(\mathbf{r}, t)$$

An infinitesimal fluid element $\delta\boldsymbol{\xi}$ at time τ become $\delta\mathbf{r}$ at time t. Thus:

$$\begin{aligned}\delta\boldsymbol{\xi} &= \boldsymbol{\xi}(\mathbf{r} + \delta\mathbf{r}, \tau) - \boldsymbol{\xi}(\mathbf{r}, \tau)\\ &\cong (\text{grad}\ \boldsymbol{\xi}) \cdot \delta\mathbf{r}\end{aligned} \tag{2.84}$$

where the grad operator in Equation 2.84 is with respect to the current coordinates $\mathbf{r}$. The tensor grad $\boldsymbol{\xi}$ is called the *relative deformation gradient*. Obviously:

$$(\text{grad}\ \boldsymbol{\xi})_{\tau=t} = \mathbf{I}, \quad \text{the unit tensor}$$

Based on grad $\boldsymbol{\xi}$, a new tensor is formed which is

$$\mathbf{C}_t(\mathbf{r}, \tau) = (\text{grad}\ \boldsymbol{\xi})^T \cdot (\text{grad}\ \boldsymbol{\xi}) \tag{2.85}$$

and is called the *relative deformation tensor* or the *relative Cauchy-Green tensor*. The tensor $\mathbf{C}_t(\mathbf{r}, \tau)$ gives the deformation of a fluid element which become $\delta\mathbf{r}$ at time t.

The history of deformation is now accounted for by the time derivatives of $\mathbf{C}_t(\mathbf{r}, t)$. Introducing:

$$\mathbf{A}_n = \left(\frac{\partial^n \mathbf{C}_t}{\partial \tau^n}\right)_{\tau=t}, \qquad n = 1, 2, 3, \ldots$$

we can develop a series expansion for $\mathbf{C}_t(\mathbf{r}, \tau)$ as:

$$\mathbf{C}_t(r, \tau) = \mathbf{C}_t(\mathbf{r}, t) + (\tau - t)\mathbf{A}_1 + \frac{1}{2!}(\tau - t)^2\mathbf{A}_2 + \ldots$$

The tensors $\mathbf{A}_n$ are known as the Rivlin-Ericksen tensors. It can be shown quite easily that the first Rivlin-Erickson tensor $\mathbf{A}_1$ is $2\mathbf{D}$. To show this, consider:

$$\begin{aligned}\delta\boldsymbol{\xi} \cdot \delta\boldsymbol{\xi} &= [(\text{grad}\ \boldsymbol{\xi}) \cdot \delta\mathbf{r}] \cdot [(\text{grad}\ \boldsymbol{\xi}) \cdot \delta\mathbf{r}]\\ &= \delta\mathbf{r} \cdot [(\text{grad}\ \boldsymbol{\xi})^T \cdot (\text{grad}\ \boldsymbol{\xi})] \cdot \delta\mathbf{r}\end{aligned}$$

by using Equation M1.19. Differentiating with respect to τ at fixed $\mathbf{r}$, we have:

$$\begin{aligned}\frac{D}{D\tau}(\delta\boldsymbol{\xi} \cdot \delta\boldsymbol{\xi})_{\tau=t} &= \frac{D}{Dt}(\delta s^2)\\ &= \delta\mathbf{r} \cdot \mathbf{C}_t \cdot \delta\mathbf{r}\end{aligned}$$

On the other hand, from Problem 1.8:

$$\frac{D}{Dt}(\delta s^2) = 2\delta\mathbf{r} \cdot \mathbf{D} \cdot \delta\mathbf{r}$$

On comparison, we have:

$$\mathbf{A}_1 = 2\mathbf{D}$$

In the same manner the second time derivative of $\delta\boldsymbol{\xi} \cdot \delta\boldsymbol{\xi}$ yields:

$$\begin{aligned}\mathbf{A}_2 &= \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1 \cdot (\text{grad}\ \mathbf{u}) + (\text{grad}\ \mathbf{u})^T \cdot \mathbf{A}_1\\ &= \text{grad}\ \mathbf{a} + (\text{grad}\ \mathbf{a})^T + 2\ (\text{grad}\ \mathbf{u})^T \cdot (\text{grad}\ \mathbf{u})\end{aligned}$$

An incompressible isotropic non-Newtonian fluid has the constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + \mu_1\mathbf{A}_1 + \mu_2(\mathbf{A}_1 \cdot \mathbf{A}_1) + \mu_3\mathbf{A}_2 \tag{2.86}$$

The fluid having Equation 2.86 as its constitutive equation is called a second order Rivlin-Erickson fluid. Here μ_1, μ_2, μ_3 are constant coefficients, where μ_1 is the Newtonian viscosity, μ_2 is the cross-viscosity, and μ_3 (<0) is the elastico-viscosity.

2.16 THERMODYNAMIC ASPECTS OF PRESSURE AND VISCOSITY

In this section, we shall review some elementary results from thermodynamics for future use in the description of fluid motion. Though a very thorough preparation in the kinetic theory of gases is not needed for the continuum fluid flow, some knowledge of the molecular properties of gases is, nevertheless, indispensable. For details, refer to Reference 2.

Let the average number of molecules per unit volume be denoted by N and let m_0 be the mass of a single molecule. The number N is called the number density. The mass density ρ is then defined as:

$$\rho = \mathrm{N m_0}$$

Denoting by m_u, the atomic mass unit:

$$m_u = 1.660565 \times 10^{-24}\ \mathrm{g}$$

the molecular weight of a substance is defined as:

$$\overline{M} = m_0/m_u$$

The $\overline{M}$ grams of a substance is termed as 1 g.mol of the substance. The molecular weight $\overline{M}$ is therefore a measure of the mass contained in a *mole* of the substance. In general, $m = n\overline{M}$, where n is the number of moles in a mass m. $\overline{M}$ is also called the *molar mass*. Similarly, the *molar volume* $\bar{v}$ is defined by $V = n\bar{v}$, where V is the volume occupied by n moles of the substance.

Let the *random* molecular velocity of any molecule be represented by the vector $\mathbf{c}$. To find the *mean* molecular velocity at a position $\mathbf{r}$ and time t, consider a small volume $\delta\nu(\mathbf{r}, t)$ surrounding the point $\mathbf{r}$. The number of molecules contained in $\delta\nu$ during a time interval δt is $N\delta\nu$, and the arithmetic sum of the velocities in $\delta\nu$ is $\Sigma\mathbf{c}$. If the sample values of $N\delta\nu$ and $\Sigma\mathbf{c}$, formed during δt, are repeated a large number of times, then their arithmetic average over the interval δt is used to define the *mean* molecular velocity $\mathbf{c}_0$ as:

$$\mathbf{c}_0 = \Sigma\mathbf{c}/N\delta\nu$$

The difference between $\mathbf{c}$ and $\mathbf{c}_0$ is called the *peculiar velocity* $\mathbf{C}$:

$$\mathbf{C} = \mathbf{c} - \mathbf{c}_0$$

The scalar value of $\mathbf{C}$ is

$$C = (c^2 - 2\mathbf{c} \cdot \mathbf{c}_0 + c_0^2)^{1/2}$$

which is the *peculiar speed*. To find the mean or expected values of C and C^2, we have to use the probability density function $P_1(C)$. The quantity $P_1(C)\, dC$ is the probability that C lies between C and $C + dC$.

Ideal Gases

The simplest of all the mathematical and physical models for the description of molecular motions in gases is the model of an *ideal gas*. A low density gas is said to form an ideal gas model if its molecules can be regarded as point masses without intermolecular forces between them, and the only interaction between the molecules is through random collisions. Most of the gases at twice their critical temperatures are ideal gases. Air for all practical purposes can be treated as an ideal gas. Extensive engineering calculations suggest that if $p \ll p_c$, or if $T > 2T_c$ and $p \leq 5p_c$, then the substance is an ideal gas. Here the subscript c stands for the critical state.

For an ideal gas, the Maxwell-Boltzmann probability density function reduces to:

$$P_1(C) = \frac{4}{\sqrt{\pi}} \alpha^{3/2} C^2 e^{-\alpha C^2} \tag{2.88}$$

where:

$$\alpha = m_0/2k_bT$$

and

$k_b = \overline{R}/N_L$, the Boltzmann constant, (numerically $1/N_L = m_u$)

$\overline{R}$ = universal gas constant*

T = absolute temperature

N_L = Avogadro constant = 6.022×10^{23} molecules/g.mol

Using Equation 2.88, we obtain the mean and the mean-square values of the peculiar speed as:

$$\overline{C} = \int_0^\infty CP_1(C)\,dC, \qquad \overline{C^2} = \int_0^\infty C^2P_1(C)\,dC$$

Using the definite integrals:

$$\int_0^\infty x^3e^{-\alpha x^2}\,dx = \tfrac{1}{2}\alpha^{-2}, \qquad \int_0^\infty x^4e^{-\alpha x^2}\,dx = \frac{3\sqrt{\pi}}{8}\alpha^{-5/2}$$

we have:

$$\overline{C} = \left(\frac{8k_bT}{\pi m_0}\right)^{1/2} \tag{2.89a}$$

$$\overline{C^2} = \frac{3k_bT}{m_0} \tag{2.89b}$$

Thus, the mean molecular kinetic energy is

$$\tfrac{1}{2}m_0\overline{C^2} = \tfrac{3}{2}k_bT \tag{2.89c}$$

* $\overline{R}$ = 8.31441 J/g.mol/K = 1545.3373 ft · lb$_f$/lb$_m$ · mol/°R; 1 K = 9/5 °R.

The stress at a point in the gas is due to the mean rate of flow of molecular momentum per unit area across an area element moving with the local mean molecular velocity $\mathbf{c}_0$. Because of this relative behavior of the momentum transfer, the peculiar speed must again be used to describe the stress. Since $\rho\mathbf{C}$ is the momentum per unit volume, the transfer of momentum is the product of $\rho\mathbf{C}$ and $\mathbf{C}$. On averaging, we have:

$$\mathbf{P} = \rho\overline{\mathbf{CC}}$$

as the stress at a point in the gas. Using Cartesian coordinates, we have:

$$p_{ij} = \rho\overline{C_i C_j}$$

from which the sum of the three normal stresses is

$$p_{11} + p_{22} + p_{33} = \rho\overline{C^2}$$

The mean value of this sum is the thermodynamic pressure:

$$p = \frac{1}{3} p_{ii} = \frac{1}{3}\rho\overline{C^2} \tag{2.90a}$$

$$= \frac{1}{3} m_0 N\overline{C^2} \tag{2.90b}$$

Thus, using Equation 2.89b, we have the following equivalent forms:

$$p = \rho k_b T/m_0 \tag{2.91a}$$

$$p = \rho \bar{R} T/\bar{M} \tag{2.91b}$$

$$p = k_b NT \tag{2.91c}$$

The ratio $\bar{R}/\bar{M} = R$ is the specific gas constant. Its use in Equation 2.91b gives:

$$p = \rho RT \tag{2.92a}$$

or:

$$pv = RT \tag{2.92b}$$

where $v = 1/\rho$ is the specific volume. If a gas of volume V and mass m is considered, then Equation 2.92b yields:

$$pV = mRT \tag{2.93a}$$

or:

$$pV = n\bar{R}T \tag{2.93b}$$

where n is the number of moles of the gas in the volume V. Any one of the Equations 2.92 and 2.93 is called the equation of state of an *ideal gas*.

Concept of Viscosity in Fluids

Viscosity in fluids is basically a molecular phenomenon. To describe this phenomenon, it is important to recognize that in gases, the molecules are in a state of random and chaotic motions. In their movements, the molecules carry and transfer their momenta to other molecules on collision, thus giving rise to forces due to the molecular transport. For a gas at rest, these forces show up as pressure; refer to Equation 2.90. On the other hand, in liquids or dense gases, the intermolecular forces are strong and the molecules are held together by strong cohesive forces with very little free molecular movements. Thus, in liquids and dense gases, the cohesive forces are dominant over the forces of free molecular momentum transport.

In any continuum fluid motion, the molecular phenomenon mentioned above continues to play its role *relative* to the continuum velocity distribution. In fact, since fluid motion always occurs under some external causes — viz., pressure gradient, body forces, heat transfer, work at the boundaries, etc. — the continuum velocity distribution is inherently nonhomogeneous. In consequence, a resistance to changes in the transport of molecular momentum in gases or a resistance to the changes in the distribution of cohesive forces in liquids comes into play to remove these nonhomogeneities in the continuum velocity distribution. These resistive effects produce the phenomenon of dynamic viscosity in the motion of fluids. The following example illustrates this point quite clearly.

Let us consider the motion of a simple gas in a unilateral direction, e.g., the x-axis, as shown in Figure 2.5.

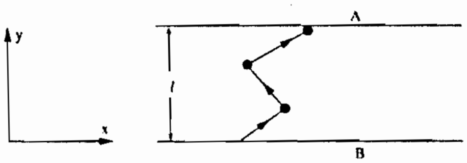

Fig. 2.5 Illustrating the transport of momentum and the consequent phenomena of viscosity in a moving gas.

The local continuum velocity is a function of y only. Consider two layers of fluid, called A and B, separated by a distance equal to the mean free path ℓ. In the kinetic theory of gases, ℓ is the mean distance traveled by a molecule in a time t between successive collisions. A simple formula for the mean free path is

$$\ell = (\sqrt{2}\,\pi N\sigma^2)^{-1} \tag{2.94}$$

where σ is the diameter of a molecule and N the number density.

Let $u_A = u(y + \ell)$ and $u_B = u(y)$ be the local continuum velocities at A and B, respectively. The mean molecular velocity which causes the movements of the molecules from A to B and vice versa is $\bar{C}$ obtained in Equation 2.89a. The rate of momentum transfer is then the shear stress given by:

$$\begin{aligned}\textbf{shear stress} &= a\bar{C}(\rho u_A - \rho u_B)\\ &= a\rho\bar{C}\ell\,\frac{\partial u}{\partial y}\end{aligned} \tag{2.95}$$

where a is a numerical constant. The coefficient of $\partial u/\partial y$, i.e.:

$$\mu = a\rho\bar{C}\ell \tag{2.96}$$

is called the coefficient of viscosity. A simple calculation from the kinetic theory of gases shows that:

$$a \simeq 0.0896\pi^{3/2}$$

Thus:

$$\begin{aligned}\mu &= 0.0896\pi^{3/2}\rho\bar{C}\ell \\ &= 0.1792\pi N\ell\sqrt{2k_b m_0 T} \\ &= 0.1792\sqrt{k_b m_0 T}/\sigma^2 \\ &= 0.1792\pi\rho\ell\sqrt{2RT}\end{aligned} \tag{2.97}$$

From the first equation in Equation 2.97, we note that:

$$\mu \sim \text{(density) (speed) (length)}$$

Sutherland Formula for Viscosity

Based on the hypothesis that the colliding molecules in a gas may be regarded as rigid spheres with attraction between them, Sutherland obtained the equation:

$$\mu = \alpha T^{1/2}\left(1 + \frac{\beta}{T}\right)^{-1} \tag{2.98}$$

where α and β are constants. This equation holds quite accurately for ideal, and quasi-ideal gases such as air. In the SI system:

$$\mu = (1.4566158)(10^{-6})T^{1/2}\left(1 + \frac{\beta}{T}\right)^{-1} \text{ kg/m-sec} \tag{2.99a}$$

where $\beta = 110.33\ K$ and T is expressed in degrees Kelvin. In the English system:

$$\mu = (7.303498)(10^{-7})T^{1/2}\left(1 + \frac{\beta}{T}\right)^{-1} \text{ lbm/ft-sec} \tag{2.99b}$$

$$= (2.27)(10^{-8})T^{1/2}\left(1 + \frac{\beta}{T}\right)^{-1} \frac{\text{slug}}{\text{ft-sec}} \text{ or } \frac{\text{lbf-sec}}{\text{ft}^2} \tag{2.99c}$$

where $\beta = 198.6°R$ and T is expressed in degrees Rankine.

Let a reference state be denoted by a subscript r. Then:

$$\mu_r = \alpha T_r^{1/2}\left(1 + \frac{\beta}{T_r}\right)^{-1}$$

Introducing the nondimensional quantities:

$$\mu^* = \mu/\mu_r, \qquad T^* = T/T_r, \qquad S_1 = \beta/T_r$$

we obtain the Sutherland formula in nondimensional form:

$$\mu^* = \frac{(1 + S_1)T^{*3/2}}{T^* + S_1} \tag{2.100a}$$

where the value of S_1 depends on the chosen reference temperature and on the chosen measurement system.

An empirical formula which closely approximates the expression in Equation 2.100a for $0.1 \leq T^* < 10$ is

$$\mu^* = A(T^*)^n \tag{2.100b}$$

where:

$$A = 0.8, \quad B = 1 \qquad \text{for } 0.1 \leq T^* \leq 1$$

and:

$$A = 1.0, \quad B = 0.67 \qquad \text{for } 1 < T^* < 10$$

2.17 EQUATIONS OF MOTION IN LAGRANGIAN COORDINATES

To derive the equations of motion in Lagrangian, or referential coordinates, one needs to work with some additional formulas and also with two different forms of the stress tensor. First, we introduce the following easily verifiable formulas:

$$Je_{ijk} = e_{\alpha\beta\gamma} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} \tag{2.101a}$$

$$Je_{\alpha\beta\gamma} = e_{ijk} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} \tag{2.101b}$$

$$J^{-1} e_{ijk} = e_{\alpha\beta\gamma} \frac{\partial X_\alpha}{\partial x_i} \frac{\partial X_\beta}{\partial x_j} \frac{\partial X_\gamma}{\partial x_k} \tag{2.101c}$$

where J has been defined in Equation 1.7 and for clarity in bookkeeping we will use the Greek and Latin indices with the Lagrangian and the Eulerian coordinates, respectively. Using 2.101a, we can deduce, by inner multiplication, the formula

$$J \frac{\partial X_\alpha}{\partial x_i} = e_{\alpha\beta\gamma} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} \tag{2.101d}$$

where i, j, and k are to be taken in the cyclic permutation of 1, 2, and 3, in this order. Again, from 2.101a, taking i = 1, j = 2, and k = 3, differentiating both sides with respect to X_ρ and using 2.101d, we obtain

$$\frac{\partial J}{\partial X_\rho} = J \frac{\partial^2 x_p}{\partial X_\alpha \partial X_\rho} \frac{\partial X_\alpha}{\partial x_p} \tag{2.102}$$

Referring to problem 1.6, we introduce the deformation tensor

$$G = Grad\ r \tag{2.103a}$$

and its inverse

$$f = G^{-1} = grad\ R \tag{2.103b}$$

Obviously,

$$G \cdot f = f \cdot G = I \tag{2.103c}$$

In the Lagrangian system, consider two intersecting elemental vectors dR and δR which are carried into the Eulerian system as intersecting elemental vectors dr and δr at r. The area element dS_0 and dS in the two systems are given by

$$NdS_0 = dR \times \delta R \tag{2.104a}$$

$$ndS = dr \times \delta r \tag{2.104b}$$

where N and n are the respective unit normal vectors. How are these area elements related? To answer the question, we write Equation 2.104a in component form:

$$\begin{aligned} N_\gamma dS_0 &= e_{\alpha\beta\gamma} dX_\alpha \delta X_\beta \\ &= e_{\alpha\beta\gamma} \frac{\partial X_\alpha}{\partial x_p} \frac{\partial X_\beta}{\partial x_q} dx_p \delta x_q \end{aligned}$$

Inner multiplication by $\dfrac{\partial X_\gamma}{\partial x_r}$ and the use of 2.101c yields

$$N_\gamma \frac{\partial X_\gamma}{\partial x_r} dS_0 = J^{-1} e_{pqr} dx_p \delta x_q \tag{2.105}$$

Next writing Equation 2.104b in component form:

$$n_r dS = e_{pqr} dx_p \delta x_q$$

and using 2.105, we have

$$n_r dS = J N_\gamma \frac{\partial X_\gamma}{\partial x_r} dS_0$$

Also, it is a straight-forward matter to check that

$$\begin{aligned} N \cdot G^{-1} &= N \cdot f \\ &= N_\gamma \frac{\partial X_\gamma}{\partial x_p} i_p \end{aligned}$$

Thus,

$$\left.\begin{aligned} ndS &= JN \cdot f dS_0 \\ &= Jf^T \cdot NdS_0 \end{aligned}\right\} \tag{2.106}$$

establishing the relation between dS and dS_0.

We now introduce two new stress tensors as developed by Piola and Kirchhoff (refer to Reference 4, p. 553) which automatically appear in the equations of motion. The first Piola–Kirchhoff tensor denoted as P is related to the Cauchy stress tensor T through the equation

$$P^T \cdot NdS_0 = T^T \cdot ndS$$

Thus P gives the actual force on dS per unit area of the undeformed area dS_0. Using 2.106, we get

$$P^T = JT^T \cdot f^T \tag{2.107a}$$

or,

$$P = Jf \cdot T \tag{2.107b}$$

or,

$$T = J^{-1}\, G \cdot P \tag{2.107c}$$

or,

$$T^T = J^{-1}P^T \cdot G^T \tag{2.107d}$$

The second Piola–Kirchhoff tensor denoted as K is defined as

$$K = P \cdot f^T \tag{2.108a}$$

or,

$$P = K \cdot G^T \tag{2.108b}$$

Using 2.108b in 2.107c, we have

$$T = J^{-1}\, G \cdot K \cdot G^T \tag{2.108c}$$

One additional result has to be obtained before we arrive at the final equations. From 2.103a

$$G = G_{p\sigma} i_p i_\sigma,\ G^T = G_{p\sigma} i_\sigma i_p$$

where

$$G_{p\sigma} = \frac{\partial x_p}{\partial X_\sigma}$$

We now consider

$$div\,(J^{-1}G^T) = \frac{\partial}{\partial x_p}(J^{-1}G_{p\sigma} i_\sigma)$$
$$= \frac{\partial}{\partial X_\lambda}(J^{-1}G_{p\theta})\frac{\partial X_\lambda}{\partial x_p} i_\theta$$

Using 2.102 we establish an important result:

$$div\ (J^{-1}G^T) = 0 \tag{2.109a}$$

or,

$$\frac{\partial}{\partial x_p}\left(J^{-1}\frac{\partial x_p}{\partial X_\sigma}\right) = 0, \quad \sigma = 1, 2, 3 \tag{2.109b}$$

To obtain the equations of motion in Lagrangian coordinates, we consider Equation 2.33 and write it as

$$\rho \ddot{r} = \rho f + div\ T^T$$

Introducing 2.107d while using 2.109b, we get

$$\rho \ddot{r} = \rho f + J^{-1}\frac{\partial P_{\sigma\lambda}}{\partial X_\sigma} i_\lambda$$

Using the continuity equation (refer to Problem 2.1), we obtain

$$\rho_0 \ddot{r} = \rho_0 f_0 + Div\ P^T \tag{2.110}$$

where

$$Div(\cdot\cdot) = \left\{\frac{\partial}{\partial X_\sigma}(\cdot\cdot)\right\}\cdot i_\sigma$$

Alternatively, using 2.108b we have

$$\rho_0 \ddot{r} = \rho_0 f_0 + Div\,(G \cdot K^T) \tag{2.111}$$

Either Equation 2.110 or Equation 2.111 is the vector equation of motion in Lagrangian coordinates.

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PROBLEMS

2.1 By using Equation 2.3 or Equations 1.29 and 2.5, prove that the equation of continuity in the Lagrangian coordinates is

$$J\rho(\mathbf{R}, t) = \rho_0$$

where:

$$\rho_0 = \rho(\mathbf{R}, t_0)$$

2.2 Using the result of Problem 2.1, show that the Lagrangian continuity equation for the Example 1.1 is

$$(1 + \tfrac{1}{2}\alpha t^2)\rho = \rho_0$$

Show that this result can also be obtained by proceeding from Equation 2.5 for one-dimensional flow and using the velocity u_1 from Example 1.1.

2.3 Obtain the continuity equation in the Eulerian coordinates by setting up a fixed rectangular control volume $dx_1 dx_2 dx_3$ with center at (x_1, x_2, x_3) and the velocity components u_1, u_2, u_3 along x_1, x_2, x_3, respectively. Check your result by writing the div($\rho\mathbf{u}$) term of Equation 2.6 in the Cartesian coordinates.

2.4 Apply the result (Equation 2.11) to a steady state and steady flow through a rigid tube of any shape and show that when the flow is considered to be uniform at every section of the tube then the mass flow rate is given by:

$$\rho u A = \text{const.} = \dot{m}$$

where A is any area of cross section and u the uniform speed at A.

2.5 For an incompressible flow, using the continuity equation establish that:

$$\begin{aligned}\Phi &= \mu\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \mu\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_j}{\partial x_i}\right)\\ &= \mu|\boldsymbol{\omega}|^2 + 2\mu\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_j}{\partial x_i}\right)\end{aligned}$$

Thus, in general show that:

$$\Phi = \mu|\boldsymbol{\omega}|^2 + 2\mu\,\text{div}[(\text{grad}\,\mathbf{u})\cdot\mathbf{u}]$$

Hint: Refer to Problem 1.5.

2.6 Consider a closed vessel full of fluid. Use the no-slip condition $\mathbf{u} = 0$ at the inner surface of the vessel, and prove that the total energy dissipation per unit time is given by:

$$\int_{V_0}\Phi\,dv = \mu\int_{V_0}|\boldsymbol{\omega}|^2\,dv$$

Hint: Use Problem 2.5.

2.7 Show that the dissipation function (Equation 2.81) can also be written as:

$$\Phi = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$$

2.8(a) Take the curl of the acceleration vector $\mathbf{a}$ defined in Equation 1.17 while using Equations M1.37, 43a and show that:

$$\text{curl } \mathbf{a} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \text{div}(\boldsymbol{\omega}\mathbf{u} - \mathbf{u}\boldsymbol{\omega}) \qquad \text{(i)}$$

(b) Use Equation M2.4 and the result of Equation i above to prove that for a material volume $\mathcal{V}(t)$:

$$\frac{D}{Dt}\int_{\mathcal{V}(t)} \boldsymbol{\omega}\, dv = \int_{\mathcal{S}(t)} [\mathbf{n} \times \mathbf{a} + \mathbf{u}(\boldsymbol{\omega} \cdot \mathbf{n})]\, dS \qquad \text{(ii)}$$

(c) Use Equation M2.4 and the result of Equation i above to show that for a fixed volume:

$$\frac{\partial}{\partial t}\int_V \boldsymbol{\omega}\, dv = \int_S [\mathbf{n} \times \mathbf{a} + \mathbf{u}(\boldsymbol{\omega} \cdot \mathbf{n}) - \boldsymbol{\omega}(\mathbf{u} \cdot \mathbf{n})]\, dS \qquad \text{(iii)}$$

2.9 Show that by setting $A = \rho$, $f = \rho\mathbf{u}$, $C = 0$, in Equation 2.55 we obtain Equation 2.6. By setting $A = \rho\mathbf{u}$, $f = \rho\mathbf{uu} - \mathbf{T}$, $C = \rho\mathbf{f}$, we obtain Equation 2.36a. By setting $A = \rho e_t$, $f = \rho e_t\mathbf{u} - \mathbf{T} \cdot \mathbf{u} + \mathbf{q}$, $C = -\rho(\text{grad } \chi) \cdot \mathbf{u}$, we obtain Equation 2.50b.

CHAPTER THREE

The Navier-Stokes Equations

3.1 FORMULATION OF THE PROBLEM

Having obtained the constitutive equation (Equation 2.73) for a viscous compressible fluid flow, we now obtain a deterministic form of the equations of motion, called the Navier-Stokes equations. Using Equation 2.60 with $\boldsymbol{\sigma}$ defined by Equation 2.72 in Equations 2.33 and 2.36a, we have the following two alternative forms of the Navier-Stokes equations:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{f} - \operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma} \tag{3.1}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\mathbf{u}) = \rho\mathbf{f} - \operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma} \tag{3.2}$$

By using the definition of D/Dt given in (Equation 1.18a) and the vector identities given in ME.1, Section 7, the acceleration $\mathbf{a} = D\mathbf{u}/Dt$ can be written in various forms as follows.

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\mathbf{u} \tag{3.3a}$$

$$= \frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad} \mathbf{u}) \cdot \mathbf{u} \tag{3.3b}$$

$$= \frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u}\mathbf{u}) - \mathbf{u}(\operatorname{div} \mathbf{u}) \tag{3.4a}$$

$$= \frac{\partial \mathbf{u}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega} \times \mathbf{u} \tag{3.4b}$$

Each form (Equations 3.3–3.4) has its own merit, both from physical and mathematical viewpoints.

The term div $\boldsymbol{\sigma}$, where:

$$\boldsymbol{\sigma} = \lambda(\operatorname{div} \mathbf{u})\mathbf{I} + 2\mu\mathbf{D}$$

is a vector. It is obvious from Equation 3.1 that $(\operatorname{div} \boldsymbol{\sigma})/\rho$ is a contribution to the acceleration $D\mathbf{u}/Dt$ due to the diffusion of velocity. In component form div $\boldsymbol{\sigma}$ can be written with reference to any coordinate system. Thus, referring to ME.1, we can write div $\boldsymbol{\sigma}$ as follows:

Cartesian Coordinates

Since:

$$\boldsymbol{\sigma} = \sigma_{mn}\mathbf{i}_m\mathbf{i}_n$$

then:

$$\operatorname{div} \boldsymbol{\sigma} = \frac{\partial}{\partial x_p} (\sigma_{mn}\mathbf{i}_m\mathbf{i}_n) \cdot \mathbf{i}_p$$
$$= \frac{\partial \sigma_{mp}}{\partial x_p} \mathbf{i}_m \tag{3.5a}$$

where σ_{mp} are the Cartesian components of $\boldsymbol{\sigma}$.

Curvilinear Coordinates

Writing:

$$\boldsymbol{\sigma} = \sigma^{ij}\mathbf{a}_i\mathbf{a}_j$$

where σ^{ij} are the contravariant components of $\boldsymbol{\sigma}$ and $\mathbf{a}_i$ are the base vectors, we have:

$$\operatorname{div} \boldsymbol{\sigma} = \frac{\partial}{\partial x_k} (\sigma^{ij}\mathbf{a}_i\mathbf{a}_j) \cdot \mathbf{a}^k$$
$$= \sigma^{ik}_{,k}\mathbf{a}_i \tag{3.5b}$$

where a comma denotes the covariant derivative. Refer to Equation M1.113. Similarly writing:

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{a}^i\mathbf{a}^j$$

we have:

$$\operatorname{div} \boldsymbol{\sigma} = \sigma_{ik,k}\mathbf{a}^i \tag{3.5c}$$

where σ_{ij} are now the covariant components of $\boldsymbol{\sigma}$.

It is important due to analytical and numerical reasons to have the expanded form of div $\boldsymbol{\sigma}$. Below it is shown that there are three such forms. Most of the formulae used here have been worked out in ME.1, Section 7.

First of all:

$$\boldsymbol{\sigma} = (\lambda \operatorname{div} \mathbf{u})\mathbf{I} + 2\mu\mathbf{D}$$
$$= (\lambda \operatorname{div} \mathbf{u})\mathbf{I} + \mu[(\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T]$$

Using the formulae:

$$\operatorname{div}(\operatorname{grad} \mathbf{u})^T = \operatorname{grad}(\operatorname{div} \mathbf{u})$$

and:

$$2\mathbf{W} \cdot \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$$

where $\mathbf{v}$ is an arbitrary vector, we obtain the *first form* as:

$$\operatorname{div} \boldsymbol{\sigma} = (\lambda + \mu)\operatorname{grad}(\operatorname{div} \mathbf{u}) + \mu\nabla^2\mathbf{u} + (\operatorname{div} \mathbf{u})(\operatorname{grad} \lambda)$$
$$+ 2(\operatorname{grad} \mathbf{u}) \cdot (\operatorname{grad} \mu) + (\operatorname{grad} \mu) \times \boldsymbol{\omega} \tag{3.6a}$$

Here:

$$\operatorname{div}(\operatorname{grad} \mathbf{u}) = \nabla^2\mathbf{u}$$

and:

$$(\text{grad } \mathbf{u}) \cdot (\text{grad } \mu) = (\text{grad } \mu \cdot \text{grad})\mathbf{u}$$

To obtain *the second* form we use the additional formulae:

$$\begin{aligned} \text{div}(\text{grad } \mathbf{u}) &= \text{grad}(\text{div } \mathbf{u}) - \text{curl } \boldsymbol{\omega} \\ &= \nabla^2 \mathbf{u} \\ 2 \text{ div } \mathbf{W} &= - \text{curl } \boldsymbol{\omega} \end{aligned}$$

thus having:

$$\begin{aligned} \text{div } \boldsymbol{\sigma} = \text{grad}[(\lambda + 2\mu)\text{div } \mathbf{u}] - \text{curl}(\mu\boldsymbol{\omega}) + 2(\text{grad } \mathbf{u}) \cdot (\text{grad } \mu) \\ + 2(\text{grad } \mu) \times \boldsymbol{\omega} - 2(\text{div } \mathbf{u})(\text{grad } \mu) \end{aligned} \tag{3.6b}$$

A *third form* can be obtained by using the vector-tensor identities of ME.1, Section 7, thus yielding:

$$\begin{aligned} \text{div } \boldsymbol{\sigma} = \text{grad}[(\lambda + 2\mu)\text{div } \mathbf{u}] - \text{curl}(\mu\boldsymbol{\omega}) \\ + 2\text{grad}[(\text{grad } \mu) \cdot \mathbf{u}] - 2\text{div}[(\text{grad } \mu)\mathbf{u}] \end{aligned} \tag{3.6c}$$

In these formulae one can introduce the bulk coefficient of viscosity μ' in place of λ by using:

$$\lambda = \mu' - 2/3\, \mu$$

We now summarize various categories of flow equations for future reference.

3.2 VISCOUS COMPRESSIBLE FLOW EQUATIONS

A. Conservation of Mass

$$\text{(i)} \quad \frac{D\rho}{Dt} + \rho \text{ div } \mathbf{u} = 0 \tag{3.7}$$

$$\text{(ii)} \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho\mathbf{u}) = 0 \tag{3.8}$$

B. Conservation of Momentum

For brevity, writing:

$$\boldsymbol{\sigma} = \lambda(\text{div } \mathbf{u})\mathbf{I} + 2\mu\mathbf{D} \tag{3.9}$$

we have:

$$\text{(i)} \quad \frac{\partial}{\partial t}(\rho\mathbf{u}) + \text{div}(\rho\mathbf{u}\mathbf{u}) = \rho\mathbf{f} - \text{grad } p + \text{div } \boldsymbol{\sigma} \tag{3.10}$$

(ii) $$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot \operatorname{grad})\mathbf{u}\right] = \rho\mathbf{f} - \operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma} \tag{3.11}$$

(iii) $$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega} \times \mathbf{u}\right] = \rho\mathbf{f} - \operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma} \tag{3.12}$$

C. Equations of Mechanical Energy

(i) $$\rho \frac{D}{Dt}(\tfrac{1}{2}|\mathbf{u}|^2) = \rho(\mathbf{u}\cdot\mathbf{f}) - \mathbf{u}\cdot \operatorname{grad} p + \operatorname{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) - \boldsymbol{\sigma} : \mathbf{D} \tag{3.13}$$

(ii) $$\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{u}|^2) = \rho(\mathbf{u}\cdot\mathbf{f} - \tfrac{1}{2}|\mathbf{u}|^2 \operatorname{div} \mathbf{u}) - \mathbf{u}\cdot \operatorname{grad} p + \operatorname{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) - \boldsymbol{\sigma} : \mathbf{D} \tag{3.14}$$

D. Equations of Internal Energy

(i) $$\rho\frac{De}{Dt} = \boldsymbol{\sigma} : \mathbf{D} - p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) \tag{3.15}$$

(ii) $$\frac{D}{Dt}(\rho e) = \boldsymbol{\sigma} : \mathbf{D} - (p + \rho e)\operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) \tag{3.16}$$

In Equations 3.15 and 3.16 we have set*:

$$\mathbf{q} = -k \operatorname{grad} T$$

where k is the fluid conductivity, and T is the absolute temperature. In deriving Equations 3.14 and 3.16 we have made use of Equation 3.7 in Equations 3.13 and 3.15.

E. Equations of Entropy and Enthalpy

(i) $$\rho T\frac{Ds}{Dt} = \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}(k \operatorname{grad} T) \tag{3.17a}$$

(ii)** $$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s\mathbf{u}) = \rho\dot{s}_{irr} + \operatorname{div}\left(\frac{k \operatorname{grad} T}{T}\right) \tag{3.17b}$$

(iii) $$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}(k \operatorname{grad} T) \tag{3.18a}$$

(iv) $$\frac{\partial}{\partial t}(\rho h) + \operatorname{div}(\rho h\mathbf{u}) = \frac{Dp}{Dt} + \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}(k \operatorname{grad} T) \tag{3.18b}$$

where $h = e + p/\rho$ is the specific enthalpy. The equation of entropy (Equation 3.17a) comes directly from Equation 3.15 by using the rate-form of the first law of thermodynamics and the equation of continuity (Equation 3.7). Again note that $k \operatorname{grad} T = -\mathbf{q}$

* To account for conductive heat transfer.

** $\rho\dot{s}_{irr} = (\Phi/T) + (k|\operatorname{grad} T|^2)/T^2$ is the rate of entropy production due to irreversible processes.

F. Equation of Energy

$$\text{(i)} \qquad \rho \frac{D}{Dt}\left(\tfrac{1}{2}|\mathbf{u}|^2 + e + \frac{p}{\rho}\right) = \rho(\mathbf{f}\cdot\mathbf{u}) + \frac{\partial p}{\partial t} + \operatorname{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) + \operatorname{div}(k \operatorname{grad} T) \qquad \textbf{(3.19a)}$$

Writing $H = h + (1/2)|\mathbf{u}|^2$ as the *total enthalpy*, Equation 3.19a is also written as:

$$\frac{\partial}{\partial t}(\rho H) + \operatorname{div}(\rho H\mathbf{u}) = \rho(\mathbf{f}\cdot\mathbf{u}) + \frac{\partial p}{\partial t} + \operatorname{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) + \operatorname{div}(k \operatorname{grad} T) \qquad \textbf{(3.19b)}$$

If $\mathbf{f} = -\operatorname{grad}\chi$ and χ is independent of time, then:

$$\text{(ii)} \qquad \rho \frac{D}{Dt}(\tfrac{1}{2}|\mathbf{u}|^2 + e + \frac{p}{\rho} + \chi) = \frac{\partial p}{\partial t} + \operatorname{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) + \operatorname{div}(k \operatorname{grad} T) \qquad \textbf{(3.20)}$$

G. Conservation of Total Kinetic Energy

Writing the total kinetic energy per unit mass as e_t, i.e.:

$$e_t = \tfrac{1}{2}|\mathbf{u}|^2 + e \qquad \textbf{(3.21)}$$

and using Equations 3.7 and 3.8 in Equation 3.19, we obtain:

$$\frac{\partial}{\partial t}(\rho e_t) + \operatorname{div}\mathbf{b} = \rho(\mathbf{f}\cdot\mathbf{u}) \qquad \textbf{(3.22a)}$$

where:

$$\mathbf{b} = (\rho e_t + p)\mathbf{u} - \boldsymbol{\sigma}\cdot\mathbf{u} - k \operatorname{grad} T \qquad \textbf{(3.22b)}$$

If

$$\mathbf{f} = -\operatorname{grad}\chi$$

and χ is *independent of time*, then Equation 3.22a can also be written as:

$$\frac{\partial}{\partial t}\{\rho(e_t + \chi)\} + \operatorname{div}\mathbf{b} = 0 \qquad \textbf{(3.23a)}$$

where:

$$\mathbf{b} = \{\rho(e_t + \chi) + p\}\mathbf{u} - \boldsymbol{\sigma}\cdot\mathbf{u} - k \operatorname{grad} T \qquad \textbf{(3.23b)}$$

For a thermally perfect gas both C_p and C_v are constants, and:

$$h = C_pT, \qquad e = C_vT$$

Thus, the equation of temperature can be obtained either from Equation 3.15 or from Equation 3.18.

3.3 VISCOUS INCOMPRESSIBLE FLOW EQUATIONS

A. Conservation of Mass

$$\operatorname{div} \mathbf{u} = 0 \tag{3.24}$$

B. Conservation of Momentum

Since for incompressible flows ρ, μ, and λ are constants, then div $\boldsymbol{\sigma}$ is directly obtained from Equation 3.6a as:

$$\operatorname{div} \boldsymbol{\sigma} = \mu \nabla^2 \mathbf{u} \tag{3.25}$$

where ∇^2 is the Laplacian. However, in place of directly depending on the formula (Equation 3.6a), one can get the same result (Equation 3.25) by proceeding as follows.

Using Equation 3.24 in Equation 2.75, we get:

$$\boldsymbol{\sigma} = 2\mu \mathbf{D}, \qquad \mu = \text{const} \tag{3.26}$$

Thus:

$$\begin{aligned} 2 \operatorname{div}(\mathbf{D}) &= \operatorname{div}(\operatorname{grad} \mathbf{u}) + \operatorname{div}(\operatorname{grad} \mathbf{u})^T \\ &= \operatorname{div}(\operatorname{grad} \mathbf{u}) + \operatorname{grad}(\operatorname{div} \mathbf{u}) \\ &= \nabla^2 \mathbf{u} \end{aligned} \tag{3.27}$$

Using Equation 3.24 in Equation M1.36, we also have:

$$\nabla^2 \mathbf{u} = -\operatorname{curl} \boldsymbol{\omega} \tag{3.28}$$

Introducing the kinematic viscosity:

$$\nu = \mu/\rho$$

the vector momentum equation for incompressible flow can be written in the following equivalent forms:

$$\text{(i)} \quad \frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u}\mathbf{u}) = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.29}$$

$$\text{(ii)} \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\mathbf{u} = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.30}$$

$$\text{(iii)} \quad \frac{\partial \mathbf{u}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega} \times \mathbf{u} = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.31}$$

$$\text{(iv)} \quad \frac{\partial \mathbf{u}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega} \times \mathbf{u} = -\frac{1}{\rho} \operatorname{grad} p - \nu \operatorname{curl} \boldsymbol{\omega} + \mathbf{f} \tag{3.32}$$

C. Equation of Vorticity

Taking the curl of each term of Equation 3.31 and assuming $\mathbf{f} = -\operatorname{grad} \chi$, we have:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega} \tag{3.33a}$$

$$= -\nu \operatorname{curl}(\operatorname{curl} \boldsymbol{\omega}) \tag{3.33b}$$

Using the vector identity:

$$\operatorname{curl}(\boldsymbol{\omega} \times \mathbf{u}) = (\mathbf{u} \cdot \operatorname{grad})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \operatorname{grad})\mathbf{u}$$

Equation 3.33a becomes:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \operatorname{grad})\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{3.34}$$

Note that the term:

$$(\boldsymbol{\omega} \cdot \operatorname{grad})\mathbf{u} = (\operatorname{grad} \mathbf{u}) \cdot \boldsymbol{\omega} = (\mathbf{D} + \mathbf{W}) \cdot \boldsymbol{\omega}$$
$$= \mathbf{D} \cdot \boldsymbol{\omega}$$

This clearly demonstrates that one part of $\mathbf{D} \cdot \boldsymbol{\omega}$ increases the vorticity by stretching the vortex line, while the remaining part contributes to the angular turning of the vortex line. Recall that D_{ij} is a stretching if $i = j$. This aspect of stretching and turning is completely absent in two dimensions in which the vorticity equation is simply:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\omega = \nu \nabla^2 \omega \tag{3.35}$$

D. Equation of Internal Energy

From Equation 3.15 for ρ = constant and k = constant, the equation of energy for an incompressible flow becomes:

$$\rho \frac{De}{Dt} = \Phi + k \nabla^2 T \tag{3.36}$$

where according to Equation 2.79

$$\Phi = 2\mu \mathbf{D} : \mathbf{D} \tag{3.37}$$

E. Equation for Pressure

It was shown in Section 2.13 that under normal conditions, i.e., div $\mathbf{u}$ not very large, the pressure in a flow field is a mean of the three normal stresses. This statement is more definitive for incompressible flow where the pressure depends only on the velocity field and vice versa, and the density remains a constant. To obtain an equation for p in terms of the velocity field we take the divergence of each term of Equation 3.29 while writing Δ = div $\mathbf{u}$:

$$-\frac{1}{\rho} \nabla^2 p = \frac{\partial \Delta}{\partial t} + \operatorname{div}[\operatorname{div}(\mathbf{u}\mathbf{u})] - \nu \nabla^2(\Delta) - \operatorname{div} \mathbf{f}$$

Using the identities, Equations M1.30, 44b, we obtain:

$$-\frac{1}{\rho} \nabla^2 p = \frac{\partial \Delta}{\partial t} + 2(\operatorname{grad} \Delta) \cdot \mathbf{u} + (\operatorname{grad} \mathbf{u})^T : (\operatorname{grad} \mathbf{u}) - \nu \nabla^2(\Delta) - \operatorname{div} \mathbf{f} + \Delta^2 \tag{3.38}$$

Using the result of Problem 1.10:

$$(\text{grad } \mathbf{u})^{\mathrm{T}} : (\text{grad } \mathbf{u}) = \mathbf{D} : \mathbf{D} - \tfrac{1}{2}|\boldsymbol{\omega}|^2$$

and introducing:

$$M_{\mathrm{T}} = |\boldsymbol{\omega}|/(2\mathbf{D} : \mathbf{D})^{1/2}$$

we get:

$$-\frac{1}{\rho}\nabla^2 p = \frac{\partial \Delta}{\partial t} + 2(\text{grad } \Delta) \cdot \mathbf{u} + (1 - M_{\mathrm{T}}^2)\mathbf{D} : \mathbf{D} - \nu\nabla^2(\Delta) - \text{div } \mathbf{f} + \Delta^2 \qquad (3.39)$$

The quantity M_{T} is a measure of the vorticity field and is called the Truesdell number.[1] A deterministic form from Equation 3.39 is obtained by setting $\Delta = 0$ according to Equation 3.24:

$$-\frac{1}{\rho}\nabla^2 p = (1 - M_{\mathrm{T}}^2)\mathbf{D} : \mathbf{D} - \text{div } \mathbf{f} \qquad (3.40)$$

as an equation for the determination of p. The main problem is to choose appropriate boundary conditions.

Note that $M_{\mathrm{T}} = 0$ for irrotational flow, while $M_{\mathrm{T}} = \infty$ for rigid bodies. Thus, $0 \leq M_{\mathrm{T}} < \infty$ for all flow fields. In passing, we restate that in Cartesian coordinates:

$$|\boldsymbol{\omega}|^2 = \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$2\mathbf{D} : \mathbf{D} = \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Refer also to Problems 1.9 through 1.11.

3.4 EQUATIONS OF INVISCID FLOW (EULER'S EQUATIONS)

The equations of inviscid (nonviscous) flow, originally derived by Euler, can directly be obtained from the equations of Section 3.2 by setting μ and λ equal to zero, or $\boldsymbol{\sigma} = 0$. Thus, the basic equations for a compressible flow of an inviscid and nonheat conducting fluid* are:

A. Conservation of Mass

$$\text{(i)} \qquad \frac{D\rho}{Dt} + \rho \, \text{div } \mathbf{u} = 0 \qquad (3.41)$$

$$\text{(ii)} \qquad \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \qquad (3.42)$$

* Also known as a perfect fluid.

B. Conservation of Momentum

(i) $$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot \text{grad})\mathbf{u}\right] = \rho\mathbf{f} - \text{grad}\, p \tag{3.43}$$

(ii) $$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \text{grad}(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega}\times\mathbf{u}\right] = \rho\mathbf{f} - \text{grad}\, p \tag{3.44}$$

(iii) $$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \text{div}(\rho\mathbf{u}\mathbf{u}) = \rho\mathbf{f} - \text{grad}\, p \tag{3.45}$$

C. Equations of Entropy and Enthalpy

(i) $$\frac{Ds}{Dt} = 0 \tag{3.46}$$

(ii) $$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} \tag{3.47}$$

D. Conservation of Energy

If $\mathbf{f} = -\text{grad}\,\chi$, where χ is independent of time, then:

$$\rho\frac{D}{Dt}(\tfrac{1}{2}|\mathbf{u}|^2 + e + \frac{p}{\rho} + \chi) = \frac{\partial p}{\partial t} \tag{3.48}$$

E. Conservation of Total Kinetic Energy

From Equations 3.22 the equation of total kinetic energy for the flow of an inviscid and non-heat-conducting fluid is

$$\frac{\partial}{\partial t}(\rho e_t) + \text{div}\,\mathbf{b} = -\rho(\text{grad}\,\chi)\cdot\mathbf{u} \tag{3.49}$$

where now:

$$\mathbf{b} = (\rho e_t + p)\mathbf{u}$$

Here χ may be any function of coordinates and time.

Inviscid Barotropic Flow

A flow in which p can be expressed as a definite function of ρ is said to be *barotropic*. For this case since:

$$d\int\frac{dp}{\rho} = \frac{dp}{\rho}$$

then:

$$\text{grad}\int\frac{dp}{\rho} = \frac{\text{grad}\, p}{\rho} \tag{3.50}$$

The equations of motion (Equations 3.43 and 3.44) with $\mathbf{f} = -\text{grad}\,\chi$ can now be written as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} + \text{grad}\left(\int \frac{dp}{\rho} + \chi\right) = 0 \tag{3.51}$$

and:

$$\frac{\partial \mathbf{u}}{\partial t} + \text{grad}\left(\tfrac{1}{2}|\mathbf{u}|^2 + \int \frac{dp}{\rho} + \chi\right) + \boldsymbol{\omega} \times \mathbf{u} = 0 \tag{3.52}$$

3.5 INITIAL AND BOUNDARY CONDITIONS

Solutions of the equations, summarized in the preceding sections, can be obtained when certain initial and boundary conditions have been prescribed in advance. The initial conditions comprise the specification of the dependent variables at a certain initial time $t = 0$. The boundary conditions are those which have to be specified at the boundaries of the fluid. For example, the fluid flow in a pipe is bounded by the pipe, and therefore conditions at the inside wall of the pipe must be prescribed. In the case of fluid flow past a body we may or may not have another surface enclosing the body. If the body is enclosed by another surface, then conditions on the dependent variables must be specified both on the outer surface of the body and on the inner surface of the enclosing body. If the enclosing body is at infinite distance from the enclosed body, or equivalently the enclosed body is placed in an infinite fluid medium, then conditions at infinity must also be specified. Conditions at a free surface with a body submerged in a fluid are also to be specified. These conditions are termed the boundary conditions.

When a viscous fluid flows past a body, the fluid particles adhere to the surface of the body. This molecular phenomenon is an experimental fact and has been found to be true in all the continuum flows as long as the Knudsen number $(K_n) < .01$. Refer to Equation 3.85. Because of the adherence, the relative velocity between the fluid and the surface of the body is zero. Since $\mathbf{u}$ is the absolute fluid velocity, the adherence condition means that:

$$\mathbf{u}_{\text{surface}} = \mathbf{V}_s \tag{3.53}$$

where $\mathbf{V}_s$ is the absolute velocity of the surface. If $\mathbf{V}_s = 0$, i.e., the body surface is at rest, then:

$$\left.\begin{aligned} \mathbf{u} \cdot \mathbf{t} &= u_t = 0 \\ \mathbf{u} \cdot \mathbf{n} &= u_n = 0 \end{aligned}\right\} \tag{3.54}$$

where $\mathbf{t}$ is the local unit tangent vector in any direction on the surface and $\mathbf{n}$ is the unit normal vector on the surface. Here u_t and u_n are the tangential and normal components, respectively, of $\mathbf{u}$ on the surface.

The conditions (Equations 3.53 and 3.54) are called the *adherence* or *no-slip* conditions. An inviscid fluid is a fluid having no viscosity, and therefore for such fluid flows past a stationary body:

$$\left.\begin{aligned} u_t &\neq 0 \\ u_n &= 0 \end{aligned}\right\} \tag{3.55}$$

so that there is a tangential slip at the surface.

3.6 MATHEMATICAL NATURE OF THE EQUATIONS

First of all the equation for the density, i.e., ρ given by Equation 2.5, is a first order partial differential equation. On integration this equation has the solution ρJ = constant. Refer to Problem 2.1. Thus the product ρJ is a constant on a particle trajectory given in Equation 1.11. These trajectories are real characteristics of the equation and thus Equation 2.5 is hyperbolic. On the other hand, the momentum equation (e.g., Equation 3.10) has second order derivatives as shown in Equation 3.6a which can be grouped as:

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\ \mathrm{grad}(\mathrm{div}\ \mathbf{u})$$

As an example, the equation for the Cartesian component u has the second derivative terms:

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

According to the classification of partial differential equations the second order terms will render the equation as elliptic if the eigenvalues μ and $\lambda + 2\mu$ of the associated quadratic form are positive. Therefore, for $\mu > 0$ and $\lambda + 2\mu > 0$ the momentum and also the energy equation for $k > 0$ are elliptic for the steady case. These equations are parabolic in time for the nonsteady case. Collectively, the Navier-Stokes system is elliptic-hyperbolic for the steady case and parabolic-hyperbolic for the nonsteady case. If $\mu = 0$, then the elliptic and parabolic properties are lost.

3.7 VORTICITY AND CIRCULATION

The equation of vorticity for an incompressible flow in Equation 3.34. To obtain the vorticity equation for a compressible flow we take the curl of each term of Equation 3.12 with $\mathbf{f} = -\mathrm{grad}\ \chi$. Using the curl of a vector product, Equation M1.43b, we have:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \mathrm{grad})\mathbf{u} - \boldsymbol{\omega}(\mathrm{div}\ \mathbf{u}) + \mathrm{curl}\left(-\frac{\mathrm{grad}\ p}{\rho} + \frac{1}{\rho}\,\mathrm{div}\ \boldsymbol{\sigma}\right) \tag{3.56}$$

where div $\boldsymbol{\sigma}$ has been defined in Equation 3.6c.

Let the circulation of velocity be defined as:

$$\Gamma = \oint_{\mathscr{C}} \mathbf{u} \cdot \delta\mathbf{r} \tag{3.57}$$

then from Equation 1.57:

$$\frac{D\Gamma}{Dt} = \oint_{\mathscr{C}} \mathbf{a} \cdot \delta\mathbf{r}$$

which by using Equation 3.1 can be written as:

$$\frac{D\Gamma}{Dt} = \oint_{\mathscr{C}} \left(-\frac{\mathrm{grad}\ p}{\rho} + \frac{1}{\rho}\,\mathrm{div}\ \boldsymbol{\sigma}\right) \cdot \delta\mathbf{r} \tag{3.58}$$

Equations 3.56 and 3.58 describe the rate of change of vorticity and cirulation, respectively, for an observer moving with the flow.

Vorticity and Circulation for Inviscid Fluids

For inviscid fluids $\sigma = 0$, and Equation 3.56 becomes:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \text{grad})\mathbf{u} - \boldsymbol{\omega}(\text{div } \mathbf{u}) + \text{curl}\left(-\frac{\text{grad } p}{\rho}\right) \tag{3.59}$$

This equation can further be simplified for a barotropic flow, viz., in which $p = p(\rho)$, and then using Equation 3.50 and the continuity equation (Equation 3.7), we get:

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \text{grad}\right)\mathbf{u} \tag{3.60a}$$

$$= (\text{grad } \mathbf{u}) \cdot \frac{\boldsymbol{\omega}}{\rho} \tag{3.60b}$$

Similarly, for an inviscid fluid, Equation 3.58 becomes:

$$\frac{D\Gamma}{Dt} = -\oint \frac{\text{grad } p}{\rho} \cdot \delta\mathbf{r} \tag{3.61}$$

For a barotropic flow:

$$\frac{\text{grad } p}{\rho} = \text{grad} \int \frac{dp}{\rho}$$

so that Equation 3.61 simply gives:

$$\frac{D\Gamma}{Dt} = 0 \tag{3.62}$$

That is, the rate of change of circulation on a simple closed curve is zero in an inviscid barotropic flow. This result is known as the Kelvin's theorem.* Under the conditions of Kelvin's theorem, we conclude from Equation 1.54 that *the strength of a vortex tube also remains constant. This result is due to Helmholtz.*** Note the conditions: "barotropic flow of inviscid fluids" for the applicability of both the Kelvin and Helmholtz results. For an exception to this case refer to Example 3.5.

Equation 3.60b can be solved explicitly by the introduction of Lagrangian reference configuration. Since at $t = t_0 : \mathbf{r} = \mathbf{R}$, $\rho = \rho_0$, the solution of Equation 3.60b is

$$\frac{\boldsymbol{\omega}}{\rho} = \frac{\partial \mathbf{r}}{\partial X^\alpha} \frac{\omega_0^\alpha}{\rho_0} \tag{3.63}$$

where $\boldsymbol{\omega}_0$ is the vorticity vector at $t = t_0$, and ω_0^α can be treated either as the Cartesian or the contravariant components. For simplicity, treating ω_0^α as the Cartesian components we can easily

* Such flows are said to be "circulation preserving."

** Also known is the third theorem of Helmboltz.

show that Equation 3.63 is a solution of Equation 3.60b. Substituting Equation 3.63 in the left-hand side of Equation 3.60b we have:

$$
\begin{aligned}
\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) &= \frac{\partial \mathbf{u}}{\partial X^\alpha}\frac{\omega_0^\alpha}{\rho_0} \\
&= \frac{\partial \mathbf{u}}{\partial x^k}\frac{\partial x^k}{\partial X^\alpha}\frac{\omega_0^\alpha}{\rho_0} \\
&= \frac{\partial \mathbf{u}}{\partial x^m}(\mathbf{i}_m \cdot \mathbf{i}_k)\frac{\partial x^k}{\partial X^\alpha}\frac{\omega_0^\alpha}{\rho_0} \\
&= (\text{grad } \mathbf{u}) \cdot \frac{\partial \mathbf{r}}{\partial X^\alpha}\frac{\omega_0^\alpha}{\rho_0} \\
&= (\text{grad } \mathbf{u}) \cdot \frac{\boldsymbol{\omega}}{\rho}
\end{aligned}
$$

This proves that Equation 3.63 is solution of Equation 3.60b.

Equation 3.63 shows that if the vorticity is zero at any instant $t = t_0$ in the flow of an inviscid barotropic flow, then the vorticity remains zero for subsequent times. This result is known as the *Lagrange-Cauchy theorem*.

Another result of much significance for inviscid and barotropic flow can be deduced on the basis of the result in Equation 3.63. At $t = t_0$ let the material element $\delta\mathbf{R}$ be tangent to a vortex line. Then $\delta\mathbf{R} \times \boldsymbol{\omega}_0 = 0$, or:

$$\delta X^\alpha = \omega_0^\alpha$$

At time $t > t_0$:

$$\delta \mathbf{r} = \frac{\partial \mathbf{r}}{\partial X^\alpha}\delta X^\alpha = \frac{\partial \mathbf{r}}{\partial X^\alpha}\omega_0^\alpha$$

Using Equation 3.63, we get:

$$\delta \mathbf{r} = \frac{\rho_0}{\rho}\boldsymbol{\omega}$$

This result states that $\delta\mathbf{r}$ is tangent to $\boldsymbol{\omega}$ and that the vortex lines are material lines. That is, *those fluid particles which formed a vortex line at $t = t_0$* will continue to form a vortex line at $t > t_0$. This result is known as the second theorem of Helmholtz. (For the first theorem of Helmholtz refer to Section 1.14.)

The Bernoulli Equation

For steady flow, Equation 3.52 is

$$\boldsymbol{\omega} \times \mathbf{u} = -\text{grad } B \tag{3.64}$$

where:

$$\tfrac{1}{2}|\mathbf{u}|^2 + \int \frac{dp}{\rho} + \chi = B \tag{3.65}$$

Let $\mathbf{t}$ be the unit tangent vector on a streamline. Since $\boldsymbol{\omega} \times \mathbf{u}$ is orthogonal to $\mathbf{t}$, the dot product by $\mathbf{t}$ on both sides of Equation 3.64 yields:

$$\frac{dB}{ds} = 0 \tag{3.66}$$

where s is the arc length along the chosen streamline. From Equation 3.66 we find that $B =$ constant along the streamline.

Equation 3.65 is the Bernoulli equation for inviscid and barotropic fluid flows, although Bernoulli originally obtained this equation only for inviscid incompressible flows. The physical interpretation of Equation 3.65 is that for a *steady inviscid and barotropic* fluid motion the sum of kinetic, pressure, and body force energies per unit mass remain constant along a streamline. It must be noted that the constant B assumes different constant values for different streamlines.

Along with the Bernoulli equation one can obtain another equation which does not demand the fluid flow to be barotropic. The energy equation for a steady inviscid fluid flow from Equation 3.48 is

$$\frac{dH}{dt} = 0 \tag{3.67}$$

where

$$\tfrac{1}{2}|\mathbf{u}|^2 + e + \frac{p}{\rho} + \chi = H \tag{3.68a}$$

or:

$$\tfrac{1}{2}|\mathbf{u}|^2 + h + \chi = H \tag{3.68b}$$

where h is the specific enthalpy. Equation 3.67 shows that $H =$ constant on a streamline. The steady-flow aspect opens a way to determine the dependence of B and H on other field values. First, taking the grad of each term in Equation 3.68a and assuming the flow to be barotropic, we have:

$$\text{grad } H = \text{grad}\left(\tfrac{1}{2}|\mathbf{u}|^2 + \int \frac{dp}{\rho} + \chi\right) + \text{grad } e + p \text{ grad } v$$

where $v = 1/\rho$ is the specific volume. Next, considering the first law of thermodynamics in the gradient form:

$$T \text{ grad } s = \text{grad } e + p \text{ grad } v$$

we get:

$$\text{grad } H = T \text{ grad } s + \text{grad } B \tag{3.69a}$$

or:

$$\text{grad } H = T \text{ grad } s - \boldsymbol{\omega} \times \mathbf{u} \tag{3.69b}$$

Equation 3.69b is called the Crocco-Vazsonyi equation, and it shows that H is the same constant for the whole flow field provided that s = constant and $\boldsymbol{\omega} = 0$. That is, both H and B are constants for the whole field in an *isentropic irrotational* steady flow.

In the case of steady flow without body forces:

$$H = h_0$$

where h_0 is the stagnation enthalpy. The Crocco-Vazsonyi equation then simply becomes:

$$\text{grad } h_0 = T \text{ grad } s - \boldsymbol{\omega} \times \mathbf{u} \tag{3.69c}$$

3.8 SOME RESULTS BASED ON THE EQUATIONS OF MOTION

Force Acting on a Solid Body

In the case of steady viscous flow past a body of surface S_b the force $\mathbf{F}$ exerted by the fluid on the body can be calculated by a direct integration of the equation of motion. For this purpose Equation 2.35b is most convenient if the body force vector $\mathbf{f} = 0$. The equation of motion is simply:

$$\text{div } \boldsymbol{\Pi} = 0 \tag{3.70}$$

where:

$$\begin{aligned} \boldsymbol{\Pi} &= \mathbf{T} - \rho\mathbf{u}\mathbf{u} \\ &= -p\mathbf{I} + \boldsymbol{\sigma} - \rho\mathbf{u}\mathbf{u} \end{aligned}$$

We now enclose the surface S_b by any arbitrary surface S_0 to have a fixed control volume V bounded by the surface $S_b \; U \; S_0$ as is shown in Figure 3.1.

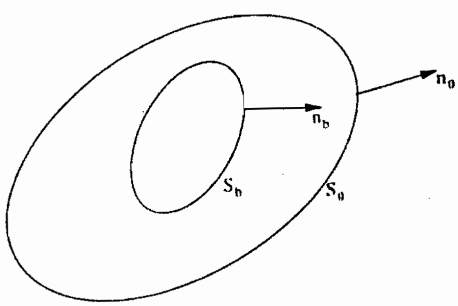

Fig. 3.1 Topology of the inner surface S_b and the outer surface S_0.

Let $\mathbf{n}_b$ and $\mathbf{n}_0$ be the outer drawn unit normal vectors to the respective surfaces. Therefore, if $\mathbf{n}_b$ is considered positive, then $\mathbf{n}_0$ will be negative with respect to $\mathbf{n}_b$.

The force exerted *by* the fluid on the body is

$$\mathbf{F} = \int_{S_b} \boldsymbol{\tau} \, dS = \int_{S_b} (-p\mathbf{I} + \boldsymbol{\sigma}) \cdot \mathbf{n}_b \, dS \tag{3.71}$$

Integrating Equation 3.70 on the volume V_0 and using the Gauss theorem we get:

$$\int_V \text{div}\,\mathbf{\Pi}\, dV = \int_{S_b} \mathbf{\Pi}\cdot\mathbf{n}_b\, dS - \int_{S_0} \mathbf{\Pi}\cdot\mathbf{n}_0\, dS = 0$$

On using the condition that the normal component of $\mathbf{u}$ at S_b is zero, i.e.:

$$\mathbf{u}\cdot\mathbf{n}|_{S_b} = 0$$

we get:

$$\mathbf{F} = \int_{S_0} [-p\mathbf{n}_0 + \boldsymbol{\sigma}\cdot\mathbf{n}_0 - \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}_0)]\, dS \tag{3.72}$$

This result is also known as the *momentum theorem*. For the total moment or torque, refer to Problem 3.2.

Stress Vector and Tensor at a Surface

It is sometimes necessary to have a formula for the stress vector $\boldsymbol{\tau}$ and tensor T evaluated at the surface of a body at rest. From the problem in Example 2.2, the stress vector $\boldsymbol{\tau}$ in terms of the coefficients of viscosity is

$$\boldsymbol{\tau} = [-p + (\lambda + 2\mu)\text{div}\,\mathbf{u}]\mathbf{n} + \mu(\boldsymbol{\omega}\times\mathbf{n}) + 2\mu(\mathbf{n}\times\text{grad})\times\mathbf{u} \tag{3.73}$$

This formula is, of course, applicable for either steady or nonsteady flows. To evaluate Equation 3.73 at the surface we use the no-slip condition:

$$\mathbf{u}|_{S_b} = 0$$

and show that the contribution of the last term in Equation 3.73 at the surface is zero. To show this, we consider a simple closed curve of arbitrary shape on the surface and use Stokes' theorem in the form given in Equation M2.10, i.e.:

$$\oint_C d\mathbf{r}\times\mathbf{u} = \int_S (\mathbf{n}\times\text{grad})\times\mathbf{u}\, dS$$

where S is the part of the surface bounded by C. However, because of the no-slip condition, the left-hand side term vanishes and so does the integrand on the right-hand side. Thus referring to Problem 3.19:

$$(\mathbf{n}\times\text{grad})\times\mathbf{u}|_{S_b} = 0$$

and:

$$\boldsymbol{\tau}|_{S_b} = [\{-p + (\lambda + 2\mu)\text{div}\,\mathbf{u}\}\mathbf{n} + \mu(\boldsymbol{\omega}\times\mathbf{n})]_{S_b} \tag{3.74a)*}$$

Also, from Example 2.2

$$D\cdot n\Big|_{S_b} = \left[\tfrac{1}{2}\omega\times n + n(\text{div}\,u)\right] \tag{3.74b}$$

The stress tensor T, defined in Equation 2.73, at S_b is

$$\mathrm{T}|_{S_b} = [-p\mathrm{I} + \lambda(\text{div}\,\mathbf{u})\mathrm{I} + 2\mu\mathrm{D}]_{S_b} \tag{3.75a}$$

* Using Equation M1.40 we have $\boldsymbol{\omega}\times\mathbf{n} = [\text{grad}\,\mathbf{u} - (\text{grad}\,\mathbf{u})^T]\cdot\mathbf{n}$.

The viscous part σ at the surface is

$$\sigma|_{S_b} = [\lambda(\text{div }\mathbf{u})\mathbf{I} + 2\mu\mathbf{D}]_{S_b} \tag{3.75b}$$

which for viscous incompressible flow is simply:

$$\sigma|_{S_b} = 2\mu\mathbf{D}|_{S_b} = \mu[\text{grad }\mathbf{u} + (\text{grad }\mathbf{u})^T]_{S_b} \tag{3.75c}$$

Vorticity Vector at the Surface

Consider a patch S bounded by a curve C on a stationary surface. On applying the usual form of the Stokes theorem to the vorticity field $\boldsymbol{\omega}$, we have:

$$\oint_c \mathbf{u}\cdot d\mathbf{r} = \int_S \boldsymbol{\omega}\cdot\mathbf{n}\, dS$$

Using the no-slip condition, we find that:

$$\boldsymbol{\omega}\cdot\mathbf{n}|_{S_b} = 0$$

This result shows that the vorticity at a stationary surface in a viscous flow is always tangential to the surface. Also the vortex tubes or filaments rest on the surface. Refer to Section 1.14.

A more practical situation is that of a body of surface S_b which moves with a velocity $\mathbf{U}$ with respect to an inertial frame, and which also rotates with an angular velocity $\boldsymbol{\Omega}$ about an axis. Let $\mathbf{r}$ be the position vector of a point of S_b from an arbitrary chosen origin 0 on the axis of rotation. Then, with respect to the inertial frame the velocity of a point of the surface is $\mathbf{U} + \boldsymbol{\Omega}\times\mathbf{r}$. Let the surface S_b be enclosed in the surface S_0 as shown in Figure 3.1. As before, let V be the region between S_b and S_0. Applying Gauss' theorem, Equation M2.4, and taking the normal $\mathbf{n}_0$ to be negative with respect to $\mathbf{n}_b$, we have:

$$\begin{aligned}\int_V \boldsymbol{\omega}\, dv &= \int_{S_b} \mathbf{n}_b \times \mathbf{u}\, dS - \int_{S_0} \mathbf{n}_0 \times \mathbf{u}\, dS \\ &= \int_{S_b} \mathbf{n}_b \times (\mathbf{U} + \boldsymbol{\Omega}\times\mathbf{r})\, dS - \int_{S_0} \mathbf{n}_0 \times \mathbf{u}\, dS \\ &= \int_{V_b} \text{curl}(\boldsymbol{\Omega}\times\mathbf{r})\, dv - \int_{S_0} \mathbf{n}_0 \times \mathbf{u}\, dS \\ &= 2\boldsymbol{\Omega}\int_{V_b} dv - \int_{S_0} \mathbf{n}_0 \times \mathbf{u}\, dS \end{aligned} \tag{3.76}$$

From Equation 3.76 we conclude that when a body is propelled in a fluid at rest at infinity (i.e., $\mathbf{u} = 0$ at S_0), then the total vorticity in the medium is $2\boldsymbol{\Omega}$ times the volume of the body. If, in particular, the body is not rotating, then the total vorticity in the medium is zero.

Rate-of-Strain Tensor at a Surface

Using the dyadic form of the Stokes' theorem, Equation M2.15,

$$\int_S (n\times grad)u\,dS = \oint_C (dr)u$$

and the no-slip condition, $u|_{S_b} = 0$, we have (refer to Problem 3.19)

$$(n \times grad)\, u = n \times (grad\, u)^T = 0 \qquad \text{at } S_b$$

Thus,

$$n \times D = n \times W \qquad \text{at } S_b$$

To obtain further results, we consider $(n \times W) \cdot n$, which can be written as

$$\begin{aligned}(n \times W) \cdot n &= n \times (W \cdot n) \\ &= \tfrac{1}{2} n \times (\omega \times n)\end{aligned}$$

Using $\omega \cdot n|_{S_b} = 0$, we have

$$n \times (W \cdot n) = \tfrac{1}{2}\omega \qquad \text{at } S_b$$

from which we conclude that

$$n \times W = \tfrac{1}{2}\omega n \qquad \text{at } S_b$$

The above equation and the equation of $D \cdot n|_{S_b}$ from Example 2.2 are simultaneously satisfied by taking

$$D\big|_{S_b} = (div\, u)nn + \tfrac{1}{2} n(\omega \times n) + \tfrac{1}{2}(\omega \times n)n$$

which is the rate-of-strain tensor at S_b under the no-slip condition.

3.9 NONDIMENSIONAL PARAMETERS IN FLUID MOTION

In the study of fluid dynamics, the dimensional considerations of all physical quantities which appear in the equations of motion are of much importance. The equations of motion themselves are dimensionally homogeneous. For example, in the equation of continuity $\partial\rho/\partial t$ has the same dimension as div(ρ**u**) which can easily be verified by choosing dimensions of length, mass, and time. In the following analysis we have adopted a different approach to expose the dimensionless groups. We choose a reference length, a reference velocity, and a series of constant "characteristic" quantities to nondimensionalize the equations. The reference and the characteristic quantities are denoted by the subscripts r and c, respectively. Following is a list of such quantities:

L_r = reference length
U_r = reference velocity
a_c = characteristic speed of sound
C_{pc} = characteristic specific heat (either C_p or C_v)
h_c = characteristic enthalpy
k_c = characteristic heat conductivity
P_c = characteristic pressure
t_c = characteristic time

T_c = characteristic temperature
ρ_c = characteristic density
μ_c = characteristic viscosity (either first or second)
χ_c = characteristic body force potential

Denoting the nondimensional terms by a superscript asterisk, we get:

$\mathbf{b} = \mathbf{b}^*\rho_c U_r^3$, vector in the energy equation
curl = curl*/L_r, curl operator
$C_p = C_p^* C_{pc}$, specific heat
div = div*/L_r, divergence operator
$\mathbf{D} = \mathbf{D}^* U_r/L_r$, rate-of-strain tensor
$e_t = e_t^* U_r^2$, sum of kinetic energies
grad = grad*/L_r, gradient operator
$h = h^* h_c$, enthalpy
$k = k^* k_c$, heat conductivity
$p = p^* P_c^*$, pressure
$\mathbf{r} = \mathbf{r}^* L_r$, position
$t = t^* t_c$, time
$T = = T^* T_c$, temperature
$\mathbf{u} = \mathbf{u}^* U_r$, velocity vector
$\rho = \rho^* \rho_c$, density
$\mu = \mu^* \mu_c$, viscosity
$\lambda = \lambda^* \mu_c$, viscosity
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* \mu_c U_r/L_r$, viscous stress tensor
$\chi = \chi^* \chi_c$, body force potential

For the purpose of uncovering the nondimensional parameters, we choose Equations 3.8, and 3.10 with the body force expressed as $\mathbf{f} = -\text{grad}\,\chi$, and Equation 3.23a. On carrying out the nondimensionalization mentioned above, we get:

$$(S_t)\frac{\partial \rho^*}{\partial t^*} + \text{div}^*(\rho^*\mathbf{u}^*) = 0 \tag{3.77}$$

$$(S_t)\frac{\partial}{\partial t^*}(\rho^*\mathbf{u}^*) + \text{div}^*(\rho^*\mathbf{u}^*\mathbf{u}^*) = -\left(\frac{1}{F_r}\right)\rho^*\text{grad}^*\chi^* - (E_u)\text{grad}^* p^* - \left(\frac{1}{R_e}\right)\text{div}^*\boldsymbol{\sigma}^* \tag{3.78}$$

$$(S_t)\frac{\partial}{\partial t^*}\left[\rho^*\left\{e_t^* + \left(\frac{1}{F_r}\right)\chi^*\right\}\right] + \text{div}^*\mathbf{b}^* = 0 \tag{3.79}$$

where:

$$\mathbf{b}^* = \left[\rho^*\left\{e_t^* + \left(\frac{1}{F_r}\right)\chi^*\right\} + (E_u)p^*\right]\mathbf{u}^* - \left(\frac{1}{R_e}\right)\left[\boldsymbol{\sigma}^*\cdot\mathbf{u}^* + \left(\frac{\alpha_0}{P_r M^2}\right)k^*\text{grad}^* T^*\right] \tag{3.80}$$

The nondimensional parameters placed in parentheses are

$$S_t = L_r/t_c U_r, \text{ the Strouhal number} \tag{3.81a}$$

$$F_r = U_r^2/\chi_c, \text{ the Froude number} \tag{3.81b}$$

Note that the body forces are usually due to gravity and in that case $\chi_c = gL_r$, where g is the force of gravity on a unit mass at the earth's surface.

$$E_u = p_c/\rho_c U_r^2, \text{ the Euler number} \tag{3.81c}$$

$$R_e = \rho_c U_c L_r/\mu_c, \text{ the Reynolds number} \tag{3.81d}$$

$$M = U_r/a_c, \text{ the Mach number} \tag{3.81e}$$

$$P_r = \mu_c C_{pc}/k_c, \text{ the Prandtl number} \tag{3.81f}$$

$$\alpha_0 = T_c C_{pc}/a_c^2 \tag{3.81g}$$

Most of the important nondimensional parameters have been listed above. Some other parameters, which usually appear in specific problems are

$$P_e = \text{Peclet number} = R_e \cdot P_r \tag{3.81h}$$

$$N_u = \text{Nusselt number} = \text{heat convection/heat conduction} \tag{3.81i}$$

$$G_r = \text{Grashof number} = \text{buoyancy parameter in convection} \tag{3.81j}$$

$$E_c = \text{Eckert number} = U_r^2/T_c C_{pc} = M^2/\alpha_0 \tag{3.81k}$$

$$B_r = \textit{Brinkman number} = \frac{\mu_c U_r^2}{k_c T_c} = P_r E_c \tag{3.81l}$$

$$1/B_r = \alpha_0/P_r M^2 \tag{3.81m}$$

A study of Equations 3.77–3.79 suggests that all quantities in them are nondimensional and when solved under some nondimensional initial and boundary conditions will yield solutions of the form:

$$\phi = \phi^*(\mathbf{r}^*, t^*, R_e, E_u, E_c, P_r, F_r, M, G_r, N_u)$$

where ϕ^* is any one of the dependent variables. The preceding analysis first of all demonstrates the existence of a series of nondimensional parameters through the use of the equations of motion. By using another scheme of nondimensionalization many parameters can be made to disappear from the equations. For example, the appearance of S_t in the equations can be completely eliminated if in place of t_c (a characteristic time) one uses L_r/U_r to nondimensionalize the time. This, however, does not mean that the effect of the Strouhal number, or in fact the effect of any other nondimensional parameter, can be altered simply by choosing other schemes of nondimensionalization. The scheme chosen here is intended only to expose some important parameters by using the equations of motion themselves.

The physical meaning of each nondimensional parameter is evident from the representations in Equation 3.81. However, in words they can be stated as follows:

S_t = reference time scale/characteristic time scale
F_r = kinetic energy/potential energy
E_u = characteristic pressure/reference pressure
R_e = inertia force/viscous force
M = reference speed/characteristic speed of sound
P_r = kinematic diffusivity/thermal diffusivity

Since in any viscous flow problem the Reynolds number R_e is of much importance, we restate its structure through the equation of motion (Equation 3.10). Noting that:

$$\operatorname{div}(\rho\mathbf{uu}) = \text{inertia force per unit volume} \sim \frac{\rho_c U_r^2}{L_r}$$

$$\operatorname{div}\boldsymbol{\sigma} = \text{viscous force per unit volume} \sim \frac{\mu U_r}{L_r^2}$$

we have:

$$R_e = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho_c U_r L_r}{\mu_c}$$

Principle of Similarity

The following question naturally arises in fluid mechanics. If the solution of the equations of motion under certain specified initial and boundary conditions or the experimental data under the same conditions are available for a specific body shape, is it possible to use these data to determine the flow field for a *geometrically similar* shape? Two bodies are said to be geometrically similar if the ratios of all linear dimensions of the two bodies and their angles are the same. From the coordinates x_i and angles θ_j of the first body, a second body is constructed by taking the coordinates as αx_i and the same angles θ_j, where $\alpha = const.$ $i = 1,2,3$, and $j = 1,2,3, \ldots$. As an immediate application, it may be noted that an ellipse of axes (A,B) is geometrically similar to an ellipse of axes (α,b) *iff* $a/A = b/B = \alpha$. The points (x_i) and (αx_i) are said to be geometrically similar. An answer to the question posed is in the affirmative provided that all the nondimensional parameters stated in Equation 3.81 have the same values for both flows. We therefore state the following, known as the principle of *dynamic similarity.*

Dynamic Similarity

Two flows past or through geometrically similar bodies are said to be dynamically similar if the field values at geometrically similar points in both flow fields are the same and the values of the nondimensional parameters S_t, F_r, E_u, R_e, M, P_r, etc. are the same for both flows.

In most practical applications the results are usually expressed in the form of one nondimensional parameter as a function of the remaining parameters, e.g.:

$$S_t = f(R_e, F_r, E_u, M, P_r) \text{ etc.}$$

Variable Nondimensional Parameters

Almost all the nondimensional parameters introduced earlier have been made to depend on the constant characteristic quantities. In a flow field these parameters, although remaining nondimensional, vary according to the field under consideration. Thus the Mach number in a field depends on the local fluid velocity and also on the local speed of sound, that is

$$M = |\mathbf{u}|/a \tag{3.82}$$

where a is the local speed of sound and is a field variable. In the same manner the Prandtl number in the field is

$$P_r = \mu C_p/k \tag{3.83a}$$

and in general is a function of temperature T. For air, an empirical formula in terms of $\gamma = C_p/C_v$ (also known as Eucken's formula) is

$$P_r = 4\gamma/(9\gamma - 5) \tag{3.83b}$$

where γ in general is a function of temperature. For a perfect gas $\gamma = 1.40$ and $P_r = 0.72$.

The Reynolds number R_e is also a field variable and defined as:

$$R_e = \rho U_r L_r/\mu \tag{3.84}$$

where both ρ and μ are field quantities. In the same manner other parameters are generally field variables.

The ratio of the mean free path ℓ and L_r is termed Knudsen's number (K_n):

$$K_n = \ell/L_r$$

If we make use of the equations (refer to Reference 2, and Section 2.16):

$$\ell = (\sqrt{2}\pi N\sigma^2)^{-1}, \qquad \mu = 0.1792\sqrt{k_b m_0 T}/\sigma^2$$

and the formula for the speed of sound in a thermally perfect gas:

$$a^2 = \gamma RT$$

we get:

$$K_n = \frac{\sqrt{\gamma}}{0.1792\sqrt{2}\pi}\frac{M}{R_e} \simeq \sqrt{\frac{\gamma\pi}{2}}\frac{M}{R_e} \tag{3.85}$$

If $0 \leq K_n \leq 0.01$ then the medium is a continuum.

Principle of Reynolds' Number Similarity

Consider two viscous incompressible flow fields:

$$x, \quad t, \quad u, \quad \rho, \quad p$$

and

$$x', \quad t', \quad u', \quad \rho', \quad p'$$

Each flow field has been measured in the same basic units, e.g., kg-m-s. Let us relate the flow field through

$$x' = \alpha x, \; t' = \beta t, \quad u' = \gamma u, \; \rho' = \sigma\rho, \quad p' = \lambda p$$

where α, β, γ, σ, and λ are nondimensional constants. By transforming the substantive derivative from the unprimed to the primed quantities, if follows that

$$\beta = \alpha/\gamma$$

The Navier–Stokes equation for incompressible flow is

$$\rho\frac{Du}{Dt} = -\,grad\,p + \mu\nabla^2 u$$

which on transformation becomes

$$\frac{\lambda}{\sigma\gamma^2}\rho'\frac{D'u'}{D't'} = -\,grad'\,p' + \frac{\lambda\alpha}{\gamma}\mu\nabla'^2 u'$$

Thus,

$$\lambda = \sigma\gamma^2, \; and \; \frac{\lambda\alpha\mu}{\gamma} = \mu'$$

or,

$$\alpha\gamma\sigma = \mu'/\mu$$

Further

$$\alpha = l'/l, \quad \gamma = q'/q, \quad \sigma = \rho'/\rho$$

where

$$q = |u|, \quad q' = |u'|, \quad l = |x|, \quad and \quad l' = |x'|$$

we have

$$\frac{l'q'\rho'}{lq\rho} = \frac{\mu'}{\mu}$$

or,

$$\frac{\rho lq}{\mu} = \frac{\rho' l' q'}{\mu'}$$

for the two flows to be dynamically similar under the Reynolds' number similarity.

3.10 COORDINATE TRANSFORMATION

To obtain a solution of the complete Navier-Stokes system of equations along with the equations of continuity and energy is the central problem in the theory of fluid flows. Starting from the times of Euler, Navier, and Stokes (1759 to 1845), a tremendous amount of effort has been expended in providing the solutions of the fluid flow equations for one situation or another. However, with the advent of the high speed digital computers, particularly in the last 15 years, the scene has shifted to the considerations of more realistic physical problems which could have never been considered earlier even in their simplest forms.

The solutions of fluid dynamic problems are always sought either for the flow of fluids past bodies or for the flow of fluids through internal passages. The first category is termed the external flow problems and the second, the internal flow problems. In any event, there is a very strong dependence of the shape of a body or of an internal passage on the flow field to be obtained through solution of the equations. Either for external or internal flow problems, the equations have to be written with reference to a coordinate system. This requires the development of a methodology of general coordinate transformation for the equations.

Orthogonal Coordinates

The simplest case in orthogonal coordinates is that of rectangular Cartesian coordinates. Let x_i ($i = 1, 2, 3$) be the Cartesian coordinates; then Equations 3.8, 3.10, and 3.22 referred to these coordinates can be written down by noting that:

$$\text{div}(\rho \mathbf{u}) = \frac{\partial}{\partial x_s}(\rho u_s) \tag{3.86a}$$

$$\text{div}(\rho \mathbf{u}\mathbf{u}) = \frac{\partial}{\partial x_s}(\rho u_r u_s)\mathbf{i}_r \tag{3.86b}$$

$$\boldsymbol{\sigma} = \sigma_{rs}\mathbf{i}_r\mathbf{i}_s \tag{3.86c}$$

$$\text{div}\,\boldsymbol{\sigma} = \frac{\partial \sigma_{rs}}{\partial x_s}\mathbf{i}_r \tag{3.86d}$$

$$\sigma_{rs} = \left(\lambda \frac{\partial u_m}{\partial x_m}\right)\delta_{rs} + \mu\left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r}\right) \tag{3.87}$$

Thus, the equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_s}(\rho u_s) = 0 \tag{3.88}$$

$$\frac{\partial}{\partial t}(\rho u_r) + \frac{\partial}{\partial x_s}(\rho u_r u_s) = -\frac{\partial p}{\partial x_r} + \frac{\partial \sigma_{rs}}{\partial x_s} + \rho f_r \tag{3.89}$$

where $r = 1, 2, 3$, and:

$$\frac{\partial}{\partial t}(\rho e_t) + \frac{\partial b_s}{\partial x_s} = \rho(\mathbf{f} \cdot \mathbf{u}) \tag{3.90}$$

where:

$$b_s = (\rho e_t + p)u_s - \sigma_{sr}u_r - k\frac{\partial T}{\partial x_s} \tag{3.91}$$

Introducing the more familiar notations:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad u_1 = u, \quad u_2 = v, \quad u_3 = w$$

$$\sigma_{11} = \sigma_{xx}, \quad \sigma_{12} = \sigma_{xy} \text{ etc.}, \; b_1 = b_x, \quad b_2 = b_y, \quad b_3 = b_z$$

we can write Equations 3.88 to 3.90 in expanded form as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{3.92}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x \tag{3.93}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y \tag{3.94}$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho w^2) = -\frac{\partial p}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z \tag{3.95}$$

$$\frac{\partial}{\partial t}(\rho e_t) + \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} = \rho(uf_x + vf_y + wf_z) \tag{3.96}$$

since σ is symmetric, $\sigma_{xz} = \sigma_{zx}$, etc. Also, equations in the form of Equation 3.11 can be written directly by using:

$$(\mathbf{u} \cdot \text{grad})\mathbf{u} = u_l \frac{\partial u_k}{\partial x_l} \mathbf{i}_k$$

or else by using the continuity equation in Equations 3.93–3.96.

We can also write Equations 3.92–3.96 as a single conservation equation with numerical vectors **E**, **F**, **G**, **H**, and **S** as:

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = \mathbf{S} \tag{3.97}$$

where:

$$\mathbf{E} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ p + \rho u^2 - \sigma_{xx} \\ \rho vu - \sigma_{yx} \\ \rho wu - \sigma_{zx} \\ b_x \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv - \sigma_{xy} \\ p + \rho v^2 - \sigma_{yy} \\ \rho wv - \sigma_{zy} \\ b_y \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} \rho w \\ \rho uw - \sigma_{xz} \\ \rho vw - \sigma_{yz} \\ p + \rho w^2 - \sigma_{zz} \\ b_z \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(uf_x + vf_y + wf_z) \end{bmatrix} \tag{3.98}$$

Next, in the xyz-space we introduce an orthogonal curvilinear coordinate system ξ_1, ξ_2, ξ_3. The position vector $\mathbf{r}$ is a function of ξ_1, ξ_2, ξ_3 where:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{3.99}$$

and $\mathbf{i} = \mathbf{i}_1$, $\mathbf{j} = \mathbf{i}_2$, $\mathbf{k} = \mathbf{i}_3$ are the constant unit vectors along x, y, z axes, respectively. The partial derivatives of $\mathbf{r}$ with respect to ξ_1, ξ_2, ξ_3 (i.e., $\partial\mathbf{r}/\partial\xi_1$, $\partial\mathbf{r}/\partial\xi_2$, $\partial\mathbf{r}/\partial\xi_3$) are tangential to the coordinate lines ξ_1, ξ_2, and ξ_3, respectively, and are called the base vectors. Let:

$$h_1 = \left|\frac{\partial\mathbf{r}}{\partial\xi_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{r}}{\partial\xi_2}\right|, \quad h_3 = \left|\frac{\partial\mathbf{r}}{\partial\xi_3}\right| \tag{3.100}$$

then:

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial\mathbf{r}}{\partial\xi_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial\mathbf{r}}{\partial\xi_2}, \quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial\mathbf{r}}{\partial\xi_3} \tag{3.101}$$

are the unit vectors along ξ_1, ξ_2, ξ_3, respectively. In terms of the unit base vectors, the velocity vector $\mathbf{u}$ is now written as:

$$\mathbf{u} = \mathbf{e}_1 u_1 + \mathbf{e}_2 u_2 + \mathbf{e}_3 u_3 \tag{3.102}$$

so that u_1, u_2, u_3 are the physical components of $\mathbf{u}$, (refer to ME.1, Section 14). The grad operator in terms of $\mathbf{e}_i$, according to our adopted convention, is

$$\text{grad}(\ ) = \nabla(\ ) = \frac{\partial(\ )}{\partial\xi_1}\frac{\mathbf{e}_1}{h_1} + \frac{\partial(\ )}{\partial\xi_2}\frac{\mathbf{e}_2}{h_2} + \frac{\partial(\ )}{\partial\xi_3}\frac{\mathbf{e}_3}{h_3} \tag{3.103}$$

where the empty space is to be filled by a scalar, vector, or tensor. Similarly the div operator is

$$\text{div}(\ ) = \nabla\cdot(\ ) = \frac{\partial(\ )}{\partial\xi_1}\cdot\frac{\mathbf{e}_1}{h_1} + \frac{\partial(\ )}{\partial\xi_2}\cdot\frac{\mathbf{e}_2}{h_2} + \frac{\partial(\ )}{\partial\xi_3}\cdot\frac{\mathbf{e}_3}{h_3} \tag{3.104}$$

The unit tensor in these coordinates is

$$\mathbf{I} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 \tag{3.105}$$

Therefore, the stress tensor $\mathbf{T}$ is given by (refer to Equation 2.73):

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \boldsymbol{\sigma} \\ &= -p(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) + \boldsymbol{\sigma} \end{aligned}$$

where

$$\sigma = (\lambda \text{ div } \mathbf{u})(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) + \mu[(\text{grad } \mathbf{u}) + (\text{grad } \mathbf{u})^T] \tag{3.106}$$

and:

$$\text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \xi_1}\frac{\mathbf{e}_1}{h_1} + \frac{\partial \mathbf{u}}{\partial \xi_2}\frac{\mathbf{e}_2}{h_2} + \frac{\partial \mathbf{u}}{\partial \xi_3}\frac{\mathbf{e}_3}{h_3} \tag{3.107a}$$

$$(\text{grad } \mathbf{u})^T = \frac{\mathbf{e}_1}{h_1}\frac{\partial \mathbf{u}}{\partial \xi_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial \mathbf{u}}{\partial \xi_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial \mathbf{u}}{\partial \xi_3} \tag{3.107b}$$

According to Equation 3.104:

$$\text{div } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \xi_1}\cdot\frac{\mathbf{e}_1}{h_1} + \frac{\partial \mathbf{u}}{\partial \xi_2}\cdot\frac{\mathbf{e}_2}{h_2} + \frac{\partial \mathbf{u}}{\partial \xi_3}\cdot\frac{\mathbf{e}_3}{h_3} \tag{3.107c}$$

Referring to Equation 3.102 we find that the evaluation of the quantities in Equations 3.106 and 3.107 involves the derivatives of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. To obtain these derivatives we proceed as follows.

Since the coordinates ξ_1, ξ_2, ξ_3 are orthogonal and also form a right-handed system, then the following identities are obviously satisfied:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \qquad \mathbf{e}_1 \cdot \mathbf{e}_3 = 0 \tag{3.108a}$$

or:

$$\frac{\partial \mathbf{r}}{\partial \xi_1}\cdot\frac{\partial \mathbf{r}}{\partial \xi_2} = 0, \qquad \frac{\partial \mathbf{r}}{\partial \xi_2}\cdot\frac{\partial \mathbf{r}}{\partial \xi_3} = 0, \qquad \frac{\partial \mathbf{r}}{\partial \xi_1}\cdot\frac{\partial \mathbf{r}}{\partial \xi_3} = 0 \tag{3.108b}$$

and

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3, \qquad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \tag{3.109}$$

Differentiating the three equations in Equation 3.108b, respectively, by that coordinate which does not appear explicitly, and adding any two and using the remaining equation, we get:

$$\frac{\partial \mathbf{r}}{\partial \xi_1}\cdot\frac{\partial^2 \mathbf{r}}{\partial \xi_2 \partial \xi_3} = 0, \qquad \frac{\partial \mathbf{r}}{\partial \xi_2}\cdot\frac{\partial^2 \mathbf{r}}{\partial \xi_1 \partial \xi_3} = 0, \qquad \frac{\partial \mathbf{r}}{\partial \xi_1}\cdot\frac{\partial^2 \mathbf{r}}{\partial \xi_2 \partial \xi_3} = 0$$

Thus, for example, the first equation yields:

$$\frac{\partial^2 \mathbf{r}}{\partial \xi_2 \partial \xi_3} = \alpha \mathbf{e}_2 + \beta \mathbf{e}_3$$

On using Equation 3.101 and noting that:

$$\frac{\partial^2 \mathbf{r}}{\partial \xi_2 \partial \xi_3} = \frac{\partial}{\partial \xi_2}\left(\frac{\partial \mathbf{r}}{\partial \xi_3}\right) = \frac{\partial}{\partial \xi_3}\left(\frac{\partial \mathbf{r}}{\partial \xi_2}\right)$$

we get:

$$\alpha = \frac{\partial h_2}{\partial \xi_3}, \qquad \beta = \frac{\partial h_3}{\partial \xi_2}, \qquad \text{etc.}$$

Using this technique, we obtain:

$$\begin{aligned}
\frac{\partial \mathbf{e}_1}{\partial \xi_2} &= \frac{\mathbf{e}_2}{h_1}\frac{\partial h_2}{\partial \xi_1}, & \frac{\partial \mathbf{e}_2}{\partial \xi_1} &= \frac{\mathbf{e}_1}{h_2}\frac{\partial h_1}{\partial \xi_2} \\
\frac{\partial \mathbf{e}_1}{\partial \xi_3} &= \frac{\mathbf{e}_3}{h_1}\frac{\partial h_3}{\partial \xi_1}, & \frac{\partial \mathbf{e}_3}{\partial \xi_1} &= \frac{\mathbf{e}_1}{h_3}\frac{\partial h_1}{\partial \xi_3} \\
\frac{\partial \mathbf{e}_2}{\partial \xi_3} &= \frac{\mathbf{e}_3}{h_2}\frac{\partial h_3}{\partial \xi_2}, & \frac{\partial \mathbf{e}_3}{\partial \xi_2} &= \frac{\mathbf{e}_2}{h_3}\frac{\partial h_2}{\partial \xi_3}
\end{aligned} \tag{3.110}$$

Similarly, differentiating the equations in Equation 3.109 and using Equation 3.110, we obtain:

$$\begin{aligned}
\frac{\partial \mathbf{e}_1}{\partial \xi_1} &= -\frac{\mathbf{e}_2}{h_2}\frac{\partial h_1}{\partial \xi_2} - \frac{\mathbf{e}_3}{h_3}\frac{\partial h_1}{\partial \xi_3} \\
\frac{\partial \mathbf{e}_2}{\partial \xi_2} &= -\frac{\mathbf{e}_3}{h_3}\frac{\partial h_2}{\partial \xi_3} - \frac{\mathbf{e}_1}{h_1}\frac{\partial h_2}{\partial \xi_1} \\
\frac{\partial \mathbf{e}_3}{\partial \xi_3} &= -\frac{\mathbf{e}_1}{h_1}\frac{\partial h_3}{\partial \xi_1} - \frac{\mathbf{e}_2}{h_2}\frac{\partial h_3}{\partial \xi_2}
\end{aligned} \tag{3.111}$$

Having obtained all the derivatives of the unit base vectors we can now use Equations 3.110 and 3.111 to obtain div **A**, curl **A**, and $\nabla^2\phi$, where **A** is an arbitrary vector and ϕ is a scalar:

$$\text{div } \mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \xi_1}(h_2h_3A_1) + \frac{\partial}{\partial \xi_2}(h_1h_3A_2) + \frac{\partial}{\partial \xi_3}(h_1h_2A_3)\right] \tag{3.112}$$

$$\begin{aligned}
\text{curl } \mathbf{A} &= \frac{\mathbf{e}_1}{h_1} \times \frac{\partial \mathbf{A}}{\partial \xi_1} + \frac{\mathbf{e}_2}{h_2} \times \frac{\partial \mathbf{A}}{\partial \xi_2} + \frac{\mathbf{e}_3}{h_3} \times \frac{\partial \mathbf{A}}{\partial \xi_3} \\
&= \frac{\mathbf{e}_1}{h_2h_3}\left\{\frac{\partial}{\partial \xi_2}(h_3A_3) - \frac{\partial}{\partial \xi_3}(h_2A_2)\right\} + \frac{\mathbf{e}_2}{h_1h_3}\left\{\frac{\partial}{\partial \xi_3}(h_1A_1) - \frac{\partial}{\partial \xi_1}(h_3A_3)\right\} \\
&\quad + \frac{\mathbf{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial \xi_1}(h_2A_2) - \frac{\partial}{\partial \xi_2}(h_1A_1)\right\}
\end{aligned} \tag{3.113}$$

$$\begin{aligned}
\nabla^2\phi &= \text{div (grad } \phi) \\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial \phi}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_3h_1}{h_2}\frac{\partial \phi}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial \phi}{\partial \xi_3}\right)\right]
\end{aligned} \tag{3.114}$$

Navier-Stokes Equations in Orthogonal Coordinates

The transformation of the continuity equation (Equation 3.8) is quite straightforward since the term div(ρ**u**) can be written according to Equation 3.112. Similarly, the term grad p in Equation 3.11 can be written using Equation 3.103. The term (**u** · grad)**u** can also be written by using the grad operator (Equation 3.103) and then operating **u** · grad on **u** while using Equation 3.102 and the derivatives in Equations 3.110 and 3.111. In terms of the basis $\mathbf{e}_i$, the tensor $\boldsymbol{\sigma}$ defined in Equation 3.106 is represented as:

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j$$

Thus div $\boldsymbol{\sigma}$ is given by:

$$\operatorname{div}\boldsymbol{\sigma} = \frac{1}{h_1}\frac{\partial}{\partial\xi_1}(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial\xi_2}(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial\xi_3}(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\mathbf{e}_3$$

Again using Equations 3.110 and 3.111, the group of terms which are the coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in div $\boldsymbol{\sigma}$ can be isolated.

Performing the operations noted above, the equations of viscous compressible flow in orthogonal curvilinear coordinates are as follows:

$$\frac{\partial\rho}{\partial t} + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(\rho h_2h_3u_1) + \frac{\partial}{\partial\xi_2}(\rho h_1h_3u_2) + \frac{\partial}{\partial\xi_3}(\rho h_1h_2u_3)\right] = 0 \quad \textbf{(3.115)}$$

$$\rho\left(\frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial\xi_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial\xi_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial\xi_3} + \frac{u_1u_2}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} - \frac{u_2^2}{h_1h_2}\frac{\partial h_2}{\partial\xi_1} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_1}{\partial\xi_3} - \frac{u_3^2}{h_1h_3}\frac{\partial h_3}{\partial\xi_1}\right)$$
$$= \frac{1}{h_1^2h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_1h_2h_3T_{\xi_1\xi_1}) + \frac{\partial}{\partial\xi_2}(h_1^2h_3T_{\xi_1\xi_2}) + \frac{\partial}{\partial\xi_3}(h_1^2h_2T_{\xi_1\xi_3})\right]$$
$$- \frac{1}{h_1^2}\frac{\partial h_1}{\partial\xi_1}T_{\xi_1\xi_1} - \frac{1}{h_1h_2}\frac{\partial h_2}{\partial\xi_1}T_{\xi_2\xi_2} - \frac{1}{h_1h_3}\frac{\partial h_3}{\partial\xi_1}T_{\xi_3\xi_3} \quad \textbf{(3.116)}$$

$$\rho\left(\frac{\partial u_2}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_2}{\partial\xi_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial\xi_2} + \frac{u_3}{h_3}\frac{\partial u_2}{\partial\xi_3} + \frac{u_1u_2}{h_1h_2}\frac{\partial h_2}{\partial\xi_1} - \frac{u_1^2}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} - \frac{u_3^2}{h_2h_3}\frac{\partial h_3}{\partial\xi_2} + \frac{u_2u_3}{h_2h_3}\frac{\partial h_2}{\partial\xi_3}\right)$$
$$= \frac{1}{h_1h_2^2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2^2h_3T_{\xi_1\xi_2}) + \frac{\partial}{\partial\xi_2}(h_1h_2h_3T_{\xi_2\xi_2}) + \frac{\partial}{\partial\xi_3}(h_1h_2^2T_{\xi_2\xi_3})\right]$$
$$- \frac{1}{h_1h_2}\frac{\partial h_1}{\partial\xi_2}T_{\xi_1\xi_1} - \frac{1}{h_2^2}\frac{\partial h_2}{\partial\xi_2}T_{\xi_2\xi_2} - \frac{1}{h_2h_3}\frac{\partial h_3}{\partial\xi_2}T_{\xi_3\xi_3} \quad \textbf{(3.117)}$$

$$\rho\left(\frac{\partial u_3}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_3}{\partial\xi_1} + \frac{u_2}{h_2}\frac{\partial u_3}{\partial\xi_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial\xi_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_3}{\partial\xi_1} - \frac{u_1^2}{h_1h_3}\frac{\partial h_1}{\partial\xi_3} + \frac{u_2u_3}{h_2h_3}\frac{\partial h_3}{\partial\xi_2} - \frac{u_2^2}{h_2h_3}\frac{\partial h_2}{\partial\xi_3}\right)$$
$$= \frac{1}{h_1h_2h_3^2}\left[\frac{\partial}{\partial\xi_1}(h_2h_3^2T_{\xi_1\xi_3}) + \frac{\partial}{\partial\xi_2}(h_1h_3^2T_{\xi_2\xi_3}) + \frac{\partial}{\partial\xi_3}(h_1h_2h_3T_{\xi_3\xi_3})\right]$$
$$- \frac{1}{h_1h_3}\frac{\partial h_1}{\partial\xi_3}T_{\xi_1\xi_1} - \frac{1}{h_2h_3}\frac{\partial h_2}{\partial\xi_3}T_{\xi_2\xi_2} - \frac{1}{h_3^2}\frac{\partial h_3}{\partial\xi_3}T_{\xi_3\xi_3} \quad \textbf{(3.118)}$$

The quantities T_{ij} for a Newtonian fluid are as follows:

$$T_{\xi_1\xi_1} = -p + 2\mu\left(\frac{1}{h_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{h_1h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{u_3}{h_1h_3}\frac{\partial h_1}{\partial \xi_3}\right) + \lambda(\text{div } \mathbf{u})$$

$$T_{\xi_2\xi_2} = -p + 2\mu\left(\frac{1}{h_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{h_1h_2}\frac{\partial h_2}{\partial \xi_1} + \frac{u_3}{h_2h_3}\frac{\partial h_2}{\partial \xi_3}\right) + \lambda(\text{div } \mathbf{u})$$

$$T_{\xi_3\xi_3} = -p + 2\mu\left(\frac{1}{h_3}\frac{\partial u_3}{\partial \xi_3} + \frac{u_1}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} + \frac{u_2}{h_2h_3}\frac{\partial h_3}{\partial \xi_2}\right) + \lambda(\text{div } \mathbf{u})$$

$$T_{\xi_1\xi_2} = \mu\left(\frac{1}{h_1}\frac{\partial u_2}{\partial \xi_1} + \frac{1}{h_2}\frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{h_1h_2}\frac{\partial h_2}{\partial \xi_1} - \frac{u_1}{h_1h_2}\frac{\partial h_1}{\partial \xi_2}\right)$$

$$T_{\xi_1\xi_3} = \mu\left(\frac{1}{h_1}\frac{\partial u_3}{\partial \xi_1} + \frac{1}{h_3}\frac{\partial u_1}{\partial \xi_3} - \frac{u_3}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{u_1}{h_1h_3}\frac{\partial h_1}{\partial \xi_3}\right)$$

$$T_{\xi_2\xi_3} = \mu\left(\frac{1}{h_2}\frac{\partial u_3}{\partial \xi_2} + \frac{1}{h_3}\frac{\partial u_2}{\partial \xi_3} - \frac{u_3}{h_2h_3}\frac{\partial h_3}{\partial \xi_2} - \frac{u_2}{h_2h_3}\frac{\partial h_2}{\partial \xi_3}\right)$$

where div **u** is written according to Equation 3.112. These equations are to be used when the conservation law form of the equations is not desired.

The equation of total enthalpy H, from Equation 3.19b is

$$\rho\left(\frac{\partial H}{\partial t} + \frac{u_1}{h_1}\frac{\partial H}{\partial \xi_1} + \frac{u_2}{h_2}\frac{\partial H}{\partial \xi_2} + \frac{u_3}{h_3}\frac{\partial H}{\partial \xi_3}\right) - \frac{\partial p}{\partial t}$$
$$= \sum_{n=1}^{3} \frac{1}{h_n}\frac{\partial}{\partial \xi_n}\left(\sigma_{n1}u_1 + \sigma_{n2}u_2 + \sigma_{n3}u_3 + \frac{k}{h_n}\frac{\partial T}{\partial \xi_n}\right)$$
$$+ \sum_{n=1}^{3} \frac{1}{h_n}\left(\sigma_{n1}u_1 + \sigma_{n2}u_2 + \sigma_{n3}u_3 + \frac{k}{h_n}\frac{\partial T}{\partial \xi_n}\right)\frac{\partial \lambda_n}{\partial \xi_n} \tag{3.119}$$

where:

$$\lambda_1 = \ell n(h_2h_3), \qquad \lambda_2 = \ell n(h_1h_3), \qquad \lambda_3 = \ell n(h_1h_2)$$

and σ_{ij} are the viscous parts of T_{ij} given above.

The equations of the next section can also be used to obtain the equations in orthogonal coordinates as a particular case.

Nonorthogonal Curvilinear Coordinates

In this section, we consider both the steady and nonsteady Eulerian curvilinear coordinate systems for the purpose of obtaining the components of velocity and acceleration vectors, and of the rate-of-strain tensor in these coordinates. It must, however, be understood from the outset that the steady coordinates are those which again form an inertial frame of reference. On the other hand, the nonsteady coordinates introduced in an inertial frame are generally noninertial.

A. Steady Eulerian Coordinates

In the inertial frame (x_1, x_2, x_3) we introduce a transformation of coordinates defined as:

$$x^i = x^i(x_1, x_2, x_3), \qquad i = 1, 2, 3$$

Every admissible transformation is invertible. In essence, this means that x^i must be differentiable at least to the first order. Assuming the transformation to be an admissible one, we have:

$$\mathbf{r} = \mathbf{r}(x^i)$$

In the following development, we shall use the definitions and results of covariant and contravariant base vectors, and tensor representations from ME.1. The fluid velocity vector **u** is representable either as:

$$\mathbf{u} = \mathbf{a}_i u^i$$

or as:

$$\mathbf{u} = \mathbf{a}^i u_i$$

Using:

$$\mathbf{u} = \frac{d\mathbf{r}}{dt}$$

we conclude that:

$$u^i = \frac{dx^i}{dt}, \qquad i = 1, 2, 3$$

are the contravariant components of **u**.

The grad operator in curvilinear coordinates is

$$\text{grad} = \nabla = \mathbf{a}^k \frac{\partial}{\partial x^k}$$

Consequently:

$$\mathbf{u} \cdot \text{grad} = u^i \frac{\partial}{\partial x^i}$$

and the acceleration vector **a** is

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{D}{Dt}(u^i \mathbf{a}_i)$$

$$= \frac{\partial}{\partial t}(u^i \mathbf{a}_i) + u^j \frac{\partial}{\partial x^j}(u^i \mathbf{a}_i)$$

Since $\mathbf{a}_i$ are independent of time, we have:

$$\mathbf{a} = \frac{\partial u^i}{\partial t}\mathbf{a}_i + u^j \frac{\partial u^i}{\partial x^j}\mathbf{a}_i + u^i u^j \frac{\partial \mathbf{a}_i}{\partial x^j}$$

Using Equation M1.88, we get:

$$\mathbf{a} = \left(\frac{\partial u^i}{\partial t} + u^j u^i_{,j}\right)\mathbf{a}_i \tag{3.120}$$

where $u^i_{,j}$ is the covariant derivative of $\mathbf{u}^i$. Thus:

$$a^i = \frac{\partial u^i}{\partial t} + u^j u^i_{,j}$$

are the contravariant components of **a**. In the same manner:

$$a_i = \frac{\partial u_i}{\partial t} + u^j u_{i,j} \tag{3.121}$$

are the covariant components of **a**.

Consider now the subject of strain and rotation with reference to a general curvilinear system of coordinates. First of all:

$$\text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^k} \mathbf{a}^k, \qquad (\text{grad } \mathbf{u})^T = \mathbf{a}^k \frac{\partial \mathbf{u}}{\partial x^k}$$

Using Equation M1.88, we get:

$$\mathbf{D} = \tfrac{1}{2}[u^k_{,i}\mathbf{a}_k\mathbf{a}^i + u^n_{,m}\mathbf{a}^m\mathbf{a}_n] \tag{3.122a}$$

$$\mathbf{W} = \tfrac{1}{2}[u^k_{,i}\mathbf{a}_k\mathbf{a}^i - u^n_{,m}\mathbf{a}^m\mathbf{a}_n] \tag{3.122b}$$

These two formulas are not of much use since we cannot take out the products of the base vectors as a common factor. To alleviate this problem, we use Equations M1.78a,b to express both **D** and **W** in the contravariant, covariant, or mixed components. Suppose we want the contravariant component forms of both **D** and **W**. Then, using

$$\mathbf{a}^i = g^{ik}\mathbf{a}_k$$

from Equation 3.122, we have:

$$\mathbf{D} = D^{kn}\mathbf{a}_k\mathbf{a}_n \tag{3.123a}$$

$$\mathbf{W} = W^{kn}\mathbf{a}_k\mathbf{a}_n \tag{3.123b}$$

where:

$$D^{kn} = \tfrac{1}{2}(u^k_{,i}g^{in} + u^n_{,m}g^{mk}) \tag{3.123c}$$

$$W^{kn} = \tfrac{1}{2}(u_{,i}g^{in} - u^n_{,m}g^{mk}) \tag{3.123d}$$

are the contravariant components. Following the same procedure, we can write the covariant and mixed forms as:

$$\mathbf{D} = D_{kn}\mathbf{a}^k\mathbf{a}^n, \qquad \mathbf{D} = D^k_{n}\mathbf{a}_k\mathbf{a}^n \tag{3.124a}$$

$$\mathbf{W} = W_{kn}\mathbf{a}^k\mathbf{a}^n, \qquad \mathbf{W} = W^k_{n}\mathbf{a}_k\mathbf{a}^n \tag{3.124b}$$

where the covariant components are

$$D_{kn} = \tfrac{1}{2}(u_{k,n} + u_{n,k}) \tag{3.124c}$$

$$W_{kn} = \tfrac{1}{2}(u_{k,n} - u_{n,k}) \tag{3.124d}$$

while the mixed components are

$$D^k_n = \tfrac{1}{2}(u^k_{.n} + u^r_{.p}g_{nr}g^{kp}) \tag{3.124e}$$

$$W^k_n = \tfrac{1}{2}(u^k_{.n} - u^r_{.p}g_{nr}g^{kp}) \tag{3.124f}$$

Some other interesting results pertain to representing the vorticity vector $\boldsymbol{\omega}$ defined in Equation 1.49. In general coordinates:

$$\boldsymbol{\omega} = \text{curl } \mathbf{u} = \mathbf{a}^i \times \frac{\partial \mathbf{u}}{\partial x^i} = -\frac{\partial \mathbf{u}}{\partial x^i} \times \mathbf{a}^i \tag{3.125}$$

Thus, as shown in Equation M1.111, the contravariant components of $\boldsymbol{\omega}$ are

$$\omega^i = (\text{curl } \mathbf{u})^i = \frac{1}{\sqrt{g}} e^{ijk} u_{k,j} \tag{3.126a}$$

where e^{ijk} is the superscripted permutation symbol. The covariant components of $\boldsymbol{\omega}$ are obtained from Equation 3.126a by using:

$$\omega_n = g_{ni}\omega^i \tag{3.126b}$$

B. Nonsteady Eulerian Coordinates

In a steady inertial frame denoted as (x_1, x_2, x_3) we now introduce a transformation of coordinates which is time dependent, i.e.:

$$x^i = x^i(x_1, x_2, x_3, t), \qquad i = 1, 2, 3 \tag{3.127a}$$

$$\tau = t \tag{3.127b}$$

If the transformation (Equation 3.127) is an admissible one, then its inversion is

$$x_m = x_m(x^1, x^2, x^3, \tau) \tag{3.128a}$$

$$t = \tau \tag{3.128b}$$

and in consequence of Equations 3.128 the base vectors $\mathbf{a}_i$ are now time dependent.

A simple example of nonsteady coordinates is provided by the Galilean transformation:

$$x^1 = x_1 - u_\infty t, \qquad x^2 = x_2, \qquad x^3 = x_3, \qquad \tau = t$$

where x^1, x^2, x^3 are rectangular Cartesian. This coordinate system, although time dependent, is inertial.* Another example is that of an arbitrarily rotating coordinate system, which is both time dependent and noninertial.

To establish a relation between the absolute velocity vector **u** and the velocity vector referred to the coordinates x^i, we go back to the definition (Equation 1.12). Thus, in view of Equations 3.128:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{\partial \mathbf{r}}{\partial \tau}\frac{d\tau}{dt} + \frac{\partial \mathbf{r}}{\partial x^i}\frac{dx^i}{dt} \\ &= \left(\frac{\partial \mathbf{r}}{\partial \tau}\right)_{(x^i)} + \mathbf{a}_i \frac{dx^i}{dt}\end{aligned} \tag{3.129}$$

Writing:

$$v^i = \frac{dx^i}{dt}, \qquad \mathbf{v} = \mathbf{a}_i v^i$$

Equation 3.129 becomes:

$$\mathbf{u} = \left(\frac{\partial \mathbf{r}}{\partial \tau}\right)_{(x^i)} + \mathbf{v} \tag{3.130}$$

where **v** is the velocity vector referred to the moving frame.

To find the first term on the right-hand side of Equation 3.130 we note that since **r** represents fixed points of space referred to the steady inertial frame, it is obvious that:

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_{(x_m)} = 0$$

Using the chain rule of partial differentiation, we have:

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_{(x_m)} = \left(\frac{\partial \mathbf{r}}{\partial \tau}\right)_{(x^i)} + \mathbf{a}_i \frac{\partial x^i}{\partial t}$$

Writing:

$$\mathbf{w} = \mathbf{a}_i \frac{\partial x^i}{\partial t} \tag{3.131}$$

which shows that $\partial x^i/\partial t$ are the contravariant components of **w**, we obtain:

$$\left(\frac{\partial \mathbf{r}}{\partial \tau}\right)_{(x^i)} + \mathbf{w} = 0 \tag{3.132a}$$

Consequently, Equation 3.130 becomes:

$$\mathbf{u} = \mathbf{v} - \mathbf{w} \tag{3.132b}$$

* Galilean invariants are acceleration, stress, pressure, temperature, and vorticity.

If the coordinates are steady, then $\mathbf{w} = 0$ and $\mathbf{v} = \mathbf{u}$.

Consider a function F. The partial derivatives of F with time in x_m and x^m are related as:

$$\left(\frac{\partial F}{\partial t}\right)_{(x_m)} = \left(\frac{\partial F}{\partial \tau}\right)_{(x^i)} + \frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial t}$$

Since:

$$\text{grad } F = \frac{\partial F}{\partial x^i}\mathbf{a}^i$$

we have:

$$\left(\frac{\partial F}{\partial t}\right)_{(x_m)} = \left(\frac{\partial F}{\partial \tau}\right)_{(x^i)} + (\text{grad } F)\cdot\mathbf{w}$$

Using the identity:

$$(\text{grad } F)\cdot\mathbf{w} = (\mathbf{w}\cdot\text{grad})F$$

we also have:

$$\left(\frac{\partial F}{\partial t}\right)_{(x_m)} = \left(\frac{\partial F}{\partial \tau}\right)_{(x^i)} + (\mathbf{w}\cdot\text{grad})F$$

In operator (vector invariant) form:

$$\left(\frac{\partial}{\partial t}\right)_{(x_m)} = \left(\frac{\partial}{\partial \tau}\right)_{(x^i)} + \mathbf{w}\cdot\text{grad} \qquad \textbf{(3.133)}$$

Three other results can be obtained from Equation 3.132 which are described below:

1. Differentiating Equation 3.132 partially with respect to x^i, we have the partial time derivative of the base vector $\mathbf{a}_i$ as:

$$\frac{\partial \mathbf{a}_i}{\partial \tau} + \frac{\partial \mathbf{w}}{\partial x^i} = 0 \qquad \textbf{(3.134)}$$

2. From Equation 3.132,

$$\left(\frac{\partial \mathbf{r}}{\partial \tau}\right)_{(x^i)} + \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial t} = 0$$

Thus, the partial derivatives of the Cartesian coordinates x_m with time are given by:

$$\frac{\partial x_m}{\partial \tau} = -\frac{\partial x_m}{\partial x^i}\frac{\partial x^i}{\partial t} \qquad \textbf{(3.135a)}$$

$$= -\frac{\partial x_m}{\partial x^i}w^i \qquad \textbf{(3.135b)}$$

3. According to Equation M1.107:

$$\text{div } \mathbf{w} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} \frac{\partial x^i}{\partial t} \right)$$

Again, from Equation 3.132:

$$\frac{\partial x^i}{\partial t} = -g^{ij} \left(\frac{\partial \mathbf{r}}{\partial \tau} \right) \cdot \mathbf{a}_j$$

Thus:

$$\text{div } \mathbf{w} = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left[(\sqrt{g}\, g^{ij}) \left(\frac{\partial \mathbf{r}}{\partial \tau} \cdot \mathbf{a}_j \right) \right]$$

Performing the differentiation on the groups of terms shown in parentheses and using Equations M1.88, M1.110a, M1.124, M1.126, and M1.127, we get the geometric conservation law (GCL):

$$\text{div } \mathbf{w} = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial t} (\sqrt{g}) \tag{3.136}$$

where $g = \det (g_{ij})$.

We now write the acceleration vector in general nonsteady coordinates. Starting from the basic equation:

$$\begin{aligned} \mathbf{a} &= \frac{D\mathbf{u}}{Dt} \\ &= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} \end{aligned}$$

we first replace the local time derivative by using Equation 3.133. Thus,

$$\begin{aligned} \mathbf{a} &= \frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{w} \cdot \text{grad})\mathbf{u} + (\mathbf{u} \cdot \text{grad})\mathbf{u} \\ &= \frac{\partial \mathbf{u}}{\partial \tau} + (\text{grad } \mathbf{u}) \cdot \mathbf{w} + (\text{grad } \mathbf{u}) \cdot \mathbf{u} \end{aligned}$$

Therefore:

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial \tau} + (\text{grad } \mathbf{u}) \cdot \mathbf{v} \tag{3.137}$$

Also, purely in terms of v and w:

$$\begin{aligned} \mathbf{a} &= \frac{\partial \mathbf{v}}{\partial \tau} + (\text{grad } \mathbf{v}) \cdot \mathbf{v} - \frac{\partial \mathbf{w}}{\partial \tau} - (\text{grad } \mathbf{w}) \cdot \mathbf{v} \\ &= \frac{\partial \mathbf{v}}{\partial \tau} + (\mathbf{v} \cdot \text{grad})\mathbf{v} - \frac{\partial \mathbf{w}}{\partial \tau} - (\mathbf{v} \cdot \text{grad})\mathbf{w} \end{aligned} \tag{3.138}$$

Substituting $\mathbf{v} = v^i\mathbf{a}_i$, $\mathbf{w} = w^i\mathbf{a}_i$ using Equation M1.88 for the partial derivative of a base vector, and the definition of the covariant derivative (cf. ME.1) while manipulating the dummy summation indices wherever needed, we finally have:

$$\mathbf{a} = \left(\frac{\partial v^i}{\partial \tau} + v^j v^i_{,j} - \frac{\partial w^i}{\partial \tau} - 2v^j w^i_{,j} + w^j w^i_{,j}\right)\mathbf{a}_i \tag{3.139}$$

Because of the appearance of $\mathbf{a}_i$ without its partial time derivative in Equation 3.139, it is tempting to introduce a substantive operator $\hat{D}/D\tau$ which selectively ignores the time variation of $\mathbf{a}_i$. Thus:

$$\mathbf{a} = \frac{\hat{D}v^i}{D\tau}\mathbf{a}_i - \left(\frac{\hat{\partial} w^i}{\partial \tau} + 2v^j w^i_{,j} - w^j w^i_{,j}\right)\mathbf{a}_i \tag{3.140a}$$

where:

$$\frac{\hat{D}}{D\tau} = \frac{\partial(\ )^i}{\partial \tau} + v^j(\ )^i_{,j}$$

and $\hat{D}/D\tau$ is the time derivative as seen by an observer fixed in the noninertial frame.

In vector form (3.140a) is written as

$$a = \frac{\hat{D}\boldsymbol{v}}{D\tau} - \frac{\hat{\partial} w}{\partial \tau} - 2(grad\, w)\cdot \boldsymbol{v} + (grad\, w)\cdot w \tag{3.140b}$$

where

$$\frac{\hat{D}}{D\tau} = \frac{\hat{\partial}}{\partial \tau} + \boldsymbol{v}\cdot grad$$

As an example, if the Cartesian frame rotates with an angular velocity Ω with respect to a rest frame then

$$w = -r_\tau = -\boldsymbol{\Omega}\times r$$

$$\frac{\hat{\partial} w}{\partial \tau} = -\dot{\boldsymbol{\Omega}}\times r$$

$$(grad\, w)\cdot \boldsymbol{v} = -\boldsymbol{\Omega}\times \boldsymbol{v}$$

$$(grad\, w)\cdot w = -\boldsymbol{\Omega}\times w$$

and (3.140b) yields

$$a = \frac{\hat{D}\boldsymbol{v}}{D\tau} + \dot{\boldsymbol{\Omega}}\times r + 2\boldsymbol{\Omega}\times \boldsymbol{v} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times r) \tag{3.140c}$$

Equations in General Coordinates

The transformed equations can now be obtained in both the steady and nonsteady general coordinates. The basic mathematical formulae to be used have already been derived in the previous subsection and in ME.1 and 4.

The velocity vector respresentation in contravariant components is

$$\mathbf{u} = u^i\mathbf{a}_i = u^1\mathbf{a}_1 + u^2\mathbf{a}_2 + u^3\mathbf{a}_3$$

To distinguish the physical components u_i used in orthogonal coordinates from the covariant components, here we have used w_i instead of u_i. Thus:

$$\mathbf{u} = w_i\mathbf{a}^i = w_1\mathbf{a}^1 + w_2\mathbf{a}^2 + w_3\mathbf{a}^3$$
$$= g^{ij}w_i\mathbf{a}_j$$

Since the pressure p is a scalar, its gradient is given by:

$$\text{grad } p = \frac{\partial p}{\partial x^i}\mathbf{a}^i = p_{,i}\mathbf{a}^i \tag{3.141a}$$

$$= g^{ik}\frac{\partial p}{\partial x^i}\mathbf{a}_k = g^{ik}p_{,i}\mathbf{a}_k \tag{3.141b}$$

Thus, the covariant components of grad p denoted as (grad p)$_i$ are given by:

$$(\text{grad } p)_i = \frac{\partial p}{\partial x^i} = p_{,i} \tag{3.142a}$$

The contravariant components of grad p denoted as (grad p)^i are given by:

$$(\text{grad } p)^i = g^{ik}\frac{\partial p}{\partial x^k} = g^{ik}p_{,k} \tag{3.142b}$$

Note that a comma preceding a Latin index implies a covariant derivative (refer to Equation M1.97). In the case of a scalar the covariant derivative is the same as the partial derivative.

If $\mathbf{u}$ is a vector, then $\rho\mathbf{u}$ is also a vector, where ρ is a scalar. Thus from Equation M1.107:

$$\text{div } \mathbf{u} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}u^i) \tag{3.143a}$$

where:

$$g = \det(g_{ij})$$

and:

$$\text{div}(\rho\mathbf{u}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\rho\sqrt{g}u^i) \tag{3.143b}$$

The stress tensor $\boldsymbol{\sigma}$ is a second order tensor. Its divergence is a vector whose components can be either covariant or contravariant. From Equation M1.113 the contravariant components are

$$(\text{div } \boldsymbol{\sigma})^i = \sigma^{ik}_{\ \ ,k}$$

$$= \frac{\partial \sigma^{ik}}{\partial x^k} + \Gamma^i_{tk}\sigma^{tk} + \Gamma^k_{tk}\sigma^{it}$$

However, from Equation M1.91:

$$\Gamma^k_{tk} = \frac{1}{2g}\frac{\partial g}{\partial x^t}$$

hence:

$$(\text{div } \boldsymbol{\sigma})^i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\,\sigma^{ik}) + \Gamma^i_{tk}\sigma^{tk} \tag{3.144}$$

The definition of Christoffel symbols of the second kind has been given in Equation M1.89, and the expanded forms of these symbols for the three-dimensional coordinates have been given in ME.1, Section 13. From Equation M1.114 the covariant components are

$$(\text{div } \boldsymbol{\sigma})_i = g^{jk}\sigma_{ij,k}$$

$$= g^{jk}\left(\frac{\partial \sigma_{ij}}{\partial x^k} - \Gamma^t_{ik}\sigma_{tj} - \Gamma^t_{jk}\sigma_{it}\right) \tag{3.145}$$

Transformation of the Navier-Stokes equations along with the equations of continuity and energy becomes a straightforward procedure if one takes into account the *invariant* vector and tensor *forms* of the equations developed in this chapter. To understand the procedure, we consider the set of Equations 3.8, 3.10, and 3.22. We assume that ρ, λ, and μ are *absolute* scalars. For brevity, we shall also use:

$$\boldsymbol{\sigma} = (\lambda \text{ div } \mathbf{u})\mathbf{I} + 2\mu\mathbf{D}$$

Equations in General Coordinates Using Contravariant Components

We first consider the transformation of equations using the contravariant components for both vectors and tensors.

(i) Continuity (Equation 3.8) Using Equation 3.143b in Equation 3.8, we have:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\rho\sqrt{g}u^i) = 0 \tag{3.146}$$

(ii) Momentum (Equation 3.10) Using Equations 3.142b and 3.144, we get:

$$\frac{\partial}{\partial t}(\rho u^i) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\rho\sqrt{g}u^iu^k) + \Gamma^i_{tk}\rho u^t u^k$$

$$= -g^{ik}\frac{\partial p}{\partial x^k} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\sigma^{ik}) + \Gamma^i_{tk}\sigma^{tk} + \rho f^i \tag{3.147}$$

(iii) Energy (Equations 3.22a)

$$\frac{\partial}{\partial t}(\rho e_t) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}b^i) = \rho(\mathbf{f}\cdot\mathbf{u}) \tag{3.148}$$

where b^i are the contravariant components of $\mathbf{b}$ given as:

$$b^i = (\rho e_t + p)u^i - g_{jr}\sigma^{ij}u^r - kg^{ij}T_{,j}$$

Equations in General Coordinates Using Covariant Components

To obtain the equations using covariant components, we replace the following quantities in Equations 3.146 through 3.148:

$$u^i = g^{ik}w_k, \qquad \sigma^{ik} = g^{ir}g^{kt}\sigma_{rt}, \qquad b^i = g^{im}b_m$$

thus having:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\rho\sqrt{g}g^{ik}w_k\right) = 0 \tag{3.149}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho w_i) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\rho\sqrt{g}g^{kt}w_iw_t) - \rho\Gamma^r_{ik}g^{kt}w_rw_t \\ = -\frac{\partial p}{\partial x^i} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}g^{kt}\sigma_{it}) - g^{kt}\Gamma^r_{ik}\sigma_{rt} + \rho f_i\end{aligned} \tag{3.150}$$

$$\frac{\partial}{\partial t}(\rho e_t) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{im}b_m) = \rho(\mathbf{f}\cdot\mathbf{u}) \tag{3.151}$$

where b_m are the covariant components of $\mathbf{b}$.

Remark: Equation 3.150 is in purely covariant component form. One way to obtain it is to perform an inner multiplication of every term of Equation 3.147 by the covariant metric tensor and carry out the necessary algebraic manipulations.

EQUATIONS IN GENERAL COORDINATES WITH VECTORS AND TENSOR DENSITIES

It was assumed earlier that ρ, μ, λ are absolute scalars. The quantities $\sqrt{g}\rho$, $\sqrt{g}\mu$, $\sqrt{g}\lambda$, are the scalar densities. We now introduce the following symbols:

$$\begin{aligned}\rho_1 &= \sqrt{g}\rho, \text{ scalar density} \\ V^i &= \sqrt{g}b^i, \text{ vector density} \\ H^{ij} &= \sqrt{g}\sigma^{ij}, \text{ tensor density} \\ M^{ij} &= \sqrt{g}\rho u^iu^j, \text{ tensor density}\end{aligned} \tag{3.152}$$

Then all the equations can be written if we assume that g is not a function of time. The steady character of g implies that the coordinates x^1, x^2, x^3 are steady in time. In this case we can multiply the equations throughout by $\sqrt{g}$ and absorb it under the partial derivative of t. The Equations 3.146–3.148 take the following forms:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x^i}(\rho_1 u^i) = 0 \tag{3.153}$$

$$\frac{\partial}{\partial t}(\rho_1 u^i) + \frac{\partial M^{ik}}{\partial x^k} + \Gamma^i_{tk} M^{tk} = -\sqrt{g}\, g^{ik}\frac{\partial p}{\partial x^k} + \frac{\partial H^{ik}}{\partial x^k} + \Gamma^i_{tk} H^{tk} + \rho_1 f^i \tag{3.154}$$

$$\frac{\partial}{\partial t}(\rho_1 e_t) + \frac{\partial V^i}{\partial x^i} = \rho_1(\mathbf{f}\cdot\mathbf{u}) \tag{3.155}$$

As a particular case, the equations in orthogonal coordinates can also be obtained from the equations written in general coordinates either from Equations 3.146–3.148, 3.149–3.151, or 3.153–3.155. In this connection it is important to take note that the covariant components w_i or the contravariant components u^i of $\mathbf{u}$ do not have the physical dimension of $\mathbf{u}$ itself. In fact, the dimensions of w_i and u^i are controlled by the dimensions of the base vectors $\mathbf{a}^i$ and $\mathbf{a}_i$, respectively. It is, however, possible to introduce a set of components, called the *physical components* of a vector, with dimensions the same as that of the vector. The introduction of physical components is particularly useful when the coordinates are orthogonal. (Refer to ME.1, Section 14.)

For orthogonal coordinates the base vectors $\mathbf{a}_i$ are also orthogonal, so that:

$$\mathbf{a}_i \cdot \mathbf{a}_j = 0, \qquad i \neq j$$

Thus:

$$g_{ij} = 0, \qquad \text{for} \quad i \neq j$$

and

$$g = g_{11} g_{22} g_{33} \tag{3.156a}$$

Also, the following Christoffel symbols are zero:

$$[12, 3] = [13, 2] = [23, 1] = 0 \tag{3.156b}$$

$$\Gamma^3_{12} = \Gamma^2_{13} = \Gamma^1_{23} = 0 \tag{3.156c}$$

It is customary to write (refer to ME.1 for details):

$$g_{11} = (h_1)^2 = (g^{11})^{-1}, \quad g_{22} = (h_2)^2 = (g^{22})^{-1}, \quad g_{33} = (h_3)^2 = (g^{33})^{-1} \tag{3.157a}$$

$$g = (h_1 h_2 h_3)^2 \tag{3.157b}$$

Using the equation:

$$w_i = g_{ik} u^k$$

we readily find that the covariant and the contravariant components in orthogonal coordinates are related as:

$$w_1 = (h_1)^2u^1, \qquad w_2 = (h_2)^2u^2, \qquad w_3 = (h_3)^2u^3 \tag{3.158}$$

Denoting the physical components of $\mathbf{u}$ by u_i (here the subscript i is meant only as a label without any tensorial significance), we get:

$$\begin{aligned} u_1 &= h_1u^1 = w_1/h_1 \\ u_2 &= h_2u^2 = w_2/h_2 \\ u_3 &= h_3u^3 = w_3/h_3 \end{aligned} \tag{3.159}$$

The magnitude of the vector $\mathbf{u}$ is then:

$$\begin{aligned} |\mathbf{u}|^2 &= (u_1)^2 + (u_2)^2 + (u_3)^2 \\ &= w_1u^1 + w_2u^2 + w_3u^3 \end{aligned} \tag{3.160}$$

Using Equations 3.156–3.160, the equations in orthogonal coordinates can be readily obtained from the set of equations as referred to above.

Equations in Nonsteady Eulerian Coordinates

The previous subsection dealt with the steady Eulerian but general coordinates. We now consider the situation when the transformation equations have an explicit dependence on time, so that they are of the form:

$$x^i = x^i(x_1, x_2, x_3, t), \qquad i = 1, 2, 3 \quad \text{and} \quad \tau = t$$

where x_1, x_2, x_3 are the rectangular Cartesian coordinates. In this regard, we recall Equation 3.132b in which $\mathbf{u}$ is the absolute velocity, $\mathbf{v}$ is the velocity with reference to the moving coordinates, and $\mathbf{w}$ is the absolute velocity of the moving coordinates. The contravariant components of $\mathbf{w}$ are $\partial x^i/\partial t$.

To obtain the equations of motion and energy in moving coordinates, it is much simpler to consider the unified conservation law, Equation 2.55. In Equation 2.55, we first replace the partial time derivative by the expression given in Equation 3.133, thus having:

$$\frac{\partial A}{\partial \tau} + (\mathbf{w} \cdot \text{grad})A + \text{div} f = C$$

or:

$$\frac{\partial A}{\partial \tau} + (\text{grad } A) \cdot \mathbf{w} + \text{div} f = C \tag{3.161}$$

where:

$$f = A\mathbf{u} + B$$

Using the identity:

$$\text{div}(A\mathbf{w}) = (\text{grad } A) \cdot \mathbf{w} + A(\text{div } \mathbf{w})$$

we get:

$$\frac{\partial A}{\partial \tau} - A \operatorname{div} \mathbf{w} + \operatorname{div}(A\mathbf{v} + B) = C$$

Using Equation 3.136, we get:*

$$\frac{\partial}{\partial \tau}(A\sqrt{g}) + \sqrt{g} \operatorname{div}(A\mathbf{v} + B) = C\sqrt{g} \tag{3.162}$$

Thus, setting:

$$A = \rho, \qquad B = 0, \qquad C = 0$$

in Equation 3.162 and using Equation 3.143b, the equation of continuity in nonsteady Eulerian coordinates becomes:

$$\frac{\partial}{\partial \tau}(\rho\sqrt{g}) + \frac{\partial}{\partial x^i}(\rho\sqrt{g}\,v^i) = 0 \tag{3.163}$$

Similarly setting:

$$A = \rho\mathbf{u}, \qquad B = -\mathbf{T} = p\mathbf{I} - \boldsymbol{\sigma}, \qquad C = \rho\mathbf{f}$$

in Equation 3.162 and using Equation M1.125 for the divergence of a tensor in conservative form, the momentum equation in nonsteady Eulerian coordinates becomes:

$$\frac{\partial}{\partial \tau}(\rho\sqrt{g}\,\mathbf{u}) + \frac{\partial}{\partial x^j}[(\rho u^i v^j - T^{ij})\sqrt{g}\,\mathbf{a}_i] = \rho\sqrt{g}\,\mathbf{f} \tag{3.164}$$

In the same manner, setting:

$$A = \rho e_t, \qquad B = -\mathbf{T}\cdot\mathbf{u} + \mathbf{q}, \qquad C = \rho(\mathbf{f}\cdot\mathbf{u})$$

and again using Equation 3.143b, the energy equation in nonsteady Eulerian coordinates becomes:

$$\frac{\partial}{\partial \tau}(\rho\sqrt{g}\,e_t) + \frac{\partial}{\partial x^j}(\sqrt{g}\,b^j) = \rho\sqrt{g}(\mathbf{f}\cdot\mathbf{u}) \tag{3.165}$$

where:

$$b^j = \rho e_t v^j - g_{ir}T^{ij}u^r + q^j$$

In both Equations 3.164 and 3.165:

$$T^{ij} = -pg^{ij} + \sigma^{ij}, \qquad q^i = -kg^{ij}\frac{\partial T}{\partial x^j}$$

* Refer also to Example 3.2 and to Problem 3.15.

Equations 3.163–3.165 are in the conservation law form and therefore can be written as a single vector equation in the manner of Equation 3.97. Thus writing:

$$\frac{\partial \mathbf{E}}{\partial \tau} + \frac{\partial \mathbf{F}}{\partial x^1} + \frac{\partial \mathbf{G}}{\partial x^2} + \frac{\partial \mathbf{H}}{\partial x^3} = \mathbf{S} \qquad (3.166\text{a})$$

where:

$$\mathbf{E} = \begin{bmatrix} \rho\sqrt{g} \\ \rho\sqrt{g}u \\ \rho\sqrt{g}v \\ \rho\sqrt{g}w \\ \rho\sqrt{g}e_t \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho\sqrt{g}v^1 \\ \sqrt{g}\lambda^{i1}\dfrac{\partial x}{\partial x^i} \\ \sqrt{g}\lambda^{i1}\dfrac{\partial y}{\partial x^i} \\ \sqrt{g}\lambda^{i1}\dfrac{\partial z}{\partial x^i} \\ \sqrt{g}b^1 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \rho\sqrt{g}v^2 \\ \sqrt{g}\lambda^{i2}\dfrac{\partial x}{\partial x^i} \\ \sqrt{g}\lambda^{i2}\dfrac{\partial y}{\partial x^i} \\ \sqrt{g}\lambda^{i2}\dfrac{\partial z}{\partial x^i} \\ \sqrt{g}b^2 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} \rho\sqrt{g}v^3 \\ \sqrt{g}\lambda^{i3}\dfrac{\partial x}{\partial x^i} \\ \sqrt{g}\lambda^{i3}\dfrac{\partial y}{\partial x^i} \\ \sqrt{g}\lambda^{i3}\dfrac{\partial z}{\partial x^i} \\ \sqrt{g}b^3 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ \rho\sqrt{g}f_x \\ \rho\sqrt{g}f_y \\ \rho\sqrt{g}f_z \\ \rho\sqrt{g}(uf_x + vf_y + wf_z) \end{bmatrix} \qquad (3.166\text{b})$$

where x, y, z are the Cartesian coordinates; u, v, w are the Cartesian components of $\mathbf{u}$; and:

$$\lambda^{ij} = \rho u^i v^j - T^{ij}$$

In Equation 3.166b, the repeated index i implies summation from 1 to 3.

As has been noted earlier, Equation 3.166 is equally applicable in steady Eulerian coordinates. In this case $v^i = u^i$ and g is independent of time. The so-called strong conservation law form of the Navier-Stokes equations (viz., Equation 3.166a) has been obtained in one situation or another by Anderson et al.,[3] Walkden[4], Viviand,[5] Vinokur,[6] Thomas and Lombard,[7] and Warsi.[8]

It must be realized that in the differential conservation equations (i.e., either in Equation 3.97 or 3.166a) the vectors **E**, **F**, **G**, and **H** have usually more elements than the dimension of the space and time together. For example, in a one-dimensional flow, the vectors **E** and **F** both have three elements but the physical space is two dimensional, i.e., the xt-space. To derive the integral conservation law based on the form of the differential conservation law (Equation 3.97), we consider two linear spaces S_1 and S_2 of dimensions m and n, respectively. Let the unit vectors in S_1 and S_2 be denoted as $\mathbf{j}_1, \ldots, \mathbf{j}_m$ and $\mathbf{i}_1, \ldots, \mathbf{i}_n$, respectively. The Cartesian product space $S_1 \otimes S_2$ is then spanned by the ordered dyadic pairs $\mathbf{j}_p\mathbf{i}_q$. Thus, an element T of $S_1 \otimes S_2$ is

$$\mathbf{T} = T_{pq}\,\mathbf{j}_p\mathbf{i}_q = \mathbf{F}_q\mathbf{i}_q$$

The divergence of **T** in the S_2-space is

$$\text{div}\ \mathbf{T} = \frac{\partial \mathbf{F}_q}{\partial x_q}$$

and the Gauss theorem in the S_2-space is

$$\int_V \frac{\partial \mathbf{F}}{\partial x_q}\,dv = \int_S \mathbf{F}_q n_q\,dS$$

From this equation follows the integral conservation law.

It is a straightforward matter to write Equation 3.166a in the nonconservation form as:

$$\frac{\partial \mathbf{E}}{\partial t} + \mathbf{A}\cdot\frac{\partial \mathbf{E}}{\partial \xi} + \mathbf{B}\cdot\frac{\partial \mathbf{E}}{\partial \eta} + \mathbf{C}\cdot\frac{\partial \mathbf{E}}{\partial \zeta} = \mathbf{S} \tag{3.166c}$$

where:

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial E_m}\,\mathbf{j}_m, \qquad \mathbf{B} = \frac{\partial \mathbf{G}}{\partial E_m}\,\mathbf{j}_m, \qquad \mathbf{C} = \frac{\partial \mathbf{H}}{\partial E_m}\,\mathbf{j}_m$$

and $E_1, \ldots, E_5$ are the elements of the vector **E**. This formulation is useful for inviscid flows. Refer to Problem 3.16.

Example 3.1

In two dimensions write the equation of momentum for an inviscid fluid flow in nonsteady Eulerian coordinates.

For simplicity writing $x^1 = \xi$, $x^2 = \eta$, and x, y as Cartesian coordinates, we first list the derivative relations:

$$x_\xi = \sqrt{g}\,\eta_y, \qquad y_\xi = -\sqrt{g}\,\eta_x, \qquad x_\eta = -\sqrt{g}\,\xi_y, \qquad y_\eta = \sqrt{g}\,\xi_x$$

where a variable subscript implies a partial derivative. If the Cartesian components of **u** are denoted as u and v, then the relations between the contravariant components and the Cartesian components of **u** are

$$u^1 = (uy_\eta - vx_\eta)/\sqrt{g}, \qquad u^2 = (vx_\xi - uy_\xi)/\sqrt{g}$$

In Equation 3.164, performing sum on j, ($j = 1, 2$), with $\mathbf{f} = 0$, and:

$$\mathbf{a}^1 = \text{grad}\ \xi, \qquad \mathbf{a}^2 = \text{grad}\ \eta$$

where:

$$\text{grad} = \mathbf{i}\partial_x + \mathbf{j}\partial_y$$

$$\mathbf{v} = \mathbf{u} + \mathbf{w}$$

with $w^1 = \xi_t$, $w^2 = \eta_t$, the equation becomes:

$$\frac{\partial}{\partial t}(\rho\sqrt{g}\mathbf{u}) + \frac{\partial}{\partial \xi}[\sqrt{g}\{\rho\mathbf{u}\xi_t + (\rho\mathbf{u}u)\xi_x + (\rho\mathbf{u}v)\xi_y + p \text{ grad } \xi\}]$$

$$+ \frac{\partial}{\partial \eta}[\sqrt{g}\{\rho\mathbf{u}\eta_t + (\rho\mathbf{u}u)\eta_x + (\rho\mathbf{u}v)\eta_y + p \text{ grad } \eta\}] = 0 \qquad \text{(i)}$$

Example 3.2

Derive the differential conservation law in nonsteady coordinates from the integral conservation law for a finite-volume, i.e., Equation 2.56.

Equation 3.166a can also be obtained from the finite-volume integral conservation law (Equation 2.56). Considering the finite-volume element as $dV = \sqrt{g}dx^1dx^2dx^3$ with reference to a time-dependent coordinate system $\tau = t$, $x^i = x^i(x, y, z, t)$ and using the vector area element $\mathbf{n}^*dS^*$ for faces on which x^1, x^2, and x^3 are fixed as $\sqrt{g}\mathbf{a}^1dx^2dx^3$, $\sqrt{g}\mathbf{a}^2dx^1dx^3$, and $\sqrt{g}\mathbf{a}^3dx^1dx^2$, respectively, we get:

$$\frac{\partial}{\partial \tau}(A\sqrt{g}) + \frac{\partial}{\partial x^i}(F \cdot \sqrt{g}\mathbf{a}^i) = \sqrt{g}C \qquad \text{(i)}$$

where:

$$F = A(\mathbf{u} - \mathbf{c}) + B$$

In the present case $\mathbf{c} = \mathbf{r}_\tau = -\mathbf{w}$ is the grid speed.

Equations in Curvilinear Coordinates with Cartesian Velocity Components

The development of the preceding procedure provides a general format for affecting the transformation of the equations detailed in Section 3.2. From a numerical standpoint, it is usually desirable to use both the contravariant and the Cartesian components of velocity in the equations. Obviously:

$$\mathbf{u} = \mathbf{a}_i u^i = \mathbf{i}_m u_m$$

where u_m are the Cartesian Components, i.e., $u_1 = u$, $u_2 = v$, $u_3 = w$, and:

$$u^i = \frac{\partial x^i}{\partial x_m} u_m \qquad \text{(3.167a)}$$

It is also a straightforward matter to check that the contravariant rate-of-strain components in terms of the components u_m are given by:

$$2D^{ik} = A_m^{ikr}\frac{\partial u_m}{\partial x^r} \qquad \text{(3.167b)}$$

where

$$A_m^{ikr} = \frac{\partial x^i}{\partial x_m} g^{rk} + \frac{\partial x^k}{\partial x_m} g^{ri}$$

and $x_1 = x$, $x_2 = y$, $x_3 = z$ are the Cartesian coordinates. Replacement of the contravariant components in terms of the Cartesian components by Equation 3.167a wherever needed, and of the contravariant components D^{ik} through Equation 3.167b in the Stokes tensor, expresses the equations in the desired form.

3.11 STREAMLINES AND STREAM SURFACES

From Chapter 1, we have the definition that a streamline is a curve in the flow field on which the tangents at a *given time* are in the directions of the local velocity vectors. Thus, the streamlines satisfy Equation 1.23a, and there cannot be a flow across a streamline. In three dimensions a contiguous collection of streamlines forms a stream surface.

In the case of steady or nonsteady incompressible flows, or steady compressible flows it is possible to identically satisfy the equation of continuity by introducing a function in two dimensions or the functions S and N in three dimensions.

Two-Dimensional Stream Function

For two-dimensional steady or nonsteady incompressible flow, the equation of continuity is

$$\operatorname{div} \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where we have set $u_1 = u$, $u_2 = v$, $x_1 = x$, and $x_2 = y$. Defining ψ through the equations:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{3.168}$$

the continuity equation is identically satisfied. In general curvilinear coordinates x^1, x^2 with the respective contravariant components u^1, u^2, the continuity equation is

$$\operatorname{div} \mathbf{u} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, u^i) = 0$$

which is satisfied by the equations:

$$\sqrt{g}\, u^1 = \frac{\partial \psi}{\partial \eta}, \qquad \sqrt{g}\, u^2 = -\frac{\partial \psi}{\partial \xi} \tag{3.169}$$

where $x^1 = \xi$, $x^2 = \eta$. In orthogonal coordinates, using Equation 3.159 we have:

$$u_1 = \frac{1}{h_2} \frac{\partial \psi}{\partial \eta}, \qquad u_2 = -\frac{1}{h_1} \frac{\partial \psi}{\partial \xi} \tag{3.170}$$

For steady compressible flow the equation of continuity is

$$\operatorname{div}(\rho \mathbf{u}) = \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0$$

in Cartesian coordinates, and:

$$\operatorname{div}(\rho \mathbf{u}) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\rho \sqrt{g}\, u^i) = 0$$

in general curvilinear coordinates. Thus:

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad \rho v = -\frac{\partial \psi}{\partial x}$$

$$\rho\sqrt{g}u^1 = \frac{\partial\psi}{\partial\eta}, \qquad \rho\sqrt{g}u^2 = -\frac{\partial\psi}{\partial\xi}$$

$$u_1 = \frac{1}{\rho h_2}\frac{\partial\psi}{\partial\eta}, \qquad u^2 = -\frac{1}{\rho h_1}\frac{\partial\psi}{\partial\xi}$$

Consider an arbitrary curve in a two-dimensional flow field. Let **t** be the unit tangent vector and **n** the unit normal vector on the curve as shown in Figure 3.2.

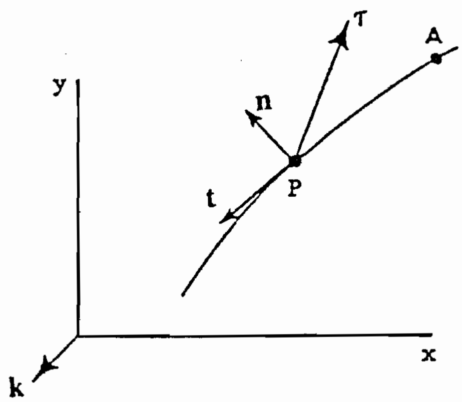

Fig. 3.2 An arbitrary curve in a two-dimensional flow field.

Now:

$$\mathbf{t} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds}$$

where s is the arc length on the curve measured from an arbitrary reference point, and according to the orientation of **t** and **n** we have:

$$\mathbf{t}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\times\mathbf{t} = \mathbf{k}$$

so that:

$$\mathbf{n} = \mathbf{i}\frac{dy}{ds} - \mathbf{j}\frac{dx}{ds}$$

These formulae enable us to form the integrals:

$$\Gamma = \int_A^P \rho\mathbf{u}\cdot\mathbf{t}\,ds = \int_A^P (\rho u\,dx + \rho v\,dy) \tag{3.171a}$$

and:

$$Q = \int_A^P \rho\mathbf{u}\cdot\mathbf{n}\,ds = \int_A^P (\rho u\,dy - \rho v\,dx) \tag{3.171b}$$

The Γ and Q represent the mass flow rates along and normal to the curve, respectively, from point A to point P. Substituting Equation 3.168 in Equation 3.171b, we have:

$$Q = \rho\int_A^P d\psi = \rho[\psi(p) - \psi(A)] \tag{3.172a}$$

Similarly, for the compressible case:

$$Q = \int_A^P d\psi = \psi(p) - \psi(A) \tag{3.172b}$$

Thus in either case, the mass flow rate across the curve between A and P is the difference of the function values of ψ between the points. (Note the difference in the dimensions of ψ for incompressible and compressible cases.)

If the curve is a streamline then $Q = 0$, and:

$$\psi(P) = \psi(A)$$

Since A and P are arbitrary points, we conclude that on a streamline $\psi(x, y) =$ constant.

Three-Dimensional Stream Functions

In the case of three-dimensional incompressible flow the equation of continuity is

$$\text{div } \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

which is applicable whether the flow is steady or nonsteady. In the case of three-dimensional steady compressible flow the continuity equation is

$$\text{div}(\rho\mathbf{u}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Let $\mathbf{A}$ be an arbitrary differentiable vector; then obviously:

$$\text{div}(\text{curl } \mathbf{A}) = 0$$

Taking curl $\mathbf{A} = \mathbf{u}$ or $\rho\mathbf{u}$ we see that the continuity equations mentioned above are identically satisfied. The form of the vector potential $\mathbf{A}$ can be decided if we note that in two-dimensional flow, the vector $\mathbf{A}$ should depend on a single scalar function.

Thus, we take:

$$\mathbf{A} = S \text{ grad } N \tag{3.173}$$

where S and N are scalar functions of the coordinates. With this choice:

$$\mathbf{u} = \text{curl}(S \text{ grad } N) \tag{3.174a}$$

or:

$$\rho\mathbf{u} = \text{curl}(S \text{ grad } N) \tag{3.174b}$$

for incompressible or compressible flow, respectively. Again note the difference in the dimensions of S and N for incompressible and compressible cases. Opening the curl in Equations 3.174 we have:

$$\mathbf{u} = (\text{grad } S) \times (\text{grad } N) \tag{3.175a}$$

for incompressible flow, and:

$$\rho \mathbf{u} = (\text{grad } S) \times (\text{grad } N) \tag{3.175b}$$

for compressible flow. Under the constraint of two dimensionality:

$$S = \psi, \qquad N = z, \qquad \mathbf{A} = \mathbf{k}\psi$$

Equations 3.175 can now be written with reference to any coordinate system. For example, in Cartesian coordinates Equation 3.175a is represented as:

$$u = S_y N_z - S_z N_y$$

$$v = S_z N_x - S_x N_z$$

$$w = S_x N_y - S_y N_x$$

where a subscript denotes a partial derivative.

The surface S = constant and N = constant are the stream surfaces. These surfaces are not unique although the intersection of the surfaces produces a unique streamline. The reason for nonuniqueness of S and N is that through a given streamline an infinity of surfaces can be passed containing contiguous streamlines. In passing we note that there are surfaces θ = constant and ϕ = constant defined by the equations:

$$u = \theta_z, \qquad v = \phi_z, \qquad w = -(\theta_x + \phi_y)$$

which also satisfy the equation div $\mathbf{u} = 0$ identically. These functions in two dimensions reduce to:

$$\theta = z\psi_y, \qquad \phi = -z\psi_x, \qquad \psi = \psi(x, y)$$

The preceding exposition was directed to establish the fact that three-dimensional flows need two stream functions to satisfy the equation of continuity. Later it will be shown that:

$$\mathbf{u} = \text{grad } \phi + \mathit{curl}\ \mathbf{A} \tag{3.176}$$

is more appropriate for the formulation of the Navier-Stokes problem if we impose the condition that $\nabla^2\phi = 0$.

3.12 NAVIER-STOKES' EQUATIONS IN STREAM FUNCTION FORM

Some numerical methods of solution of the incompressible and compressible Navier-Stokes equations use the stream function form. The simplicity of this formulation lies in the fact that the continuity equation is automatically satisfied. As follows we have first considered a general formulation covering both the two-dimensional and axially symmetric fluid motions. The next subsection deals with the use of scalar and vector potentials in the three-dimensional formulation. Refer also to the exercises at the end of this chapter.

Two-Dimensional and Axially Symmetric Flows

Let (ξ_1, ξ_2, ξ_3), *in this order*, be an orthogonal curvilinear coordinate system. It must be noted that in the matter of enumeration of coordinates we always follow the right-hand rule. Thus, in this sense the coordinates (ξ_2, ξ_3, ξ_1) and (ξ_3, ξ_1, ξ_2), in these orders, are also right-handed.

Let u_1, u_2, u_3 be the physical components of **u** along ξ_1, ξ_2, ξ_3, respectively. If the quantities are independent of one of the coordinates, then the motion is either two dimensional or axially symmetric. If, for example, the quantities are independent of ξ_3, then (ξ_1, ξ_2) in this order form a right-handed system.

For the purpose of demonstration we consider the case in which the quantities are independent of ξ_3. In the two-dimensional case the flow is the same for every ξ_3 = constant, and the velocity component $u_3 = 0$ and $h_3 = 1$. On the other hand, in the axially symmetric case ξ_3 is the azimuthal angle, and the derivatives with respect to ξ_3 are zero although u_3 may or may not be zero, and $h_3 \neq 1$.

Referring to Equation 3.115, we find that for the incompressible axially symmetric motion with ξ_3 as the azimuthal angle, the continuity equation is identically satisfied by the equations:

$$h_3 u_1 = \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2}, \qquad h_3 u_2 = -\frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1} \tag{3.177}$$

Writing:

$$h_3 u_3 = w$$

we find that the components of vorticity from Equation 3.113 are

$$\omega_1 = \frac{1}{h_2 h_3}\frac{\partial w}{\partial \xi_2}, \qquad \omega_2 = -\frac{1}{h_1 h_3}\frac{\partial w}{\partial \xi_1}, \qquad \omega_3 = -\frac{1}{h_3} D^2\psi$$

where the differential operator D^2 is

$$D^2 = \frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial \xi_2}\right)\right]$$

We now consider Equation 3.118 and the vorticity equation (Equation 3.34) which yield the two equations:

$$\frac{\partial w}{\partial t} - \frac{1}{h_1 h_2 h_3}\frac{\partial(\psi, w)}{\partial(\xi_1, \xi_2)} = \nu D^2 w \tag{3.178a}$$

$$\frac{\partial}{\partial t}(D^2\psi) + \frac{2w}{h_1 h_2 h_3^2}\frac{\partial(w, h_3)}{\partial(\xi_1, \xi_2)} - \frac{1}{h_1 h_2 h_3}\frac{\partial(\psi, D^2\psi)}{\partial(\xi_1, \xi_2)} + \frac{2D^2\psi}{h_1 h_2 h_3^2}\frac{\partial(\psi, h_3)}{\partial(\xi_1, \xi_2)} = \nu D^4\psi \tag{3.178b}$$

where:

$$\frac{\partial(\alpha, \beta)}{\partial(\xi, \eta)} = \frac{\partial \alpha}{\partial \xi}\frac{\partial \beta}{\partial \eta} - \frac{\partial \alpha}{\partial \eta}\frac{\partial \beta}{\partial \xi}$$

for arbitrary differentiable α and β.

In two dimensions $w = 0$, $h_3 = 1$, and D^2 becomes the Laplacian ∇^2, viz.:

$$\nabla^2 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}\left(\frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\right)\right]$$

and then Equation 3.178b becomes:

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{h_1 h_2}\frac{\partial(\psi, \nabla^2\psi)}{\partial(\xi_1, \xi_2)} = \nu\nabla^4\psi \tag{3.179a}$$

and:

$$\nabla^2\psi = -\omega \tag{3.179b}$$

Three-Dimensional Flows

It was noted earlier that the velocity defined by Equation 3.176 is acceptable provided that:

$$\nabla^2\phi = 0 \tag{3.180}$$

where ϕ is a scalar potential. Also, without any loss of generality, we can impose the condition on the vector potential **A** that it should be solenoidal, viz.:

$$\text{div }\mathbf{A} = 0 \tag{3.181}$$

This assumption is justifiable on the basis of the form of **A** given in Equation 3.173 in which S and N are not unique, for one can choose S and N such that:

$$\begin{aligned}\text{div }\mathbf{A} &= \text{div}(S \text{ grad } N)\\ &= S\nabla^2 N + (\text{grad } S)\cdot(\text{grad } N)\\ &= 0\end{aligned}$$

That is, N is chosen such that $\nabla^2 N = 0$ and the surfaces S = constant and N = constant are chosen to be orthogonal.

Using the curl of Equation 3.176, we get:

$$\text{div(grad }\mathbf{A}) = -\boldsymbol{\omega} \tag{3.182}$$

The complete formulation of the three-dimensional vector potential method is thus comprised of Equations 3.34 and 3.180 through 3.182.

Example 3.3

Obtain the Navier-Stokes equations for flows past arbitrary shaped bodies of revolution.

Some important aerodynamic results are associated with the flow past bodies of revolution. The surface of a body of revolution is obtained by rotating an arbitrary curve about a fixed straight line in space. The fixed straight line is the axis of symmetry (or axis of revolution) of the body. Taking the x-axis as the axis of revolution, we consider a body of revolution in which ϕ is the azimuthal angle as measured from a reference plane as shown in the figure. Further, we take $r_0(x)$ as the local radius drawn from the x-axis to the body surface.

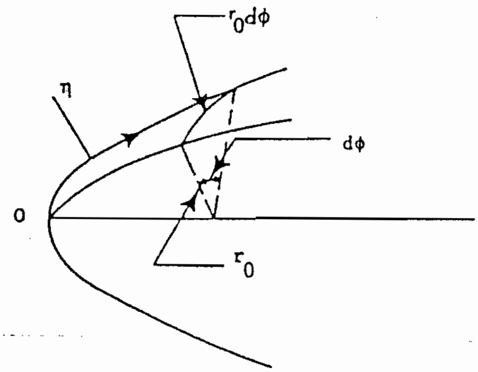

Fig. Example 3.3 Body of revolution.

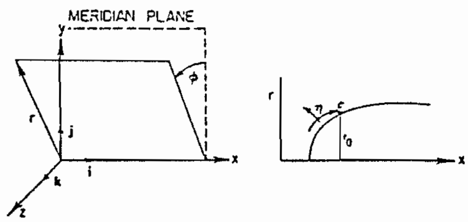

Fig. Example 3.3 Curvilinear coordinates in a meridian plane.

For the purpose of obtaining the Navier-Stokes equations for this case, we have to decide about a three-dimensional coordinate system. A natural choice is to take a coordinate ξ, which follows the body contour and the azimuthal angle ϕ as the surface coordinates. The coordinate η is taken as the transverse coordinate. Thus we take (ξ, η, ϕ), in this order, to represent the needed coordinate system. Here, as shown in the second figure of Example 3.3 ξ follows the body contour, while η is the coordinate drawn away from the surface. The coordinate system ξ, η lies in a meridian plane.

Referring to the figure for coordinates, we have:

$$y = r\cos\phi, \qquad z = r\sin\phi, \qquad y^2 + z^2 = r^2$$

$$x_\phi = 0, \qquad y_\phi = -z, \qquad z_\phi = y, \qquad r_\phi = 0$$

$$r_\xi^2 = y_\xi^2 + z_\xi^2, \qquad r_\eta^2 = y_\eta^2 + z_\eta^2$$

Based on these relations, the metric coefficients g_{ij}, g^{ij} and the Christoffel symbols Γ^i_{jk} in three dimensions are

$$\begin{aligned}
g_{11} &= x_\xi^2 + r_\xi^2 & g^{33} &= 1/r^2 \\
g_{22} &= x_\eta^2 + r_\eta^2 & g^{12} &= -r^2 g_{12}/g \\
g_{33} &= r^2 = y^2 + z^2 & g^{13} &= g^{23} = 0 \\
J &= x_\xi r_\eta - x_\eta r_\xi & g^{11} &= r^2 g_{22}/g \\
g_{12} &= x_\xi x_\eta + r_\xi r_\eta & g^{22} &= r^2 g_{11}/g \\
g_{13} &= g_{23} = 0 \\
g &= r^2[g_{11}g_{22} - (g_{12})^2] = r^2J^2 & & \qquad\qquad \text{(i)}
\end{aligned}$$

$$\Gamma^1_{11} = \frac{r^2}{2g}\left[g_{22}\frac{\partial g_{11}}{\partial \xi} - g_{12}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right)\right] \qquad \Gamma^3_{13} = \Gamma^3_{23} = 0$$

$$\Gamma^1_{12} = \frac{r^2}{2g}\left[g_{22}\frac{\partial g_{11}}{\partial \eta} - g_{12}\frac{\partial g_{22}}{\partial \xi}\right] \qquad \Gamma^2_{33} = \frac{r}{g}(g_{12}r_\xi - g_{11}r_\eta)$$

$$\Gamma^1_{13} = \Gamma^1_{23} = 0 \qquad \Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{12} = \Gamma^3_{33} = 0$$

$$\Gamma^1_{22} = \frac{r^2}{2g}\left[g_{22}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right) - g_{12}\frac{\partial g_{22}}{\partial \eta}\right] \qquad \Gamma^3_{13} = \frac{r_\xi}{r}, \quad \Gamma^3_{23} = \frac{r_\eta}{r}$$

$$\Gamma^1_{33} = \frac{r^3}{g}(g_{12}r_\eta - g_{22}r_\xi)$$

$$\Gamma^2_{11} = \frac{r^2}{2g}\left[g_{11}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right) - g_{12}\frac{\partial g_{11}}{\partial \xi}\right]$$

$$\Gamma^2_{22} = \frac{r^2}{2g}\left[g_{11}\frac{\partial g_{22}}{\partial \eta} - g_{12}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right)\right]$$

$$\Gamma^2_{12} = \frac{r^2}{2g}\left(g_{11}\frac{\partial g_{22}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \eta}\right) \tag{ii}$$

The base vectors $\mathbf{a}_i$ are

$$\mathbf{a}_1 = \mathbf{i}\,x_\xi + \mathbf{j}\,r_\xi\cos\phi + \mathbf{k}\,r_\xi\sin\phi$$

$$\mathbf{a}_2 = \mathbf{i}\,x_\eta + \mathbf{j}\,r_\eta\cos\phi + \mathbf{k}\,r_\eta\sin\phi$$

$$\mathbf{a}_3 = r(\mathbf{k}\cos\phi - \mathbf{j}\sin\phi) \tag{iii}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the Cartesian coordinates x, y, z, respectively.

In place of considering the problem of transformation in a piecemeal fashion, we consider a general transformation so that particular cases can be written down easily. First of all, we write Equations 3.8, 3.10, and 3.22a as follows:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0 \tag{iv}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \operatorname{div}\mathbf{X} = 0 \tag{v}$$

$$\frac{\partial(\rho e_t)}{\partial t} + \operatorname{div}\mathbf{b} = 0 \tag{vi}$$

where:

$$\mathbf{X} = \rho\mathbf{u}\mathbf{u} + p\mathbf{I} - \boldsymbol{\sigma}$$
$$= -\boldsymbol{\Pi}$$

Here $\boldsymbol{\sigma}$, e_t, and $\mathbf{b}$ have already been defined in Equations 3.9, 3.21, and 3.22b, respectively. The transformed equations are

$$\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x^i}(\sigma u^i) = 0 \tag{vii}$$

$$\frac{\partial}{\partial t}(\sigma\mathbf{u}) + \frac{\partial}{\partial x^k}(X^{ik}\mathbf{a}_i) = 0 \tag{viii}$$

$$\frac{\partial E}{\partial t} + \frac{\partial Y^k}{\partial x^k} = 0 \qquad \textbf{(ix)}$$

where introducing the following abbreviations:

$$\sigma = \sqrt{g}\rho, \qquad P = \sqrt{g}p, \qquad \chi = \sqrt{g}\lambda(\text{div}\ \mathbf{u}), \qquad E = \sqrt{g}\rho e,$$

we have:

$$X^{ik} = \sigma u^i u^k + g^{ik}(P - \chi) - 2\sqrt{g}\mu D^{ik}$$

$$Y^k = (E + P)u^k - \psi^k - k\sqrt{g}g^{kj}\frac{\partial T}{\partial x^j}$$

$$\psi^k = \chi u^k + \mu\sqrt{g}(g^{ik}g_{jm}u^j_{,i} + u^k_{,m})u^m \qquad \textbf{(x)}$$

Equation viii has to be opened by using the summation convention. If the first term on the left of Equation viii is expressed in terms of the Cartesian components, then the three equations of motion can be written down as the coefficients of the unit vectors **i**, **j**, **k**. The end result is to have equations which have both the Cartesian and the contravariant components of **u**. A résumé of the interrelationships between the various components is given below.
Let, U, V, W be the Cartesian components of **u**. Then:

$$U = u^1 x_\xi + u^2 x_\eta$$

$$V = S\cos\phi - ru^3\sin\phi$$

$$W = S\sin\phi + ru^3\cos\phi$$

where:

$$S = u^1 r_\xi + u^2 r_\eta$$

$$= V\cos\phi + W\sin\phi$$

In turn, then:

$$u^1 = (Ur_\eta - Sx_\eta)/J$$

$$u^2 = (Sx_\xi - Ur_\xi)/J$$

$$u^3 = (W\cos\phi - V\sin\phi)/r$$

Also, writing the physical components of **u** as u, v, w we have:

$$u = u^1\sqrt{g_{11}}, \qquad v = u^2\sqrt{g_{22}}, \qquad w = u^3\sqrt{g_{33}} = ru^3$$

The velocity component $w = ru^3$ is called the *swirl velocity*. It must also be noted that in the second term on the left of Equation viii the expansion process yields new terms which can further be abbreviated as follows:

$$XM^{11} = X^{11}x_\xi + X^{12}x_\eta$$

$$XM^{12} = X^{12}x_\xi + X^{22}x_\eta$$

$$\begin{aligned}
XM^{13} &= X^{13}x_\xi + X^{23}x_\eta \\
XM^{21} &= (X^{11}r_\xi + X^{12}r_\eta)\cos\phi - rX^{13}\sin\phi \\
XM^{22} &= (X^{12}r_\xi + X^{22}r_\eta)\cos\phi - rX^{23}\sin\phi \\
XM^{23} &= (X^{13}r_\xi + X^{23}r_\eta)\cos\phi - rX^{33}\sin\phi \\
XM^{31} &= (X^{11}r_\xi + X^{12}r_\eta)\sin\phi - rX^{13}\cos\phi \\
XM^{32} &= (X^{12}r_\xi + X^{22}r_\eta)\sin\phi - rX^{23}\cos\phi \\
XM^{33} &= (X^{13}r_\xi + X^{23}r_\eta)\sin\phi - rX^{33}\cos\phi
\end{aligned} \qquad \text{(xi)}$$

The previous operations show that there are many different forms in which the equations can be stated. When solving the complete Navier-Stokes equations for flows past axially symmetric bodies and in conditions when it is not possible to keep the η-coordinate orthogonal to the ξ-coordinate, it may be preferable to have a mixed Cartesian and contravariant representation of the velocity vector in the equations. Further, by cross multiplications the equations for V and W can be reduced to two separate equations for S and u^3, where ru^3 is the swirl velocity.

To demonstrate the form of Equation vii for the case when ξ and η are orthogonal (ϕ-coordinate is orthogonal to the meridian plane), first we have:

$$g_{11} = (h)^2, \qquad g_{22} = 1, \qquad g_{33} = (r)^2, \qquad g = (hr)^2$$

Thus, the physical components are

$$u = hu^1, \qquad v = u^2, \qquad \frac{w}{r} = u^3 \qquad \text{(xii)}$$

Making these substitutions in Equation vii, we have:

$$\frac{\partial\rho}{\partial t} + \frac{1}{rh}\left[\frac{\partial}{\partial\xi}(\rho ru) + \frac{\partial}{\partial\eta}(\rho rhv) + \frac{\partial}{\partial\phi}(\rho hw)\right] = 0 \qquad \text{(xiii)}$$

Example 3.4

List the important terms leading to the equations of fluid dynamics in cylindrical and spherical coordinates.

(a) Cylindrical coordinates: (r, ϕ, z), (ϕ, z, r) or, (z, r, ϕ).

Taking $r = \xi_1$, $\phi = \xi_2$, $z = \xi_3$ and $u_r = u_1$, $u_\phi = u_2$, $u_z = u_3$, we have the following results:

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z; \qquad h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1$$

$$\text{div}\,\mathbf{u} = \Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial\phi^2} + \frac{\partial^2 F}{\partial z^2}$$

$$\mathbf{e}_r = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi, \qquad \mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi, \qquad \mathbf{e}_z = \mathbf{k}$$

$$\text{curl}\,\mathbf{u} = \left\{\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right\}\mathbf{e}_r + \left\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right\}\mathbf{e}_\phi + \frac{1}{r}\left\{\frac{\partial}{\partial r}(ru_\phi) - \frac{\partial u_r}{\partial\phi}\right\}\mathbf{e}_z$$

$$T_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} + \lambda\Delta$$

$$T_{\phi\phi} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}\right) + \lambda\Delta$$

$$T_{zz} = -p + 2\mu \frac{\partial u_z}{\partial z} + \lambda\Delta$$

$$T_{r\phi} = \mu\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right) = T_{\phi r}$$

$$T_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = T_{zr}$$

$$T_{\phi z} = \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}\right) = T_{z\phi}$$

(b) Spherical coordinates: (r, θ, ϕ), (θ, ϕ, r), or (ϕ, r, θ).

Taking $r = \xi_1$, $\theta = \xi_2$, $\phi = \xi_3$ and $u_r = u_1$, $u_\theta = u_2$, $u_\phi = u_3$, we have the following results:

$$x = r \sin\theta \cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta;\quad h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$$

$$\operatorname{div} \mathbf{u} = \Delta = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial \theta}(r u_\theta \sin\theta) + \frac{\partial}{\partial \phi}(r u_\phi)\right]$$

$$\nabla^2 F = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial F}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial F}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial F}{\partial \phi}\right)\right]$$

$$\mathbf{e}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$$

$$\mathbf{e}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta$$

$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$$

$$\operatorname{curl} \mathbf{u} = \left\{\frac{\partial}{\partial \theta}(r u_\phi \sin\theta) - \frac{\partial}{\partial \phi}(r u_\theta)\right\}\frac{\mathbf{e}_r}{r^2 \sin\theta}$$
$$+ \left\{\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(r u_\phi \sin\theta)\right\}\frac{\mathbf{e}_\theta}{r\sin\theta} + \left\{\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right\}\frac{\mathbf{e}_\phi}{r}$$

$$T_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} + \lambda\Delta$$

$$T_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \lambda\Delta$$

$$T_{\phi\phi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}\right) + \lambda\Delta$$

$$T_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) = T_{\theta r}$$

$$T_{r\phi} = \mu\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right) = T_{\phi r}$$

$$T_{\theta\phi} = \mu\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi \cot\theta}{r}\right) = T_{\phi\theta}$$

Example 3.5

Find the conditions under which a viscous incompressible flow is circulation preserving.

If curl $\boldsymbol{\omega}$ = grad σ and $\mathbf{f} = -$grad χ, then according to Equation 3.32:

$$\mathbf{a} = -\text{grad } G$$

where:

$$G = \frac{p}{\rho} + \nu\sigma + \chi$$

The result follows by using Equation 1.57.

Example 3.6

Consider a curvilinear volume element in an orthogonal curvilinear coordinate system ξ, η, ζ with metric coefficients $h_1(\xi, \eta, \zeta)$, $h_2(\xi, \eta, \zeta)$, $h_3(\xi, \eta, \zeta)$. Derive the equation of continuity for a compressible flow using this elemental volume.

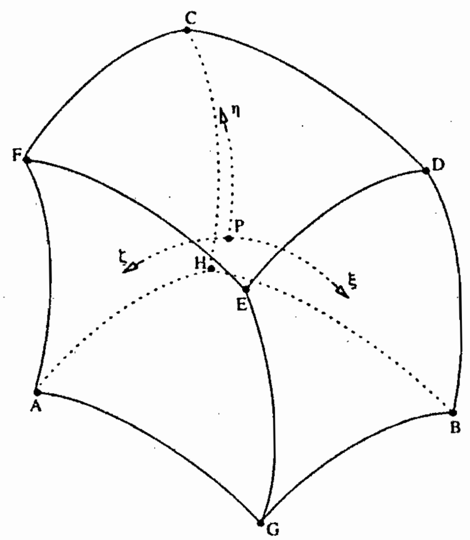

Fig. Example 3.6 Curvilinear element of volume.

The area of the faces AFCH and GEDB, AHBG and FCDE, HCDB and AFEG, respectively, are

$$h_2(\xi \mp \tfrac{1}{2}d\xi, \eta, \zeta)\, d\eta \cdot h_3(\xi \mp \tfrac{1}{2}d\xi, \eta, \zeta)\, d\zeta$$

$$h_1(\xi, \eta \mp \tfrac{1}{2}d\eta, \zeta)\, d\xi \cdot h_3(\xi, \eta \mp \tfrac{1}{2}d\eta, \zeta)\, d\zeta$$

$$h_1(\xi, \eta, \zeta \mp \tfrac{1}{2}d\zeta)\, d\xi \cdot h_2(\xi, \eta, \zeta \mp \tfrac{1}{2}d\zeta)\, d\eta$$

Writing $\rho u_1 = M$, $\rho u_2 = N$, and $\rho u_3 = R$, where M, N, and R are functions of ξ, η, ζ; performing Taylor's expansion; and retaining only the first order terms in $d\xi$, $d\eta$, and $d\zeta$, we find that the mass flow rate through the element in the ξ-direction is

$$-\left[h_2h_3\frac{\partial M}{\partial \xi} + M\frac{\partial}{\partial \xi}(h_2h_3)\right] d\xi\, d\eta\, d\zeta = -\frac{\partial}{\partial \xi}(Mh_2h_3)\, d\xi\, d\eta\, d\zeta$$

Performing similar analysis for the two remaining pairs and adding, we get:

$$\frac{\partial}{\partial t}(\rho h_1h_2h_3)\, d\xi\, d\eta\, d\zeta = -\left[\frac{\partial}{\partial \xi}(Mh_2h_3) + \frac{\partial}{\partial \eta}(Nh_1h_3) + \frac{\partial}{\partial \zeta}(Rh_1h_2)\right] d\xi\, d\eta\, d\zeta$$

Thus:

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \xi}(\rho u_1h_2h_3) + \frac{\partial}{\partial \eta}(\rho u_2h_1h_3) + \frac{\partial}{\partial \zeta}(\rho u_3h_1h_2)\right] = 0$$

Example 3.7

Deduce the differential equation for the kinematic condition on an impervious boundary.

At an impervious boundary S_b with the equation $G(\mathbf{r}_s, t) = 0$, the basic kinematic condition is that:

$$\mathbf{u} \cdot \mathbf{n}|_{S_b} = \mathbf{V}_s \cdot \mathbf{n}$$

where $\mathbf{n}$ is the unit normal vector on S_b which is moving with velocity $\mathbf{V}_s$. Since $dG/dt = 0$, and $d\mathbf{r}_s/dt = \mathbf{V}_s$, we have:

$$\frac{\partial G}{\partial t} + \mathbf{V}_s \cdot \text{grad } G = 0$$

or:

$$\frac{\partial G}{\partial t} + (\mathbf{V}_s \cdot \mathbf{n})|\text{grad } G| = 0$$

Using the basic kinematic conditon noted above, we have:

$$\frac{\partial G}{\partial t} + \mathbf{u} \cdot \text{grad } G = 0 \qquad \text{on} \quad S_b \tag{i}$$

Thus, on an impervious boundary, Equation i must be satisfied.

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PROBLEMS

3.1 (a) What is the acceleration vector $\mathbf{a}$ for steady flow in which $\boldsymbol{\omega} \times \mathbf{u} = 0$?

(b) Use the equation of continuity (Equation 2.6) and show that for steady flow the no-slip condition at a surface S demands that:

$$\operatorname{div} \mathbf{u}|_S = 0 \tag{i}$$

(c) Use Equation 3.74a to show that the normal stress acting against the surface S_b under the condition $\mu' = 0$ is

$$p_n = \left(p + \frac{4\mu}{3\rho}\frac{\partial \rho}{\partial t}\right)\Big|_{S_b}$$

3.2 Use Equation 2.35c for steady flow and without body forces, i.e., $\mathbf{f} = 0$, to show that the total torque exerted by the fluid on the surface S_b of Figure 3.1 is

$$\mathbf{M} = \int_{S_b} (\mathbf{r} \times \boldsymbol{\tau})\, dS \tag{i}$$

$$= \int_{S_0} \mathbf{r} \times (-p\mathbf{n}_0 + \boldsymbol{\sigma} \cdot n_0)\, dS - \int_{S_0} (\mathbf{r} \times \rho\mathbf{u})(\mathbf{u} \cdot \mathbf{n}_0)\, dS \tag{ii}$$

Hint: Use the condition $\mathbf{u} \cdot \mathbf{n}|_{S_b} = 0$.

3.3* The requirement that $\Phi \geqq 0$ has been proved in Section 2.14. Another method of proving the same is to use Equation 3.17a. Divide Equation 3.17a by T, and show that the second law of thermodynamics for a material volume $\mathcal{V}(t)$, viz.:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho s\, dv \geqq \int_{\mathcal{V}(t)} \operatorname{div}\left(\frac{k \operatorname{grad} T}{T}\right) dv$$

demands that $\Phi \geqq 0$.

3.4 Show that the equations of continuity and momentum with respect to a frame of reference rotating with a constant angular velocity $\boldsymbol{\Omega}$ are given by:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \tag{i}$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} - \operatorname{grad}(\tfrac{1}{2}\,\Omega^2 R^2)\right] = \rho\mathbf{f} + \operatorname{div} \mathbf{T} \tag{ii}$$

or:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\left(\frac{1}{2}|\mathbf{v}|^2 - \frac{\Omega^2 R^2}{2}\right) + (2\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{v}\right] = \rho\mathbf{f} + \operatorname{div} \mathbf{T} \tag{iii}$$

where now $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ and grad, div are with reference to the rotating frame. (Refer to the Problem 1.12.)

3.5* Assuming $e = e(T, \rho)$, we have:

* Refer to the material in Section 4.6.

$$\frac{de}{dt} = \left(\frac{\partial e}{\partial T}\right)_{\rho}\frac{dT}{dt} + \left(\frac{\partial e}{\partial \rho}\right)_{T}\frac{d\rho}{dt} \tag{i}$$

Use Equation i in Equation 3.15 to show that the equation of temperature is

$$\rho C_v \frac{DT}{Dt} = \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}(k \text{ grad } T) + \left[\rho^2\left(\frac{\partial e}{\partial \rho}\right)_T - p\right] \operatorname{div} \mathbf{u} \tag{ii}$$

From Equation ii show that the equation of temperature for an ideal gas is

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}(k \text{ grad } T) \tag{iii}$$

Hint: Use the ideal gas equation.

3.6** Use Equation 3.47 to show that for a thermally perfect inviscid gas:

$$\frac{Dh}{Dt} + a^2 \operatorname{div} \mathbf{u} = 0, \qquad a = \text{speed of sound}$$

3.7 Consider Equation 3.97 for the case of a two-dimensional flow. Let it be desired to transform the coordinates from x, y to ξ, η, where $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ and also $\xi = \xi(x, y)$, $\eta = \eta(x, y)$. Use the chain rule of partial differentiation and Equation M1.131a with $\sqrt{g} = J$ to obtain the equation:

$$\frac{\partial}{\partial t}(\mathbf{E}/\hat{J}) + \frac{\partial}{\partial \xi}(\xi_x \mathbf{F}/\hat{J} + \xi_y \mathbf{G}/\hat{J}) + \frac{\partial}{\partial \eta}(\eta_x \mathbf{F}/\hat{J} + \eta_y \mathbf{G}/\hat{J}) = \mathbf{S}/\hat{J}$$

where:

$$\hat{J} = 1/J = \xi_x \eta_y - \xi_y \eta_x$$

3.8 (a)** Use Equation 2.51 to show that the energy equation for a fixed control volume, modified by the inclusion of the shaft work and heat input, in a steady-state process is

$$\dot{Q}_{c.v} = \dot{W}_{c.v} + \int_{S^*} \rho u_n(\tfrac{1}{2}|\mathbf{u}|^2 + h + \chi)\, dS^* \tag{i}$$

where:

$$\dot{Q}_{c.v} = \text{additional heat input rate} - \int_{S^*} \mathbf{q} \cdot \mathbf{n}^* \, dS^*$$

$$\dot{W}_{c.v} = \text{useful work output rate} - \int_{S^*} \mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}^*)\, dS^*$$

Also show that for a single steady-state–steady-flow (SSSF) process, Equation i with $|\mathbf{u}| = u_n = u$ becomes:

$$\dot{Q}_{c.v} + \dot{m}(\tfrac{1}{2}u_i^2 + h_i + \chi_i) = \dot{m}(\tfrac{1}{2}u_e^2 + h_e + \chi_e) + \dot{W}_{c.v} \tag{ii}$$

** Refer to the material in Section 4.6.

where the subscripts i and e denote inlet and exit sections, respectively, and:

$$\dot{m} = \rho_i u_i A_i = \rho_e u_e A_e$$

(b) Show by using the results of Problem 1.12 that for an accelerating and rotating control volume (angular velocity $\mathbf{\Omega}$), the vector A appearing in Equation 2.38 is

$$\mathbf{A} = \ddot{\mathbf{p}} + 2\mathbf{\Omega} \times \mathbf{u}_r + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \dot{\mathbf{\Omega}} \times \mathbf{r}$$

3.9 Let (ξ, η, ζ), in this order, be an orthogonal right-handed curvilinear coordinate system. Use Equation 3.100 to show that:

$$h_1^2 = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \qquad \text{(i)}$$

$$h_2^2 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 \qquad \text{(ii)}$$

$$h_3^2 = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \qquad \text{(iii)}$$

where $\xi = \xi_1$, $\eta = \xi_2$, $\zeta = \xi_3$, $x = x_1$, $y = x_2$, $z = x_3$. Next use Equations i–iii to obtain the following results:

(a) Plane polar coordinates (r, θ, z = constant):
Set:

$$\xi = r, \qquad \eta = \theta, \qquad \zeta = z = \text{constant}$$

and thus:

$$x = \xi \cos \eta, \qquad y = \xi \sin \eta, \qquad z = \zeta = \text{constant}$$

Show that:

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1 \qquad \text{(iv)}$$

and then use Equation 3.170 to show that:

$$u_1 = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_2 = u_\theta = -\frac{\partial \psi}{\partial r}, \qquad u_3 = u_z = 0 \qquad \text{(v)}$$

where u_r, u_θ, u_z are the physical components of **u** along r, θ, and z, respectively.

(b) Cylindrical polar coordinates (z, r, ϕ):
Set:

$$\xi = z, \qquad \eta = r, \qquad \zeta = \phi$$

and thus:

$$x = \eta\cos\zeta, \qquad y = \eta\sin\zeta, \qquad z = \xi$$

Show that:

$$h_1 = 1, \qquad h_2 = 1, \qquad h_3 = r \tag{vi}$$

Use Equation 3.177 to obtain:

$$u_1 = u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad u_2 = u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad u_3 = u_\phi = \frac{w}{r} \tag{vii}$$

where u_z, u_r, u_ϕ are the physical components of **u** along z, r, ϕ, respectively.

(c) Spherical coordinates (r, θ, ϕ):
Set:

$$\xi = r, \qquad \eta = \theta, \qquad \zeta = \phi$$

and thus:

$$x = \xi\sin\eta\cos\zeta, \qquad y = \xi\sin\eta\sin\zeta, \qquad z = \xi\cos\eta \tag{viii}$$

Show that:

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta \tag{ix}$$

and then use Equation 3.177 to obtain:

$$u_1 = u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_2 = u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad u_3 = u_\phi = w/r\sin\theta$$

3.10 Use Equation 3.175a to show that in orthogonal coordinates, the physical components of **u** in terms of S and N are

$$u_1 = \frac{1}{h_2h_3}\left(\frac{\partial S}{\partial\eta}\frac{\partial N}{\partial\zeta} - \frac{\partial S}{\partial\zeta}\frac{\partial N}{\partial\eta}\right)$$

$$u_2 = \frac{1}{h_1h_3}\left(\frac{\partial S}{\partial\zeta}\frac{\partial N}{\partial\xi} - \frac{\partial S}{\partial\xi}\frac{\partial N}{\partial\zeta}\right)$$

$$u_3 = \frac{1}{h_1h_2}\left(\frac{\partial S}{\partial\xi}\frac{\partial N}{\partial\eta} - \frac{\partial S}{\partial\eta}\frac{\partial N}{\partial\xi}\right) \tag{i}$$

From Equation i verify the following particular cases:

(a) $\xi = x$, $\eta = y$, $\zeta = z$, $S = \psi(x, y)$, $N = z$.
Then:

$$h_1 = h_2 = h_3 = 1$$

and:

$$u_1 = u = \frac{\partial\psi}{\partial y}, \qquad u_2 = v = -\frac{\partial\psi}{\partial x}, \qquad u_3 = w = 0 \tag{ii}$$

(b) $\xi = r, \eta = \theta, \zeta = z, S = \psi(r, \theta), N = z.$
Then:

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1$$

and:

$$u_1 = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_2 = u_\theta = -\frac{\partial \psi}{\partial r}, \qquad u_3 = w = 0 \tag{iii}$$

(c) $\xi = z, \eta = r, \zeta = \phi, S = \psi(z, r), N = \phi.$
Then:

$$h_1 = h_2 = 1, \qquad h_3 = r$$

and:

$$u_1 = u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad u_2 = u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad u_3 = u_\phi = 0 \tag{iv}$$

(d) $\xi = r, \eta = \theta, \zeta = \phi, S = \psi(r, \phi), N = \phi.$
Then:

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r \sin \theta$$

and:

$$u_1 = u_r = \frac{1}{r^2 \sin \theta}\frac{\partial \psi}{\partial \theta}, \quad u_1 = u_\theta = -\frac{1}{r \sin \theta}\frac{\partial \psi}{\partial r}, \quad u_3 = u_\phi = 0 \tag{v}$$

3.11 Use the results of Section 3.10 to show that the components of the rate-of-strain tensor D in orthogonal curvilinear coordinates are

$$\begin{aligned}
D_{\xi_1\xi_1} = D_{11} &= \frac{1}{h_1}\frac{\partial u_1}{\partial \xi} + \frac{u_2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial \zeta} \\
D_{\xi_2\xi_2} = D_{22} &= \frac{1}{h_2}\frac{\partial u_2}{\partial \eta} + \frac{u_1}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{u_3}{h_2 h_3}\frac{\partial h_2}{\partial \zeta} \\
D_{\xi_3\xi_3} = D_{33} &= \frac{1}{h_3}\frac{\partial u_3}{\partial \zeta} + \frac{u_1}{h_1 h_3}\frac{\partial h_3}{\partial \xi} + \frac{u_2}{h_2 h_3}\frac{\partial h_3}{\partial \eta} \\
D_{\xi_1\xi_2} = D_{12} &= \frac{1}{2}\left(\frac{1}{h_2}\frac{\partial u_1}{\partial \eta} + \frac{1}{h_1}\frac{\partial u_2}{\partial \xi} - \frac{u_1}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{u_2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right) \\
D_{\xi_1\xi_3} = D_{13} &= \frac{1}{2}\left(\frac{1}{h_3}\frac{\partial u_1}{\partial \zeta} + \frac{1}{h_1}\frac{\partial u_3}{\partial \xi} - \frac{u_1}{h_1 h_3}\frac{\partial h_1}{\partial \zeta} - \frac{u_3}{h_1 h_3}\frac{\partial h_3}{\partial \xi}\right) \\
D_{\xi_2\xi_3} = D_{23} &= \frac{1}{2}\left(\frac{1}{h_3}\frac{\partial u_2}{\partial \zeta} + \frac{1}{h_2}\frac{\partial u_3}{\partial \eta} - \frac{u_2}{h_2 h_3}\frac{\partial h_2}{\partial \zeta} - \frac{u_3}{h_2 h_3}\frac{\partial h_3}{\partial \eta}\right)
\end{aligned} \tag{i}$$

Also, the components of vorticity ω are

$$\omega_{\xi_1} = \omega_1 = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial \eta}(h_3 u_3) - \frac{\partial}{\partial \zeta}(h_2 u_2)\right\}$$

$$\omega_{\xi_2} = \omega_2 = \frac{1}{h_3 h_1}\left\{\frac{\partial}{\partial \zeta}(h_1 u_1) - \frac{\partial}{\partial \xi}(h_3 u_3)\right\}$$

$$\omega_{\xi_3} = \omega_3 = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial \xi}(h_2 u_2) - \frac{\partial}{\partial \eta}(h_1 u_1)\right\} \tag{ii}$$

3.12 (a) Transform the two-dimensional incompressible Navier-Stokes equations written in the rectangular Cartesian coordinates (x, y) to the stream function form by taking:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad f_x = -\frac{\partial \chi}{\partial x}, \quad f_y = -\frac{\partial \chi}{\partial y}$$

to have:

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = \nu\nabla^4\psi \tag{i}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \nabla^2(\nabla^2)$$

and:

$$\frac{\partial(f, g)}{\partial(x, y)} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

Show that the components of the vorticity are

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega = \omega_z = -\nabla^2\psi \tag{ii}$$

(b) Show that the stream function form of the equation in plane polar coordinates, by taking:

$$f_r = -\frac{\partial \chi}{\partial r}, \quad f_\theta = -\frac{1}{r}\frac{\partial \chi}{\partial \theta}$$

is

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)} = \nu\nabla^4\psi \tag{iii}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{iv}$$

Also show that the components of vorticity are

$$\omega_r = 0, \quad \omega_\theta = 0$$

$$\omega = \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$= -\nabla^2\psi \tag{v}$$

(c) In cylindrical coordinates (z, r, ϕ) for a flow having symmetry about the zr-plane and with $u_\phi = 0$, obtain the equation:

$$\frac{\partial}{\partial t}(D^2\psi) - r\frac{\partial\left(\psi, \frac{1}{r^2}D^2\psi\right)}{\partial(z, r)} = \nu D^4\psi \tag{vi}$$

where:

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and:

$$D^4 = D^2(D^2)$$

Show also that the vorticity components are given by:

$$\omega_r = 0, \qquad \omega_\phi = \frac{-1}{r}D^2\psi, \qquad \omega_z = 0 \tag{vii}$$

(d) In spherical polar coordinates (r, θ, ϕ) for a flow having symmetry about the $r\theta$-plane and with $u_\phi = 0$, obtain the equation:

$$\frac{\partial}{\partial t}(D^2\psi) - (1 - \alpha^2)\frac{\partial\left[\psi, \frac{1}{r^2(1 - \alpha^2)}D^2\psi\right]}{\partial(\alpha, r)} = \nu D^4\psi \tag{viii}$$

where:

$$\alpha = \cos\theta$$

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1 - \alpha^2}{r^2}\frac{\partial^2}{\partial \alpha^2}$$

$$D^4 = D^2(D^2) \tag{ix}$$

Show also that the vorticity components are

$$\omega_r = 0, \qquad \omega_\phi = 0, \qquad \omega_\phi = \frac{-1}{r\sqrt{1 - \alpha^2}}D^2\psi \tag{x}$$

3.13 Consider the transformed Navier-Stokes equations (Equations 3.115–3.119) for the case of a two-dimensional flow in the $\xi\eta$-plane.

(a) First show that if ξ and η are conformal coordinates, then:

$$h_2 = h_1 = h$$

(b) use the result in (a) and set:

$$v_1 = hu_1, \qquad v_2 = hu_2$$

and show that the Navier-Stokes equations for incompressible two-dimensional flow in conformal coordinates are

$$\frac{\partial v_1}{\partial \xi} + \frac{\partial v_2}{\partial \eta} = 0 \tag{i}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + Jv_k\frac{\partial v_i}{\partial x^k} + \frac{1}{2}(v_1^2 + v_2^2)\frac{\partial J}{\partial x^i}\right] + \frac{\partial p}{\partial x^i} = \mu\left[J\nabla^2 v_i + \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right)\frac{\partial J}{\partial x^k}\right] \tag{ii}$$

where the repeated index implies summation and:

$$x^1 = \xi, \quad x^2 = \eta, \quad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}, \quad J = \frac{1}{h^2} \quad \omega = J\left(\frac{\partial v_2}{\partial \xi} - \frac{\partial v_1}{\partial \eta}\right) \tag{iii}$$

3.14 Using the results of Example 3.4, show that the equations of incompressible flow in cylindrical and spherical coordinates are as follows:

(a) Cylindrical coordinates
Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0$$

Momentum Equations:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right)$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial \phi} + \nu\left(\nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right)$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Vorticity Components:

$$\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}, \qquad \omega_\phi = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \qquad \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_\phi) - \frac{1}{r}\frac{\partial u_r}{\partial \phi}$$

(b) Spherical coordinates
Continuity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0$$

Momentum Equations:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{2u_\theta \cot\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2 \cot\theta}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{r\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right)$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot\theta}{r}$$

$$= -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right)$$

where:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Vorticity components:

$$\omega_r = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right\}$$

$$\omega_\theta = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\phi)$$

$$\omega_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

3.15 Referring to Equation 3.97, verify that the conservation equation in Cartesian coordinates can also be stated as:

$$\frac{\partial \mathbf{E}}{\partial t} + \operatorname{div} \mathbf{\Sigma} = \mathbf{S} \qquad \text{(i)}$$

where:

$$\mathbf{\Sigma}\cdot\mathbf{i}_1 = \mathbf{F}, \qquad \mathbf{\Sigma}\cdot\mathbf{i}_2 = \mathbf{G}, \qquad \mathbf{\Sigma}\cdot\mathbf{i}_3 = \mathbf{H} \qquad \text{(ii)}$$

and **E**, **F**, **G**, **H**, **S** are numerical vectors each having five components in three dimensions. Further, show that on coordinate transformation to general nonsteady coordinates $x^i = x^i(x, y, z, t)$, $\tau = t$, Equation i becomes:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial}{\partial x^i}[\sqrt{g}(\mathbf{E}w^i + \mathbf{\Sigma}\cdot\mathbf{a}^i)] = \sqrt{g}\mathbf{S} \qquad \text{(iii)}$$

where:

$$\mathbf{U} = \sqrt{g}\mathbf{E}$$

$$w^i = \frac{\partial x^i}{\partial t}$$

$$\Sigma \cdot \mathbf{a}^i = \mathbf{F}\frac{\partial x^i}{\partial x} + \mathbf{G}\frac{\partial x^i}{\partial y} + \mathbf{H}\frac{\partial x^i}{\partial z}$$

3.16 For an inviscid and non-heat-conducting fluid show that the vectors **F**, **G**, and **H** defined in Equation 3.166b are

$$\mathbf{R}^i = \sqrt{g}\left[\rho v^i,\quad \rho u v^i + p\frac{\partial x^i}{\partial x},\quad \rho v v^i + p\frac{\partial x^i}{\partial y},\quad \rho w v^i + p\frac{\partial x^i}{\partial z},\quad \rho e_t v^i + p u^i\right]^T$$

where:

$$\mathbf{R}^1 = \mathbf{F}, \qquad \mathbf{R}^2 = \mathbf{G}, \qquad \mathbf{R}^3 = \mathbf{H}$$

3.17 Use Equation 3.74 to show that the viscous stress on the surface S_h defined as x^2 = constant, along an arbitrary curve with the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds$ is

$$\tau \cdot \mathbf{t}\big|_{S_h} = \frac{\mu}{\sqrt{g g^{22}}}\{\mathbf{a}_1(\omega \cdot \mathbf{a}_3) - \mathbf{a}_3(\omega \cdot \mathbf{a}_1)\} \cdot \mathbf{t}\big|_{S_h} \tag{i}$$

From Equation i show that the frictional stresses on S_h along the x^1 and x^3 directions, respectively, are

$$F_r = \frac{\mu}{\sqrt{g g^{22} g_{rr}}}\{g_{r1}(\omega \cdot \mathbf{a}_3) - g_{r3}(\omega \cdot \mathbf{a}_1)\}\big|_{S_h}$$

where there is no sum on r, and r = 1, 3. Thus, the resultant frictional stress is

$$F = (F_1^2 + F_3^2 + 2F_1F_3\cos\theta)^{1/2}, \qquad \cos\theta = g_{13}/\sqrt{g_{11}g_{33}}$$

3.18 Use Equation 2.72 to show that the magnitude of the viscous stress normal to an arbitrary curve with unit tangent vector $\mathbf{t} = d\mathbf{r}/ds$ is

$$|\sigma \cdot \mathbf{t}| = \left[\lambda^2\Delta^2 + 4\mu\lambda\Delta D^{ij}\frac{dx^r}{ds}\frac{dx^s}{ds}g_{ir}g_{js} + 4\mu^2 D^{ij}D^{mn}\frac{dx^r}{ds}\frac{dx^s}{ds}g_{ir}g_{jm}g_{ns}\right]^{1/2}$$

where $\Delta = div\ \mathbf{u}$. Taking, in turn, the unit vector $\mathbf{t}$ along the coordinates x_r (r = 1, 2, 3), show that the viscous stress normal to x_r is

$$|\sigma \cdot \mathbf{t}|_r = [\lambda^2\Delta^2 + 4\mu\lambda\Delta D^{ij}g_{ir}g_{jr}/g_{rr} + 4\mu^2 D^{ij}D^{mn}g_{ir}g_{jm}g_{nr}/g_{rr}]^{1/2}$$

where there is no sum on r, and r = 1, 2, 3.

3.19 To establish the result

$$(\mathbf{n}\times\text{grad})\times\mathbf{u}\big|_{S_h} = 0 \qquad \text{when } \mathbf{u}\big|_{S_h} = 0$$

one way to proceed is as follows:

(a) Use the covariant components of **n** and the contravariant components of **u** to express $(\mathbf{n} \times \text{grad}) \times \mathbf{u}$ in general curvilinear coordinates x^i. Evaluate the expression so obtained on any fixed arbitrary surface S_b using the no-slip condition.

(b) Consider S_b as the surface x^2 = const., and referring to Equations M1.119, 120 first establish that the covariant components of **n** are

$$n_1 = 0,\ n_2 = \left(\sqrt{g/G_2}\right)_{x^2 = \text{const.}},\ n_3 = 0$$

Next using the above components establish the result

$$(\mathbf{n} \times \text{grad}) \times \mathbf{u}\Big|_{x^2 = \text{const.}} = 0 \qquad \text{whenever } \mathbf{u}\Big|_{x^2 = \text{const}} = 0$$

From this result one may conclude that the same result is obtained for the surfaces x^1 = const., and x^3 = const., and thus for any arbitrary surface S_b.

3.20 The equation of vorticity magnitude is simply obtained by taking the dot product of Equation 3.34 by ω. Show that after rearranging the terms the required equation is

$$\frac{D}{Dt}|\omega|^2 = 2\omega \cdot D \cdot \omega + \nu\nabla^2\left(|\omega|^2\right) - 2\nu(\text{grad } \omega) : (\text{grad } \omega)$$

From this equation we conclude that the vorticity magnitude increases (decreases) in the direction of an eigenvector of D if the corresponding eigenvalue is positive (negative).

Note: The integral of the vorticity magnitude over a domain is called the *enstrophy* of the domain.

CHAPTER FOUR

Flow of Inviscid Fluids

4.1 INTRODUCTION

Inviscid fluids are hypothetical fluids in which the coefficient of viscosity is zero. Although such fluids do not occur in nature, their dynamics provide useful results in regions away from a solid surface. The neglect of viscosity simplifies the equations considerably, and this was the main reason why most of the solutions from the time of Bernoulli, Euler, and Lagrange up to the early part of the 20th century were obtained for such fluids. Recently, there has been a renewed interest in inviscid fluid motion because the present day computers are capable of solving inviscid equations without further simplification for flows past or through bodies of practical shapes. It is interesting to note here that due to $R_e = \infty$, the model of an inviscid fluid according to Equation 3.85 forms a perfect continuum.

The equations obtained by neglecting the viscous terms, i.e., setting $\mu = 0$ in the Navier-Stokes equations, are called the Euler equations. Euler's equations have been stated in Chapter 3, Equations 3.41–3.49. For low speed flows, the density changes are negligibly small and ρ is considered a constant. In high speed flows the effect of compressibility on fluid motion is significant, and thus ρ is a function of both space coordinates and time.

Most of the basic theory of inviscid fluid motion and its application to aeronautical problems is available in various books. The theory of inviscid incompressible flow is available in Lamb,[1] Milne-Thompson,[2] Serrin,[3] Batchelor,[4] Karamcheti,[5] Loitsyanskii,[6] and the theory of inviscid compressible flow is available in Shapiro,[7] Liepmann and Roshko,[8] Thompson,[9] and Anderson.[10] The purpose of this chapter is to systematically establish those results through the equations of motion which provide an overall understanding of the phenomena of both the incompressible and compressible inviscid fluid motion.

The case of steady motion of inviscid gases was briefly considered in Sections 3.4 and 3.7. To have a clear understanding of the interplay of the Bernoulli function B and the enthalpy function H and to obtain further results, we consider Equations 3.64 and 3.67. By taking the dot product with $\mathbf{u}$ on both sides of Equation 3.64, we get:

$$\mathbf{u} \cdot \operatorname{grad} B = 0$$

where B has been defined in Equation 3.65. Also, for steady flow Equation 3.67 is

$$\mathbf{u} \cdot \operatorname{grad} H = 0$$

where H has been defined in Equations 3.68. Thus, for steady flow of inviscid gases both B and H are constants on streamline; and, in general, both constants are different. Only for isentropic flow, i.e., s = constant, since the first law of thermodynamics yields:

$$h = \int \frac{dp}{\rho}$$

then the constants B and H are equal on streamlines. Note also that Equation 3.68 is an integral of the energy equation under the steady-state conditions. In the absence of the body forces, it takes the form

$$h + \frac{1}{2}|\mathbf{u}|^2 = h_0 \tag{4.1}$$

where h_0 is the stagnation enthalpy. If in addition, the gas is a thermally perfect gas, then $h = C_pT$ and:

$$T + \frac{1}{2C_p}|\mathbf{u}|^2 = T_0 \tag{4.2}$$

Part I: Inviscid Incompressible Flow

4.2 THE BERNOULLI CONSTANT

For steady flow Equation 3.64 is valid, and due to the incompressibility condition the parameter B given by:

$$B = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \chi \tag{4.3}$$

is a constant on each streamline. Equation 4.3 is the Bernoulli equation for steady inviscid and incompressible flows. Since, in general, $\boldsymbol{\omega} \neq 0$, then the parameter B must somehow depend on the distribution of vorticity in the field. To establish the dependence of B on $\boldsymbol{\omega}$ for a streamline, we consider Equation 3.64 and use Equation 3.175a for $\mathbf{u}$ in a three-dimensional flow. After some simplification, we get:

$$\boldsymbol{\omega} \cdot [(\text{grad } N)(\text{grad } S) - (\text{grad } S)(\text{grad } N)] = -\text{grad } B \tag{4.4}$$

Equation 4.4 provides the components of $\boldsymbol{\omega}$ in terms of the derivatives of N, S, and B. In particular, for a two-dimensional flow:

$$N = z, \qquad S = \psi(x, y), \qquad \boldsymbol{\omega} = \mathbf{k}\omega$$

so that:

$$\omega \text{ grad } \psi = -\text{grad } B$$

which implies:

$$\omega = -\frac{dB}{d\psi} \tag{4.5}$$

Equation 4.5 shows that different values of B on different streamlines are due to the vorticity in the field. Only when $\omega = 0$, the parameter B is a global constant for the whole flow field. Such flows are called irrotational.

4.3 IRROTATIONAL FLOWS

If $\boldsymbol{\omega} = 0$, then the velocity field is irrotational. Since $\boldsymbol{\omega} = \text{curl } \mathbf{u}$, then the condition $\boldsymbol{\omega} = 0$ implies the existence of a scalar ϕ through the equation:

$$\boldsymbol{\omega} = \text{curl } \mathbf{u} = \text{curl(grad } \phi) \equiv 0$$

or:

$$\mathbf{u} = \text{grad } \phi \tag{4.6}$$

Thus, Equation 3.44 becomes:

$$\text{grad}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \chi\right] = 0$$

which on integration yields:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \chi = f(t) \tag{4.7}$$

where $f(t)$ is an arbitrary function of time. Equation 4.7 is the nonsteady form of the Bernoulli equation.

As is obvious from these derivations, Equations 4.3 and 4.7 are solutions of the equations of motion (Equation 3.44) when ρ = constant. The main effort now is to calculate a proper ϕ and then determine the velocity distribution $\mathbf{u}$ through Equation 4.6. Finally, the substitution of $|\mathbf{u}|$ in either Equation 4.3 or 4.7 yields the required pressure distribution. The determination of B or $f(t)$ is accomplished through the given boundary conditions or initial conditions of the problem.

An independent equation for ϕ is obtained through the continuity equation:

$$\text{div } \mathbf{u} = \text{div}(\text{grad } \phi) = 0$$

or:

$$\nabla^2 \phi = 0 \tag{4.8}$$

which is the Laplace equation in ϕ, and it has to be solved under some proper boundary conditions. Because the flow is ultimately determined by solving Equation 4.8, it is also called a *potential flow*.

Boundary Conditons

Since the fluid under consideration is inviscid, one cannot apply the no-slip condition. The only condition for flow past a body of surface S is that the normal component of $\mathbf{u}$ at S should be equal to the normal component of the velocity of the body. Thus:

$$\mathbf{u} \cdot \mathbf{n}|_S = \mathbf{n} \cdot \text{grad } \phi|_S = v_{nb}$$

or:

$$\left.\frac{\partial \phi}{\partial n}\right|_S = v_{nb}$$

Thus, the boundary value problem for ϕ is a Neumann problem:

$$\begin{aligned} \nabla^2 \phi &= 0 \quad \text{in } V \\ \frac{\partial \phi}{\partial n} &= v_{nb} \quad \text{on } S \end{aligned} \tag{4.9}$$

Example 4.1

From among all flows satisfying the same boundary conditions, the potential flow has the least kinetic energy.

Let v be any other flow in V such that:

$$\mathbf{v} \cdot \mathbf{n}|_S = \left.\frac{\partial \phi}{\partial n}\right|_S$$

and:

$$\text{div } \mathbf{v} = 0 \text{ throughout } V$$

Consider the identity:

$$\begin{aligned} \text{div}(\phi \mathbf{v}) &= \phi \text{ div } \mathbf{v} + \mathbf{v} \cdot \text{grad } \phi \\ &= \mathbf{v} \cdot \text{grad } \phi \end{aligned}$$

Integrating throughtout V and using Gauss' theorem, we have:

$$\int_S \phi(\mathbf{v} \cdot \mathbf{n})\, dS = \int_V (\mathbf{v} \cdot \text{grad } \phi)\, d\nu$$

or:

$$\int_S \phi \frac{\partial \phi}{\partial n}\, dS = \int_V (\mathbf{v} \cdot \text{grad } \phi)\, d\nu$$

Using Equation M5.11, we get:

$$\int_V (\mathbf{v} \cdot \text{grad } \phi)\, d\nu = \int_V |\text{grad } \phi|^2\, d\nu \qquad \text{(i)}$$

Now, since:

$$\int_V |\mathbf{v} - \text{grad } \phi|^2\, d\nu \geqslant 0$$

then using Equation i, we get:

$$\int_V |\mathbf{v}|^2\, d\nu \geqslant \int_V |\text{grad } \phi|^2\, d\nu = \int_V |\mathbf{u}|^2\, d\nu$$

which proves the result.

Two-Dimensional Irrotational Flows

For two-dimensional steady irrotational inviscid and incompressible flows the most powerful method of solution is the method of complex variables. First of all, the introduction of the stream function ψ through Equation 3.168 satisfies the equation of continuity identically. Also using Equation 4.6, we have:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{4.10}$$

The equations:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{4.11}$$

are the Cauchy-Riemann equations for the real and imaginary parts of the complex function:

$$w(z) = \phi(x, y) + i\psi(x, y)$$

where $i = \sqrt{-1}$ and $z = x + iy$. Since the cosine of the angle between ϕ = constant and ψ = constant is

$$(\text{grad } \phi) \cdot (\text{grad } \psi)/|\text{grad } \phi| \, |\text{grad } \psi|$$

we note that it is zero due to Equations 4.11. Thus the streamlines and the equipotential lines intersect each other orthogonally in the xy-plane. Further, by cross differentiation of Equations 4.11 one has:

$$\nabla^2 \phi = 0, \qquad \nabla^2 \psi = 0 \tag{4.12}$$

Thus, both ϕ and ψ, which are the real and imaginary parts of the analytic function $w(z)$, satisfy the Laplace equation. Therefore, if analytic functions satisfying proper geometrical and dynamical conditions can be constructed, then in effect the solutions of the Laplace equations under these conditions become readily available without actually solving Equations 4.12.

Let C be an arbitrary closed contour in the xy-plane. Consider the contour integral:

$$\oint_C \frac{dw}{dz} dz = \oint_C (udx + vdy) + i \oint_C (udy - vdx)$$
$$= -\Gamma + iQ$$

Here Γ is considered positive in the clockwise sense. In what follows, we have conformed to the convention that the contour C is traversed counterclockwise with the positive normal drawn away from the enclosed region. (Refer to Equation M3.17). The quantity Γ is the circulation of velocity around C, and Q is the rate of volume flow per unit height of C. Using the residue theorem,[11] we have:

$$\oint_C \frac{dw}{dz} dz = 2\pi i \left(\text{sum of the residues at the poles of } \frac{dw}{dz}\right)$$
$$= -\Gamma + iQ \tag{4.14}$$

We now define the complex velocity:

$$V = u + iv$$

and its complex conjugate:

$$\bar{V} = u - iv$$

as the conjugate velocity. Since:

$$\left.\begin{aligned}\frac{dw}{dz} &= \frac{\partial\phi}{\partial x} + i\,\frac{\partial\psi}{\partial x} \\ &= \frac{\partial\psi}{\partial y} - i\,\frac{\partial\phi}{\partial y}\end{aligned}\right\} = u - iv$$

we have:

$$\frac{dw}{dz} = \overline{V} \tag{4.15}$$

Examples of Analytic Functions for Inviscid Flows

(i) Uniform Flow. The linear analytic function:

$$w(z) = \overline{V}_\infty z \tag{4.16}$$

where the conjugate velocity is

$$\overline{V}_\infty = u_\infty - iv_\infty = |\overline{V}_\infty|e^{-i\alpha}, \qquad \alpha = \tan^{-1}(v_\infty/u_\infty)$$

describes a uniform inviscid flow V_∞ from left to right inclined at an angle α with the x-axis.

(ii) Flow Due to a Source or Sink. The logarithmic function:

$$w = A\ell nz = A\ell n(re^{i\theta}) \tag{4.17}$$

with A as a real constant yields:

$$\phi = A\ell nr, \qquad \psi = A\theta$$

Thus, ψ = constant are rays emanating from the origin. If A > 0 then there is a source at the origin, while if $A < 0$ then there is a sink at the origin. The conjugate velocity:

$$\frac{dw}{dz} = \overline{V} = \frac{A}{z}$$

has a simple pole at the origin, with residue A. Thus, from Equation 4.14:

$$-\Gamma + iQ = 2\pi iA$$

and thus $\Gamma = 0$, $Q = 2\pi A$. We call Q the strength of the source or sink at the origin. In terms of Q, Equation 4.17 is

$$w = \frac{Q}{2\pi}\ell nz \tag{4.18a}$$

and:

$$|\overline{V}| = \frac{|Q|}{2\pi r} \tag{4.18b}$$

If the position of the source or sink is shifted to $z_o = x_0 + iy_0$, then:

$$w(z) = \frac{Q}{2\pi} \ell n(z - z_0) \tag{4.19}$$

(iii) Flow Due to a Point Vortex. In Equation 4.17 if A is purely imaginary (i.e., $A = iB$) where B is real, then:

$$\phi = -B\theta, \qquad \psi = B\ell nr$$

Thus ψ = constant are concentric circles with center $z = 0$, describing the circulatory motion about a point vortex. In this case, following the previous procedure, we have:

$$-\Gamma + iQ = -2\pi B$$

and thus $\Gamma = 2\pi B$ and $Q = 0$. Therefore, $B = \Gamma/2\pi$, and the potential function becomes:

$$w = \frac{-\Gamma}{2\pi i} \ell nz \tag{4.20a}$$

and:

$$|V| = \frac{\Gamma}{2\pi r}, \qquad \Gamma > 0 \tag{4.20b}$$

If the position of the point vortex is at $z = z_0$, then:

$$w = \frac{-\Gamma}{2\pi i} \ell n(z - z_0) \tag{4.21}$$

From Equation 4.21:

$$\psi = \frac{\Gamma}{2\pi} \ell n|z - z_0| \tag{4.22a}$$

or:

$$\psi(x, y) = \frac{\Gamma}{2\pi} \ell n \{(x - x_0)^2 + (y - y_0)^2\}^{1/2} \tag{4.22b}$$

The flow due to a point vortex is everywhere irrotational except at the point of the vortex, viz., at $z = 0$ or $z = z_0$.

(iv) Flow Due to a Dipole or Doublet. A combination of a source and sink both of the same strength Q, when the distance between the two approaches zero, is called a *dipole* or *doublet*.

Let a source of strength Q be situated at a distance h to the left of the origin on the x-axis and a sink of the same strength be situated at a distance h to the right of the origin on the x-axis. The superposition of the two is then:

$$\frac{Q}{2\pi} \ell n(z + h) - \frac{Q}{2\pi} \ell n(z - h) = \frac{Q}{2\pi}\left(\frac{2h}{z} + \frac{2h^3}{3z^3} + \ldots\right), \qquad Q > 0 \tag{4.23}$$

Let:

$$\lim_{\substack{h \to 0 \\ Q \to \infty}} 2hQ = m$$

then taking the limit of Equation 4.23 as $h \to 0$, we obtain the complex potential:

$$w(z) = \frac{m}{2\pi z} \tag{4.24}$$

m is called the *strength or moment of the dipole.*

(v) Flow Past a Circular Cylinder. The flow past a *circular cylinder* of a radius a and center at the origin can be obtained by placing a doublet at the origin in a uniform flow. This scheme can work provided that the streamline $\psi = 0$ is a circle of radius a.

The complex potential due to a uniform flow is

$$w_1(z) = u_\infty z$$

where u_∞ is the uniform velocity at infinity parallel to the x-axis. The complex potential due to a doublet at the center of the cylinder is

$$w_2(z) = \frac{m}{2\pi z}$$

The superposition of w_1 and w_2 gives:

$$w(z) = w_1(z) + w_2(z)$$

or:

$$w(z) = u_\infty z + \frac{m}{2\pi z}$$

The stream function ψ is then given by:

$$\psi = \left[u_\infty - \frac{m}{2\pi(x^2 + y^2)}\right]y$$

In order that the streamlines ψ = constant given by the above be the streamlines for the flow past a circular cylinder, the *null streamline* $\psi = 0$ must be the equation of a circle of radius a. This amounts to the fact that the velocity *at* the surface of the cylinder must be everywhere tangential since the velocity normal to a stationary cylinder must be zero. Besides $y = 0$, the other null streamline is obtained if:

$$x^2 + y^2 = a^2$$

and then:

$$m = 2\pi a^2 u_\infty$$

Thus, the complex potential for the flow past a stationary circular cylinder placed in a uniform stream u_∞ parallel to the x-axis is

$$w(z) = u_\infty\left(z + \frac{a^2}{z}\right) \tag{4.25}$$

where a is the radius of the cylinder. The conjugate velocity is

$$\bar{V} = u_\infty\left(1 - \frac{a^2}{z^2}\right) \tag{4.26}$$

To illustrate the geometric aspect of the flow past a circular cylinder, we consider the plane polar coordinates as shown in Figure 4.1.

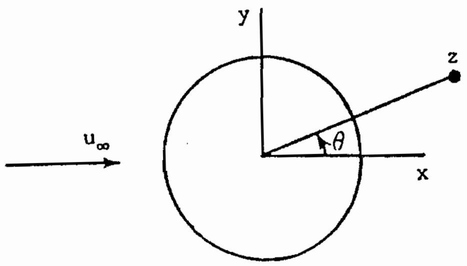

Fig. 4.1 Flow from left to right past a circular cylinder.

Writing:

$$z = re^{i\theta}$$

in Equation 4.25, we find that the stream function in polar coordinates is given by:

$$\psi = u_\infty\left(r - \frac{a^2}{r}\right)\sin\theta \tag{4.27}$$

Using Equation v of Problem 3.9, we have:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}$$

so that:

$$u_r = u_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{4.28a}$$

$$u_\theta = -u_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta \tag{4.28b}$$

In the boundary layer theory the angle is usually measured clockwise from the forward stagnation point in a flow from left to right. To find the velocity components from Equations 4.28 when θ is measured clockwise, replace θ by $2\pi - \theta$. Thus:

$$u_r = u_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{4.29a}$$

$$u_\theta = u_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta \qquad \textbf{(4.29b)}$$

and:

$$u_r = 0 \qquad \text{at} \quad r = a$$

$$u_\theta = 2u_\infty \sin\theta \qquad \text{at} \quad r = a \qquad \textbf{(4.30)}$$

The stagnation points are those points where $\overline{V} = 0$, which according to Equation 4.26 are at $z = \pm a$, or $\theta = 0$, $\theta = \pi$. The maximum velocity is $2u_\infty$ at $\theta = \pi/2$. The pressure at the surface is given by the Bernoulli equation:

$$p + \frac{1}{2}\rho u_\theta^2 = p_\infty + \frac{1}{2}\rho u_\infty^2$$

from which the pressure coefficients c_p defined as:

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho u_\infty^2}$$

is

$$c_p = 1 - 4\sin^2\theta$$

As is obvious from the stream function (Equation 4.27) the flow is symmetrical about the x-axis.

(vi) Flow Past a Circular Cylinder at an Angle of Attack. The flow past a circular cylinder when a uniform velocity at infinity is inclined at an angle α with the x-axis can be obtained from the preceding results. The complex velocity at infinity is

$$V_\infty = |V_\infty| e^{i\alpha}$$

As shown in Figure 4.2, let new rectangular axes X, Y be obtained by rotating the original axes through an angle α keeping the origin of the new axes the same as of the old axes.

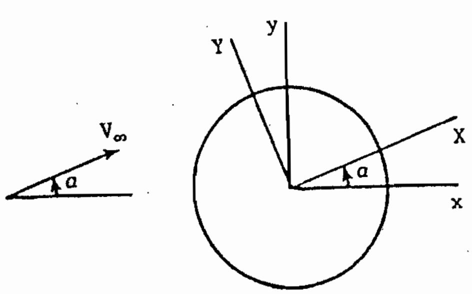

Fig. 4.2 Flow past a circular cylinder at an angle of attack.

Let Z be the complex variable in the XY-plane. In the Z-plane the flow past the circular cylinder is given by Equation 4.25 as:

$$w(Z) = |V_\infty|\left(Z + \frac{a^2}{Z}\right)$$

Now, since:

$$z = Ze^{i\alpha}$$

we have:

$$w(z) = |V_\infty|\left(ze^{-i\alpha} + \frac{a^2e^{i\alpha}}{z}\right)$$

or:

$$w(z) = \overline{V}_\infty z + \frac{a^2V_\infty}{z} \tag{4.31}$$

is the required potential flow in the z-plane.

(vii) Flow with Circulation. If in addition to the uniform flow parallel to the x-axis past a circular cylinder we superimpose on it the flow due to a vortex filament which lies along the axis of the cylinder, then we get the flow past a circular cylinder with circulation. The complex potential for this case is

$$w(z) = u_\infty\left(z + \frac{a^2}{z}\right) - \frac{\Gamma}{2\pi i}\ell nz \tag{4.32}$$

where Γ is the strength of the vortex filament. The complex conjugate velocity is then:

$$\overline{V} = u_\infty\left(1 - \frac{a^2}{z^2}\right) - \frac{\Gamma}{2\pi iz} \tag{4.33}$$

To obtain the position of the stagnation points we set Equation 4.33 to zero. The roots of the quadratic equation are

$$z = \frac{-i\Gamma}{4\pi u_\infty} \pm \left(a^2 - \frac{\Gamma^2}{16\pi^2u_\infty^2}\right)^{1/2} \tag{4.34}$$

Therefore, the stagnation points depend both on the magnitude and sign of the circulation Γ.

The stream function for this case is now given by:

$$\psi = u_\infty\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ell n(r) \tag{4.35}$$

Because of the circulation Γ the flow pattern is not symmetrical with respect to the x-axis.

Blasius Formulae for Force and Moment

Let a body with a definite contour C lie completely immersed in an inviscid fluid flow as shown in Figure 4.3.

With θ defined in Figure 4.3, the components of the unit tangent vector **t** and of the unit external normal **n** are, ($\mathbf{n} = \mathbf{t} \times \mathbf{k}$):

$$t_x = \cos\theta, \qquad t_y = \sin\theta$$

$$n_x = \sin\theta, \qquad n_y = -\cos\theta \tag{4.36a}$$

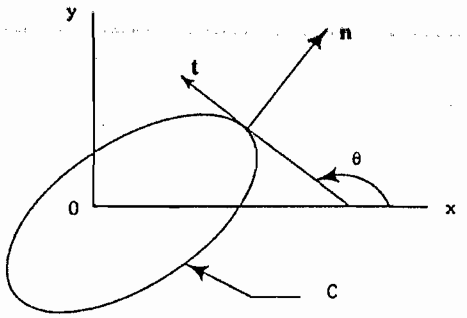

Fig. 4.3 Force and moment on a contour C in an inviscid flow.

Thus:

$$dz = e^{i\theta}\, ds \tag{4.36b}$$

In the absence of viscosity the stress vector $\boldsymbol{\tau}$ is simply:

$$\boldsymbol{\tau} = -p\mathbf{n}$$

so that the force exerted by the fluid on the contour C is

$$\mathbf{F} = -\oint_C p\mathbf{n}\, ds \tag{4.37}$$

where ds is the element of arc length on C. Now, from the Bernoulli equation:

$$p + \tfrac{1}{2}\rho|V|^2 = \text{constant}$$

hence, Equation 4.37 becomes:

$$\mathbf{F} = \tfrac{1}{2}\rho \oint_C |V|^2\, \mathbf{n}\, ds \tag{4.38}$$

where for a closed contour:

$$\oint_C \mathbf{n}\, ds = 0$$

Writing the components of Equation 4.38 and forming the complex force function, we have:

$$F_x - i\, F_y = \tfrac{1}{2}\, i\rho \oint_C |V|^2 e^{-i\theta}\, ds \tag{4.39}$$

Equation 4.39 shows that we need $|V|$ on the contour C. Since the body is at rest, the fluid velocity on C is tangential to the contour. Now on C:

$$V = |V| e^{i\theta}$$

so that:

$$\overline{V} = |V| e^{-i\theta}$$

and:

$$|V|^2 = \overline{V}^2 e^{2i\theta}$$

Thus, Equation 4.39 on using Equation 4.36b becomes:

$$\begin{aligned} F_x - i\,F_y &= \tfrac{1}{2}\, i\rho \oint_C \overline{V}^2\, dz \\ &= \tfrac{1}{2}\, i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz \end{aligned} \tag{4.40}$$

which is the famous *Blasius* formula for the force acting on a two-dimensional body in an inviscid flow.

The moment of pressure forces about the origin 0 is

$$\begin{aligned} \mathbf{k}M &= \oint_C (\mathbf{r} \times \boldsymbol{\tau})\, ds \\ &= \oint_C (\mathbf{n} \times \mathbf{r}) p\, ds \end{aligned}$$

Thus:

$$\begin{aligned} M &= \oint_C (y n_x - x n_y) p\, ds \\ &= \tfrac{1}{2}\rho\, \mathrm{Real}\left[\oint_C \overline{V}^2\, z\, dz\right] \\ &= \tfrac{1}{2}\rho\, \mathrm{Real}\left[\oint_C \left(\frac{dw}{dz}\right)^2 z\, dz\right] \end{aligned} \tag{4.41}$$

Formulae 4.40 and 4.41 are the *Blasius* formulae for the force and moment.

If there are no singularities in the flow field exterior to the contour C, then $\overline{V}$ is an analytic function of z. It can therefore be expanded as a Laurent series which satisfies the conditions at infinity. This expansion is

$$\overline{V} = \frac{dw}{dz} = \overline{V}_\infty + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \tag{4.42}$$

where a_1, a_2, ... are the complex constants to be determined. The coefficient a_1 is determined by direct integration. This gives:

$$\begin{aligned} \oint_C \frac{dw}{dz}\, dz &= -\Gamma + iQ \\ &= 2\pi i a_1 \end{aligned}$$

However, $Q = 0$ for a fixed closed contour; hence:

$$\Gamma = -2\pi i a_1 \tag{4.43}$$

To find the force acting on the contour C, we substitute Equation 4.42 in Equation 4.40 and by direct integration we get:

$$F_x - iF_y = -i\rho\overline{V}_\infty\Gamma$$

Therefore, if the flow at infinity is parallel to the x-axis and of magnitude u_∞, then:

$$F_x = 0, \qquad F_y = +\rho u_\infty \Gamma \tag{4.44}$$

To find the moment M, we substitute Equation 4.42 in Equation 4.41 and find by integration that:

$$M = -\pi\rho\ \mathrm{Real}[i(a_1^2 + 2a_2\overline{V}_\infty] \tag{4.45}$$

where the value of a_2 depends on the shape of the body.

Thus in any problem of the plane potential flow past a body, the resultant force and moment are given by the first three terms in a Laurent expansion of the analytic function $\overline{V}$ of the complex variable z. Formula 4.44 shows that in the absence of circulation there is no resultant force acting on the body. This result is known as *d'Alembert's paradox*.

4.4. METHOD OF CONFORMAL MAPPING IN INVISCID FLOWS

Until now the method of complex variables has been used only in the construction of simple potential functions $w(z)$ for sources, sinks, and doublets. A major use of these solutions has been in the construction of potential functions for the inviscid flow past a circular cylinder with and without an angle of attack. The real use of the method of complex variables lies in the technique of conformal transformation through which arbitrary shaped contours in the physical plane are transformed to the circular boundaries in the transformed plane. The transformation from the z- to the ζ-plane:

$$z = f(\zeta) \tag{4.46}$$

where $z = x + iy$ and $\zeta = \xi + i\eta$ cannot be conformal at points where either $f'(\zeta) = 0$ or ∞; and these points are called the critical and singular points, respectively. Let $w(z)$ be the potential function for a profile in the z-plane. Denoting the transformed functions in the ζ-plane by a subscript asterisk (*) we have on using Equation 4.46:

$$w(z) = w[f(\zeta)] = w^*(\zeta)$$

Also:

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \frac{dw^*}{d\zeta}\frac{1}{f'(\zeta)} \tag{4.47}$$

Thus:

$$\overline{V}^* = \overline{V}f'(\zeta) \tag{4.48}$$

In problems of *external flow* past wing profiles and other shapes the transformation, besides being conformal, should satisfy the following conditions:

1. The transformation function should be such that points at infinity in the z-plane should transform to points at infinity in the ζ-plane.
2. The direction of velocity at infinity in the transformed plane (i.e., ζ-plane) should be the same as the direction of velocity at infinity in the physical plane (i.e., z-plane).

Applying condition 2. to Equation 4.48, we have the result that in:

$$\bar{V}^*_\infty = \bar{V}_\infty f'(\zeta_\infty) \tag{4.49}$$

$f'(\zeta_\infty)$ should be a real number. Another result of much importance is obtained by considering the contour integral:

$$\oint_C \frac{dw}{dz}\, dz = -\Gamma + iQ$$

Using Equation 4.47 we have:

$$\oint_C \frac{dw}{dz}\, dz = \oint_C \frac{dw^*}{d\zeta} \frac{1}{f'(\zeta)}\, dz$$
$$= \oint_{C^*} \frac{dw^*}{d\zeta}\, d\zeta$$

Thus:

$$\oint_C \bar{V}\, dz = \oint_{C^*} \bar{V}^*\, d\zeta$$

so that:

$$\Gamma = \Gamma^*, \qquad Q = Q^* \tag{4.50}$$

which means that the circulation and flux along the contours in the physical and transformed planes are the same.

The whole methodology on the use of conformal transformation in inviscid external flow problems is now summarized as follows. Referring to Figure 4.4, let a profile C be exposed to an inviscid stream inclined at an angle α.

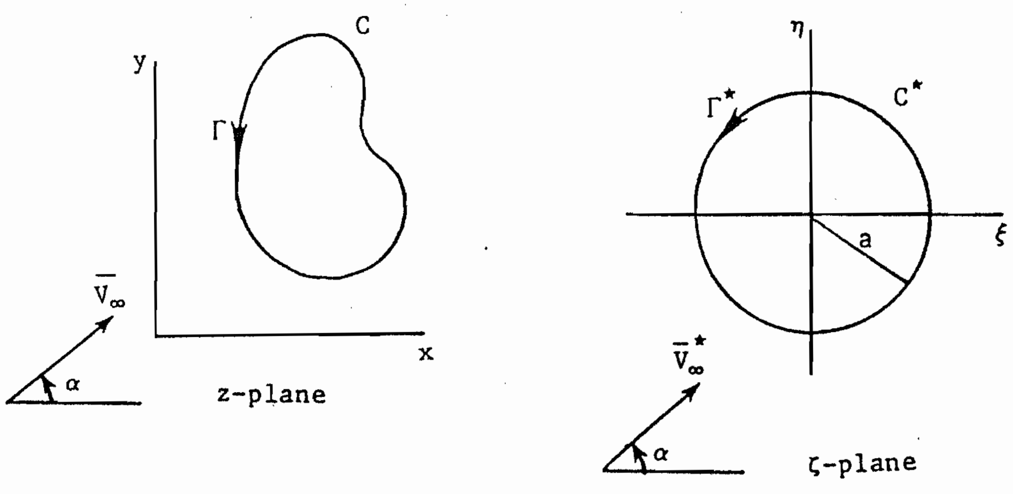

Fig. 4.4 The physical z-plane and the transformed ζ- plane.

Let Γ and Γ^* be the circulations on C and on C^*, respectively. Of course, $\Gamma^* = \Gamma$. Let the circular boundary C^* be the result of a conformal transformation:

$$z = f(\zeta)$$

applied to C. From Equation 4.31, the potential of the flow past the circle of radius a with circulation Γ^* is given by:

$$w^*(\zeta) = \bar{V}_\infty^*\zeta + \frac{V_\infty^* a^2}{\zeta} - \frac{\Gamma^*}{2\pi i}\,\ell n\zeta$$

Using Equations 4.49 and 4.50, we have:

$$w^*(\zeta) = \left(\bar{V}_\infty\zeta + \frac{V_\infty a^2}{\zeta}\right)f'(\zeta_\infty) - \frac{\Gamma}{2\pi i}\,\ell n\zeta \tag{4.51}$$

Therefore, if the conformal transformation $z = f(\zeta)$ is known, then from Equation 4.51 the potential function $w(z)$ can be obtained.

Kutta-Joukowskii's Transformation

The transformation:

$$z = \zeta + \frac{c^2}{\zeta} \tag{4.52}$$

where c is a constant is called the Kutta-Joukowskii transformation. In this case:

$$f'(\zeta_\infty) = 1 \tag{4.53}$$

First, let us consider a circle in the ζ-plane of radius c on which:

$$\zeta = ce^{i\theta}$$

Then Equation 4.52 becomes

$$z = c(e^{i\theta} + e^{-i\theta}) = 2c\cos\theta$$

consequently:

$$x = 2c\cos\theta, \qquad y = 0$$

which represents a segment on the real axis (x-axis) with end points $(-2c, 0)$, $(2c, 0)$. Thus the tranformation of a segment $-2c \leq x \leq 2c$ in the z-plane through the function (Equation 4.52) is a circle of radius c in the ζ-plane. Next we consider another circle, concentric with the basic circle of radius c, which is given by

$$\zeta = ae^{i\theta}, \qquad a > c$$

Using Equation 4.52, we get:

$$x = \left(a + \frac{c^2}{a}\right)\cos\theta$$

$$y = \left(a - \frac{c^2}{a}\right) \sin\theta \tag{4.54}$$

Thus, each circle a = constant in the ζ-plane represents an ellipse in the z-plane whose semimajor and semiminor axes are, respectively:

$$m_1 = a + \frac{c^2}{a}, \qquad m_2 = a - \frac{c^2}{a}$$

Thus:

$$m_1^2 - m_2^2 = 4c^2 \tag{4.55a}$$

$$m_1 + m_2 = 2a \tag{4.55b}$$

The transformation (Equation 4.52) therefore transforms confocal ellipses in the z-plane to the concentric circles in the ζ-plane. Using Equation 4.53, and Equation 4.55b in Equation 4.51, we find that the potential flow past an ellipse in the z-plane is given by:

$$w(z) = \frac{1}{2}\bar{V}_\infty \{z + (z^2 - 4c^2)^{1/2}\} + \frac{1}{8} V_\infty \frac{(m_1 + m_2)^2}{c^2} \{z - (z^2 - 4c^2)^{1/2}\}$$
$$- \frac{\Gamma}{2\pi i} \ell n[\tfrac{1}{2}\{z + (z^2 - 4c^2)^{1/2}\}] \tag{4.56}$$

where a plus sign has been chosen before the square root in:

$$\zeta = \frac{1}{2}[z - (z^2 - 4c^2)^{1/2}]$$

so that the *exterior* of an ellipse is mapped onto the *exterior* of a circle.

Two particular cases of Equation 4.56 are

1. $m_1 = m_2 = a, \qquad c = 0$
2. $m_2 = 0, \qquad m_1 = 2a = 2c$

Case 1 is that of a potential flow for a circle in the z-plane which is exposed to an inclined free-stream flow, see Equation 4.31. In this case note that:

$$\lim_{c \to 0} \frac{z - (z^2 - 4c^2)^{1/2}}{c^2} = \frac{2}{z}$$

Case 2 is that of a potential flow for a finite flat plate of length $4c$. Writing:

$$V_\infty = u_\infty + iv_\infty$$

we get the potential function as:

$$w(z) = u_\infty z - iv_\infty(z^2 - 4c^2)^{1/2} - \frac{\Gamma}{2\pi i} \ell n[\tfrac{1}{2}\{z + (z^2 - 4c^2)^{1/2}\}]$$

The conjugate velocity for the flow past a finite flat plate is

$$\overline{V}(z) = u_\infty - \left(iv_\infty z + \frac{\Gamma}{2\pi i}\right)(z^2 - 4c^2)^{-1/2} \tag{4.57}$$

Thus the velocity is singular both at $z = 2c$ and at $z = -2c$. However, if the value of Γ is chosen to be:

$$\Gamma = +4\pi c v_\infty \tag{4.58}$$

then the velocity will not be singular at $z = 2c$. To see the behavior at $z = -2c$, we substitute Equation 4.58 in Equation 4.57 and after simplification, we get:

$$\overline{V}(z) = u_\infty - iv_\infty\left(\frac{z - 2c}{z + 2c}\right)^{1/2} \tag{4.59}$$

which shows that the flow is still singular at $z = -2c$. Thus singularities at both the leading and trailing edges cannot be removed simultaneously.

Pure Circulatory Motion Around a Plate

If $u_\infty = v_\infty = 0$, then from Equation 4.57:

$$\overline{V}(z) = \frac{-\Gamma}{2\pi i}(z^2 - 4c^2)^{-1/2}$$

For $y = 0$:

$$U = \frac{\pm\Gamma}{2\pi}(4c^2 - x^2)^{-1/2} \tag{4.60}$$

where $z = x$ on a flat plate and $-2c \leqslant x \leqslant 2c$. The velocity distribution (Equation 4.60) is a consequence of pure circulatory motion around the plate. If $\Gamma > 0$ (i.e., clockwise), then on the upper surface of the plate $U = U_u > 0$, while on the lower surface $U = U_\ell < 0$. These two velocities are different in sign and their difference is

$$U_u - U_\ell = \frac{\Gamma}{\pi}(4c^2 - x^2)^{-1/2} \tag{4.61}$$

Thus, the velocity is discontinuous in traversing from the lower to the upper side.

The preceding formula is equally applicable if we consider the segment $-2c \leqslant x \leqslant 2c$ as a layer of fluid rather than a plate. This layer of fluid is a line of discontinuity in fluid motion and is termed a *vortex sheet*; it is as if the whole segment is continuously dotted with point vortices. The strength of the vortex sheet is given by Equation 4.61 itself, and the total strength is obtained by integrating Equation 4.61 from $-2c$ to $2c$ which gives Γ.

Flow Past a Wing Profile

The theory of plane irrotational flow of an inviscid fluid has been used to its maximum potential in the design of two-dimensional airfoil sections. Consider a profile with a *sharp* (not rounded) trailing edge in an incompressible flow as shown in Figure 4.5(a). Let the incident stream of magnitude $|V_\infty|$ be inclined with the x-axis at an angle α. The contour of the profile is denoted as C in the z-plane.

Let us *assume* that a conformal transformation:

$$z = f(\zeta), \qquad \zeta = \xi + i\eta$$

exists which transforms the profile C in the z-plane into a circle C^* in the ζ-plane, as shown in Figure 4.5(b).

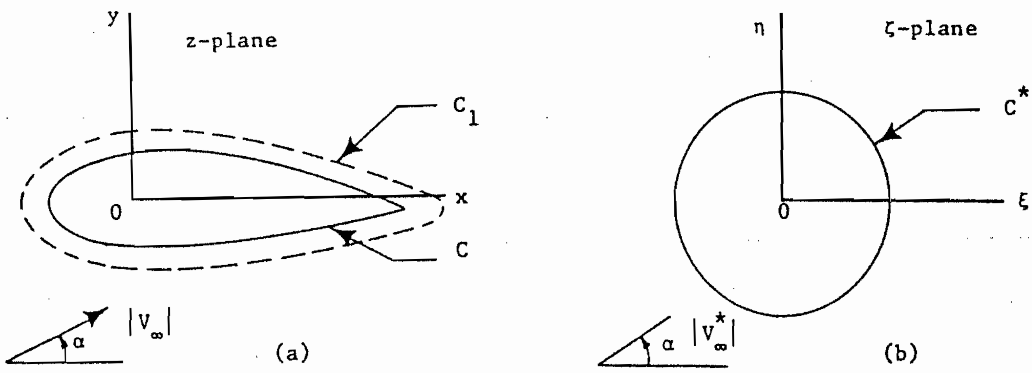

Fig. 4.5 Transformation of an airfoil to a circle.

We shall denote all quantities in the ζ-plane by a superscript asterisk. The important conditions to be satisfied by the desired transformation have already been discussed earlier and listed as Equations 4.49, 4.50, and 4.51.

The complex potential (Equation 4.51) for an airfoil with a *sharp trailing edge* is not unique because it depends on an unspecified circulation Γ. Thus an infinite number of solutions for the same airfoil are possible although physically only one solution should exist. The non-uniqueness of the solution is due to the neglect of viscosity; and no matter how small the viscosity of the fluid is, it has a very important influence near the surface of the airfoil. The amount of circulation we are looking for is not arbitrary but is determined by the action of viscosity which has been completely neglected in inviscid potential flows.

In the early part of the 20th century, Kutta in Germany and *Joukowski* and Chaplygin in Russia were concerned with the problem of determining the amount of circulation which actually exists in a practical situation for an airfoil with a *sharp* trailing edge. The following postulate known as the Kutta-*Joukowski* condition, or simply as the Kutta condition, was proposed on an experimental basis.

Kutta Conditon. "Out of the infinite number of possible inviscid potential flows past an airfoil with a sharp trailing edge, the flow that physically occurs is the one in which there is no velocity discontinuity at the trailing edge." This implies that there is a smooth flow with finite velocity past the trailing edge.

We utilize the Kutta conditon to determine the right amount of circulation for an airfoil with a sharp trailing edge which includes a small angle δ. Referring to Figure 4.6, let the trailing edge T of the airfoil correspond to a point T^* on the circle C^* in the ζ-plane.

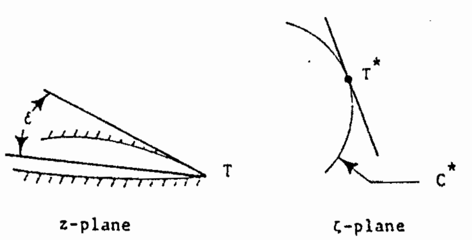

Fig. 4.6 Transformation of an airfoil trailing edge from the z-plane to the ζ-plane.

Consider the conformal mapping of the small region exterior to point T in the z-plane onto the region exterior to the point T^* of the ζ-plane. The angle exterior to the trailing edge is $2\pi - \delta$ so that the conformal mapping is

$$z - z_T = \kappa(\zeta - \zeta_{T^*})^{(2\pi-\delta)/\pi} \tag{4.62}$$

where κ is a real constant. Since the angle at T is $(2\pi - \delta)$ while at T^* the angle is π, the transformation (Equation 4.62) is not conformal right at the trailing edge but is conformal in the region exterior to T. Now:

$$\begin{aligned}
\overline{V}_{T^*} &= \left(\frac{dw^*}{d\zeta}\right)_{\zeta=\zeta_{T^*}} \\
&= \left(\frac{dw}{dz}\right)_{z=z_T} \cdot \left(\frac{dz}{d\zeta}\right)_{\zeta=\zeta_{T^*}} \\
&= \overline{V}_T\kappa\left(\frac{2\pi-\delta}{\pi}\right)[(\zeta-\zeta_{T^*})^{(\pi-\delta)/\pi}]_{\zeta=\zeta_{T^*}}
\end{aligned} \tag{4.63}$$

However, $\delta < \pi$ is an acute angle so that the right-hand side of Equation 4.63 is zero. Since according to Kutta's postulate $\overline{V}_T$ is finite, then:

$$\overline{V}_{T^*} = 0 \tag{4.64}$$

That is, the point on the circle in the ζ-plane which corresponds to the trailing edge T in the z-plane is a stagnation point. We now use Equation 4.64 to calculate the right amount of circulation for an airfoil with a sharp trailing edge. On differentiating the complex potential (Equation 4.51), using Equation 4.64, and setting $\zeta = \zeta_{T^*}$ we get:

$$\overline{V}_{T^*} = \left(\overline{V}_\infty - \frac{V_\infty a^2}{\zeta_{T^*}^2}\right) f'(\zeta_\infty) - \frac{\Gamma}{2\pi i\zeta_{T^*}} = 0$$

so that

$$\Gamma = -2\pi i f'(\zeta_\infty)\left[\frac{V_\infty a^2}{\zeta_{T^*}} - \overline{V}_\infty \zeta_{T^*}\right] \tag{4.65}$$

On the circle in the transformed plane the point ζ_{T^*} can be represented as:

$$\zeta_{T^*} = ae^{i\lambda}$$

where λ is the polar angle measured from a reference axis in the ζ-plane. Further, the free-stream velocity is

$$V_\infty = |V_\infty|e^{i\alpha}$$

Thus, the circulation Γ from Equation 4.65 is given by:

$$\Gamma = 4\pi a|V_\infty| f'(\zeta_\infty)\sin(\alpha - \lambda) \tag{4.66}$$

Note that $f'(\zeta_\infty)$, λ, and a — which are all real can only be obtained from the transformation $z = f(\zeta)$ — which is a separate problem. Using Equation 4.51, we now obtain the conjugate velocity $\overline{V}(z)$ in the z-plane which on substituting Γ from Equation 4.66 takes the form:

$$\bar{V} = \frac{dw}{dz}$$

$$= f'(\zeta_\infty)\left[\bar{V}_\infty - \frac{a^2 V_\infty}{\zeta^2} + \frac{2ia}{\zeta}|V_\infty|\sin(\alpha - \lambda)\right] \Big/ \frac{dz}{d\zeta} \tag{4.67}$$

To find $\bar{V}_T$ we again use the transformation (Equation 4.62), and on taking the limit as $z \to z_T$ and $\zeta \to \zeta_T$, we find that:

$$\lim_{z \to z_T} \bar{V} = \bar{V}_T = 0$$

which shows that the trailing edge with $\delta \neq 0$ is a *stagnation point.* If the trailing edge is cusped (i.e., $\delta = 0$), then it is not a stagnation point. The angle $(\alpha - \lambda)$ is called the theoretical angle of attack for an airfoil. Measurements have shown that the calculated value of $(\alpha - \lambda)$ is slightly greater than the measured value. It must be noted that $\Gamma = 0$ when $\alpha - \lambda$. If the trailing edge is rounded rather than sharp, then neither the Kutta condition nor the calculated value of Γ in Equation 4.66 are applicable and the determination of circulation remains an open problem.

An Iterative Method for the Numerical Generation of $z = f(\zeta)$

After the publication of Theodorsen's paper in 1931, there have been a number of potential flow solutions available for different airfoil sections. A complete work on airfoil sections is available in Reference 13. Here we consider a method, developed originally in Russia and reported in Reference 6, which is very suitable for the construction of conformal transformations for arbitrary shaped bodies.

The function $z = f(\zeta)$, which conformally maps the region of the z-plane exterior to a specified profile C onto the region of the ζ-plane exterior to the circle C^* of radius a, can be represented as a series:

$$z = \zeta + p_0 + iq_0 + \sum_{n=1}^{\infty} (p_n + iq_n)\left(\frac{a}{\zeta}\right)^n \tag{4.68}$$

For points on the circumference of the circle C^*:

$$\zeta = ae^{i\theta}, \qquad 0 \leq \theta \leq 2\pi \tag{4.69}$$

where θ is measured counterclockwise from the ξ-axis.

The main problem now is the computation of the coefficients $p_0, q_0, \ldots, p_n$, and q_n. To develop a method for the evaluation of these coefficients, we substitute Equation 4.69 in Equation 4.68 and equate the real and imaginary parts. Thus:

$$x(\theta) = p_0 + (p_1 + a)\cos\theta + q_1 \sin\theta + \sum_{n=2}^{\infty} (p_n \cos n\theta + q_n \sin n\theta)$$

$$y(\theta) = q_0 + (a - p_1)\sin\theta + q_1 \cos\theta + \sum_{n=2}^{\infty} (q_n \cos n\theta - p_n \sin n\theta) \tag{4.70}$$

Although the coordinates (x, y) for the airfoil contour are known either in a tabular or in a functional form, the functions $x(\theta)$ and $y(\theta)$ are unknown. Thus an iterative method has to be used to compute $x(\theta)$, $y(\theta)$ along with the coefficients appearing in Equation 4.70.

First, by using the orthogonality conditions for the trigonometrical functions, we have:

$$p_0 = \frac{1}{2\pi}\int_0^{2\pi} x(\theta)\, d\theta$$

$$p_1 + a = \frac{1}{\pi}\int_0^{2\pi} x(\theta)\cos\theta\, d\theta$$

$$a - p_1 = \frac{1}{\pi}\int_0^{2\pi} y(\theta)\ \sin\theta\, d\theta$$

$$p_n = -\frac{1}{\pi}\int_0^{2\pi} y(\theta)\sin n\theta\, d\theta, \qquad n > 1$$

$$q_n = \frac{1}{\pi}\int_0^{2\pi} y(\theta)\cos n\theta\, d\theta, \qquad n \geq 0$$

Thus:

$$a = \frac{1}{2\pi}\int_0^{2\pi} [x(\theta)\cos\theta + y(\theta)\sin\theta]\, d\theta$$

$$p_1 = \frac{1}{2\pi}\int_0^{2\pi} [x(\theta)\cos\theta - y(\theta)\sin\theta]\, d\theta$$

We now choose the zeroth approximation for $x(\theta)$ as:

$$x^{(0)}(\theta) = \alpha + \beta\cos\theta$$

where α and β are arbitrary. From the expressions for p_0 and $p_1 + a$, we then have:

$$p_0^{(0)} = \alpha, \qquad p_0^{(0)} + a^{(0)} = \beta$$

Corresponding to the abscissae $x^{(0)}$ the ordinates $y^{(1)}$ are selected from the given tabular or functional forms, and then the coefficients:

$$a^{(1)} - p_1^{(1)}, \qquad p_n^{(1)}, \qquad q_n^{(1)}$$

are computed from the given integral relations. Based on these coefficients the abscissae and then the ordinates are calculated and the process is continued. For example, in the m-th iteration:

$$x^{(m-1)}(\theta) = \alpha + \beta\cos\theta + q_1^{(m-1)}\sin\theta + \sum_{n=2}^{\infty}(p_n^{(m-1)}\cos n\theta + q_n^{(m-1)}\sin n\theta)$$

$$p_1^{(m)} + a^{(m)} = \beta$$

$$a^{(m)} - p_1^{(m)} = \frac{1}{\pi}\int_0^{2\pi} y^{(m)}(\theta)\sin\theta\, d\theta$$

from which:

$$a^{(m)} = \frac{\beta}{2} + \frac{1}{2\pi}\int_0^{2\pi} y^{(m)}(\theta)\sin\theta\, d\theta$$

The preceding iterative method is simple to execute on a computer. The only extra calculation subroutine to be developed is that of interpolation so that in each sweep new values of the ordinates corresponding to the calculated values of abscissae become immediately available. The method converges quite rapidly.

4.5. SOURCES, SINKS, AND DOUBLETS IN THREE DIMENSIONS

The definition of irrotational flows given in Section 4.3 is equally applicable to both the two- and three-dimensional flows. Again, the Laplace equation (Equation 4.8) has to be solved under some boundary conditions so as to form a properly posed problem. Let us consider the problem of external flow of an incompressible inviscid fluid past a body. Two obvious cases are

1. Body at rest in a fluid stream of uniform velocity at infinity
2. Body moving in a stationary fluid of infinite extent

For case 1:

$$\mathbf{u} \cdot \mathbf{n}|_S = 0$$

However:

$$\mathbf{u} = \text{grad } \phi$$

hence:

$$\left.\frac{\partial \phi}{\partial n}\right|_S = 0 \tag{4.71a}$$

At infinity:

$$\text{grad } \phi \rightarrow \mathbf{V}_\infty \tag{4.71b}$$

The condition (Equation 4.71a) in the case of Cartesian coordinates (x, y, z) implies that:

$$\frac{\partial \phi}{\partial x} = V_\infty \cos(\mathbf{V}_\infty, x), \qquad \frac{\partial \phi}{\partial y} = V_\infty \cos(\mathbf{V}_\infty, y), \qquad \frac{\partial \phi}{\partial z} = V_\infty \cos(\mathbf{V}_\infty, z)$$

as:

$$r = (x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$$

For case 2:

$$\mathbf{u} \cdot \mathbf{n}|_s = V_n$$

or:

$$\left.\frac{\partial \phi}{\partial n}\right|_S = V_n \tag{4.72a}$$

and:

$$\text{grad } \phi \rightarrow 0 \text{ at infinity} \tag{4.72b}$$

which in the Cartesian coordinates imply that:

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z} \to 0 \quad \text{as} \quad r \to \infty$$

Sources and Sinks in Three Dimensions

If from a point P in space fluid flows out or goes in radially, then we call point P a source or a sink, respectively. Surrounding point P by a sphere of arbitrary radius with P at the center and denoting the radial component of velocity by u_r, we find that the quantity:

$$4\pi r^2 u_r = Q$$

is an invariant. The invariancy of Q is obviously due to the conservation of mass. The quantity Q is called the strength of the source or sink, and:

$$u_r = \frac{Q}{4\pi r^2} \tag{4.73}$$

If $Q > 0$, then it is a source; if $Q < 0$, then it is a sink. The physical components of grad ϕ in terms of the polar coordinates (r, θ, ω) are (here we use the azimuthal angle as ω):

$$u_r = \frac{\partial \phi}{\partial r}, \qquad u_\theta = \frac{\partial \phi}{r \partial \theta}, \qquad u_\omega = \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \omega}$$

Because of symmetry both u_θ and u_ω are zero, and we have:

$$\frac{\partial \phi}{\partial r} = \frac{Q}{4\pi r^2}$$

which on integration yields:

$$\phi = \frac{-Q}{4\pi r} \tag{4.74}$$

as the potential function for a source or sink. It is shown in ME.5, that in three dimensions $1/r$ is a fundamental solution of the Laplace equation provided that $r \neq 0$. Thus, the ϕ in Equation 4.74 satisfies the Laplace equation (Equation 4.8). From Equation 4.74, we find that a source or sink of strength Q at $r = 0$ produces a velocity potential ϕ in the surrounding space. In two dimensions (refer to Equation 4.18a) we have already studied that the potential of a source at $r = 0$ is given by:

$$\phi = \frac{Q}{2\pi} \ell n r = \frac{-Q}{2\pi} \ell n \left(\frac{1}{r}\right) \tag{4.75}$$

Let V be an arbitrary volume in space in which there is a continuous distribution of sources. Let the density of sources or the source strength per unit volume be denoted as ρ. Then $\rho\, d\nu$ is the strength of sources in a volume $d\nu$. Each source of strength $\rho\, d\nu$ at the point (ξ, η, ζ) produces a potential at (x, y, z) given by:

$$\Delta\phi = \frac{-\rho(\xi, \eta, \zeta)\, d\nu\, (\xi)}{4\pi r} \tag{4.76}$$

where:

$$r = [(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{1/2}$$

However, since the Laplace equation is linear, then the solution at a point (x, y, z) is the limit sum of the contributions of the form (Equation 4.76). Thus the total potential from all the sources in V is given by:

$$\phi = -\frac{1}{4\pi}\int_V \frac{\rho(\xi, \eta, \zeta)\, dv\, (\boldsymbol{\xi})}{r} \tag{4.77}$$

where dv depends on the variable point $(\xi, \eta, \zeta) = \boldsymbol{\xi}$. Comparing Equation 4.77 with Equation M5.37 we find that ϕ satisfies the Poisson equation:

$$\nabla^2\phi = \rho(\boldsymbol{\xi})$$

within the volume V. Outside of V, ϕ satisfies the Laplace equation:

$$\nabla^2\phi = 0$$

Doublets in Three Dimensions

Let PP' be an arbitrary direction in space as shown in Figure 4.7. Let us place a source of strength Q at P and a sink of equal strength at P', where $\delta s = PP'$ is a small distance.

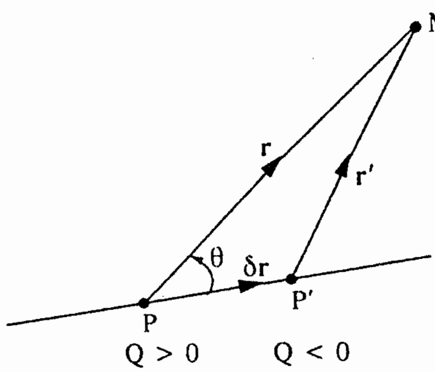

Fig. 4.7 Combination of a source and an equal sink in three dimensions.

Our purpose is to find the potential at point M in the limit when the distance $\delta s \rightarrow 0$ while $Q \rightarrow \infty$. The resulting combination is then called a *doublet* or *dipole* in three dimensions.

According to the construction in Figure 4.7 and the linearity of Equation 4.8, the potential at M is

$$\phi = \frac{Q}{4\pi r'} - \frac{Q}{4\pi r}$$

or:

$$\phi = \frac{Q(\delta s)}{4\pi}\frac{(r - r')}{rr'(\delta s)}$$

Writing:

$$m = \lim_{\substack{\delta s \to 0 \\ Q \to \infty}} Q(\delta s)$$

using the trigonometric formula:

$$r'^2 = r^2 + (\delta s)^2 - 2r(\delta s) \cos \theta$$

and taking the limit as $\delta s \to 0$, we get:

$$\phi = \frac{m}{4\pi} \frac{\cos \theta}{r^2} \tag{4.78a}$$

which is the potential due to a doublet of strength or moment m. Notice that if the source is placed at P' and the sink at P, then the sense of m is reversed. We can also define:

$$\boldsymbol{\mu} = \lim_{\substack{\delta r \to 0 \\ Q \to \infty}} Q\,\delta \mathbf{r}, \qquad |\boldsymbol{\mu}| = m$$

and then Equation 4.78a will be written as:

$$\phi = \frac{\boldsymbol{\mu} \cdot \mathbf{r}}{4\pi r^3} \tag{4.78b}$$

Induced Velocities Due to Line and Sheet Vortices

The definitions of line and sheet vortices have been given in Chapter 1. Here we wish to develop the necessary formulae for the induced velocities due to line and sheet vortices.

The basic equation for the calculation of the velocity distribution in space due to a specified distribution of vorticity $\boldsymbol{\omega}$ is Equation M5.43. This equation states that for a solenoidal velocity field $\mathbf{u}_s$, i.e.:

$$\text{div } \mathbf{u}_s = 0$$

and a given vorticity field:

$$\boldsymbol{\omega} = \text{curl } \mathbf{u}_s$$

the velocity at any space point $\mathbf{x}$ is given by:

$$\mathbf{u}_s(\mathbf{x}) = \frac{-1}{4\pi} \int_V \frac{(\mathbf{x} - \boldsymbol{\xi}) \times \boldsymbol{\omega}(\boldsymbol{\xi})}{r^3}\, dv(\boldsymbol{\xi}) \tag{4.79}$$

where $\boldsymbol{\xi}$ is the variable of integration, and:

$$r = |\mathbf{x} - \boldsymbol{\xi}|$$

We first consider a single line vortex, on which let ℓ be a unit tangent vector. We consider a small cylindrical volume δv of cross-sectional area $\delta\sigma$ which encloses a length $\delta\ell$ of the line vortex. Then:

$$\delta v = (\delta\ell)(\delta\sigma)$$

Thus, the elemental induced velocity, from Equation 4.79, is

$$\delta\mathbf{u}_s = -\frac{1}{4\pi}\frac{(\mathbf{x}-\boldsymbol{\xi})\times\boldsymbol{\omega}(\boldsymbol{\xi})}{r^3}(\delta\ell)(\delta\sigma) \qquad \textbf{(4.80)}$$

Writing $\boldsymbol{\omega} = \omega\boldsymbol{\ell}$ in Equation 4.80 and noting that:

$$\Gamma = -\omega(\delta\sigma)$$

is the strength of the line vortex, we obtain:

$$\delta\mathbf{u}_s = \frac{-\Gamma}{4\pi}\frac{\boldsymbol{\ell}\times(\mathbf{x}-\boldsymbol{\xi})}{r^3}\delta\ell$$

Consequently, the induced velocity is obtained as a line integral along the line vortex as:

$$\mathbf{u}_s(\mathbf{x}) = \frac{-\Gamma}{4\pi}\oint\frac{\boldsymbol{\ell}\times(\mathbf{x}-\boldsymbol{\xi})}{r^3}\,d\ell$$

As an example, let the line vortex be a straight line extending from $-\infty$ to $+\infty$ in the z-direction. Then $\boldsymbol{\ell} = \mathbf{k}$, $d\ell = d\zeta$, and the formula for the induced velocity is

$$\mathbf{u}_s(\mathbf{x}) = \frac{-\Gamma\boldsymbol{\lambda}}{4\pi}\int_{-\infty}^{\infty}\frac{d\zeta}{[\lambda^2 + (z-\zeta)^2]^{3/2}} = \frac{-\Gamma\boldsymbol{\lambda}}{2\pi\lambda^2} \qquad \textbf{(4.81a)}$$

where:

$$\boldsymbol{\lambda} = (x-\xi)\mathbf{j} - (y-\eta)\mathbf{i}$$
$$\lambda = |\boldsymbol{\lambda}|$$

Another formula for the induced velocity due to a closed line vortex (Reference 4, p. 95) is

$$\mathbf{u}_s(\mathbf{x}) = +\frac{\Gamma}{4\pi}\,\mathrm{grad}\,\Omega(\mathbf{x}) \qquad \textbf{(4.81b)}$$

where:

$$\Omega(\mathbf{x}) = \int\frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{n}}{r^3}\,dS(\xi)$$

and $\mathbf{n}$ is the positive unit normal vector on any open surface bounded by the line vortex.

We now consider a vortex sheet which is a surface on which the vorticity is quite large. Thus, there is a surface concentration of vorticity at every point of the surface of a vortex sheet. A mathematical idealization of such a surface is a surface across which the tangential component of velocity is discontinuous. Denoting the two sides of the surface as $-$ and $+$ and the unit normal to the surface directed from side $-$ to side $+$ by $\mathbf{n}$, we have:

$$(\mathbf{u}^- - \mathbf{u}^+)\cdot\mathbf{n} = 0$$

and:

$$(\mathbf{u}^- - \mathbf{u}^+)\cdot\mathbf{t} \neq 0$$

where $\mathbf{t}$ is a surface tangent vector. From these equations we conclude that:

$$\mathbf{u}^- - \mathbf{u}^+ = \boldsymbol{\omega}\delta n$$
$$= \boldsymbol{\Gamma}$$

where δn is an elemental distance along $\mathbf{n}$. The vector $\boldsymbol{\Gamma}$ defined above has the dimension of velocity and is a measure of the local concentration of vorticity. To find the velocity induced by a sheet vortex, we again consider Equation 4.79. Let δS be an elemental surface in the sheet vortex; then:

$$\delta \nu \cong (\delta n)(\delta S)$$

and consequently, from Equation 4.79 we have:

$$\delta \mathbf{u}_s = \frac{-1}{4\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}) \times \boldsymbol{\Gamma}(\boldsymbol{\xi})}{r^3} \delta S$$

Thus the induced velocity is

$$\mathbf{u}_s(\mathbf{x}) = \frac{1}{4\pi} \int_S \frac{\boldsymbol{\Gamma}(\boldsymbol{\xi}) \times (\mathbf{x} - \boldsymbol{\xi})}{r^3} dS(\boldsymbol{\xi}) \tag{4.82}$$

where the integration is to be performed on the whole sheet.

Part II: Inviscid Compressible Flow

4.6 BASIC THERMODYNAMICS

Thermodynamics is concerned with the behavior of substances with physical properties charactertized by certain variables, called the *state variables*. The main aim of thermodynamics is to establish functional relations among the state variables so that if some state variables are known, then the others can be evaluated through these functional relations. Before we proceed further, it is instructive to list a series of simple definitions.

System — Any arbitrary collection of matter which has definable state properties is called a *system*. A system is said to be *closed* when no mass enters or leaves the system, vis-à-vis material volume. A system is said to be *open* if mass is allowed to leave or enter the system, a control volume. Every system has a surrounding, and the changes (in general) either in the system or its surrounding have a mutual effect on each other. If, however, changes occur only within the system without any appreciable effect from or to the surrounding, then the system is termed to be *isolated*.

Process — A single operation or action (or a series a operations or actions), which consequently produces changes in a given state is called a *process*.

Cycle — Suppose a known state 1 has been changed to a new state 2 and is then brought back to the original state 1. The totality of all the processes from 1 to 2 and then from 2 to 1 is then said to form a complete *cycle*.

Equilibrium — The thermodynamic state of a system, viz., its temperature, pressure, density, etc. changes or can be changed in two ways: first, by allowing nonhomogeneities to exist in the state variables within the system and second, by producing changes in the surroundings. For an ideal equilibrium state to exist, it is first of all necessary that the causes which can produce changes within the system are absent. This condition is attainable when at a given time all the state variables assume uniform values throughout the system, e.g., there are no spatial gradients in the state variables. The second requirement for the existence of an equilibrium state is that the changes in the surrounding are of infinitesimal nature so that the system is allowed to assume an equilibrium state without producing intense nonhomogeneities. Some slight variations in any one of the two conditions given above may be called a quasi-equilibrium state. All fluid dynamic problems are of a quasi-equilibrium nature.

Reversible processes — Suppose a thermodynamic system has changed from an initial state 1 to a new state 2. By producing changes in the surrounding, it is possible to reverse the state from 2 to 1. However, this type of reversal generally will be at the expense of a change in the surrounding. On the other hand, a *reversible process* is an ideal process, which having once taken place, can be reversed without producing any change in either the system or the surrounding. In actual practice, this means that the infinitesimal changes have been carried out so slowly that both the system and the surrounding pass through a succession of equilibrium states.

Sign convention for work and heat — In any process, the work done *by* a system and the head *added to* a system are taken as positive, while the work done *on* a system and the heat *taken from* a system are taken as negative.

State variables — The state variables of a thermodynamic system are the: pressure p, absolute temperature T, density ρ or specific volume $v = 1/\rho$, internal energy e per unit mass, and entropy s per unit mass. The last two state variables are defined through the two thermodynamic laws, which are described next.

First Law of Thermodynamics

Let a system having a unit mass of the substance undergo a cyclic process from state 1 to state 2 and then back to state 1, as shown in Figure 4.8. The total amounts of work and heat in this cyclic process per unit mass can be represented as line integrals. The first law of thermodynamics states that for *any* cycle the cyclic integral of the work is proportional to the cyclic integral of the heat. Thus:

$$J_e \oint \delta q = \oint \delta w \tag{4.83}$$

where δq, δw are the infinitesimal amounts of heat and work per unit mass and J_e is a constant, called Joule's constant. The units of work are $ft \cdot lb_f$, erg, or joule (J) while those of heat are Btu, or calories. The value of the Joule constant is

$$\begin{aligned} J_e &= 778.26 \text{ ft} \cdot \text{lb}_f/\text{Btu} \\ &= 4.1857 \text{ J/cal} = 4.1857 \times 10^7 \text{ ergs/cal} \end{aligned}$$

At an international conference held in 1929, it was agreed to regard both heat and work as equivalent forms of energy with common units for measurement. Thus, it was decided to take:

$$1 \text{ Btu} = 778.26 \text{ ft} \cdot \text{lb}_f$$

and:

$$1 \text{ cal} = 4.1857 \text{ J}$$

With this choice $J_e = 1$ and Equation 4.83 takes the form:

$$\oint \delta q = \oint \delta w \tag{4.84}$$

Equation 4.84 has never been found to be in disagreement with any experiment for a cyclic process. Since an integral is the limit of an algebraic sum, the results in Equation 4.84 can be stated as: the algebraic sum of δq and δw in passing from state 1 to state 2 through a curve C_1 and then passing from state 2 to state 1 through a curve C_2 are always equal. Referring to Figure 4.8, this process is known as passing through a complete cycle.

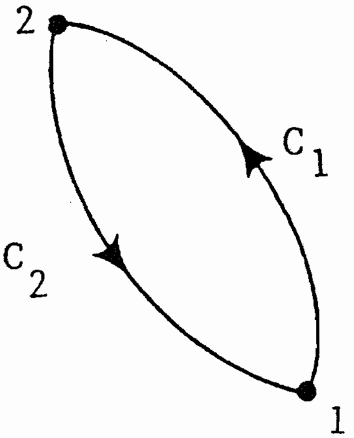

Fig. 4.8 A thermodynamic cycle.

Mathematically $C_1 \cup C_2$ forms a closed curve and the aforementioned experimental evidence is written as:

$$\oint (\delta q - \delta w) = 0 \tag{4.85}$$

From the properties of the line integrals, we find that there exists a point function ϵ, such that:

$$d\epsilon = \delta q - \delta w \tag{4.86}$$

Equation 4.85 shows that although neither δq nor δw is a perfect differential, the quantity $d\epsilon$ is a perfect differential. Here ϵ is the sum of the internal, kinetic, and potential energies. Thus:

$$\epsilon = e + KE + PE$$

where e, KE, and PE are the internal, kinetic, and potential energies respectively, per unit mass. In most cases of interest in this book the differential change:

$$d(KE + PE) \simeq 0$$

so that Equation 4.86 is written in the form:

$$de = \delta q - \delta w \tag{4.87}$$

Second Law of Thermodynamics

The second law of thermodynamics is stated either as a Kelvin-Planck statement or as a *Clausius* statement. An impotant corollary of these statements is the *Clausius inequality* which states that for any possible cycle:

$$\oint \frac{\delta q}{T} \leq 0 \tag{4.88}$$

where δq is the heat transfer per unit mass in a cyclic process, and T is the absolute temperature. For a reversible cycle:

$$\oint \frac{\delta q}{T} = 0 \tag{4.89a}$$

and for an irreversible cycle:

$$\oint \frac{\delta q}{T} < 0 \tag{4.89b}$$

Equation 4.89a enables us to define another state variable, called entropy. Again, using the properties of the line integrals, we find that Equation 4.89a implies the existence of a differentiable point function s such that:

$$ds = \frac{\delta q}{T} \tag{4.90a}$$

More precisely:

$$ds = \frac{(\delta q)_{rev}}{T} \tag{4.90b}$$

where $(\delta q)_{rev}$ is the reversible heat transfer.* The quantity s is called the entropy per unit mass,** and is necessarily a state variable. For a process proceeding from state 1 to state 2, we have from Equation 4.90b:

$$s_2 - s_1 = \int_1^2 \frac{(\delta q)_{rev}}{T} \tag{4.90c}$$

It is important to note that the property called entropy becomes known only through a *reversible* process as the difference between two states. Since it is a state property, the difference $s_2 - s_1$ remains the same whether the path from 1 to 2 is reversible or irreversible. To exploit this conclusion, we add an irreversible path C_3 in Figure 4.8 connecting points 1 and 2. Let the cycle $C_r = C_1 \cup C_2$ be reversible while the cycle $C_{irr} = C_1 \cup C_3$ is irreversible. Then, according to Equations 4.89a, b we have:

$$(C_1) \int_1^2 \frac{\delta q}{T} + (C_2) \int_2^1 \frac{\delta q}{T} = 0$$

and:

$$(C_1) \int_1^2 \frac{\delta q}{T} + (C_3) \int_2^1 \frac{\delta q}{T} < 0$$

Thus:

$$(C_2) \int_2^1 \frac{\delta q}{T} > (C_3) \int_2^1 \frac{\delta q}{T}$$

* For a reversible heat transfer, the difference in temperature between the working substance and the surrounding must be infinitesimal.

** The units of s are J/kg/K, Btu/lb$_m$/°R.

Since C_2 is a reversible path and entropy is a property:

$$(C_2)\int_2^1 \frac{\delta q}{T} = (C_2)\int_2^1 ds$$

$$= (C_3)\int_2^1 ds$$

Thus:

$$(C_3)\int_2^1 ds > (C_3)\int_2^1 \frac{\delta q}{T}$$

$$ds > \frac{\delta q}{T} \tag{4.91}$$

Generally, for any process:

$$ds \geqslant \frac{\delta q}{T} \tag{4.92}$$

where the equality sign holds only for a reversible process. Equation 4.92 can be written as an equality:

$$ds = \frac{\delta q}{T} + \frac{\delta q_\ell}{T}$$

where δq_ℓ is the lost heat due to the irreversibilities. Equivalent to $\delta q_\ell/T$ an entropy change can be found, so that:

$$ds = \frac{\delta q}{T} + ds_{irr} \tag{4.93}$$

where $ds_{irr} \geqslant 0$, and δq is the *actual* heat transfer. It is obvious from Equations 4.90 that ds is positive (entropy increases) when the heat transfer is positive and ds is negative (entropy decreases) when the heat transfer is negative. However, there will also be a change in the direction opposite to that of the system in the surrounding. If ds_{sys} and ds_{surr} are the differential changes in the system and the surrounding, respectively, then it can be shown quite simply that for all processes:

$$ds_{sys} + ds_{surr} \geqslant 0 \tag{4.94a}$$

If the system is completely isolated, then:

$$ds_{sys} \geqslant 0 \tag{4.94b}$$

The quality sign holds only for a reversible process. Equations 4.94a, b describe the principle of increase of entropy. In other words, Equations 4.94a, b show that in any physical process, the entropy either will remain the same or can only increase.

Deductions from the Two Thermodynamic Laws

As mentioned earlier, the main aim of thermodynamics is to establish relations among the state variables. This effort is particularly fruitful for the case of those substances in which the specification of any two state variables is enough to obtain the remaining unknown variables. Such substances are called *pure substances*. Consider a pure substance for which we specify p and T as functions of the specific volume v and the entropy s. Then:

$$p = p(v, s)$$
$$T = T(v, s)$$

where we have denoted the function by the dependent variable itself. Eliminating s between the two functions, we have:

$$f(p, v, T) = 0 \tag{4.95}$$

A familiar formula of the form (Equation 4.95) is the equation of state for a ideal gas obtained earlier in Equation 2.92.

In fluid mechanics, the work done by a unit mass of the fluid against the pressure forces is

$$\delta w = p\, d\left(\frac{1}{\rho}\right) = p\, dv$$

If δq is the heat added to the system from outside, then according to the first law (Equation 4.87) the change in the internal energy is

$$de = \delta q - p\, dv$$

or:

$$\delta q = de + p\, dv \tag{4.96}$$

Using Equation 4.90a, we can also express Equation 4.96 as:

$$T\, ds = de + p\, dv \tag{4.97}$$

which is obviously the first law of thermodynamics for a reversible thermodynamic process. For an observer moving with the flow, the rate form of Equation 4.97 is

$$T\frac{Ds}{dt} = \frac{De}{dt} + p\frac{Dv}{dt} \tag{4.98}$$

Example 4.2

Let f, g, h be point functions, viz., they assume definite values at each point in a given domain. Further, let:

$$f = f(g, h), \qquad g = g(f, h), \qquad h = h(f, g)$$

Then show that:

$$\left(\frac{\partial f}{\partial h}\right)_g \cdot \left(\frac{\partial g}{\partial f}\right)_h \cdot \left(\frac{\partial h}{\partial g}\right)_f = -1$$

$$\left(\frac{\partial f}{\partial g}\right)_h \cdot \left(\frac{\partial g}{\partial h}\right)_f \cdot \left(\frac{\partial h}{\partial f}\right)_g = -1$$

From the ensuing analysis conclude that:

$$\left(\frac{\partial f}{\partial g}\right)_h = 1\Big/\left(\frac{\partial g}{\partial f}\right)_h, \quad \left(\frac{\partial g}{\partial h}\right)_f = 1\Big/\left(\frac{\partial h}{\partial g}\right)_f, \quad \left(\frac{\partial h}{\partial f}\right)_g = 1\Big/\left(\frac{\partial f}{\partial h}\right)_g$$

Since f, g, h, are point functions, their perfect differentials are

$$df = \left(\frac{\partial f}{\partial g}\right)_h dg + \left(\frac{\partial f}{\partial h}\right)_g dh$$

$$dg = \left(\frac{\partial g}{\partial f}\right)_h df + \left(\frac{\partial g}{\partial h}\right)_f dh$$

$$dh = \left(\frac{\partial h}{\partial f}\right)_g df + \left(\frac{\partial h}{\partial g}\right)_f dg$$

Using the third equation in the first and equating coefficients, we get:

$$\left(\frac{\partial f}{\partial h}\right)_g \left(\frac{\partial h}{\partial f}\right)_g = 1$$

$$\left(\frac{\partial f}{\partial g}\right)_h + \left(\frac{\partial f}{\partial h}\right)_g \left(\frac{\partial h}{\partial g}\right)_f = 0$$

Thus:

$$\left(\frac{\partial f}{\partial h}\right)_g = 1\Big/\left(\frac{\partial h}{\partial f}\right)_g$$

so that:

$$\left(\frac{\partial f}{\partial h}\right)_g \left(\frac{\partial g}{\partial f}\right)_h \left(\frac{\partial h}{\partial g}\right)_f = -1$$

Similarily, other results can be proved.

Specific Heats

The specific heat is defined as the amount of heat required to raise the temperature of a unit mass of a substance by unity. Thus, the specific heat is

$$C = \frac{\delta q}{dT} \tag{4.99a}$$

Assuming T to be a function of p and v, we have:

$$dT = \left(\frac{\partial T}{\partial p}\right)_v dp + \left(\frac{\partial T}{\partial v}\right)_p dv \tag{4.99b}$$

where *a* subscript outside the parentheses denotes the variable held fixed in the partial differentiation. similarly assuming e to be a function of p and v and using Equation 4.97, we have:

$$\delta q = \left(\frac{\partial e}{\partial p}\right)_v dp + \left(\frac{\partial e}{\partial v}\right)_p dv + p\,dv \tag{4.99c}$$

Introducing the expressions (Equations 4.99b, c) in Equation 4.99a, we get:

$$C = \frac{\delta q}{dT} = \frac{\left(\frac{\partial e}{\partial p}\right)_v dp + \left(\frac{\partial e}{\partial v}\right)_p dv + p\,dv}{\left(\frac{\partial T}{\partial p}\right)_v dp + \left(\frac{\partial T}{\partial v}\right)_p dv}$$

Thus, C is not unique, for it depends on the derivative dp/dv. We can, however, define two specific heats, called the principal specific heats: one for $dp = 0$ (p = constant), and the other for $dv = 0$ (v = constant). Thus:

$$C_p = \left(\frac{\partial q}{dT}\right)_{dp=0} = \frac{1}{\left(\frac{\partial T}{\partial v}\right)_p}\left[\left(\frac{\partial e}{\partial v}\right)_p + p\right]$$

or, referring to Example 4.1, we have:

$$C_p = \left(\frac{\partial e}{\partial T}\right)_p + p\left(\frac{\partial v}{\partial T}\right)_p \tag{4.100}$$

and:

$$C_v = \left(\frac{\delta q}{dT}\right)_{dv=0} = \left(\frac{\partial e}{\partial p}\right)_v \Big/ \left(\frac{\partial T}{\partial p}\right)_v$$

or referring to Example 4.1:

$$C_v = \left(\frac{\partial e}{\partial T}\right)_v \tag{4.101}$$

The C_p and C_v are called the specific heats at constant pressure and constant volume, respectively. Both C_p and C_v are also the state variables.

Enthalpy

We introduce another state variable, called the enthalpy or total heat. The enthalpy per unit mass or the specific enthalpy is defined as:

$$h = e + pv \tag{4.102}$$

It must be noted that in an actual computation, the quantity pv must be divided by J_e to have the units of heat.

Differentiating Equation 4.102 with respect to T while keeping p constant, we get the same expression as in Equation 4.100. Thus:

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{4.103}$$

In terms of h, the combined thermodynamic law (Equation 4.97) is

$$T\,ds = dh - v\,dp \tag{4.104}$$

Maxwell Equations

Two functions F and G defined as:

$$F = e - Ts \tag{4.105a}$$

and:

$$G = F + pv = h - Ts \tag{4.105b}$$

are called the Helmholtz and Gibbs free energy functions, respectively. Taking the differentials of Equations 4.105 and using Equations 4.97 and 4.102, we have:

$$dF = -p\,dv - s\,dT \tag{4.106a}$$

$$dG = v\,dp - s\,dT \tag{4.106b}$$

Similarly from Equations 4.97 and 4.104, we have:

$$de = T\,ds - p\,dv \tag{4.107a}$$

$$dh = T\,ds + v\,dp \tag{4.107b}$$

From Equations 4.106 and 4.107 we conclude that by taking:

$$F = F(v, T), \qquad G = G(p, T), \qquad e = e(s, v), \qquad h = h(s, p)$$

forming their perfect differentials and comparing coefficients with Equations 4.106 and 4.107, we have:

$$\left(\frac{\partial F}{\partial v}\right)_T = -p, \qquad \left(\frac{\partial F}{\partial T}\right)_v = -s \tag{4.108a}$$

$$\left(\frac{\partial G}{\partial p}\right)_T = v, \qquad \left(\frac{\partial G}{\partial T}\right)_p = -s \tag{4.108b}$$

$$\left(\frac{\partial h}{\partial p}\right)_s = v \qquad \left(\frac{\partial h}{\partial s}\right)_p = T \tag{4.109a}$$

$$\left(\frac{\partial e}{\partial v}\right)_s = -p \qquad \left(\frac{\partial e}{\partial s}\right)_v = T \tag{4.109b}$$

Differentiating each variable on the right-hand side of Equations 4.108 and 4.109 with respect to the corresponding variable which was held fixed earlier on the left-hand side, and equating terms:

$$\frac{\partial^2 F}{\partial T\,\partial v} = \frac{\partial^2 F}{\partial v\,\partial T}\ \text{etc.},$$

we get:

$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial s}{\partial v}\right)_T \tag{4.110a}$$

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T \tag{4.110b}$$

$$\left(\frac{\partial v}{\partial s}\right)_p = \left(\frac{\partial T}{\partial p}\right)_s \tag{4.110c}$$

$$\left(\frac{\partial p}{\partial s}\right)_v = -\left(\frac{\partial T}{\partial v}\right)_s \tag{4.110d}$$

Equations 4.110 are called the Maxwell equations.

As an immediate application of one of the Maxwell equations, we reconsider the definitions of C_p and C_v. From Equations 4.90a and 4.99a:

$$C_p = \left(\frac{\delta q}{dT}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_p \tag{4.111a}$$

and:

$$C_v = \left(\frac{\delta q}{\partial T}\right)_v = T\left(\frac{\partial s}{\partial T}\right)_v \tag{4.111b}$$

Taking $s = s(T, v)$, $s = s(T, p)$, $v = v(T, p)$, and their differentials, we find:

$$(ds)_p = \left[\left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv\right]_{p = \text{const.}} \tag{4.112a}$$

$$(ds)_p = \left(\frac{\partial s}{\partial T}\right)_p dT \tag{4.112b}$$

$$(dv)_p = \left(\frac{\partial v}{\partial T}\right)_p dT \tag{4.112c}$$

Using Equations 4.112b, c in Equation 4.112a and the definitions in Equations 4.111a, b, we get:

$$C_p - C_v = T\left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p \tag{4.113}$$

which expresses the difference of the specific heats as a function of T, s, and v. Using the first Maxwell equation (Equation 4.110a), we have:

$$C_p - C_v = T\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial v}{\partial T}\right)_p \tag{4.114}$$

which expresses the difference of the specific heats as a function of T, p, v.

Isentropic State

From Equation 4.90a, we see that for a *reversible adiabatic* process $\delta q = 0$, which implies:

$$ds = 0, \qquad \text{or,} \qquad s = \text{constant} \tag{4.115}$$

The state of a system in which the entropy is a constant is called an ***isentropic state***.

Speed of Sound

Another thermodynamic variable of much importance in fluid dynamics is the speed of sound a defined as:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.116}$$

also called the isentropic speed of sound. Introducing $\rho = 1/v$ in Equation 4.116, we also have the form:

$$a^2 = -v^2\left(\frac{\partial p}{\partial v}\right)_s = -v^2\bigg/\left(\frac{\partial v}{\partial p}\right)_s \tag{4.117}$$

Differentiating a^2 given in Equation 4.117 with respect to p, we obtain:

$$\frac{\partial a^2}{\partial p} = 2v(\Gamma - 1) \tag{4.118}$$

where:

$$\Gamma = \frac{a^4}{2v^3}\left(\frac{\partial^2 v}{\partial p^2}\right)_s \tag{4.119}$$

is an important thermodynamic variable in gas dynamics.

Thermodynamic Relations for an Ideal Gas

For an ideal gas, the equation of state has been given in Equations 2.92 and using these in Equation 4.114, we get the result:

$$C_p - C_v = R \tag{4.120}$$

Thus, although C_p and C_v are functions of the state variables, their differences are a constant. Also, from Equation 4.99a, we have:

$$(\delta q)_p = C_p\, dT, \qquad (\delta q)_v = C_v\, dT$$

Thus, Equation 4.96 yields:

$$C_p\, dT = de + pd\left(\frac{1}{\rho}\right) = dh \tag{4.121a}$$

since p = constant and:

$$C_v\, dT = de \tag{4.121b}$$

Also, from Equations 4.97 and 4.120:

$$T\, ds = C_v\, dT - T(C_p - C_v)\frac{d\rho}{\rho} \tag{4.122}$$

Introducing the ratio:

$$\gamma = C_p/C_v > 1$$

Equation 4.122 becomes

$$\frac{T\,ds}{C_v} = dT - T(\gamma - 1)\frac{d\rho}{\rho} \tag{4.123}$$

In general, for an ideal gas both C_p and C_v are functions of temperature as has been demonstrated by Joule's classic experiment. An empirical formula expressing C_p as a function of absolute temperature T for air is

$$C_p/R = A_0 + A_1T + A_2T^2 + A_3T^3 + A_4T^4 = f(T) \tag{4.124}$$

where:

$$A_0 = 3.65359, \qquad A_1 = -(1.33736)(10^{-3})/°K$$

$$A_2 = (3.29421)(10^{-6})/(°K)^2, \qquad A_3 = -(1.91142)(10^{-9})/(°K)^3$$

$$A_4 = (0.275462)(10^{-12})/(°K)^4$$

Thus, using Equation 4.120, we have an expression for γ (also called Eucken's formula):

$$\gamma(T) = \frac{f(T)}{f(T) - 1} \tag{4.125}$$

In terms of γ, the equation of state for an ideal gas (Equation 2.92a) can be written as:

$$T = \frac{p}{\rho C_v(\gamma - 1)} \tag{4.126}$$

Perfect Gases

An ideal gas is said to be perfect if the specific heats C_p and C_v are constants. Thus, for a perfect gas γ = constant and Equation 4.123 can directly be integrated to have:

$$T\rho^{1-\gamma}\exp\left(\frac{-s}{C_v}\right) = \text{constant} \tag{4.127a}$$

If use is made of Equation 4.126, then:

$$p\rho^{-\gamma}\exp\left(\frac{-s}{C_v}\right) = \text{constant} \tag{4.127b}$$

Equations 4.127a, b are two different forms for the equation of state for a perfect gas; the first one in T, ρ, s and the second in p, ρ, s. Under the adiabatic condition (viz., $\delta q = 0$), we noted in Equation 2.49 that s = constant for a reversible process. Thus, Equations 4.115 yield, respectively:

$$T = K_0\rho^{\gamma-1}$$

$$p = K\rho^{\gamma} \tag{4.128}$$

where K_0 and K are constants. Equation 4.128 is called the *adiabatic or isentropic equation of* *state for a perfect gas*. Based on any one of the equations (Equations 4.127 or 4.128), the speed of sound in a perfect gas is

$$
\begin{aligned}
a^2 &= \frac{\gamma p}{\rho} \\
&= \gamma RT \\
&= (\gamma - 1)C_pT
\end{aligned}
$$

Also, from Equations 4.121 since C_p and C_v are constants, we get:

$$h = C_pT, \qquad e = C_vT$$

4.7 SUBSONIC AND SUPERSONIC FLOW

In what follows we shall denote the magnitude of the velocity vector by q (i.e., $|\mathbf{u}| = q$) and by a, the local speed of sound. The ratio of q and a at a point in the flow is called the local Mach number $M = q/a$. Also, from the formula:

$$a = \sqrt{\gamma RT}$$

where T is the absolute temperature, we conclude on comparison with Equation 2.89a that the speed of sound is proportional to the mean molecular speed at a point. The Mach number M is then the ratio of the continuum speed to the mean molecular speed at a point in the flow. A flow is called subsonic if $M < 1$ and supersonic if $M > 1$. A transonic flow is the one in which the Mach number is close to unity, i.e., $M \cong 1$.

To study the nature of the subsonic and supersonic flows we consider a gas in steady motion. Let the gas be slightly perturbed at a *fixed point* 0 as shown in Figure 4.9(a). The disturbance so produced will propagate with the speed of sound *relative* to the gas. In effect, the disturbance is "carried along" by the gas with the velocity of the gas and is then propagated with the speed of sound a. To fix ideas, we assume that the velocity vector $\mathbf{u}$ of the gas with respect to a stationary coordinate system is uniform. Since the propagation of disturbances can be in any direction, we introduce a unit vector $\mathbf{n}$ such that $a\mathbf{n}$ is the velocity of propagation with respect to the gas.

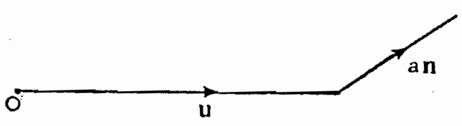

Fig. 4.9(a) Disturbance at a fixed point O in a uniform flow.

Thus, $\mathbf{u} + a\mathbf{n}$ is the velocity with which the disturbance propagates with respect to the stationary frame. Now two cases arise depending on whether $q < a$ or $q > a$. If $q < a$, then the disturbance will eventually reach every point both upstream and downstream as shown in Figure 4.9(b).

If $q > a$, then the disturbance can propagate in a restricted region as shown in figure 4.9(c). That is, the direction of the vector $\mathbf{u} + a\mathbf{n}$ cannot be outside of the conical region whose vertex is at the fixed point 0. The effect of the disturbance therefore remains confined in the conical region, and the fluid both upstream of point 0 and that which lies outside the cone remains unaware of the disturbance. Note that Figures 4.9 have been drawn to scale for the velocity prevalent at 0.

Let the semivertex angle of the cone be α, then from geometry:

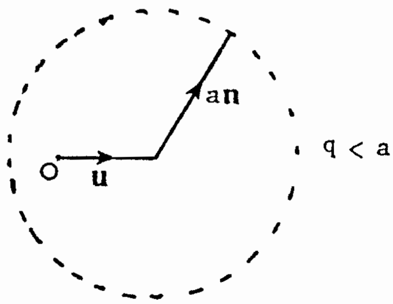

Fig. 4.9(b) Propagation of a disturbance in subsonic uniform flow.

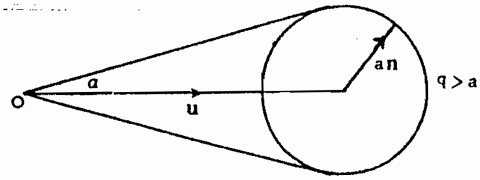

Fig. 4.9(c) Propagation of a disturbance in supersonic uniform flow.

$$\sin \alpha = a/q = 1/M$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) \quad \textbf{(4.129a)}$$

The angle α is called the Mach angle, while the surface of the cone is called the Mach surface. Each generator of the cone is then a Mach line. In two dimensions the cone is replaced by a wedge whose planar trace is a pair of Mach lines. In Figure 4.9(c) let $\mathbf{n}_M$ be a normal to the Mach line; then it is obvious that:

$$\mathbf{u} \cdot \mathbf{n}_M = \pm q \cos\left(\frac{\pi}{2} - \alpha\right)$$

or

$$\mathbf{u} \cdot \mathbf{n}_M = \pm a \quad \textbf{(4.129b)}$$

The positive or negative sign depends on the positive direction of the normal $\mathbf{n}_M$. Satisfaction of Equation 4.129b is, therefore, a fundamental requirement for a Mach line. Based on this property we state that a Mach line is the envelope of those sound wave fronts which have been generated due to movement of the disturbance with the gas in a supersonic flow.

4.8 CRITICAL AND STAGNATION QUANTITIES

A special situation arises when the local flow speed is imagined to have been brought adiabatically and reversibly (without friction) equal to the local sound speed. All the quantities describing the flow are then called "critical" quantities. We shall denote such quantities by a subscript.* Thus:

$$q_* = a_*$$

and a_* is called the critical sound speed. We may define a normalized speed using a_* as:

$$M_* = q/a_*$$

so that $M_* = 1$ when $q = q_* = a_*$.

Another special situation arises when the local flow speed is imagined to have been brought to rest by an adiabatic and reversible process. In this case the thermodynamic quantities are termed the isentropic stagnation quantities. We shall denote the isentropic stagnation quantities by a zero subscript, e.g., p_0, T_0, ρ_0, etc. Disregarding the body force potential χ and imagining that a fluid particle has been brought to a state of rest through an adiabatic and reversible process (i.e., isentropic), Equation 3.68b becomes:

$$h + \tfrac{1}{2} q^2 = h_0 \tag{4.130}$$

It must be noted that Equation 4.130 is also valid when a fluid particle is brought to rest adiabatically in an actual process, which is generally irreversible. Thus, under the condition of adiabaticity, the stagnation enthalpy is the same both under the reversible and irreversible conditions. For generally irreversible flows with heat transfer, the value of h_0 will then be different at each point so that, for two points 1 and 2, $h_{01} \neq h_{02}$.

For an isentropic flow with uniform stagnation properties Equation 4.130 yields:

$$\frac{a^2}{\gamma - 1} + \tfrac{1}{2} q^2 = \frac{a_0^2}{\gamma - 1} \tag{4.131}$$

Thus, from Equation 4.131 and $h = a^2/(\gamma - 1)$ we have:

$$a_* = a_0\sqrt{\frac{2}{\gamma + 1}}$$

$$h_0 = \frac{a_*^2(\gamma + 1)}{2(\gamma - 1)}$$

and:

$$h_* = \frac{a_*^2}{\gamma - 1} = \frac{2h_0}{\gamma + 1}$$

Further, since $h = C_pT$, we also have:

$$T_* = \frac{2T_0}{\gamma + 1}$$

Now, from the isentropic equation of state and the equation of state for an ideal gas $p = \rho RT$ we get:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{1/(\gamma - 1)}$$

Hence:

$$\frac{\rho_*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma - 1)}, \quad \frac{p_*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma - 1)}$$

The formulae given above clearly demonstrate that the critical quantities are solely determined by the stagnation quantities.

4.9 ISENTROPIC IDEAL GAS RELATIONS

A series of relations between the Mach number M and the thermodynamic variables can be obtained for an isentropic flow of an ideal gas. The fundamental (pivotal) equation is Equation 4.131 which can be arranged as:

$$\frac{a^2}{a_0^2} = 1 - \frac{(\gamma - 1)}{2}\frac{q^2}{a_0^2}$$

By recalling the definition of the speed of sound:

$$a^2 = \gamma RT, \qquad a_0^2 = \gamma RT_0$$

we immediately obtain:

$$\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2}\frac{q^2}{a_0^2}$$

$$= 1 - \frac{\gamma - 1}{2}\frac{q^2}{a^2}\frac{a^2}{a_0^2}$$

so that:

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \qquad (4.132)$$

Using the equation of state $p = \rho RT$ and the isentropic equation of state $p = K\rho^\gamma$, we easily obtain:

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma - 1)} \qquad (4.133)$$

and:

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \qquad (4.134)$$

In terms of critical Mach number M_*:

$$\frac{T}{T_0} = 1 - \frac{\gamma - 1}{\gamma + 1} M_*^2$$

$$M_*^2 = \frac{\gamma + 1}{(\gamma - 1) + 2/M^2}$$

$$M^2 = \frac{2}{(\gamma + 1)/M_*^2 - (\gamma - 1)}$$

$$\frac{\rho}{\rho_0} = \left(1 - \frac{\gamma - 1}{\gamma + 1} M_*^2\right)^{1/(\gamma - 1)}$$

$$\frac{p}{p_0} = \left(1 - \frac{\gamma - 1}{\gamma + 1} M_*^2\right)^{\gamma/(\gamma - 1)}.$$

For $M \rightarrow \infty$:

$$M_* \rightarrow \left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2}$$

so that:

$$0 \leqslant M_* \leqslant \left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2}$$

In passing, we also note that the speed q in an isentropic ideal gas flow is given by:

$$q^2 = \frac{2\gamma p_0}{(\gamma - 1)\rho_0}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}\right] \tag{4.135a}$$

$$= \frac{2\gamma p_0}{(\gamma - 1)\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{(\gamma - 1)/\gamma}\right] \tag{4.135b}$$

For further details on the isentropic flow of gases through converging-diverging nozzles, refer to References 10 and 14.

4.10 ONE-DIMENSIONAL UNSTEADY INVISCID COMPRESSIBLE FLOW

In this section we shall discuss the one-dimensional unsteady inviscid compressible flow which automatically involves the concept of characteristics and their use in solving inviscid flow problems. First of all, it is important to introduce the concept of characteristics, particularly in relation to the solution of the partial differential equations of the hyperbolic type. Here we have introduced the concept of characteristics through a series of simple example problems. Some basic results in regard to the hyperbolic equations and characteristics have also been stated in Chapter 5.

Example 4.3

Consider a single linear or quasilinear partial differential equation (PDE) in one dependent variable u and two independent variables x, y in the form:

$$Pu_x + Qu_y = R \tag{i}$$

In Equation i, P, Q, R are functions of x, y, and also of u if the equation is quasilinear. Further, the variable subscripts imply partial derivatives. Define the characteristics of Equation i and obtain the equations for their determination.

In addition to the problem stated in Equation i, let Γ be a curve in the xy-plane on which u has been specified. Specification of the values on Γ along with Equation i is called a *Cauchy problem*. To introduce the idea of a characteristic in a simple way, we transform Equation i from (x, y) to another permissible coordinate system (ξ, η), thus, having:

$$(P\xi_x + Q\xi_y)u_\xi + (P\eta_x + Q\eta_y)u_\eta = R \quad \textbf{(ii)}$$

Suppose Γ is itself one of the coordinate curves, for example, ξ = constant. then u_η can directly be calculated because u has been specified on ξ = constant. Based on the available u_η it is possible to calculate u_ξ through Equation ii provided that:

$$P\xi_x + Q\xi_y \neq 0$$

On the other hand, suppose:

$$P\xi_x + Q\xi_y = 0 \quad \textbf{(iii)}$$

then it is *not* possible to compute u_ξ, and in this situation the curve Γ or ξ = constant is called a characteristic of the PDE. From ξ = constant we also have:

$$\xi_x\, dx + \xi_y\, dy = 0 \quad \textbf{(iv)}$$

Eliminating ξ_x/ξ_y between Equations iii and iv we find that the solutions or the trajectories of the differential equation:

$$\frac{dy}{dx} = \frac{Q}{P} \quad \textbf{(v)}$$

provide the characteristics of Equation i. Thus we reach the conclusion that a characteristic curve is that on which the Cauchy data cannot be used to obtain the solution of Equation i in the neighboring region.

The preceding analysis has been used to demonstrate the concept of characteristic curves. As follows we give an analytic method to obtain the solution of Equation i.

Let the parametric equations of a curve in the xy-plane be $x = x(\sigma)$, $y = y(\sigma)$, where σ is a parameter. Thus, on this curve:

$$u = u(x(\sigma), y(\sigma)) = u(\sigma)$$

Now:

$$\frac{du}{d\sigma} = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma}$$

so that *if*:

$$\frac{dx}{d\sigma} = P, \qquad \frac{dy}{dx} = Q \quad \textbf{(vi)}$$

then from Equation i:

$$\frac{du}{d\sigma} = R \quad \textbf{(vii)}$$

Note that Equations vi lead to Equation v. The development of the method of solution depending on Equations vi and vii is demonstrated by the following examples.

Example 4.4

Solve the following Cauchy problem:

$$u_t + \alpha u_x = 0, \qquad \alpha = \text{constant}$$

$$u(x, 0) = f(x)$$

The given PDE is linear and the curve Γ is the axis $t = 0$. In the xt-plane we consider a curve $x = x(\sigma)$, $t = t(\sigma)$ so that:

$$\frac{du}{d\sigma} = u_t \frac{dt}{d\sigma} + u_x \frac{dx}{d\sigma}$$

Therefore, if:

$$\frac{dt}{d\sigma} = 1, \qquad \frac{dx}{d\sigma} = \alpha$$

then:

$$\frac{du}{d\sigma} = 0$$

Solving the three simple ODEs, we get:

$$t = \sigma - \sigma_0, \qquad x = \alpha\sigma + k, \qquad u(\sigma) = C = \text{constant}$$

where $\sigma = \sigma_0$ corresponds to $t = 0$, and $k = $ constant. Eliminating σ between the first two equations, we get the characteristics:

$$x - \alpha t = \text{constant}$$

which are straight lines. Now:

$$u(\sigma_0) = C$$

so that using the initial condition, we get:

$$u(\sigma_0) = f(\alpha\sigma_0 + k)$$

writing $k = x - \alpha\sigma$, we get:

$$u(\sigma) = f(\alpha\sigma_0 + x - \alpha\sigma)$$

Thus:

$$u(x, t) = f(x - \alpha t)$$

is the solution of the given equation. It is easily seen that $u = $ constant on the characteristics $x - \alpha t = $ constant. Note also that since $\alpha = $ constant, the characteristics in the xt-plane do not intersect.

Example 4.5

Solve the following Cauchy problem:

$$u_t + uu_x = 0$$

$$u(x, 0) = f(x)$$

Here the given PDE is quasilinear. Following the method of characteristics, we have:

$$\frac{dt}{d\sigma} = 1, \qquad \frac{dx}{d\sigma} = u, \qquad \frac{du}{d\sigma} = 0$$

From the first and third equation, we have:

$$t = \sigma - \sigma_0, \qquad u(\sigma) = \text{constant} = C$$

so that from the second equation:

$$x = C\sigma + k$$

Now:

$$u(\sigma_0) = f(C\sigma_0 + k) = C$$

so that:

$$C = f(C\sigma_0 + k)$$

Replacing C by u and k by $x - C\sigma$, we obtain the implicit solution of the given equation as:

$$u = f(x - ut) \tag{i}$$

For each u = constant, the straight line $x - ut$ = constant represents a characteristic in the xt-plane. In essence the characteristics are

$$x - ut = \beta, \qquad u = f(\beta) \tag{ii}$$

where β is a parameter. Unlike the linear case, the characteristics can now intersect. Let $\beta = \beta_1$, $\beta = \beta_2$ be two different values of β and let $f(\beta_1) = u_1$, $f(\beta_2) = u_2$. Then the characteristics intersect in the xt-plane at the point:

$$t = (\beta_1 - \beta_2)/(u_2 - u_1), \qquad x = (\beta_1 u_2 - \beta_2 u_1)/(u_2 - u_1)$$

This behavior is due to the quasilinear nature of the equation.

Example 4.6

For the problem of Example 4.5, find the least time at which the characteristics begin to intersect.

In Example 4.5, we found that because of the quasilinear nature of the equation the characteristics are bound to intersect at some point in the xt-plane. Since each distinct characteristic has a constant value of u on it, then the intersection of characteristics implies nonuniqueness of solution. Using the solution in Equation i of Example 4.5, our purpose here is to find the least time at which the many-valuedness begins to develop.

First of all, the intersecting family of lines will have an envelope which is obtained by eliminating the parameter β from the following two equations (obtained from Equation ii of Example 4.5):

$$x = \beta + tf(\beta)$$
$$0 = 1 + tf'(\beta)$$

It must be noted from this second equation that an envelope for $t > 0$ is possible when $f'(\beta) < 0$; in other words, when $f'(x - ut) < 0$. The least time τ when the many-valuedness in u first appears (the time the envelope begins to form) is then:

$$\tau = -\frac{1}{f'(\beta)} = -\frac{1}{f'(x - ut)}$$

Also since at $t = \tau$:

$$u_t = \frac{f'(x - u\tau)}{1 + \tau f'(u - u\tau)} = \infty$$

we have:

$$x_u = \frac{1 + \tau f'(x - u\tau)}{f'(x - u\tau)} = 0$$

Thus, for $t < \tau$ the continuous solution is given by $u = f(x - ut)$, and for $t > \tau$ no continuous solution exists. To continue the solution in the region $t > \tau$ we must permit the existence of discontinuities or shocks in the solution of the given equation.

The actual determination of τ can be accomplished within the framework of the continuous solution $u = f(x - ut)$. First, noting that:

$$(x_{uu})_{t=\tau} = \frac{\tau f''(x - u\tau)}{[f'(x - u\tau)]^2}$$

we conclude that the vanishing of the second derivative, viz.:

$$(x_{uu})_{t=\tau} = 0, \quad \text{or,} \quad f''(x - u\tau) = 0$$

implies a cusp at $t = \tau$ (i.e., the two lines on either side of the point of intersection are tangent to the envelope). If the function f is such that $f''(x - u\tau)$ is nonvanishing, then for the determination of τ we can also have $(u)_{t=\tau} = 0$. Thus the equations for the determination of τ are

$$(x_u)_{t=\tau} = 0, \quad \text{and} \quad (x_{uu})_{t=\tau} = 0 \quad \text{or} \quad (u)_{t=\tau} = 0$$

Example 4.7

Find the least time τ for the following Cauchy problem:

$$u_t + uu_x = 0$$

$$u(x, 0) = e^{-x^2}$$

In this case:

$$u = f(x - ut)$$
$$= e^{-(x-ut)^2}$$

$$f(\beta) = e^{-\beta^2}, \qquad f'(\beta) = -2\beta e^{-\beta^2}, \qquad f'' = (-2 + 4\beta^2)e^{-\beta^2}$$

According to the result in Example 4.6:

$$(x_u)_{t=\tau} = 0, \text{ implies } 1 - 2u\tau(x - u\tau) = 0 \tag{i}$$

$$(x_{uu})_{t=\tau} = 0, \text{ implies } f''(\beta) = 0 \tag{ii}$$

Thus, from Equation ii:

$$\beta = x - u\tau = 1/\sqrt{2}$$

and from Equation i:

$$u\tau = 1/\sqrt{2}$$

From:

$$u = f(\beta) = e^{-\beta^2}; \qquad \beta = x - u\tau$$

we have:

$$u = 1/\sqrt{e}$$

so that:

$$\tau = \sqrt{e/2}$$

Having introduced the idea of characteristics through Examples 4.3–4.7, we now consider the one-dimensional, nonsteady, constant entropy (homentropic) compressible flow of a gas. In contrast to the case of small disturbances, the present case pertains to the propagation of finite disturbances. A simple theory of finite intensity disturbances evolves when both the density and the local speed of sound are considered to be definite functions of one state variable. We take the pressure p as the state variable and introduce a new function $P(p)$ through the integral:

$$P(p) = \int_{p_0}^{p} \frac{dp}{\rho a}$$

Note that this integral is definite because the entropy is constant throughout. Obviously:

$$dP = \frac{dp}{\rho a}$$

The one-dimensional Euler equations are

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

Using the chain rule of partial differentiation we write:

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial P}\frac{\partial P}{\partial t} = \frac{\rho}{a}\frac{\partial P}{\partial t}$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial P}\frac{\partial P}{\partial x} = \frac{\rho}{a}\frac{\partial P}{\partial x}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial P}\frac{\partial P}{\partial x} = a\rho\frac{\partial P}{\partial x}$$

Substituting these partial derivatives in the Euler's equations and first adding and then subtracting the two equations, we obtain:

$$\frac{D^+}{Dt}(u + P) = 0 \tag{4.136a}$$

$$\frac{D^-}{Dt}(u - P) = 0 \tag{4.136b}$$

where:

$$\frac{D^+}{Dt} = \frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}$$

is the time rate of change as viewed by an observer moving at velocity $u + a$, and similarly:

$$\frac{D^-}{Dt} = \frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}$$

is the time rate of change as viewed by an observer moving at velocity $u - a$. Following the method used in Examples 4.3–4.6 we introduce the absolute velocities, i.e.:

$$\frac{dx}{dt} = u + a \tag{4.137a}$$

in Equation 4.136a, and:

$$\frac{dx}{dt} = u - a \tag{4.137b}$$

in Equation 4.136b. It then immediately becomes apparent that the quantities:

$$R^+ = u + P$$

and:

$$R^- = u - P$$

are conserved along the solution curves of Equations 4.137a and b, respectively. The solution curves of Equation 4.137a form one family of characteristics called the right running characteristics and are denoted by C^+. Similarly, the solution curves of Equation 4.137b form another family of characteristics called the left running characteristics and are denoted by C^-. The quantities R^+ and R^- are called the Riemann invariants.

From Equations 4.137 we note that the velocity of propagation of the waves along the C^+ family (or C^- family) with respect to the gas is $+a$ (or respectively, $-a$), where a is the local speed of sound in the gas. The wave motion considered here is termed nonsimple since the disturbances are finite. Both u and a are not constants, and the integration of Equations 4.137 can only be performed in a step-by-step fashion to obtain the characteristics. In contrast, in the case of small disturbances (acoustic theory), the speed of sound is constant and $u = 0$. Thus, the characteristics can immediately be obtained. Such waves are called *simple waves*. It can be shown quite simply that besides the acoustic theory, simple wave motion also occurs when disturbances propagate in one direction into a fluid at rest or in a state of uniform flow. The most common example, which produces concrete results, is that of the withdrawal of a piston from an open-ended pipe. We consider the case of an impulsive start of the piston from rest at time $t = 0$. We set the origin $x = 0$ at the position of the piston at $t = 0$. Initially the gas to the right of the piston is at rest. Since the gas has to adjust to the state of the finite velocity at the surface of the piston in negligible time, the origin becomes a singular point from which a centered expansion fan emanates. As shown in Figure 4.10, there are now three distinct regions in the xt-plane at $t > 0$. The first is the undisturbed region, the second is the expansion fan, and the third is the uniform region.

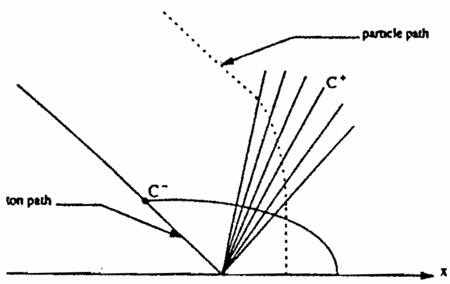

Fig. 4.10 Impulsive piston withdrawal with a finite speed X.

The C^- characteristics originate in the undisturbed region, where $u = 0$ and $a = a_0$. Since for a perfect gas:

$$P = \frac{2}{\gamma - 1} a$$

we have:

$$R^- = u - \frac{2}{\gamma - 1} a = \text{constant everywhere}$$

$$= -\frac{2}{\gamma - 1} a_0$$

Thus:

$$a = a_0 + \frac{\gamma - 1}{2} u$$

The C^+ characteristics are straight lines on each of which both a and $u = \dot{X}$ are constants. Note that for impulsive piston withdrawal $\dot{X} < 0$ and is a constant. Thus, from Equation 4.137a:

$$\frac{x}{t} = u + a$$

Using the expression for a obtained above:

$$\frac{u}{a_0} = \frac{2}{\gamma + 1}\left(\frac{x}{a_0 t} - 1\right)$$

and then:

$$\frac{a}{a_0} = 1 + \frac{\gamma - 1}{\gamma + 1}\left(\frac{x}{a_0 t} - 1\right)$$

This solution is applicable in the expansion fan region which is defined by:

$$1 + \frac{\gamma + 1}{2}\frac{\dot{X}}{a_0} \leq \frac{x}{a_0 t} \leq 1$$

The particle path in this region is obtained by solving the equation:

$$u = \frac{dx}{dt} = \frac{-2}{\gamma + 1} a_0 + \frac{2}{\gamma + 1}\frac{x}{t}$$

which is:

$$x = \frac{-2a_0}{\gamma - 1} t + kt^{2/(\gamma + 1)}$$

the constant k is determined by imposing the initial condition at some time $t = t_0$.

4.11 STEADY PLANE FLOW OF INVISCID GASES

We now consider the steady two-dimensional flow of ideal gases. The fundamental equations for this case are

$$\operatorname{div}(\rho \mathbf{u}) = 0 \tag{4.138a}$$

$$(\mathbf{u} \cdot \operatorname{grad})\mathbf{u} = \operatorname{grad}(\tfrac{1}{2} q^2) + \boldsymbol{\omega} \times \mathbf{u}$$

$$= -\frac{1}{\rho} \operatorname{grad} p \tag{4.138b}$$

$$p = p(\rho, s) \tag{4.138c}$$

where $q = |\mathbf{u}|$ and s is the entropy per unit mass.

In a plane the Cartesian coordinates x and y with unit vector $\mathbf{i}$ and $\mathbf{j}$ can always be introduced. Thus the velocity and the vorticity vectors are

$$\mathbf{u} = \mathbf{i}u + \mathbf{j}v$$

$$\boldsymbol{\omega} = \omega \mathbf{k}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Stream Function Formulation

Introducing the stream function ψ, we write:

$$\rho u = \psi_y, \qquad \rho v = -\psi_x$$

where a variable subscript implies partial derivative. Thus:

$$\rho q = (\psi_x^2 + \psi_y^2)^{1/2}$$

From Equation 4.138c:

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

Now:

$$\left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

thus, writing:

$$\left(\frac{\partial p}{\partial s}\right)_\rho = b$$

we have:

$$dp = a^2\, d\rho + b\, ds$$

so that:

$$\text{grad } p = a^2 \text{ grad } \rho + b \text{ grad } s$$

Further since:

$$\frac{\partial}{\partial x}\left(\frac{\rho^2 q^2}{2}\right) = \rho^2 \frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) + \rho q^2 \frac{\partial \rho}{\partial x}$$

and:

$$\frac{\partial}{\partial y}\left(\frac{\rho^2 q^2}{2}\right) = \rho^2 \frac{\partial}{\partial y}\left(\frac{q^2}{2}\right) + \rho q^2 \frac{\partial \rho}{\partial y}$$

we get:

$$\text{grad}(\tfrac{1}{2} q^2) = \frac{1}{\rho^2} \text{grad}\left(\frac{\rho^2 q^2}{2}\right) - \frac{q^2}{\rho} \text{grad } \rho \tag{4.139a}$$

Note also that:

$$\boldsymbol{\omega} \times \mathbf{u} = \frac{\omega}{\rho} \text{grad } \psi \tag{4.139b}$$

Using Equations 4.139a in Equation 4.138b, we get:

$$\text{grad } \rho = \frac{1}{\rho(q^2 - a^2)} [\rho\omega \text{ grad } \psi + \rho b \text{ grad } s + \tfrac{1}{2} \text{grad}(\psi_x^2 + \psi_y^2)] \tag{4.140a}$$

Considering the definition of ω, we have:

$$-\rho\omega = \nabla^2\psi - \frac{1}{\rho} (\text{grad } \psi) \cdot (\text{grad } \rho) \tag{4.140b}$$

Substituting grad ρ from Equation 4.140a in Equation 4.140b, we have:

$$(a^2 - u^2)\psi_{xx} - 2uv\psi_{xy} + (a^2 - v^2)\psi_{yy} = -\rho\left(a^2\omega + q^2 b \frac{ds}{d\psi}\right) \tag{4.141}$$

where we have used the equation:

$$\text{grad } s = (\text{grad } \psi) \frac{ds}{d\psi}$$

Equation 4.141 is a general equation for plane motion of an inviscid gas. If the flow is both homentropic (s is the same constant for all streamlines) and irrotational ($\omega = 0$), then:

$$(a^2 - u^2)\psi_{xx} - 2uv\psi_{xy} + (a^2 - v^2)\psi_{yy} = 0 \tag{4.142}$$

Irrotational Flow of an Inviscid Gas

Equation 4.142 is the stream function equation for an irrotational flow of an inviscid gas. For irrotational flow there also exists a potential function ϕ such that:

$$\mathbf{u} = \text{grad } \phi$$

To find the equation for ϕ we first use Equation 4.138a in the expanded form:

$$\rho \text{ div } \mathbf{u} + \mathbf{u} \cdot \text{grad } \rho = 0 \tag{4.143a}$$

For homentropic flow (shown earlier):

$$a^2 \text{ grad } \rho = \text{grad } p \tag{4.143b}$$

and for an irrotational flow, from Equation 4.138b:

$$\text{grad}(\tfrac{1}{2} q^2) = -\frac{1}{\rho} \text{grad } p \tag{4.143c}$$

Using Equations 4.143b, c in Equation 4.143a, we get:

$$a^2\nabla^2\phi - u(u\phi_{xx} + v\phi_{yy}) - v(u\phi_{xy} + v\phi_{yy}) = 0$$

or:

$$(a^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (a^2 - v^2)\phi_{yy} = 0 \tag{4.144}$$

which is the potential equation for an ideal gas.

Equations 4.142 and 4.144 are quasilinear partial differential equations in ψ and ϕ, respectively. They are uniformly elliptic or uniformly hyperbolic if for an arbitrary vector **w** the quadratic form:

$$(a^2 - u^2)w_1^2 - 2uvw_1w_2 + (a^2 - v^2)w_2^2$$

is positive-definite or negative-definite, respectively. Because of the quasilinearity these equations cannot be solved in closed form and numerical methods have to be used to obtain solutions in specific cases. The boundary conditions for the solution of Equation 4.144 are stated below.

We consider a uniform subsonic flow of an ideal fluid past a finite body of surface S_b in an unbounded region. For the solution of Equation 4.144 we need the boundary conditions at the surface and at infinity. The surface boundary condition is obtained by the requirement that the normal component of the velocity at S_b should be zero, so that from the definition $\mathbf{u} = \text{grad}\ \phi$, we have:

$$u_n = \frac{\partial \phi}{\partial n} = 0 \quad \text{at} \quad S_b \tag{4.145a}$$

For the boundary conditions at infinity, we refer to a paper by Finn and Gilbarg[15] which shows that at infinity:

$$\phi = u_\infty(X \cos \alpha + Y \sin \alpha) + \frac{m}{4\pi\beta}\ \ell n(\bar{r})^2 + \frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{\beta y}{x}\right) \tag{4.145b}$$

where u_∞ is the constant velocity at infinity inclined at an angle α with the X-axis, m/β is the source strength of the flow, Γ is the circulation:

$$\beta^2 = 1 - M_\infty^2 > 0$$

and:

$$z = Ze^{-i\alpha}$$

$$\bar{r} = (x^2 + \beta^2 y^2)^{1/2}$$

Note that the axes $(x, y) = z$ are oriented such that x lies along u_∞.

The potential function ϕ in Equation 4.145b yields the velocity components u and $\mathbf{v}$ by differentiation. Even this solution can be used to calculate the force exerted on an obstacle by using a control volume approach. (Refer to Problem 4.12.)

Case of Small Perturbations

If the geometric configuration of an obstacle is such that the flow past the body is only slightly different from the uniform conditions at infinity, then both Equations 4.142 and 4.144 can be linearized. Let the free-stream flow of constant thermodynamic properties be directed along the x-axis with a constant velocity u_∞. Then writing:

$$u = u_\infty + u', \quad v = v', \quad p = p_\infty + p', \quad \rho = \rho_\infty + \rho', \quad a = a_\infty + a'$$

$$\phi = u_\infty x + \phi' \tag{4.146a}$$

$$\psi = u_\infty y + \psi' \tag{4.146b}$$

in Equations 4.142 and 4.144 and neglecting the squares of the perturbation quantities, we get:

$$(1 - M_\infty^2)\psi'_{xx} + \psi'_{yy} = 0 \tag{4.147}$$

$$(1 - M_\infty^2)\phi'_{xx} + \phi'_{yy} = 0 \tag{4.148}$$

where:

$$M_\infty = u_\infty / a_\infty$$

Equations 4.147 and 4.148 are elliptic if $M_\infty < 1$, and hyperbolic if $M_\infty > 1$. The boundary conditions for an external flow problem past a body in an unbounded fluid are obtained from Equations 4.146a, b. Taking $\psi = 0$ at the surface of the body S_b, we have:

$$\psi' = -u_\infty y(x) \quad \text{at} \quad S_b$$

$$\begin{aligned}\frac{\partial \phi'}{\partial n} &= -u_\infty(\mathbf{i} \cdot \mathbf{n}) \\ &= -u_\infty \cos(\mathbf{i}, \mathbf{n}) \quad \text{at} \quad S_b\end{aligned}$$

and

$$\psi' = 0, \qquad \phi' = 0 \text{ at infinity}$$

Subsonic Flow

Let $M_\infty < 1$ and $\beta = (1 - M_\infty^2)^{1/2}$. Then the transformation

$$\xi = x, \qquad \eta = \beta y$$

transforms Equation 4.147 and 4.148 to the Laplace equations in the $\xi\eta$-plane, i.e.:

$$\psi'_{\xi\xi} + \psi'_{\eta\eta} = 0 \tag{4.149a}$$

$$\phi'_{\xi\xi} + \phi'_{\eta\eta} = 0 \tag{4.149b}$$

The same body when placed in an incompressible flow satisfies the perturbation equations in the xy-plane, i.e.:

$$\bar{\psi}'_{xx} + \bar{\psi}'_{yy} = 0 \tag{4.150a}$$

$$\bar{\phi}'_{xx} + \bar{\phi}'_{yy} = 0 \tag{4.150b}$$

where $\bar{\psi}'$, $\bar{\phi}'$ are the perturbation quantities for the incompressible flow problem. If the solutions of Equations 4.150 under the same boundary conditions as those of Equations 4.149 are known, then the subsonic compressible problem in the $\xi\eta$-plane is solved. In this situation:

$$\psi'(\xi, \eta) = \bar{\psi}'(x, y); \qquad \bar{\psi}'_\xi = \bar{\psi}'_x; \qquad \psi'_\eta = \bar{\psi}'_y$$

$$\phi'(\xi, \eta) = \bar{\phi}'(x, y); \qquad \bar{\phi}'_\xi = \bar{\phi}'_x; \qquad \phi'_\eta = \bar{\phi}'_y$$

From Problem 4.11 we then find that the pressure coefficients:

$$c_p = (p - p_\infty)/\tfrac{1}{2}\rho_\infty u_\infty^2, \quad \bar{c}_p = (\bar{p} - p_\infty)/\tfrac{1}{2}\rho_\infty u_\infty^2$$

of the subsonic compressible and incompressible flows, respectively, are related as:

$$c_p = \bar{c}_p/(1 - M_\infty^2)^{1/2}$$

which is called the Prandtl-Glauert rule.

Supersonic Flow

If $M_\infty > 1$, then writing $\beta^2 = M_\infty^2 - 1$, we have the equations:

$$\beta^2\psi'_{xx} - \psi'_{yy} = 0 \tag{4.151a}$$

$$\beta^2\phi'_{xx} - \phi'_{yy} = 0 \tag{4.151b}$$

which are hyperbolic partial differential equations. A general solution of any of these equations, for example, Equation 4.151a, is

$$\psi' = F_1(x - \beta y) + F_2(x + \beta y)$$

It is immediately noted that the function value of F_1 is unchanged along the lines:

$$x - \beta y = \text{constant}$$

and similarly the function value of F_2 is unchanged along the lines:

$$x + \beta y = \text{constant}$$

These lines are the characteristics of the hyperbolic equation (Equation 4.151a). They are called the *Mach lines* or *Mach waves*. In the xy-plane with x as the horizontal axis the gradients of these lines are given as:

$$\tan\theta = \pm\frac{1}{\beta} = \pm\frac{1}{(M_\infty^2 - 1)^{1/2}}$$

Thus:

$$\theta = \pm\sin^{-1}\left(\frac{1}{M_\infty}\right)$$

(For problems refer to Problems 4.12 and 4.13.)

Example 4.8

Using the integral form of the conservation equations, Equations 2.13, 2.38, and 2.51, derive the inviscid one-dimensional flow equations for a time dependent variable area duct.

The derivation of the equations for inviscid flow in a variable area duct has already been given in Problem 4.10 by using a control volume approach. The derivation given below is applicable to an area which is both a function of x and t. Here we follow the same technique but because of a deforming control volume, we consider Equations 2.13, 2.38, and 2.51 with appropriate deletions of some terms. Note also that the term u_r is now denoted as u since the control volume is nonrotating. The equations in integral form are

$$\frac{d}{dt}\int_{V^*(t)} \rho dv^* + \int_{S^*} \rho\,(u-c)\cdot n^*\,dS^* = 0, \tag{i}$$

$$\frac{d}{dt}\int_{V^*(t)} \rho u dv^* + \int_{S^*} \rho\, u\,(u-c)\cdot n^*\,dS^* = -\int_{S^*} pn^*\,dS^*, \tag{ii}$$

$$\frac{d}{dt}\int_{V^*(t)} \rho\, e_t dv^* + \int_{S^*} \rho\, e_t\,(u-c)\cdot n^*\,dS^* = -\int_{S^*} pu\cdot n^*dS^*, \tag{iii}$$

where $c\ (r, t)$ is the velocity of the deforming control surface.†

We propose to apply the integral equations i – iii to a control volume at the instant t which is bounded on the left by $A\ (x, t)$, on the right by $A\ (x + dx, t)$, and by the lateral surface. (Refer to figure of Problem 4.10) It is important to note that now

$$(u - c) \cdot n^* \,|_{Lateral\ surface} = 0.$$

As mentioned, the technique is the same as used in Problem 4.10. The only new thing which appears in this problem is that the R.H.S. of Equation iii on the lateral surface is not zero. This term, when evaluated for the control volume at time t, is

$$-\int_{S^*} pu\cdot n^*\,dS^* = -\left[(pu\cdot n^*)_1 A_1 + (pu\cdot n^*)_2 A_2 + \int_{S^* Lateral} pu\cdot n^*\,dS^*\right], \tag{iv}$$

where the subscripts 1 and 2 denote the values at x and $x + dx$. Since the flow is one-dimensional, we approximate the last term in (iv) as

$$\int_{S^* Lateral} pu\cdot n^*\,dS^* \simeq p(c\cdot n^*)(l\,dx) = p\frac{\delta n}{\delta t}\,l\,dx,$$

where l is the perimeter enclosing the area A and δn is the displacement along the normal to the lateral surface in time δt. Thus,

$$\int_{S^* Lateral} pu\cdot n^*\,dS^* \simeq p\frac{(l\,\delta n)}{\delta t}\,dx = p\,\frac{\partial A}{\partial t}\,dx.$$

Thus, the final equations are

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}\left[\rho(u-c_x)A\right] = 0, \tag{v}$$

$$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}\left[\{\rho(u-c_x)+p\}\,A\right] = p\,\frac{\partial A}{\partial x}, \tag{vi}$$

$$\frac{\partial}{\partial t}(\rho e_t A) + \frac{\partial}{\partial x}\left[\{\rho e_t(u-c_x)+pu\}\,A\right] = -p\,\frac{\partial A}{\partial t}\,, \tag{vii}$$

†The algebra becomes simpler if one writes $\mathbf{u} - \mathbf{c} = \mathbf{v}$.

where c_x is the x – component of c.

4.12 THEORY OF SHOCK WAVES

It is a well-known experimental fact that in a real gas flow there exist surfaces, formed of the material particles of the gas, across which steeply high gradients of pressure, density, temperature, and velocity occur. In the case of an inviscid gas, which is devoid of viscosity, these steep gradients become finite discontinuities in passing from one side to the other side of the surface. In the case of real gases the transition from one side to the other side occurs through a thin layer of fluid, called the shock layer. (Refer to Section 5.27.)

This phenomenon of shock waves is also corroborated by the available solutions of the gas dynamic equations, and the phenomenon of the shock layers, by the solutions of the Navier-Stokes equations. The gas dynamic equations are quasilinear partial differential equations. An equation is called quasilinear when the highest order derivatives are linear with coefficients which are formed of the undifferentiated dependent variables and their lower order derivatives. It has been well established both analytically and numerically that the gas dynamic equations develop finite discontinuities in the course of their solutions. (Refer to Example 4.5.)

To establish the subject of shock waves for inviscid gases from the first principle we start from the equations referred to a control volume. All specific cases, e.g., normal shocks either stationary or nonstationary, oblique shocks, etc. are then the particular cases of the general results.

Shock Relations for an Arbitrarily Moving Shock

We consider a control volume V^* of surface S^* moving with a velocity $\mathbf{c}$ as is shown in Figure 4.11. Let Σ be a surface in V^* which is also moving with the control volume with the same velocity $\mathbf{c}$.

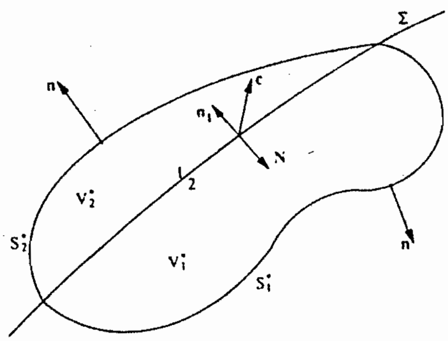

Fig. 4.11 Control volume approach to the development of shock relations.

Let a function $F(\mathbf{x}, t)$ be discontinuous in the variables $\mathbf{x}$ at all points of $\Sigma(t)$; then the surface Σ is said to be singular relative to the function $F(\mathbf{x}, t)$. The surface Σ divides the volume V^* into two volumes V_1^* and V_2^* with surfaces S_1^* and S_2^*, respectively. Let the side of Σ bordering with V_1^* be called side 1 and that bordering with V_2^* be called side 2. Details of the normals are shown in Figure 4.11.

The mass, momentum, and energy balance equations using a control volume have earlier been obtained as Equations 2.13, 2.37, and 2.51, respectively. Following the same procedure, the entropy inequality can be established. All four equations for the flow of an inviscid gas are

$$\frac{d}{dt}\int_{V^*} \rho \, dv^* + \int_{S^*} \rho(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* = 0 \tag{4.152}$$

$$\frac{d}{dt}\int_{V^*} \rho\mathbf{u} \, dv^* + \int_{S^*} \rho\mathbf{u}(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* + \int_{S^*} p\mathbf{n}^* \, dS^* = 0 \tag{4.153}$$

$$\frac{d}{dt}\int_{V^*} \rho e_t \, dv^* + \int_{S^*} \rho e_t(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* + \int_{S^*} p(\mathbf{u} - \mathbf{n}^*) \, dS^* = 0 \tag{4.154}$$

$$\frac{d}{dt}\int_{V^*} \rho s \, dv^* + \int_{S^*} \rho s(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* \geq 0 \tag{4.155}$$

where:

$$e_t = e + \tfrac{1}{2}|\mathbf{u}|^2$$

is total energy per unit mass, and s is the entropy per unit mass. We next consider the limiting case when $V^* \to 0$ and Σ tends to a limiting area Σ_0. For any function F:

$$\lim_{V^* \to 0} \frac{d}{dt}\int_{V^*} F \, dv^* \to 0$$

which can be shown to be valid at any fixed time t. Further, in this situation we also have the following limiting values.

As $V^* \to 0$: $\Sigma \to \Sigma_0$,

$$S_1^* \to \Sigma_0 \quad \text{with} \quad \mathbf{n}^* = \mathbf{N}, \qquad F = F_2$$

$$S_2^* \to \Sigma_0 \quad \text{with} \quad \mathbf{n}^* = \mathbf{n}_1 = -\mathbf{N}, \qquad F = F_1$$

where F is any flow variable, including the velocity vector.

First Shock Condition

We first write Equation 4.152 as:

$$\frac{d}{dt}\int_{V^*} \rho \, dv^* + \int_{V_1} \rho(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* + \int_{V_2} \rho(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^* \, dS^* = 0$$

and then apply the limiting behaviors stated above. Thus:

$$\rho_2(\mathbf{u}_2 - \mathbf{c}) \cdot \mathbf{N} - \rho_1(\mathbf{u}_1 - \mathbf{c}) \cdot \mathbf{N} = 0$$

Writing:

$$G = \mathbf{c} \cdot \mathbf{n}_1 = \mathbf{c} \cdot (-\mathbf{N})$$

and:

$$\mathbf{u}_1 \cdot \mathbf{N} = -u_{1n}, \qquad \mathbf{u}_2 \cdot \mathbf{N} = -u_{2n} \tag{4.156}$$

we get:

$$\rho_1(G - u_{1n}) = \rho_2(G - u_{2n})$$

which is the first shock condition.

Second Shock Condition

First, writing Equation 4.153 as:

$$\frac{d}{dt}\int_{v^*} \rho \mathbf{u}\, dv^* + \int_{S_1^*} \rho \mathbf{u}(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^*\, dS^* + \int_{S_2^*} \rho \mathbf{u}(\mathbf{u} - \mathbf{c}) \cdot \mathbf{n}^*\, dS^*$$

$$+ \int_{S_1^*} p\mathbf{n}^*\, dS^* + \int_{S_2^*} p\mathbf{n}^*\, dS^* = 0$$

and then applying the limiting procedure, we get:

$$\rho_2\mathbf{u}_2(\mathbf{u}_2 - \mathbf{c}) \cdot \mathrm{N} - \rho_1\mathbf{u}_1(\mathbf{u}_1 - \mathbf{c}) \cdot \mathrm{N} + p_2\mathrm{N} - p_1\mathrm{N} = 0$$

Using Equation 4.156, we get:

$$(p_2 - p_1)\mathrm{N} = \rho_1\mathbf{u}_1(G - u_{1n}) - \rho_2\mathbf{u}_2(G - u_{2n})$$

which is the second shock condition.

Third Shock Condition

Applying the preceding procedure to Equation 4.154, we obtain:

$$p_1u_{1n} - \rho_1e_{t1}(G - u_{1n}) = p_2u_{2n} - \rho_2e_{t2}(G - u_{2n})$$

which is the third shock condition.

Fourth Shock Condition

Applying the preceding procedure to the inequality Equation 4.155, we obtain:

$$\rho_2s_2(G - u_{2n}) - \rho_1s_1(G - u_{1n}) \geq 0$$

which is the fourth shock condition.

We now introduce the relative velocity vector:

$$\mathbf{w} = \mathbf{u} - \mathbf{c}$$

and use Equation 4.156 to write the normal components as:

$$\mathbf{w}_{1n} = \mathbf{w}_1 \cdot \mathrm{N} = G - u_{1n} > 0$$

$$\mathbf{w}_{2n} = \mathbf{w}_2 \cdot \mathrm{N} = G - u_{2n} > 0 \qquad \textbf{(4.157)}$$

The four basic shock equations are then:

$$\rho_1w_{1n} = \rho_2w_{2n} = \dot{m} \qquad \textbf{(4.158)}$$

$$(p_2 - p_1)\mathrm{N} = \rho_1\mathbf{u}_1w_{1n} - \rho_2\mathbf{u}_2w_{2n} \qquad \textbf{(4.159a)}$$

or using Equation 4.158:

$$(p_2 - p_1)\mathbf{N} = \dot{m}(\mathbf{u}_1 - \mathbf{u}_2) \tag{4.159b}$$

$$p_1 u_{1n} - \rho_1 e_{t1} w_{1n} = p_2 u_{2n} - \rho_2 e_{t2} w_{2n} \tag{4.160}$$

$$\rho_2 s_2 w_{2n} - \rho_1 s_1 w_{1n} \geq 0 \tag{4.161a}$$

or using Equation 4.158:

$$\dot{m}(s_2 - s_1) \geq 0 \tag{4.161b}$$

Some further relations can be obtained by taking the dot product of Equation 4.159a with **N**. Thus:

$$p_2 - p_1 = \rho_2 u_{2n} w_{2n} - \rho_1 u_{1n} w_{1n}$$

Adding and subtracting $\rho_1 G w_{1n}$ and $\rho_2 G w_{2n}$ on the right-hand side and using Equation 4.158, we get:

$$p_1 + \rho_1 w_{1n}^2 = p_2 + \rho_2 w_{2n}^2 \tag{4.162}$$

Also:

$$p_2 - p_1 = \dot{m}(w_{1n} - w_{2n}) \tag{4.163}$$

Substituting the pressure difference from Equation 4.163 in Equation 4.159b, we get:

$$\mathbf{u}_1 - \mathbf{u}_2 = (w_{1n} - w_{2n})\mathbf{N} \tag{4.164}$$

which states that the *velocity change across a shock is normal to the shock front and is of magnitude* $(w_{1n} - w_{2n})$.

Let **t** be the unit tangent vector in the surface so that $\mathbf{t} \cdot \mathbf{N} = 0$. Taking the dot product of Equation 4.159a with **t**, we get:

$$\rho_1 u_{1t} w_{1n} = \rho_2 u_{2t} w_{2n} \tag{4.165}$$

where:

$$u_{1t} = \mathbf{u}_1 \cdot \mathbf{t}, \qquad u_{2t} = \mathbf{u}_2 \cdot \mathbf{t}$$

Using Equation 4.158 in Equation 4.165, we have:

$$\dot{m}(u_{1t} - u_{2t}) = 0 \tag{4.166}$$

In summary, the shock relations in the normal components are

$$\rho_1 w_{1n} = \rho_2 w_{2n} = \dot{m}$$

$$p_1 + \rho_1 w_{1n}^2 = p_2 + \rho_2 w_{2n}^2$$

$$p_2 - p_1 = \dot{m}(w_{1n} - w_{2n})$$

$$p_1 u_{1n} - \rho_1 e_{t1} w_{1n} = p_2 u_{2n} - \rho_2 e_{t2} w_{2n}$$

$$s_2 \geqslant s_1 \tag{4.167}$$

Shock Surface, Slip Surface, and Contact Discontinuity

From Equation 4.166 we have three distinct possibilities:

1. If $\dot{m} \neq 0$, i.e., there is mass flow rate across the surface Σ_0, then:

   $$u_{1t} = u_{2t}$$

   and the tangential components of the absolute velocities on the two sides of the surface are continuous. This surface is called a shock or a *shock surface*.

2. If $\dot{m} = 0$ then there is no mass flow rate across Σ_0; then from Equation 4.158:

   $$u_{1n} = u_{2n} = G$$

   and from Equation 4.159a:

   $$p_2 = p_1$$

   In this case the tangential velocity components and density on both sides of the surface can differ by any amount. Such a surface is called a *slip surface*.

3. If $\dot{m} = 0$ and also $u_{1t} = u_{2t}$, then only the density and the specific internal energy on both sides of the surface can differ by any amount. Such a surface is called the *surface of contact discontinuity*.

Energy Equation for a Shock Surface

From Equation 4.156 we note that:

$$\mathbf{u} = -u_n \mathbf{N} + u_t \mathbf{t}$$

For a shock $u_{1t} = u_{2t} = u_t$, and therefore using Equation 4.157 we get:

$$\frac{1}{2}|\mathbf{u}_1|^2 - \frac{1}{2}|\mathbf{u}_2|^2 = \frac{1}{2}w_{1n}^2 - \frac{1}{2}w_{2n}^2 - G(w_{1n} - w_{2n})$$

To find the energy equation, we start from Equation 4.160 and first replace e_t by the equivalent expression:

$$e_t = h - \frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2$$

Using Equations 4.157 and 4.159b, we finally get:

$$h_1 + \frac{1}{2}w_{1n}^2 = h_2 + \frac{1}{2}w_{2n}^2 \tag{4.168}$$

which is another important shock relation.

Hugonoit Equation

To obtain the Hugonoit equation, we start from Equation 4.168 and write it as:

$$h_2 - h_1 = \frac{1}{2}(w_{1n}^2 - w_{2n}^2)$$

$$= \frac{1}{2} w_{1n} w_{2n} \left(\frac{w_{1n}}{w_{2n}} - \frac{w_{2n}}{w_{1n}} \right)$$

$$= \frac{w_{1n} w_{2n}}{2\rho_1 \rho_2} (\rho_2 - \rho_1)(\rho_2 + \rho_1)$$

Also, from Equation 4.162:

$$p_2 - p_1 = \frac{\rho_1 w_{1n}^2}{\rho_2} (\rho_2 - \rho_1)$$

Thus:

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \tag{4.169a}$$

Alternatively:

$$e_2 - e_1 = \frac{1}{2}(p_1 + p_2)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \tag{4.169b}$$

Equation 4.169a is called the Hugonoit equation. Note that the derivation of this equation does not depend on any specific equation of state.

Summary of all Shock Relations

For the purpose of reference we now collect all the shock relations in a concise form. We introduce the jump symbol δ to mean:

$$\delta(f) = f_2 - f_1$$

Also recall the following notation:

$$\mathbf{w} = \mathbf{u} - \mathbf{c}$$

$$\mathbf{c} \cdot \mathbf{N} = -G$$

$$\mathbf{u} \cdot \mathbf{N} = -u_n$$

$$w_n = G - u_n$$

Case I: Shock Relations Without Using an Equation of State.

(I) $\quad \delta(\rho w_n) = 0$

(II) $\quad -\delta(p)\mathbf{N} = \delta(\rho \mathbf{u} w_n)$

(IIIa) $\quad \delta(p) = \delta(\rho u_n w_n)$

$$\text{(IIIb)} \qquad \delta(p) = \rho_1 w_{1n} \delta(u_n)$$

$$\text{(IIIc)} \qquad \delta(p) = \frac{\rho_1 w_{1n}^2}{\rho_2} \delta(\rho)$$

$$\text{(IVa)} \qquad \delta(\mathbf{u}) = -\mathbf{N}\delta(u_n)$$

$$\text{(IVb)} \qquad \delta(\mathbf{u}) = \frac{-\mathbf{N}\delta(p)}{\rho_1 w_{1n}}$$

$$\text{(IVc)} \qquad \delta(\mathbf{u}) = \frac{-w_{1n}}{\rho_2} \delta(\rho)\mathbf{N}$$

$$\text{(V)} \qquad \delta(h + \tfrac{1}{2}|\mathbf{u}|^2) = G\delta(u_n)$$

$$\text{(VIa)} \qquad -\delta(h) = \tfrac{1}{2}\delta(w_n^2)$$

$$\text{(VIb)} \qquad \delta(h) = \tfrac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\delta(p) \tag{4.170}$$

Note that:

$$\mathbf{u} = -u_n\mathbf{N} + u_t\mathbf{t}; \qquad |\mathbf{u}|^2 = u_n^2 + u_t^2; \qquad u_{2n} > u_{1n}; \qquad w_{2n} < w_{1n}$$

Case II: Shock Relations While using an Equation of State. It is a simple exercise to rewrite the shock relations given in Equation 4.170 for a perfect gas, viz., with the use of the equation of state $p = \rho RT$. Assuming a thermally and calorically perfect gas, the shock relations are obtained in the following forms.

For a perfect gas:

$$h = \frac{\gamma p}{\rho(\gamma - 1)}$$

$$\text{(I)} \qquad \delta(\mathbf{u}) = \mathbf{N}\delta(w_n) = \frac{2(\rho_1 w_{1n}^2 - \gamma p_1)\mathbf{N}}{-\rho_1 w_{1n}(\gamma + 1)}$$

$$\text{(II)} \qquad \delta(p) = \frac{2(\rho_1 w_{1n}^2 - \gamma p_1)}{\gamma + 1}$$

$$\text{(III)} \qquad \delta(\rho) = \frac{2\rho_1(\rho_1 w_{1n}^2 - \gamma p_1)}{2\gamma p_1 + \rho_1 w_{1n}^2(\gamma - 1)}$$

$$\text{(IV)} \qquad \frac{p_2}{p_1} = \frac{(\gamma + 1)\rho_2 - (\gamma - 1)\rho_1}{(\gamma + 1)\rho_1 - (\gamma - 1)\rho_2}$$

$$\text{(V)} \qquad \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma - 1)p_2 + (\gamma + 1)p_1} \tag{4.171}$$

The relation V in Equation 4.171 is known as the Rankine-Hugonoit equation.

It is sometimes convenient to express the shock relations in terms of the shock Mach numbers. We define the shock Mach numbers in front and behind the shock as:

$$M_{1n} = w_{1n}/a_1, \qquad M_{2n} = w_{2n}/a_2$$

Using the shock relations it is a straight algebraic problem to show that:

$$M_{2n}^2 = \frac{2 + (\gamma - 1)M_{1n}^2}{2\gamma M_{1n}^2 - (\gamma - 1)}$$

For other shock relations in terms of the Mach numbers refer to Problem 4.15.

The Role of Entropy

On the basis of the fundamental thermodynamic considerations our conclusion based on Equation 4.161b is that there is also a jump in entropy and that the entropy behind the shock is always greater than that in the front, viz., $s_2 \geq s_1$. This can be very simply demonstrated for compressive shocks, i.e., $\rho_2 > \rho_1$, $p_2 > p_1$, by using the equation of state given as Equation 4.127b and showing that:

$$\frac{p_2}{p_1} \geq \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

since this implies that $s_2 - s_1 \geq 0$. To show this, use IV in Equation 4.171 and define a function $y(x)$ with $x = \rho_2/\rho_1$ as:

$$y(x) = \frac{(\gamma + 1)x - (\gamma - 1)}{(\gamma + 1) - (\gamma - 1)x} - x^{\gamma}$$

where $\gamma > 1$. By a simple numerical evaluation of $y(x)$ it is easy to show that $y(x) > 0$ for any x in the interval:

$$1 \leq x < \frac{\gamma + 1}{\gamma - 1}$$

This proves that $s_2 \geq s_1$.

The increase in entropy can also be justified either by considering a stationary shock in an inertial frame or by attaching the reference frame to a steadily moving shock. In either case it is well known that the stagnation pressure behind the shock is less than the stagnation pressure ahead of the shock. (Refer to Problem 4.17.) That is:

$$p_{01} > p_{02} \tag{4.172a}$$

On the other hand, the stagnation enthalpy on both sides of the shock remains unchanged for an observer moving with a shock, i.e.:

$$h_{01} = h_{02} \tag{4.172b}$$

or:

$$T_{01} = T_{02} \tag{4.172c}$$

To prove Equation 4.172a we consider a thermally and calorically perfect gas. Using the combined thermodynamic law:

$$T\,ds = C_v\,dT + pd\left(\frac{1}{\rho}\right)$$

we get on integration:

$$s = \text{constant} + C_p \ell n T - R \ell n p$$

Thus:

$$s_{02} - s_{01} = s_2 - s_1 = C_p \ell n \frac{T_{02}}{T_{01}} - R \ell n \frac{p_{02}}{p_{01}}$$

Using Equation 4.172c, we get:

$$\delta(s) = -R \ell n \frac{p_{02}}{p_{01}}$$

This equation proves that if $s_2 > s_1$ then $p_{02} < p_{01}$. Refer to Problem 4.16 for an expression for $\delta(s)$ in terms of the shock Mach number M_{1n}.

Stationary Shocks

The shock relations developed so far represent the dynamic relations which connect the discontinuities of flow variables on the two sides of the surface or curve Σ. These relations are simply a consequence of the continuities of mass, momentum, and energy fluxes and therefore hold for any material medium. Further, since the fluid acceleration does not appear in any of these relations, these equations must hold for any arbitrary motion of the shock surface or curve Σ. Therefore, if the velocity of the shock and of the flow in front of it are known, the velocity and other thermodynamic quantities become known through the developed shock relations.

There are many physical situations in which a shock is stationary with respect to an observer in an inertial frame. An example is the shock developing in front of a uniformly moving projectile in a steady atmosphere with the observer attached to the projectile. The shock relations for such cases are simply obtained by setting the shock velocity G equal to zero. In the following two subsections we consider the shock relations for a stationary normal and a stationary oblique shock.

Stationary Normal Shock

Setting $G = 0$ in the relations given in Equation 4.170, we have:

$$\delta(\rho u_n) = 0$$

$$\delta(p + \rho u_n^2) = 0$$

$$\delta(h + \tfrac{1}{2}|\mathbf{u}|^2) = 0$$

$$\delta(u_t) = 0$$

Note that the symbol u is reserved for the absolute velocity.

Stationary Oblique Shocks

Let a straight oblique shock be at rest in an inertial frame. without any loss of generality we can take an absolute fluid velocity ahead of the shock (side 1) as directed along the positive x-axis as shown in Figure 4.12.

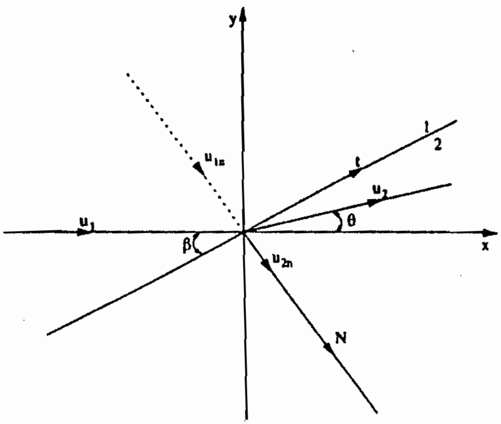

Fig. 4.12 Analysis of an oblique shock.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the fluid velocity vectors on sides 1 and 2, respectively. Further let β be the angle between $\mathbf{u}_1$ and the shock and θ be the acute angle between $\mathbf{u}_1$ and $\mathbf{u}_2$. Then the angle β is called the *shock angle*, while the angle θ is called the *turning angle*. Denoting by u_1 and u_2 the magnitudes of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, it is obvious geometrically that:

$$u_{1n} = u_1 \cos\left(\frac{\pi}{2} - \beta\right) = u_1 \sin \beta$$

$$u_{2n} = u_2 \cos\left[\frac{\pi}{2} - (\beta - \theta)\right] = u_2 \sin(\beta - \theta)$$

$$u_{1t} = u_1 \cos \beta$$

$$u_{2t} = u_2 \cos(\beta - \theta) \tag{4.173a}$$

For a normal shock $\beta = \pi/2$, $\theta = 0$. Because of the continuity of the tangential velocity, we have:

$$u_1 \cos \beta = u_2 \cos(\beta - \theta) \tag{4.173b}$$

Now:

$$u_{1x} = u_1, \qquad u_{1y} = 0 \quad \text{(by choice)}$$

$$u_{2x} = u_2 \cos \theta, \qquad u_{2y} = u_2 \sin \theta$$

hence:

$$u_{1n} = u_1 \sin \beta \tag{4.174a}$$

$$u_{2n} = u_{2x} \sin \beta - u_{2y} \cos \beta \tag{4.174b}$$

Also:

$$u_1 \cos\beta = u_{2x}\cos\beta + u_{2y}\sin\beta \tag{4.174c}$$

From the first two equations in Equation 4.173a:

$$\frac{u_{2n}}{u_{1n}} = \frac{u_2}{u_1}\frac{\sin(\beta - \theta)}{\sin\beta}$$

where from Equation 4.173b:

$$\frac{u_2}{u_1} = \frac{\cos\beta}{\cos(\beta - \theta)}$$

Thus:

$$u_{2n} - u_{1n} = u_{1n}\left[\frac{\tan(\beta - \theta) - \tan\beta}{\tan\beta}\right]$$

Introducing the Mach number M_1 as:

$$M_1 = \frac{u_1}{a_1}$$

we get:

$$\frac{u_{1n} - u_{2n}}{a_1} = \frac{M_1 \tan\theta}{\cos\beta + \sin\beta\tan\theta} \tag{4.175a}$$

Since:

$$M_{1n} = \frac{u_{1n}}{a_1}$$

then, using Equation 4.174a, we have:

$$M_{1n} = M_1 \sin\beta \tag{4.175b}$$

Using Equation ii from Problem 4.15 with Equation 4.175b in Equation 4.175a, we get:

$$\tan\theta = \frac{2(M_1^2\sin^2\beta - 1)\cot\beta}{(\gamma + 1)M_1^2 - 2(M_1^2\sin^2\beta - 1)} \tag{4.176}$$

For a given value of M_1 Equation 4.176 gives the turning angle θ as a function of β. Note that from Equation 4.176, $\theta = 0$ either when $\beta = \pi/2$, or when $\beta = \sin^{-1}(1/M_1)$. In the first case we have a normal shock while in the second case we have a Mach wave.

Prandtl's Relation

A useful relation for a stationary shock is due to Prandtl and is derived as follows. The isentropic energy equation is

$$h + \frac{1}{2}|\mathbf{u}|^2 = \frac{\gamma + 1}{2(\gamma - 1)} a_*^2$$

where a_* is the speed of sound at Mach one. Also, for a perfect gas:

$$h = \frac{\gamma p}{(\gamma - 1)\rho} = \frac{a^2}{\gamma - 1}$$

Thus the energy equation can be written in the following form:

$$\frac{p}{\rho} = \frac{(\gamma - 1)}{\gamma}\left[\frac{\gamma + 1}{2(\gamma - 1)} a_*^2 - \frac{1}{2}|\mathbf{u}|^2\right]$$

from which the expression for a^2 can be written by using $a^2 = \gamma p/\rho$. Thus:

$$\frac{p_1}{\rho_1} = \frac{(\gamma - 1)}{\gamma}\left[\frac{\gamma + 1}{2(\gamma - 1)} a_*^2 - \frac{1}{2} u_1^2\right]$$

$$\frac{p_2}{\rho_2} = \frac{(\gamma - 1)}{\gamma}\left[\frac{\gamma + 1}{2(\gamma - 1)} a_*^2 - \frac{1}{2} u_2^2\right]$$

$$a_1^2 = (\gamma - 1)\left[\frac{\gamma + 1}{2(\gamma - 1)} a_*^2 - \frac{1}{2} u_1^2\right]$$

$$a_2^2 = (\gamma - 1)\left[\frac{\gamma + 1}{2(\gamma - 1)} a_*^2 - \frac{1}{2} u_2^2\right] \qquad \textbf{(4.177)}$$

From the last two equations of Equation 4.177:

$$a_*^2 = \frac{(\gamma - 1)u_1^2 - 2a_1^2}{\gamma + 1} = \frac{(\gamma - 1)u_2^2 + 2a_2^2}{\gamma + 1}$$

The shock relations for the normal components of the velocities are

$$\rho_1 u_{1n} = \rho_2 u_{2n}$$

$$p_1 + \rho_1 u_{1n}^2 = p_2 + \rho_2 u_{2n}^2$$

Thus, forming the difference:

$$u_{1n} - u_{2n} = \frac{p_2}{\rho_2 u_{2n}} - \frac{p_1}{\rho_1 u_{1n}}$$

and using the first two equations of Equation 4.177 after some simplification we get:

$$u_{1n}u_{2n} = \frac{\gamma + 1}{2\gamma} a_*^2 + \frac{(\gamma - 1)(u_1^2 u_{2n} - u_2^2 u_{1n})}{2\gamma(u_{1n} - u_{2n})}$$

Now:

$$u_1^2 = u_{1n}^2 + u_{1t}^2, \qquad u_2^2 = u_{2n}^2 + u_{2t}^2, \qquad u_{1t} = u_{2t} = u_t$$

hence:

$$u_{1n}u_{2n} = a_*^2 - \frac{\gamma - 1}{\gamma + 1} u_t^2$$

which is called the Prandtl relation.

Shock Polar for Stationary Oblique Shocks

From Equation 4.174c:

$$u_1 = u_{2x} + u_{2y} \tan \beta \tag{4.178}$$

and from Equations 4.174a, b:

$$u_{1n}u_{2n} = u_1(u_{2x} \sin^2 \beta - u_{2y} \sin \beta \cos \beta)$$

hence, from the Prandtl relation, we have:

$$u_1(u_{2x} \tan^2 \beta - u_{2y} \tan \beta) = a_*^2(1 + \tan^2 \beta) - \frac{\gamma - 1}{\gamma + 1} u_1^2 \tag{4.179}$$

where we have used:

$$u_t = u_1 \cos \beta$$

Eliminating $\tan \beta$ between Equations 4.178 and 4.179, we get:

$$U_{2y}^2 = \frac{(U_1 - U_{2x})^2 (U_1 U_{2x} - 1)}{\dfrac{2U_1^2}{\gamma + 1} - U_1 U_{2x} + 1} \tag{4.180}$$

where:

$$U_1 = \frac{u_1}{a_*}, \qquad U_{1x} = \frac{u_{1x}}{a_*} \text{ etc.}$$

If the function (Equation 4.180) is plotted for a fixed value of U_1 with rectangular axes U_{2x} and U_{2y}, then the resulting curve will be a strophoid intersecting the U_{2x} axis at two points:

$$U_{2x} = U_1, \quad \text{or,} \quad u_{2x} = u_1 \tag{i}$$

and:

$$U_{2x} = \frac{1}{U_1}, \quad \text{or,} \quad u_{2x} = \frac{a_*^2}{u_1} \tag{ii}$$

According to the Prandtl relation, the condition in Equation ii corresponds to a normal shock condition. The condition in Equation i corresponds to a Mach wave. A line drawn from the origin at an angle θ intersects the strophoid at two points in the range of U_{2x} given by:

$$\frac{a_*^2}{u_1} \leq u_{2x} = u_1$$

The first and second points of intersection correspond to a subsonic and supersonic flow, respectively, behind an oblique shock of turning angle θ. In general, the flow behind an oblique shock is supersonic; therefore, the second intersection provides the Mach number behind a shock for a given u_1. The maximum turning angle θ_m for a given u_1 is the angle which is formed by a straight line from the origin tangent to the strophoid.

The construction of a strophoid in a problem greatly facilitates the analysis of the oblique shock structure and also in the analysis of flows past wedges. (Refer to, e.g., Reference 7.)

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PROBLEMS

4.1 This exercise aims at obtaining the general solution of the Laplace equation $\nabla^2\phi = 0$ in plane polar coordinates.

(a) Show that the Laplace equation in the plane polar coordinates (r, θ) given by:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{i}$$

(b) Using the method of separation of variables, viz., writing:

$$\phi = F(\theta)G(r), \qquad 0 \leq \theta \leq 2\pi$$

show that the general solution of Equation i is

$$\phi = a_0 + a_1 \ell n r + a_2\theta + a_3\theta \ell n r$$
$$+ \sum_{n=1}^{\infty} \left[\left(A_n r^n + \frac{B_n}{r^n}\right) \cos n\theta + \left(C_n r^n + \frac{D_n}{r^n}\right) \sin n\theta\right] \qquad \textbf{(ii)}$$

Find the velocity components $u_r = \partial\phi/\partial r$, $u_\theta = \partial\phi/r\partial\theta$.

(c) Use the solution (Equation ii) to find the solution for the flow past a circular cylinder under the following conditions:

(i) At $r = a : \dfrac{\partial \phi}{\partial r} = 0$

(ii) At $r = \infty : \dfrac{\partial \phi}{\partial r} = u_\infty \cos \theta, \qquad \dfrac{\partial \phi}{r\partial \theta} = -u_\infty \sin \theta$

(d) Use the Bernouili equation to show that the pressure on the circular cylinder is given by:

$$p(a, \theta) = \text{constant} - 2\rho U_\infty^2 \sin^2 \theta + 2\rho u_\infty\left(\frac{a_2}{a}\right) \sin \theta$$

4.2 **(a)** Let a circular cylinder of radius a be moving nonuniformly along the x-axis in an unbounded fluid at rest at infinity. At any instant of time the center of the cylinder is given by:

$$x = \int u_x(t)\, dt, \qquad y = 0$$

Attach a moving coordinate system with the center of the cylinder, and denote the coordinates by x', y' where:

$$x' = x - \int u_x(t)\, dt, \qquad y' = y$$

The potential function ϕ referred to the moving coordinate system is then given by:

$$\phi = \frac{-a^2 u_\infty}{r'} \cos \theta' = \frac{-a^2 u_\infty x'}{r'^2}$$

Using this potential flow and the nonsteady Bernoulli equation (Equation 4.7) with $\chi = 0$ show that the pressure on the cylinder is given by

$$\frac{p}{\rho} = f(t) - \tfrac{1}{2}u_\infty^2 + a\frac{du_\infty}{dt}\cos\theta' + u_\infty^2 \cos 2\theta'$$

and that the x-component of the force on the cylinder is

$$F_x = -\pi a^2 \rho \frac{du_x}{dt}$$

(b) For a sphere of radius a the potential function is

$$\phi = -\frac{a^3 u_\infty \cos\theta'}{2r'^2} = -\frac{a^3 u_\infty x'}{2r'^3}$$

where θ' is the angle between the x'-axis and the normal to the sphere. Repeat the steps of Problem 4.2a and show that the pressure distribution on the surface of the sphere is

$$\frac{p}{\rho} = f(t) + \frac{1}{2}a\frac{du_\infty}{dt}\cos\theta' - \frac{1}{16}u_\infty^2 + \frac{9}{16}u_\infty^2\cos 2\theta'$$

and the x'-component of the force is

$$F_x = -\frac{2}{3}\rho\pi a^3\frac{du_\infty}{dt}$$

4.3 Consider an ellipse with semimajor and minor axes, respectively, as m_1 and m_2. The *elliptic coordinates*, which are natural to the ellipse, are defined by the transformation:

$$z = x + iy = -C\cosh(\xi_2 - i\xi_1)$$

where the elliptic coordinate ξ_1 is considered to be positive from the left stagnation point.

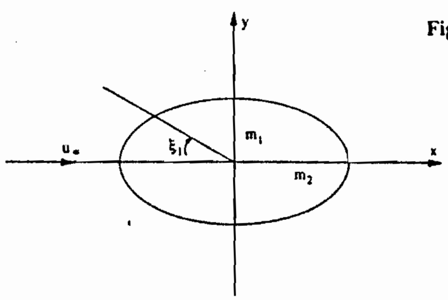


Fig. problem 4.3 Ellipse exposed to a uniform flow.

Let $\xi_2 = \alpha$ be the ellipse itself, then:

$$C = (m_1^2 - m_2^2)^{1/2}, \qquad m_1 + m_2 = Ce^{\alpha}$$

(a) Show that the potential function for the inviscid flow past the ellipse with the free stream parallel to the major axis (x-axis) is given by:

$$w = -\frac{1}{2}u_\infty\left[C\exp(\xi_2 - i\xi_1) + \frac{(m_1 + m_2)^2}{C}\exp(-\xi_2 + i\xi_1)\right]$$

where u_∞ is the free-stream velocity.

(b) Using the equations:

$$u_1 = \frac{1}{h_2}\frac{\partial\psi}{\partial\xi_2}, \qquad u_2 = -\frac{1}{h_1}\frac{\partial\psi}{\partial\xi_1}$$

where h_1 and h_2 are the scale factors, show that at the surface of the ellipse ($\xi_2 = \alpha$):

$$(u_1)_{\xi_2 \to a} = \frac{(m_1 + m_2)u_\infty \sin \xi_1}{[m_1^2 \sin^2 \xi_1 + m_2^2 \cos^2 \xi_1]^{1/2}}$$

$$(u_2)_{\xi_2 \to a} = 0$$

4.4 (a) Show that the Laplace equation in three dimensions referred to spherical polar coordinate (r, θ, ω), where:

$$x = r \sin \theta \cos \omega, \qquad y = r \sin \theta \sin \omega, \qquad z = r \cos \theta$$

is given by:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 \phi}{\partial \omega^2} = 0 \tag{i}$$

(b) A flow is called *axially symmetric* if there is no dependence of quantities on ω. In the case of axial symmetry, solve Equation i by the method of separation of variables:

$$\phi = F(\theta)G(r), \qquad 0 \leq \theta \leq \pi$$

The two resulting equations are

$$\frac{1}{\sin \theta}\frac{d}{d\theta}\left(\sin \theta \frac{dF}{d\theta}\right) + kF = 0 \tag{ii}$$

$$\frac{d^2 G}{dr^2} + \frac{2}{r}\frac{dG}{dr} - \frac{kG}{r^2} = 0 \tag{iii}$$

where k is an arbitrary parameter.

In Equation ii make the transformation:

$$\eta = \cos \theta$$

and show that Equation ii becomes a Legendre equation if we choose $k = n(n + 1)$; $n = 0, 1, 2, \ldots$.

(c) Show that the general solution of Equation i is then given by:

$$\phi = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\eta) \tag{iv}$$

where $P_n(\eta)$ are the Legendre polynomials.

4.5 Use the solution (Equation iv) of Problem 4.4 to find the flow past a sphere of radius a placed in a uniform stream U_∞ under the following conditions:

(i) At $r = a$: $\dfrac{\partial \phi}{\partial r} = 0$

(ii) At $r = \infty$: $\dfrac{\partial \phi}{\partial r} = U_\infty \cos \theta, \qquad \dfrac{\partial \phi}{r\partial \theta} = -U_\infty \sin \theta$

Show that the potential function is given as:

$$\phi = U_\infty\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

Use this ϕ to show that the velocity on the surface of the sphere is given by:

$$u_r = \frac{\partial\phi}{\partial r} = 0$$

$$u_\theta = \frac{\partial\phi}{r\partial\theta} = -\tfrac{3}{2}U_\infty \sin\theta$$

and that the pressure on the surface is

$$p(a, \theta) = \text{constant} - \tfrac{9}{8}\rho U_\infty^2 \sin^2\theta$$

4.6 It was proved in ME.5 that for a solenoidal vector $\mathbf{u}_s$, viz., a vector $\mathbf{u}_s$ for which div $\mathbf{u}_s = 0$, the equation:

$$\text{curl } \mathbf{u}_s = \omega$$

defines the velocity field $\mathbf{u}_s$ as:

$$\mathbf{u}_s = -\frac{1}{4\pi}\int_V \frac{(\mathbf{x} - \xi) \times \omega(\xi)}{r^3}\, d\nu(\xi)$$

where:

$$r = |\mathbf{x} - \xi|^{1/2}$$

In a small volume element $\delta\nu(\xi)$ surrounding the point ξ, the vorticity is $\omega\delta\nu(\xi)$ which induces a velocity $\delta\mathbf{u}(\mathbf{x})$ at $\mathbf{x}$. For a thin vortex filament consider a volume element:

$$\delta\nu(\xi) = (\delta\sigma)(\delta\ell)$$

where $\delta\sigma$ is the sectional area of the filament and $\delta\ell$ an elemental length along the filament. Denoting by $\ell(\equiv\mathbf{n})$ as a unit vector along the filament and using Equation M2.7 for $S \rightarrow 0$ show that the induced velocity at $\mathbf{x}$ is given by:

$$\delta\mathbf{u}_s = \frac{\Gamma}{4\pi}\frac{\ell \times (\mathbf{x} - \xi)}{r^3}\delta\ell \qquad \text{(i)}$$

where Γ is the circulation around the filament. Write the magnitude of $\delta\mathbf{u}_s$. This formula is called the *Biot-Savart law* of induced velocity due to a vortex filament.

4.7 Using the equation of state for a perfect gas, $p = \rho RT$, and the thermodynamic equations:

$$T\,ds = de + pd\left(\frac{1}{\rho}\right)$$

$$de = C_v\,dT$$

establish the equation of state.

$$p = \text{constant}\ e^{s/C_v} \cdot \rho^\gamma \qquad \text{(i)}$$

where C_p, C_v and $\gamma = C_p/C_v$ are constants. From Equation i it is obvious to conclude that if the flow is homentropic/isentropic then Equation i reduces to the isentropic equation of state:

$$p = K\rho^{\gamma} \qquad \textbf{(ii)}$$

where K is a constant.

4.8 Use Equation 4.131 with $|\mathbf{u}| = q$ and the isentropic equation of state (i.e., Equation ii of Problem 4.7) to prove that:

$$q^2 - \frac{2\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty}\left[1 - \left(\frac{p}{p_\infty}\right)^{(\gamma-1)/\gamma}\right] = q_\infty^2$$

where p_∞, ρ_∞, and q_∞ are constant reference values.

4.9 Take the definition of the speed of sound, viz.:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = -v^2\Big/\left(\frac{\partial v}{\partial p}\right)_s$$

where $v = 1/\rho$. Differentiate this equation with respect to ρ and obtain:

$$\left(\frac{\partial a^2}{\partial \rho}\right)_s = 2v(\Gamma - 1)$$

where:

$$\Gamma = \frac{a^4}{2v^3}\left(\frac{\partial^2 v}{\partial p^2}\right)_s > 0$$

is called the fundamental gas dynamic derivative. Show that the value of Γ for a perfect gas is $(\gamma + 1)/2$, and in this case the function P as defined in Equations 4.136 is $2a/(\Gamma - 1)$.

4.10 Use the control volume formulation for the equations of fluid dynamics, e.g., Equations 4.152–4.154, to show that the one-dimensional gas dynamic equations for a duct of variable area are

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0 \qquad \textbf{(i)}$$

$$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}[A(\rho u^2 + p)] = p\frac{dA}{dx} \qquad \textbf{(ii)}$$

$$\frac{\partial}{\partial t}(\rho e_t A) + \frac{\partial}{\partial x}[A(\rho e_t u + pu)] = 0 \qquad \textbf{(iii)}$$

where $A = A(x)$ is the cross section of the duct, ρ the density, u the velocity along x, and e_t is the total energy of the gas, i.e.:

$$e_t = \tfrac{1}{2}u^2 + e$$

4.11 Use the small perturbation analysis given by Equations 4.146 in the result of Problem 4.8 to show that after neglecting the second order terms of the perturbations, we have:

$$p' = -\rho_\infty u_\infty u'$$

and hence:

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{-2u'}{u_\infty}$$

4.12 From Equation 3.72 we have the formula for the force **F** exerted by the fluid on a body of surface S_b in terms of an integral on a fixed control surface S_0 which, for a perfect fluid, is simply:

$$\mathbf{F} = -\int_{S_0} [p\mathbf{n}_0 + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}_0)]\ dS \qquad \text{(i)}$$

where $\mathbf{n}_0$ is the unit external normal to the control surface S_0. We now wish to apply Equation i to calculate the force exerted on an arbitrary shaped *two-dimensional* profile in an ideal gas which produces small perturbations in an otherwise uniform flow (u_∞, p_∞). For simplicity we take the control surface as a circle of a large radius r in a region where conditons at infinity are applicable. Taking the unit tangent vector **t** on the circle in the clockwise direction, we have:

$$\mathbf{n}_0 = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

$$\mathbf{t} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

(a) Evaluate u' and v' from Equation 4.145b verify that that the circulation Γ and the source strength m/β, respectively, are given by the integrals:

$$\Gamma = \oint \mathbf{u}\cdot\mathbf{t}\ dS = \int_0^{2\pi} (-u'\sin\theta + v'\cos\theta)\, r\, d\theta \qquad \text{(ii)}$$

$$m/\beta = \oint \mathbf{u}\cdot\mathbf{n}_0\ dS = \int_0^{2\pi} (u'\cos\theta + v'\sin\theta)\, r\, d\theta \qquad \text{(iii)}$$

where:

$$\mathbf{u} = \mathbf{i}(u_\infty + u') + \mathbf{j}v'$$

(b) Using the Bernoulli equation for a barotropic flow:

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 = \text{constant}$$

where:

$$q^2 = (u_\infty + u')^2 + v'^2, \qquad q_\infty = u_\infty$$

we can use a Taylor expansion for $p(q)$ as:

$$p = p_\infty + \frac{dp}{dq}(q - q_\infty) + 0(q - q_\infty)^2$$

$$= q_\infty - \rho q(q - q_\infty)$$

$$\cong p_\infty - \rho u(u - u_\infty) \qquad \textbf{(iv)}$$

where $u = u_\infty + u'$.

Write the Cartesian components F_x and F_y from Equation i and substituting the expression for the pressure from Equation iv, show that:

$$F_x = -\rho_\infty u_\infty m/\beta$$

$$F_y = -\rho_\infty u_\infty \Gamma$$

If the source strength is zero, then the result is the usual D'Alembert paradox.

4.13 (a) The perturbation velocity components u', v' of a subsonic or supersonic irrotational flow satisfy the equation:

$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} = 0$$

Show that:

$$u' = \frac{1}{1 - M_\infty^2}\frac{\partial \psi'}{\partial y}, \qquad v' = -\frac{\partial \psi'}{\partial x} \qquad \textbf{(i)}$$

satisfy the Equation 4.147 if $\beta^2 = 1 - M_\infty^2 > 0$, and Equations 4.151 if $\beta^2 = M_\infty^2 - 1 > 0$.

(b) It has been shown that the C^+ characteristic solution is given by:

$$\psi' = F_1(x - \beta y)$$

and the C^- characteristic solution is given by:

$$\psi' = F_2(x + \beta y)$$

where remembering that $\beta = \sqrt{M_\infty^2 - 1}$. Show by using Equation i that for C^+:

$$\frac{u'}{v'} = -\frac{1}{\beta}$$

while for C^-:

$$\frac{u'}{v'} = \frac{1}{\beta}$$

(c) Consider a very thin airfoil in a supersonic stream. The presence of the C^+ and C^- Mach lines produces a deflection of the streamlines of small amount given by the equation:

$$\frac{v}{u} = \tan\theta$$

Shown that if θ is very small, then:

$$\frac{v}{u} \cong \frac{v'}{u_\infty} = \theta$$

and this θ is also the inclination of the characteristic lines. If the equations of the upper and lower surfaces of the airfoil are $y = f_1(x)$ and $y = f_2(x)$, respectively, then for the C^+ characteristics:

$$\frac{v'}{u_\infty} = \theta_1(x) = \frac{df_1}{dx}$$

and for the C^+ characteristics:

$$\frac{v'}{u_\infty} = \theta_2(x) = \frac{df_2}{dx}$$

Sketch the ideas given above. Finally show that:

$$\frac{u'}{u_\infty} = \frac{\theta}{\beta} = \frac{\theta}{\sqrt{M_\infty^2 - 1}} \qquad \text{(ii)}$$

4.14 A shock advances into a stationary fluid, i.e., $\mathbf{u}_1 = 0$. Using Equation 4.164 establish the direction of motion immediately behind the shock front, viz., the direction of motion of the compressed fluid.

4.15 Define the Mach numbers M_{1n} and M_{2n} as:

$$M_{1n} = w_{1n}/a_1, \qquad M_{2n} = w_{2n}/a_2$$

where a_1 and a_2 are the local sound velocities in the respective regions. Use the shock relations for a perfect gas given in Equation 4.171 and show that:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_{1n}^2 - 1) \qquad \text{(i)}$$

$$w_{1n} - w_{2n} = \frac{2a_1}{\gamma + 1}\left(M_{1n} - \frac{1}{M_{1n}}\right) \qquad \text{(ii)}$$

$$\frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma + 1}\left(1 - \frac{1}{M_{1n}^2}\right) \qquad \text{(iii)}$$

$$\frac{T_2}{T_1} = \frac{\{2\gamma M_{1n}^2 - (\gamma - 1)\}\{(\gamma - 1)M_{1n}^2 + 2\}}{(\gamma + 1)^2 M_{1n}^2} \qquad \text{(iv)}$$

4.16 From Equation 4.107a we have the result that:

$$\delta(s) = C_p \ell n \frac{T_2}{T_1} - R \ell n \frac{p_2}{p_1}$$

hence, using Equations i and iv of Problem 4.15 show that:

$$\exp\frac{\delta(s)}{R} = \left(\frac{2\gamma}{\gamma + 1}M_{1n}^2 - \frac{\gamma - 1}{\gamma + 1}\right)^{1/(\gamma - 1)}\left\{\frac{2}{(\gamma + 1)M_{1n}^2} + \frac{\gamma - 1}{\gamma + 1}\right\}^{\gamma/(\gamma - 1)}$$

4.17 A normal shock moves steadily with a speed of 2000 ft/s in a stationary atmosphere in which:

$$\bar{p}_1 = 14.7 \text{ lbf/in}^2, \qquad \bar{T}_1 = 600° \text{ R}, \qquad \bar{u}_1 = 0$$

as shown in Figure i:

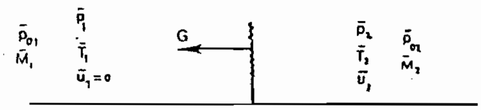
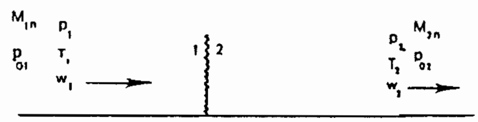


Fig. problem 4.17 (i) Moving shock, (ii) stationary shock.

Attaching the reference frame with the shock we have the configuration of Figure ii:



Note that:

$$w_1 = G - \bar{u}_1$$

$$w_2 = G - \bar{u}_2$$

$$p_1 = \bar{p}_1, \qquad T_1 = \bar{T}_1, \qquad a_1 = \bar{a}_1, \qquad p_{01} = \bar{p}_1$$

$$p_2 = \bar{p}_2, \qquad a_2 = \bar{a}_2$$

For air, taking $\gamma = +1.4$, $R = 1716.16$ ft · lbf/slug/°R, first calculate the velocity of sound a_1 and the Mach number M_{1n}. Using this value of M_{1n} calculate M_{2n}, and p_2/p_1, T_2/T_1 from Equations i and iv of Problem 4.15 and hence p_2 and T_2. Knowing T_2 calculate a_2 and hence w_2 (from $w_2 = M_{2n}a_2$), $\bar{u}_2$ and $\bar{M}_2$. Using the isentropic equation (Equation 4.134) show that $\bar{p}_{02} = 65$ psi, $p_{01} = 68.91$ psi, $p_{02} = 59.93$ psi. From these values we conclude that $p_{01} > p_{02}$, while $\bar{p}_{01} < \bar{p}_{02}$.

4.18 In Problem 4.13, the deflection of a Mach line in a small perturbation flow has been obtained as:

$$\frac{v'}{u'} = \frac{\theta}{\sqrt{M_\infty^2 - 1}}$$

It is completely justifiable to use this expression, even without the assumption of small perturbations, to have an expression in passing from one Mach line to a very close Mach line. Therefore, if q is the local velocity in a supersonic flow and dq is a small change in q with $d\theta$ as a small change in the direction of the Mach line, then:

$$\frac{dq}{q} = \frac{d\theta}{\sqrt{M^2 - 1}} \tag{i}$$

where $q = |\mathbf{u}|$, and $M = q/a$.

We now consider the development of a supersonic flow due to a flow past a convex corner. This problem is known as the Prandtl-Meyer flow. Without any loss of generality we consider a sonic flow parallel to a horizontal wall which meets a convex corner at the point 0.

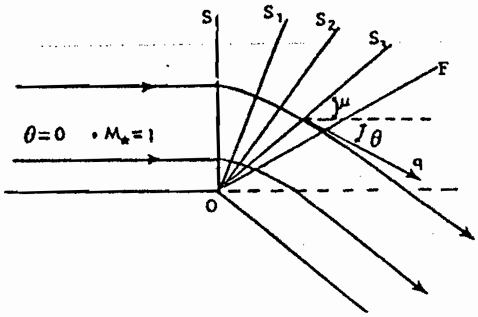


Fig. problem 4.17 Convex corner; Prandlt-Meyer flow.

A sudden expansion causes the flow to become supersonic and after passing through a series of Mach lines emanating from point 0 the flow finally becomes parallel to the inclined wall. The Mach lines between the Mach lines OS and OF form an *expansion fan*.

To find the motion in the expansion fan we make use of Equation i by introducing:

$$M_* = \frac{q}{a_*}$$

to have:

$$\frac{dM_*}{M_*} = \frac{d\theta}{\sqrt{M^2 - 1}} \tag{ii}$$

Using the expression of M in terms of M_* (Sect. 4.9) in Equation ii, we get:

$$d\theta = \left(\frac{M_*^2 - 1}{1 - \alpha M_*^2}\right)^{1/2} \frac{dM_*}{M_*} \tag{iii}$$

where:

$$\alpha = \frac{\gamma - 1}{\gamma + 1}$$

Integrating Equation iii and using the condition $\theta = 0$ at $M_* = 1$, show that:

$$\theta = \frac{1}{\sqrt{\alpha}} \tan^{-1}\left\{\frac{\alpha(M_*^2 - 1)}{1 - \alpha M_*^2}\right\}^{1/2} - \tan^{-1}\left\{\frac{M_*^2 - 1}{1 - \alpha M_*^2}\right\}^{1/2} \tag{iv}$$

The Mach angle μ, viz., the angle made by a Mach line with the local direction of flow is given by:

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

$$= \tan^{-1}\left\{\frac{1 - \alpha M_*^2}{M_*^2 - 1}\right\}^{1/2} \tag{v}$$

Thus, show that the angle ϕ made by any Mach line with the initial Mach line OS is given by:

$$\phi = \frac{\pi}{2} + \theta - \mu = \frac{1}{\sqrt{\alpha}} \tan^{-1}\left\{\left[\frac{\alpha\left(M_*^2 - 1\right)}{1 - \alpha M_*^2}\right]^{1/2}\right\} \tag{vi}$$

The maximum value of θ denoted as θ_{max} is obtained for $M \to \infty$ which corresponds to an outflow in a complete vacuum. Show that:

$$\theta_{max} = \frac{\pi}{2}\left(\frac{1}{\sqrt{\alpha}} - 1\right)$$

4.19 This exercise is meant to fix ideas in regard to the transonic flows past airfoils. To start with, a subsonic flow regime is the one in which the presence of the body is felt well ahead of the body by the oncoming flow. Consequently, the deflection of the free stream begins before the fluid reaches the body. In passing over the surface of the airfoil the fluid accelerates but for the free-stream Mach number $M_\infty \leq 0.8$ the flow remains subsonic throughout the airfoil surface.

In the regime of free-stream Mach number $0.8 < M_\infty < 1$, the acceleration (and expansion) of the flow on the airfoil surface causes the formation of pockets of supersonic flows both on the upper and lower surfaces. These pockets of supersonic flows are terminated by normal shocks on the downstream side as is shown in the Figure i:

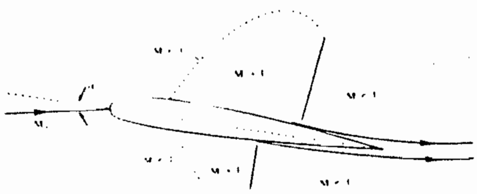


Fig. problem 4.19 Illustration of a transonic flow past an airfoil.

In the regime of free-stream Mach number $1 < M_\infty < 1.2$, there forms a bow shock in front of the airfoil behind which there is a region of subsonic flow. This subsonic region is then followed by an extensive region of supersonic flow. In the range of $1 < M_\infty < 1.2$, the supersonic flow is terminated by the trailing edge shocks.

It has become customary to designate the regime $0.8 < M_\infty < 1.2$ as the *transonic flow regime*.

(a) For the transonic flow in which $M_\infty \sim 1$, the shock wave produced behind the supersonic pocket is usually weak. In the last equation (VIb) of Equation 4.170, i.e., in the Hugoniot equation, introduce the symbol:

$$v = 1/\rho$$

so that:

$$\delta(h) = v_1\delta(p) + \frac{1}{2}\delta(v)\delta(p) \tag{i}$$

Now take $h = h(s, p)$, $v = v(s, p)$ along with the formulae:

$$T = \left(\frac{\partial h}{\partial s}\right)_p, \qquad v = \left(\frac{\partial h}{\partial p}\right)_s$$

and perform the Taylor expansions:

$$h(s_2, p_2) = h(s_1 + \delta(s), p_1 + \delta(p))$$

and:

$$v(s_2, p_2) = v(s_1 + \delta(s), p_1 + \delta(p))$$

Substitute these expansions correct to orders $\delta(s)$ and $(\delta(p))^3$ in Equation i and show that:

$$\delta(s) = \frac{1}{12T_1}(\delta(p))^3\left(\frac{\partial^2 v}{\partial p^2}\right)_1 \quad \text{(ii)}$$

Equation ii shows that for weak shocks the entropy jump is proportional to the third power of the pressure jump. Since for weak shocks $\delta(p)$ is small, $\delta(s)$ is practically negligible. Such shocks are called *isentropic shocks*.

(b) Using Equations 4.143b, c in Equation 4.143a, we have:

$$a^2 \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \operatorname{grad}(\tfrac{1}{2} q^2) = 0 \quad \text{(i)}$$

For the case of irrotational flow, Equation i can be written purely in terms of the potential ϕ by using the definition $\mathbf{u} = \operatorname{grad} \phi$. The equation so obtained is the exact potential flow model for inviscid compressible flows including the transonic flow. Note that the local speed of sound a is given by:

$$a^2 = a_0^2 - \tfrac{1}{2}(\gamma - 1)q^2 \quad \text{(ii)}$$

The vanishing of the normal component of velocity at an impermeable surface is given by

$$\mathbf{u} \cdot \mathbf{n} = (\operatorname{grad} \phi) \cdot \mathbf{n} = \frac{\partial \phi}{\partial n} = 0 \quad \text{(iii)}$$

For obtaining the equation for transonic *small-disturbance* flow, let us introduce Cartesian coordinates so that the primary flow is along the x-axis. Considering a thin body of surface S in the flow, we write:

$$\phi(x, y, z) = U_\infty[x + \epsilon\Phi(x, y, z)] \quad \text{(iv)}$$

So that Equation ii is written as:

$$a^2 = a_\infty^2 + \frac{U_\infty^2}{2}(\gamma - 1)[1 - \{(1 + \epsilon\Phi_x)^2 + \epsilon^2\Phi_y^2 + \epsilon^2\Phi_z^2\}] \quad \text{(v)}$$

Substitute Equations iv and v in the full potential equation and show that the equation containing all the nonlinear terms in the coefficient of Φ_{xx}, while neglecting the ϵ^2 terms in the rest of the coefficients, is

$$\begin{aligned}&[1 - M_\infty^2 - (\gamma + 1)\epsilon M_\infty^2\Phi_x(1 + \tfrac{1}{2}\epsilon\Phi_x) - \tfrac{1}{2}(\gamma - 1)\epsilon^2 M_\infty^2(\Phi_y^2 + \Phi_z^2)]\Phi_{xx} \\ &+ [1 - (\gamma - 1)\epsilon M_\infty^2\Phi_x]\Phi_{yy} + [1 - (\gamma - 1)\epsilon M_\infty^2\Phi_x]\Phi_{zz} \\ &- 2\epsilon M_\infty^2(\Phi_y\Phi_{xy} + \Phi_z\Phi_{xz}) = 0 \end{aligned} \quad \text{(vi)}$$

If now $\epsilon\Phi_x$, $\epsilon\Phi_y$, and $\epsilon\Phi_z$ are assumed to be very small, then it amounts to the two cases:

1. $$\Phi_x, \Phi_y, \Phi_z \ll 1, \quad \epsilon = 0(1)$$

2. $$\Phi = 0(1), \quad \epsilon \ll 1$$

In either case, Equation vi with the last term retained is

$$[1 - M_\infty^2 - (\gamma + 1)\epsilon M_\infty^2 \Phi_x]\Phi_{xx} + \Phi_{yy} + \Phi_{zz} - 2\epsilon M_\infty^2(\Phi_y\Phi_{xy} + \Phi_z\Phi_{xz}) = 0 \quad \textbf{(vii)}$$

If the last term in Equation vii is neglected, then we get the classical small-perturbation form of the equation:

$$[1 - M_\infty^2 - (\gamma + 1)\epsilon M_\infty^2 \Phi_x]\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad \textbf{(viii)}$$

In two dimensions Equation viii is

$$[1 - M_\infty^2 - (\gamma + 1)\epsilon M_\infty^2 \Phi_x]\Phi_{xx} + \Phi_{yy} = 0 \quad \textbf{(ix)}$$

In Equation ix introduce a new small parameter δ defined through the equation:

$$\epsilon = \delta^{2/3}/M_\infty^2$$

and the scaled variable:

$$\bar{y} = \delta^{1/3} y$$

to have the conservative form:

$$\frac{\partial}{\partial x}[K\Phi_x - \tfrac{1}{2}(\gamma + 1)\Phi_x^2] + \Phi_{\bar{y}\bar{y}} = 0 \quad \textbf{(x)}$$

where:

$$K = (1 - M_\infty^2)/\delta^{2/3}$$

is called the transonic similarity parameter. Equation x is called the Murmann-Krupp equation.

The numerical method of solution of Equation x as given in Reference 16 is a type-dependent differencing technique. Thus, in the subsonic region where the equation is elliptic the central difference formulae for the derivatives are used. In the supersonic region where the equation is hyperbolic the upwind difference formulae are used.

From Equation iii the exact boundary condition is

$$(1 + \epsilon\Phi_x)n_1 + \epsilon\Phi_y n_2 + \epsilon\Phi_z n_3 = 0 \quad \textbf{(xi)}$$

where n_1, n_2, n_3 are the components of **n**.

CHAPTER FIVE

Laminar Viscous Flow

Part I: Exact Solutions

5.1 INTRODUCTION

The fundamental equations for the case of incompressible viscous flow have already been summarized in Chapter 3, viz., Equations 3.24 and 3.29–3.32. These equations are in the vector invariant form and therefore can be written in any general coordinate system. With reference to a rectangular Cartesian system x, y, z with the respective components, u, v, w, the equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{5.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \tag{5.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \tag{5.4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity and ∇^2 is the Laplacian. Because of the incompressibility, ρ = constant or the Mach number is vanishingly small, i.e., $M \ll 1$. The components of vorticity and of the viscous stress are

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5.5}$$

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}, \qquad \sigma_{yy} = 2\mu\frac{\partial v}{\partial y}, \qquad \sigma_{zz} = 2\mu\frac{\partial w}{\partial z},$$

$$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad \sigma_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \qquad \sigma_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{5.6}$$

Equations of motion, the components of vorticity and of the viscous stress, for other standard coordinate systems can be written with the help of Section 3.10. Refer also to Problem 3.14. Equations of motion, the components of vorticity and of the viscous stress, for other standard coordinate systems can be written with the help of Section 3.10. Refer also to Problem 3.14 and to References 1 through 3.

5.2 EXACT SOLUTIONS

(i) Flow on an Infinite Plate

The simplest problem of viscous fluid motion is the flow on an infinite plate under a constant pressure. If x is the direction of main flow then $u \neq 0$, $v = w = 0$. The continuity equation is then simply

$$\frac{\partial u}{\partial x} = 0$$

If y is taken normal to the plate, then we conclude that $u = u(y)$. The equations of motion reduce to one equation which is

$$\frac{\partial^2 u}{\partial y^2} = 0, \qquad p = \text{constant}$$

Using the no-slip condition:

$$y = 0 : u = 0$$

and denoting the wall shear as $\tau_w = \mu(\partial u/\partial y)_{y=0}$, we obtain:

$$u = \frac{\tau_w}{\mu} y \tag{5.7}$$

which shows that the distribution of velocity is linear in y.

(ii) Flow Between Two Infinite Parallel Plates

Let the two infinite plates be placed symmetrically with respect to the x-axis as shown in Figure 5.1(a).

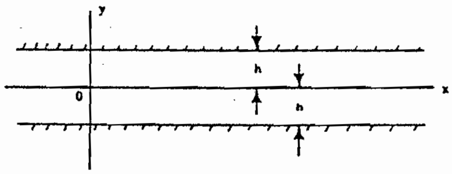


Fig. 5.1(a) Flow between two parallel plates.

Let the upper and lower plates be each at distance h from the x-axis. The upper plate is then at $y = h$ and the lower plate at $y = -h$. If the flow takes place along the x-axis under the action of a constant pressure gradient, then the equations yield the solution:

$$u = \frac{y^2}{2\mu}\frac{dp}{dx} + Ay + B \tag{5.8a}$$

Applying the no-slip conditions:

$$u = 0 \quad \text{at} \quad y = \pm h$$

we have the plane Poiseuille flow:

$$u = -\frac{1}{2\mu}\frac{dp}{dx}(h^2 - y^2) \tag{5.8b}$$

The flux of fluid through a section normal to the x-axis and of unit width in the z-direction is

$$Q = \int_{-h}^{+h} u\, dy = -\frac{2h^3}{3\mu}\frac{dp}{dx}$$

and the average velocity over a section is

$$u_m = \frac{Q}{2h} = \frac{-h^2}{3\mu}\frac{dp}{dx} \tag{5.9}$$

Integrating Equation 5.9 with respect to x, we have:

$$p = \frac{-3\mu Q}{2h^3}x + p_0, \qquad p_0 = \text{const.}, \qquad Q = \text{const.}$$

$$= \frac{-3\mu u_m}{h^2}x + p_0$$

The shear stress at the walls are

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=\pm h} = \pm h\frac{dp}{dx} = \mp\frac{3\mu u_m}{h} \tag{5.10}$$

which shows that τ_w is directed to the right on both plates. The coefficient of friction is defined as:

$$\lambda = \frac{\tau_w}{\frac{1}{2}\rho u_m^2} \tag{5.11}$$

Substituting Equation 5.10 in Equation 5.11, we have:

$$\lambda = \frac{6}{R_e}, \qquad R_e = \frac{u_m h}{\nu} \tag{5.12}$$

The case of flow between moving parallel plates is similarly treated. Let both plates move with different velocities in the direction of the positive x-axis. The velocity of the upper plate is assumed to be U_1 and that of the lower plate to be U_2. Substituting these values in Equation 5.8a we get the solution:

$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right) + \frac{U_1 - U_2}{2}\frac{y}{h} + \frac{U_1 + U_2}{2} \tag{5.13}$$

called plane Couette flow. A simple Couette flow is obtained by setting $U_2 = 0$, $dp/dx = 0$.

(iii) Flow Between Rotating Coaxial Cylinder (Circular Couette Flow)

Consider the flow between two infinite coaxial cylinders when the motion is produced in the annulus due to the rotation of the cylinders. Let r_1 and r_2 ($r_2 > r_1$) be the radii of the cylinders rotating with constant angular velocities ω_1 and ω_2, respectively, about the common axis of the cylinders. Using cylindrical coordinates r, ϕ, z, where r is the radial distance in the plane of rotation, ϕ the azimuthal angle, and z along the rotation axis. Because of the symmetry of the problem, we have:

$$u_r = 0, \qquad u_\phi = v(r, \phi), \qquad u_z = 0, \qquad \text{and} \qquad p = p(r) \tag{5.14a}$$

Using the continuity equation we have:

$$\frac{\partial v}{\partial \phi} = 0, \qquad \text{or,} \qquad v = v(r) \tag{5.14b}$$

Substituting Equations 5.14a, b in the equations of motion we find that for steady flow:

$$\frac{dp}{dr} = \frac{\rho v^2}{r} \tag{5.15a}$$

and:

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0 \tag{5.15b}$$

The solution of Equation 5.15b is

$$v(r) = Ar + \frac{B}{r} \tag{5.15c}$$

where A and B are constants to be determined by the wall boundary conditions. Substituting Equation 5.15c in Equation 5.15a and integrating, we obtain:

$$p = \rho\left(\frac{A^2r^2}{2} - \frac{B^2}{2r^2} + 2AB\ell nr\right) + C$$

where C is a constant of integration. The no-slip conditions in this case are the boundary conditions:

$$v = \omega_1 r_1 \quad \text{at} \quad r = r_1$$

$$v = \omega_2 r_2 \quad \text{at} \quad r = r_2$$

Substituting these boundary conditions in Equation 5.15c, we obtain:

$$A = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}$$

$$B = \frac{(\omega_1 - \omega_2)r_1^2 r_2^2}{r_2^2 - r_1^2}$$

The frictional force per unit area on either of the cylinders can now be computed. The frictional stress on the inner cylinder is

$$(\tau_{r\phi})_{r=r_1} = \frac{-2\mu(\omega_1 - \omega_2)r_2^2}{r_2^2 - r_1^2}$$

so that the force acting on unit length of the cylinder is

$$(F_{r\phi})_{r=r_1} = \frac{-4\pi\mu(\omega_1 - \omega_2)r_1 r_2^2}{r_2^2 - r_1^2}$$

and the moment of this force is

$$(M)_{r=r_1} = \frac{-4\pi\mu(\omega_1 - \omega_2)r_1^2 r_2^2}{r_2^2 - r_1^2}$$

(iv) Steady Flow Through a Cylindrical Pipe (Hagen-Poiseuille Flow)

Let us consider the steady flow of a viscous fluid through a cylindrical pipe of an arbitrary cross section. Let the axis of the pipe be along the z-axis and any cross section normal to the axis be in the xy-plane, as shown in Figure 5.1(b).

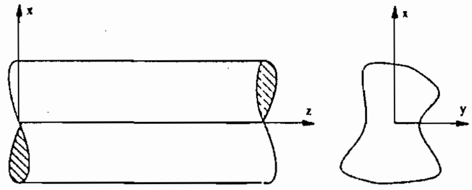


Fig. 5.1(b) Flow through a cylinder of arbitrary section.

Since the fluid flows in the z-direction, $u = v = 0$ and the equations of continuity and motion reduce to:

$$\frac{\partial w}{\partial z} = 0, \qquad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz}$$

Since w is not a function of z, and p is not a function of x and y, then:

$$\frac{dp}{dz} = \frac{-\Delta p}{\ell} = \text{constant}$$

where Δp is the pressure difference in a length ℓ of the pipe. The mathematical problem of steady unidirectional flow in a uniform pipe of section contour C is then to solve the system:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-\Delta p}{\mu \ell} \tag{5.16a}$$

$$w|_C = 0 \tag{5.16b}$$

The problem posed in Equations 5.16 can be solved only numerically for arbitrary section contours C, although the problem of fluid flow in an elliptic or circular pipe is solvable in closed form. If C is an ellipse, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the solution of Equations 5.16 is envisaged as:

$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \qquad A = \text{constant}$$

Substituting this form of w in Equation 5.16a, we determine the constant A and the solution becomes:

$$w = \frac{\Delta p}{2\mu\ell} \cdot \frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{5.17}$$

The maximum velocity w_{max} is on the pipe axis and from Equation 5.17:

$$w_{max} = \frac{\Delta p}{2\mu\ell} \cdot \frac{a^2b^2}{a^2 + b^2} \tag{5.18a}$$

The rate of volume flow is

$$Q = \int_S w \; dS = w_{max} \int_S \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx \; dy$$

Letting $x = ax'$, $y = by'$, $r' = (x'^2 + y'^2)^{1/2}$, $x' = r' \cos \theta$, $y' = r' \sin \theta$, we get:

$$Q = abw_{max} \int_0^1 \int_0^{2\pi} (1 - r'^2)r' \; dr' \; d\theta$$
$$= \frac{\pi ab}{2} w_{max}$$

The mean velocity over the elliptic section is then:

$$w_m = \frac{Q}{\pi ab} = \frac{1}{2} w_{max} \tag{5.18b}$$

i.e., the mean velocity is half of the maximum velocity.
For a circular pipe of radius a, from Equation 5.17:

$$w(r) = \frac{a^2\Delta p}{4\mu\ell}\left(1 - \frac{r^2}{a^2}\right) \tag{5.19}$$

where:

$$r = (x^2 + y^2)^{1/2}$$

Similarly:

$$Q = \frac{\pi a^4 \Delta p}{8\mu\ell}, \qquad w_m = \frac{a^2\Delta p}{8\mu\ell} \tag{5.20}$$

The coefficient of resistance λ of a pipe of length ℓ and diameter $d = 2a$ is related to the drop of pressure necessary to obtain a given rate of volume flow, and is defined through the equation:

$$\Delta p = \lambda \frac{\ell}{d} \frac{\rho w_m^2}{2} \tag{5.21}$$

Substituting Δp from the second equation in Equation 5.20, we obtain the elegant formula:

$$\lambda = \frac{64}{R_e} \tag{5.22}$$

where:

$$R_e = \frac{w_m d}{\nu}$$

Equation 5.22 is called the law of resistance for steady laminar pipe flow. Experiments have demonstrated that for $R_e < 2300$ the flow is always laminar but transition to turbulence is a possibility for $R_e \geqq 2300$.

Flow in the Entrance Region of a Circular Pipe

After the transients have died out, the flow in the entrance region, preceding the fully developed laminar flow region in which the solution is given by Equations 5.19, is approximately described by the equation[3a]

$$w_m \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

where

$$w_m = \frac{2}{a^2}\int_0^a rwdr$$

is the mean (average over the section) velocity. The first equation has the solution

$$w(r,z) = 2w_m\left(1 - \frac{r^2}{a^2}\right) - 4w_m\sum_{n=1}^{\infty}\frac{1}{\eta_n^2}\left[1 - \frac{J_0\left(\frac{r}{a}\eta_n\right)}{J_0(\eta_n)}\right]e^{-\lambda_n z}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{8\nu w_m}{a^2}\left[1 + 1/2\sum_{n=1}^{\infty} e^{-\lambda_n z}\right]$$

where

$$\lambda_n = \frac{2\eta_n^2}{aR_e}, \quad R_e = \frac{2w_m a}{\nu}$$

and J_0 is the Bessel's function of order zero (check by substitution). By using the definition of w_m described above and the following formulae involving the Bessel's functions

$$\zeta J_0(\zeta) = \frac{d}{d\zeta}[\zeta J_1(\zeta)]$$

$$J_2(\zeta) = \frac{2}{\zeta}J_1(\zeta) - J_0(\zeta)$$

we find that η_n are the roots of the equation

$$J_2(\eta_n) = 0$$

By numerical evaluation it can be established that the six roots

$$\eta_1 = 5.135622,\ \eta_2 = 8.417245,\ \eta_3 = 11.619807,$$
$$\eta_4 = 14.797853,\ \eta_5 = 17.878305,\ \eta_6 = 32.247696$$

give quite accurate results. Also the solution $w\ (r, z)$ should be used in the range $0.008\ R_e \leq z/a \leq 0.16\ R_e$, where R_e can have any value in the laminar flow regime.

(v) Nonsteady Unidirectional Flow

As in problem (i), $u \neq 0$, $v = w = 0$, p = constant or a function of time. The equation becomes:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{5.23}$$

It is also interesting to note that in this case the only nonzero component of vorticity is $\omega_z = \omega = -\partial u/\partial y$. Thus Equation 5.23 on differentiation with respect to y also yields:

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2} \tag{5.24}$$

Equation 5.24 gives the physical interpretation of the diffusion of vorticity in the y-direction due to viscosity. It is instructive to note that the solution:

$$\chi(y, t) = \frac{1}{2(\pi \nu t)^{1/2}} \exp\left(-\frac{y^2}{4\nu t}\right) \tag{5.25}$$

satisfies Equation 5.23 identically, and is called the fundamental solution of Equation 5.23. It satisfies the following conditions:

$$\chi \rightarrow 0 \quad \text{as} \quad t \rightarrow 0 \quad \text{when} \quad y \neq 0$$

$$\chi \rightarrow \infty \quad \text{as} \quad t \rightarrow 0 \quad \text{when} \quad y = 0$$

$$\int_{-\infty}^{\infty} \chi\, dy = 1$$

where the last property is due to the formula:

$$\int_{-\infty}^{\infty} \exp(-\xi^2)\, d\xi = \pi^{1/2}$$

Let the fluid occupy the region $-\infty < y < \infty$ and let the velocity at time $t = 0$ be a prescribed function of y, i.e., $u(y, 0) = f(y)$, where $f(y)$ is a continuous function of y. The solution of Equation 5.23 for $t > 0$ is then given by:

$$u(y, t) = \int_{-\infty}^{\infty} f(\alpha)\chi(y - \alpha, t)\, d\alpha \tag{5.26}$$

Note: Make the substitution $\alpha = y - 2(\nu t)^{1/2}\beta$ in Equation 5.26 and show that in the limit as $t \rightarrow 0 : u(y, 0) = f(y)$.

Stokes Problems

PROBLEM I: Impulsive Motion of an Infinite Plate Consider an infinite plate and impose the following conditions on the fluid flow (this problem is also known as the Rayleigh problem):

(a) $u = 0$ for all y at $t = 0$

(b) $u = 0$ for all t as $y \rightarrow \infty$

(c) $u = U_\infty$ at $y = 0$ for $t > 0$

Because of the conditions (a) and (b), we introduce a new variable η defined as:

$$\eta = y/2(\nu t)^{1/2} \tag{5.27}$$

which combines both conditions into one condition $\eta = \infty$. Making the transformation, Equation 5.27, in Equation 5.23 and defining:

$$u = U_\infty f(\eta)$$

we get:

$$f'' + 2\eta f' = 0 \tag{5.28}$$

The boundary conditions are

$$\begin{aligned} \eta &= 0 : f = 1 \\ \eta &= \infty : f = 0 \end{aligned} \tag{5.29}$$

The solution of Equation 5.28 under the conditions of Equation 5.29 is

$$u(y, t) = U_\infty \mathrm{erfc}(\eta) \tag{5.30}$$

where:

$$\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta)$$

$$\mathrm{erf}(\eta) = \frac{2}{\pi^{1/2}} \int_0^\eta e^{-\alpha^2}\, d\alpha$$

The vorticity diffuses from the plate into the fluid as:

$$\omega = \frac{U_\infty}{(\nu \pi t)^{1/2}} \exp\left(\frac{-y^2}{4\nu t}\right)$$

and the wall shear is given by:

$$\begin{aligned} \tau_w &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\mu\omega(0, t) \\ &= -U_\infty\left(\frac{\rho\mu}{\pi t}\right)^{1/2} \end{aligned}$$

The distance through which the effect of viscosity penetrates due to the motion of the plate is a function of time and is defined as:

$$\delta(t) = \frac{1}{U_\infty}\int_0^\infty u(y, t)\, dy = 2\left(\frac{\nu t}{\pi}\right)^{1/2}$$

Thus $\delta \sim t^{1/2}$ and increases with time.

PROBLEM II: Harmonic Oscillation of an Infinite Plate The fluid motion generated by the periodic oscillation of an infinite plate in its own plane can be obtained by imposing the following boundary conditons on Equation 5.23:

$$u = U_\infty \cos \omega t \quad \text{at} \quad y = 0$$

$$u = 0 \quad \text{as} \quad y \to \infty$$

where ω is the circular frequency of oscillation.

The solution of the problem can conveniently be obtained by introducing complex exponentials. Thus, assuming:

$$u = U_\infty e^{i\omega t} f(y), \qquad i = \sqrt{-1}$$

such that $f(0) = 1$ and $f(\infty) = 0$, we obtain:

$$\begin{aligned} u(y, t) &= \text{Real}[U_\infty e^{i\omega t} \cdot e^{-(i\omega/\nu)^{1/2} y}] \\ &= U_\infty \exp\left\{-y\left(\frac{\omega}{2\nu}\right)^{1/2}\right\} \cos\left\{\omega t - y\left(\frac{\omega}{2\nu}\right)^{1/2}\right\} \end{aligned} \tag{5.31}$$

The wall shear is

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = U_\infty(\mu\rho\omega)^{1/2} \sin\left(\omega t - \frac{\pi}{4}\right)$$

which shows that between the velocity and shear at the wall, there is a lag in phase of 45°.

Ekman Layer Problem

Consider a body of fluid on a surface when the surface is rotating at a constant angular velocity Ω, e.g., the Earth. We consider the surface to be almost flat and the fluid to have a horizontal free surface. On the free surface, we introduce a rectangular Cartesian system x, y, z, where x and y are in the free surface and z is measured positive vertically upward with z = 0 at the free surface. Following the notation of Reference 3 (Chapter 2), let the free surface be subjected to a constant shearing force μS along the x-direction. Since the motion is steady, the velocity distribution is

$$u = u(z), \quad \upsilon = \upsilon(z), \quad w = 0$$

Further, by absorbing the centripetal and the body forces, a modified pressure P is defined as

$$P = p + \rho\chi - \tfrac{1}{2}\rho\Omega^2 R^2$$

where χ is the body force potential ($f = -\,grad\,\chi$) and R is the perpendicular distance of a point from the axis of rotation. Since the velocity distribution is a function of z only, we have

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0.$$

Thus, the governing equations are obtained from Equation ii (Problem 3.4), which are

$$-2\Omega_3 \upsilon = \nu \frac{d^2 u}{dz^2}$$

$$2\Omega_3 u = \nu \frac{d^2 \upsilon}{dz^2}$$

The pressure gradient in the z-direction is balanced by the Coriolis force, i.e.,

$$2(\Omega_1 \upsilon - \Omega_2 u) = -\frac{1}{\rho}\frac{dP}{dz}$$

Note that $\Omega_3 = \Omega\cos\theta$, where θ is the angle between the vector $\boldsymbol{\Omega}$ and the unit vector **k** along the z-coordinate.

The solution of the coupled equations can simply be obtained by multiplying the second equation by $i = \sqrt{-1}$ and adding the two equations. Thus, writing

$$\lambda = u + i\upsilon, \quad \alpha^2 = 2i\Omega_3/\nu$$

we have

$$\frac{d^2\lambda}{dz^2} - \alpha^2\lambda = 0$$

Since $\lambda(-\infty) = 0$, the acceptable solution is

$$\lambda = Ae^{\alpha z}$$

where

$$\alpha = k(1 + i), \quad k = (\Omega_3/\nu)^{1/2}$$

Using the condition at the free surface

$$z = 0 \; : \; \mu\frac{du}{dz} = \mu S, \quad \mu\frac{d\upsilon}{dz} = 0$$

we obtain

$$A = \frac{S(1-i)}{2k}$$

The solution is then given by

$$u = \frac{S}{k\sqrt{2}}e^{kz}\cos\left(kz - \frac{\pi}{4}\right)$$

$$\upsilon = \frac{S}{k\sqrt{2}}e^{kz}\sin\left(kz - \frac{\pi}{4}\right)$$

Two important features of the Ekman's problem are as follows:

(a) The model equations exhibit a balance of the viscous and Coriolis forces.

(b) At the free surface ($z = 0$) the velocity vector is

$$\mathbf{u} = \frac{S}{2k}(\mathbf{i} - \mathbf{j})$$

and at a depth π/k (or $z = -\pi/k$) the velocity vector is

$$u = -\frac{S}{2k}e^{-\pi}(i - j)$$

Thus, at a depth π/k the velocity vector has decreased by a factor of $e^{-\pi}$ and its direction has become opposite to that at the free surface. The depth

$$\pi/k = \pi(\nu/\Omega_3)^{1/2}$$

is a measure of the Ekman layer thickness. Note that $\Omega_3 = \Omega\cos\theta$.

(vi) Motion Produced Due to a Vortex Filament

Let us consider the flow of a viscous fluid produced by a vortex filament in a fluid at rest. The streamlines are then concentric circles with their center on the vortex filament. The velocity field is purely circumferential and varies with the distance r from the center. The equations of motion are obtained by setting:

$$u_r = u_z = 0, \qquad u_\phi = v(r, t)$$

so that the equation of motion becomes:

$$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} \tag{5.32}$$

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{5.33}$$

The term $\partial p/\partial\phi = 0$, because if it is not zero then purely circumferential flow will not be possible.

For a nonviscous fluid the vortex filament is preserved indefinitely and the velocity field given by Equation 4.20b is

$$v = \frac{\Gamma_\infty}{2\pi r} \tag{5.34}$$

where Γ_∞ is the constant circulation of the vortex filament. In the case of a viscous fluid because of the internal friction a continuous supply of energy is essential to maintain the vortex filament or else it will decay with time.

Let us consider the motion produced by a vortex filament starting from a time $t = 0$, which amounts to assuming that the vortex filament came in existence instantaneously at $t = 0$. It is convenient to consider the vorticity equation in place of Equation 5.33 because of the ease in appyling the boundary conditions. The vorticity is defined as (refer to Problem 3.12b):

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(rv)$$

which when used in Equation 5.33 yields the equation:

$$\frac{\partial \omega}{\partial t} = \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right) \tag{5.35}$$

Because the vortex filament is assumed to have come into existence at $t = 0$ and the fluid is at rest at infinity, we prescribe the following conditions:

$$\text{(a)} \quad \omega = 0 \quad \text{when} \quad t = 0 \quad \text{for} \quad r > 0$$

$$\text{(a)} \quad \omega \to 0 \quad \text{when} \quad t \to \infty \quad \text{for} \quad r > 0 \tag{5.36}$$

The solution of Equation 5.35 under the initial conditions Equation 5.36 is

$$\omega = \frac{\alpha_0}{t} \exp\left(\frac{-r^2}{4\nu t}\right)$$

so that:

$$v = \frac{2\alpha_0 \nu}{r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$

Since at $t = 0$ the velocity is given by Equation 5.34, then $\alpha_0 = \Gamma_\infty/4\pi\nu$. Thus the velocity and vorticity are given by:

$$v = \frac{\Gamma_\infty}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right] \tag{5.37a}$$

$$\omega = \frac{\Gamma_\infty}{4\pi\nu t} \exp\left(-\frac{r^2}{4\nu t}\right) \tag{5.37b}$$

From Equations 5.37 we find that at time $t = 0$ the flow is irrotational ($\omega = 0$) and the velocity is given by the formula in Equation 5.34. For $t > 0$, the solution is rotational and the vorticity decays with increasing r. Further the vortex filament which is at $r = 0$, decays with increasing time. The solution of Equation 5.37 obtained above yields infinite values of the angular momentum and the kinetic energy for unbounded domains. However, there exists yet another solution of Equations 5.35 and 5.36 which removes these defects and was obtained by Taylor (cf. Reference 114). It is easy to verify that the other solution is

$$\omega = \frac{\beta_0}{t^2}\left(1 - \frac{r^2}{4\nu t}\right) \exp\left(-\frac{r^2}{4\nu t}\right) \tag{5.38a}$$

$$v = \frac{\beta_0 r}{2t^2} \exp\left(-\frac{r^2}{4\nu t}\right) \tag{5.38b}$$

If the total angular momentum of the fluid in the region is $\rho\Lambda$, then using:

$$\rho\Lambda = \int_0^{2\pi}\int_0^{\infty} \rho r v \cdot r \, d\theta \, dr$$

we get:

$$\beta_0 = \frac{\Lambda}{8\pi\nu^2}$$

(vii) Two-Dimensional Stagnation Point Flow (Hiemenz Flow)

Consider a fluid flowing uniformly along the direction of the negative y-axis toward a plane vertical wall, with $y = 0$ being the wall itself. One of the streamlines of the uniform flow coming parallel to the y-axis impinges the wall normally and the stream spreads both along the positive and negative x-direction. The point of impingement at $x = 0$, $y = 0$ is the stagnation point. For an inviscid fluid the components of velocity in the neighborhood of the stagnation point are obtained by applying the continuity, irrotationality and the inviscid boundary condition. The velocity components are

$$u_e = \alpha x, \qquad v_e = -\alpha y \tag{5.39}$$

where α is a constant. To obtain the viscous solution, we set:

$$y = \eta\left(\frac{\nu}{\alpha}\right)^{1/2}, \qquad u = \alpha x f'(\eta), \qquad v = -(\nu\alpha)^{1/2} f(\eta), \qquad -\frac{1}{\rho}\frac{\partial p}{\partial x} = \alpha^2 x \tag{5.40}$$

where a prime denotes differentiation with respect to η. The introduction of Equation 5.40 in Equation 5.2 yields a single ordinary differential equation:

$$f''' + ff'' + 1 - f'^2 = 0 \tag{5.41}$$

with the boundary conditions:

$$\begin{aligned} &\eta = 0 : f = f' = 0 \\ &\eta \to \infty : f' \to 1 \end{aligned} \tag{5.42}$$

Equation 5.41 can only be solved numerically, and the tabulated values of $f(\eta)$ are available in Reference 1, p. 90. Knowing $f(\eta)$ and $f'(\eta)$ one can predict the velocity distribution in the neighborhood of the stagnation point. The skin friction coefficient c_f defined as:

$$c_f = \tau_w / \tfrac{1}{2}\rho u_e^2$$

is

$$c_f = 2f''(0)(\nu/\alpha)^{1/2}/x, \qquad x \neq 0$$

where $f''(0) = 1.2326$. Having obtained $f(\eta)$, the second equation of motion (viz., Equation 5.3) yields $-1/\rho\, \partial p/\partial y$. Thus using the last equation of Equation 5.40, the pressure distribution becomes known as:

$$p = p_0 - \tfrac{1}{2}\rho\alpha^2 x^2 - \rho\nu\alpha(\tfrac{1}{2}f^2 + f')$$

(viii) Axially Symmetric Stagnation Point Flow (Homann Flow)

In this case, a fluid stream approaches an axially symmetric blunt nosed body at a zero angle of attack. Using cylindrical coordinates r, ϕ, z, with z as the axis of symmetry along which the impinging streamline is directed, the stagnation point is at the nose of the cylindrical body where $r = z = 0$. Because of the axial symmetry, all quantities are independent of ϕ and we also set $u_\phi = 0$.

The inviscid fluid velocity components in the neighborhood of the stagnation point are

$$u_{re} = \alpha r, \qquad u_{ze} = -2\alpha z \tag{5.43}$$

where α is a constant. For the viscous solution we use the equations in cylindrical coordinates and set:

$$z = \left(\frac{\nu}{2\alpha}\right)^{1/2}\eta, \qquad u_r = \alpha r f'(\eta), \qquad u_z = -(2\nu\alpha)^{1/2}f(\eta), \qquad -\frac{1}{\rho}\frac{\partial p}{\partial r} = \alpha^2 r \tag{5.44}$$

in the equation for u_r and obtain an ordinary differential equation:

$$f''' + ff'' + \tfrac{1}{2}(1 - f'^2) = 0 \tag{5.45}$$

The boundary conditions are

$$\begin{aligned} \eta &= 0 : f = f' = 0 \\ \eta &\rightarrow \infty : f' \rightarrow 1 \end{aligned} \tag{5.46}$$

Numerical solution of Equation 5.45 is also available in Reference 1, p. 90. The coefficient of skin friction c_f is

$$c_f = 2f''(0)(2\nu/\alpha)^{1/2}/r, \qquad r \neq 0$$

where $\sqrt{2}\,f''(0) = 1.312$. Having obtained $f(\eta)$, the equation of motion for u_z yields $-1/\rho\ \partial p/\partial z$, which when used with the last equation of Equation 5.44 yields the pressure near the stagnation point as $p = p_0 - {}^1/_2\, \rho\alpha^2 r^2 - 2\rho\nu\alpha({}^1/_2 f^2 + f')$.

(ix) Motion Between Two Inclined Plates

Let two infinite flat plates intersect along the z-axis at an angle 2α. We choose the x-axis as the line along which the two plates are symmetrically placed. The motion within the region thus formed is two-dimensional and is shown in Figure 5.2.

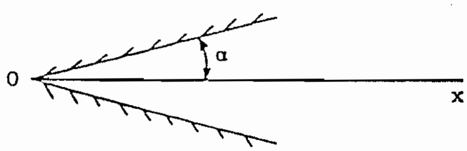


Fig. 5.2 Flow between two inclined plates.

An important point to note is that the flow between the plates is possible only when the point of intersection is either a sink or a source. If 0 is a sink, then fluid flows radially inward thus forming a converging channel flow. If 0 is a source, then fluid flows radially outward thus forming a diverging channel flow. (Refer to the configuration of Figure 5.2.) Choosing cylindrical coordinates r, ϕ, and z we have:

$$u_r = v(r, \phi), \qquad u_\phi = 0, \qquad u_z = 0$$

The equations are then:

$$\frac{\partial}{\partial r}(rv) = 0 \tag{5.47a}$$

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{5.47b}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \phi} = \frac{2\nu}{r}\frac{\partial v}{\partial \phi} \tag{5.47c}$$

The boundary conditions are

$$v = 0 \quad \text{at} \quad \phi = \pm\alpha \tag{5.47d}$$

Equation 5.47a shows that rv is independent of r so that:

$$rv = f = f(\phi) \tag{5.48a}$$

Substituting Equation 5.48a in Equation 5.47c and integrating with respect to ϕ, we get:

$$\frac{p}{\rho} = \frac{2\nu}{r^2}f + \psi(r) \tag{5.48b}$$

Substituting Equations 5.48a, b in Equation 5.47b, we obtain:

$$\frac{d^2f}{d\phi^2} + 4f + \frac{f^2}{\nu} = \frac{r^3}{\nu}\frac{d\psi}{dr}$$

Since the left-hand side is a function of ϕ and the right-hand side is a function of r, then both must be equal to the same constant, say A. Thus:

$$\frac{d^2f}{d\phi^2} + 4f + \frac{f^2}{\nu} = A \tag{5.49}$$

$$\frac{r^3}{\nu}\frac{d\psi}{dr} = A \tag{5.50}$$

Integrating Equation 5.50, the pressure given in Equation 5.48b becomes:

$$\frac{p}{\rho} = \frac{\nu}{r^2}\left(2f - \frac{A}{2}\right) + \text{constant} \tag{5.51}$$

To integrate Equation 5.49, we first nondimensionalize it as follows. Let $f_0 = f(0) =$ constant, then we set:

$$u = f/f_0, \qquad \eta = \phi/\alpha, \qquad R_e = \alpha f_0/\nu$$

On transformation Equation 5.49 becomes:

$$u'' + 4\alpha^2 u + \alpha R_e u^2 = B \tag{5.52a}$$

where $B = \alpha^2 A/f_0$. Multiplying Equation 5.52a by u' and integrating once, we have:

$$\begin{aligned} u'^2 &= \frac{-2\alpha R_e}{3}u^3 - 4\alpha^2u^2 + 2Bu + C \\ &= P, \text{ (say)} \end{aligned} \tag{5.52b}$$

Thus:

$$\eta = \pm \int \frac{du}{\sqrt{P}} + D \tag{5.53}$$

The boundary conditions are determined through Equation 5.47d as:

$$u = 0 \quad \text{for} \quad \eta = \pm 1$$

These, however, are not enough to determine the three constants B, C, and D. To overcome this problem we note that since the maximum of u (which is 1) occurs at $\eta = 0$, we must have:

$$u' = 0 \quad \text{at} \quad \eta = 0$$

Thus the cubic in Equation 5.52b must have one of its zeros at $u = 1$. Writing $C = k$, we have:

$$2B = \frac{2\alpha R_e}{3} + 4\alpha^2 - k$$

where now the constant B is expressed as a function of $C = k$. From Equation 5.52b we also note that at $\eta = \pm 1$, $u'^2 = k$, so that k must be a nonnegative constant. Based on these arguments, we rewrite Equation 5.52b as

$$u'^2 = (1 - u)\left\{\frac{2\alpha R_e}{3}(u^2 + u) + 4\alpha^2 u + k\right\}$$

Thus, considering the upper half of the channel, we have:

$$\eta = \int_u^1 \frac{dy}{(1 - y)^{1/2}\left\{\frac{2\alpha R_e}{3}(y^2 + y) + 4\alpha^2 y + k\right\}^{1/2}} \tag{5.54}$$

Equation 5.54 gives the functional dependence of the velocity distribution on η. The difficulty here is that although both α and R_e can be prescribed, the constant k remains an unknown. The constant k is related with the radial pressure gradient at the wall and according to:

$$u'^2 = k \quad \text{at} \quad \eta = \pm 1$$

it must be determined as part of the solution. Only when $|R_e| \rightarrow \infty$, is it possible to express k in terms of R_e. To determine k under this condition, we set $u = 0$ in Equation 5.54 and have:

$$1 = \int_0^1 \frac{dy}{(1 - y)^{1/2}\left\{\frac{2\alpha R_e}{3}(y^2 + y) + 4\alpha^2 y + k\right\}^{1/2}} \tag{5.55}$$

In the case of converging channel flow, $f_0 < 0$. Thus, for $R_e \rightarrow -\infty$ the integral (Equation 5.55) can be nonzero only when the quadratic expression in the integrand tends to zero as $y \rightarrow 1$ so that the integral becomes improper. Thus:

$$k \rightarrow -\tfrac{4}{3}\alpha R_e, \qquad R_e < 0$$

and then Equation 5.54 becomes:

$$\eta \cong \int_u^1 \left(\frac{-2\alpha R_e}{3}\right)^{-1/2} \frac{dy}{(1 - y)(y + 2)^{1/2}} \tag{5.56}$$

or:

$$1 - \eta \cong (-\tfrac{1}{2}\alpha R_e)^{-1/2} \left\{\tanh^{-1}\left(\frac{u + 2}{3}\right)^{1/2} - \tanh^{-1}\left(\frac{2}{3}\right)^{1/2}\right\}$$

The formula for u then becomes:

$$u = 3 \tanh^2\{(-\tfrac{1}{2}\alpha R_e)^{1/2}(1 - \eta) + \tanh^{-1}(\tfrac{2}{3})^{1/2}\} - 2 \tag{5.57}$$

which is a weak solution of $\eta = \phi/\alpha$. We therefore conclude that a converging laminar flow for high Reynolds number is, for the most part, a potential flow. However, near the wall region ($\eta \cong 1$) the velocity changes rapidly, thus indicating that the effect of viscosity is confined to the wall region.

In the case of diverging channel flow, $f_0 > 0$ and thus $R_e > 0$. The symmetry of motion, along with the maximum $u = 1$ at the axis, can be achieved only for selected pairs (R_e, k). For $k = 0$, Equation 5.55 yields:

$$\left(\frac{2\alpha R_e}{3}\right)^{1/2} = \int_0^1 \frac{dy}{\{y(1 - y)(1 + y + 6\alpha R_e^{-1})\}^{1/2}}$$

Assuming R_e to be fairly large, we have:

$$\left(\frac{2\alpha R_e}{3}\right)^{1/2} \cong \int_0^1 \frac{dy}{\{y(1 - y^2)\}^{1/2}}$$

which gives:

$$\alpha R_e \cong 10.31 \tag{5.58}$$

The foregoing analysis shows that the diverging channel flow for large R_e does not become a potential flow in any region of the channel. Further, the solution for large R_e under the imposed conditions exists for a given α when the estimate (Equation 5.58) is satisfied.

5.3 EXACT SOLUTIONS IN SLOW MOTION

In some cases of physical interest the nonlinear and local acceleration terms are very small in comparison with the pressure and viscous terms. This implies that the inertia forces are very small in comparison with the viscous and pressure forces. To carry out this type of approximation we consider the incompressible equations (Equations 3.24 and 3.30) with the body force vector $\mathbf{f} = 0$. Let L be a characteristic length of the body and U a characteristic velocity of the flow. Using the parameters L, U, μ, and ρ we introduce the following nondimensionalized variables:

$$t^* = \frac{Ut}{L}, \quad \mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad p^* = \frac{L(p - p_\infty)}{\mu U}$$

$$\mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad R_e = \frac{UL}{\nu}$$

where p_∞ is the pressure at infinity. It must be noted that we have defined the nondimensional pressure p^* through $\mu U/L$, which is a representative viscous stress, in place of the reference dynamic pressure $1/2\ \rho U^2$, because for slow motion the viscous and pressure forces are of the same order of magnitude. Carrying out the nondimensionalization, we have:

$$\text{div}^*\mathbf{u}^* = 0 \tag{5.59a}$$

$$R_e\left\{\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \text{grad}^*)\mathbf{u}^*\right\} = -\text{grad}^* p^* + \nabla^{*2}\mathbf{u}^* \tag{5.59b}$$

$$\mathbf{u}^* = 0 \text{ at the body surface} \tag{5.59c}$$

$$\mathbf{u}^* \to \mathbf{U}/U \quad \text{as} \quad |\mathbf{x}^*| \to \infty \tag{5.59d}$$

Here:

$$R_e = \frac{UL}{\nu} = \frac{\rho U^2}{L}\Big/\frac{\mu U}{L^2}$$

$$= \text{Inertial forces/viscous forces}$$

If the Reynolds number $R_e \ll 1$, then the left-hand side of Equation 5.59b is negligible in comparison to the right-hand side. Reverting to the dimensional variables, we have:

$$\text{div}\ \mathbf{u} = 0 \tag{5.60a}$$

$$\text{grad}\ p = \mu\nabla^2\mathbf{u} \tag{5.60b}$$

The set of Equations 5.60 is called the Stokes system.[2] From Equations 5.60 we also have:

$$\nabla^2\boldsymbol{\omega} = 0 \tag{5.61a}$$

where:

$$\text{div}\ \boldsymbol{\omega} = 0 \tag{5.61b}$$

Also:

$$\nabla^2 p = 0 \tag{5.62}$$

Flow Past a Rigid Sphere

The case of flow past a sphere in an unbounded fluid was first considered by Stokes. We choose the origin of the spherical polar coordinates at the center of the sphere and the axis $\theta = 0$ in the direction of the incident stream $\mathbf{U}$. The problem can be treated easily in terms of the stream function ψ and taking into consideration the axial symmetry of the flow. The set of Equations 5.60a, b reduce to one equation in ψ as (refer to Equation viii of Problem 3.12):

$$D^4\psi = 0 \tag{5.63}$$

where:

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1-\alpha^2}{r^2}\frac{\partial^2}{\partial \alpha^2}; \qquad \alpha = \cos\theta$$

The velocity components along the radius and at right angles to it (i.e., u_r and u_θ, respectively) are given by:

$$u_r = \frac{-1}{r^2}\frac{\partial\psi}{\partial\alpha} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$u_\theta = \frac{-1}{\sqrt{1-\alpha^2}}\frac{\partial\psi}{\partial r} = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

The boundary conditions for the fourth order equation (Equation 5.63) are obtained from the conditions:

$$u_r = u_\theta = 0 \quad \text{at} \quad r = a$$

$$u_r = U\cos\theta, \qquad u_\theta = -U\sin\theta, \quad \text{at} \quad r = \infty \tag{5.64}$$

where a is the radius of the sphere and U the magnitude of the velocity along the axis $\theta = 0$. From the boundary conditions at an infinite distance from the sphere we conclude that:

$$\psi = \tfrac{1}{2}Ur^2\sin^2\theta \quad \text{as} \quad r \to \infty \tag{5.65}$$

Based on this we *envisage* a solution of Equation 5.63 in the form:

$$\psi = \sin^2\theta f(r) \tag{5.66}$$

which when substituted in Equation 5.63 gives the equation for $f(r)$ as:

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2f}{dr^2} - \frac{2f}{r^2}\right) = 0$$

A trial solution of the form r^n gives:

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4 \tag{5.67}$$

where A, B, C, D are constants. Substituting Equation 5.67 in Equation 5.66, we obtain:

$$u_r = 2\left(\frac{A}{r^3} + \frac{B}{r} + C + Dr\right)\cos\theta$$

$$u_\theta = -\left(-\frac{A}{r^3} + \frac{B}{r} + 2C + 3Dr\right)\sin\theta \tag{5.68}$$

Applying the boundary conditions (Equation 5.64) we get:

$$A = \frac{Ua^3}{4}, \qquad B = \frac{-3aU}{4}, \qquad C = \frac{U}{2}, \qquad D = 0$$

Consequently:

$$\psi = \frac{r^2}{2}U\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\sin^2\theta \tag{5.69}$$

is the solution of Equation 5.63 which satisfies all the boundary conditions, including Equation 5.65. The velocity components are

$$u_r = U\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]\cos\theta \tag{5.70a}$$

$$u_\theta = -U\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]\sin\theta \tag{5.70b}$$

The solutions (Equations 5.69 and 5.70) are for the case when the sphere is at rest and the velocity of the fluid at infinity is U. To obtain the solution when the fluid is at rest at infinity and the sphere is moving with velocity $-U$, we subtract $U\cos\theta$ and $-U\sin\theta$ from Equations 5.70a, b, respectively, and have:

$$u_r = -U\left[\frac{3a}{2r} - \frac{a^3}{2r^3}\right]\cos\theta \tag{5.71a}$$

$$u_\theta = U\left[\frac{3a}{4r} + \frac{a^3}{4r^3}\right]\sin\theta \tag{5.71b}$$

The corresponding stream function is then:

$$\psi = \frac{-3Uar}{4}\left(1 - \frac{a^2}{3r^2}\right)\sin^2\theta \tag{5.72}$$

The components of velocity with reference to a Cartesian coordinate system can now be obtained quite easily. Let the origin of coordinates as before be at the center of the sphere such that the x_3-axis corresponds to the ray $\theta = 0$. If u_1, u_2, and u_3 are the velocity components along x_1, x_2, and x_3, respectively, then the relations between the Cartesian and the spherical polar components with $u_\phi = 0$ are

$$\begin{aligned} u_1 &= \frac{x_1 u_r}{r} + \frac{x_1 x_3}{r\sqrt{x_1^2 + x_2^2}}\, u_\theta \\ u_2 &= \frac{x_2 u_r}{r} + \frac{x_2 x_3}{r\sqrt{x_1^2 + x_2^2}}\, u_\theta \\ u_3 &= \frac{x_3 u_r}{r} - \frac{\sqrt{x_1^2 + x_2^2}}{r}\, u_\theta \end{aligned} \tag{5.73}$$

where:

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$\theta = \tan^{-1}\frac{\sqrt{x_1^2 + x_2^2}}{x_3}$$

$$\phi = \tan^{-1}\frac{x_2}{x_1}$$

Substituting u_r and u_θ from Equation 5.70, the Cartesian components of velocity for a sphere at rest in a stream of fluid of velocity U parallel to the x_3-axis are

$$u_1 = \frac{3U}{4}\frac{ax_1x_3}{r^3}\left(\frac{a^2}{r^2} - 1\right) \tag{5.74a}$$

$$u_2 = \frac{3U}{4}\frac{ax_2x_3}{r^3}\left(\frac{a^2}{r^2} - 1\right) \tag{5.74b}$$

$$u_3 = U\left\{1 - \frac{3a}{4r} - \frac{a^3}{4r^3} + \frac{3a}{4}\frac{x_3^2}{r^3}\left(\frac{a^2}{r^2} - 1\right)\right\} \tag{5.74c}$$

Having obtained the velocity distribution, we can find the pressure p through the use of Equation 5.60b which is

$$p = p_\infty - \frac{3\mu U a x_3}{2r^3} \tag{5.75}$$

where p_∞ is the pressure at infinity ($r = \infty$). To calculate the drag force, we proceed as follows.

Any of the Cartesian velocity components given in Equations 5.74 can be obtained from the expression:

$$u_i = \frac{3U}{4}\frac{ax_3x_i}{r^3}\left(\frac{a^2}{r^2} - 1\right) + U\delta_{i3}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) \tag{5.76}$$

where δ_{ij} is the Kronecker delta, and $i = 1, 2, 3$. We can now easily find the force per unit area on the surface of the sphere in the direction i as:

$$(\tau_i)_{r=a} = (n_jT_{ij})_{r=a} = \left[n_j\left\{-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}\right]_{r=a} \tag{5.77}$$

Using Equations 5.75 and 5.76 in Equation 5.77 and noting that:

$$n_j = x_j/r$$

we get the i-th component of the force as:

$$(\tau_i)_{r=a} = -p_\infty n_i + \frac{3\mu U\delta_{i3}}{2a} \tag{5.78}$$

The drag force is a vector whose i-th component is obtained by integrating Equation 5.78 over the surface of the sphere. Denoting the force vector by $\mathbf{F}$, we have:

$$F_i = a^2\int_0^{2\pi} d\phi \int_0^{\pi} (\tau_i)_{r=a} \sin\theta \, d\theta$$

By using the relations:

$$x_j = an_j, \qquad x_1 = a\sin\theta\cos\phi$$

$$x_2 = a\sin\theta\sin\phi, \qquad x_3 = a\cos\theta$$

over the surface of the sphere, we get:

$$F_1 = F_2 = 0$$

$$F_3 = 6\pi\mu U a \tag{5.79}$$

The force $D = F_x = 6\pi\mu Ua$ is the drag acting on a sphere due to the streaming flow past it with velocity U. First, the total force is only due to the viscosity of the fluid. Further, note that the streamlines are symmetrically disposed with the direction of the streaming flow, or in the direction of motion of the sphere if it is moving in a fluid at rest. Thus there is no wake behind the sphere which is always experimentally expected on the leeward side of bluff bodies.

From the value of drag, we can obtain the drag coefficient C_D which is a nondimensional coefficient defined as the drag force divided by $1/2\ \rho U^2$ and by the area of the body projected on a plane normal to the direction of the velocity U. Thus the drag coefficient for a sphere becomes:

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2} = 12 \bigg/ \frac{Ua}{\nu} = \frac{24}{R_e} \tag{5.80}$$

where $R_e = 2Ua/\nu$ is the Reynolds number based on the diameter of the sphere.

Flow Past a Rigid Circular Cylinder

The case of flow past a cylinder of infinite length normal to its axis was also considered by Stokes. Because of the two-dimensional nature of the flow, it is natural to introduce plane polar coordinates r and θ. The stream function ψ for the slow motion is then given by Equation iii of Problem 3.12(b):

$$\nabla^4\psi = 0 \tag{5.81}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The velocity components along the directions of r and θ are, respectively:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}$$

The boundary conditions for the fourth order Equation 5.81 are obtained from the conditions:

$$\begin{aligned} &u_r = u_\theta = 0 \quad \text{at} \quad r = a \\ &u_r = U\cos\theta, \qquad u_\theta = -U\sin\theta, \quad \text{at} \quad r = \infty \end{aligned} \tag{5.82}$$

where a is the radius of the cylinder and U is the magnitude of the velocity along the axis $\theta = 0$. From the boundary conditions at infinity we conclude that:

$$\psi = rU\sin\theta \quad \text{as} \quad r \rightarrow \infty \tag{5.83}$$

Based on this form of ψ, we envisage a solution of Equation 5.81 in the form:

$$\psi = UF(r)\sin\theta$$

which when substituted in Equation 5.81 gives:

$$\frac{d^2F_1}{dr^2} + \frac{1}{r}\frac{dF_1}{dr} - \frac{F_1}{r^2} = 0 \tag{5.84a}$$

where:

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{F}{r^2} = F_1 \tag{5.84b}$$

Solving first the homogeneous Equation 5.84a and then the nonhomogeneous Equation 5.84b we get:

$$\psi = U\left(\frac{A}{r} + Br + Cr\ell nr + Dr^3\right)\sin\theta \tag{5.85}$$

where A, B, C, and D are constants to be determined by the prescribed boundary conditions. If we first apply the boundary conditions at infinity, then we get $C = D = 0$, so that:

$$\psi = U\left(\frac{A}{r} + r\right)\sin\theta$$

By applying the boundary conditions at the wall we get two different values for the same constant A so that a unique solution of the Stokes problem for a circular cylinder does not exist. This is called the Stokes paradox.

A useful solution for the slow motion past a circular cylinder can be obtained by first applying the exact wall boundary conditions and then determining the remaining unknown by a better approximation than provided by the Stokes equations. Taking $D = 0$ and applying the wall conditions, we get:

$$u_r = \frac{CU}{2}\left[\frac{a^2}{r^2} - 1 - 2\ell n\frac{a}{r}\right]\cos\theta \tag{5.86a}$$

$$u_\theta = \frac{CU}{2}\left[\frac{a^2}{r^2} - 1 + 2\ell n\frac{a}{r}\right]\sin\theta \tag{5.86b}$$

where C remains a free constant to be determined later. Let x_1, x_2 be the Cartesian coordinates with the origin at the center of the cylinder such that the free-stream flow occurs in the x_1-direction. The velocity components along x_1 and x_2 are then given by:

$$u_1 = u_r\cos\theta - u_\theta\sin\theta$$

$$u_2 = u_r\sin\theta + u_\theta\cos\theta$$

so that the velocity distribution for a cylinder at rest in a streaming flow U in the direction of x_1 is

$$u_1 = CU\left[x_1^2\left(\frac{a^2}{r^4} - \frac{1}{r^2}\right) - \left(\frac{a^2}{2r^2} - \frac{1}{2}\right) - \ell n\frac{a}{r}\right] \tag{5.87a}$$

$$u_2 = CUx_1x_2\left(\frac{a^2}{r^4} - \frac{1}{r^2}\right) \tag{5.87b}$$

where $r^2 = x_1^2 + x_2^2$. The pressure is given by:

$$p_\infty - p = \frac{-2\mu CUx_1}{r^2} \tag{5.87c}$$

To obtain the velocity distribution when the fluid is at rest at infinity and the cylinder moves with a velocity $-U$ in the direction of x_1, we simply subtract U from Equation 5.87a so that the expression for u_2 remains as in Equation 5.87b but Equation 5.87a becomes:

$$u_1 = -U + CU\left[x_1\left(\frac{a^2}{r^4} - \frac{1}{r^4}\right) - \left(\frac{a^2}{2r^2} - \frac{1}{2}\right) - \ell n \frac{a}{r}\right] \tag{5.88}$$

the expression for pressure remains the same as Equation 5.87c.

Using the velocity distribution and the expression for the pressure we can calculate the force per unit area acting on the cylinder by the formula:

$$(\tau_i)_{r=a} = \left[-pn_i + \mu n_j\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]_{r=a}$$

The magnitude of the drag on the circular cylinder per unit length is then given by:

$$D = 4\pi\mu UC \tag{5.89}$$

The difficulties associated in obtaining the constant C by the given conditions of the problem can now be analyzed. For large values of r the viscous terms can be estimated to be

$$|\mu\nabla^2\mathbf{u}| \sim \frac{\mu UC}{r^2}$$

while the estimate of the inertia terms which have been neglected is

$$|\rho(\mathbf{u}\cdot\text{grad})\mathbf{u}| \sim \frac{\rho U^2C^2}{r}\ell n\frac{r}{a}$$

Thus, the ratio of the inertia and viscous forces is of the order of

$$\frac{CR_e}{2}\left(\frac{r}{a}\ell n\frac{r}{a}\right)$$

where $R_e = 2aU/\nu$. This shows that for a given value of r, say r_0, a Reynolds number exists such that the ratio of the inertia and viscous terms is small and Stokes' equations form a valid approximation. However, for the same value of R_e, if a value of $r > r_0$ is chosen then the Stokes approximation fails to exist. Thus Stokes' approximation is *not a uniformly valid approximation*. Even in the case of a sphere, where we had no difficulty in obtaining the solution, an estimate of the ratio of the inertia and viscous terms is of the order $rR_e/2a$, which for a fixed small R_e increases with increasing r. *Therefore, the Stokes solution for a sphere is also not a uniformly valid solution*. The nonuniformity of the Stokes solutions at infinity led Oseen to include some nonlinear effects in the Stokes equations. Details of Oseen's solution are available in Lamb,[3] and on this basis, the constant C in Equation 5.89 is obtained as:

$$C = \left[\ell n\left(\frac{7.4}{R_e}\right)\right]^{-1}, \quad R_e = \frac{2aU}{\nu}$$

In Figure 5.3 measured drag coefficients for spheres and cylinders have been shown to demonstrate their behavior for varying Reynolds' numbers. In particular, the purpose here is to emphasize the wide range of R_e where the pressure and inertia are the main contributors to C_D.

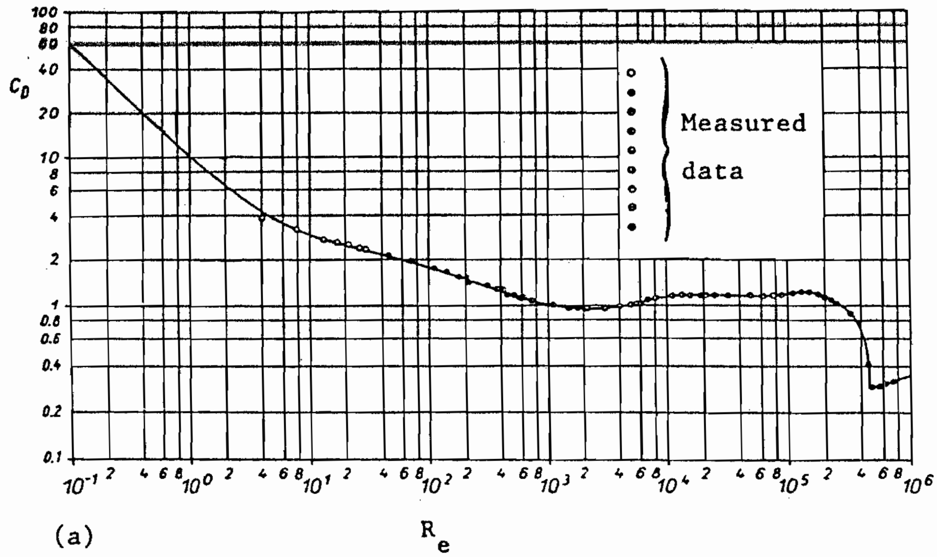


Fig. 5.3(a) Measured drag coefficients for cylinders. $c_D = D/\frac{1}{2}\rho U^2 A$, $A = 2aL$, $L = 1$; $c_D = 8\pi C/R_e$ for $R_e \leq 1$. (From Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1968. With permission.)

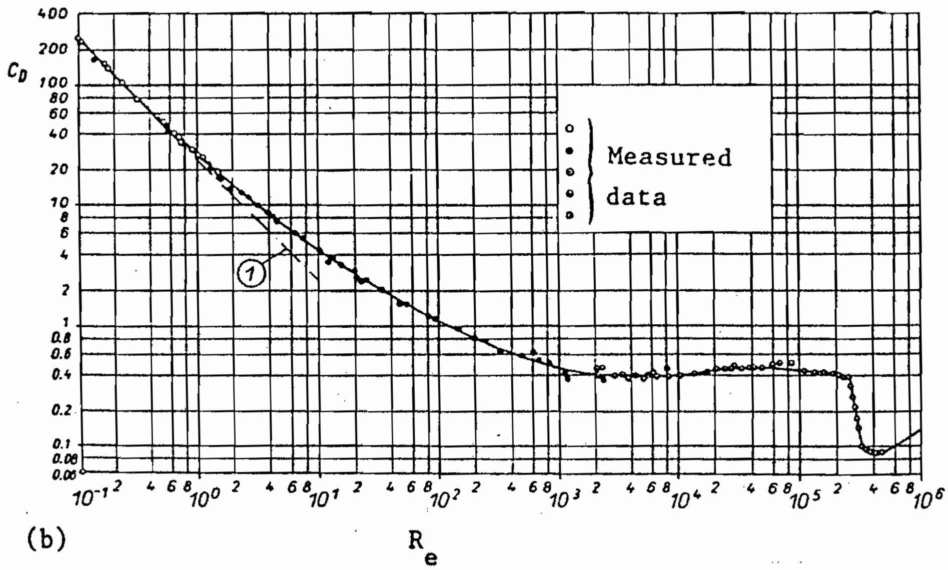


Fig. 5.3(b) Measured drag coefficients for spheres. $c_D = D/\frac{1}{2}\rho U^2 A$, $A = \pi a^2$; $c_D = 24/R_e$ for $R_e \leq 1$, curve ①. (From Schlichting, H., *Boundary Layer Theory*, McGraw-Hill, New York, 1968. With permission.)

This discussion is available in many standard texts, e.g., Reference 1. A summary discussion is given here to emphasize the role of pressure, inertia, and viscous forces. To fix ideas we take Figure 5.3 (b).

1. For $R_e \leq 1$, Equation 5.80 is applicable. In this case there is no flow separation and the drag is the friction drag.

2. For $1 < R_e < 10^3$ the drag coefficient C_D drops continuously. In this regime the flow separates and, in consequence, a wake is formed which creates a pressure field. Thus, the total drag (also

called the profile drag) is the sum of the friction and pressure drags. Experiments have shown that at $R_e \simeq 10^3$ the friction drag is about 5% of the profile or the total drag.

3. For $10^3 < R_e < (3)(10^5)$ the C_D curve is relatively flat. In this regime the wake flow changes from laminar to turbulent state, but the attached viscous layer (the boundary layer) in the forward portion of the sphere is laminar, which separates just upstream of 90°. In the turbulent wake the pressure is essentially constant, but lower than the pressure in the forward part of the sphere. This pressure difference is the main contributor to the pressure or form drag.
4. For R_e just larger than $(3)(10^5)$ the viscous layer in the forward portion undergoes a transition from laminar to turbulent state. The point of separation moves downward, thus reducing the size of the wake. This causes an abrupt reduction in C_D.
5. The drag reduction associated with the turbulent viscous layer in the forward part of the sphere does not occur at a unique value of R_e. For very smooth spheres it can be $(4)(10^5)$, and for rough spheres it can be as low as $(5)(10^3)$.

Example 5.1

A steel ball of radius a and density ρ_S is dropped in a fluid of density ρ and viscosity μ. The terminal speed of the ball is defined to be the speed when its acceleration is zero. Use the C_D vs. R_e curve for spheres (Figure 5.3 (b)) and determine the terminal speed v of the ball.

Given:

$$\rho = 1.94 \text{ slug/ft}^3, \quad \mu = 0.000021 \text{ slug/ft-s}, \quad a = 0.05 \text{ ft}$$

$$\rho_S = 14.6 \text{ slug/ft}^3.$$

At the terminal speed v:

$$\text{Drag force} + \text{Buoyance force} = \text{Weight}$$

or,

$$1/2\rho\pi v^2 a^2 \, C_D + 4/3\pi a^3 \rho g = 4/3\pi a^3 \rho_S g$$

Thus

$$v^2 = agk_1/C_D$$

where

$$k_1 = (8/3)\,(\rho_S/\rho - 1)$$

Further

$$R_e = 2av\rho/\mu$$

or,

$$R_e = k_2/C_D^{1/2} \tag{i}$$

where

$$k_2 = 2a\rho\,(agk_1)^{1/2}/\mu$$

Thus, for the given data, the pair ($R_e \approx (7)(10^4)$, $C_D = 0.5$) on the curve in Figure 5.3(b) satisfies Equation i. This yields the terminal speed v = 7.5445 ft/s.

Part II: Boundary Layers

5.4 INTRODUCTION

The subject of fluid dynamics before the opening decade of the 20th century was already firmly established as a separate branch of theoretical physics. All the basic laws of fluid motion and their physical and mathematical interpretations were discovered and further enriched by the pioneering works of Euler, Cauchy, Lagrange, Bernoulli, Stokes, Thompson, Helmholtz, Reynolds, and Lamb. Each one of these pioneer researchers attempted to solve the equations of motion in one situation or another, but it was soon realized that this aspect is the most difficult part of the whole subject. The main difficulty in obtaining the solutions was (and is) the basic nonlinearity of the equations which cannot be removed except in some very specialized situations. However, it was found that a class of solutions of the nonlinear equations can be obtained, in an indirect manner, if it is assumed a priori that the viscous effects are negligible and the fluid moves without vorticity (i.e., the fluid is inviscid and the motion is irrotational). We have already discussed this aspect of flow in Chapter 4.

The inviscid irrotational fluid theory occupied the attention of many researchers for a long time. Some very important and useful results were established for two-dimensional inviscid irrotational fluid flow problems through an effective use of the theory of functions of complex variables. Despite the exact nature of these solutions, all such solutions predicted that the force required in translating a body in steady motion through an infinite expanse of ideal incompressible fluid is zero. This result, which is quite contrary to physical experience, was obtained much earlier by Lagrange and d'Alembert and is usually known as d'Alembert's paradox. The explantion of this paradox, of course, lies in the complete neglect of fluid viscosity.

The d'Alembert paradox, although a stunning blow to the ideal irrotational flow theory, acted as a catalyst in the development of more realistic and physically acceptable theories and solutions of the equations of motion. Based on this paradox and other independent observations it was concluded that no matter how small the viscosity of the fluid may be, its effects cannot be neglected altogether. Further, based on the much debated adherence (no-slip) condition it was concluded that the most likely place where the viscosity effect cannot be neglected is the region in the neighborhood of the surface of a body. This conclusion gave rise to important developments in the design of practical airfoil shapes based on the theories and solutions already available for the inviscid irrotational flows. The important modification which was introduced in the inviscid fluid theory was the inclusion of a layer of negligible thickness on the surface of the airfoil where the vorticity was assumed to be concentrated. The layer of concentrated vorticity is called the vortex sheet; its effect in the ideal fluid theory was taken indirectly through the circulation produced around an airfoil having a sharp trailing edge. Details of this subject have been discussed earlier in Chapter 4.

The other major exploitation of the conclusion, that the viscosity effects are not negligible near the surface of the body, was made by Prandtl in 1904. Prandtl's idea was that in problems of flow of slightly viscous fluids past bodies the frictional effects are confined to a thin layer of fluid adjacent to the surface of the body. In his classical paper, Prandtl[4] showed that the motion of a small amount of fluid involved in this boundary layer (German: Grenzschicht) decides such important matters as the frictional drag, heat transfer, and transfer of momentum between the body and the fluid. The postulate of a thin boundary layer in fluid motion past bodies simplifies the flow problems to a certain extent and permits a division of the flow field in two regions — a flow field outside the boundary layer assumed devoid of viscosity and capable of being described by the inviscid fluid theory, and a flow field in the immediate vicinity of the body where viscous effects are important and described appropriately by the simplified Navier-Stokes equations. The problem of motion in the whole flow field therefore reduces to finding the motion in the boundary layer subject to the conditions of zero velocity at the surface (the no-slip condition) and an inviscid predetermined known flow at the outer edge of the boundary layer. The theory involving these simplifications is known as the *boundary layer theory* and has received ever increasing attention since its inception. Boundary layer theory is so rich in its content and meaning that it has received much consideration both from the theoretical and experimental viewpoints. The literature on boundary layer theory is quite extensive and we shall therefore give a very thorough fundamental sketch by quoting only the most important literature.

5.5 FORMULATION OF THE BOUNDARY LAYER PROBLEM

Boundary layer theory is applicable to those problems in fluid flow in which the fluid is only slightly viscous, i.e., when the coefficient of viscosity μ or the kinematic viscosity ν tends to zero. The most appropriate parameter which combines the relative orders of magnitude of the inertia and viscosity is the Reynolds number $R_e = \rho U_\infty L/\mu$, where L and U_∞ are the representative length and velocity of the flow, respectively, and ρ the density of the fluid. Therefore, boundary layer theory is applicable when the Reynolds number is very large: *the largeness of R_e being implicitly understood to be due to the smallness of μ or ν.*

To be able to expose the inherent nature of a boundary layer, we shall first consider the equations for the two-dimensional incompressible flow in a Cartesian coordinate system. From Equations 5.1–5.3:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.90}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{5.91}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{5.92}$$

We shall assume that the direction of main flow is along the direction of positive x, and y is everywhere normal to it. Further, u and v are the velocity components along x and y, respectively.

Since the flow is two-dimensional, the direction of the vorticity vector is along the positive direction of z, where x, y, z form a right-handed coordinate system. The only nonzero component of vorticity is ω which is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Now we introduce a flat body (a flat plate, a body of very large radius of curvature) in the flow such that x is measured along the surface and y everywhere normal to it. The no-slip condition then demands that:

$$\text{at the surface } y = 0 : u = v = 0 \tag{5.93a}$$

The flow condition at upstream infinity is due to a uniform flow u_∞ so that:

$$\text{as } x \longrightarrow -\infty \text{ and/or } y \longrightarrow \infty : u \longrightarrow u_\infty \tag{5.93b}$$

We supplement Equations 5.90–5.92 with another set of equations obtained by setting $\nu = 0$ in the Navier-Stokes equations, or equivalently setting $R_e = \infty$. The equations so obtained are the Euler equations. Denoting all quantities by a subscript e, the Euler equations are

$$\frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} = 0 \tag{5.94a}$$

$$\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + v_e \frac{\partial u_e}{\partial y} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} \tag{5.94b}$$

$$\frac{\partial v_e}{\partial t} + u_e \frac{\partial v_e}{\partial x} + v_e \frac{\partial v_e}{\partial y} = -\frac{1}{\rho}\frac{\partial p_e}{\partial y} \tag{5.94c}$$

$$\omega_e = \frac{\partial v_e}{\partial x} - \frac{\partial u_e}{\partial y}$$

The Euler equations (Equations 5.94b, c) are of lower order than the Navier-Stokes equations; as such, all the boundary conditions stated in Equations 5.93a, b cannot be satisfied. In particular, because of the neglect of viscous forces, we cannot satisfy the no-slip condition so that the requisite boundary conditions now become:

$$\text{at } y = 0 : v_e = 0 \tag{5.95a}$$

$$\text{at infinity} : u_e \rightarrow u_\infty \tag{5.95b}$$

According to the definition of a boundary layer, the effect of viscosity is confined in a small neighborhood of the wall and outside of it the flow is essentially nonviscous. Thus the tangential velocity u which is zero at the surface rises to a value equal to that of the inviscid flow within a short distance normal to the surface. Consequently, the gradients in the direction of the normal to the surface must be much larger than in the tangential direction. With this physical basis in mind we assess the order of magnitude of each term in the equations by introducing the order symbol -0. We say that a quantity α is of the same order as another quantity β if the limit of the ratio α/β is a finite nonzero quantity and denote it as $\alpha = 0(\beta)$.

To evolve a pattern for the comparison of various quantities, we introduce a quantity δ which is assumed to be of the order of thickness of the viscous layer, i.e., the boundary layer. Thus according to the physical basis of a boundary layer, we conclude that y is at most of the order δ and:

$$\frac{\partial}{\partial y} = 0\left(\frac{1}{\delta}\right) \tag{5.96a}$$

The derivatives $\partial/\partial t$ and $\partial/\partial x$, the velocity component u, and the pressure p take part in a slowly varying process and therefore are taken as operations and quantities of order one, i.e.:

$$\frac{\partial}{\partial t} = 0(1), \quad \frac{\partial}{\partial x} = 0(1), \quad u = 0(1)$$

$$\frac{\partial u}{\partial t} = 0(1), \quad \frac{\partial u}{\partial x} = 0(1), \quad \frac{\partial p}{\partial x} = 0(1) \tag{5.96b}$$

We are now in a position to assess the order of magnitude of the velocity component v normal to the surface. From Equation 5.90 we have:

$$v = -\int_0^y \frac{\partial u}{\partial x}\, dy$$

since $v = 0$ at $y = 0$. Using the mean-value theorem of the integral calculus we have:

$$v = -y\left(\frac{\partial u}{\partial x}\right)_{y=y_1}$$

where y_1 is a number between 0 and y. Since:

$$\frac{\partial u}{\partial x} = 0(1)$$

we conclude that v is at most of the order of δ. Consequently:

$$v = 0(\delta), \qquad \frac{\partial v}{\partial y} = 0(1) \tag{5.96c}$$

Assessing each term of Equation 5.91 according to the order of magnitude approximation given in Equations 5.96a, b, c, we find that for the viscous terms to be really important the quantity δ must be $0(\sqrt{\nu})$. Thus, the term $\partial^2u/\partial y^2$ is of a higher order of magnitude than the term $\partial^2u/\partial x^2$, which is therefore neglected. If we carry out the same type of order of magnitude analysis for each term of Equation 5.92, then:

$$\frac{\partial p}{\partial y} = 0(\delta)$$

The complete set of approximated equations are then:

$$\frac{\partial u_b}{\partial x} + \frac{\partial v_b}{\partial y} = 0 \tag{5.97a}$$

$$\frac{\partial u_b}{\partial t} + u_b\frac{\partial u_b}{\partial x} + v_b\frac{\partial u_b}{\partial y} = -\frac{1}{\rho}\frac{\partial p_b}{\partial x} + \nu\frac{\partial^2 u_b}{\partial y^2} \tag{5.97b}$$

$$\frac{\partial p_b}{\partial y} = 0(\delta) \tag{5.97c}$$

where a subscript b denotes the boundary layer variable.

In the same manner, the boundary layer approximation of the vorticity ω is

$$\omega_b = -\frac{\partial u_b}{\partial y} \tag{5.97d}$$

The important point to realize now is that unlike the Euler equations (Equations 5.94) the boundary layer equations (Equations 5.97a, b) satisfy the exact surface boundary conditions of the viscous flow, i.e.:

$$\text{at } y = 0 : u_b = v_b = 0 \tag{5.97e}$$

In obtaining the boundary conditions at the outer (extreme) edge of the boundary layer, a number of valid possibilities come to mind. Here it is important to emphasize that Prandtl's boundary layer approximation applies for $R_e \to \infty$ or $\nu \to 0$. Thus, the boundary layer is thin, and as a first approximation *the velocity u_b at the edge should be equal to the tangential slip velocity of the inviscid flow at the surface.*

One of the possibilities mentioned above is to take:

$$u_b(x, \delta, t) = u_e(x, \delta, t)$$

Let us explore this idea through an example of flow past of flow past a circular cylinder. Using Equations 4.29b we have:

$$u_\theta(x, \delta) = u_e(x, \delta) = 2u_\infty\left(1 - \frac{\delta}{a} + \ldots\right) \sin\frac{x}{a}$$

$$u_r(x, \delta) = v_e(x, \delta) = 2u_\infty\left(\frac{\delta}{a} + \ldots\right) \cos\frac{x}{a}$$

where $\theta = x/a$ is considered positive in the clockwise sense. Neglecting terms $0(\delta)$, we get:

$$u_b(x, \delta) = u_e(x, 0)$$

$$v_e(x, \delta) = 0$$

Another way to arrive at the same result is to invoke the natural requirement that the tangential velocity u_b and the vorticity ω_b must match with their corresponding counterparts of the outer flow at a distance $y \geqslant \delta$. Further, this matching must be asymptotically smooth. These requirements imply that:

$$\text{as } y \to \delta : u_e - u_b \to 0, \qquad \omega_e - \omega_b \to 0 \tag{5.98a}$$

and

$$\frac{\partial^n}{\partial y^n}(u_e - u_b) \to 0 \quad \text{for} \quad n \geqslant 1 \tag{5.98b}$$

Thus, according to Equation 5.97d and the definition of ω_e, we have:

$$\frac{\partial v_e}{\partial x} \to \frac{\partial}{\partial y}(u_e - u_b) \quad \text{as} \quad y \to \delta$$

Using the smoothness condition (Equation 5.98b), we conclude that:

$$\frac{\partial v_e}{\partial x} \to 0 \quad \text{as} \quad y \to \delta \tag{5.98c}$$

This condition can only be satisfied if $v_e \equiv 0$. However, $v_e = 0$ only at the wall $y = 0$; therefore if the Euler solution u_e is evaluated at the wall $y = 0$, then the above condition becomes automatically satisfied. That is, the velocity u_e with which the tangential component u_b matches at the outer edge must be a function of x and t only. With this stipulation, the condition (Equation 5.98b) yields:

$$\frac{\partial^n u_b}{\partial y^n} \to 0 \quad \text{as} \quad y \to \delta \quad \text{for all } n \geqslant 1 \tag{5.98d}$$

Thus, we reach the important conclusion that the *boundary layer solution matches at the outer edge with that Euler solution which has been evaluated at the surface of the body.* This conclusion is consistent with Equation 5.97c, and thus:

$$p_b \cong p_e(x, 0, t)$$

Consequently, from Equation 5.94b:

$$-\frac{1}{\rho}\frac{\partial p_e}{\partial x} = \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x}$$

The important conclusion drawn from these equations is that the pressure in the boundary layer is prescribed independently of the vicous flow field. Because of this property, we sometimes emphasize this point by saying that the pressure is *impressed* on the boundary layer by the external (outer) inviscid flow field.

Another important conclusion for two-dimensional boundary layers is based on Equation 5.98c and $u_e = u_e(x, 0, t)$ which implies that $\omega_e = 0$. That is, the external flow in a two-dimensional boundary layer is a *potential* flow, i.e., the flow is irrotational.

We now summarize the boundary layer equations for two-dimensional incompressible flow in terms of the dimensional variables.

1. Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.99}$$

2. Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.100}$$

3. Pressure:

$$\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} \tag{5.101}$$

4. Boundary conditions:

$$\text{at } y = 0 : u = v = 0$$

$$\text{at } y \rightarrow \delta : u \rightarrow u_e(x, t) \tag{5.102}$$

Note that for convenience sake we have dropped the subscript b because any solution that satisfies Equations 5.99–5.102 must be a boundary layer solution. It is shown later that Equations 5.99 and 5.100 are also valid for a longitudinally curved surface when x is the actual curvilinear distance along the surface.

According to the standard classification of the partial differential equations Equation 5.100 is a parabolic quasilinear partial differential equation. Therefore, beside the boundary conditions (Equation 5.102) one must also specify an intial condition at a fixed time for nonsteady flow, or, at a fixed x for steady flow.

Method of Inner and Outer Limits

The order of magnitude analysis as followed in the previous section is not sufficiently rigorous. A more rigorous point of view on the derivation of the boundary layer equations is contained in the researches of Kaplun[5] Lagerstrom,[6] and Van Dyke,[7] which we shall now follow. As before, we call the region exterior to the boundary layer as the *external or outer region*. The flow in this region is governed by the Euler equations of motion. This means that in the external region the viscous effects are so small in comparison with those of inertia that we can set $\nu = 0$ directly wherever it appears explicitly. The other region is that of the boundary layer, also called the inner region, where the effects of viscosity and inertia are equally important and thus both are retained. To see the inner structure of the boundary layer, we require a magnifying technique, which in effect is provided by stretching the boundary layer coordinate y, obtained by dividing y by a small parameter. The most appropriate parameter which is small enough for our purpose and dictated by the order of magnitude method is

$$\epsilon = 1/\sqrt{R_e}, \qquad R_e = u_\infty L/\nu$$

which defines the stretched coordinate as:

$$\bar{y} = y/\epsilon$$

Note that $\nu \rightarrow 0$ implies $\epsilon \rightarrow 0$.

Now we define two limit processes, called the outer and inner limits of any flow variable. Let $f(x, y, \nu)$ be a flow variable (velocity component, pressure, etc.). The outer limit is defined as the limit of the function f as $\nu \rightarrow 0$ while keeping the point (x, y) fixed. In symbolic form:

$$\text{outer limit} = \lim_{\substack{\nu \to 0 \\ x, y \text{ fixed}}} f(x, y, \nu) = f_e(x, y) \tag{5.103a}$$

Thus, the solution of the Euler equations is the outer limit of the Navier-Stokes solution at a point (x, y) away from the surface. The inner limit is obtained by first affecting a transformation through the stretched variable $\bar{y}$ and then the limit of the function is taken as $\nu \rightarrow 0$ by keeping x and $\bar{y}$ fixed. Symbolically:

$$\text{inner limit} = \lim_{\substack{\nu \to 0 \\ x, \bar{y} \text{ fixed}}} f(x, \bar{y}, \nu)$$

$$= f_i(x, \bar{y}) \tag{5.103b}$$

Note that the requirement, $\bar{y}$ fixed as $\nu \rightarrow 0$, amounts to approaching more and more toward the surface. Thus, to obtain the boundary layer equations, we first make the stretching transformation:

$$\bar{y} = \frac{y}{\epsilon} \quad \text{and} \quad \bar{v} = \frac{v}{\epsilon} \tag{5.104}$$

in the continuity and the Navier-Stokes equations and then take the inner limit of each term of the equations. The end result will be to obtain Equations 5.99 and 5.100, which are the boundary layer equations as derived earlier.

The final point to be clarified is about the matching of the inner and outer solution. It is quite natural to think that both solutions will be equally valid in an overlap region. In the overlap region the inner limit of the outer solution must be equal to the outer limit of the inner solution. The outer solution is $u_e(x, y)$ and the inner solution is $u_b(x, \bar{y})$, so that in the overlap region:

$$\lim_{\substack{\nu \to 0 \\ x, \bar{y} \text{ fixed}}} u_e(x, \epsilon\bar{y}) = \lim_{\substack{\nu \to 0 \\ x, y \text{ fixed}}} u_b\left(x, \frac{y}{\epsilon}\right)$$

Thus:

$$u_e(x, 0) = u_b(x, \infty) \tag{5.105}$$

Note that we have obtained the previous result that the boundary layer solution at the outer edge of the layer matches with the solution of the Euler equations evaluated at the surface. The result (Equation 5.105) also justifies the use of the ∞ in place of δ for the outer edge of the boundary layer. However, it must be emphasized that the use of ∞ in Equation 5.105 signifies the attainment of nonviscous conditions, although in practice a finite value α of $\bar{y}$ serves the purpose of ∞. Thus the outer conditions are achieved at:

$$\bar{y} = \alpha$$

showing that the thickness of the shear layer is proportional to $\epsilon = R_e^{-1/2}$.

The inner limit of the vorticity ω and the stream function ψ are as follows:

$$\omega_b = \frac{1}{\epsilon} \lim_{\substack{\nu \to 0 \\ x, \bar{y} \text{ fixed}}} (\epsilon\omega)$$

$$\psi_b = \epsilon \lim_{\substack{\nu \to 0 \\ x, \bar{y} \text{ fixed}}} (\psi / \epsilon)$$

5.6 BOUNDARY LAYER ON TWO-DIMENSIONAL CURVED SURFACES

Up to this stage we have carried out the boundary layer approximation to the Navier-Stokes equations written only in the Cartesian coordinates x, y. We shall now obtain the boundary layer equations for the viscous flow past a two-dimensional longitudinally curved surface by using orthogonal curvilinear coordinates. The Navier-Stokes equations in the orthogonal curvilinear coordinates have already been discussed in Chapter 3. (Refer to Equations 3.115–3.118.)

Let us choose ξ along the surface and η everywhere normal to it with $\eta = 0$ as defining the surface itself. Because of the two-dimensional flow, the velocity component $u_3 = 0$ and all derivatives with respect to ζ vanish identically. Further since η is measured normal to the surface, the scale factor $h_2 = 1$ but $h_1 = h_1(\xi, \eta)$. To obtain the expression for $h_1(\xi, \eta)$ we consider Figure 5.4 and find that for the two curves parallel to the body contour C_B we have:

$$d\theta = \frac{h_1\, d\xi}{R_C} = \frac{\left(h_1 + \dfrac{\partial h_1}{\partial \eta} d\eta\right) d\xi}{R_C + d\eta}$$

$$= \frac{h_1\, d\xi}{R_C}\left(\frac{1 + \dfrac{1}{h_1}\dfrac{\partial h_1}{\partial \eta} d\eta}{1 + \dfrac{d\eta}{R_C}}\right)$$

Thus:

$$\frac{1}{R_C} = \frac{1}{h_1}\frac{\partial h_1}{\partial \eta}$$

The longitudinal curvature of the body $k(\xi)$ is then:

$$k(\xi) = \frac{1}{R_B} = \left(\frac{1}{h_1}\frac{\partial h_1}{\partial \eta}\right)_{\eta=0}$$

which is satified by taking:

$$h_1(\xi, \eta) = h_1(\xi, 0)[1 + \eta k(\xi)]$$

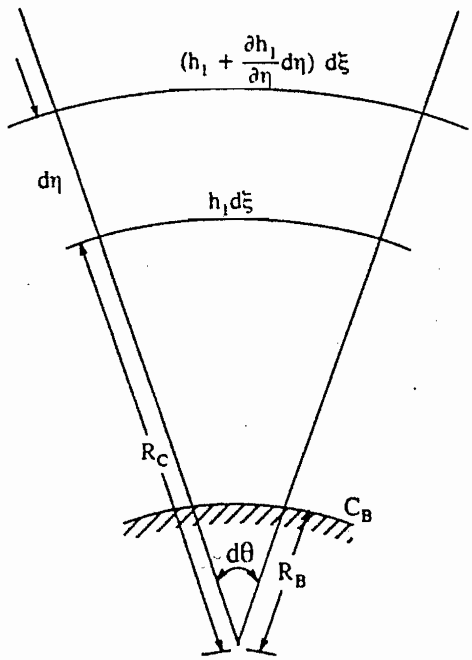


Fig. 5.4 Curves parallel to the basic body curve.

Writing:

$$h_1(\xi, 0) = h_{1w}, \qquad 1 + \eta k(\xi) = h \tag{5.106a}$$

we then have:

$$h_1(\xi, \eta) = hh_{1w} \tag{5.106b}$$

In the Navier-Stokes equations (Equations 3.115–3.118), we now set h_1 as defined in Equation 5.106b, $h_2 = 1$, $u_1 = u$, $u_2 = v$, $\eta = y$, $u_3 = 0$, $\partial/\partial\zeta = 0$. Further, we define the arc length along the body surface as x, where:

$$x = \int_0^{\xi} h_{1w}\, d\xi \tag{5.107a}$$

thus:

$$\frac{\partial}{\partial x} = \frac{1}{h_{1w}} \frac{\partial}{\partial \xi} \tag{5.107b}$$

The process of obtaining the boundary layer equations is now quite straightforward. We can use either the order of magnitude method or the method of inner limit and have the equations:

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(hv) = 0 \tag{5.108}$$

$$\frac{\partial u}{\partial t} + \frac{u}{h}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho h}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.109}$$

$$\frac{ku^2}{h} = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{5.110}$$

where h has already been defined in Equation 5.106a.

Now two cases arise depending on the behavior of the curvature $k(x)$ of the surface. If $k(x)$ is small (R_c large), then $h = 1 + ky$ can be taken as $h = 1$. However, if $k(x)$ is large ($R_C = 0(\epsilon)$), then the form of h given in Equation 5.106a must be retained. It is interesting to note that since for small and moderately curved surfaces $h \cong 1$, the boundary layer equations (Equations 5.108 and 5.109) become exactly of the same form as obtained earlier in Cartesian coordinates (refer to Equations 5.99 and 5.100). The interpretation for the curved surfaces is that now x is a curvilinear distance measured along the surface and y is everywhere normal to the surface with $y = 0$ being the surface itself. Refer to Problem 5.7(b).

The important difference between the boundary layer equations for a curved and a flat surface actually lies in the variation of pressure across the boundary layer. Referring to Equation 5.110, we find that even for $h \cong 1$, the pressure gradient $\partial p/\partial y$ is a quantity of the 0(1) in contrast to it being of the $0(\delta)$ in the case of a flat surface (refer to Equation 5.97c). However, the variation of pressure can still be ignored provided that $k(x)$ is not excessively large. Most cylindrical shapes, such as airfoils, ellipses, circles, etc. have moderate curvatures; and the pressure variation across the thin boundary layer developing over these surface can be ignored.

Boundary Layer Parameters

Discussion regarding the conditions at the outer edge makes it clear that the external conditions for the boundary layer solution are approached asymptotically as $y \to \infty$ on the boundary layer scale. Alternatively, the asymptotic conditions are also available when $y/\epsilon \to \infty$, but now y is a finite physical distance measured from the surface. Both these definitions defy a precise definition of the actual distance $y = \delta$ where the external conditions are applicable. The magnitude δ is called the boundary layer thickness. A definition of δ which is usually adopted in experimental work or in numerical solution of the equations is to take δ as that value of y where the boundary layer tangential velocity is about 0.99 of the external velocity.Despite such a loose definition of the boundary layer thickness, it is conceptually an important length parameter of the boundary layer.

Besides the boundary layer thickness, there are other length parameters which can be precisely defined to fix the structure of a boundary layer. The first such length is defined on the basis of the mass flow rate through the boundary layer along the direction of the main flow. The mass flow rate through a boundary layer of thickness δ is

$$(\text{unit length along } z) \int_0^\delta \rho u_b \, dy$$

and it cannot be equal to:

$$\text{(unit length along } z) \; \rho_e u_e \delta$$

because the fluid has been slowed down by viscosity within the boundary layer, thus producing a deficiency in the mass flow. However, there exists a length δ^* such that:

$$\int_0^\delta \rho_e u_e \, dy - \int_0^\delta \rho u \, dy = \rho_e u_e \delta^*$$

The length parameter δ^* is called the *displacement thickness* of the boundary layer and according to the above balance equation:

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \tag{5.111}$$

The quantity $\rho_e u_e \delta^*$ is the mass flow defect due to the action of viscosity. This mass flow defect causes the external-flow streamline to be displaced away by a distance δ^*. Because of this displacement effect, the fluid particles in the free stream consider their motion to be past a modified body rather than past the original body. This modified body is obtained by augumenting the original body contour in the y-direction by the local displacement thickness δ^*. For thin boundary layers the aforementioned modification to the body contour is usually not carried out. In those cases where it is necessary to incorporate this modification, the solution obtaining process can only be an iterative one. (Refer to Section 5.20.)

The second length parameter is defined on the basis of the rate of transfer of momentum through the boundary layer. The rate of momentum transfer across the boundary layer cannot be equal to the rate of momentum transfer if there had been no viscosity. Consequently, there exists a length θ such that:

$$\int_0^\delta (\rho u) u_e \, dy - \int_0^\delta \rho u^2 \, dy = \rho_e u_e^2 \theta$$

From this balance equation we have:

$$\theta = \int_0^\delta \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e}\right) dy \tag{5.112}$$

where θ is called the momentum thickness.

Similarly, following these arguments:

$$\int_0^\delta (\rho u) u_e^2 \, dy - \int_0^\delta \rho u^3 \, dy = \rho_e u_e^3 \Theta$$

from which:

$$\Theta = \int_0^\delta \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u^2}{u_e^2}\right) dy \tag{5.113}$$

Θ is called the energy thickness of the boundary layer.

The four length parameters, δ, δ^*, θ, and Θ are very useful in describing the structure of a boundary layer developing over the surface of a body. Besides the length parameters discussed above, there are other parameters discussed as follows:

1. The viscous shear stress exerted by the moving fluid past an object is obtained by evaluating σ_{xy} on the surface of the body:

$$(\sigma_{xy})_{y=0} = \left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]_{y=0} = \left[\mu\left(\frac{\partial u}{\partial y}\right)\right]_{y=0} \text{ (force/unit area)}$$

We shall denote the wall shear by a subscript w as:

$$\tau_w = \left[\mu\left(\frac{\partial u}{\partial y}\right)\right]_{y=0} \tag{5.114}$$

which is obviously a function of the surface coordinate x. The coefficient of skin-friction is then defined as:

$$c_f = \frac{\tau_w}{1/2\rho u_e^2} \tag{5.115}$$

2. The dissipation of energy per unit volume per unit time due to viscosity has already been defined in Equation 2.79. Under the boundary layer approximation the dissipation of energy is given by:

$$\Phi \cong \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{5.116}$$

A boundary layer problem is said to be solved when the velocity profile u/u_e; the length parameters δ^*, θ, Θ; the skin-friction coefficient c_f; and the dissipation function Φ are known. This information may be obtained by experimental measurements or by solution of the equations. For incompressible flow, $\rho_e = \rho =$ constant and $\mu =$ constant.

5.7 SEPARATION OF THE TWO-DIMENSIONAL STEADY BOUNDARY LAYERS

The action of viscosity in the phenomenon of fluid motion past a surface is to retard the velocity and consequently decrease the fluid momentum within the boundary layer. Although according to the boundary layer theory, the retarding action of viscosity is dominant only near the surface; nevertheless, it exerts a very strong influence on the overall flow behavior past the surface. Realizing that the fluid motion is governed by the pressure distribution impressed on the boundary layer, we must take into consideration the interplay of the retarding action of viscosity and the imposed pressure distribution. If the pressure distribution is such that it decreases along the direction of the main flow, then the boundary layer will remain attached to the surface. On the other hand if there is a pressure increase in the direction of the main flow, then because of the reduced fluid momentum within the boundary layer, the boundary layer cannot remain attached to the surface. The detachment of the boundary layer from the surface is known as *boundary layer separation*. Once separation has taken place, the vorticity which was initially confined in the boundary layer now finds its way into the outside streaming flow, thus disturbing the outer flow. The separation of the boundary layer also causes the formation of a wake behind the body in which there usually occurs eddying or turbulent motion. The formation of wake behind the body causes a substantial rise in the overall drag of the body. The part of total drag due to wake formation is known as the "form drag".

Besides the overall disturbance which the boundary layer separation creates in the flow field around the body, there is an end of the boundary layer theory itself after separation. The reason lies in the fact that after separation the approximations on which the boundary layer theory was based are no longer applicable. Sometimes this difficulty is referred to as a *singularity* of the governing differential equations. Next we define the boundary layer separation on simple physical concepts and then prove some assertions analytically.

We shall consider the boundary layer Equations 5.108–5.110 with $h = 1$ for the steady flow, which are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.117}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.118}$$

$$u_e\frac{\partial u_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5.119}$$

$$\text{at } y = 0 : u = v = 0$$

$$\text{at } y \to \infty : u \to u_e(x) \tag{5.120}$$

where x is the curvilinear distance along the surface and y is normal to it.

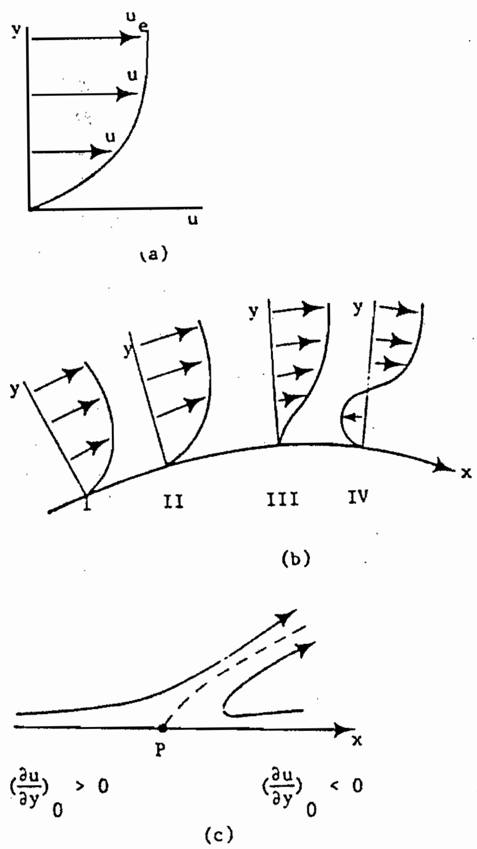


Fig. 5.5 Shapes of boundary layer velocity profiles along a surface.

Figure 5.5(a) shows a sketch of the boundary layer velocity profile before separation at a particular point on the surface. This profile is concave toward the y-axis and satisfies the boundary conditions (Equation 5.120). Since $u = 0$ at $y = 0$ and $u \to u_e$ as $y \to \infty$, $\partial u/\partial y$ decreases monotonically from its maximum at the surface through positive values until it becomes zero at the outer edge. The second derivative $\partial^2 u/\partial y^2$ is, therefore, definitely negative for points near the outer edge. The velocity profiles for this case are I and II of Figure 5.5(b).

Next we evaluate all terms of Equation 5.118 at $y = 0$ to have:

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5.121}$$

If now $\partial p/\partial x \leq 0$ (pressure decreasing along the x-direction, favorable), then $\partial^2 u/\partial y^2$ is negative right across the boundary layer. On the other hand if $dp/dx > 0$ (pressure increasing along the x-direction, adverse), then:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} > 0$$

and since it must in any case become negative near the outer edge, it changes sign somewhere in the boundary layer. The point along y where the sign changes is the point where $\partial^2 u/\partial y^2 = 0$, showing that at this value of y the velocity profile has a point of inflection. (See profile III of Figure 5.5b.) The fact that $\partial^2 u/\partial y^2$ is positive for a distance along y in the adverse pressure gradient implies that $\partial u/\partial y$ is no longer a monotonically decreasing function. The distribution of $\partial u/\partial y$ at a point on the surface in the adverse pressure region is such that it predicts a lower value at the surface, increases for a distance along y (to allow $\partial^2 u/\partial y^2$ to become positive), and finally decreases to zero toward the outer edge. At subsequent downstream positions on the surface it predicts lower values of $(\partial u/\partial y)_{y=0}$. This mechanism continues until at a point of the surface $(\partial u/\partial y)_{y=0}$ becomes zero. The point where $(\partial u/\partial y)_{y=0} = 0$ or the skin-friction $\tau_w = 0$ is the point of separation. Beyond this point $(\partial u/\partial y)_{y=0}$ becomes negative showing that u has reversed its direction. This situation has been depicted in Figure 5.5(c), where P is the point on the surface where $(\partial u/\partial y)_{y=0} = 0$. The broken curve is a streamline which distinguishes between the streamlines of the separated and the back flow. Profile IV of Figure 5.5(b) is a typical velocity profile just after the separation point P.

To get a better appreciation of the phenomenon under consideration, we must insist that what has been discussed until now is peculiar only to the separation of boundary layers and not to the separation of general viscous flows past bodies. By the phrase "general viscous flow" we have in mind the common phenomenon of fluid separation past bodies which occurs even for quite low Reynolds number (even $R_e = 10$) where no boundary layer theory is applicable. Therefore, the use of terms such as "imposed pressure gradient", "singularity", etc. must be looked at within the framework of the boundary layer theory.

The first important point to note in boundary layer separation is that the position of separation, say $x = x_s$, does not depend on any particular value of the Reynolds number R_e. This conclusion is supported by the fact that the boundary layer velocity profiles are similar with respect to the Reynolds number. (See Problem 5.4.) Therefore, x_s does not depend on R_e but only on the pressure gradient $\partial p/\partial x$. The second important point is in regard to the behavior of the boundary layer solution near the separation point and to find why the equations are singular at this point.

In answer to these questions Goldstein[8] was the first to examine the boundary layer solution at the separation point and found that the velocity distribution at separation has an algebraic singularity. Details of Goldstein's results are also available in Reference 9. The occurrence of singularity at separation and the behavior of the velocity components u and v at that point was also studied by Landau and Lifshitz.[10] Their analysis with a slight change is described next.

The main outcome of the boundary layer approximation is that the velocity component v normal to the surface is of a much smaller magnitude than the tangential component throughout the boundary layer, (refer to Equations 5.96c and 5.104). As the point of separation is approached the boundary layer becomes thicker, thus extending itself more toward the mainstream. This means that the normal velocity component plays a greater role due to its increased magnitude, and finally in the neighborhood of the separation point it becomes of the same order of magnitude as the velocity component u. According to Equation 5.104, for v to be of the same order as u means that v has increased by a factor of $1/\epsilon = R_e^{-1/2}$. Thus on the boundary layer scale we assert that near separation the velocity v has become infinite.

Let $x = x_s$ be the separation point on the surface; then according to the above discussion:

$$v(x_s, y) \to \infty, \qquad y \neq 0$$

and:

$$\left(\frac{\partial v}{\partial y}\right)_{x=x_s} \to \infty, \qquad y \neq 0$$

From the continuity equation we therefore conclude that:

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_s} \to \infty$$

or:

$$\left(\frac{\partial x}{\partial u}\right)_{u=u_0} \to 0 \qquad \textbf{(5.122a)}$$

where $u_0(y) = u(x_s, y)$ is the velocity profile at $x = x_s$. In the neighborhood of the separation point $x = x_s$ both $x_s - x$ and $u - u_0$ are assumed to be small. Writing:

$$\chi = x_s - x, \qquad \xi = u - u_0$$

we can expand the function $\chi = \chi(\xi)$ in a McLaurin series. Thus:

$$\chi(\xi) = \chi(0) + \xi\left(\frac{\partial \chi}{\partial \xi}\right)_0 + \frac{\xi^2}{2}\left(\frac{\partial^2 \chi}{\partial \xi^2}\right)_0 + \ldots$$

or:

$$x_s - x = (x_s - x)_{u=u_0} - (u - u_0)\left(\frac{\partial x}{\partial u}\right)_{u=u_0} - \frac{(u - u_0)^2}{2}\left(\frac{\partial^2 x}{\partial u^2}\right)_{u=u_0} + \ldots \qquad \textbf{(5.122b)}$$

Using Equation 5.122a in Equation 5.122b, we get:

$$u(x, y) = u_0(y) + \alpha(y)(x_s - x)^{1/2} \qquad \textbf{(5.122c)}$$

where:

$$\alpha(y) = \left[-1/2\left(\frac{\partial^2 x}{\partial u^2}\right)\right]_{u=u_0}^{-1/2}$$

Using the equation of continuity, it is easy to show that:

$$v = \frac{\beta(y)}{(x_s - x)^{1/2}} \qquad \textbf{(5.122d)}$$

where:

$$\beta(y) = 1/2 \int \alpha(y)\, dy$$

The velocity distribution near the separation point is therefore given by Equations 5.122c and 5.122d. the singularity at separation is completely reflected in the v distribution.

To find $\alpha(y)$, we resort to the boundary layer equation (Equation 5.118) and argue that near separation the pressure gradient and the viscous forces are much smaller than the inertial forces. Substituting Equations 5.122c, d in the left-hand side of Equation 5.118 we get:

$$\left\{\beta \frac{du_0}{dy} - \frac{1}{2}\alpha u_0\right\} + \left[\beta \frac{d\alpha}{dy} - \frac{\alpha^2}{2}\right](x_s - x)^{1/2}$$
$$= [\text{pressure} + \text{viscous contributions}](x_s - x)^{1/2}$$

Thus for the right-hand side to be as small as $x \rightarrow x_s$, we must have:

$$\beta \frac{du_0}{dy} = \frac{1}{2}\alpha u_0$$

which can be satisfied by writing:

$$\beta = \frac{1}{2}Au_0(y)$$

where A = constant provided that:

$$\alpha = A\frac{du_0}{dy}$$

Finally, the velocity distribution near the separation point is

$$u(x, y) = u_0(y) + A\frac{du_0}{dy}(x_s - x)^{1/2}$$

$$v(x, y) = \frac{Au_0(y)}{2(x_s - x)^{1/2}}$$

Now at $y = 0$, both u and v have to be zero: therefore:

$$u_0(0) = 0, \qquad \left(\frac{du_0}{dy}\right)_{y=0} = 0$$

which proves our previous assertion that at separation both u and $\partial u/\partial y$ have to be zero.

The foregoing analysis exposes the following properties of a steady two-dimensional boundary layer near separation:

1. At separation $(\partial u/\partial y)_{\substack{y=0\\x=x_s}} = 0$.
2. The u-component varies as $(x_s - x)^{1/2}$ while the v-component varies as $(x_s - x)^{-1/2}$ near separation.
3. The solution is singular at $x = x_s$ and cannot be continued beyond separation.

5.8 TRANSFORMED BOUNDARY LAYER EQUATIONS

The boundary layer equations in the Cartesian form, viz., Equations 5.117 and 5.118 or Equations 5.108 and 5.109 with $h = 1$, have the coordinate x as the actual curvilinear distance measured along the surface and y as the actual distance measured normal to the surface. A large volume of classic literature (see, e.g., Reference 1) and also many numerical and analytical techniques of solution of these equations can be stated in a unified way by effecting a coordinate transformation. Various particular transformations developed by different authors over a period of time can be obtained as the particular cases of the following general transformation.

Introducing the stream function ψ in Equations 5.99 and 5.100 through:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

yields the single equation:

$$\frac{\partial^2 \psi}{\partial t\, \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + \nu \frac{\partial^3 \psi}{\partial y^3} \tag{5.123}$$

The boundary conditions are

$$\text{at } y = 0 : \psi = \frac{\partial \psi}{\partial y} = 0$$

$$\text{at } y \to \infty : \frac{\partial \psi}{\partial y} \to u_e(x, t) \tag{5.124}$$

For nonsteady flow the initial condition at $t = 0$ must also be prescribed. In what follows, we shall consider the steady form of Equation 5.123.

Let $\xi = \xi(x)$ be a general nondimensional function of x. Further:

$$u_e(x) = u_e[x(\xi)] = u_e(\xi)$$

and we propose to introduce a nondimensional function η and a nondimensional stream function $f(\xi, \eta)$ through:

$$y = \eta N_e(\xi)/u_e(\xi) \tag{5.125}$$

$$\psi = N_e(\xi) f(\xi, \eta) \tag{5.126}$$

so that:

$$\frac{\partial \psi}{\partial y} = u = u_e(\xi)\frac{\partial f}{\partial \eta} = u_e(\xi) f' \tag{5.127a}$$

Here a prime denotes differentation with respect to η, and $N_e(\xi)$ is yet undetermined. Using Equations 5.125 and 5.126, the other derivatives are

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{u_e^2}{N_e} f'' \tag{5.127b}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{u_e^3}{N_e^2} f''' \tag{5.127c}$$

$$\frac{\partial \psi}{\partial x} = \frac{d\xi}{dx}\left[\frac{dN_e}{d\xi} f + N_e \frac{\partial f}{\partial \xi} + \frac{N_e^2}{u_e}\frac{d}{d\xi}\left(\frac{u_e}{N_e}\right)\eta f'\right] \tag{5.127d}$$

$$\frac{\partial^2 \psi}{\partial x\, \partial y} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\left[\frac{du_e}{d\xi} f' + u_e \frac{\partial f'}{\partial \xi} + N_e \frac{d}{d\xi}\left(\frac{u_e}{N_e}\right)\eta f''\right] \tag{5.127e}$$

Substituting Equations 5.127 in Equations 5.123 and 5.124, the equation becomes:

$$f''' + \gamma(\xi) f f'' + \beta(\xi)(1 - f'^2) = \chi(\xi)\left\{f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi}\right\} \tag{5.128}$$

with the boundary conditions:

$$\left.\begin{aligned} &\text{at } \eta = 0 : f = f' = 0 \\ &\text{as } \eta \to \infty : f' \to 1 \end{aligned}\right\} \tag{5.129}$$

The coefficients are

$$\gamma(\xi) = \frac{N_e}{\nu u_e}\frac{dN_e}{d\xi}\frac{d\xi}{dx} = \frac{N_e}{\nu u_e}\frac{dN_e}{dx} \tag{5.130a}$$

$$\beta(\xi) = \frac{N_e^2}{\nu u_e^2}\frac{du_e}{d\xi}\frac{d\xi}{dx} = \frac{N_e^2}{\nu u_e^2}\frac{du_e}{dx} \tag{5.130b}$$

$$\chi(\xi) = \frac{N_e^2}{\nu u_e}\frac{d\xi}{dx} \tag{5.130c}$$

In terms of the variables ξ and η, the wall shear τ_w is given by:

$$\tau_w = \frac{\mu u_e^2(\xi)}{N_e(\xi)} f''(0, \xi) \tag{5.130d}$$

In future, we shall use Equation 5.128 for different transformation $\xi = \xi(x)$ and $\eta = \eta(x, y)$.

Similar Boundary Layers

The essential idea of similarity lies in recognizing an inherent property of *some* boundary layers which is such that when the boundary layer coordinate y is nondimensionalized in a proper manner, then the nondimensional boundary layer profile u/u_e has the same shape independent of the position x on the surface. Following Schlichting[1] we search for a function $G = G(x, \nu)$, such that for two positions $x = x_1$ and $x = x_2$ we have:

$$\frac{u[x_1, y/G(x_1, \nu)]}{u_e(x_1)} = \frac{u[x_2, y/G(x_2, \nu)]}{u_e(x_2)} \tag{5.131}$$

The identity (Equation 5.131) is a necessary condition for similar boundary layers. It is satisfied if in Equation 5.128 we consider f to be a function of η only and both β and γ are taken as constants. Taking $\xi = x$ and $\gamma = 1$, Equation 5.128 becomes:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{5.132}$$

$$\text{at } \eta = 0 : f = f' = 0$$

$$\text{as } \eta \to \infty : f' \to 1$$

where β = constant. Equation 5.132 is called the Falkner-Skan equation. Here $G(x, \nu)$ is related with N_e as shown below.

To obtain the forms of N_e and η in this case, we use Equations 5.130a, b. First of all γ = 1 and β = constant imply, respectively:

$$\frac{1}{\nu u_e}\frac{d}{dx}(N_e^2) = 2 \tag{5.133a}$$

$$\frac{N_e^2}{\nu u_e^2}\frac{du_e}{dx} = \beta = \text{constant} \tag{5.133b}$$

Thus:

$$\frac{d}{dx}\left(\frac{N_e^2}{u_e}\right) = \nu(2 - \beta)$$

which on integration gives:

$$N_e^2 = \nu(2 - \beta)xu_e \tag{5.133c}$$

Note that from the definition of η given in Equation 5.125, the function G appearing in Equation 5.131 is $G = \{\nu(2 - \beta)x/u_e\}^{1/2}$.

On using Equation 5.133c in Equation 5.133b and integrating, we get:

$$u_e(x) = Ax^m \tag{5.134}$$

where A is a constant and:

$$m = \frac{\beta}{2 - \beta}, \qquad \beta = \frac{2m}{m + 1}, \qquad 2 - \beta = \frac{2}{m + 1}$$

Thus the outer flow for which a similar boundary layer exists must have the form of Equation 5.134.

It is of interest to know the body shapes for which the Euler solution when evaluated at the surface has the form of Equation 5.134. To do this we look for the potential flow past a semi-infinite wedge which is known to be given by the stream function:

$$\psi_e = Im\left[\frac{U_\infty K(2 - \beta)}{2} z^{2/(2-\beta)} e^{-i\pi\beta/(2-\beta)}\right]$$

where $z = X + iY$ is the complex variable, $\pi\beta$ is the included angle of the wedge, and K is an arbitrary real constant; see Figure 5.6.

Therefore, the tangential velocity u_e is given by:

$$u_e(r, \theta) = \frac{\partial\psi_e}{r\partial\theta} = KU_\infty r^{\beta/(2-\beta)}\exp\left[\frac{(2\theta - \pi\beta)i}{2 - \beta}\right]$$

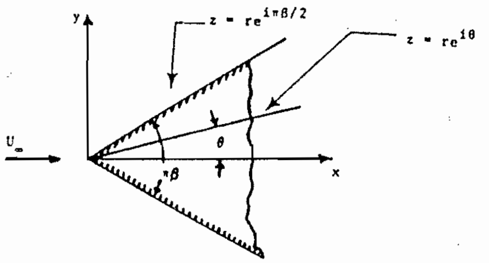


Fig. 5.6(a) Flow past a wedge.

Evaluating the velocity at the wedge surface, i.e., $\theta = \pi\beta/2$, we get:

$$u_r(r) = KU_\infty r^{\beta/(2-\beta)}$$

Writing $r = x$ as the distance measured from the apex along the wedge surface, we get the desired potential flow (Equation 5.134) with $A = KU_\infty$.

It must be noted that the potential solution ψ_e given above is singular because the constant K cannot be determined from the conditions at upstream infinity. In fact, there is no solution for the potential flow past a wedge which has a finite value at infinity, except when $\beta = 0$. The potential flow past the wedge as discussed embraces the flow past a semi-infinite plate ($\beta = 0$), a vertical wall ($\beta = 1$), etc. (Refer also to Equations 5.41.)

Boundary Layer on a Semi-Infinite Plate

Let a semi-infinite plate with a sharp (cuspidal) leading edge be placed in a high Reynolds number (i.e., $\nu \rightarrow 0$) flow of constant free stream velocity u_∞. A boundary layer develops on the plate starting from the leading edge. We take a Cartesian coordinate system having the origin at the leading edge with x along the plate in the direction of u_∞ and y as normal to the plate, as shown in Figure 5.7.

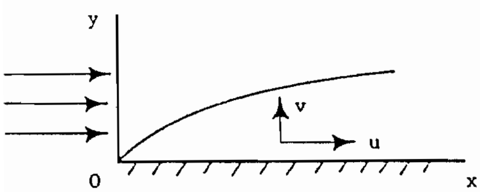


Fig. 5.6(b) Boundary layer on a semi-infinite plate.

The simplicity of the present problem is that $u_e = u_\infty$ which is a constant. Thus the pressure gradient is zero and:

$$\beta = 0, \qquad m = 0, \qquad A = u_\infty$$

Also from Equations 5.133c, 5.125, and 5.126:

$$N_e = \sqrt{2\nu x u_\infty}, \qquad \eta = y\sqrt{\frac{u_\infty}{2\nu x}}, \qquad \psi = \sqrt{2\nu x u_\infty}\, f(\eta)$$

Thus Equation 5.132 becomes:

$$f''' + ff'' = 0 \tag{5.135}$$

with the boundary conditions:

$$\eta = 0 : f = f' = 0$$
$$\eta \to \infty : f' \to 1 \qquad (5.136)$$

Equation 5.135 is called the Blasius equation. (Refer also to Problem 5.3.)
Equations 5.135 and 5.136 can also be written in two other forms.

1. Let $f(\eta) = 1/\sqrt{2}\,\phi(\xi)$, $\eta = \xi/\sqrt{2}$, then:

$$2\phi''' + \phi\phi'' = 0$$

$$\text{at } \xi = 0 : \phi = \phi' = 0$$
$$\text{as } \xi \to \infty : \phi' \to 1 \qquad (5.137a)$$

2. Let $f(\eta) = 1/\sqrt{2}\,\chi(\lambda)$, $\eta = \sqrt{2}\,\lambda$, then:

$$\chi''' + \chi\chi'' = 0$$

$$\text{at } \lambda = 0 : \chi = \chi' = 0$$
$$\text{as } \lambda \to \infty : \chi' \to 2 \qquad (5.137b)$$

Solution of the Blasius Equation

The solution of Equation 5.135 under the boundary conditions (Equation 5.136) was initially obtained by Blasius. Because of the boundary condition at $\eta = \infty$, called the asymptotic boundary condition, Blasius solved the problem in two steps. First, he used a series expansion for $f(\eta)$ for small values of η as:

$$f(\eta) = \sum_{n=0}^{\infty} \frac{a_n \eta^n}{n!}$$

Substituting this series in Equation 5.135, a series of recursive relations among the constants a_n are obtained. Using the wall boundary conditions (i.e., at $\eta = 0$) and writing $a_2 = \alpha$, we have the series:

$$f(\eta) = \frac{\alpha\eta^2}{2!} - \frac{\alpha^2\eta^5}{5!} + \frac{11\alpha^3\eta^8}{8!} - \frac{375\alpha^4\eta^{11}}{11!} + \frac{27897\alpha^5\eta^{14}}{14!} - \cdots$$

The constant α, which as can be seen from above is $f''(0) = \alpha$, can be obtained only by using the asymptotic boundary condition. To obtain α, Blasius introduced the asymptotic form of $f(\eta)$ for large values of η as:

$$f(\eta) \cong \eta - \beta + f_1(\eta)$$

where $f_1(\infty) = f_1'(\infty) = 0$, and β is yet another constant. Thus for $\eta \to \infty$:

$$f(\eta) \to \eta - \beta$$

Substituting the asymptotic form in Equation 5.135 and neglecting $f_1 f_1''$ in comparison with $(\eta - \beta)f_1''$, we obtain the equation:

$$f_1''' + (\eta - \beta)f_1'' = 0$$

Solving this equation by using the conditions at $\eta = \infty$, the Blasius function for large η becomes:

$$f(\eta) \cong \eta - \beta + \gamma \int_{\infty}^{\eta} d\eta' \int_{\infty}^{\eta'} \exp\{-1/2(\eta'' - \beta)^2\}\, d\eta''$$

where γ is a constant of integration.

The constants α, β, and γ are now obtained by matching the values of f, f', and f'' at a value of η where both the series and the asymptotic solutions are equally valid. This matching then yields the following values:

$$f''(0) = \alpha = 0.4696, \qquad \beta = 1.2168, \qquad \gamma = 0.331$$

Tabulated values of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ are available in Reference 9, p. 224. Tabulated values of the Blasius function ϕ given by Equation 5.137a are available in Reference 1, p. 129. Note that $\phi''(0) = 0.332$. Numerical values for χ, governed by the Equation 5.137b, can be generated easily by using either $f(\eta)$ or $\phi(\zeta)$. In this case $\chi''(0) = 1.328$.

A method for solving Equations 5.135 and 5.136, which is very suitable for digital computers, is to cast the system as an initial-value problem. Introducing the transformation:

$$\zeta = k\eta, \qquad \phi(\zeta) = \frac{1}{k} f(\eta)$$

where k = constant, Equation 5.135 becomes:

$$\frac{d^3\phi}{d\zeta^3} + \phi \frac{d^2\phi}{d\zeta^2} = 0$$

Since:

$$\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0} = \alpha$$

then we have:

$$\left(\frac{d^2\phi}{d\zeta^2}\right)_{\zeta=0} = \frac{\alpha}{k^3}$$

If we set $k = \alpha^{1/3}$, then:

$$\left(\frac{d^2\phi}{d\zeta^2}\right)_{\zeta=0} = 1$$

Consequently, we pose the following initial-value problem:

$$\frac{d^3\phi}{d\zeta^3} + \phi \frac{d^2\phi}{d\zeta^2} = 0 \tag{5.138a}$$

$$\text{at } \zeta = 0 : \phi = \frac{d\phi}{d\zeta} = 0, \qquad \frac{d^2\phi}{d\zeta^2} = 1 \tag{5.138b}$$

The initial-value problem (Equations 5.138) was proposed by Weyl[11] and can be solved numerically by the Runge-Kutta method detailed in Problem 5.5. Based on the numerical solution, the values of $d\phi/d\zeta$ are generated and are continued with increasing ζ until a constant value is obtained. Then using the condition:

$$\left(\frac{df}{d\eta}\right)_{\eta\to\infty} \to 1$$

we find that:

$$\left(\frac{d\phi}{d\zeta}\right)_{\zeta\to\infty} \to \frac{1}{\alpha^{2/3}}$$

Consequently, the value of α is given by:

$$\alpha = \left[\operatorname*{Lim}_{\zeta\to\infty} \frac{d\phi}{d\zeta}\right]^{-3/2}$$

Note that finding α was the main problem in the solution of the Blasius equation. Once α and thus $k = \alpha^{1/3}$ becomes known, one can generate $f(\eta), f'(\eta), f''(\eta)$.

Based on the numerical solution of the Blasius equation one can use numerical quadratures to obtain:

$$\int_0^\infty (1 - f')\, d\eta = 1.2168$$

$$\int_0^\infty f'(1 - f')\, d\eta = 0.4696 = \alpha$$

Thus:

$$\delta^* = 1.7208\left(\frac{\nu x}{u_\infty}\right)^{1/2}$$

$$\theta = 0.664\left(\frac{\nu x}{u_\infty}\right)^{1/2}$$

$$\delta = 3.6\left(\frac{2\nu x}{u_\infty}\right)^{1/2}$$

$$\frac{\delta^*}{\theta} = 2.5915$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} = 0.664 R_x^{-1/2}$$

where:

$$R_x = \frac{u_\infty x}{\nu}$$

These results show that for a laminar boundary layer on a flat plate the lengths δ, δ^*, and θ increase (grow) as $x^{1/2}$ and the ratio of the displacement and momentum thickness is a constant.

The solution obtained for a semi-infinite plate is not applicable to a plate of finite length along the x-axis. The reason is that in obtaining the Blasius solution the boundary conditions at the trailing edge have not been taken into consideration. Recall also that the similarity of the Blasius solution is also due to assumption of the semi-infinite nature of the plate. Despite these limitations we may use the Blasius solution for a plate of length L so as to have an approximate estimation of the total drag experienced by the plate. The drag force D experienced on one side of a plate of length L and of unit width is

$$D = \int_0^L \tau_w(x)\, dx = 0.664 \rho u_\infty^2 L R_e^{-1/2}$$

where $R_e = u_\infty L/\nu$. The drag coefficient C_D is then:

$$C_D = \frac{D}{\frac{1}{2}\rho L u_\infty^2} = 1.328 R_e^{-1/2}$$

All of these formulae have been confirmed experimentally.[1] The velocity distribution in the boundary layer is obviously given by:

$$u = \frac{\partial \psi}{\partial y} = u_\infty f'(\eta) \tag{5.139a}$$

$$v = -\frac{\partial \psi}{\partial x} = \sqrt{\frac{\nu u_\infty}{2x}}\,(\eta f' - f) \tag{5.139b}$$

Two discrepancies of the classical boundary layer theory must be noted at this stage. First, the asymptotic behavior of the stream function ψ by using $f \simeq \eta - \beta$ as $\eta \rightarrow \infty$ is

$$\psi = u_\infty y - \beta\sqrt{2\nu u_\infty x} = u_\infty(y - \delta^*) \tag{5.140a}$$

This result shows that the boundary layer stream function does not merge smoothly with the inviscid stream function at its edge which is $u_\infty y$, the error being $0(\sqrt{\nu})$. The second result pertains to the behavior of v at infinity. From Equation 5.139b or 5.140a, we have:

$$(v)_{\eta\to\infty} = \beta\sqrt{\frac{\nu u_\infty}{2x}}, \qquad \beta = 1.2168 \tag{5.140b}$$

This result first of all shows that $(v)_{\eta\to\infty} \neq 0$; moreover, at the leading edge ($x = 0$) it is infinite, a result which contradicts the assumption made in the beginning that at the leading edge the velocity is uniform, i.e., $u = u_\infty$ and $v = 0$. Thus the Blasius solution is singular throughout the line $x = 0$, and the solution is not usable there. The formula (Equation 5.140b) for v also shows that the solution is not applicable for $x < 0$.

Boundary Layer on a Wedge

Although the boundary layer problem on a wedge is not of much practical importance, nevertheless, it forms a very important model for boundary layers with pressure gradient. As discussed earlier, the boundary layer on a wedge is *similar* and is described by an ordinary differential equation (viz., Equation 5.132) with the pressure gradient parameter β as a constant. By changing the value of β or m, one can obtain a class of boundary layer solutions under both favorable and adverse pressure gradients. Obviously, $\beta = 0$ corresponds to the boundary layer flow on a semi-infinite plate and $\beta = 1$ corresponds to the two-dimensional stagnation point flow. For general values of β (the range of β will be stated shortly), the flow is considered to be past a semi-infinite wedge of included angle $\pi\beta$. Each solution corresponds to a fixed β; and, because of the similarity, in each case the velocity profile is the same for all x-positions. The solutions of Equation 5.132 can also be interpreted as the boundary layer solutions at localized points on a body which is not a wedge. Thus, for an airfoil the flow in the vicinity of the front stagnation point is given by Equation 5.132 with $\beta = 1$, while the solution at the point of minimum pressure on an airfoil is given by Equation 5.132 with $\beta = 0$. The significance of the parameter m or β can be better understood by using Equation 5.134 which gives:

$$
\begin{aligned}
-\frac{1}{\rho}\frac{dp}{dx} &= u_e \frac{du_e}{dx} \\
&= mA^2x^{2m-1}
\end{aligned}
$$

Therefore, if $m > 0$ ($\beta > 0$), then the pressure is decreasing or favorable; while if $m < 0$ ($\beta < 0$), then the pressure is increasing or unfavorable.*

Numerical solution of Equation 5.132 was first obtained by Hartree,[12] and later by others. Tabulated values are available in articles by Schlichting,[1] Rosenhead,[9] and Loitsyanskii.[13] It was found that the solution of the equation can be obtained only for values of β in the range:

$$-0.1988 \leqslant \beta \leqslant 2$$

The solution for $\beta = -0.198837735$ yields $f''(0) \approx 0$, signifying the boundary layer separation. The value of m corresponding to $\beta = -0.1988$ is -0.0904. Thus separation occurs when the external flow follows the law:

$$u_e = Ax^{-0.0904}$$

Before we explain a numerical method of solving Equation 5.132, it is instructive to summarize the following definitions. First, using the external velocity u_e (Equation 5.134 in Equations 5.133c, 5.125, 5.126) and noting that:

$$2 - \beta = \frac{2}{m+1}$$

we have:

$$
\begin{aligned}
\eta &= y\left\{\frac{(m+1)Ax^{m-1}}{2\nu}\right\}^{1/2} \\
\psi &= \left\{\frac{2\nu Ax^{m+1}}{m+1}\right\}^{1/2} f(\eta) \\
u &= Ax^m f'(\eta) \\
v &= \left\{\frac{\nu(m+1)Ax^{m-1}}{2}\right\}^{1/2} \{(1-\beta)\eta f' - f\}
\end{aligned}
\qquad \textbf{(5.141a)}
$$

The other quantities of interest are

1. Local skin-friction coefficient c_f:

* Note that a negative β represents a hypothetical wedge!

$$c_f = \left(\frac{2\nu(m+1)}{Ax^{m+1}}\right)^{1/2} f''(0) \tag{5.141b}$$

2. Displacement thickness δ^*:

$$\delta^* = d_1(\beta)\left(\frac{2\nu x^{1-m}}{A(m+1)}\right)^{1/2} \tag{5.141c}$$

where:

$$d_1(\beta) = \int_0^\infty (1 - f')\, d\eta \tag{5.141d}$$

3. Momentum thickness θ:

$$\theta = d_2(\beta)\left(\frac{2\nu x^{1-m}}{A(m+1)}\right)^{1/2} \tag{5.141e}$$

where:

$$d_2(\beta) = \int_0^\infty f'(1 - f')\, d\eta \tag{5.141f}$$

Numerical Solution of the Falkner-Skan Equation

The boundary-value problem posed by the Falkner-Skan equation (Equation 5.132) has been a subject of much numerical and mathematical research. In addition to the development of various methods of numerical solution of this equation, some fundamental results in the existence and uniqueness of its solution have also been obtained. Refer to Reference 9. Besides those of Hartree,[12] numerical methods have been developed by Nachtsheim and Swigert[14] and Cebeci (refer to Reference 15).

The most important step in the numerical solution of Equation 5.132 is to devise an algorithm in which the asymptotic condition:

$$\lim_{\eta \to \infty} f''(\eta; \alpha) \to 0$$

is also used. Once a fixed value of β has been chosen, the solution can symbolically be written as:

$$f = \mathrm{f}(\eta; \alpha(\beta)), \qquad \alpha = f''(0; \beta)$$

Let η_e be the value of η where the asymptotic conditions are satisfied. Then:

$$f'_e(\alpha) = f'(\eta_e; \alpha) = 1$$

$$f''_e(\alpha) = f''(\eta_e; \alpha) = 0$$

The shooting method, which involves the solution of an initial value problem, seems to be a much simpler approach to solving the Falkner-Skan equation. Starting from an initial guess for α and η_e and using a nonuniform grid size $h(\eta)$:

$$h(\eta) = h_i + (h_f - h_i)\eta/\eta_e \tag{5.142a}$$

Equation 5.132 has been solved using the Runge-Kutta method by Warsi and Koomullil.[16] Here h_i and h_f are the initial and final step sizes, respectively. The values $h_i = 0.0005$ and $h_f = 0.01$ seem to be quite appropriate. Next, α is updated by solving the equation:

$$f_e'(\alpha) = g(\alpha) = 1$$

by the secant method which yields:

$$\Delta\alpha^{(n+1)} = \frac{\Delta\alpha^{(n)}[1 - g(\alpha^{(n)})]}{g(\alpha^{(n)}) - g(\alpha^{(n-1)})} \tag{5.142b}$$

where:

$$\Delta\alpha^{(k)} = \alpha^{(k)} - \alpha^{(k-1)}, \qquad k \geq 1$$

Simultaneously η_e is updated through the equation:

$$(\eta_e)_{new} = (\eta_e)_{old}\,[1 + \gamma f''\{(\eta_e)_{old};\ \alpha^{(n)}\}] \tag{5.142c}$$

where γ is arbitrary. Very accurate solutions of Equation 5.132 have been obtained by using the method comprised of Equations 5.142, $f''(0) = 10^{-6}$ for $\beta = -0.198837735$. Based on the numerical solution, the values of $d_1(\beta)$, $d_2(\beta)$ defined in Equation 5.141d, f and of α for various β are given in Table 1.

TABLE 1. Falkner-Skan Solution for Various β (correct to three decimals)

β	$d_1(\beta)$	$d_2(\beta)$	α
−0.198837735	2.359	0.585	0.0000
−0.1900	2.007	0.577	0.0860
−0.1800	1.871	0.568	0.1285
−0.1600	1.708	0.552	0.1905
−0.1400	1.597	0.539	0.2395
−0.1000	1.444	0.515	0.3191
0.00	1.217	0.470	0.4696
0.20	0.984	0.408	0.6869
0.40	0.853	0.367	0.8542
0.50	0.804	0.350	0.9277
1.00	0.648	0.292	1.2326
2.00	0.498	0.231	1.6870

$$\alpha = (1+\beta)d_2 + \beta d_1 \tag{5.142d}$$

Nonsimilar Boundary Layers

It had been deduced earlier that the *similarity* solutions of the steady boundary layer equations exist only for a class of external flows which have the form of Equation 5.134. Only for such external flows is it possible to reduce the transformed equation (Equation 5.128) to an ordinary differential equation. For arbitrary shaped bodies the external flow for all x-positions cannot be of the form of Equation 5.134, and therefore a partial differential equation of the form of Equation 5.128 has to be solved. A finite difference approximation of the boundary layer equations either in the primitive variable form or in the transformed form can be carried out and solved on a computer. We shall discuss this method in Section 5.11. Here we discuss Gortler's method of series expansion for the solution of nonsimilar boundary layers.

Gortler[17] proposed a transformation, which in effect implies taking:

$$\xi = \frac{1}{\nu}\int_0^x u_e(x)\,dx \tag{5.143}$$

and:

$$\gamma(\xi) = 1$$

in Equation 5.128. Thus from Equations 5.130 we have:

$$N_e(\xi) = \nu\sqrt{2\xi}$$

$$\beta(\xi) = \frac{2\xi}{u_e}\frac{du_e}{d\xi}$$

$$\chi(\xi) = 2\xi$$

Consequently, from Equations 5.125 and 5.126:

$$\eta = \frac{yu_e(\xi)}{\nu\sqrt{2\xi}}$$

$$\psi = \nu\sqrt{2\xi}\,f(\xi, \eta)$$

and Gortler's equation becomes

$$f''' + ff'' + \beta(1 - f'^2) = 2\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \tag{5.144}$$

$$\beta = \frac{2\xi}{u_e}\frac{du_e}{d\xi}$$

with the boundary conditions:

$$\text{at } \eta = 0 : f = f' = 0$$

$$\text{as } \eta \to \infty : f' \to 1$$

The purpose of any f and η-transformation is, first of all, to express the boundary layer equation in dimensionless form. The characteristic body shape information is contained in the potential flow $u_e(x)$ which is different for different body shapes. The appearance of the nondimensional parameters makes the presentation of results more meaningful. Besides these advantages the most important advantage of the η-transformation is that the variation of the boundary layer thickness with the position on the surface is minimized to such an extent that whether the boundary layer is thin or thick the value of η does not change by more than a factor of 2.

Gortler's Series Solution

In terms of Gortler's variables described above the boundary layer equation assumes the form of Equation 5.144. In any boundary layer problem the external velocity $u_e(x)$ is assumed to be known, where x is the curvilinear distance along the body contour. The point $x = 0$ is either a point on a sharp edge (cuspidal point) or a point on a rounded edge (stagnation point). Gortler's main assumption is that the velocity $u_e(x)$ is expressible as a convergent power series in x in a certain interval $0 \leqslant x \leqslant x_1$. Thus, for a body with a cuspidal point the power series of $u_e(x)$ is

$$u_e(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad u_0 \neq 0 \tag{5.145a}$$

where u_k are constants. For a body with a stagnation point at $x = 0$, the flow may be symmetric or antisymmetric. The symmetric flow arises about a symmetric body when the axis of symmetry coincides with the free stream direction. In the symmetric case the power series for $u_e(x)$ is

$$u_e(x) = \sum_{k=0}^{\infty} u_{2k+1}x^{2k+1}, \qquad u_1 \neq 0 \tag{5.145b}$$

while for the antisymmetric case:

$$u_e(x) = \sum_{k=1}^{\infty} u_k x^k, \qquad u_1 \neq 0 \tag{5.145c}$$

The two most successful cases covered by Gortler's method are those in which the external flows are described by Equations 5.145a, b. Here we shall be concerned only with these cases and leave the antisymmetric case (Equation 5.145c) for reference to the original paper of Gortler. For the purpose of minimizing the algebraic manipulations we shall introduce the nondimensional external velocity ϕ through the equation:

$$u_e(x) = cu_\infty\phi(\bar{x}) \tag{5.146}$$

where $\bar{x} = x/L$. Here L is a characteristic body length, u_∞ is the free stream velocity, and c is a nondimensional constant. Thus the series for $\phi(\bar{x})$ corresponding to Equation 5.145a, b are, respectively:

$$\phi(\bar{x}) = 1 + \sum_{k=1}^{\infty} a_k\bar{x}^k \tag{5.147a}$$

and:

$$\phi(\bar{x}) = \bar{x} + \sum_{k=1}^{\infty} a_k\bar{x}^{2k+1} \tag{5.147b}$$

Using Equation 5.146, the Gortler variable ξ from Equation 5.143 is

$$\xi = \alpha\int_0^{\bar{x}} \phi(\bar{x})\, d\bar{x} \tag{5.148}$$

where:

$$\alpha = \frac{cu_\infty L}{\nu}$$

Gortler now assumes a series expansion for $\beta(\xi)$ as:

$$\beta(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \ldots$$

To determine the constants β_i $(i = 0, 1, \ldots)$, we first write:

$$2\,\frac{du_e}{dx}\int_0^x u_e(x)\, dx = u_e^2(x)\beta(\xi)$$

On using Equations 5.146 and 5.148 and the expansion for β, we have:

$$2\phi'(\bar{x}) \int_0^{\bar{x}} \phi(\bar{x})\, d\bar{x} = \phi^2(\bar{x}) \sum_{k=0}^{\infty} \beta_k \alpha^k \left(\int_0^{\bar{x}} \phi(\bar{x})\, d\bar{x} \right)^k \tag{5.149}$$

Substituting Equation 5.147a in Equation 5.149 and comparing coefficients of different powers of $\bar{x}$, we get:

$$\begin{aligned}
\beta_0 &= 0 \\
\beta_1 &= 2a_1/\alpha \\
\beta_2 &= 4(-a_1^2 + a_2)/\alpha^2 \\
\beta_3 &= 2(4a_1^3 - 7a_1a_2 + 3a_3)/\alpha^3 \\
\beta_4 &= 1/3(-48a_1^4 + 118a_1^2a_2 - 66a_1a_3 - 28a_2^2 + 24a_4)/\alpha^4 \\
\beta_5 &= 1/6(192a_1^5 - 605a_1^3a_2 + 375a_1^2a_3 + 320a_1a_2^2 - 192a_1a_4 \\
&\quad - 150a_2a_3 + 60a_5)/\alpha^5 \text{ etc.}
\end{aligned} \tag{5.150}$$

In the same manner for the case of Equation 5.147b, we have:

$$\begin{aligned}
\beta_0 &= 1 \\
\beta_1 &= 3a_1/\alpha \\
\beta_2 &= (-13a_1^2 + \tfrac{40}{3} a_2)/\alpha^2 \\
\beta_3 &= (54a_1^3 - 96a_1a_2 + 42a_3)/\alpha^3 \\
\beta_4 &= (-221a_1^4 + \tfrac{1648}{3} a_1^2a_2 - 312a_1a_3 - \tfrac{1184}{9} a_2^2 + \tfrac{576}{5} a_4)/\alpha^4 \\
\beta_5 &= (898a_1^5 - \tfrac{8560}{3} a_1^3a_2 + 1790a_1^2a_3 + 1520a_1a_2^2 \\
&\quad - 928a_1a_4 - 720a_2a_3 + \tfrac{880}{3} a_5)/\alpha^5 \text{ etc.}
\end{aligned} \tag{5.151}$$

To solve Equation 5.144, Gortler assumes a series expansion for $f(\xi, \eta)$ as:

$$f(\xi, \eta) = \sum_{n=0}^{\infty} \xi^n f_n(\eta) \tag{5.152}$$

Substitution of the β- and f-series into Equation 5.144 results in a series of equations which, except for f_0, all depend on the various constants $\beta_n's$. To make them independent of $\beta_n's$ we use the method of splitting as follows:

$$\begin{aligned}
f_1(\eta) &= \beta_1 F_1(\eta) \\
f_2(\eta) &= \beta_1^2 F_{11}(\eta) + \beta_2 F_2(\eta) \\
f_3(\eta) &= \beta_1^3 F_{111}(\eta) + \beta_1\beta_2 F_{12}(\eta) + \beta_3 F_3(\eta) \\
f_4(\eta) &= \beta_1^4 F_{1111}(\eta) + \beta_1^2\beta_2 F_{112}(\eta) + \beta_1\beta_3 F_{13}(\eta) + \beta_2^2 F_{22}(\eta) + \beta_4 F_4(\eta)
\end{aligned}$$

$$f_5(\eta) = \beta_1^5 F_{11111}(\eta) + \beta_1^3\beta_2 F_{1112}(\eta) + \beta_1^2\beta_3 F_{113}(\eta)$$
$$+ \beta_1\beta_2^2 F_{122}(\eta) + \beta_1\beta_4 F_{14}(\eta) + \beta_2\beta_3 F_{23}(\eta) + \beta_5 F_5(\eta), \text{ etc.} \qquad \textbf{(5.153)}$$

The splitting method produces equations for various functions appearing in Equation 5.153 which are independent of $\beta_n's$, so that these equations can be solved once and for all. As an example, the equations for $f_0(\eta)$ and $F_1(\eta)$ are

$$f_0''' + f_0 f_0'' + \beta_0\left(1 - f'^2\right) = 0$$

$$F_1''' + f_0 F_1'' + 3f_0'' F_1 - 2\left(1+\beta_0\right) f_0' F_1' = f_0'^2 - 1$$

The boundary conditions are

$$\text{at } \eta = 0 : f_0 = f_0' = 0,\ F_1 = F_1' = 0$$

$$\text{as } \eta \to \infty : f_0' \to 1,\ F_1' \to 0$$

Similarly, equations for F_{11}, F_2 etc. are written. These ordinary differential equations up to $F_5(\eta)$ were solved at the Computation Laboratory, Harvard University.[18] Values of the second derivatives of the functions up to F_5 are given in Table 2.

TABLE 2. Second Derivatives of Some of the Gortler Functions at the Surface for Cuspidal and Symmetric Bodies

Case $\beta_0 = 0$ (Sharp edge)

$f_0''(0) = 0.4696$	$F_{111}''(0) = -2.313327$	$F_{113}''(0) = 2.450686$
$F_1''(0) = 1.032361$	$F_{12}''(0) = 2.775762$	$F_{122}''(0) = 2.355199$
$F_{11}''(0) = -0.714746$	$F_{13}''(0) = -1.047926$	$F_{14}''(0) = -0.949480$
$F_2''(0) = 0.908119$	$F_{22}''(0) = -0.505493$	$F_{23}''(0) = -0.899401$
$F_{111}''(0) = 1.103512$	$F_4''(0) = 0.774210$	$F_5''(0) = 0.731424$
$F_{12}''(0) = -1.191046$	$F_{1111}''(0) = 5.600941$	
$F_3''(0) = 0.829995$	$F_{112}''(0) = -7.803477$	

Case $\beta_0 = 1$ (Stagnation)

$f_0''(0) = 1.232587$	$F_{111}''(0) = -0.008272$	$F_{113}''(0) = 0.052941$
$F_1''(0) = 0.493840$	$F_{12}''(0) = 0.058722$	$F_{122}''(0) = 0.051995$
$F_{11}''(0) = -0.077205$	$F_{13}''(0) = -0.124239$	$F_{14}''(0) = -0.114928$
$F_2''(0) = 0.464540$	$F_{22}''(0) = -0.061283$	$F_{23}''(0) = -0.112469$
$F_{111}''(0) = 0.022415$	$F_4''(0) = 0.0424639$	$F_5''(0) = 0.409895$
$F_{12}''(0) = -0.136636$	$F_{1111}''(0) = 0.003560$	
$F_3''(0) = 0.442383$	$F_{112}''(0) = -0.029057$	

For some simple external flows, it is possible to solve the boundary layer problem without even using Equation 5.150 or 5.151. For problems refer to Problem 5.9.

Example 5.1

Use a five-term Gortler series solution of steady incompressible laminar boundary layer to find the coefficient of local skin friction on a flat plate that is under the influence of an adverse pressure gradient in which the external flow $u_e(x)$ is

$$u_e(x) = u_\infty\left(1 - \frac{a_0 x}{L}\right)$$

Here a_0 is an assigned parameter and L a characteristic length. Find the separation point for the cases $a_0 = 1$ and $1/8$.

We will solve this problem without using Equation 5.150. First, from Equation 5.143:

$$\xi = \alpha\left(\frac{x}{L} - \frac{a_0 x^2}{2L^2}\right)$$

where $\alpha = u_\infty L/\nu$. Thus:

$$\frac{x}{L} = \frac{1}{a_0}\left[1 - \left(1 - 2a_0\frac{\xi}{\alpha}\right)^{1/2}\right]$$

$$u_e(\xi) = u_\infty\left(1 - 2a_0\frac{\xi}{\alpha}\right)^{1/2}$$

and:

$$\beta(\xi) = \frac{-2a_0\xi}{\alpha}\left(1 - 2a_0\frac{\xi}{\alpha}\right)^{-1}$$

For brevity writing $B = 2a_0/\alpha$ and using the binomial expansion for the function $\beta(\xi)$, we get:

$$\beta_0 = 0, \qquad \beta_n = -B^n, \qquad n \geq 1$$

Thus, the five-term Gortler series with $\zeta = B\xi$ is

$$\begin{aligned} f(\xi, \eta) = f_0(\eta) &- \zeta F_1(\eta) - \zeta^2[F_2(\eta) - F_{11}(\eta)] \\ &- \zeta^3[F_{111}(\eta) - F_{12}(\eta) + F_3(\eta)] \\ &- \zeta^4[F_{112}(\eta) + F_4(\eta) - F_{1111}(\eta) - F_{13}(\eta) - F_{22}(\eta)] \\ &- \zeta^5[F_{11111}(\eta) + F_{113}(\eta) + F_{122}(\eta) + F_5(\eta) - F_{1112}(\eta) - F_{14}(\eta) - F_{23}(\eta)] \end{aligned}$$

The skin friction coefficient is

$$c_f = \sqrt{\frac{2}{\xi}}f''(0, \xi)$$

With the aid of Table 2, we then have:

$$\begin{aligned} c_f = \sqrt{\frac{2}{\xi}}\,[0.4696 &- 1.032361\zeta - 1.622865\zeta^2 \\ &- 3.124553\zeta^3 - 7.416718\zeta^4 - 20.790608\zeta^5] \end{aligned}$$

Note that $\zeta = 2a_0\xi/\alpha$ and $\alpha = u_\infty L/\nu$.

To obtain the separation point we set $c_f = 0$ and solve the fifth degree equation. Thus:

$$\zeta = 0.2536 \quad \text{at separation}$$

For $a_0 = 1$, $\xi/\alpha = 0.1268$ which gives $x/L = 0.1361$

For $a_0 = \frac{1}{8}$, $\xi/\alpha = 1.0144$ which gives $x/L = 1.0884$

The case $a_0 = 1$ was first solved by Howarth[19] also by a series expansion method. The separation point predicted by Howarth is $x/L = 0.12$.

5.9 MOMENTUM INTEGRAL EQUATION

It is quite clear that the basic nonlinearity of Navier-Stokes' equations is not disturbed by affecting the boundary layer approximation. Because of the nonlinearity, it is not possible to obtain closed form solutions either of Navier-Stokes' or of Prandtl's boundary layer equations. This difficulty alone has given rise to many approximate methods of solutions; chief among them has been the "momentum integral method" initiated in the early 1920s. This method does not yield a detailed solution, as can be obtained nowadays by solving the differential equations of the boundary layer on a digital computer. Nevertheless, it provides a useful method for a quick estimation of the averaged boundary layer properties.

The momentum integral equation is obtained by integrating each term of Equation 5.100 throughout the boundary layer from $y = 0$ to $y = \delta$ (the outer edge) to have:

$$\int_0^\delta \frac{\partial u}{\partial t}\,dy + \int_0^\delta u\frac{\partial u}{\partial x}\,dy + \int_0^\delta v\frac{\partial u}{\partial y}\,dy = \frac{\partial u_e}{\partial t}\,\delta + u_e\frac{\partial u_e}{\partial x}\,\delta + \nu\int_0^\delta \frac{\partial^2 u}{\partial y^2}\,dy$$

where δ is a function of both x and t, i.e., $\delta = \delta(x, t)$. Integrating the third term on the left and using the continuity equation (Equation 5.99), and similarly integrating the last term on the right and using the condition:

$$\left(\frac{\partial u}{\partial y}\right)_{y=\delta} = 0$$

we get:

$$\int_0^\delta \frac{\partial u}{\partial t}\,dy + u_e(v)_{y=\delta} + 2\int_0^\delta u\frac{\partial u}{\partial x}\,dy = \frac{\partial u_e}{\partial t}\,\delta + u_e\frac{\partial u_e}{\partial x}\,\delta - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

A second application of the continuity equation in the form:

$$(v)_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x}\,dy$$

and the use of the formula of differentiation of an integral** with variable limits yields:

$$-\frac{\partial}{\partial t}\int_0^\delta u\,dy + u_e\frac{\partial \delta}{\partial t} + u_e\frac{\partial}{\partial x}\int_0^\delta u\,dy - \frac{\partial}{\partial x}\int_0^\delta u^2\,dy = -\frac{\partial u_e}{\partial t}\,\delta - u_e\frac{\partial u_e}{\partial x}\,\delta + \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

Since:

$$u_e\frac{\partial}{\partial x}\int_0^\delta u\,dy = \frac{\partial}{\partial x}\left\{u_e\int_0^\delta u\,dy\right\} - \frac{\partial u_e}{\partial x}\int_0^\delta u\,dy$$

** $d/dx \int_{\alpha(x)}^{\beta(x)} f(x, \xi)\,d\xi = \int_\alpha^\beta \partial f/\partial x\,d\xi + f[x; \beta(x)]\,d\beta/dx - f[x; \alpha(x)]\,d\alpha/dx.$

then:

$$\frac{\partial}{\partial t}\int_0^\delta (u_e - u)\,dy + \frac{\partial}{\partial x}\int_0^\delta u(u_e - u)\,dy + \frac{\partial u_e}{\partial x}\int_0^\delta (u_e - u)\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

Using the expressions for the displacement thickness (Equation 5.111) and of the momentum thickness (Equation 5.112), we write:

$$\frac{1}{u_e^2}\frac{\partial}{\partial t}(u_e\delta^*) + \frac{\partial\theta}{\partial x} + \frac{1}{u_e}\frac{\partial u_e}{\partial x}(2\theta + \delta^*) = \frac{\nu}{u_e^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{5.154}$$

Equation 5.154 is the momentum integral equation for the unsteady two-dimensional incompressible boundary layers. For steady flow we have:

$$\frac{\partial\theta}{\partial x} + \frac{1}{u_e}\frac{\partial u_e}{\partial x}(2\theta + \delta^*) = \frac{\nu}{u_e^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{5.155}$$

Note that the right-hand side of Equation 5.154 or 5.155 is the nondimensional wall shear, i.e.:

$$\frac{\nu}{u_e^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\tau_w}{\rho u_e^2} = (1/2)c_f$$

Another integral representation can be obtained by multiplying the boundary layer equation (Equation 5.100) by u throughout and then performing the integration from $y = 0$ and $y = \delta$. Using the expressions for the energy thickness (Equation 5.113) and of the rate of energy dissipation (Equation 5.116) we get:

$$\frac{\partial}{\partial t}(u_e^2\theta) + u_e^2\frac{\partial\delta^*}{\partial t} + \frac{\partial}{\partial x}(u_e^3\Theta) = \frac{2\int_0^\delta \Phi\,dy}{\rho} \tag{5.156}$$

For steady flow we simply have:

$$\frac{\partial}{\partial x}(u_e^3\Theta\rho) = 2\int_0^\delta \Phi\,dy \tag{5.157}$$

Equation 5.156 or 5.157 is the energy integral equation for the incompressible two-dimensional boundary layers.

Solution of the Momentum Integral Equation

Among the various approximate methods of solution of the boundary layer equations, the simplest is based on Equation 5.155. The earliest method of solving this equation is due to Karman[20] and Pohlhausen and has been described fully in Schlichting.[1] Here we shall describe a simpler method due to Waltz,[21] Thwaites,[22] and Loitsyanskii.[13]

We consider Equation 5.155 in the form:

$$\frac{d\theta}{dx} + \frac{\theta u_e'}{u_e}(2 + H) = \frac{\tau_w}{\rho u_e^2} \tag{5.158}$$

where:

δ^*, the displacement thickness

θ, the momentum thickness

$H = \dfrac{\delta^*}{\theta}$, the shape parameter

$u_e = u_e(x)$, the external potential flow

$u_e' = \dfrac{du_e}{dx}$

$\tau_w = \mu\left(\dfrac{\partial u}{\partial y}\right)_{y=0}$, the wall shear

Equation 5.158 is a first order ordinary differential equation in *three* unknowns, δ^*, θ, and τ_w. To express this equation in *one* unknown, we have to prescribe the *form* of the velocity profile u/u_e having the indicated unknown as a parameter. The motivation for this approach is derived from the definitions of δ^* and θ. To proceed, we first multiply Equation 5.158 by θ/ν and introduce the *form* parameter:

$$\lambda(x) = u_e' Z \tag{5.159a}$$

where:

$$Z = \theta^2/\nu \tag{5.159b}$$

On simplification, we have:

$$\frac{dZ}{dx} = \frac{F}{u_e} \tag{5.160a}$$

where:

$$F = \frac{2\tau_w\theta}{\mu u_e} - 2(2 + H)\lambda \tag{5.160b}$$

We now envisage a single-parameter family of velocity profiles in the form:

$$\frac{u}{u_e} = G\left(\frac{y}{\theta}; \lambda\right) \tag{5.161a}$$

where the function G need not be specified at this stage. In principle:

$$H = \frac{\delta^*}{\theta} = \int_0^\infty \left[1 - G\left(\frac{y}{\theta}; \lambda\right)\right] d\left(\frac{y}{\theta}\right) = H(\lambda) \tag{5.161b}$$

and:

$$\left\{\frac{\partial\left(\dfrac{u}{u_e}\right)}{\partial\left(\dfrac{y}{\theta}\right)}\right\}_{y=0} = \frac{\theta}{u_e}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\tau_w\theta}{\mu u_e} = \zeta(\lambda) \tag{5.161c}$$

Both Equations 5.161b, c define H and ζ as functions of the shape parameter λ, and thus from Equation 5.160:

$$F(\lambda) = 2\zeta(\lambda) - 2[2 + H(\lambda)]\lambda \tag{5.162}$$

The success of the method depends on how simply the function $F(\lambda)$ in Equation 5.162 can be specified. Fortunately, based on the available solutions of the Falkner-Skan equation, and also on the solutions for various cases by the Karman-Pohlhausen method, it was concluded that $F(\lambda)$ can be represented as a linear function of λ in the form:*

$$F(\lambda) = a - b\lambda \tag{5.163}$$

where the constant a is usually taken from 0.44 to 0.47, while b from 5.35 to 6.0. For definiteness we shall take $a = 0.44$ and $b = 5.6774$. In the case of Falkner-Skan solutions (refer also to Problem 5.13):

$$\lambda = \beta d_2^2(\beta), \qquad H = d_1(\beta)/d_2(\beta), \qquad \zeta = \alpha(\beta)d_2(\beta)$$

and the reader should verify the validity of Equation 5.163 by using these expressions. Substituting Equation 5.163 in Equation 5.160a, a first order ordinary linear differential equation is obtained which can be solved to have:

$$Z = \frac{a}{u_e^b}\int_0^x u_e^{b-1}(x)\,dx \tag{5.164}$$

The constant of integration is set equal to zero to avoid the singularity at $x = 0$. Using L'Hospital's rule we find that for a stagnation point:

$$\lambda(0) = u_e'(0)Z(0) = a/b$$

and for a cuspidal leading edge:

$$\lambda(0) = 0$$

The simplicity of the present method lies in the fact that the whole problem of approximate solution has been reduced to a numerical quadrature. Once Z has been computed, the thickness θ and also the parameter λ (from Equation 5.159a) become known for all x.

In practical applications the external flow is usually represented in the form:

$$u_e(x) = cu_\infty\phi(\bar{x}), \qquad \bar{x} = \frac{x}{L}$$

Thus defining:

$$\bar{Z}(\bar{x}) = \frac{cu_\infty}{L}Z(x)$$

we have, from Equation 5.164:

* Thwaites[22] uses $m = -\lambda$ and takes $a = 0.45$ and $b = 6.0$.

$$\bar{Z}(\bar{x}) = \frac{a}{\phi^b(\bar{x})} \int_0^{\bar{x}} \phi^{b-1}(\bar{x})\, d\bar{x} \tag{5.165}$$

and then:

$$\lambda(\bar{x}) = \bar{Z}\phi'(\bar{x})$$

For numerical computations it is convenient to write Equation 5.165 in a recursive form as:

$$\phi^b(\bar{x}_n)\bar{Z}(x_n) = \phi^b(\bar{x}_{n-1})\bar{Z}(x_{n-1}) + a\int_{\bar{x}_{n-1}}^{\bar{x}_n} \phi^{b-1}(\bar{x})\, d\bar{x} \tag{5.166}$$

where $n = 1, 2, \ldots$ give the successive positions starting from $\bar{x}_0 = 0$. The value of λ near separation is about $-.088$.

This method has been found to give quite accurate results for favorable ($\lambda > 0$) and moderately unfavorable ($\lambda < 0$) pressure gradients, but becomes increasingly inaccurate as the separation point is approached.

Choice of the Velocity Profile

Having determined λ for a given external velocity u_e or $\phi(\bar{x})$, it is now possible to obtain the velocity profile by specifying the function G in Equation 5.161a. Any polynomial or transcendental function having the requisite properties of a boundary layer can be chosen. The chosen profile should somehow be made to depend on the parameter λ. Although the basis of the method depends on a profile of the form in Equation 5.161a, it is much simpler to specify the function G in which the normalizing length is the boundary layer thickness δ rather than the momentum thickness θ. Many choices are possible but for illustration we take one example from Loitsyaniskii,[13] which is

$$\frac{u}{u_e} = 1 - a_1\left(1 - \frac{y}{\delta}\right)^n - a_2\left(1 - \frac{y}{\delta}\right)^{n+1} - a_3\left(1 - \frac{y}{\delta}\right)^{n+2} \tag{5.167}$$

where using the conditions at $y = 0$: $u = 0$, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{-u_e u_e'}{\nu}$, $\dfrac{\partial^3 u}{\partial y^3} = 0$, we get

$$a_1 = \tfrac{1}{6}(n + 1)(n + 2) - \tfrac{1}{2}\Lambda$$

$$a_2 = -\tfrac{1}{3}(n - 1)(n + 2) + \frac{n}{n + 1}\Lambda$$

$$a_3 = \tfrac{1}{6}n(n - 1) - \frac{n - 1}{n + 1}\frac{\Lambda}{2}$$

$$n = 4 + 0.15\Lambda. \text{ (empirical choice)}$$

Here $\Lambda = u_e'\delta^2/\nu$ is the Pohlhausen shape parameter. The relation between Λ and λ is

$$\lambda = \left(\frac{\theta}{\delta}\right)^2 \Lambda \tag{5.168a}$$

where from Equation 5.167:

$$\frac{\theta}{\delta} = \frac{a_1}{n + 1} + \frac{a_2}{n + 2} + \frac{a_3}{n + 3} - \frac{a_1^2}{2n + 1} - \frac{a_1 a_2}{n + 1}$$

$$- \frac{(a_2^2 + 2a_1a_3)}{2n + 3} - \frac{a_2a_3}{n + 2} - \frac{a_3^2}{2n + 5} \qquad (5.168b)$$

Thus for a given λ, Equation 5.168a must be solved for A and then used in Equation 5.167 to obtain the desired profile. The solution of Equation 5.168a is quite complicated, but can be easily accomplished on a computer. As soon as the solution of Equation 5.168a is avilable we can express y/δ in Equation 5.167 as $y/\theta \cdot \theta/\delta$ through the use of Equation 5.168b, and then:

$$G\left(\frac{y}{\theta}, \lambda\right) = 1 - A_1\left(\frac{\delta}{\theta} - \frac{y}{\theta}\right)^n - A_2\left(\frac{\delta}{\theta} - \frac{y}{\theta}\right)^{n+1} - A_3\left(\frac{\delta}{\theta} - \frac{y}{\theta}\right)^{n+2}$$

where:

$$A_1 = a_1\left(\frac{\theta}{\delta}\right)^n, \qquad A_2 = a_2\left(\frac{\theta}{\delta}\right)^{n+1}, \qquad A_3 = a_3\left(\frac{\theta}{\delta}\right)^{n+2}$$

Example 5.2

Consider the external flow of Example 5.1. Use the solution of the momentum integral equation to express λ as a function $\bar{x}$. Taking the value of λ at separation as $-.0852$, find the value of $\bar{x}$ at separation for any a_0.

In this case:

$$\phi(\bar{x}) = 1 - a_0\bar{x}$$

and the integral in Equation 5.165 can be readily evaluated. Thus there is no need to resort to Equation 5.166. It can be readily verified that in this case:

$$\begin{aligned} \lambda(\bar{x}) &= \phi'(\bar{x})\bar{Z}(\bar{x}) \\ &= -\frac{a}{b}[(1 - a_0\bar{x})^{-b} - 1] \end{aligned}$$

For $\lambda = -.0852$, with $a = 0.45$, $b = 5.35$, we have:

$$a_0\bar{x} = 0.12257$$

Note: In this case the values of $\bar{x}$ for $a_0 = 1$ and $a_0 = 1/8$ are very close to those obtained by the Gortler method in Example 5.1.

5.10 FREE BOUNDARY LAYERS

The concept of a boundary layer is not always associated with the existence of a surface. There are problems in fluid mechanics in which even in the absence of a surface, the velocity gradients normal to the flow direction are much more important then those parallel to it. For example, when a fluid jet is issued into an otherwise stationary medium of fluid, then extensive transfer of momentum between the layers of moving and stationary fluid occurs. The process of approximation for the Navier-Stokes equations describing the flow in this and similar situations is then the same as for the flow past a surface. The equations become exactly of the boundary layer type but the boundary conditions are different.

Flow in the Wake of a Flat Plate

The flow in the wake of a flat plate of finite length is an example of the free boundary layer flow. The flow just downstream of the trailing edge has been considered by Goldstein,[23] and

Stewartson.[24] On the other hand, the far wake region has been considered by Tollmien,[25] which we discuss later.

Referring to Figure 5.7, we consider a plate of length L with x again measured from the leading edge. The flow after $x = L$ is the wake flow. The boundary layer equations of the wake are the same as those for the flow past a plate because the ambient pressure is again constant. However, the boundary conditions are now:

$$\text{at } y = 0 : \frac{\partial u}{\partial y} = v = 0, x > L$$

$$\text{as } y \to \infty : u \to u_\infty$$

The condition $\partial u/\partial y = 0$ at $y = 0$ is due to the symmetry of flow on both sides of the line $y = 0$. At a sufficiently large downstream distance x, the velocity is only slightly different from the free-stream value u_∞. If u' is the difference in the x-component of velocity, then in the wake:

$$u = u_\infty - u', \qquad v = 0$$

Using these equations in the flat plate boundary layer equation and neglecting the product terms of u', we get the equation:

$$u_\infty \frac{\partial u'}{\partial x} = \nu \frac{\partial^2 u'}{\partial y^2} \tag{5.169a}$$

and the boundary conditions are then:

$$\text{at } y = 0 : \frac{\partial u'}{\partial y} = 0$$

$$\text{as } y \to \infty : u' \to 0 \tag{5.169b}$$

The solution of Equation 5.169a under the boundary condition (Equation 5.169b) is

$$u' = \frac{Au_\infty}{\sqrt{x}} \exp\left(-\frac{u_\infty y^2}{4\nu x}\right)$$

where A is a constant. This constant is now obtained by using the condition that the decrease of flow of momentum in the wake is equal to the friction drag of the plate, i.e.:

$$2D = \rho \int_{-\infty}^{\infty} u(u_\infty - u)\, dy$$

per unit width of the plate. For the far wake region:

$$2D = \rho u_\infty \int_{-\infty}^{\infty} u'\, dy$$

Using the drag formula for a plate of length L in Section 5.8, we then obtain:

$$\rho u_\infty \int_{-\infty}^{\infty} u'\, dy = 1.328\, \rho u_\infty^2 \left(\frac{\nu L}{u_\infty}\right)^{1/2}$$

On substituting the expression for u', we get:

$$A = 0.664 \left(\frac{L}{\pi}\right)^{1/2}$$

Thus the velocity distribution in the far wake is

$$\frac{u}{u_\infty} = 1 - 0.664 \left(\frac{L}{\pi x}\right)^{1/2} \exp\left(-\frac{u_\infty y^2}{4\nu x}\right)$$

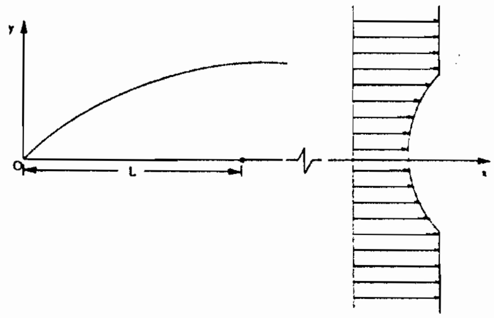


Fig. 5.7 An example of a far-wake profile in the wake of a flat plate.

Two-Dimensional Jet

Another example of the free boundary layer is that of a two-dimensional laminar jet of fluid issuing into a stationary medium of the same fluid. The equations of motion are again the boundary layer equations for a flat plate because the ambient pressure is constant. The physical situation is that of a jet issuing from an infinitely long narrow slit, a section of which is shown in Figure 5.8.

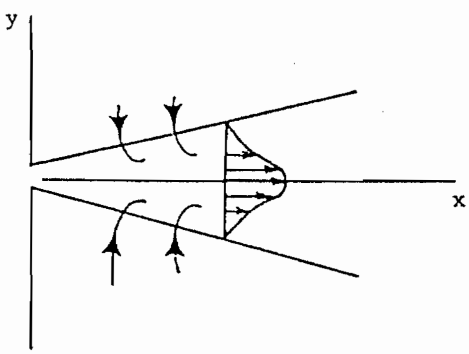


Fig. 5.8 Flow through a long slit forming a two-dimensional jet.

The equations of motion are the same as of the flat plate boundary layer:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{5.170a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.170b}$$

The boundary conditions are

$$\text{at } y = 0 \text{ (axis of the jet)} : \frac{\partial u}{\partial y} = 0,\ v = 0 \tag{5.170c}$$

$$\text{as } y \rightarrow \pm\infty : u \rightarrow 0 \tag{5.170d}$$

Integrating Equation 5.170a with respect to y from $-\infty$ to ∞ and using Equation 5.170b and the boundary conditions (Equations 7.170c, d) we get:

$$\int_{-\infty}^{\infty} \rho u^2\, dy = J_0 = \text{constant} \tag{5.170e}$$

Equation 5.170e provides an additional condition (and the most important) for the solution of Equations 5.170a, b. It states a conservation law: the momentum flux across a transverse section of the jet is the same for all sections along x.

A transformation of the equations is now effected which is obtained by noting that there is no characteristic velocity involved in the problem. An artificial introduction of a velocity V and then its elimination leads to:

$$\psi = \left(\frac{\nu x J_0}{\rho}\right)^{1/3} \phi(\eta)$$

where:

$$\eta = y\left(\frac{J_0}{\rho \nu^2 x^2}\right)^{1/3}$$

Introducing these variables in Equations 5.170a, b, we get:

$$\phi''' + \frac{1}{3}(\phi'^2 + \phi\phi'') = 0 \tag{5.171a}$$

a prime denotes differentiation with respect to η. The boundary conditions are

$$\left.\begin{array}{l} \text{at } \eta = 0 : \phi = \phi'' = 0 \\ \text{as } \eta \rightarrow \pm\infty : \phi' \rightarrow 0 \end{array}\right\} \tag{5.171b}$$

The conservation law (Equation 5.170e) becomes:

$$\int_{-\infty}^{\infty} \phi'^2(\eta)\, d\eta = 1 \tag{5.171c}$$

Integrating Equation 5.171a once and using the boundary conditions at $\eta = 0$ given in 5.171b, we get:

$$\phi'' + \frac{1}{3}\phi\phi' = 0$$

Let ϕ be a constant ϕ_0 when $\phi' = 0$. Then integrating this equation while using Equation 5.171c, and the standard integrals:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right), \qquad \int_{-\infty}^{\infty} \operatorname{sech}^4 x \, dx = \frac{4}{3}$$

we get:

$$\psi = 1.651 \left(\frac{\nu x J_0}{\rho}\right)^{1/3} \tanh\left\{0.2752 y \left(\frac{J_0}{\rho \nu^2 x^2}\right)^{1/3}\right\} \tag{5.172}$$

as the solution of the problem under consideration. The mass flux across any section of the jet is equal to:

$$\dot{m} = 2 \int_0^{\infty} \rho u \, dy = 3.3019 \rho \left(\frac{J_0 \nu x}{\rho}\right)^{1/3}$$

Axially Symmetric Jet

This is another example of a free boundary layer in the absence of a pressure gradient. A jet flows through a small orifice and the motion is symmetrical about the axis of the jet. Using the equations in cylindrical coordinates in which x is taken along the axis and r as normal to the axis, and affecting the boundary layer approximations, we have:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \tag{5.173a}$$

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{5.173b}$$

The boundary conditions are

$$\text{at } r = 0 : \frac{\partial u}{\partial r} = 0, \; v = 0$$

$$\text{as } r \to \infty : u \to 0 \tag{5.173c}$$

The conservation of momentum is expressed by the equation:

$$2\pi\rho \int_0^{\infty} u^2 r \, dr = \text{constant} = J_0 \tag{5.173d}$$

Again using the condition that there is not characteristic length involved in the problem, we obtain the transformation:

$$\psi = \nu x \phi(\eta), \qquad \eta = \frac{r}{x\sqrt{\nu}}$$

Introducing these variables in Equation 5.173a and noting that:

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}$$

we have:

$$F'' + \frac{(1 + \phi)}{\eta} F' + F^2 = 0 \tag{5.173e}$$

where $F = \phi'/\eta$. The boundary conditions are

$$\text{at } \eta = 0 : \phi = \phi' = 0 \tag{5.173f}$$

Integrating Equation 5.173c under the conditions in Equation 5.173f, we have:

$$\phi(\eta) = \frac{a^2\eta^2}{1 + \frac{1}{4}a^2\eta^2}$$

where a is an arbitrary constant, which is obtained by using Equation 5.173d. Since:

$$u = \frac{\phi'}{x\eta}$$

the constant is give by:

$$a = \left(\frac{3J_0}{18\pi\mu}\right)^{1/2}$$

Thus the solution of the problem is

$$\psi = \frac{\nu x a^2\eta^2}{1 + \frac{1}{4}a^2\eta^2}$$

so that:

$$u = \frac{2a^2}{x(1 + \frac{1}{4}a^2\eta^2)^2} \tag{5.174}$$

The flux at any section x is

$$\dot{m} = 2\pi\rho \int_0^\infty ru\, dr = 8\pi\mu x$$

which is independent of J_0.

5.11. NUMERICAL SOLUTION OF THE BOUNDARY LAYER EQUATION

Since the advent of high speed digital computers, there has been a rapid increase in the development of a variety of numerical techniques for the solution of the boundary layer and also the Navier-Stokes equations. In recent times the subject has become quite extensive, and here we shall discuss only the basic ideas in the context of steady incompressible two-dimensional boundary layers. Numerical methods for other cases have been discussed at appropriate places.

Among the various techniques of numerical solution we shall discuss only the method of finite differences. Although the analysis to follow is self-contained, it is expected that the reader has some knowledge of numerical analysis and of numerical finite differences of partial derivatives. For a thorough understanding of various approximations and difference schemes refer to, e.g., Reference 26. Refer also to ME.8.

The boundary layer equation, whether steady or nonsteady, is a quasilinear *parabolic* partial differential equation which has to be solved along with the equation of continuity under given initial and boundary conditions. The simplest parabolic equation is the nonsteady rectilinear heat conduction equation in a homogeneous medium or the diffusion equation (Equation 5.23). Although the boundary layer equation is quasilinear, the diffusion equation (Equation 5.23) serves as a prototype for the development of ueful numerical methods for the solution of the boundary layer equation.

Numerical Solution of the Diffusion Equation

Let the following initial and boundary value problem be posed for Equation 5.23:

$$u(0, y) = \phi(y); \qquad 0 \leq y \leq h \tag{5.175a}$$

$$u(t, 0) = U_\infty, \quad u(t, h) = 0; \qquad t > 0 \tag{5.175b}$$

where h is a constant. To set up a finite difference approximation of Equation 5.23 we divide the region of the ty-plane into a grid system of mesh sizes Δt and Δy. Thus we choose an integer J such that $\Delta y = h/J$. Let the integers n and j, respectively, denote the positions along t and y, respectively, such that $t_n = (n - 1)\Delta t$ and $y_j = (j - 1)\Delta y$. We shall denote the value $u(t_n, y_j)$ by u_j^n. If we represent the time derivative by a forward difference and the spatial derivative by a central difference, then the finite difference approximation of Equation 5.23 is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta y)^2} \tag{5.176a}$$

The initial and boundary values (Equations 5.175a, b) are

$$u_j^1 = \phi(y_j), \qquad j = 1, 2, \ldots, J + 1 \tag{5.176b}$$

$$u_1^n = U_\infty, \qquad u_J^n = 0, \qquad n = 1, 2, \ldots \tag{5.176c}$$

From Equation 5.176a we find that the solution for each new time $n + 1$ and for all j can be determined by knowing the solution at the previous time n. Because of this, such a difference scheme is known as an *explicit* scheme.

Errors: Truncation and Round-Off

At this stage we must recognize the types of errors made in representing a continuous system (Equation 5.23) in the form of a discrete system (Equation 5.176a). Use of a Taylor expansion readily shows that the error in the representation (Equation 5.176a) is of the order:

$$0(\Delta t) + 0[(\Delta x)^2]$$

called the *truncation error*. Hopefully by choosing very small step sizes Δt and Δx we can make this error as small as we please. Unfortunately this refinement cannot in general be carried out, because there is another error, known as the "round-off" error, which is inevitable in any numerical computation. Every computing machine can retain only a number of significant digits in an arithmetic operation so that each number taking part in the operation is rounded-off to a desired accuracy. Thus for example, in solving the explicit system (Equation 5.176a), suppose a computer retains 15 digits in each arithmetic operation. If the accumulation of the errors of round-off do not spoil the solution, then we can choose as small steps as are necessary for an accurate solution. However, there are difference schemes in which the buildup of the round-off errors is checked by a restriction on the step sizes. The problems *associated with round-off errors and not with the truncation errors* are categorized under the topic of "stability of a difference scheme". Thus a scheme is stable if the round-off errors do not accumulate in such a manner that it is impossible to solve the difference system. The "stability analysis" due to Von Neumann is quite simple to perform.

For brevity writing:

$$\lambda = \frac{\nu \Delta t}{(\Delta y)^2}$$

the difference scheme (Equation 5.176a) is

$$u_j^{n+1} = u_j^n + \lambda(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

When this recursive equation is solved numerically, an approximate solution $\bar{u}_j^n$ is obtained, so that:

$$u_j^n = \bar{u}_j^n + \delta_j^n$$

where δ_j^n is the round-off error. Thus δ_j^n satisfies the equation:

$$\delta_j^{n+1} = \delta_j^n + \lambda(\delta_{j+1}^n - 2\delta_j^n + \delta_{j-1}^n)$$

Let:

$$\delta_j^n = \epsilon_n \exp(i(j\,\Delta y)q), \qquad i = \sqrt{-1}$$

which means that we are considering the error as a Fourier term with amplitude ϵ_n and wave number q. On substitution we get:

$$\begin{aligned}\epsilon_{n+1} &= \epsilon_n[1 + \lambda\{\exp(iq\Delta y) - 2 + \exp(-iq\Delta y)\}] \\ &= \epsilon_n\{1 - 2\lambda(1 - \cos q\Delta y\}\end{aligned}$$

Consequently, if the error does not grow with time then:

$$\left|\frac{\epsilon_{n+1}}{\epsilon_n}\right| = |1 - 2\lambda(1 - \cos q\Delta y)| \leq 1$$

which implies:

$$-1 \leq 1 - 2\lambda(1 - \cos q\Delta y) \leq 1$$

or:

$$1 - 2\lambda(1 - \cos q\Delta y) \geq -1$$

and:

$$1 - 2\lambda(1 - \cos q\Delta y) \leq 1$$

The first inequality is satisfied for any q if $\lambda \leq 1/2$, while the second inequality is satisfied for any positive λ and any q. Thus we reach the conclusion that the explicit scheme (Equation 5.176a) is stable if:

$$\lambda = \frac{\nu \, \Delta t}{(\Delta y)^2} \leq \frac{1}{2} \tag{5.177}$$

A finite difference approximation of Equation 5.23 in the form:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \nu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta y)^2} \tag{5.178}$$

involves a simultaneous solution of $J - 1$ algebraic equations for each time step. Such schemes are known as *implicit*. Stability analysis of Von Neumann shows that this difference scheme is unconditionally stable. No restrictions on step sizes are necessary for this case, and therefore we can choose convenient step sizes to achieve the desired accuracy.

A generalization of Equation 5.176a and 5.178 is the following scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\nu}{(\Delta y)^2} \{S(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (1 - S)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\} \tag{5.179}$$

where S is a number such that $0 \leq S \leq 1$. Thus for $S = 0$, we get the explicit and for $S = 1$, the implicit representation. It has been established that for $1/2 \leq S \leq 1$, the scheme (Equation 5.179) is unconditionally stable.

Three of the most popular stable schemes which are used in the solution of boundary layer equations are listed as follows. (Refer also to ME.8.)

Crank and Nicolson

The Crank-Nicolson scheme is obtained by putting $S = 1/2$ in Equation 5.179. This scheme is implicit and unconditionally stable. The truncation error E is of the order:

$$E = 0[(\Delta t)^2] + 0[(\Delta y)^2]$$

Dufort and Frankel

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \nu \frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{(\Delta y)^2}$$

This scheme is explicit and always stable. The truncation error is of the order:

$$E = 0(\Delta t) + 0[(\Delta y)^2] + 0\left(\frac{\Delta t}{\Delta y}\right)$$

Thus, we must have $\Delta t/\Delta y \to 0$ as $\Delta y \to 0$ and $\Delta t \to 0$ for accurate solutions.

Three-Point Scheme

$$\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} = \nu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta y)^2}$$

This scheme is always stable. The error is

$$E = 0[(\Delta t)^2] + 0[(\Delta y)^2]$$

Solution of the Boundary Layer Equation

The basic ideas and the nomenclature developed so far will now be used in the finite difference formulation of the steady boundary layer equations. In this connection the reader is particularly referred to a survey article by Blottner.[27] The methods of solution discussed below are only suggestive in nature and are intended merely to lay down the basic approach to the problem. For the purpose of finite difference, we can use the equations either in untransformed physical coordinates or in the transformed nondimensional coordinates. For the untransformed equations we refer to Krause and Paskonov (refer to Reference 27). The untransformed equations are Equations 5.99 and 5.100 which under the steady state conditions are Equations 5.117–5.120.

Let Δx and Δy be the size of a mesh in the xy-plane and let k and j be the integers denoting positions along x and y, respectively, as shown in Figure 5.9. The value of $u(x, y)$ at the position $x_k = (k - 1)\Delta x$ and $y_j = (j - 1)\Delta y$ is $u(x_k, y_j)$ which we denote as $u_{k,j}$. We shall also use average quantities, i.e., all evaluated at the mesh center (x).

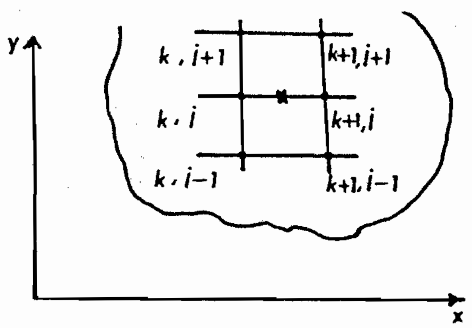


Fig. 5.9 A two-dimensional mesh with nodes(x)

Thus:

$$u_{k+1/2,j} = 1/2(u_{k+1,j} + u_{k,j}) \tag{5.180a}$$

$$V_{k+1/2,j} = 1/2(v_{k+1,j} + v_{k,j}) \tag{5.180b}$$

Denoting $u_e\, du_e/dx$ by $P(x)$, we also have:

$$P_{k+1/2} = 1/2(P_{k+1} + P_k) \tag{5.180c}$$

We now set up an implicit difference scheme for Equation 5.118. Note that the implicit difference scheme for the linear heat conduction equation was found to be unconditionally stable. Since the boundary layer equations are quasilinear, no such proof of stability for them is available. However, based on numerical experimentation and also due to the manner* in which we solve the difference equations, it is safe to say that the implicit schemes for boundary layers are also unconditionally stable.

The implicit scheme for Equation 5.118 correct to the second order** in the truncation error is

$$\frac{u_{k+1/2,j}}{\Delta x}(u_{k+1,j} - u_{k,j}) + \frac{V_{k+1/2,j}}{2\Delta y}[S(u_{k+1,j+1} - u_{k+1,j-1})$$

* The iterative process to be described makes the difference equations linear.

** $(f(x + h) - f(x))/h = f'(x + h/2) + h^2/24\, f'''(x + h/2)$.

$$+ (1 - S)(u_{k,j+1} - u_{k,j-1})] = P_{k+1/2} + \frac{\nu}{(\Delta y)^2}[S(u_{k+1,j+1} - 2u_{k+1,j}$$

$$+ u_{k+1,j-1}) + (1 - S)(u_{k,j+1} - 2u_{k,j} + u_{k,j-1})] \qquad \textbf{(5.181)}$$

where $0 \leq S \leq 1$. Note that for $S = 0$ we get an explicit scheme and for $S \geq 1/2$ an implicit scheme. It is important to note that Equation 5.180 is a nonlinear difference scheme. To be able to solve it as a system of linear algebraic equations, we introduce the idea of an iterative process built into the solution-obtaining process. The convergence of this iterative process will imply the availability of the solutions of Equation 5.181 itself. The iterative process also serves another purpose in that since the equations to be solved are treated as linear equations in each cycle, then the unconditional stability criteria for $S \geq 1/2$ is assured.

Let p denote the iteration number (iteration counter) at the $(k + 1)$th station. To make the difference equation linear, we replace the values $u_{k+1,j}$ and $v_{k+1,j}$ appearing in Equation 5.180a, b as available in the $(p - 1)$th cycle. Denoting the quantities to be calculated at the p-th cycle be a superscript p, Equation 5.181 can be arranged as:

$$-A_{k,j}u^{(p)}_{k+1,j+1} + B_{k,j}u^{(p)}_{k+1,j} - C_{k,j}u^{(p)}_{k+1,j-1} = D_{k,j} \qquad \textbf{(5.182)}$$

where the coefficients $A_{k,j}$, etc. depend on the values already available at the k-th station and on the $(p - 1)$th iterates at the $(k + 1)$ station. The set of equations for all values of j are linear algebraic equations of the tridiagonal form. To solve these equations, we introduce:

$$u^{(p)}_{k+1,j} = E_{k,j}u^{(p)}_{k+1,j+1} + F_{k,j} \qquad \textbf{(5.183)}$$

into the last term on the left-hand side of Equation 5.182 to obtain:

$$E_{k,j} = \frac{A_{k,j}}{B_{k,j} - C_{k,j}E_{k,j-1}} \qquad \textbf{(5.184a)}$$

$$F_{k,j} = \frac{D_{k,j} + C_{k,j}F_{k,j-1}}{B_{k,j} - C_{k,j}E_{k,j-1}} \qquad \textbf{(5.184b)}$$

Denoting by $j = 1$ the position $y = 0$, the boundary condition:

$$u^{(p)}_{k+1,1} = 0 \text{ (for all } k)$$

implies that:

$$E_{k,1} = F_{k,1} = 0 \qquad \textbf{(5.184c)}$$

Using Equation 5.184c we can generate the values of $E_{k,j}$ and $F_{k,j}$ recursively through Equations 5.184a, b. Let $j = M$ (an integer) denote the outer edge of the boundary layer at the $(k + 1)$th station. The solution of Equation 5.183 can be obtained by setting $j = M - 1$ and working backward toward the wall $y = 0$. Based on the newly calculated values of $u_{k+1,j}$, we use the continuity equation (Equation 5.117) to calculate the new values of v by the difference scheme:

$$1/2\left[\frac{u_{k+1,j} - u_{k,j}}{\Delta x} + \frac{u_{k+1,j+1} - u_{k,j+1}}{\Delta x}\right] + 1/2\left[\frac{v_{k+1,j+1} - v_{k+1,j}}{\Delta y} + \frac{v_{k,j+1} - v_{k,j}}{\Delta y}\right] = 0$$

This equation gives new values of v at the $(k + 1)$ station based on the recently calculated values of u. The whole cycle of iteration is terminated when a prescribed small difference between two consecutive iterations is achieved.

Because of the boundary layer growth, the outer condition at the edge of the boundary layer is not achieved in the same number of Δy steps for all values of k. Suppose at station k we required $L + 1$ number of Δy steps to match the external velocity $u_e(k)$. For the $(k + 1)$ station we first set:

$$u_{k+1,L+1} = u_e(k + 1)$$

and then calculate:

$$u_{k+1,L} = E_{k,L}u_{k+1,L+1} + F_{k,L}$$

and:

$$u_{k+1,L-1} = E_{k,L-1}u_{k+1,L} + F_{k,L-1}$$

the superscript p has been removed for brevity. Choosing a small positive number ϵ, we test the inequality:

$$|u_{k+1,L} - u_{k+1,L-1}| < \epsilon$$

If this is not satisfied then we add one step Δy at the $(k + 1)$ station and calculate $E_{k,L+1}$, $F_{k,L-1}$ based on the external velocity $u_e(k + 1)$ and then $u_{k+1,L-2}$ and $u_{k+1,L+1}$, every time testing the inequality. The process is terminated when the desired tolerance is attained.

The last point to be discussed is in regard to the initial data which must go as an input for starting the marching process. At a suitable point, say $x = 0$, we need to specify the u-profile quite accurately but the specification of the v-profile is a matter of convenience.

For the v-profile we can specify arbitrary function values, even zeros. The reason for this convenience is that we are using an iterative process which corrects itself before going over to the next station. In each cycle a new v-profile is calculated.

For the transformed boundary layer equations, we can follow the same type of finite difference approximation. As an example we may take the equation obtained by using Gortler's transformation, Equation 5.144. The advantages with this transformed form are (1) the continuity equation is not to be considered and (2) the outer condition is usually available for the same fixed values of η_δ. Some authors keep the continuity equation but transform the equations by introducing the similarity variable η.

The equation of the form given in Equation 5.144 contains a third order derivative, in which on using the finite difference, a pentadiagonal form of equations is obtained. Equations of this form have been considered by Fussell and Hellums.[28] An alternative to the preceding approach is to introduce:

$$f(\eta) = \int_0^\eta F(\eta)\, d\eta$$

into Equation 5.144 so as to get the tridiagonal form of equations in F. In each cycle of iteration the values of $f(\eta)$ are updated by simple quadratures.

The Box Method

Beside the finite difference methods of Krause and Paskonov, the method developed by Keller and Cebeci[29] (see other references in Reference 15) is quite efficient and accurate. The Box method is second order accurate in both coordinate step sizes.

Consider the boundary layer equations after use has been made of Falkner's transformation:[30]

$$\xi = x, \quad N_e(x) = \sqrt{\nu x u_e}, \quad \gamma(x) = \tfrac{1}{2}(1 + \beta), \quad \beta = \frac{x}{u_e}\frac{du_e}{dx}, \quad \chi(x) = x$$

Then the equation is

$$f''' + \tfrac{1}{2}(1 + \beta)ff'' + \beta(1 - f'^2) = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{5.185a}$$

The boundary conditions are

$$\begin{aligned} &\text{at } \eta = 0 : f = f' = 0 \\ &\text{as } \eta \to \eta_e : f' \to 1 \end{aligned} \tag{5.185b}$$

where $\eta_e = \eta_e(x)$ is the value of η where the external conditions are satisfied. The first step of the Box method is to write three first order partial differential equations in place of Equation 5.185a. Introducing U and V defined as follows, we have the three equations:

$$\frac{\partial f}{\partial \eta} = U \tag{5.185c}$$

$$\frac{\partial U}{\partial \eta} = V \tag{5.185d}$$

$$\frac{\partial V}{\partial \eta} = x\left(U\frac{\partial U}{\partial x} - V\frac{\partial f}{\partial \eta}\right) - \tfrac{1}{2}(1 + \beta)fV - \beta(1 - U^2) \tag{5.185e}$$

The boundary conditions are

$$\begin{aligned} &\text{at } \eta = 0 : f = 0, U = 0 \\ &\text{as } \eta \to \eta_e : U \to 1 \end{aligned} \tag{5.185f}$$

Besides the boundary conditions Equations 5.185b, f it is assumed that the solution is known at some upstream station $x = x_0$ from where the solution is to be started. Thus the functions:

$$f_0(\eta) = f(x_0, \eta), \quad U_0(\eta) = U(x_0, \eta), \quad V_0(\eta) = V(x_0, \eta)$$

are assumed to be given. As before let k and j denote the integer indices for the steps along x and η, respectively. Consider a rectangular mesh similar to Figure 5.9 with corners at the points (x_{k-1}, η_{j-1}), (x_k, η_{j-1}), (x_k, η_j), and (x_{k-1}, η_j), respectively. Consider a nonuniform mesh spacing in both x and η, defined by:

$$\begin{aligned} x_k &= x_{k-1} + r_k, \qquad 1 \le k \le I \\ \eta_j &= \eta_{j-1} + h_j, \qquad 1 \le j \le J, \qquad \eta_0 = 0 \end{aligned}$$

The unknown vector w is then:

$$w = \begin{bmatrix} f \\ U \\ V \end{bmatrix}$$

Keller and Cebeci use the following difference approximations:

$$w_{k,j-1/2} = 1/2(w_{k,j} + w_{k,j-1})$$

$$\left(\frac{\partial w}{\partial \eta}\right)_{k,j-1/2} = \frac{1}{h_j}(w_{k,j} - w_{k,j-1})$$

$$\left(\frac{\partial w}{\partial x}\right)_{k-1/2,j-1/2} = \frac{1}{r_k}(w_{k,j-1/2} - w_{k-1,j-1/2})$$

$$= 1/2\left[\frac{(w_{k,j} - w_{k-1,j})}{r_k} + \frac{w_{k,j-1} - w_{k-1,j-1}}{r_k}\right]$$

$$\left(\frac{\partial w}{\partial \eta}\right)_{k-1/2,j-1/2} = 1/2\left[\left(\frac{\partial w}{\partial \eta}\right)_{k,j-1/2} + \left(\frac{\partial w}{\partial \eta}\right)_{k-1,j-1/2}\right]$$

Thus:

$$\left(\frac{\partial w}{\partial \eta}\right)_{k-1/2,j-1/2} = +\ 1/2\left[\frac{(w_{k,j} - w_{k,j-1})}{h_j} + \frac{(w_{k-1,j} - w_{k-1,j-1})}{h_j}\right]$$

and:

$$w_{k-1/2,j-1/2} = 1/4(w_{k,j} + w_{k,j-1} + w_{k-1,j} + w_{k-1,j-1})$$

In the Box method, the Equations 5.185c, d are approximated at the point $(k, j - 1/2)$, while Equation 5.185e is approximated at the center $(k - 1/2, j - 1/2)$ as follows:

$$\left(\frac{\partial f}{\partial \eta}\right)_{k,j-1/2} = U_{k,j-1/2}$$

or:

$$f_{k,j} = f_{k,j-1} + 1/2h_j(u_{k,j} + U_{k,j-1}) \quad \textbf{(5.186a)}$$

$$\left(\frac{\partial U}{\partial \eta}\right)_{k,j-1/2} = V_{k,j-1/2}$$

or:

$$U_{k,j} = U_{k,j-1} + 1/2h_j(V_{k,j} + V_{k,j-1}) \quad \textbf{(5.186b)}$$

$$\left(\frac{\partial V}{\partial \eta}\right)_{k-1/2,j-1/2} = x_{k-1/2}\left[U_{k-1/2,j-1/2}\left(\frac{\partial U}{\partial x}\right)_{k-1/2,j-1/2} - V_{k-1/2,j-1/2}\left(\frac{\partial f}{\partial x}\right)_{k-1/2,j-1/2}\right]$$

$$-\left[\frac{(1+\beta)}{2}fV - \beta(1 - U^2)\right]_{k-1/2,j-1/2} \quad \textbf{(5.186c)}$$

The boundary conditions are

$$f_{k,0} = 0, \qquad U_{k,0} = 0, \qquad U_{k,J} = 1$$

It is immediately seen that the difference approximation Equation 5.186c results in nonlinear equations. Keller and Cebeci solve the system of equations (Equation 5.186) by Newton's method, which is an iterative method. Let the superscript enclosed index p on either f, U, V

(denoted as w) stand for the iteration number and the solution at the station $x = x_{k-1}$ is known. Now we are required to find the solution at $x = x_k$. To start the iterative process let us set;

$$f_{k,j}^{(0)} = f_{k-1,j}, \qquad U_{k,j}^{(0)} = U_{k-1,j}, \qquad V_{k,j}^{(0)} = V_{k-1,j}$$

in the field for all values of j in the interval $1 \leq j \leq J - 1$. On the body surface $j = 0$:

$$f_{k,0}^{(0)} = 0, \qquad U_{k,0}^{(0)} = 0, \qquad V_{k,0}^{(0)} = V_{k-1,0}$$

On the outer boundary $j = J$:

$$f_{k,J}^{(0)} = f_{k-1,J}, \qquad U_{k,J}^{(0)} = 1, \qquad V_{k,J}^{(0)} = V_{k-1,J}$$

Keller and Cebeci now take:

$$w_{k,j}^{(p+1)} = w_{k,j}^{(p)} + \delta w_{k,j}^{(p)}$$

and substitute in Equations 5.186 and neglect the second and higher order terms in $\delta w_{k,j}$ thus having a linear system of algebraic equations in the error terms. When these equations are arranged, they form a linear block-tridiagonal system of the form:

$$A_{k,j}\begin{bmatrix}\delta f\\ \delta U\\ \delta V\end{bmatrix}_{k,j-1} + B_{k,j}\begin{bmatrix}\delta f\\ \delta U\\ \delta V\end{bmatrix}_{k,j} + C_{k,j}\begin{bmatrix}\delta f\\ \delta U\\ \delta V\end{bmatrix}_{k,j+1} = D_{k,j}$$

where $A_{k,j}$, $B_{k,j}$, $C_{k,j}$, are 3×3 matrices having elements which are either known from the previous x-station or from the previous iteration. An efficient method of solving the system of equations in block-tridiagonal form has been given by Keller.[31]

5.12 THREE-DIMENSIONAL BOUNDARY LAYERS

The derivation of the boundary layer equations in three dimensions, when the Navier-Stokes' equations are written in rectangular Cartesian coordinates, is quite straightforward and requires no new additional concept. Following the basic ideas developed in Section 5.5, the resulting boundary layer equations are simple but of very little practical importance, because the shape of the body where these equations can be applied is only a flat plate. The simplicity of the equations in the Cartesian form is a very attractive feature. Fortunately, there is a class of body shapes for which, despite the use of curvilinear coordinates, the equations are in the *Cartesian form*. To arrive at these equations, and also to cover those in which it is not possible to obtain the equations in the Cartesian form, we have to consider the equations in general orthogonal coordinates.

The Navier-Stokes equations with reference to an inertial orthogonal curvilinear coordinate system were obtained in Section 3.10. In the derivation to follow we shall consistently use the convention that the coordinates ξ_1 and ξ_3 are defined on the surface and $\xi_2 = 0$ defines the surface itself. Further, we shall confine our attention only to that case in which the curvatures of the coordinate lines ξ_1 and ξ_3 are not excessively large. To derive the boundary layer equations we introduce the stretched variables:

$$\bar{\xi}_2 = \xi_2/\epsilon, \qquad \bar{u}_2 = u_2/\epsilon \tag{5.187}$$

where, as before:

$$\epsilon = R_e^{-1/2}$$

and $\epsilon \rightarrow 0$ means that $\nu \rightarrow 0$. Introducing the transformation (Equation 5.187) in Equations 3.115–3.118) with ρ = constant* and then taking the limit as $\nu \rightarrow 0$ while keeping ξ_1, $\bar{\xi}_2$, ξ_3 fixed, we obtain the boundary layer equations. Despite the simplicity of this process, there are a few points which must be taken into consideration before applying the inner limit process.

The Metric Coefficients

In an orthogonal curvilinear coordinate system the scale factors h_1, h_2, and h_3 appear in the definition of the length element ds according to the formula:

$$ds^2 = h_1^2\, d\xi_1^2 + h_2^2\, d\xi_2^2 + h_3^2\, d\xi_3^2$$

Here the h's are purely geometric quantities which do not depend on the flow field or the kinematic viscosity ν. However, the boundary layer approximation, by its very nature, is such that it disturbs the metric coefficients along with the other quantities. To see how the scale factors are disturbed, we introduce the stretched variable defined in Equation 5.187 in the scale factors to have:

$$h_i(\xi_1, \epsilon\bar{\xi}_2, \xi_3), \qquad i = 1, 2, 3$$

Taking the inner limit, we get:

$$\lim_{\substack{\nu \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} h_i(\xi_1, \epsilon\bar{\xi}_2, \xi_3) = h_i(\xi_1, 0, \xi_3)$$

The right-hand side of this equation shows that after the inner limit *has been applied, the scale factors will depend only on the surface coordinates* ξ_1 and ξ_3.

In the derivation of two-dimensional boundary layer equations we obtained $\partial u_1/\partial\xi_2 = 0(1/\epsilon)$. The same is true in three dimensions, i.e.:

$$\frac{\partial u_1}{\partial \xi_2} = 0\left(\frac{1}{\epsilon}\right), \qquad \frac{\partial u_3}{\partial \xi_2} = 0\left(\frac{1}{\epsilon}\right)$$

From this result it is *wrong* to conclude that the derivative of any of the scale factors with respect to ξ_2 is also of the order $1/\epsilon$. For:

$$\frac{\partial h_i}{\partial \xi_2} = \frac{\partial h_i(\xi_1, \epsilon\bar{\xi}_2, \xi_3)}{\partial(\epsilon\bar{\xi}_2)} = h_i'(\xi_1, \epsilon\bar{\xi}_2, \xi_3)$$

where a prime denotes differentiation with respect to $\epsilon\bar{\xi}_2$. In the limit as $\epsilon \rightarrow 0$ we simply have:

$$\lim_{\substack{\nu \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} h_i'(\xi_1, \epsilon\bar{\xi}_2, \xi_3) = h_i'(\xi_1, 0, \xi_3) = 0(1)$$

which shows that the first derivative of h_i, $i = 1, 2, 3$, with respect to ξ_2 are quantities of the order of one and not of the order of $1/\epsilon$.

* For compressible boundary layers, refer to Section 5.22.

The Matching Conditions

We shall now discuss the *matching conditions* which the boundary layer solution must satisfy at the outer edge of the boundary layer. For the present discussion, we again use another subscript b to distinguish between the boundary layer and the Euler solutions. The boundary layer solution is denoted as:

$$u_{1b}, \quad u_{2b}, \quad u_{3b}, \quad p_b$$

The solution of the Euler equations, which is the outer limit of the Navier-Stokes solution, is denoted as before by:

$$u_{1e}, \quad u_{2e}, \quad u_{3e}, \quad p_e$$

To be specific, we shall denote the values of the scale factors evaluated at the surface by a subscript w as:

$$h_{iw} = h_i(\xi_1, 0, \xi_3), \qquad i = 1, 2, 3$$

With these notations, the vorticity components of the boundary layer and the external flow are

$$\omega_{1b} = \frac{1}{\epsilon} \lim_{\substack{\nu \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} (\epsilon\omega_1) = \frac{1}{h_{2w}h_{3w}} \frac{\partial}{\partial \xi_2}(h_{3w}u_{3b}) \tag{5.188a}$$

$$\omega_{2b} = \frac{1}{h_{1w}h_{3w}} \left\{ \frac{\partial}{\partial \xi_3}(h_{1w}u_{1b}) - \frac{\partial}{\partial \xi_1}(h_{3w}u_{3b}) \right\} \tag{5.188b}$$

$$\omega_{3b} = \frac{1}{\epsilon} \lim_{\substack{\nu \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} (\epsilon\omega_3) = \frac{-1}{h_{1w}h_{2w}} \frac{\partial}{\partial \xi_2}(h_{1w}u_{1b}) \tag{5.188c}$$

$$\omega_{1e} = \frac{1}{h_2h_3} \left\{ \frac{\partial}{\partial \xi_2}(h_3u_{3e}) - \frac{\partial}{\partial \xi_3}(h_2u_{2e}) \right\} \tag{5.189a}$$

$$\omega_{2e} = \frac{1}{h_1h_3} \left\{ \frac{\partial}{\partial \xi_3}(h_1u_{1e}) - \frac{\partial}{\partial \xi_1}(h_3u_{3e}) \right\} \tag{5.189b}$$

$$\omega_{3e} = \frac{1}{h_1h_2} \left\{ \frac{\partial}{\partial \xi_1}(h_2u_{2e}) - \frac{\partial}{\partial \xi_2}(h_1u_{1e}) \right\} \tag{5.189c}$$

At the outer edge of the boundary layer we have the requirement that:

$$u_{1b} \to u_{1e}, \qquad u_{3b} \to u_{3e}, \qquad \omega_{1b} \to \omega_{1e}, \qquad \omega_{3b} \to \omega_{3e}$$

Application of this principle to the vorticity terms given in Equations 5.189a, c immediately shows that for matching the Euler solution must be evaluated at the surface, because it is only at $\xi_2 = 0$ that $u_{2e} = 0$. With the requirement that the Euler solution be evaluated at the surface, the vorticity components ω_{1e} and ω_{3e} are zero.* Consequently, we obtain the additional conditions that:

* By this we mean that first the Euler solution is evaluated at $\xi_2 = 0$ and then put into Equation 5.189.

$$\frac{\partial u_{1b}}{\partial \xi_2} \rightarrow 0, \qquad \frac{\partial u_{3b}}{\partial \xi_2} \rightarrow 0$$

as the outer edge is approached, i.e., $\bar{\xi}_2 \rightarrow \infty$.

To apply the matching principle from a rigorous point of view, we restate that the inner (boundary layer) and the outer (Euler) solutions are equally valid in an overlap region in which:

the inner limit of the outer solution = the outer limit of the inner solution

In symbolic form, let $f_e(\xi_1, \xi_2, \xi_3)$ and $f_b(\xi_1, \bar{\xi}_2, \xi_3)$ be the respective outer and inner solutions. Then in the overlap region:

$$\lim_{\substack{\epsilon \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} f_e(\xi_1, \epsilon\bar{\xi}_2, \xi_3) = \lim_{\substack{\epsilon \to 0 \\ \xi_1, \xi_2, \xi_3 \text{ fixed}}} f_b\left(\xi_1, \frac{\xi_2}{\epsilon}, \xi_3\right)$$

or:

$$f_e(\xi_1, 0, \xi_3) = f_b(\xi_1, \infty, \xi_3)$$

a result which was obtained earlier in Section 5.4.

Since the Euler solution which matches with the boundary layer solution has been evaluated at the surface, then from Equations 5.188b and 5.189b we find that:

$$\text{as } \bar{\xi}_2 \rightarrow \infty : \omega_{2b} \rightarrow \omega_{2e}$$

This condition does *not* demand that ω_{2e} should be zero for matching. *Thus we conclude that in three-dimensional boundary layers it is not necessary for the outer flow to be irrotational.* The only requirement is that ω_{1e} and ω_{3e} be zero. In some specific situations ω_{2e} may become zero, and then the external flow is said to be irrotational or potential.

The last point about which we have to convince ourselves is in regard to the boundary layer pressure p_b. To investigate the nature of the boundary layer pressure distribution p_b, we apply the transformation (Equation 5.187) to the equation for u_2 and take the inner limit. Thus:

$$\lim_{\substack{\epsilon \to 0 \\ \xi_1, \bar{\xi}_2, \xi_3 \text{ fixed}}} \left\{ \epsilon \left(\frac{u_1^2}{h_1 h_2} h_1' + \frac{u_3^2}{h_2 h_3} h_3' \right) \right\} = \frac{1}{\rho h_{2w}} \frac{\partial p_b}{\partial \bar{\xi}_2}$$

where h_i' have been defined earlier. This equation shows, in fact, that:

$$\frac{\partial p_b}{\partial \bar{\xi}_2} = 0(\epsilon)$$

and:

$$\frac{\partial p_b}{\partial \xi_2} = 0(1)$$

provided that either h_1' or h_3' or both are not extremely large. It has already been shown that h_1' and h_3' are quantities of 0(1) as far as the boundary layer approximation is concerned. However, from a geometric standpoint both h_1' and h_3' can become extremely large, such as near a sharp corner. We exclude this possibility and then reach the conclusion that the pressure varies very slowly across the thin boundary layer. In fact, as in two dimensions, we write:

$$\frac{\partial p_b}{\partial \xi_2} = 0$$

Thus, the boundary layer pressure is independent of the boundary layer coordinate ξ_2 which, in effect, means that it is obtained through the Euler equations evaluated at the surface, i.e.:

$$p_b = p_{ew}$$

The Euler equations which determine the pressure at the wall (i.e., at $\xi_2 = 0$) are

$$\frac{\partial u_{1e}}{\partial t} + \frac{u_{1e}}{h_{1w}}\frac{\partial u_{1e}}{\partial \xi_1} + \frac{u_{3e}}{h_{3w}}\frac{\partial u_{3e}}{\partial \xi_3} + \frac{u_{1e}u_{3e}}{h_{1w}h_{3w}}\frac{\partial h_{1w}}{\partial \xi_3} - \frac{u_{3e}^2}{h_{1w}h_{3w}}\frac{\partial h_{3w}}{\partial \xi_1} = -\frac{1}{\rho h_{1w}}\frac{\partial p_e}{\partial \xi_1} \tag{5.190a}$$

$$\frac{\partial u_{3e}}{\partial t} + \frac{u_{1e}}{h_{1w}}\frac{\partial u_{3e}}{\partial \xi_1} + \frac{u_{3e}}{h_{3w}}\frac{\partial u_{3e}}{\partial \xi_3} + \frac{u_{1e}u_{3e}}{h_{1w}h_{3w}}\frac{\partial h_{3w}}{\partial \xi_1} - \frac{u_{1e}^2}{h_{1w}h_{3w}}\frac{\partial h_{1w}}{\partial \xi_3} = -\frac{1}{\rho h_{3w}}\frac{\partial p_e}{\partial \xi_3} \tag{5.190b}$$

where all the metric quantities appearing in Equations 5.190 do not depend on ξ_2. It must also be noted that these equations by themselves do not imply that the external flow is irrotational. If, however, the flow is irrotational, i.e., $\omega_{2e} = 0$, then for steady flow Equations 5.190 can be integrated to give the Bernoulli equation:

$$p_e = \text{constant} - \frac{\rho}{2}(u_{1e}^2 + u_{3e}^2)$$

Taking into consideration the limiting behaviors of the quantities as discussed above and dropping the subscript w from the metric coefficients, the three-dimensional boundary layer equations in orthogonal curvilinear coordinates are

$$\frac{\partial}{\partial \xi_1}(h_2h_3u_1) + \frac{\partial}{\partial \xi_2}(h_1h_3u_2) + \frac{\partial}{\partial \xi_3}(h_1h_2u_3) = 0 \tag{5.191}$$

$$\frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial \xi_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial \xi_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{u_3^2}{h_1h_3}\frac{\partial h_3}{\partial \xi_1}$$
$$= \frac{-1}{\rho h_1}\frac{\partial p_e}{\partial \xi_1} + \frac{1}{\rho h_2}\frac{\partial}{\partial \xi_2}\left(\frac{\mu}{h_2}\frac{\partial u_1}{\partial \xi_2}\right) \tag{5.192}$$

$$\frac{\partial u_3}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_3}{\partial \xi_1} + \frac{u_2}{h_2}\frac{\partial u_3}{\partial \xi_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial \xi_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{u_1^2}{h_1h_3}\frac{\partial h_1}{\partial \xi_3}$$
$$= \frac{-1}{\rho h_3}\frac{\partial p_e}{\partial \xi_3} + \frac{1}{\rho h_2}\frac{\partial}{\partial \xi_2}\left(\frac{\mu}{h_2}\frac{\partial u_3}{\partial \xi_2}\right) \tag{5.193}$$

The pressure gradients in Equations 5.192 and 5.193 are obtained through the Euler equations (Equations 5.190).

Equations in Rotating Coordinates

The equations of motion referred to a rotating coordinate system have already been discussed in Problem 3.4. Refer also to Problem 1.12. For the case of incompressible flow without body forces, the equations are

$$\text{div}\,\mathbf{u} = 0 \tag{5.194a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} - \text{grad}\left(\frac{\Omega^2 R^2}{2}\right) = -\frac{1}{\rho}\,\text{grad}\, p + \nu\nabla^2\mathbf{u} \tag{5.194b}$$

An alternative form of Equation 5.194b is

$$\frac{\partial \mathbf{u}}{\partial t} + \text{grad}\,\frac{|\mathbf{u}|^2}{2} + (2\mathbf{\Omega} + \boldsymbol{\omega}) \times \mathbf{u} - \text{grad}\left(\frac{\Omega^2 R^2}{2}\right) = -\frac{1}{\rho}\,\text{grad}\, p + \nu\nabla^2\mathbf{u} \tag{5.194c}$$

In Equations 5.194, all quantities have been referred to a rotating coordinate system which is rotating with a constant angular velocity Ω about an arbitrary axis in space. The vector $\mathbf{\Omega}$ points in the direction of the axis of rotation of the rotating system. Further, R is the perpendicular distance of a fluid particle from the axis of rotation, and ρ the fluid density. It will be helpful for future discussions to have available the Euler equations by putting $\nu = 0$ in either Equation 5.194b or Equation 5.194c. Setting $\nu = 0$ in Equation 5.194c we get:

$$\text{div}\,\mathbf{u}_e = 0 \tag{5.195a}$$

$$\frac{\partial \mathbf{u}_e}{\partial t} + \text{grad}\,\frac{|\mathbf{u}_e|^2}{2} + (2\mathbf{\Omega} + \boldsymbol{\omega}_e) \times \mathbf{u}_e - \text{grad}\left(\frac{\Omega^2 R^2}{2}\right) = -\frac{1}{\rho}\,\text{grad}\, p_e \tag{5.195b}$$

where a subscript e as usual denotes the Euler or external solution.

The essential difference in the form of equations in rotating and nonrotating coordinates is the appearance of two additional terms in the former due to the rotation. These terms are $\mathbf{\Omega} \times \mathbf{u}$, the Coriolis acceleration, and $-\text{grad}(\Omega^2R^2/2)$, the centripetal acceleration.

The boundary layer equations are obtained by first introducing a coordinate system on the rotating body and then writing the vector equations (Equations 5.194a, b) in their component forms. The whole process of taking the inner limit as discussed earlier is then applied. The resulting equations in orthogonal curvilinear coordinate (ξ_1, ξ_2, ξ_3) are then:

$$\frac{\partial}{\partial \xi_1}(h_2h_3u_1) + \frac{\partial}{\partial \xi_2}(h_1h_3u_2) + \frac{\partial}{\partial \xi_3}(h_1h_2u_3) = 0 \tag{5.196}$$

$$\begin{aligned}\frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial \xi_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial \xi_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{u_3^2}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} + 2\Omega_2u_3 - \frac{\Omega^2}{2h_1}\frac{\partial R^2}{\partial \xi_1} \\ = -\frac{1}{\rho h_1}\frac{\partial p_e}{\partial \xi_1} + \frac{\nu}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{h_2}\frac{\partial u_1}{\partial \xi_2}\right)\end{aligned} \tag{5.197}$$

$$\begin{aligned}\frac{\partial u_3}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_3}{\partial \xi_1} + \frac{u_2}{h_2}\frac{\partial u_3}{\partial \xi_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial \xi_3} + \frac{u_1u_3}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{u_1^2}{h_1h_3}\frac{\partial h_1}{\partial \xi_3} - 2\Omega_2u_1 - \frac{\Omega^2}{2h_3}\frac{\partial R^2}{\partial \xi_3} \\ = -\frac{1}{\rho h_3}\frac{\partial p_e}{\partial \xi_3} + \frac{\nu}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{h_2}\frac{\partial u_3}{\partial \xi_2}\right)\end{aligned} \tag{5.198}$$

As before, all scale factors appearing in Equations 5.196–5.198 have been evaluated at the body surface $\xi_2 = 0$. Further, the distance R is now a function of the surface coordinates ξ_1, ξ_3 only; and we can state that in the boundary layer equations R is the perpendicular distance of a point from the axis of rotation in the plane of rotation. There is no loss of generality by setting $h_2 = 1$ in Equations 5.196–5.198.

The matching conditions for the boundary layer solution are obtained through solution of the Euler equations and afterward evaluation at the surface. The pressure p_e is given by the equations:

$$\frac{\partial u_{1e}}{\partial t} + \frac{u_{1e}}{h_1}\frac{\partial u_{1e}}{\partial \xi_1} + \frac{u_{3e}}{h_3}\frac{\partial u_{1e}}{\partial \xi_3} + \frac{u_{1e}u_{3e}}{h_1h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{u_{3e}^2}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} + 2\Omega_2 u_{3e} - \frac{\Omega^2}{2h_1}\frac{\partial R^2}{\partial \xi_1}$$
$$= -\frac{1}{\rho h_1}\frac{\partial p_e}{\partial \xi_1} \tag{5.199a}$$

$$\frac{\partial u_{3e}}{\partial t} + \frac{u_{1e}}{h_1}\frac{\partial u_{3e}}{\partial \xi_1} + \frac{u_{3e}}{h_3}\frac{\partial u_{3e}}{\partial \xi_3} + \frac{u_{1e}u_{3e}}{h_1h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{u_{1e}^2}{h_1h_3}\frac{\partial h_1}{\partial \xi_3} - 2\Omega_2 u_{1e} - \frac{\Omega^2}{2h_3}\frac{\partial R^2}{\partial \xi_3}$$
$$= -\frac{1}{\rho h_1}\frac{\partial p_e}{\partial \xi_3} \tag{5.199b}$$

Therefore, the boundary conditions for Equations 5.196–5.198 are

$$\text{at } \xi_2 = 0 : u_1 = u_2 = u_3 = 0$$
$$\text{as } \xi_2 \to \infty : u_1 \to u_{1e},\ u_3 \to u_{3e} \tag{5.200}$$

Choice of Surface Coordinates

In the *derivation* of boundary layer equations the surface coordinates played no role whatsoever. The choice of surface coordinates is therefore completely arbitrary, and ways must be found to evolve the simplest form of the equations of motion. In this connection it is important to bring some definitions from the differential geometry of surfaces and curves, particularly the definitions of the geodesic curves and that of the Gaussian curvature of a surface. We refer to a general discussion of these topics as given in ME.7, but quote here only the requisite formulae for immediate use.

For orthogonal curvilinear coordinates ξ_1, ξ_3 in a surface the geodesic curvatures of the curves are

$$(k_g)_{\xi_3 = \text{constant}} = \frac{1}{h_1h_3}\frac{\partial h_1}{\partial \xi_3}, \qquad \xi_1\text{-curve} \tag{5.201a}$$

$$(k_g)_{\xi_1 = \text{constant}} = \frac{-1}{h_1h_3}\frac{\partial h_3}{\partial \xi_1}, \qquad \xi_3\text{-curve} \tag{5.201b}$$

The Gaussian or the total curvature at any point on the surface is

$$K = -\frac{1}{h_1h_3}\left\{\frac{\partial}{\partial \xi_1}\left(\frac{1}{h_1}\frac{\partial h_3}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_3}\left(\frac{1}{h_3}\frac{\partial h_1}{\partial \xi_3}\right)\right\} \tag{5.201c}$$

A curve in a surface is called a geodesic curve or a surface geodesic if its geodesic curvature is zero. Thus, from Equations 5.201a, b the ξ_1-curve is a geodesic if $\partial h_1/\partial \xi_3 = 0$, and the ξ_3-curve is a geodesic if $\partial h_3/\partial \xi_1 = 0$.

There is a class of surfaces which has the property that the total curvature is zero everywhere on the surface. Such surfaces are called developable surfaces. If suitable cuts have been made, then a developable surface can be rolled out without stretching into a plane. Next we list the conditions under which $K = 0$, and then interpret the results from a geometric viewpoint. Referring to Equation 5.201c, we find that $K = 0$ under the following conditions:

(i) $$h_1 = h_3 = 1$$

(ii)(a) $$\frac{\partial h_1}{\partial \xi_3} = 0, \qquad h_3 = 1$$

(ii)(b) $\qquad \dfrac{\partial h_3}{\partial \xi_1} = 0, \qquad h_1 = 1$

(iii) $\qquad \dfrac{\partial h_1}{\partial \xi_3} = 0, \qquad \dfrac{\partial h_3}{\partial \xi_1} = 0$

(iv)(a) $\qquad \dfrac{\partial h_1}{\partial \xi_3} = 0, \qquad \dfrac{\partial}{\partial \xi_1}\left(\dfrac{1}{h_1}\dfrac{\partial h_3}{\partial \xi_1}\right) = 0$

(iv)(b) $\qquad \dfrac{\partial h_3}{\partial \xi_1} = 0, \qquad \dfrac{\partial}{\partial \xi_3}\left(\dfrac{1}{h_3}\dfrac{\partial h_1}{\partial \xi_3}\right) = 0$

Condition i corresponds to the simplest case of a plane in which the coordinates ξ_1 and ξ_3 are rectangular Cartesian.

Condition iia states that the ξ_3-curves are straight lines, which are necessarily geodesics. Further, when the surface is rolled out into a plane, the ξ_1-curves become straight lines. The distance along these straight lines is the actual distance along the geodesic curve in the surface. Case iib is just the reverse of iia.

Condition iii states that both curves are the surface geodesics, but none of them is a straight line in the surface. Straight lines are again obtained when the surface is rolled out into a plane.

Conditions ii and iii imply that when a developable surface is rolled out into a plane and the Cartesian coordinates x and z are introduced, then these straight lines represent the actual (parametric) distances along the respective geodesics in the surface. Further, when the surface is rolled back to its original shape, the coordinate curves will be the geodesic curves of the surface. *We therefore conclude that the boundary layer equations can be written in the Cartesian form when the surface is developable and the chosen coordinate curves are the surface geodesics.* Both conditions ii and iii in the general case imply that:

$$h_1 = h_1(\xi_1), \qquad h_3 = h_3(\xi_3)$$

We are now in a position to introduce the arc distances x and z, respectively, along the ξ_1 and ξ_3 curve as:

$$x = \int h_1(\xi_1)\, d\xi_1, \qquad z = \int h_3(\xi_3)\, d\xi_3$$

Thus, taking $\xi_2 = y$ and $h_2 = 1$, the boundary layer equations (Equations 5.191–5.193) and the Euler equations (Equations 5.190), respectively, take the following forms:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.202}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.203}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p_e}{\partial z} + \nu\frac{\partial^2 w}{\partial y^2} \tag{5.204}$$

$$\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + w_e\frac{\partial u_e}{\partial z} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} \tag{5.205}$$

$$\frac{\partial w_e}{\partial t} + u_e\frac{\partial w_e}{\partial x} + w_e\frac{\partial w_e}{\partial z} = -\frac{1}{\rho}\frac{\partial p_e}{\partial z} \tag{5.206}$$

The boundary conditions are

$$\text{at } y = 0 : u = v = w = 0$$
$$\text{as } y \to \infty : u \to u_e,\ w \to w_e \qquad (5.207)$$

These equations are exactly in the Cartesian form; however, it must be emphasized that x and z are the curvilinear arc distances in the surface. In the same manner, the equations of a boundary layer on a rotating developable surface in Cartesian form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (5.208)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + 2\Omega_2 w - \frac{\Omega^2}{2}\frac{\partial R^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (5.209)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\Omega_2 u - \frac{\Omega^2}{2}\frac{\partial R^2}{\partial z} = -\frac{1}{\rho}\frac{\partial p_e}{\partial z} + \nu\frac{\partial^2 w}{\partial y^2} \qquad (5.210)$$

The boundary conditions are

$$\text{at } y = 0 : u = v = w = 0$$
$$\text{as } y \to \infty : u \to u_e,\ w \to w_e \qquad (5.211)$$

Internal Cartesian Coordinates

For writing the equations in the Cartesian form, the condition that both the surface curves be geodesics is a little restrictive. Utilizing the condition iva we shall show that for writing the equations in the Cartesian form, only the ξ_1-curves must be the surface geodesics; however, the other curves ξ_3, orthogonal to them, need not be geodesics. The analysis given next is essentially due to Howarth.[32]

The first equation in condition iva implies that $h_1 = h_1(\xi_1)$ and thus introducing the arc distance ξ along ξ_1 through:

$$\xi = \int h_1(\xi_1)\, d\xi_1$$

the second equation in condition iva yields:

$$\frac{\partial^2 h_3}{\partial \xi^2} = 0$$

Thus:

$$h_3 = \xi f(\xi_3) + g(\xi_3)$$

The element of length ds on the surface is then:

$$ds^2 = d\xi^2 + \ + h_3^2\, d\xi_3^2$$

Defining a new variable η through:

$$\eta = \int f(\xi_3)\, d\xi_3$$

we have:

$$ds^2 = d\xi^2 + [\xi + k(\eta)]^2\, d\eta^2$$

where $k = g/f$. We now introduce the new coordinates x and z through the equations:

$$x = \xi \sin\eta + \int k(\eta)\cos\eta\, d\eta$$

$$z = \xi \cos\eta - \int k(\eta)\sin\eta\, d\eta \tag{5.212}$$

Using Equation 5.212 it is easy to show that:

$$ds^2 = dx^2 + dz^2$$

which is exactly the formula for the length element in the Cartesian coordinates. Because of this property, the coordinates x and z defined in Equation 5.212 are termed the *internal Cartesian* coordinates. Based on Equation 5.212 it is easy to show by using the chain rule that:

$$\frac{1}{h_1}\frac{\partial}{\partial \xi_1} = \sin\eta\frac{\partial}{\partial x} + \cos\eta\frac{\partial}{\partial z}$$

$$\frac{1}{h_3}\frac{\partial}{\partial \xi_3} = \cos\eta\frac{\partial}{\partial x} - \sin\eta\frac{\partial}{\partial z}$$

Further, the velocity components u_1 and u_3 are related with their transformed counterparts u and w through the equations:

$$\frac{u_1}{h_1} = u\frac{\partial \xi_1}{\partial x} + w\frac{\partial \xi_1}{\partial z}$$

$$\frac{u_3}{h_3} = u\frac{\partial \xi_3}{\partial x} + w\frac{\partial \xi_3}{\partial z}$$

Introducing these expressions in Equations 5.196–5.198 with $h_2 = 1$, $\xi_2 = y$ and $\partial h_1/\partial \xi_2 = 0$, we recover Equations 5.202–5.204.

This analysis is applicable only to a developable surface. As an example let us consider a circular cone for which the geodesic coordinates are taken to be the cone generators (ξ_1-curves) and the coordinates orthogonal to the geodesics as the circles traced or swept by the meridional angle (ξ_3-curves). Thus although h_1 can be eliminated by introducing the parametric curves x, h_3 nevertheless appears. Following the method of internal Cartesian coordinates as described earlier, we can also eliminate the scale factor h_3.

Nondevelopable Surfaces

If the surface is not developable, then as pointed out by Moore[33] it is still a great simplification to choose one curve (e.g., ξ_1) as a geodesic* and the other (ξ_3-curve) as orthogonal to it. In this case at least $\partial h_1/\partial \xi_3 = 0$, and then only the scale factor h_3 has to be specified. Introducing the parametric curve x, the element of length on the surface is given by

$$ds^2 = dx^2 + h_3^2\, d\xi_3^2$$

* Recall that the geodesic curves exist whether the surface is developable or not.

The previous discussion is purely geometric without any consideration of the flow field past the body. Actually a judicious choice of the surface coordinates is made based both on the geometric and dynamic considerations. Sometimes on pure geometrical considerations we can simplify the differential equations but the consequent boundary conditions may become complicated. Therefore, we have to make a complete study of the body geometry, nature of the incident flow, and the boundary conditions before the solution of a boundary layer problem. Consider as an example the flow past a circular cylinder when the incident flow is inclined to the cylinder axis. According to the geometric considerations, the best choice of surface coordinates are the orthogonal geodesics formed by the circles and the cylinder generators. However, the boundary conditions at the external edge of the boundary layer become quite complicated in this case because the incident flow is inclined to the circular geodesics. These difficulties are quite common in three-dimensional boundary layer theory and no universal rule is available which can be applied to all cases. Based on physical and geometric considerations, Hayes,[34] Timman[35] and Squire[36] proposed a coordinate system (called the *intrinsic coordinate* system); this has been used quite successfully in solving some difficult problems. However, before we define the intrinsic coordinate systems, it is important to describe some physical aspects of a three-dimensional boundary layer.

Physical Consequences of Three Dimensionality

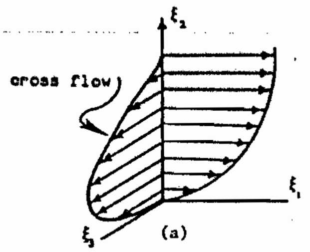
There are two distinguishing features of a three-dimensional boundary layer flow which were absent in two dimensions. One is the existence of a *cross flow*, and the other is the *thinning or thickening* of the boundary layer. Both features are associated with the nature of external flow. The shape of a body and the imposed flow conditions determine an external flow in which the resulting streamlines are curved when observed in a plane parallel to the body surface. As an example, the external flow on a flat plate under certain imposed conditions may result in parabolic streamlines.

The curved nature of an external streamline can be maintained only when there is a pressure gradient in the transverse direction of the streamline to balance the centrifugal force. Therefore, since the external pressure is impressed on the boundary layer, then the boundary layer flow occurs under the influence of pressure gradients both along and transverse to the external streamlines. The transverse pressure gradient gives rise to a velocity component within the boundary layer normal to the external streamlines, which is called the *cross flow*, or the *secondary flow*. Note that this aspect was completely absent in two-dimensional boundary layers. In traversing across the boundary layer from the outer edge toward the surface, the directions of the external streamlines are not maintained; see Figure 5.10(b). The reason for this is that within the boundary layer the fluid velocity has been decreased due to the effect of viscosity; and being under the constraint of the external centrifugal force, the streamlines must attain increased curvature with respect to the external streamline. This, of course, is another explanation for the existence of the cross flow.

The other feature of a three-dimensional boundary layer is the divergence or convergence of the external streamlines on a body surface. For example, in the flow past a convex body which is curved transversely to the flow and extends laterally downstream, the boundary layer becomes thinner as it grows. If, on the other hand, the body is concave, then there is a convergence of external streamlines and the boundary layer grows in thickness.

Intrinsic Coordinates

The intrinsic coordinate system for the description of boundary layer flow is obtained by choosing the surface projections of the external streamlines as one family of coordinate curves, and the curves orthogonal to them as the other coordinate curves. The simplicity afforded by the intrinsic coordinates is that now at the outer edge of the boundary layer one of the velocity components is zero. Let the ξ_1-curves be the surface projections of the external streamlines and the ξ_3-curves orthogonal to them be the other set of coordinates. With this choice, $u_{3_e} = 0$, and Equations 5.190 become:



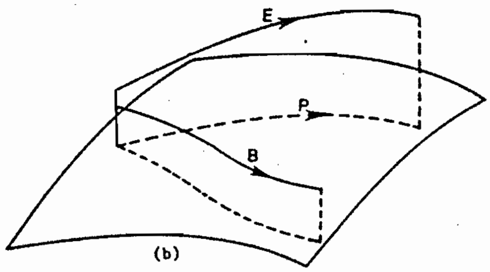


Fig. 5.10 (a) Cross-flow profile, (b) Orientation of external and boundary layer streamlines in traversing along the normal to the surface. E: external streamline, P: projection of E on the surface, B: boundary layer streamline.

$$\frac{\partial u_{1e}}{\partial t} + \frac{u_{1e}}{h_1}\frac{\partial u_{1e}}{\partial \xi_1} = -\frac{1}{\rho h_1}\frac{\partial p_e}{\partial \xi_1} \tag{5.213a}$$

$$\frac{u_{1e}^2}{h_1}\frac{\partial h_1}{\partial \xi_3} = \frac{1}{\rho}\frac{\partial p_e}{\partial \xi_3} \tag{5.213b}$$

The boundary conditions for Equations 5.191–5.193 now become:

$$\text{at } \xi_2 = 0 : u_1 = u_2 = u_3 = 0$$

$$\text{as } \xi_2 = \infty : u_1 \rightarrow u_{1e},\ u_3 \rightarrow 0$$

Note that the cross flow velocity u_3 is zero both at the surface and at the outer edge although it is nonzero within the layer. This is because the pressure gradient in Equation 5.193 and given by Equation 5.213b is nonzero. If on the other hand $\partial h_1/\partial \xi_3 = 0$ (i.e., the ξ_1-curves are surface geodesics), then $\partial p/\partial \xi_3 = 0$ and the solution of Equation 5.193 under the given boundary conditions is simply $u_3 \equiv 0$. *This result shows that if the streamlines of the external flow are geodesics of the surface, then the cross flow velocity vanishes identically*. In this situation the flow direction in the boundary layer coincides with the direction of the *primary* or *mainstream flow*, and the equations are just those of a two-dimensional boundary layer. The preceding result has also been used in those cases where the cross flow is small but not zero because then it is possible to solve the problem in an iterative manner.

Domains of Dependence and Influence

A study of the three-dimensional boundary layer equations suggests that they have dual physical properties. First, there is diffusion of vorticity across the boundary layer. This phenomenon is controlled by the second derivative of velocity with respect to the boundary layer coordinate ξ_2. It can be rigorously proved that because of this term the equations are parabolic. Second, because there are no second derivative terms with respect to either of the surface coordinates the left-hand side of the equations considered alone represent advection, or propagation of properties. As is well known, the propagation phenomenon is governed by the hyperbolic type of equations. Thus, although the three-dimensional boundary layer equations are essentially parabolic in character, physically they represent the diffusion due to viscosity along the boundary layer coordinate ξ_2 as well as the convective propagation along streamlines through each point ξ_2 = constant with a finite speed.

The propagation phenomenon in three-dimensional boundary layers as described above was mathematically explained by Raetz[37] and by Wang.[38] The contributions of these authors have been of much importance in obtaining the numerical solutions of three-dimensional boundary layer equations. Specifically, it tells us that there is a definite region upstream of a point P in the boundary layer which influences the solution at P; and correspondingly, there is a definite region downstream which is influenced by the conditions at P. This principle is known as that of *dependence* and *influence*. Referring to Figure 5.11, we consider points on the boundary layer coordinate AB which extends from the wall to the outer edge. Conditions at any point on AB (such as P) depend only on conditions in an upstream wedge-like domain called the domain of dependence D. This domain is formed by the intersection in the line AB of the two characteristic surfaces S_1 and S_2 which have the vertical diffusion lines as characteristics. These surfaces enclose all those streamlines or subcharacteristics that eventually cross the line AB and also envelope all those outmost streamlines that may leave the volume of D. In essence, streamlines cannot enter through S_1 and S_2, but they can leave. The extension of this domain downstream of the line AB defines the range of influence I.

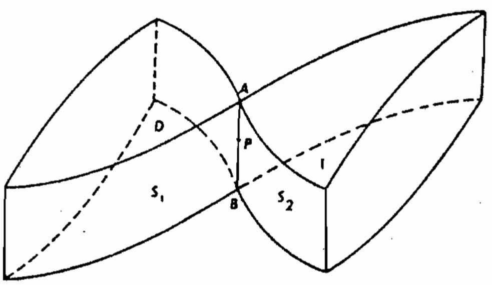


Fig. 5.11 Domain of dependence and range of influence in a three-dimensional boundary layer.

5.13 MOMENTUM INTEGRAL EQUATIONS IN THREE DIMENSIONS

The momentum integral equations for the two-dimensional boundary layer flow were obtained in Section 5.9. In this section we folow the same procedure to obtain the equations for the three-dimensional flow. The integral equations for three-dimensional cases can be obtained in various forms depending on the use of the external velocity u_{1e} or u_{3e} in defining various thicknesses of the boundary layer. We shall, however, follow Mager[39] in obtaining the equations which are even applicable to the cases when either u_{1e} or u_{3e} vanishes. Because of an increased dimension, there are two kinds of displacement and four kinds of momentum thicknesses. Denoting the displacement thickness by δ^* and the momentum thickness by θ, we have:

$$Q_e\delta_1^* = \int_0^\delta (u_{1e} - u_1)\, d\xi_2, \qquad Q_e\delta_3^* = \int_0^\delta (u_{3e} - u_3)\, d\xi_2$$

$$Q_e^2\theta_{11} = \int_0^\delta (u_{1e} - u_1)u_1\, d\xi_2, \qquad Q_e^2\theta_{13} = \int_0^\delta (u_{1e} - u_1)u_3\, d\xi_2$$

$$Q_e^2\theta_{31} = \int_0^\delta (u_{3e} - u_3)u_1\, d\xi_2, \qquad Q_e^2\theta_{33} = \int_0^\delta (u_{3e} - u_3)u_3\, d\xi_2$$

where δ is the boundary layer thickness and:

$$Q_e^2 = u_{1e}^2 + u_{3e}^2$$

Integrating Equations 5.197 and 5.198 with respect to ξ_2 from 0 to δ and using the continuity equation (Equation 5.196) to eliminate u_2, we have:

$$\frac{1}{Q_r^2}\frac{\partial}{\partial t}(Q_r\delta_1^*) + \frac{1}{h_1}\frac{\partial\theta_{11}}{\partial\xi_1} + \frac{1}{h_3}\frac{\partial\theta_{13}}{\partial\xi_3} + \theta_{11}F_1 + \theta_{13}F_3$$
$$- \frac{(\theta_{11}+\theta_{33})}{h_1h_3}\frac{\partial h_3}{\partial\xi_1} + \frac{\delta_1^*}{h_1Q_r}\frac{\partial u_{1r}}{\partial\xi_1} + \frac{\delta_3^*}{Q_r}\left(\frac{1}{h_1}\frac{\partial u_{3r}}{\partial\xi_1} + \omega_{2r}\right) = \frac{\tau_{1w}}{\rho Q_r^2} \tag{5.215}$$

$$\frac{1}{Q_r^2}\frac{\partial}{\partial t}(Q_r\delta_3^*) + \frac{1}{h_1}\frac{\partial\theta_{31}}{\partial\xi_1} + \frac{1}{h_3}\frac{\partial\theta_{33}}{\partial\xi_3} + \theta_{31}F_1 + \theta_{33}F_3$$
$$- \frac{(\theta_{11}+\theta_{33})}{h_1h_3}\frac{\partial h_1}{\partial\xi_3} + \frac{\delta_3^*}{h_3Q_r}\frac{\partial u_{3r}}{\partial\xi_3} + \frac{\delta_1^*}{Q_r}\left(\frac{1}{h_3}\frac{\partial u_{1r}}{\partial\xi_3} - \omega_{2r}\right) = \frac{\tau_{3w}}{\rho Q_r^2} \tag{5.216}$$

where w_{2r} has already been defined in Equation 5.189b, and:

$$\tau_{1w} = \mu\left(\frac{\partial u_1}{\partial\xi_2}\right)_{\xi_2=0}, \qquad \tau_{3w} = \mu\left(\frac{\partial u_3}{\partial\xi_2}\right)_{\xi_2=0}$$

$$F_1 = \frac{1}{h_1}\frac{\partial}{\partial\xi_1}(n(Q_r^2h_3^2)$$

$$F_3 = \frac{1}{h_3}\frac{\partial}{\partial\xi_3}(n(Q_r^2h_1^2)$$

Equations 5.215 and 5.216 represent the momentum integral equations for the incompressible three-dimensional boundary layer flow.

5.14 SEPARATION AND ATTACHMENT IN THREE DIMENSIONS

A three-dimensional boundary layer starts from either a line or point on the surface, called a line or point of attachment. It is an experimental fact that after following the surface for some distance the boundary layer separates and forms a region of reversed flow on the surface and a wake behind the surface. The main difficulty in evolving a comprehensive theory of this subject is that the conditions under which a three-dimensional boundary layer separation occurs are many and varied. Despite these difficulties we shall try to outline this subject in a simple manner following the works of Maskell[40] and Lighthill.[41] The first thing to note is that the three-dimensional separation is such that the two-dimensional concepts cannot simply be generalized to three dimensions to explain the whole phenomenon. Restricting ourselves only to introducing the basic notions of the subject, we proceed as follows.

There are two broad categories of the three-dimensional boundary layer separation. In the first category there are those cases in which the boundary layers developing on different portions of the same surface meet along a curve in the surface to form a single layer of fluid which may not remain attached to the surface. An example of this type is the boundary layer developing on the surface of a sphere rotating in a quiescent fluid. The boundary layers developing on the two hemispheres coalesce at the equator and leave the surface forming a jet of fluid.

The second category is comprised of those cases in which the boundary layer separates under the combined effects of viscosity and adverse pressure gradient. It is actually this category which is important from the boundary-layer standpoint and is being considered here. We shall exclusively be concerned with the steady-state boundary layers on developable surfaces. Referring to Equations 5.202–5.204, we have u, v, and w as the velocity components along x, y, and z, respectively. As before $y = 0$ at the surface, where because of the no-slip condition, $u = v = w = 0$. Denoting the viscous shear stress at the wall by τ_{0x} and τ_{0z} defined as:

$$\tau_{0x} = \mu\left(\frac{\partial u}{\partial y}\right)_0, \qquad \tau_{0z} = \mu\left(\frac{\partial w}{\partial y}\right)_0$$

we expand each velocity component near the wall by Taylor's expansion to have:

$$u = \frac{y\tau_{0x}}{\mu} + 0(y^2) \tag{5.217a}$$

$$w = \frac{y\tau_{0z}}{\mu} + 0(y^2) \tag{5.217b}$$

$$v = -\frac{y^2\nabla}{2\mu} + 0(y^3) \tag{5.217c}$$

where:

$$\nabla = \frac{\partial\tau_{0x}}{\partial x} + \frac{\partial\tau_{0z}}{\partial z}$$

and μ is the coefficient of viscosity. Equation 5.217c has been obtained by using Equations 5.217a, b in the continuity equation and integrating once with respect to y. Based on Equations 5.217 we conclude that if at a point both wall shear stresses τ_{0x} and τ_{0z} are not simultaneously zero, then:

$$\operatorname*{Lim}_{y\to 0} \frac{v}{\sqrt{u^2 + w^2}} \to 0 \tag{5.217d}$$

Therefore, v is of a lower order of magnitude than either u or w. All those points in the surface where Equation 5.217d holds are known as the *regular points* of the flow. Note that if one of the wall shear stresses is zero, then the point is still a regular point of the flow. However, if both the wall shear stresses vanish simultaneously at a point in the surface, then:

$$\operatorname*{Lim}_{y\to 0} \frac{v}{\sqrt{u^2 + w^2}} = 0(1) \tag{5.217e}$$

and then the velocity component v is of the same order as the other two. (Recall that in the event of two-dimensional boundary layer separation, the velocity component normal to the surface became of the same order of magnitude as the tangential component.) Points in the surface where Equation 5.217e holds are called the *singular points* of the flow.

In the event of a singularity at a point, two cases arise depending on the sign of Δ. Referring to Equation 5.217c we find that if $\Delta < 0$, then the fluid will leave the surface at an angle; while if $\Delta > 0$, then the boundary layer attaches itself to the surface. From this result we conclude that in three dimensions reattachment is as important as separation. In either case, simultaneous vanishing of τ_{0x} and τ_{0z} give rise to either separation or reattachment of the flow. As mentioned earlier, this condition is satisfied, in general, at isolated points of the surface. In quite exceptional circumstances, such as in two-dimensional flows, both wall shear stresses vanish all along a line and then we have a line of separation or reattachment.

Limiting Streamlines and Vortex Lines

The equations of a streamline in the flow are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Therefore, the streamlines in the surface are given by:

$$\frac{dx}{dz} = \operatorname{Lim}_{y \to 0} \frac{u}{w} \tag{5.218a}$$

and are called the *limiting streamlines*. Using Equations 5.217a, b in Equation 5.218a, we find that the differential equation for the limiting streamlines is

$$\frac{dx}{\tau_{0_x}} = \frac{dz}{\tau_{0_z}} \tag{5.218b}$$

or, alternatively:

$$\frac{dx}{\tau_x} = \frac{dz}{\tau_z}, \qquad y = 0 \tag{5.218c}$$

From Equation 5.218a we conclude that a limiting streamline is a curve whose directions coincide with that of the vanishing fluid velocity at the surface.

In the same manner, the differential equation for vortex lines in the surface is obtained by evaluating the vorticity components at the surface to have:

$$\frac{dx}{\omega_{0_x}} = \frac{dz}{\omega_{0_z}} \tag{5.219}$$

where $\omega_{0_x} = \tau_{0_z}/\mu$ and $\omega_{0_z} = -\tau_{0_x}/\mu$. In vector form, the differential equations of the limiting streamlines and vortex lines in the surface, respectively, are

$$d\mathbf{r} \times \boldsymbol{\tau}_0 = 0 \tag{5.220a}$$

$$d\mathbf{r} \times \boldsymbol{\omega}_0 = 0 \tag{5.220b}$$

Since:

$$\boldsymbol{\tau}_0 \cdot \boldsymbol{\omega}_0 = 0$$

both systems of lines cover the surface completely and intersect each other orthogonally. At a regular point there is just one limiting streamline intersecting one surface vortex line. In the situation when $\tau_{0_x} = \tau_{0_z} = 0$, the point is a singular or branch point of both the differential equations (Equations 5.220). A classification of such branch points (refer to ME.6) is conducted by studying the sign of Jacobian J defined as:

$$J = \frac{\partial \omega_{0_z}}{\partial z}\frac{\partial \omega_{0_x}}{\partial x} - \frac{\partial \omega_{0_z}}{\partial x}\frac{\partial \omega_{0_x}}{\partial z}$$

$$= \left(\frac{\partial \tau_{0_x}}{\partial x}\frac{\partial \tau_{0_z}}{\partial z} - \frac{\partial \tau_{0_z}}{\partial x}\frac{\partial \tau_{0_x}}{\partial z}\right)\Big/\mu^2$$

If $J < 0$, then the singular point is a "saddle" point; if $J > 0$, then the singular point is a "nodal" point (see Figure 5.12).

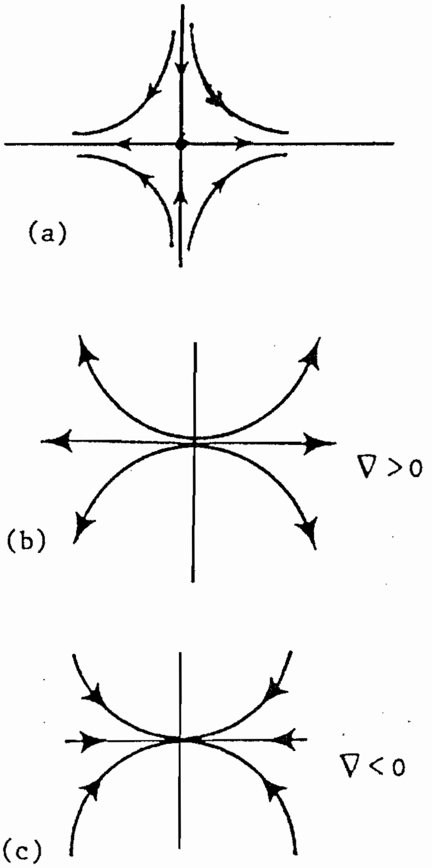


Fig. 5.12 Pattern of skin friction lines near isolated singular points; (a) saddle point, (b) nodal point of attachment, (c) nodal point of separation.

On the basis of the results obtained, we can form a conceptual model of the separation phenomenon as follows. In the event of simultaneous vanishing of both wall shear stresses at a point or a group of points on the surface, there exists a separated region which is inaccessible to the viscous upstream region. The two regions are separated by a surface called the "separation surface". The trace or curve of intersection of the separation surface with the body is a separation line. Since no limiting streamline can pass through the separation line, the separation line must be the envelope of limiting streamlines (Equations 5.218). On the separation line there are singular points, or points which belong to both the separated and unseparated regions. This common occurrence of points in the two regions can happen only when they are the branch points, i.e., saddle or nodal points (refer to ME.6). Further, at these points the velocity component normal to the surface is of the same order as the other two components; therefore, the boundary layer leaves the surface at an angle.

In this discussion we have postulated the basic notions regarding the phenomenon of singular separation. The whole subject is quite extensive, and further reading of the literature cited is essential (e.g., refer to Reference 42).

5.15 BOUNDARY LAYERS ON BODIES OF REVOLUTION AND YAWED CYLINDERS

The boundary layer equations for incompressible flows past bodies of revolution can be obtained as a particular case of Equations 5.191–5.193. Referring to Example 3.3, we consider an orthogonal coordinate system ξ_1, ξ_2 in any meridian plane and $\xi_3 = \phi$ as the azimuthal angle. Thus:

$$h_1 = h_1(\xi_1), \qquad h_2 = 1, \qquad h_3 = r(\xi_1, \xi_2)$$

Again introducing the arc distances:

$$x = \int_0^{\xi_1} h_1(\xi_1)\, d\xi_1, \qquad y = \xi_2$$

and writing:

$$u_1 = u, \qquad u_2 = v, \qquad u_3 = w$$

we have:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) + \frac{\partial w}{\partial \phi} = 0 \tag{5.221}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{w}{r}\frac{\partial u}{\partial \phi} - \frac{w^2}{r}\frac{\partial r}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.222}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{w}{r}\frac{\partial w}{\partial \phi} + \frac{uw}{r}\frac{\partial r}{\partial x} = -\frac{1}{\rho r}\frac{\partial p_e}{\partial \phi} + \nu\frac{\partial^2 w}{\partial y^2} \tag{5.223}$$

Here w is the swirl velocity. For slender bodies:**

$$\begin{aligned} r(\xi_1, \xi_2) &= r_0(\xi_1) + \xi_2 \cos\beta(\xi_1) \\ &= r_0(x) + y\cos\beta(x) \end{aligned} \tag{5.224}$$

where $\beta(x)$ is the angle between the local tangent to the body surface and the axis of revolution.*** For thick bodies (i.e., $r_0(x) \gg \delta$) where δ is the boundary layer thickness, we can replace r by $r_0(x)$. For axially symmetric flow, viz., for axial flow past a body of revolution, the derivatives with respect to ϕ are zero. The simplest case arises when the swirl velocity is also zero. Thus for an axial flow with zero swirl we have:

$$\frac{\partial}{\partial x}(r_0 u) + \frac{\partial}{\partial y}(r_0 v) = 0 \tag{5.225a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.225b}$$

The boundary conditions are obviously:

** If δ^*/r_0 is small, then

$$r_0\delta^* = \int_0^{\delta}\left(1 - \frac{\rho u}{\rho_e u_e}\right) r\, dy, \qquad r_0\theta = \int_0^{\delta}\frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) r\, dy$$

*** Note the difference between the x as the arc distance defined above and the Cartesian x-axis in Example 3.3.

$$y = 0 : u = v = 0$$

$$y \rightarrow \infty : u \rightarrow u_e(x)$$

Mangler's Tranformation

Another simplification of Equations 5.225 occurs for *steady* flow in which the equations reduce to those for steady two-dimensional boundary layers. Following Mangler,[43] we introduce new variables s and n as:

$$s = \int_0^x \frac{r_0^2}{\ell^2}\,dx, \qquad n = \frac{r_0 y}{\ell}$$

$$u(x, y) = U(s, n)$$

where ℓ is a characteristic length. The problem is now to obtain a function $V(s, n)$, where:

$$V(s, n) = f\{u(x, y), v(x, y)\}$$

such that both Equations 5.225 reduce exactly to the two-dimensional forms. Introducing this noted transformation, we obtain:

$$\frac{\partial U}{\partial s} + \frac{\ell}{r_0}\frac{\partial}{\partial n}\left(v + \frac{n}{\ell}\frac{dr_0}{ds}u\right) = 0$$

Since r_0 is a function of s only, we set:

$$V = \frac{\ell}{r_0}\left(v + \frac{n}{\ell}\frac{dr_0}{ds}u\right)$$

and obtain:

$$\frac{\partial U}{\partial s} + \frac{\partial V}{\partial n} = 0 \tag{5.226a}$$

and:

$$U\frac{\partial U}{\partial s} + V\frac{\partial U}{\partial n} = -\frac{1}{\rho}\frac{\partial p_e}{\partial s} + \nu\frac{\partial^2 U}{\partial n^2} \tag{5.226b}$$

Equations 5.226 constitute the boundary layer equations for steady axial flow past an axially symmetric body. When the solution of these equations becomes available, the solution in terms of the original variables x and y is obtained through the use of the transformation relations noted above.

Boundary Layer on Yawed Cylinders

The set of Equations 5.202–5.204 are in general coupled. There is, however, a situation in which two of the velocities can be determined independently of the third; this is the case provided by the boundary layers on infinite yawed cylinders.

A cylinder of arbitrary cross section is a developable surface; therefore, one can always choose the equations of motion in rectangular Cartesian form. Referring to Equations 5.202–5.204 and to Figure 5.13, we choose x as a the distance measured along the surface normal to the leading edge, z the coordinate measured along the generators, and y as the boundary layer coordinate measured along the local normals to the surface.

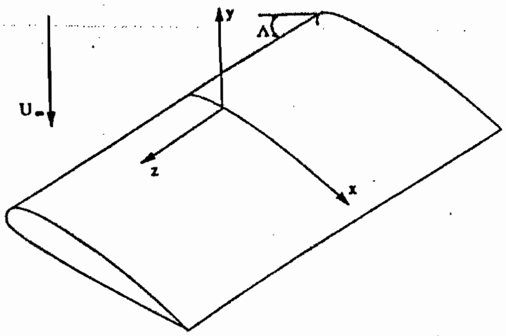


Fig. 5.13 Boundary layer on a yawed infinite wing.

Since the cylinder is assumed to be of infinite length, there is no dependence of quantities on the z-coordinate. With this simplification, the external velocity components u_e and w_e are functions only of x, and the pressure gradients are given by the equations

$$-\frac{1}{\rho}\frac{\partial p_e}{\partial x} = u_e\frac{\partial u_e}{\partial x}, \qquad -\frac{1}{\rho}\frac{\partial p_e}{\partial z} = u_e\frac{\partial w_e}{\partial x} \tag{5.227}$$

The type of external flow satisfying Equation 5.227 is rotational. For irrotational flow we must have:

$$u_e = u_e(x), \qquad w_e = w_\infty = \text{constant}$$

and then the external pressure p_e satisfies the Bernoulli equation:

$$p_e = \text{constant} - \frac{\rho}{2}(u_e^2 + w_\infty^2)$$

The boundary layer equations for an infinite yawed cylinder are obtained from Equations 5.202–5.204 and are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.228a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.228b}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2} \tag{5.228c}$$

The boundary conditions are

$$\text{at } y = 0: u = v = w = 0$$

$$\text{as } y \to \infty: u \to u_e(x),\ w \to w_\infty = \text{constant} \tag{5.228d}$$

Note that Equations 5.228a, b can now be solved independently of Equation 5.228c. This is a consequence of the assumption that the flow is laminar and the cylinder is of infinite length. Once the velocity components u and v have been determined, the equation for w can be solved under its own boundary conditions. The nondependence of the velocity components u and v on w is also known as the *principle of independence*; for references refer to Schlichting.[1]

Cross Flow

In the case of a yawed infinite wing the problem of cross flow within the boundary layer can be studied quite simply. Referring to Figure 5.14, let U_∞ be the free-stream constant velocity which makes an angle Λ (yaw angle) with the normal to the leading edge. Let u_∞ and w_∞ be the resolved parts of U_∞ perpendicular to and along the leading edge, respectively. The free-stream components u_∞ and w_∞ give rise to the external potential flow components which have been denoted by $u_e(x)$ and $w_e(x) = w_\infty$. The external flow is then $U_e(x) = \sqrt{u_e^2 + w_\infty^2}$. The potential external velocity $U_e(x)$ is along the direction of the free streamlines which are generally curved in the lateral direction unless the wing is a flat plate. We project these external streamlines in the viscous flow region and take these lines as a reference system for following the direction of velocity vectors within the boundary layer. One such laterally curved line is shown in the boundary layer on a wing in Figure 5.14.

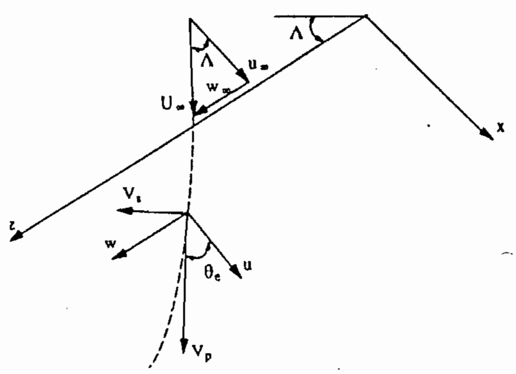


Fig. 5.14 Projection of a laterally curved external streamline somewhere in the boundary layer on a yawed wing.

At a point in the boundary layer we resolve the velocity along and perpendicular to the projected external streamline. The velocity component along the external streamline is the primary flow V_e, and the component perpendicular to it is the secondary flow V_s. Let $\theta_e(x)$ be the angle between the local direction (tangent) of the external streamline and the u-direction; then:

$$V_s = w \cos \theta_e - u \sin \theta_e$$

$$V_p = w \sin \theta_e + u \cos \theta_e$$

Introducing the relations:

$$u_e = U_e \cos \theta_e, \qquad w_\infty = U_e \sin \theta_e$$

we get:

$$\frac{V_s}{U_e} = \frac{1}{2}\left(\frac{w}{w_\infty} - \frac{u}{u_e}\right) \sin 2\theta_e$$

Also since:

$$\tan \Lambda = \frac{w_\infty}{u_\infty} = \frac{u_e}{u_\infty} \cdot \frac{w_\infty}{u_e}$$

$$= \frac{u_e}{u_\infty} \tan \theta_e$$

we have:

$$\frac{V_s}{U_e} = \frac{(u_e/u_\infty)\tan\Lambda}{u_e^2/u_\infty^2 + \tan^2\Lambda}\left(\frac{w}{w_\infty} - \frac{u}{u_e}\right)$$

It is obvious that the secondary flow vanishes both at the surface and at the edge of the boundary layer.

Transformed Equations for Yawed Cylinders

The Equations 5.228a, b are exactly the two-dimensional boundary layer equations. We can take advantage of Gortler's transformation (Equation 5.143) to write all the equations in dimensionless form. Introducing Gortler's transformation and:

$$w = w_\infty g(\xi, \eta)$$

Equations 5.228 become:

$$f''' + ff'' + \beta(\xi)(1 - f'^2) = 2\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right)$$

$$g'' + fg' = 2\xi\left(f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi}\right)$$

$$\text{at } \eta = 0 : f = f' = g = 0$$

$$\text{at } \eta \to \infty : f' \to 1,\ g \to 1$$

These equations can be solved for a given potential flow (cylinder shape) by either the series expansion or numerical methods of two-dimensional boundary layers.

5.16 THREE-DIMENSIONAL STAGNATION POINT FLOW

The problem of three-dimensional stagnation point flow in steady state has been considered by Howarth.[44] Howarth showed that near the stagnation point the same solution is valid both for the developable and nondevelopable surfaces; the only requirement being that the stagnation point must be a regular point of the surface. The external flow near the stagnation point is governed by two requirements: (1) the velocity components are zero at the stagnation point and (2) the flow is irrotational there. Choosing the origin of coordinates at the stagnation point, the behavior of h_1 and h_3 near the stagnation point can be studied by a Taylor expansion. Thus near the stagnation point:

$$h_1 = 1 + \xi_1\left(\frac{\partial h_1}{\partial \xi_1}\right)_0 + \ldots$$

$$h_3 = 1 + \xi_1\left(\frac{\partial h_3}{\partial \xi_1}\right)_0 + \ldots$$

The condition of irrotationality is

$$\frac{\partial}{\partial \xi_3}(h_1 u_{1e}) = \frac{\partial}{\partial \xi_1}(h_3 u_{3e}) \tag{5.229}$$

If we now expand u_{1e} and u_{3e} in powers of ξ_1 and ξ_3 and retain only the first order terms, we find that the irrotationality condition (Equation 5.230) is satisfied provided that near the stagnation point:

$$u_{1e} = a\xi_1 + b\xi_3, \qquad u_{3e} = b\xi_1 + c\xi_3 \tag{5.230}$$

where a, b; and c are arbitrary constants. Substituting Equation 5.230 in Equations 5.190, we find that for the pressure gradient to remain of the first order, the curvature terms must vanish. This can happen only when $h_1 = h_3 = 1$. These approximations suggest that the region near the stagnation point can be considered as a plane in which Cartesian coordinates can be introduced. Consequently, we can adjust the coordinates $\xi_1 = x$ and $\xi_3 = z$ such that:

$$u_{1e} = u_e = Ax, \qquad u_{3e} = w_e = Bz \tag{5.231a}$$

where A and B are arbitrary constants. Introducing the transformation:

$$\eta = y\left(\frac{A}{\nu}\right)^{1/2}, \qquad u = Axf'(\eta), \qquad w = Bzg'(\eta) \tag{5.231b}$$

in Equations 5.202–5.207, we get:

$$f''' + ff'' + \alpha gf'' - f'^2 + 1 = 0 \tag{5.231c}$$

$$\frac{1}{\alpha}g''' + gg'' + \frac{1}{\alpha}fg'' - g'^2 + 1 = 0 \tag{5.231d}$$

where a prime denotes differentiation with respect to η and $\alpha = B/A$. The boundary conditions are

$$\begin{aligned} &\text{at } \eta = 0 : f = f' = g = g' = 0 \\ &\text{at } \eta \to \infty : f' \to 1 \text{ and } g' \to 1 \end{aligned} \tag{5.231e}$$

The ordinary differential equations (Equations 5.231c, d) have been solved numerically for various values of α and are available in Reference 44. Note that for $\alpha = 1$ and $f = g$ the solution corresponds to the flow near the stagnation point of a body or revolution.

5.17 BOUNDARY LAYER ON ROTATING BLADES

The problem of boundary layer on the rotor blades of helicopters and turbomachines is of much importance in engineering. Below we formulate such problems in those cases in which the blade span is sufficiently large. The boundary layer equations for a developable surface in rotating coordinates have been stated as Equations 5.208–5.211. In these equations, $\Omega = |\mathbf{\Omega}|$ is the angular velocity of the coordinate system, Ω_2 is the component of $\mathbf{\Omega}$ in the y-direction, and $R(x, z)$ is the perpendicular distance of a fluid particle in the boundary layer from the axis of rotation. The boundary layer equations in rotating orthogonal curvilinear coordinates are Equations 5.196–5.198. As before, all scale factors appearing in Equations 5.196–5.198 have been evaluated at the body surface $\xi_2 = 0$. Further, the distance $R(\xi_1, \xi_3)$ is the perpendicular distance of a point from the axis of rotation in the plane of rotation, and Ω_2 is the component of $\mathbf{\Omega}$ along the ξ_2-coordinate. There is no loss of generality by setting $h_2 = 1$.

An important class of external flows, which is of sufficient practical importance, occurs when the flow produced by a rotating blade and observed with respect to a nonrotating (inertial) coordinate frame is irrotational. Referring to Equation 5.195b, we find that this happens when:

$$2\Omega + \omega_e = 0 \tag{5.232}$$

In this case, for steady flow, we can directly integrate Equation 5.195b to obtain the Bernoulli equations:

$$\frac{P_e}{\rho} = \text{constant} - \frac{1}{2}|\mathbf{u}_e|^2 + \frac{\Omega^2 R^2}{2}$$

where $\mathbf{u}_e$ is the inviscid velocity vector. Note that Equation 5.232 is not always true, particularly if a blade moves in the wake produced by the one moving ahead of it.

With reference to ξ_1, ξ_2, ξ_3, the three components of Equation 5.232 are

$$-2\Omega_1 = \omega_{1e} = \frac{1}{h_2h_3}\left\{\frac{\partial}{\partial\xi_2}(h_3u_{3e}) - \frac{\partial}{\partial\xi_3}(h_2u_{2e})\right\}$$

$$-2\Omega_2 = \omega_{2e} = \frac{1}{h_1h_3}\left\{\frac{\partial}{\partial\xi_3}(h_1u_{1e}) - \frac{\partial}{\partial\xi_1}(h_3u_{3e})\right\}$$

$$-2\Omega_3 = \omega_{3e} = \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\xi_1}(h_2u_{2e}) - \frac{\partial}{\partial\xi_2}(h_1u_{1e})\right\} \tag{5.233}$$

Under the boundary layer approximation we use the external flow and the scale factors evaluated at the surface $\xi_2 = 0$, so that:

$$\Omega_1 = \Omega_3 = 0, \qquad \Omega_2 \neq 0$$

That is, the components Ω_1 and Ω_3 have no influence on the boundary layer solution, although it must be remembered that Ω_1, Ω_3 may not be zero away from the surface. The component Ω_2 is such that it is related with u_{1e} $(\xi_1, 0, \xi_3)$ and u_{3e} $(\xi_1, 0, \xi_3)$, according to the second equation in Equation 5.233.

5.18 NUMERICAL SOLUTION OF 3-D BOUNDARY LAYER EQUATIONS

Numerical solution of the three-dimensional boundary layer equations by the method of finite difference approximation is now almost a routine matter. A body of literature is available in a large number of research papers, and some methods have been published in books, e.g., Cebeci and Bradshaw[15,45] and Patanker and Spalding.[46] Here, for the purpose of introducing the subject, we shall consider a finite difference method which in essence is similar to the one considered in Section 5.11 for two-dimensional boundary layers. The present method of solution is applicable to the boundary layer equations for developable surfaces, viz., when the equations can be written in the Cartesian form, and was developed by Dwyer.[47] It has, however, been shown through an exercise (refer to Problem 5.16) that the present method can also be used after a transformation has been effected to the boundary layer equations written in orthogonal curvilinear coordinates.

We first summarize the various difference approximations used for the derivatives appearing in the equations. Let Δx, Δy, and Δz be the step sizes along x, y, and z, respectively, and i, j, and k be integers so that the distances are given by:

$$x_i = i\,\Delta x, \qquad y_j = j\,\Delta y, \qquad z_k = k\,\Delta z$$

We take x and z as the surface coordinates and y as the boundary layer coordinate. Let the solution of Equations 5.202–5.204 be known at the position (x_i, z_{k-1}) for all values of y_j, and it is desired to obtain the solution at (x_{i+1}, z_k) for all y_j.

Let ψ be the surrogate variable representing either u or w, then the derivative approximations are taken as follows:

$$\left(\frac{\partial\psi}{\partial x}\right)_{i,j,k} = \frac{\psi_{i+1,j,k} - \psi_{i,j,k}}{\Delta x} + 0(\Delta x)$$

$$\left(\frac{\partial\psi}{\partial y}\right)_{i,j,k} = \frac{\psi_{i,j+1,k} - \psi_{i,j-1,k}}{2\Delta y} + 0(\Delta y)^2$$

$$\left(\frac{\partial\psi}{\partial z}\right)_{i,j,k} = \frac{\psi_{i,j,k} - \psi_{i,j,k-1}}{\Delta z} + 0(\Delta z)$$

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{i,j,k} = \frac{\psi_{i,j+1,k} - 2\psi_{i,j,k} + \psi_{i,j-1,k}}{(\Delta y)^2} + 0(\Delta y)^2$$

The difference analogue given above are then used to form averaged derivatives in such a manner that every term in the steady form of Equations 5.203 and 5.204 is evaluated at:

$$p = [(i + \tfrac{1}{2})\,\Delta x,\ j\,\Delta y,\ (k - \tfrac{1}{2})\,\Delta z]$$

Thus the derivatives in the Equations 5.203 and 5.204 are taken as:

$$\left(\frac{\partial\psi}{\partial x}\right)_p = \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)_{i,j,k} + \left(\frac{\partial\psi}{\partial x}\right)_{i,j,k-1}\right]$$

$$\left(\frac{\partial\psi}{\partial z}\right)_p = \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial z}\right)_{i+1,j,k} + \left(\frac{\partial\psi}{\partial z}\right)_{i,j,k}\right]$$

$$\left(\frac{\partial\psi}{\partial y}\right)_p = \frac{1}{4}\left[\left(\frac{\partial\psi}{\partial y}\right)_{i+1,j,k} + \left(\frac{\partial\psi}{\partial y}\right)_{i,j,k} + \left(\frac{\partial\psi}{\partial y}\right)_{i+1,j,k-1} + \left(\frac{\partial\psi}{\partial y}\right)_{i,j,k-1}\right]$$

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_p = \frac{1}{4}\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)_{i+1,j,k} + \left(\frac{\partial^2\psi}{\partial y^2}\right)_{i,j,k} + \left(\frac{\partial^2\psi}{\partial y^2}\right)_{i+1,j,k-1} + \left(\frac{\partial^2\psi}{\partial y^2}\right)_{i,j,k-1}\right]$$

The terms appearing without differentiation are also averaged as:

$$\bar{\psi} = \frac{1}{2}(\psi_{i,j,k} + \psi_{i+1,j,k-1})$$

Dwyer considers the steady boundary layer equations for a developable surface. For ease in handling the integration along the boundary layer coordinate a similarity type variable η defined as:

$$\eta = y\sqrt{\frac{(u_e^2 + w_e^2)}{2\nu x}}$$

is introduced which keeps the boundary layer thickness nearly uniform. Finite difference approximation based on the above formulae is then carried out on both the momentum equations. Continuity equation is approximated in the same way. To make the system of difference equations linear, an iterative process is used in which the coefficients of the nonlinear terms at the current station $(i + 1, j, k)$ are replaced by the values available from the previous iteration. The end result is to get two systems of tridiagonal form of linear algebraic equations which have to be solved according to the method discussed in Section 5.11. In each cycle of iteration the value of the velocity component v is updated by using the difference form of the continuity equation.

5.19 UNSTEADY BOUNDARY LAYERS

Until now our studies have been directed to steady boundary layers. In a variety of physical situations (for example, boundary layers developing on an unsteadily moving body or boundary layers on stationary surfaces under unsteady external streams) are inherently time dependent. We have already obtained some nonsteady solutions of the linearized Navier-Stokes equations in the beginning of this chapter. The reader is particularly urged to look back at the Rayleigh solution for the impulsive start of a flat plate, and the Stokes solution for an oscillating plate. The Rayleigh solution is of much interest since it illustrates the manner in which the viscous effects diffuse in a shear layer with the passage of time. It is also worth recalling that the choice of the variable $\eta = y/2\sqrt{\nu t}$ is essentially directed by the initial condition at $t = 0$ and the boundary condition at $y = \infty$.

Unsteady boundary layers can be divided into two categories:

1. Purely nonsteady
2. Periodic

The first category comprises the impulsively started and accelerated motions of the body so that it is always nonperiodic. In the second category, the oscillations of the body and the harmonic perturbations of the external flow are the obvious problems. We consider both categories separately.

Purely Unsteady Boundary Layers

Consider unsteady flow in the boundary layer in a coordinate system attached to the body. Let, as usual, the coordinate x be measured along the surface of the body and y normal to it. The boundary layer equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.234}$$

For two-dimensional incompressible flows the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.235a}$$

while for the axially symmetric flows it is

$$\frac{\partial}{\partial x}(r_0 u) + \frac{\partial}{\partial y}(r_0 v) = 0 \tag{5.235b}$$

where $r_0 = r_0(x)$. Note that the unsteady axially symmetric boundary layer equations (viz., Equations 5.234 and 5.235b) cannot be reduced to the equations of the two-dimensional form, because Mangler's transformation is valid only for steady flows. The initial and boundary conditions for flows started from rest are

$$\text{for } t < 0 \text{ and } x \geqslant 0,\ y \geqslant 0 : u = v = 0 \tag{5.236a}$$

$$\text{at } y = 0 \text{ and } x \geq 0 \; (x > 0 \text{ for a flat plate}),\; t > 0 : u = v = 0 \tag{5.236b}$$

$$\text{at } t = 0 \text{ and } x > 0 \; (x \geq 0 \text{ for a flat plate}),\; y > 0 : u = u_e(x, 0) \tag{5.236c}$$

$$\text{as } y \to \infty \text{ and } x \geq 0,\; t \geq 0 : u \to u_e(x, t) \tag{5.236d}$$

We now consider a particular form of the external flow in which $u_e(x, t)$ is represented as:

$$u_e(x, t) = V(t)F(x)$$

As in the Rayleigh problem, we can take advantage of the initial condition (Equation 5.236c) and the boundary condition (Equation 5.236d) to introduce a variable η defined as:

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

so that for $y \to \infty$ or $t \to 0$ we recover the external velocity u_e.

It will be advantageous to consider one equation for the stream function ψ which is applicable to both the axially symmetric and two-dimensional cases. For this purpose we consider the continuity equation in the form:

$$\frac{\partial}{\partial x}(r_0^\epsilon u) + \frac{\partial}{\partial y}(r_0^\epsilon v) = 0$$

where $\epsilon = 0$ for the two-dimensional and $\epsilon = 1$ for the axially symmetric flows, respectively. The stream function ψ is then defined as:

$$u = \frac{1}{r_0^\epsilon}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{1}{r_0^\epsilon}\frac{\partial \psi}{\partial x}$$

We now introduce the stream function ψ defined as:

$$\psi = 2r_0^\epsilon\sqrt{\nu t}\,V(t)F(x)\phi(x, \eta, t)$$

and the variable η defined above, and have:

$$\phi''' + 2\eta\phi'' + \frac{4t}{V}\frac{dV}{dt}(1 - \phi') - 4t\frac{\partial \phi'}{\partial t} = 4Vt\left\{\frac{dF}{dx}(\phi'^2 - \phi\phi'' - 1) + F\left(\phi'\frac{\partial \phi'}{\partial x} - \frac{\epsilon}{r_0}\frac{dr_0}{dx}\phi\phi'' - \phi''\frac{\partial \phi}{\partial x}\right)\right\} \tag{5.237}$$

where a prime denotes differentiation with respect to η. The boundary conditions are

$$\text{at } \eta = 0 : \phi = \phi' = 0$$
$$\text{as } \eta \to \infty : \phi' \to 1$$

The solution of Equation 5.237, for the case $V(t) = At^\alpha$, by series expansion in ascending powers of $t^{\alpha+1}$, i.e.:

$$\phi = \phi_0(\eta) + At^{\alpha+1}\left\{\frac{dF}{dx}\phi_{11} + \epsilon F r_0^{\epsilon-2}\frac{dr_0}{dx}\phi_{12}\right\} + \dots$$

has been attempted by Blasius,[48] Goldstein and Rosenhead,[49] and Watson[50] for two-dimensional, and by Boltze[51] and Warsi[52] for the axially symmetric cases. Here A and α are arbitrary positive constants. The method of series expansion gives rise to a series of ordinary differential equations; the first three of them are listed below:

$$\phi_0''' + 2\eta\phi_0'' + 4\alpha(1 - \phi_0') = 0$$

$$\phi_{11}''' + 2\eta\phi_{11}'' - 4(2\alpha + 1)\phi_{11}' = 4(\phi_0'^2 - 1 - \phi_0\phi_0'')$$

$$\phi_{12}''' + 2\eta\phi_{12}'' - 4(2\alpha + 1)\phi_{12}' = -4\phi_0\phi_0''$$

The boundary conditions are

$$\text{at } \eta = 0 : \phi_0 = \phi_0' = 0,\ \phi_{11} = \phi_{11}' = 0,\ \phi_{12} = \phi_{12}' = 0$$

$$\text{as } \eta \to \infty : \phi_0' \to 1,\ \phi_{11}' \to 0,\ \phi_{12}' \to 0$$

The solution of the first equation can be obtained by transforming it to the form of the equation of "parabolic cylinder" functions.[53] The parabolic cylinder function $D_{-2\alpha-1}(\sqrt{2}\,\eta)$, for α not to be a negative integer, is defined as:

$$D_{-2\alpha-1}(\sqrt{2}\,\eta) = \frac{2^{\alpha+1/2}}{\Gamma(2\alpha + 1)} e^{\eta^2/2} \int_\eta^\infty (\tau - \eta)^{2\alpha} e^{-\tau^2}\, d\tau$$

where Γ is the Gamma function. The solution of the first equation under its boundary conditions is

$$\phi_0'(\eta) = 1 - 2^{\alpha+1/2}\pi^{-1/2}\Gamma(\alpha + 1)e^{-\eta^2/2}D_{-2\alpha-1}(\sqrt{2}\,\eta)$$

Using the "duplication formula" of the Gamma function:[53,54]

$$\Gamma(\alpha)\Gamma(\alpha + 1/2) = 2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha)$$

we can prove that the solution satisfies the boundary conditions at $\eta = 0$. The solution of the equation next in hierarchy is then obtained, and the same process is applied to all the succeeding equations. This process, though quite laborious, at least in principle shows that all the solutions can be expressed in terms of the parabolic cylinder function and its integrals.

Periodic Boundary Layers

The boundary layer problems arising due to the periodic motion of bodies can be traced back to Stokes who considered the problem of viscous flow due to an oscillating plate. The boundary layer problem due to a harmonically oscillating stream without a superposed mean flow was initially considered by Schlichting,[55] which we shall now discuss. (Refer also to Reference 56.)

Let us consider the motion in the boundary layer referred to a coordinate system fixed with the body past which a harmonically oscillating potential flow $u_e(x, t)$ of small amplitude occurs. The concept of a small amplitude arises in relation to a characteristic length of the body and is used as a basis for the development of an approximate method of solution. According to Schlichting, let ℓ be a characteristic length and U_m the maximum value of $u_e(x, t)$. Then:

$$\left| u_e \frac{\partial u_e}{\partial x} \right| \sim \frac{U_m^2}{\ell}$$

If A is the amplitude and ω the frequency of oscillations, then:

$$U_m \sim \omega A$$

and:

$$\left|\frac{\partial u_e}{\partial t}\right| \sim U_m \omega$$

Consequently:

$$\left|u_e \frac{\partial u_e}{\partial x}\right| \Big/ \left|\frac{\partial u_e}{\partial t}\right| \sim \frac{A}{\ell}$$

Therefore, if the amplitude A is small then as a zero order approximation we can retain $\partial u/\partial t$ in comparison with both $u\ \partial u/\partial x + v\ \partial u/\partial y$ and $u_e\ \partial u_e/\partial x$ in Equation 5.234. The equation of the zero order approximation is then:

$$\frac{\partial u_0}{\partial t} = \frac{\partial u_e}{\partial t} + \nu \frac{\partial^2 u_0}{\partial y^2}$$

where the continuity equation is

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

The first order approximation is obtained by considering:

$$u = u_0 + u_1 + \ldots, \qquad v = v_0 + v_1 + \ldots$$

resulting in the nonhomogeneous equation:

$$\frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = u_e \frac{\partial u_e}{\partial x} + \nu \frac{\partial^2 u_1}{\partial y^2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0$$

Similarly we can generate higher order approximations.

Let $U_0(x)$ be the external potential flow velocity due to the stationary flow past a cylinder. Then for the harmonic oscillation we assume:

$$u_e = U_0(x) \cos \omega t$$

The boundary conditions for Equation 5.234 are

$$\text{at } y = 0 : u = v = 0$$

$$\text{as } y \to \infty : u \to u_e(x, t) = U_0 \cos \omega t$$

In the problem under consideration it is simpler to work with complex quantities. Therefore, we shall write:

$$u_e(x, t) = Re\{U_0(x)e^{i\omega t}\}, \qquad i = \sqrt{-1}$$

This form and the linearity of the zero order equation suggests that we take:

$$u_0 = U_0(x)f_0'(\eta)e^{i\omega t}$$

where:

$$\eta = y\sqrt{\frac{\omega}{2\nu}}$$

Thus, we have the equation:

$$2if_0' - f_0''' = 2i$$

$$\eta = 0 : f_0 = f_0' = 0$$

$$\eta \rightarrow \infty : f_0' \rightarrow 1$$

where a prime denotes differentiation with respect to η. The solution of the equation is

$$f_0(\eta) = \eta - 1/2(1 - i)\{1 - e^{-(1+i)\eta}\}$$

Therefore:

$$u_0 = U_0(x)\{1 - e^{-(1+i)\eta}\}e^{i\omega t}$$

$$v_0 = -\sqrt{\frac{2\nu}{\omega}}\frac{dU_0}{dx}f_0(\eta)e^{i\omega t}$$

In real variables, the solution is

$$u_0 = 1/2\ U_0\{f_0'e^{i\omega t} + \bar{f}_0'e^{-i\omega t}\}$$

$$v_0 = -\sqrt{\frac{\nu}{2\omega}}\frac{dU_0}{dx}\{f_0e^{i\omega t} + \bar{f}_0e^{-i\omega t}\}$$

where an overhead bar denotes the complex conjugate. Based upon the forms of u_0 and v_0 and the equation for u_1, we assume a real form of u_1 as:

$$u_1 = \frac{U_0}{4\omega}\frac{dU_0}{dx}\{f_1'e^{2i\omega t} + \bar{f}_1'e^{-2i\omega t} + 2f_2'\} \tag{5.238}$$

where $f_2(\eta)$ is a real function of η. Substituting Equation 5.238 in the first order equation and equating to zero the coefficients of different powers of $e^{i\omega t}$, we have the equations:

$$f_1''' - 4if_1' = 2(f_0'^2 - f_0f_0'' - 1) \tag{5.239a}$$

$$f''' = 2f_0'\bar{f}_0' - (f_0\bar{f}_0'' + \bar{f}_0f_0'') - 2 \tag{5.239b}$$

with the boundary conditions:

$$\text{at } \eta = 0 : f_1 = f_1' = 0, f_2 = f_2' = 0$$

$$\text{as } \eta \to \infty : f_1' \to 0, f_2' \to 0$$

The equation for $\bar{f}_1$ is the same as Equation 5.239a with all quantities being replaced by their conjugates. The solutions of Equations 5.239 are

$$f_1(\eta) = \frac{-(1+i)}{2\sqrt{2}} + (1+i)\left\{\frac{1}{2\sqrt{2}} e^{-\sqrt{2}\eta} \cos \sqrt{2}\eta + \frac{\eta}{2} e^{-\eta}(\sin \eta + \cos \eta)\right\}$$

$$+ (1-i)\left\{\frac{1}{2\sqrt{2}} e^{-\sqrt{2}\eta} \sin \sqrt{2}\eta + \frac{\eta}{2} e^{-\eta}(\sin \eta - \cos \eta)\right\}$$

$$f_2(\eta) = \frac{13}{4} - \frac{3\eta}{4} - 1/4e^{-2\eta} - e^{-\eta}\{3 \cos \eta + (2+\eta) \sin \eta\}$$

We substitute the above equation in Equation 5.238, take the limit as $\eta \to \infty$, and find that:

$$\lim_{\eta \to \infty} u_1 = \frac{-3}{4\omega} U_0 \frac{dU_0}{dx} \tag{5.240}$$

Thus the first approximation to the tangential velocity does not vanish outside the boundary layer. It must be noted that Equation 5.240 is independent of viscosity and is also steady despite the fact that the forcing field is periodic. This result could not have been anticipated in advance, because it is due to an interaction between the boundary layer and the external periodic flow. The steady potential flow (Equation 5.240) is called *secondary* or *streaming* motion at the outer edge of the boundary layer, and its existence has been confirmed by the experiments of Andrade[57] and Schlichting.[1] An extension of this result to three dimensions has been made by Warsi.[58]

It is important to understand the conditions under which the steady streaming flow outside the boundary layer can occur. First, recall that we linearized the equations by assuming that the amplitude A is small in comparison to a body dimension ℓ, i.e., $A/\ell \ll 1$. Based on this assumption, we form a Reynolds number $R_A = U_m A/\nu$, which is also small due to the smallness of A. However, $U_m = (\omega A)$; hence $R_A = U_m^2/\nu\omega$. Therefore, the analysis which yielded the steady streaming outside the boundary layer is valid only when R_A is small; in other words, the amplitude is small and the frequency ω is very large. On the other hand, if R_A is large, then as shown by Stuart,[59] there exists a second outer boundary layer at the outer edge in which there is no steady streaming. For details on this aspect, i.e., R_A large, refer to Stuart[59] and Rayleigh.[60]

There are many other interesting examples of boundary layers in unsteady flow. The reader is particularly referred to Lighthill[61] and the special articles by Stewartson,[62] Stuart,[59] Rott,[63] Schlichting,[1] and Telinois.[64] Refer also to Cousteix.[65]

Separation of Unsteady Boundary Layers

In Section 5.7 we have already discussed the separation of the *steady* two-dimensional boundary layers wherein it was shown that the separation is a phenomenon in which the boundary layer leaves the body surface and that it occurs at a point on the body surface where the wall shear is zero. It was also shown that the steady boundary layer separation is characterized by a Goldstein[8]-type of singularity of boundary layer equations in which the velocity component v normal to the wall increases according to $(x_s - x)^{-1/2}$, where x_s is the stationary separation point. Thus we have the result that at separation:

$$u = 0, \qquad \frac{\partial u}{\partial y} = 0$$

both being evaluated at the body surface.

In the case of nonsteady flows, it was pointed out by Moore,[66] Rott,[67] and Sears[68] that the definition of steady boundary layer separation cannot be adopted to define the unsteady boundary layer separation. It is because of the fact that the separation point is not stationary with respect to the coordinates attached to the wall on which the boundary layer forms. The principle of unsteady boundary layer separation as formulated by Moore, Rott, and Sears (also called the M-R-S principle) is stated as follows: "The unsteady two-dimensional boundary layer separation is characterized by the condition of simultaneous vanishing of both the velocity and the shear stress at some point in the boundary layer when observed with respect to a coordinate system attached to the moving separation point." It was shown later by Sears and Telionis[69] and Shen[70] that the M-R-S condition is also characterized by a Goldstein-type singularity of the boundary layer equations. Although the mathematical details on the singularity have been expounded in the above references, it can simply be perceived in the form of extremely slow convergence or complete breakdown of the numerical solution while numerically solving the boundary layer equations near the separation point.

Mathematical Formulation of the M-R-S Principle

A mathematical formulation of the M-R-S principle can be obtained if Equations 5.234 and 5.235a are transformed to a coordinate system attached with the moving separation point. Let $u_s(t)$ be the velocity of the separation point. Note that $u_s(t)$, in general, is not known a priori but is obtained as part of the solution. We now attach a coordinate system with the moving separation point so that the required transformation equations are

$$\begin{aligned} &\bar{x} = x - x_s(t), \qquad \bar{y} = y, \qquad \bar{t} = t \\ &\bar{u} = u - u_s, \qquad \bar{v} = v, \qquad \bar{u} = u_e - u_s \end{aligned} \tag{5.241}$$

Introducing Equation 5.241 in Equations 5.234 and 5.235a and noting that:

$$\frac{\partial x}{\partial t} = 0, \qquad \frac{\partial y}{\partial t} = 0, \qquad \frac{\partial \bar{x}}{\partial t} = -u_s, \qquad \frac{\partial \bar{y}}{\partial t} = 0$$

we have:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial \bar{u}_e}{\partial \bar{t}} + u_e\frac{\partial \bar{u}_e}{\partial \bar{x}} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\frac{\partial \bar{u}_e}{\partial \bar{x}} + \frac{\partial \bar{v}_e}{\partial \bar{y}} = 0$$

The boundary conditions are now:

$$\bar{y} = 0 : \bar{u} = -u_s, \bar{v} = 0$$

$$\bar{y} \to \infty : \bar{u} = \bar{u}_e(\bar{x}, \bar{t}) = u_e(x, t) - u_s(t)$$

Thus, the M-R-S principle states that in the case of adverse pressure gradients there exists a point in the flow such that:

$$\frac{\partial \bar{u}}{\partial \bar{y}} = 0 \quad \text{at} \quad \bar{u} = 0 \tag{5.242}$$

Williams and Johnson[71] have used the above transformed equations to establish Equation 5.242 for the case when the external potential flow is a function of $Ax + Bt$ and the velocity $u_s = -B/A$ = constant, where A and B are constants.

Numerical Method of Solution of Unsteady Equations

The most efficient way of solving the unsteady boundary layer equations is through the method of finite difference approximations. Almost all numerical solutions show a breakdown of the numerical procedures at separation, indicating Goldstein's singularity. This provides a useful means for the determination of the separation point. In nonsteady flows, flow reversal may occur without any singular behavior of the solution and a breakdown of the numerical procedure. Therefore, the numerical scheme must be capable of taking account of the direction of velocity to incorporate the "upstream" influence in a proper manner. Here we follow the scheme given in Reference 70.

We denote the grid locations along x, y, and t, respectively, by i, j, n and the step sizes as:

$$\Delta t_n = t_n - t_{n-1}$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_j = y_j - y_{j-1}$$

Referring to Equations 5.234 and 5.235a, we write:

$$u(x_i, y_j, t_n) = u^n_{i,j}$$

$$v(x_i, y_j, t_n) = v^n_{i,j}$$

and use the following finite difference approximations:

1. Backward time differencing:

$$\left.\frac{\partial u}{\partial t}\right|^n_{i,j} = \frac{u^n_{i,j} - u^{n-1}_{i,j}}{\Delta t_n}$$

2. Upwind differencing for the convective derivative:

$$\left.\frac{\partial u}{\partial x}\right|^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x_{i+1}}$$

3. Alternate differencing for the convective derivative:

$$\left.\frac{\partial u}{\partial x}\right|^n_{i,j} = \frac{1}{2}\left[\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x_i} + \frac{u^{n-1}_{i+1,j} - u^{n-1}_{i,j}}{\Delta x_{i+1}}\right]$$

4. Central differencing for the y-derivatives:

$$\left.\frac{\partial u}{\partial y}\right|^n_{i,j} = \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j}^{n} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2}$$

When these difference approximations are substituted in the boundary layer equations (Equations 5.234 and 5.235a) there results a nonlinear difference scheme which can be solved by iteration. For details refer to Reference 70 and to the method of finite difference for steady boundary layers discussed previously in this chapter. Refer also to Telinois.[64]

5.20 SECOND ORDER BOUNDARY LAYER THEORY

The process of boundary layer approximation as introduced clearly indicates that the solution of boundary layer equations form an asymptotic first approximation to the solution of the Navier-Stokes equations for small values of ν or large Reynolds' numbers. With this in mind it is natural to think of developing an asymptotic series solution for the complete Navier-Stokes equations with the boundary layer solution as the first term of the series. If this scheme is viable, then it should be possible to determine any desired number of approximations in succession. Ironically this straightforward scheme is not a feasible approach to the problem under consideration.

There are many examples in mathematics and mechanics where the straightforward asymptotic series solution can be generated without any significant difficulty. Such problems are known as *regular perturbation* problems. Side-by-side there are examples where this direct approach is not successful and such problems are known as *singular perturbation* problems. Singular perturbation problems usually arise in those cases where the leading solution to be perturbed is given by a lower order differential equation and the next approximation is sought through a higher order differential equation. For details on this subject refer to Van Dyke[72] and Cole.[73] A simple example is provided by Panton[74] which is called the Friedrichs' problem[75] and is stated as follows:

Solve the following problem for $\epsilon \rightarrow 0$:

$$\epsilon u'' + u' = -\frac{3}{2}(1 - 3\epsilon)e^{-3y}$$
$$u(0) = 0, \; u(\infty) = 1$$

where a prime denotes differentiation with reference to y. The exact solution of the problem is

$$u = \frac{3}{2}(1 - e^{-y/\epsilon}) - \frac{1}{2}(1 - e^{-3y})$$

If the expansion process is used, then a singular perturbation problem results. Taking the limit of the equation as $\epsilon \rightarrow 0$ and fixed y, we get:

$$u' = -\frac{3}{2}e^{-3y}$$

so that the outer solution is

$$u = 1 + \frac{1}{2}e^{-3y}$$

Similarly, first using the stretching transformation $\eta = y/\epsilon$ and then taking the limit of the equation as $\epsilon \rightarrow 0$ and fixed η (refer to Section 5.4 on outer and inner limits), we get:

$$\frac{d^2u}{d\eta^2} + \frac{du}{d\eta} = 0$$

so that the boundary layer solution is

$$u = A(1 - e^{-\eta})$$

Using the previously used principle:

inner limit of the outer solution = outer limit of the inner solution

we get $A = 3/2$. Thus, the boundary layer solution of the Friedrichs' problem is

$$u_b = \frac{3}{2}(1 - e^{-y/\epsilon})$$

while the external solution is

$$u_e = 1 + \frac{1}{2}e^{-3y}$$

The problem of perturbing the boundary layer solution to obtain the successive approximations for the Navier-Stokes solution is a problem of singular perturbation. Some initial insight into this problem is provided by the following observations. For simplicity, consider the two-dimensional steady Navier-Stokes and the corresponding boundary layer equations in the stream function form; the Navier-Stokes equation is of the fourth order while the boundary layer equation is of the third order. This observation is sufficient to warn us that we have to deal with a singular perturbation problem. The other observation pertains to a defect in the boundary layer solution. First, note that the inviscid vertical velocity v_e which is consistent with the surface speed $u_e(x, 0)$ is obtained by integrating Equation 5.94a. Thus:

$$v_e = -y\frac{du_e}{dx} \qquad \text{as } y \to 0$$

is the vertical velocity near a wall in the absence of the boundary layer. Now though $u_b \to u_e(x, 0)$ at the boundary layer edge, the vertical velocity v_b does not correspondingly match with the velocity given above. This can be seen by integrating Equation 5.97a with respect to y, thus obtaining:

$$\begin{aligned} v_b &= -\int_0^y \frac{\partial u_b}{\partial x}\,dy \\ &= \frac{d}{dx}\int_0^y (u_e - u_b)\,dy - y\frac{du_e}{dx} \end{aligned}$$

Thus:

$$\begin{aligned} (v_b)_{y\to\delta} &= v_b(x, \infty) \\ &= \frac{d}{dx}(u_e\delta^*) - y\frac{du_e}{dx} \end{aligned}$$

The first term on the right-hand side is an additional vertical flow at the edge of the boundary layer. This outflow is of little consequence as far as the solution of thin boundary layers is concerned but becomes of increasing importance when either the higher approximations are desired or the interaction between the boundary layer flow and the external inviscid flow is of importance.

Method of Matched Asymptotic Expansion

In this section we shall consider the method of "matched asymptotic expansion" as developed by Van Dyke[72] to illustrate the manner in which the higher approximations for a viscous problem can be obtained by starting from the classical boundary layer solution as a first approximation. To present the essential ideas, it suffices to consider the case of steady two-dimensional viscous incompressible flow past a given body and consider only the second order theory.

The Navier-Stokes equations in dimensionless form as obtained by referring all lengths to a characteristic length L, velocities to a characteristic speed U_∞, and pressure to ρU_∞^2 are

$$\nabla \cdot \mathbf{q} = 0 \tag{5.243a}$$

$$\nabla(\tfrac{1}{2}\,\mathbf{q} \cdot \mathbf{q}) + (\nabla \times \mathbf{q}) \times \mathbf{q} + \nabla p = -\epsilon^2\nabla \times (\nabla \times \mathbf{q}) \tag{5.243b}$$

$$(\mathbf{q} \cdot \nabla)\omega = \epsilon^2\nabla^2\omega \tag{5.243c}$$

where:

$$\epsilon^2 = R_e^{-1} = \frac{\nu}{U_\infty L}$$

Further, $\mathbf{q}$, ω, and p are the dimensionless velocity vector, vorticity, and pressure, respectively. The dependence of $\mathbf{q}$, ω, and p on position and the Reynolds number is symbolically represented as:

$$\mathbf{q}(\mathbf{x}, \epsilon), \qquad \omega(\mathbf{x}, \epsilon), \qquad p(\mathbf{x}, \epsilon)$$

For two-dimensional velocity field $\mathbf{q}$ there exists a stream function $\psi = \psi(\mathbf{x}, \epsilon)$, which at a large distance from the body is represented as ψ_∞. The boundary conditions for Equations 5.243a, b are those of the conditions at upstream infinity and of the nop-slip at the surface, i.e.,

$$\mathbf{q}|_S = 0$$

where S is the body surface. In this chapter for the sake of simplicity we have adopted the convention that the small case letters will stand for the first approximation and the capitals for the second approximation. Further, the subscript e will denote the external or outer region, and b the boundary layer and its approximations.

Outer Expansion

As the Reynolds number R_e increases, viz., $\epsilon \rightarrow 0$, the motion at a fixed point in the field away from the surface S approaches an inviscid flow. Kaplun[5] called this the outer or Euler limit. The outer expansions of $\mathbf{q}$, ω, p, and ψ are

$$\mathbf{q}(\mathbf{x}, \epsilon) = \mathbf{q}_e(\mathbf{x}) + \epsilon\mathbf{Q}_e(\mathbf{x}) + \ldots$$

$$\omega(\mathbf{x}, \epsilon) = \omega_e(\mathbf{x}) + \epsilon\Omega_e(\mathbf{x}) + \ldots$$

$$p(\mathbf{x}, \epsilon) = p_e(\mathbf{x}) + \epsilon P_e(\mathbf{x}) + \ldots$$

$$\psi(\mathbf{x}, \epsilon) = \psi_e(\mathbf{x}) + \epsilon \Psi_e(\mathbf{x}) + \ldots$$

Substituting these expansions in Equations 5.243a, b and equating to zero the coefficients of different powers of ϵ, we get:

$$\nabla \cdot \mathbf{q}_e = 0 \tag{5.244a}$$

$$\nabla[\tfrac{1}{2}\mathbf{q}_e \cdot \mathbf{q}_e + p_e] + (\nabla \times \mathbf{q}_e) \times \mathbf{q}_e = 0 \tag{5.244b}$$

$$\nabla \cdot \mathbf{Q}_e = 0 \tag{5.245a}$$

$$\nabla[\mathbf{q}_e \cdot \mathbf{Q}_e + P_e] + (\nabla \times \mathbf{Q}_e) \times \mathbf{q}_e + (\nabla \times \mathbf{q}_e) \times \mathbf{Q}_e = 0, \text{ etc.} \tag{5.245b}$$

The set of Equations 5.244 comprise the Euler equations for the basic inviscid flow, while the set of Equations 5.245 represent the second approximation. Equations having the second derivatives begin to appear with the thid approximation. Since the outer expansion is not valid at the surface S, the only boundary condition which can be satisfied is

$$\mathbf{q}_e \cdot \mathbf{n}|_S = 0$$

where $\mathbf{n}$ is the positive unit normal vector to S.

Let $\mathbf{k}$ be the positive unit constant vector normal to the plane of the two-dimensional flow field, then:

$$\mathbf{k}\omega_e = \nabla \times \mathbf{q}_e \tag{5.246a}$$

$$\mathbf{k}\Omega_e = \nabla \times \mathbf{Q}_e, \text{ etc.} \tag{5.246b}$$

Thus, the Euler equation (Equation 5.244b) is written as:

$$\nabla(\tfrac{1}{2}q_e^2 + p_e) + \omega_e \mathbf{k} \times \mathbf{q}_e = 0$$

or:

$$\omega_e \mathbf{k} \times \mathbf{q}_e = -\nabla B_1 \tag{5.247a}$$

where:

$$B_1 = p_e + \tfrac{1}{2}q_e^2 \tag{5.247b}$$

and:

$$q_e = |\mathbf{q}_e|$$

In two dimensions:

$$\mathbf{k} \times \mathbf{q}_e = \nabla\psi_e \tag{5.248a}$$

$$\mathbf{k} \times \mathbf{Q}_e = \nabla \Psi_e \tag{5.248b}$$

Using Equation 5.248a in Equation 5.247a we can remove the ∇ or grad operator to have:

$$\omega_e = -\frac{dB_1}{d\psi_e} = -B_1'(\psi_e) \tag{5.248c}$$

The Bernoulli function $B_1(\psi_e)$ is a constant for irrotational flow, but for rotational inviscid flow it has to be calculated from the upstream conditions.

Similarly, by using Equations 5.246b and 5.248b in Equation 5.245b, we get:

$$\begin{aligned}\nabla(P_e + \mathbf{q}_e \cdot \mathbf{Q}_e) &= -\omega_e \nabla \Psi_e - \Omega_e \nabla \psi_e \\ &= B_1' \nabla \Psi_e - \Omega_e \nabla \psi_e\end{aligned}$$

Using the formula:

$$\nabla B_1'(\psi_e) = B_1''(\psi_e) \nabla \psi_e$$

we find that the grad operator ∇ can be removed provided that:

$$\Omega_e = -\Psi_e B_1''(\psi_e) \tag{5.249a}$$

Thus:

$$P_e + \mathbf{q}_e \cdot \mathbf{Q}_e = \Psi_e B_1'(\psi_e) \tag{5.249b}$$

In order to analyze and establish the various approximations of the external and the boundary layer flows and also their matching conditions, we choose orthogonal coordinates ξ and η with scale factors h_1 and h_2, respectively. Further, the coordinate system is considered to be body conforming, i.e., the coordinate line $\eta = 0$ of the ξ-family follows the given body contour. As described in Section 5.6, we take:

$$h_1 = hh_1(\xi, 0), \qquad h_2 = 1$$

where:

$$h = 1 + \eta k(\xi)$$

Defining the arc distance x according to Equation 5.107a and taking $\eta = y$, the nondimensional Navier-Stokes equations for 2-D steady flow are

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(hv) = 0 \tag{5.250}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial}{\partial y}(hu) &= -\frac{\partial p}{\partial x} \\ &+ \frac{\epsilon^2}{h}\left[2h\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial h}{\partial y}\right)\right. \\ &\left.+ \frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial x} + h^2\frac{\partial u}{\partial y} - uh\frac{\partial h}{\partial y}\right)\right]\end{aligned} \tag{5.251}$$

$$u \frac{\partial v}{\partial x} + hv \frac{\partial v}{\partial y} - u^2 \frac{\partial h}{\partial y} = -h \frac{\partial p}{\partial y}$$

$$+ \epsilon^2 \left[\frac{\partial}{\partial x} \left(\frac{1}{h} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{u}{h} \frac{\partial h}{\partial y} \right) \right. \tag{5.252}$$

$$\left. + 2h \frac{\partial^2 v}{\partial y^2} + 2 \frac{\partial h}{\partial y} \left(\frac{\partial v}{\partial y} - \frac{1}{h} \frac{\partial u}{\partial x} - \frac{v}{h} \frac{\partial h}{\partial y} \right) \right]$$

As before:

$$\epsilon^2 = R_e^{-1}$$

and the vorticity component is

$$\omega = \frac{1}{h} \frac{\partial v}{\partial x} - \frac{1}{h} \frac{\partial}{\partial y} (hu) \tag{5.253}$$

The physical components of the velocities $\mathbf{q}_e$ and Q_e with respect to the (x, y) system defined above are (u_e, v_e) and (U_e, V_e), respectively. In terms of the stream functions ψ_e and Ψ_e they are

$$u_e = \frac{\partial \psi_e}{\partial y}, \qquad v_e = -\frac{1}{h} \frac{\partial \psi_e}{\partial x} \tag{5.254a}$$

$$U_e = \frac{\partial \Psi_e}{\partial y}, \qquad V_e = -\frac{1}{h} \frac{\partial \Psi_e}{\partial x} \tag{5.254b}$$

Substituting Equation 5.254a in the formula for the vorticity of the external flow (Equation 5.253) and using Equation 5.248c, we get the elliptic equation:

$$\frac{\partial}{\partial x} \left(\frac{1}{h} \frac{\partial \psi_e}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \psi_e}{\partial y} \right) = hB_1'(\psi_e) \tag{5.255}$$

Equation 5.255 is to be solved under the boundary conditions:

$$\psi_e|_S = 0; \qquad \psi_e \rightarrow \psi_\infty \text{ at upstream infinity}$$

Here $B_1(\psi_e)$ and ψ_∞ are known functions. The elliptic problem (Equation 5.255) provides the basic inviscid flow past the given body and is equivalent to solving the Euler equations.

In the same manner, the second approximation is obtained by substituting Equation 5.254b in Equation 5.253 and using Equation 5.249a to have:

$$\frac{\partial}{\partial x} \left(\frac{1}{h} \frac{\partial \Psi_e}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \Psi_e}{\partial y} \right) = \Psi_e B_1''(\psi_e) \tag{5.256}$$

Since the upstream flow has been assumed to be independent of the Reynolds number, one of the boundary conditions for Equations 5.226 is

$$\Psi_e \rightarrow 0 \text{ at upstream infinity}$$

The other boundary condition for the solution of Equation 5.256 cannot be prescribed at this stage. This missing boundary condition will, however, be specified after the effect of viscosity on the outer flow has properly been considered. (Refer to Equations 5.267 and 5.269).

Some Important Derivatives at the Wall

In the course of our investigation we will need the derivatives of u_e, v_e, and p_e with respect to y at $y = 0$, which have been collected below. First of all, from Equations 5.247b and 5.249b, the first and second order pressures at the wall where $v_e = V_e = 0$ are

$$p_e(x, 0) = B_1(0) - \frac{1}{2} u_e^2(x, 0) \tag{5.257}$$

$$P_e(x, 0) = \Psi_e(x, 0)B_1'(0) - u_e(x, 0)U_e(x, 0) \tag{5.258}$$

To obtain $(\partial v_e v/\partial y)_0$, we consider Equation 5.244a and transform it to the (x, y) system which will take the form of Equation 5.250. Evaluating every term at $y = 0$, we get:

$$\left(\frac{\partial v_e}{\partial y}\right)_0 = -\frac{\partial u_e(x, 0)}{\partial x} \tag{5.259a}$$

To obtain $(\partial p_e/\partial y)_0$, we consider Equation 5.244b which on transformation to x, y takes the form of Equations 5.251 and 5.252 with $\epsilon = 0$. Evaluating the second equation at $y = 0$, we get:

$$\left(\frac{\partial p_e}{\partial y}\right)_0 = k(x)u_e^2(x, 0) \tag{5.259b}$$

Finally, to obtain $(\partial u_e/\partial y)_0$ we consider Equation 5.247b, which is

$$B_1(\psi_e) = p_e(x, y) + \frac{1}{2}(u_e^2 + v_e^2)$$

Differentiating with respect to y and setting $y = 0$ while using Equation 5.259b, we get:

$$\left(\frac{\partial u_e}{\partial y}\right)_0 = B_1'(0) - k(x)u_e(x, 0) \tag{5.259c}$$

Inner Expansion

In the coordinate system x, y the solution of the Navier-Stokes equations (5.250–5.252) is symbolically represented as:

$$u(x, y, \epsilon), \qquad v(x, y, \epsilon), \qquad \psi(x, y, \epsilon)$$

For $\epsilon \to 0$, we introduce the following stretched variables in the Navier-Stokes equations:

$$\bar{y} = y/\epsilon, \qquad \bar{v} = v/\epsilon, \qquad \bar{\psi} = \psi/\epsilon$$

The solution will then be a function of x, $\bar{y}$, and ϵ. We now perform an expansion in powers of ϵ as:

$$u(x, \bar{y}, \epsilon) = u_b(x, \bar{y}) + \epsilon U_b(x, \bar{y}) + \ldots$$

$$\bar{v}(x, \bar{y}, \epsilon) = \bar{v}_b(x, \bar{y}) + \epsilon \bar{V}_b(x, \bar{y}) + \ldots$$

$$p(x, \bar{y}, \epsilon) = p_b(x, \bar{y}) + \epsilon P_b(x, \bar{y}) + \ldots$$

$$\bar{\psi}(x, \bar{y}, \epsilon) = \bar{\psi}_b(x, \bar{y}) + \epsilon \bar{\Psi}_b(x, \bar{y}) + \ldots \tag{5.260}$$

From the expressions Equation 5.260 it is obvious that the boundary layer solution is obtained by taking the limit as $\epsilon \to 0$ while keeping x and $\bar{y}$ fixed (called the inner limit). In this sense it is important to note that:

$$v_b = \epsilon \bar{v}_b = \epsilon \lim_{\substack{\epsilon \to 0 \\ x, \bar{y} \text{ fixed}}} \frac{v(x, \bar{y}, \epsilon)}{\epsilon}$$

and:

$$\psi_b = \epsilon \bar{\psi}_b = \epsilon \lim_{\substack{\epsilon \to 0 \\ x, \bar{y} \text{ fixed}}} \frac{\psi(x, \bar{y}, \epsilon)}{\epsilon}$$

This means that the boundary layer approximations v_b and ψ_b are not obtained by applying the inner limit to v and ψ, but by applying to v/ϵ and ψ/ϵ and aferward multiplying them by ϵ.

The First and Second Order Boundary Layer Problems

To obtain the equations of the first and second order boundary layer approximation, we consider equations 5.250–5.252 and first transform them in terms of the inner variables $\bar{y}$ and $\bar{v}$. Substituting the expansions given in Equation 5.260 and equating to zero the coefficients of ϵ^0 and ϵ, we get:

$$\frac{\partial u_b}{\partial x} + \frac{\partial \bar{v}_b}{\partial \bar{y}} = 0 \tag{5.261a}$$

$$u_b \frac{\partial u_b}{\partial x} + \bar{v}_b \frac{\partial u_b}{\partial \bar{y}} = -\frac{\partial p_b}{\partial x} + \frac{\partial^2 u_b}{\partial \bar{y}^2} \tag{5.261b}$$

$$0 = -\frac{\partial p_b}{\partial \bar{y}} \tag{5.261c}$$

$$\frac{\partial U_b}{\partial x} + \frac{\partial \bar{V}_b}{\partial \bar{y}} = -\frac{\partial}{\partial \bar{y}}(k\bar{y}\bar{v}_b) \tag{5.262a}$$

$$u_b \frac{\partial U_b}{\partial x} + U_b \frac{\partial u_b}{\partial x} + \bar{v}_b \frac{\partial U_b}{\partial \bar{y}} + \bar{V}_b \frac{\partial u_b}{\partial \bar{y}}$$

$$= -\frac{\partial P_b}{\partial x} + \frac{\partial^2 U_b}{\partial \bar{y}^2} + k(x)\left[\bar{y}\frac{\partial^2 u_b}{\partial \bar{y}^2} + \frac{\partial u_b}{\partial \bar{y}} - \bar{y}\bar{v}\frac{\partial u_b}{\partial \bar{y}} - u_b \bar{v}_b\right] \tag{5.262b}$$

$$k(x)u_b^2 = \frac{\partial P_b}{\partial \bar{y}} \tag{5.262c}$$

The set of equations (Equations 5.261) represent the usual first order boundary layer problem. It may be noted that these equations do not show any term containing the surface curvature $k(x)$. The set of equations (Equations 5.262) represent the second order boundary layer problem, and they contain terms depending both on the first order boundary layer theory and also the curvature $k(x)$. To complete the formulation, we have to investigate the boundary conditions and also the matching conditions with the inviscid flow. The surface boundary conditions are obviously obtained by using the no-slip condition and are

$$\text{at } \bar{y} = 0 : u_b = \bar{v}_b = 0; \; U_b = \bar{V}_b = 0 \tag{5.263}$$

To complete the formulation we have to investigate the matching conditions of the inner and outer solutions.

Matching of Inner and Outer Solutions

The inner and outer solutions match in an overlap region where the limiting behavior of the outer solution in terms of the inner variables is the same as the limiting behavior of the inner solution in terms of the outer variables. The two limiting behaviors are explained as follows.

The principle of the *inner limit of the outer solution* is based on McLaurin's expansion. Let $f(x, y)$ be a function depending on the outer variables. Then:

$$f(x, y) = f(x, \epsilon\bar{y}) = f(x, 0) + \epsilon\bar{y}\left(\frac{\partial f}{\partial y}\right)_0 + \ldots$$

Thus, the inner limit of the outer solution is

$$\lim_{\substack{\epsilon \to 0 \\ x, \bar{y} \text{ fixed}}} f(x, y) = f(x, 0)$$

The principle of the *outer limit of the inner solution* is based on an asymptotic expansion. Let $\phi(x, \bar{y})$ be a function depending on the inner variables. Then:

$$\phi(x, \bar{y}) = \phi(x, y/\epsilon) = a_0 + \frac{a_1}{\bar{y}} + \frac{a_2}{\bar{y}^2} + \ldots$$

Thus:

$$a_0 = \phi(x, \infty)$$

$$a_1 = \lim_{\bar{y} \to \infty} [\bar{y}\{\phi(x, \bar{y}) - a_0\}]$$

$$= -\lim_{\bar{y} \to \infty} \left(\bar{y}^2 \frac{\partial \phi}{\partial \bar{y}}\right) \quad \text{etc.}$$

For the matching of first approximations of the outer and inner solutions, a simple matching principle is

$$\text{inner limit of the outer solution} = \text{outer limit of the inner solution} \tag{I}$$

which has earlier been used in the first order boundary layer theory.

For higher order approximations we need a generalized matching principle, which according to Lagerstrom and Cole[76] is

$$m\text{-term inner expansion of } (p\text{-term outer expansion})$$
$$= p\text{-term outer expansion of } (m\text{-term inner expansion}) \tag{II}$$

Thus, principle I is a particular case of principle II. Usually $m = p$ or $m = p + 1$. Principle I yields the matching conditions for the first order boundary layer solution as has been done earlier. The matching conditions are

$$u_e(x, 0) = u_b(x, \infty); \qquad p_e(x, 0) = p_b(x, \infty)$$

For the evaluation of the boundary conditions for U_b we take $m = p = 2$. Alternatively, we may consider the limits correct to $0(\epsilon)$ as:

$$\text{inner expansion of order } \epsilon \text{ of } u_e(x, y) + \epsilon \text{ inner limit of } U_e(x, y)$$
$$= \text{outer limit of } u_b(x, \bar{y}) + \epsilon \text{ outer limit of } U_b(x, \bar{y})$$

so that:

$$U_b(x, \infty) = \bar{y}\left(\frac{\partial u_e}{\partial y}\right)_0 + U_e(x, 0), \quad \text{as } \bar{y} \to \infty \tag{5.264}$$

Using Equation 5.259c in Equation 5.264, we have:

$$U_b(x, \infty) = U_e(x, 0) + \bar{y}[B_1'(0) - k(x)u_e(x, 0)], \quad \text{as } \bar{y} \to \infty \tag{5.265}$$

To find the matching conditions for the vertical velocity, we take $m = 1$ and $p = 2$. Alternatively, we may consider the limits of the terms as follows:

$$\text{inner limit of } v_e(x, y) + \epsilon \text{ inner limit of } V_e(x, y) = \epsilon \text{ outer limit } \bar{v}_b(x, \bar{y})$$

Note that the outer limit of $\bar{v}_b(x, \bar{y})$ is not $\bar{v}_b(x, \infty)$ but is $\lim\limits_{\bar{y}\to\infty} [\bar{v}_b - \bar{y}(\partial\bar{v}_b/\partial\bar{y})]$. Thus:

$$V_e(x, 0) = \lim_{\bar{y}\to\infty}\left[\bar{v}_b - \bar{y}\frac{\partial \bar{v}_b}{\partial \bar{y}}\right] \tag{5.266}$$

Further since:

$$V_e(x, 0) = -\frac{\partial \Psi_e(x, 0)}{\partial x}$$

and:

$$\bar{v}_b(x, \bar{y}) = -\frac{\partial \bar{\psi}_b}{\partial x}$$

Equation 5.266 becomes:

$$\Psi_e(x, 0) = \lim_{\bar{y}\to\infty}\left[\bar{\psi}_b - \bar{y}\frac{\partial \bar{\psi}_b}{\partial \bar{y}}\right] \tag{5.267}$$

Recall that for the solution of Equation 5.256 we were short of the wall boundary condition, and now Equation 5.267 provides the needed value.
To explore the physical outcome of Equation 5.267, we use the equation of continuity (Equation 5.261a) in Equation 5.266 and have:

$$V_e(x, 0) = \int_0^\infty \left[\frac{\partial v_b}{\partial \bar{y}} + \frac{\partial u_e(x, 0)}{\partial x}\right] d\bar{y}$$

$$= \frac{d}{dx}\int_0^\infty [u_e(x, 0) - u_b(x, \bar{y})]\, d\bar{y} \tag{5.268}$$

Using:

$$V_e(x, 0) = -\frac{\partial \Psi_e(x, 0)}{\partial x}$$

in Equation 5.268 we obtain:

$$\epsilon \Psi_e(x, 0) = \int_0^\infty [u_b - u_e(x, 0)]\, dy \tag{5.269}$$

The quantity on the right of Equation 5.269 is proportional to the displacement thickness and therefore the perturbation stream function Ψ_e is called the flow due to the displacement thickness.

Following the same procedure for the pressure as has been done for the velocity (i.e., taking $m = p = 2$) and using Equation 5.259b and 5.258, we get:

$$P_b(x, \infty) = B_1'(0)\Psi_e(x, 0) - u_e(x, 0)U_e(x, 0) + \bar{y}k(x)u_e^2(x, 0), \quad \text{as } \bar{y} \to \infty \tag{5.270}$$

The second order pressure in Equation 5.270 is the value in the inviscid region. To find the behavior of the second order pressure in the viscous region, i.e., $P_b(x, \bar{y})$, we have to resort to Equation 5.262c. Integrating Equation 5.262c and using Equation 5.270 (in which $\bar{y}$ is $(\bar{y})_\infty$), we get:

$$P_b(x, \bar{y}) = \bar{y}k(x)u_e^2(x, 0) + k(x)\int_{\bar{y}}^\infty [u_e^2(x, 0) - u_b^2]\, d\bar{y} + P_e(x, 0) \tag{5.271}$$

where:

$$P_e(x, 0) = B_1'(0)\Psi_e(x, 0) - u_e(x, 0)U_e(x, 0)$$

Thus, $\partial P_b/\partial x$ appearing in Equation 5.262b is obtained by differentiating Equation 5.271 with respect to x.

In summary, the second order boundary layer solution is obtained by solving Equation 5.262b under the boundary conditions (Equations 5.263 and 5.265) with the pressure gradient derived from Equation 5.271. The $U_e(x, 0)$ appearing in Equation 5.265 is obtained by first solving Equation 5.256 for Ψ_e and then:

$$U_e(x, 0) = \left(\frac{\partial \Psi_e}{\partial y}\right)_{y=0}$$

The preceding analysis was first conducted by Van Dyke,[7] and later used by him and others (for a detailed reference list, cf., Reference 72) in various fluid flow problems. Refer also to Reference 77.

A Unified Second Order Correct Viscous Model

The perturbation scheme of Van Dyke as discussed in the preceding subsection provides much insight into the basic structure of the boundary layers and is viewed as a fundamental contribution to the subject of high Reynolds' number viscous flows. Although the scheme provides a mathematically consistent method of obtaining higher approximations to the classical boundary layer theory, it is computationally cumbersome to execute. It involves the following steps:

1. Compute the inviscid flow.
2. Compute the first order boundary layer flow.
3. Compute the flow due to the displacement thickness.
4. Compute the solution of the second order boundary layer equations.

In this section, we propose to discuss a method due to Davis[78] which blends the first and second approximations into a single model and thus the solution will be second order correct. The starting point of the present approach is again the nondimensional Navier-Stokes equations (Equations 5.250–5.252). First, for high Reynolds' number ($\epsilon \rightarrow 0$) and for points away from the solid surface, the second order correct external flow is

$$\begin{aligned} u_{es}(x, y) &= u_e(x, y) + \epsilon U_e(x, y) \\ v_{es}(x, y) &= v_e(x, y) + \epsilon V_e(x, y) \\ p_{es}(x, y) &= p_e(x, y) + \epsilon P_e(x, y) \end{aligned} \tag{5.271}$$

We now introduce the stretched variables:

$$\bar{y} = y/\epsilon, \qquad \bar{v} = v/\epsilon$$

into Equations 5.250–5.252 and without performing the inner limit process simply retain terms which are independent of ϵ. Such equations will be correct to the second order in the longitudinal surface curvature, and are

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial \bar{y}}(h\bar{v}) = 0 \tag{5.272}$$

$$u\frac{\partial u}{\partial x} + \bar{v}\frac{\partial}{\partial \bar{y}}(hu) = -\frac{\partial p}{\partial x} + h\frac{\partial}{\partial \bar{y}}\left[\frac{1}{h}\frac{\partial}{\partial \bar{y}}(hu)\right] \tag{5.273}$$

$$\frac{u^2}{h}\frac{\partial h}{\partial \bar{y}} = \frac{\partial p}{\partial \bar{y}} \tag{5.274}$$

It must be noted that since:

$$h = 1 + \epsilon\bar{y}k(x)$$

we have used the condition $\partial^2 h/\partial \bar{y}^2 = 0$ in obtaining Equation 5.273. In stretched variable, the vorticity is

$$\epsilon\omega = -\frac{1}{h}\frac{\partial}{\partial \bar{y}}(hu) \tag{5.275}$$

Matching

For large values of $\bar{y}$ the fundamental requirement on the solution is that is irrotational. This situation is available at the outer extreme of the viscous layer, where the solution is denoted as:

$$u_\delta(x, \bar{y}), \qquad v_\delta(x, \bar{y}), \qquad p_\delta(x, \bar{y}) \tag{5.276}$$

Because of the irrotationality it is obvious from Equation 5.275 that:

$$\frac{\partial}{\partial \bar{y}}(hu_\delta) = 0 \tag{5.277a}$$

which implies:

$$hu_\delta = F(x, \epsilon) \tag{5.277b}$$

We shall now show that $F(x, \epsilon)$ is a second order correct surface speed and that it has the effect of the displacement thickness of the boundary layer. Starting from the condition that:

$$u_\delta \rightarrow u_{es} \quad \text{as} \quad \epsilon \rightarrow 0 \tag{5.278a}$$

we make an inner expansion of the second order correct velocity u_{es} as follows:

$$\begin{aligned} u_{es}(x, y) &= u_{es}(x, \epsilon\bar{y}) \\ &= u_{es}(x, 0) + \epsilon\bar{y}\left(\frac{\partial u_{es}}{\partial y}\right)_0 + \ldots \\ &= u_e(x, 0) + \epsilon U_e(x, 0) + \epsilon\bar{y}\left(\frac{\partial u_e}{\partial y}\right)_{y=0} + \ldots \end{aligned} \tag{5.278b}$$

On the other hand, from Equation 5.277b:

$$\begin{aligned} u_\delta(x, \bar{y}) &= F(x, \epsilon)(1 + \epsilon\bar{y}k)^{-1} \\ &= [F_1(x) + \epsilon F_2(x) + \ldots](1 + \epsilon\bar{y}k)^{-1} \\ &= F_1(x) + \epsilon F_2(x) - \epsilon k\bar{y}F_1(x) + \ldots \end{aligned} \tag{5.278c}$$

On the basis of Equation 5.278a, we compare Equations 5.278b and c and have:

$$u_e(x, 0) = F_1(x)$$

$$U_e(x, 0) = F_2(x)$$

$$\left(\frac{\partial u_e}{\partial y}\right)_0 = -kF_1(x), \text{ etc.}$$

Thus finally:

$$F(x, \epsilon) = F_1(x) + \epsilon F_2(x) = u_e(x, 0) + \epsilon U_e(x, 0) \tag{5.279}$$

which proves that $F(x, \epsilon)$ is a second order correct surface speed. Therefore, the boundary conditions for Equations 5.272–5.274 are

$$\text{at } \bar{y} = 0 : u = \bar{v} = 0$$

$$\text{as } \bar{y} \rightarrow \infty : u \rightarrow u_\delta = \frac{F}{h}$$

As in Van Dyke's second order equation, the formulation of the pressure term is again important in the unified model. Applying Equation 5.277a to Equation 5.273, we get:

$$p_\delta(x, \bar{y}) = p_\infty + \frac{1}{2} - \frac{1}{2} u_\delta^2 \tag{5.280}$$

We now integrate Equation 5.274 from a fixed point $\bar{y}_0 = y_0/\epsilon$ to a point $\bar{y}$ and have:

$$p(x, \bar{y}) = p(x, \bar{y}_0) + \int_{\bar{y}_0}^{\bar{y}} \frac{u^2}{h} \frac{\partial h}{\partial \bar{y}} \, d\bar{y} \tag{5.281a}$$

Thus, formally for large $\bar{y}$:

$$p_\delta(x, \bar{y}) = p_\delta(x, \bar{y}_0) + \int_{\bar{y}_0}^{\bar{y}} \frac{u_\delta^2}{h} \frac{\partial h}{\partial \bar{y}} \, d\bar{y} \tag{5.281b}$$

Subtracting Equation 5.281b from Equation 5.281a, we have:

$$p(x, \bar{y}) = p_\delta(x, \bar{y}) + \{p(x, \bar{y}_0) - p_\delta(x, \bar{y}_0)\} + \int_{\bar{y}_0}^{\bar{y}} \frac{1}{h} \frac{\partial h}{\partial \bar{y}} (u^2 - u_\delta^2) \, d\bar{y} \tag{5.282}$$

Note that $\bar{y}_0 = y_0/\epsilon$ (where y_0 is a fixed point $\neq 0$) can be made as large as we please by letting $\epsilon \rightarrow 0$. Taking the inner limit of Equation 5.282, viz., $\epsilon \rightarrow 0$ while keeping $\bar{y}$ fixed, we automatically make $\bar{y}_0 \rightarrow \infty$ so that:

$$p(x, \bar{y}) = p_\delta(x, \bar{y}) + \int_{\bar{y}}^{\infty} \frac{1}{h} \frac{\partial h}{\partial \bar{y}} (u_\delta^2 - u^2) \, d\bar{y} \tag{5.283}$$

where p_δ is given in Equation 5.280. The wall pressure coefficient is conventiently obtained by using Equation 5.283 and noting that $u_\delta(x, 0) = F(x, \epsilon)$. Thus:

$$c_{p_w} = 2(p_w - p_\infty) = 1 - F^2(1 - 2I)$$

where:

$$I = \int_0^\infty \frac{1}{h^3} \frac{\partial h}{\partial \bar{y}} \left(1 - \frac{h^2 u^2}{F^2}\right) d\bar{y}$$

Werle and Wornom[79] have successfully used the preceding model for the calculation of flow past a circular cylinder and have found results which compare well with the Navier-Stokes solutions for the Reynolds number as low as 100.

The main problem in the calculation of second order correct viscous flows is the determination of $F(x, \epsilon)$, i.e., the second order correct surface speed. One method to determine this speed is first to solve the classical boundary layer equations for a given body and determine the displacement thickness. The original body is then thickened with the calculated displacement thickness and a new potential flow is calculated for this modified body shape. The solution of the modified body evaluated at the surface will provide the required $F(x, \epsilon)$ correct to the second order. Werle and Wornom have used an iterative method which is worth investigating.

5.21 INVERSE PROBLEMS IN BOUNDARY LAYERS

From the time of the original proposal by Prandtl on boundary layer theory until the late 1960s, the boundary layer equations were solved under prescribed boundary conditions of no-slip at the wall and of external boundary conditions at the boundary layer edge. The complete set of equations has been written out at various places in this book, e.g., Equations 5.99–5.102. Such boundary layer formulations are called *direct problems*. However, under certain important circumstances, to be described in this section, it is desirable to delete Equation 5.101 and instead prescribe either the wall shear or the displacement thickness. If this is done, then the external velocity u_e is obtained as part of the overall solution. Such formulations of the boundary layer are called *inverse problems*.

The importance of inverse problems was felt after the publication of a paper by Catherall and Mangler.[80] The main thrust of this paper is that if the displacement thickness is prescribed then the boundary layer equations can be solved through the point of separation and beyond without any indication of a *singularity* at separation. Recall from Section 5.7 that the separation of a *steady* boundary layer is signified by the vanishing of the wall shear at a point on the wall. Goldstein[8] showed mathematically that the steady boundary layer separation occurs as a singularity of the equations. Earlier work by Hartree[12] and later by Leigh[81] showed that the numerical solution of boundary layer equations in the direct formulation cannot be continued at and aft of the separation point, thus confirming Goldstein's anticipated singularity. A complete review on the separating boundary layers has been given by Brown and Stewartson[82] and Williams.[83]

It was discussed in Section 5.7 that as the point of separation is approached, the velocity component v normal to the wall increases rapidly and at separation becomes of the same order of magnitude as the longitudinal component of velocity u. Since the classical boundary layer theory is based on the assumption that $v \ll u$, the behavior $v \propto u$ near separation erodes the very foundation of the theory, thus implying in an indirect way toward a singularity or a complete breakdown of the equations themselves. There are, however, many situations in fluid flows in which a separation occurs at a point where $\partial u/\partial y = 0$ and aft of it there is a narrow region of back flow followed by a reattachment. This aspect of flow separation has been shown in Figure 5.15. This thin recirculating region on the surface is called a separation bubble.

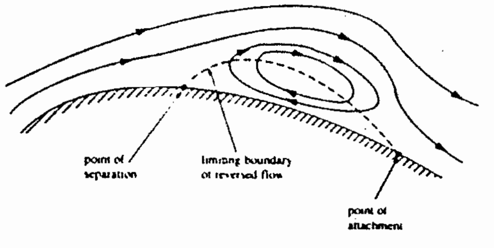


Fig. 5.15 Separation bubble and the limiting boundary of the reversed flow.

Thus the argument is that if the separation bubble is thin and nowhere its thickness exceeds $0(\sqrt{\nu})$, then the boundary layer assumptions are not violated and Prandtl's equations must be valid throughout including a separation bubble. A direct boundary layer problem cannot be used to solve at and past the separation point. However, the work of Catherall and Mangler showed that if proper considerations are given to the direction of flow in the *thin* separation bubble, then an inverse boundary layer problem can successfully be used to pass the separation point.

A question which naturally arises is why an inverse formulation does not show any singularity in the solution at the separation point which is always present in a direct formulation. In this connection it is also important to mention that the available complete Navier-Stokes numerical solutions never show any singularity at the separation point. In answer to this question, recall that the potential velocity u_e which is used as a boundary condition in the direct problems is that solution of the Euler equations which has been evaluated at the wall. Now, because of the existence of the boundary layer, the streamlines of the external inviscid flow are displaced into the inviscid region. The displacement of the external streamlines is therefore due to viscosity, and this does not produce any serious problems unless there is a significant interaction between the viscous and inviscid regions. Obviously, prior to the separation and in the separated region there must be a strong interaction between the viscous and the inviscid regions which was not taken into account while calculating the imposed longitudinal velocity u_e. On the other hand, in an inverse problem since u_e is not prescribed, the proper values of u_e are calculated provided that the solution algorithm has a mechanism of iteration on the displacement thickness distribution.

Inverse formulations have also been used to design the body shapes which have a specified wall shear stress distribution. A knowledge of u_e from the inverse solution determines a unique shape. Inverse formulations have also been used without separation and in viscous-inviscid interactions.

Inverse Formulation with Assigned Displacement Thickness

The following formulation is essentially due to Carter.[84] We consider the nondimensional boundary layer equations in stretched variables as given in Equations 5.261. Introducing the stream function $\bar{\psi}_b$, we have:

$$u_b = \frac{\partial \bar{\psi}_b}{\partial \bar{y}}, \qquad \bar{v}_b = -\frac{\partial \bar{\psi}_b}{\partial x} \tag{5.284}$$

Defining the stretched nondimensional displacement thickness as:

$$\Delta^* = \delta^*/\epsilon$$

we have:

$$u_e\Delta^* = \int_0^{\bar{y}_e} (u_e - u_b)\, d\bar{y}, \qquad \bar{y}_e \sim \infty$$

Using the first equation of Equation 5.284, we get:

$$\bar{\psi}_{be} = u_e(\bar{y}_e - \Delta^*) \tag{5.285}$$

We now introduce the transformation $\zeta = \bar{y}/\Delta^*$ in Equation 5.261b and have:

$$\Delta^{*2}u_b \frac{\partial u_b}{\partial x} - \Delta^* \frac{\partial u_b}{\partial \zeta}\frac{\partial \bar{\psi}_b}{\partial x} = \Delta^{*2}\beta + \frac{\partial^2 u_b}{\partial \zeta^2} \tag{5.286}$$

where $\beta = u_e\, du_e/dx$. Based on the form (Equation 5.285), we introduce a new stream function χ, with $\chi_e = 0$, defined as:

$$\chi = \frac{1}{\sqrt{2x}}[\bar{\psi}_b - u_b(\bar{y} - \Delta^*)] \tag{5.287}$$

in Equation 5.286:

$$\Delta^{*2}u_b \frac{\partial u_b}{\partial x} - \Delta^* \frac{\partial u_b}{\partial \zeta}\frac{\partial}{\partial x}[\sqrt{2x}\,\chi + u_b(\zeta - 1)\Delta^*] = \Delta^{*2}\beta + \frac{\partial^2 u_b}{\partial \zeta^2} \tag{5.288}$$

Also, from the first equation in Equation 5.284 and from Equation 5.287:

$$\frac{\partial \chi}{\partial \zeta} = \frac{\Delta^*}{\sqrt{2x}}(1 - \zeta)\frac{\partial u_b}{\partial \zeta} \tag{5.289}$$

The boundary conditions for Equations 5.288 and 5.289 are

$$\zeta = 0 : u_b = 0, \chi = 0$$

$$\zeta \rightarrow \zeta_e : \chi = 0$$

Here ζ_e is the value of ζ at the edge of the boundary layer. When Equations 5.288 and 5.289 are solved in the inverse mode, the velocity u_e appearing in the function β is not the specified $u_e(x, 0)$, as is normally done in the direct boundary layer problems. The simultaneous solution of Equations 5.288 and 5.289 besides providing u_b and χ also provides β from which $u_e = (u_e)_{vis}$ is to be calculated. To have a complete iterative procedure we must have a scheme for updating the displacement thickness Δ^*. Carter has proposed the formula:

$$\Delta^{*(p+1)} = \Delta^{*(p)}\frac{(u_e)_{vis}}{(u_e)_{inv}} \tag{5.290}$$

for updating the displacment thickness. In Equation 5.290, p is the iterative index and $(u_e)_{inv} = u_e(x, 0)$ is the correct inviscid speed at the wall. On convergence we must have:

$$|(u_e)_{vis} - (u_e)_{inv}| < \epsilon$$

To calculate the proper value of $(u_e)_{inv}$, we note that since $\psi_e(x, 0) = 0$, and then from Equation 5.269:

$$(\psi)_{inv}|_S = \epsilon\Psi_e(x, 0) = -\epsilon u_e(x, 0)\Delta^*(x)$$

However, since the proper $u_e(x, 0)$ is not available, we then take:

$$(\psi)_{inv}|_S = -(u_e)_{vis}\delta^*$$

which provides the wall boundary condition for the Euler equations in the stream function form. If the Euler equations are taken in the velocity formulation, then:

$$(v)_{inv}|_S = \epsilon\frac{d}{dx}[(u_e)_{vis}\Delta^*]$$

This noted iterative process is continued until convergence. For details on the numerical scheme, particularly for the difference approximation technique in the reversed flow region, refer to Reference 84.

5.22 FORMULATION OF THE COMPRESSIBLE BOUNDARY LAYER PROBLEM

The basic rules for the derivation of incompressible boundary layer equations, both in two and three dimensions, have already been stated and used in the previous sections. The same basic rules and concepts are applicable to compressible fluid motion. As has been demonstrated earlier, the method of inner and outer limits is the simplest way to obtain boundary layer equations, which we intend to use here in the derivation of boundary layer equations for viscous compressible flow.

In this section we shall derive boundary layer equations for the viscous compressible flows past arbitrary surfaces. All particular cases (e.g., two-dimensional, and three-dimensional boundary layer equations in orthogonal coordinates) then follow from the general equations obtained here.

In the surface, we envisage a nonorthogonal curvilinear coordinate system (x^1, x^3) which is needed to predict the boundary layers on surfaces of practical shapes, e.g., finite wing, fuselage of an airplane, ship hull, etc. However, without any loss of generality the boundary layer coordinate x^2 can be taken to be a local orthogonal curve on the surface. Thus, (x^1, x^2, x^3), in this order, form a right-handed coordinate system. The reader who is interested in the development of the ideas leading to the equations obtained here is referred to the previous work by Lin,[85] Mager,[86] Hansen,[87] and Cebeci et al.[88]

We consider the Navier-Stokes equations for viscous compressible flow in the vector invariant form as stated earlier as Equations 3.8 and 3.10 along with the equation of energy, Equations 3.23. For simplicity we drop the body force **f** and so also the body force potential χ. We now write these equations in the contravariant components by using the techniques developed in Section 3.10. The equations of motion (Equation 3.147) along with the equation of total enthalpy are

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\rho \sqrt{g} u^j) = 0 \tag{5.291}$$

$$\rho\left(\frac{\partial u^i}{\partial t} + u^j u^i_{,j}\right) = g^{ij} \frac{\partial}{\partial x^j} (\lambda \Delta - p) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g}\, \mu d^{ij}) + \mu \Gamma^i_{jt} d^{jt} \tag{5.292}$$

$$\begin{aligned} &\rho\left(\frac{\partial H}{\partial t} + u^j \frac{\partial H}{\partial x^j}\right) \\ &\quad = \frac{\partial p}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} \sigma^{jt} g_{ti} u^i) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(k \sqrt{g} g^{jt} \frac{\partial T}{\partial x^t}\right) \end{aligned} \tag{5.293}$$

where $u^i_{,j}$ is the covariant derivative of the contravariant components u^i, viz.:

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jt} u^t$$

Γ^i_{jt} are the Christoffel symbols (refer to ME.1), and :

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^n} (\sqrt{g} u^n)$$

$$d^{ij} = (g^{im} u^j_{,m} + g^{jn} u^i_{,n}) = 2D^{ij}$$

$$\sigma^{jt} = \lambda\, \Delta g^{jt} + \mu d^{jt}$$

μ and λ are the first and second coefficients of viscosity, respectively; k, p, T, ρ are the conductivity, pressure, temperature, density, respectively, and:

$$H = h + \tfrac{1}{2} |\mathbf{u}|^2$$

is the total enthalpy.

For the purpose of adopting a set of simple notation (particularly after the terms in Equations 5.291–5.293 have been expanded using the summation convention on repeated indices), we write:

$$\sqrt{g_{11}} = h_1, \qquad \sqrt{g_{22}} = h_2, \qquad \sqrt{g_{33}} = h_3, \qquad \sqrt{G_2} = J$$

$$x^1 = \xi, \qquad x^2 = \eta, \qquad x^3 = \zeta \tag{5.294}$$

$$u = h_1 u^1, \qquad v = h_2 u^2, \qquad w = h_3 u^3$$

Since we have taken η to be orthogonal to the surface, i.e., to both ξ and ζ, we have $g^{12} = g^{23} = 0$, so that $g = g_{22}G_2$, where $G_2 = g_{11}g_{33} - (g_{13})^2$. Also, $g^{22} = G_2/g = 1/g_{22}$, and $\sqrt{g} = h_2 J$.

Let μ_∞ be the free-stream coefficient of viscosity. To implement the method of inner limiting procedure on the equations we introduce the stretched variables:

$$\bar{\eta} = \eta/\sqrt{\mu_\infty}, \qquad \bar{u}^2 = u^2/\sqrt{\mu_\infty}$$

in Equations 5.291–5.293 and then take the limit of each term as $\mu_\infty \to 0$ while keeping ξ, $\bar{\eta}$, ζ fixed. In this connection we have to be careful in the estimation of the derivatives of the metric coefficients g_{11} and g_{33} with respect to η. For surfaces which are not excessively curved either in the direction of ξ or ζ, all the metric coefficients (g^{ij}, g_{ij}, g) which remain in the final equations are those which have been evaluated at the surface, i.e., at $\eta = 0$. The same result applies to the Christoffel symbols, even those which have derivatives of g_{ij} with respect to η. Recall from Section 5.12 that the derivative of a metric coefficient with respect to η is a quantity of $0(1)$ and *not* of the $0(1/\sqrt{\mu_\infty})$. However, there are some geometric shapes which have excessive curvatures of the $0(1/\sqrt{\mu_\infty})$. In these cases we have to keep either $\partial g_{11}/\partial\eta$, $\partial g_{13}/\partial\eta$, or $\partial g_{33}/\partial\eta$ or all three. Since our purpose here is to obtain a *general* form of the approximation, we shall presently keep all the terms.

Estimation of the Viscous Terms

The viscous term in Equation 5.292 is

$$\underset{\text{(I)}}{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\,\mu d^{ij})} + \underset{\text{(II)}}{\mu\Gamma^i_{jk}d^{jk}}$$

We first consider the case when either $\partial g_{11}/\partial\eta$, $\partial g_{33}/\partial\eta$, or $\partial g_{13}/\partial\eta$ or all are of the $0(1/\sqrt{\mu_\infty})$. This case arises when the curvature of the surface of the $0(1/\delta)$ is very large, where δ is the boundary layer thickness. In these cases only terms of $0(1)$ must be retained. Thus on expansion, we find that term I approximates to:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}(\sqrt{g}\,\mu d^{i2}), \qquad i = 1, 3$$

while the term II approximates to

$$g^{22}\mu\left(\Gamma^i_{12}\frac{\partial u^1}{\partial\eta} + \Gamma^i_{32}\frac{\partial u^3}{\partial\eta}\right), \qquad i = 1, 3$$

On substituting the expressions for d^{12}, Γ^1_{12}, Γ^1_{32} for the case $i = 1$ and d^{32}, Γ^3_{12}, Γ^3_{32} for the case $i = 3$ and retaining only the terms of $0(1)$, we find that the viscous terms for the two boundary layer equations are, respectively:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left[\mu\sqrt{g}\,g^{22}\left\{\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)+\frac{u}{2gh_1}\left(g_{22}g_{33}\frac{\partial g_{11}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{13}}{\partial\eta}\right)\right.\right.$$

$$\left.\left.+\frac{w}{2gh_1}\left(g_{22}g_{33}\frac{\partial g_{13}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{33}}{\partial\eta}\right)\right\}\right]$$

$$+\frac{\mu}{2g}\left[\left(g_{22}g_{33}\frac{\partial g_{11}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{13}}{\partial\eta}\right)\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right.$$

$$\left.+\left(g_{22}g_{33}\frac{\partial g_{13}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{33}}{\partial\eta}\right)\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)\right]g^{22} \tag{5.295a}$$

and:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left[\mu\sqrt{g}\,g^{22}\left\{\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)+\frac{u}{2gh_1}\left(g_{11}g_{22}\frac{\partial g_{13}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{11}}{\partial\eta}\right)\right.\right.$$

$$\left.\left.+\frac{w}{2gh_3}\left(g_{11}g_{22}\frac{\partial g_{33}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{13}}{\partial\eta}\right)\right\}\right]$$

$$+\frac{\mu}{2g}\left[\left(g_{11}g_{22}\frac{\partial g_{13}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{11}}{\partial\eta}\right)\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right.$$

$$\left.+\left(g_{11}g_{22}\frac{\partial g_{33}}{\partial\eta}-g_{13}g_{22}\frac{\partial g_{13}}{\partial\eta}\right)\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)\right]g^{22} \tag{5.295b}$$

We now consider the case when the curvature of the surface is not large. In this case:

$$\frac{\partial g_{11}}{\partial\eta},\quad \frac{\partial g_{13}}{\partial\eta},\quad \frac{\partial g_{33}}{\partial\eta}$$

are each of the O(1). Consequently, Equations 5.295a, b are, respectively:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left[\mu\sqrt{g}\,g^{22}\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right] \tag{5.296a}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left[\mu\sqrt{g}\,g^{22}\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)\right] \tag{5.296b}$$

Further, in this case the metric coefficient g_{11} and g_{33} are evaluated at the surface so that h_1 and h_3 can be taken outside of the differentiation with respect to η in Equations 5.296.

We are now in a position to write the three-dimensional boundary layer equations in general coordinates for either the case of excessive surface curvatures or of moderate surface curvatures. Below we write only equations for the case when curvatures are not large.

$$\frac{\partial\rho}{\partial t}+\frac{1}{h_2J}\left[\frac{\partial}{\partial\xi}\left(\frac{\rho h_2Ju}{h_1}\right)+\frac{\partial}{\partial\eta}(\rho Jv)+\frac{\partial}{\partial\zeta}\left(\frac{\rho h_2Jw}{h_3}\right)\right]=0 \tag{5.297}$$

where J is a function of ξ, ζ only.

$$\rho\left[\frac{\partial u}{\partial t}+\frac{u}{h_1}\frac{\partial u}{\partial\xi}+\frac{v}{h_2}\frac{\partial u}{\partial\eta}+\frac{w}{h_3}\frac{\partial u}{\partial\zeta}-\frac{g_{13}K_1}{h_1h_3}(u)^2+K_3(w)^2+K_{13}uw\right]$$

$$= \frac{-h_1}{G_2}\left(g_{33}\frac{\partial p}{\partial \xi} - g_{13}\frac{\partial p}{\partial \zeta}\right) + \frac{1}{h_2}\frac{\partial}{\partial \eta}\left(\frac{\mu}{h_2}\frac{\partial u}{\partial \eta}\right) \tag{5.298}$$

$$\rho\left[\frac{\partial w}{\partial t} + \frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{v}{h_2}\frac{\partial w}{\partial \eta} + \frac{w}{h_3}\frac{\partial w}{\partial \zeta} + K_1(u)^2 - \frac{g_{33}K_3}{h_1h_3}(w)^2 + K_{31}uw\right]$$

$$= \frac{-h_3}{G_2}\left(g_{11}\frac{\partial p}{\partial \zeta} - g_{13}\frac{\partial p}{\partial \xi}\right) + \frac{1}{h_2}\frac{\partial}{\partial \eta}\left(\frac{\mu}{h_2}\frac{\partial w}{\partial \eta}\right) \tag{5.299}$$

$$\rho\left[\frac{\partial H}{\partial t} + \frac{u}{h_1}\frac{\partial H}{\partial \xi} + \frac{v}{h_2}\frac{\partial H}{\partial \eta} + \frac{w}{h_3}\frac{\partial H}{\partial \zeta}\right] - \frac{\partial p}{\partial t}$$

$$= \frac{1}{h_2}\frac{\partial}{\partial \eta}\left[\frac{\mu}{h_2P_r}\frac{\partial H}{\partial \eta} - \frac{\mu(1-P_r)}{2h_2P_r}\frac{\partial}{\partial \eta}(u_r)^2\right] \tag{5.300}$$

$$p = \rho RT \tag{5.301}$$

where:

$$K_1 = \frac{h_3}{g_{11}}\Upsilon^3_{11}$$

$$K_3 = \frac{h_1}{g_{33}}\Upsilon^1_{33}$$

$$K_{13} = \frac{2}{h_3}\Upsilon^1_{13} - \frac{1}{h_1h_3}\frac{\partial h_1}{\partial \zeta}$$

$$K_{31} = \frac{2}{h_1}\Upsilon^3_{13} - \frac{1}{h_1h_3}\frac{\partial h_3}{\partial \xi}$$

$$H = c_pT + \frac{1}{2}(u_r)^2$$

$$(u_r)^2 = (u)^2 + (w)^2$$

$$P_r = \mu C_p/k \text{ (the Prandtl number)} \tag{5.302}$$

It must be noted that $\Upsilon^\alpha_{\beta\gamma}$ are the surface Christoffel symbols for the surface η = constant. (Refer to ME.7 for a thorough discussion on surface geometry.) The coefficients K_1 and K_3 are related with the geodesic curvatures of surface curves ζ = constant and ξ = constant, respectively, through the following equations.

Geodesic curvature of the curve ζ = constant:

$$= (k_g)_{\zeta=\text{constant}} = \frac{JK_1}{h_1h_3}$$

and, geodesic curvature of the curve ξ = constant:

$$= (k_g)_{\xi=\text{constant}} = -\frac{JK_3}{h_1h_3}$$

Cebeci et al. write:

$$g_{13} = h_1 h_3 \cos\theta$$

so that:

$$J = h_1 h_3 \sin\theta$$

where $\theta(\xi, \zeta)$ is the angle between the surface coordinates. In terms of θ the coefficients are

$$Y^3_{11} = \frac{h_1}{(h_3 \sin\theta)^2}\left[\frac{\partial}{\partial\xi}(h_3\cos\theta) - \frac{\partial h_1}{\partial\zeta}\right]$$

$$Y^1_{33} = \frac{h_3}{(h_1 \sin\theta)^2}\left[\frac{\partial}{\partial\zeta}(h_1\cos\theta) - \frac{\partial h_3}{\partial\xi}\right]$$

$$Y^1_{13} = \frac{1}{h_1 \sin^2\theta}\left[\frac{\partial h_1}{\partial\zeta} - \frac{\partial h_3}{\partial\xi}\cos\theta\right]$$

$$Y^3_{13} = \frac{1}{h_3 \sin^2\theta}\left[\frac{\partial h_3}{\partial\xi} - \frac{\partial h_1}{\partial\zeta}\sin\theta\right]$$

$$K_1 = \frac{1}{h_1 h_3 \sin^2\theta}\left[\frac{\partial}{\partial\xi}(h_3\cos\theta) - \frac{\partial h_1}{\partial\zeta}\right]$$

$$K_3 = \frac{1}{h_1 h_3 \sin^2\theta}\left[\frac{\partial}{\partial\zeta}(h_1\cos\theta) - \frac{\partial h_3}{\partial\xi}\right]$$

$$K_{13} = \frac{1}{h_1 h_3 \sin^2\theta}\left[(1 + \cos^2\theta)\frac{\partial h_1}{\partial\zeta} - 2\frac{\partial h_3}{\partial\xi}\cos\theta\right]$$

$$K_{31} = \frac{1}{h_1 h_3 \sin^2\theta}\left[(1 + \cos^2\theta)\frac{\partial h_3}{\partial\xi} - 2\frac{\partial h_1}{\partial\zeta}\cos\theta\right]$$

External Flow Equations and the Boundary Conditions

Applying the external limiting procedure to Equations 5.292 and 5.293, viz., taking the limit of each term as $\mu_\infty \to 0$ while keeping ξ, η, ζ fixed we get the Euler equations which when evaluated at the wall ($\eta = 0$), satisfy the equations:

$$\rho_e\left[\frac{\partial u_e}{\partial t} + \frac{u_e}{h_1}\frac{\partial u_e}{\partial\xi} + \frac{w_e}{h_3}\frac{\partial u_e}{\partial\zeta} - \frac{g_{13}}{h_1 h_3}K_1(u_e)^2 + K_3(w_e)^2 + K_{13}u_e w_e\right]$$
$$= \frac{-h_1}{G_2}\left(g_{33}\frac{\partial p}{\partial\xi} - g_{13}\frac{\partial p}{\partial\zeta}\right) \tag{5.303}$$

$$\rho_e\left[\frac{\partial w_e}{\partial t} + \frac{u_e}{h_1}\frac{\partial w_e}{\partial\xi} + \frac{w_e}{h_3}\frac{\partial w_e}{\partial\zeta} + K_1(u_e)^2 - \frac{g_{13}}{h_1 h_3}K_3(w_e)^2 + K_{31}u_e w_e\right]$$
$$= \frac{-h_3}{G_2}\left(g_{11}\frac{\partial p}{\partial\zeta} - g_{13}\frac{\partial p}{\partial\xi}\right) \tag{5.304}$$

$$p = \rho_e R T_e \tag{5.305}$$

where the subscript e denotes the external or outer solution. The boundary conditions for Equations 5.297–5.300 are at $\eta = 0 : u = w = 0,\ v = 0$ or $v = v_0$ (suction or injection)

$$T = T_0 \quad \text{or} \quad k\left(\frac{\partial T}{\partial \eta}\right)_0 \quad \text{prescribed} \tag{5.306}$$

as $\eta \to \infty : u \to u_e(\xi, \zeta),\ w \to w_e(\xi, \zeta),\ T \to T_e(\xi, \zeta)$

As before the metric coefficients appearing in Equations 5.303 and 5.304 have been evaluated at the surface $\eta = 0$.

The coefficient of viscosity μ appearing in Equations 5.298–5.300 is assumed to be a function of temperature. This functional form is provided by Sutherland's law, Equation 2.100a and its particular form, Equation 2.100b.

Particular Cases

Case A — The retention of g_{22} in the Equations 5.297–5.300 is optional. One can set $g_{22} = 1$ and then η becomes the actual normal distance from the surface.

Case B — For orthogonal coordinates on the surface the coefficient $g_{13} = 0$, or $\theta = \pi/2$. Only this consideration yields the three-dimensional boundary layer equations in the orthogonal coordinates. Further, it becomes obvious from Equations 5.302 that in the case of orthogonal coodinates the coefficients K_1 and K_2 become the geodesic curvatures of the curves $\zeta =$ constant and $\xi =$ constant, respectively.

Case C — Equations 5.297 through 5.300 can also be used for axially symmetric boundary layers. In this case $h_3 = r$, and as such Equation 5.296 must be written accordingly.

Case D — For the case of two-dimensional boundary layers, the velocity $w = 0$, and there is no dependence on ζ. Further, taking $g_{22} = g_{33} = 1$ and $g_{13} = 0$ and using the general equations, we get:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{5.307}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{5.308}$$

$$\rho\left(\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) - \frac{\partial p}{\partial t} = \frac{\partial}{\partial y}\left[\frac{\mu}{P_r}\frac{\partial H}{\partial y} - \frac{\mu(1 - P_r)}{2P_r}\frac{\partial}{\partial y}(u)^2\right] \tag{5.309}$$

where x is the curvilinear distance along the surface and y is the distance normal to the surface.

Note that following the procedure used in deriving Equation 5.155, the momentum integral equation for steady flow with the steady isentropic external flow, i.e., Equation 3.68b with $\chi = 0$ and $H =$ constant is

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}\left(2 + \frac{\delta^*}{\theta} - M_e^2\right) = \frac{\mu_w}{\rho_e u_e^2}\left(\frac{\partial u}{\partial y}\right)_0$$

where:

$$M_e = u_e/a_e$$

Numerical Solution of Compressible Boundary Layer Equations

It is always preferable to transform the boundary layer equations before attempting to solve the equations. Further, the continuity equation can also be identically satisfied by the introduction of a two-component vector potential discussed in Section 3.11. Let ψ and ϕ be the components of the vector potential, defined through the equations:

$$\frac{\rho h_2 J u}{h_1} = \frac{\partial \psi}{\partial \eta}, \qquad \frac{\rho h_2 J w}{h_3} = \frac{\partial \phi}{\partial \eta} \tag{5.310a}$$

Substituting Equation 5.310a in Equation 5.297, we find that:

$$\rho J v = -\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial \phi}{\partial \zeta}\right) \tag{5.310b}$$

Let us introduce a transformation of coordinates somewhat similar to that of Howarth-Dorodnitsyn (for the transformations of Howarth-Dorodnitsyn, Lees-Levy,[89] and Illingworth-Stewartson,[90] refer to Stewartson:[91]

$$\xi = \xi, \qquad \zeta = \zeta, \qquad \bar{\eta} = \int_0^{\eta} \left(\frac{u_e}{\rho_e \mu_e s}\right)^{1/2} \rho h_2 \, d\eta \tag{5.311}$$

where a subscript e as before denotes a quantity at the edge of the boundary layer, and:

$$s = \int_0^{\xi} h_1(\xi, \zeta)\, d\xi, \qquad \zeta = \text{constant}$$

We now introduce the nondimensional counterparts $f(\xi, \bar{\eta}, \zeta)$ and $g(\xi, \bar{\eta}, \zeta)$ of ψ and ϕ, respectively, through the equations:

$$u = u_e f', \qquad w = w_e g'$$

where a prime denotes differentiation with respect to $\bar{\eta}$. It is easy to see that:

$$\psi = \frac{J}{h_1} (\rho_e \mu_e u_e s)^{1/2} f(\xi, \bar{\eta}, \zeta)$$

$$\phi = \frac{J w_e}{h_3 u_e} (\rho_e \mu_e u_e s)^{1/2} g(\xi, \bar{\eta}, \zeta)$$

Introducing the transformation (Equation 5.311) in the Equations 5.298–5.300 and noting the differentiations:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u_e}{\partial \xi} f' + u_e \frac{\partial f'}{\partial \xi} + u_e f'' \frac{\partial \bar{\eta}}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = u_e f'' \frac{\partial \bar{\eta}}{\partial \eta} = u_e h_2 \rho \left(\frac{u_e}{\rho_e \mu_e s}\right)^{1/2} f''$$

etc., etc.

the equations become:*

$$(bf'')' + m_1 ff'' - m_2 (f')^2 - m_5 f' g' + m_6 f'' g - m_8 (g')^2 + m_{11} \frac{\rho_e}{\rho}$$
$$= m_{10}\left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi}\right) + m_7 \left(g' \frac{\partial f'}{\partial \zeta} - f'' \frac{\partial g}{\partial \zeta}\right) \tag{5.312}$$

* On substitution the terms containing $\partial\bar{\eta}/\partial\xi$ and $\partial\bar{\eta}/\partial\zeta$ cancel out.

$$(bg'')' + m_1 fg'' - m_4 f'g' - m_3(g')^2 + m_6 gg'' - m_9(f')^2 + m_{12}\frac{\rho_e}{\rho}$$
$$= m_{10}\left(f'\frac{\partial g'}{\partial \xi} - g''\frac{\partial f}{\partial \xi}\right) + m_7\left(g'\frac{\partial g'}{\partial \zeta} - g''\frac{\partial g}{\partial \zeta}\right) \tag{5.313}$$

For the equation of energy we introduce:

$$E = H/H_e$$

and after transformation obtain the equation:

$$(cE')' + n_1E' + n_2' = m_{10}\left(f'\frac{\partial E}{\partial \xi} - E'\frac{\partial f}{\partial \xi}\right) + m_7\left(g'\frac{\partial E}{\partial \zeta} - E'\frac{\partial g}{\partial \zeta}\right) \tag{5.314}$$

In Equations 5.312–5.314:

$$b = \frac{\rho\mu}{\rho_e\mu_e}, \qquad c = \frac{b}{P_r}$$

and all m's and n's are functions of the external quantities. The boundary conditions are

$$\text{at } \bar{\eta} = 0 : f = f' = g = g' = 0,\ E' = 0$$
$$\text{as } \bar{\eta} \to \infty : f = 1,\ g = 1,\ E = 1 \tag{5.315}$$

A numerical method of solution of Equations 5.312–5.314 under the boundary conditions (Equation 5.315) has been developed by Cebeci.[88] In this method the equations are first written as a system of first order equations as:

$$\begin{aligned} f' &= h \\ h' &= q \\ g' &= r \\ r' &= t \\ E' &= G \end{aligned} \tag{5.316a}$$

$$(bq)' + m_1 fq - m_2(h)^2 - m_5 hr + m_6 qg - m_8(r)^2 + m_{11}\frac{\rho_e}{\rho}$$
$$= m_{10}\left(h\frac{\partial h}{\partial \xi} - q\frac{\partial f}{\partial \xi}\right) + m_7\left(r\frac{\partial h}{\partial \zeta} - q\frac{\partial g}{\partial \zeta}\right) \tag{5.316b}$$

$$(bt)' + m_1 ft - m_4 hr + m_3(r)^2 + m_6 gt - m_9(h)^2 + m_{12}\frac{\rho_e}{\rho}$$
$$= m_{10}\left(h\frac{\partial r}{\partial \xi} - t\frac{\partial f}{\partial \xi}\right) + m_7\left(r\frac{\partial r}{\partial \zeta} - t\frac{\partial g}{\partial \zeta}\right) \tag{5.316c}$$

$$(cG)' + n_1G + n_2' = m_{10}\left(h\frac{\partial E}{\partial \xi} - G\frac{\partial f}{\partial \xi}\right) + m_7\left(r\frac{\partial E}{\partial \zeta} - G\frac{\partial g}{\partial \zeta}\right) \tag{5.316d}$$

Next consider a three-dimensional mesh system as shown in Figure 5.16.

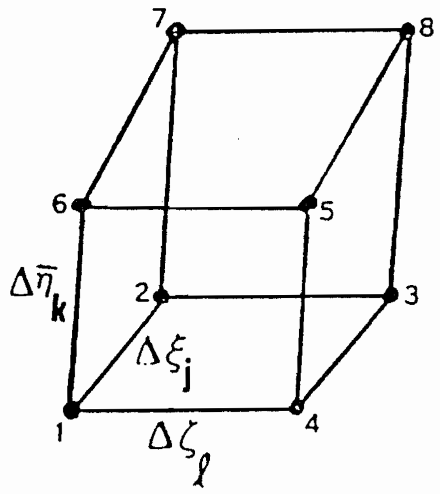


Fig. 5.16 A computational module for three-dimensional boundary layers.

Points 1 to 8 in Figure 5.16 have the following designations:

$$\begin{aligned}
\text{Point 1} &= j-1, k-1, \ell-1\\
\text{Point 2} &= j, k-1, \ell-1\\
\text{Point 3} &= j, k-1, \ell\\
\text{Point 4} &= j-1, k-1, \ell\\
\text{Point 5} &= j-1, k, \ell\\
\text{Point 6} &= j-1, k, \ell-1\\
\text{Point 7} &= j, k, \ell-1\\
\text{Point 8} &= j, k, \ell
\end{aligned}$$

The step sizes along ξ, $\bar{\eta}$, ζ, respectively, are $\Delta\xi_j$, $\Delta\bar{\eta}_k$, $\Delta\zeta_\ell$ so that:

$$\begin{aligned}
\xi_0 &= 0, \quad \xi_j = \xi_{j-1} + \Delta\xi_j, \quad j = 1, 2, \ldots, J\\
\bar{\eta}_0 &= 0, \quad \bar{\eta}_k = \bar{\eta}_{k-1} + \Delta\bar{\eta}_k, \quad k = 1, 2, \ldots, K\\
\zeta_0 &= 0, \quad \zeta_\ell = \zeta_{\ell-1} + \Delta\zeta_\ell, \quad \ell = 1, 2, \ldots, L
\end{aligned}$$

The difference equations for Equations 5.316a are written at the midpoint (ξ_j, $\eta_{k-1/2}$, ζ_ℓ) as:

$$\frac{f_k^{j,\ell} - f_{k-1}^{j,\ell}}{\Delta\bar{\eta}_k} = h_{k-1/2}^{j,\ell}$$

$$\frac{h_k^{j,\ell} - h_{k-1}^{j,\ell}}{\Delta\bar{\eta}_k} = q_{k-1/2}^{j,\ell}$$

$$\frac{g_k^{j,\ell} - g_{k-1}^{j,\ell}}{\Delta\bar{\eta}_k} = r_{k-1/2}^{j,\ell}$$

$$\frac{r_k^{j,\ell} - r_{k-1}^{j,\ell}}{\Delta\bar{\eta}_k} = t_{k-1/2}^{j,\ell}$$

where:

$$\lambda_{k-1/2}^{j,\ell} = \frac{1}{2}(\lambda_k^{j,\ell} + \lambda_{k-1}^{j,\ell})$$

Similarly, Equations (5.316b–d) are written by using an averaged form of the backward differences for all the derivatives. For example q' is written as:

$$(\bar{q}_k - \bar{q}_{k-1})/\Delta\bar{\eta}_k$$

where:

$$\bar{q}_k = \frac{1}{4}(q_k^{j,\ell} + q_k^{j,\ell-1} + q_k^{j-1,\ell-1} + q_k^{j-1,\ell})$$

while the term $m_7\, r\, \partial h/\partial\zeta$ is written as:

$$(m_7)^{j-1/2,\ell-1/2}\, \bar{r}_{k-1/2}\left(\frac{\bar{h}_\ell - \bar{h}_{\ell-1}}{\Delta\zeta_\ell}\right)$$

where:

$$(m_7)^{j-1/2,\ell-1/2} = \frac{1}{4}[(m_7)^{j,\ell} + (m_7)^{j,\ell-1} + (m_7)^{j-1,\ell} + (m_7)^{j-1,\ell-1}]$$

$$\bar{h}_\ell = \frac{1}{4}(h_k^{j,\ell} + h_{k-1}^{j,\ell} + h_k^{j-1,\ell} + h_{k-1}^{j-1,\ell})$$

If now the initial conditions at $\xi = \xi_j$ and $\zeta = \zeta_\ell$ are known, then the coupled system of equations can be solved by iteration and block elimination method for all values of k, $0 \leq k \leq K$. For the block elimination method, refer to Reference 15.

Part III: Navier-Stokes' Formulation

5.23 INCOMPRESSIBLE FLOW

The problem of numerical solution of the complete set of Navier-Stokes' equations is of much practical and theoretical importance. A study in "computational fluid dynamics" (CFD) involves a thorough knowledge in the numerical analysis; numerical aspects of solving the partial and ordinary differential equations; various topics in linear algebra, especially those pertaining to matrices; and numerical iterative methods. Besides, one should also be well versed with the programming aspects for implementing these numerical methods on large-scale digital computers. Computational fluid dynamics is a growing and fertile area of current research interests. Various monographs and books have been written on this subject. References 92–95 provide a good starting basis to this extensive and growing field of activity. In this section we shall discuss only briefly the fundamental ideas which are important in the formulation of a problem for the numerical solution.

There are two distinct ways to proceed in the formulation for numerical solution for incompressible flows. The first is to proceed with the pressure-velocity formulation, also known as the primitive variable method; and the second is the vorticity-potential or stream function method. In the next subsections we shall discuss both these formulations.

Formulation of the Problem in Primitive Variables

The equations of motion for a viscous incompressible flow have been stated in Section 3.3. The equation of continuity (Equation 3.24) and Equation 3.29 with the body force terms equal to zero are considered here. For the sake of completeness we also state the equation for the temperature T, which from Equation 3.36 is

$$\frac{DT}{Dt} = \frac{\Phi}{\rho C} + \kappa \nabla^2 T$$

where C is the constant specific heat, $\kappa = k/\rho C$ is the thermal diffusivity, and Φ is the dissipation function:

$$\Phi = 2\mu \mathbf{D} : \mathbf{D}$$

Every correct solution must satisfy Equations 3.24 and 3.29 exactly. The important point to note here is that for the case of an ideal incompressible flow there is neither an evolution equation for the determination of pressure nor an equation of state which connects the pressure with temperature. Thus, the equation of temperature noted above is of no use to determine the pressure. The present difficulties can formally be presented if we consider the thermodynamic equations:

$$e = e(\rho, s), \qquad p = \rho^2\left(\frac{\partial e}{\partial \rho}\right)_s, \qquad T = \left(\frac{\partial e}{\partial s}\right)_\rho$$

Since for an incompressible flow ρ = constant, and hence $e = e(s)$. Therefore, the pressure cannot be determined by using the second equation. The only thermodynamic relation comes from the last equation, i.e., $T = de/ds$, which implies that $T = T(s)$ or $s = s(T)$. Thus $e = e(T)$ and $de = C\, dT$, where C is the specific heat.

Now realizing that in an incompressible flow the velocity of sound is infinite, it is concluded that the pressure signals are transmitted instantaneously in all directions. Consequently, the pressure p must be satisfying an elliptic equation; and in fact it does, as is shown by taking the divergence of each term in Equation 3.29 and using Equation 3.24. Thus, referring to Equation 3.38 with $\Delta = 0$, we have either Equation 3.40, or:

$$\nabla^2 p = -\rho(\text{grad } \mathbf{u})^{\mathrm{T}} : (\text{grad } \mathbf{u}) \tag{5.317a}$$

The scheme of numerical solution is to solve Equations 3.24, 3.29, and 5.317 simultaneously. The boundary conditions must be those of no-slip at the solid surfaces and those consistent with the geometry of flow, viz., external flow, internal flow, etc. Use of the no-slip condition on a solid surface, i.e., $\mathbf{u}|_{S_b} = 0$, in Equation 3.29 yields:

$$\text{grad } p|_{S_b} = \mu \nabla^2 \mathbf{u}|_{S_b}$$

Using the vector identity:

$$\nabla^2 \mathbf{u} = \text{grad}(\text{div } \mathbf{u}) - \text{curl } \boldsymbol{\omega}$$

where $\boldsymbol{\omega}$ = curl $\mathbf{u}$, the preceding equation becomes:

$$\text{grad } p|_{S_b} = -\mu\, \textit{curl}\ \boldsymbol{\omega}|_{S_b}$$

Taking the dot product with $\mathbf{n}$, the unit outward drawn normal to S_b, we get:

$$\left.\frac{\partial p}{\partial n}\right|_{S_b} = -\mu \mathbf{n} \cdot (\text{curl } \boldsymbol{\omega})\Big|_{S_b} \tag{5.317b}$$

Equations 5.317 form a Neumann problem for the Poïsson equation. (Refer to Problems 5.17c and d.)

A direct numerical solution on the preceding lines has not been entirely successful. Usually the solution of Poisson's equation (Equation 5.317a) with the nonlinear velocity derivative terms under the normal boundary conditions (Equation 5.317b) creates most of the problems. Some ad hoc variants of the above noted exact formulation have been proposed which have been successful in some situations and are described as follows.

Ad Hoc Modifications

Let the divergence of the velocity vector **u** be denoted as:

$$\Delta = \text{div } \mathbf{u}$$

Some authors, e.g., Ghia,[96] suggest the use of the equation:

$$\nabla^2 p = -\rho(\text{grad } \mathbf{u})^T : (\text{grad } \mathbf{u}) - \frac{\partial \Delta}{\partial t}\rho \tag{5.318a}$$

Equation 5.318a makes use of $\Delta = 0$ everywhere in Equation 3.38 except for the first term having the partial time derivative. In solving Equation 5.318a at a new time step, the pressure for which $\Delta = 0$ is taken as the right pressure. In this method only Equations 3.29 and 5.318a are solved simultaneously under the proper boundary conditions. Another version of the pressure equation with $\Delta = 0$ everywhere except for the first term in Equation 3.39 is

$$\nabla^2 p = -\rho\frac{\partial \Delta}{\partial t} - \rho(1 - M_T^2)\, \mathbf{D} : \mathbf{D} \tag{5.318b}$$

where M_T is Truesdell's number.

The other ad hoc proposal is due to Chorin[97] in which the kinematic condition $\Delta = 0$ is replaced by the equation:

$$\beta\frac{\partial p}{\partial t} + \text{div } \mathbf{u} = 0 \tag{5.319}$$

where β is an artificial compressibility and:

$$p = 1/\beta$$

is an artificial equation of state. Obviously, the parameter β is very small and, in fact, is zero for an *ideal* incompressible flow. In essence, Chorin does not consider an ideal incompressible flow but allows the fluid to be slightly compressible. Incompressibility is then achieved by a dynamic relaxation in time so that $\partial p/\partial t \rightarrow 0$ and a steady state is attained. The simplicity of the method is that only Equations 3.29 and 5.319 are solved, and the problem of solving a Poisson equation for pressure under the normal derivative condition is dropped. This technique has proved useful in many flow situations, particularly those which tend to their steady states. An equation similar to Equation 5.319 has also been used by Steger, et al.[98]

Formulation of the Problem in Vorticity/Potential Form

The lack of an evolution equation for pressure and the simultaneous satisfaction of the kinematic condition (Equation 3.24) to solve Equation 3.29 were recognized quite early as the major hurdles in a pressure-velocity formulation, e.g., Fromm.[99] An alternative is to remove this consideration from the formulation of the problem. In two dimensions, the introduction of a stream function ψ satisfies the kinematic condition (Equation 3.24) identically, while a cross differentiation eliminates the pressure so that Equation 3.29 is replaced by the equation of the scalar vorticity ω. Similarly, the introduction of a vector potential **A** whose curl is the velocity vector **u** again satisfies Equation 3.24 identically. In the following two subsections we consider these two formulations separately.

Vorticity-Stream Function Formulation

In a two-dimensional flow we introduce the stream function ψ defined by the vector equation:

$$\mathbf{u} = \text{curl}\,(\mathbf{k}\psi)$$

where **k** is the unit constant vector normal to the plane of flow, and ψ is the stream function. (Refer to Section 3.11). Since the divergence of a curl is identically zero, the kinematic equation (Equation 3.24) is identically satisfied. Moreover, since the vorticity vector in two dimensions has only one component along the direction of **k**, viz.:

$$\boldsymbol{\omega} = \mathbf{k}\omega$$

the vorticity equation (Equation 3.35) is a scalar equation. Taking the curl of **u**, we obtain the scalar Poisson equation:

$$\nabla^2\psi = -\omega \tag{5.320}$$

The vorticity-stream function formulation is, therefore, comprised of Equations 3.35 and 5.320. The boundary conditions for the solution need some consideration. To fix ideas, let it be required to solve the problem in a doubly connected region R bounded by the curves C_b and C_∞, where C_b is the contour of a solid body and C_∞ is the contour of a curve enclosing the body. Let:

$$C = C_b \cup C_\infty$$

and let **n** and **t** be the unit normal and tangent vectors, respectively, on C. Further let $\mathbf{u}|_C = \mathbf{v}$, where div $\mathbf{v} = 0$. Note that by the use of the divergence theorem the admissible **v**'s are those which satisfy:

$$\int_C \mathbf{v}\cdot\mathbf{n}\,d\ell = 0$$

Now:

$$\mathbf{u}\cdot\mathbf{n}|_C = \mathbf{v}\cdot\mathbf{n} = v_n$$

implies that the tangential derivative of ψ is v_n, i.e.:

$$\left.\frac{\partial\psi}{\partial s}\right|_C = v_n$$

where s is the arc length along C. If the body is impervious, then $v_n|_{C_b} = 0$ and we have ψ = constant on C_b. Similar arguments can be used to specify ψ on other segments of C. For the no-slip condition at the contour of the solid body, we have:

$$\mathbf{u}\cdot\mathbf{t}|_{C_b} = v_t|_{C_b}$$

which implies that $\partial\psi/\partial n|_{C_b} = v_t|_{C_b}$. These are too many boundary conditions for Equation 5.320. Thus the solution algorithm must use one, and then it should be made certain that the other is satisfied.

For the solution of Equation 3.35, the boundary conditions at C_∞ are easy to specify. Usually C_∞ is chosen to be that contour enclosing the body, where $\omega = 0$. On the other hand, the specification of ω on C_b is dictated by the solution itself and is a numerical problem. Refer to Roache[92] for further details and also the recent work of Abdallah.[100,101]

Vorticity-Potential Function Formulation

The vorticity-potential formulation for computational purposes was initiated by Aziz and Hellums.[102] Refer also to Richardson and Cornish[103] for a mathematical justification of the method.

In three dimensions a single stream function is not enough to describe the flow. In this connection refer also to Section 3.11. In place of ψ, we now define a vector potential A and a scalar potential ϕ such that the velocity vector is now defined as:

$$\mathbf{u} = \text{grad } \phi + \text{curl } \mathbf{A} \tag{5.321}$$

Thus, the continuity equation is identically satisfied if:

$$\nabla^2\phi = 0 \tag{5.322}$$

in the region of the flow field. Without any loss of generality, we can also impose the condition that the vector potential A is divergence free or solenoidal, i.e.:

$$\text{div } \mathbf{A} = 0 \tag{5.323}$$

Taking the curl of Equation 5.321 and using Equation 5.323, we get:*

$$\nabla^2\mathbf{A} = -\boldsymbol{\omega} \tag{5.324}$$

which is a vector Poisson equation, and $\boldsymbol{\omega} = \text{curl } \mathbf{u}$. The vorticity equation is Equation 3.34.

The whole problem of a viscous incompressible flow in three dimensions is to solve Equations 5.322, 5.324, and 3.34 simultaneously taking due care in satisfying the appropriate boundary conditions. For simplicity let us consider a three-dimensional finite body in an unbounded fluid medium. Let the body be stationary and the fluid flowing past it have a prescribed free-stream velocity. Referring to Figure 5.17, we denote the fluid domain by Ω with boundary S, where:

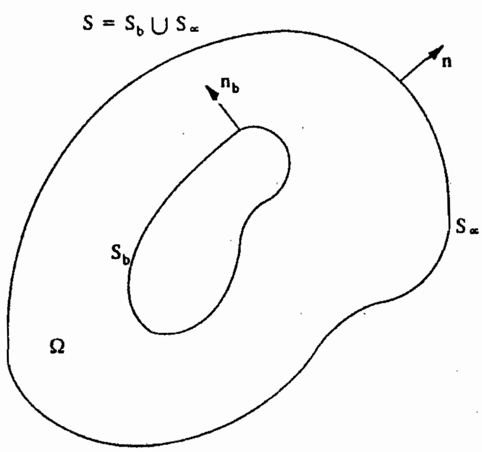


Fig. 5.17 Finite closed body in three-dimensions enclosed by an outer surface.

* The Laplacian of a vector must always be interpreted as $\nabla^2\mathbf{A} = \text{div}(\text{grad } \mathbf{A})$.

$$S = S_b \cup S_\infty$$

Since div $\mathbf{u} = 0$, from the Gauss divergence theorem:

$$\int_S \mathbf{u} \cdot \mathbf{n}\, dS = \int_S \mathbf{v} \cdot \mathbf{n}\, dS = 0 \qquad \textbf{(5.325a)}$$

where $\mathbf{u}|_S = \mathbf{v}$, and $\mathbf{n}$ is the unit normal vector on the boundary S. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be the mutually orthogonal unit tangent vectors in the surface. Thus, from Equation 5.321:

$$\frac{\partial \phi}{\partial n} + (\text{curl } \mathbf{A}) \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} \quad \text{on } S$$

Taking:

$$(\text{curl } \mathbf{A}) \cdot \mathbf{n}|_S = 0 \qquad \textbf{(5.325b)}$$

we first determine the scalar potential ϕ by the equations:

$$\nabla^2 \phi = 0 \quad \text{in } \Omega \qquad \textbf{(5.326)}$$

$$\frac{\partial \phi}{\partial n} = \mathbf{v} \cdot \mathbf{n} \quad \text{on } S$$

Thus, ϕ is determined to within an additive constant. The condition (Equation 5.325b) plays an important role in the determination of the boundary conditions for $\mathbf{A}$. First of all, it is identically satisfied if the tangential components vanish, i.e., if:

$$\mathbf{A} \cdot \mathbf{t}_1 = \mathbf{A} \cdot \mathbf{t}_2 = 0 \qquad \textbf{(5.327a)}$$

and then from the solenoidal equation (Equation 5.323) the normal component satisfies the derivative condition implicit in div $\mathbf{A}$. In other words:

$$\text{div } \mathbf{A} = 0 \quad \text{on } S \qquad \textbf{(5.327b)}$$

The other case in which Equation 5.325b is *implicitly* satisfied is when:

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } S \qquad \textbf{(5.327c)}$$

$$(\text{curl } \mathbf{A}) \cdot \mathbf{t}_1 = \mathbf{v} \cdot \mathbf{t}_1 - (\text{grad } \phi) \cdot \mathbf{t}_1 \quad \text{on } S \qquad \textbf{(5.327d)}$$

$$(\text{curl } \mathbf{A}) \cdot \mathbf{t}_2 = \mathbf{v} \cdot \mathbf{t}_2 - (\text{grad } \phi) \cdot \mathbf{t}_2 \quad \text{on } S \qquad \textbf{(5.327e)}$$

One can easily verify these two cases by choosing a local orthogonal coordinate system, with the right-handed coordinate axes having $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{n}$ as unit vectors.

It has been shown mathematically by Richardson and Cornish,[103] that if the domain Ω is simply connected then the appropriate boundary conditions for $\mathbf{A}$ are given by Equations 5.327a, b; while if it is multiply connected (has holes), then the boundary conditions are given by Equations 5.327c–e. Note that all the boundary conditions stated above take due care of the no-slip condition wherever it is needed.

Integro-Differential Formulation

A method similar to that discussed above, but which does not explicitly require the solutions of Equations 5.322 and 5.324 was developed by Wu et al., e.g., Reference 104. The basis of the method is compiled from the vorticity equation (Equation 3.34) in the form:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \text{curl}\,(\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega} \tag{5.328}$$

and the vector Poisson equation:

$$\nabla^2 \mathbf{u} = -\text{curl}\,\boldsymbol{\omega} \tag{5.329}$$

where the last equation is a result of using the solenoidal condition div $\mathbf{u} = 0$ in the identity:

$$\nabla^2 \mathbf{u} = -\text{grad}\,(\text{div}\,\mathbf{u}) - \text{curl}\,\boldsymbol{\omega}$$

Let $\boldsymbol{\xi}$ be a variable point and $\mathbf{x}$ be the point at which the velocity is sought through Equation 5.329. Defining:

$$r = |\boldsymbol{\xi} - \mathbf{x}|$$

and using the integral representations derived in Equations M5.16, 17 and applied to Equation 5.329, we have:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega} \frac{\text{curl}\,\boldsymbol{\omega}(\boldsymbol{\xi})}{r}\, dv(\boldsymbol{\xi}) + \frac{1}{4\pi}\int_{S}\left[\frac{1}{r}\frac{\partial \mathbf{u}(\boldsymbol{\xi})}{\partial n} - \mathbf{u}(\boldsymbol{\xi})\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS(\boldsymbol{\xi}) \tag{5.330}$$

in three dimensions, and:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \frac{1}{2\pi}\int_{R} \ell n\left(\frac{1}{r}\right) \text{curl}\,\boldsymbol{\omega}(\boldsymbol{\xi})\, dR(\boldsymbol{\xi}) \\ &+ \frac{1}{2\pi}\int_{\Gamma}\left[\ell n\left(\frac{1}{r}\right)\frac{\partial \mathbf{u}(\boldsymbol{\xi})}{\partial n} - \mathbf{u}(\boldsymbol{\xi})\frac{\partial}{\partial n}\,\ell n\left(\frac{1}{r}\right)\right] d\Gamma(\boldsymbol{\xi}) \end{aligned} \tag{5.331}$$

in two dimensions. In Equation 5.330 Ω is a three-dimensional region bounded by the surface S, while in Equation 5.331 R is a two-dimensional region bounded by a curve Γ. In either case the normal derivative is given by:

$$\frac{\partial}{\partial n} = \mathbf{n} \cdot \text{grad} \tag{5.332a}$$

where the grad operator has partial derivatives with respect to the coordinates at variable point $\boldsymbol{\xi}$. Thus:

$$\text{grad}_{\xi} r = \frac{(\boldsymbol{\xi} - \mathbf{x})}{r} \tag{5.332b}$$

and:

$$\begin{aligned} \frac{\partial \mathbf{u}(\boldsymbol{\xi})}{\partial n} &= (\mathbf{n} \cdot \text{grad})\mathbf{u}(\boldsymbol{\xi}) \\ &= (\text{grad}_{\xi}\mathbf{u}) \cdot \mathbf{n} \end{aligned} \tag{5.332c}$$

For further simplifications of Equation 5.330 or 5.331 we need the following result. First, it can be shown that:

$$(\text{grad } \mathbf{u}) \cdot \mathbf{n} = \boldsymbol{\omega} \times \mathbf{n} + (\mathbf{n} \times \text{grad}) \times \mathbf{u} + \mathbf{n}(\text{div } \mathbf{u}) \qquad \textbf{(5.332d)}$$

Second, a corollary of the Gauss theorem states that (cf. Eq. M2.4):

$$\int_\Omega \text{curl } \mathbf{P} \, dv = -\int_S (\mathbf{P} \times \mathbf{n}) \, dS \qquad \textbf{(5.332e)}$$

Thus writing:

$$\mathbf{P} = \frac{\boldsymbol{\omega}}{r}$$

in Equation 5.332e and using Equation 5.332b, we get:

$$\int_\Omega \frac{\text{curl}_\xi \boldsymbol{\omega}}{r} \, dv = -\int_S \frac{\boldsymbol{\omega} \times \mathbf{n}}{r} \, dS + \int_\Omega \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^3} \, dv \qquad \textbf{(5.332f)}$$

Similarly writing:

$$\mathbf{P} = \boldsymbol{\omega} \ell n\left(\frac{1}{r}\right)$$

in Equation 5.332e, we get:

$$\int_R \ell n\left(\frac{1}{r}\right) \text{curl}_\xi \boldsymbol{\omega} \, dR = -\int_\Gamma (\boldsymbol{\omega} \times \mathbf{n}) \ell n\left(\frac{1}{r}\right) d\Gamma + \int_R \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^2} \, dR \qquad \textbf{(5.332g)}$$

Using Equations 5.332b–d and 5.332f in Equation 5.330, we get:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^3} \, dv(\boldsymbol{\xi}) + \frac{1}{4\pi} \int_S \left[\frac{(n \times \text{grad}_\xi) \times \mathbf{u}}{r} + \frac{\mathbf{n} \cdot (\boldsymbol{\xi} - \mathbf{x})}{r^3} \mathbf{u}(\boldsymbol{\xi}) \right] dS \qquad \textbf{(5.333)}$$

Similarly, using Equations 5.332b–d and 5.332g in Equation 5.331, we get:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{2\pi} \int_R \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^2} \, dR + \frac{1}{2\pi} \int_\Gamma \left[(n \times \text{grad}_\xi) \times \mathbf{u} \ell n\left(\frac{1}{r}\right) + \frac{\mathbf{n} \cdot (\boldsymbol{\xi} - \mathbf{x})}{r^2} \mathbf{u}(\boldsymbol{\xi}) \right] d\Gamma \qquad \textbf{(5.334)}$$

Application of the Boundary Conditions

For viscous flow we apply the no-slip condition:

$$\mathbf{u}|_{S_b} = 0$$

and as before choose a local orthogonal coordinate frame on S_b. The no-slip condition then yields:

$$(\mathbf{n} \times \text{grad}) \times \mathbf{u}|_{S_b} = 0$$

For interior flow Equations 5.333 and 5.334 can be written in the combined form:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{A}\int_{\Omega} \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^d}\, d\nu \qquad (5.335)$$

where for three dimensions $A = 4\pi$, $d = 3$, and for two dimensions $A = 2\pi$, $d = 2$ with $\Omega = R$.

For flow exterior to a solid surface with the free-stream velocity $\mathbf{V}_\infty$ at infinity, we consider S as the union of the body surface S_b and a large sphere/circle as S_∞. In this case, since:

$$\lim_{r\to\infty} \int_{S_\infty} \frac{(\mathbf{n} \times \text{grad}_\xi) \times \mathbf{u}}{r}\, dS = 0$$

we obtain:

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_\infty + \frac{1}{A}\int_{\Omega} \frac{(\boldsymbol{\xi} - \mathbf{x}) \times \boldsymbol{\omega}}{r^d}\, d\nu \qquad (5.336)$$

The substitution of either Equation 5.335 or 5.336 in Equation 5.328 for $\mathbf{u}$ gives rise to the integro-differential formulation.

Basic Computational Aspects

The mathematical models for the description of incompressible visous flows as discussed in the preceding subsections can only be solved by numerical methods. In this regard the reader is referred to the specific methods in the quoted references and to ME.8.

5.24 COMPRESSIBLE FLOW

In this section, we shall be concerned with those problems of compressible flows in which both the viscosity and thermal conductivity are of importance. Here we have discussed the problem of obtaining the full Navier-Stokes solution from the point of view of theoretical and numerical formulations. For the purpose of solving the Navier-Stokes equations for compressible viscous** flow past bodies of arbitrary shapes, we consider the nondimensional equations of motion in the invariant tensor form.

A dimensional quantity is denoted by a superscript * and the value of a quantity at free stream by a subscript ∞. The equations have been nondimensionalized by using the following free-stream values:

L^* = all lengths

U^*_∞ = the velocity vector

ρ^*_∞ = density

$\rho^*_\infty U^{*2}_\infty$ = pressure and energy per unit volume

T^*_∞ = temperature

μ^*_∞ = viscosity

$C^*_{p\infty}$ = specific heat at constant pressure

** The present formulation is for an ideal gas (also called thermally perfect).

The nondimensional form of the conservation Equations 3.8, 3.10, and 3.22a are

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \mathbf{u}) = 0 \tag{5.337}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \text{div}\,\mathbf{X} = 0 \tag{5.338}$$

$$\frac{\partial}{\partial t} E_t + \text{div}\,\mathbf{b} = 0 \tag{5.339}$$

where:

$$\mathbf{X} = \rho \mathbf{u}\mathbf{u} + p\mathbf{I} - \epsilon \boldsymbol{\sigma}$$

$$\boldsymbol{\sigma} = \lambda \Delta \mathbf{I} + \mu \mathbf{d}$$

$$\Delta = \text{div}\,\mathbf{u}$$

$$\mathbf{d} = \text{def}\,\mathbf{u} = 2\mathrm{D}$$

$$\mathbf{b} = (E_t + p)\mathbf{u} - \epsilon \boldsymbol{\sigma} \cdot \mathbf{u} - \epsilon S \mu \text{ grad } \mathbf{T}$$

$$e_t = e + \tfrac{1}{2}|\mathbf{u}|^2, \quad E_t = \rho e_t$$

$$\epsilon = 1/R_e = \mu_\infty^* / \rho_\infty^* U_\infty^* L^*$$

$$S = T_\infty^* C_{p\infty}^* C_p / P_r U_\infty^{*2}$$

$$P_r = \mu^* C_p^* / k^*$$

The set of Equations 5.337–5.339 form a closed system of equations when the following additional equations are appended with them:

1. The equation of state:

$$p = C_1(\gamma - 1)\rho T C_p / \gamma$$

where:

$$C_1 = C_{p\infty}^* T_\infty^* / U_\infty^{*2}$$

C_p is the nondimensional specific heat at constant pressure and γ is the ratio of specific heats. Both C_p and γ are functions of the temperature T. Note that here λ, μ and C_p are nondimensional.

2. Sutherland's viscosity law:

$$\mu = (1 + \beta_1)T^{3/2}/(T + \beta_1)$$

where:

$$\beta_1 = \beta / T_\infty^*, \qquad \beta = 110.33^\circ\,\mathrm{K}$$

3. Stokes' law:

$$3\lambda + 2\mu = 0$$

4. Eucken's formula:

$$P_r(T) = 4\gamma/(9\gamma - 5)$$

Determination of Temperature

On the mole basis, the specific heat at constant pressure of an ideal gas in the range of 300 to 1000 K as given by Zucrow[105] is

$$\bar{C}_p^* = \bar{R}(A_0 + A_1T^* + A_2T^{*2} + A_3T^{*3} + A_4T^{*4})$$

where A_i are constants having the values: $A_0 = 3.65359$, $A_1 = -1.33763 \times 10^{-3}/\text{K}$, $A_2 = 3.29421 \times 10^{-6}/(\text{K})^2$, $A_3 = -1.91142 \times 10^{-9}/(\text{K})^3$, $A_4 = 0.275462 \times 10^{-12}/(\text{K})^4$, and $\bar{R}$ is the universal gas constant; $\bar{R} = 8314.3$ J/kg.mol/K. Introducing:

$$C_p^* = \frac{\bar{C}_p^*}{\bar{M}}, \qquad R = \frac{\bar{R}}{\bar{M}}$$

where $\bar{M}$ is the molecular weight, and on nondimensionalization:

$$T = \frac{T^*}{T_\infty^*}, \qquad C_p = \frac{C_p^*}{C_{p\infty}^*}$$

we obtain:

$$C_p = \frac{f(T)}{f(1)}$$

where:

$$f(T) = A_0 + (A_1T_\infty^*)T + (A_2T_\infty^{*2})T^2 + (A_3T_\infty^{*3})T^3 + (A_4T_\infty^{*4})T^4$$

Note also that:

$$C_{p\infty}^* = Rf(1)$$

Using the equations:

$$\gamma = C_p^*/C_v^*, \qquad C_p^* - C_v^* = R$$

and the previous equations, we get:

$$\gamma(T) = \frac{f(T)}{f(T) - 1}$$

The determination of temperature is governed by the thermodynamic equation of an ideal gas, which is

$$de^* = C_v^*\, dT^*$$

On nondimensionalization it becomes:

$$de = \frac{T_\infty^* C_{p\infty}^*}{U_\infty^{*2}} \frac{C_p}{\gamma} dT$$

Thus:

$$de = C_2[f(T) - 1]\, dT$$

where:

$$C_2 = C_3/f(1)$$

$$C_3 = \frac{RT_\infty^*}{U_\infty^{*2}} f(1)$$

Substituting $f(T)$ in the above equation, we get $e(T)$ on integration. Using $e(T)$ in the definition of e_t, we get:

$$(A_0 - 1)T + \tfrac{1}{2}(A_1T_\infty^*)T^2 + \tfrac{1}{3}(A_2T_\infty^{*2})T^3 + \tfrac{1}{4}(A_3T_\infty^{*3})T^4 + \tfrac{1}{5}(A_4T_\infty^{*4})T^5$$
$$= \frac{1}{C_2}\left(e_t - \tfrac{1}{2}|\mathbf{u}|^2\right) \qquad \textbf{(5.340)}$$

as the equation for the determination of temperature. For a thermally and calorically perfect gas:

$$f(T) = f(1) = A_0, \qquad \gamma = A_0/(A_0 - 1), \qquad C_p = 1$$

$$a_\infty^{*2} = \gamma RT_\infty^* = C_p^*(\gamma - 1)T_\infty^*$$

Thus, for a thermally and calorically perfect gas:

$$p = \frac{\rho T}{\gamma M_\infty^2} \qquad \textbf{(5.341a)}$$

$$T = \gamma(\gamma - 1)M_\infty^2\left(e_t - \tfrac{1}{2}|\mathbf{u}|^2\right) \qquad \textbf{(5.341b)}$$

where M_∞ is the free-stream Mach number.

For a numerical solution of Equations 5.337–5.339 with reference to a general coordinate system, we have to transform these equations. Transformation to general coordinates has already been discussed in Section 3.10, and the transformed equations are Equations 3.163–3.165. The nondimensional counterpart of the set of Equations 3.163–3.165 in time-dependent coordinates is

$$\frac{\partial \sigma}{\partial \tau} + \frac{\partial}{\partial x^j}(\sigma v^j) = 0 \qquad \textbf{(5.342)}$$

$$\frac{\partial}{\partial \tau}(\sigma \mathbf{u}) + \frac{\partial}{\partial x^k}(X^{ik}\mathbf{a}_i) = 0 \qquad \textbf{(5.343)}$$

$$\frac{\partial}{\partial \tau} E + \frac{\partial Y^k}{\partial x^k} = 0 \tag{5.344}$$

where:

$$\sigma = \sqrt{g}\, \rho, \qquad E = \sqrt{g}\, E_t, \qquad E_t = \rho e_t$$

$$X^{ik} = \sigma u^i v^k + \sqrt{g}\, p g^{ik} - \epsilon \sqrt{g}\, \sigma^{ik}$$

$$\sigma^{ik} = \lambda \Delta g^{ik} + 2\mu D^{ik}$$

$$2D^{ik} = u^i_{,s} g^{sk} + u^k_{,r} g^{ri}$$

$$u^i_{,s} = \frac{\partial u^i}{\partial x^s} + \Gamma^i_{s\ell} u^\ell, \quad \text{etc.},$$

$$Y^k = E v^k + \sqrt{g}\, p u^k - \epsilon \sqrt{g}\, g_{ir} \sigma^{kj} u^r - \epsilon S \mu \sqrt{g}\, g^{kj} \frac{\partial T}{\partial x^j} \tag{5.345}$$

Further:

$$g = \det(g_{ij})$$

v is the relative velocity, and **u** is the absolute velocity. Both are related as:

$$\mathbf{u} = \mathbf{v} - \mathbf{w}$$

As has been noted in Chapter 3, the contravariant components of **w** are $\partial x^i/\partial t$. If x_m $(m = 1, 2, 3)$ are the rectangular Cartesian coordinates then:

$$\frac{\partial x_m}{\partial \tau} + w^j \frac{\partial x_m}{\partial x^j} = 0 \tag{5.346}$$

By using the covariant and the contravariant base vectors $\mathbf{a}_i$ and $\mathbf{a}^i$, respectively, we can express the contravariant components of the velocity vector **u** in terms of the Cartesian components and vice versa. For example:

$$\mathbf{u} = u^i \mathbf{a}_i$$

and thus:

$$\begin{aligned} u^i &= \mathbf{u} \cdot \mathbf{a}^i \\ &= \mathbf{u} \cdot (\text{grad } x^i) \end{aligned}$$

Writing the Cartesian components of **u** as u, v, w, we have:

$$u^i = u \frac{\partial x^i}{\partial x_1} + v \frac{\partial x^i}{\partial x_2} + w \frac{\partial x^i}{\partial x_3}, \qquad i = 1, 2, 3 \tag{5.347a}$$

Similarly:

$$u = u^i \frac{\partial x_1}{\partial x^i}, \qquad v = u^i \frac{\partial x_2}{\partial x^i}, \qquad w = u^i \frac{\partial x_3}{\partial x^i} \tag{5.347b}$$

The contravariant components D^{ik} of the rate-of-strain tensor in terms of the Cartesian velocity components are

$$D^{ik} = \frac{1}{2}\left(\frac{\partial x^i}{\partial x_m} g^{pk} + \frac{\partial x^k}{\partial x_m} g^{pi}\right)\frac{\partial u_m}{\partial x^p} \tag{5.347c}$$

where u_m are the Cartesian components and x_m, the Cartesian coordinates. On the other hand, in terms of the contravariant velocity components u^i:

$$D^{ik} = \frac{1}{2}(u^i_{,s} g^{sk} + u^k_{,r} g^{ri}) \tag{5.347d}$$

where:

$$u^i_{,s} = \frac{\partial u^i}{\partial x^s} + \Gamma^i_{sp} u^p$$

Another set of the Navier-Stokes equations for numerical solution can be written in which only the terms of Equation 5.343 are to be rearranged. This set is

$$\frac{\partial \sigma}{\partial \tau} + \frac{\partial}{\partial x^j}(\sigma v^j) = 0 \tag{5.348}$$

$$\frac{\partial}{\partial \tau}(\sigma \mathbf{u}) + \frac{\partial}{\partial x^k}(\sigma \mathbf{u} v^k + \sqrt{g}\, p \mathbf{a}^k) = \epsilon \frac{\partial}{\partial x^k}(\sqrt{g}\, \sigma^{ik} \mathbf{a}_i) \tag{5.349}$$

$$\frac{\partial}{\partial \tau} E + \frac{\partial Y^k}{\partial x^k} = 0 \tag{5.350}$$

Either the set of Equations 5.342–5.344 or 5.348–5.350 can be used for numerical purposes. From Equation 5.343 or 5.349 three scalar equations in three dimensions or two scalar equations in two dimensions can be obtained by writing:

$$\mathbf{u} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

$$\mathbf{a}_i = \mathbf{i}\frac{\partial x_1}{\partial x^i} + \mathbf{j}\frac{\partial x_2}{\partial x^i} + \mathbf{k}\frac{\partial x_3}{\partial x^i}$$

and:

$$\mathbf{a}^k = \text{grad } x^k$$

so that:

$$\mathbf{a}^k = \mathbf{i}\frac{\partial x^k}{\partial x_1} + \mathbf{j}\frac{\partial x^k}{\partial x_2} + \mathbf{k}\frac{\partial x^k}{\partial x_3}$$

If the coordinates are steady, then $\mathbf{w} = 0$, $\mathbf{v} = \mathbf{u}$, and $\mathbf{Y} = \mathbf{b}$.

Equations 5.342–5.344 or 5.348–5.350 have to be solved under the prescribed initial and boundary conditions. The boundary conditions for the Navier-Stokes solution are comprised of the no-slip condition at the surface of the body and of the free-stream condition at a large distance of the body including the condition at a downstream station far from the body. The boundary conditions can exactly be satisfied by choosing the coordinates x^i as body fitted, viz., those coordinates which are natural to the body and to the far-distant boundaries. We shall discuss this topic in Section 5.26.

It must be noted that the set of Equations 5.342–5.344 and 5.348–5.350 are in the strong conservation law form. The basis of the conservation law form is, of course, the physical conservation principle, but the motivation here is from a numerical viewpoint. Numerical solutions of the Navier-Stokes equations have also been obtained successfully when not *all* the equations are in the strong conservation law form, e.g., Warsi et al.[106]

5.25 HYPERBOLIC EQUATIONS AND CONSERVATION LAWS

First, the importance and use of conservation laws in the numerical computations can be appreciated by using the following simple example. Consider the conservation equation:

$$\operatorname{div} \mathbf{u} = Q$$

in a plane. Using Gauss' divergence theorem, we have:

$$\int_C \mathbf{u} \cdot \mathbf{n}\, ds = \int_R Q\, dv$$

First, we consider the case of Cartesian coordinates in which the equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Q \tag{5.351a}$$

where u and v are the Cartesian components of $\mathbf{u}$. We consider a rectangular mesh system with positive direction of traverse and the unit normals as shown in Figure 5.18(a).

Thus:

$$\begin{aligned}\int_C \mathbf{u} \cdot \mathbf{n}\, ds &= \int_{ab} \mathbf{u} \cdot \mathbf{n}\, ds + \int_{bc} \mathbf{u} \cdot \mathbf{n}\, ds + \int_{cd} \mathbf{u} \cdot \mathbf{n}\, ds + \int_{da} \mathbf{u} \cdot \mathbf{n}\, ds \\ &= -\int_{ab} v\, dx - \int_{bc} u\, dy + \int_{cd} v\, dx + \int_{da} u\, dy\end{aligned}$$

Considering the mesh to be small, we approximate the line integrals by the average values at the centers shown by heavy dots in Figure 5.18(a). Thus:

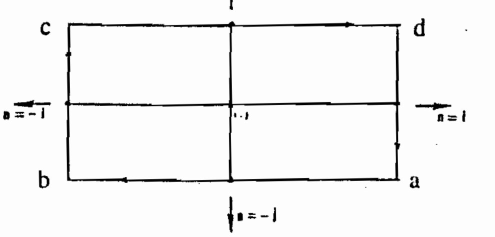


Fig. 5.18(a) Rectangular mesh in the xy-plane.

$$-v_{i,j-1/2}\,\Delta x - u_{i-1/2}\,\Delta y + v_{i,j+1/2}\,\Delta x + u_{i+1/2,j}\,\Delta y = Q_{i,j}\,\Delta x\,\Delta y$$

Replacing the values, e.g., $v_{i,j-1/2}$ by $(v_{i,j} + v_{i,j-1})/2$, etc. we get:

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = Q_{i,j} \tag{5.351b}$$

On the other hand, the central difference approximation at the mesh center (i, j) of Equation 5.351a is exactly the same as Equation 5.351b. Thus, the conservation law is preserved in the numerical solution of Equation 5.351a.

We now consider the case of general coordinates in which Equation 5.351a becomes:

$$\frac{\partial}{\partial x^1}(\sqrt{g}\,u^1) + \frac{\partial}{\partial x^2}(\sqrt{g}\,u^2) = \sqrt{g}\,Q \tag{5.352a}$$

$$\int_C (u^1 n_1 + u^2 n_2)\,ds = \int_R f^k_{,k}\,dv = \int_R \sqrt{g}\,Q\,dv \tag{5.352b}$$

For a positive system shown in Figure 5.18(b):

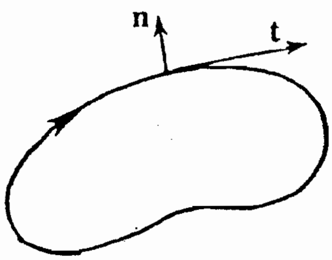


Fig. 5.18(b) Orientation of positive normal and the positive traverse on a closed curve.

the components of **n** are

$$n_1 = -\sqrt{g}\,\frac{dx^2}{ds}, \qquad n_2 = \sqrt{g}\,\frac{dx^1}{ds}$$

In the x^1x^2-plane we again erect a mesh system of the form shown in Figure 5.18(a). Noting that:

$$\int_C \mathbf{u}\cdot\mathbf{n}\,ds = \int_C (\sqrt{g}\,u^2\,dx^1 - \sqrt{g}\,u^1\,dx^2)$$

and on ab: $dx^2 = 0$, on bc: $dx^1 = 0$, etc., we obtain the difference approximation of Equation 5.352b as:

$$\frac{(\sqrt{g}\,u^1)_{i+1,j} - (\sqrt{g}\,u^1)_{i-1,j}}{2\Delta x^1} + \frac{(\sqrt{g}\,u^2)_{i,j+1} - (\sqrt{g}\,u^2)_{i,j-1}}{2\Delta x^2} = (\sqrt{g}\,Q)_{i,j} \tag{5.352c}$$

Since the central difference approximation of Equation 5.352a is the same as Equation 5.352c, we conclude that in the case of general coordinates the conservation law is preserved if the differential equation is written in the form of Equation 5.352a. In contrast, if we open the differentiation in Equation 5.352a to have:

$$\sqrt{g}\left(\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2}\right) + \frac{1}{2\sqrt{g}}\left(u^1\frac{\partial g}{\partial x^1} + u^2\frac{\partial g}{\partial x^2}\right) = \sqrt{g}\, Q$$

then its central derivative approximation is *not* the same as Equation 5.352c and thus the conservation principle is violated. From this example we conclude that in the case of general coordinates the conservation law form of the equations preserves the physical conservation principles in case the equations are solved by the numerical finite differencing.

The other important concept associated with the conservation law equations is that of a *weak* or *generalized* solution. A weak solution is that solution which has jump discontinuities in certain regions of space or of space-time continuum. Thus, a weak solution is only piecewise smooth, and smooth solution in the ordinary sense does not exist for all space-time points. However, it must be kept in mind that although the solution has jump discontinuities, the conservation principle must invariably be satisfied. The reason for this is that the basic conservation laws are always stated and obtained in the integral form, Equation 2.53. Therefore, on both sides of the interfaces at which the jump discontinuities occur the conserved quantitites have the same values. An important physical example of weak solutions is afforded by the gas dynamic shock waves. If only the smooth part of the solution is desired, then jump conditions must be supplied along with initial data for the solution of conservation law equations. Fortunately because of the validity of the conservation laws across a jump discontinuity we can devise a unified approach to the solution of the conservation law equations provided that we allow the existence of generalized derivatives in place of the ordinary derivatives, wherever they are needed.

To illustrate this point, we consider the one-dimensional unsteady motion of an inviscid gas along the x-axis. Let ρ, u, p, e be the density, velocity, pressure, and internal energy per unit mass, respectively. All these quantities are functions of x and time t. The momentum and the total energy per unit volume are

$$m = \rho u, \qquad \mathbf{m} = \mathbf{i}\rho u, \qquad E_t = \rho e_t$$

$$E_t = \rho e + \tfrac{1}{2}\rho u^2$$

where $\mathbf{i}$ is the unit vector along the direction of x. We now consider a material volume bounded by $x = a(t)$, $x = b(t)$ with the positive unit vectors drawn outward from the surface, as is shown in Figure 5.19. The total mass, momentum vector, and energy contained in $V(t)$, respectively, are

$$M = \int_a^b \rho\, dx, \qquad \mathbf{P} = \int_a^b \mathbf{m}\, dx, \qquad E = \int_a^b E_t\, dx \tag{5.353a}$$

Also since $V(t)$ is a material volume, then we have:

$$\frac{da}{dt} = u(a, t), \qquad \frac{db}{dt} = u(b, t) \tag{5.353b}$$

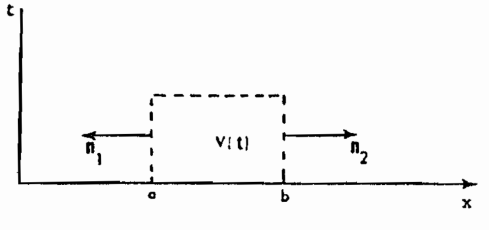


Fig. 5.19 Material volume (shown dotted) in a one-dimensional inviscid flow.

According to the basic principles of mechanics:

$$\frac{dM}{dt} = 0$$

$$\frac{d\mathbf{P}}{dt} = \text{sum of the pressure forces on the part of the fluid between } x = a, x = b$$

$$= -p(a, t)\mathbf{n}_1 - p(b, t)\mathbf{n}_2$$

$$\frac{dE}{dt} = \text{rate of work done by the pressure forces}$$

$$= (-p(a, t)\mathbf{n}_1) \cdot \mathbf{i}u(a, t) + (-p(b, t)\mathbf{n}_2) \cdot \mathbf{i}u(b, t) \tag{5.353c}$$

However, $\mathbf{n}_1 = -\mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i}$; hence:

$$\frac{dP}{dt} + p(b, t) - p(a, t) = 0 \tag{5.353d}$$

$$\frac{dE}{dt} + p(b, t)u(b, t) - p(a, t)u(a, t) = 0 \tag{3.353e}$$

Using Equation 5.353a in Equations 5.353c–e along with the Reynolds transport theorem (Equation 1.32c) and Equations 5.353b, we can write:

$$\int_{a(t)}^{b(t)} \frac{\partial A}{\partial t}\, dx + (Au + B)\Big|_{a(t)}^{b(t)} = 0 \tag{5.354}$$

which is the one-dimensional version of the integral conservation law (Equation 2.53), where A and B are the surrogate variables such that the pair (A, B) has the values:

$$(\rho, 0), \qquad (m, p), \qquad (E_t, \frac{mp}{\rho})$$

If $Au + B$ is at least C^1 spatially differentiable, then Equation 5.354 can be written as:

$$\int_a^b \left[\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au + B)\right] dx = 0$$

If further, the integrand is continuous, we have:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au + B) = 0 \tag{5.355}$$

Thus the basic integral conservation law for gas dynamic equations is Equation 5.354, while the basic differential conservation law is Equation 5.355. Substituting the forms of A and the corresponding B, we can write Equation 5.355 in vector form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{5.356a}$$

where:

$$\mathbf{U} = \begin{bmatrix} \rho \\ m \\ E_t \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} m \\ p + \dfrac{m^2}{\rho} \\ (E_t + p)\dfrac{m}{\rho} \end{bmatrix} \tag{5.356b}$$

Note that the three equations obtained from Equations 5.356 have to be supplemented with an equation of state:

$$p = p(e, \rho)$$

which can also be written as:

$$p = p\left[\frac{1}{\rho}\left(E_t - \frac{1}{2}\frac{m^2}{\rho}\right), \rho\right] \tag{5.357}$$

In particular, for a thermally and calorically perfect gas:

$$\begin{aligned} p &= (\gamma - 1)\rho e \\ &= (\gamma - 1)\left(E_t - \frac{1}{2}\frac{m^2}{\rho}\right) \end{aligned}$$

In Equation 5.357, p has been expressed as a function of the variables ρ, m, and E_t; and these three quantities form the column vector $\mathbf{U}$. Thus, in essence:

$$\mathbf{F} = \mathbf{F}(\mathbf{U}) \tag{5.358}$$

Proceeding along the same lines, we can also write the equations of motion of an inviscid gas in two dimensions by introducing:

$$n = \rho v$$

and then:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \tag{5.359}$$

where now:

$$\mathbf{U} = \begin{bmatrix} \rho \\ m \\ n \\ E_t \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} m \\ p + \dfrac{m^2}{\rho} \\ \dfrac{mn}{\rho} \\ (E_t + p)\dfrac{m}{\rho} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} n \\ \dfrac{mn}{\rho} \\ p + \dfrac{n^2}{\rho} \\ (E_t + p)\dfrac{n}{\rho} \end{bmatrix}$$

Also because of the equation of state (Equation 5.357):

$$\mathbf{F} = \mathbf{F}(\mathbf{U}), \quad \mathbf{G} = \mathbf{G}(\mathbf{U}) \tag{5.360}$$

A system of equations in any number of dependent and independent variables of the form of Equations 5.356 or 5.359 is called a *system of conservation laws*. The contributions of Lax,[107,108] both to the mathematical and the computational aspects of the system of conservation equations, have stimulated much interest and research in such equations.

The forms of the flux vectors in Equations 5.358 and 5.360 can be utilized to write Equation 5.356 and 5.359 for the case of smooth solutions, as a system of quasilinear equations, respectively:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \cdot \frac{\partial \mathbf{U}}{\partial x} = 0 \tag{5.361a}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \cdot \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \cdot \frac{\partial \mathbf{U}}{\partial y} = 0 \tag{5.361b}$$

where $\mathbf{A}$ and $\mathbf{B}$ are flux Jacobian matrices formed by the partial derivatives of $\mathbf{F}(\mathbf{U})$ and $\mathbf{G} = \mathbf{G}(\mathbf{U})$ with respect to the components of $\mathbf{U}$. We can also interpret $\mathbf{A}$ and $\mathbf{B}$ as second order tensors, and then the dot in Equations 5.361 implies the scalar product of a second order tensor with a vector. Consider these elements as belonging to a linear space which is spanned* by the constant unit vectors $\mathbf{j}_n$. Then:

$$\mathbf{A} = \frac{\partial F_m}{\partial U_n} \mathbf{j}_m \mathbf{j}_n$$

where from a matrix viewpoint the first index (m) corresponds to rows while the second index (n) corresponds to columns. Writing the column vector:

$$\frac{\partial \mathbf{U}}{\partial x} = \frac{\partial U_r}{\partial x} \mathbf{j}_r$$

we get:

$$\mathbf{A} \cdot \frac{\partial \mathbf{U}}{\partial x} = \frac{\partial F_m}{\partial U_n} \frac{\partial U_r}{\partial x} \delta_{nr} \mathbf{j}_m$$

$$= \frac{\partial F_m}{\partial U_n} \frac{\partial U_n}{\partial x} \mathbf{j}_m$$

which represents a column vector. It is easily verified that the result obtained above is the same as that due to the multiplication of Jacobian matrix $\mathbf{A}$ with column vector $\partial \mathbf{U}/\partial x$.

To investigate the role of jump discontinuities in relation to the conservation laws, we consider Equation 5.359a in region Ω of the xyt-space. Let Σ be a surface in Ω where jump discontinuities in the variables occur. We take Σ as dividing region Ω into two distinct parts Ω_1 and Ω_2 with surfaces S_1 and S_2, respectively, as shown in Figure 5.20.

Let $\mathbf{\Phi}$ be a vector-valued test function which is everywhere continuous in Ω including the surface Σ and which vanishes at the boundaries S_1 and S_2 of Ω, i.e., $\mathbf{\Phi}$ is of compact support in Ω. Multiplying every term of Equation 5.359 scalarly by $\mathbf{\Phi}$, integrating over Ω_1 and Ω_2 separately, and using Gauss' divergence theorem we get:

* Refer to the material preceding Example 3.1.

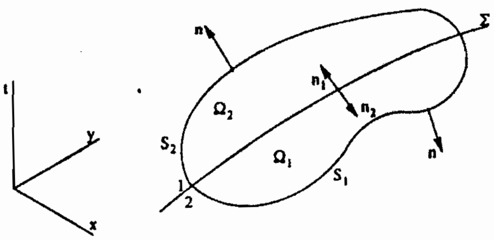


Fig. 5.20 A region in the xyt-space containing a singular surface Σ.

$$\int_{\Omega_1}\left(\frac{\partial\Phi}{\partial t}\cdot \mathrm{U} + \frac{\partial\Phi}{\partial x}\cdot \mathrm{F} + \frac{\partial\Phi}{\partial y}\cdot \mathrm{G}\right) d\nu = \int_{\Sigma}(n_{1t}\mathrm{U} + n_{2x}\mathrm{F} + n_{1y}\mathrm{G})\cdot \Phi\, dS \tag{5.362a}$$

$$\int_{\Omega_2}\left(\frac{\partial\Phi}{\partial t}\cdot \mathrm{U} + \frac{\partial\Phi}{\partial x}\cdot \mathrm{F} + \frac{\partial\Phi}{\partial y}\cdot \mathrm{G}\right) d\nu = \int_{\Sigma}(n_{2t}\mathrm{U} + n_{2x}\mathrm{F} + n_{2y}\mathrm{G})\cdot \Phi\, dS \tag{5.362b}$$

since Φ has been taken to be zero on both S_1 and S_2. Also in Equations 5.362 n_{1t}, n_{1x}, n_{1y} are the components of $\mathbf{n}_1$ while n_{2t}, n_{2x}, n_{2y} are the components of $\mathbf{n}_2$. Since:

$$\mathbf{n}_1 = -\mathbf{n}_2$$

we have:

$$n_{1t} = -n_{2t}, \qquad n_{1x} = -n_{2x}, \qquad n_{1y} = -n_{2y} \tag{5.362c}$$

Adding Equations 5.362 and using Equation 5.362c, we get:

$$\int_{\Omega}\left(\frac{\partial\Phi}{\partial t}\cdot \mathrm{U} + \frac{\partial\Phi}{\partial x}\cdot \mathrm{F} + \frac{\partial\Phi}{\partial y}\cdot \mathrm{G}\right) d\nu$$

$$= \int_{\Sigma}[n_{2t}(\mathrm{U}_2 - \mathrm{U}_1) + n_{2x}(\mathrm{F}_2 - \mathrm{F}_1) + n_{2y}(\mathrm{G}_2 - \mathrm{G}_1)]\cdot \Phi\, dS \tag{5.364}$$

However, the integral on the right of Equation 5.364 should be zero to preserve conservation; and since Φ is arbitrary, the jump conditions are given by:

$$n_{2t}\delta(\mathrm{U}) + n_{2x}\delta(\mathrm{F}) + n_{2y}\delta(\mathrm{G}) = 0 \tag{5.365}$$

where $\delta(\ )$ means the difference between the values on the two sides of Σ. Because of Equation 5.365 we have:

$$\int_{\Omega}\left(\frac{\partial\Phi}{\partial t}\cdot \mathrm{U} + \frac{\partial\Phi}{\partial x}\cdot \mathrm{F} + \frac{\partial\Phi}{\partial y}\cdot \mathrm{G}\right) d\nu = 0 \tag{5.366}$$

which is the condition for the existence of weak solutions for any test function Φ of compact support in Ω.

As an example we again consider the one-dimensional gas flow. In this case Σ is a curve in the xt-plane and:

$$n_{2t} = \frac{\dot{x}}{(1 + \dot{x}^2)^{1/2}}, \qquad n_{2x} = \frac{-1}{(1 + \dot{x}^2)^{1/2}}$$

where $\dot{x} = dx/dt$. In this case the jump condition Equation 5.365 is

$$\dot{x}\delta(\mathbf{U}) - \delta(\mathbf{F}) = 0$$

which yields the shock conditions for a shock moving unsteadily with speed $\dot{x}$. Writing the components of U and the corresponding components of F from Equation 5.356b, we get:

$$(u_1 - \dot{x})\rho_1 = (u_2 - \dot{x})\rho_2 = m$$

$$mu_2 - mu_1 = p_1 - p_2$$

$$m(e_2 + \frac{1}{2}u_2^2 - e_1 - \frac{1}{2}u_1^2) = p_1u_1 - p_2u_2$$

which are the shock relations for a shock moving with the speed $\dot{x}$.

For two- and three-dimensional motions, let N be the unit normal vector on the shock directed from side 1 to side 2. Further, let V_s be the shock speed. Then the shock relations are

$$(u_{n1} - V_s)\rho_1 = (u_{n2} - V_s)\rho_2 = m \tag{5.367a}$$

$$m(\mathbf{u}_2 - \mathbf{u}_1) = (p_1 - p_2)\mathbf{N} \tag{5.367b}$$

$$m(e_2 + \frac{1}{2}|\mathbf{u}_2|^2 - e_1 - \frac{1}{2}|\mathbf{u}_1|^2) = p_1u_{n1} - p_2u_{n2} \tag{5.367c}$$

where:

$$u_{n1} = \mathbf{u}_1 \cdot \mathbf{N}, \qquad u_{n2} = \mathbf{u}_2 \cdot \mathbf{N}$$

From Equation 5.367b we conclude that:

$$m(\mathbf{N} \times \mathbf{u}_2 - \mathbf{N} \times \mathbf{u}_1) = 0 \tag{5.367d}$$

and:

$$m(u_{n2} - u_{n1}) = p_1 - p_2 \tag{5.367e}$$

Equation 5.367d can be satisfied in the following three cases:

1. Only $m = 0$
2. Only $\mathbf{N} \times \mathbf{u}_1 = \mathbf{N} \times \mathbf{u}_2$
3. Both 1 and 2 are simultaneously satisfied.

Case 1 occurs at a *slip surface* and then the tangential component of velocity ($\mathbf{N} \times \mathbf{u}$) can be discontinuous by any amount. Also because $m = 0$:

$$u_{n1} = u_{n2} = V_s$$

and thus from Equation 5.367e the pressure is continuous. Hence, for a slip surface $\mathbf{N} \times \mathbf{u}$ is discontinuous, $\mathbf{N} \cdot \mathbf{u}$ is continuous, p is continuous, and the total energy per unit mass is discontinuous. Case 2 occurs at a shock surface and on it the tangential component of velocity is continuous. Case 3 occurs at a surface of "contact discontinuity", where according to Equations 5.367a, c, the density and the internal energy (temperature) can be discontinuous by any amount. (Refer to Chapter 4 for further details on the shock waves.)

System of Quasilinear Equations from the Conservation Equations

Although the conservation law form of the equations, (e.g., Equation 5.359) is preferable for numerical purposes, it is important to rewrite these equations in the quasilinear form for analytical reasons. As was noted earlier, because of the functional form (Equation 5.360), it is possible to write the equation in the quasilinear form. In the component form Equation 5.361a is

$$\frac{\partial U_i}{\partial t} + A_{ij}\frac{\partial U_j}{\partial x} = 0 \tag{5.368}$$

where $i = 1, 2, \ldots, n$:

$$A_{ij}(\mathbf{U}) = \frac{\partial F_i}{\partial U_j}$$

are the elements of the Jacobian matrix **A**, and the repeated index j in Equation 5.368 implies summation. To classify the system of equations (Equation 5.368), we first consider the case when the A_{ij}'s are constants so that Equation 5.368 forms a system of first order equations with constant coefficients. This system of equations is called *hyperbolic* if **A** has all real eigenvalues and a *complete* set of eigenvectors.

We next consider the case when A_{ij}'s are functions of (x, t) but not of **U**, viz., the equations are linear. In this case, the system of equations is called hyperbolic if $\mathbf{A}(x, t)$ has all real eigenvalues and a complete set of eigenvectors in a region R of the xt-plane.

For quasilinear equations we perturb a basic solution U_i by a small variation, viz., let:

$$U_i = U_i + \delta U_i$$

where δU_i are small. Substituting this solution in Equation 5.368 and neglecting squares of the perturbations, we get a linear set of equations which can be classified as hyperbolic according to the rules for linear equations with variable coefficients.

An important property of the hyperbolic system of equations is that the equations can be written in characteristic form in which differentiations appear only in characteristic directions. To proceed on this problem we consider the vector form (Equation 5.361a of the equations. Let $\mathbf{V}_k$ ($k = 1, 2, \ldots, n$) be n-component vectors which are the eigenvectors of the transpose matrix $\mathbf{A}^T$. That is

$$(\mathbf{A}^T - \lambda_{(k)}\mathbf{I}) \cdot \mathbf{V}_k = 0 \tag{5.369a}$$

where there is no summation on k, **I** is the unit matrix, and $\lambda_{(k)}$ are the eigenvalues. Recalling that the eigenvalues of **A** and $\mathbf{A}^T$ are the same and that:

$$\mathbf{A}^T \cdot \mathbf{V}_k = \mathbf{V}_k \cdot \mathbf{A}$$

Equation 5.369a yields:

$$\mathbf{V}_k \cdot \mathbf{A} = \lambda_{(k)}\mathbf{V}_k \tag{5.369b}$$

Premultiplying each term of Equation 5.361a by $\mathbf{V}_k$ and using Equation 5.369b, we get:

$$\mathbf{V}_k \cdot \left[\frac{\partial \mathbf{U}}{\partial t} + \lambda_{(k)}\frac{\partial \mathbf{U}}{\partial x}\right] = 0 \tag{5.370a}$$

which is a system of real equations; the number of equations are determined by the number of eigenvalues $\lambda_{(k)}$. The characteristic curves are obtained by solving the equations:

$$\frac{dx}{dt} = \lambda_k(\mathbf{U}, x, t) \tag{5.370b}$$

Another method of defining a hyperbolic system of equations is through the use of "similarity transformation" of matrices. In this connection it is important to state some definitions from the theory of matrices (see Reference 109):

1. The eigenvalues λ of the matrix $\mathbf{A}$ are obtained by solving the equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

2. The eigenvectors of $\mathbf{A}$ satisfy the equation:

$$(\mathbf{A} - \lambda_{(k)}\mathbf{I}) \cdot \mathbf{v}_k = 0$$

   where, as in Equation 5.369a, the use of parenthesis on k in λ implies that there is no sum on the repeated k index.

3. A matrix $\mathbf{M}$ whose columns comprise the elements of n linearly independent eigenvectors of a given matrix $\mathbf{A}$, viz., $\mathbf{v}_k$, is called a modal matrix of $\mathbf{A}$. Thus:

$$\mathbf{M} = \begin{bmatrix} v_{11} & v_{21} & \cdot & \cdot & \cdot & v_{n1} \\ v_{12} & v_{22} & \cdot & \cdot & \cdot & v_{n2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ v_{1n} & v_{2n} & & & & v_{nn} \end{bmatrix}$$

4. The premultiplication of $\mathbf{A}$ by $\mathbf{M}^{-1}$ and the postmultiplication by $\mathbf{M}$ produces a diagonal matrix $\mathbf{\Lambda}$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \lambda_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \lambda_n \end{bmatrix}$$

   where:

$$\mathbf{M}^{-1} \cdot \mathbf{A} \cdot \mathbf{M} = \mathbf{\Lambda}, \qquad \mathbf{M}^{-1} \cdot \mathbf{A} = \mathbf{\Lambda} \cdot \mathbf{M}^{-1}$$

   This transformation of $\mathbf{A}$ to $\mathbf{\Lambda}$ is a similarity transformation, i.e., $\mathbf{A}$ and $\mathbf{\Lambda}$ are similar. It can be shown easily that:

$$\mathbf{M} \cdot \mathbf{\Lambda} \cdot \mathbf{M}^{-1} = \mathbf{A}$$

Based on the definition 1–4 we now state that the system of equations (Equation 5.361a) is hyperbolic at the point $(x, t, \mathbf{U})$ if all the eigenvalues λ_k are real and the norms of $\mathbf{M}$ and $\mathbf{M}^{-1}$

are uniformly bounded. The characteristic form of the equations are obtained as in Equation 5.370a where $\mathbf{V}_k$ are the eigenvectors of the matrix $\mathbf{A}^T$. For fixed k, $\mathbf{V}_k$ forms a row vector of $\mathbf{M}^{-1}$.

Example 5.2

Write the one-dimensional equations of gas dynamics in characteristic form by using the algorithm developed in Equations 5.369 and 5.370.

The fundamental equations of one-dimensional gas dynamics are

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 \tag{i}$$

$$\frac{du}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{ii}$$

$$\frac{de}{dt} + \frac{p}{\rho}\frac{\partial u}{\partial x} = 0 \tag{iii}$$

$$p = p(\rho, e) \tag{iv}$$

where:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

If use is made of the combined thermodynamic law:

$$de + pd\left(\frac{1}{\rho}\right) = T\, ds$$

where s is the specific entropy, then by using Equations i and iii we obtain an equation for entropy in place of the internal energy. Thus an equivalent set of equations is

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 \tag{v}$$

$$\frac{du}{dt} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial s}{\partial x} = 0 \tag{vi}$$

$$\frac{ds}{dt} = 0 \tag{vii}$$

$$p = p(\rho, s) \tag{viii}$$

where a is the speed of sound:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equations v–viii can now be written in the form of Equation 5.361a with:

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} \rho \\ u \\ s \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} u & \rho & 0 \\ \dfrac{a^2}{\rho} & u & \dfrac{1}{\rho}\dfrac{\partial p}{\partial s} \\ 0 & 0 & u \end{bmatrix}$$

The eigenvalues are the solution of the equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

which are

$$\lambda_1 = u + a, \qquad \lambda_2 = u - a, \qquad \lambda_3 = u$$

Thus, the eigenvectors as given by Equation 5.369a are

$$\frac{V_{k1}}{\begin{vmatrix} A_{22} - \lambda_{(k)} & A_{32} \\ A_{23} & A_{33} - \lambda_{(k)} \end{vmatrix}} = \frac{V_{k2}}{\begin{vmatrix} A_{32} & A_{12} \\ A_{33} - \lambda_{(k)} & A_{13} \end{vmatrix}} = \frac{V_{k3}}{\begin{vmatrix} A_{12} & A_{22} - \lambda_{(k)} \\ A_{13} & A_{23} \end{vmatrix}} = \mu$$

so that:

$$V_{k1} = \mu(u - \lambda_k)^2$$

$$V_{k2} = -\rho\mu(u - \lambda_k)$$

$$V_{k3} = \mu \frac{\partial p}{\partial s}$$

where μ = constant. Using these eigenvectors and the corresponding eigenvalues in Equation 5.370a we have the equations:

$$\frac{D^+p}{Dt} + \rho a \frac{D^+u}{Dt} = 0 \tag{ix}$$

$$\frac{D^-p}{Dt} - \rho a \frac{D^-u}{Dt} = 0 \tag{x}$$

$$\frac{Ds}{Dt} = 0 \tag{xi}$$

where:

$$\frac{D^{\pm}}{Dt} = \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}$$

A widely studied case is that of homentropic flow in which s = constant throughout the field.* Thus, Equation xi is identically satisfied, and from Equation viii:

$$p = p(\rho)$$

Defining a quantity P as (refer to Section 4.10):

$$P = \int \frac{dp(\rho)}{\rho a(\rho)}$$

we find that:

$$\frac{D^{\pm} P}{Dt} = \frac{1}{\rho a} \frac{D^{\pm} p}{Dt}$$

so that Equations ix and x can be written as:

$$\frac{D^{\pm}}{Dt}(u \pm P) = 0$$

The quantities $u \pm P$ are called the Riemann invariants, and they remain constants along the + and − characteristics.

Hyperbolic Equations in Higher Dimensions

It was noted earlier that the conservation law equation (Equation 5.359) in the spatial variables x and y can be rewritten in the quasilinear form (Equation 5.361b) if $\mathbf{F} = \mathbf{F}(\mathbf{U})$ and $\mathbf{G} = \mathbf{G}(\mathbf{U})$. The writing of Equation 5.361b and of the similar equations in higher spatial dimensions in the characteristic form requires some additional considerations. Here we elaborate only the problem of two spatial dimensions. Let a new matrix $\mathbf{P}$ be defined as:

$$\mathbf{P} = k_1\mathbf{A} + k_2\mathbf{B}$$

where k_1 and k_2 are arbitrary real numbers. The system of equations (Equation 5.361b) is hyperbolic at the point $(x, y, t, \mathbf{U})$ if there exists a similarity transformation which is such that:

$$\mathbf{M}^{-1} \cdot \mathbf{P} \cdot \mathbf{M} = \mathbf{\Lambda}$$

where:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

and the eigenvalues λ of $\mathbf{P}$ are all real. Further, the norms of $\mathbf{M}$ and $\mathbf{M}^{-1}$ are uniformly bounded.

* If s = constant in a flow, then the flow is smooth and free from shocks.

5.26 NUMERICAL TRANSFORMATION AND GRID GENERATION

In fluid mechanics and also in many other branches of mechanics and physics, the boundary-value problems are solved by specifying certain boundary values on closed curves or surfaces. For example, in viscous flow problems the no-slip condition is prescribed on a closed curve or surface which is the boundary curve or surface of the body past which the flow takes place. Since the Navier-Stokes solutions are generally obtained by numerical methods, the satisfaction of the no-slip condition becomes a serious problem in those cases when the equations are solved numerically in a coordinate system which is not natural to the body shape. However, the same conditions can be exactly satisfied if the Navier-Stokes equations are first transformed to a coordinate system which is natural to the body shape, and then solved numerically. By the term natural we mean that one coordinate line in the two-dimensional case and two coordinate lines in the three-dimensional case follow the exact shape of the body past which the flow takes place. Such a coordinate system is also termed body oriented or body fitted. Recall from Part II of this chapter that the boundary layer equations either in two or three dimensions have coordinates which are natural to the body shape.

Starting with a paper by Winslow,[110] in the period from 1970 to 1990 there has been a large body of research in the numerical methods of coordinate generation with particular applications to fluid dynamic problems. This research interest has been sustained by the availability of new and powerful techniques for solving a system of nonlinear partial differential equations on high speed digital computers. The literature on the subject is quite extensive, and we refer to Thompson et al.[111] for reference.

At present there are various methods available for the generation of curvilinear coordinates in two- and three-dimensional domains which can be broadly categorized into the following topics:

1. Methods based on solving partial differential equations (PDEs), particularly the elliptic PDEs
2. Methods based on algebraic techniques, particularly the transfinite interpolation

In this section we shall be concerned only with the methods based on elliptic PDEs for the sole reason that diverse topological domains bounded by arbitrary surfaces or curves can be filled by coordinate curves which are solutions of the elliptic equations and thus are continuous at least to the second order.

Equations for Grid Generation

The derivation of the elliptic PDE model for grid generation is quite straightforward and can be obtained conveniently by going through ME.4, Section 6. Equation M4.59 is an equation carrying within it the Laplacians of curvilinear coordinates x^i. Introducing a second order differential operator D as:

$$D = gg^{ij}\frac{\partial^2}{\partial x^i\,\partial x^j}$$

while using Equations M1.121 and writing $x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$, we obtain:

$$D = G_1\partial_{\xi\xi} + G_2\partial_{\eta\eta} + G_3\partial_{\zeta\zeta} + 2G_4\partial_{\xi\eta} + 2G_5\partial_{\xi\zeta} + 2G_6\partial_{\eta\zeta} \tag{5.371a}$$

In two dimensions, D becomes:

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} \tag{5.371b}$$

Thus, in three dimensions the three scalar equations as obtained from Equation M4.59 are

$$\begin{aligned} Dx &= -g(x_\xi \nabla^2\xi + x_\eta \nabla^2\eta + x_\zeta \nabla^2\zeta) \\ Dy &= -g(y_\xi \nabla^2\xi + y_\eta \nabla^2\eta + y_\zeta \nabla^2\zeta) \\ Dz &= -g(z_\xi \nabla^2\xi + z_\eta \nabla^2\eta + z_\zeta \nabla^2\zeta) \end{aligned} \tag{5.371c}$$

The equations in Equations 5.371c can be converted into a closed system of equations for x, y, and z if we prescribe the Laplacian of coordinates. A general form of such a specification is

$$\nabla^2\xi = g^{ij}P^1_{ij}, \qquad \nabla^2\eta = g^{ij}P^2_{ij}, \qquad \nabla^2\zeta = g^{ij}P^3_{ij} \tag{5.372}$$

where $P^k_{ij} = P^k_{ji}$ are symmetric functions of the coordinates and are supposed to be user specified.* The role of control functions is to produce a desired distribution of coordinates in the region under consideration. A modest specification will be to take all $P^k_{ij} = 0$ except P^1_{11}, P^2_{22}, and P^3_{33}. Thus writing $P^1_{11} = P$, $P^2_{22} = Q$, and $P^3_{33} = R$, we have the generating system as:

$$\begin{aligned} \nabla^2\xi &= g^{11}P = \frac{G_1}{g}P \\ \nabla^2\eta &= g^{22}Q = \frac{G_2}{g}Q \\ \nabla^2\zeta &= g^{33}R = \frac{G_3}{g}R \end{aligned} \tag{5.373}$$

With this choice the generating system of PDEs becomes:

$$\begin{aligned} Dx + (G_1Px_\xi + G_2Qx_\eta + G_3Rx_\zeta) &= 0 \\ Dy + (G_1Py_\xi + G_2Qy_\eta + G_3Ry_\zeta) &= 0 \\ Dz + (G_1Pz_\xi + G_2Qz_\eta + G_3Rz_\zeta) &= 0 \end{aligned} \tag{5.374}$$

In two dimensions the system is

$$\begin{aligned} Dx + (g_{22}Px_\xi + g_{11}Qx_\eta) &= 0 \\ Dy + (g_{22}Py_\xi + g_{11}Qy_\eta) &= 0 \end{aligned} \tag{5.375}$$

Equations 5.374 in three dimensions and Equations 5.375 in two dimensions form the elliptic PDE models based on the Poisson equations for grid generation. That is, these equations generate the physical x, y, z in three dimensions or x, y in two dimensions as functions of the computational coordinates ξ, η, ζ or ξ, η, respectively, under some specified Dirichlet boundary conditions.

The benefit of solving the inverted form of Equations 5.374 over that of solving the Poisson equations (Equation 5.373) is significant. For example, in the two-dimensional case the Poisson equations:

$$\nabla^2\xi = g^{11}P, \qquad \nabla^2\eta = g^{22}Q$$

yield $\xi(x, y)$ and $\eta(x, y)$ at discrete (x, y) points. However, to obtain the curves ξ = constant

* Some authors simply take $\nabla^2\xi = P$, $\nabla^2\eta = Q$, $\nabla^2\zeta = R$.

and η = constant one needs to do extensive interpolation between the (x, y) points. This process is computationally time-consuming because the lines ξ = constant and η = constant are almost always needed at nonequispaced (x, y) points. The same argument goes with the three-dimensional case.

Gaussian Equations for Grid Generation

Another model which generates both surface and three-dimensional grids which is also elliptic has been developed by Warsi.[112] This model is based on the Beltramians of surface coordinates rather than on the Laplacians as was the case in the previous subsection.

The starting point of this model is the formulae of Gauss, Equation M7.2. These formulae can further be manipulated in two ways depending on whether the aim is to have a model for the 3-*D* coordinates or to have a model for the generation of coordinates in a given surface. If a surface is given and the aim is to generate coordinates, then as has been shown in ME.7, Section 6 the equation for the generation of the coordinates ξ and η on the surface ζ = constant are

$$
\begin{aligned}
L^{(3)}x &= X^{(3)}R^{(3)} \\
L^{(3)}y &= Y^{(3)}R^{(3)} \\
L^{(3)}z &= Z^{(3)}R^{(3)}
\end{aligned}
\tag{5.376}
$$

where:

$$
\begin{aligned}
L^{(3)} &= g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} + S\partial_{\xi} + T\partial_{\eta} \\
R^{(3)} &= G_3(k_I^{(3)} + k_{II}^{(3)})
\end{aligned}
$$

and:

$$
\begin{aligned}
S &= g_{22}P^1_{11} - 2g_{12}P^1_{12} + g_{11}P^1_{22} \\
T &= g_{22}P^2_{11} - 2g_{12}P^2_{12} + g_{11}P^2_{22}
\end{aligned}
$$

It is, however, a common practice to take all the $P^{\alpha}_{\beta\gamma}$ as zero except P^1_{11} and P^2_{22}.

Based on the x, y, z of the given surface, $F(x, y, z) = 0$, the sum of the principal curvatures $k_I^{(3)} + k_{II}^{(3)}$ has to be modeled. Refer to Reference 112. Recent results have shown that Equations 5.376 also satisfy the equations of Weingarten (Equation M7.7).

5.27 NUMERICAL ALGORITHMS FOR VISCOUS COMPRESSIBLE FLOWS

Recent advances in computer technology have been instrumental in the development of some accurate and fast compressible Navier-Stokes and gas dynamic codes for the solution of complex flow problems. In this connection the analysis and discussion of the conservation laws as developed in Section 5.25 of this chapter are quite important in the formulation and testing of computational algorithms.

In the beginning our aim is to let the reader appreciate the significance of high gradients which usually appear in important regions of the flow field and which are a consequence of retaining viscous effects in the flow. In the design of any numerical method the aim is usually to resolve such high gradients. To understand such effects we study two example problems which are amenable to exact analytical treatment.

Problem 5.3

Consider the one-dimensional steady viscous compressible flow of a gas. The effect of viscosity is to create a high gradient layer, called the *shock layer.** Find the solution of the shock layer and estimate its thickness.

If the effect of viscosity is retained in the equations of motion of a high speed gas flow, then the finite discontinuities are replaced by steep gradients across a thin layer of fluid, called the shock layer. This problem was originally solved by Gilbarg and Paolucci[113] and has also been discussed by Serrin.[114]

We consider a unidirectional steady flow of a viscous gas. The necessary equations are

$$\frac{d}{dx}(\rho u) = 0 \tag{i}$$

$$\frac{d}{dx}(\rho u^2) = \frac{d}{dx}\left\{-p + (\lambda + 2\mu)\frac{du}{dx}\right\} \tag{ii}$$

$$\frac{d}{dx}(\rho u e) = \left\{-p + (\lambda + 2\mu)\frac{du}{dx}\right\}\frac{du}{dx} + \frac{d}{dx}\left(k\frac{dT}{dx}\right) \tag{iii}$$

where in Equation iii, use has been made of Equation i. Noting that Equations i and ii are directly integrable, we have:

$$\rho u = m = \text{constant} \tag{iv}$$

$$\rho u^2 - m\alpha = -p + (\lambda + 2\mu)\frac{du}{dx}$$

or:

$$m(u - \alpha) = -p + (\lambda + 2\mu)\frac{du}{dx} \tag{v}$$

where α is constant. Using Equation v in Equation iii, we can again integrate and have:

$$m\{e - \tfrac{1}{2}(u - \alpha)^2 + \beta\} = k\frac{dT}{dx}$$

where β is yet another constant. Thus a set of three equations for further consideration is

$$\rho u = m \tag{vi}$$

$$p + m(u - \alpha) = (\lambda + 2\mu)\frac{du}{dx} \tag{vii}$$

$$m\{e - \tfrac{1}{2}(u - \alpha)^2 + \beta\} = k\frac{dT}{dx} \tag{viii}$$

Ahead and behind the shock layer there exist regions 1 and 2, respectively, where the conditions are uniform and the derivatives with respect to x are zero. On the scale of the shock layer thickness, we may denote these regions as $-\infty$ and $+\infty$, respectively. In any one of these regions Equations vii and viii become:

* Physically consider a finite disturbance traveling steadily in a gas with the coordinates attached to the wave.

$$p + mu = m\alpha \tag{ix}$$

or:

$$p + \rho u^2 = m\alpha = \text{constant} \tag{x}$$

and:

$$e - \frac{1}{2}(u - \alpha)^2 + \beta = 0$$

which can be written, on using Equation ix as:

$$e + \frac{p}{\rho} + \frac{1}{2}u^2 = \frac{1}{2}\alpha^2 - \beta \tag{xi}$$

Equations vi, x, and xi then yield the usual Hugoniot equations:

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = e_2 + \frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

For an ideal gas:

$$p = \rho RT = \frac{mRT}{u}$$

and Equation vii becomes:

$$(\lambda + 2\mu)u\frac{du}{dx} = m(RT + u^2 - \alpha u) \tag{xii}$$

Assuming the gas to be thermally and calorically perfect, Equation xi becomes:*

$$C_pT + \frac{1}{2}u^2 = \frac{1}{2}\alpha^2 - \beta$$

Eliminating T between this equation and Equation xii, we get:

$$(\lambda + 2\mu)u\frac{du}{dx} = A(u - u_1)(u - u_2) \tag{xiii}$$

where A is a constant. Since $u_2 \leqslant u \leqslant u_1$, from Equation xiii we conclude that in the shock layer:

$$\frac{du}{dx} < 0$$

* The same equation is obtained if Equations viii and xii are added, and it is assumed that $k = C_p(\lambda + 2\mu)$.

and there is a steep decrease of u in passing through it from side 1 to side 2. We now nondimensionalize Equation xiii and assume $\lambda + 2\mu$ = constant** Thus taking:

$$f = \frac{u}{u_1}, \qquad \theta = \frac{u_2}{u_1}, \qquad \eta = \frac{Ax}{\lambda + 2\mu}$$

we have:

$$f\frac{df}{d\eta} = -(1 - f)(f - \theta) \tag{xiv}$$

We now choose the point $\eta = 0$ as that point in the shock layer where the supersonic speed has decreased to the sonic speed a_*, that is

$$\eta = 0 \quad \text{when} \quad f = \frac{a_*}{u_1}$$

However, from the Prandtl relation for a normal shock (refer to Chapter 4):

$$a_* = (u_1 u_2)^{1/2}$$

hence we have:

$$\text{at } \eta = 0 : f = \sqrt{\theta} \tag{xv}$$

It must also be noted that:

$$\text{at } \eta = -\infty : f = 1$$

$$\text{at } \eta = +\infty : f = \theta \tag{xvi}$$

Solution of Equation xiv satisfying the conditions in Equations xv and xvi can easily be found by the separation of variables and is

$$\frac{1 - f}{(f - \theta)^\theta} = c \exp[(1 - \theta)\eta]$$

where:

$$c = (1 - \sqrt{\theta})/(\sqrt{\theta} - \theta)^\theta$$

Thus, for a given θ the function can be tabulated for various η-values. The thickness of the shock layer can be defined as:

$$\Delta = \Delta_1 + \Delta_2$$

where:

$$\Delta_1 = \frac{1}{1 - \sqrt{\theta}} \int_{-\infty}^{0} (1 - f)\, d\eta$$

** If $\lambda + 2\mu = f(T)$, then Equation xiii can be integrated only numerically.

$$\Delta_2 = \frac{1}{\sqrt{\theta} - \theta}\int_0^{\infty}(f - \theta)\,d\eta$$

which is very small, of the order of $\lambda + 2\mu$.

Example 5.4

Consider the viscous Burger equation. Solve this equation under a prescribed initital condition to demonstrate the nature of the viscous and nonlinear terms in the Navier-Stokes equations.

The Burger equation in nondimensional form is

$$u_t + uu_x = \epsilon u_{xx} \tag{i}$$

where ϵ is the inverse of the Reynolds number. We wish to solve Equation i under the initital condition:

$$u(x, 0) = \phi(x), \qquad -\infty < x < \infty \tag{ii}$$

The solution of the problem has been obtained by Hopf.[115]

Equation i can be written as:

$$u_t = (\epsilon u_x - \tfrac{1}{2}u^2)_x$$

Let:

$$P_x = u, \qquad P_t = \epsilon u_x - \tfrac{1}{2}u^2$$

then:

$$P_t = \epsilon P_{xx} - \tfrac{1}{2}P_x^2 \tag{iii}$$

Further, let:

$$P(x, t) = -2\epsilon \ell n F(x, t), \qquad F > 0$$

then Equation iii becomes

$$F_t = \epsilon F_{xx} \tag{iv}$$

which is just the linear heat conduction equation. The initital condition for Equation iv is then:

$$F(x, 0) = f(x) \tag{v}$$

where:

$$f(x) = \exp\left(-\int_{x_0}^{x}\frac{\phi(\eta)}{2\epsilon}\,d\eta\right)$$

Thus, the solution of the problem posed in Equations iv and v is

$$u(x, t) = \frac{\int_{-\infty}^{\infty} (x - \xi) f(\xi) g(x - \xi, t)\, d\xi}{t \int_{-\infty}^{\infty} f(\xi) g(x - \xi, t)\, d\xi} \tag{vi}$$

$$g(x - \xi, t) = \frac{1}{2(\pi \epsilon t)^{1/2}} \exp\left[\frac{-(x - \xi)^2}{4\epsilon t}\right]$$

Had we considered the linearized equation:

$$u_t = \epsilon u_{xx}, \qquad u(x, 0) = \phi(x)$$

the solution would have been:

$$u(x, t) = \int_{-\infty}^{\infty} \phi(\xi) g(x - \xi, t)\, d\xi \tag{vii}$$

For an impulsive initial condition:

$$\phi(x) = \delta(x)$$

we have:

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad x \leq 0 \\ &= \exp\left(-\frac{1}{2\epsilon}\right) \quad \text{for} \quad x > 0 \end{aligned}$$

The solution (Equation vi) after some simplification becomes:

$$u(x, t) = 2\left(\frac{\epsilon}{\pi t}\right)^{1/2} \frac{\exp(-x^2/4\epsilon t)}{\coth(1/4\,\epsilon) - \operatorname{erf}(x/2\sqrt{\epsilon t})} \tag{viii}$$

On the other hand, the solution of the linearized problem in Equation vii becomes:

$$u(x, t) = \frac{1}{2\epsilon}\left(\frac{\epsilon}{\pi t}\right)^{1/2} \exp(-x^2/4\epsilon t) \tag{ix}$$

For $t \to \infty$ the behavior of the solution in Equation viii is

$$u(x, t) = 2\left(\frac{\epsilon}{\pi t}\right)^{1/2} \tanh(1/4\,\epsilon) \exp(-x^2/4\,\epsilon t)$$

which has the same mathematical structure as that of Equation ix except for a diminished amplitude $2 \tan h(1/4\,\epsilon)$ in comparison to $1/2\,\epsilon$.

The main conclusions to be drawn from the preceding example are as follows: the nonlinear term uu_x has the effect of steepening the velocity profile; however, the presence of viscosity keeps the velocity profile from becoming discontinuous in space and time. Further, the presence of viscosity produces a tendency toward isotropy so that after a large time the velocity profile is practically isotropic although with a diminished amplitude. From these results we conclude that for certain initital conditions the Burger equation with $\epsilon = 0$ is bound to have discontinuous solutions. Refer to Examples 4.5 and 4.6.

Nature of the Difference Schemes

The example solution 5.4 of the Burger equation has clearly demonstrated the role played by the viscous or the dissipation term in obtaining smooth solutions. Thus, to obtain smooth solutions of the nonlinear equations by numerical methods (the solution can be very steep but not discontinuous), it is desirable to develop those computational algorithms in which some sort of dissipation or artificial viscosity can be introduced. This line of attack was initiated by Von Neumann and Richtmyer,[116] Lax,[108] and Lax and Wendroff.[117]

The artificial viscosity can be introduced explicitly either by modifying the terms of the equations being solved or by the development of those finite differences which are inherently dissipative. In Reference[116] the artificial viscosity was introduced in the 1-D gas dynamic equations (Equation 5.356b) as follows:

$$\mathbf{F} = \begin{bmatrix} m \\ p + q + \dfrac{m^2}{\rho} \\ (E_t + p + q)\dfrac{m}{\rho} \end{bmatrix}$$

where q is an additional term having the dimension of pressure and defined as:

$$q = -\rho(b_1 \Delta x)^2 |u_x| u_x$$

Here b_1 is the nondimensional adjustable constant. It is quite apparent that when q is differentiated with respect to x, the end result is the availability of a dissipative term proportional to u_{xx}. It has been found that the addition of the q-term smears the shock over a few grid steps and suppresses the post shock oscillations in the solution. After the original work of Von Neumann a number of different forms of q have been tried by various authors.

In place of modifying the terms of a differential equation to incorporate the effect of dissipation, the other alternative is to devise dissipative numerical schemes without disturbing the original equations. Most of the recent developments in the solution algorithms are in this direction. To understand the *fundamental basis* of the dissipative schemes we consider the formulation of Lax and Wendroff[117] as referred to earlier. For simplicity we consider the one-dimensional hyperbolic conservation law (Equation 5.356a) with $\mathbf{F} = \mathbf{F}(\mathbf{U})$. The quasilinear form is then Equation 5.361a. First of all, from Equation 5.356a:

$$\begin{aligned}\frac{\partial^2 \mathbf{U}}{\partial t^2} &= -\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}}{\partial x}\right)\\ &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial t}\right)\\ &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial U'}\frac{\partial U'}{\partial t}\right)\\ &= -\frac{\partial}{\partial x}\left(\mathbf{A}\cdot\frac{\partial \mathbf{U}}{\partial t}\right)\\ &= \frac{\partial}{\partial x}\left(\mathbf{A}\cdot\frac{\partial \mathbf{F}}{\partial x}\right)\end{aligned} \tag{5.377}$$

Now consider the Taylor expansion of $\mathbf{U}(x, t + \Delta t)$ and use Equations 5.361a and 5.377 to have:

$$U(x, t + \Delta t) = U(x, t) - (\Delta t)A \cdot \frac{\partial U}{\partial x} + \frac{1}{2}(\Delta t)^2 \frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right) + \ldots$$

Rewriting in a continuous time derivative form, i.e.:

$$\frac{U(x, t + \Delta t) - U(x, t)}{\Delta t} = \frac{\partial U}{\partial t}$$

we then have:

$$\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} = \frac{1}{2}(\Delta t) \frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right) + 0(\Delta t)^2 \tag{5.378}$$

Looking at Equation 5.378 we find that a finite difference approximation of U at $t + \Delta t$ gives rise to the original equation (Equation 5.361a) plus a dissipative term on the right-hand side of the equation. This observation forms the basis of the Lax-Wendroff method which we now consider.

The Lax-Wendroff scheme of solving Equation 5.356a is a two-step explicit method and is correct to $0(\Delta t)^2$. In the first step, provisional values are calculated at the center of a mesh in the xt-plane through the equation:

$$\frac{U_{j+1/2}^{n+1/2} - U_{j+1/2}^{n}}{\Delta t/2} = -\frac{1}{2(\Delta x/2)}(F_{j+1}^{n} - F_{j}^{n})$$

or, writing $\lambda = \Delta t/\Delta x$, we have:

$$U_{j+1/2}^{n+1/2} = \frac{1}{2}(U_{j+1}^{n} + U_{j}^{n}) - \frac{\lambda}{2}(F_{j+1}^{n} - F_{j}^{n}) \tag{5.379a}$$

In the second step, the final values are calculated through the equation:

$$\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} = -\frac{1}{2\left(\frac{\Delta x}{2}\right)}(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2})$$

or:

$$U_{j}^{n+1} = U_{j}^{n} - \lambda(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}) \tag{5.379b}$$

To understand the implications of the final equation (Equation 5.379b) we expand the $F_{j+1/2}^{n+1/2}$ in a Taylor series in time. That is:

$$\begin{aligned} F_{j+1/2}^{n+1/2} &\cong F_{j+1/2}^{n} + \frac{1}{2}(\Delta t)\left(\frac{\partial F}{\partial t}\right)_{j+1/2}^{n} \\ &= F_{j+1/2}^{n} + \frac{1}{2}(\Delta t)\left(A \cdot \frac{\partial U}{\partial t}\right)_{j+1/2}^{n} \\ &= F_{j+1/2}^{n} - \frac{1}{2}(\Delta t)\left(A \cdot \frac{\partial F}{\partial x}\right)_{j+1/2}^{n} \end{aligned} \tag{5.380a}$$

Similarly:

$$F^{n+1/2}_{j-1/2} = F^n_{j-1/2} - \frac{1}{2}(\Delta t)\left(A \cdot \frac{\partial F}{\partial x}\right)^n_{j-1/2} \tag{5.380b}$$

Substituting Equations 5.380a, b in Equation 5.379b, we get:

$$U^{n+1}_j = u^n_j - \frac{\lambda}{2}(F^n_{j+1} - F^n_{j-1}) + \frac{\lambda^2}{2}[A^n_{j+1/2} \cdot (F^n_{j+1} - F^n_j) - A^n_{j-1/2} \cdot (F^n_j - F^n_{j-1})] \tag{5.381}$$

Thus, Equations 5.379a and 5.381 form an explicit two-step scheme for the solution of Equation 5.356a. This scheme is second order correct in both space and time. It is immediately seen that if in Equation 5.378 we approximate the time derivative by a forward difference formula and the spatial derivatives by the central difference formula, then we will obtain Equation 5.381. The extensions to two- and three-dimensional equations are straightforward. It has been proved by Lax and Wendroff that for the stability of the scheme (Equation 5.381) the Courant number defined as:

$$C_r = (|u| + a)\frac{\Delta t}{\Delta x}$$

where a is the local speed of sound, should be such that:

$$\max(C_r) \leq 1$$

This condition is also known as the Courant-Friedrichs-Lewy (CFL) condition.

Other two-step explicit schemes for one-dimensional conservation laws are due to Godunov,[118] MacCormack,[119] and Rusanov.[120] Godunov's and MacCormack's schemes are stated as follows:

1. Godunov's scheme accurate to the first order:

$$U^{n+1}_{j+1/2} = \frac{1}{2}(U^n_{j+1} + U^n_j) - \lambda(F^n_{j+1} - F^n_j)$$

$$U^{n+1}_j = U^n_j - \lambda(F^{n+1}_{j+1/2} - F^{n+1}_{j-1/2}) \tag{5.382}$$

$$\max(C_r) \leq 1$$

2. MacCormack's scheme accurate to the second order:
   The MacCormack scheme can be stated in the following two forms:
   (a) Forward-backward form

$$\hat{U}^{n+1}_j = U^n_j - \lambda(F^n_{j+1} - F^n_j)$$

$$U^{n+1}_j = \frac{1}{2}(U^n_j + \hat{U}^{n+1}_j) - \frac{\lambda}{2}(\hat{F}^{n+1}_j - \hat{F}^{n+1}_{j-1}) \tag{5.383a}$$

$$\max(C_r) \leq 1, \quad \lambda = \Delta t/\Delta x$$

   (b) Backward-forward form

$$\hat{U}^{n+1}_j = U^n_j - \lambda(F^n_j - F^n_{j-1})$$

$$U^{n+1}_j = \frac{1}{2}(U^n_j + \hat{U}^{n+1}_j) - \frac{\lambda}{2}(\hat{F}^{n+1}_{j+1} - \hat{F}^{n+1}_j) \tag{5.383b}$$

Note: $\mathbf{F}^n_{j+1} = \mathbf{F}(\mathbf{U}^n_{j+1})$, $\hat{\mathbf{F}}^{n+1}_{j+1} = \mathbf{F}(\hat{\mathbf{U}}^{n+1}_{j+1})$, etc.

The first equation in either Equation 5.383a or b is called a predictor, while the second equation in either of the equations is called a corrector. Note that in Equation 5.383a the predictor is a forward difference while in Equation 5.383b it is a backward difference, or one-sided or upwind difference. The utility of these classifications can be understood through the following analysis due to Steger and Warming.[121]

Consider the one-dimensional quasilinear equations (Equation 5.368). For the purpose of analysis, let us consider $\mathbf{A} = (A_{ij})$ to be a *constant matrix*. In vector form the equations are

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \cdot \frac{\partial \mathbf{U}}{\partial x} = 0$$

Premultiplying by $\mathbf{M}^{-1}$ and introducing the vector $\mathbf{w}$ as:

$$\mathbf{w} = \mathbf{M}^{-1} \cdot \mathbf{U}$$

we obtain:

$$\frac{\partial w_k}{\partial t} + \lambda_{(k)} \frac{\partial w_k}{\partial x} = 0$$

as the equations along the characteristic directions. For brevity dropping the index k, we have:

$$\frac{\partial w}{\partial t} + \lambda \frac{\partial w}{\partial x} = 0 \tag{5.384}$$

Equation 5.384 can be written as an ordinary differential equation if the partial derivative $\partial w/\partial x$ is approximated by a difference. Using a forward or a backward difference, we have the following two alternative representations:

$$\frac{dw_j}{dt} + \frac{\lambda}{\Delta x}(w_{j+1} - w_j) = 0 \tag{5.385a}$$

$$\frac{dw_j}{dt} + \frac{\lambda}{\Delta x}(w_j - w_{j-1}) = 0 \tag{5.385b}$$

For a study of the stability we assume:

$$w_j = a(t) \exp[i\theta j]$$

where:

$$i = \sqrt{-1}, \qquad \theta = k\,\Delta x, \qquad k = \text{wave number}$$

Thus, Equations 5.385a, b, respectively, yield on solution:

$$a(t) = a(0) \exp(\overline{\sigma} t)$$
$$a(t) = a(0) \exp(-\sigma t)$$

where:

$$\sigma = \frac{\lambda}{\Delta x}\left[2\sin^2\frac{\theta}{2} + i\sin\theta\right]$$

and $\bar{\sigma}$ is the complex conjugate of σ. We thus see that for stability of Equation 5.385a $\lambda < 0$ while for the stability of Equation 5.385b $\lambda > 0$. The case $\lambda < 0$ implies that the wave moves to the left while $\lambda > 0$ implies that the wave moves to the right. From this we conclude that if the eigenvalues λ are positive, then for the stability of the difference scheme we must use a backward (one-sided or upwind) difference. The opposite is true for the case when the eigenvalues are negative.

Formulation for Compressible Navier-Stokes' Equations

In recent years much progress has been made in solving the complete Navier-Stokes equations for compressible flows when the equations have been written in the strong conservation law form. In general coordinates we consider Equations 5.342–5.344. However, for simplicity we consider the case of steady Eulerian coordinates with the following abbreviations:

$$E = \sqrt{g}\,E_t, \qquad P = \sqrt{g}\,p$$

The conservation form of the equations is then:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} + \frac{\partial \mathbf{N}}{\partial \zeta} = 0 \tag{5.386}$$

Using the definition of the covariant and contravariant base vectors and the quantities defined earlier in Equation 5.345 with $\mathbf{v} = \mathbf{u}$, we can write the vector $\mathbf{U}$ and the flux vectors in appropriate forms. Thus:

$$\mathbf{U} = \begin{bmatrix} \sigma \\ \sigma u \\ \sigma v \\ \sigma w \\ E \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \sigma u^1 \\ X^{11}x_\xi + X^{21}x_\eta + X^{31}x_\zeta \\ X^{11}y_\xi + x^{21}y_\eta + X^{31}y_\zeta \\ X^{11}z_\xi + X^{21}z_\eta + X^{31}z_\zeta \\ Y^1 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \sigma u^2 \\ X^{12}x_\xi + X^{22}x_\eta + X^{32}x_\zeta \\ X^{12}y_\xi + x^{22}y_\eta + X^{32}y_\zeta \\ X^{12}z_\xi + X^{22}z_\eta + X^{32}z_\zeta \\ Y^2 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} \sigma u^3 \\ X^{13}x_\xi + X^{23}x_\eta + X^{33}x_\zeta \\ X^{13}y_\xi + x^{23}y_\eta + X^{33}y_\zeta \\ X^{13}z_\xi + X^{23}z_\eta + X^{33}z_\zeta \\ Y^3 \end{bmatrix} \tag{5.387}$$

In Equation 5.387 u, v, w are the Cartesian components while u^1, u^2, u^3 are the contravariant components of the velocity vector **u**, and x, y, z are the Cartesian coordinates. In contrast to Equation 5.359 the most important point to note in Equation 5.386 is that **F**, **G**, and **N** are *not* purely functions of **U** since the terms X^{ik} also contain the first partial derivatives due to viscous transport processes besides that of convection. However, noting that the Cartesian and the contravariant components of the velocity vector are related through the equations:

$$u = u^1 x_\xi + u^2 x_\eta + u^3 x_\zeta$$

$$v = u^1 y_\xi + u^2 y_\eta + u^3 y_\zeta$$

$$w = u^1 z_\xi + u^2 z_\eta + u^3 z_\zeta$$

$$u^1 = u\alpha^1 + v\beta^1 + w\gamma^1$$

$$u^2 = u\alpha^2 + v\beta^2 + w\gamma^2$$

$$u^3 = u\alpha^3 + v\beta^3 + w\gamma^3$$

where:

$$\alpha^j = g^{jk}\frac{\partial x}{\partial x^k}, \qquad \beta^j = g^{jk}\frac{\partial y}{\partial x^k}, \qquad \gamma^k = g^{jk}\frac{\partial z}{\partial x^k}$$

we can bifurcate the vectors **F**, **G**, **N** each in a hyperbolic part and a parabolic part as follows:

$$\mathbf{F} = \mathbf{F}_H - \mathbf{F}_P, \qquad \mathbf{G} = \mathbf{G}_H - \mathbf{G}_P, \qquad \mathbf{N} = \mathbf{N}_H - \mathbf{N}_P$$

Introducing the notations:

$$m = \sigma u$$

$$n = \sigma v$$

$$q = \sigma w$$

$$a = \sigma(u)^2 + P = \frac{(m)^2}{\sigma} + P$$

$$b = \sigma(v)^2 + P = \frac{(n)^2}{\sigma} + P$$

$$c = \sigma(w)^2 + P = \frac{(q)^2}{\sigma} + P$$

$$d = \sigma uv = \frac{mn}{\sigma}$$

$$f = \sigma vw = \frac{nq}{\sigma}$$

$$w = \sigma uw = \frac{mq}{\sigma}$$

$$h = (E + P)u = \frac{(E + P)m}{\sigma}$$

$$r = (E + P)v = \frac{(E + P)n}{\sigma}$$

$$s = (E + P)w = \frac{(E + P)q}{\sigma}.$$

we have:

$$\mathbf{U} = \begin{bmatrix} \sigma \\ m \\ n \\ q \\ E \end{bmatrix}$$

$$\mathbf{F}_H = \begin{bmatrix} m\alpha^1 + n\beta^1 + q\gamma^1 \\ a\alpha^1 + d\beta^1 + w\gamma^1 \\ d\alpha^1 + b\beta^1 + f\gamma^1 \\ w\alpha^1 + f\beta^1 + c\gamma^1 \\ h\alpha^1 + r\beta^1 + s\gamma^1 \end{bmatrix}$$

$$\mathbf{G}_H = \begin{bmatrix} m\alpha^2 + n\beta^2 + q\gamma^2 \\ a\alpha^2 + d\beta^2 + w\gamma^2 \\ d\alpha^2 + b\beta^2 + f\gamma^2 \\ w\alpha^2 + f\beta^2 + c\gamma^2 \\ h\alpha^2 + r\beta^2 + s\gamma^2 \end{bmatrix}$$

$$\mathbf{N}_H = \begin{bmatrix} m\alpha^3 + n\beta^3 + q\gamma^3 \\ a\alpha^3 + d\beta^3 + w\gamma^3 \\ d\alpha^3 + b\beta^3 + f\gamma^3 \\ w\alpha^3 + f\beta^3 + c\gamma^3 \\ h\alpha^3 + r\beta^3 + s\gamma^3 \end{bmatrix}$$

Introducing the symbols X_P^{ik} and Y_P^i as:

$$X_P^{ik} = \epsilon\sqrt{g}\,(\lambda\,\Delta g^{ik} + 2\mu D^{ik}) \qquad \textbf{(5.388a)}$$

$$Y_P^i = \epsilon\sqrt{g}\left(g_{jr}\sigma^{ij}u^r + S\mu g^{ik}\frac{\partial T}{\partial x^k}\right) \qquad \textbf{(5.388b)}$$

where S has been defined after Equation 5.339, we have:

$$\mathbf{F}_P = \begin{bmatrix} 0 \\ X_P^{11}x_\xi + X_P^{21}x_\eta + X_P^{31}x_\zeta \\ X_P^{11}y_\xi + X_P^{21}y_\eta + X_P^{31}y_\zeta \\ X_P^{11}z_\xi + X_P^{21}z_\eta + X_P^{31}z_\zeta \\ Y_P^1 \end{bmatrix}$$

$$\mathbf{G}_P = \begin{bmatrix} 0 \\ X_P^{12}x_\xi + X_P^{22}x_\eta + X_P^{32}x_\zeta \\ X_P^{12}y_\xi + X_P^{22}y_\eta + X_P^{32}y_\zeta \\ X_P^{12}z_\xi + X_P^{22}z_\eta + X_P^{32}z_\zeta \\ Y_P^2 \end{bmatrix}$$

$$\mathbf{N}_P = \begin{bmatrix} 0 \\ X_P^{13}x_\xi + X_P^{23}x_\eta + X_P^{33}x_\zeta \\ X_P^{13}y_\xi + X_P^{23}y_\eta + X_P^{33}y_\zeta \\ X_P^{13}z_\xi + X_P^{23}z_\eta + X_P^{33}z_\zeta \\ Y_P^3 \end{bmatrix} \tag{5.389}$$

Noting that the terms in Equations 5.388 are linear, we can further split each term of Equation 5.389 to have groupings of the derivatives with respect to ξ, η, and ζ.* Writing:

$$\mathbf{F}_P = \mathbf{F}_{P\xi} + \mathbf{F}_{P\eta} + \mathbf{F}_{P\zeta}$$

$$\mathbf{G}_P = \mathbf{G}_{P\xi} + \mathbf{G}_{P\eta} + \mathbf{G}_{P\zeta}$$

$$\mathbf{N}_P = \mathbf{N}_{P\xi} + \mathbf{N}_{P\eta} + \mathbf{N}_{P\zeta}$$

where the second subscript means that the vector is composed of elements having derivatives with respect to the indicated variable, viz.:

$$\mathbf{F}_{P\xi} = fn(\mathbf{U},\mathbf{U}_\xi) \text{ etc.},$$

the conservation law form (Equation 5.386) is now rewritten as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_H}{\partial \xi} + \frac{\partial \mathbf{G}_H}{\partial \eta} + \frac{\partial \mathbf{N}_H}{\partial \zeta} = \frac{\partial}{\partial \xi}(\mathbf{F}_{P\xi} + \mathbf{F}_{P\eta} + \mathbf{F}_{P\zeta})$$
$$+ \frac{\partial}{\partial \eta}(\mathbf{G}_{P\xi} + \mathbf{G}_{P\eta} + \mathbf{G}_{P\zeta}) + \frac{\partial}{\partial \zeta}(\mathbf{N}_{P\xi} + \mathbf{N}_{P\eta} + \mathbf{N}_{P\zeta}) \tag{5.390}$$

Equation 5.390 is a mixed hyperbolic-parabolic equation. The hyperbolic parts contained in **F**, **G**, **N** are such that:

$$\mathbf{F}_H = fn(\mathbf{U}), \qquad \mathbf{G}_H = fn(\mathbf{U}), \qquad \mathbf{N}_H = fn(\mathbf{U})$$

The main stipulation in this statement is that the pressure $P = \sqrt{g}\, p$ is obtained from an equation of state which is such that P can be expressed as a function of **U**. In particular, for a perfect gas:

$$P = (\gamma - 1)\left[E - \frac{1}{2\sigma}\{(m)^2 + (n^2) + (q)^2\}\right]$$

where $\gamma = C_p/C_v$.

In the case when ξ, η, ζ are the Cartesian coordinates we find that:

* Refer to Problem 5.25.

$$\alpha^1 = 1, \quad \alpha^2 = 0, \quad \alpha^3 = 0$$
$$\beta^1 = 0, \quad \beta^2 = 1, \quad \beta^3 = 0$$
$$\gamma^1 = 0, \quad \gamma^2 = 0, \quad \gamma^3 = 1$$

In the case of curvilinear coordinates all coefficients α^i, β^i, γ^i, are functions of ξ, η, ζ. The flux terms appearing in Equation 5.390 can be written in any desired form. The most prevalent forms are those in which the mixed Cartesian-contravariant velocity components are used. In this connection, the writing of the parabolic parts are usually quite involved algebraically and help can be obtained from Equation 3.167b.

Finite difference and finite element methods for the solution of Equation 5.386 are being pursued vigorously, particularly by the fluid dynamists. The two most prevalent methods from a finite difference point of view are the McCormack explicit hybrid method,[122] and the factored implicit scheme of Warming and Beam.[123] The reader is referred, e.g., to Reference 95 for an in-depth understanding of the construction of various types of flow codes.

5.28 THIN-LAYER NAVIER-STOKES' EQUATIONS (TLNS)

From a computational viewpoint it is usually the case that for high Reynolds' number flows maximum computational effort is expended in resolving the flow field in the direction across the streamwise direction. Based on this observation Baldwin and Lomax[124] suggested dropping the streamwise derivatives in the viscous terms if they are expected to be small. The resulting equations are simpler and require less computational time and storage than the full equations. This type of approximation, although it looks similar to the boundary layer approximation, is entirely different in character. First, the three equations (two in two dimensions) are fully active in this approximation; second, the pressure is not "impressed" on the viscous layer. Thus the TLNS system is expected to provide solutions of the same generic type as those of the Navier-Stokes equations.

The derivation of TLNS equations can be accomplished in a straightforward manner from Equation 5.390. According to the TLNS approximation we neglect all the ξ- and ζ-derivatives of the viscous terms, where ξ and ζ are the coordinates along a surface and η is the outgoing coordinate from a surface. Thus, Equation 5.390 becomes:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_H}{\partial \xi} + \frac{\partial \mathbf{G}_H}{\partial \eta} + \frac{\partial \mathbf{N}_H}{\partial \zeta} = \frac{\partial}{\partial \eta} \mathbf{G}_{P_\eta} \tag{5.391}$$

Equation 5.391 represents a mixed set of hyperbolic-parabolic partial differential equations. The method of solution in Equation 5.391 is the same as any of the time-marching methods for the Navier-Stokes equations. (Refer to Problem 5.25 for the method of obtaining the vector $\mathbf{G}_{P_\eta}$.)

Parabolized Navier-Stokes' Equations (PNS)

The parabolized Navier-Stokes equations were devised earlier than the TLNS equations but with the same aim, viz., to devise a flow model which needs substantially less computational work than the full Navier-Stokes system. Originally work on the PNS equations was initiated by Rubin (cf. references which follow).

Three important restrictions on the Navier-Stokes equations are imposed to obtain the PNS equations:

1. Steady flow.
2. Only the streamwise viscous diffusion terms are neglected.

3. Streamwise velocity is always positive, i.e., there is no flow reversal in the streamwise direction.

Thus, the PNS equation is simply:

$$\frac{\partial \mathbf{F}_H}{\partial \xi} + \frac{\partial \mathbf{G}_H}{\partial \eta} + \frac{\partial \mathbf{N}_H}{\partial \zeta} = \frac{\partial}{\partial \eta}\mathbf{G}_{P\eta} + \frac{\partial}{\partial \zeta}\mathbf{H}_{P\zeta} \tag{5.392}$$

where, in expanding Equation 5.392 in component form the streamwise derivatives appearing in $\mathbf{G}_{P\eta}$ and $\mathbf{H}_{P\zeta}$ should be discarded. This requires a space marching numerical method along the streamwise direction.

The most important problem to be resolved in a PNS model is in regard to the streamwise pressure gradient. Taking ξ to be the streamwise coordinate, the pressure term appearing in vector $\mathbf{F}_H$ has been the most important consideration for research. Next we summarize the various formulations which have proved to be successful:

1. Rudman and Rubin[125] have completely discarded the $\partial p/\partial \xi$ in ξ-momentum equation, but $\partial p/\partial \eta$ and $\partial p/\partial \zeta$ are retained in their respective equations.
2. Lubard and Helliwell[126] have retained the respective pressure gradients in all the momentum equations.

If the streamwise pressure gradient $\partial p/\partial \xi$ is retained in the ξ-momentum equation, then two cases arise. First, the pressure may be calculated simultaneously with the flow field; second, the pressure may be calculated from the external supersonic flow and then "impressed" on the subsonic viscous flow near the body surface. If the first course is taken, then in the subsonic viscous layer the downstream influence can occur and in this situation the parabolic marching technique is not well posed. Exponentially growing solutions, known as the *departure solutions*, then occur and the solution fails to exist. To circumvent this problem Vigneron et al.[127] developed a method in which only part of $\partial p/\partial \xi$ is modeled as $\omega\ \partial p/\partial \xi$ and the remaining part $(1 - \omega)$ $\partial p/\partial \xi$ is either discarded or modeled through the values of p available in the supersonic outer flow. If the second course is taken, then the pressure as available in the external supersonic stream is also taken as the pressure in the subsonic viscous flow near the body surface, i.e., as if the pressure has been impressed on the shear layer near the body surface. This implies that in the subsonic viscous layer $\partial p/\partial \eta \cong 0$. In this connection refer to Lin and Rubin[128] and to Schiff and Steger.[129]

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PROBLEMS

5.1 (a) Verify that, by disregarding any boundary or initial condition, an *exact solution* of the two-dimensional incompressible nonsteady Navier-Stokes equations written in the Cartesian coordinates is given by:

$$u = u_0 e^{-\lambda t} \sin ax \sin by$$

$$v = v_0 e^{-\lambda t} \cos ax \cos by$$

$$\frac{p}{\rho} = \frac{p_0}{\rho} + \frac{1}{4} u_0^2 e^{-2\lambda t} \cos 2ax - \frac{1}{4} v_0^2 e^{-2\lambda t} \cos 2by$$

where u_0, v_0, p_0, a, and b are constants and:

$$v_0 = \frac{au_0}{b}, \qquad \lambda = \nu(a^2 + b^2)$$

(b) Show that the vorticity, deformation rate, and dissipation function are, respectively, given as follows.

$$\omega = \frac{-u_0\lambda}{\nu b} e^{-\lambda t} \sin ax \cos by$$

$$D_{12} = \frac{u_0(b^2 - a^2)}{2b} e^{-\lambda t} \sin ax \cos by$$

$$\phi = \mu u_0^2 e^{-2\lambda t}\left(4a^2 \cos^2 ax \sin^2 by + \frac{(b^2 - a^2)^2}{b^2} \sin^2 ax \cos^2 by\right)$$

(c) Using the Hiemenz flow solution, i.e., Equation 5.40, verify the expression of viscous pressure near a stagnation point. How can one obtain the inviscid pressure near a stagnation point from the viscous pressure?

5.2 (a) Suppose we impose the following conditions on the solution of Equation 5.23:

$$\text{at } t = 0 \text{ for } 0 < y < \infty : u(y, 0) = 0$$

$$\text{at } t > 0 \text{ for } y = 0 : u(0, t) = g(t)$$

Show that the solution of Equation 5.23 is then given as:

$$u(y, t) = \frac{1}{2\sqrt{\pi\nu}} \int_0^t g(\tau) \frac{y}{(t - \tau)^{3/2}} \exp\left\{\frac{-y^2}{4\nu(t - \tau)}\right\} d\tau$$

(b) Let $g(t)$ be a step function:

$$g(t) = 0 \quad \text{for } t < 0$$
$$= U_\infty \text{ for } t > 0$$

then verify that the solution in (a) is the Stokes solution (Equation 5.30).

(c) A viscous flow field is given by

$$u = (a \sin z + c \cos y) \exp(-t/Re)$$

$$v = (b \sin x + a \cos z) \exp(-t/Re)$$

$$w = (c \sin y + b \cos x) \exp(-t/Re)$$

where a,b,c are nondimensional constants and Re is the Reynolds' number.

Use the incompressible Navier–Stokes equations to find the requisite nondimensional pressure field for the given flow. Next, write the differential equations for the streamlines. Take a = b = c = 1 and find the coordinates on a streamline which passes through the point x = 0, y = 0.3, z = 0.5. Use a numerical method (for example, the Runge–Kutta method) and take x as the independent variable in the range $0.0 \leq x \leq 1.0$ to solve the problem.

5.3 For the boundary layer on a semi-infinite plate, write the stream function form of Equation 5.123 for steady flow with $u_e = u_\infty$ = constant. Also rewrite the boundary conditions (Equation 5.124). To establish the validity (and also the uniqueness) of the forms of η and ψ used in Equation 5.135 first note the neither the equation nor its boundary conditions depend on any *characteristic length.*

(a) Introduce an artificial length L in the equations through the transformation:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}\sqrt{\frac{R_e}{2}}, \quad \bar{\psi} = \frac{\psi}{\sqrt{2\nu u_\infty L}}, \quad R_e = \frac{u_\infty L}{\nu}$$

and show that the resulting equation in again independent of L, and that:

$$\psi(x, y) = \sqrt{2\nu u_\infty L}\, \bar{\psi}(\bar{x}, \bar{y})$$

(b) How can the dependence of $\psi(x, y)$ on L in (a) be made to disappear?

5.4 Transform the Navier-Stokes Equations 5.90–5.92 by using the following nondimensional variables:

$$t^* = tU_\infty/L, \quad x^* = x/L, \quad \overline{y^*} = y/L\epsilon, \quad u^* = u/U_\infty$$

$$\overline{v^*} = v/\epsilon U_\infty, \quad p^* = p/\rho U_\infty^2, \quad \epsilon = R_e^{-1/2} = (\nu/U_\infty L)^{1/2}$$

(a) Show that the nondimensional Navier-Stokes equations have different solutions for different values of ϵ.

(b) Use the same transformation in the boundary layer equations (Equations 5.97b) and show that the nondimensional boundary layer equations *do not* explicitly depend on ϵ. Explain.

5.5 (a) Develop a computer program for the numerical solution of the initial-value problem:

$$\phi''' + \phi\phi'' = 0$$

$$\phi(0) = \phi'(0) = 0, \qquad \phi''(0) = 1$$

by using the Runge-Kutta method given below. Here $\phi = \phi(\zeta)$. Suppose the solution at $\zeta_n = nh$, $(n = 0, 1, 2, \ldots, n)$ is known. Writing:

$$\phi_n = \phi(\zeta_n), \qquad \phi_n' = \phi'(\zeta_n), \qquad \phi_n'' = \phi''(\zeta_n)$$

compute the following quantities:

$$A = -\frac{h^3}{6}\phi_n\phi_n''$$

$$B = -\frac{h^3}{6}(\phi_n + 0.5h\phi_n' + 0.125h^2\phi_n'' + 0.125A)\left(\phi_n'' + \frac{3A}{h^2}\right)$$

$$C = -\frac{h^3}{6}(\phi_n + 0.5h\phi_n' + 0.125h^2\phi'' + 0.125A)\left(\phi'' + \frac{3B}{h^2}\right)$$

$$D = -\frac{h^3}{6}(\phi_n + h\phi_n' + 0.5h^2\phi_n'' + C)\left(\phi_n'' + \frac{6C}{h^2}\right)$$

Then:

$$\phi_{n+1} = \phi_n + h\phi_n' + \frac{h^2}{2}\phi_n'' + \frac{1}{20}(9A + 6B + 6C - D)$$

$$\phi_{n+1}' = \phi_n' + h\phi_n'' + \frac{1}{h}(A + B + C)$$

$$\phi_{n+1}'' = \phi_n'' + \frac{1}{h^2}(A + 2B + 2C + D)$$

(b) Check:

$$|\phi_{n+1}' - \phi_n'| \leqslant \epsilon$$

If not, then proceed one more step and repeat the procedure. Stop calculations when the above inequality is satisfied.

(c) Based on the generated solution find α defined as $f''(0) = \alpha$, and then tabulate the Blasius function $f(\eta)$, and $f'(\eta)$, $f''(\eta)$ for each η.

5.6 For similar boundary layers, show that:

$$\frac{d}{dx}\left(\frac{u_e\delta^*}{\nu}\right) = \frac{d_1^2}{\delta^*}$$

where δ^* is the displacement thickness and d_1 has been defined in Equation 5.141d. For similar boundary layers also show that:

(a)
$$\lambda(x) = \beta d_2^2(\beta)$$

where:

$$\lambda(x) = \frac{\theta^2}{\nu}\frac{du_e}{dx}$$

(b)
$$c_f = 2f''(0;\alpha)d_2(\beta)/R_\theta$$

where $\alpha = \alpha(\beta)$, c_f is the local coefficient of friction and:

$$R_\theta = \frac{u_e\theta}{\nu}$$

(c) Show that

$$\frac{\delta^*}{\tau_w}\frac{dp}{dx} = \frac{-\beta d_1}{\alpha}$$

5.7 (a) Starting from the continuity equation (Equation 5.97a) establish the result:

$$v_b(x, t, \delta) = -\frac{\partial u_e}{\partial x}\delta + \frac{\partial}{\partial x}(u_e\delta^*)$$

(b) Write a computer program for the calculation of surface distance $\bar{x}$ if surface Cartesian coordinates $\bar{X}$ and $\bar{Y}$ are given. Use the simple formula:

$$\bar{x}_i = \bar{x}_{i-1} + \{(\bar{X}_i - \bar{X}_{i-1})^2 + (\bar{Y}_i - \bar{Y}_{i-1})^2\}^{1/2}$$

(Note: $\bar{x} = x/L$, $\bar{X} = X/L$, $\bar{Y} = Y/L$; L = characteristic length.)

(c) Let a boundary layer velocity profile be chosen as:

$$\frac{u}{u_e} = \sin\left(\frac{\pi y}{2\delta}\right), \qquad 0 \leq y \leq \delta$$

$$= 1, \qquad y \geq \delta$$

Show that:

$$\frac{\theta}{\delta} = \frac{4-\pi}{2\pi}, \qquad \frac{\delta^*}{\theta} = \frac{2(\pi-2)}{4-\pi}$$

5.8 Let a boundary layer velocity profile be represented as:

$$\frac{u}{u_e} = \tanh\left(\frac{\alpha y}{\delta}\right), \qquad 0 \leq y \leq \delta$$

where α is a suitably chosen constant such that at $y = \delta : u \cong u_e$. Based on this show that another form of the profile is

$$\frac{u}{u_e} = \tanh\left(\frac{Ay}{\theta}\right)$$

where $A = \ell n(\cosh\alpha) + \tanh\alpha - \alpha$.

5.9 For the following external flows use the Gortler variable ξ defined in Equation 5.143 and then develop *analytically* the expansion for $\beta(\xi)$ in powers of ξ up to ξ^5. Compare the coefficients obtained with those in either Equation 5.150 or 5.151, as applicable, for these flows.

(a) Flow along a finite plate in a convergent channel:

$$u_e(x) = \frac{u_\infty L}{L - x}, \qquad 0 \leqslant x \leqslant L$$

where L is the distance of the leading edge from a downstream sink producing the flow.

(b) Flow along a semi-infinite plate in a diverging channel:

$$u_e(x) = \frac{u_\infty L}{L + x}, \qquad x \geqslant 0$$

where L is the distance of the leading edge from an upstream source producing the flow.

(c) Flow past a circular cylinder:

$$u_e(x) = 2u_\infty \sin \frac{x}{R}$$

where R is the radius.

5.10 In continuation of Problem 5.9, find the first five coefficients in the expansion of $\beta(\xi)$ for the following experimentally determined external flow past a circular cylinder in water. Hiemenz in 1911 obtained the empirical formula:

$$u_e(x) = cu_\infty \phi(\bar{x})$$

$$\phi(\bar{x}) = \bar{x} - 0.149462016\,\bar{x}^3 - 0.026065748\bar{x}^5$$

where:

$$\bar{x} = x/R, \quad (R = \text{radius})$$

$$c = 1.815683594$$

$$u_\infty = 19.2 \text{ cm/s}$$

$$R = 4.875 \text{ cm}$$

In the experiment $\nu = 0.01$ cm²/s, and thus:

$$\alpha = \frac{cRu_\infty}{\nu} = 1.69948 \times 10^4$$

Note: In this problem use the formulae given in Equation 5.151.

5.11 Use the values of Table 2 to find the coefficient of skin friction c_f for Problems 5.9 and 5.10 correct to ξ^5.

5.12 Find the approximate separation points ξ/α and then $\bar{x}$ for Problems 5.9(b) and (c), and 5.10. Explain why the separation points for Problems 5.9c and 5.10 are different: (Accurate value of $\bar{x}$ at separation is 1.82387 rad.)

5.13 **(a)** Develop a computer program for the solution of the momentum integral equation based on the recursive relation (Equation 5.166) and then use it for the external flows given in Problems 5.9 and 5.10. Calculate $\lambda(\bar{x})$ for a number of $\bar{x}$-steps. If the solution is to be started from a stagnation point, then it is preferable to calculate λ close to the stagnation point through the formula:

$$\lambda(\bar{x}) = \frac{a + .0472}{b} \phi'(\bar{x})$$

(b) Having calculated $\lambda(\bar{x})$ for a given problem, solve Equation 5.168a numerically for Λ for each λ and then plot u/u_e vs. y/δ for each location.

5.14 The following problem clarifies the role of suction or injection for the prevention of steady boundary layer separation. For details on the boundary layer control (BLC) refer to Schlichting[1]; and Lachman.[130]

Consider the steady boundary layer equations for a flat plate. The equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Let $v_0(x)$ be a prescribed velocity normal to the plate surface at the plate $y = 0$. The function $v_0(x)$ can be either positive or negative. If $v_0(x) > 0$, then it implies injection of fluid; while if $v_0(x) < 0$, then it implies the suction of fluid from the surface. The boundary conditions are

$$\text{at } y = 0 : u = 0,\ v = v_0(x)$$

$$\text{as } y \to \infty : u \to u_\infty$$

(a) Let the plate be infinite in its plane so that there is no dependence on x and then v_0 has to be a constant. Write the equation for u and show that its solution is

$$u = u_\infty(1 - e^{yv_0/\nu}), \qquad v_0 < 0$$

That is, for an infinite plate *only* suction is possible. This velocity profile is called the *asymptotic suction profile.*

(b) For a semi-infinite plate with the leading edge at $x = 0$, introduce the transformation:

$$\eta = y\sqrt{\frac{u_\infty}{2\nu x}}, \qquad \psi = \sqrt{2\nu x u_\infty}\, f(x, \eta)$$

and show that the boundary layer equation takes the form:

$$f''' + ff'' = 2x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right)$$

$$\text{at } \eta = 0 : f = f_0(x),\ f' = 0$$

$$\text{as } \eta \to \infty : f' \to 1$$

Show that $f_0(x)$ in terms of $v_0(x)$ is given by:

$$f_0(x) = -\frac{1}{\sqrt{2\nu x u_\infty}}\int_0^x v_0(x)\, dx$$

$f_0(x) > 0$ suction

$f_0(x) < 0$ injection

It has been found (refer to Reference 1) that the suction profile for a semi-infinite plate tends to the asymptotic suction profile given in (a) when $f_0(x) \geqslant \sqrt{2}$.

(c) Develop a series expansion for $f(x, \eta)$ as:

$$f(x, \eta) = f_0(x) + \frac{\alpha(x)\eta^2}{2!} + \ldots$$

and show that for $f_0(x) = 0$, an approximation for $\alpha(x)$ near $x = 0$ is

$$\alpha(x) = [0.46 f_0 + (0.21 f_0^2 + 0.61)^{1/2}]^3$$

From this expression we conclude that for $f_0(x) > 0$ the boundary layer never separates, while for $f_0(x) < 0$ the boundary layer can separate.

5.15 A useful theorem, known as "Prandtl's Transposition Theorem" is stated as follows: If $u(x, y)$, $v(x, y)$ is a solution of the steady incompressible boundary layer equations, then $U(X, Y)$, $V(X, Y)$ is also a solution where:

$$X = x, \quad Y = y + f(x), \qquad U(X, Y) = u(x, y)$$

Here $f(x)$ is an arbitrary function of x.

(a) Use the equation of continuity to obtain:

$$V(X, Y) = v(x, y) + f'(x)u(x, y)$$

(b) Verify the validity of the theorem by direct substitution into the boundary layer equations.

5.16 Consider the steady-state three-dimensional boundary layer equations in orthogonal curvilinear coordinates, viz., Equations 5.191–5.193, along with the external flow equations. Show that by using the transformation:

$$h_2 = 1, \quad \xi = \xi_1, \quad \eta = \xi_2/h_1h_3, \quad \zeta = \xi_3 \tag{i}$$

and writing:

$$u_1 = u/h_1h_3^2, \quad u_2 = \left(v - u\frac{\partial\eta}{\partial\xi_1} - w\frac{\partial\eta}{\partial\xi_3}\right)\Big/ h_1h_3, \quad u_3 = w/h_1^2h_3$$

the equations take the following form:

$$\frac{\partial u}{\partial\xi} + \frac{\partial v}{\partial\eta} + \frac{\partial w}{\partial\zeta} = 0 \tag{i}$$

$$u\frac{\partial u}{\partial\xi} + v\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial\zeta} - \alpha_1u^2 + \alpha_2uw - \alpha_3w^2$$

$$= u_e\frac{\partial u_e}{\partial\xi} + w_e\frac{\partial u_e}{\partial\zeta} - \alpha_1u_e^2 + \alpha_2u_ew_e - \alpha_3w_e^2 + \nu\frac{\partial u^2}{\partial\eta^2} \tag{ii}$$

$$u\frac{\partial w}{\partial\xi} + v\frac{\partial u}{\partial\eta} + w\frac{\partial w}{\partial\zeta} - \beta_1w^2 + \beta_2uw - \beta_3u^2$$

$$= u_e \frac{\partial w_e}{\partial \xi} + w_e \frac{\partial w_e}{\partial \zeta} - \beta_1 w_e^2 + \beta_2 u_e w_e - \beta_3 u_e^2 + \nu \frac{\partial^2 w}{\partial \eta^2} \qquad \text{(iii)}$$

where:

$$\alpha_1 = \frac{\partial}{\partial \xi} \ell n(h_1 h_3^2), \quad \alpha_2 = \frac{\partial}{\partial \zeta} \ell n\left(\frac{1}{h_3^2}\right), \quad \alpha_3 = \left(\frac{h_3}{h_1}\right)^2 \frac{\partial}{\partial \xi} \ell n(h_3)$$

$$\beta_1 = \frac{\partial}{\partial \zeta} \ell n(h_1^2 h_3), \quad \beta_2 = \frac{\partial}{\partial \xi} \ell n\left(\frac{1}{h_3^2}\right), \quad \beta_3 = \left(\frac{h_1}{h_3}\right)^2 \frac{\partial}{\partial \zeta} \ell n(h_1)$$

Refer to Section 5.18 for the use of these equations.

5.17 (a) In the vorticity-stream function formulation the values of the wall vorticity ω_w are not known a priori. The values of ω_w are determined by the no-slip condition applied to Equation 5.320. If rectangular Cartesian coordinates x and y are used such that x is along the direction of the main flow and y normal to the wall with $y = 0$ as the wall itself, show by expanding $\psi(x, \Delta y)$ in powers of Δy that the wall vorticity is given by:

$$\omega_w = \frac{-2\psi(x, \Delta y)}{(\Delta y)^2} + 0(\Delta y)$$

where $\psi(x, 0) = 0$, and $(\partial\psi/\partial y)_0 = 0$.

(b) Again use the series expansion of $\psi(x, \Delta y)$, and this time retain terms up to the third order. Differentiate:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

with respect to y and show that another expression for the wall vorticity is

$$\omega_w = \frac{-3\psi(x, \Delta y)}{(\Delta y)^2} - \frac{1}{2}\, \omega(x, \Delta y) + 0(\Delta y)^2$$

(c) Evaluating Equation 3.22 on a solid surface S_b under the no-slip condition $\mathbf{u}|_{S_b} = 0$, we get

$$\text{grad } p = -\mu \text{ (curl } \omega) \qquad \text{at } S_b \qquad \text{(i)}$$

Write the components of Equation i with reference to a rectangular Cartesian system (x_i) and show that if $x_2 = 0$ is the flat surface S_b then

$$\mu \frac{\partial \hat{\omega}}{\partial x_2} = i_2 \times (grad\, p) \qquad \text{at } x_2 = 0 \qquad \text{(ii)}$$

where

$$\hat{\omega} = i_1 \omega_1 + i_3 \omega_3$$

(d) Take the cross product of Equation i with **n** (the unit normal vector on S_b) and by using Equation M1.40 establish the formula

$$\mu\left[\frac{\partial\omega}{\partial n}-n\cdot(grad\ \omega)\right]=n\times(grad\ p) \qquad \text{at } S_b \tag{iii}$$

where

$$\omega\cdot n\Big|_{S_b}=0$$

5.18 With reference to a curvilinear coordinate system let a body contour be defined by $\eta = 0$ with the coordinate ξ being taken along the contour. Let u^1, u^2 be the contravariant components of the velocity vector **u**. Then from the equation of continuity:

$$\frac{\partial}{\partial\xi}(\sqrt{g}\,u^1) + \frac{\partial}{\partial\eta}(\sqrt{g}\,u^2) = 0$$

we have:

$$\sqrt{g}\,u^1 = \frac{\partial\psi}{\partial\eta}, \qquad \sqrt{g}\,u^2 = -\frac{\partial\psi}{\partial\xi}$$

which define the stream function ψ. The no-slip condition is

$$\frac{\partial\psi}{\partial\xi} = 0, \qquad \frac{\partial\psi}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0 \tag{i}$$

(a) Use the formula for the Laplacian in curvilinear coordinates, (Equation M4.58) and other formulae from ME.4 as needed, to transform the equation:

$$\nabla^2\psi = -\omega$$

as:

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{\sqrt{g}}\left(g_{22}\frac{\partial\psi}{\partial\xi} - g_{12}\frac{\partial\psi}{\partial\eta}\right)\right\} + \frac{\partial}{\partial\eta}\left\{\frac{1}{\sqrt{g}}\left(g_{11}\frac{\partial\psi}{\partial\eta} - g_{12}\frac{\partial\psi}{\partial\xi}\right)\right\} = -\sqrt{g}\,\omega \tag{ii}$$

(b) Expand $\psi(\xi, \Delta\eta)$ in powers of $\Delta\eta$ and use the condition $\psi(\xi, 0) = 0$ and Equations i and ii to show that the wall vorticity correct to the $0(\Delta\eta)$ is given by:

$$\omega_w = \frac{-2\psi(\xi,\Delta\eta)}{(\Delta\eta)^2}\left(\frac{g_{11}}{g}\right)_w + 0(\Delta\eta)$$

5.19 Using the metric coefficients for an axially symmetric body as developed in ME.7, show that the Navier-Stokes equations with longitudinal and transverse curvature effects are

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rhv) = 0$$

$$\frac{u}{h}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + \frac{kuv}{h} = -\frac{1}{\rho h}\frac{\partial p}{\partial x}$$

$$+ \nu\left[\frac{1}{h^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{rh}\left\{\frac{\partial}{\partial x}\left(\frac{r}{h}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(rh)\frac{\partial u}{\partial y}\right\}\right]$$

$$\frac{u}{h}\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + \frac{ku^2}{h} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$+ \nu\left[\frac{1}{h^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{rh}\left\{\frac{\partial}{\partial x}\left(\frac{r}{h}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}(rh)\frac{\partial v}{\partial y}\right\}\right]$$

5.20 When an infinite plate is started impulsively from rest at a high speed, the ensuing motion of the fluid is a compressible flow. The solution of the equivalent incompressible flow was obtained earlier as Equation 5.30. The problem considered here is also known as the Rayleigh problem. The present solution is valid even when $P_r \neq 1$.

Denoting a dimensional quantity by an asterisk, the pertinent equations are

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial y^*}(\rho^* v^*) = 0 \qquad \text{(i)}$$

$$\rho^*\left(\frac{\partial u^*}{\partial t^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = \frac{\partial}{\partial y^*}\left(\mu^*\frac{\partial u^*}{\partial y^*}\right) \qquad \text{(ii)}$$

$$\rho^* C_p^*\left(\frac{\partial T^*}{\partial t^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = \frac{\partial}{\partial y^*}\left(k^*\frac{\partial T^*}{\partial y^*}\right) + \mu^*\left(\frac{\partial u^*}{\partial y^*}\right)^2 \qquad \text{(iii)}$$

$$p^* = R\rho^* T^* = \text{constant} \qquad \text{(iv)}$$

If ρ_∞^*, T_∞^*, and μ_∞^* are the constant values far away from the plate, then from Equation iv:

$$\rho^* T^* = \rho_\infty^* T_\infty^* \qquad \text{(v)}$$

Further, assuming μ^* to be a linear function of T^*, we have:

$$\mu^*/\mu_\infty^* = T^*/T_\infty^* \qquad \text{(vi)}$$

The initital and the boundary conditions for an isothermal plate are

$$u^*(y^*, 0) = 0, \quad (y^* \neq 0); \qquad u^*(\infty, t^*) = 0, \quad u^*(0, t^*) = u_w^*$$

$$T^*(y^*, 0) = T_\infty^*, \quad (y^* \neq 0); \qquad T^*(\infty, t^*) = T_\infty^*, \quad T^*(0, t^*) = T_w^*$$

here u_w^* and T_w^* are the values at the plate surface.

(a) Introduce the nondimensionalization:

$$t = \frac{t^* u_w^*}{L^*}, \quad T = \frac{T^*}{T_\infty^*}, \quad \mu = \frac{\mu^*}{\mu_\infty^*}, \quad \rho = \frac{\rho^*}{\rho_\infty^*}, \quad u = \frac{u^*}{u_w^*}$$

$$y = \frac{y^*}{L^*}\sqrt{R_e}, \quad v = \frac{v^*}{u_w^*}\sqrt{R_e}, \quad R_e = u_w^* L^* \rho_\infty^*/\mu_\infty^*$$

where L^* is a characteristic length. Next introduce the Howarth-Dorodnitsyn transformation:[91]

$$\eta = \int_0^y \rho(y, t)\, dy, \qquad \tau = t$$

and show that the equations are

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2} \qquad \text{(vii)}$$

$$\frac{\partial T}{\partial \tau} = \frac{1}{P_r}\frac{\partial^2 T}{\partial \eta^2} + (\gamma - 1)M_\infty^2\left(\frac{\partial u}{\partial \eta}\right)^2 \qquad \text{(viii)}$$

where $P_r = \mu^* C_p^*/k^*$ and $\gamma = C_p^*/C_v^*$ *are considered to be constants, and* $M_\infty = u_\infty^*/a_\infty^*$. The initital and the boundary conditions for Equations vii and viii are

$$u(\eta, 0) = 0, \quad \eta \neq 0; \qquad u(\infty, \tau) = 0, \quad u(0, \tau) = 1$$

$$T(\eta, 0) = 1, \quad \eta \neq 0; \qquad T(\infty, \tau) = 1, \quad T(0, \tau) = T_w$$

(b) Introduce the transformation:

$$z = \eta/2\sqrt{\tau}$$

and show that the velocity field is given by:

$$u = \text{erfc}(z)$$

while the temperature equation is

$$\frac{d^2T}{dz^2} + 2P_r z\frac{dT}{dz} = \frac{-4(\gamma - 1)M_\infty^2 P_r}{\pi} e^{-2z^2} \qquad \text{(ix)}$$

with:

$$T = T_w \quad \text{at} \quad z = 0$$

$$T = 1 \quad \text{at} \quad z = \infty$$

(c) Introduce:

$$\zeta = z\sqrt{P_r}$$

and show that the solution of Equation ix is

$$T = [1 - T_w + \phi(\infty)]\,\text{erf}(\zeta) + T_w - \phi(\zeta) \qquad \text{(x)}$$

where $\phi(\zeta)$ is a particular solution of Equation ix such that $\phi(0) = 0$. Show that $\phi(\zeta)$ is given as:

$$\phi(\zeta) = \frac{2(\gamma - 1)M_\infty^2}{\sqrt{k_1\pi}}\int_0^\zeta e^{-\theta^2}\,\text{erf}(\sqrt{k_1}\,\theta)\,d\theta$$

where:

$$k_1 = \frac{2}{P_r} - 1$$

5.21 The coefficient of heat transfer on a solid wall is given by the formula:

$$C_H = \frac{-k_w^*\left(\dfrac{\partial T^*}{\partial n^*}\right)_w}{\rho_\infty^* u_\infty^*(H_w^* - H_\infty^*)}$$

where an asterisk denotes a dimensional quantity. k^* is the coefficient of conductivity ($W/m - K$), H^* is the total enthalpy:

$$H^* = e_t^* + \frac{p^*}{\rho^*}$$

and ρ_∞^*, u_∞^* are the free-stream values. On nondimensionalization:

$$n = n^*/L^*, \quad p = p^*/\rho_\infty^* u_\infty^{*2}, \quad \rho = \rho^*/\rho_\infty^*, \quad e_t = e_t^*/u_\infty^{*2}, \quad \mu = \mu^*/\mu_\infty^*$$

show that for a thermally perfect gas:

$$C_H = \frac{\gamma}{P_r R_e} \frac{\mu_w \left(\frac{\partial e_t}{\partial n}\right)_w}{\left[\left(e_t + \frac{p}{\rho}\right)_w - \left(e_t + \frac{p}{\rho}\right)_\infty\right]}$$

where:

$$\gamma = C_p^*/C_v^*, \qquad P_r = \frac{\mu C_p^*}{k^*} \quad \text{(Prandtl number; assumed constant)}$$

$$R_e = \rho_\infty^* u_\infty^* L^*/\mu_\infty^*$$

5.22 In two dimensions, the shear stress at the wall is given by:

$$\tau_w = \mu_w \left(\frac{\partial u_t}{\partial n}\right)_w$$

where u_t is the component of velocity tangential to the surface. For a 2-D profile, choosing the body fitted coordinates ξ, η with $\eta = 0$ as the surface itself, show on using the no-slip condition that the formula for τ_w is

$$\tau_w = \mu_w \left(\eta_y \frac{\partial u}{\partial \eta} - \eta_x \frac{\partial v}{\partial \eta}\right)_{\eta=0}$$

where u and v are the Cartesian components of **u** and a variable subscript denotes a partial derivative. (Refer also to Problem 3.17.)

5.23 If the covariant components $\zeta_i^{(m)}$ of a vector $\boldsymbol{\zeta}^{(m)}$ satisfy the equations:

$$\zeta_{i,k}^{(m)} + \zeta_{k,i}^{(m)} = 0$$

then the vector $\boldsymbol{\zeta}^{(m)}$ is said to be a Killing vector.

(a) Using the definitions:

$$\zeta_{i,k}^{(m)} = \frac{\partial \zeta_i^{(m)}}{\partial x^k} - \Gamma_{ik}^t \zeta_t^{(m)}$$

$$\zeta_{k,i}^{(m)} = \frac{\partial \zeta_k^{(m)}}{\partial x^i} - \Gamma_{ik}^t \zeta_t^{(m)}$$

verify the following results:

(i) For rectangular Cartesian coordinates:

$m = 1$: $\zeta_1 = 1, \ \zeta_2 = 0, \ \zeta_3 = 0$
$m = 2$: $\zeta_1 = 0, \ \zeta_2 = 1, \ \zeta_3 = 0$
$m = 3$: $\zeta_1 = 0, \ \zeta_2 = 0, \ \zeta_3 = 1$

(ii) For cylindrical coordinates:

$m = 1$: $\zeta_1 = \cos\phi, \ \zeta_2 = -r\sin\phi, \ \zeta_3 = 0$
$m = 2$: $\zeta_1 = \sin\phi, \ \zeta_2 = r\cos\phi, \ \zeta_3 = 0$
$m = 3$: $\zeta_1 = 0, \ \zeta_2 = 0, \ \zeta_3 = 1$

(iii) For spherical coordinates:

$m = 1$: $\zeta_1 = \sin\theta\cos\phi, \ \zeta_2 = r\cos\theta\cos\phi, \ \zeta_3 = -r\sin\theta\sin\phi$
$m = 2$: $\zeta_1 = \sin\theta\sin\phi, \ \zeta_2 = r\cos\theta\sin\phi, \ \zeta_3 = r\sin\theta\cos\phi$
$m = 3$: $\zeta_1 = \cos\phi, \ \zeta_2 = -r\sin\phi, \ \zeta_3 = 0$

(b) Show that the differential conservation law form in terms of the Killing vectors[131] for each m is

$$\frac{\partial}{\partial \tau}(\sqrt{g}A^i\zeta_i) + \frac{\partial}{\partial x^k}(\sqrt{g}\zeta_i f^{ik}) = C^i\zeta_i\sqrt{g}$$

5.24 Suppose it is desired to obtain the three-dimensional incompressible boundary layer equations referred to general steady coordinates in the strong conservation law form. The starting point is then naturally the Navier-Stokes equations in the form of Equation 5.349. For incompressible flow with time-independent coordinates and in dimensional variables the equation is

$$\frac{\partial}{\partial t}(\sqrt{g}\mathbf{u}) + \frac{\partial}{\partial x^j}[\sqrt{g}\{\mathbf{u}u^j + \frac{p}{\rho}\,\mathrm{grad}\,x^j - \Pi^{ij}\mathbf{a}_i\}] = 0 \qquad \textbf{(i)}$$

where:

$$\Pi^{ij} = \nu(u^i_{,m}g^{mj} + u^j_{,n}g^{ni})$$

(a) Write the three equations from Equation i using:

$$\mathbf{u} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$
$$\mathbf{a}_i = \mathbf{i}\frac{\partial x}{\partial x^i} + \mathbf{j}\frac{\partial y}{\partial x^i} + \mathbf{k}\frac{\partial z}{\partial x^i}, \qquad i = 1, 2, 3$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the constant unit vectors and u, v, w are the rectangular Cartesian coordinates. The continuity equation along with the three equations from Equation i can be written in the divergence form:

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial \xi}(\mathbf{F} + \mathbf{F}_p) + \frac{\partial}{\partial \eta}(\mathbf{G} + \mathbf{G}_p) + \frac{\partial}{\partial \zeta}(\mathbf{H} + \mathbf{H}_p) = 0 \qquad \textbf{(ii)}$$

where $x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$ and:

$$\mathbf{E} = \sqrt{g}\begin{bmatrix} 0 \\ u \\ v \\ w \end{bmatrix}$$

$$\mathbf{F} = \sqrt{g} \begin{bmatrix} u^1 \\ uu^1 + \frac{p}{\rho}\xi_x \\ vu^1 + \frac{p}{\rho}\xi_y \\ wu^1 + \frac{p}{\rho}\xi_z \end{bmatrix}$$

$$\mathbf{G} = \sqrt{g} \begin{bmatrix} u^2 \\ uu^2 + \frac{p}{\rho}\eta_x \\ vu^2 + \frac{p}{\rho}\eta_y \\ wu^2 + \frac{p}{\rho}\eta_z \end{bmatrix}$$

$$\mathbf{H} = \sqrt{g} \begin{bmatrix} u^3 \\ uu^3 + \frac{p}{\rho}\zeta_x \\ vu^3 + \frac{p}{\rho}\zeta_y \\ wu^3 + \frac{p}{\rho}\zeta_z \end{bmatrix}$$

$$\mathbf{F}_p = -\sqrt{g} \begin{bmatrix} 0 \\ \Pi^{i1}\frac{\partial x}{\partial x^i} \\ \Pi^{i1}\frac{\partial y}{\partial x^i} \\ \Pi^{i1}\frac{\partial z}{\partial x^i} \end{bmatrix}$$

$$\mathbf{G}_p = -\sqrt{g} \begin{bmatrix} 0 \\ \Pi^{i2}\frac{\partial x}{\partial x^i} \\ \Pi^{i2}\frac{\partial y}{\partial x^i} \\ \Pi^{i2}\frac{\partial z}{\partial x^i} \end{bmatrix}$$

$$\mathbf{H}_p = -\sqrt{g} \begin{bmatrix} 0 \\ \Pi^{i3}\frac{\partial x}{\partial x^i} \\ \Pi^{i3}\frac{\partial y}{\partial x^i} \\ \Pi^{i3}\frac{\partial z}{\partial x^i} \end{bmatrix}$$

Note that:

$$u = u^i \frac{\partial x}{\partial x^i}, \qquad v = u^i \frac{\partial y}{\partial x^i}, \qquad w = u^i \frac{\partial z}{\partial x^i}$$

$$u^m = \mathbf{u} \cdot \mathbf{a}^m = \mathbf{u} \cdot \text{grad } x^m, \qquad m = 1, 2, 3$$

(b) Now envisage a body in which the surface coordinates are denoted as ξ, ζ and on which η is the outgoing coordinate. Introduce $\overline{u^2} = u^2/\sqrt{\nu}$ and $\overline{\eta}$ *in all the flow quantities. On taking the inner limit show that the boundary layer equations in divergence form can be written as:*

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \xi} + \frac{\partial}{\partial \eta}(\mathbf{G} + \mathbf{G}_p) + \frac{\partial \mathbf{H}}{\partial \zeta} = 0 \qquad \text{(ii)}$$

(c) Write also the boundary layer approximation of **G**. Note that Equation ii also represents four equations.

5.25 (a) Consider the formulae for the covariant derivative of contravariant components (Equation M1.97) and of the divergence of a vector (Equation M1.107). Use Equations M1.90 and M1.91 to establish the following results:

$$u^i_{,j} = \frac{\partial \mathbf{u}}{\partial x^j} \cdot a^i \qquad \text{(i)}$$

$$\operatorname{div} \mathbf{u} = \Delta = \frac{\partial \mathbf{u}}{\partial x^j} \cdot a^j \qquad \text{(ii)}$$

$$2D^{ik} = \frac{\partial \mathbf{u}}{\partial x^s} \cdot \mathbf{a}^i g^{sk} + \frac{\partial \mathbf{u}}{\partial x^r} \cdot \mathbf{a}^k g^{ri} \qquad \text{(iii)}$$

(b) Write Equation i–iii in two dimensions when **u** is expressed in terms of its Cartesian components.

(c) Using the result in (b), write X_p^{ik} ($i = 1, 2, k = 1, 2$) defined in Equation 5.388a by using Stokes' formula $3\lambda + 2\mu = 0$.

(d) Using the result in (c) write the expressions for:

$$X_p^{12} x_\xi + X_p^{22} x_\eta$$

and:

$$X_p^{12} y_\xi + X_p^{22} y_\eta$$

of the vector $\mathbf{G}_p$ defined in Equation 5.389. Separately, write the parabolic parts $\mathbf{G}_{p\xi}$ and $\mathbf{G}_{p\eta}$ having the derivatives with respect to ξ and η, respectively.

CHAPTER SIX

Turbulent Flow

Part I: Stability Theory and the Statistical Description of Turbulence

6.1 INTRODUCTION

The most prevalent form of fluid flow in nature is of an irregular and chaotic form. If in addition to it being irregular and chaotic the flow field is also diffusive and dissipative, then the fluid motion is said to form a turbulent flow field. The evaluation or computation of such flow fields by deterministic methods, which in principle are expected to resolve the minutest details of a random field, is extremely tedious and too detailed for engineering purposes (see, e.g., Reference 1). Thus, suitable statistical methods have to be used to analyze the turbulent flow fields. However, before we start on the use of statistical methods, it is important to consider various subsidiary problems so as to provide a connected and well-rounded treatment to the problems of formulation and solution of turbulent flow equations.

6.2 STABILITY OF LAMINAR FLOWS

In real flows, a number of disturbing sources exist which interact with the main flow field in such a manner that small disturbances may not remain small under certain given conditions of the flow. These disturbances are practically unavoidable and, therefore, are always present in every problem. On the other hand, with the exception of some flow regions, the state of laminar flow is simply a theoretical idealization of a real flow field whose results may or may not coincide with the results of actual observations. The problem, then, is to analyze the effects of these disturbances on a given state of laminar flow field and determine whether these disturbances decay or grow in time or in space. Let a laminar incompressible flow field under the free-stream Reynolds number R_e be available. We now assume that the imposed disturbances are infinitesimal (very small) to start with but they may decay or grow depending on the given value of R_e. If for a given R_e the disturbances decay or grow with the passage of time or in space, then the laminar flow is said to be *stable* or *unstable*, respectively, under the influence of an initial set of small disturbances.

The theory of stability does not provide an answer to the problem of transition from laminar to turbulent flow. However, it yields some qualitative indications in the direction of transition which have been used in devising a few empirical transition models for boundary layer flows, and free flows in wakes and jets. The stability theory which uses the concept of infinitesimal disturbances and which does not go beyond the first approximation is termed the *linear stability theory*. The subject of linear stability theory as it stands today is due to the contributions of Tollmien,[2] Lin,[3] and Schlichting.[4] Refer also to an article by Shen.[5] (The importance of the linear theory of stability is that it ensures nonlinear instability whenever there is linear instability.)

Formulation of the Problem

The Navier-Stokes equations for an incompressible flow in the Cartesian tensor notation are

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i \tag{6.1}$$

$$\frac{\partial u_k}{\partial x_k} = 0 \tag{6.2}$$

where $j = 1, 2, 3$. Let a *steady* laminar solution (U_j, P) be already available, satisfying the equations:

$$U_k \frac{\partial U_j}{\partial x_k} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \nabla^2 U_j \tag{6.3}$$

$$\frac{\partial U_k}{\partial x_k} = 0 \tag{6.4}$$

and that *this solution satisfies all the imposed conditions*, e.g., no-slip, free-stream conditions, etc. of the problem. We now consider the superposition of infinitesimal disturbances (u_j', p') on (U_j, P) and write:

$$u_j = U_j + u_j', \qquad p = P + p' \tag{6.5}$$

Because of the infinitesimal nature of u_j' and p' we can neglect their squares and higher powers wherever they occur. Substituting Equation 6.5 in Equations 6.1 and 6.2 and using Equations 6.3 and 6.4, we get the linear equations:

$$\frac{\partial u_j'}{\partial t} + U_k \frac{\partial u_j'}{\partial x_k} + u_k' \frac{\partial U_j}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \nabla^2 u_j' \tag{6.6}$$

$$\frac{\partial u_k'}{\partial x_k} = 0 \tag{6.7}$$

where $j = 1, 2, 3$. Since the imposed boundary conditions on the flow have all been satisfied by the solution (U_j, P), the boundary conditions for Equations 6.6 and 6.7 are necessarily *homogeneous.*

We now consider the case when perturbations are described by natural sinusoidal oscillations with small amplitudes. Taking advantage of the linearity of Equations 6.6 and 6.7 and the fact that the laminar solution (U_j, P) is independent of time t, we write:

$$\begin{aligned} u_j' &= e^{-i\sigma t} f_j(x_1, x_2, x_3) \\ p' &= e^{-i\sigma t} g(x_1, x_2, x_3) \end{aligned} \tag{6.8}$$

where $i = (-1)^{1/2}$ and σ is a complex parameter:

$$\sigma = \sigma_r + i\sigma_i$$

having the dimension $(\text{time})^{-1}$. Substituting Equation 6.8 in Equation 6.6 and 6.7 we obtain:

$$-i\sigma f_j + U_k \frac{\partial f_j}{\partial x_k} + f_k \frac{\partial U_j}{\partial x_k} = -\frac{1}{\rho}\frac{\partial g}{\partial x_j} + \nu \nabla^2 f_j \tag{6.9}$$

$$\frac{\partial f_k}{\partial x_k} = 0 \tag{6.10}$$

The boundary conditions for f_j are also homogeneous. Equations 6.9 and 6.10 under the homogeneous boundary conditions now carry σ as a parameter. A nontrivial solution of the equations will be possible for some selected values of the parameter σ; and, therefore, an eigenvalue problem for σ and f_i results. Based on the form of perturbations in Equation 6.8 we can easily determine the growth or decay of solutions with time by checking the signs of the imaginary part σ_i. Since:

$$e^{-i\sigma t} = e^{\sigma_i t} \cdot e^{-i\sigma_r t}$$

then it follows that:

1. For $\sigma_i > 0$, the disturbances grow with time.
2. For $\sigma_i < 0$, the disturbances decay with time.
3. For $\sigma_i = 0$, the disturbances neither grow nor decay.

Therefore if the eigenvalue σ is real ($\sigma_i = 0$), then we have a neutral temporal stability condition. For the investigation of the spatial stability conditions in a simple way, we consider the disturbances as an oblique traveling wave in the x_1x_3-plane so that the forms of f_j and g are

$$\begin{aligned} f_j(x_1, x_2, x_3) &= \psi_j(x_2)e^{i(k_1x_1 + k_3x_3)} \\ g(x_1, x_2, x_3) &= h(x_2)e^{i(k_1x_1 + k_3x_3)} \end{aligned} \tag{6.11}$$

where k_1 and k_3 are the complex wave numbers along x_1 and x_3, respectively, having the dimension of $(\text{length})^{-1}$. Substituting Equation 6.11 in Equations 6.9 and 6.10, we have:

$$\begin{aligned} &i(-\sigma + k_1U_1 + k_3U_3)\psi_j + U_2\frac{\partial\psi_j}{\partial x_2} + \psi_k\frac{\partial U_j}{\partial x_k} \\ &= -\frac{1}{\rho}\frac{\partial h}{\partial x_j}\delta_{2j} - \frac{ih}{\rho}(k_1\delta_{1j} + k_3\delta_{3j}) + \nu\frac{\partial^2\psi_j}{\partial x_2^2} - \nu(k_1^2 + k_3^2)\psi_j \end{aligned} \tag{6.12}$$

$$\frac{\partial\psi_2}{\partial x_2} + i(k_1\psi_1 + k_3\psi_3) = 0 \tag{6.13}$$

where $\delta_{\alpha j}$ is the Kronecker delta. For a fixed value of σ, Equations 6.12 and 6.13 form an eigenvalue problem for wave numbers k_1 and k_3. For negative imaginary parts of either k_1 or k_3 the condition of spatial instability results.

We now nondimensionalize Equations 6.12 and 6.13 by using a characteristic length L and a characteristic velocity U_∞ through the following scheme:

$$\begin{aligned} &x = \frac{x_1}{L}, \quad y = \frac{x_2}{L}, \quad z = \frac{x_3}{L}, \quad \alpha = Lk_1, \quad \beta = Lk_3 \\ &\tau = \frac{U_\infty t}{L}, \quad U = \frac{U_1}{U_\infty}, \quad V = \frac{U_2}{U_\infty}, \quad W = \frac{U_3}{U_\infty} \\ &\phi_1 = \frac{\psi_1}{U_\infty}, \quad \phi_2 = \frac{\psi_2}{U_\infty}, \quad \phi_3 = \frac{\psi_3}{U_\infty} \\ &\chi = \frac{h}{\rho U_\infty^2}, \quad \omega = \frac{\sigma L}{U_\infty}, \quad R_e = \frac{U_\infty L}{\nu} \end{aligned} \tag{6.14}$$

On nondimensionalization and introducing $\gamma^2 = \alpha^2 + \beta^2$, we get:

$$i\phi_1(-\omega + \alpha U + \beta W) + V\frac{d\phi_1}{dy} + \phi_1\frac{\partial U}{\partial x} + \phi_2\frac{\partial U}{\partial y} + \phi_3\frac{\partial U}{\partial z}$$

$$= -i\alpha\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_1 \tag{6.15}$$

$$i\phi_2(-\omega + \alpha U + \beta W) + V\frac{d\phi_2}{dy} + \phi_1\frac{\partial V}{\partial x} + \phi_2\frac{\partial V}{\partial y} + \phi_3\frac{\partial V}{\partial z}$$

$$= -\frac{d\chi}{dy} + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_2 \tag{6.16}$$

$$i\phi_3(-\omega + \alpha U + \beta W) + V\frac{d\phi_3}{dy} + \phi_1\frac{\partial W}{\partial x} + \phi_2\frac{\partial W}{\partial y} + \phi_3\frac{\partial W}{\partial z}$$

$$= -i\beta\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_3 \tag{6.17}$$

$$\frac{d\phi_2}{dy} + i(\alpha\phi_1 + \beta\phi_3) = 0 \tag{6.18}$$

6.3 FORMULATION FOR PLANE PARALLEL LAMINAR FLOWS

The set of equations formed by Equations 6.15–6.18, although linear in character, are quite complicated to solve. We, therefore, proceed to consider a simplified version of these equations without disturbing the central idea of the problem. First, we interpret x, y, z as a local general orthogonal coordinate system with y as the coordinate perpendicular to the main flow direction or orthogonal to a wall in boundary layer problems. Second, the variation of laminar velocity components in the directions of x and z is assumed to be much smaller than the variation along the y-direction. Thus locally the laminar flow is considered to be a function of the coordinate y only, i.e.,

$$U = U(y), \qquad V = V(y), \qquad W = W(y)$$

From the continuity equation, we then conclude that $V = 0$. This approximation is an exact formulation for a laminar flow past an infinite plate. With the locally parallel approximation Equations 6.15–6.17 become:

$$i\phi_1(-\omega + \alpha U + \beta W) + \phi_2\frac{dU}{dy} = -i\alpha\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_1 \tag{6.19}$$

$$i\phi_2(-\omega + \alpha U + \beta W) = -\frac{d\chi}{dy} + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_2 \tag{6.20}$$

$$i\phi_3(-\omega + \alpha U + \beta W) + \phi_2\frac{dW}{dy} = -i\beta\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_3 \tag{6.21}$$

Equations 6.18–6.21 form the stability problem for three-dimensional perturbations superposed on three-dimensional laminar flow fields. As a further simplification, we consider the two-dimensional laminar flow for which $W = 0$. The equations then simplify to:

$$i\phi_1(-\omega + \alpha U) + \phi_2 \frac{dU}{dy} = -i\alpha\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_1 \tag{6.22}$$

$$i\phi_2(-\omega + \alpha U) = -\frac{d\chi}{dy} + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_2 \tag{6.23}$$

$$i\phi_3(-\omega + \alpha U) = -i\beta\chi + \frac{1}{R_e}\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_3 \tag{6.24}$$

$$\frac{d\phi_2}{dy} + i(\alpha\phi_1 + \beta\phi_3) = 0 \tag{6.25}$$

which are the equations for the three-dimensional perturbations superposed on the two-dimensional laminar flow. Eliminating χ between Equations 6.22 and 6.23 and using Equation 6.25 for ϕ_1, we obtain a fourth order equation containing ϕ_2 and ϕ_3. Next, eliminating χ between Equations 6.23 and 6.24 we get another equation containing ϕ_2 and ϕ_3. Using the second equation in the first equation we obtain a single differential equation in $\phi_2(y)$, which is

$$(U - c)\left(\frac{d^2}{dy^2} - \gamma^2\right)\phi_2 - \phi_2\frac{d^2U}{dy^2} = \frac{-i}{\alpha R_e}\left(\frac{d^4}{dy^4} - 2\gamma^2\frac{d^2}{dy^2} + \gamma^4\right)\phi_2 \tag{6.26}$$

where:

$$c = \frac{\omega}{\alpha} = c_r + ic_i$$

is a complex nondimensional velocity called the phase speed of perturbation. The dimensional complex velocity is

$$v = \frac{\sigma}{k_1} = v_r + iv_i$$

so that:

$$c = \frac{v}{U_\infty} = \frac{\omega}{\alpha}$$

A three-dimensional disturbance of the form:

$$\exp[i(\alpha x + \beta z - \omega\tau)]$$

with the amplitude as a function of y is an oblique wave with the wave front direction lying obliquely in the xz-plane. This type of sinusoidal disturbance is called a Tollmien-Schlichting wave. Following Dunn and Lin,[6] it is possible to devise a transformation in the xz-plane such that one coordinate ξ is chosen along the direction of motion of the wave front and the other coordinate η as orthogonal to it in the same plane, which necessarily falls along the front as shown in Figure 6.1.

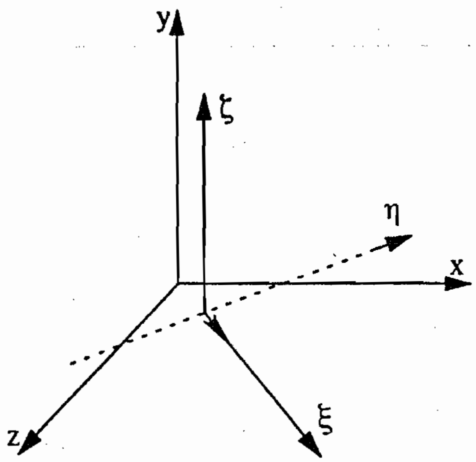


Fig. 6.1 Oblique wave propagating in the xz-plane.

This transformation is equivalent to setting $\beta = 0$ and then interpreting ξ as x. In this case Equation 6.25 becomes:

$$i\alpha\phi_1 + \frac{d\phi_2}{dy} = 0$$

which is identically satisfied by introducing a function $\phi(y)$ as:

$$\phi_2 = -i\alpha\phi, \qquad \phi_1 = \frac{d\phi}{dy}$$

Thus, Equation 6.26 becomes:

$$(U - c)(\phi'' - \alpha^2\phi) - \phi U'' = \frac{-i}{\alpha R_e}(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi) \tag{6.27}$$

where primes and iv denote differentiation with respect to y. Equation 6.27 is called the *Orr-Sommerfeld equation.*

Note that the above derivation is general and extends the applicability of Equation 6.27. A short way is to consider the 2-*D* Navier-Stokes equation in ψ and write $\psi = \Psi(y) + \psi'(x, y, t)$. Further, writing $\psi' = \phi(y)\exp\{i\alpha(x - ct)\}$, one gets Equation 6.27.

Historically (refer to Reference 4 or 7) the Orr-Sommerfeld equation was obtained for the two-dimensional perturbations superposed on a two-dimensional laminar flow to simplify the mathematics involved in the formulation. However, the simplifications introduced at various stages in the preceding derivation, leading again to the same equation (Equation 6.27), show that the applicability of the Orr-Sommerfeld equation goes beyond its originally intended bounds. In essence, one can now interpret the Orr-Sommerfeld equation to be applicable to the case of three-dimensional perturbations superposed on two-dimensional laminar flows if the coordinate x is interpreted as aligned along the direction of motion of the wave front. The complex velocity c is then interpreted as the *velocity of wave propagation.*

Squire's Theorem:

In Equation 6.26 if we make the substitution:

$$\alpha R_e = \gamma \bar{R}_e$$

then we get:

$$(U - c)(\phi_2'' - \gamma^2\phi_2) - \phi_2 U'' = \frac{-i}{\gamma \overline{R}_c}(\phi_2'''' - 2\gamma^2\phi_2'' + \gamma^4\phi_2) \tag{6.28}$$

which is exactly of the same form as Equation 6.27. Noting that:

$$\overline{R}_c = \frac{\alpha R_c}{(\alpha^2 + \beta^2)^{1/2}}$$

so that:

$$\overline{R}_c < R_c$$

If the solution of Equation 6.26 at a given value of R_c is unstable, then the solution of Equation 6.28 is unstable at a lower Reynolds number, i.e., at $\overline{R}_c$. Consequently, the problem of three-dimensional perturbations at a given Reynolds number R_c is equivalent to the problem of two-dimensional perturbations at a lower Reynolds number $\overline{R}_c$. In other words, if the direction of the wave propagation is along the direction of the main laminar flow, then this flow is likely to be unstable at a lower Reynolds number in comparison to the case when the wave front is oblique to the main flow direction. Thus, the growth rate of an unstable 3-D wave is equal to that of an unstable 2-D wave at a lower R_c. This is known as Squire's theorem.[8]

Temporal and Spatial Instabilities

In the Orr-Sommerfeld equation (Equation 6.27) the function $\phi(y)$ is the nondimensional amplitude of the perturbation stream function which is related to the velocity components u' and v' along x- and y-axes, respectively, as:

$$u' = \frac{u_1'}{U_\infty} = \frac{d\phi}{dy} e^{i(\alpha x - \omega\tau)} \tag{6.29a}$$

$$v' = \frac{u_2'}{U_\infty} = -i\alpha\phi e^{i(\alpha x - \omega\tau)} \tag{6.29b}$$

The nondimensional perturbation stream function defined through

$$u' = \frac{\partial\psi'}{\partial y}, \qquad v' = -\frac{\partial\psi'}{\partial x}$$

is then:

$$\psi'(x, y, \tau) = \phi(y)e^{i(\alpha x - \omega\tau)} \tag{6.29c}$$

where a prime denotes a perturbation quantity. In general, both α and ω are complex, i.e.:

$$\alpha = \alpha_r + i\alpha_i, \qquad \omega = \omega_r + i\omega_i$$

and both are obtained as part of the solution of an eigenvalue problem. If $\alpha_i = 0$ and α_r is a prescribed constant, then ω is obtained by solving the Orr-Sommerfeld equation and the amplitude will vary with time as $\exp(\omega_i\tau)$. If $\omega_i > 0$, then the instability is called the temporal instability. If $\omega_i = 0$ and ω_r is a prescribed constant, then α is obtained by solving the Orr-

Sommerfeld equation and the amplitude will vary with x as $\exp(-\alpha_i x)$. If $\alpha_i < 0$, then the instability is called the spatial instability.

Boundary Conditions for the Orr-Sommerfeld Equation

It was pointed out earlier that boundary conditions for the perturbations are necessarily homogeneous. To demonstrate the formulation of boundary conditions, we consider the following physical situations. It must be noted that a prime on ϕ is a derivative with respect to y.

(i) Flow Between Parallel Plates. Let $y = y_1$ and $y = y_2$ be two parallel plates. Since on a rigid wall the perturbation velocities are zero, from Equations 6.29:

$$y = y_1 : \phi = \phi' = 0$$

$$y = y_2 : \phi = \phi' = 0 \tag{6.30}$$

(ii) Flow on a Plate with Disturbances Symmetric about a Line. Let $y = y_2$ be the line about which the disturbances are symmetric and $y = y_1$ be the surface of the plate. Performing a Taylor expansion about $y = y_2$, we have:

$$u'(y_2 + h) = u'(y_2) + h\left(\frac{du'}{dy}\right)_{y=y_2} + \frac{h^2}{2}\left(\frac{d^2u'}{dy^2}\right)_{y=y_2} + \dots$$

$$u'(y_2 - h) = u'(y_2) - h\left(\frac{du'}{dy}\right)_{y=y_2} + \frac{h^2}{2}\left(\frac{d^2u'}{dy^2}\right)_{y=y_2} + \dots$$

Because of symmetry:

$$u'(y_2 + h) = u'(y_2 - h)$$

hence:

$$\text{at } y = y_2 : \frac{du'}{dy} = 0$$

also:

$$\text{at } y = y_2 : v' = 0$$

Thus, the complete boundary conditions for ϕ are,

$$y = y_1 : \phi = 0,\ \phi' = 0$$

$$y = y_2 : \phi = 0,\ \phi'' = 0 \tag{6.31}$$

(iii) Flow on a Plate with Disturbances Antisymmetric about a Line. Performing the Taylor expansion and because of antisymmetry:

$$u'(y_2 + h) = -u'(y_2 - h)$$

we have:

$$\text{at } y = y_2 : u' = 0,\ \frac{d^2u'}{dy^2} = 0$$

Hence, the complete boundary conditions are

$$y = y_1 : \phi = 0,\ \phi' = 0$$
$$y = y_2 : \phi' = 0,\ \phi''' = 0 \tag{6.32}$$

(iv) Boundary Conditions for Boundary Layer Problems. Let $y = \delta$ be the boundary layer edge where obviously:

$$U \to U_e$$

and:

$$U'' \to 0$$

Thus, in this region Equation 6.27 becomes:

$$(U_e - c)(\phi'' - \alpha^2\phi) + \frac{i}{\alpha R_e}(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi) = 0$$

Since U_e is a local constant, the above equation is a linear ordinary differential equation with constant coefficients. Arranging the terms, we have:

$$\phi^{iv} - (\alpha^2 + \xi^2)\phi'' + \alpha^2\xi^2\phi = 0$$

where:

$$\xi^2 = \alpha^2 + i\alpha R_e(U_e - c)$$

Assuming a solution of the form $\phi = \exp(my)$, the general solution is

$$\phi = A_1e^{\alpha y} + A_2e^{-\alpha y} + A_3e^{\xi y} + A_4e^{-\xi y}$$

where A_1, A_2, etc. are arbitrary constants. As $y \to \infty$ (on the boundary layer scale) or δ, the positive exponentials become unbounded, hence, it is natural to set A_1 and A_3 equal to zero. In place of directly setting A_1 and A_3 equal to zero, it is much more revealing to satisfy this requirement by first solving the system of four equations with ϕ, ϕ', ϕ'', ϕ''' evaluated at $y = \delta$ as the right-hand sides of the equations. The requirement $A_1 = 0$ is then equivalent to:

$$\begin{vmatrix} \phi & 1 & 1 & 1 \\ \phi' & -\alpha & \xi & -\xi \\ \phi'' & \alpha^2 & \xi^2 & \xi^2 \\ \phi''' & -\alpha^3 & \xi^3 & -\xi^3 \end{vmatrix} = 0$$

and the requirement $A_3 = 0$ is equivalent to:

$$\begin{vmatrix} 1 & 1 & \phi & 1 \\ \alpha & -\alpha & \phi' & -\xi \\ \alpha^2 & \alpha^2 & \phi'' & \xi^2 \\ \alpha^3 & -\alpha^3 & \phi''' & -\xi^3 \end{vmatrix} = 0$$

Writing D for a prime (differentiation) we have:

$$\left.\begin{aligned} Z &= (D^2 - \xi^2)(D + \alpha)\phi = 0 \\ \zeta &= (D^2 - \alpha^2)(D + \xi)\phi = 0 \end{aligned}\right\} \text{ as } y \to \delta \tag{6.33a}$$

Equations 6.33a then give the relations that must be satisfied by the solution ϕ when $y \to \delta$. The boundary conditions at the wall $y = 0$ in a boundary layer problem are obviously given by:

$$y = 0 : u' = 0, v' = 0$$

which in terms of ϕ are

$$y = 0 : \phi = \phi' = 0 \tag{6.33b}$$

As stated earlier, solution of the Orr-Sommerfeld equation (Equation 6.27) under homogeneous boundary conditions is an eigenvalue problem. Although the equation is linear, it cannot be solved in a closed form once and for all which could be applied to all cases. Only numerical methods can successfully be used to solve Equation 6.27 for each specific case. Numerical methods of solution started receiving attention after the pioneering work of Lin (refer to the original references in Reference 7). From then on, many have contributed to this field. The effort in this section is to expose the basic methodology of the numerical approach for the Orr-Sommerfeld equation.

First, one must identify the basic parameters of Equation 6.27, which are

$U(y)$: the local laminar velocity profile

α : the nondimensional wave number (complex)

R_e : the Reynolds number

c : the velocity of wave propagation (complex)

Note that, in general:

$$c = \frac{\omega}{\alpha} = c_r + ic_i$$

where:

$$c_r = \frac{\omega_r\alpha_r + \omega_i\alpha_i}{\alpha_r^2 + \alpha_i^2}$$

$$c_i = \frac{\omega_i\alpha_r - \omega_r\alpha_i}{\alpha_r^2 + \alpha_i^2}$$

For temporal stability problems $\alpha_i = 0$ and α_r is a prescribed constant, and for spatial stability problems $\omega_i = 0$ and ω_r is a prescribed constant.

Temporal Stability

In this subsection we consider the temporal stability problem so that in any given problem $U(y)$,

R_e and a *real* α have to be prescribed. The differential equation on solution then furnishes one eigenfunction ϕ and one complex eigenvalue c for each pair (α, R_e), so that:

$$c_r = c_r(\alpha, R_e)$$

$$c_i = c_i(\alpha, R_e)$$

If $c_i < 0$, the disturbances decay in time; while for $c_i > 0$, the disturbances grow in time. It is of interest to know those pairs of values (α, R_e) for which $c_i = 0$ thus obtaining a curve in the αR_e-plane which separates the regions of stability and instability. These curves are known as the curves of *neutral stability*. The point on this curve at which R_e is lowest is called the *critical Reynolds number*, R_{cr}. A typical neutral stability curve has been shown in Figure 6.2. The significance of R_{cr} is that below this Reynolds number the motion is stable for all values of α_r.

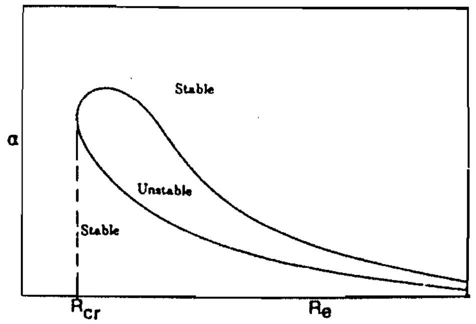


Fig. 6.2 Typical curve of neutral stability in the αR_e-plane.

Since the instability usually occurs at high Reynolds' numbers, then the case $R_e \to \infty$ is of interest. It happens that even for $R_e \to \infty$ there exists a definite range of wave numbers α for which the flow is stable. According to Lin,[9] the behavior of the stability curve for $R_e \to \infty$ is as follows:

Upper branch: $\alpha_r = (601.2116/R_e)^{1/11}$, $\quad c_r = 0.2667\alpha_r^2$

Lower branch: $\alpha_r = (211.7088/R_e)^{1/7}$, $\quad c_r = 0.611\,\alpha_r^2$

6.4 TEMPORAL STABILITY AT INFINITE REYNOLDS' NUMBER

Setting $R_e = \infty$ in Equation 6.27, we have:

$$(U - c)(\phi'' - \alpha^2\phi) - \phi U'' = 0 \qquad \textbf{(6.34a)}$$

The boundary conditions are

$$y = y_1 : \phi = 0; \quad y = y_2 : \phi = 0 \qquad \textbf{(6.34b)}$$

It is obvious that by solving Equation 6.34a one cannot obtain a critical Reynolds number. We can, however, use this equation to establish a connection between the nature of the velocity profile with the stability problem and the behavior of ϕ for $R_e \to \infty$. The problem of velocity profiles was solved by Rayleigh in 1880 (refer to References 4 and 7), and are known as the Rayleigh theorems.

Rayleigh's First Theorem

The existence of a point of inflection in the velocity profile $U(y)$ is a necessary condition for instability.

A point of inflection means that $U''(y) = 0$ at some value of y in the flow. Let ϕ^* be the complex conjugate of ϕ. Then the equations for ϕ and ϕ^* are

$$\phi'' - \alpha^2\phi - \frac{U''}{U - c}\phi = 0 \tag{6.35a}$$

$$\phi^{*''} - \alpha^2\phi^* - \frac{U''}{U - c^*}\phi^* = 0 \tag{6.35b}$$

Multiplying the first equation by ϕ^* and the second by ϕ and subtracting, we get:

$$\frac{d}{dy}(\phi'\phi^* - \phi\phi^{*\prime}) = \frac{2ic_iU''}{|U - c|^2}|\phi|^2 \tag{6.36a}$$

Integrating from y_1 to y_2 while using the boundary conditions Equation 6.34b we get:

$$2ic_i\int_{y_1}^{y_2}\frac{U''|\phi|^2}{|U - c|^2}dy = 0 \tag{6.36b}$$

For instability $c_i > 0$ so that for the satisfaction of Equation 6.36b, the values of U'' should be partly positive and partly negative in the range $y_1 \leqslant y \leqslant y_2$. This implies that $U''(y) = 0$ at some point in the range $y_1 \leqslant y \leqslant y_2$, proving the first theorem of Rayleigh. Thus the existence of a point of inflection in the velocity profile is a necessary condition for the occurrence of instability. Later Tollmien proved that the existence of a point of inflection in the laminar velocity profile is both necessary and sufficient condition. (On the inviscid basis one arrives at the erroneous conclusion that the Blasius boundary layer and the Poiseuille flows are stable for all values of R_e.)

Rayleigh's Second Theorem

The velocity of propagation of neutral disturbances ($c_i = 0$) is smaller than the maximum of the laminar flow velocity.

For a clear understanding we consider the second theorem of Rayleigh for velocity profiles with and without a point of inflection separately.

First, according to the second theorem for a velocity profile having a point of inflection, there must exist one neutral disturbance ($c_i = 0$) for which the wave propagation velocity is equal to the velocity of laminar flow. The value of y where both velocities are equal is said to be a point in a layer of fluid called the *critical layer*. Let $y = y_c$ be in the critical layer. In Equation 6.36a the quantity:

$$\phi'\phi^* - \phi\phi^{*\prime}$$

is purely imaginary. Writing:

$$W = \frac{-i}{2}(\phi'\phi^* - \phi\phi^{*\prime})$$

in Equation 6.36a, we get:

$$\frac{dW}{dy} = \frac{c_iU''|\phi|^2}{(U - c_r)^2 + c_i^2} \tag{6.37a}$$

Let $c_i = 0$; then $dW/dy = 0$ or $W =$ constant everywhere except when $U = c_r$ at $y = y_c$. In the latter case we integrate Equation 6.37a between $y_c - \delta$ to $y_c + \delta$, where δ is an arbitrary number. The change in W is

$$[W] = W(y_c + \delta) - W(y_c - \delta) = \int_{y_c - \delta}^{y_c + \delta} \frac{c_i U''|\phi|^2}{(U - c_r)^2 + c_i^2} dy$$

Now:

$$U' = \frac{dU}{dy}, \quad \text{or} \quad dy = \frac{dU}{U'}$$

hence:

$$[W] = \int_{U_1}^{U_2} \frac{c_i U''|\phi|^2 \, dU}{U'[(U - c_r)^2 + c_i^2]} \tag{6.37b}$$

where:

$$U_1 = U(y_c - \delta), \qquad U_2 = U(y_c + \delta)$$

The purpose is now to find the limiting value of Equation 6.37b when $c_i \rightarrow 0$. To find the limit, we use a result from mathematical analysis which states that:

$$\operatorname*{Lim}_{\lambda \to 0} \int_{\xi_1}^{\xi_2} \frac{\lambda f(\xi) \, d\xi}{(\xi - x)^2 + \lambda^2} = \pm \pi f(x)$$

Using this result in Equation 6.37b we find that:

$$\operatorname*{Lim}_{c_i \to 0} [W] = \pm \pi \left(\frac{U''_c}{U'_c}\right) |\phi_c|^2 \tag{6.38}$$

Equation 6.38 describes the discontinuous behavior of W near the critical point. Since in any event the total change in W must be zero, then either $U''_c = 0$ or $\phi_c = 0$. However, $\phi_c = 0$ amounts to pressing another boundary condition on ϕ and is therefore ruled out. Thus, the result is that $U''_c = 0$. That is, in the neutral case the velocity of wave propagation is equal to the velocity of laminar flow exactly at the point of inflection, i.e.:

$$U(y_c) = c_r$$

This proves the second theorem, since $U(y_c) < U_{max}$. The preceding result also shows that in the neutral case $U = c_r$ precisely at $y = y_c$.

In the second case, we consider a convex velocity profile so that $U(y)$ has no point of inflection. Multiplying Equation 6.35a by ϕ^* and Equation 6.35b by ϕ and adding, we get:

$$\frac{d}{dy}(\phi'\phi^* + \phi\phi^{*\prime}) - 2|\phi'|^2 - 2\alpha^2|\phi|^2 - \frac{2U''}{U - c_r}|\phi|^2 = 0$$

On integration from y_1 to y_2 and using the boundary conditions, we get:

$$\int_{y_1}^{y_2} \left[|\phi'|^2 + \left(\alpha^2 + \frac{U''}{U - c_r}\right)|\phi|^2\right] dy = 0 \tag{6.39}$$

Let $U_m = \max U(y)$. If $c_r > U_m$, then noting that $U'' < 0$ throughout the shear layer the condition Equation 6.39 can never be satisfied. Thus, c_r must be less than U_m, which proves the second theorem. Further, there is a point $y = y_c$ where $U = c_r$. However, for the existence of the integral in Equation 6.39 the limit:

$$\lim_{y \to y_c} \left[\frac{U''}{U - c_r}\right]$$

must exist, which is impossible because U'' does not tend to zero anywhere in the range $y_1 \leq y \leq y_2$. This points to a discontinuity across the critical layer giving rise to oscillations, also known as Tollmien-Schlichting waves.

From the two theorems of Rayleigh we conclude that the most attractive features of the inviscid stability theory are (1) the velocity profiles with a point of inflection are bound to become unstable for $R_e \to \infty$ and (2) the existence of a *critical layer* $y = y_c$. From the second conclusion we find that a general solution of Equation 6.34a will be of interest only near $y = y_c$. Since Equation 6.34a is of the second order, then we seek a solution near the critical point in the form of a series expansion as:

$$\phi = a_0(y - y_c)^\lambda + a_1(y - y_c)^{\lambda+1} + \ldots \tag{6.40a}$$

where λ is a parameter to be determined. Further, Equation 6.34a is a homogeneous linear equation, so that any constant multiple of ϕ is also a solution. We therefore take $a_0 = 1$. Near the critical point we also expand U in powers of $y - y_c$, which is

$$U = c_r + (y - y_c)U'_c + \frac{1}{2}(y - y_c)^2 U''_c + \ldots \tag{6.40b}$$

Substituting Equations 6.40 in Equation 6.34a, we have the indicial equation:

$$\lambda(\lambda - 1) = 0$$

or:

$$\lambda = 0, \qquad \lambda = 1$$

Since the roots differ by an integer, then the solution corresponding to one of the roots remains indeterminate. Taking $\lambda = 1$, one formal solution of Equation 6.34a is

$$\phi_1 = (y - y_c) + a_1(y - y_c)^2 + \ldots$$

which is a regular power series. To find the other independent solution we set:

$$\phi_2 = V\phi_1$$

in Equation 6.34a for ϕ and obtain:

$$\phi_1 V'' + 2\phi_1' V' = 0$$

where primes denote differentiations with respect to y. The solution of this equation is

$$\begin{aligned} V' &= \text{constant}/\phi_1^2 \\ &= a_0 + \frac{d_0}{(y - y_c)^2} + \frac{d_1}{y - y_c} + d_2(y - y_c) + \ldots \end{aligned}$$

Thus, finally:

$$\phi_2 = 1 + b_1(y - y_c) + \ldots + \phi_1 d_1 \ell n(y - y_c)$$

where the arbitrary leading constant has been set equal to 1 without any loss of generality. Near $y = y_c$ this solution behaves like:

$$\phi_2 \cong 1 + d_1(y - y_c)\ell n(y - y_c)$$

and:

$$U - c_r \cong (y - y_c)U'_c$$

Substituting this in Equation 6.34a, we get

$$d_1 = \frac{U''_c}{U'_c}$$

Thus the two independent solutions of Equation 6.34a are

$$\phi_1 = (y - y_c) + a_1(y - y_c)^2 + \ldots$$

$$\phi_2 = \{1 + b_1(y - y_c) + \ldots\} + \frac{U''_c}{U'_c}\phi_1 \ell n(y - y_c)$$

The logarithmic singularity in the second solution is a consequence of neglecting viscosity. The main utility of these solutions lies in providing matching conditions for the complete Orr-Sommerfeld equation for the case $R_e \rightarrow \infty$. Refer to Reference 10.

6.5 NUMERICAL ALGORITHM FOR THE ORR-SOMMERFELD EQUATION

Over the years, starting from the work of Lin[7] and the references contained therein, there have come into existence a number of methods for the solution of the complete Orr-Sommerfeld equation. Refer also to references contained in References 4 and 10. For some of the recent work, particularly carried out in the 1960s and 1970s, refer to References 11 and 12. Refer also to Reference 13.

The purpose of this subsection is to provide a format for the numerical solution of Equation 6.27. The reader, after having attained some understading of the complexities of the problem at hand, should be able to choose a method for a given problem.

We consider the problem of stability of a boundary layer for which Equation 6.27 has to be solved under the boundary conditions (Equations 6.33). Equation 6.27 is a fourth order ordinary linear differential equation and, therefore, must have four independent solutions which can be linearly combined to form a complete solution. An effective method of solving linear differential equations is to use the method of *fundamental solutions*. Let $\psi_j(y)$, $j = 1, 2, 3, 4$ be the four fundamental solutions of Equation 6.27; then the solution ϕ is given by:

$$\phi(y) = C_1\psi_1(y) + C_2\psi_2(y) + C_3\psi_3(y) + C_4\psi_4(y)$$

The fundamental solutions $\psi_j(y)$ satisfy the following initial conditions:

$$\psi_1(0) = 1, \quad \psi'_1(0) = 0, \quad \psi''_1(0) = 0, \quad \psi'''_1(0) = 0$$

$$\psi_2(0) = 0, \quad \psi_2'(0) = 1, \quad \psi_2''(0) = 0, \quad \psi_2'''(0) = 0$$

$$\psi_3(0) = 0, \quad \psi_3'(0) = 0, \quad \psi_3''(0) = 1, \quad \psi_3'''(0) = 0$$

$$\psi_4(0) = 0, \quad \psi_4'(0) = 0, \quad \psi_4''(0) = 0, \quad \psi_4'''(0) = 1$$

Thus, using the boundary conditions (Equation 6.33b) we find that:

$$C_1 = 0, \qquad C_2 = 0$$

Also since the differential equation (Equation 6.27) is homogeneous, the function ϕ is determined to within a constant multiplier and we can set $C_3 = 1$. Thus:

$$\phi(y) = \psi_3(y) + C_4\psi_4(y)$$

Substituting this form in Equation 6.27 we pose two initial value problems by choosing an initial guess for c:

$$(U - c)(\psi_3'' - \alpha^2\psi_3) - U''\psi_3 = \frac{-i}{\alpha R_e}(\psi_3'''' - 2\alpha^2\psi_3'' + \alpha^4\psi_3) \qquad \textbf{(6.41a)}$$

$$\psi_3(0) = 0, \quad \psi_3'(0) = 0, \quad \psi_3''(0) = 1, \quad \psi_3'''(0) = 0 \qquad \textbf{(6.41b)}$$

$$(U - c)(\psi_4'' - \alpha^2\psi_4) - U''\psi_4 = \frac{-i}{\alpha R_e}(\psi_4'''' - 2\alpha^2\psi_4'' + \alpha^4\psi_4) \qquad \textbf{(6.42a)}$$

$$\psi_4(0) = 0, \quad \psi_4'(0) = 0, \quad \psi_4''(0) = 0, \quad \psi_4'''(0) = 1 \qquad \textbf{(6.42b)}$$

The problem posed in Equations 6.41 and 6.42 should yield nontrivial solutions because of the inhomogeneous initial conditions. At $y = \delta$:

$$\phi(\delta) = \psi_3(\delta) + C_4\psi_4(\delta)$$

Substituting $\phi(\delta)$ in ζ given in Equation 6.33a, we can compute C_4 through the equation:

$$C_4 = -\frac{\psi_3'''(\delta) + \xi\psi_3''(\delta) - \alpha^2\{\psi_3'(\delta) + \xi\psi_3(\delta)\}}{\psi_4'''(\delta) + \xi\psi_4''(\delta) - \alpha^2\{\psi_4'(\delta) + \xi\psi_4(\delta)\}}$$

Having determined C_4 we now pose the initial value problem for ϕ through Equation 6.27 under the initial conditions:

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) = 1, \quad \phi'''(0) = C_4$$

The solution $\phi(y)$ thus obtained is continued to $y = \delta$; and the complex values of $\phi(\delta)$, $\phi'(\delta)$, $\phi''(\delta)$, $\phi'''(\delta)$ are then used in Z given in Equation 6.33a to form two real equations:

$$Z_r = 0, \qquad Z_i = 0$$

which are to be solved for the eigenvalues c and α.

To be specific, we consider the problem of temporal stability in which one is usually interested in having curves in the $\alpha_r R_e$-plane for a number of c_i-values starting from $c_i = 0$, and the corresponding eigenfunctions. With this aim we prescribe $\alpha_i = 0$ and set the real and imaginary parts of Z in Equation 6.33a equal to zero, i.e.:

$$Z_r(c_r, \alpha_r) = 0, \quad Z_i(c_r, \alpha_r) = 0$$

which are to be solved for c_r and α_r. In the particular case under consideration these two equations come from Z in Equation 6.33a, i.e.:

$$\xi^2 = \frac{\phi'''(\delta) + \alpha_r\phi''(\delta)}{\phi'(\delta) + \alpha_r\phi(\delta)} = A + iB, \quad \text{(say)}$$

so that from the definition of ξ^2:

$$c_r = U_e - \frac{B}{\alpha_r R_e} \tag{6.43a}$$

$$\alpha_r^2 + c_i\alpha_r R_e - A = 0 \tag{6.43b}$$

It is obvious from Equations 6.41a, 6.42a, and also 6.27 that for solving these equations we need to prescribe guess values for c_r and α_r to start the solution. Thus the solution procedure must form an iteration loop in which improved values of c_r and α_r are calculated at the end of each iteration through Equations 6.43.

The temporal stability curves for the boundary layer on a flat plate taken from Reference 11 have been shown in Figure 6.3. Here the horizontal axis is the Reynolds number based on the displacement thickness:

$$R_{\delta^*} = \frac{u_\infty \delta^*}{\nu}$$

while the vertical axis is the nondimensional wave number based on δ^*:

$$\alpha_{r1} = k_1\delta^*$$

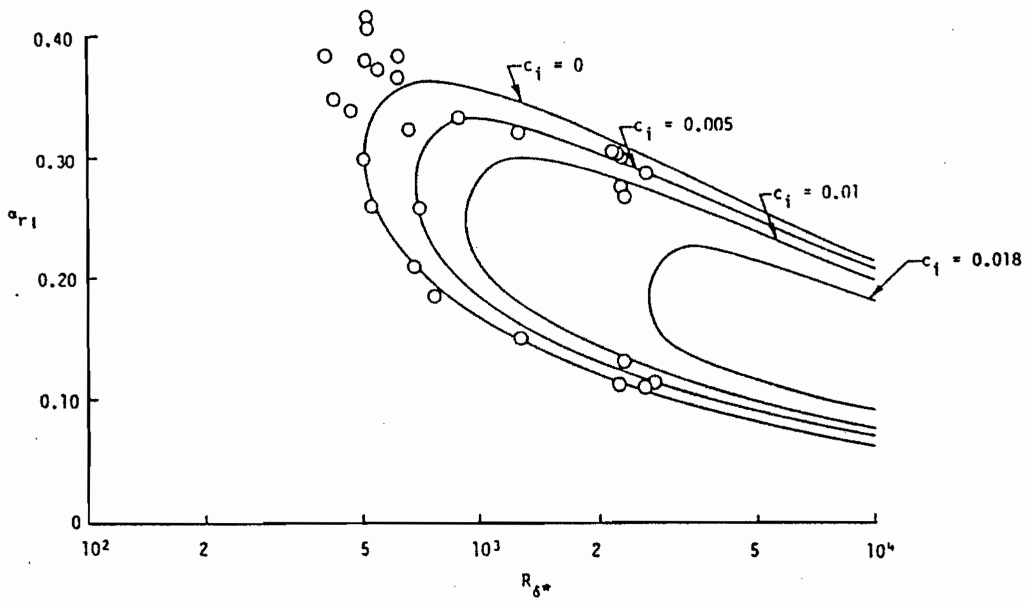


Fig. 6.3 Relation between a_{r1} and R_{δ^*} for various values of c_i. (Taken from Cebeci, T. and Bradshaw, P., *Momentum Transfer in Boundary Layers*, Hemisphere, New York, 1977. With permission.)

In Figure 6.3, the small circles are the experimental data points taken from Schubauer and Skramstad.[14]

For obtaining the spatial instability curves we prescribe α_i while taking $\omega_i = 0$. As before, R_e is also prescribed. Starting with an arbitrary guess for ω_r and α_r, the iterative procedure is started and then the corrected values are obtained from Z in Equation 6.33a.

6.6 TRANSITION TO TURBULENCE

It was stated at the beginning of this chapter that the linear stability theory does not provide complete conditions for a breakdown of a given laminar flow to attain the state of a fully developed turbulent flow. Despite the meager information available from the linear stability theory, researchers have used the results of spatial and temporal amplification factors to devise empirical formulae for the prediction of transition. A well-known rule proposed by Smith and Gamberoni[15] is called the e^9-rule. Since $e^9 = 8103.08$, the e^9-rule means that fully developed turbulent flow will occur when the amplitude of the initial small disturbance has increased roughly by a factor of 8100. For details refer to the original reference and also to References 4 and 16. To understand the basis of the e^9-rule, we follow Cebeci and Bradshaw.[11] First of all, from Equation 6.29c we obtain the magnitude of spatial perturbation $|\psi'|$ which is

$$|\psi'| = e^{-\alpha_i x}|\phi(y)|$$

Thus:

$$\frac{d}{dx}\,\ell n|\psi'| = -\alpha_i$$

or:

$$\frac{d|\psi'|}{|\psi'|} = -\alpha_i\, dx$$

To establish a functional form of dx in a simple way, Cebeci and Bradshaw[11] considered similar boundary layers in which by using Equations 5.134 and 5.141c, one finds:

$$\frac{dR_{\delta^*}}{dx} = \frac{Ld_1^2}{\delta^*}$$

where L is a characteristic length. Thus:

$$\frac{d|\psi'|}{|\psi'|} = -\frac{1}{Ld_1^2}\,\alpha_i\delta^*\, dR_{\delta^*}$$

which on integration from a reference state denoted by a subscript 1 yields:

$$a = \frac{|\psi'|}{|\psi'|_1} = \exp\left(-\frac{1}{Ld_1^2}\int_{(R_{\delta^*})_1}^{R_{\delta^*}} \alpha_i\delta^*\, dR_{\delta^*}\right)$$

The quantity a is the amplification with respect to an initial state 1. The e^9-method consists of evaluation of the above integral. The stability calculations begin from $R_{\delta^*} > R_{\delta^*,cr}$. Cebeci's calculations show that the transition begins when $\max[\ell na] \cong 8.98$, and thus the name e^9-rule.

Natural transition has been found as a result of appearance of turbulent spots in a flow near a wall which grow both along the longitudinal and lateral directions. Under the influence of strong disturbances these spots burst and mix with the otherwise laminar flow. Repeated bursting of turbulent spots and the high frequency of their appearance establish the transition to turbulence. For a basic understanding of this phenomenon, refer to Emmons[17] and Dhawan and Narasimha.[18] Some empirical formulae which have been proposed by Cebeci and Smith[16] for external boundary layer transition are in the form of a intermittency factor γ_{tr}. For two-dimensional flows the expression for γ_{tr} is

$$\gamma_{tr} = 1 - \exp\left[\frac{-3R_0^{0.66}U_e^{1.66}S_{tr}^{-1.34}(S - S_{tr})}{C^2}\int_{S_{tr}}^{S}\frac{dS}{U_e}\right] \tag{6.44a}$$

where:

$$R_0 = \frac{U_\infty L}{\nu}, \qquad U_e = \frac{u_e}{U_\infty}, \qquad S = \frac{x}{L}$$

u_e is the external potential flow of the boundary layer:

$$C = 60, \quad \text{for incompressible flow}$$

$$C = 60 + 4.86\,M_e^{1.92}, \qquad 0 < M_e < 5, \quad \text{for compressible flow}$$

M_e is the Mach number based on external velocity; and U_∞, L are the characteristic velocity and length, respectively. For axially symmetric bodies, Equation 6.44a is

$$\gamma_{tr} = 1 - \exp\left[\frac{-3R_0^{0.66}U_e^{1.66}S_{tr}^{-1.34}\bar{r}_0(S_{tr})}{C^2}\left(\int_{S_{tr}}^{S}\frac{dS}{\bar{r}_0}\right)\left(\int_{S_{tr}}^{S}\frac{dS}{U_e}\right)\right] \tag{6.44b}$$

where $\bar{r}_0 = r_0(x)/L$ is the local radius from the axis of symmetry to the boundary contour. For two-dimensional flows, an empirical relation between the transitional values of R_x and R_θ has been developed in Reference 16. This formula is based on Michel's method[19] and the e^9-rule, and is

$$R_\theta = 1.174\left(1 + \frac{22400}{R_x}\right)R_x^{0.46} \tag{6.45}$$

Therefore, if R_θ is known as a function of R_x or vice versa, then $(R_\theta)_{tr}$ or $(R_x)_{tr}$ become known. For example, the functional relation between R_θ and R_x for the Blasius boundary layer is

$$R_\theta = 0.664R_x^{1/2}$$

Thus, from Equation 6.45, $(R_x)_{tr} \approx 2.027 \times 10^6$.

6.7 STATISTICAL METHODS IN TURBULENT CONTINUUM MECHANICS

An initially laminar flow becomes turbulent after passing through a nonlinear state of interaction, called the *transition*. The transition from laminar to turbulent flow occurs under the influence of a number of factors. As alluded to in Section 6.6, our knowledge of the transition phenomenon is for the most part based on empirical experimental data, linear stability theory, and to some extent on nonlinear stability theory, e.g., Reference 20. For the purpose of a connected discussion on the statistical methods applied to a turbulent field, we shall assume from here onward that a fully developed turbulent field is available without any regard as to how this field has been generated.

The following is a listing of a number of properties which every turbulent flow field must satisfy:

1. A turbulent field is a *random* field.
2. Turbulence is a continuum pheonmenon. Thus it is not possible to treat turbulence purely on the basis of the kinetic theory of gases in which molecules are considered as discrete particles with mean-free paths.
3. Turbulence is *diffusive*. Diffusiveness of turbulence is responsible for the rapid mixing of various physical and chemical properties. Because of this property, the rates of mass and momentum transfer are much higher in turbulent than in laminar flow fields.
4. Turbulence is *dissipative*. There is an inherent dissipative mechanism in a turbulence field which establishes a connection between the large-scale eddies and the small-scale eddies. It must, however, be stated here that practically all the dissipation of turbulence takes place through the small-scale eddies which are affected by the fluid viscosity.
5. Turbulence occurs at high Reynolds numbers.
6. Turbulence is always three dimensional.

These six properties are important for a field to be classified as a turbulent field. Added to these properties is the basic postulate that the Navier-Stokes equations are exactly satisfied at every point in the space of a turbulent fluid flow.

Average or Mean of Turbulent Quantities

Because of the random behavior of a turbulence field, we have to devise an averaging process to obtain deterministic quantities from some available experimental or theoretical data. A natural averaging procedure is the one which requires the use of a probability function. To describe this concept let us consider a scalar function $\phi(\mathbf{x}, t)$, where $\mathbf{x}$ represents spatial coordinates and t is time. For example, let $\phi(\mathbf{x}, t)$ be equal to $u(\mathbf{x}, t)$ which is the x-component of the velocity vector $\mathbf{u}$ at the space-time point $(\mathbf{x}, t)$. If the field is turbulent, we call $\phi(\mathbf{x}, t)$ as a random function of $\mathbf{x}$ and t. By this we mean that with the repetition of an experiment under identical macroscopic conditions the same value of ϕ is not realized, but the distribution of ϕ-values follows certain laws of probability. Let α be a random variable. The probability that ϕ lies between α and $\alpha + d\alpha$ is denoted as $P(\alpha)\, d\alpha$, where $P(\alpha)$ is called the probability density. The probability average of $\phi(\mathbf{x}, t)$ is then defined as:

$$\bar{\phi} = \int_{-\infty}^{\infty} \alpha P(\alpha)\, d\alpha$$

The probability average is equal to the ensemble average. In essence, the ensemble average is the arithmetic mean of all numerical values of ϕ recorded in a large number of similar setups. A knowledge of $P(\alpha)$, particularly for continuum mechanics, is a very difficult task and we have to devise other simpler ways to obtain the averages.

Time and Space Averaging

The most practical way of obtaining averages is to integrate the function ϕ with respect to the space coordinates and time along with a weighting function as multiplier. A general defintion of such an average is (refer to Reference 21):

$$\overline{\phi(\mathbf{x}, t)} = \iiiint_{-\infty}^{\infty} \phi(\mathbf{x} - \boldsymbol{\xi}, t - \tau)\, w(\boldsymbol{\xi}, \tau)\, d\boldsymbol{\xi}\, d\tau \tag{6.46a}$$

where w is the weighting fucntion, $d\xi$ represents a volume element centered at the position ξ, and:

$$\iiiint_{-\infty}^{\infty} w(\xi, \tau)\, d\xi\, d\tau = 1 \tag{6.46b}$$

Time Average

In time averaging, the time interval $-\infty \leq t \leq \infty$ is replaced by $-T \leq t \leq T$, where T is a large time interval to be defined later. We now define the weighting function $w(\xi, \tau)$ as:

$$w(\xi, \tau) = \frac{1}{2T}\delta(\xi)$$

where:

$$\delta(\xi) = \delta(\xi_1)\delta(\xi_2)\delta(\xi_3)$$

is the Dirac delta function. (For the operational properties of the delta function, refer to M⁻ 5.) Introducing the above weight w in Equation 6.46a and using the operational properties of the delta function just referred to, we get:

$$\overline{\phi(\mathbf{x}, t)} = \frac{1}{2T}\int_{-T}^{T} \phi(\mathbf{x}, t - \tau)\, d\tau \tag{6.47a}$$

$$= \frac{1}{2T}\int_{-T}^{T} \phi(\mathbf{x}, t + \tau)\, d\tau \tag{6.47b}$$

$$= \frac{1}{2T}\int_{t-T}^{t+T} \phi(\mathbf{x}, \tau)\, d\tau \tag{6.47c}$$

Other alternative definitions are

$$\overline{\phi(\mathbf{x}, t)} = \frac{1}{T}\int_{t-T/2}^{t+T/2} \phi(\mathbf{x}, \tau)\, d\tau \tag{6.47d}$$

$$= \frac{1}{T}\int_{t}^{t+T} \phi(\mathbf{x}, \tau)\, d\tau \tag{6.47e}$$

Ensemble Average

For the purpose of comparison, we also state the ensemble average as follows. Let there be N realizations of ϕ at the same space-time point $(\mathbf{x}, t)$. Then the ensemble average of ϕ is

$$\overline{\phi(\mathbf{x}, t)} = \frac{1}{N}\sum_{k=1}^{N} \phi_k \tag{6.48}$$

provided that N is sufficiently large; and more appropriately N should tend to infinity for $\overline{\phi}$ to exist. The hypothesis which allows us to state that the results of Equation 6.47 and of Equation 6.48 are the same is called the *ergodic hypothesis*. It has been established that when ϕ is statistically steady, then the time and ensemble averages are really equal.*

Statistically stationary processes have been defined in Section 6.8.

To establish the equality of time and ensemble averages for a steady flow** in a simple way we refer to Figure 6.4 where a trace of the fluctuating u-component of velocity due to a flow past a flat plate has been shown. This trace is a record of a hot wire placed 0.02 in. above the surface and 56 in. behind the leading edge of the flat plate.

The concept of ensemble average is to read off the values of u at the end of a chosen time interval h, say .01 or .001 s. Each realization is recorded and the arithmetic sum is formed by using Equation 6.48 as:

$$\bar{u} = \frac{1}{N}\sum_{k=1}^{N} u_k \tag{6.49a}$$

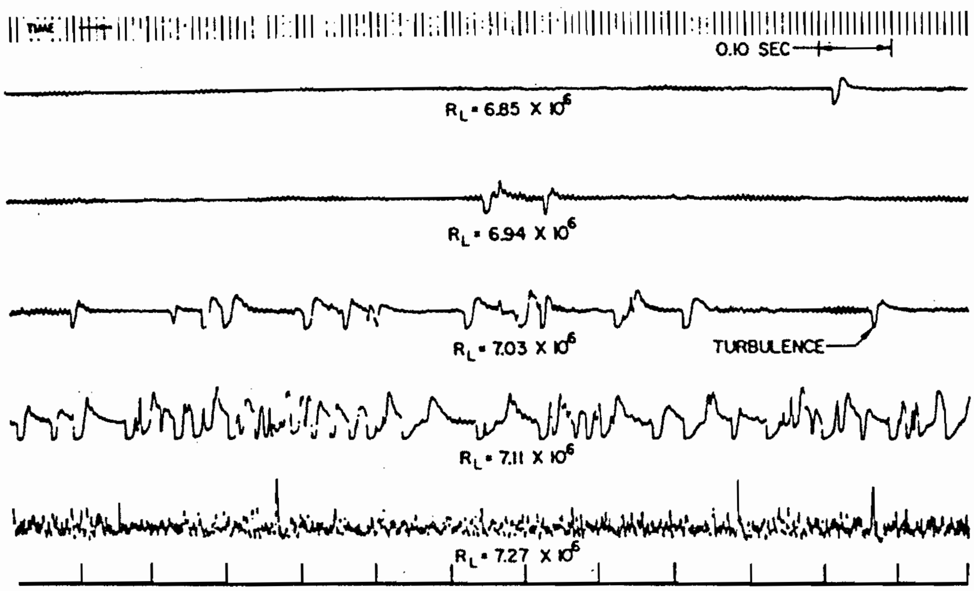


Fig. 6.4 Hot-wire anemometer records showing the growth of laminar oscillations, their breakdown into turbulent spots, and finally the development of fully developed turbulent flow. (Taken from Cebeci, T. and Smith, A. M. O., *Analysis of Turbulent Boundary Layers*, Academic Press, New York, 1974. With permission.)

Since we are considering a steady flow, the time average of u from Equation 6.47e is

$$\bar{u} = \frac{1}{T}\int_0^T u(x, \tau)\, d\tau \tag{6.49b}$$

Using Riemann's definition of an integral, we write Equation 6.49b as:

$$\bar{u} \cong \frac{1}{T}\sum_{k=1}^{N} u_k(\tau_k - \tau_{k-1})$$

where:

$$\tau_0 = 0, \qquad \tau_N = T$$

Taking a uniform time interval:

** Steady flow is certainly statistically steady in time.

$$h = \tau_k - \tau_{k-1} \quad \text{for all } k$$
$$= T/N$$

we have:

$$\bar{u} = \frac{h}{T}\sum_{k=1}^{N} u_k = \frac{1}{N}\sum_{k=1}^{N} u_k$$

which is exactly the ensemble average (Equation 6.49a). From this simple analysis we conclude that the average values obtained by Equations 6.49, are the same provided that N is sufficiently large. Further, $T = Nh$, i.e.:

$$T \text{ (period of averaging)} = \begin{matrix} N \text{ (number of} \\ \text{realizations)} \end{matrix} \times \begin{matrix} h \text{ (time interval between} \\ \text{two successive realizations)} \end{matrix} \tag{6.49c}$$

Note that this simple analysis, besides providing the validity of the ergodic hypothesis for steady flow, also gives a method for the determination of the averaging period T through the formula (Equation 6.49c). Thus for $N = 1000$ and $h = .001$ s the time period $T = 1$ s. The same hypothesis is used for quasisteady flows but now the determination of T is also influenced by another time scale due to the nonsteadiness of the large-scale motion. In any event, period T must be larger than any time scale of turbulent fluctuations but smaller than the time scale of large-scale motions. As an example, if the overall fluid motion is oscillatory, then the averaging period T must be smaller than the period of oscillations of large-scale motions.

Space Average

In space averaging the infinite integration is replaced by an integration over a finite but large volume V_0. Thus we introduce the weighting function w as:

$$w(\xi, \tau) = \frac{1}{V_0}\,\delta(\tau)$$

Using this weight in Equation 6.46a and using the operational properties of the delta function, we get:

$$\overline{\phi(\mathbf{x}, t)} = \frac{1}{V_0}\iiint_{V_0} \phi(\mathbf{x} - \xi, t)\, d\xi \tag{6.50}$$

Equation 6.50 is called the space average. Arguments similar to those in the time average show that for a process statistically stationary in space, the space average and the ensemble average are the same. It must be noted that choices of the forms of w taken above are due to a basic property of delta functions (Equation M5.18iii) which is used in satisfying Equation 6.46b. Thus:

$$\iiint_{-\infty}^{\infty}\int_{-T}^{T} \frac{1}{2T}\,\delta(\xi)\, d\xi\, d\tau = \frac{1}{2T}\int_{-T}^{T} d\tau = 1$$

and also:

$$\iiint_{V_0}\int_{-\infty}^{\infty} \frac{1}{V_0}\,\delta(\tau)\, d\xi\, d\tau = \frac{1}{V_0}\iiint_{V_0} d\xi = 1$$

Equation 6.46a has recently become the centerpiece of a powerful method of finding the solutions of turbulent flow equations. The rationale of the method is that when complete equations for obtaining flow solutions are solved on a numerical grid, then this grid system (no matter how fine) can resolve only a certain portion of a fluctuating variable. The part which can be resolved is called the large-scale component or the average, and the remaining part is called the subgrid scale component. If Equation 6.46a or equivalently:

$$\overline{\phi}(\mathbf{x}, t) = \iiiint_{-\infty}^{\infty} \phi(\xi, \tau)\, w(\mathbf{x} - \xi, t - \tau)\, d\xi\, d\tau$$

is considered as a space-time filter, then $\overline{\phi}$ is the large-scale or filtered part obtained by removing the high frequency and high wave number components. The filter or the weight function is generally written as:

$$w(\mathbf{x}, t) = w_t(t) \prod_{i=1}^{n} w_i(x_i)$$

where $w_t(t)$ is the time-dependent part while $w_i(x_i)$ depend on the space coordinates. Here n is the number of the spatial dimension. The common choice of w_t is based on Reynolds' time averaging,[22] and is

$$w_t(t) = \frac{1}{\Delta_t}, \quad \text{when} \quad -\frac{\Delta_t}{2} \leq t \leq \frac{\Delta_t}{2}$$
$$= 0 \quad \text{otherwise}$$

The spatial filtering functions w_i due to Smagorinsky[23] and Deardorff[24] is

$$w_i(x_i) = \frac{1}{\Delta_i}, \quad \text{when} \quad -\frac{\Delta_i}{2} \leq x_i \leq \frac{\Delta_i}{2}$$
$$= 0 \quad \text{otherwise}$$

The form proposed by Leonard[25] is

$$w_i(x_i) = \sqrt{\frac{\gamma}{\pi}}\frac{1}{\Delta_i} \exp\left(\frac{-\gamma x_i^2}{\Delta_i^2}\right)$$

where $\gamma \cong 6$.

Basic Axioms of Averaging

As noted earlier, the averaging period T must be large in comparison with the characteristic periods of random turbulent motions but must be small in comparison with any time scale or period of the large-scale motions. Further, we assume the averaging process to be *linear*; therefore, the basic axioms of the averaging process are as follows.

Let f and g be two random functions, then the operation of averaging satisfies the following three axioms, denoted as A(i), A(ii), A(iii):

A(i) $\quad \overline{f + g} = \overline{f} + \overline{g}$

A(ii) $\quad \overline{af} = a\overline{f}, \quad a = \text{constant}$

A(iii) $\quad \bar{a} = a, \quad a = \text{constant}$

(iv) $$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}, \quad \text{where } s \text{ is either } x_1, x_2, x_3 \text{ or } t$$

(v) $$\overline{\bar{f}\, g} = \bar{f}\,\bar{g}$$

From v we arrive at an important result by writing $g = 1$, so that:

(vi) $$\bar{\bar{f}} = \bar{f}$$

Property vi allows us to write any random quantity f as the sum of a mean and a fluctuation as:

$$f = \bar{f} + f'$$

so that by using property vi we have:

(vii) $$\overline{f'} = 0$$

6.8 STATISTICAL CONCEPTS

For the purpose of providing a rigorous theoretical basis to all considerations in turbulence theory it is important to relate various sorts of averages and correlations, etc. to those justifiable by the probability theory. With the use of the ergodic hypothesis we can then establish a rapport between the time or space averages with the expected values from the probability theory. These considerations also lead us to functional differential equations from the Navier-Stokes equations for the determination of probability functions of continuum turbulent flows. In this section we shall only touch upon a few basic definitions and formulations of this wide field of statistical hydrodynamics. For a deeper understanding of the results from a hydrodynamic viewpoint the reader is referred to, e.g., Reference 26.

Probability Distribution Functions

Let u be a random variable which can take any value between $-\infty$ to $+\infty$. The function that specifies the probability that an event lies between $-\infty$ and u is denoted as $F_1(u)$ and is called the one-dimensional *probability distribution function*. The statement that u lies between $-\infty$ and ∞ means that $-\infty \leq u < \infty$. Since the event must lie somewhere on the real axis, it is obvious that:

$$\lim_{u \to -\infty} F_1(u) = 0 \qquad \text{no chance}$$

$$\lim_{u \to +\infty} F_1(u) = 1 \qquad \text{certain}$$

The probability that u lies between a and b is the difference:

$$F_1(b) - F_1(a)$$

and if $b > a$, the difference is positive. Thus $F_1(u)$ is a monotonically increasing function of u. In a similar fashion, we can define an n-dimensional probability distribution function:

$$F_n(u_1, u_2, \ldots, u_n)$$

which is the probability that u_1 lies between $-\infty$ and u_1, u_2 lies between $-\infty$ and u_2, and so on. Further:

$$\text{Lim } F_n = 0 \quad \text{as} \quad u_n \to -\infty \text{ for } all\ n$$

Consider a two-dimensional distribution function $F_2(u, v)$. The probability that u lies between a and b and that v lies between c and d is obtained as follows. Consider first $F_2(u, c)$ and $F_2(u, d)$ which are one-dimensional functions in u. For any u, the probability that v lies between c and d is

$$\phi(u) = F_2(u, d) - F_2(u, c)$$

The probability that u lies between a and b is, therefore:

$$\phi(b) - \phi(a)$$

so that:

$$F_2(b, d) - F_2(b, c) - F_2(a, d) + F_2(a, c)$$

is the probability that u lies between a and b and that v lies between c and d.

Consider a turbulent flow in which the x-component of velocity at various points, say at $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ is to be recorded. Then:

$$F_n(u_1, u_2, \ldots, u_n)$$

is the probability distribution function for this case, where u_i is the x-component of velocity at position $\mathbf{x}_i$. Thus F_n is the probability that u_1 lies between $-\infty$ and u_1, u_2 lies between $-\infty$ and u_2, and so on. If we pass on from the discrete set of points to a continuous distribution of points, then the probability distribution function becomes $F[u(\mathbf{x})]$, which is a functional for the velocity component u. We can further generalize by considering all the components (u, v, w) for all the points by taking the probability distribution function as the functional:

$$F[\mathbf{u}(\mathbf{x})] = F[u(\mathbf{x}), v(\mathbf{x}), w(\mathbf{x})]$$

This functional represents the entire field, such that $u(\mathbf{x})$ lies between $-\infty$ and $u(\mathbf{x})$, $v(\mathbf{x})$ lies between $-\infty$ and $v(\mathbf{x})$, and $w(\mathbf{x})$ lies between $-\infty$ and $w(\mathbf{x})$.

Probability Density

From the definition of probability distribution function $F_1(u)$ given earlier we find the probability that u lies between u and $u + du$ is

$$F_1(u + du) - F_1(u)$$

Using a first order Taylor expansion, we have:

$$F_1(u + du) - F_1(u) = \frac{dF_1}{du}\, du$$

The quantity:

$$P_1(u) = \frac{dF_1}{du} \tag{6.51a}$$

is called the one-dimensional *probability density*. Thus the probability that an event lies between u and $u + du$ is simply $P_1(u)\,du$. Also from Equation 6.51a:

$$F_1(u) = \int_{-\infty}^{u} P_1(\alpha)\,d\alpha$$

and since:

$$\lim_{u \to \infty} F_1(u) = 1$$

we have:

$$\int_{-\infty}^{\infty} P_1(\alpha)\,d\alpha = 1 \tag{6.51b}$$

In the same manner:

$$P_2(u, v) = \frac{\partial^2 F_2}{\partial u\,\partial v}$$

$$P_n(u_1, u_2, \ldots, u_n) = \frac{\partial^n F_n}{\partial u_1\,\partial u_2 \ldots \partial u_n}$$

and:

$$F_n = \int_{-\infty}^{u_1} \cdots \int_{-\infty}^{u_n} P_n(\alpha_1, \alpha_2, \ldots, \alpha_n)\,d\alpha_1\,d\alpha_2 \ldots d\alpha_n \tag{6.51c}$$

Mathematical Expectation

The mathematical expectation of a function $g(u)$ of random variable u is defined as:

$$\overline{g(u)} = \int_{-\infty}^{\infty} g(u)P_1(u)\,du$$

If $g(u) = u$, then the mean is

$$\bar{u} = \int_{-\infty}^{\infty} uP_1(u)\,du \tag{6.52}$$

In general, if g is a function of n random variables, then its mathematical expectation is

$$\overline{g(u_1, u_2, \ldots, u_n)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(u_1, u_2, \ldots, u_n)P_1(u_1, u_2, \ldots, u_n)\,du_1\,du_2 \ldots du_n$$

Moments of various orders denoted as $\overline{u^n}$ can now be defined through the equation:

$$\overline{u^n} = \int_{-\infty}^{\infty} u^n P_1(u)\,du$$

where for $n = 1$ we have the usual mean (Equation 6.52), for $n = 2$ the mean square value, and so on. The variance σ^2 is defined as:

$$\sigma^2 = \overline{u^2} - \overline{u}^2$$
$$= \int_{-\infty}^{\infty} (u - \overline{u})^2 P_1(u)\, du \tag{6.53}$$

while σ is the standard deviation.

Correlation Functions

Given the second order probability density $P_2(u_1, u_2)$ we form the mathematical expectation or average of the product $u_1^p u_2^q$, where p and q are the integral powers of u_1 and u_2, respectively, as:

$$\overline{u_1^p u_2^q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1^p u_2^q P_2(u_1, u_2)\, du_1\, du_2 \tag{6.54}$$

which is called the double *correlation*. For example, in a turbulent flow let u_1 and u_2 be the x-component of velocities at points $\mathbf{x}_1$ and $\mathbf{x}_2$ at times t_1 and t_2, respectively. Then the correlation function is

$$R(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = \overline{u(\mathbf{x}_1, t_1)u(\mathbf{x}_2, t_2)} \tag{6.55}$$

where for fixed values of $\mathbf{x}_1$, $\mathbf{x}_2$, t_1, t_2 we have a correlation of the type in Equation 6.54. Correlations of various orders can be defined as, e.g.:

$$\overline{u_1^p u_2^q \ldots u_n^r} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_1^p u_2^q \ldots u_n^r P_n(u_1, u_2, \ldots, u_n)\, du_1\, du_2 \ldots du_n \tag{6.56}$$

Let $u_1, u_2, \ldots, u_n$ be the x-components of velocities at n-points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ at times $t_1, t_2, \ldots, t_n$, respectively. Then the correlation function is

$$R(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \ldots; \mathbf{x}_n, t_n) = \overline{u(\mathbf{x}_1, t_1)u(\mathbf{x}_2, t_2) \ldots u(\mathbf{x}_n, t_n)} \tag{6.57}$$

where for fixed values of $x_1, x_2, \ldots, t_1, t_2, \ldots,$ we have a correlation of the type in Equation 6.56.

Stationary Processes

We shall now define stationary processes in the context of correlation functions (Equation 6.57). For stationary processes the probability densities are also stationary.

A process is called *stationary in time* if its correlation function (Equation 6.57) does not depend on the absolute times but depends only on the time differences. Thus for a stationary in time process:

$$R(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \ldots; \mathbf{x}_n, t_n) = F(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; t_2 - t_1, \ldots, t_n - t_1) \tag{6.58}$$

A process is called *stationary in space* if its correlation function (Equation 6.57) does not depend on the absolute positions in space but depends only on the relative dispositions of the points. Thus for a process stationary in space:

$$R(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \ldots; \mathbf{x}_n, t_n) = F(\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \ldots, \mathbf{x}_n - \mathbf{x}_1; t_1, t_2, \ldots, t_n) \tag{6.59}$$

Characteristic Functions

The Fourier transform of probability density is called the *characteristic function*. Thus the function $M_1(v)$ defined as:

$$M_1(v) = \int_{-\infty}^{\infty} e^{iuv} P_1(u)\, du \tag{6.60}$$

is the characteristic function. The form of Fourier's transform chosen here is simple to work within the present context. Expanding the exponential in series we have:

$$\begin{aligned} M_1(v) &= \sum_{n=0}^{\infty} \frac{(iv)^n}{n!} \int_{-\infty}^{\infty} u^n P_1(u)\, du \\ &= \sum_{n=0}^{\infty} \frac{(iv)^n}{n!} \overline{u^n} \end{aligned} \tag{6.61a}$$

An important result which now emerges is that if all the moments $\overline{u^n}$ are known and the series (Equation 6.61a) is convergent, then $M_1(v)$ can formally be constructed. Knowing $M_1(v)$ we can use the Fourier inversion formula to obtain $P_1(u)$ as:

$$P_1(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_1(v) e^{-iuv}\, dv \tag{6.61b}$$

Some important obvious conclusions based on Equations 6.60 and 6.61 follow:

$$M_1(0) = 1$$

$$\left.\frac{d^n M_1(v)}{dv^n}\right|_{v=0} = i^n \overline{u^n}$$

which means that the n-th moment is obtained by differentiating the characteristic function n times.

Taking the logarithm of Equation 6.61a we have:

$$\ell n M_1(v) = \ell n \left[1 + \sum_{n=1}^{\infty} \frac{(iv)^n}{n!} \overline{u^n} \right]$$

which on expansion can be arranged as:

$$\ell n M_1(v) = \sum_{n=1}^{\infty} \frac{(iv)^n}{n!} \lambda_n .$$

where:

$$\begin{aligned} \lambda_1 &= \bar{u} \\ \lambda_2 &= \overline{u^2} - \bar{u}^2 = \sigma^2 \\ \lambda_3 &= \overline{u^3} - 3\bar{u}\overline{u^2} + 2\bar{u}^3 \\ &\ldots\ldots\ldots \end{aligned}$$

The quantities λ_n are called the semi-invariants or cumulants.

Characteristic functions for the correlations of various orders can similarly be obtained, e.g.:

$$M_n(v_1, v_2, \ldots, v_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mathbf{u}\cdot\mathbf{v}} P_n(u_1, u_2, \ldots, u_n)\, d\mathbf{u} \qquad (6.62)$$

where:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

and:

$$\left.\frac{\partial^\alpha}{\partial v_1^\alpha} \frac{\partial^\beta}{\partial v_2^\beta} \cdots \frac{\partial^\gamma}{\partial v_n^\gamma} M_n(v_1, v_2, \ldots, v_n)\right|_{\substack{v_1 = 0 \\ v_2 = 0 \\ \vdots \\ v_n = 0}} = (i)^{\alpha+\beta+\ldots+\gamma}\, \overline{u_1^\alpha u_1^\beta \ldots u_n^\gamma}$$

Gaussian Distribution

In a one-dimensional process if the random variable u_1 is distributed about its mean value $\bar{u}_1$ in such a way that the probability density function $P_1(u_1)$ has the formulation:

$$P_1(u_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(u_1 - \bar{u}_1)^2/2\sigma^2] \qquad (6.63)$$

where σ is the standard deviation of u_1, then the process is called Gaussian or normal. In an n-dimensional Gaussian process, the probability density function is

$$P_n(u_1, u_2, \ldots, u_n) = \frac{1}{A^{1/2}(2\pi)^{n/2}} \exp\left[-\frac{1}{2A}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}(u_j - \bar{u}_j)(u_k - \bar{u}_k)\right]$$

where A is the determinant:

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$a_{jk} = \overline{(u_j - \bar{u}_j)(u_k - \bar{u}_k)}$$

and A_{jk} is the cofactor of a_{jk} in A.

Following the same limiting procedure, the discrete characteristic function (Equation 6.62) in the limit of a continuous distribution takes the form:

$$M[v(t)] = \int_R \exp\left[i \int u(t)v(t)\, dt\right] P[u(t)]\, du(t)$$

6.9 INTERNAL STRUCTURE IN THE PHYSICAL SPACE

The most important part of the subject of fluid turbulence is to analyze and understand the internal structure of the turbulent flow fields. It is only through the availability of statistical methods that it becomes possible to analyze the internal structure of turbulence through a study of various velocity correlations between points of the field. This mode of attack provides a quantitative measure of the length and time scales of a hierarchy of turbulent eddies. Following this line of approach a number of definitive results for homogeneous and isotropic turbulence have been established. In this section we shall study the formation of dynamic equations of various correlations and then analyze the structure of homogeneous and isotropic turbulence. Books by Batchelor,[27] Townsend,[28] and Hinze[29] contain some important formulations and results on the theories of homogeneous and isotropic turbulence and must be consulted. Refer also to Reference 21.

The importance of correlation functions lies in the following observations. Physically, turbulence is composed of motions having a wide range of length scales. Qualitatively, turbulence is described as the interaction of eddies of various sizes, where an eddy is defined as a structure in which the fluctuating velocity components are correlated.

Second and Third Order Correlations

The most general two-point velocity correlation of the second order in perturbation velocity components is

$$R_{ij}(\mathbf{x}, \mathbf{y}, t, t_1) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{y}, t_1)} \tag{6.64a}$$

Here $\mathbf{x}$ and $\mathbf{y}$ are two points in the fluid medium where the i-th Cartesian component of velocity is recorded at time t and the j-th Cartesian component of velocity is recorded at time t_1, respectively. Introducing the separation distance $\mathbf{r}$ through:

$$\mathbf{r} = \mathbf{y} - \mathbf{x}$$

and the delay time τ through:

$$\tau = t_1 - t$$

we have:

$$R_{ij}(\mathbf{x}, \mathbf{r}, t, \tau) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t + \tau)} \tag{6.64b}$$

For three points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, writing:

$$\mathbf{r} = \mathbf{y} - \mathbf{x}, \qquad \mathbf{s} = \mathbf{z} - \mathbf{x}$$

and:

$$\tau = t_1 - t, \qquad \tau_1 = t_2 - t$$

the three-point third order correlation is

$$T_{ijk}(\mathbf{x}, \mathbf{r}, \mathbf{s}, t, \tau, \tau_1) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t + \tau)u_k'(\mathbf{x} + \mathbf{s}, t + \tau_1)} \tag{6.64c}$$

Correlations of high order can similarly be established.

A random field is said to be statistically stationary in time if R_{ij} does not depend on the absolute times t and t_1 but depends only on the time difference τ, i.e.:

$$R_{ij}(\mathbf{x}, \mathbf{r}, \tau) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t + \tau)} \tag{6.65a}$$

where the appearance of t in Equation 6.65a is simply for reference purpose and can conveniently be set equal to zero. A random field is said to be statistically stationary in space if R_{ij} does not depend on the absolute positions in space but depends only on the relative distance between the points, i.e.:

$$R_{ij}(\mathbf{r}, t, \tau) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t + \tau)} \tag{6.65b}$$

If observations are made at the same time t, then Equation 6.65b becomes:

$$R_{ij}(\mathbf{r}, t) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t)} \tag{6.65c}$$

The correlations (Equations 6.65b, c) are applicable to a homogeneous field. In the same fashion, the three-point third order correlation for a homogeneous field, when observations have been made at the same time t, is

$$T_{ijk}(\mathbf{r}, \mathbf{s}, t) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t)u_k'(\mathbf{x} + \mathbf{s}, t)} \tag{6.65d}$$

Similarly the two-point third order velocity correlation for a homogeneous field is

$$T_{ijk}(0, \mathbf{r}, t) = \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x}, t)u_k'(\mathbf{x} + \mathbf{r}, t)} \tag{6.65e}$$

A symmetry relation for homogeneous fields can simply be obtained by using Equation 6.65c. First, interchanging i and j, we have:

$$R_{ji}(\mathbf{r}, t) = \overline{u_j'(\mathbf{x}, t)u_i'(\mathbf{x} + \mathbf{r}, t)}$$

Thus:

$$\begin{aligned} R_{ji}(-\mathbf{r}, t) &= \overline{u_j'(\mathbf{x}, t)u_i'(\mathbf{x} - \mathbf{r}, t)} \\ &= \overline{u_i'(\mathbf{d}, t)u_j'(\mathbf{d} + \mathbf{r}, t)} \end{aligned}$$

Therefore:

$$R_{ji}(-\mathbf{r}, t) = R_{ij}(\mathbf{r}, t) \tag{6.66}$$

since the correlation does not depend on the absolute position.

The pressure-velocity correlation in a homogeneous field with pressure at x and velocity at y is

$$\begin{aligned} \overline{p'(\mathbf{x}, t)u_j'(\mathbf{y}, t)} &= \overline{p'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t)} \\ &= P_j(\mathbf{r}, t) \end{aligned} \tag{6.67a}$$

If the pressure at y and the velocity at x is taken then:

$$\begin{aligned} \overline{p'(\mathbf{y}, t)u_i(\mathbf{x}, t)} &= \overline{p'(\mathbf{y}, t)u_i(\mathbf{y} - \mathbf{r}, t)} \\ &= P_i(-\mathbf{r}, t) \end{aligned} \tag{6.67b}$$

Dynamic Equation of Correlations

The Navier-Stokes equations for incompressible flow are

$$\operatorname{div} \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{uu}) = -\operatorname{grad} p + \nu\nabla^2\mathbf{u}$$

where $\mathbf{u}$ is the total or instantaneous velocity vector, and p is the total kinematic pressure (pressure/density). The components of instantaneous velocity $\mathbf{u}$ with respect to a Cartesian coordiante system (x_1, x_2, x_3) are (u_1, u_2, u_3). In subscript component notation the Navier-Stokes equations for a point $\mathbf{x}$ and time t are

$$\frac{\partial}{\partial x_j} u_j(\mathbf{x}, t) = 0 \qquad \textbf{(6.68a)}$$

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_j} [u_j(\mathbf{x}, t)u_i(\mathbf{x}, t)] = \nu\nabla_x^2 u_i(\mathbf{x}, t) - \frac{\partial}{\partial x_i} p(\mathbf{x}, t) \qquad \textbf{(6.68b)}$$

where:

$$\nabla_x^2 = \frac{\partial^2}{\partial x_j\, \partial x_j}$$

For another point $\mathbf{y}$ but at the *same* time t the equations are

$$\frac{\partial}{\partial y_j} u_j(\mathbf{y}, t) = 0 \qquad \textbf{(6.69a)}$$

$$\frac{\partial}{\partial t} u_k(\mathbf{y}, t) + \frac{\partial}{\partial y_j} [u_j(\mathbf{y}, t)u_k(\mathbf{y}, t)] = \nu\nabla_y^2 u_k(\mathbf{y}, t) - \frac{\partial}{\partial y_k} p(\mathbf{y}, t) \qquad \textbf{(6.69b)}$$

where:

$$\nabla_y^2 = \frac{\partial^2}{\partial y_j\, \partial y_j}$$

Points $\mathbf{x}$ and $\mathbf{y}$ are independent of each other.* Therefore multiplying first Equation 6.68a by $u_k(\mathbf{y}, t)$ and Equation 6.69a by $u_k(\mathbf{x}, t)$ and performing the average, we have, respectively:

$$\frac{\partial}{\partial x_j} \overline{u_j(\mathbf{x}, t)u_k(\mathbf{y}, t)} = 0 \qquad \textbf{(6.70a)}$$

$$\frac{\partial}{\partial y_j} \overline{u_j(\mathbf{y}, t)u_k(\mathbf{x}, t)} = 0 \qquad \textbf{(6.70b)}$$

Again using the independence of $\mathbf{x}$ and $\mathbf{y}$, we also have:

$$\frac{\partial}{\partial x_j} \overline{u_j(\mathbf{x}, t)p(\mathbf{y}, t)} = 0 \qquad \textbf{(6.71a)}$$

* In other words, the derivatives of $u_j(\mathbf{x}, t)$ at $\mathbf{x}$ are independent of $u_k(\mathbf{y}, t)$.

$$\frac{\partial}{\partial y_j}\overline{u_j(\mathbf{y},t)p(\mathbf{x},t)} = 0 \tag{6.71b}$$

In the same manner multiplying Equation 6.68b by $u_k(\mathbf{y}, t)$ and Equation 6.69b by $u_i(\mathbf{x}, t)$ and then adding and taking the average of each term, we get:

$$\frac{\partial}{\partial t}\overline{u_i(\mathbf{x},t)u_k(\mathbf{y},t)} + \frac{\partial}{\partial x_j}\overline{u_j(\mathbf{x},t)u_i(\mathbf{x},t)u_k(\mathbf{y},t)} + \frac{\partial}{\partial y_j}\overline{u_j(\mathbf{y},t)u_k(\mathbf{y},t)u_i(\mathbf{x},t)}$$

$$= \nu\nabla_x^2\,\overline{u_i(\mathbf{x},t)u_k(\mathbf{y},t)} + \nu\nabla_y^2\,\overline{u_k(\mathbf{y},t)u_i(\mathbf{x},t)} - \frac{\partial}{\partial x_i}\overline{p(\mathbf{x},t)u_k(\mathbf{y},t)}$$

$$- \frac{\partial}{\partial y_k}\overline{p(\mathbf{y},t)u_i(\mathbf{x},t)} \tag{6.72}$$

Equation 6.72 does not form a closed system of equations for the second order correlations because of the the appearance of third order correlations. Various closure techniques have to be devised to solve these equations. Another important point to note here is that for general flow fields the instantaneous velocity components $u_i(\mathbf{x}, t)$ and the pressure $p(\mathbf{x}, t)$ are expressible as:

$$u_i(\mathbf{x},t) = \overline{u}_i(\mathbf{x},t) + u_i'(\mathbf{x},t)$$

$$p(\mathbf{x},t) = \overline{p}(\mathbf{x},t) + p'(\mathbf{x},t)$$

where:

$$\overline{u_i'} = 0, \qquad \overline{p'} = 0$$

Thus, using the above decomposition in Equations 6.68b and 6.69b and taking the average of each term while using $\overline{u_i'} = 0$ and $\overline{p'} = 0$, we get two averaged equations. Next, we multiply the first averaged equation by $\overline{u}_k(\mathbf{y}, t)$ and the second by $\overline{u}_i(\mathbf{x}, t)$ and then add these two equations. Introducing the decompositions in Equation 6.72 and using the equation derived above while using Equations 6.70, we get:

$$\frac{\partial}{\partial t}\overline{u_i'(\mathbf{x},t)u_k'(\mathbf{y},t)} + \overline{u}_j(\mathbf{x},t)\frac{\partial}{\partial x_j}\overline{u_i'(\mathbf{x},t)u_k'(\mathbf{y},t)} + \overline{u}_j(\mathbf{y},t)\frac{\partial}{\partial y_j}\overline{u_i'(\mathbf{x},t)u_k'(\mathbf{y},t)}$$

$$+ \frac{\partial}{\partial x_j}\overline{u_i'(\mathbf{x},t)u_j'(\mathbf{x},t)u_k'(\mathbf{y},t)} + \frac{\partial}{\partial y_j}\overline{u_i'(\mathbf{x},t)u_j'(\mathbf{y},t)u_k'(\mathbf{y},t)} + \frac{\partial\overline{u}_i(\mathbf{x},t)}{\partial x_j}\overline{u_j'(\mathbf{x},t)u_k'(\mathbf{y},t)}$$

$$+ \frac{\partial\overline{u}_k(\mathbf{y},t)}{\partial y_j}\overline{u_i'(\mathbf{x},t)u_j'(\mathbf{y},t)} = \nu\nabla_x^2\,\overline{u_i'(\mathbf{x},t)u_k'(\mathbf{y},t)} + \nu\nabla_y^2\,\overline{u_i'(\mathbf{x},t)u_k'(\mathbf{y},t)}$$

$$- \frac{\partial}{\partial x_i}\overline{p'(\mathbf{x},t)u_k'(\mathbf{y},t)} - \frac{\partial}{\partial y_k}\overline{p'(\mathbf{y},t)u_i'(\mathbf{x},t)} \tag{6.73}$$

This equation is the dynamic equation for second order correlations of the perturbation velocity and, as was the case with Equation 6.72, it also cannot be solved unless the third order correlations can be made to depend on the second order correlations.

Homogeneous Turbulence

A field of homogeneous turbulence is statistically stationary in space. Thus, the following requirements must be met for a field to be classified as of homogeneous turbulence.

1. In a homogeneous turbulence the mean velocity $\overline{u}_i(\mathbf{x}, t)$ must be a constant.
2. Velocity correlations or pressure-velocity correlations of two or more points must be independent of the absolute positions of the points and should depend on the separation distance between one and all other points.

To apply condition 1 in a convenient way we take the mean velocity to be zero. A particular case of condition 2 occurs when the separation distance between two points is zero, i.e., $\mathbf{r} = 0$, and then from Equation 6.65c:

$$R_{ij}(0, t) = \overline{u'_i(\mathbf{x}, t)u'_j(\mathbf{x}, t)}$$

from which:

$$R_{11} = \overline{u_1'^2}, \qquad R_{22} = \overline{u_2'^2}, \qquad R_{33} = \overline{u_3'^2}$$

Since they must not depend on $\mathbf{x}$, the mean values $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_3'^2}$, although different among themselves, have the same values at a given time throughout the field. For two points $\mathbf{x}$ and $\mathbf{y}$, we have:

$$\mathbf{r} = \mathbf{y} - \mathbf{x}$$

and since $\mathbf{x}$ and $\mathbf{y}$ are independent, we can write the differentiations as:

$$\frac{\partial}{\partial x_j} = -\frac{\partial}{\partial r_j}, \quad \frac{\partial}{\partial y_j} = \frac{\partial}{\partial r_j}; \qquad \nabla_x^2 = \nabla_y^2 = \frac{\partial^2}{\partial r_j\, \partial r_j} = \nabla^2$$

Using the definitions of various correlations given in Equations 6.65c to 6.65e, Equations 6.70 to 6.73 are written with zero mean values as:

$$\frac{\partial}{\partial r_j} R_{jk}(\mathbf{r}, t) = 0 \tag{6.74a}$$

$$\frac{\partial}{\partial r_j} R_{jk}(-\mathbf{r}, t) = 0 \tag{6.74b}$$

$$\frac{\partial}{\partial r_j} P_j(-\mathbf{r}, t) = 0 \tag{6.75a}$$

$$\frac{\partial}{\partial r_j} P_j(\mathbf{r}, t) = 0 \tag{6.75b}$$

$$\frac{\partial}{\partial t} R_{ik}(\mathbf{r}, t) - \frac{\partial}{\partial r_j} T_{jik}(0, \mathbf{r}, t) + \frac{\partial}{\partial r_j} T_{jki}(0, -\mathbf{r}, t)$$

$$= 2\nu\nabla^2 R_{ik}(\mathbf{r}, t) + \frac{\partial}{\partial r_i} P_k(\mathbf{r}, t) - \frac{\partial}{\partial r_k} P_i(-\mathbf{r}, t) \tag{6.76}$$

Equation 6.76 is the equation describing the dynamics of a homogeneous turbulence field, while Equations 6.74 are the solenoidal conditions for second order correlations. It is implicitly assumed that all correlations tend to zero as $|\mathbf{r}| \to \infty$. Using the continuity equation at $\mathbf{y} = \mathbf{x} + \mathbf{r}$, it is easy to show that:

$$\frac{\partial}{\partial r_k} T_{ijk}(0, \mathbf{r}, t) = 0 \tag{6.77}$$

Homogeneous Shear Turbulence

A field of *homogeneous shear turbulence* is a turbulent field in which the mean velocity is of the form:

$$\bar{u}_1 = ax_2, \qquad \bar{u}_2 = \bar{u}_3 = 0$$

so that there is a constant shear in only one direction. Any other mean velocity distribution which is more general than this would destroy the preferred directionality and also the homogeneous character of the flow. For homogeneous shear turbulence Equation 6.73 is written as:

$$\frac{\partial}{\partial t} R_{ik}(\mathbf{r}, t) - \bar{u}_j(\mathbf{x}, t)\frac{\partial}{\partial r_j} R_{ik}(\mathbf{r}, t) + \bar{u}_j(\mathbf{y}, t)\frac{\partial}{\partial r_j} R_{ik}(\mathbf{r}, t) + \frac{\partial \bar{u}_i(\mathbf{x}, t)}{\partial x_j} R_{jk}(\mathbf{r}, t)$$

$$+ \frac{\partial \bar{u}_k(\mathbf{y}, t)}{\partial y_j} R_{ij}(\mathbf{r}, t) - \frac{\partial}{\partial r_j} T_{jik}(0, \mathbf{r}, t) + \frac{\partial}{\partial r_i} T_{jki}(0, -\mathbf{r}, t)$$

$$= 2\nu\nabla^2 R_{ik}(\mathbf{r}, t) + \frac{\partial}{\partial r_i} P_k(\mathbf{r}, t) - \frac{\partial}{\partial r_k} P_i(-\mathbf{r}, t)$$

where use has been of the definition in Equation 6.65d. In this equation only the mean velocity distribution given above should be used.

Isotropic Turbulence

An isotropic turbulence field satisfies all the necessary criteria of a homogeneous turbulence field and in addition satisfies the following properties:

1. The mean-square values of the velocity components at all points in space at any given time t are all equal to one another, i.e.:

$$\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2} = \overline{u^2}, \quad \textit{say}. \tag{6.78a}$$

   Thus $\overline{u^2}$ though a spatial constant, is a function of time.

2. The double velocity correlation between two points is symmetric, i.e.:

$$R_{ij}(\mathbf{r}, t) = R_{ji}(\mathbf{r}, t) \tag{6.78b}$$

   because in the isotropic case the correlations are independent of the direction of $\mathbf{r}$.

3. The correlation between the velocity and the pressure fluctuations at two points is zero, i.e.:

$$P_j(\mathbf{r}, t) = \overline{p'(\mathbf{x}, t)u_j'(\mathbf{y} + \mathbf{r}, t)} = 0 \tag{6.78c}$$

Example 6.1

Prove the validity of Equation 6.78c.

Multiply both sides of Equation 6.78c by r_j/r to have:

$$\frac{P_j r_j}{r} = \frac{r_j}{r}\overline{p'(\mathbf{x}, t)u_j'(\mathbf{x} + \mathbf{r}, t)}$$

$$= (\overline{p^2})^{1/2}(\overline{u^2})^{1/2} F(r, t)$$

where F is the scalar function of r and t. The above equation is satisfied by taking:

$$P_j = \frac{r_j}{r}(\overline{p^2})^{1/2}(\overline{u^2})^{1/2}F(r, t)$$

Using the equation of continuity (Equation 6.75b), we get:

$$\frac{3F}{r} + r\frac{\partial}{\partial r}\left(\frac{F}{r}\right) = 0$$

so that:

$$F = \frac{\text{const.}}{r^2}$$

However, F must be finite as $r \rightarrow 0$; hence $F \equiv 0$, which proves the result.

Analysis of Isotropic Turbulence

Basic analysis of isotropic turbulence has been performed by Taylor,[30] Karman and Howarth,[31] Robertson,[32] Kovaszany,[33] Batchelor,[27] and Townsend.[28] Referring to Figure 6.5, let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be arbitrary unit vectors at points $\mathbf{x}$ and $\mathbf{x} + \mathbf{r}$, respectively, where $\mathbf{u}'(\mathbf{x}, t)$ and $\mathbf{u}'(\mathbf{x} + \mathbf{r}, t)$ are the perturbation velocity vectors. Let ϕ and θ be the angles made by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively, with the direction of $\mathbf{r}$. The correlation between the resolved part of $\mathbf{u}'(\mathbf{x}, t)$ in the direction of $\boldsymbol{\lambda}$ and of $\mathbf{u}'(\mathbf{x} + \mathbf{r}, t)$ in the direction of $\boldsymbol{\mu}$, using the summation convention on repeated indices, is

$$\overline{\lambda_i u_i'(\mathbf{x}, t)\mu_j u_j'(\mathbf{x} + \mathbf{r}, t)} = \lambda_i\mu_j R_{ij}(\mathbf{r}, t)$$

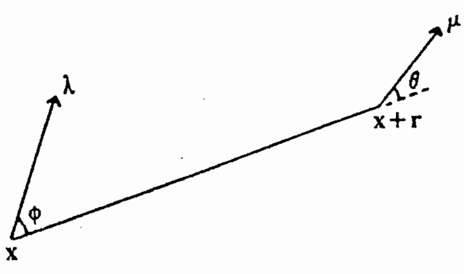


Fig. 6.5 A geometrical sketch for the description of isotropic turbulence.

If the flow field is isotropic, then arbitrary translations or rotations of the configuration formed by $\mathbf{r}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as a rigid body should not change the value of $\lambda_i\mu_j R_{ij}$. It can, therefore, depend on the scalars formed by the components r_i, μ_i, λ_i, and on the time t. This means that $\lambda_i\mu_j R_{ij}$ is a scalar invariant depending on the scalars r^2, $\boldsymbol{\lambda} \cdot \boldsymbol{\mu}$, $\boldsymbol{\lambda} \cdot \mathbf{r}$, $\boldsymbol{\mu} \cdot \mathbf{r}$, $(\boldsymbol{\lambda} \times \boldsymbol{\mu}) \cdot \mathbf{r}$, and other triple scalar products. Thus:

$$\lambda_i\mu_j R_{ij}(\mathbf{r}, t) = F(r^2, \boldsymbol{\lambda} \cdot \boldsymbol{\mu}, \boldsymbol{\lambda} \cdot \mathbf{r}, \boldsymbol{\mu} \cdot \mathbf{r}, (\boldsymbol{\lambda} \times \boldsymbol{\mu}) \cdot \mathbf{r}, \ldots)$$

From the left-hand side of this equation, we note that function F must be a homogeneous bilinear function of components λ_i, μ_j; further, since triple scalar products differ only in sign, then the form of F must be:

$$\lambda_i\mu_j R_{ij}(\mathbf{r}, t) = \lambda_i r_i \mu_j r_j a(r, t) + \lambda_i\mu_j\delta_{ij}b(r, t) + e_{ijk}\lambda_i\mu_j r_k c(r, t)$$

where a, b, and c are scalar functions of r and t. Further for isotropic turbulence:

$$R_{ij} = R_{ji}$$

so that the function $c \equiv 0$. Consequently:

$$R_{ij} = r_i r_j a(r, t) + \delta_{ij} b(r, t) \tag{6.79}$$

which is the representation of a symmetric second order Cartesian tensor.

For the formulation of third order two-point correlations in isotropic turbulence we again proceed from Figure 6.5 with three arbitrary unit vectors $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$, where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are chosen at the first point and $\boldsymbol{\nu}$ at the second point. Thus:

$$\lambda_i \mu_j \nu_k T_{ijk}(\mathbf{r}, t) = \overline{\lambda_i u_i'(\mathbf{x}, t)\mu_j u_j'(\mathbf{x}, t)\nu_k u_k'(\mathbf{x} + \mathbf{r}, t)}$$

The left-hand side of this equation must be a trilinear function of the scalars of the form:

$$\lambda_i r_i,\ \mu_i r_i,\ \nu_i r_i,\ r^2,\ \lambda_i \mu_i,\ \lambda_i \nu_i,\ \mu_i \nu_i$$

Hence:

$$T_{ijk}(\mathbf{r}, t) = r_i r_j r_k a_1(r, t) + r_i \delta_{jk} a_2(r, t)$$
$$+ r_j \delta_{ik} a_3(r, t) + r_k \delta_{ij} a_4(r, t)$$

Further, since T_{ijk} is symmetric in the first two indices, then:

$$a_3 = a_2$$

and using the notation:

$$a_1 = A, \qquad a_2 = B, \qquad a_4 = C$$

we have:

$$T_{ijk}(\mathbf{r}, t) = r_i r_j r_k A(r, t) + (r_i \delta_{jk} + r_j \delta_{ik}) B(r, t) + r_k \delta_{ij} C(r, t) \tag{6.80}$$

If $\mathbf{r}$ is changed to $-\mathbf{r}$, then:

$$T_{ijk}(-\mathbf{r}, t) = -T_{ijk}(\mathbf{r}, t)$$

so that T_{ijk} is an odd function of components r_i.

To establish further relations between $a(r, t)$ and $b(r, t)$ of the second order correlation (Equation 6.79) and between $A(r, t)$, $B(r, t)$, and $C(r, t)$ of the third order correlation (Equation 6.80), we use the divergence-free conditions in Equations 6.74a and 6.77. Thus:

$$\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial}{\partial r_i}[r_i r_j a(r, t) + \delta_{ij} b(r, t)] = 0$$

Using the chain rule:

$$\frac{\partial a}{\partial r_i} = \frac{\partial a}{\partial r}\frac{\partial r}{\partial r_i} = \frac{r_i}{r}\frac{\partial a}{\partial r}, \quad \text{etc.}$$

we get:

$$r\frac{\partial a}{\partial r} + \frac{1}{r}\frac{\partial b}{\partial r} + 4a = 0 \tag{6.81}$$

Similarly using Equation 6.80 in Equations 6.78, we get:

$$r\frac{\partial A}{\partial r} + \frac{2}{r}\frac{\partial B}{\partial r} + 5A = 0$$

$$r\frac{\partial C}{\partial r} + 2B + 3C = 0$$

Both A and B can be expressed in terms of C. From the last equation:

$$B = -\tfrac{1}{2}\left(3C + r\frac{\partial C}{\partial r}\right) \tag{6.82a}$$

and then:

$$A = \frac{1}{r}\frac{\partial C}{\partial r} \tag{6.82b}$$

Longitudinal and Lateral Correlations

The second and third order correlations, Equations 6.79 and 6.80, respectively, have been written with reference to an arbitrary rectangular Cartesian coordinate system. It is convenient to have their expressions available with reference to a coordinate system in which the axes are directed along $\mathbf{r}$ and perpendicular to $\mathbf{r}$ as shown in Figure 6.6.

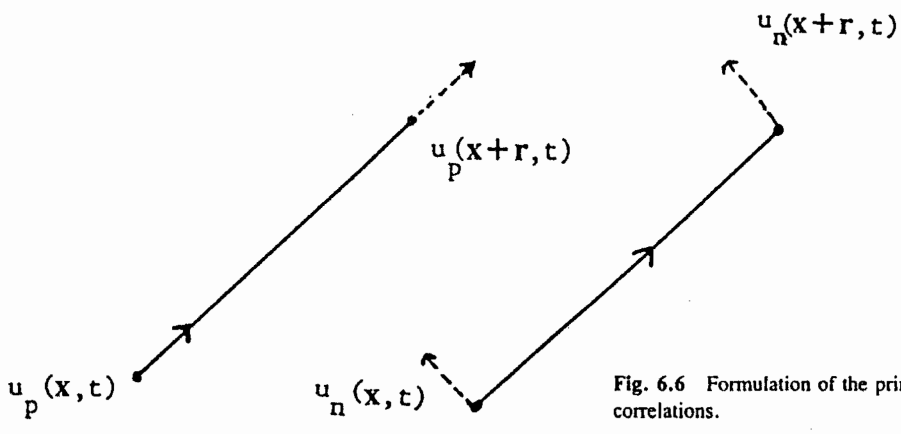


Fig. 6.6 Formulation of the principal and normal correlations.

The axes p and n are called the "principal" and "normal" axes. The unit vector in the direction of $\mathbf{r}$ is

$$\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{r}$$

while the unit vector in the direction of n is $\mathbf{n}$ with $|\mathbf{n}| = 1$. Second order correlation functions along p and n, which are called the longitudinal and lateral or transverse correlations, respectively, can now be formed as:

$$\overline{u^2}f(r, t) = \overline{u_p'(\mathbf{x}, t)u_p'(\mathbf{x} + \mathbf{r}, t)} \tag{6.83a}$$

$$\overline{u^2}g(r, t) = \overline{u_n'(\mathbf{x}, t)u_n'(\mathbf{x} + \mathbf{r}, t)} \tag{6.83b}$$

where $\overline{u^2}$ defined in Equation 6.78a is only a function of time. Inner multiplication of R_{ij} by the unit vector along p must be equal to the right-hand side of Equation 6.83a. Similarly, the inner multiplication of R_{ij} by the unit vector along n must be equal to the right-hand side of Equation 6.83b. Thus:

$$\begin{aligned}\overline{u^2}f &= \frac{r_i}{r}\frac{r_j}{r} R_{ij} \\ &= r^2 a(r, t) + b(r, t)\end{aligned}$$

Similarly:

$$\overline{u^2}g = n_i n_j R_{ij} = b(r, t)$$

Using the second equation in the first, we have:

$$\frac{\overline{u^2}(f - g)}{r^2} = a(r, t)$$

Introducing the last two equations in Equation 6.81, we obtain an expression of g in terms of f as:

$$g = f + \frac{1}{2} r \frac{\partial f}{\partial r} \tag{6.84}$$

In terms of f and g the second order correlation R_{ij} becomes:

$$R_{ij} = \frac{\overline{u^2} r_i r_j}{r^2}(f - g) + \overline{u^2}\delta_{ij} g \tag{6.85}$$

Following the preceding procedure, we form the two-point third order longitudinal and lateral correlations:

$$(\overline{u^2})^{3/2}k(r, t) = \overline{u_p'^2(\mathbf{x}, t)u_p'(\mathbf{x} + \mathbf{r}, t)} \tag{6.86a}$$

$$(\overline{u^2})^{3/2}h(r, t) = \overline{u_n'^2(\mathbf{x}, t)u_p'(\mathbf{x} + \mathbf{r}, t)} \tag{6.86b}$$

$$(\overline{u^2})^{3/2}q(r, t) = \overline{u_p'(\mathbf{x}, t)u_n'(\mathbf{x}, t)u_n'(\mathbf{x} + \mathbf{r}, t)} \tag{6.86c}$$

Hence:

$$(\overline{u^2})^{3/2}k(r, t) = \frac{r_i r_j r_k}{r^3} T_{ijk} = -2rC(r, t)$$

$$(\overline{u^2})^{3/2}h(r, t) = \frac{n_i n_j n_k}{r^3} T_{ijk} = rC(r, t)$$

$$(\overline{u^2})^{3/2}q(r,\,t) = \frac{r_i n_j r_k}{r} T_{ijk} = rB = \frac{-3rC}{2} - \frac{r^2}{2}\frac{\partial C}{\partial r}$$

Using Equations 6.82 along with the equations given above, we get:

$$A(r,\,t) = \frac{(u^2)^{3/2}}{r^3}(k - 2q - h)$$

$$B(r,\,t) = (\overline{u^2})^{3/2}\frac{q}{r}$$

$$C(r,\,t) = (\overline{u^2})^{3/2}\frac{h}{r}$$

Thus:

$$\frac{k - 2q - h}{r^3} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{h}{r}\right)$$

and:

$$\frac{2q}{r} = \frac{-3h}{r} - r\frac{\partial}{\partial r}\left(\frac{h}{r}\right)$$

From the above two equations we conclude that:

$$k = -2h$$

$$q = -h - \frac{r}{2}\frac{\partial h}{\partial r}$$

Finally, Equation 6.80 becomes:

$$T_{ijk}(0,\,\mathbf{r},\,t) = (\overline{u^2})^{3/2}\left[\frac{k - 2q - h}{r^3} r_i r_j r_k + (r_i\delta_{jk} + r_j\delta_{ik})\frac{q}{r} + r_k\delta_{ij}\frac{h}{r}\right] \tag{6.87}$$

Approximate Analysis

For small values of $|\mathbf{r}| = r$, the correlation function (Equation 6.83a) can be expanded in a Taylor expansion as:

$$\begin{aligned}
\overline{u^2}f(r,\,t) &= \overline{u_p'(\mathbf{x},\,t)u_p'(\mathbf{x} + \mathbf{r},\,t)} \\
&= \overline{u_p'(0,\,t)u_p'(r,\,t)} \\
&= \overline{u_p'^2(0,\,t)} + \overline{ru_p'(0,\,t)\left(\frac{\partial u_p'}{\partial r}\right)_0} + \frac{r^2}{2}\overline{u_p'(0,\,t)\left(\frac{\partial^2 u_p'}{\partial r^2}\right)_0} + \ldots
\end{aligned}$$

Now:

$$\overline{u_p'^2(0,\,t)} = \overline{u^2} = \text{a function of time only}$$

hence:

$$\overline{u_p'(0, t)\left(\frac{\partial u_p'}{\partial r}\right)_0} = \frac{1}{2}\left[\frac{\partial}{\partial r}\overline{(u_p'^2)}\right]_0 = 0$$

$$\overline{u_p'(0, t)\left(\frac{\partial^2 u_p'}{\partial r^2}\right)_0} = \frac{\partial^2}{\partial r^2}\overline{\left(\frac{u_p'^2}{2}\right)} - \overline{\left(\frac{\partial u_p'}{\partial r}\right)_0^2}$$

$$= -\overline{\left(\frac{\partial u_p'}{\partial r}\right)_0^2}$$

Therefore, for small values of r:

$$\overline{u^2}f(r, t) = \overline{u^2} - \frac{r^2}{2}\overline{\left(\frac{\partial u_p'}{\partial r}\right)_0^2}$$

Writing:

$$\frac{1}{\lambda_T^2} = \frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u_p'}{\partial r}\right)_0^2}$$

we have:

$$f(r, t) = \left(1 - \frac{r^2}{2\lambda_T^2}\right) \tag{6.88a}$$

Note that λ_T has the dimension of length. Using Equation 6.88a in Equation 6.84, we have:

$$g(r, t) = 1 - \frac{r^2}{\lambda_T^2} \qquad \text{for small } r \tag{6.88b}$$

Thus:

$$f(0, t) = g(0, t) = 1$$

and:

$$f''(0, t) = -\frac{1}{\lambda_T^2}$$

This analysis describes the behavior of f and g for two points which are very close to one another.

We now assume that both f and g approach zero when the distance between points $r \to \infty$. This assumption is also supported by experimental observations. With this assumption the various moments such as:

$$\int_0^\infty r^m f(r, t)\, dr, \qquad m \geq 0$$

have finite values. From Equation 6.84, we then have:

$$\int_0^\infty r^m g(r, t)\, dr = \frac{1 - m}{2}\int_0^\infty r^m f(r, t)\, dr$$

so that:

$$\int_0^\infty rg(r, t)\, dr = 0$$

From these equations, we conclude that for $m > 1$ the moments of f and g have opposite signs. Since $f(r, t)$ is positive for all values of r, then g must be negative for large values of r. Refer to Figure 6.7. It is also of interest to plot function g of Equation 6.88b for small values of r, which is a parabola intersecting the r-axis at $r = \pm\lambda_T$.

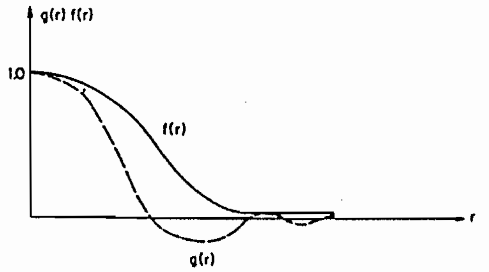


Fig. 6.7 Plot of the functions $f(r)$ and $g(r)$. (Taken from Stanisic, M. M., *The Mathematical Theory of Turbulence*. Springer-Verlag, New York, 1985. With permission.)

Taylor's expansions of the correlations given in Equation 6.86, and using the conditions:

$$\overline{u_p'^3(0, t)} = 0$$

$$\overline{u_p'^2\left(\frac{\partial u_p'}{\partial r}\right)_0} = \frac{1}{3}\left[\frac{\partial}{\partial r}(\overline{u_p'^3})\right]_0 = 0$$

and:

$$\left[\overline{u_p'^2\left(\frac{\partial^2 u_p'}{\partial r^2}\right)}\right]_0 = \left[\frac{\partial}{\partial r}\overline{\left(u_p'^2\frac{\partial u_p'}{\partial r}\right)}\right]_0 - \left[\overline{\frac{\partial u_p'}{\partial r}\frac{\partial}{\partial r}(u_p'^2)}\right]_0 = 0$$

yield the following behavior of k, h, and q for small r:

$$(\overline{u^2})^{3/2}k(r, t) = \sigma r^3$$

$$(\overline{u^2})^{3/2}h(r, t) = -\frac{1}{2}\sigma r^3$$

$$(\overline{u^2})^{3/2}q(r, t) = \frac{5}{4}\sigma r^3 \tag{6.89}$$

where:

$$\sigma = \frac{1}{6}\overline{u_p'^2(0, t)\left(\frac{\partial^3 u_p'}{\partial r^3}\right)_0}$$

Dynamic Equation for Isotropic Turbulence

Having obtained the forms of needed correlations, we can now form the equation for the dynamics of isotropic turbulence. First, we note from Equation 6.78c that for isotropic turbulence P_j = 0 so that Equation 6.76 becomes:

$$\frac{\partial}{\partial t} R_{ij}(\mathbf{r}, t) - \frac{\partial}{\partial r_\ell} T_{\ell ij}(0, \mathbf{r}, t) + \frac{\partial}{\partial r_\ell} T_{\ell ji}(0, -\mathbf{r}, t) = 2\nu\nabla^2 R_{ij}(\mathbf{r}, t)$$

Substituting Equations 6.85 and 6.87 in the above equation, after some simplification we get:

$$\frac{\partial}{\partial t}(\overline{u^2}f) = (\overline{u^2})^{3/2}\left(\frac{\partial k}{\partial r} + \frac{4k}{r}\right) + 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{6.90}$$

which is the dynamic equation for isotropic turbulence.

For small values of r, substituting Equations 6.88a and 6.89 in Equation 6.90 and neglecting terms of order r^2, we get:

$$\frac{d}{dt}\overline{u^2} = \frac{-10\nu\overline{u^2}}{\lambda_T^2} \tag{6.91}$$

In the case of isotropic turbulence, the turbulence energy per unit mass is

$$K = \frac{1}{2}(\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}) = \frac{3}{2}\overline{u^2}$$

and the rate of dissipation of energy ϵ is

$$\epsilon = -\frac{dK}{dt}$$

Thus, from Equation 6.91 we have:

$$\epsilon = \frac{15\nu\overline{u^2}}{\lambda_T^2} \tag{6.92}$$

The expression for the rate of dissipation of energy ϵ as given in Equation 6.92 was first obtained by Taylor.[34] The length scale λ_T which takes part in the dissipation process is called the *dissipation length scale*, or also as Taylor's microlength.

6.10. INTERNAL STRUCTURE IN THE WAVE NUMBER SPACE

The technique of Fourier transform has been found to be extremely useful both in the theoretical and experimental analyses of turbulent flows. The concept of eddies of various sizes and scales takes a more precise form through the use of the Fourier transform method. In this section, although we have developed the basic methodology of the Fourier transform for general turbulent flow, the detailed application of this technique is described only for isotropic turbulence.

Some General Definitions

We shall denote the wave number space by the vector $\mathbf{k}$ where $\mathbf{k} = (k_1, k_2, k_3)$. The purpose of the Fourier transform is to map various functions from the physical space $\mathbf{x} = (x_1, x_2, x_3)$ to the wave number space $\mathbf{k} = (k_1, k_2, k_3)$. In this connection we first consider a second order correlation formed by the velocity components at $\mathbf{x}$ and $\mathbf{x} + \mathbf{r}$ at the same time t, i.e.:

$$R_{j\ell}(\mathbf{x}, \mathbf{r}, t) = \overline{u_j'(\mathbf{x}, t)u_\ell'(\mathbf{x} + \mathbf{r}, t)}$$

As discussed earlier, the correlation $R_{j\ell}$ does not depend on the absolute position $\mathbf{x}$ only when the field is homogeneous and isotropic.

We now denote the Fourier transform of $R_{j\ell}(\mathbf{x}, \mathbf{r}, t)$ by $\Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t)$, through the equation:

$$\Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{j\ell}(\mathbf{x}, \mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{6.93a}$$

and its inverse by:

$$R_{j\ell}(\mathbf{x}, \mathbf{r}, t) = \iiint_{-\infty}^{\infty} \Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{k} \tag{6.93b}$$

with:

$$i = \sqrt{-1}, \quad d\mathbf{r} = dx_1 \, dx_2 \, dx_3, \quad d\mathbf{k} = dk_1 \, dk_2 \, dk_3$$

For $\mathbf{r} = 0$:

$$R_{j\ell}(\mathbf{x}, 0, t) = \iiint_{-\infty}^{\infty} \Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t) \, d\mathbf{k} \tag{6.94}$$

where $R_{j\ell}(\mathbf{x}, 0, t)$ is the autocorrelation tensor having the components:

$$\overline{u_1'^2}, \quad \overline{u_2'^2}, \quad \overline{u_3'^2}, \quad \overline{u_1'u_2'}, \quad \overline{u_1'u_3'}, \quad \overline{u_2'u_3'}$$

at position $\mathbf{x}$ and time t. The first three are the normal stresses; the last three are the shear stresses and are called the Reynolds stresses. The first three also define the turbulence energy per unit mass:

$$K = \tfrac{1}{2} (\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2})$$

A complete determination of $R_{j\ell}(\mathbf{x}, 0, t)$ through Equations 6.94 would permit us to evaluate the energy and the Reynolds shear stresses associated with various velocity fluctuations; this is why $R_{j\ell}(\mathbf{x}, 0, t)$ is called the *energy tensor* and $\Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t)$, the *energy spectrum tensor*. Note also that $\Phi_{j\ell} \, d\mathbf{k} = \Phi_{j\ell} \, dk_1 \, dk_2 \, dk_3$ is the contribution to $R_{j\ell}$ in the range $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$ or k_1 and $k_1 + dk_1$, k_2 and $k_2 + dk_2$, k_3 and $k_3 + dk_3$. To obtain a tensor from $\phi_{j\ell}$ which is a function of the wave number magnitude k:

$$k = (\mathbf{k} \cdot \mathbf{k})^{1/2} = |\mathbf{k}|$$

we integrate $\Phi_{j\ell}$ over a sphere of radius k to have:

$$E_{j\ell}(\mathbf{x}, k, t) = \iint_S \Phi_{j\ell}(\mathbf{x}, \mathbf{k}, t) \, dS(k) \tag{6.95}$$

which is called the *averaged energy spectrum tensor*. Thus $E_{j\ell} \, dk$ is the contribution to $R_{j\ell}$ in the range k and $k + dk$. Based on $E_{j\ell}$ we now define a three-dimensional *energy spectrum function* by contraction of indices as:

$$E(\mathbf{x}, \mathbf{k}, t) = \frac{1}{2} E_{jj}(\mathbf{x}, \mathbf{k}, t)$$
$$= \frac{1}{2}(E_{11} + E_{22} + E_{33}) \tag{6.96}$$

From Equation 6.95, since:

$E_{11}\, dk$ is the contribution to R_{11} in the range k and $k + dk$

$E_{22}\, dk$ is the contribution to R_{22} in the range k and $k + dk$

$E_{33}\, dk$ is the contribution to R_{33} in the range k and $k + dk$

then:

$$\frac{1}{2}(E_{11} + E_{22} + E_{33})\, dk = E(\mathbf{x}, k, t)\, dk$$

is the contribution to K in the range k and $k + dk$. Consequently:

$$K(\mathbf{x}, t) = \int_0^\infty E(\mathbf{x}, k, t)\, dk \tag{6.97}$$

and therefore $E(\mathbf{x}, k, t)$ is the energy density. For homogeneous and isotropic turbulence, K is a function of time only. Further, for isotropic turbulence:

$$K = \frac{3}{2}\overline{u^2}$$

hence:

$$\frac{3}{2}\overline{u^2} = \int_0^\infty E(k, t)\, dk \tag{6.98}$$

Dynamic Equation of Homogeneous Turbulence in the k-Space

From here onward we shall study only the properties of homogeneous and isotropic turbulence in the wave number space. Because of homogeneity and isotropy, all quantities are independent of the absolute position $\mathbf{x}$. The dynamic equation for homogeneous turbulence has already been derived earlier in Equation 6.76. We rewrite it in a different form as:

$$\frac{\partial}{\partial t} R_{j\ell} - T_{j\ell} = 2\nu\nabla^2 R_{j\ell} + P_{j\ell} \tag{6.99a}$$

where:

$$T_{j\ell}(\mathbf{r}, t) = \frac{\partial}{\partial r_m}\left[\overline{u'_m(\mathbf{x}, t)u'_j(\mathbf{x}, t)u'_\ell(\mathbf{x} + \mathbf{r}, t)} - \overline{u'_m(\mathbf{x} + \mathbf{r}, t)u'_\ell(\mathbf{x} + \mathbf{r}, t)u'_j(\mathbf{x}, t)}\right] \tag{6.99b}$$

$$P_{j\ell}(\mathbf{r}, t) = \frac{\partial}{\partial r_j}\overline{p'(\mathbf{x}, t)u'_\ell(\mathbf{x} + \mathbf{r}, t)} - \frac{\partial}{\partial r_\ell}\overline{p'(\mathbf{x} + \mathbf{r}, t)u'_j(\mathbf{x}, t)} \tag{6.99c}$$

The solenoidal condition is

$$\frac{\partial}{\partial r_j} R_{j\ell}(\mathbf{r}, t) = 0 \tag{6.99d}$$

We now take the Fourier transform of each term and have:

$$\frac{\partial}{\partial t} \Phi_{j\ell}(\mathbf{k}, t) - \Gamma_{j\ell}(\mathbf{k}, t) = \Pi_{j\ell}(\mathbf{k}, t) - 2\nu k^2 \Phi_{j\ell}(\mathbf{k}, t) \tag{6.100a}$$

$$k_j \Phi_{j\ell}(k, t) = 0 \tag{6.100b}$$

where:

$$\Phi_{j\ell}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{j\ell}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{6.101a}$$

$$\Gamma_{j\ell}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} T_{j\ell}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{6.101b}$$

$$\Pi_{j\ell}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} P_{j\ell}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{6.101c}$$

Equation 6.100b is called the orthogonality condition in the wave number space. Further, conclusions on the preceding analysis are as follows. From Equation 6.99b we find that:

$$T_{j\ell}(0, t) = 0$$

hence, the inverse of Equation 6.101b yields the important result:

$$\iiint_{-\infty}^{\infty} \Gamma_{j\ell}(\mathbf{k}, t)\, d\mathbf{k} = 0 \tag{6.102a}$$

Note also the inverse relations:

$$T_{j\ell}(r, t) = \iiint_{-\infty}^{\infty} \Gamma_{j\ell}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}}\, dk \tag{6.102b}$$

$$P_{j\ell}(r, t) = \iiint_{-\infty}^{\infty} \Pi_{j\ell}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} \tag{6.102c}$$

Analysis of Isotropic Turbulence in the k-Space

The dynamic equation for isotropic turbulence in the wave number space can be written by using Equation 6.99a, in which because of the result (Equation 6.78c) the term $P_j = 0$. Taking the Fourier transform of each term we have the equation:

$$\frac{\partial}{\partial t} \Phi_{j\ell}(\mathbf{k}, t) - \Gamma_{j\ell}(\mathbf{k}, t) = -2\nu k^2 \Phi_{j\ell}(\mathbf{k}, t) \tag{6.103}$$

To find the form of $\Phi_{j\ell}$ in the case of isotropic turbulence we note that the Fourier transform (Equation 6.101a) involves only the scalar product $\mathbf{k} \cdot \mathbf{r}$. Hence, $\Phi_{j\ell}$ must possess the same symmetry and invariance properties as $R_{j\ell}$ and must also be an isotropic tensor. Based on these observations we assume the form of $\Phi_{j\ell}$ as:

$$\Phi_{j\ell} = k_j k_\ell \Phi_1 + \delta_{j\ell}\Phi_2$$

where Φ_1 and Φ_2 are functions of k and t. Using the orthogonality condition (Equation 6.100b) we obtain:

$$k^2 k_\ell \Phi_1 + \delta_{j\ell} k_j \Phi_2 = 0$$

or:

$$k^2 \Phi_1 + \Phi_2 = 0$$

Thus:

$$\Phi_{j\ell}(\mathbf{k}, t) = \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right)\Phi_2 \tag{6.104a}$$

To proceed further, we contract the indices in Equation 6.104a and obtain:

$$\Phi_{jj} = 2\Phi_2(k, t)$$

so that from Equation 6.95:

$$E_{jj} = \iint_S \Phi_{jj}\, dS(k)$$

$$= 2(4\pi k^2)\Phi_2 \tag{6.104b}$$

because on the surface of a sphere k = constant. Using Equation 6.104b in Equation 6.96, we get:

$$E(k, t) = \frac{1}{2} E_{jj} = 4\pi k^2 \Phi_2$$

so that:

$$\Phi_2 = \frac{E(k, t)}{4\pi k^2} \tag{6.104c}$$

Thus, Equation 6.104a is finally written as:

$$\Phi_{j\ell}(\mathbf{k}, t) = \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right) \frac{E(k, t)}{4\pi k^2} \tag{6.105}$$

An important consequence of Equation 6.105 is that:

$$\Phi_{jj}(\mathbf{k}, t) = \frac{E(k, t)}{2\pi k^2} \tag{6.106a}$$

so that:

$$\frac{E(k, t)}{2\pi k^2} = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{ii}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{6.106b}$$

Since $\Gamma_{j\ell}$ has the same symmetry and invariance properties as Φ_{ij} we take $\Gamma_{j\ell}$ in the form of Equation 6.105 and write it as:

$$\Gamma_{j\ell}(\mathbf{k}, t) = \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right) \frac{T(k, t)}{4\pi k^2} \tag{6.107}$$

where the newly introduced function $T(k, t)$ is called the *transfer function*. Substituting Equations 6.105 and 6.107 in Equation 6.103 we get:

$$\frac{\partial E}{\partial t} = T(k, t) - 2\nu k^2 E(k, t) \tag{6.108}$$

which is the *basic equation for isotropic turbulence*.

To establish an important property of $T(k, t)$, we substitute Equation 6.107 in Equation 6.102a and get:

$$\iiint_{-\infty}^{\infty} \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right) \frac{T(k, t)}{4\pi k^2} \, dk_1 \, dk_2 \, dk_3 = 0$$

Introducing spherical coordinates in the wave number space:

$$k_1 = k \sin\theta \cos\phi, \quad k_2 = k \sin\theta \sin\phi, \quad k_3 = k \cos\theta$$

we get:

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} [\delta_{j\ell} - f_{j\ell}(\theta, \phi)] T(k, t) \sin\theta \, d\theta \, d\phi \, dk = 0$$

Thus:

$$\int_0^\infty T(k, t) \, dk = 0 \tag{6.109}$$

We now integrate Equation 6.108 from 0 to ∞ with respect to k while using Equation 6.109. Thus:

$$\frac{\partial}{\partial t} \int_0^\infty E(k, t) \, dk = -2\nu \int_0^\infty k^2 E(k, t) \, dk$$

Using Equation 6.98 we have the result:

$$\begin{aligned} -\frac{dK}{dt} &= 2\nu \int_0^\infty k^2 E(k, t) \, dt \\ &= \epsilon \end{aligned} \tag{6.110}$$

where:

$$K = \frac{3}{2}\overline{u^2}$$

The results in Equations 6.109 and 6.110 show that the total turbulence energy is in no way changed by the inertial terms. The inertial terms only serve to transfer the spectral resolution of kinetic energy from one wave number range to the other. Only viscosity dissipates the total energy to heat.

Physically k has the dimension of (length)$^{-1}$. Hence, small and large eddies are represented by large and small wave numbers, respectively. Because of the term $k^2E(k, t)$ in Equation 6.110, most of the dissipation occurs at high waves numbers (small eddies) while most of the energy is contained in low wave numbers (large eddies). Typical distributions of E and k^2E are shown in Figure 6.8.

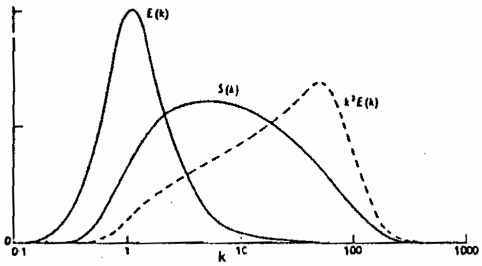


Fig. 6.8 Distribution of E, S, and k^2E in isotropic turbulence. (Taken from Townsend, A. A., *The Structure of Turbulent Shear Flow*, Cambridge University Press, London, 1956, 27. With permission.)

The function S is defined as $S(k, \tau) = -\int_0^k T(k, \tau)\, d\kappa$

Connection Between $\overline{u^2}f(r, t)$ and $E(k, t)$

From Equation 6.85:

$$\begin{aligned} R_{jj} &= R_{11} + R_{22} + R_{33} \\ &= \overline{u^2}\,(f + 2g) \\ &= 2R(r, t), \quad \text{say} \end{aligned} \tag{6.111a}$$

Substituting Equation 6.111a in Equation 6.106b, we get:

$$\frac{E(k, t)}{2\pi k^2} = \frac{1}{4\pi^3}\iiint_{-\infty}^{\infty} R(r, t)e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{6.111b}$$

Introducing spherical coordinates in the physical space $\mathbf{r}$:

$$r_1 = r\cos\theta\cos\phi, \quad r_2 = r\sin\theta\cos\phi, \quad r_3 = r\cos\theta$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{r}$, so that:

$$\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$$

thus:

$$E(k, t) = \frac{k^2}{\pi}\int_0^\infty\int_0^\pi r^2R(r, t)e^{-ikr\cos\theta}\, dr\, d\theta$$

However,

$$\int_0^{\pi} kr \sin\theta\, e^{-ikr\cos\theta}\, d\theta = 2 \sin kr$$

hence:

$$E(k, t) = \frac{2k}{\pi} \int_0^{\infty} rR(r, t) \sin kr\, dr \tag{6.112a}$$

Thus, the Fourier transform of $rR(r, t)$ is $E(k, t)/k$. To find the inverse of Equation 6.112, we note that:

$$R_{ij}(\mathbf{r}, t) = \iiint_{-\infty}^{\infty} \Phi_{ij} e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k}$$

Using Equations 6.106a and 6.111a we get:

$$R(r, t) = \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{E(k, t)}{k^2} e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k}$$

We can further simplify this equation by introducing spherical coordinates in the wave space with:

$$\mathbf{k} \cdot \mathbf{r} = kr \cos\theta$$

$$k_1 = k \sin\theta \cos\phi, \quad k_2 = k \sin\theta \sin\phi, \quad k_3 = k \cos\theta$$

Thus:

$$R(r, t) = \frac{1}{2} \int_0^{\infty} \int_0^{\pi} \frac{E(k, t)}{kr} kr \sin\theta\, e^{ikr\cos\theta}\, dk\, d\theta$$

However:

$$\int_0^{\pi} kr \sin\theta\, e^{ikr\cos\theta}\, d\theta = 2 \sin kr$$

Hence:

$$rR(r, t) = \int_0^{\infty} \frac{E(k, t)}{k} \sin kr\, dk \tag{6.112b}$$

which is the inversion of Equation 6.112a. Now using Equation 6.84 in Equation 6.111a, we have:

$$R = \frac{\overline{u^2}}{2r^2} \frac{\partial}{\partial r} (r^3 f)$$

Thus Equations 6.112 yield the results:

$$E(k, t) = \frac{1}{\pi}\int_0^\infty \overline{u^2}krf(r, t)(\sin kr - kr\cos kr)\,dr \tag{6.113a}$$

and:

$$\overline{u^2}f(r, t) = 2\int_0^\infty \frac{E(k, t)}{(kr)^3}(\sin kr - kr\cos kr)\,dk \tag{6.113b}$$

Therefore, a knowledge of $u^2f(r, t)$ gives the energy spectrum function $E(k, t)$.

Taylor's dissipation length parameter λ_T defined through Equation 6.92 can now be expressed in terms of $E(k, t)$ by using Equations 6.98 and 6.110 as:

$$\frac{5}{\lambda_T^2} = \left[\int_0^\infty k^2E(k, t)\,dk\right] \Big/ \int_0^\infty E(k, t)\,dk \tag{6.114}$$

Thus, a knowledge of $E(k, t)$ enables one to determine λ_T.

Formulation of One-Dimensional Spectrum

From the time of Taylor's original contributions to the theory and experiments on isotropic turbulence there have been many experimental verifications of the proposed theory and its results, e.g., Reference 28. Much impetus to the theory was given by the contributions of Karman and Howarth, which for the first time emphasized the tensor character of the correlations and their transforms. The tensor theory makes the analysis much easier to comprehend, as has been followed throughout this chapter. In this subsection we shall derive the formulae for the measurement of longitudinal and lateral spectrum functions. Despite the simplicity of these one-dimensional measurements, these results are of profound importance in interpreting those results which are in no way one dimensional.

A one-dimensional spectrum has dependence only on one component of the wave vector **k**, say on k_1. There are two obvious ways to write the spectrum function in one dimension:

$$\text{(i)}\qquad \phi_{j\ell}(k_1, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{j\ell}(r_1, 0, 0, t)e^{-ik_1r_1}\,dr_1 \tag{6.115a}$$

or:

$$\text{(ii)}\qquad \phi_{j\ell}(k_1, t) = \iint_{-\infty}^{\infty} \Phi_{j\ell}(k_1, k_2, k_3, t)\,dk_2\,dk_3 \tag{6.115b}$$

Both representations must obviously yield the same result. It is, however, simpler to take Equation 6.115a for future use. Taking the principal direction as r_1 and the transverse direction r_2, we have:

$$R_{11}(r_1, 0, 0, t) = \overline{u^2}f(r_1, t)$$

$$R_{22}(r_1, 0, 0, t) = \overline{u^2}g(r_1, t)$$

and since both f and g are even functions of the spatial argument, we have:

$$\phi_{11}(k_1, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u^2}f(r_1, t)\cos k_1r_1\, dr_1 \tag{6.116a}$$

$$\phi_{22}(k_1, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u^2}g(r_1, t)\cos k_1r_1\, dr_1 \tag{6.116b}$$

and:

$$\int_{-\infty}^{\infty} f(r_1, t)\sin k_1r_1\, dr_1 = \int_{-\infty}^{\infty} g(r_1, t)\sin k_1r_1 = 0 \tag{6.116c}$$

The inverse transforms of Equations 6.116 are

$$\overline{u^2}f(r_1, t) = \int_{-\infty}^{\infty} \phi_{11}(k_1, t)\cos k_1r_1\, dk_1 \tag{6.117a}$$

$$\overline{u^2}g(r_1, t) = \int_{-\infty}^{\infty} \phi_{22}(k_1, t)\cos k_1r_1\, dk_1 \tag{6.117b}$$

Differentiating Equation 6.117a with respect to r_1, substituting in Equation 6.84, and using Equation 6.117b, we get:

$$\phi_{22}(k_1, t) = \frac{1}{2}\left[\phi_{11}(k_1, t) - k_1\frac{d\phi_{11}}{dk_1}\right] \tag{6.118}$$

where use has been made of the condition:

$$\lim_{k_1\to\infty} k_1\phi_{11}(k_1, t) = 0$$

Differentiating Equations 6.116a, b with respect to k_1 and using Equations 6.84 and 6.113a, we get:

$$\frac{d\phi_{11}}{dk_1} + 2\frac{d\phi_{22}}{dk_1} = \frac{-E(k_1, t)}{k_1}$$

which, on substituting ϕ_{22} from Equation 6.118, becomes:

$$k_1^3\frac{d}{dk_1}\left(\frac{1}{k_1}\frac{d\phi_{11}}{dk_1}\right) = E(k_1, t) \tag{6.119a}$$

The solution of this equation is

$$\phi_{11}(k_1, t) = \frac{1}{2}\int_{k_1}^{\infty}\left(1 - \frac{k_1^2}{\xi^2}\right)\frac{E(\xi, t)}{\xi}\, d\xi \tag{6.119b}$$

Thus, from Equation 6.118:

$$\phi_{22}(k_1, t) = \frac{1}{4}\int_{k_1}^{\infty}\left(1 + \frac{k_1^2}{\xi^2}\right)\frac{E(\xi, t)}{\xi}\, d\xi \tag{6.120}$$

The function $\phi_{11}(k_1, t)$ is called the one-dimensional longitudinal spectrum function. *If both ϕ_{11} and ϕ_{22} can be measured,* then from Equation 6.118:

$$\frac{d\phi_{11}}{dk_1} = \frac{\phi_{11} - 2\phi_{22}}{k_1}$$

yielding the derivative, and from Equation 6.119a:

$$E(k_1, t) = k_1^3 \frac{d}{dk_1}\left[\frac{\phi_{11} - 2\phi_{22}}{k_1^2}\right] \tag{6.121}$$

Therefore, only one differentation of the experimental data yields the energy spectrum function $E(k_1, t)$.

The importance of the preceding analysis can be perceived if k_1 appearing in $E(k_1, t)$ is interpreted as $k = (k_1^2 + k_2^2 + k_3^2)^{1/2}$; then simply by making the one-dimensional measurements of ϕ_{11} and ϕ_{22}, one has determined $E(k, t)$ which is the energy spectrum function of a three-dimensional turbulence field.

Taylor's Formulae

Taylor reduced the problem of measurement of isotropic turbulence to a much simpler level by introducing a hypothesis which is known as *Taylor's hypothesis.* It states that if the mean velocity of flow, which acts as the carrier of turbulent fluctuations, is sufficiently larger than the fluctuating velocity (i.e., the fluctuations are too weak to induce any significant motion of their own), then one may assume the sequence of changes in velocity at a *fixed* point to be simply due to the passage of unchanging pattern of turbulent motion past the point. Let U_1 be the mean velocity along x_1. Then Taylor's hypothesis means that:

$$u_i(\mathbf{x}, t) = u_i(x_1 - U_1 t, x_2, x_3, 0)$$

and:

$$R_{j\ell}(\mathbf{x}, t) = R_{j\ell}(r_1 - U_1 t, r_2, r_3, 0)$$

The time is, therefore, given by:

$$t = \frac{x_1}{U_1}$$

and:

$$\frac{\partial}{\partial x_1} = \frac{1}{U_1}\frac{\partial}{\partial t}$$

These relations are used to determine the spatial variations in the mean flow direction by an analysis of the velocity fluctuations at a fixed point in time.

If n is the frequency in cycles/s, then

$$U_1 = n\lambda_1 = \frac{2\pi n}{k_1}$$

or:

$$k_1 = \frac{2\pi n}{U_1}$$

Thus, Equations 6.116 and 6.117 become:

$$\phi_{11}\left(\frac{2\pi n}{U_1}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u^2} f(r_1) \cos\frac{2\pi n r_1}{U_1}\, dr_1$$

$$\phi_{22}\left(\frac{2\pi n}{U_1}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u^2} g(r_1) \cos\frac{2\pi n r_1}{U_1}\, dr_1$$

$$\overline{u^2} f(r_1) = \frac{2\pi}{U_1}\int_{-\infty}^{\infty} \phi_{11}\left(\frac{2\pi n}{U_1}\right) \cos\frac{2\pi n r_1}{U_1}\, dn$$

$$\overline{u^2} g(r_1) = \frac{2\pi}{U_1}\int_{-\infty}^{\infty} \phi_{22}\left(\frac{2\pi n}{U_1}\right) \cos\frac{2\pi n r_1}{U_1}\, dn$$

To have an exact correspondence with the formulae of Taylor, we set:

$$\frac{2\pi}{U_1}\phi_{11}\left(\frac{2\pi n}{U_1}\right) = \frac{1}{2}\psi_{11}(n)$$

$$\frac{2\pi}{U_1}\phi_{22}\left(\frac{2\pi n}{U_1}\right) = \frac{1}{2}\psi_{22}(n)$$

Then:

$$\psi_{11}(n) = \frac{4}{U_1}\int_0^{\infty} \overline{u^2} f(r_1) \cos\frac{2\pi n r_1}{U_1}\, dr_1$$

$$\psi_{22}(n) = \frac{4}{U_1}\int_0^{\infty} \overline{u^2} g(r_1) \cos\frac{2\pi n r_1}{U_1}\, dr_1$$

$$\overline{u^2} f(r_1) = \int_0^{\infty} \psi_{11}(n) \cos\frac{2\pi n r_1}{U_1}\, dn$$

$$\overline{u^2} g(r_1) = \int_0^{\infty} \psi_{22}(n) \cos\frac{2\pi n r_1}{U_1}\, dn$$

To prove his hypothesis, Taylor has measured $U_1\psi_{11}(n)$ as a function of n/U_1 for different values of U_1.

6.11 THE THEORY OF UNIVERSAL EQUILIBRIUM

Since the late 1930s there has been an all-out effort to understand and isolate those basic physical and structural aspects of turbulence phenomena which remain invariant or quasi-invariant in practically any given turbulent field. In 1941 Kolmogorov[69] proposed the following two hypotheses for general turbulent flows:

1. For large Reynolds' numbers, the small-scale components (small eddies) are statistically steady, isotropic (no sense of directionality), and independent of the detailed structure of the large-scale motion (large eddies). This range of wave numbers is called the ***equilibrium range***.

2. When the Reynolds number is large enough for the energy-containing eddies to be much larger than the viscous dissipation eddies, then there exists a considerable *subrange* in the neighborhood of the lower end of equilibrium range in which negligible dissipation occurs and the transfer of energy by the inertial forces is the dominant process. Both hypotheses define particular ranges of the wave numbers as shown in Figure 6.9(a).

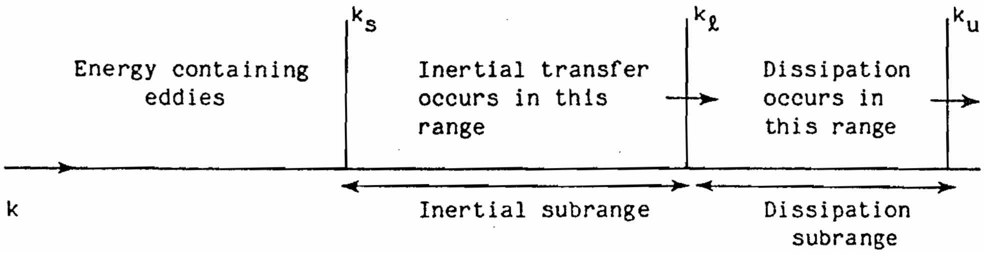


Fig. 6.9(a) Sketch showing different wave number range based on Kolmogorov's hypothesis.

Inertial subrange is, therefore, a part of the equilibrium range. According to the above hypotheses energy is removed by viscous dissipation over the whole equilibrium range but mainly at the upper end of the range where $k \sim k_u$. Energy is inserted by the inertial transfer at the lower end where $k \sim k_\ell$. Thus:

$$k_s \ll k_\ell \ll k_u$$

The removal and insertion of energy proceed at the same rate ϵ_d which is nearly ϵ since the flow is statistically isotropic. Thus ϵ is one of the structural parameters. The other parameter is the kinematic viscosity ν since the dissipation of energy depends on ν. Therefore, the controlling parameters of the equilibrium range are ϵ and ν. The dimensions of ϵ and ν are

$$\epsilon = (\text{length})^2/(\text{time})^3$$

$$\nu = (\text{length})^2/\text{time}$$

Beased on the dimensions of ϵ and ν, Kolmogorov defined a length and a velocity scale as:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \cong 1/k_u \tag{6.122}$$

$$v = (\nu\epsilon)^{1/4} \tag{6.123}$$

The length η is called the Kolmogorov length scale parameter, or the microlength of turbulence. This is the smallest length scale in turbulence, although it is quite large in comparison to the mean-free path. Based on dimensional considerations we find that the effect of a change in either ϵ and ν will be to change the effective length and time scales of motion. Thus both the postulates of Kolmogorov are applicable to any turbulent flow in the range k_s to k_u. This range is called the *universal equilibrium range*.

Determination of $E(k, t)$ Based on Kolmogorov's Hypothesis

Based on the first hypothesis the energy density E in the universal equilibrium range must also be a universal form and must depend on ϵ and ν. We, therefore, write it as:

$$E(k, t) = AE_0(\eta k)$$

where E_0 is a nondimensional function of ηk and A must have the dimension of energy density which is

$$E = (\text{length})^3/(\text{time})^2$$

Based on the dimensions of ϵ and ν we set:

$$A = (\epsilon\nu^5)^{1/4}$$

so that:

$$E(k, t) = (\epsilon\nu^5)^{1/4}E_0(\eta k) \tag{6.124a}$$

$$= \eta v^2 E_0(\eta k) \tag{6.124b}$$

In the inertial subrange which is governed by postulate 2, viscosity effect is insignificant in comparison to ϵ. Hence in the inertial subrange, E should not *explicitly* depend on ν, so that $E_0(\eta k)$ must be proportional to $\nu^{-5/4}$. Thus:

$$E_0(\eta k) = K_0(\eta k)^{-5/3} \tag{6.125a}$$

in the subrange, where K_0 is a universal constant. Substituting Equation 6.125a in Equation 6.124a we have:

$$E(k, t) = K_0(\epsilon)^{2/3}k^{-5/3} \tag{6.125b}$$

Thus in the inertial subrange the energy density behaves as:

$$E(k, t) \sim k^{-5/3} \tag{6.125c}$$

The two hypotheses of Kolmogorov enable us to determine the behavior of small-scale motion in the inertial subrange but do not determine $E_0(\eta k)$ for large wave numbers where dissipation occurs. To resolve this problem various *transfer theories* have been proposed by Heisenberg,[35] Kovaszany,[33] Obukhoff,[36] Pao,[37] and others. Next, we discuss these transfer theories.

Transfer Theories

The basic equation of isotropic turbulence is Equation 6.108, which is one equation in two unknowns, viz., $E(k, t)$ and $T(k, t)$. The function $T(k, t)$ arises due to the basic nonlinearity of the equations of motion and has the property given in Equation 6.109. Thus the nature of T is that it neither adds nor destroys the energy of turbulence. It only plays a role in the redistribution of energy at various levels of k. Because of this property, we call $T(k, t)$ as the *transfer function.* We now define another function $S(k, t)$ as:

$$S(k, t) = -\int_0^k T(k, t)\, dk$$

which on using Equation 6.109 can be written as:

$$S(k, t) = \int_k^\infty T(k, t)\, dk \tag{6.126}$$

Thus, $S(k, t)$ defines the transfer of energy from eddies in the range 0 to k to those in the range k to ∞.

Introducing Equation 6.126 in Equation 6.108, we get:

$$\frac{\partial E}{\partial t} = -\frac{\partial S}{\partial k} - 2\nu k^2 E(k, t) \tag{6.127}$$

which forms the starting point for various transfer theories. For the purpose of demonstration we consider only the transfer theories of Heisenberg and Pao.

Heisenberg's Transfer Theory. Heisenberg assumed the form of $S(k, t)$* as:

$$S(k, t) = 2\alpha_H\left(\int_k^\infty k^{-3/2}E^{1/2}\,dk\right)\left(\int_0^k k^2 E\,dk\right) \tag{6.128}$$

where α_H is a constant. Note that $S(\infty, t) = 0$ as it should be. Integrating Equation 6.127 from 0 to k, we have:

$$\frac{\partial}{\partial t}\int_0^k E(k, t)\,dk = -S(k, t) - 2\nu\int_0^k k^2E(k, t)\,dk \tag{6.129}$$

Let us now restrict the solution of Equation 6.129 to the universal equilibrium range so that the value of k is sufficiently large. We can, therefore, write:

$$\frac{\partial}{\partial t}\int_0^k E(k, t)\,dk \cong \frac{\partial}{\partial t}\int_0^\infty E(k, t)\,dk = -\epsilon$$

so that Equation 6.129 is rewritten as:

$$\epsilon = S(k, t) + 2\nu\int_0^k k^2E(k, t)\,dk \tag{6.130}$$

On using Equation 6.128 in Equation 6.130, we get:

$$\epsilon = 2\left[\nu + \alpha_H\int_k^\infty k^{-3/2}E^{1/2}\,dk\right]\left[\int_0^k k^2E(k, t)\,dk\right]$$

Writing:

$$\phi = \int_0^k k^2E(k, t)\,dk$$

we have:

$$\frac{\epsilon}{2\phi} = \nu + \alpha_H\int_k^\infty k^{-5/2}\left(\frac{d\phi}{dk}\right)^{1/2} dk \tag{6.131}$$

Differentiating Equation 6.131 with respect to k and noting that ϵ is not a function of k, we have:

$$\frac{d\phi}{\phi^4} = \frac{4\alpha_H^2}{(\epsilon)^2}\frac{dk}{k^5}$$

Note that S has the dimension (length)^2/(time)^3.

Integrating once:

$$-\frac{1}{3\phi^3} = \frac{-\alpha_H^2}{(\epsilon)^2}\frac{1}{k^4} + \text{constant}$$

and since for $k \to \infty$, $\phi \to \epsilon/2\nu$, then we have:

$$\phi^{-3} = \left(\frac{\epsilon}{2\nu}\right)^{-3} + \frac{3\alpha_H^2}{(\epsilon)^2}k^{-4}$$

Then since:

$$E(k, t) = \frac{1}{k^2}\frac{d\phi}{dk}$$

we obtain:

$$E(k, t) = \left(\frac{8\epsilon}{9\alpha_H}\right)^{2/3} k^{-5/3}\left(1 + \frac{8\nu^3}{3\epsilon\alpha_H^2}k^4\right)^{-4/3} \tag{6.132}$$

which is the solution of Equation 6.127 for large values of k, as obtained by Heisenberg. It is immediately seen that for k not very large, i.e., in the inertial subrange, Equation 6.132 approximates to:

$$E(k, t) \sim \left(\frac{8\epsilon}{9\alpha_H}\right)^{2/3} k^{-5/3} \tag{6.133a}$$

which is the same result as obtained through Kolmogorov's hypotheses. On the other hand, a result which could not have been predicted through Kolmogorov's hypotheses is that as $k \to \infty$ (small eddies), Equation 6.132 approximates to:

$$E(k, t) \sim k^{-7} \tag{6.133b}$$

Pao's Transfer Theory. Pao also proposed an energy transfer mechanism for large Reynolds' numbers with a view toward establishing a relation between T and E. He argued that the transfer of energy is a cascade process in which the spectral energy is continuously transferred to ever larger wave numbers.

Let the rate of energy spectral transfer at the wave number k be denoted as σ. Then Pao takes:

$$S(k, t) = E(k, t)\,\sigma$$

where σ is a function of ϵ and k. Based on dimensional considerations we then take:

$$\sigma = K_0^{-1}(\epsilon)^{1/3}k^{5/3}$$

where K_0 is the constant appearing in Kolmogorov's Equation 6.125a. Thus:

$$S(k, t) = K_0^{-1}(\epsilon)^{1/3}k^{5/3}E(k, t) \tag{6.134}$$

Differentiating Equation 6.130 with respect to k:

$$0 = \frac{dS}{dk} + 2\nu k^2 E(k, t)$$

in which using Equation 6.134, we get:

$$\frac{d}{dk}[K_0^{-1}(\epsilon)^{1/3}k^{5/3}E] = -2\nu k^2 E$$

Integrating, we get:

$$E(k, t) = Nk^{-5/3}\exp\left[-\frac{3K_0}{2}\nu(\epsilon)^{-1/3}k^{4/3}\right] \tag{6.135}$$

where N is a function independent of k. Using Equation 6.135 in the definition:

$$\epsilon = 2\nu\int_0^\infty k^2 E(k, t)\, dk$$

we obtain:

$$N = K_0(\epsilon)^{2/3}$$

Thus, the energy density obtained by Pao is given by:

$$E(k, t) = K_0(\epsilon)^{2/3}k^{-5/3}\exp[-\tfrac{3}{2}K_0\nu(\epsilon)^{-1/3}k^{4/3}] \tag{6.136}$$

We now define a new length scale:

$$\lambda_k = (\tfrac{1}{2}K_0)^{3/4}\eta \tag{6.137a}$$

and call it the *dissipation length scale of Kolmogorov*. Using the definition of η from Equation 6.122, we have:

$$\lambda_k = (\tfrac{3}{2}K_0\nu)^{3/4}(\epsilon)^{-1/4} \tag{6.137b}$$

With the definition in Equation 6.137b, Equation 6.136 can be rewritten as:

$$E(k, t) = K_0(\epsilon)^{2/3}k^{-5/3}\exp[-(\lambda_K k)^{4/3}] \tag{6.138}$$

For moderate values of k, which are appropriate for the inertial subrange, we have $\lambda_K k \ll 1$ so that Equation 6.138 in the subrange reduces to the form:

$$E(k, t) = K_0(\epsilon)^{2/3}k^{-5/3}$$

which is exactly of the form in Equation 6.125b. For large values of k (small eddies), Equation 6.138 predicts an exponential decay unlike the algebraic decay (k^{-7}) predicted by Heisenberg.

Figure 6.9(b) shows the distribution of $E(k, t)$ for various ranges of the wave number k, while Figure 6.9(c) shows an experimental verification of the spectrum function.

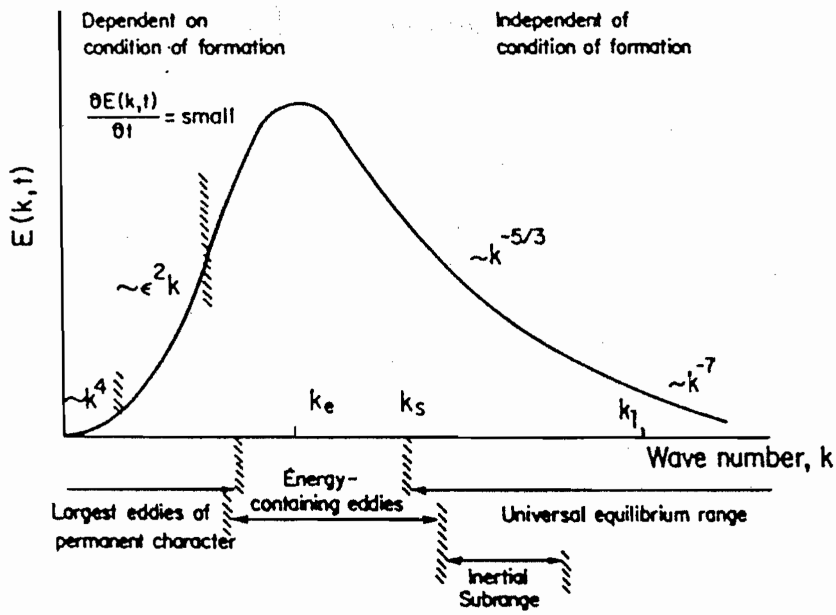


Fig. 6.9(b) Distribution of $E(k, t)$ for various range in isotropic turbulence. (Taken from Hinze, J. O., *Turbulence, and Introduction to its Mechanism and Theory*, McGraw-Hill, New York, 1959. With permission.)

Comparison of Taylor's and Kolmogorov's Dissipation Lengths

The length parameter of Kolmogorov (viz., η or λ_K) represents the scales of dissipating eddies. It is customary to call η as a microlength since it is the lowest eddy size in turbulence. On the other hand, the dissipation length parameter of Taylor denoted as λ_T is considerably larger than the scale of dissipating eddies. On dimensional considerations Rotta[39] has introduced another length scale λ_R through the equation:

$$\epsilon = \frac{BK^{3/2}}{\lambda_R}, \quad \text{or,} \quad K = (\epsilon\lambda_R/B)^{2/3} \tag{6.139}$$

where λ_R is nearly equal to an integral length of turbulence and B is a constant. From Equation 6.92, we also have a parallel expression:

$$\epsilon = \frac{10\nu K}{\lambda_T^2}$$

Substituting the expression of K obtained above, we get:

$$\lambda_T = (10)^{1/2}\eta^{2/3}(\lambda_R/B)^{1/3}$$

which establishes a relation between η, λ_R, and λ_T. Further, substituting η from Equation 6.137a, we get:

$$\lambda_T = C\lambda_K^{2/3}\lambda_R^{1/3}; \qquad C = \left(\frac{20}{3K_0B^{2/3}}\right)^{1/2} \tag{6.140}$$

Equation 6.140 establishes a relation between the three fundamental length scales, viz., of Kolmogorov, Taylor, and Rotta. According to Townsend[28] the value of B is $(2/3)^{1/2}$; and from the measurements of Grant et al.[40] the value of K_0 is about 1.44. Thus $C \cong 2.3021$. In the

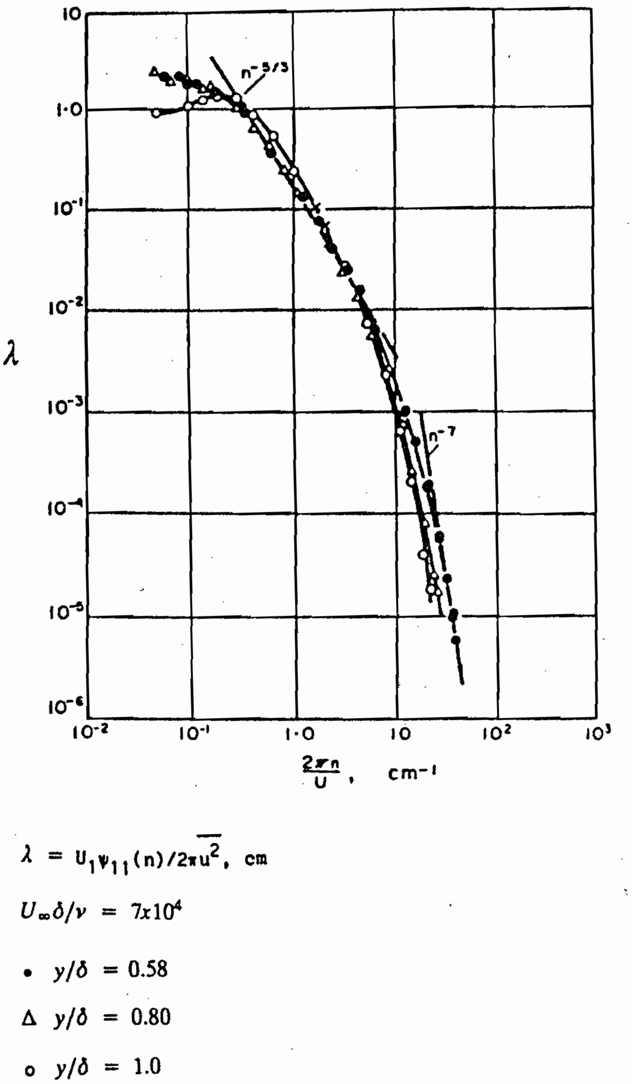


Fig. 6.9(c) Experimental validity of the results of Kolmogorov and Heisenberg. Shown in the figure is the frequency spectrum function $\psi_{11}(n)$ of the longitudinal velocity fluctuation of the turbulent boundary layer flow on a flat plate due to Klebanoff. (Taken from Rotta, J. C., *Prog. Aeronaut. Sci.*, 2, 1, 1962. With permission.)

transfer theories discussed previously the value of K_0 has been taken in the range $1.44 \leqslant K_0 \leqslant 2.2$. Thus, the values of C lie in the range $1.8625 \leqslant C \leqslant 2.3021$.

Integral Length and Time Scales

The length scales λ_T and λ_K introduced by Taylor and Kolmogorov, respectively, and studied in this chapter each have a sound physical and theoretical basis. Both of these length scales take part in the process of energy dissipation and, therefore, should be appropriately termed as the *dissipation length scales*. It must also be noted that λ_K is considerably smaller than λ_T.

Based on the behavior of the second order velocity correlation function (R_{ij}) we can also define other length scales. An important behavior of $R_{ij}(\mathbf{x}, \mathbf{r}, t)$ is that it approaches zero for some separation distance $\mathbf{r} = \mathbf{r}^*$, which implies that the fluid velocities are not correlated when the separation distance is $\mathbf{r}^*$. Thus the magnitude $r^* = |\mathbf{r}^*|$ can be taken as a length scale for large-scale motions. It is, however, customary to define the length scales based on the integral of correlation functions and call them integral scales.

Starting from the correlation function (Equation 6.64b) with $\tau = 0$, we first define a normalized correlation:

$$\hat{R}_{ij}(\mathbf{x}, \mathbf{r}, t) = \frac{R_{ij}(\mathbf{x}, \mathbf{r}, t)}{p_{(i)}q_{(j)}} \tag{6.141}$$

where:

$$p_{(i)}^2 = \overline{u_i^2(\mathbf{x}, t)}$$

$$q_{(j)}^2 = \overline{u_j^2(\mathbf{x} + \mathbf{r}, t)}$$

and there is *no* summation on the repeated i and j indices. A general integral length is now defined as:

$${}^{ij}L_m = \int_{-\infty}^{\infty} \hat{R}_{ij}(\mathbf{x}, r_1, r_2, r_3, t)\, dr_m \tag{6.142}$$

where m is 1, 2, or 3. The length scales which are commonly used in turbulence theory are the longitudinal and lateral integral scales based on velocity correlations having similar components at both points. Let r_1 and r_2 be the longitudinal and lateral coordinates. Then a longitudinal length scale is

$${}^{11}L_1 = \int_{-\infty}^{\infty} \hat{R}_{11}(\mathbf{x}, r_1, r_2, r_3, t)\, dr_1 \tag{6.143a}$$

while a lateral length scale is

$${}^{11}L_2 = \int_{-\infty}^{\infty} \hat{R}_{11}(\mathbf{x}, r_1, r_2, r_3, t)\, dr_2 \tag{6.143b}$$

where:

$$\hat{R}_{11} = \frac{R_{11}}{(\overline{u_1^2})^{1/2}(\overline{u_2^2})^{1/2}}$$

In the same manner, an integral time scale can be defined as:

$${}^{ij}T = \int_{-\infty}^{\infty} \hat{R}_{ij}(\mathbf{x}, \mathbf{r}, t)\, dt \tag{6.144}$$

Part II: Development of the Averaged Equations

6.12 INTRODUCTION

The equations of mean turbulent flow are obtained through the Navier-Stokes' equations by performing the ensemble or the time average on each term of the equations. For future reference we collect all the pertinent equations along with their physical interpretations in subsequent sections. The starting point of such a program is, of course, the set of Navier-Stokes' equations and the equation of energy. It is imperative to exploit all available equations for a thorough understanding of the physical meaning of a number of new terms appearing in the averaged equations.

6.13 AVERAGED EQUATIONS FOR INCOMPRESSIBLE FLOW

Let $\mathbf{u}$ be the instantaneous velocity and p the instantaneous pressure in an incompressible flow. Our basic postulate is that all instantaneous quantities satisfy the equation of continuity and the Navier-Stokes equations of motion. Thus, we use Equations 3.24 and 3.29 without body forces in the ensuing analysis.

Denoting a mean quantity by an overhead bar and a fluctuating quantity by a prime, we write:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \tag{6.145a}$$

$$p = \bar{p} + p' \tag{6.145b}$$

$$\sigma = \bar{\sigma} + \sigma' \tag{6.145c}$$

$$\omega = \bar{\omega} + \omega' \tag{6.145d}$$

Substituting Equation 6.145a in Equation 3.24, we have:

$$\operatorname{div} \bar{\mathbf{u}} + \operatorname{div} \mathbf{u}' = 0$$

Thus, on averaging and noting that since:

$$\overline{\operatorname{div} \bar{\mathbf{u}}} = \operatorname{div} \bar{\mathbf{u}}$$

then we have:

$$\operatorname{div} \bar{\mathbf{u}} = 0 \tag{6.146}$$

and hence:

$$\operatorname{div} \mathbf{u}' = 0 \tag{6.147}$$

Substituting Equations 6.145a–b in Equation 3.29, we get:

$$\begin{aligned}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \frac{\partial \mathbf{u}'}{\partial t} &+ \operatorname{div}(\bar{\mathbf{u}}\bar{\mathbf{u}}) + \operatorname{div}(\bar{\mathbf{u}}\mathbf{u}') + \operatorname{div}(\mathbf{u}'\bar{\mathbf{u}}) + \operatorname{div}(\mathbf{u}'\mathbf{u}') \\ &= -\operatorname{grad}\left(\frac{\bar{p}}{\rho}\right) - \operatorname{grad}\left(\frac{p'}{\rho}\right) + \nu\nabla^2\bar{\mathbf{u}} + \nu\nabla^2\mathbf{u}'\end{aligned} \tag{6.148}$$

Taking the average of each term, the averaged Navier-Stokes equation in vector form is given by:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \operatorname{div}(\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\operatorname{grad}\left(\frac{\bar{p}}{\rho}\right) + \nu\nabla^2\bar{\mathbf{u}} - \operatorname{div}((\overline{\mathbf{u}'\mathbf{u}'}) \tag{6.149}$$

Equation 6.149 is also known as Reynolds' averaged equation. Subtracting Equation 6.149 from Equation 6.148, we get an equation for the perturbations as:

$$\frac{\partial \mathbf{u}'}{\partial t} + \operatorname{div}(\bar{\mathbf{u}}\mathbf{u}') + \operatorname{div}(\mathbf{u}'\bar{\mathbf{u}}) + \operatorname{div}(\mathbf{u}'\mathbf{u}')$$

$$= -\text{grad}\left(\frac{p'}{\rho}\right) + \nu\nabla^2\mathbf{u}' + \text{div}(\overline{\mathbf{u}'\mathbf{u}'}) \tag{6.150}$$

which will be used subsequently. We now substitute Equations 6.145a, b, d in the second form of the Navier-Stokes equation (Equation 3.31) and take the average of each term, thus having:

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \text{div}(\overline{\mathbf{u}}\overline{\mathbf{u}}) = -\text{grad}\,\overline{p}_m + \nu\nabla^2\overline{\mathbf{u}} + \mathbf{T} \tag{6.151}$$

where:

$$\begin{aligned}
\text{div}(\overline{\mathbf{u}}\overline{\mathbf{u}}) &= \text{grad}(\tfrac{1}{2}|\overline{\mathbf{u}}|^2) + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}} \\
\overline{p}_m &= K + \frac{\overline{p}}{\rho} \\
K &= \tfrac{1}{2}\overline{\mathbf{u}' \cdot \mathbf{u}'} \\
\mathbf{T} &= \overline{\mathbf{u}' \times \boldsymbol{\omega}'} \\
\boldsymbol{\omega}' &= \text{curl}\,\mathbf{u}'
\end{aligned} \tag{6.152}$$

The term $\text{div}(-\overline{\mathbf{u}'\mathbf{u}'})$ apearing in Equation 6.149 is the divergence of the unknown correlation $-\overline{\mathbf{u}'\mathbf{u}'}$. *The dyad* $-\rho\overline{\mathbf{u}'\mathbf{u}'}$ *is called the Reynolds stress tensor.* On the other hand, in Equation 6.151, the *new* mean pressure $\overline{p}_m$ is composed of the mean pressure $\overline{p}$ and the quantity K which is the turbulence kinetic energy per unit mass. Further, the term $\mathbf{T}$ is a vector which has the characteristic of a body force per unit mass. Either Equation 6.149 or 6.151 can be used to solve a problem in turbulence; however, the first equation needs a specification of $-\rho\overline{\mathbf{u}'\mathbf{u}'}$ in terms of the mean velocity field while the second equation needs the specifications of both K and $\mathbf{T}$. We shall discuss these points further in later sections.

The vector invariant nature of the equations deloped so far is of much importance in practical computations. Thus, if a problem in turbulence requires the solution of averaged equations for flows past arbitrary shaped bodies or for flows in arbitrary shaped internal passages, then these equations can directly be used to write equations in the chosen coordinates by the method developed in Chapter 3. In particular, for Cartesian coordinates (x_1, x_2, x_3) using the index notation and the convention of summation over repeated indices, we can write Equations 6.146, 6.149, and 6.151, respectively, as:

$$\frac{\partial\overline{u}_j}{\partial x_j} = 0 \tag{6.153}$$

$$\frac{\partial\overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x_i} + \nu\nabla^2\overline{u}_i - \frac{\partial}{\partial x_j}(\overline{u_i'u_j'}) \tag{6.154}$$

$$\frac{\partial\overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\overline{p}_m}{\partial x_i} + \nu\nabla^2\overline{u}_i + e_{ijk}(\overline{u_j'\omega_k'}) \tag{6.155}$$

We now consider the equation of vorticity (Equation 3.34) and on using Equation 6.145a, b, d, we have:

$$\frac{\partial\overline{\boldsymbol{\omega}}}{\partial t} + \frac{\partial\boldsymbol{\omega}'}{\partial t} + (\text{grad}\,\overline{\boldsymbol{\omega}})\cdot\overline{\mathbf{u}} + (\text{grad}\,\boldsymbol{\omega}')\cdot\overline{\mathbf{u}} + (\text{grad}\,\overline{\boldsymbol{\omega}})\cdot\mathbf{u}'$$

$$+ (\text{grad}\ \boldsymbol{\omega}') \cdot \mathbf{u}' - (\text{grad}\ \bar{\mathbf{u}}) \cdot \bar{\boldsymbol{\omega}} - (\text{grad}\ \mathbf{u}') \cdot \bar{\boldsymbol{\omega}}$$

$$- (\text{grad}\ \bar{\mathbf{u}}) \cdot \boldsymbol{\omega}' - (\text{grad}\ \mathbf{u}') \cdot \boldsymbol{\omega}' = \nu\nabla^2\bar{\boldsymbol{\omega}} + \nu\nabla^2\boldsymbol{\omega}' \qquad (6.156)$$

On taking the average of each term we get the averaged vorticity equation:

$$\frac{\partial\bar{\boldsymbol{\omega}}}{\partial t} + (\text{grad}\ \bar{\boldsymbol{\omega}}) \cdot \bar{\mathbf{u}} - (\text{grad}\ \bar{\mathbf{u}}) \cdot \bar{\boldsymbol{\omega}} = \nu\nabla^2\bar{\boldsymbol{\omega}} + \text{div}(\overline{\mathbf{u}'\boldsymbol{\omega}'} - \overline{\boldsymbol{\omega}'\mathbf{u}'}) \qquad (6.157)$$

where for the last term we have used Equation M1.43a. Further, since the divergence of $\boldsymbol{\omega}$ is identically zero, we have:

$$\text{div}\ \bar{\boldsymbol{\omega}} = 0$$

$$\text{div}\ \boldsymbol{\omega}' = 0 \qquad (6.158)$$

Subtracting Equation 6.157 from Equation 6.156, we get the perturbation equation for the vorticity:

$$\frac{\partial\boldsymbol{\omega}'}{\partial t} + (\text{grad}\ \boldsymbol{\omega}') \cdot \bar{\mathbf{u}} + (\text{grad}\ \bar{\boldsymbol{\omega}}) \cdot \mathbf{u}' + (\text{grad}\ \boldsymbol{\omega}') \cdot \mathbf{u}' - (\text{grad}\ \mathbf{u}') \cdot \bar{\boldsymbol{\omega}}$$

$$- (\text{grad}\ \bar{\mathbf{u}}) \cdot \boldsymbol{\omega}' - (\text{grad}\ \mathbf{u}') \cdot \boldsymbol{\omega}' = \nu\nabla^2\boldsymbol{\omega}' + \text{div}(\overline{\boldsymbol{\omega}'\mathbf{u}' - \mathbf{u}'\boldsymbol{\omega}'}) \qquad (6.159)$$

Equation 6.157 is also in the vector invariant form. In the case of Cartesian coordinates it is written as:

$$\frac{\partial\bar{\omega}_i}{\partial t} + \bar{u}_j \frac{\partial\bar{\omega}_i}{\partial x_j} - \bar{\omega}_j \frac{\partial\bar{u}_i}{\partial x_j} = \nu \frac{\partial^2\bar{\omega}_i}{\partial x_j\, \partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_i'\omega_j'} - \overline{\omega_i'u_j'}) \qquad (6.160)$$

Equation of Turbulence Kinetic Energy

We now define:

$$q^2 = \mathbf{u}' \cdot \mathbf{u}' \qquad (6.161)$$

as twice the instantaneous energy of turbulence per unit mass, while:

$$K = \frac{1}{2}\overline{\mathbf{u}' \cdot \mathbf{u}'} = \frac{1}{2}\overline{q^2} \qquad (6.162)$$

as the mean turbulence energy per unit mass. Our purpose is to have the rate equation for K. Multiplying Equation 6.150 scalarly by $\mathbf{u}'$ and then taking the average of each term while using Equations M1.30, 6.146, and 6.147, we get:

$$\overline{\mathbf{u}' \cdot \frac{\partial\mathbf{u}'}{\partial t}} + \overline{\mathbf{u}' \cdot \{(\text{grad}\ \bar{\mathbf{u}}) \cdot \mathbf{u}'\}} + \overline{\mathbf{u}' \cdot \{(\text{grad}\ \mathbf{u}') \cdot \bar{\mathbf{u}}\}}$$

$$+ \overline{\mathbf{u}' \cdot \{(\text{grad}\ \mathbf{u}') \cdot \mathbf{u}'\}} = \overline{-\mathbf{u}' \cdot \text{grad}\left(\frac{p'}{\rho}\right)} + \nu\overline{\mathbf{u}' \cdot \nabla^2\mathbf{u}'}$$

In the Cartesian tensor notation Equations 6.161 and 6.162 are

$$q^2 = u_i'^2$$

$$K = \tfrac{1}{2}\overline{q^2} = \tfrac{1}{2}\overline{u_i'^2}$$

and then using the transformation techniques given in ME.1, the above equation in Cartesian coordinates becomes:

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = -\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left\{\overline{u_j'\left(\frac{p'}{\rho} + \tfrac{1}{2}q^2\right)}\right\} + \nu\overline{u_i'\nabla^2 u_i'} \tag{6.163}$$

where use has been made of continuity Equations 6.146 and 6.147.

Another form of the turbulent energy equation can be obtained by manipulating the last term in Equation 6.163. Using the continuity equation $\partial u_j'/\partial x_j = 0$, we easily get the form:

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j}$$

$$= -\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left\{\overline{u_j'\left(\frac{p'}{\rho} + \frac{1}{2}q^2\right)}\right\} + \nu\frac{\partial}{\partial x_j}\left\{\overline{u_i'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right\} - \epsilon_d \tag{6.164}$$

where:

$$\epsilon_d = \bar{e}_d = \nu\overline{\frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \tag{6.165a}$$

$$e_d = \nu\frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{6.165b}$$

A third form of the energy equation can be obtained by considering:

$$\frac{\partial K}{\partial x_j} = \tfrac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_i'^2}) = \overline{u_i'\frac{\partial u_i'}{\partial x_j}}$$

so that:

$$\overline{u_i'\nabla^2 u_i'} = \nabla^2 K - \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

Further:

$$\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} = \tfrac{1}{2}\overline{u_i'u_j'}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Using the above results in Equation 6.163 and writing:

$$\epsilon = \bar{e} = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{6.166a}$$

$$e = \nu\left(\frac{\partial u_i'}{\partial x_j}\right)^2 \tag{6.166b}$$

we get:

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = -\tfrac{1}{2}\overline{u_i' u_j'}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left\{\overline{u_j'\left(\frac{p'}{\rho} + \tfrac{1}{2}q^2\right)}\right\} + \nu\nabla^2 K - \epsilon \quad (6.167)$$

which is the rate equation for the turbulence energy K.

At this stage it is important to establish the connection between ϵ_d and ϵ introduced in Equations 6.164 and 6.167, respectively. A general expression for the dissipation rate has been given in Equation 2.79. Since the dissipation of turbulence energy ϵ_d occurs through the action of viscosity, it is obvious that the dissipation per unit mass must be given by:

$$e_d = \frac{1}{\rho}\boldsymbol{\sigma}' : \mathbf{D}'$$

where $\boldsymbol{\sigma}'$ is the fluctuating part of the Stokesian stress $\boldsymbol{\sigma}$, and $\mathbf{D}'$ is the fluctuating rate-of-strain tensor. Thus, for incompressible flow:

$$e_d = \nu\{\text{grad } \mathbf{u}' + (\text{grad } \mathbf{u}')^T\} : \text{grad } \mathbf{u}'$$

$$= \nu \frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

Using the continuity equation and the definition of e given in Equation 6.166b, we have:

$$e_d = e + \nu \frac{\partial^2}{\partial x_i\, \partial x_j}(u_i' u_j')$$

and

$$\bar{e}_d = \epsilon_d$$

$$\epsilon_d = \epsilon + \nu \frac{\partial^2}{\partial x_i\, \partial x_j}(\overline{u_i' u_j'}) \quad (6.168a)$$

Note also that based on the results obtained above,

$$\epsilon_d = \nu\left[\nabla^2 K + \frac{\partial^2}{\partial x_i\, \partial x_j}(\overline{u_i' u_j'}) - \overline{u_i' \nabla^2 u_i'}\right] \quad (6.168b)$$

$$= \nu\overline{|\boldsymbol{\omega}'|^2} + 2\nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}} \quad (6.168c)$$

From Equation 6.168a we note that ϵ is not the complete dissipation function unless the second derivatives of the Reynolds stress terms are either zero or negligible in comparison with ϵ. For high Reynolds number flows, $\epsilon \cong \epsilon_d$.

Both ϵ and ϵ_d are exactly equal for isotropic turbulence. In one of his basic papers, G. I. Taylor[xl] deduced that for isotropic turbulence, the mean square values of the first partial derivatives can all be expressed in terms of a single derivative. Using subscripted notation without summation convention, these relations can be expressed as follows:

(i) $\overline{\left(\dfrac{\partial u_i'}{\partial x_i}\right)^2}$, $\quad i = 1, 2, 3$, are all equal

(ii) $\overline{\left(\dfrac{\partial u_i'}{\partial x_j}\right)^2} = 2\overline{\left(\dfrac{\partial u_1'}{\partial x_1}\right)^2}$, $\quad i \neq j, \quad i, j$ being 1, 2, 3

(iii) $\overline{\dfrac{\partial u_i'}{\partial x_j}\dfrac{\partial u_j'}{\partial x_i}} = -\tfrac{1}{2}\overline{\left(\dfrac{\partial u_1'}{\partial x_1}\right)^2}$, $\quad i \neq j, \quad i, j$ being 1, 2, 3

These relations lead to the equation

$$\epsilon = 15\nu\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}$$

Compare with Equation 6.92.

Equation of Mean-Square Vorticity Fluctuations

The mean-square vorticity fluctuations (MSVF) is defined as:

$$\phi = \overline{\psi} \tag{6.169a}$$

where

$$\psi = \frac{1}{2}\,\boldsymbol{\omega}' \cdot \boldsymbol{\omega}' \tag{6.169b}$$

In the case of Cartesian coordinates, we have:

$$\psi = \frac{1}{2}\,\omega_i'^2$$

$$\phi = \frac{1}{2}\,|\overline{\boldsymbol{\omega}'^2}| \tag{6.169c}$$

Considering the identity:

$$\frac{\partial}{\partial x_j}\,(\overline{\omega_i'^2}) = 2\overline{\omega_i'\,\frac{\partial \omega_i'}{\partial x_j}}$$

we have:

$$\nabla^2\phi = \overline{\omega_i\,\frac{\partial^2 \omega_i'}{\partial x_j\,\partial x_j}} + \overline{\left(\frac{\partial \omega_i'}{\partial x_j}\right)^2} \tag{6.170}$$

To obtain the rate equation for MSVF, ϕ, we multiply Equation 6.159 scalarly by $\boldsymbol{\omega}'$ and then average each term. Thus:

$$\overline{\boldsymbol{\omega}' \cdot \frac{\partial \boldsymbol{\omega}'}{\partial t}} + \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \boldsymbol{\omega}') \cdot \overline{\mathbf{u}}\}} + \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \overline{\boldsymbol{\omega}}) \cdot \mathbf{u}'\}} + \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \boldsymbol{\omega}') \cdot \mathbf{u}'\}}$$

$$- \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \mathbf{u}') \cdot \overline{\boldsymbol{\omega}}\}} - \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \overline{\mathbf{u}}) \cdot \boldsymbol{\omega}'\}} - \overline{\boldsymbol{\omega}' \cdot \{(\text{grad}\ \mathbf{u}') \cdot \boldsymbol{\omega}'\}} = \nu\overline{\boldsymbol{\omega}' \cdot \nabla^2\boldsymbol{\omega}'}$$

In Cartesian tensor form this equation, while using Equations 6.169 and 6.170, becomes:

$$\frac{\partial \phi}{\partial t} + \overline{u}_j\,\frac{\partial \phi}{\partial x_j} = -\,\overline{\omega_i' u_j'}\,\frac{\partial \overline{\omega}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\,(\overline{u_j'\psi}) + \overline{\omega_i'\omega_j' D_{ij}'}$$

$$+ \overline{\omega_i'\omega_j'}\,\overline{D}_{ij} + \overline{\omega}_j\overline{\omega_i'\,\frac{\partial u_i'}{\partial x_j}} + \nu\nabla^2\phi - \nu\overline{\left(\frac{\partial \omega_i'}{\partial x_j}\right)^2} \tag{6.171}$$

where:

$$D'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)$$

and:

$$\overline{D}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

are the instantaneous and the mean rate-of-strain tensor components, respectively.

Rate Equation for the Reynolds Stresses

The dyad $-\rho\overline{\mathbf{u}'\mathbf{u}'}$ is called the Reynolds stress tensor whose Cartesian components are $-\rho\overline{u'_i u'_j}$. First, note that this tensor is symmetric; and based on dimensional considerations we find that the term $-\rho\overline{u'_i u'_j}$ has the dimension of force per unit area, viz., a stress. Second, since:

$$-\rho\overline{u'_i u'_j} = -\overline{(\rho u'_i)u'_j}$$

it can also be interpreted as transfer of the i-th component of the fluctuating momentum per unit volume by the j-th component of the fluctuating velocity. According to Newton's law, a transfer of momentum gives rise to a force; hence, we again reach the conclusion that $-\rho\overline{u'_i u'_j}$ is a stress. The components of this tensor can also be arranged as a 3 × 3 matrix as:

$$-\rho\overline{\mathbf{u}'\mathbf{u}'} = \begin{bmatrix} -\rho\overline{u_1'^2} & -\rho\overline{u'_1 u'_2} & -\rho\overline{u'_1 u'_3} \\ -\rho\overline{u'_1 u'_2} & -\rho\overline{u_2'^2} & -\rho\overline{u'_2 u'_3} \\ -\rho\overline{u'_1 u'_3} & -\rho\overline{u'_2 u'_3} & -\rho\overline{u_3'^2} \end{bmatrix}$$

The main diagonal terms contribute to the pressure at a point in the flow, while the nondiagonal terms are the tangential or shear stress components.

As defined, the Reynolds stresses seem to depend only on the turbulent fluctuating velocity components. Intrinsically, the Reynolds stress also depends on turbulent fluctuating vorticity components; this becomes apparent if we compare Equations 6.149 and 6.151. This comparison yields the equation:

$$\mathbf{T} = \text{grad}\, K - \text{div}(\overline{\mathbf{u}'\mathbf{u}'}) = \text{div}(K\mathbf{I} - \overline{\mathbf{u}'\mathbf{u}'}) \tag{6.172}$$

where $\mathbf{I}$ is the unit tensor. Recalling from Equation 6.152 the expression for $\mathbf{T}$, we conclude that velocity and vorticity correlations are expressible in terms of derivatives of the Reynolds stresses. Writing the Cartesian components of Equation 6.172, we have:

$$T_i = e_{ijk}\overline{u'_j\omega'_k} = \frac{\partial K}{\partial x_j}\delta_{ij} - \frac{\partial}{\partial x_j}(\overline{u'_i u'_j})$$

Thus:

$$\begin{aligned} T_1 &= \overline{u'_2\omega'_3} - \overline{u'_3\omega'_2} = \frac{\partial K}{\partial x_1} - \frac{\partial}{\partial x_j}(\overline{u'_1 u'_j}) \\ T_2 &= \overline{u'_3\omega'_1} - \overline{u'_1\omega'_3} = \frac{\partial K}{\partial x_2} - \frac{\partial}{\partial x_j}(\overline{u'_2 u'_j}) \\ T_3 &= \overline{u'_1\omega'_2} - \overline{u'_2\omega'_1} = \frac{\partial K}{\partial x_3} - \frac{\partial}{\partial x_j}(\overline{u'_3 u'_j}) \end{aligned} \tag{6.173}$$

Equations 6.173 establish the intrinsic relationship between the Reynolds stresses and the fluctuating vorticity field.

To find the rate equation for the Reynolds stresses, consider Equation 6.150 in the Cartesian component form:

$$\frac{\partial u_i'}{\partial t} + u_k' \frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k \frac{\partial u_i'}{\partial x_k} + u_k' \frac{\partial u_i'}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \, \partial x_k} + \frac{\partial}{\partial x_k}(\overline{u_i' u_k'}) \tag{6.174}$$

Write another equation similar to Equation 6.174 by replacing i by j. Multiplying the first equation by u_j' and the second by u_i', we then add and take the average of each term. The resulting equation will be

$$\frac{\partial \bar{\tau}_{ij}}{\partial t} + \bar{u}_k \frac{\partial \bar{\tau}_{ij}}{\partial x_k} = P_{ij} + Q_{ij} + F_{ij} - \epsilon_{ij} + \nu \frac{\partial^2 \bar{\tau}_{ij}}{\partial x_k \, \partial x_k} \tag{6.175}$$

where:

$$\begin{aligned}
\bar{\tau}_{ij} &= \overline{u_i' u_j'} \\
P_{ij} &= -\left(\bar{\tau}_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \bar{\tau}_{jk} \frac{\partial \bar{u}_i}{\partial x_k}\right) \\
Q_{ij} &= \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \\
F_{ij} &= -\frac{\partial}{\partial x_k}\left[\overline{u_i' u_j' u_k'} + \overline{\frac{p'}{\rho}(u_j' \delta_{ik} + u_i' \delta_{jk})}\right] \\
\epsilon_{ij} &= 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}
\end{aligned} \tag{6.176a}$$

It is an easy matter to show, using the summation convention, that:

$$\begin{aligned}
P &= \tfrac{1}{2} P_{ii} = -\bar{\tau}_{jk} \frac{\partial \bar{u}_j}{\partial x_k} = -\tfrac{1}{2}\bar{\tau}_{jk}\left(\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_j}\right) \\
D &= \tfrac{1}{2} F_{ii} = -\frac{\partial}{\partial x_j}\left\{\overline{u_j'\left(\tfrac{1}{2} q^2 + \frac{p'}{\rho}\right)}\right\} \\
\epsilon &= \tfrac{1}{2}\epsilon_{ii} = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}
\end{aligned} \tag{6.176b}$$

With the quantities defined in Equations 6.176, the energy equation (Equation 6.167) can simply be written as:

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = P + D - \epsilon + \nu \nabla^2 K \tag{6.177}$$

Rate Equation for the Dissipation ϵ

Differentiating Equation 6.174 with respect to x_j, performing the inner multiplication by $\partial u_i'/\partial x_j$, and then averaging each term, we get:

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_k \frac{\partial \epsilon}{\partial x_k} = -2\nu\left[\frac{\partial^2 \bar{u}_i}{\partial x_\ell \, \partial x_k}\overline{\left(u_k' \frac{\partial u_i'}{\partial x_\ell}\right)} + \frac{\partial \bar{u}_i}{\partial x_k}\left\{\overline{\left(\frac{\partial u_i'}{\partial x_\ell}\frac{\partial u_k'}{\partial x_\ell}\right)} + \overline{\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}\right)}\right\}\right.$$

$$+ \overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial u_j'}{\partial x_\ell}\frac{\partial u_k'}{\partial x_\ell}} + \frac{1}{2}\frac{\partial}{\partial x_k}\left\{\overline{u_k'\frac{\partial u_j'}{\partial x_\ell}\frac{\partial u_j'}{\partial x_\ell}}\right\} + \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial u_k'}{\partial x_\ell}\frac{\partial p'}{\partial x_\ell}}\right)$$

$$+ \nu\overline{\left(\frac{\partial^2 u_j'}{\partial x_k\,\partial x_\ell}\right)^2}\Bigg] + \nu\nabla^2\epsilon \qquad (6.178)$$

which is the rate equation for the dissipation rate ϵ.

Physical Interpretation of the Terms

A number of terms involving Reynolds' stresses, gradients of mean velocity, pressure, and velocity correlations, etc. have appeared in Equations 6.167 or 6.177, 6.171, and 6.175. The left-hand side of any of these equations is, of course, the rate of change of quantity following the motion, or the usual substantive derivative. The convective part of this derivative is also called the *advection* of transported quantity. Thus:

$$\bar{u}_j\frac{\partial K}{\partial x_j}, \qquad \bar{u}_j\frac{\partial \phi}{\partial x_j}, \qquad \bar{u}_k\frac{\partial}{\partial x_k}(\overline{u_i'u_j'})$$

are the advections of K, ϕ, and $\overline{u_i'u_j'}$, respectively. The terms in energy Equation 6.167 have been regrouped in Equation 6.177 and have the following significance: term P is the sum of all products of the Reynolds stresses and the mean rate-of-strain, which amounts to the rate of doing work by the Reynolds stresses in deformation of the mean flow field. This rate of doing work decreases the energy of mean motion and shows up as a positive contribution to the energy of turbulent fluctuations. Thus we call P the *production* term. The term D is the divergence of a scalar multiplied by the tubulent fluctuation velocity vector, i.e.:

$$D = -\text{div}\left[\overline{\mathbf{u}'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right]$$

To interpret this term let us consider a closed surface S of volume V_0 and apply Gauss' divergence theorem, thus having:

$$-\int_{V_0} D\,dv = \int_S \overline{\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)\mathbf{u}'}\cdot\mathbf{n}\,dS$$

Hence:

$$\overline{\mathbf{u}'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}$$

represents a flux vector. Its divergence integrated over the volume represents the net flux out of a closed surface. Consequently, the contribution of D is that of the *diffusion* of energy due to the pressure and velocity fluctuations. The term D is called the *diffusion* of energy. In the same manner the term:

$$\nu\nabla^2 K = \nu\,\text{div}(\text{grad}\,K)$$

is a diffusion of energy due to the molecular transport of K. This contribution is usually small in comparison to the turbulent diffusion D. The term ϵ is the dissipation function which is always positive and signifies a destruction of energy. In summary, the important budgeting of turbulent energy can be written in words as:

Substantive transport of energy = production + diffusion − dissipation

Following the same short of physical reasoning, we can list the physical interpretations of terms in Equations 6.171 and 6.175:

Equation 6.171

$-\overline{\omega_i' u_j'}\dfrac{\partial \overline{\omega}_i}{\partial x_j}$: Production of ϕ

$\dfrac{\partial}{\partial x_j}(\overline{u_j'\psi})$: Transport of ψ by turbulent velocity fluctuations

$\overline{\omega_i'\omega_j' D_{ij}'}$: Production of ϕ by turbulent stretching of turbulent vorticity

$\overline{\omega_i'\omega_j'}\, \overline{D}_{ij}$: Production/removal of turbulent vorticity by stretching/squeezing of vorticity fluctuations by the mean rate-of-strain

$\overline{\omega}_j \overline{\omega_i' \dfrac{\partial u_i'}{\partial x_j}}$: Mixed production

$\nu\nabla^2\phi$: Viscous diffusion of ϕ

$\nu\overline{\left(\dfrac{\partial \omega_i'}{\partial x_j}\right)^2}$: Dissipation of ϕ to heat **(6.179)**

Equation 6.175

P_{ij} : Production of the Reynolds stresses
Q_{ij} : Pressure-strain correlation
F_{ij} : Turbulent diffusion of the Reynolds stresses
ϵ_{ij} : Dissipation (decay or destruction) of the Reynolds stresses

Analysis of the Pressure-Strain Correlation

The term Q_{ij} appearing in Equation 6.175 is the pressure-strain correlation and it plays a very important role in field distribution of the Reynolds stresses. To understand the correct behavior of term Q_{ij} it is helpful to write it in terms of its constituents. This can be achieved by starting from the Poisson-type equations for both the mean pressure $\overline{p}$ and the fluctuating pressure p'. Taking the divergence of Equation 6.149 while using Equation 6.146, we get:

$$-\frac{1}{\rho}\nabla^2\overline{p} = \operatorname{div}[(\operatorname{grad}\overline{\mathbf{u}})\cdot\overline{\mathbf{u}} + \operatorname{div}(\overline{\mathbf{u}'\mathbf{u}'})]$$

Using the identities, Equations M1.44b and M1.38, we get:

$$-\frac{1}{\rho}\nabla^2\overline{p} = (\operatorname{grad}\overline{\mathbf{u}})^{\mathrm{T}} : (\operatorname{grad}\overline{\mathbf{u}}) + \operatorname{div}[\operatorname{div}(\overline{\mathbf{u}'\mathbf{u}'})] \tag{6.181}$$

In passing, we note that Equation 6.181 in the Cartesian tensor notation is

$$-\frac{1}{\rho}\nabla^2\overline{p} = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial^2 \overline{\tau}_{ij}}{\partial x_i\,\partial x_j}$$

while its general tensor form is

$$-\frac{1}{\rho}\nabla^2\overline{p} = \overline{u}^j{}_{,i}\overline{u}^i{}_{,j} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}A^k)$$

where a comma preceding an index represents a covariant derivative, $\sqrt{g}$ is the transformation Jacobian, x^k are the general coordinates, and A^k are the contravariant components of the vector div $\bar{\tau}$ defined as:

$$A^k = (\text{div } \bar{\tau})^k = \frac{\partial \bar{\tau}^{kj}}{\partial x^j} + \Gamma^k_{\ell j}\bar{\tau}^{\ell j} + \frac{1}{2g}\frac{\partial g}{\partial x^\ell}\bar{\tau}^{k\ell}$$

To find the differential equation for the fluctuating pressure p' we take the divergence of Equation 6.148 while using Equations 6.146, 6.147, and 6.181 thus, having:

$$-\frac{1}{\rho}\nabla^2 p' = 2(\text{grad } \bar{\mathbf{u}})^{\mathrm{T}} : (\text{grad } \mathbf{u}') + \text{div}(\text{div } \boldsymbol{\tau}) - \text{div}(\text{div } \bar{\boldsymbol{\tau}}) \tag{6.182}$$

where we recall that:

$$\boldsymbol{\tau} = \mathbf{u}'\mathbf{u}', \quad \tau_{ij} = u'_i u'_j$$
$$\bar{\boldsymbol{\tau}} = \overline{\mathbf{u}'\mathbf{u}'}, \quad \bar{\tau}_{ij} = \overline{u'_i u'_j}, \text{ etc.}$$

In Cartesian tensor notation, Equation 6.182 is

$$-\frac{1}{\rho}\nabla^2 p' = 2\frac{\partial \bar{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} + \frac{\partial^2 \tau_{ij}}{\partial x_i\,\partial x_j} - \frac{\partial^2 \bar{\tau}_{ij}}{\partial x_i\,\partial x_j} \tag{6.183}$$

Denoting the right-hand side of Equation 6.183 by $f'(\xi)$, we obtain the solution as (refer to Equation M5.17):

$$p'(\mathbf{x}) = \frac{\rho}{4\pi}\int_\Omega \frac{f'(\xi)}{r}\,d\nu(\xi) + \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial p'(\xi)}{\partial n} - p'(\xi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS(\xi) \tag{6.184}$$

where Ω is the region of space integration with S as the boundary. The expression for $p'(\mathbf{x})$ obtained in Equation 6.184 can now be used to establish various components which contribute to the pressure-strain correlation $Q_{k\ell}$. Denoting by $(\ldots)_x$ as a quantity evaluated at point $\mathbf{x}$, we multiply Equation 6.184 throughout by:

$$\left(\frac{\partial u'_k}{\partial x_\ell}\right)_{\mathbf{x}}$$

and perform the average of each term. Thus:

$$\overline{\frac{p'}{\rho}\frac{\partial u'_k}{\partial x_\ell}} = \frac{1}{4\pi}\int_\Omega\left[\overline{\left(\frac{\partial^2 \tau_{ij}}{\partial x_i\,\partial x_j}\right)_\xi\left(\frac{\partial u'_k}{\partial x_\ell}\right)_{\mathbf{x}}} + 2\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)_\xi\overline{\left(\frac{\partial u'_j}{\partial x_i}\right)_\xi\left(\frac{\partial u'_k}{\partial x_\ell}\right)_{\mathbf{x}}}\right]\frac{d\nu(\xi)}{r} + \chi_{k\ell} \tag{6.185a}$$

where $\chi_{k\ell}$ is the contribution from the surface integral. Interchanging k and ℓ in Equation 6.185a, we have:

$$\overline{\frac{p'}{\rho}\frac{\partial u'_\ell}{\partial x_k}} = \frac{1}{4\pi}\int_\Omega\left[\overline{\left(\frac{\partial^2 \tau_{ij}}{\partial x_i\,\partial x_j}\right)_\xi\left(\frac{\partial u'_\ell}{\partial x_k}\right)_{\mathbf{x}}} + 2\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)_\xi\overline{\left(\frac{\partial u'_j}{\partial x_i}\right)_\xi\left(\frac{\partial u'_\ell}{\partial x_k}\right)_{\mathbf{x}}}\right]\frac{d\nu(\xi)}{r} + \chi_{\ell k} \tag{6.185b}$$

Writing the first integral on the right-hand side of Equation 6.185a as $\phi_{k\ell}$ and the second as $\psi_{k\ell}$, we find by adding Equations 6.185a, b:

$$Q_{k\ell} = \overline{\frac{p'}{\rho}\left(\frac{\partial u_k'}{\partial x_\ell} + \frac{\partial u_\ell'}{\partial x_k}\right)}$$

$$= \phi_{k\ell} + \phi_{\ell k} + \psi_{k\ell} + \psi_{\ell k} + \chi_{k\ell} + \chi_{\ell k} \tag{6.186}$$

Writing:

$$\Phi_{k\ell} = \phi_{k\ell} + \phi_{\ell k}$$

$$\Psi_{k\ell} = \psi_{k\ell} + \psi_{\ell k}$$

$$W_{k\ell} = \chi_{k\ell} + \chi_{\ell k}$$

we have:

$$Q_{k\ell} = \Phi_{k\ell} + \Psi_{k\ell} + W_{k\ell} \tag{6.187}$$

Note that $\Phi_{k\ell}$ involves only the fluctuating quantities, while $\Psi_{k\ell}$ has an explicit dependence on the gradients of mean velocity. The term $W_{k\ell}$ is dependent on the boundary values only.

The preceding analysis provides a rational basis for the modeling of the term Q_{ij} which will be described later. Part $\Psi_{k\ell}$ is called the *rapid term* while part $\Phi_{k\ell}$ is called the *return term.*

6.14 AVERAGED EQUATIONS FOR COMPRESSIBLE FLOW

In the preceding section we have exclusively been concerned with the averaged equations for a constant density fluid flow. For a compressible flow the fluctuations in density, enthalpy, temperature, and viscosity also have to be considered. In the conventional time averaging due to Reynolds, the physical quantities, $\mathbf{u}$, ρ, p, h, T, and μ etc. are written as:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \quad p = \bar{p} + p', \quad \rho = \bar{\rho} + \rho'$$

$$H = \bar{H} + H', \quad T = \bar{T} + T', \quad \mu = \bar{\mu} + \mu'$$

Substitution of the these quantities in compressible flow Equations 3.8, 3.10, 3.19b, or 3.22a and afterward performing the time averaging yield the averaged equations. These equations are available in Reference 41, and reference may also be made to Reference 16. The resulting equations are different in form from the averaged equations of the constant density case. As an example, the averaged continuity equation for compressible flow with the conventional time averaging becomes:

$$\frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\bar{\rho}\bar{\mathbf{u}} + \overline{\rho'\mathbf{u}'}) = 0$$

In 1965, Favre[42] proposed a mass weighted average which when used produces averaged equations of about the same form as were obtained for the constant density case. Rubesin and Rose[43] have also documented Favre's method in a lucid manner. Refer also to Reference 16.

In mass weighted averaging the physical quantities $\mathbf{u}$, H, T, μ etc. are resolved as:

$$\mathbf{u} = \dot{\bar{\mathbf{u}}} + \mathbf{u}'', \quad H = \dot{\bar{H}} + H'', \quad T = \dot{\bar{T}} + T'', \quad \mu = \dot{\bar{\mu}} + \mu'' \tag{6.188}$$

but both p and ρ are written in the same manner as in Reynolds' averaging. The overhead :

implies a mass weighted average,* and a double prime (") denotes the fluctuations about this average. Let α and β be any of the physical variables in Equation 6.188. Then the mass weighted average is defined as:

$$\dot{\bar{\alpha}} = \frac{\overline{\rho\alpha}}{\bar{\rho}} \tag{6.189a}$$

or:

$$\bar{\rho}\dot{\bar{\alpha}} = \overline{\rho\alpha} \tag{6.189b}$$

Writing:

$$\alpha = \dot{\bar{\alpha}} + \alpha''$$

multiplying by ρ, taking the average, and using equation 6.189b, we get:

$$\overline{\rho\alpha''} = 0 \tag{6.190}$$

Since:

$$\rho = \bar{\rho} + \rho'$$

Equation 6.190 yields:

$$\overline{\alpha''} = -\frac{\overline{\rho'\alpha''}}{\bar{\rho}} \tag{6.191a}$$

which is not zero. Since:

$$\alpha = \bar{\alpha} + \alpha' = \dot{\bar{\alpha}} + \alpha''$$

then:

$$\alpha'' = \bar{\alpha} + \alpha' - \dot{\bar{\alpha}}$$

Further from:

$$\bar{\alpha} = \dot{\bar{\alpha}} + \overline{\alpha''}$$

we have:

$$\alpha'' = \alpha' + \overline{\alpha''}$$

Using Equation 6.191a, we get:

$$\alpha'' = \alpha' - \frac{\overline{\rho'\alpha''}}{\bar{\rho}} \tag{6.191b}$$

* Also called the Favre average.

Multiplying Equation 6.191b by ρ and averaging gives:

$$\overline{\rho\alpha''} = \overline{\rho'\alpha'} - \overline{\rho'\alpha''}$$

and thus, because of Equation 6.190:

$$\overline{\rho'\alpha'} = \overline{\rho'\alpha''} \tag{6.191c}$$

Using Equation 6.191c in Equation 6.191b, we get:

$$\alpha'' = \alpha' - \frac{\overline{\rho'\alpha'}}{\bar{\rho}} \tag{6.192}$$

so that the relation between Reynolds' and Favre's averaging is

$$\dot{\bar{\alpha}} = \bar{\alpha} + \frac{\overline{\rho'\alpha'}}{\bar{\rho}} \tag{6.193a}$$

Next consider the term:

$$\overline{\rho\alpha''\beta''}$$

Using Equation 6.192, we get:

$$\dot{\overline{\alpha''\beta''}} = \overline{\alpha'\beta'} + \frac{\overline{\rho'\alpha'\beta'}}{\bar{\rho}} - \frac{\overline{\rho'\alpha'}\cdot\overline{\rho'\beta'}}{(\bar{\rho})^2} \tag{6.193b}$$

We now apply the mass weighted average to Equations 3.8, 3.10, 3.19b, and 3.22a. First, for the sake of brevity we write the heat flux vector as $\mathbf{q}$, where $\mathbf{q} = -k$ grad T and also drop the body force vector $\mathbf{f}$. Second, we must clarify the meaning of the terms $\bar{\sigma}$ and H.

We write:

$$\sigma = \bar{\sigma} + \sigma', \qquad \mathbf{q} = \bar{\mathbf{q}} + \mathbf{q}'$$

where:

$$\sigma = \lambda(\text{div } \mathbf{u})\mathbf{I} + 2\mu\mathbf{D}$$

The meaning of $\bar{\sigma}$ is to write $\mathbf{u} = \dot{\bar{\mathbf{u}}} + \mathbf{u}''$, $\lambda = \dot{\bar{\lambda}} + \lambda''$, and $\mu = \dot{\bar{\mu}} + \mu''$ and then to perform the time average. For the total enthalpy H:

$$H = \dot{\bar{h}} + h'' + \tfrac{1}{2}\dot{\bar{\mathbf{u}}}\cdot\dot{\bar{\mathbf{u}}} + \dot{\bar{\mathbf{u}}}\cdot\mathbf{u}'' + \tfrac{1}{2}\mathbf{u}''\cdot\mathbf{u}''$$

so that on averaging ρH and dividing by $\bar{\rho}$, we get:

$$\dot{\bar{H}} = \dot{\bar{h}} + \tfrac{1}{2}\dot{\bar{\mathbf{u}}}\cdot\dot{\bar{\mathbf{u}}} + \tfrac{1}{2}\frac{\overline{\rho\mathbf{u}''\cdot\mathbf{u}''}}{\bar{\rho}}$$

Consequently, from $H = \dot{\bar{H}} + H''$, we have:

$$H'' = h'' - \frac{1}{2}\frac{\overline{\rho \mathbf{u}'' \cdot \mathbf{u}''}}{\overline{\rho}} + \dot{\bar{\mathbf{u}}} \cdot \mathbf{u}'' + \frac{1}{2}\mathbf{u}'' \cdot \mathbf{u}''$$

The equations are then as follows:

$$\frac{\partial \overline{\rho}}{\partial t} + \operatorname{div}(\overline{\rho}\dot{\bar{\mathbf{u}}}) = 0 \tag{6.194}$$

$$\frac{\partial}{\partial t}(\overline{\rho}\dot{\bar{\mathbf{u}}}) + \operatorname{div}(\overline{\rho}\dot{\bar{\mathbf{u}}}\dot{\bar{\mathbf{u}}}) = -\operatorname{grad}\overline{p} + \operatorname{div}(\overline{\boldsymbol{\sigma}} - \overline{\rho \mathbf{u}''\mathbf{u}''}) \tag{6.195}$$

The equation of total enthalpy is

$$\frac{\partial}{\partial t}(\overline{\rho}\dot{\overline{H}}) - \frac{\partial \overline{p}}{\partial t} + \operatorname{div}[\overline{\rho}\dot{\overline{H}}\dot{\bar{\mathbf{u}}} + \overline{\mathbf{q}} + \overline{\rho h''\mathbf{u}''}$$

$$- \overline{\boldsymbol{\sigma}} \cdot \dot{\bar{\mathbf{u}}} + \overline{\rho \mathbf{u}''\mathbf{u}''} \cdot \dot{\bar{\mathbf{u}}} - \overline{(\boldsymbol{\sigma} - \frac{1}{2}\rho \mathbf{u}''\mathbf{u}'') \cdot \mathbf{u}''}] = 0 \tag{6.196}$$

In the same manner the equations of $\dot{\bar{e}}_t$ and $\overline{E}_t$ are

$$\frac{\partial}{\partial t}(\overline{\rho}\dot{\bar{e}}_t) + \operatorname{div}[(\overline{\rho}\dot{\bar{e}}_t + \overline{p})\dot{\bar{\mathbf{u}}}$$

$$- \overline{\boldsymbol{\sigma}} \cdot \dot{\bar{\mathbf{u}}} + \overline{\mathbf{q}} + \overline{\rho e_t''\mathbf{u}''} + \overline{p\mathbf{u}''} - \overline{\boldsymbol{\sigma} \cdot \mathbf{u}''}] = 0 \tag{6.197}$$

Note that:

$$E_t = \rho e_t$$

and that:

$$E_t = \rho(\dot{\bar{e}}_t + e_t'')$$

thus:

$$\overline{E}_t = \overline{\rho}\dot{\bar{e}}_t + \overline{\rho e_t''} = \overline{\rho}\dot{\bar{e}}_t$$

Because of this result:

$$E_t = \overline{E}_t + E_t'$$

The equation for $\overline{E}_t$ comes directly from Equation 6.197 and is

$$\frac{\partial \overline{E}_t}{\partial t} + \operatorname{div}[(\overline{E}_t + \overline{p})\dot{\bar{\mathbf{u}}} - \overline{\boldsymbol{\sigma}} \cdot \dot{\bar{\mathbf{u}}} + \overline{\mathbf{q}} + \overline{E_t'\mathbf{u}''} + \overline{p\mathbf{u}''} - \overline{\boldsymbol{\sigma} \cdot \mathbf{u}''}] = 0 \tag{6.198}$$

Equation of the Turbulence Energy and the Reynolds Stresses*

Similar to the incompressible case we write:

* For a derivation of the Reynolds stress equations, refer to Problems 6.11 and 6.12.

$$q^2 = \frac{\rho \mathbf{u}'' \cdot \mathbf{u}''}{\bar{\rho}}$$

from which the turbulence kinetic energy per unit mass is

$$K = \tfrac{1}{2}\overline{q^2} = \frac{\overline{\rho \mathbf{u}'' \cdot \mathbf{u}''}}{2\bar{\rho}} = \tfrac{1}{2}\dot{\overline{\mathbf{u}'' \cdot \mathbf{u}''}} \tag{6.199a}$$

In Cartesian tensor notation:

$$K = \frac{1}{2\bar{\rho}}\overline{\rho u_i'' u_i''} = \tfrac{1}{2}\dot{\overline{u_i'' u_i''}} \tag{6.199b}$$

The turbulence kinetic energy equation is

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{\rho}K) + \frac{\partial}{\partial x_j}(\bar{\rho}K\dot{\bar{u}}_j) &= -(\overline{\rho u_i'' u_j''})\frac{\partial \dot{\bar{u}}_i}{\partial x_j} - \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{\rho q^2 u_j''}) \\ &- \frac{\partial}{\partial x_j}(\overline{u_j'' p}) + \overline{p\frac{\partial u_i''}{\partial x_i}} + \frac{\partial}{\partial x_j}(\overline{u_i''\sigma_{ij}}) - \overline{\sigma_{ij}\frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{6.199c}$$

The rate equation for the Reynolds stress is written below by using the notation:

$$\tau_{ij} = u_i'' u_j''$$

Thus:

$$\begin{aligned}\frac{\partial}{\partial t}(\overline{\rho\tau_{ik}}) + \frac{\partial}{\partial x_j}(\dot{\bar{u}}_j\overline{\rho\tau_{ik}}) &= -\overline{\rho\tau_{ij}}\frac{\partial \dot{\bar{u}}_k}{\partial x_j} - \overline{\rho\tau_{kj}}\frac{\partial \dot{\bar{u}}_i}{\partial x_j} \\ &- \frac{\partial}{\partial x_j}(\overline{\rho\tau_{ik}u_j''}) - \frac{\partial}{\partial x_i}(\overline{p u_k''}) - \frac{\partial}{\partial x_k}(\overline{p u_i''}) + \overline{p\left(\frac{\partial u_k''}{\partial x_i} + \frac{\partial u_i''}{\partial x_k}\right)} \\ &+ \frac{\partial}{\partial x_j}(\overline{\sigma_{ij}u_k''}) + \frac{\partial}{\partial x_j}(\overline{\sigma_{kj}u_i''}) - \overline{\sigma_{ij}\frac{\partial u_k''}{\partial x_j}} - \overline{\sigma_{kj}\frac{\partial u_i''}{\partial x_j}}\end{aligned} \tag{6.199d}$$

It must be noted that both in Equations 6.199c, d the pressure p and the viscous stress tensor σ_{ij} are instantaneous values. Writing:

$$p = \bar{p} + p', \qquad \sigma_{ij} = \bar{\sigma}_{ij} + \sigma_{ij}'$$

in Equations 6.199c, d gives rise to additional terms involving quantities like $\bar{\sigma}_{ij}\,\overline{u_k''}$, etc. For the purpose of completeness we restate these equations below, noting that from Equation 6.192 $\overline{\alpha''\beta'} = \overline{\alpha'\beta'}$:

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{\rho}K) + \frac{\partial}{\partial x_j}(\bar{\rho}K\dot{\bar{u}}_j) &= -\overline{\rho u_i'' u_j''}\frac{\partial \dot{\bar{u}}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left\{\overline{u_j''(p' + \tfrac{1}{2}\rho q^2)}\right\} \\ &+ \frac{\partial}{\partial x_j}(\overline{\sigma_{ij}' u_i'}) - \overline{\sigma_{ij}'\frac{\partial u_i'}{\partial x_j}} + \left[-\frac{\partial}{\partial x_j}(\bar{p}\overline{u_j''}) + \bar{p}\overline{\frac{\partial u_i''}{\partial x_i}}\right. \\ &\left.+ \frac{\partial}{\partial x_j}(\bar{\sigma}_{ij}\overline{u_i''}) - \bar{\sigma}_{ij}\overline{\frac{\partial u_i''}{\partial x_j}}\right]\end{aligned} \tag{6.200a}$$

$$\frac{\partial}{\partial t}(\overline{\rho\tau_{ik}}) + \frac{\partial}{\partial x_j}(\dot{\bar{u}}_j\overline{\rho\tau_{ik}}) = -\overline{\rho\tau_{ij}}\frac{\partial\dot{\bar{u}}_k}{\partial x_j} - \overline{\rho\tau_{kj}}\frac{\partial\dot{\bar{u}}_i}{\partial x_j}$$

$$+ \overline{p'\left(\frac{\partial u_k''}{\partial x_i} + \frac{\partial u_i''}{\partial x_k}\right)} - \frac{\partial}{\partial x_j}\left\{\overline{\rho\tau_{ik}u_j''} + \overline{p'(u_k''\delta_{ij} + u_i''\delta_{kj})}\right\}$$

$$+ \frac{\partial}{\partial x_j}\overline{(\sigma_{ij}'u_k')} + \frac{\partial}{\partial x_j}\overline{(\sigma_{kj}'u_i')} - \overline{\sigma_{ij}'\frac{\partial u_k'}{\partial x_j}} - \overline{\sigma_{kj}'\frac{\partial u_i'}{\partial x_j}}$$

$$+\left[-\frac{\partial}{\partial x_j}(\bar{p}\overline{u_k''}\delta_{ij} + \bar{p}\overline{u_i''}\delta_{kj}) + \bar{p}\left(\frac{\partial\overline{u_k''}}{\partial x_i} + \frac{\partial\overline{u_i''}}{\partial x_k}\right)\right.$$

$$\left. + \frac{\partial}{\partial x_j}(\bar{\sigma}_{ij}\overline{u_k''} + \bar{\sigma}_{kj}\overline{u_i''}) - \bar{\sigma}_{ij}\frac{\partial\overline{u_k''}}{\partial x_j} - \bar{\sigma}_{kj}\frac{\partial\overline{u_i''}}{\partial x_j}\right] \tag{6.200b}$$

In Equations 6.200 we have written the terms similar to Equations 6.164 and 6.175, and the additional terms due to compressibility have been put together in square brackets. The physical meaning of some of the terms are as follows:

$\overline{\sigma_{ij}'\dfrac{\partial u_i'}{\partial x_j}}$	:	Viscous dissipation of K
$\dfrac{\partial}{\partial x_j}\overline{(\sigma_{ij}'u_i')}$	:	Transport of K by viscous forces
$\dfrac{\partial}{\partial x_i}\overline{(p'u_k'')} + \dfrac{\partial}{\partial x_k}\overline{(p'u_i'')}$	:	Diffusion of the Reynolds stresses due to turbulent fluctuations

Dissipation Function

In the case of compressible flow if we identify the molecular quantities λ and μ as functions of the mass weighted temperature, then we should write:

$$\lambda = \dot{\bar{\lambda}} + \lambda'', \qquad \mu = \dot{\bar{\mu}} + \mu''$$

Thus, using the viscous stress tensor:

$$\boldsymbol{\sigma} = \lambda\Delta\mathbf{I} + 2\mu\mathbf{D}, \qquad \Delta = \text{div}\ \mathbf{u}$$

and the Favre decomposition, we get:

$$\boldsymbol{\sigma}' = \dot{\bar{\lambda}}(\Delta'' - \overline{\Delta''})\mathbf{I} + \dot{\bar{\Delta}}(\lambda'' - \overline{\lambda''})\mathbf{I} + (\lambda''\Delta'' - \overline{\lambda''D''})\mathbf{I}$$

$$+ 2\dot{\bar{\mu}}(\mathbf{D}'' - \overline{\mathbf{D}''}) + 2(\mu'' - \overline{\mu''})\dot{\bar{\mathbf{D}}} + 2(\mu''\mathbf{D}'' - \overline{\mu''\mathbf{D}''})$$

If the Reynolds decomposition is used, then:

$$\boldsymbol{\sigma}' = (\bar{\lambda}\Delta' + \lambda'\bar{\Delta})\mathbf{I} + (\lambda'\Delta' - \overline{\lambda'\Delta'})\mathbf{I} + 2(\bar{\mu}\mathbf{D}' + \mu'\bar{\mathbf{D}}) + 2(\mu'\mathbf{D}' + \overline{\mu'\mathbf{D}'})$$

The viscous dissipation of turbulence energy is

$$\bar{\rho}\epsilon_d = \overline{\sigma_{ij}'\frac{\partial u_i'}{\partial x_j}} = \overline{\boldsymbol{\sigma}' : \text{grad}\ \mathbf{u}'} = \overline{\boldsymbol{\sigma}' : \mathbf{D}'} \tag{6.201a}$$

Using the Reynolds form of σ' in Equation 6.201a and keeping only the principal terms, we have a simpler formula:

$$\bar{\rho}\epsilon_d = \lambda\overline{\Delta'^2} + 2\mu\overline{\mathbf{D}' : \mathbf{D}'} \tag{6.201b}$$

On using the result of Problem 1.10 in Equation 6.201b and the Stokes relation:

$$3\bar{\lambda} + 2\bar{\mu} = 0$$

we get:

$$\bar{\rho}\epsilon_d = \frac{4}{3}\bar{\mu}\overline{\Delta'^2} + 2\bar{\mu}[\overline{(\text{grad } \mathbf{u}')^T : (\text{grad } \mathbf{u}')} - \overline{\Delta'^2}] + \bar{\mu}\overline{|\omega'|^2} \tag{6.201c}$$

For high Reynolds number flows, the middle bracketed term on the right-hand side of Equation 6.201c is usually very small and thus we define the compressible dissipation ϵ_c as

$$\bar{\rho}\epsilon_c = \tfrac{4}{3}\bar{\mu}\overline{\Delta'^2} + \bar{\mu}\overline{|\omega'|^2} \tag{6.201d}$$

The above formula is for the dissipation of turbulence energy in terms of the vorticity magnitude and expansion. Note that for incompressible flow, we recover Equation 6.165a from Equation 6.201c. For compressible flow, it is the term ϵ_c for which a model equation is sought. In incompressible flow, the part ϵ of ϵ_d has an equation (Equation 6.178) which is a direct consequence of the Navier-Stokes equations. Such an equation is not available for compressible flow. To proceed, the present trend is to take the model equation for the last term of Equation 6.201d as the incompressible ϵ — equation with variable density, and to model the first term of Equation 6.201d separately.

6.15 TURBULENT BOUNDARY LAYER EQUATIONS

Boundary layer flow is a nonisotropic wall-bound or free-shear flow. For a clarity of exposition we first consider the derivation of boundary layer equations for two-* and three-dimensional flows as referred to rectangular Cartesian coordinates. It must, however, be emphasized again that the equations in rectangular Cartesian coordinates are of wide applicability; because, as we recall from Part II in Chapter 5, one of the coordinates (e.g., x) can be interpreted as a curvilinear coordinate measured along the surface and the other as normal to the surface.

Equations in Rectangular Cartesian Coordinates

From Equations 6.153, 6.154, and 6.167, the two-dimensional equations are as follows:

$$\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} = 0 \tag{6.202a}$$

$$\frac{\partial \bar{u}_1}{\partial t} + u_1\frac{\partial \bar{u}_1}{\partial x_1} + \bar{u}_2\frac{\partial \bar{u}_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \nu\nabla^2\bar{u}_1 - \frac{\partial}{\partial x_1}(\overline{u_1'^2}) - \frac{\partial}{\partial x_2}(\overline{u_1'u_2'}) \tag{6.202b}$$

$$\frac{\partial \bar{u}_2}{\partial t} + u_1\frac{\partial \bar{u}_2}{\partial x_1} + \bar{u}_2\frac{\partial \bar{u}_2}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} + \nu\nabla^2\bar{u}_2 - \frac{\partial}{\partial x_1}(\overline{u_1'u_2'}) - \frac{\partial}{\partial x_2}(\overline{u_2'^2}) \tag{6.202c}$$

$$\frac{\partial K}{\partial t} + \bar{u}_1\frac{\partial K}{\partial x_1} + \bar{u}_2\frac{\partial K}{\partial x_2} = -\overline{u_1'^2}\frac{\partial \bar{u}_1}{\partial x_1} - \overline{u_2'^2}\frac{\partial \bar{u}_2}{\partial x_2}$$

* Note that turbulence is always three dimensional but the mean flow field can be either two or three dimensional.

$$- \overline{u_1' u_2'} \left(\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1} \right) - \frac{\partial}{\partial x_1} \left\{ \overline{u_1' \left(\frac{1}{2} q^2 + \frac{p'}{\rho} \right)} \right\} - \frac{\partial}{\partial x_2} \left\{ \overline{u_2' \left(\frac{1}{2} q^2 + \frac{p'}{\rho} \right)} \right\}$$

$$+ \nu \nabla^2 K - \epsilon \tag{6.202d}$$

where:

$$q^2 = u_i'^2, \qquad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Let L and δ be the characteristic lengths and U and V be the characteristic velocitites along coordinate axes x_1 and x_2, respectively. We take x_1-axis as the direction of the main flow. Denoting the friction velocity by u_τ (Equation 6.228), we introduce the following quantities in the two-dimensional equations (Equations 6.202):

$$T = \frac{tU}{L}, \qquad u = \frac{\bar{u}_1}{U}, \qquad v = \frac{\bar{u}_2}{V}, \qquad p = \frac{\bar{p}}{\rho U^2}$$

$$R_{ij} = \overline{u_i' u_j'}/u_\tau^2, \qquad E = K/u_\tau^2, \qquad S_i = \frac{1}{u_\tau^3} \overline{u_i' \left(\frac{1}{2} q^2 + \frac{p'}{\rho} \right)},$$

$$x = \frac{x_1}{L}, \quad y = \frac{x_2}{\delta}$$

The nondimensional form of Equations 6.202 is then:

$$\frac{U\delta}{VL} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.203a}$$

$$\frac{\partial u}{\partial T} + u \frac{\partial u}{\partial x} + \frac{V}{U} \frac{L}{\delta} v \frac{\partial u}{\partial y}$$

$$= - \frac{\partial p}{\partial x} + \frac{1}{R_e} \left(\frac{\partial^2 u}{\partial x^2} + \frac{L^2}{\delta^2} \frac{\partial^2 u}{\partial y^2} \right) - \frac{u_\tau^2}{U^2} \frac{\partial R_{11}}{\partial x} - \frac{u_\tau^2}{U^2} \frac{L}{\delta} \frac{\partial R_{12}}{\partial y} \tag{6.203b}$$

$$\frac{V}{U} \left(\frac{\partial v}{\partial T} + u \frac{\partial v}{\partial x} + \frac{V}{U} \frac{L}{\delta} v \frac{\partial v}{\partial y} \right)$$

$$= - \frac{L}{\delta} \frac{\partial p}{\partial y} + \frac{1}{R_e} \frac{V}{U} \left(\frac{\partial^2 v}{\partial x^2} + \frac{L^2}{\delta^2} \frac{\partial^2 v}{\partial y^2} \right) - \frac{u_\tau^2}{U^2} \frac{\partial R_{12}}{\partial x} - \frac{u_\tau^2}{U^2} \frac{L}{\delta} \frac{\partial R_{22}}{\partial y} \tag{6.203c}$$

$$\frac{\partial E}{\partial T} + u \frac{\partial E}{\partial x} + \frac{V}{U} \frac{L}{\delta} v \frac{\partial E}{\partial y} = -R_{11} \frac{\partial u}{\partial x} - \frac{V}{U} \frac{L}{\delta} R_{22} \frac{\partial v}{\partial y}$$

$$- \frac{L}{\delta} R_{12} \frac{\partial u}{\partial y} - \frac{V}{U} R_{12} \frac{\partial v}{\partial x} - \frac{u_\tau}{U} \left(\frac{\partial S_1}{\partial x} + \frac{L}{\delta} \frac{\partial S_2}{\partial y} \right) + \frac{1}{R_e} \left(\frac{\partial^2 E}{\partial x^2} + \frac{L^2}{\delta^2} \frac{\partial^2 E}{\partial y^2} \right) - \frac{L}{U u_\tau^2} \epsilon \tag{6.203d}$$

where:

$$R_e = \frac{UL}{\nu}$$

We now assume that $R_e \to \infty$ and $U \gg V$, $L \gg \delta$, i.e., $V = 0(\delta)$ and

$$\frac{U\delta}{VL} = 1$$

or:

$$\frac{V}{U} = \frac{\delta}{L}$$

As in the laminar boundary layer approximation, if the inertia and viscous terms in Equation 6.203b are of the same order of magnitude, then we must have:

$$\frac{L^2}{R_e\delta^2} = 0(1)$$

so that:

$$\frac{\delta}{L} = 0\left(\frac{1}{\sqrt{R_e}}\right)$$

Also, if the Reynolds stress terms in Equation 6.203b are of the same order of magnitude as the inertia terms, then:

$$\frac{u_\tau^2}{U^2}\frac{L}{\delta} = 0(1)$$

so that:

$$\frac{u_\tau^2}{U^2} = 0\left(\frac{\delta}{L}\right)$$

and:

$$\frac{u_\tau^2}{U^2} = 0\left(\frac{V}{U}\right)$$

Using these estimates in Equation 6.203b, we find that only the terms:

$$\frac{-\partial p}{\partial x}, \quad \frac{L^2}{R_e\delta^2}\frac{\partial^2 u}{\partial y^2}, \quad \frac{-u_\tau^2}{U^2}\frac{L}{\delta}\frac{\partial R_{12}}{\partial y}$$

must be retained on the right-hand side of the equation. Similarly, from Equation 6.203c we find that the pressure gradient estimate must be such as to balance the highest order term, which is the last term in Equation 6.203c. Thus:

$$\frac{\partial p}{\partial y} = \frac{-u_\tau^2}{U^2}\frac{\partial R_{22}}{\partial y}$$

Following the preceding procedure of estimation in Equation 6.203d, we retain the terms:

$$-\frac{L}{\delta}R_{12}\frac{\partial u}{\partial y}, \quad \frac{-u_\tau}{U}\frac{L}{\delta}\frac{\partial S_2}{\partial y}, \quad \frac{L^2}{R_e\delta^2}\frac{\partial^2 E}{\partial y^2}, \quad \frac{L}{Uu_\tau^2}\epsilon$$

Although the retained terms are not exactly of the same order of magnitude, we have to keep them on the physical basis of production, diffusion, and dissipation in a turbulent boundary layer equation.

The preceding order of magnitude analysis has established a rule for writing the boundary layer equations in Cartesian coordinates simply by inspection. The rule is that the derivatives of mean quantities with respect to the coordinate normal to the surface (viz., the boundary layer coordinate) must always be retained. In the following we collect the boundary layer equations in two and three dimensions as referred to a rectangular Cartesian system in dimensional physical variables.

Two-Dimensional Equations

Using the substantive operator:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_1\frac{\partial}{\partial x_1} + \bar{u}_2\frac{\partial}{\partial x_2}$$

the equations are

$$\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} = 0 \tag{6.204}$$

$$\frac{D\bar{u}_1}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\nu\frac{\partial \bar{u}_1}{\partial x_2} - \bar{\tau}_{12}\right) \tag{6.205}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} - \frac{\partial}{\partial x_2}(\overline{u_2'^2}) \tag{6.206}$$

$$\frac{DK}{Dt} = -\bar{\tau}_{12}\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial}{\partial x_2}\left\{\nu\frac{\partial K}{\partial x_2} - \overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} - \epsilon \tag{6.207}$$

$$\frac{D}{Dt}(\bar{\tau}_{12}) = \overline{-u_2'^2}\frac{\partial \bar{u}_1}{\partial x_2} + \overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)}$$

$$-\frac{\partial}{\partial x_2}\left\{\overline{u_1'\left(u_2'^2 + \frac{p'}{\rho}\right)}\right\} + \nu\frac{\partial^2}{\partial x_2^2}(\bar{\tau}_{12}) - \epsilon_{12} \tag{6.208}$$

where:

$$\bar{\tau}_{12} = \overline{u_1'u_2'}, \qquad \epsilon_{12} = 2\nu\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}}$$

Three-Dimensional Equations

Using the substantive operator:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_1\frac{\partial}{\partial x_1} + \bar{u}_2\frac{\partial}{\partial x_2} + \bar{u}_3\frac{\partial}{\partial x_3}$$

the equations are

$$\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} + \frac{\partial \bar{u}_3}{\partial x_3} = 0 \tag{6.209}$$

$$\frac{D\bar{u}_1}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial \bar{u}_1}{\partial x_2} - \bar{\tau}_{12}\right) \tag{6.210}$$

$$\frac{D\bar{u}_3}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_3} + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial \bar{u}_3}{\partial x_2} - \bar{\tau}_{23}\right) \tag{6.211}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} - \frac{\partial}{\partial x_2}(\overline{u_2'^2}) \tag{6.212}$$

$$\frac{DK}{Dt} = -\bar{\tau}_{12}\frac{\partial \bar{u}_1}{\partial x_2} - \bar{\tau}_{23}\frac{\partial \bar{u}_3}{\partial x_2} + \frac{\partial}{\partial x_2}\left\{\nu\frac{\partial K}{\partial x_2} - \overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} - \epsilon \tag{6.213}$$

$$\frac{D}{Dt}(\bar{\tau}_{12}) = -\overline{u_2'^2}\frac{\partial \bar{u}_1}{\partial x_2} + \overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)}$$

$$- \frac{\partial}{\partial x_2}\left\{\overline{u_1'\left(u_2'^2 + \frac{p'}{\rho}\right)}\right\} + \nu\frac{\partial^2}{\partial x_2^2}(\bar{\tau}_{12}) - \epsilon_{12} \tag{6.214}$$

$$\frac{D}{Dt}(\bar{\tau}_{23}) = -\overline{u_2'^2}\frac{\partial \bar{u}_3}{\partial x_2} + \overline{\frac{p'}{\rho}\left(\frac{\partial u_2'}{\partial x_3} + \frac{\partial u_3'}{\partial x_2}\right)}$$

$$- \frac{\partial}{\partial x_2}\left\{\overline{u_3'\left(u_2'^2 + \frac{p'}{\rho}\right)}\right\} + \nu\frac{\partial^2}{\partial x_2^2}(\bar{\tau}_{23}) - \epsilon_{23} \tag{6.215}$$

where:

$$\bar{\tau}_{12} = \overline{u_1'u_2'}, \quad \bar{\tau}_{23} = \overline{u_2'u_3'}, \quad \epsilon_{12} = \overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}}, \quad \epsilon_{23} = \overline{\frac{\partial u_2'}{\partial x_k}\frac{\partial u_3'}{\partial x_k}}$$

Equations 6.206 and 6.212 on integration yield, respectively:

$$\bar{p}(x_1, x_2, t) = \bar{p}_0(x_1, t) - \rho\overline{u_2'^2}$$

$$\bar{p}(x_1, x_2, x_3, t) = \bar{p}_0(x_1, x_3, t) - \rho\overline{u_2'^2}$$

The contribution of $\overline{u_2'^2}$ in the above equations is usually small except near the separation region of the boundary layer. Neglecting $\overline{u_2'^2}$ we calculate $\bar{p}(x_1, 0, t)$ and $\bar{p}(x_1, 0, x_3, t)$ through the Euler equations at the surface which are

$$\frac{\partial u_{1e}}{\partial t} + u_{1e}\frac{\partial u_{1e}}{\partial x_1} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} \tag{6.216}$$

for the two-dimensional flows, and:

$$\frac{\partial u_{1e}}{\partial t} + u_{1e}\frac{\partial u_{1e}}{\partial x_1} + u_{3e}\frac{\partial u_{1e}}{\partial x_3} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} \tag{6.217a}$$

$$\frac{\partial u_{3e}}{\partial t} + u_{1e}\frac{\partial u_{3e}}{\partial x_1} + u_{3e}\frac{\partial u_{3e}}{\partial x_3} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_3} \tag{6.217b}$$

for the three-dimensional flows. The boundary conditions for Equations 6.204–6.208 are

$$\text{at } x_2 = 0 : \bar{u}_1 = \bar{u}_2 = 0,\ K = 0,\ \bar{\tau}_{12} = 0$$

$$\text{as } x_2 \to \infty : \bar{u}_1 \to u_{1e}(x_1, t),\ K \to 0 \text{ or } K_\infty,\ \bar{\tau}_{12} \to 0$$

where K_∞ is the free-stream turbulence energy. The boundary conditions for Equations 6.209–6.215 are

$$\text{at } x_2 = 0 : \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0,\ K = 0,\ \bar{\tau}_{12} = \bar{\tau}_{23} = 0$$

$$\text{as } x_2 \to \infty : \bar{u}_1 \to u_{1e},\ \bar{u}_3 \to \bar{u}_{3e},\ K \to 0 \text{ or } K_\infty,\ \bar{\tau}_{12} = \bar{\tau}_{23} \to 0$$

Equations in Orthogonal Curvilinear Coordinates

The boundary layer equations as referred to general curvilinear coordinates can be obtained by applying the boundary layer approximation to the full averaged Reynolds equations written in the curvilinear coordinates. Refer to the technique of coordinate transformation as discussed in Section 5.12. In this section we write only the boundary layer equations obtained in orthogonal curvilinear coordinates. For orthogonal coordinate transformation refer again to Chapter 3.

We take ξ_1, ξ_2, ξ_3 as an orthogonal curvilinear coordinate system with ξ_1, ξ_3 lying in the surface and ξ_2 as the coordinate extending out of the surface in the normal direction. Let $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, u'_1, u'_2, u'_3 be the physical components of the average and the perturbation velocity vectors $(\bar{\mathbf{u}}, \mathbf{u}')$; the equations are

$$\frac{\partial}{\partial \xi_1}(h_2 h_3 \bar{u}_1) + \frac{\partial}{\partial \xi_2}(h_1 h_3 \bar{u}_2) + \frac{\partial}{\partial \xi_3}(h_1 h_2 \bar{u}_3) = 0 \tag{6.218}$$

$$\begin{aligned} &\frac{\partial \bar{u}_1}{\partial t} + \frac{\bar{u}_1}{h_1}\frac{\partial \bar{u}_1}{\partial \xi_1} + \frac{\bar{u}_2}{h_2}\frac{\partial \bar{u}_1}{\partial \xi_2} \\ &\quad + \frac{\bar{u}_3}{h_3}\frac{\partial \bar{u}_1}{\partial \xi_3} + \frac{\bar{u}_1 \bar{u}_3}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{(\bar{u}_3)^2}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} \\ &\quad = -\frac{1}{\rho h_1}\frac{\partial \bar{p}}{\partial \xi_1} + \frac{1}{\rho h_2}\frac{\partial}{\partial \xi_2}\left(\frac{\mu}{h_2}\frac{\partial \bar{u}_1}{\partial \xi_2} - \rho \bar{\tau}_{12}\right) \end{aligned} \tag{6.219}$$

$$\begin{aligned} &\frac{\partial \bar{u}_3}{\partial t} + \frac{\bar{u}_1}{h_1}\frac{\partial \bar{u}_3}{\partial \xi_1} + \frac{\bar{u}_2}{h_2}\frac{\partial \bar{u}_3}{\partial \xi_2} \\ &\quad + \frac{\bar{u}_3}{h_3}\frac{\partial \bar{u}_3}{\partial \xi_3} + \frac{\bar{u}_1 \bar{u}_3}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{(\bar{u}_1)^2}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} \\ &\quad = -\frac{1}{\rho h_3}\frac{\partial \bar{p}}{\partial \xi_3} + \frac{1}{\rho h_2}\frac{\partial}{\partial \xi_2}\left(\frac{\mu}{h_2}\frac{\partial \bar{u}_3}{\partial \xi_2} - \rho \bar{\tau}_{23}\right) \end{aligned} \tag{6.220}$$

$$\begin{aligned} &\frac{\partial K}{\partial t} + \frac{\bar{u}_1}{h_1}\frac{\partial K}{\partial \xi_1} + \frac{\bar{u}_2}{h_2}\frac{\partial K}{\partial \xi_2} + \frac{\bar{u}_3}{h_3}\frac{\partial K}{\partial \xi_3} \\ &\quad = -\frac{\bar{\tau}_{12}}{h_2}\frac{\partial \bar{u}_1}{\partial \xi_2} - \frac{\bar{\tau}_{23}}{h_2}\frac{\partial \bar{u}_3}{\partial \xi_2} - \frac{1}{h_2}\frac{\partial}{\partial \xi_2}\overline{\left\{u'_2\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)\right\}} \\ &\quad + \frac{\nu}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{h_2}\frac{\partial K}{\partial \xi_2}\right) - \epsilon \end{aligned} \tag{6.221}$$

where as has been discussed in Chapter 5, all the metric coefficients h_i have been evaluated at the surfaces, i.e.:

$$h_i = h_i(\xi_1, 0, \xi_3)$$

Part III: Basic Empirical and Boundary Layer Results in Turbulence

6.16 THE CLOSURE PROBLEM

The appearance of second order correlations (the Reynolds stresses) in the averaged Navier-Stokes equations, e.g., Equation 6.149, and of other unknowns in the subsequent Equations 6.167, 6.175, etc. call for a technique to specify these unknowns in terms of dependent variables of the equations. These equations are, therefore, never closed. The problem of specification of these unknowns as functions of the dependent variables is known as the "closure problem".

The earliest known attempt to model the Reynolds stresses in terms of the derivatives of mean velocity components is due to Boussinesq.[44] Consider a unidirectional flow on an infinite plate. Let x, y, z be the rectangular Cartesian system with $\bar{u}$, $\bar{v}$, $\bar{w}$, as the mean velocity components and u', v', w' as the fluctuating turbulent components along x, y, z, respectively. The direction of unidirectional flow on the plate is taken as the x-axis so that $\bar{u}_x = 0$ and $\bar{v} = \bar{w} = 0$. Boussinesq argued that just as in the case of viscous shear stress having the form $\mu\ \partial u/\partial y$, the turbulent shear stress may be modeled as:

$$-\rho\overline{u'v'} = \mu_T \frac{\partial \bar{u}}{\partial y} \tag{6.222}$$

In Equation 6.222 the coefficient μ_T is called the "eddy" or "turbulent" viscosity. Note that the coefficient of molecular viscosity μ is a property of the fluid with values known irrespective of the flow conditions. On the other hand, μ_T is supposed to be a property of the turbulent flow; and thus it must depend on the *mechanism* of transport of turbulent momentum.

The idea of Boussinesq was not implemented for a long time until Prandtl[45] used it in his mixing length hypothesis. In recent times Boussinesq's formulation in an extended and generalized form due to Kolmogorov is in extensive use for the prediction of mean turbulent flow fields.

6.17 PRANDTL'S MIXING LENGTH HYPOTHESIS

Prandtl's mixing length hypothesis has provided some significant results in the prediction of simple flows, such as the turbulent flow past flat plates, unseparated boundary layers (here $\partial\bar{u}/\partial y$ is zero only at the outer edge), far regions in wakes and jets, pipe flows, etc. The original derivation of Prandtl's formula is available in Durand[46] and Schlichting.[4] It is, however, instructive to consider Prandtl's derivation from a point of view which establishes some kind of similarity between the turbulent and laminar transport processes.

Prandtl's mixing length hypothesis can effectively be divided into two parts. In the first part we follow the reasoning of the molecular transport in gases. In Chapter 2, we have already stated the simplest model for the transport of molecular momentum in gases. We restate Equations 2.95 and 2.96 below for the purpose of comparison:

$$\text{viscous shear stress} = a\rho\bar{C}\ell \frac{\partial u}{\partial y}$$

so that the coefficient of viscosity μ is given as:

$$\mu = a\rho\bar{C}\ell$$

Here $a = 0.0896\pi^{3/2}$, ρ is the fluid density, $\bar{C}$ is the average molecular speed, and ℓ is the mean-free path. In the first part of Prandtl's theory let us set up variables in the turbulent case which resemble those given above in the molecular transport. Consider a unidirectional flow in which v' is the fluctuating component normal to the main flow direction. Similar to $\bar{C}$, we define a macroscopic speed as:

$$v_T = (\overline{v'^2})^{1/2} \tag{6.223a}$$

In analogy with the expression for μ, we write:

$$\mu_T = a_1\rho v_T\ell_T \tag{6.223b}$$

where a_1 is a nondimensional constant and ℓ_T is a length scale similar to but essentially different in scope from ℓ. For the determination of v_T and ℓ_T we now consider the second part of Prandtl's hypothesis. Prandtl argues that v_T is proportional to the difference between mean velocities at $y + \ell_T$ and y. Thus:

$$v_T = \alpha|\bar{u}(y + \ell_T) - \bar{u}(y)|$$

which on Taylor's expansion yields:

$$v_T \cong \alpha\ell_T\left|\frac{\partial\bar{u}}{\partial y}\right| \tag{6.223c}$$

Substituting Equation 6.223c in Equation 6.223b and defining a new length scale ℓ as:

$$\ell^2 = \alpha a_1\ell_T^2$$

we get:

$$\mu_T = \rho\ell^2\left|\frac{\partial\bar{u}}{\partial y}\right| \tag{6.224}$$

The length ℓ defined above and used in Equation 6.224 is called Prandtl's mixing length. Finally, first substituting Equation 6.223b and next Equation 6.224 in Equation 6.222, we obtain two representations for the Reynolds stress:

$$-\rho\overline{u'v'} = a_1\rho v_T\ell_T\frac{\partial\bar{u}}{\partial y} \tag{6.225a}$$

$$-\rho\overline{u'v'} = \rho\ell^2\frac{\partial\bar{u}}{\partial y}\left|\frac{\partial\bar{u}}{\partial y}\right| \tag{6.225b}$$

This derivation is simple and exposes the essential elements of Prandtl's theory in a direct fashion. The specification of the mixing length ℓ in Prandtl's formula remains an open problem. The specification of ℓ as a function of coordinates depends on the geometry of flow, e.g., wall-bound flow, free flow, etc.

Turbulent Flow Near a Wall

We first consider the application of Prandtl's formula (Equation 6.225b) for a wall-bound turbulent flow past an infinite plate. Besides providing a solution of the averaged Navier-Stokes equations for an infinite plate, this problem also introduces certain quantities which are of much importance in both theoretical and experimental turbulent flow problems. As before, taking the x-axis as the direction of main flow on the xz-plane and y normal to it, the averaged Navier-Stokes equations for an infinite plate simply reduce to one equation which is

$$\mu \frac{d^2\bar{u}}{dy^2} + \frac{d\bar{\tau}}{dy} = 0$$

where:

$$\bar{\tau} = -\rho\overline{u'v'}$$

Integrating once, we have:

$$\mu \frac{d\bar{u}}{dy} + \bar{\tau} = \tau_w \tag{6.226}$$

where since $\bar{\tau} = 0$ at $y = 0$, the quantity τ_w must be the wall shear stress:

$$\tau_w = \mu\left(\frac{d\bar{u}}{dy}\right)_{y\to 0}$$

Then very close to the wall viscous stresses dominate over turbulent stresses, and thus Equation 6.226 is approximated as:

$$\mu \frac{d\bar{u}}{dy} = \tau_w \quad \text{for } y \to 0$$

which on integration yields:

$$\bar{u} = \frac{\tau_w}{\mu} y \tag{6.227}$$

The region very close to the wall where the linear distribution (Equation 6.227) is applicable is called the *viscous sublayer region*. Away from the wall there is a turbulent core of fine turbulence in which turbulent stresses dominate over viscous stresses, and therefore in the core region Equation 6.226 is approximated as:

$$\bar{\tau} = \tau_w, \quad \text{for } y > 0$$

Using Equation 6.225b (on a flat plate $\partial u/\partial y > 0$), we get:

$$\frac{d\bar{u}}{dy} = \frac{1}{\ell}\sqrt{\tau_w/\rho}$$

Since the dimension of τ_w/ρ is (velocity)^2, then we introduce the velocity:

$$u_\tau = \sqrt{\tau_w/\rho} \tag{6.228}$$

So that:

$$\frac{d\bar{u}}{dy} = \frac{u_\tau}{\ell} \tag{6.229a}$$

Prandtl then notes that since the distance from the wall is the only characteristic length of the problem, it is logical to take:

$$\ell = \kappa y \tag{6.229b}$$

where κ is a constant. Using Equation 6.229b in Equation 6.229a and integrating, we get:

$$\bar{u}(y) = \frac{u_\tau}{\kappa} \ell n y + C^* \tag{6.230}$$

where C^* is a constant of integration. It is obvious that the no-slip condition at the wall $y = 0$ cannot be used in Equation 6.230 to determine C^*. The constant C^* can only be determined by matching the solutions in Equations 6.227 and 6.230 in an overlap region where both solutions are equally valid.

The quantity u_τ, defined in Equation 6.228 and having the dimension of velocity, is called the *friction velocity*. Based on ν and u_τ we can also define a friction length:

$$\ell_\tau = \frac{\nu}{u_\tau} \tag{6.231}$$

Basic parameters for the viscous sublayer are ρ, μ, and τ_w. Based on dimensional considerations these three parameters must define the thickness of the sublayer as a linear function of ℓ_τ while the velocity at the edge of the sublayer must be a linear function of u_τ. Thus the thickness of the sublayer δ_τ must be given by:

$$\delta_\tau = \alpha \tau_w^a \rho^b \mu^c$$

where α, a, b, c are dimensionless constants. On dimensional considerations we easily find that:

$$a = -\frac{1}{2}, \qquad b = -\frac{1}{2}, \qquad c = 1$$

so that:

$$\delta_\tau = \alpha \ell_\tau$$

with α remaining an unknown parameter. In the same manner we can show that velocity at the edge of the sublayer is

$$U_\tau = \beta u_\tau$$

where β is an unknown parameter. However, at the edge of the sublayer using Equation 6.227, we have:

$$U_\tau = \frac{\tau_w}{\mu} \delta_\tau$$

Substituting the expression for U_τ given above, we get:

$$\beta = \alpha$$

Thus:

$$\delta_\tau = \alpha \ell_\tau$$

$$U_\tau = \alpha u_\tau$$

Taking the matching condition that the velocity at the edge of the sublayer and the lower edge of the core region must be the same, Equation 6.230 yields:

$$U_\tau = \frac{u_\tau}{\kappa} \ell n(\alpha \ell_\tau) + C^* = \alpha u_\tau$$

Thus:

$$C^* = u_\tau \left\{ \alpha - \frac{1}{\kappa} \ell n\left(\frac{\alpha \nu}{u_\tau}\right) \right\}$$

With this value of C^*, Equation 6.230 becomes:

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \ell n\left(\frac{y u_\tau}{\nu}\right) + \left(\alpha - \frac{1}{\kappa} \ell n \alpha\right) \qquad \textbf{(6.232a)}$$

where α and κ are the unknown constants which can be determined only through experiments. Changing the logarithm to base ten, Equation 6.232a becomes:

$$\frac{\bar{u}}{u_\tau} = \frac{2.303}{\kappa} \log\left(\frac{y u_\tau}{\nu}\right) + \alpha - \frac{2.303}{\kappa} \log \alpha \qquad \textbf{(6.232b)}$$

In the early 1930s Nikuradse[4] performed a series of repeatable experiments on pipe flows. He found that irrespective of the Reynolds number of mean flow, the velocity distribution for *smooth pipes* is always given by the formula:

$$\frac{\bar{u}}{u_\tau} = 5.75 \log\left(\frac{y u_\tau}{\nu}\right) + 5.5 \qquad \textbf{(6.233)}$$

On comparing Equations 6.232b and 6.233, we find that:

$$\kappa = 0.40, \qquad \alpha = 11.637$$

Since the velocity distribution in Equation 6.233 is valid for all Reynolds' numbers for which turbulent flow near a wall exists, then the logarithmic velocity distribution is said to be a universal velocity distribution near a wall. The constant κ is independent of the Reynolds number. It is therefore a universal constant of turbulence and also is known as Karman's constant. It will be seen later that constant α is different for boundary layers, but κ consistently lies between 0.40 and 0.41.

Experimental Determination of u_τ

Taking Equation 6.233 as an accepted velocity distribution for a near wall flow, we can set up simple formulae to calculate the friction velocity u_τ for a given problem. Let $\bar{u}_m$ be the average cross-sectional velocity of flow; then Equation 6.233 can be rewritten as:

$$\frac{\bar{u}}{\bar{u}_m} = 5.75\, S \log\left(\frac{y \bar{u}_m}{\nu} S\right) + 5.5\, S$$

where $S = u_\tau/\bar{u}_m$. In an experimental setup for a pipe flow the mean velocity distribution $\bar{u}(y)$ is calculated for a number of y-values. The cross-sectional average $\bar{u}_m$ is also calculated. The values $\bar{u}/\bar{u}_m$ are then plotted on a semilog paper with respect to $y\bar{u}_m/\nu$. The available graph is then compared with the above equation for different values of S. The best fit yields the value of S and hence of u_τ.

Application of the Logarithmic Formula in Pipe Flow

In a fully developed laminar or mean turbulent flow through a pipe the velocity is constant on every streamline, and thus there is a complete absence of inertia forces. Therefore, the frictional resistance acting on the pipe wall must balance the resultant pressure force acting on any chosen cylindrical element of fluid. For a pipe of diameter $d = 2a$, if Δp is the pressure difference in a length ℓ of the pipe in the pipe, then the pressure force is $(\Delta p)\ \pi d^2/4$; on the other hand, the frictional resistance on the fluid element is $\tau_w \pi \ell d$, where τ_w is the wall shear stress. Equating these two, we find that:

$$\Delta p = \frac{4\ell}{d}\tau_w = \frac{2\ell}{a}\tau_w \tag{6.234a}$$

From Equation 6.234a we conclude that the drop of pressure in a segment of length $\ell = a/2$ is equal to the friction stress per unit area of the pipe surface. The most important quantity to be considered in pipe flows is the resistance coefficient λ defined earlier in the laminar flow through a pipe as:

$$\lambda = \frac{d}{\ell}\frac{\Delta p}{\frac{1}{2}\rho\bar{u}_{av}^2} \tag{6.234b}$$

where $\bar{u}_{av}$ is the cross-sectional average velocity in the tube. Substituting Δp from Equation 6.234a, we get:

$$\frac{\bar{u}_{av}}{u_\tau} = \sqrt{8/\lambda} \tag{6.234c}$$

Measuring the y-coordinate from the wall of the tube, we find from the logarithmic law (Equation 6.233) that the maximum velocity in the pipe is given by:

$$\frac{\bar{u}_{max}}{u_\tau} = 5.75\log\left(\frac{au_\tau}{\nu}\right) + 5.5 \tag{6.235a}$$

Thus:

$$\frac{\bar{u}_{max} - \bar{u}}{u_\tau} = 5.75\log\frac{a}{y} = \frac{1}{\kappa}\ell n\frac{a}{y} \tag{6.235b}$$

and:

$$\bar{u} = \bar{u}_{max} + \frac{u_\tau}{\kappa}\ell n\frac{y}{a} \tag{6.235c}$$

Using polar coordinates (r, θ) with r measured from the axis, we have:

$$\bar{u}_{av} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \bar{u}\, r\, d\theta\, dr$$

Making the transformation:

$$r = a - y$$

we obtain:

$$\bar{u}_{av} = 2\int_0^1 \left(1 - \frac{y}{a}\right)\bar{u}\left(\frac{y}{a}\right) d\left(\frac{y}{a}\right)$$

which on substituting Equation 6.235c yields:

$$\frac{\bar{u}_{max} - \bar{u}_{av}}{u_\tau} = \frac{3}{2\kappa} = 3.75 \tag{6.235d}$$

Writing Equation 6.235a as:

$$\frac{\bar{u}_{max} - \bar{u}_{av}}{u_\tau} + \frac{\bar{u}_{av}}{u_\tau} = 5.75 \log\left(\frac{\bar{u}_{av}d}{\nu} \cdot \frac{u_\tau}{2\bar{u}_{av}}\right) + 5.5$$

and using Equations 6.234c and 6.235d, we obtain:

$$\frac{1}{\sqrt{\lambda}} = 2.0329 \log(R\sqrt{\lambda}) - 0.9112 \tag{6.236}$$

where:

$$R = \frac{\bar{u}_{av}d}{\nu}$$

Based on a number of measurements it has been found that the equation:

$$\frac{1}{\sqrt{\lambda}} = 2.0 \log(R\sqrt{\lambda}) - 0.8 \tag{6.237a}$$

satisfies the experimental data best.[4] Equation 6.237a is called the *resistance law* for flow through a pipe and is shown plotted along with the experimental data in Figure 6.10.

A fairly good approximation to Equation 6.237a as proposed by Nikuradse is

$$\lambda = 0.0032 + \frac{0.221}{R^{0.237}}$$

Another resistance law proposed by Blasius is

$$\lambda = \frac{0.3164}{R^{0.25}} \tag{6.237b}$$

which is not of universal applicability but holds only for $R < 10^5$.

Power Laws for the Velocity Distribution

Besides the logarithmic formulae as discussed above, there are some purely empirical formulae called the power laws. A power law velocity distribution should give rise to a power law for the friction coefficient. Thus for the power law:

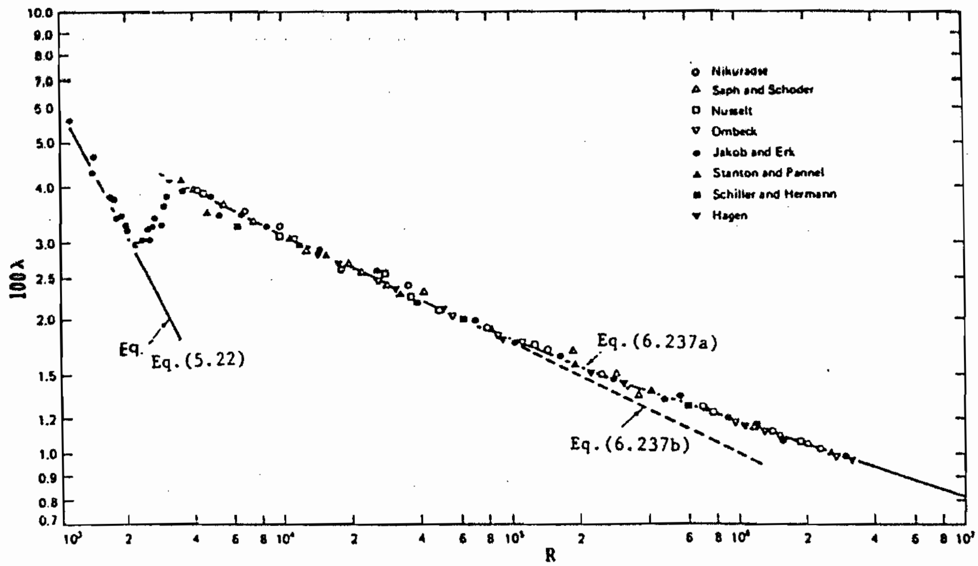


Fig. 6.10 Comparison of the experimental and theoretical results on the friction coefficient for a circular pipe. (Taken from Schetz, J. A., *Foundations of Boundary Layer Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1984. With permission.)

$$\frac{\bar{u}}{\bar{u}_{max}} = \left(\frac{y}{a}\right)^n$$

we must have:

$$\lambda = \frac{\text{const.}}{R^m}$$

To establish a relation between m and n we form the average velocity based on the given power law distribution and have:

$$\frac{\bar{u}_{av}}{\bar{u}_{max}} = \frac{2}{(n + 1)(n + 2)}$$

Next using the conditions at the edge of the sublayer:

$$y = \frac{\alpha \nu}{u_\tau} : \bar{u} = \alpha u_\tau$$

we get:

$$\frac{\alpha u_\tau}{\bar{u}_{max}} = \left(\frac{\alpha \nu}{a u_\tau}\right)^n$$

or:

$$\frac{u_\tau}{\bar{u}_{av}} \cdot \frac{\bar{u}_{av}}{\bar{u}_{max}} = \frac{2^n \alpha^{n-1}}{R^n} \left(\frac{\bar{u}_{av}}{u_\tau}\right)^n$$

where $R = \bar{u}_{av}d/\nu$. Using Equation 6.234c we finally have:

$$\lambda = C/R^m$$

where:

$$m = \frac{2n}{n + 1}$$

$$C = 2^{(5n+1)/(n+1)} \cdot \alpha^{(2n-2)/(n+1)} \cdot [(n + 1)(n + 2)]^{2/(n+1)}$$

In civil engineering the 1/7th power law for the velocity distribution has widely been used. If we put $n = 1/7$, then $m = 1/4$.

Rough Pipes

Let h be the average roughness height of grains attached to the tube wall. The value of h, usually expressed in *mm*, is called the absolute roughness. The ratio h/a is called the relative roughness which usually varies from 0.2 to 7%. The purpose here is only to categorize the roughness conditions. For extensive results the reader is referred to Schlichting.[4] If $h \ll \delta_\tau$ where δ_τ is the thickness of the sublayer, then the pipe is said to be hydraulically smooth. If $h \sim \delta_\tau$ or $h \gg \delta_\tau$, then the pipe is said to be hydraulically rough. Some important experimental observations of flows in rough pipes are as follows:

1. A limiting roughness height exists for which there is no effect on the critical Reynolds number for transition from laminar to turbulent flow. At and above the limiting roughness heights, the velocity distribution of laminar flows is more susceptible to instability and, hence, to an earlier transition to turbulence.
2. For very small values of h/a, the data fairly satisfy the smooth pipe data. The data for h/a between 3 and 4% are markedly different from the smooth pipe data.
3. At sufficiently high Reynolds numbers, the coefficient of resistance λ ceases to depend on R but depends solely on the relative roughness. The logarithmic velocity distribution in rough pipes is an empirical formula which is

$$\frac{\bar{u}}{u_\tau} = 5.75 \log \frac{y}{h} + \phi\left(\frac{hu_\tau}{\nu}\right)$$

where:

$$\phi\left(\frac{hu_\tau}{\nu}\right) = 3.75 - 5.75 \log \frac{a}{h} + \sqrt{8/\lambda}$$

and:

$$\lambda = \left(1.68 + 2 \log \frac{a}{h}\right)^{-2}$$

6.18 WALL-BOUND TURBULENT FLOWS

Both Prandtl[48] and Karman[49], in their early research, observed that close to a wall the boundary layer satisfies a similarity principle in the velocity distribution of the form:

$$u^+ = \phi(y^+) \tag{6.238}$$

where:

$$u^+ = \bar{u}/u_\tau, \qquad y^+ = yu_\tau/\nu$$

Further, $u_\tau = \sqrt{\tau_w/\rho}$, and for convenience sake we have set:

$$\overline{u_1} = \bar{u}, \qquad x_2 = y$$

Figure 6.11 demonstrates an experimental check on the similarity principle[38] in which it is shown that in the sublayer region the function ϕ is linear, i.e.:

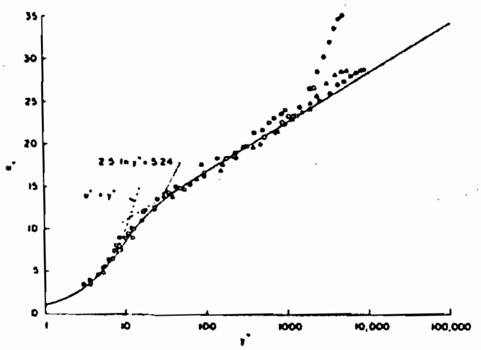


Fig. 6.11 Mean velocity distribution near a smooth wall. (Taken from Cebeci, T. and Smith, A. M. O., *Analysis of Turbulent Boundary Layers*, Academic Press, New York, 1974. With permission.)

$$u^+ = y^+ \tag{6.239}$$

while in the turbulent core region it is logarithmic, i.e.:

$$u^+ = \frac{1}{\kappa}\,\ell n y^+ + C \tag{6.240}$$

where C is a universal constant. For boundary layers, the values $C = 4.9$ and $\kappa = 0.41$ have correlated the experimental data quite well[53]. If $\kappa = 0.40$, then $C = 5.24$ seems to be a better choice as is shown in Figure 6.11.

The foregoing results, i.e., Equations 6.239 and 6.240, were also obtained earlier in connection with Prandtl's mixing length hypothesis. Derivation of the logarithmic law (Equation 6.230) was based on solution of the averaged equation of motion for flow past an infinite plate along with the use of Prandtl's mixing length formula. These restrictions, however, are not necessary to obtain the logarithmic velocity distribution in the turbulent core region as demonstrated below.

Analysis of the near-wall region depends on the parameters ρ, μ, and τ_w. The characteristic length naturally is Equation 6.231 which is used to define the similarity variable $y^+ = y/\ell_\tau$ appearing in Equation 6.238. The viscous sublayer is governed mainly by viscosity, but the turbulent core region is governed by turbulence viscosity. Thus, the turbulent core region should not explicitly depend on viscosity. To exploit this feature of the flow field we differentiate Equation 6.238 once with respect to y and have:

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau^2}{\nu}\frac{d\phi}{dy^+}$$

To remove the explicit effect of viscosity in the core region we postulate that:

$$\frac{d\phi}{dy^+} \propto \frac{\nu/u_\tau}{y}$$

or:

$$\frac{d\phi}{dy^+} = \frac{\nu}{\kappa u_\tau y} \tag{6.241a}$$

$$= \frac{1}{\kappa y^+} \tag{6.241b}$$

where $\kappa = 0.41$ is Karman's constant. Thus, in the turbulent core region:

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{\kappa y} \tag{6.241c}$$

and there is no explicit appearance of ν in Equation 6.241c. This derivative formula is fundamental to the description of flow in the turbulent core region. Integration of Equation 6.241c yields:

$$\bar{u} = \frac{u_\tau}{\kappa} \ell n y + C^*$$

while the integration of Equation 6.241b along with the use of Equation 6.238 yields:

$$u^+ = \frac{1}{\kappa} \ell n y^+ + C$$

which is Equation 6.240. Here both C^* and C are constants.

As a further check on the validity of the results in Equations 6.239 and 6.240 we consider the boundary layer equation (Equation 6.205) very close to the surface on which the boundary layer forms. Neglecting the inertia and pressure forces we again obtain:

$$\frac{\partial \tau_t}{\partial y} = 0$$

where:

$$\tau_t = \mu \frac{\partial \bar{u}}{\partial y} + (-\rho\overline{u'v'})$$

is the total shear stress. Integrating the equation once and using the condition $\tau_w = \mu(\partial\bar{u}/\partial y)_{y=0}$, we get:

$$\tau_t = \tau_w \tag{6.242}$$

Because of Equation 6.242, Equation 6.241c can also be stated as:

$$\frac{\partial \overline{u}}{\partial y} = \frac{1}{\kappa y}(\tau_r/\rho)^{1/2}$$

Equation 6.242 states that the sum of viscous and Reynolds' stresses in the vicinity of the boundary is equal to the shear stress τ_w. The region where this statement is true is called the region of constant shearing, also called the *inner region*. Very close to the wall ($y \rightarrow 0$, $y^+ \rightarrow 0$) the turbulent stress is small in comparison to the viscous stress, and this consideration leads one to the formula (Equation 6.239) for velocity in the viscous sublayer. The thickness of the sublayer region, according to Figure 6.11 is about $y^+ \cong 5$. Away from the sublayer ($y \rightarrow 0$, $y^+ \rightarrow \infty$) the viscous shearing is insignificant in comparison with the Reynolds stress. This region is called the turbulent core region. The velocity distribution in this region is logarithmic. Further, since in the core region:

$$-\rho\overline{u'v'} \cong \tau_w$$

then:

$$(-\overline{u'v'})^{1/2} \cong u_\tau$$

There is considerable overlapping between the viscous sublayer and the turbulent core region, and this is called the *buffer region*. (Refer to Examples 6.2 and 6.3.) Using Figure 6.11 one can divide the inner region into three regimes as follows:

$0 \leq y^+ < 5$ — sublayer region

$5 \leq y^+ < 30$ — buffer layer*

$30 \leq y^+ < 10^4$ — turbulent core region

Example 6.2

Show that the viscous shear stress at $y^+ = 60$ is only 4% of the wall shear stress τ_w.

Using the turbulent core equation (Equation 6.241c) or alternatively differentiating Equation 6.240 with y, we get:

$$\mu\frac{\partial \overline{u}}{\partial y} = \frac{\tau_w}{\kappa y^+}$$

from which it follows that at $y^+ = 60$:

$$\mu\frac{\partial \overline{u}}{\partial y} = 0.0406\ \tau_w.$$

Example 6.3

By using the mixing length hypothesis show that the velocity distribution in the inner region is given by:

$$u^+ = 2\int_0^{y^+} \frac{dy^+}{1 + (1 + 4\ell^{+2})^{1/2}} \tag{i}$$

where $\ell^+ = \ell u_\tau/\nu$ is the normalized mixing length. Write particular cases for the (1) sublayer,

* Approximately $u^+ = 5\ \ell n\ y^+ - 3.05$ in the buffer zone.

(2) buffer layer, and (3) logarithmic layer. Also evaluate Equation i for the case when $\ell^+ = \alpha_0^2 y^+(1 + \alpha_0^4 y^{+2})^{1/2}$, where $\alpha_0 = 0.3$.

According to the mixing length hypothesis:

$$-\rho\overline{u'v'} = \rho\ell^2\left(\frac{\partial\bar{u}}{\partial y}\right)^2$$

Substituting this expression in Equation 6.242 and on nondimensionalization we get:

$$\ell^{+2}\left(\frac{\partial u^+}{\partial y^+}\right)^2 + \frac{\partial u^+}{\partial y^+} - 1 = 0$$

The result to be obtained, i.e., Equation i, follows directly from the above equation. Particular cases are for:

1. The sublayer $\ell^+ \cong 0$.

2. and 3. Both for the buffer region and the logarithmic region

$$\ell^+ = \kappa y^+[1 - \exp(-y^+/A^+)]; \qquad A^+ \cong 26.0$$

and the velocity can be obtained through Equation i only by numerical quadrature. Evaluation of the integral for the given ℓ^+ gives: $u^+ = (1/\alpha_0^2)\tan^{-1}(\alpha_0^2 y^+)$. This solution is a fairly good approximation of u^+ in the sublayer and the buffer zone.

Example 6.4

Use Equation 6.242 to show that the maximum production of turbulence energy is $u_\tau^4/4\nu$.

Multiply both sides of Equation 6.242 by $\partial\bar{u}/\partial y$ and form the equation:

$$P = u_\tau^2\frac{\partial\bar{u}}{\partial y} - \nu\left(\frac{\partial\bar{u}}{\partial y}\right)^2$$

where:

$$P = -\overline{u'v'}\frac{\partial\bar{u}}{\partial y}$$

is the production. Differentiating P with respect to $\partial\bar{u}/\partial y$, we find that the maximum occurs when:

$$\frac{\partial\bar{u}}{\partial y} = \frac{u_\tau^2}{2\nu}$$

Thus:

$$P_{max} = \frac{u_\tau^4}{4\nu}$$

Example 6.5

Use the Prandtl's mixing length hypothesis (Equation 6.225b) to solve the problem of two-dimensional constant velocity stream mixing with the stationary ambient air to produce a turbulent flow field.

The problem of a two-dimensional turbulent mixing in a stationary medium was solved by Tollmien (refer to Durand, Ref. 46). This problem provides a major application of Prandtl's mixing length formula. The pertinent equations for this case are obtained from Equations 6.204 and 6.205 with the simplifying conditions that the ambient pressure is constant and the effect of viscosity is negligible. The equations are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{i}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(-\rho\overline{u'v'}\right) \tag{ii}$$

Further, there is an inflow or entrainment of the surrounding fluid in the mixing region due to the velocity $\bar{v}$ at the lower end of the mixing region as is shown in Fig. Example 6.5

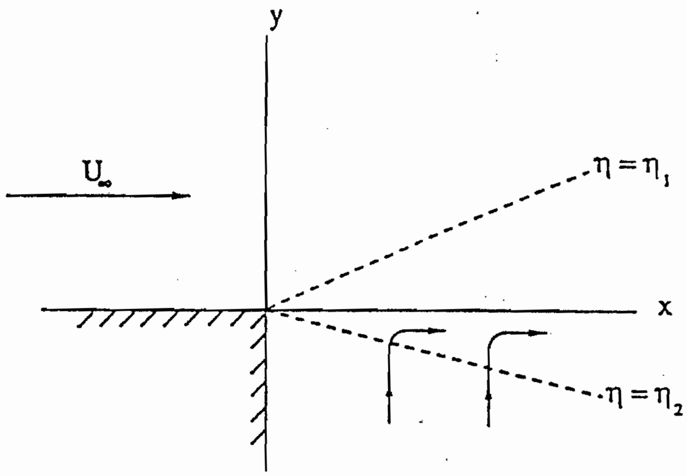


Fig. Example 6.5

At the boundary of the mixing zone and the free system:

$$\bar{u} = u_\infty, \bar{v} = 0, \frac{\partial \bar{u}}{\partial y} = 0 \tag{iii}$$

and at the boundary of the mixing zone and the stationary air:

$$\bar{u} = 0, \frac{\partial \bar{u}}{\partial y} = 0 \tag{iv}$$

Using the Prandtl's formula, we have

$$\left(-\rho\overline{u'v'}\right) = \rho l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{v}$$

where now $l = cx$ is the mixing length.

Introducing the stream function $\overline{\psi}$ through

$$\overline{u} = \frac{\partial \overline{\psi}}{\partial y}, \quad \overline{v} = -\frac{\partial \overline{\psi}}{\partial x}$$

the Equations i through iv become

$$\frac{\partial \overline{\psi}}{\partial y}\frac{\partial^2 \overline{\psi}}{\partial x\,\partial y} - \frac{\partial \overline{\psi}}{\partial x}\frac{\partial^2 \overline{\psi}}{\partial y^2} = 2l^2 \frac{\partial^2 \overline{\psi}}{\partial y^2}\frac{\partial^3 \overline{\psi}}{\partial y^3} \tag{vi}$$

$$\begin{aligned} &as\ y \to \infty : \overline{\psi} = u_{\infty} y, \quad \frac{\partial \overline{\psi}}{\partial y} = u_{\infty} \\ &as\ y \to -\infty : \overline{u} = \frac{\partial \overline{\psi}}{\partial y} = 0 \end{aligned} \tag{vii}$$

On dimensional considerations, and also noting that neither the equation nor the boundary conditions depend on any characteristic length, we take

$$\overline{\psi} = u_{\infty}\, x\phi(\eta), \quad \eta = y/x$$

Thus

$$\overline{u} = u_{\infty}\, x\phi' \ and\ \overline{v} = u_{\infty}(\eta\phi' - \phi)$$

and Equation vi becomes

$$\phi''(c^2\phi''' + \phi) = 0$$

giving the equations

$$\phi'' = 0 \tag{viii}$$

$$c^2\phi''' + \phi = 0 \tag{ix}$$

Equation viii describes the undisturbed region and therefore should be used to provide the boundary conditions for Equation ix. First of all, $\phi'' = 0$ is satisfied by

$$\phi(\eta) = \eta, \quad \phi'(\eta) = 1 \ and\ also\ \phi'(\eta) = 0$$

Let $\eta = \eta_1$ be the smallest positive root of $\phi'(\eta) = 1$ and $|\eta_2|$ be the smallest value of the negative root of $\phi'(\eta) = 0$. Thus, $\eta = \eta_1$ is the upper boundary and $\eta = \eta_2$ is the lower boundary. Introducing $\alpha = c^{-2/3}$, the general solution of Equation ix is

$$\phi(\eta) = a_1 e^{-\alpha\eta} + e^{1/2\alpha\eta}\left(a_2 \cos\frac{\sqrt{3}\alpha\eta}{2} + a_3 \sin\frac{\sqrt{3}\alpha\eta}{2}\right) \tag{x}$$

Boundary Conditions:

At the upper boundary ($\eta = \eta_1$)

$$\phi'(\eta_1) = 1 \quad (\textit{corresponds to } \bar{u} = u_{-})$$
$$\phi(\eta_1) = \eta_1 \quad (\textit{corresponds to } \bar{v} = 0,\ or,\ \phi - \eta\phi'' = 0)$$
$$\phi''(\eta) = 0 \quad (\textit{corresponds to } \frac{\partial \bar{u}}{\partial y} = 0)$$

At the lower boundary ($\eta = \eta_2$)

$$\phi'(\eta_2) = 0 \quad (\textit{corresponds to } \bar{u} = 0)$$
$$\phi''(\eta_2) = 0 \quad (\textit{corresponds to } \frac{\partial \bar{u}}{\partial y} = 0)$$

Writing

$$\zeta = \alpha\eta$$
$$d_1 = \alpha a_1,\ d_2 = \alpha a_2,\ d_3 = \alpha a_3$$
$$d_4 = \alpha\left(a_2 + \sqrt{3}a_3\right),\ d_5 = \alpha\left(a_3 - \sqrt{3}a_2\right)$$
$$d_6 = \alpha\left(a_2 - \sqrt{3}a_3\right),\ d_7 = \alpha\left(a_3 + \sqrt{3}a_2\right)$$
$$\lambda_1 = \frac{\sqrt{3}\zeta_1}{2},\ \lambda_2 = \frac{\sqrt{3}\zeta_2}{2}$$

we find by algebraic manipulations

$$d_3 = d_2 \tan \lambda_2$$

and substituting the algebraic expressions for d_2 and d_3 in

$$d_4 = d_2 + \sqrt{3}d_3,\quad d_5 = d_3 - \sqrt{3}d_2$$
$$d_6 = d_2 - \sqrt{3}d_3,\quad d_7 = d_3 + \sqrt{3}d_2$$

and writing

$$\chi = \zeta_2 - \zeta_1$$

the transcendental equation to be solved is

$$\cos\left(\frac{\sqrt{3}\chi}{2}\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}\chi}{2}\right) - e^{\frac{3\chi}{2}} = 0 \tag{xi}$$

Knowing χ, we have

$$\zeta_2 = \zeta_1 + \chi$$

Next, by imposing the boundary conditions, d_1, d_2 *and* d_3 are obtained. The final solution is

$$\frac{\bar{u}}{u_\infty} = -d_1 e^{-\zeta} + 1/2e^{1/2\zeta}\left(d_4 \cos\frac{\sqrt{3}\zeta}{2} + d_5 \sin\frac{\sqrt{3}\zeta}{2}\right)$$

and

$$\frac{\alpha\bar{v}}{u_\infty} = -d_1(1+\zeta)e^{-\zeta} + (1/2d_4\zeta - d_2)e^{1/2\zeta}\cos\frac{\sqrt{3}\zeta}{2}$$
$$+ (1/2d_5\zeta - d_3)e^{1/2\zeta}\sin\frac{\sqrt{3}\zeta}{2}$$

The value of α has to be obtained by experiments.

Prandtl's experiments have shown that

$$l = 0.0174x$$

and the width b(x) of the mixing layer is

$$b(x) = 0.255x$$

Thus c = 0.024607316 and α = 11.82017305 while $\zeta_1 - \zeta_2 = -\chi = 3.014144128$, though from Equation xi, $\chi = -3.016744$. On numerical solution

$$\begin{aligned}\zeta_1 &= 0.981409,\ \zeta_2 = -2.03534\\ d_1 &= -0.0165348,\ d_2 = 0.133541,\ d_3 = 0.687491\\ d_4 &= 1.32431,\ d_5 = 0.456191.\end{aligned}$$

6.19 ANALYSIS OF TURBULENT BOUNDARY LAYER VELOCITY PROFILES

Having completed the important analysis of the near-wall region or the inner region of a boundary layer, the next step is to consider the remaining part, called the outer region of a boundarv layer. Experimental data on turbulent boundary layers point out that the inner region is about 20% of the boundary layer thickness δ. The remaining 80% has been observed to be essentially independent of the direct effects of viscosity, and its mechanics is chiefly governed by the Reynolds stresses. The *effective* viscosity in the outer region is, therefore, much higher (by an order of magnitude) in comparison to that in the inner region. Besides the difference in *effective* viscous effects, the length scales ℓ_τ and δ associated with the inner and the outer regions, respectively, are also quite different. These observations point to the fact that the analysis of turbulent boundary layers has to be based on a combination of empirical hypotheses and physical laws.

To expose the essential physical and analytical aspects of turbulent boundary layers it is advisable to utilize, as far as possible, the available data on turbulent boundary layer on a flat plate to expose the underlying turbulence mechanisms. In this respect a report by Klebanoff[50] is quite valuable and must be consulted. In a review article, Clauser[51] carried out a very systematic study of the available data on turbulent boundary layers. His main conclusions were as follows:

1. A universal velocity distribution law holds for boundary layers *near* smooth walls, which is independent of the pressure gradient imposed on the boundary layer. Thus the results obtained in Section 6.18 are applicable to all turbulent boundary layers including that on a flat plate ($dp/dx = 0$).
2. For a flat plate boundary layer (i.e., $dp/dx = 0$, and $u_e = u_\infty$) if the velocity defect $(u_\infty - \bar{u})/u_\tau$ is plotted vs. y/δ, then all the data points from various stations lie on a single curve. The same curve holds whether the plate is smooth or rough and is plotted in Figure 6.12.

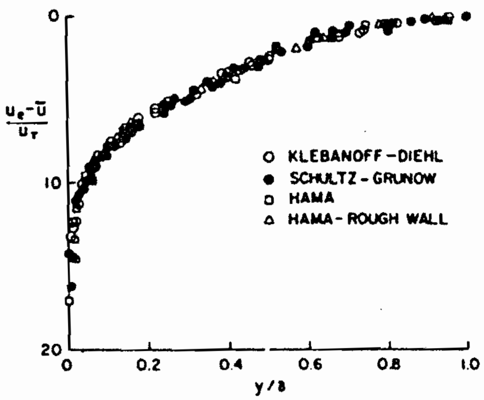


Fig. 6.12 Plot of the normalized velocity defect versus y/δ on both smooth and rough plates. (Taken from Cebeci, T., and Smith, A. M. O., *Analysis of Turbulent Boundary Layers*, Academic Press, Inc., 1974. With permission.)

Thus, for a flat plate:

$$\frac{u_\infty - \bar{u}}{u_\tau} = f\left(\frac{y}{\delta}\right) \qquad \text{outer region}$$

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa}\,\ell n\left(\frac{yu_\tau}{\nu}\right) + C \qquad \text{inner region}$$

where $C = 4.9$. Comparing the two solutions in the *overlap region* (where both are valid) one immediately concludes that:

$$f\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa}\,\ell n\frac{y}{\delta} + B; \qquad \kappa = 0.41$$

Thus the velocity defect formula in the overlap region is

$$\frac{u_\infty - \bar{u}}{u_\tau} = -\frac{1}{\kappa}\,\ell n\frac{y}{\delta} + B \tag{6.243}$$

where B = constant. Clauser plotted the experimental velocity defect data from various sources for a flat plate as are shown in Figure 6.13. It is obvious that in the overlap region the experimental data satisfy Equation 6.243 if B is chosen to be 2.5. It is also obvious from Figure 6.13 that Equation 6.243 is *not* valid for the whole boundary layer, and after the end of the overlap region the velocity defect is given approximately by:

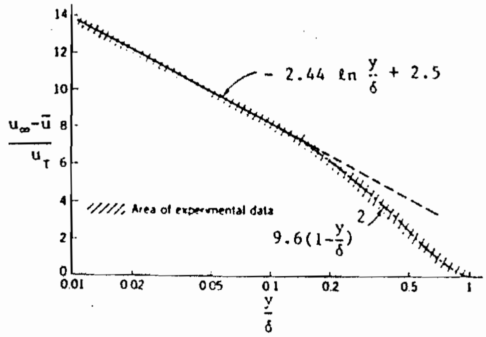


Fig. 6.13 Logarithmic plot of velocity distribution in the outer part of the turbulent boundary layer on a flat plate. (Taken from Hinze, J. O., *Turbulence, an Introduction to Its Mechanism and Theory*, McGraw-Hill, New York, 1959. With permission.)

$$\frac{u_\infty - \bar{u}}{u_\tau} = 9.6\left(1 - \frac{y}{\delta}\right)^2$$

The skin friction law for a flat plate is now obtained by eliminating $\bar{u}$ and y between Equations 6.240 and 6.243. This can simply be done by adding the two equations, thus obtaining:

$$\sqrt{2/c_f} = \frac{1}{\kappa} \ell n(R_\delta \sqrt{c_f/2}) + B + C \tag{6.244}$$

where:

$$R_\delta = u_\infty \delta/\nu, \quad B = 2.6, \quad C = 4.9, \quad \kappa = 0.41$$

As has been noted earlier the velocity profile (Equation 6.243) is valid only in the overlap region. To have an expression for the whole boundary layer, Coles[52] investigated a number of experimental results and arrived at the following empirical formulation.

In the sublayer:

$$\frac{\bar{u}}{u_\tau} = \frac{y u_\tau}{\nu}$$

and for all regions excluding the sublayer:

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \ell n\left(\frac{y u_\tau}{\nu}\right) + C + \frac{\Pi(x)}{\kappa} w\left(\frac{y}{\delta}\right) \tag{6.245}$$

where both $\Pi(x)$ and $w(y/\delta)$ are nondimensional. In general, Π is a function of x and is called the *profile parameter*. The function $w(y/\delta)$ is called the *wake function*, and according to Coles it has the following simple form:

$$w\left(\frac{y}{\delta}\right) = 1 - \cos\frac{\pi y}{\delta} \tag{6.246}$$

Coles' velocity profile can also be written in the following form by evaluating Equation 6.245 at $y = \delta$ and subtracting from Equation 6.245. On simplification and using Equation 6.246, we get:

$$\frac{\bar{u}}{u_e} = 1 + \frac{u_\tau}{\kappa u_e} \ell n\left(\frac{y}{\delta}\right) - \frac{2\Pi(x) u_\tau}{\kappa u_e} \cos^2\left(\frac{\pi y}{2\delta}\right) \tag{6.247}$$

Further results are obtained by using the velocity profile (Equation 6.247). Thus, using Equation 6.247 in the definitions of δ^* and θ, we get:

$$\frac{\kappa\delta^* u_e}{\delta u_\tau} = 1 + \Pi(x) \tag{6.248a}$$

and:

$$\frac{\kappa^2(\delta^* - \theta)u_e^2}{\delta u_\tau^2} = 1.5\Pi^2 + 3.18\Pi + 2 \tag{6.248b}$$

so that:

$$\frac{\theta}{\delta} = \frac{u_\tau}{\kappa u_e}(1 + \Pi) - \frac{u_\tau^2}{\kappa^2 u_e^2}(1.5\Pi^2 + 3.18\Pi + 2) \tag{6.248c}$$

where in Equation 6.248b use has been made of the integral:

$$Si(\pi) = \int_0^\pi \frac{\sin z}{z}\,dz = 1.8519$$

Substituting u_e/u_τ from Equation 6.248a in Equation 6.248b, we get:

$$\kappa(H - 1)(\Pi + 1)u_e = Hu_\tau(1.5\Pi^2 + 3.18\Pi + 2) \tag{6.248d}$$

where $H = \delta^*/\theta$ is the *shape parameter*. From Equation 6.245 the coefficient of friction c_f implicitly depends on $\Pi(x)$ according to:

$$\sqrt{2/c_f} = \frac{1}{\kappa}\ell n R_\tau + C + \frac{2\Pi(x)}{\kappa} \tag{6.248e}$$

where:

$$c_f = \frac{2u_\tau^2}{u_e^2}, \quad R_\tau = \frac{\delta u_\tau}{\nu}, \quad C = 4.9, \quad \kappa = 0.41$$

The formulae given in Equations 6.248 are valid for boundary layers with or without pressure gradients. It is obvious from Equation 6.248d that the determination of $\Pi(x)$ depends on knowledge of H and c_f which may be either prescribed or determined by other means. However, in the case of boundary layer on a flat plate ($u_e = u_\infty$) under high Reynolds numbers, i.e., $R_\theta > 5000$, the parameter Π is a constant. Its value is obtained simply by evaluating Equation 6.245 at $y = \delta$ and comparing with Equation 6.244. Thus, $\Pi = 0.533$, although Coles recommends the value $\Pi = 0.55$. If $R_\theta < 5000$, then according to Coles (refer to Reference 16) the variation of Π with R_θ is approximated as:

$$\Pi = 0.55[1 - \exp(-0.243\sqrt{z_1} - 0.298z_1)]$$

where:

$$z_1 = (R_\theta/425 - 1)$$

Knowing Π for a zero pressure gradient boundary layer, it is possible to list a number of formulae for this case. All the results given below are for $\Pi = 0.55$ and $\kappa = 0.41$. From Equation 6.248c:

$$\frac{R_\theta}{R_\delta} = 3.78\sqrt{c_f/2} - 25.0(c_f/2) \qquad \textbf{(6.249a)}$$

Eliminating R_δ between Equations 6.244 and 6.249a, we get:

$$\sqrt{2/c_f} = \frac{1}{\kappa}\,\ell n R_\theta - \frac{1}{\kappa}\,\ell n(1 - 6.6138\sqrt{c_f/2}) + 4.2568 \qquad \textbf{(6.249b)}$$

where $c_f = 2u_\tau^2/u_\infty^2$. It can be verified that for a given R_θ the simpler expression:

$$c_f = 0.0131 R_\theta^{-1/6} \qquad \textbf{(6.249c)}$$

yields about the same values of c_f as given by Equation 6.249b. Thus Equation 6.249c can be used in the momentum integral equation for a flat plate:

$$\frac{d\theta}{dx} = \frac{1}{2}c_f$$

to have:

$$R_\theta = 0.01533 R_x^{6/7}$$

or:

$$\theta = 0.01533 x R_x^{-1/7} \qquad \textbf{(6.249d)}$$

representing θ as a function of x for a flat plate.

Example 6.6

(a) For a flat plate boundary layer show that:

(b) If at a point on a plate $R_{\delta^*} = 1.6(10^4)$, find the values of c_f, R_δ, R_θ, and H at this location.

The answer of part a comes directly from Equation 6.248d by taking $\Pi = 0.55$ and $\kappa = 0.41$.

To answer part b we start from Equation 6.248a written as:

$$\frac{R_{\delta^*}}{R_\tau} = \frac{1 + \Pi}{\kappa}, \qquad R_\tau = \frac{\delta u_\tau}{\nu}$$

Taking $\Pi = 0.55$ and $\kappa = 0.41$, we get:

$$R_\tau = 4232.26$$

Using Equation 6.248e we get:

$$c_f = 0.00256$$

Then:

$$R_\tau = R_\delta \sqrt{\frac{c_f}{2}}$$

hence:

$$R_\delta = 118295.26$$

From Equation 6.249a:

$$R_\theta = 12212.49$$

Also:

$$H = 1.31$$

Besides the zero pressure gradient case the other case in which the value of Π can be established without knowing H and c_f is that of the *equilibrium* or *self-preserving* flows. These types of flows are the turbulent counterpart of the Falkner-Skan flows in laminar boundary layers. Equilibrium boundary layers have been investigated by Clauser,[53] and they form a class of boundary layers in which the pressure gradient is such that the parameter β defined as:

$$\beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}}{dx}$$

remains a constant for every position on the body surface. Further, when the velocity defect $(u_e - \bar{u})/u_\tau$ is plotted vs. y/δ, then the solutions for all x-positions fall on a single curve. Thus the equilibrium boundary layers form a one-parameter family of velocity profiles of the form

$$\frac{u_e - \bar{u}}{u_\tau} = f\left(\frac{y}{\delta}; \beta\right)$$

For equilibrium boundary layers $\Pi = \Pi(\beta)$, and this functional dependence can be established empirically.[54]

Another empirical formula for correlating the velocity distribution with the experimental data for a turbulent boundary layer has been proposed by Whitfield.[55] In terms of c_f and R_θ it is stated as:

$$\frac{\bar{u}(\eta)}{u_e} = A \tan^{-1}(\Gamma\eta) + B[\tanh(b\eta^*)]^{1/2} \tag{6.250}$$

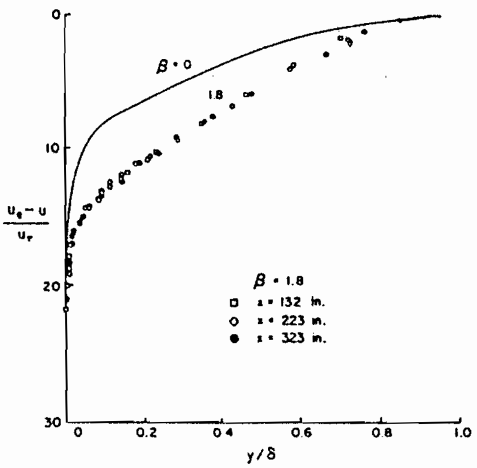


Fig. 6.14 Equilibrium turbulent boundary layers for $\beta = 0$ and $B = 1.8$. (Taken from Cebeci, T. and Smith, A. M. O., *Analysis of Turbulent Boundary Layers*, Academic Press, Inc., 1974. With permission.)

where:

$$\eta = \frac{y}{\theta}, \quad \Gamma = \alpha_0^2 R_\theta \sqrt{c_f/2}, \quad A = R_\theta c_f/2\Gamma, \quad B = 1 - \frac{A\pi}{2}, \quad \alpha_0 = 0.3$$

The first term takes care of a part of the inner layer including the sublayer (refer to example 6.3), and thus Equation 6.250 is valid right from the wall to the outer edge of the boundary layer. (In contrast, recall that the Coles correlation [Equation 6.247] is not applicable to the sublayer.) Given c_f, H, and R_θ the velocity profile at any point on the body surface can be determined if parameters a and b can be computed. This is done by forming two equations from Equation 6.250 by evaluating it at $\eta = 2$ and $\eta = 5$ and using the following empirical formulae:

$$\frac{\bar{u}(2)}{u_e} = 1.723e^{-0.6H}\left(1 + \frac{50}{R_\theta}\right)$$

$$\frac{\bar{u}(5)}{u_e} = 0.87 + 0.08e^{-2.6(H-1.95)^2}$$

where $\bar{u}(2) = (\bar{u})_{y=2\theta}$, etc. Refer to Problem 6.18 for the computation of a and b. Although Equation 6.250 correlates the experimental data both for favorable and adverse pressure gradients very efficiently, it suffers a defect in that it cannot be used in calculation of the Reynolds stresses through the momentum equation* (refer to Problem 6.16). Further, it does not yield the desired second derivative of velocity at the wall. These difficulties can be circumvented by introducing exponential factors whose effects diminish rapidly with η. A modified form of Equation 6.250 can be

$$\frac{\bar{u}(\eta)}{u_e} = A\tan^{-1}(\Gamma\eta) + B(1 - e^{-\Gamma^2\eta^2})[\tanh(b\eta^a)]^{1/2} + \Gamma_1\eta^2 e^{-\Gamma\eta/\alpha_0^{1.2}} \qquad \textbf{(6.251)}$$

where:

* Both first and second derivatives are singular at $\eta = 0$ if $a < 2$, and the second derivative is singular at $\eta = 0$ if a lies in the range $2 < a < 4$.

$$\Gamma_1 = \frac{\beta R_\theta c_f}{4H}, \qquad \beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}}{dx}$$

Note that Equation 6.251 depends explicitly on β.

Numerical computations were performed by evaluating both a and b by using the method detailed in Problem 6.18. In this process it was found that the expression for $\bar{u}(2)$ given as:

$$\frac{\bar{u}(2)}{u_e} = \frac{1}{2}\left[1.723e^{-0.6H}\left(1 + \frac{50}{R_\theta}\right) + \left\{\frac{2(H-1)}{H(H+1)}\right\}^{(H-1)/2}\right]$$

gives slightly better results than the one proposed earlier. The correlated velocity profiles using Equation 6.251 are about the same as obtained by using Equation 6.250.

In cases where only H and R_θ have been provided, one may use the Ludweig-Tillman formula[56] to compute c_f:

$$c_f = 0.246(10)^{-0.678H}R_\theta^{-0.268} \tag{6.252}$$

Law of the Wall for Compressible Flow

We briefly consider the law of the wall in high-speed compressible turbulent flow of air in the vicinity of a solid wall. Accordingly, the results here pertain to the inner region of a compressible turbulent boundary layer flow. The mechanics of flow in the near wall region is assumed to follow the gradient law behavior according to Equation 6.241c but with variable density. Thus, in the turbulent core region:

$$\frac{\partial \bar{u}}{\partial y} = \frac{1}{\kappa y}\sqrt{\tau_w/\bar{\rho}} = \frac{u_\tau}{\kappa y}\sqrt{\rho_w/\bar{\rho}} \tag{6.253a}$$

where ρ_w is the density at the wall and $u_\tau = \sqrt{\tau_w/\rho_w}$ is the friction velocity. In the literature, e.g., Cebeci and Smith[16]; Equation 6.253a is interpreted as a mixing length model and is known as Van Driest-I.[56] In Van Driest-I the Crocco energy integral[16] has been taken to provide the needed functional relationship between velocity and density. The Crocco energy integral stated in terms of the total enthalpy H is

$$\frac{H - H_w}{H_e - H_w} = \frac{\bar{u}}{u_e}$$

which can be rewritten as:

$$\frac{\bar{T}}{T_w} = 1 + B\left(\frac{\bar{u}}{u_e}\right) - A^2\left(\frac{\bar{u}}{u_e}\right)^2 \tag{6.253b}$$

Here w and e denote the wall and the external edge of the boundary layer, and:

$$A = u_e(2C_pT_w)^{-1/2}$$

$$B = \frac{T_{aw}}{T_w} - 1$$

where T_{aw} is the adiabatic wall temperature given by

$$T_{aw} = T_e + r\frac{u_e^2}{2C_p}$$

where r is the recovery factor. However, the Crocco integral exists under the condition $P_r = 1$ and $r = 1$, and thus the T_{aw} appearing in B is that in which $r = 1$.

Introducing the speed of sound $a_e = \sqrt{\gamma R T_e}$ and $M_e = u_e/a_e$, Van Driest found that a better matching with the data can be achieved by taking

$$A = \left(\frac{\gamma - 1}{2} M_e^2 \frac{T_e}{T_{aw}}\right)^{1/2}$$

$$B = \frac{T_e}{T_w}\left(1 + r\frac{\gamma - 1}{2} M_e^2\right) - 1$$

Under the boundary layer approximation $\partial p/\partial y \cong 0$, and therefore the equation of state for a perfect gas is

$$\frac{\rho_w}{\bar{\rho}} = \frac{\bar{T}}{T_w} \tag{6.253c}$$

Using Equations 6.253b, c in Equation 6.253a and performing the integration we get:

$$\frac{U^*}{u_\tau} = \frac{1}{\kappa}\ell n y^+ + C \tag{6.254a}$$

where $y^+ = y u_\tau/\nu_w$ and U^* is a generalized velocity defined as

$$U^* = \frac{u_e}{A}\left[\sin^{-1}\left(\frac{2A^2\dfrac{\bar{u}}{u_e} - B}{(B^2 + 4A^2)^{1/2}}\right) + \sin^{-1}\left(\frac{B}{(B^2 + 4A^2)^{1/2}}\right)\right] \tag{6.254b}$$

Thus, in the compressible turbulence case the generalized velocity U^* given by Equation 6.254b, rather than the actual velocity $\bar{u}$, follows a logarithmic law in the turbulent core region.

Maise and McDonald[57] have extended Coles' correlation, Equation 6.245, to the compressible case by using the equation

$$(U_e^* - U^*)/u_\tau = -2.5\,\ell n(y/\delta) + 1.25[2 - w(y/\delta)]$$

where U_e^* is the value of U^* at $y = \delta$. This equation seems to correlate the experimental data up to $M_e \cong 5$ quite well. However, when the actual velocity defect is plotted with y/δ, then the error increases with increasing Mach number M_e.

6.20 MOMENTUM INTEGRAL METHODS IN BOUNDARY LAYERS

Either Equation 6.247, 6.250, or 6.251 can be used to generate the velocity distribution at any point if the quantities R_θ, H, and c_f can be calculated by an independent method. In this regard the integral method of boundary layer solution seems to be the best possible choice. We have already introduced a momentum integral method, called the Loitsyanskii-Thwaites method, for laminar boundary layers in Chapter 5, Part II. The attractive feature of the method is that the whole problem of boundary layer solution reduces to a simple numerical quadrature. Here we shall first explore the possbility of using a method similar to that discussed in Chapter 5 for turbulent boundary layers. Besides this method we shall briefly consider the methods of Truckenbrodt and Head. For a detailed exposition on the momentum integral methods refer to Schlichting[4] and Loitsyanskii.[58]

In the case of laminar flow we introduced the *form* parameter λ defined in Equation 5.159a which can also be written as:

$$\lambda = \frac{u_e'\theta}{u_e} R_\theta$$

In the same manner the wall shear function ζ defined in Equation 5.161c can be rewritten as:

$$\zeta = \frac{\tau_w}{\rho u_e^2} R_\theta$$

In the above forms both λ and ζ are exhibited with an explicit factor of R_θ. In the case of turbulent flow we envisage these parameters with an explicit factor of $G(R_\theta)$ as:

$$\lambda = \frac{u_e'\theta}{u_e} G(R_\theta) \tag{6.255a}$$

$$\zeta = \frac{\tau_w}{\rho u_e^2} G(R_\theta) \tag{6.255b}$$

and, as in the laminar case, assume the shape parameter H as:

$$H = H(\lambda)$$

In Equations 6.255, $G(R_\theta)$ is a function of R_θ and has to be specified.

We consider the momentum integral equation (refer to Problem 6.17) for steady flow and multiply each term by $G(R_\theta)$ to have:

$$G(R_\theta)\frac{d\theta}{dx} + (2 + H)\lambda = \zeta \tag{6.256a}$$

Now consider the identity:

$$\frac{d}{dx}\left(\frac{\lambda u_e}{u_e'}\right) = \frac{d}{dx}(\theta G)$$

Opening the differentation on the right-hand side and introducing:

$$m(R_\theta) = \frac{G' R_\theta}{G}$$

we get:

$$(1 + m)G\frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{\lambda u_e}{u_e'}\right) - m\lambda \tag{6.256b}$$

Substituting $G\, d\theta/dx$ from Equation 6.256b in Equation 6.256a, we get:

$$\frac{d\lambda}{dx} = \frac{u_e'}{u_e} F(\lambda) + \frac{u_e''}{u_e'}\lambda \tag{6.257}$$

where:

$$F(\lambda) = (1 + m)\zeta - [3 + m + (1 + m)H]\lambda \tag{6.258}$$

From Equation 6.258 it is obvious that for $m = 1$ we recover the laminar flow form of F, i.e., Equation 5.160b.

Similar to the laminar case, we set:

$$\lambda = u_e' Z \tag{6.259a}$$

where:

$$Z = \frac{\theta G(R_\theta)}{u_e} \tag{6.259b}$$

On substituting Equation 6.259a in Equation 6.257, we get:

$$\frac{dZ}{dx} = \frac{F}{u_e} \tag{6.260}$$

which is exactly the same as Equation 5.160a but with F as defined in Equation 6.258.

As in the case of laminar boundary layers, *if* $F(\lambda)$ can be expressed as a linear function of the parameter λ, then Equation 6.260 can be integrated exactly. Let:

$$F(\lambda) = A - B\lambda$$

where A and B are constants; then Equation 6.260 becomes:

$$\frac{d}{dx}(Zu_e^B) = Au_e^{B-1}$$

Thus:

$$Z(x) = \frac{A}{u_e^B}\int_{x_{tr}}^{x} u_e^{B-1}(\xi)\, d\xi + \frac{C_1}{u_e^B} \tag{6.261}$$

where x_{tr} is the value of x at the transition point. The constant C_1 is then obtained as:

$$C_1 = u_e^{B-1}(x_{tr})[\theta G(R_\theta)]_{tr}$$

The success of the integral method discussed above depends on how well $F(\lambda)$ can be approximated as a linear function of λ. To analyze this aspect and to see clearly the limitations involved in a linear representation of $F(\lambda)$, we consider two power law variations for the function $G(R_\theta)$:

1. $G(R_\theta) = R_\theta^{1/6}$
2. $G(R_\theta) = R_\theta^{1/4}$

Corresponding to case 1 we see from Equation 6.249c that for a flat plate:

$$\frac{\tau_w}{\rho u_e^2} = 0.00655 R_\theta^{-1/6}$$

while corresponding to the case 2 we refer to Schlichting,[4] and have:

$$\frac{\tau_w}{\rho u_e^2} = 0.0128 R_\theta^{-1/4}$$

Thus, for case 1: $\zeta = 0.00655$, $m = 1/6$, and:

$$F(\lambda) = 0.0076 - \left(\frac{19}{6} + \frac{7H}{6}\right)\lambda$$

For case 2: $\zeta = 0.0128$, $m = 1/4$, and:

$$F(\lambda) = 0.016 - \left(\frac{13}{4} + \frac{5H}{4}\right)\lambda$$

In either case it is apparent that a constant H has to be specified to obtain a constant B. Referring to Reference 4, we note that Buri in 1931 determined the numerical values for A and B based on Nikuradse's data. Based on values for both the favorable and unfavorable pressure gradients, Schlichting suggests the choice:

$$A = 0.016, \qquad b = 5.0$$

which is consistent with the 1/4th power law and for $H \cong 1.4$. The form of G in case 2 and the values of the constants quoted above yield values of θ close to the experimental data for favorable and mildly adverse pressure gradients. The method does not yield correct results under adverse pressure gradient conditions. Separation occurs at $\lambda = -.06$. As Schlichting has noted, it is remarkable that in spite of using the shear stress formula for a flat plate and a constant value of H, the method still predicts acceptable values of θ up to mildly adverse pressure gradients.

The main reason for failure of the power law method for adverse pressure gradients is that the *shape parameter* $H = \delta^*/\theta$ increases as the separation point is approached, thus disabling the use of a constant value of B. The variation of the shape factor H in turbulent boundary layer flow is usually in the range $1.125 < H < 2.6$. Separation occurs in the range $1.8 < H < 2.6$. The flat plate boundary layer has $H \cong 1.3$. On the other hand, in the laminar boundary layer $2.2 < H < 3.5$, with a definitive value $H = 2.59$ for a flat plate boundary layer.

The preceding method yields the values of Z which depend on C_1. The constant C_1 appearing in Equation 6.261 depends on the value $x = x_{tr}$ at which transition to turbulence occurs. This value may either be given based on some experimental data or be calculated from the empirical formula Equation 6.45:

$$(R_\theta)_{tr} = 1.174\left(1 + \frac{22400}{(R_x)_{tr}}\right)(R_x)_{tr}^{0.46}$$

Effective use of this formula can be made when R_θ is known or when R_θ and R_x can be functionally related through the laminar solution. Using this formula, the corresponding $(R_x)_{tr}$ and hence x_{tr} can be calculated.

Method of Truckenbrodt

The method of Truckenbrodt[50a,b] is based on both the momentum integral and the energy integral equations. The energy integral equation is derived as follows. To obtain the steady energy integral equation for turbulent boundary layers* we have to consider both the equation of

* In contrast to the momentum integral equation, the energy integral equation is different for laminar and turbulent boundary layers.

momentum (Equation 6.205) and the equation of energy (Equation 6.207). In the present nomenclature we replace $\bar{u}_1$, $\bar{u}_2$, u_1', u_2', x_1, and x_2 in the equations mentioned above by $\bar{u}$, $\bar{v}$, u', v', x, and y, respectively. Multiplying Equation 6.205 by $\bar{u}$ throughout and combining terms, we get:

$$\bar{u}\frac{\partial}{\partial x}\left(\frac{\bar{u}^2}{2}\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{\bar{u}^2}{2}\right) - \bar{u}\frac{d}{dx}\left(\frac{u_e^2}{2}\right) - \overline{u'v'}\frac{\partial \bar{u}}{\partial y}$$
$$+ \frac{\partial}{\partial y}(\bar{u}\overline{u'v'}) - \nu\frac{\partial^2}{\partial y^2}\left(\frac{\bar{u}^2}{2}\right) + \nu\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = 0 \qquad \textbf{(6.262a)}$$

Adding Equations 6.262a and 6.207, we have:

$$\bar{u}\frac{\partial E}{\partial x} + \bar{v}\frac{\partial E}{\partial y} - \bar{u}\frac{d}{dx}\left(\frac{u_e^2}{2}\right) + \frac{\partial}{\partial y}\left\{\bar{u}\overline{u'v'} + \overline{v'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\}$$
$$= \nu\frac{\partial^2 E}{\partial y^2} - Y \qquad \textbf{(6.262b)}$$

where:

$$E = K + \frac{\bar{u}^2}{2}$$

and:

$$Y = \nu\left(\frac{\partial \bar{u}}{\partial y}\right)^2 + \epsilon$$

is the total viscous dissipation. Integrating Equation 6.262b with respect to y from $y = 0$ to δ, and using the condition:

$$(K)_{y=\delta} = 0$$

we get:

$$\frac{d}{dx}(u_e^3\Theta) = 2\int_0^\delta Y\,dy + 2\frac{d}{dx}\int_0^\delta \bar{u}K\,dy$$

where the energy thickness Θ is

$$\Theta = \int_0^\delta \frac{\bar{u}}{u_e}\left(1 - \frac{\bar{u}^2}{u_e^2}\right)dy$$

Schlichting (Reference 4, p. 635) introduces the following notation:

$$d_1 = \int_0^\delta Y\rho\,dy$$

$$= \text{rate of dissipation of energy per unit volume}$$

and:

$$t_1 = \frac{d}{dx}\int_0^\delta \rho \bar{u} K \, dy$$

= the rate of energy convection per unit volume

In these notations the above equation becomes:

$$\frac{1}{u_e^3}\frac{d}{dx}(u_e^3\Theta) = \frac{2(d_1 + t_1)}{\rho u_e^3} \tag{6.263}$$

Truckenbrodt uses energy integral Equation 6.263 and the previous work of Wieghardt and Tillman[60] and of Rotta.[61] Weighardt and Tillman have shown that by using the power law:

$$\frac{\bar{u}}{u_e} = \left(\frac{y}{\delta}\right)^n$$

a relation between the energy thickness Θ and the momentum thickness θ of the form:

$$\hat{H} = \frac{\Theta}{\theta} = \frac{\alpha H}{\beta + H} \tag{6.264a}$$

can be established. Here α and β are constants, and based on experimental data their values are

$$a = 1.269, \qquad \beta = -0.379$$

Rotta has shown that the energy dissipation d_1 is related to R_θ as:

$$\frac{d_1}{\rho u_e^3} = \frac{0.56 \times 10^{-2}}{R_\theta^{1/6}} \tag{6.264b}$$

Truckenbrodt now neglects t_1 in comparison with d_1, so that Equation 6.263 becomes:

$$\frac{1}{u_e^3}\frac{d}{dx}(\hat{H}\theta u_e^3) = \frac{1.12 \times 10^{-2}}{R_\theta^n} \tag{6.265}$$

where $n = 1/6$. Now two cases arise: one in which both H and $\hat{H}$ are assumed as constants and the other in which it is not so.

Case I: $\hat{H}$ = Constant. In this case Equation 6.265 becomes:

$$\frac{1}{u_e^3}\frac{d}{dx}(\theta u_e^3) = \frac{1.12 \times 10^{-2}}{\hat{H}R_\theta^n}, \qquad n = \frac{1}{6}$$

Opening the differentiation on the left and considering the term $d/dx\ (\theta R_\theta^n)$, we obtain:

$$\frac{d}{dx}(\theta R_\theta^n) + (2n + 3)\theta R_\theta^n \frac{u_e'}{u_e} = \frac{1.12 \times 10^{-2}}{\hat{H}}(1 + n)$$

which is a linear ordinary differential equation in θR_θ^n. Its integral is then:

$$\theta R_\theta^n u_e^{2n+3} = \frac{1.12 \times 10^{-2}}{\hat{H}} (n + 1) \int_{x_t}^{x} u_e^{2n+3}(\xi)\, d\xi + C \tag{6.266}$$

Setting $n = 1/6$ we have:

$$\theta R_\theta^{1/6} u_e^{10/3} = \frac{7(1.12 \times 10^{-2})}{6\hat{H}} \int_{x_t}^{x} u_e^{10/3}(\xi)\, d\xi + C$$

The constant C is to be evaluated by the values of θ and u_e at the transition point. Taking the average value of H as 1.4, the value of $\hat{H}$ from Equation 6.264a is 1.73. With this value the coefficient of the integral in the above equation is 0.0076.

The formula in Equation 6.266 for the calculation of θ is quite satisfactory for the case of favorable pressure gradients. However, after the point of zero pressure gradient, values of both H and H change quite significantly, particularly as the separation point is approached. The analysis of Truckenbrodt given below thus pertains to the region starting from the zero pressure gradient all through the adverse pressure gradient regime.

Case II: $\hat{H}$ not a Constant. In this case both the momentum integral and the energy integral equations have to be used. First, writing:

$$\Theta = \theta\hat{H}$$

in Equation 6.263, we obtain:

$$\theta \frac{d\hat{H}}{dx} + \hat{H}\frac{d\theta}{dx} + \frac{3\theta\hat{H}}{u_e} u_e' = \frac{2(d_1 + t_1)}{\rho u_e^3}$$

Substituting $d\theta/dx$ from the momentum integral equation:

$$\frac{d\theta}{dx} + \frac{\theta u_e'}{u_e}(2 + H) = \frac{\tau_w}{\rho u_e^2}$$

we get:

$$\theta\frac{d\hat{H}}{dx} - (H - 1)\frac{\theta\hat{H}}{u_e} u_e' + \frac{\hat{H}\tau_w}{\rho u_e^2} = \frac{2(d_1 + t_1)}{\rho u_e^3}$$

Multiplying each term by R_θ^n and using the following abbreviations:

$$\lambda = \frac{\theta u_e'}{u_e} R_\theta^n$$

$$f_1(\hat{H}) = (H - 1)\hat{H}$$

$$f_2(\hat{H}) = \left[\frac{2(d_1 + t_1)}{\rho u_e^3} - \frac{\hat{H}\tau_w}{\rho u_e^2}\right] R_\theta^n \tag{6.267}$$

we have:

$$\theta R_\theta^n \frac{d\hat{H}}{dx} = \lambda f_1(\hat{H}) + f_2(\hat{H}) \tag{6.268}$$

Truckenbrodt now introduces another shape factor $L(\hat{H})$ through the equation:

$$L(\hat{H}) = \int_{\hat{H}_p}^{\hat{H}} \frac{d\hat{H}}{f_1(\hat{H})}$$

where $\hat{H}_p$ is the value of $\hat{H}$ at the point of zero pressure gradient. Taking $H_p = 1.268$ the value of $\hat{H}_p$ from Equation 6.264a is $\hat{H}_p \sim 1.81$. Then:

$$\frac{dL}{dx} = \frac{dL}{d\hat{H}}\frac{d\hat{H}}{dx} = \frac{1}{f_1(\hat{H})}\frac{d\hat{H}}{dx}$$

hence, Equation 6.268 becomes:

$$\theta R_\theta^n \frac{dL}{dx} = \lambda + \overline{K}(L) \tag{6.269}$$

where:

$$\overline{K}(L) = f_2(\hat{H})/f_1(\hat{H})$$

Using the wall shear formula in Equation 6.252 and Equation 6.264b in the third equation of Equation 6.267, we find that for $n = 1/6$ the function $\overline{K}(L)$ can be approximated as a linear function of L (refer to Reference 4) as:

$$\overline{K}(L) = a(L - b) \tag{6.270}$$

where:

$$a = 0.0304$$

$$b(x) = 0.0304\,\ell n R_\theta - 0.23$$

so that:

$$R_\theta = e^{\chi(x)}$$

where:

$$\chi(x) = \frac{b(x) + 0.23}{0.0304}$$

Using the R_θ given above, we form a new function $\phi(x)$ as:

$$\theta R_\theta^n = \frac{\nu}{u_e} e^{(n+1)\chi} = \phi(x) \tag{6.271a}$$

As an approximation we use Equation 6.266 for θR_θ^n, and obtain:

$$\phi(x) = \left[A(n)\int_{x_1}^{x} u_e^{2n+3}(\xi)\, d\xi + C\right] \Big/ u_e^{2n+3} \tag{6.271b}$$

where:

$$A(n) = \frac{1.12 \times 10^{-2}(n + 1)}{\hat{H}}$$

and consequently with $\hat{H} = 1.73$:

$$A(\tfrac{1}{6}) = 0.0076$$

Given the velocity $u_e(x)$, Equations 6.271 establish the values of $b(x)$ for each x. Now, using Equation 6.270 in Equation 6.269, we get:

$$\frac{dL}{dx} - \frac{a}{\phi}L = \frac{u_e'}{u_e} - \frac{ab(x)}{\phi}$$

Defining the transformation:

$$\eta = \exp\left(-\int \frac{a}{\phi}\,dx\right)$$

the above equation transforms to:

$$\frac{d}{d\eta}(\eta L) = \frac{d}{d\eta}(\eta \ell n u_e) + b(\eta) - \ell n u_e$$

On integration we get:

$$L(\eta) = \frac{\eta_t}{\eta}L_t + \ell n \frac{u_e(\eta)}{u_e(\eta_t)} + \frac{1}{\eta}\int_{\eta_t}^{\eta}\left[b(\xi) - \ell n \frac{u_e(\xi)}{u_e(\eta_t)}\right] d\xi \qquad \textbf{(6.272)}$$

The algorithm formed by using Equations 6.271a, b and 6.272 solves the problem as formulated by Truckenbrodt.

Method of Head

Head[62] has developed a simpler method based on the momentum integral equation for the calculation of turbulent boundary layers. The important concept in Head's method is that of *entrainment velocity*. Entrainment velocity is defined as the velocity normal to the boundary layer edge directed inward as shown in Figure 6.15, and given by the expression:

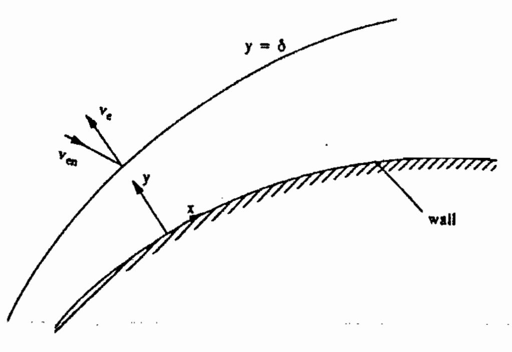


Fig. 6.15 Sketch showing the existence of the entrainment speed.

$$v_{en} = \frac{d}{dx}\int_0^{\delta} \bar{u}\, dy$$

This is a velocity defect at a position in the outer part of the boundary layer. The relation between v_e and v_{en} is given by the continuity equations. Since:

$$\bar{v} = -\int_0^{y} \frac{\partial \bar{u}}{\partial x}\, dy$$

then:

$$\bar{v}_e = -\int_0^{\delta} \frac{\partial \bar{u}}{\partial x}\, dy$$

Combining these equations, we get:

$$v_{en} = -v_e + u_e \frac{d\delta}{dx}$$

Another expression for v_{en} is obtained by differentiating the definition of displacement thickness:

$$u_e\delta^* = \int_0^{\delta} (u_e - \bar{u})\, dy$$

$$= u_e\delta - \int_0^{\delta} \bar{u}\, dy$$

Thus:

$$\frac{d}{dx}\{u_e(\delta - \delta^*)\} = v_{en} \tag{6.273a}$$

The starting point of Head's method is Equation 6.273a. Introducing the form parameter:

$$H_1 = (\delta - \delta^*)/\theta$$

in Equation 6.273a, we have:

$$\frac{v_{en}}{u_e} = \frac{1}{u_e}\frac{d}{dx}(u_e\theta H_1) \tag{6.273b}$$

Head now assumes that v_{en}/u_e is a function of the form parameter H_1, i.e.:

$$\frac{d}{dx}(u_e\theta H_1) = u_e F(H_1) \tag{6.274}$$

Besides Equation 6.274 we also have the momentum integral equation:

$$\frac{d\theta}{dx} + (2 + H)\frac{\theta u_e'}{u_e} = \frac{1}{2}c_f \tag{6.275}$$

Head's method rests on solving Equations 6.274 and 6.275 simultaneously. These two equations have four unknowns: θ, H, c_f, and H_1. First of all, Head assumed that:

$$H_1 = G(H)$$

Next, Equation 6.252 provides a relation between c_f, H, and R_θ. If the function $F(H_1)$ appearing in Equation 6.274 is also specified, then a closed system of equations becomes available. In Reference 11 the following empirical relations have been stated:

$$F(H_1) = 0.0306(H_1 - 3.0)^{-0.6169}$$

$$H_1 = G(H) = 0.8234(H - 1.1)^{-1.287} + 3.3, \qquad H \leqslant 1.6$$

$$= 1.5501(H - 0.6778)^{-3.064} + 3.3, \qquad H \geqslant 1.6$$

Solution of ordinary differential Equations 6.274 and 6.275 by using the empirical functions given above can be obtained by numerical methods. A computer program on Head's method is available in Cebeci and Bradshaw.[11]

6.21 DIFFERENTIAL EQUATION METHODS IN TWO-DIMENSIONAL BOUNDARY LAYERS

With the advent of high speed digital computers the numerical methods of solution of the coupled system of boundary layer equations are among the most efficient and accurate methods for the prediction of boundary layers. Although integral methods have slowly been replaced by differential methods, the former are still the most inexpensive methods for estimating momentum and displacement thicknesses and local skin friction coefficient. In some cases even a hand calculator is all one needs to obtain these parameters.

In this section we shall discuss two representative methods for solving boundary layer equations by the method of finite difference. For a series of other solutions the reader is referred to Kline et al.[63] An important concept which has to be developed before discussing any method of boundary layer solution is that of "intermittency" at the outer edge of the boundary layer. The outer edge of a boundary layer is not a single sharply defined edge curve for all instants. There are intermittent regions of turbulent and nonturbulent flows in the extreme outer parts of a boundary layer as shown in Figure 6.16.

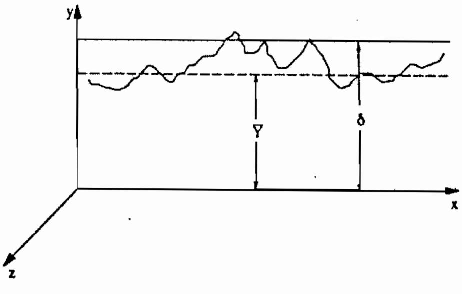


Fig. 6.16 Intermittent behavior at the outer edge of a boundary layer.

The analysis to be described is due to Klebanoff[64] and is based on the boundary layer measurements on a flat plate. Let the instantaneous distance from the wall to the free-stream boundary be denoted by $Y(x, z, t)$. Thus, Y is a random variable and is statistically stationary in z and t. The intermittency factor $\gamma(x, y)$ is the fraction of time the flow is turbulent at a particular position. Thus:

$$\gamma(x, y) = \frac{\text{time for which the flow is turbulent}}{\text{total time of observation}}$$

and $\gamma(x, y)$ is the probability that the boundary layer edge is at a distance greater than Y. Obviously, $\gamma(x, y)$ decreases to zero for large values of y and tends to one in the main boundary layer. Thus, $-\partial\gamma/\partial y > 0$ is the probability density, and the average location of the free-stream boundary is

$$\bar{Y}(x) = \int_0^\infty -y\frac{\partial\gamma}{\partial y}\,dy = \int_0^\infty \gamma\,dy$$

The standard deviation is

$$\sigma(x) = \left[-\int_0^\infty (y - \bar{Y})^2 \frac{\partial\gamma}{\partial y}\,dy\right]^{1/2}$$

Experimental results due to Klebanoff have shown that $-\partial\gamma/\partial y$ has a Gaussian distribution, viz.:

$$-\frac{\partial\gamma}{\partial y} = \frac{1}{\sigma\sqrt{2\pi}}\exp[-(y - \bar{Y})^2/2\sigma^2]$$

and according to the measurements:

$$\bar{Y} = 0.78\delta, \qquad \sigma = 0.14\delta, \qquad \sqrt{2}\sigma = 0.2\delta$$

Hence:

$$-\frac{\partial\gamma}{\partial y} = \frac{1}{.2\delta\sqrt{\pi}}\exp\left[-25\left(\frac{y}{\delta} - 0.78\right)^2\right]$$

Letting:

$$\zeta = 5\left(\frac{y}{\delta} - 0.78\right)$$

we have:

$$\gamma = \frac{1}{\sqrt{\pi}}\int_\zeta^\infty e^{-\zeta^2}\,d\zeta = \frac{1}{2}\,\text{erfc}(\zeta) \tag{6.276}$$

Note that:

$$\text{erfc}(0) = 1, \qquad \text{erfc}(-\infty) = 2, \qquad \text{erfc}(\infty) = 0$$

For $\dfrac{y}{\delta} = 0$ (wall) : $\zeta = -3.9$, so that $\gamma \cong 1$;

for $\dfrac{y}{\delta} = 1.2 : \zeta = 2.1$, so that $\gamma \cong 0$.

The following algebraic formula has been used by Cebeci and Smith (refer to Reference 63) in lieu of the exact expression (Equation 6.276):

$$\gamma = \left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1} \tag{6.277}$$

Zero-Equation Modeling in Boundary Layers

We now consider the zero-equation modeling for boundary layer flows. In the zero-equation modeling only the continuity equation (Equation 6.204) and the momentum equation (Equation 6.205) are used. In two dimensions the specification of a single Reynolds stress component $\overline{u_1'u_2'}$ is done by considering the eddy viscosity μ_T for the inner and outer regions. As before, we shall use $\bar{u}$, $\bar{v}$, u', v', x, y in place of $\bar{u}_1$, $\bar{u}_2$, u_1', u_2', x_1 x_2, respectively.

The eddy viscosity is considered to be composed of an inner part $(\mu_T)_i$ and an outer part $(\mu_T)_0$. The inner part depends on a length scale having the effect of viscous damping as proposed by Van Driest[65] and is

$$\ell = \kappa y\{1 - \exp(-y^+/A^+)\} \tag{6.278}$$

$$y^+ = \frac{yu_\tau}{\nu}$$

$$\kappa = 0.41, \quad A^+ \approx 26$$

The outer eddy viscosity $(\mu_T)_0$ is composed of the y-independent part $\alpha\rho u_e\delta^*$ with $\alpha = 0.0168$, and multiplied by γ as defined in Equation 6.277. The explanation of this modeling comes from the experimental data of Klebanoff. These experiments have established that the vanishing of the outer eddy viscosity $(\mu_T)_0$ as the outer edge is approached is due to the intermittency factor γ. Therefore, if the effect of intermittency is factored out from $(\mu_T)_0$, then the y-independent $\alpha\rho u_e\delta^*$ will result.

The formulation of the zero-equation model to be discussed in this section is due to Cebeci and Smith.[16] Introducing the transformation:

$$x = x, \qquad \eta = y\sqrt{u_e/\nu x}, \qquad \psi = (\nu x u_e)^{1/2} f(x, \eta)$$

$$\nu_T = -\overline{u'v'}\Big/\frac{\partial \bar{u}}{\partial y}, \qquad \nu_T^+ = \nu_T/\nu, \qquad \beta = \frac{x}{u_e}\frac{du_e}{dx}$$

$$f' = \bar{u}/u_e, \qquad b = 1 + \nu_T^+$$

in the boundary layer equations, we get:

$$(bf'')' + \beta(1 - f'^2) + \frac{1+\beta}{2} ff'' = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right)$$

where a prime denotes differentiation with respect to η. The boundary conditions are

$$\text{at } \eta = 0: f' = 0, f = 0 \quad \text{(no mass transfer)}$$

or

$$f' = 0, \quad f = -\frac{1}{(\nu x u_e)^{1/2}}\int_0^x v_w(x)\, dx \quad \text{(with mass transfer)}$$

where ν_w is the wall suction or injection velocity, and

$$\text{as } \eta \to \infty : f' \to 1$$

The inner and outer specifications of ν_T for a fully developed turbulent flow are

$$(\nu_T)_i = \ell^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \tag{6.279a}$$

$$(\nu_T)_o = \gamma\alpha(R_\theta)u_e\delta^* \tag{6.279b}$$

where ℓ is defined in Equation 6.278 and γ is the intermittency factor defined in Equation 6.277. According to Reference 16:

$$\alpha(R_\theta) = 0.0168, \quad \text{if } R_\theta > 5000$$

$$= 0.0168 \frac{1.55}{1 + \Pi}, \quad \text{if } R_\theta < 5000$$

where:

$$\Pi = 0.55[1 - \exp(-0.243z_1^{1/2} - 0.298z_1)]$$
$$z_1 = R_\theta/425 - 1$$

The transformed boundary layer equation along with the above specifications has been solved by Keller's Box method which was discussed in detail in Chapter 5.

Taking into account the pressure gradient and also the effect of suction or blowing, the nondimensional damping length A^+ has been modified[16,66] as follows:

$$A^+ = 26\left[\exp(11.8v_w^+) - \frac{p^+}{v_w^+}\{1 - \exp(11.8v_w^+)\}\right]^{-1/2}$$

where:

$$p^+ = \frac{\nu}{\rho u_\tau^3}\frac{dp}{dx}, \qquad v_w^+ = \frac{v_w}{u_\tau}$$

For $v_w^+ = 0$:

$$A^+ = 26(1 + 11.8p^+)^{-1/2}$$

For a unified treatment of laminar, transitional, and turbulent flows, the eddy viscosity Equation 6.279 should be multiplied by a transition factor, e.g., one given in Equation 6.44.

One-Equation Model of Glushko

The basic model due to Glushko[67] depends on the assumption that diffusion and dissipation terms in energy Equation 6.207 are functions of K and an integral length scale ℓ via a Reynolds number:

$$R = \frac{\sqrt{K}\,\ell}{\nu}$$

The form of these functions was chosen on dimensional considerations. Glushko also assumed an eddy viscosity model for the Reynolds stress, viz.:

$$-\rho\overline{u'v'} = \rho\nu_T \frac{\partial \bar{u}}{\partial y}$$

Glushko assumed that

$$\frac{\nu_T}{\nu} = f(R)$$

The integral length scale was evaluated from the two-point correlation function of the longitudinal velocity components for a flat plate. The resulting form of ℓ was then assumed to follow the equation:

$$\frac{\ell}{\delta} = \phi\left(\frac{y}{\delta}\right)$$

where the function ϕ is defined in Equation 6.293. With ℓ so determined, Glushko expressed the function $f(R)$ as:

$$f(R) = \alpha R H(R)$$

where $H(R)$ has been defined later in Equation 6.294. With this specification:

$$-\rho\overline{u'v'} = \mu\alpha R H(R) \frac{\partial \bar{u}}{\partial y}$$

The modeling of dissipation and diffusion was done as follows:

$$\epsilon = \nu C\left\{1 + \frac{\nu_T(\kappa R)}{\nu}\right\}\frac{K}{\ell^2}$$

$$\frac{\partial}{\partial y}\left\{\nu\frac{\partial K}{\partial y} - \overline{v'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} = \frac{\partial}{\partial y}\left\{\nu\left[1 + \frac{\nu_T(\kappa R)}{\nu}\right]\frac{\partial K}{\partial y}\right\}$$

where:

$$\frac{\nu_T(\kappa R)}{\nu} = \alpha\kappa R H(\kappa R)$$

$$\kappa = 0.41$$

Beckwith and Bushnell[68] have programmed Glushko's model for numerical computation. They first transformed the equations as follows:

$$F = \frac{\bar{u}}{u_e}, \qquad E = \frac{K}{u_e^2}$$

$$M = 1 + \alpha R H(R) = 1 + \alpha H(R)\phi\sqrt{E}\, R_\delta$$

$$D = 1 + \alpha\kappa R H(\kappa R) = 1 + \alpha\kappa H(\kappa R)\phi\sqrt{E}\, R_\delta$$

where:

$$R_\delta = \frac{u_e \delta}{\nu}$$

Introducing the similarity type variables:

$$\eta(x, y) = \frac{u_e y}{\nu (2\xi)^m}$$

$$\xi(x) = \int_0^x \frac{u_e(x)}{\nu} \, dx$$

where m, in general, is a function of ξ. However, for m = constant the equations transform as follows:

1. Continuity

$$(2\xi)^{2m} \frac{\partial F}{\partial \xi} + \frac{\partial V}{\partial \eta} \frac{m(2\xi)^{2m} F}{\xi} = 0$$

$$V = \frac{\nu(2\xi)^{2m}}{u_e} F \frac{\partial \eta}{\partial x} + (2\xi)^m \frac{\bar{v}}{u_e}$$

2. Momentum

$$(2\xi)^{2m} F \frac{\partial F}{\partial \xi} + V \frac{\partial F}{\partial \eta} = \frac{(2\xi)^{2m}}{u_e} \frac{du_e}{d\xi} (1 - F^2) + \frac{\partial}{\partial \eta}\left(M \frac{\partial F}{\partial \eta}\right)$$

3. Energy

$$(2\xi)^{2m} F \frac{\partial E}{\partial \xi} + V \frac{\partial E}{\partial \eta}$$

$$= \frac{-2(2\xi)^{2m}}{u_e} \frac{du_e}{d\xi} EF + (M - 1)\left(\frac{\partial F}{\partial \eta}\right)^2 + \frac{\partial}{\partial \eta}\left(D \frac{\partial E}{\partial \eta}\right) - \frac{C(2\xi)^{2m}}{R_\delta^2} \frac{DE}{\phi^2}$$

The boundary conditions are

$$\eta = 0 : F = E = 0, \; V = 0$$

$$\eta \to \infty : F \to 1, \; E \to 0 \text{ or } E_e$$

Beckwith and Bushnell have used an implicit finite difference approximation resulting in a tridiagonal matrix form for the solution of the above equations. The values of the constants chosen by them are

$$\alpha = 0.2, \quad \kappa = 0.4, \quad C = 3.93, \quad R_0 = 110, \quad m = 0.5$$

Three-dimensional boundary layers are discussed in Part IV.

Part IV: Turbulence Modeling

6.22 GENERALIZATION OF BOUSSINESQ'S HYPOTHESIS

Our considerations in Part III have been confined to the simplest statement of Boussinesq's hypothesis and to its applications to the wall-bound flows and flows through pipes. These simple applications have brought forth the concept of viscous sublayer and of friction velocity. A generalization of Boussinesq hypothesis for both compressible and incompressible shear layers can be made along the lines of Stokes' law of friction discussed in Chapter 2. By a "shear layer", we mean a region of fluid with motion that can be analyzed only by using the complete Reynolds equations of motion. If the viscous and the turbulent diffusive terms in the equations of motion along streamwise direction are neglected, then it is called a "thin shear layer". Following Kolmogorov[69a] and referring to a Cartesian coordinate system, we postulate that the Reynolds stress tensor has the form:

$$-\rho\overline{u_i'u_j'} = A\delta_{ij} + \mu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

On contracting the indices, i.e., writing $j = i$ and adding on i from 1 to 3, we find that:

$$A = -\tfrac{2}{3}(\rho K + \mu_T\bar{\Delta}), \qquad \bar{\Delta} = \operatorname{div}\bar{u}$$

where:

$$K = \tfrac{1}{2}\overline{u_i'^2}$$

is the turbulence energy per unit mass. Thus for an incompressible flow:

$$-\rho\overline{u_i'u_j'} = -\tfrac{2}{3}\rho K\delta_{ij} + \mu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{6.280a}$$

Introducing the notation:

$$\bar{\tau}_{ij} = \overline{u_i'u_j'}, \qquad \bar{D}_{ij} = \tfrac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \qquad \nu_T = \mu_T/\rho$$

we have:

$$\bar{\tau}_{ij} = \tfrac{2}{3}K\delta_{ij} - 2\nu_T\bar{D}_{ij} \tag{6.280b}$$

where ν_T is the eddy kinematic viscosity. The tensor invariant form whose Cartesian components are given by Equation 6.280b is

$$\bar{\boldsymbol{\tau}} = \overline{\mathbf{u}'\mathbf{u}'} = \tfrac{2}{3}K\mathbf{I} - \nu_T\{\text{grad}\,\bar{\mathbf{u}}) + (\text{grad}\,\bar{\mathbf{u}})^T\} \tag{6.280c}$$

The Newtonian constitutive equations (Equations 6.280) have played an important role in the modeling of simple turbulent flows and were first suggested by Kolmogorov[69] in these forms. The structure of Equations 6.280 suggests that the principal axes of the Reynolds stress tensor $\bar{\tau}_{ij}$ are parallel to the principal axes of the mean rate-of-strain tensor $\bar{D}_{ij}$. However, experiments have suggested that, in general, this is not true. The reason is that $\bar{D}_{ij}$ can be changed instantaneously by an application of forces due to pressure, but the Reynolds stresses require some time to relax to the new values of $\overline{D}_{ij}$. It is, therefore, expected that Kolmogorov's constitutive equation (Equations 6.280) cannot yield the observed mean velocity field in those cases in which the mean flow streamlines undergo sudden curvature changes (such as in the case of recirculating flows, sudden contractions, or turbulent vortex flows).

To solve the flow equations, viz., Equations 6.153 and 6.154, one has to come up with a specification of μ_T in terms of the other averaged quantities. Depending on the level of modeling, the specification of μ_T is achieved based on physical, dimensional, and intuitive reasonings. The simplest of all these specifications is when μ_T depends only on the mean rate-of-strain. Thus, a direct generalization of Equation 6.224 is

$$\mu_T = \rho\ell^2(2\overline{D}_{ij}\overline{D}_{ij})^{1/2} \tag{6.281a}$$

or alternatively as:

$$\mu_T = \rho\ell^2|\overline{\omega}| \tag{6.281b}$$

where:

$$|\overline{\omega}| = (\overline{\omega}\cdot\overline{\omega})^{1/2}$$

is the vorticity magnitude of the mean flow. Also recall that:

$$2\overline{D}_{ij}\overline{D}_{ij} = \frac{\partial \overline{u}_i}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

Refer also to Equation 6.357 for another formulation.

In Equations 6.281 ℓ is a length scale which has to be specified, and after this specification the flow Equations 6.153 and 6.154 can be solved independently of any other turbulence equations. It must, however, be realized that the substitution of Equations 6.280 in the averaged equations of motion gives rise to a modified pressure p^*, where:

$$\overline{p}^* = \overline{p} + \tfrac{2}{3}\rho K$$

which is obtained as part of the solution. The turbulence modeling of the type in Equations 6.280 is called the zero-equation modeling.

Two alternatives which present themselves in the zero-equation modeling are

1. Use either Equation 6.281a or b throughout the shear layer but then come up with a proper specification of the length scale valid for the whole layer.

2. Use either Equation 6.281a or b only in the "inner region" of a shear layer, and then devise another specification for the "outer region". The inner and outer eddy viscosities are denoted as $(\mu_T)_i$ and $(\mu_T)_o$, respectively. The modeling of $(\mu_T)_o$ is not based on $(\mu_T)_i$ as was shown in Equations 6.279. Later, Baldwin and Lomax[70] modified Equations 6.279 for use in the calculation of thin layers which will be discussed in the sequel.

Specification of the Length Scale

Let n be the actual normal distance from a surface on which a shear layer is developing. The mixing length ℓ due to Van Driest[65] has been used with much advantage, because it takes due account of the effect of viscosity in the near wall region. Van Driest's formula in terms of the normal distance n is

$$\ell = \kappa n\{1 - \exp(-nu_\tau/26\nu)\}$$
$$= \kappa n\{1 - e^{-n/A}\}$$

Thus:

$$A = 26\nu/u_\tau = 26\mu\sqrt{\rho\tau_w}\ \cdot$$

is the damping length. Alternatively, written in the form of Equation 6.278:

$$\ell = \kappa n\{1 - \exp(-n^+/A^+)\} \tag{6.282}$$

where:

$$n^+ = \frac{nu_\tau}{\nu}$$
$$\kappa = 0.41$$
$$A^+ = 26.0$$

The Van Driest formula (Equation 6.282) is applicable only to the near wall region. In the turbulent core region Equation 6.282 reduces to Prandtl's mixing length, $\ell = \kappa n$. For regions which are outside the turbulent core region another formula for ℓ has also been used. Bradshaw[71] has developed an expression for the outer region, which although devised for boundary layers, may also be applied to thin shear layers. The formula is

$$\frac{\ell}{n_\delta} = 0.089 \tanh\left(4.6067 \frac{n}{n_\delta}\right) \tag{6.283}$$

where n_δ is the thickness of the shear layer. In the case of boundary layers, n_δ is the boundary layer thickness where boundary layer velocity matches with the external flow to within a specified tolerance. In the case of averaged Navier-Stokes' solutions, a convenient way to obtain n_δ is to make a check on the values of vorticity while traversing out of the shear layer. If the vorticity magnitude is less than a prescribed tolerance, the normal distance from the wall is taken as n_δ. The common point of applicability of the two formulae can be obtained by numerical computation. Let n_m be the matching point, then Equation 6.282 is applicable in the range $0 \leqslant n \leqslant n_m$, while Equation 6.283 is applicable in the range $n_m \leqslant n \leqslant n_\delta$. Thus the complete specification of ℓ is

$$\begin{aligned}\ell &= \kappa n\{1 - \exp(-n^+/A^+)\}, & 0 \leqslant n \leqslant n_m\\ &= 0.089 n_\delta \tanh\left(4.6067 \frac{n}{n_\delta}\right), & n_m \leqslant n \leqslant n_\delta\\ &= 0.089 n_\delta & n > n_\delta\end{aligned} \tag{6.284}$$

In normal circumstances $n_m \sim 0.1 n_\delta$. A typical ℓ distribution in a boundary layer has been shown in Figure 6.17.

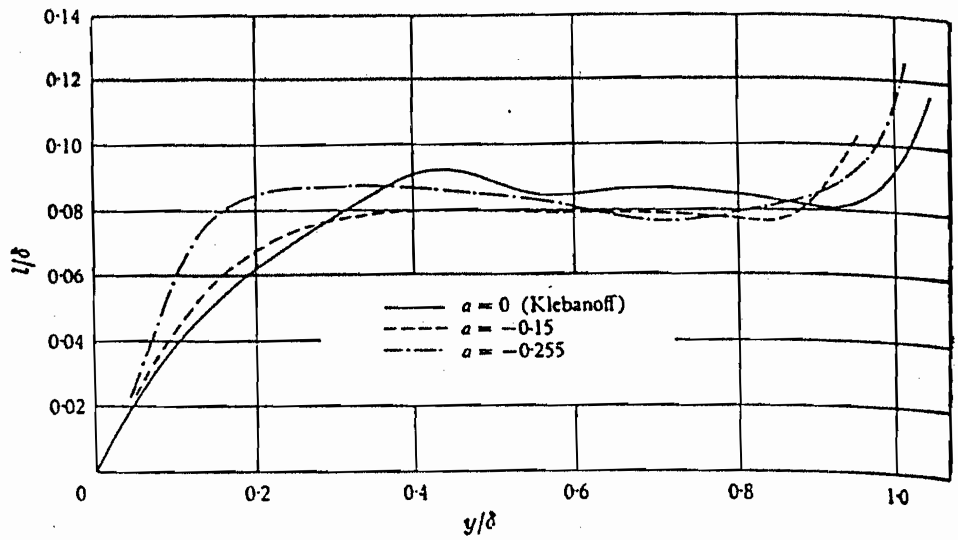


Fig. 6.17 Distribution of the mixing length ℓ for three values of the exponent 'a' in the external velocity $u_e = x^a$ in a turbulent boundary layer. (Taken from Bradshaw, P. S., *J. Fluid Mech.*, 29, 629, 1967. With permission.)

6.23 ZERO-EQUATION MODELING IN SHEAR LAYERS

Two cases of flows which have been predicted with much success through the use of the Boussinesq-Kolmogorov type of modeling for $(\mu_T)_i$, with Van Driest's mixing length ℓ, are the boundary layers and the thin shear layers. The thin shear layer models are those which are obtained by neglecting the streamwise diffusion terms in the complete Navier-Stokes equations. The main point of difference between a boundary layer and a thin shear layer is that in the former the pressure is specified independently of the viscous flow field, but in the latter the pressure is dependent on the viscous flow field and therefore must be obtained as part of the solution.

Thin Shear Layers

A model which does not require the calculation of boundary layer thickness has been developed by Baldwin and Lomax.[70] For nonseparated flows, Baldwin and Lomax have adopted the following model which is applicable to thin shear layers:

$$(\mu_T)_i = \rho\ell^2|\overline{\omega}|, \qquad n \leq n_{match} \tag{6.285a}$$

where $|\overline{\omega}|$ is the vorticity and ℓ is the first equation in Equation 6.284. In the outer region of a shear layer for wall-bound flow:

$$(\mu_T)_o = \frac{0.0269\rho n_m F_m}{1 + 5.5\left(\dfrac{0.3n}{n_m}\right)^6}, \qquad n \geq n_{match} \tag{6.285b}$$

where n_{match} is the value of n at which the inner and outer viscosities are equal. Further:

$$F_m = \mathrm{Max}[n|\overline{\omega}|\{1 - \exp(-n^+/A^+)\}]$$

and n_m is the value of n at which the maximum F_m is attained. In the outer region of a shear layer, in the wake, or free flow:

$$(\mu_T)_0 = \frac{0.0067\rho n_m(\Delta U)^2}{\left[1 + 5.5\left(\frac{0.3n}{n_m}\right)^6\right]f_m} \tag{6.285c}$$

where:

$$f_m = \text{Max}[n|\overline{\omega}|]$$

and:

$$\Delta U = |\overline{\mathbf{u}}|_{\text{Max}} - |\overline{\mathbf{u}}|_{\text{Min}}$$

As before, n_m is the value of n at which the maximum f_m is attained. Recent calculations have shown that the constants appearing in Equations 6.285b, c depend strongly on the Mach number and the coefficient of skin friction.

6.24 ONE-EQUATION MODELING

The basis of one-equation modeling lies in the suggestion of both Prandtl and Kolmogorov that on physical considerations the eddy viscosity μ_T should depend on the turbulence kinetic energy K. Besides K it can also be made to depend on other turbulence quantities. If only K is to be considered, then on dimensional consideration a turbulence length scale ℓ has to be introduced so that μ_T for the high turbulent Reynolds number part of the flow field, i.e., for $\sqrt{K}\,\ell/\nu \gg 1$, is given as:

$$\mu_T = \rho b_1 \ell K^{1/2} \tag{6.286}$$

where b_1 is a nondimensional constant. Equation 6.286 forms the basis of the one-equation model if the only turbulence equation one is prepared to solve is that of the turbulence energy K, and ℓ is specified either by Equation 6.284 or by some other means. Considering either Equations 6.164 or 6.167 we also note that for solving the energy equation one has to model the dissipation rate ϵ and the diffusion D. The modeling of these terms proceeds as follows.

Rotta[72] argued that since most of the energy of turbulence to be dissipated resides in the larger eddies where the effect of viscosity is negligible, the expression for ϵ must not explicitly depend on ν. Thus, on dimensional considerations he proposed the form:

$$\epsilon = \frac{b_3 K^{3/2}}{\ell} \tag{6.287}$$

where b_3 is a nondimensional constant. Note that b_3 can be absorbed in the definition of ℓ. Rotta further argued that since the dissipation of energy to heat occurs at small eddies (large wave numbers), there must exist a process of nonlinear transfer in which energy is transmitted to smaller eddies through a cascade mechanism. That is, the larger eddies transmit their energy to the eddies next to their size by a nonlinear transfer. These eddies then transfer their energy to the next in size, and so on down to the eddies of the smallest size. (Refer to Part I of this chapter, specifically to the Taylor and Kolmogorov length scales, transfer functions, etc.)

The term:

$$\overline{u_j'\left(\frac{q^2}{2} + \frac{p'}{\rho}\right)}$$

physically represents the transport of the quantity $q^2/2 + p'/\rho$ by the j-th component of velocity.

Since the term D in Equation 6.177 is the divergence of a flux vector, we model the diffusive flux as:

$$\overline{\mathbf{u}'\left(\frac{q^2}{2} + \frac{p'}{\rho}\right)} = -b_5\nu_T \text{ grad } K$$

or:

$$\overline{u_j'\left(\frac{q^2}{2} + \frac{p'}{\rho}\right)} = -b_5\nu_T \frac{\partial K}{\partial x_j}$$

Thus:

$$D = \text{div}(b_5\nu_T \text{ grad } K) = \frac{\partial}{\partial x_j}\left(b_5\nu_T \frac{\partial K}{\partial x_j}\right) \tag{6.288}$$

where b_5 is a nondimensional constant. It is customary to write $b_5 = 1/\sigma_k$. We now have all the important ingredients of a one-equation model which are Equations 6.286–6.288. The main problem now is to select the constants b_1, b_3, and b_5 which are almost universal.

Choice of the Constants b_1, b_3, and b_5

To obtain the values of the constants, we consider the turbulent *core region* of a shear layer in the neighborhood of a wall. As before, let n be the actual distance normal to the surface. From Equations 6.281 we then simply have:

$$\mu_T = \rho\ell^2 \left|\frac{\partial \bar{u}}{\partial n}\right|$$

where $\bar{u}$ is the mean velocity along the direction of the main flow. In the core region:

$$\ell = \kappa n$$

and:

$$\bar{u} = \frac{u_\tau}{\kappa} \ell n(n) + C^*$$

so that:

$$\mu_T = \rho\kappa n u_\tau$$

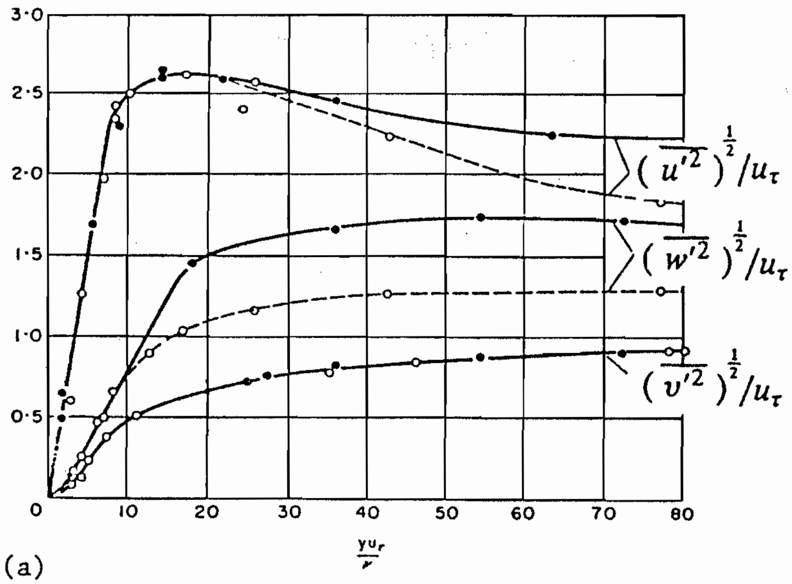
Also from Equation 6.286:

$$\mu_T = \rho\kappa b_1 n K^{1/2}$$

Equating these two different expressions of μ_T, we get:

$$K/u_\tau^2 = 1/b_1^2$$

A majority of experiments of the measured values of K in the boundary layers, pipe, and channel flows etc. have demonstrated that the value of K/u_τ^2 falls between 3.34 and 3.43 in the lower part of the core region next to the sublayer, i.e., $y^+ = 7.5$. One such measurement due to Laufer[73] has been shown in Figure 6.18(a).



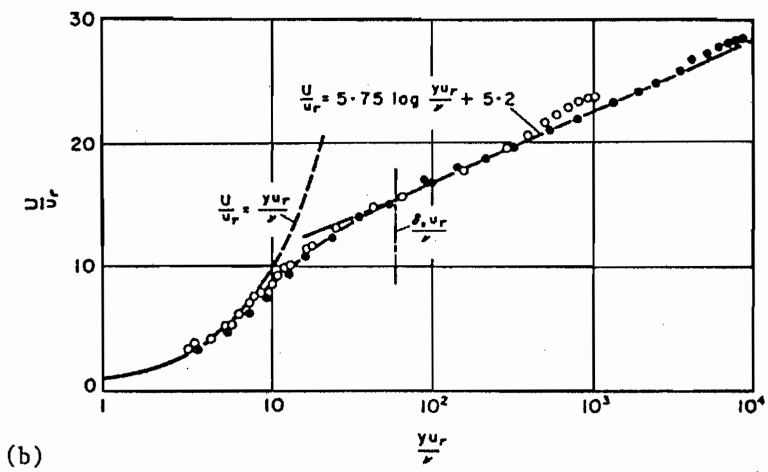


Fig. 6.18 (a) Distribution of the normalized normal stresses near a wall; (b) Velocity distribution near a wall. Note: Data ○ Reynolds number = 50,000 ● 500,000. The energy at $n^+ = 10$ is used to establish the constant b_1. (Taken from Rotta, J. C., *Prog. Aeronaut. Sci.*, 2, 1, 1962. With permission.)

Thus, $b_1^2 \cong 0.3$. In most of the turbulence modelings the value of b_1^2 has been chosen between 0.29 and 0.33. For definitiveness, we shall take $b_1^2 = 0.3$ and from here onward use the notation α_0 for b_1^2, where:

$$b_1^2 = \alpha_0 = 0.3$$

Thus, next to the sublayer

$$K = \frac{u_\tau^2}{\alpha_0}$$

To determine b_3, we first note that in the core region Equation 6.287 becomes:

$$\epsilon = \frac{b_3 K^{3/2}}{\kappa n}$$

Also, in the core region there is a sort of structural equilibrium, which means that in energy Equation 6.177 the production and dissipation terms balance each other, i.e., $P = \epsilon$. In the core region:

$$P = \frac{\mu_T}{\rho}\left(\frac{\partial \bar{u}}{\partial n}\right)^2 = \frac{b_1 u_\tau^2 K^{1/2}}{\kappa n}$$

Using $P = \epsilon$, we get:

$$b_3 = b_1^3 = \alpha_0^{3/2}$$

Thus, next to the sublayer:

$$\epsilon = \frac{u_\tau^3}{\kappa n}$$

The gradient diffusion model for the velocity-pressure correlation is not directly verifiable by experiments. Thus the constant $b_5 = 1/\sigma_k$ appearing in Equation 6.288 cannot be ascertained accurately for its numerical values. Various authors have used the values of σ_k in the range $.718 \leqslant \sigma_k \leqslant 1$ without any serious distortion of the computed K profiles. A value of $\sigma_k \sim 1$ may be chosen as an optimum value. The quantity σ_k is also called the Prandtl-Schmidt number. It has become a common practice to write $\alpha_0^2 = C_\mu = 0.09$.

Modifications Due to the Explicit Effects of Viscosity

The region of flow very close to the wall is directly influenced by viscosity. To bring the low turbulent Reynolds number effects in the modeling we have to modify the previous formulae so that they become valid for the whole flow field including the sublayer. At present there are two computationally proven methods to bring the viscous effects explicitly in the modeling which we describe below.

Method I. Reynolds et al. (refer to Reference 66) modified the eddy viscosity to incorporate viscous damping according to the formula:

$$\mu_T = \rho b_1 \ell K^{1/2}\{1 - \exp(-b_2\sqrt{K}n/\nu)\} \tag{6.289}$$

with $b_2 = 0.02$. Further, the dissipation function (Equation 6.287) suggested by Rotta is modified as:

$$\epsilon = \frac{b_3 K^{3/2}}{\ell}\left(1 + \frac{\nu b_4}{\ell\sqrt{K}}\right) \tag{6.290}$$

In both Equations 6.289 and 6.290 the length scale ℓ retains the simple form of Prandtl right to the wall, i.e.:

$$\ell = \kappa n$$

The constant b_4 is directly determinable from energy Equation 6.177. Very close to the wall the energy equation is

$$-\epsilon + \nu \frac{d^2K}{dn^2} = 0$$

Also, very close to the wall (refer to Problem 6.14):

$$K = \lambda n^2, \qquad \ell = \kappa n$$

where λ is a constant. Thus, using Equation 6.290, we have:

$$b_4 = \frac{2\kappa^2}{b_3} = 2\kappa^2(\alpha_0)^{-3/2} = 2.046$$

In summary, the one-equation model with Reynolds' modifications for viscous effects is stated as:

$$\begin{aligned}
\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} &= P + D - \epsilon + \nu\nabla^2 K \\
P &= -\bar{\tau}_{ij}\bar{D}_{ij} = 2\nu_T \bar{D}_{ij}\bar{D}_{ij} \\
D &= \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial K}{\partial x_j}\right) \\
\epsilon &= \frac{b_3 K^{3/2}}{\ell}\left(1 + \frac{\nu b_4}{\ell\sqrt{K}}\right) \\
\nu_T &= b_1\sqrt{K}\ell\{(1 - \exp(-b_2\sqrt{K}n/\nu))\}
\end{aligned} \qquad \textbf{(6.291)}$$

The constants in terms of $\alpha_0 = 0.3$ are

$$b_1 = \alpha_0^{1/2}, \quad b_2 = \alpha_0^{3.25}, \quad b_3 = \alpha_0^{3/2}, \quad b_4 = 2\kappa^2(\alpha_0)^{-3/2}, \quad \sigma_k \cong 1, \quad \kappa = 0.41$$

Method II. The following method is originally due to Glushko.[67] Later Beckwith and Bushnell[68] reformulated the boundary layer problem using Glushko's method and made a number of test runs both for favorable and adverse pressure gradients.

Glushko assumes a form similar to Equation 6.286 as:

$$\mu_T = \mu\alpha R H(R) \qquad \textbf{(6.292)}$$

where $R = \sqrt{K}\ell/\nu$ is a turbulence Reynolds number with ℓ an integral scale of turbulence. The integral scale was evaluated from the two-point correlation function of longitudinal velocity components for a flat plate. A representative empirical fit taken from Viegas and Coakley[74] in our notation is

$$\begin{aligned}
\frac{\ell}{n_\delta} &= \frac{n}{n_\delta}, & 0 \leq \frac{n}{n_\delta} < 0.23 \\
&= \left(\frac{n}{n_\delta} + 0.37\right)\Big/ 2.61 & 0.23 \leq \frac{n}{n_\delta} < 0.57 \\
&= \left(1.48 - \frac{n}{n_\delta}\right)\Big/ 2.52, & 0.57 \leq \frac{n}{n_\delta} \leq 1.48
\end{aligned} \qquad \textbf{(6.293)}$$

The function H in Equation 6.292 is defined as:

$$H(R) = \frac{R}{R_0}, \qquad 0 \leqslant \frac{R}{R_0} < 0.75$$

$$= \frac{R}{R_0} - \left(\frac{R}{R_0} - 0.75\right)^2, \qquad 0.75 \leqslant \frac{R}{R_0} < 1.25$$

$$= 1, \qquad 1.25 \leqslant \frac{R}{R_0} < \infty \tag{6.294}$$

The modeling of dissipation and diffusion terms are done as:

$$\epsilon = \nu C\{1 + \alpha\kappa RH(\kappa R)\}\frac{K}{\ell^2} \tag{6.295}$$

$$D = \nu\frac{\partial}{\partial x_j}\left[\{1 + \alpha\kappa RH(\kappa R)\}\frac{\partial K}{\partial x_j}\right] \tag{6.296}$$

The constant appearing in Equations 6.294–6.296 are as follows:

$$\alpha = 0.2, \qquad \kappa = 0.41, \qquad C = 3.93, \quad R_0 = 110$$

6.25 TWO-EQUATION (K-ϵ) MODELING

The analysis in this and the following section depends on the research of Hanjalic and Launder,[75] Launder et al.,[76] and Hanjalic and Launder.[77] A fuller discussion of the basis of the two-equation model occurs in Section 6.26.

In the two-equation or the K-ϵ model, the equations of energy K and of dissipation rate ϵ are the two turbulence equations. The Reynolds stress is modeled through Equations 6.280, and the eddy viscosity μ_T is expressed in terms of both K and ϵ. Using Equation 6.286 and eliminating ℓ through Equation 6.287, we get:

$$\mu_T = \rho\alpha_0^2\frac{K^2}{\epsilon}, \qquad \alpha_0^2 = 0.09 \tag{6.297}$$

To incorporate the viscous effects, Speziale et al.[78] have recently proposed the modification:

$$\mu_T = \rho\alpha_0^2\frac{K^2}{\epsilon}(1 + b_7R_T^{-1/2})\tanh(n^+/n_c^+) \tag{6.298}$$

where: $70 < n_c^+ < 120$, $b_7 = 3.45$, and

$$R_T = K^2/\nu\epsilon$$

$$\frac{\partial K}{\partial t} + \bar{u}_j\frac{\partial K}{\partial x_j} = P - \epsilon + \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial K}{\partial x_j}\right) + \nu\nabla^2 K \tag{6.299}$$

where ν_T comes from Equation 6.298 and $\sigma_k \cong 1$. In Equation 6.299, the turbulence diffusion D has been modeled in an intuitive manner. Refer to Equation 6.309 and the discussion which follows it.

Modeling of the Dissipation Rate Equation

The modeling of terms in the dissipation rate equation (Equation 6.178), have been accomplished by Hanjalic et al.[75-77] in the papers quoted earlier in this section. Below we first state the physical meaning of the group of terms in Equation 6.178 and write their modelings (in this connection refer also to References 79 and 80):

$$2\nu \frac{\partial \bar{u}_j}{\partial x_k}\left(\overline{\frac{\partial u_j'}{\partial x_\ell}\frac{\partial u_k'}{\partial x_\ell}} + \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}}\right) = \text{generation of the dissipation rate } \epsilon$$

$$= \left(c_{\epsilon 1}\epsilon \frac{\bar{\tau}_{jk}}{K} + c_{\epsilon 11}\epsilon\delta_{jk}\right)\frac{\partial \bar{u}_j}{\partial x_k}$$

$$= -c_{\epsilon 1}\frac{\epsilon}{K}P + c_{\epsilon 11}\epsilon\bar{\Delta} \tag{6.300a}$$

The term in parentheses on the left represents the generation rate of vorticity fluctuations through the self-stretching action of turbulence. The constant $c_{\epsilon 1}$ lies in the range 1.063–1.45, and for incompressible flow $\bar{\Delta} = 0$.

$$2\left\{\nu \overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial u_j'}{\partial x_\ell}\frac{\partial u_k'}{\partial x_\ell}} + \overline{\left(\nu \frac{\partial^2 u_j'}{\partial x_k \partial x_\ell}\right)^2}\right\} = \begin{array}{l}\text{net energy transfer from the low to}\\ \text{the high wave numbers}\end{array}$$

$$= c_{\epsilon 2}\frac{(\epsilon)^2}{K} \tag{6.300b}$$

where $c_{\epsilon 2}$ lies in the range 1.68–1.92:

$$\frac{\partial}{\partial x_k}\left\{\overline{u_k'e} + \frac{2\nu}{\rho}\overline{\frac{\partial u_k'}{\partial x_\ell}\frac{\partial p'}{\partial x_\ell}}\right\} = \begin{array}{l}\text{diffusional transport of } \epsilon \text{ by pressure}\\ \text{fluctuations}\end{array}$$

$$= -\frac{\partial}{\partial x_k}\left(c_\epsilon \frac{K}{\epsilon}\bar{\tau}_{k\ell}\frac{\partial \epsilon}{\partial x_\ell}\right) \tag{6.300c}$$

and where $c_\epsilon = 0.15$:

$$2\nu \frac{\partial^2 \bar{u}_j}{\partial x_\ell \partial x_k}\overline{\left(u_k' \frac{\partial u_j'}{\partial x_\ell}\right)} = \text{mean-field generation of } \epsilon$$

$$= -c_{\epsilon 3}\frac{\nu K}{\epsilon}\bar{\tau}_{ij}\frac{\partial^2 \bar{u}_k}{\partial x_i \partial x_\ell}\frac{\partial^2 \bar{u}_k}{\partial x_j \partial x_\ell} \tag{6.300d}$$

The complete dissipation rate equation with explicit viscous effects included is then given by using the expressions in Equations 6.300 as:

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_j\frac{\partial \epsilon}{\partial x_j} = c_\epsilon \frac{\partial}{\partial x_k}\left(\frac{K}{\epsilon}\bar{\tau}_{k\ell}\frac{\partial \epsilon}{\partial x_\ell}\right) + c_{\epsilon 1}\frac{\epsilon P}{K}$$

$$-c_{\epsilon 2}f_\epsilon(R_T)\frac{\epsilon}{K}\left[\epsilon - 2\nu\left(\frac{\partial\sqrt{K}}{\partial n}\right)^2\right] + \nu c_{\epsilon 3}\bar{\tau}_{ij}\frac{K}{\epsilon}\frac{\partial^2 \bar{u}_k}{\partial x_i \partial x_\ell}\frac{\partial^2 \bar{u}_k}{\partial x_j \partial x_\ell} + \nu\nabla^2\epsilon \tag{6.301}$$

The extra factor in the square brackets on the right-hand side has been inserted to remove the infinite singularity of ϵ^2/K at a wall, n being the coordinate normal to the surface. Further:

$$f_\epsilon(R_T) = 1 - \tfrac{2}{9}\exp(-R_T^2/36)$$

$$R_T = K^2/\nu\epsilon$$

$$c_\epsilon = 0.15, \quad c_{\epsilon 1} = 1.44, \quad c_{\epsilon 2} = 1.92, \quad c_{\epsilon 3} = 2.0$$

Equation 6.299 and 6.301 form the two-equation (K-ϵ) model.

Modeling for Separated Flows

The eddy viscosity formula (Equation 6.297) is valid only for attached shear layers. Goldberg[81] has found a modification of Equation 6.297 for separated flows. First of all, in the separated region, the sublayer also detaches from the surface; therefore, the mechanics of the separated region should be governed by a velocity scale other than u_τ. The length scale must also be compatible with the size of the separation bubble. The separation bubble is bounded by the bubble edge or the dividing streamline which starts from the separation point and ends at the reattachment point. Within the bubble there is the detached viscous sublayer, and within the sublayer is the back-flow region with a boundary that has zero tangential velocity. Goldberg's formulation, tested by numerical computation, can be summarized as follows:

$$\mu_T = \rho F\left(\frac{n}{n_b}\right)\frac{K^2}{\epsilon}, \qquad 0 \leq n \leq n_b$$

$$F\left(\frac{n}{n_b}\right) = \frac{U_s n_b}{2\sqrt{2}\beta^2}\left[A\left(\frac{n}{n_b}\right) + B\right]G^{1/2}$$

$$A = -\tfrac{1}{2}(\beta_0^2)^{9/5}, \qquad B = (\tfrac{1}{2}\beta_0^2)^{3/5} - A$$

$$G = 2.541494\left[1 - \exp\left\{-\frac{1}{2}\left(\frac{n}{n_b}\right)^2\right\}\right]$$

$$\beta = 0.2292529 + 0.770747\left(\frac{n_v}{n_b}\right)^2$$

$$= K_v/K_b$$

$$\beta_0 = \sqrt{0.7}$$

$$U_s = \max|\overline{\tau}|$$

where:

n_b = distance from the wall to the edge of the back-flow region

n_v = distance from the wall to the edge of the viscous sublayer region

K_b = turbulence kinetic energy at the edge of the back-flow region

K_v = turbulence kinetic energy at the edge of the sublayer region

6.26 REYNOLDS' STRESS EQUATION MODELING

It was mentioned earlier that Boussinesq-Kolmogorov's hypothesis, which results in a Newtonian constitutive equation for the Reynolds stress tensor, is not an accurate model for the prediction of flows with strong streamline curvatures. In such cases it is preferable to suspend the use of the Newtonian constitutive equation and instead solve the complete set of Reynolds' stress rate equations. The modeling given in this section can be used to achieve this aim. The other important outcome of the Reynolds stress equation modeling will be to have another K-equation which seems to better represent the actual physics than does Equation 6.299. It must be mentioned again that the two-equation model depends on the Newtonian constitutive equation.

To proceed, we have to model the rate equations for the Reynolds stresses, Equation 6.175. The terms P_{ij}, Q_{ij}, D_{ij}, and ϵ_{ij} appearing in Equation 6.175 and defined in Equations 6.176 have the following interpretations:

P_{ij} = production rate of the Reynolds stresses

Q_{ij} = pressure and rate-of-strain correlation

F_{ij} = turbulent diffusion rate of the Reynolds stresses

ϵ_{ij} = dissipation (decay) rate of the Reynolds stresses

Recall also that $\bar{\tau}_{ij} = \overline{u_i' u_j'}$, where $-\rho\bar{\tau}_{ij}$ are the Cartesian components of the Reynolds stress tensor. The left-hand side of Equation 6.175 represents the convection of Reynolds' stresses.

The pressure and rate-of-strain correlation was originally formulated by Rotta[38] in the form:

$$Q_{ij} = \gamma_1 \frac{\epsilon}{K} \left(\tfrac{2}{3} K\delta_{ij} - \bar{\tau}_{ij}\right) \tag{6.302}$$

where $\gamma_1 \cong 4.5$. This model is based on the concept that pressure fluctuations and local fluctuating rate-of-strains interact in such a manner that the resulting effect is to produce changes toward isotropy in small-scale structures, i.e.:

$$\bar{\tau}_{ij} = \tfrac{2}{3} K\delta_{ij}$$

Thus, both the magnitude and the sign of the expression in Equation 6.302 give a measure of the local anisotropy among small-scale structures.

Any realistic modeling of the pressure and the rate-of-strain correlation Q_{ij} must depend on interpretation of the terms in Equation 6.187. In Equation 6.187 the term Φ_{ij} involves only the fluctuating quantities and is called the *return term*, while the term Ψ_{ij} has an explicit dependence on the gradients of mean velocity and is called the *rapid term*. The remaining term W_{ij} is dependent on the wall boundary condition.

In Hanjalic-Launder's analysis, the return term is formulated similar to that of Rotta, Equation 6.302. Thus:

$$\Phi_{ij} = 3c_1 \frac{\epsilon}{K} \left(\tfrac{2}{3} K\delta_{ij} - \bar{\tau}_{ij}\right) \tag{6.303a}$$

where c_1 is a constant. Considering the second terms in Equations 6.185a, b and making the approximation that the mean velocity gradients are to be evaluated at the position **x**, it is possible to express the remaining space integrals in terms of the Reynolds stresses. Using the symmetry constraints and the kinematic condition (divergence condition), Lauder et al.[76] modeled ψ_{ij} as:

$$\Psi_{ij} = \alpha\left(\tfrac{2}{3} P\delta_{ij} - P_{ij}\right) + \beta\left(\tfrac{2}{3} P\delta_{ij} - \Delta_{ij}\right) - \gamma K\bar{D}_{ij} \tag{6.303b}$$

where:

$$\alpha = (6.666c_2 + 8)/11$$

$$\beta = (53.333c_2 - 2)/11$$

$$\gamma = 4(100c_2 - 1)/55$$

and c_2 is another constant. Further:

$$\Delta_{ij} = -\left(\bar{\tau}_{ik}\frac{\partial \bar{u}_k}{\partial x_j} + \bar{\tau}_{jk}\frac{\partial \bar{u}_k}{\partial x_i}\right)$$

In a similar fashion, the wall function W_{ij} is modeled as:

$$W_{ij} = \frac{\alpha_0^{3/2}}{\kappa}\left[c_1\frac{\epsilon}{K}(\bar{\tau}_{ij} - \tfrac{2}{3}K\delta_{ij}) + c_2(P_{ij} - \Delta_{ij})\right]\frac{K^{3/2}}{\epsilon n} \tag{6.303c}$$

where as before $\alpha_0 = 0.3$ and $\kappa = 0.41$. The complete formulation of Q_{ij} is the sum of Equations 6.303a, b, c, i.e.:

$$Q_{ij} = \Phi_{ij} + \Psi_{ij} + W_{ij}$$

Hanjalic and Launder have also modeled the decay function ϵ_{ij} and the diffusion function F_{ij}. The decay function having the necessary viscous damping is modeled as:

$$\epsilon_{ij} = \tfrac{2}{3}\epsilon\left\{(1 - f_s)\delta_{ij} + \frac{3\bar{\tau}_{ij}}{2K}f_s\right\} \tag{6.304}$$

where:

$$f_s(R_T) = (1 + 0.1R_T)^{-1}$$

$$R_T = \frac{K^2}{\nu\epsilon}$$

For flows away from the wall the term $f_s \rightarrow 0$, and we obtain:

$$\epsilon_{ij} = \tfrac{2}{3}\epsilon\delta_{ij} \tag{6.305}$$

which signifies an isotropic decay of low-scale motions.

In the modeling of diffusion term F_{ij}, the pressure transport has been neglected by Hanjalic and Launder and only the triple correlation is formulated. Thus:

$$F_{ij} = c_s\frac{\partial}{\partial x_k}\left\{\frac{K}{\epsilon}\left(\bar{\tau}_{k\ell}\frac{\partial \bar{\tau}_{ij}}{\partial x_\ell} + \bar{\tau}_{j\ell}\frac{\partial \bar{\tau}_{ki}}{\partial x_\ell} + \bar{\tau}_{i\ell}\frac{\partial \bar{\tau}_{jk}}{\partial x_\ell}\right)\right\} \tag{6.306}$$

where $c_s \cong 0.11$. The pressure transport is not negligible in the near-wall region and must be included in any proposed modification of Equation 6.306.*

* The constants c_1 and c_2 are the c_1', c_2' of Launder et al.[76]

Determination of the Constants c_1 and c_2

The constants c_1 and c_2 quoted in Equations 6.303 have been evaluated by using the experimental data of Uberoi,[82] Tucker and Reynolds,[83] and Champagne et al.[84] The available data pertain to the measured values of normal and shear stresses in a plane homogeneous shear layer and in the vicinity of a wall.

(i) Plane Homogeneous Shear Layer. In this case neglecting the convective and diffusive terms in Equation 6.175, we simply have:

$$P_{ij} + Q_{ij} - \epsilon_{ij} = 0 \tag{6.307a}$$

Further by neglecting the anisotropy of the dissipation rate we have

$$P \cong \epsilon \tag{6.307b}$$

and obviously for plane shear flow

$$P = -\overline{\tau}_{12} \frac{\partial \overline{u}_1}{\partial x_2} \tag{6.307c}$$

Putting $i = j = 1$, 2, and 3 and using the modelings in Equations 6.303a, b and 6.305, we get:

$$(\overline{u_1'^2} - \tfrac{2}{3}K)/K = 8(1 + 10c_2)/99c_1$$

$$(\overline{u_2'^2} - \tfrac{2}{3}K)/K = 2(1 - 100c_2)/99c_1$$

$$(\overline{u_3'^2} - \tfrac{2}{3}K)/K = 10(12c_2 - 1)/99c_1$$

$$-\overline{\tau}_{12}/K = \left[\frac{200c_2 - 2}{165c_1} - \frac{(53.333c_2 - 2)}{33c_1}\frac{\overline{u_1'^2}}{K} + \frac{(3 - 6.666c_2)}{33c_1}\frac{\overline{u_2'^2}}{K}\right]^{1/2}$$

For a homogenous shear layer the experimental data is[76]

$$\overline{u_1'^2}/K = 0.9667,\ \overline{u_2'^2}/K = 0.4867,\ \overline{u_3'^2}/K = 0.5467,\ -\overline{\tau}_{12}/K = 0.33 \tag{6.307d}$$

The experimental values given above are close to those represented by the model if we choose $c_1 = 0.5$ and $c_2 = 0.06$.

(ii) Near-Wall Turbulent Shear Layer. In this case the governing equation is again Equation 6.307a. However, because of the near-wall effects, we must include the term W_{ij} (Equation 6.303c) in Q_{ij}. Again using Equation 6.307b and:

$$\epsilon = \frac{(\alpha_0 K)^{3/2}}{\kappa n}$$

we obtain:

$$(\overline{u_1'^2} - \tfrac{2}{3}K)/K = 8(1 + 18.25c_2)/66c_1$$

$$(\overline{u_2'^2} - \tfrac{2}{3}K)/K = 2(1 - 133c_2)/66c_1$$

$$(\overline{u_3'^2} - \tfrac{2}{3}K)/K = 2(60c_2 - 5)/66c_1$$

$$-\bar{\tau}_{12}/K = \left[\frac{100c_2 - 1}{55c_1} + \frac{(3 + 4.334c_2)}{22c_1}\frac{\overline{u_2'^2}}{K} - \frac{(64.333c_2 - 2)}{22c_1}\frac{\overline{u_1'^2}}{K}\right]^{1/2}$$

The experimental data for near-wall turbulences in Figure 6.18(a) is

$$\overline{u_1'^2}/K = 1.1767, \quad \overline{u_2'^2}/K = 0.2467, \quad \overline{u_3'^2}/K = 0.5767, \quad -\overline{u_1'u_2'}/K = 0.24 \qquad \textbf{(6.307e)}$$

The same model constants, i.e., $c_1 = 0.5$ and $c_2 = 0.06$, satisfy the experimental values quite closely.

The complete Reynolds' stress rate equation with the viscous terms included is then:

$$\begin{aligned}
\frac{\partial \bar{\tau}_{ij}}{\partial t} + \bar{u}_k \frac{\partial \bar{\tau}_{ij}}{\partial x_k} &= P_{ij} - \tfrac{2}{3}\epsilon\left\{(1 - f_s)\delta_{ij} + \tfrac{3}{2}\frac{\bar{\tau}_{ij}}{K}f_s\right\} \\
&+ 3c_1\frac{\epsilon}{K}(\tfrac{2}{3}K\delta_{ij} - \bar{\tau}_{ij}) + \alpha(\tfrac{2}{3}P\delta_{ij} - P_{ij}) + \beta(\tfrac{2}{3}P\delta_{ij} - \Delta_{ij}) - \gamma K\bar{D}_{ij} \\
&+ \frac{\alpha_0^{3/2}}{\kappa}\left[c_1\frac{\epsilon}{K}(\bar{\tau}_{ij} - \tfrac{2}{3}K\delta_{ij}) + c_2(P_{ij} - \Delta_{ij})\right]\frac{K^{3/2}}{\epsilon n} \\
&+ c_s\frac{\partial}{\partial x_k}\left\{\frac{K}{\epsilon}\left(\bar{\tau}_{k\ell}\frac{\partial \bar{\tau}_{ij}}{\partial x_\ell} + \bar{\tau}_{j\ell}\frac{\partial \bar{\tau}_{ki}}{\partial x_\ell} + \bar{\tau}_{i\ell}\frac{\partial \bar{\tau}_{jk}}{\partial x_\ell}\right)\right\} + \nu\nabla^2\bar{\tau}_{ij}
\end{aligned} \qquad \textbf{(6.308)}$$

The constants α, β, γ have been stated after Equation 6.303b, and are $\alpha = 0.76$, $\beta = 0.11$, $\gamma = 0.36$. These values are based on $c_1 = 0.5$ and $c_2 = 0.06$.

In the Reynolds stress closure, the equations of continuity and momentum are to be solved along with the equations of Reynolds stress, energy, and dissipation (viz., Equations 6.299, 6.301, and 6.308, respectively).

Another Modeling of the Energy Equation

The proper form of the energy equation to be solved along with the other turbulence equations can now be obtained simply by setting $j = i$ in Equation 6.308 and then adding on all i's. Noting that:

$$\overline{u_{ii}'^2} = \bar{\tau}_{ii} = 2K, \qquad P_{ii} = 2P, \qquad \Delta_{ii} = 2P$$

we then have:

$$\frac{\partial K}{\partial t} + \bar{u}_k\frac{\partial K}{\partial x_k} = P - \epsilon + c_s\frac{\partial}{\partial x_k}\left\{\frac{K}{\epsilon}\left(\bar{\tau}_{k\ell}\frac{\partial K}{\partial x_\ell} + \bar{\tau}_{i\ell}\frac{\partial \bar{\tau}_{ik}}{\partial x_\ell}\right)\right\} + \nu\nabla^2 K \qquad \textbf{(6.309)}$$

where $c_s \cong 0.11$.

As has been noted above, a natural choice for the energy equation is Equation 6.309 despite the fact that Equation 6.299 is much simpler than Equation 6.309. Using the eddy viscosity formula (Equation 6.297) in Equation 6.299 and comparing the two equations we find that the contribution of the diffusion term in either Equation 6.299 or Equation 6.309 is the same provided that the Reynolds stresses satisfy the equations:

$$\bar{\tau}_{j\ell}\frac{\partial K}{\partial x_\ell} + \bar{\tau}_{i\ell}\frac{\partial \bar{\tau}_{ij}}{\partial x_\ell} = \frac{\alpha_0^2}{c_s\sigma_k} K \frac{\partial K}{\partial x_j}$$

where $j = 1, 2, 3$. In particular, these equations are satisfied in the case of plane homogeneous shear turbulent flow. Taking $\bar{\tau}_{mn} = \alpha_{mn}K$ where α_{mn} are constants and considering only the case when $j = 2$, we get:

$$\alpha_{22} + \alpha_{22}^2 + \alpha_{12}^2 = \frac{\alpha_0^2}{c_s\sigma_k}$$

From Equation 6.307d, $\alpha_{22} = 0.4667$; and since $\alpha_0 = 0.3$, $c_s = 0.11$. and $\sigma_k = 1$, we get $\alpha_{12} = -0.3656$ which is close to the value given in the last equation of Equation 6.307d. From this we conclude that Equation 6.309 has the eddy viscosity representation of the diffusion term if $\bar{\tau}_{ij}$ can be considered as a linear function of K. It is also of interest to see that Equation 6.301 has an eddy viscosity representation of the diffusion term if:

$$\bar{\tau}_{j\ell}\frac{\partial \epsilon}{\partial x_\ell} = \frac{\alpha_0^2}{c_\epsilon\sigma_\epsilon} K \frac{\partial \epsilon}{\partial x_j}$$

where σ_ϵ is a constant, called the Prandtl-Schmidt number. Again considering the case $j = 2$, x_2 being normal to the streamwise direction, and taking $\sigma_\epsilon = 1.3$, we have:

$$\bar{\tau}_{22} = \frac{\alpha_0^2}{c_\epsilon\sigma_\epsilon} K \simeq \alpha_{22}K$$

Thus, in essence the eddy viscosity representation of the turbulent diffusion term in either Equation 6.301 or Equation 6.309 is strictly possible in those cases when the Reynolds' stresses can be expressed as linear functions of K.

The Wall Boundary Conditions

The boundary condition for K at a wall is obviously $K_w = 0$. The infinity boundary condition is also quite straightforward since one has to prescribe the intensity of turbulence (which may be zero) from which K can be constructed.

The wall boundary condition for ϵ comes directly from Equation 6.309 which is

$$\epsilon_w = (\nu \nabla^2 K)_w \tag{6.310a}$$

the correct boundary condition for ϵ. Using the near-wall behavior of K (refer to Problem 6.14), an equivalent condition is

$$\epsilon_w = 2\nu\left(\frac{\partial\sqrt{K}}{\partial n}\right)^2_{n=0} \tag{6.310b}$$

where n is the normal distance from a wall.

Some numerical programs have, however, taken either $\epsilon_w = 0$ or ∞. If $\epsilon_w = 0$ is adopted then in order to have a consistent turbulence energy equation, one should replace ϵ in Equations 6.299 or 6.309 by

$$\epsilon + 2\nu\left(\frac{\partial\sqrt{K}}{\partial n}\right)^2$$

which in the near-wall region takes the form

$$\epsilon + \frac{2\nu K}{n^2}$$

This line of approach has been adopted by Chien.[85] On the other hand, Daly and Harlow[86] have suggested the following emperical formula for smooth walls:

$$\epsilon = 2\nu C n^{-0.5628} \quad \text{as} \quad n \to 0 \tag{6.310c}$$

where the constant C has to be computed numerically. According to some recent direct numerical simulation results $\epsilon_w = 0.26u_\tau^2/\nu$. Refer to Problem 6.14 and the references cited therein. Also note the result of Example 6.4.

6.27 APPLICATION TO TWO-DIMENSIONAL THIN SHEAR LAYERS

In a two-dimensional thin shear layer only one component of Reynolds' shear stress significantly affects the mean flow. If x_2 is taken as the coordinate normal to the mean flow streamwise direction x_1 then only the Reynolds' stress $-\rho\bar{\tau}_{12}$ is of importance. From Equation 6.308, the differential equation for $\bar{\tau}_{12}$ with $\alpha = 0.76$, $\beta = 0.11$, and $\gamma = 0.36$ is

$$\begin{aligned}\frac{D\bar{\tau}_{12}}{Dt} &= -(3c_1 + f_s)\frac{\epsilon}{K}\bar{\tau}_{12} - (0.24\bar{\tau}_{22} - 0.11\bar{\tau}_{11} + 0.18K)\frac{\partial \bar{u}_1}{\partial x_2}\\ &+ \frac{\alpha_0^{3/2}}{\kappa}\left[c_1\frac{\epsilon}{K}\bar{\tau}_{11} + c_2(\bar{\tau}_{11} - \bar{\tau}_{22})\frac{\partial \bar{u}_1}{\partial x_2}\right]\frac{K^{3/2}}{\epsilon x_2}\\ &+ \frac{\partial}{\partial x_2}\left\{\frac{Kc_s}{\epsilon}\left(2\bar{\tau}_{22}\frac{\partial \bar{\tau}_{12}}{\partial x_2} + \bar{\tau}_{12}\frac{\partial \bar{\tau}_{22}}{\partial x_2}\right) + \nu\frac{\partial \bar{\tau}_{12}}{\partial x_2}\right\}\end{aligned}$$

where $c_1 = 0.5$, $c_2 = 0.06$, $\alpha_0 = 0.3$, and $\kappa = 0.41$, and D/Dt is the streamwise convective operator. Now, from Figure 6.18a,

$$\overline{u_3'^2} = \bar{\tau}_{33} \cong \frac{1}{2}(\bar{\tau}_{11} + \bar{\tau}_{22})$$

hence

$$K = \frac{3}{4}(\bar{\tau}_{11} + \bar{\tau}_{22})$$

Further, from Figure 6.19:

$$-\bar{\tau}_{12}/(\bar{\tau}_{11}\bar{\tau}_{22})^{1/2} = 0.47$$

hence neglecting $(\bar{\tau}_{12}/K)^4$ and higher powers we get:

$$\bar{\tau}_{22} = 3.4(\bar{\tau}_{12})^2/K \tag{6.311a}$$

Also,

$$\bar{\tau}_{11} = 4K/3 - 3.4(\bar{\tau}_{12})^2/K \tag{6.311b}$$

Thus, the equation for $\bar{\tau}_{12}$ is

$$\frac{D\bar{\tau}_{12}}{Dt} = -(3c_1 + f_s)\frac{\epsilon}{K}\bar{\tau}_{12} + \frac{\partial}{\partial x_2}\left[\left\{\nu + 6.8c_s\frac{2(\bar{\tau}_{12})^2}{\epsilon}\right\}\frac{\partial\bar{\tau}_{12}}{\partial x_2}\right]$$

$$-K\left[1.19\left(\frac{\bar{\tau}_{12}}{K}\right)^2 + 0.033\right]\frac{\partial\bar{u}_1}{\partial x_2} + \left[\left\{-3.4c_1\epsilon\left(\frac{\bar{\tau}_{12}}{K}\right)^2 + \frac{4c_1}{3}\epsilon\right\}\right.$$

$$\left. + c_2\left\{\frac{4}{3}K - 6.8\frac{(\bar{\tau}_{12})^2}{K}\right\}\frac{\partial\bar{u}_1}{\partial x_2}\right]\frac{(\alpha_0 K)^{3/2}}{\epsilon x_2 \kappa} \tag{6.312}$$

The energy and dissipation equations from Equations 6.309 and 6.301 by using Equations 6.311 are (refer to Equation 6.310b) and the discussion preceding it):

$$\frac{DK}{dt} = P - \epsilon + \frac{\partial}{\partial x_2}\left[\left\{\nu + 3.9c_s\frac{(\bar{\tau}_{12})^2}{\epsilon}\right\}\frac{\partial K}{\partial x_2}\right] \tag{6.313}$$

$$\frac{D\epsilon}{dt} = c_{\epsilon 1}\frac{\epsilon P}{K} - c_{\epsilon 2}f_\epsilon\frac{\epsilon}{K}\left[\epsilon - 2\nu\left(\frac{\partial\sqrt{K}}{\partial x_2}\right)^2\right]$$

$$+ \frac{\partial}{\partial x_2}\left[\left\{\nu + 3.4c_\epsilon\frac{(\bar{\tau}_{12})^2}{\epsilon}\right\}\frac{\partial\epsilon}{\partial x_2}\right] + 3.4c_{\epsilon 3}\frac{(\bar{\tau}_{12})^2}{\epsilon}\nu\left(\frac{\partial^2\bar{u}_1}{\partial x_2^2}\right)^2 \tag{6.314}$$

where note that under the thin layer approximation:

$$P = -\bar{\tau}_{12}\frac{\partial\bar{u}_1}{\partial x_2}, \qquad P_{12} = -\bar{\tau}_{22}\frac{\partial\bar{u}_1}{\partial x_2}, \qquad \Delta_{12} = -\bar{\tau}_{11}\frac{\partial\bar{u}_1}{\partial x_2}$$

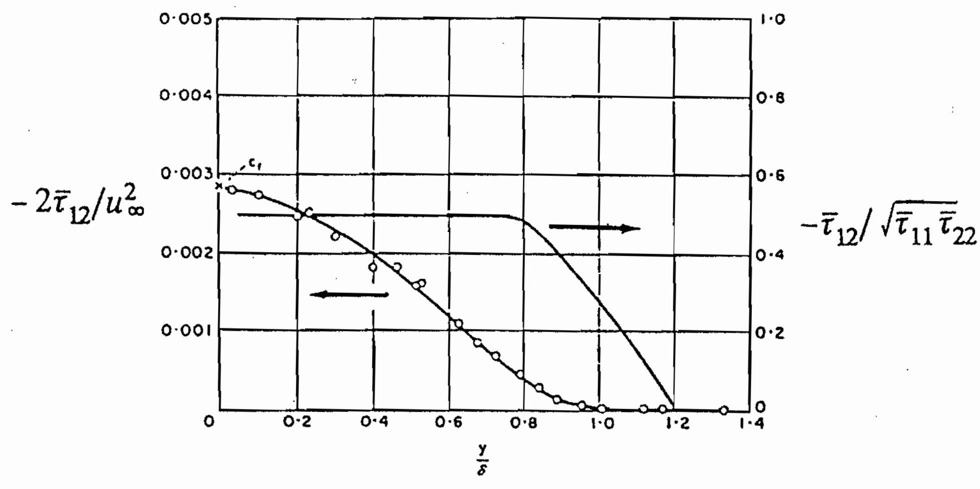
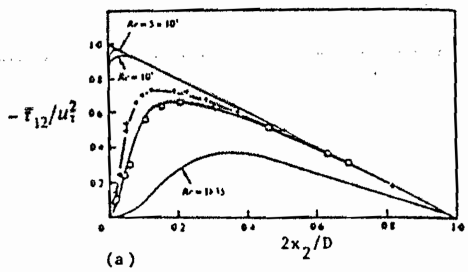


Fig. 6.19 Distribution of the normalized Reynold's stress in the fully developed flow on a flat plate. (From Klebanoff, P. S., NACA TN, 3178, 1954.)

The solution of the thin shear layer equations, i.e., Equations 6.312–6.314, have been obtained by Hanjalic and Launder;[77a] and the results for distribution of the Reynolds stress and the turbulence energy in a channel of width D are reproduced as follows.



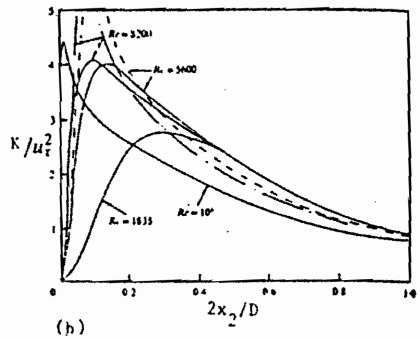


Fig. 6.20 (a)Distribution of normalized Reynolds' stress in channel; (b)Distribution of normalized energy in a channel. Experimental data of Eckelmann[87] is shown by + ▪·:Data --•-- Reynolds number 8200; o, --- Reynolds number 5600. (Taken from Hànjalic, K. and Launder, B. E., *J. Fluid Mech.*, 74, 593, 1976. With permission.)

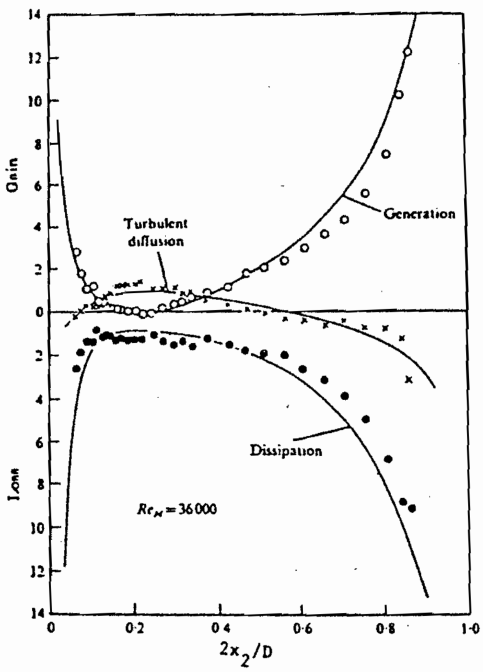


Fig. 6.21 Turbulence energy balance in asymmetric channel flow. Smooth curves are the calculated values. Data o production, • dissipation, × diffusion. (Taken from Hanjalic, K. and Launder, B. E., *J. Fluid Mech.*, 52, 609, 1972. With permission.)

6.28 ALGEBRAIC REYNOLDS' STRESS CLOSURE

Algebraic Reynolds' stress closure is a technique to generate Reynolds' stresses without solving differential equations for the stresses. An attractive feature of the technique to be developed is that the resulting closure model is a rigorous consequence of the rate equations for the stresses themselves. The whole computational procedure, therefore, evolves around the generation of the Reynolds stresses through the algebraic model while solving only Equations 6.301 and 6.309 for ϵ and K, respectively.

The original idea for algebraic stress closure is due to Rodi.[88] Later, Warsi et al.[89] used the same technique to establish a nonlinear constitutive equation by deductive iteration.

The fundamental equation of the method is Equation 6.175, which we rewrite as:

$$\frac{D\bar{\tau}_{ij}}{Dt} = P_{ij} + Q_{ij} - \epsilon_{ij} + d_{ij} \tag{6.315a}$$

where:

$$d_{ij} = \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \bar{\tau}_{ij}}{\partial x_k}\right) + F_{ij} \tag{6.315b}$$

and P_{ij}, Q_{ij}, ϵ_{ij}, and F_{ij} have been defined in Equations 6.176. Recall that:

$$P_{ii} = 2P, \qquad Q_{ii} = 0, \quad \epsilon_{ii} = 2\epsilon, \qquad F_{ii} = 2D$$

Contracting the indices in Equations 6.315, we get the energy equation:

$$\frac{DK}{Dt} = P - \epsilon + d \tag{6.315c}$$

where:

$$d = \tfrac{1}{2} d_{ii} = \frac{\partial}{\partial x_k}\left(\nu \frac{\partial K}{\partial x_k}\right) + D$$

We now write:

$$T_{ij} = \bar{\tau}_{ij}/K$$

in Equation 6.315a and use Equation 6.315c to have:

$$T_{ij}(P - \epsilon + d) + K\frac{DT_{ij}}{Dt} = P_{ij} + Q_{ij} - \epsilon_{ij} + d_{ij}$$

If the derivatives of T_{ij} are assumed to be small in comparison with the other terms, further assuming that:

$$d_{ij} = T_{ij} d \tag{6.315d}$$

or by neglecting viscosity:

$$F_{ij} = T_{ij} D \tag{6.315e}$$

then the above equation becomes:

$$T_{ij}(P - \epsilon) = P_{ij} + Q_{ij} - \epsilon_{ij} \tag{6.316}$$

Equation 6.316 provides an implicit algebraic (nondifferential) equation for the determination of $\bar{\tau}_{ij}$ once the modelings given in Equations 6.303 and 6.304 are inserted in place of Q_{ij} and ϵ_{ij}. Rodi, however, uses the following modelings:

$$Q_{ij} = 3c_1 \frac{\epsilon}{K} (\tfrac{2}{3} K\delta_{ij} - \bar{\tau}_{ij}) + 10c_2(\tfrac{2}{3} P\delta_{ij} - P_{ij})$$

$$\epsilon_{ij} = \tfrac{2}{3} \epsilon\delta_{ij}$$

for the region away from a wall and obtains:

$$T_{ij} = \tfrac{2}{3} \delta_{ij} + \frac{\gamma_0}{P + a_0\epsilon} (P_{ij} - \tfrac{2}{3} P\delta_{ij}) \tag{6.317}$$

where:

$$\gamma_0 = 1 - 10c_2, \quad a_0 = 3c_1 - 1, \quad c_1 = 0.5, \quad c_2 = 0.06$$

For the modeling Q_{ij}, refer to Reference 76.

Besides the modelings of Q_{ij} and ϵ_{ij} there are two main assumptions involved in the derivation of either Equation 6.316 or 6.317. The first one is that the derivatives of T_{ij} are vanishingly small. This assumption is a restatement of the so-called structural equilibrium hypothesis, also known as Townsend's[28] hypothesis. In essence this hypothesis states that for thin shear layers the shear stress $-\rho\bar{\tau}_{12}$ is linearly proportional to the kinetic energy K. Donaldson[90] and Bradshaw[91] have also utilized this hypothesis with much advantage. The second assumption, viz., Equation 6.315d, can be only weakly justified if one takes the diffusion formulation of F_{ij} of Rodi. Hence, if one takes:

$$-\overline{u_i'u_j'u_k'} = c_S \frac{K}{\epsilon} \bar{\tau}_{k\ell} \frac{\partial \bar{\tau}_{ij}}{\partial x_\ell}; \qquad c_S = 0.25 \tag{6.318a}$$

and thus:

$$F_{ij} = \frac{\partial}{\partial x_k} \left(c_S \frac{K}{\epsilon} \bar{\tau}_{k\ell} \frac{\partial \bar{\tau}_{ij}}{\partial x_\ell} \right) \tag{6.318b}$$

then on substituting $\bar{\tau}_{ij} = KT_{ij}$ in Equation 6.318b and again neglecting the derivatives of T_{ij}, we get the result given in Equation 6.315d. The weak justification comes from the observation that unlike the left-hand side, the right-hand side of Equation 6.318a alters under the permutation of the indices i, j, and k. This defect does not appear if one uses the formulation (Equation 6.306) but then the simple result (Equation 6.315d) cannot be obtained without further assumptions. Despite these problems with the diffusion term the simplicity of Equation 6.317 is attractive and can be used for a calculation of the Reynolds stresses away from the wall. If wall effects are desired, it is preferable to use Equation 6.316 with proper modelings. In any event, the determination of T_{ij} through either Equation 6.316 or 6.317 has to be an iterative one.

6.29 DEVELOPMENT OF A NONLINEAR CONSTITUTIVE EQUATION

Here we use implicit Equation 6.317 to obtain zero, first, and second approximations by deductive iteration. To carry out this program we introduce the quantity:

$$M_{ij} = \frac{K}{\epsilon} \frac{\partial \bar{u}_i}{\partial x_j}$$

and:

$$T_{ij} = \bar{\tau}_{ij}/K$$

in P_{ij} and P defined in Equations 6.176a, b. Thus:

$$P_{ij} = -\epsilon(T_{ik}M_{jk} + T_{jk}M_{ik})$$

$$P = -\epsilon T_{k\ell}M_{k\ell}$$

Substituting these expressions in Equation 6.317, we get:

$$(a_0 - T_{k\ell}M_{k\ell})T_{ij} = \tfrac{2}{3}\{a_0 - (1 - \gamma_0)T_{k\ell}M_{k\ell}\}\delta_{ij} - \gamma_0(T_{ik}M_{jk} + T_{jk}M_{ik}) \qquad \textbf{(6.319)}$$

Equation 6.319 is a nonlinear algebraic equation for the determination of T_{ij}. We propose to set up the following iterative scheme for the determination of T_{ij}:

$$a_0T_{ij}^{(n+1)} = T_{ij}^{(n)}T_{k\ell}^{(n)}M_{k\ell} + \tfrac{2}{3}\{a_0 - (1 - \gamma_0)T_{k\ell}^{(n)}M_{k\ell}\}\delta_{ij} - \gamma_0(T_{ik}^{(n)}M_{jk} + T_{jk}^{(n)}M_{ik}) \qquad \textbf{(6.320)}$$

when (n) is the iteration index. For the zero order approximation we take the isotropic form (refer to equation after Equation 6.302):

$$T_{ij}^{(0)} = \tfrac{2}{3}\,\delta_{ij} \qquad \textbf{(6.321a)}$$

which means that $\overline{\tau}_{ij}^{(0)} = 2/3K\delta_{ij}$. Substituting Equation 6.321a in Equation 6.320 and using continuity equation $M_{ii} = 0$, we get:

$$T_{ij}^{(1)} = \tfrac{2}{3}\,\delta_{ij} - A(M_{ij} + M_{ji}) \qquad \textbf{(6.321b)}$$

where:

$$A = \frac{2\gamma_0}{3a_0}$$

is a constant. In terms of the familiar variables, Equation 6.321b is

$$\overline{\tau}_{ij} = \tfrac{2}{3}\,K\delta_{ij} - \frac{AK^2}{\epsilon}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \qquad \textbf{(6.321c)}$$

which is immediately recognized as the Boussinesq-Kolmogorov eddy viscosity hypothesis with:

$$\nu_T = \frac{AK^2}{\epsilon}$$

However, on comparison with Equation 6.297 for high Reynolds numbers we find that constant A is not equal to α_0^2, since for $\gamma_0 = 0.4$ and $a_0 = 0.5$ (values suggested by Launder) the value of $A = 0.53$. If, on the other hand, we take the value of $a_0 = 1.86 = (2/.7\text{-}1)$ as suggested by Rotta[38a] and $\gamma_0 = 0.3$, then $A = 0.11$ (which is close to $\alpha_0^2 = 0.09$). Despite this discrepancy in constant A the form of Equation 6.321c is a valid approximation of the Reynolds stress equation, although both Boussinesq and Kolmogorov proposed it on a heuristic basis. We can go a step further and find the second approximation by substituting Equation 6.321b in Equation 6.320 and neglecting the third order terms in M_{ij}. Thus:

$$T_{ij}^{(2)} = \tfrac{2}{3}\delta_{ij} - A(M_{ij} + M_{ji}) + \tfrac{3}{2}A^2[(M_{ik} + M_{ki})M_{jk}$$

$$+ (M_{jk} + M_{kj})M_{ik} - \tfrac{2}{3}\delta_{ij}(M_{k\ell} + M_{\ell k})M_{k\ell}]$$

In terms of the rate-of-strain components, i.e.:

$$\overline{D}_{ij} = \tfrac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

we have:

$$\bar{\tau}_{ij} = \tfrac{2}{3}K\delta_{ij} - \frac{2AK^2}{\epsilon}\overline{D}_{ij}$$

$$+ 3A^2\frac{K^3}{\epsilon^2}\left[\overline{D}_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{D}_{jk}\frac{\partial \bar{u}_i}{\partial x_k} - \tfrac{2}{3}\delta_{ij}\overline{D}_{k\ell}\frac{\partial \bar{u}_k}{\partial x_\ell}\right] \qquad \textbf{(6.322)}$$

which is a nonlinear constitutive equation for the Reynolds stresses. The coordinate-invariant form of Equation 6.322 is

$$\bar{\boldsymbol{\tau}} = \tfrac{2}{3}K\mathbf{I} - \frac{2AK^2}{\epsilon}\overline{\mathbf{D}} + \frac{3A^2K^3}{\epsilon^2}[(\text{grad }\bar{\mathbf{u}})\cdot\overline{\mathbf{D}} + \{(\text{grad }\bar{\mathbf{u}})\cdot\overline{\mathbf{D}}\}^T - \tfrac{2}{3}(\overline{\mathbf{D}}:\overline{\mathbf{D}})\mathbf{I}] \qquad \textbf{(6.323)}$$

Constitutive Equation 6.323 is one of the many nonlinear models proposed by other researchers, e.g., Lumley,[92] Saffman[93] (Refer to Problem 6.19 for the Saffman model.) An important point to note here is that Equation 6.323 is a direct consequence of the rate equation of Reynolds' stress, i.e., Equation 6.175. Other proposals are usually of an ad hoc nature; and they basically follow the procedures used in obtaining the constitutive equations for non-Newtonian fluids, e.g., Reference 94. Recently, Speziale[95] has proposed a nonlinear constitutive equation of the following form:

$$\bar{\tau}_{ij} = \frac{2}{3}K\delta_{ij} - \frac{2\alpha_0^2K^2}{\epsilon}\overline{D}_{ij} - \frac{4\alpha_0^4\gamma_1K^3}{\epsilon^2}(\overline{D}_{im}\overline{D}_{mj}$$

$$-\frac{1}{3}\overline{D}_{mn}\overline{D}_{mn}\delta_{ij}) - \frac{4\alpha_0^4\gamma_2K^3}{\epsilon^2}(\overset{o}{\overline{D}}_{ij} - \frac{1}{3}\overset{o}{\overline{D}}_{mm}\delta_{ij}) \qquad \textbf{(6.324)}$$

where $\alpha_0 = 0.3$, $\gamma_1 \equiv \gamma_2 \equiv 1.68$. Further $\overset{o}{\overline{D}}_{ij}$ is the upper convected Oldroyd derivative[94] defined as:

$$\overset{o}{\overline{D}}_{ij} = \frac{\partial \overline{D}_{ij}}{\partial t} + \bar{u}_k\frac{\partial \overline{D}_{ij}}{\partial x_k} - \overline{D}_{kj}\frac{\partial \bar{u}_i}{\partial x_k} - \overline{D}_{ki}\frac{\partial \bar{u}_j}{\partial x_k}$$

From a rational mechanics point of view an important difference between Equation 6.322 and 6.324 is that the former is not invariant to arbitrary moving reference frames although it is Galilean invariant, as are the equations of motion of fluid dynamics. On the other hand, Equation 6.324 is invariant to arbitrary frames, since it has been constructed this way. Refer to ME.9 for frame-invariant conditions. It is, however, interesting to note that if the substantive derivative part in $\overset{o}{\overline{D}}_{ij}$ and the coefficient of γ_1 are neglected then 6.324 reduces to 6.322.

Extension to Compressible Flow

For the case of compressible flow we use the Favre averaging as described in Section 6.14. To recapitulate, if α and β are random variables then

$$\begin{aligned}\alpha &= \overline{\alpha} + \alpha' \\ &= \dot{\overline{\alpha}} + \alpha''\end{aligned}$$

where $\dot{\overline{\alpha}}$ is the Favre average and $\overline{\alpha''} \neq 0$ but $\overline{\rho\alpha''} = 0$. Two important results in this regard are

$$\overline{\alpha'\beta''} = \overline{\alpha'\beta'}$$

and

$$\overline{\rho\alpha''\beta''} = \overline{\rho}\,\dot{\overline{\alpha''\beta''}}$$

where an overhead bar is the usual Reynolds average. The relation between the Reynolds and Favre averages is

$$\overline{\alpha} = \dot{\overline{\alpha}} - \frac{\overline{\rho'\alpha'}}{\overline{\rho}}$$

Equations 6.194 and 6.195 written in Cartesian tensor form are

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho}\dot{\overline{u}}_j\right) = 0 \qquad \text{(6.325a)}$$

$$\frac{\partial}{\partial t}\left(\overline{\rho}\dot{\overline{u}}_i\right) + \frac{\partial}{\partial x_j}\left(\overline{\rho}\dot{\overline{u}}_i\dot{\overline{u}}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\sigma}_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(-\overline{\rho}\dot{\overline{\tau}}_{ij}\right) \qquad \text{(6.325b)}$$

where

$$\begin{aligned}\overline{\sigma} &= \frac{-2}{3}\overline{\mu}\overline{\Delta}\delta_{ij} + \overline{\mu}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \\ \overline{\Delta} &= \frac{\partial \overline{u}_k}{\partial x_k} \\ \tau_{ij} &= u_i'' u_j''\end{aligned}$$

The Reynolds stress equation, Equation 6.200b, is as follows:

$$\frac{\partial}{\partial t}\left(\overline{\rho}\dot{\overline{\tau}}_{ij}\right) + \frac{\partial}{\partial x_k}\left(\overline{\rho}\dot{\overline{u}}_k\dot{\overline{\tau}}_{ij}\right) = A_{ij} + B_{ij} + C_{ij} - \overline{\rho}\epsilon_{ij} - \overline{\rho}g_{ij}$$

$$+\frac{2}{3}\overline{p'\Delta'}\,\delta_{ij}-\left(\overline{u_i''}\frac{\partial\bar{p}}{\partial x_j}+\overline{u_j''}\frac{\partial\bar{p}}{\partial x_i}\right)$$
$$+\overline{u_i''}\frac{\partial\bar{\sigma}_{jk}}{\partial x_k}+\overline{u_j''}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} \tag{6.326}$$
$$+\frac{\partial}{\partial x_k}\left(\overline{\sigma'_{ik}u'_j+\sigma'_{jk}u'_i}\right)$$

where

$$\overline{\sigma'_{ik}\frac{\partial u'_j}{\partial x_k}+\sigma'_{jk}\frac{\partial u'_i}{\partial x_k}}=\bar{\rho}\epsilon_{ij}+\bar{\rho}g_{ij} \tag{6.327a}$$

and

$$A_{ij}=-\bar{\rho}\left(\dot{\bar{\tau}}_{ik}\frac{\partial\dot{\bar{u}}_j}{\partial x_k}+\dot{\bar{\tau}}_{ik}\frac{\partial\dot{\bar{u}}_i}{\partial x_k}\right) \tag{6.327b}$$

$$B_{ij}=\overline{p'\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}-\frac{2}{3}\overline{p'\Delta'\delta_{ij}} \tag{6.327c}$$

$$C_{ij}=-\frac{\partial}{\partial x_k}\left[\bar{\rho}\,\overline{\dot{\tau}_{ij}u_k''}+\overline{p'\left(u_i'\delta_{jk}+u_j'\delta_{ik}\right)}\right] \tag{6.327d}$$

Turbulence Energy Equation

We define

$$q^2=\frac{\rho u''\cdot u''}{\bar{\rho}},\quad K=\tfrac{1}{2}\overline{q^2}$$

Thus

$$K=\frac{1}{2\bar{\rho}}\overline{\rho u''\cdot u''}=\tfrac{1}{2}\overline{\dot{u}''\cdot u''} \tag{6.328}$$

which is the turbulence energy per unit mass. In Equations 6.327 writing j = i and forming the sum, we get

$$\left.\begin{aligned} A_{ii} &= -2\bar{\rho}\dot{\tilde{\tau}}_{ik}\frac{\partial\dot{\tilde{u}}_i}{\partial x_k} = 2A \\ B_{ii} &= 0 \\ C_{ii} &= -\frac{\partial}{\partial x_k}\left[\overline{\rho q^2 u_k''} + 2\overline{p' u_k'}\right] \\ &= -2\frac{\partial}{\partial x_k}\left[\overline{u_k''\left(p' + \tfrac{1}{2}\rho q^2\right)}\right] = 2C \end{aligned}\right\} \tag{6.329a}$$

Note that

$$\bar{\rho}\,\dot{\overline{\tau_{ii}u_k''}} = \overline{\rho\tau_{ii}u_k''} = \overline{\rho q^2 u_k''}$$

Similarly

$$\begin{aligned} \bar{\rho}\epsilon_{ii} + \bar{\rho}g_{ii} &= 2\overline{\sigma_{ik}'\frac{\partial u_i'}{\partial x_k}} \\ &= 2\bar{\rho}\epsilon_c \end{aligned} \tag{6.329b}$$

where ϵ_c is the compressible isotropic dissipation and defined in Equation 6.201d.

Writing j = i in Equation 6.326, we get the turbulence energy equation

$$\begin{aligned} \frac{\partial}{\partial t}(\bar{\rho}K) + \frac{\partial}{\partial x_k}\left(\bar{\rho}K\dot{\tilde{u}}_k\right) &= \left(A - \bar{\rho}\epsilon_c\right) + C + \overline{p'\Delta'} - \overline{u_i''}\frac{\partial\bar{p}}{\partial x_i} \\ &+ \overline{u_i''}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{\sigma_{ik}'u_i'}\right) \end{aligned} \tag{6.330}$$

On using Equation 6.325a in 6.326, we have

$$\begin{aligned} \bar{\rho}\frac{D}{Dt}\dot{\tilde{\tau}}_{ij} = A_{ij} &+ B_{ij} + C_{ij} - \bar{\rho}\epsilon_{ij} - \bar{\rho}g_{ij} \\ &+ \frac{2}{3}\overline{p'\Delta'}\delta_{ij} - \left(\overline{u_i''}\frac{\partial\bar{p}}{\partial x_j} + \overline{u_j''}\frac{\partial\bar{p}}{\partial x_i}\right) \\ &+ \overline{u_i''}\frac{\partial\bar{\sigma}_{jk}}{\partial x_k} + \overline{u_j''}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} \\ &+ \frac{\partial}{\partial x_k}\left(\overline{\sigma_{ik}'u_j'} + \overline{\sigma_{jk}'u_i'}\right) \end{aligned} \tag{6.331}$$

and Equation 6.330 as

$$\bar{\rho}\frac{DK}{Dt} = \left(A - \bar{\rho}\epsilon_c\right) + C + \overline{p'\Delta'} - \overline{u_k''}\frac{\partial\bar{p}}{\partial x_k} + \overline{u_m''}\frac{\partial\bar{\sigma}_{mk}}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{\sigma_{mk}'u_m'}\right) \tag{6.332}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\bar{u}}_k \frac{\partial}{\partial x_k}$$

We now introduce the ratio

$$T_{ij} = \frac{\bar{\tau}_{ij}}{K} \tag{6.333}$$

then

$$\begin{aligned}\bar{\rho}\frac{D}{Dt}\dot{\bar{\tau}}_{ij} &= \bar{\rho}\frac{D}{Dt}\left(KT_{ij}\right) \\ &= \bar{\rho}T_{ij}\frac{DK}{Dt} + \bar{\rho}K\frac{D}{Dt}T_{ij}\end{aligned} \tag{6.334}$$

On using Equation 6.334, 6.326 in Equation 6.331, we get

$$\begin{aligned}\bar{\rho}K\frac{D}{Dt}T_{ij} = & -T_{ij}\left[\left(A - \bar{\rho}\epsilon_c\right) + C + \overline{p'\Delta'} - \overline{u_k''}\frac{\partial\bar{p}}{\partial x_k} + \overline{u_m''}\frac{\partial\sigma_{mk}}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{\sigma'_{mk}u'_m}\right)\right] \\ & + A_{ij} + B_{ij} + C_{ij} - \bar{\rho}\epsilon_{ij} - \bar{\rho}g_{ij} + \frac{2}{3}\overline{p'\Delta'}\delta_{ij} \\ & - \left(\overline{u_i''}\frac{\partial\bar{p}}{\partial x_j} + \overline{u_j''}\frac{\partial\bar{p}}{\partial x_k}\right) + \overline{u_i''}\frac{\partial\bar{\sigma}_{jk}}{\partial x_k} + \overline{u_j''}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} \\ & + \frac{\partial}{\partial x_k}\left(\overline{\sigma'_{ik}u'_j} + \overline{\sigma'_{jk}u'_i}\right)\end{aligned} \tag{6.335}$$

Assumptions To Be Justified

We *a priori* make the following assumptions.

$$\left.\begin{aligned}&\text{(i)} && \frac{D}{Dt}T_{ij} = 0 \\ & && T_{ij}\left[C + \overline{p'\Delta'} - \overline{u_k''}\frac{\partial\bar{p}}{\partial x_k} + \overline{u_m''}\frac{\partial\sigma_{mk}}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{\sigma'_{mk}u'_m}\right)\right] \\ &\text{(ii)} && = C_{ij} - \bar{\rho}g_{ij} + \frac{2}{3}\overline{p'\Delta'}\delta_{ij} - \left(\overline{u_i''}\frac{\partial\bar{p}}{\partial x_j} + \overline{u_j''}\frac{\partial\bar{p}}{\partial x_i}\right) \\ & && + \overline{u_i''}\frac{\partial\bar{\sigma}_{jk}}{\partial x_k} + \overline{u_j''}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{\sigma'_{ik}u'_j} + \overline{\sigma'_{jk}u'_j}\right)\end{aligned}\right\} \tag{6.336}$$

Thus, Equation 6.335 becomes

$$T_{ij}\left(A - \bar{\rho}\epsilon_c\right) = A_{ij} + B_{ij} - \bar{\rho}\epsilon_{ij} \tag{6.337}$$

Implicit Algebraic Stress Model

We now introduce the following modelings which are exactly similar to the incompressible case:

$$\bar{\rho}\epsilon_{ij} = \frac{2}{3}\bar{\rho}\epsilon_c\delta_{ij} \tag{6.338a}$$

$$B_{ij} = 3c_1\frac{\epsilon_c}{K}\bar{\rho}\left(\frac{2}{3}K\delta_{ij} - \dot{\tau}_{ij}\right) + 10c_2\left(\frac{2}{3}A\delta_{ij} - A_{ij}\right) \tag{6.338b}$$

where

$$A = 1/2A_{mm},\ c_1 = 0.5,\ c_2 = 0.06$$

Substituting Equations 6.338a and b in Equation 6.337, we get

$$\begin{aligned} T_{ij}\left[A + (3c_1 - 1)\bar{\rho}\epsilon_c\right] &= \frac{2}{3}\left[A + (3c_1 - 1)\bar{\rho}\epsilon_c\right]\delta_{ij} \\ &\quad + (1 - 10c_2)A_{ij} - \frac{2}{3}(1 - 10c_2)A\delta_{ij} \end{aligned} \tag{6.339}$$

Thus

$$T_{ij} = \frac{2}{3}\delta_{ij} + \frac{(1 - 10c_2)\left(A_{ij} - \frac{2}{3}A\delta_{ij}\right)}{A + (3c_1 - 1)\,\bar{\rho}\epsilon_c}$$

or, writing

$$\gamma_n = 1 - 10c_2,\quad a_n = 3c_1 - 1$$

we have

$$T_{ij} = \frac{2}{3}\delta_{ij} + \frac{\gamma_n}{A + a_n\,\bar{\rho}\epsilon_c}\left(A_{ij} - \frac{2}{3}A\delta_{ij}\right) \tag{6.340}$$

Equation 6.340 provides an implicit algebraic stress model for the compressible flow case.

Explicit Algebraic Stress Model

Let

$$M_{ij} = \frac{K}{\epsilon_c}\frac{\partial \dot{\bar{u}}_i}{\partial x_j}$$

then from Equation 6.327b

$$A_{ij} = -\bar{\rho}\epsilon_c (T_{ik} M_{jk} + T_{jk} M_{ik})$$

and

$$A = -\bar{\rho}\epsilon_c T_{lk} M_{lk}$$

Consequently, Equation 6.340 is written as

$$\left(a_o - T_{lk} M_{lk}\right) T_{ij} = \frac{2}{3}\left\{a_o - \left(1-\gamma_o\right) T_{lk} M_{lk}\right\} \delta_{ik} \\ -\gamma_o \left(T_{ik} M_{jk} + T_{jk} M_{ik}\right) \tag{6.341}$$

By using the earlier method of successive iteration, the zero, first, and second order approximations as obtained from Equation 6.341 are

$$\dot{\bar{\tau}}_{ij}^{(0)} = \frac{2}{3} K \delta_{ij} \tag{6.342}$$

$$\dot{\bar{\tau}}_{ij}^{(1)} = \frac{2}{3} K \delta_{ij} - \frac{2AK^2}{\epsilon_c} \dot{\bar{D}}_{ij} \tag{6.343}$$

$$\dot{\bar{\tau}}_{ij}^{(2)} = \frac{2}{3} K \delta_{ij} - \frac{2AK^2}{\epsilon_c} \dot{\bar{D}}_{ij}$$

$$+ 3A^2 \frac{K^3}{\epsilon_c^2} \left[\dot{\bar{D}}_{ik} \frac{\partial \dot{\bar{u}}_j}{\partial x_k} + \dot{\bar{D}}_{jk} \frac{\partial \dot{\bar{u}}_i}{\partial x_k} - \frac{2}{3} \delta_{ij} \dot{\bar{D}}_{kl} \frac{\partial \dot{\bar{u}}_k}{\partial x_l} \right] \tag{6.344}$$

where

$$\dot{\bar{D}}_{ij} = ½ \left(\frac{\partial \dot{\bar{u}}_i}{\partial x_j} + \frac{\partial \dot{\bar{u}}_j}{\partial x_i} \right)$$

and

$$A = \frac{2\gamma_0}{3a_0} = 0 \cdot 11$$

Equation 6.344 is the second-order explicit nonlinear stress model for the compressible flow case. It must be emphasized that Equation 6.344 provides a form for the nonlinear turbulence modeling. The constants are not universal and have to be adjusted for cases which are of strongly nonhomogeneous turbulent in nature.

The Dissipation Equation

Following Sarkar et al. (Reference 95d), we take

$$\epsilon_c = \epsilon_S\left(1+\alpha_1 M_T^2\right) \tag{6.345}$$

where ϵ_S is the dilatation dissipation and M_T is the turbulence Mach number defined as

$$\begin{aligned} M_T^2 &= \frac{2K}{\gamma R\dot{\bar{T}}} \\ \gamma &= c_p / c_v \\ R &= \textit{specific gas constant.} \end{aligned} \tag{6.346}$$

The constant $\alpha_1 = 1.0$.

The differential equation for ϵ_s from Ref. 95d is

$$\begin{aligned} \frac{\partial}{\partial t}\left(\bar{\rho}\epsilon_s\right) + \frac{\partial}{\partial x_l}\left(\bar{\rho}\epsilon_s\dot{\bar{u}}_k\right) &= -C_{\epsilon 1}\frac{\epsilon_s}{K}\bar{\rho}\dot{\bar{\tau}}_{ij}\frac{\partial \dot{\bar{u}}_i}{\partial x_j} \\ &\quad -C_{\epsilon 2}\bar{\rho}\frac{\epsilon_S^2}{K} + \frac{\partial}{\partial x_k}\left(C_\epsilon \frac{\bar{\rho}K\dot{\bar{\tau}}_{kl}}{\epsilon_S}\frac{\partial \epsilon_S}{\partial x_l}\right) \end{aligned} \tag{6.347}$$

where

$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.90, \quad C_\epsilon = 0.15$$

The Total Energy Equation

The instantaneous energy per unit mass is

$$\begin{aligned} e_t &= e + \tfrac{1}{2}u\cdot u \\ &= e + \tfrac{1}{2}\dot{\bar{u}}\cdot\dot{\bar{u}} + \dot{\bar{u}}\cdot u'' + \tfrac{1}{2}u''\cdot u'' \end{aligned}$$

Multiplying by ρ and performing the Reynolds averaging of each term and then dividing by $\bar{\rho}$, we get

$$\dot{\bar{e}}_t = \dot{\bar{e}} + \tfrac{1}{2}\,\dot{\bar{u}}\cdot\dot{\bar{u}} + \tfrac{1}{2}\,\dot{\overline{u''\cdot u''}}$$

The equation of total energy $\dot{\bar{e}}_t$ is Equation 6.197, which is written as

$$\begin{aligned} \frac{\partial}{\partial t}\left(\bar{\rho}\dot{\bar{e}}_t\right) + \frac{\partial}{\partial x_k}\left(\bar{\rho}\dot{\bar{u}}_k\dot{\bar{e}}_t\right) &= \frac{\partial}{\partial x_k}\left(\bar{\sigma}_{jk}\dot{\bar{u}}_j - \bar{p}\dot{\bar{u}}_k + \bar{\kappa}\frac{\partial \dot{\bar{T}}}{\partial x_k}\right) \\ &\quad + \frac{\partial}{\partial x_k}\left(\overline{\sigma'_{jk}u'_j} - \overline{p'u'_k} - \bar{\rho}\dot{\overline{u''_k e''_t}}\right) \end{aligned} \tag{6.348}$$

where $\bar{\kappa}$ is the averaged thermal conductivity. Coupled with the preceding equations is the ideal gas equation

$$\bar{p} = \bar{\rho} R \dot{\bar{T}} \tag{6.349}$$

Modeling of the Correlations in the Total Energy Equation

First of all for high Reynolds number flows we set

$$\overline{\sigma'_{jk} u'_j} \simeq 0, \ for \ k = 1, 2, 3$$

Next the turbulent heat flux is modeled as[95d]

$$\overline{\dot{T}'' u''_k} = \frac{-\alpha_0^2 K^2}{\epsilon_c \sigma_T} \frac{\partial \dot{\bar{T}}}{\partial x_k}$$

where ε_c is the total dissipation as given in Equation 6.345. Further, $\alpha_0 = 0.3$ and the turbulent Prandtl number $\sigma_T = 0.7$. Next, by using $\dot{\bar{e}}_t$ in the definition of e_t we get

$$e''_t = e'' + \dot{\bar{u}} \cdot \dot{u}'' + \tfrac{1}{2} u'' \cdot u'' - \tfrac{1}{2} \overline{\dot{u}'' \cdot u''}$$

Using the ideal gas formula

$$e'' = c_v T''$$

where c_v is the coefficient of specific heat at constant volume, we get the turbulent energy flux as

$$\begin{aligned}\overline{\dot{e}''_t u''_k} &= c_v \overline{\dot{T}'' u''_k} + \dot{\bar{u}}_j \overline{\dot{u}''_j u''_k} + \tfrac{1}{2} \overline{\dot{u}''_j u''_j u''_k} \\ &= c_v \overline{\dot{T}'' u''_k} + \dot{\bar{u}}_j \dot{\bar{\tau}}_{jk} + \tfrac{1}{2} \overline{\dot{u}''_j \tau_{jk}}\end{aligned}$$

Finally, to convert the Favre average velocity to the Reynolds average velocity, we need to have a model for the turbulent mass flux $\overline{\rho' u'_k}$. Again, following Reference 95d, we model this term as

$$\overline{\rho' u'_k} = -\frac{\alpha_n^2 K^2}{\epsilon_c \sigma_S} \frac{\partial \bar{\rho}}{\partial x_k}$$

6.30 CURRENT APPROACHES TO NONLINEAR MODELING

Since the early 1990s there has been a renewed attempt to develop a nonlinear Reynolds stress model which can universally be used for turbulent shear layers of strongly nonhomogeneous character. A series of papers written by Speziale, Gatski, and Abid with coworkers (References 95a through 95d) have appeared. For brevity, we shall use the abbreviation SGA for the methodology and the model developed by these authors. A common theme of the SGA method and the consequent model is to make the model depend on the anisotropy of the Reynolds stress and on the interactions of the turbulent field quantities with the rate-of-strain and the vorticity fields of the mean flow.

In place of just reproducing the SGA model, it will be much more beneficial to a reader to see the similarities and differences in the quantities and the model developed in Sections 6.28 and 6.29 with the corresponding SGA quantities and the model. From this one can then usefully compare the two approaches. The discussion to follow is centered only around the incompressible mean turbulent fields.

In the SGA model, the anisotropy tensor b_{ij} defined as

$$b_{ij} = \frac{\overline{\tau}_{ij}}{2K} - \tfrac{1}{3}\delta_{ij} \tag{6.350}$$

is used. Note that in the notation of Section 6.28

$$b_{ij} = \tfrac{1}{2}T_{ij} - \tfrac{1}{3}\delta_{ij}$$

Further, the usual mean rate-of-strain and the vorticity tensors are

$$\overline{D}_{ij} = \tfrac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

$$\overline{W}_{ij} = \tfrac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

Using these quantities, the production term P_{ij} in Equation 6.315a becomes

$$P_{ij} = -2K\left(b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik}\right) - 2K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) - \frac{4}{3}K\overline{D}_{ij} \tag{6.351a}$$

For incompressible flow and further noting that

$$b_{ik}\overline{W}_{ik} = 0$$

the energy production term becomes

$$P = -2Kb_{ik}\overline{D}_{ik} \tag{6.351b}$$

Using Equation 6.315a in Equation 6.316, we get

$$\begin{aligned} b_{ij}(P-\epsilon) = & -\frac{2}{3}K\overline{D}_{ij} - K\left(b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik} - \frac{2}{3}b_{mn}\overline{D}_{mn}\delta_{ij}\right) \\ & -K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) + \tfrac{1}{2}Q_{ij} \end{aligned} \tag{6.352}$$

which is the Equation 6.316 in the terminology of SGA. Next, we first write the modeling of the pressure rate-of-strain term Q_{ij} adopted in Section 6.29 and stated preceeding Equation 6.317. In terms of SGA terminology

$$Q_{ij} = -6c_1\epsilon b_{ij} + \frac{40}{3}c_2 K\overline{D}_{ij} + 20c_2 K\left(b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik} - \frac{2}{3}b_{mn}\overline{D}_{mn}\delta_{ij}\right) + 20c_2 K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) \tag{6.353}$$

On the other hand, in the SGA method, the term Q_{ij} is modeled as

$$Q_{ij} = -C_1\epsilon b_{ij} + C_2 K\overline{D}_{ij} + C_3 K\left(b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik} - \frac{2}{3}b_{mn}\overline{D}_{mn}\delta_{ij}\right) + C_4 K\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) \tag{6.354}$$

On comparison we note that both forms are exactly the same but the constants are different. That is, according to the choice of $c_1 = 0.5$ and $c_2 = 0.06$.

$$6c_1 = 3.0, \quad \frac{40}{3}c_2 = 0.8, \quad 20c^2 = 1.2$$

But the comparable constants in Equation 6.354 chosen in SGA are

$$C_1 = 6.8, \quad C_2 = 0.36, \quad C_3 = 1.25, \quad C_4 = 0.4$$

Thus, the first basic difference between the implicit models of Section 6.28 and SGA is in the choice of constants in the modeling of Q_{ij}.

We now describe the linerization method of SGA. First, Equation 6.354 is substituted in Equation 6.352 to have

$$b_{ij} = \tfrac{1}{2} g\tau\left(C_2 - \frac{4}{3}\right)\overline{D}_{ij} + (C_3 - 2)\left(b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik} - \frac{2}{3}b_{mn}\overline{D}_{mn}\delta_{ij}\right) + (C_4 - 2)\left(b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right) \tag{6.355a}$$

where

$$\tau = \frac{K}{\epsilon}, \quad g = \left(\frac{C_1}{2} + \frac{P}{\epsilon} - 1\right)^{-1}$$

which is an implicit equation in b_{ij}. In the SGA methods, this equation is solved by iteration; the $(n + 1)st$ iterate is explicitly given as

$$\left[\left(\frac{P}{\epsilon}\right)^{(n)} - 1\right]b_{ij}^{n+1} = -\frac{2}{3}\tau^{(n)}\overline{D}_{ij}^{(n+1)} - \tau^{(n)}\left\{b_{ik}\overline{D}_{jk} + b_{jk}\overline{D}_{ik} - \frac{2}{3}b_{mn}\overline{D}_{mn}\delta_{ij}\right\}^{(n+1)} - \tau^{(n)}\left\{b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}\right\}^{(n+1)} + \tfrac{1}{2}\frac{Q_{ij}^{(n+1)}}{\epsilon^{(n)}} \tag{6.355b}$$

This equation can be written in terms of the non-dimensional rescaled variables

$$\overline{D}_{ij}^{*} = \tfrac{1}{2} g \tau (2 - C_3) \overline{D}_{ij}$$

$$\overline{W}_{ij}^{*} = \tfrac{1}{2} g \tau (2 - C_4) \overline{W}_{ij}$$

$$b_{ij}^{*} = \left(\frac{C_3 - 2}{C_2 - \frac{4}{3}} \right) b_{ij}$$

to have

$$\begin{aligned} b_{ij}^{*(n+1)} = & -\frac{3}{3 - 2\eta^2 + 6\xi^2} \Big\{ \overline{D}_{ij}^{*(n+1)} + \overline{D}_{ik}^{*(n+1)} \overline{W}_{kj}^{*(n+1)} \\ & + \overline{D}_{jk}^{*(n+1)} \overline{W}_{ki}^{*(n+1)} \\ & -2\left(\overline{D}_{ik}^{*(n+1)} \overline{D}_{kj}^{*(n+1)} - \tfrac{1}{3} \overline{D}_{mn}^{*(n+1)} \overline{D}_{mn}^{*(n+1)} \delta_{ij} \right) \Big\} \end{aligned} \tag{6.356}$$

where

$$\overline{D}_{ij}^{*(n+1)} = \tfrac{1}{2} g^{(n)} \tau^{(n)} (2 - C_3) \overline{D}_{ij}^{(n+1)}$$

$$\overline{W}_{ij}^{*(n+1)} = \tfrac{1}{2} g^{(n)} \tau^{(n)} (2 - C_4) \overline{W}_{ij}^{(n+1)}$$

$$b_{ij}^{*(n+1)} = \left(\frac{C_3 - 2}{C_2 - \frac{4}{3}} \right) b_{ij}^{(n+1)}$$

$$\xi = \left[\overline{W}_{ij}^{*(n+1)} \overline{W}_{ij}^{*(n+1)} \right]^{1/2}$$

$$\eta = \left[\overline{D}_{ij}^{*(n+1)} \overline{D}_{ij}^{*(n+1)} \right]^{1/2}$$

To remove the singularity in the coefficient appearing in Equation 6.356, a Padé-type regularization is used to have

$$\frac{3}{3 - 2\eta^2 + 6\xi^2} \simeq \frac{3(1 + \eta)}{3 + \eta^2 + 6\xi^2 + 6\xi^2 \eta^2}$$

Equation 6.356 is the nonlinear Reynolds stress model according to Speziale, Gatski, Abid et al. The second important difference between the model developed in Sections 6.28 and 6.29 and the SGA model is in the iteration techniques employed. In Equation 6.322, all terms on the right are those which are evaluated in the first iteration. From Equation 6.321c

$$b_{ij}^{(1)} = -A \frac{K}{\epsilon} \overline{D}_{ij}$$

so that the model equation 6.322 in terms of b_{ij}, $\overline{D}_{ij}$, and $\overline{W}_{ij}$ takes the form

$$\begin{aligned} b_{ij}^{(2)} = b_{ij}^{(1)} & - \frac{3A}{2} \frac{K}{\epsilon} \Big[\left(b_{ik}^{(1)} \overline{D}_{jk} + b_{jk}^{(1)} \overline{D}_{ik} - \frac{2}{3} b_{kl}^{(1)} \overline{D}_{kl} \delta_{ij} \right) \\ & + \left(b_{ik}^{(1)} \overline{W}_{jk} + b_{jk}^{(1)} \overline{W}_{ik} \right) \Big] \end{aligned} \tag{6.357}$$

6.31 HEURISTIC MODELING

In heuristic modeling the transport equations for energy, dissipation, and Reynolds' stresses are formed intuitively on an ad hoc basis without resorting to the traditional equations of mean turbulence, except those of continuity and momentum. This line of approach was initiated by Kolmogorov[69a] and Saffman.[93] Further developments were made by Wilcox and Chambers[96] and Rubesin.[97] The important turbulence quantities for further consideration are K, ω, and $\overline{\tau}_{ij}$, where ω is the specific dissipation, i.e.:

$$\omega = \epsilon/K \tag{6.358a}$$

Thus, the eddy viscosity (from Equation 6.297) is expressed as:

$$\mu_T = \frac{\rho\alpha_0^2 K}{\omega} \tag{6.358b}$$

The model equation for the Reynolds stress is taken as:

$$\frac{\partial\overline{\tau}_{ij}}{\partial t} + \overline{u}_k\frac{\partial\overline{\tau}_{ij}}{\partial x_k} = P_{ij} + \gamma_1\omega(\tfrac{2}{3}K\delta_{ij} - \overline{\tau}_{ij}) + \frac{\gamma_2 K}{\omega}(\tfrac{1}{3}\delta_{ij}\overline{D}_{k\ell}\overline{D}_{k\ell} - \overline{D}_{ik}\overline{D}_{kj})$$
$$- \tfrac{2}{3}K\omega(1 - \gamma_3\lambda + \gamma_4\lambda^2)\delta_{ij} + \frac{\partial}{\partial x_k}\left\{\left(\nu + \frac{\alpha_0^2 K}{2\omega}\right)\frac{\partial\overline{\tau}_{ij}}{\partial x_k}\right\} \tag{6.359}$$

where:

$$\lambda = (2\overline{D}_{k\ell}\overline{D}_{k\ell})^{1/2}/\omega$$

and:

$$\gamma_1 = 4.5, \quad \gamma_2 = 4\alpha_0^2, \quad \gamma_3 = \alpha_0, \quad \gamma_4 = \alpha_0^2$$

Contracting the indices in Equation 6.359, we get the equation of energy:

$$\frac{\partial K}{\partial t} + \overline{u}_k\frac{\partial K}{\partial x_k} = \left\{\alpha_0(2\overline{D}_{ij}\overline{D}_{ij})^{1/2} - \omega\right\}K$$
$$- \left(\overline{\tau}_{ij} + \frac{2\alpha_0^2 K}{\omega}\overline{D}_{ij}\right)\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_k}\left\{\left(\nu + \frac{\alpha_0^2 K}{2\omega}\right)\frac{\partial K}{\partial x_k}\right\} \tag{6.360}$$

The equation of ω^2 as modeled by Saffman is

$$\frac{\partial\omega^2}{\partial t} + \overline{u}_k\frac{\partial\omega^2}{\partial x_k} = \left\{\alpha_1\left(\frac{\partial\overline{u}_i}{\partial x_j}\frac{\partial\overline{u}_i}{\partial x_j}\right)^{1/2} - \beta_1\omega\right\}\omega^2 + \frac{\partial}{\partial x_k}\left\{\left(\nu + \frac{\alpha_0^2 K}{2\omega}\right)\frac{\partial\omega^2}{\partial x_k}\right\} \tag{6.361}$$

where:

$$\alpha_1 = \alpha_0\beta_1 - 2\kappa^2$$

and:

$$\alpha_0 = 0.3, \quad \kappa = 0.41, \quad \beta_1 = \tfrac{5}{3}$$

The choice of constants, particularly of γ_3 and γ_4, is governed by the behavior of production and dissipation of energy in the near-wall region. Let n be the coordinate normal to a wall; then the eddy viscosity formulation of the Reynolds stress, Equation 6.358b, yields:

$$-\bar{\tau}_{12} = \frac{\alpha_0^2 K}{\omega}\frac{\partial \bar{u}}{\partial n} \qquad \text{as} \qquad n \rightarrow 0$$

Further, from the near-wall results described in Equation 6.307e we also have:

$$-\bar{\tau}_{12} = \alpha_0 K$$

Equating the two results, we obtain:

$$\omega = \alpha_0 \frac{\partial \bar{u}}{\partial n} \qquad \text{as} \qquad n \rightarrow 0 \tag{6.362a}$$

and thus:

$$\lambda = \frac{1}{\alpha_0} \qquad \text{as} \qquad n \rightarrow 0 \tag{6.362b}$$

Since in the near-wall region the production and dissipation of energy are nearly equal, the choice of $\gamma_3 = \alpha_0$, $\gamma_4 = \alpha_0^2$ and the behaviors given in Equations 6.362 are consistent with the terms of energy Equation 6.360.

Having established all the structural constants, we now have two options. The first is the availability of another two-equation model in which Equations 6.360 and 6.361 provide the differential equations for K and ω. In this case the Reynolds stress is modeled through the Kolmogorov eddy viscosity hypothesis (Equations 6.280) with μ_τ from Equation 6.358b. The second is the availability of a Reynolds stress closure model in which Equations 6.359–6.361 are used without using Kolmogorov's hypothesis.

The wall boundary conditions for ω can be directly obtained from the near-wall analysis considered earlier. For a smooth wall:

$$\omega = \alpha_0 \frac{\partial u}{\partial n}, \qquad \text{as} \qquad n \rightarrow 0$$

Hence using the logarithmic velocity distribution, we have:

$$\omega \rightarrow \frac{\alpha_0 u_\tau}{\kappa n} \qquad \text{as} \qquad n \rightarrow 0$$

6.32 MODELING FOR COMPRESSIBLE FLOW

The modeling concepts developed so far have mostly been confined to incompressible turbulent flows, except for a brief digression in Section 6.30. In the case of compressible flow with turbulence other correlations appear which are due to the random fluctuations in the density, temperature, enthalpy, and total energy. In this section, we propose to incorporate those modifications in some of the already modeled terms which are necessary to take account of the effect of compressibility. The equations to be considered are the mass-weighted equations derived in Section 6.14.

The technique of mass weighted averaging* requires the resolution of velocity, enthalpy, total enthalpy, temperature, viscosity, total energy, and conductivity into the mass weighted averages and the perturbations, respectively, as:

$$\mathbf{u} = \dot{\bar{\mathbf{u}}} + \mathbf{u}'', \quad h = \dot{\bar{h}} + h'', \quad H = \dot{\bar{H}} + H'', \quad T = \dot{\bar{T}} + T''$$

$$\mu = \dot{\bar{\mu}} + \mu'', \quad \lambda = \dot{\bar{\lambda}} + \lambda'', \quad e_t = \dot{\bar{e}}_t + e_t'', \quad k = \dot{\bar{k}} + k''$$

On the other hand, the pressure, density, heat flux vector, and viscous stress tensor are resolved into their time averages and perturbations, respectively, as:

$$p = \bar{p} + p', \quad \rho = \bar{\rho} + \rho', \quad \mathbf{q} = \bar{\mathbf{q}} + \mathbf{q}', \quad \sigma = \bar{\sigma} + \sigma'$$

Here we will confine the discussion up to two-equation modeling.

Stokes' Law of Friction

Introducing the above resolutions in Stokes' law of friction:

$$\sigma = \lambda\Delta\mathbf{I} + 2\mu\mathbf{D}; \qquad \Delta = \operatorname{div}\mathbf{u}$$

and keeping only the principal terms by disregarding the unknown averages, we get:

$$\bar{\sigma} = \dot{\bar{\lambda}}\dot{\bar{\Delta}}\mathbf{I} + 2\dot{\bar{\mu}}\dot{\bar{\mathbf{D}}}; \qquad \dot{\bar{\Delta}} = \operatorname{div}\dot{\bar{\mathbf{u}}} \tag{6.363a}$$

and:

$$\overline{\rho'\sigma'} = \overline{\rho\lambda''\Delta''}\mathbf{I} + 2\overline{\rho\mu''\mathbf{D}''} \tag{6.363b}$$

The coefficients of viscosity and heat conduction are assumed to be functions of the temperature, i.e.:

$$\dot{\bar{\lambda}} = \lambda(\dot{\bar{T}}), \qquad \dot{\bar{\mu}} = \mu(\dot{\bar{T}}), \qquad \dot{\bar{k}} = k(\dot{\bar{T}})$$

From here onward we shall drop the $\dot{-}$ on λ, μ, and k and consider them as available from some functional relations in the sense defined above (e.g., the Sutherland viscosity function [Equation 2.98] with $\dot{\bar{T}}$ replace by T).

Complete Stress Tensor

The complete stress tensor appearing in the averaged equation of motion (Equation 6.195) is now written as:

$$\Sigma = \bar{p}\mathbf{I} - \bar{\sigma} - (-\overline{\rho\mathbf{u}''\mathbf{u}''})$$

If the Reynolds stress is modeled by using the turbulent eddy viscosities λ_T and μ_T, then according to the Boussinesq hypothesis:

$$-\overline{\rho\mathbf{u}''\mathbf{u}''} = A\mathbf{I} + \lambda_T\dot{\bar{\Delta}}\mathbf{I} + 2\mu_T\dot{\bar{\mathbf{D}}}$$

* The choice of mass weighted or time averaging as applied to obtain the mean equations should predict the same results.

Writing this equation in the component form and then performing the inner sum, we get:

$$A = -\tfrac{2}{3}\bar{\rho}K - (\lambda_T + \tfrac{2}{3}\mu_T)\dot{\bar{\Delta}}$$

Substitution of A in the above formula for the Reynolds stress shows that terms containing λ_T cancel out and we simply have:

$$-\overline{\rho \mathbf{u}''\mathbf{u}''} = -\tfrac{2}{3}\bar{\rho}K\mathbf{I} + \mu_T(2\dot{\bar{\mathbf{D}}} - \tfrac{2}{3}\dot{\bar{\Delta}}\mathbf{I}) \tag{6.364}$$

Thus, the complete stress tensor, on using the Stokes condition:

$$3\lambda + 2\mu = 0$$

and Equation 6.364 is written as:

$$\Sigma = \Sigma_L + \Sigma_T \tag{6.365a}$$

where:

$$\Sigma_L = \bar{p}\mathbf{I} - \mu(2\dot{\bar{\mathbf{D}}} - \tfrac{2}{3}\dot{\bar{\Delta}}\mathbf{I}) \tag{6.365b}$$

$$\Sigma_T = \tfrac{2}{3}\bar{\rho}K\mathbf{I} - \mu_T(2\dot{\bar{\mathbf{D}}} - \tfrac{2}{3}\dot{\bar{\Delta}}I) \tag{6.365c}$$

Heat Flux

A proper budgeting of heat flux due to turbulent fluctuations can be done by examining the correlation terms appearing in energy Equation 6.197. First of all, it is important to recall that:

$$\dot{\bar{e}}_t = \dot{\bar{e}} + \tfrac{1}{2}\dot{\bar{\mathbf{u}}}\cdot\dot{\bar{\mathbf{u}}} + K$$

and:

$$e''_t = e'' + \dot{\bar{\mathbf{u}}}\cdot\mathbf{u}'' + \tfrac{1}{2}\mathbf{u}''\cdot\mathbf{u}'' - K$$

Thus, the last three terms under the divergence operation in Equation 6.197 can be grouped as:

$$\bar{\mathbf{q}}_T + \overline{\rho\mathbf{u}''\mathbf{u}''}\cdot\dot{\bar{\mathbf{u}}}$$

where:

$$\bar{\mathbf{q}}_T = \overline{(\rho e'' + p)\mathbf{u}''} - \overline{\boldsymbol{\sigma}\cdot\mathbf{u}''} + \tfrac{1}{2}\overline{\rho(\mathbf{u}''\cdot\mathbf{u}'')\mathbf{u}''}$$

is the balance of heat and work associated with the fluctuations. The heat flux vectors are modeled in the usual way, viz.:

$$\bar{\mathbf{q}} = -k \operatorname{grad}\dot{\bar{T}}, \qquad \bar{\mathbf{q}}_T = -k_T \operatorname{grad}\dot{\bar{T}}$$

where k_T is the eddy conductivity. For a thermally perfect gas $e = C_v T$ and hence:

$$\bar{\mathbf{q}} = -\frac{\mu\gamma}{P_r}\text{grad}\,\dot{\bar{e}} \tag{6.366a}$$

$$\bar{\mathbf{q}}_T = -\frac{\mu_T\gamma}{P_{rT}}\text{grad}\,\dot{\bar{e}} \tag{6.366b}$$

where $\gamma = C_p/C_v$ and:

$$P_r = \frac{\mu C_p}{k}, \qquad P_{rT} = \frac{\mu_T C_p}{k_T}$$

are the Prandtl numbers. For air $P_r = 0.72$ and the turbulent Prandtl number $P_{rT} = 0.92$.

Production of Turbulence Energy

Similar to the case of incompressible flow, the production of turbulence energy per unit mass is again denoted by P. Thus:

$$\bar{\rho}P = -\Sigma_T : \text{grad}\,\dot{\bar{\mathbf{u}}}$$

Using Equation 6.365c and introducing:

$$S = 2\dot{\bar{\mathbf{D}}} : \dot{\bar{\mathbf{D}}} - \tfrac{2}{3}\dot{\bar{\Delta}}^2$$

we get:

$$\bar{\rho}P = -\tfrac{2}{3}\bar{\rho}K\dot{\bar{\Delta}} + \mu_T S \tag{6.367}$$

Model Equations

Using the previous analysis, we now summarize the complete averaged equations of continuity, momentum, and total energy as:

$$\frac{\partial\bar{\rho}}{\partial t} + \text{div}(\bar{\rho}\dot{\bar{\mathbf{u}}}) = 0 \tag{6.368}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\dot{\bar{\mathbf{u}}}) + \text{div}(\bar{\rho}\dot{\bar{\mathbf{u}}}\dot{\bar{\mathbf{u}}} + \Sigma) = 0 \tag{6.369}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\dot{\bar{e}}_t) + \text{div}(\bar{\rho}\dot{\bar{e}}_t\dot{\bar{\mathbf{u}}} + \Sigma\cdot\dot{\bar{\mathbf{u}}} + \bar{\mathbf{q}} + \bar{\mathbf{q}}_T) = 0 \tag{6.370}$$

and the heat flux vectors have been defined in Equations 6.366. If the Boussinesq hypothesis is used, then the complete stress tensor is modeled as shown in Equations 6.365. With this done we now have the capability of solving Equations 6.368–6.370 for zero-equation, one-equation, or two-equation closures. For a two-equation closure we have to couple the above equations with the scalar equations of energy and dissipation which can be written compactly as:

$$\frac{\partial}{\partial t}(\bar{\rho}K) + \text{div}(\bar{\rho}K\dot{\bar{\mathbf{u}}} + \mathbf{F}) = H_k \tag{6.371}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\epsilon) + \text{div}(\bar{\rho}\epsilon\dot{\bar{\mathbf{u}}} + \mathbf{G}) = H_\epsilon \tag{6.372}$$

The terms **F**, H_k, and **G**, H_ϵ can be written by comparison with Equations 6.301 and 6.309, respectively. Writing $\tau_{ij} = u_i''u_j''$, the k-th component of the vectors **F** and **G** and other quantities are

$$F_k = -\mu\frac{\partial K}{\partial x_k} - \frac{Kc_s}{\epsilon}\left\{\overline{\rho\tau_{k\ell}}\frac{\partial K}{\partial x_\ell} + \frac{\overline{\rho\tau_{i\ell}}}{\bar{\rho}}\frac{\partial}{\partial x_\ell}(\overline{\rho\tau_{ik}})\right\}$$

$$G_k = -\mu\frac{\partial\epsilon}{\partial x_k} - \frac{Kc_\epsilon}{\epsilon}\overline{\rho\tau_{k\ell}}\frac{\partial\epsilon}{\partial x_\ell}$$

$$H_k = \bar{\rho}(P - \epsilon_c) \qquad \text{(Refer to Equation 6.201d)}$$

$$H_\epsilon = c_{\epsilon 1}S\frac{\bar{\rho}\epsilon P}{K} - c_{\epsilon 2}f_\epsilon(R_T)\frac{\epsilon}{K}\left(\bar{\rho}\epsilon - 2\mu\left(\frac{\partial\sqrt{K}}{\partial n}\right)^2\right)$$

$$+ \frac{\mu}{\rho}c_{\epsilon 3}\overline{\rho\tau_{ij}}\frac{K}{\epsilon}\frac{\partial^2\dot{\bar{u}}_k}{\partial x_i\,\partial x_\ell}\frac{\partial^2\dot{\bar{u}}_k}{\partial x_j\,\partial x_\ell}$$

$$\mu_T = \frac{\alpha_0^2\bar{\rho}K^2}{\epsilon}(1 + b_7R_T^{-1/2})\tanh(n^+/n_c^+)$$

$$f_\epsilon(R_T) = 1 - \tfrac{2}{9}\exp(-R_T^2/36)$$

$$R_T = \frac{\bar{\rho}K^2}{\mu\epsilon}, \qquad S = 1 - a\frac{K\dot{\bar{\Delta}}}{P}, \qquad b_7 = 3.45,$$

$$70 < n_c^+ < 120 \tag{6.373}$$

where $c_{\epsilon 11}$ has not yet been estimated. The other constants are

$$c_s = 0.11, \quad c_\epsilon = 0.15, \quad c_{\epsilon 1} = 1.45, \quad c_{\epsilon 2} = 2.0, \quad c_{\epsilon 3} = 2.0.$$

The ϵ appearing in Equations 6.373 is the vorticity part of 6.201d. Some authors have used a much simpler form of F_k and G_k in comparison with those given in Equation 6.373, and they are are

$$F_k = -\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial K}{\partial x_k}$$

$$G_k = -\left(\mu + \frac{\mu_T}{\sigma_\epsilon}\right)\frac{\partial\epsilon}{\partial x_k}$$

where:

$$\sigma_k = 1.0, \qquad \sigma_\epsilon = 1.3$$

The boundary conditions for Equations 6.371 and 6.372 have already been discussed in Section 6.26. If ϵ is assumed to vanish at the wall, then the term H_k appearing in Equation 6.373 should be modified as:

$$H_t = \bar{\rho}(P - \epsilon_c) - 2\mu\left(\frac{\partial\sqrt{K}}{\partial n}\right)^2$$

Justification of the Modeling Constants for Compressible Flow

Accumulated experimental results on compressible turbulent boundary layer and wake flows have shown that the direct effect of density fluctuations on turbulence up to certain speeds is negligibly small. This aspect of turbulence physics is presented in the form of a hypothesis, known as Morkovin's hypothesis.[98] It states that the turbulence structures are not significantly affected by the density fluctuations as long as the free-stream Mach number M_e is less than 5 for the boundary layers and M_e is less than 1.5 for the jets and wakes. By the term "turbulence structures" we mean the length scales, correlation coefficients, spectrum shapes, modeling constants, etc. (Refer also to Reference 99.) The rationale behind the choice of modeling constants in Equation 6.373, as being the same as those used in the incompressible flow, rests on the accepted validity of Morkovin's hypothesis.

The part of compressibility effect which is not covered by Morkovin's hypothesis is the spatial and temporal variation of the averaged density $\bar{\rho}$. It has been established experimentally that the variation of $\bar{\rho}$ produces significant changes in comparison to the incompressible flow. This fact is demonstrated in Figure 6.22, where the variation of the normalized skin friction with the free-stream Mach number has been evaluated by Van Driest.[100]

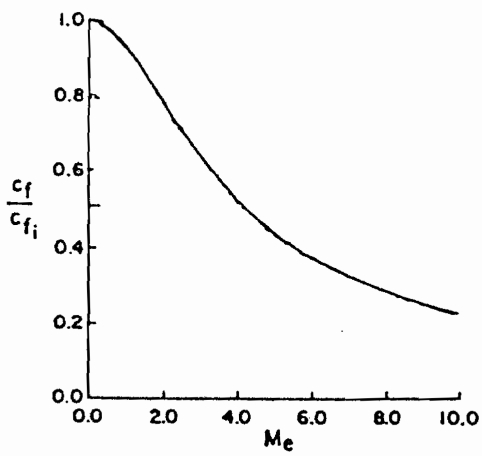


Fig. 6.22 Effect of compressibility on the skin friction coefficient normalized with the incompressible value, on an adiabatic flat plate.

The significant effect of density variation in high speed turbulent flows of $M_e < 5$ for the wall-bound flows and $M_e < 1.5$ for wake and jet flows is, therefore, analyzed simply by using the compressible averaged equations. For zero-equation modeling the local averaged density is used in the definition of eddy viscosity, and all other correlations involving density fluctuations are ignored. For a good grasp of the proposed modelings and their validation with computational results, the reader is referred to a well-written review by Marvin.[101]

6.33 THREE-DIMENSIONAL BOUNDARY LAYERS

As has usually been the case, the important consideration in solving a three-dimensional boundary layer problem is in the modeling of Reynolds' stress terms. Once this problem has been resolved, the calculation methods for three-dimensional turbulent boundary layers are essentially based on the same techniques as those of the laminar two- and three-dimensional boundary layers. In this section we shall study the modeling of Reynolds' stress transport equations, which besides providing a closure model for solving Equations 6.209–6.215 also provides a rational way of obtaining an eddy viscosity model for three-dimensional boundary layers.

The most important terms, from a modeling point of view, in Equations 6.214 and 6.215 are the pressure-strain terms. Experiments of Johnston [102] have clearly established that in a turbulent boundary layer the *vector of shear stress* is, in general, not parallel to the *vector of mean velocity gradient*. (It must be noted that the word "vector" in place of "tensor" has been used above because only two components of each tensor could be retained under the boundary layer approximation.) This experimental guidance must form a basis for the modeling of pressure-strain correlation in such a manner that the stress vector and the strain vector must not be parallel. We, however, turn the problem around and ask this question. Under what type of modeling of pressure-strain correlations is the experimental result of Johnston contradicted? To answer this question, we start from Townsend's "structural equilibrium hypothesis" (refer to Reference 103). Let:

$$\tau = (\bar{\tau}_{12}, \bar{\tau}_{23}), \ \bar{\tau}_{12} = \bar{\tau}_{21}$$

$$\mathbf{R} = \left(\frac{\partial \bar{u}_1}{\partial x_2}, \frac{\partial \bar{u}_3}{\partial x_2}\right)$$

be the vector of shear stress and the vector of mean velocity gradient, respectively. Townsend's hypothesis states that:

$$|\tau| = \alpha_0 K, \qquad \alpha_0 = 0.3 \tag{6.374}$$

Now multiply each term of Equation 6.213 by $\alpha_0^2 K$ and use the definition in Equation 6.374. Next multiply the first equation (Equation 6.214) by $\bar{\tau}_{12}$ and the second (Equation 6.215) by $\bar{\tau}_{23}$ and add the two equations. Comparison of the two resulting equations in $|\tau|$ yields the pressure-strain correlations

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)} = \frac{-\epsilon\bar{\tau}_{12}}{K} + (\overline{u_2'^2} - \alpha_0^2 K)\frac{\partial \bar{u}_1}{\partial x_2}$$

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_3'}{\partial x_2} + \frac{\partial u_2'}{\partial x_3}\right)} = \frac{-\epsilon\bar{\tau}_{23}}{K} + (\overline{u_2'^2} - \alpha_0^2 K)\frac{\partial \bar{u}_3}{\partial x_2} \tag{6.375}$$

Similarly, we have

$$\frac{\partial}{\partial x_2}\left\{\overline{u_1'\left(u_2'^2 + \frac{p'}{\rho}\right)}\right\} = \frac{\bar{\tau}_{12}}{K}\frac{\partial}{\partial x_2}\left\{\overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\}$$

$$\frac{\partial}{\partial x_2}\left\{\overline{u_3'\left(u_2'^2 + \frac{p'}{\rho}\right)}\right\} = \frac{\bar{\tau}_{23}}{K}\frac{\partial}{\partial x_2}\left\{\overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} \tag{6.376}$$

The viscous and decay terms are insignificant for high Reynolds' number flows away from the wall and have been ignored. With the formulations in Equations 6.375 and 6.376 the rate equations for Reynolds' stresses, Equations 6.214 and 6.215, become:

$$\frac{D}{Dt}(\bar{\tau}_{12}) = -\alpha_0^2 K\frac{\partial \bar{u}_1}{\partial x_2} - \frac{\bar{\tau}_{12}}{K}\frac{\partial}{\partial x_2}\left\{\overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} - \frac{\epsilon\bar{\tau}_{12}}{K}$$
$$+ \text{ viscous and decay terms} \tag{6.377a}$$

$$\frac{D}{Dt}(\bar{\tau}_{23}) = -\alpha_0^2 K\frac{\partial \bar{u}_3}{\partial x_2} - \frac{\bar{\tau}_{23}}{K}\frac{\partial}{\partial x_2}\left\{\overline{u_2'\left(\frac{1}{2}q^2 + \frac{p'}{\rho}\right)}\right\} - \frac{\epsilon\bar{\tau}_{23}}{K}$$

$$+ \text{ viscous and decay terms} \tag{6.377b}$$

Now the flow near the surface but outside the sublayer can be investigated based on the available shear transport equations. Outside the sublayer but close to the surface, the convective, diffusive, and decay terms are negligible in comparison with the production and dissipation terms. Thus, in the wall region Equations 6.377 approximate as:

$$\alpha_0^2 K \frac{\partial \bar{u}_1}{\partial x_2} = \frac{-\epsilon \bar{\tau}_{12}}{K} \tag{6.378a}$$

$$\alpha_0^2 K \frac{\partial \bar{u}_3}{\partial x_2} = \frac{-\epsilon \bar{\tau}_{23}}{K} \tag{6.378b}$$

From Equations 6.378 we conclude that the assumption of an expression for stress vector in terms of turbulence energy in the form of Equation 6.374 leads one to the Boussinesq eddy viscosity hypothesis. Substituting ϵ from Equation 6.287 with $b_3 = \alpha_0^{3/2}$, Equations 6.378 yield:

$$\begin{aligned} \bar{\tau}_{12} &= -\ell|\tau|^{1/2} \frac{\partial \bar{u}_1}{\partial x_2} \\ \bar{\tau}_{23} &= -\ell|\tau|^{1/2} \frac{\partial \bar{u}_3}{\partial x_2} \end{aligned} \tag{6.379}$$

from which it is easy to show by substitution that:

$$|\tau \times \mathbf{R}| = \left| \bar{\tau}_{23} \frac{\partial \bar{u}_1}{\partial x_2} - \bar{\tau}_{12} \frac{\partial \bar{u}_3}{\partial x_2} \right| = 0$$

Thus, the assumption of Townsend's structural equilibrium hypothesis, or the pressure-strain modeling of the form in Equation 6.375, implies that the vectors of shear stress and of the mean velocity gradient are parallel, which is not supported by the experimental data of Johnston.

We now consider a modeling of shear stress transport equations due to Rotta.[104] Based on the integral representation of the fluctuating pressure p' and then of the pressure-strain correlation (Equation 6.186), Rotta arrived at the following formulations:

$$\begin{aligned} &\overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)} \\ &\quad = \left(f_{11} \frac{\partial \bar{u}_1}{\partial x_2} + f_{13} \frac{\partial \bar{u}_3}{\partial x_2}\right)\overline{u_2'^2} - k_p \frac{K^{1/2}}{\ell} \bar{\tau}_{12} \end{aligned} \tag{6.380a}$$

$$\begin{aligned} &\overline{\frac{p'}{\rho}\left(\frac{\partial u_3'}{\partial x_2} + \frac{\partial u_2'}{\partial x_3}\right)} \\ &\quad = \left(f_{31} \frac{\partial \bar{u}_1}{\partial x_2} + f_{33} \frac{\partial \bar{u}_3}{\partial x_2}\right)\overline{u_2'^2} - k_p \frac{K^{1/2}}{\ell} \bar{\tau}_{23} \end{aligned} \tag{6.380b}$$

where:

$$f_{11} = (k_{11}\overline{u_1^2} + k_{33}\overline{u_3^2})/\overline{u_r^2}$$

$$f_{13} = f_{31} = (k_{11} - k_{33})\bar{u}_1\bar{u}_3/\bar{u}_r^2$$

$$f_{33} = (k_{11}\bar{u}_3^2 + k_{33}\bar{u}_1^2)/\bar{u}_r^2$$

where:

$$\bar{u}_r = (\bar{u}_1^2 + \bar{u}_3^2)^{1/2}$$

and k_p, k_{11}, k_{33} are adjustable parameters. To eliminate the consideration of $\overline{u_2'^2}$, Rotta assumes that $\overline{u_2'^2}$ can be expressed in terms of K through the equation:

$$\overline{u_2'^2} = \frac{Ka_p}{1 - k_{11}} \tag{6.381a}$$

where a_p is a nondimensional parameter. Further, introducing the dimensionless parameter T as:

$$T = (1 - k_{33})/(1 - k_{11}) \tag{6.381b}$$

and the following quantities:

$$a_{p11} = a_p(\bar{u}_1^2 + T\bar{u}_3^2)/\bar{u}_r^2 = (1 - f_{11})\overline{u_2'^2}/K$$

$$a_{p13} = a_p(1 - T)\bar{u}_1\bar{u}_3/\bar{u}_r^2 = -\overline{u_2'^2}f_{13}/K$$

$$a_{p33} = a_p(\bar{u}_3^2 + T\bar{u}_1^2)/\bar{u}_r^2 = (1 - f_{33})\overline{u_2'^2}/K$$

we find that Equations 6.380 can be written as:

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)} = (\overline{u_2'^2} - a_{p11}K)\frac{\partial \bar{u}_1}{\partial x_2} - a_{p13}K\frac{\partial \bar{u}_3}{\partial x_2} - k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{12} \tag{6.382a}$$

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_3'}{\partial x_2} + \frac{\partial u_2'}{\partial x_3}\right)} = (\overline{u_2'^2} - a_{p33}K)\frac{\partial \bar{u}_3}{\partial x_2} - a_{p13}K\frac{\partial \bar{u}_1}{\partial x_2} - k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{23} \tag{6.382b}$$

Substituting Equations 6.382 in Equations 6.214 and 6.215 and neglecting the viscous terms, we obtain the modeled equations:

$$\frac{D\bar{\tau}_{12}}{Dt} = -\left(a_{p11}\frac{\partial \bar{u}_1}{\partial x_2} + a_{p13}\frac{\partial \bar{u}_3}{\partial x_2}\right)K - k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{12} + \frac{\partial}{\partial x_2}\left\{k_d K^{1/2}\ell\frac{\partial \bar{\tau}_{12}}{\partial x_2}\right\} \tag{6.383}$$

$$\frac{D\bar{\tau}_{23}}{Dt} = -\left(a_{p13}\frac{\partial \bar{u}_1}{\partial x_2} + a_{p33}\frac{\partial \bar{u}_3}{\partial x_2}\right)K - k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{23} + \frac{\partial}{\partial x_2}\left\{k_d K^{1/2}\ell\frac{\partial \bar{\tau}_{23}}{\partial x_2}\right\} \tag{6.384}$$

where a gradient model has been used for the diffusion terms. Rotta suggests the values of a_p in the range:

$$0.2 < a_p < 0.25$$

and

$$a_p = 0.267, \qquad T = 1 \text{ for isotropic turbulence.}$$

Also:

$$k_p = a_p/\sqrt{\alpha_0}, \qquad \alpha_0 = 0.3$$

and:

$$k_d \cong 0.38, \qquad T \cong 0.5$$

Eddy Viscosity Approach to 3-D Boundary Layers

From Equation 6.286, the scalar eddy viscosity ν_T is given by:

$$\nu_T = \ell\sqrt{\alpha_0 K}, \qquad \alpha_0 = 0.3 \tag{6.385}$$

In the place of Kolmogorov's hypothesis (Equation 6.280) for Reynolds' stresses, we can now define the tensor eddy viscosity through the use of Equations 6.383 and 6.384. Neglecting the convection and diffusion terms, we obtain:

$$\left(a_{p11}\frac{\partial \bar{u}_1}{\partial x_2} + a_{p13}\frac{\partial \bar{u}_3}{\partial x_2}\right)K = -k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{12}$$

$$\left(a_{p13}\frac{\partial \bar{u}_1}{\partial x_2} + a_{p33}\frac{\partial \bar{u}_3}{\partial x_2}\right)K = -k_p\frac{K^{1/2}}{\ell}\bar{\tau}_{23}$$

Substituting $\ell = \nu_T/\sqrt{\alpha_0 K}$ in these equations and writing k_p in terms of a_p, we get:

$$-\bar{\tau}_{12} = (\nu_T)_{11}\frac{\partial \bar{u}_1}{\partial x_2} + (\nu_T)_{13}\frac{\partial \bar{u}_3}{\partial x_2} \tag{6.386}$$

$$-\bar{\tau}_{23} = (\nu_T)_{13}\frac{\partial \bar{u}_1}{\partial x_2} + (\nu_T)_{33}\frac{\partial \bar{u}_3}{\partial x_2} \tag{6.387}$$

where:

$$(\nu_T)_{11} = \nu_T(\bar{u}_1^2 + T\bar{u}_3^2)/\bar{u}_r^2$$

$$(\nu_T)_{13} = (\nu_T)_{31} = \nu_T(1 - T)\bar{u}_1\bar{u}_3/\bar{u}_r^2$$

$$(\nu_T)_{33} = \nu_T(\bar{u}_3^2 + T\bar{u}_1^2)/\bar{u}_r^2$$

are the components of the tensor eddy viscosity. Equations 6.386 and 6.387 provide the tensor eddy viscosity model for the Reynolds stresses. The scalar eddy viscosity ν_t is to be obtained from Equation 6.385. Thus, the tensor eddy viscosity formulation (Equations 6.386 and 6.387) is a one-equation model in which energy Equation 6.213 also has to be solved.

To obtain the mixing length formula, Rotta approximates energy Equation 6.213 in the form:

$$P \cong \epsilon$$

Further, using Equation 6.385 in Equation 6.287, the dissipation ϵ is written as:

$$\epsilon = \nu_T^3/\ell^4$$

so that:

$$\frac{\nu_T^3}{\ell^4} = -\bar{\tau}_{12}\frac{\partial \bar{u}_1}{\partial x_2} - \bar{\tau}_{23}\frac{\partial \bar{u}_3}{\partial x_2}$$

Substituting Equations 6.386 and 6.387, we get on simplification:

$$\nu_T = \ell^2\left[\left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 + \left(\frac{\partial \bar{u}_3}{\partial x_2}\right)^2 + \frac{(T-1)}{\bar{u}_r^2}\left(\bar{u}_3\frac{\partial \bar{u}_1}{\partial x_2} - \bar{u}_1\frac{\partial \bar{u}_3}{\partial x_2}\right)^2\right]^{1/2} \tag{6.388}$$

which is a generalization of the classical mixing length formula. In Equation 6.388 the length scale ℓ must be interpreted as an appropriate mixing length and $T = 0.5$.

As has usually been the procedure for two-dimensional boundary layers, we take Equation 6.388 to provide the inner eddy kinematic viscosity, $(\nu_t)_i$. Rotta suggests the value $T = 0.5$. The complete specification for ν_T in three-dimensional boundary layers is then obtained when we develop an expression similar to that in two dimensions for the outer eddy viscosity, $(\nu_T)_0$. Disregarding the transition factor, we take:

$$(\nu_T)_0 = \gamma\alpha \int (u_{re} - \bar{u}_r)\, dx_2 \tag{6.389}$$

where:

$$u_{re} = (u_{1e}^2 + u_{3e}^2)^{1/2}$$

and where γ and α retain the same interpretation as in two dimensions.

Based on the preceding development, we can propose a new formula for μ_T which may be used in solving the averaged Navier-Stokes equations (in lieu of Equations 6.281) as:

$$\mu_T = \rho\ell^2[2\bar{\mathrm{D}}:\bar{\mathrm{D}} + b(\bar{\omega}\cdot\bar{\mathrm{u}}/|\bar{\mathrm{u}}|)^2]^{1/2} \tag{6.390}$$

where $b = T - 1$. Under the boundary layer approximation, Equation 6.390 reduces to Equation 6.355.

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PROBLEMS

6.1 Differentiate the time average with respect to time t and show that:

$$\frac{\partial \overline{\phi}}{\partial t} = \overline{\frac{\partial \phi}{\partial t}}$$

which is property 4 in the basic axioms of averaging; Section 6.7.

6.2 In the final stages of the decay of isotropic turbulence the equation takes the simpler form:

$$\frac{\partial F}{\partial t} = 2\nu\left(\frac{\partial^2 F}{\partial r^2} + \frac{4}{r}\frac{\partial F}{\partial r}\right) \qquad \text{(i)}$$

where:

$$F(r, t) = \overline{u^2} f(r, t)$$

Multiply Equation i by r^4, integrate with respect to r from 0 to ∞, and use the conditions:

$$\lim_{r \to \infty} r^4 F(r, t) = 0$$

$$\lim_{r \to \infty} r^4 \frac{\partial F}{\partial r} = 0$$

to show that:

$$\int_0^\infty r^4 F(r, t)\, dr = m = \text{const.} \qquad \text{(ii)}$$

The fundamental solution of Equation i is

$$F(r, t) = \frac{A}{(\nu t)^{5/2}} \exp\left(-\frac{r^2}{8\nu t}\right)$$

(verify by direct substitution), and show by using Equation ii that:

$$\overline{u^2} f(r, t) = \frac{m}{48(2\pi)^{1/2}(\nu t)^{5/2}} \exp\left(-\frac{r^2}{8\nu t}\right) \qquad \text{(iii)}$$

Based on Equation iii show that the longitudinal length scale is

$$L_r = 2(2\pi\nu t)^{1/2}$$

6.3 A laboratory method of producing an isotropic flow is to place a screen formed of wires at the entrance of the working section of a wind tunnel. The two important quantities are $\overline{u^2}$ and L, where L is a longitudinal length scale. Based on the dimensions of $\overline{u^2}$ and L and noting that the effect of viscosity is negligible in the *initial stages of the decay of isotropic turbulence*, we conclude that the rates of change in time of $\overline{u^2}$ and L must be given by the equations:

$$\frac{d}{dt}(\overline{u^2})^{1/2} = \frac{-\alpha \overline{u^2}}{L} \qquad \text{(i)}$$

$$\frac{dL}{dt} = \beta(\overline{u^2})^{1/2} \qquad \text{(ii)}$$

where α and β are nondimensional constants. If U is the constant mean velocity in the tunnel, then:

$$t = \frac{x}{U}$$

where x is the longitudinal coordinate. Thus Equations i and ii can be rewritten as:

$$\frac{d}{dx}(\overline{u^2})^{1/2} = \frac{-\alpha\overline{u^2}}{UL} \tag{iii}$$

$$\frac{dL}{dx} = \frac{\beta(\overline{u^2})^{1/2}}{U} \tag{iv}$$

where $\overline{u^2} = fn(x/U)$. If the screen is placed at $x = x_0$, then:

$$\text{at } x = x_0 : (\overline{u^2})^{1/2} = (\overline{u^2})_0^{1/2}, \qquad L = \gamma d \tag{v}$$

where $(\overline{u_0^2})^{1/2}$ is the initital tunnel turbulence intensity, d is the diameter of the mesh wires, and γ is a proportionality constant.

Solve Equations iii and iv under the initial conditions of Equation v and show that the solution is

$$\frac{(\overline{u^2})_0^{1/2}}{(\overline{u^2})^{1/2}} = \left[1 + (\alpha + \beta)\frac{(\overline{u^2})_0^{1/2}}{\gamma U}\frac{(x - x_0)}{d}\right]^{\alpha/(\alpha+\beta)}$$

$$\frac{L}{d} = \gamma\left[1 + (\alpha + \beta)\frac{(\overline{u^2})_0^{1/2}}{\gamma U}\frac{(x - x_0)}{d}\right]^{\beta/(\alpha+\beta)}$$

6.4 Prove that for a one-dimensional Gaussian process the characteristic function is

$$M_1(v_1) = \exp[iv_1\bar{u}_1 - \tfrac{1}{2}(\sigma v_1)^2]$$

where:

$$\sigma^2 = \overline{(u_1 - \bar{u}_1)^2}$$

Also prove that:

$$\begin{aligned} \overline{(u_1 - \bar{u}_1)^n} &= 1.3.5\ldots(n - 1)\sigma^n, && \text{if } n \text{ is even} \\ &= 0 && \text{, if } n \text{ is odd} \end{aligned}$$

and that only two semi-invariants λ_1 and λ_2 are nonzero, viz.:

$$\lambda_1 = \bar{u}_1$$

$$\lambda_2 = \sigma^2$$

$$\lambda_j = 0, \quad j \geq 3$$

6.5 The problem of taking the Fourier transform of random functions requires some further considerations. If $u(\mathbf{x}, t)$ is a random function of $\mathbf{x}$ and t, then its Fourier transform either with $\mathbf{x}$ or time t does not exist, in general. For simplicity let us first consider $\mathbf{x}$ fixed. Then we define a truncated function:

$$\begin{aligned} u_T(\mathbf{x}, t) &= u(\mathbf{x}, t) && \text{for } |t| \leq T \\ &= 0 && \text{for } |t| > T \end{aligned} \tag{i}$$

and that:

$$\lim_{T\to\infty} u_T = u$$

Because of the definition (Equation i), the Fourier transform:

$$\phi_T(x, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_T(\mathbf{x}, t)e^{-i\omega t}\, dt \tag{ii}$$

exists. However the limit:

$$\lim_{T\to\infty} \phi_T(\mathbf{x}, \omega)$$

does not exist. We shall, therefore, average ϕ_T over a suitable frequency range $\omega_1 \leq \omega \leq \omega_2$ by integration:

$$\int_{\omega_1}^{\omega_2} \phi_T(\mathbf{x}, \omega)\, d\omega$$

and also define:

$$Z(\omega_2) - Z(\omega_1) = \lim_{T\to\infty} \int_{\omega_1}^{\omega_2} \phi_T(\mathbf{x}, \omega)\, d\omega \tag{iii}$$

Show that if $\omega_1 = \omega$ and $\omega_2 = \omega + d\omega$ where $d\omega$ is an infinitesimal, then

$$\frac{dZ(\omega)}{d\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\mathbf{x}, t)e^{-i\omega t}\, dt \tag{iv}$$

so that:

$$u(\mathbf{x}, t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{v}$$

where $dZ(\omega)$ is a random function of variable ω.

6.6 (a) The distribution of E and k^2E in Figure 6.8 shows that their peaks are attained at different values of k. This suggests that there is *separation* between the group of energy-containing eddies (small wave numbers) and the group of dissipation eddies (large wave numbers). The wave number range between the upper bound of small wave numbers and the lower bound of large wave numbers thus defines the *inertial* subrange which acts as a sort of barrier between the two groups of eddies.

To investigate the nature of energy-containing eddies, let the spectrum function in the low wave number range be denoted as:

$$E(k) = V^2LF(kL) \tag{i}$$

where V and L are the characteristic velocity and length, respectively, in this range. Since in the subrange (Equation i) must coincide with Equations 6.124b and 6.125a, then in the subrange:

$$F(kL) \sim K_0(Lk)^{-5/3}$$

Then, show that:

$$\frac{v}{V} = \left(\frac{\eta}{L}\right)^{1/3} \qquad \text{(ii)}$$

Denoting $X = kL$ and $x = k\eta$ and by k_* the value of k in the overlap region, show from the formulae:

$$K = \int_0^\infty E(k)\, dk$$

$$\epsilon = 2\nu \int_0^\infty k^2 E(k)\, dk$$

that:

$$\frac{K}{V^2} = \int_0^\infty F(X)\, dX + \left(\frac{v}{V}\right)^2 o\,(x_*^{-2/3}) \qquad \text{(iii)}$$

and:

$$\frac{\epsilon\eta^2}{2\nu v^2} = \left(\frac{v}{V}\right)^4 o\,(X_*^{4/3}) + \int_0^\infty x^2 E(x)\, dx \qquad \text{(iv)}$$

where o is the order symbol. From Equations iii and iv conclude that:

$$\left(\frac{v}{V}\right)^2 x_*^{-2/3} \ll 1$$
$$\left(\frac{v}{V}\right)^4 X_*^{4/3} \ll 1$$

and that the energy-containing eddies and the dissipation eddies are separated.

6.6 (b) By using Equation 6.113b show that corresponding to $E \sim k^{-5/3}$ the function $f(r) \sim r^{2/3}$.

6.7 To understand the role of vorticity in isotropic turbulence, consider vorticicty Equation 3.34 in Cartesian coordinates. In the *absence of mean flow* the perturbations exactly satisfy the vorticity equation. The equations for the fluctuating vorticity components are:

$$\frac{\partial \omega_i'}{\partial t} + u_j' \frac{\partial \omega_i'}{\partial x_j} - \omega_j' \frac{\partial u_i'}{\partial x_j} = \nu \nabla^2 \omega_i' \qquad \text{(i)}$$

where $i = 1, 2, 3$, and:

$$\omega_i' = e_{ijk} \frac{\partial u_k'}{\partial x_j}$$

Taking the inner product of Equation i by ω_i', introducing the vorticity density ϕ as:

$$\psi = \omega_i'^2 = \omega^2; \qquad \phi = \overline{\omega^2}$$

and noting that for isotropic turbulence ψ and ϕ are functions of time only, show that the vorticity density equation for isotropic turbulence is

$$\frac{\partial \phi}{\partial t} = 2\overline{\omega_i'\omega_j'\frac{\partial u_i'}{\partial x_j}} - 2\nu\overline{\left(\frac{\partial \omega_i'}{\partial x_j}\right)^2} \quad \text{(ii)}$$

The first term on the right of Equation ii represents the growth of vorticity due to random stretching of the vortex lines, while the second term represents the viscous dissipation.

6.8 (a) Consider points x and x + r in the field of isotropic turbulence. Form the vorticity correlation $\overline{\omega_i'\omega_j'}$ and show that:

$$\overline{\omega_i'\omega_j'} = -e_{ik\ell}e_{jmn}\frac{\partial^2}{\partial r_k\, \partial r_m} R_{\ell n}(\mathbf{r}, t) \quad \text{(i)}$$

where $\mathbf{r} = \mathbf{y} - \mathbf{x}$.

(b) Set $i = j$ in Equation i and $\mathbf{r} = 0$ to show that:

$$\overline{\omega^2} = \overline{\phi} = -(\nabla_r^2 R_{ii})_{r=0}$$

where:

$$\nabla_r^2 = \frac{\partial^2}{\partial r_j\, \partial r_j}$$

Using Equations 6.84 and 6.85 show that:

$$\overline{\omega^2} = \frac{15\overline{u^2}}{\lambda_T^2}$$

and thus, from Equation 6.92:

$$\epsilon = \nu\phi$$

which provides another expression for the dissipation function ϵ.

6.9 For the case of a turbulent flow referred to cylindrical polar coordinate system (r, ϕ, z) with the velocity components u, v, w, respectively, show that the averaged Navier-Stokes equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{1}{r}\frac{\partial \bar{v}}{\partial \phi} + \frac{\partial \bar{w}}{\partial z} = 0 \quad \text{(i)}$$

$$\frac{D\bar{u}}{Dt} - \frac{\bar{v}^2}{r} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} + \nu\left(\nabla^2\bar{u} - \frac{2}{r^2}\frac{\partial v}{\partial \phi} - \frac{\bar{u}}{r^2}\right)$$
$$- \frac{\partial}{\partial r}(\overline{u'^2}) - \frac{1}{r}\frac{\partial}{\partial \phi}(\overline{u'v'}) - \frac{\partial}{\partial z}(\overline{u'w'}) - \frac{1}{r}(\overline{u'^2} - \overline{v'^2}) \quad \text{(ii)}$$

$$\frac{D\bar{v}}{Dt} + \frac{\bar{u}\bar{v}}{r} = -\frac{1}{\rho r}\frac{\partial \bar{p}}{\partial \phi} + \nu\left(\nabla^2\bar{v} + \frac{2}{r^2}\frac{\partial \bar{u}}{\partial \phi} - \frac{\bar{v}}{r^2}\right) - \frac{\partial}{\partial r}(\overline{u'v'})$$
$$- \frac{1}{r}\frac{\partial}{\partial \phi}\overline{v'^2} - \frac{\partial}{\partial z}(\overline{v'w'}) - \frac{2\overline{u'v'}}{r} \quad \text{(iii)}$$

$$\frac{D\bar{w}}{dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \nu\nabla^2\bar{w} - \frac{1}{r}\frac{\partial}{\partial r}(\overline{ru'w'}) - \frac{1}{r}\frac{\partial}{\partial \phi}(\overline{v'w'}) - \frac{\partial}{\partial z}(\overline{w'^2}) \quad \text{(iv)}$$

where:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial r} + \frac{\bar{v}}{r}\frac{\partial}{\partial \phi} + \bar{w}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

6.10 Show by direct expansion that the total dissipation ϵ_d is given by:

$$\epsilon_d = \nu\left[2\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v'}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w'}{\partial z}\right)^2} + \overline{\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)^2}\right]$$

while:

$$\epsilon = \nu\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial y}\right)^2} + \overline{\left(\frac{\partial w'}{\partial z}\right)^2} + \overline{\left(\frac{\partial u'}{\partial y}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z}\right)^2} + \overline{\left(\frac{\partial v'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z}\right)^2} + \overline{\left(\frac{\partial w'}{\partial x}\right)^2} + \overline{\left(\frac{\partial w'}{\partial y}\right)^2}\right]$$

6.11 Write Equation 6.195 in Cartesian tensor form, and then by using suitable algebraic manipulations obtain the equation of the second moment:

$$\frac{\partial}{\partial t}(\bar{\rho}\dot{\bar{u}}_i\dot{\bar{u}}_k) + \frac{\partial}{\partial x_j}(\bar{\rho}\dot{\bar{u}}_i\dot{\bar{u}}_j\dot{\bar{u}}_k) + \dot{\bar{u}}_k\frac{\partial \bar{p}}{\partial x_i} + \dot{\bar{u}}_i\frac{\partial \bar{p}}{\partial x_k} - \dot{\bar{u}}_k\frac{\partial}{\partial x_j}(\bar{\sigma}_{ij} - \overline{\rho u_i'' u_j''}) - \dot{\bar{u}}_i\frac{\partial}{\partial x_j}(\bar{\sigma}_{kj} - \overline{\rho u_k'' u_j''}) = 0 \tag{i}$$

Use Equation i to obtain:

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\bar{\rho}\dot{\bar{u}}_i\dot{\bar{u}}_i) + \frac{\partial}{\partial x_j}(\dot{\bar{u}}_j\tfrac{1}{2}\bar{\rho}\dot{\bar{u}}_i\dot{\bar{u}}_i) = -\dot{\bar{u}}_i\frac{\partial \bar{p}}{\partial x_i} + \dot{\bar{u}}_i\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j}(\dot{\bar{u}}_i\overline{\rho u_i'' u_j''}) - \left(-\overline{\rho u_i'' u_j''}\frac{\partial \dot{\bar{u}}_i}{\partial x_j}\right) \tag{ii}$$

Equation ii is the equation of kinetic energy of the mean motion. What is the significance of the last term on the right-hand side of Equation ii?

6.12 Consider the complete Navier-Stokes equation in the Cartesian tensor form and first obtain the equation of the second moment in instantaneous values, viz.:

$$\frac{\partial}{\partial t}(\rho u_i u_k) + \frac{\partial}{\partial x_j}(\rho u_i u_j u_k) + u_k\frac{\partial p}{\partial x_i} + u_i\frac{\partial p}{\partial x_k} - u_k\frac{\partial \sigma_{ij}}{\partial x_i} - u_i\frac{\partial \sigma_{kj}}{\partial x_j} = 0 \tag{i}$$

Next replace the instantaneous velocity u_i by $\dot{\bar{u}}_i + u_i''$, and perform the mass weighted averaging of each term. Using Equation i of Problem 6.11 obtain the Reynolds stress transport equation (Equation 6.200b). Contract the indices in Equation 6.200b to obtain the equation of turbulence energy K, viz., Equation 6.200a.

6.13 On the basis of the wall law we have the result that in the sublayer region:

$$\frac{\bar{u}}{u_\tau} = \frac{yu_\tau}{\nu}$$

The sublayer is subjected to some vigorous disturbances from the outer turbulence; hence, its Reynolds' number must be at the lower critical limit for parallel shear flows. Based on the linear stability theory this Reynolds' number must be about 50, viz.:

$$\frac{\bar{u}_c y_c}{\nu} \cong 50$$

where the subscript c mean "critical". Show that:

$$\frac{y_c u_\tau}{\nu} \cong 7$$

This result provides a limiting value. Experimental plots, on the average, show that the sublayer extends to about $yu_\tau/\nu \cong 5$. Thus, we may take the sublayer thickness to be about $5\nu/u_\tau$.

6.14 In the immediate vicinity of a wall all fluctuation velocity components are small. For simplicity taking u', v', w' as fluctuating components along x, y, z, respectively, where y is the coordinate normal to the wall, we have:

$$u' = v' = w' = 0 \quad \text{at} \quad y = 0$$

and also:

$$\frac{\partial u'}{\partial x} = \frac{\partial u'}{\partial z} = \frac{\partial v'}{\partial x} = \frac{\partial v'}{\partial z} = \frac{\partial w'}{\partial x} = \frac{\partial w'}{\partial z} = 0 \quad \text{at} \quad y = 0 \tag{i}$$

Using the equation of continuity:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

we find that:

$$\frac{\partial v'}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{ii}$$

Based on this information *show that at* $y = 0$:

$$\frac{\partial^2 v'}{\partial y^2} = -\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial w'}{\partial y}\right)$$

$$\frac{\partial}{\partial y}(u'v') = 0$$

$$\frac{\partial^2}{\partial y^2}(u'v') = 0$$

$$\frac{\partial^3}{\partial y^3}(u'v') = 3\frac{\partial u'}{\partial y}\frac{\partial^2 v'}{\partial y^2} \qquad \text{(iii)}$$

Develop a power series for the perturbations near a wall, i.e.,

$$u' = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots$$

$$v' = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \ldots$$

$$w' = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \ldots$$

where a_i, b_i, c_i are functions of x, z and t. Using the results in Equations i, ii, and iii show that:

$$\overline{(u'v')} = \overline{a_1 b_2}y^3 + (\overline{a_2 b_2} + \overline{a_1 b_3})y^4 + \ldots$$

$$K = \tfrac{1}{2}(\overline{a_1^2} + \overline{c_1^2})y^2 + (\overline{a_1 a_2} + \overline{c_1 c_2})y^3 + \ldots$$

$$\overline{v'\tfrac{1}{2}q^2} = \tfrac{1}{2}\overline{b_2(a_1^2 + c_1^2)}y^4 + \ldots$$

$$\epsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = \nu(\overline{a_1^2} + \overline{c_1^2}) + 4\nu(\overline{a_1 a_2} + \overline{c_1 c_2})y + \ldots$$

$$\left[\frac{\partial}{\partial y}\sqrt{K}\right]^2 = \tfrac{1}{2}(\overline{a_1^2} + \overline{c_1^2}) + 2(\overline{a_1 a_2} + \overline{c_1 c_2})y + \ldots \qquad \text{(iv)}$$

From these expansions near a wall conclude that for $y \to 0$

$$\overline{(u'v')}\frac{d\bar{u}}{dy} = 0(y^3)$$

$$\frac{\partial}{\partial y}\overline{(v'\tfrac{1}{2}q^2)} = 0(y^3) \qquad \text{(v)}$$

$$\nu\left(\frac{d^2K}{dy^2}\right)_{y=0} = \left[\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}\right]_{y=0} = \nu(\overline{a_1^2} + \overline{c_1^2})$$

$$\epsilon \simeq \frac{2\nu K}{y^2}, \qquad \left.\frac{K}{\epsilon y^2}\right|_{y=0} = \frac{1}{2\nu}$$

$$\epsilon - 2\nu\left[\frac{\partial}{\partial y}\sqrt{K}\right]^2 = 0(y^2)$$

Also use the energy equation near a wall to show that

$$\frac{d}{dy}\left(\frac{\overline{v'p'}}{\rho}\right) = 0(y) \tag{vi}$$

Note: According to some recent direct numerical simulations and measurements, [105] – [107], $\overline{a_1^2} + \overline{c_1^2}$ varies from $0.166\ u_\tau^4/\nu^2$ to $0.26\ u_\tau^4/\nu^2$, and $\overline{a_1b_2}$ varies from $-0.00072\ u_\tau^5/\nu^3$ to $-0.0013\ u_\tau^5/\nu^3$.

6.15 For a turbulent boundary layer on a flat plate below $R_\delta < 10^5$ it has been found that a simple power law velocity distribution of the *similarity* type

$$u/u_\infty = (y/\delta)^n \tag{i}$$

yields useful results. Corresponding to Equation i the resistance law is

$$c_f = CR_\delta^{-m} \tag{ii}$$

where $m = 2n/(n + 1)$.

(a) Use Equation i to show that:

$$\frac{\theta}{\delta} = \frac{n}{(n + 1)(2n + 1)}$$

(b) Use the momentum integral equation:

$$\frac{d\theta}{dx} = \frac{1}{2}c_f$$

to obtain:

$$\theta(x) = \frac{nx}{(n + 1)(2n + 1)}\left[\frac{C(2n + 1)(3n + 1)}{2n}\right]^{(n+1)/(3n+1)} \cdot R_x^{-2n/(3n+1)}$$

where $R_x = u_\infty x/\nu$.

(c) Take $n = 1/7$, $C = 0.045$ to show that:

$$\theta(x) = 0.036xR_x^{-1/5}$$

$$\delta(x) = 0.37xR_x^{-1/5}$$

$$\frac{\delta_{\text{Turb.}}}{\delta_{\text{Lam.}}} \cong 0.074R_x^{3/10}$$

6.16 Suppose the measured velocity distribution $\bar{u}(y)$ in a boundary layer at a fixed location x and for all values of y is available. The external velocity $u_e(x)$ is also known. Set up an algorithm and the necessary formulae to calculate the wall friction $\tau_w(x)$.

For a steady two-dimensional turbulent boundary layer flow the equations are

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{i}$$

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} + \nu\frac{\partial^2\bar{u}}{\partial y^2} + \frac{\partial}{\partial y}(-\overline{u'v'}) \tag{ii}$$

The boundary conditions are

$$\text{at } y = 0: \bar{u} = \bar{v} = 0, \overline{u'v'} = 0$$
$$\text{as } y \to \infty: \bar{u} \to u_e, \overline{u'v'} \to 0 \quad \text{(iii)}$$

The momentum integral equation is

$$\frac{d\theta}{dx} = \frac{1}{2}\left[1 + \beta\left(1 + \frac{2}{H}\right)\right]c_f \quad \text{(iv)}$$

where δ^*, θ are the displacement and momentum thicknesses, respectively, and:

$$c_f = 2u_\tau^2/u_e^2$$
$$\beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}}{dx}$$
$$= \frac{-2H}{c_f}\frac{\theta}{u_e}\frac{du_e}{dx}$$
$$H = \delta^*/\theta \quad \text{(v)}$$

(a) Let $\eta = y/\theta$ be the nondimensional boundary layer coordinate. Using Equations ii and iii show that the boundary conditions become:

$$\text{at } \eta = 0: \frac{\bar{u}}{u_e} = 0, \quad \frac{\partial}{\partial\eta}\left(\frac{\bar{u}}{u_e}\right) = \frac{R_\theta c_f}{2}, \quad \frac{\partial^2}{\partial\eta^2}\left(\frac{\bar{u}}{u_e}\right) = \frac{\beta R_\theta c_f}{2H}$$
$$\text{as } \eta \to \infty: \frac{\bar{u}}{u_e} \to 1, \quad \frac{\partial}{\partial\eta}\left(\frac{\bar{u}}{u_e}\right) = \frac{\partial^2}{\partial\eta^2}\left(\frac{\bar{u}}{u_e}\right) = \ldots \to 0 \quad \text{(vi)}$$

(b) Suppose by some method we have the velocity profile available in the form (refer to Equation 6.251):

$$\frac{\bar{u}}{u_e} = f'(R_\theta; \eta), \qquad \left(f' = \frac{\partial f}{\partial\eta}\right) \quad \text{(viia)}$$

where:

$$\eta = \frac{y}{\theta} \quad \text{(viib)}$$

Perform the transformation, Equations viia, b, on Equations i and ii and use Equation iv to show that the Reynolds stress, on using $\gamma = \frac{R_\theta}{2}\left[1 + \beta\left(1 + \frac{1}{H}\right)\right]c_f$, is

$$\frac{-\overline{u'v'}}{u_e^2} = \frac{\beta\eta c_f}{2H} - \frac{1}{2}c_f\left(1 + \beta + \frac{\beta}{H}\right)ff' + \frac{1}{2}c_f(1+\beta)\int_0^\eta f'^2\,d\eta$$
$$+ \gamma\int_0^\eta\left(f'\frac{\partial f'}{\partial R_\theta} - f''\frac{\partial f}{\partial R_\theta}\right)d\eta + \frac{1}{2}c_f - \frac{1}{R_\theta}f'' \quad \text{(viii)}$$

6.17 Following the steps in the derivation of two-dimensional momentum integral equation for laminar boundary layer flow (Chapter 5), show that exactly the same equation is obtained for the turbulent flow, i.e.:

$$\frac{1}{u_e^2}\frac{\partial}{\partial t}(u_e\delta^*) + \frac{\partial\theta}{\partial x} + \frac{1}{u_e}\frac{\partial u_e}{\partial x}(2\theta + \delta^*) = \frac{\tau_w}{\rho u_e^2}$$

where τ_w is the wall friction due to the turbulent flow.

6.18 This exercise pertains to the numerical evaluation of parameters a and b appearing in Equation 6.250. To fix ideas, it must be mentioned at the outset that computation of a and b depends on the specified values of c_f, R_θ, and H.

(a) Denoting $\bar{u}(2)/u_e = U_2$ and $\bar{u}(5)/u_e = U_5$, use Equation 6.250 to show that:

$$a = \ell n\left[\frac{\ell n(1 + A_2^2) - \ell n(1 - A_2^2)}{\ell n(1 + A_1^2) - \ell n(1 - A_1^2)}\right] \bigg/ \ell n(2.5)$$

$$b = 2^{-(a+1)}[\ell n(1 + A_1^2) - \ell n(1 - A_1^2)]$$

where:

$$A_1 = [U_2 - A\tan^{-1}(2\Gamma)]/B$$

$$A_2 = [U_5 - A\tan^{-1}(5\Gamma)]/B$$

(b) Make a computer program to compute parameters a and b for each of the following sets of H, c_f, R_θ:

β	H	c_f	R_θ	
−0.270	1.297	0.00264	14410	Favorable pressure gradient
0.000	1.350	0.00284	6870	Zero pressure gradient
0.000	1.386	0.00294	4440	Zero pressure gradient
1.497	1.515	0.00152	9980	Adverse pressure gradient
5.499	1.594	0.00124	34190	Adverse pressure gradient
2.996	1.796	0.00088	36840	Adverse pressure gradient
14.541	1.837	0.00091	17370	Adverse pressure gradient
18.939	1.869	0.00062	73200	Adverse pressure gradient
182.776	2.455	0.00018	26330	Adverse pressure gradient
44.577	2.546	0.00028	9290	Adverse pressure gradient
78.988	2.566	0.00031	12190	Adverse pressure gradient

(c) Having calculated a and b, compute $\bar{u}(\eta)/u_e$ from Equation 6.250 for each data set. Form:

$$\eta_G = 1 - \left[\frac{\bar{u}(\eta)}{u_e}\right]^2_{\eta=1}$$

called the Gruschwitz parameter. Plot η_G vs. H. What is your conclusion?

6.19 Saffman has proposed a second order constitutive equation for the Reynolds stresses which is

$$\overline{\tau}_{ij} = \frac{2}{3}K\delta_{ij} - 2A\frac{K}{\omega}\overline{D}_{ij} - \frac{\lambda_0 K}{\omega^2}(e_{ikm}\overline{D}_{jk} + e_{jkm}\overline{D}_{ik})\overline{\omega}_m; \quad A = \alpha_0^2$$

where $\overline{\omega}_m$ is the mean vorticity component. Using the equations:

$$e_{ikm}\overline{\omega}_m = -2\overline{W}_{ik}$$

and:

$$M_{ij} = \frac{1}{\omega}\frac{\partial \overline{u}_i}{\partial x_j}$$

show that:

$$T_{ij} = \overline{\tau}_{ij}/K = \frac{2}{3}\delta_{ij} - A(M_{ij} + M_{ji})$$
$$+ \frac{\lambda_0}{2}[(M_{jk} + M_{kj})(M_{ik} - M_{ki}) + (M_{ik} + M_{ki})(M_{jk} - M_{kj})]$$

where ω has been defined in Equation 6.325a and λ_0 is a constant. Taking $\lambda_0 = 0.02$ find the values of $T_{11} - 2/3$, $T_{22} - 2/3$, $T_{33} - 2/3$, and of T_{12} for a plane flow in which $\overline{u}_3 = 0$ and all quantities are independent of x_3.

6.20 Experiments have shown that isotropic turbulence decays at large time as $K(t) \simeq t^{-1.1}$. Use Equations 6.301 and 6.309 for isotropic turbulence in which $K = K(t)$, and $K(0) = K_0$, $\epsilon(0) = \epsilon_0$ to obtain $K(t)$ as a function of time. From this expression, justify the choice of $c_{\epsilon 2} \simeq 1.92$.

MATHEMATICAL EXPOSITION 1

BASE VECTORS AND VARIOUS REPRESENTATIONS

1. INTRODUCTION

For an analytical description of fluid motion it is imperative from the outset to have a consistent set of notations and a good working knowledge of some basic rules of mathematical manipulations and concepts. In this ME we shall establish the rules regarding vector and tensor manipulations in rectangular Cartesian coordinates and also in general coordinates. Throughout the text we have used a right-handed system of coordinates which is exemplified by the direction of arrows in Figure M1.1.

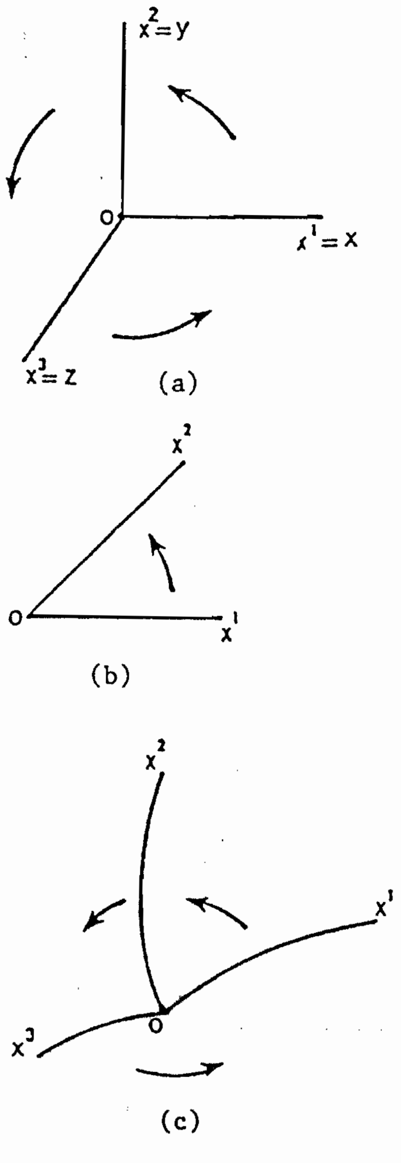


Fig. M1.1 (a) Right-handed rectangular Cartesian system; (b) Non-rectangular Cartesian system; (c) General curvilinear system.

For rectangular Cartesian coordinates we shall use the notation x_1, x_2, x_3 or simply x_i (i = 1, 2, 3); and for general coordinates we shall use x^1, x^2, x^3 or simply x^i (i = 1, 2, 3). In the latter case the superscripts serve only as labels and not powers. When there will be no need of sub- or superscripted notation for coordinates, then we shall write:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z; \qquad x^1 = \xi, \quad x^2 = \eta, \quad x^3 = \zeta$$

If the coordinates are rectangular Cartesian, then unit vectors of constant directions along x_1, x_2, x_3 are denoted as $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, respectively; while for x, y, z they are denoted as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively. Thus the *base vectors* for rectangular Cartesian coordinates are

$$\mathbf{i}_1 = \mathbf{i} = (1, 0, 0), \quad \mathbf{i}_2 = \mathbf{j} = (0, 1, 0), \quad \mathbf{i}_3 = \mathbf{k} = (0, 0, 1)$$

For a concise representation of entitites we shall use the *summation convention* on repeated indices. The rule of summation is that if in a certain term or in a multiplication of terms the same index appears twice, then a sum is implied over the range of index values. Thus the position vector $\mathbf{r}$ drawn from the origin (0, 0, 0) to point (x_1, x_2, x_3) is represented as:

$$\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = x_k\mathbf{i}_k \tag{1}$$

where k is a dummy index implying a sum from k = 1 to 3. Any other vector $\mathbf{v}$ is also represented as:

$$\mathbf{v} = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3 = v_k\mathbf{i}_k \tag{2}$$

For quantities in general coordinates we shall use both the sub- and superscript index notations which have definite geometric meanings. *The main exception will be that in general coordinates the summation will be applied on repeated indices when one index is a subscript while the other is a superscript.* Examples of summation convention follow:

1. In rectangular Cartesian coordinates:

$$A_m a_m, \quad T_{mn}a_n, \quad T_{ii}, \quad \frac{\partial u_k}{\partial x_k}, \quad \text{etc. imply sums}$$

2. In general coordinates:

$$A^m a_m, \quad T_{mn}a^m, \quad T^{mn}a_m, \quad T^i_i, \quad \frac{\partial u^k}{\partial x^k} \quad \text{etc. imply sums}$$

It must be mentioned here that all results which are valid in a general coordinate system reduce to equally valid results in a rectangular Cartesian system.

2. REPRESENTATIONS IN RECTANGULAR CARTESIAN SYSTEMS

The base vectors $\mathbf{i}_k$ are independent orthonormal vectors. Thus:

$$\mathbf{i}_j \cdot \mathbf{i}_k = \delta_{jk} \tag{3}$$

where δ_{jk} is the *Cartesian* Kronecker symbol defined as:

$$\delta_{jk} = 1 \quad \text{if} \quad j = k$$

$$= 0 \quad \text{if} \quad j \neq k \tag{4}$$

Further, following the right-hand rule:

$$\mathbf{i}_k \times \mathbf{i}_m = e_{kmn}\mathbf{i}_n \quad \text{(sum on } n\text{)} \tag{5}$$

where e_{kmn} is called the permutation symbol and is defined as follows:

1. If the number of transpositions required to bring k, m, n in the form 1, 2, 3 is even, then $e_{kmn} = 1$.
2. If the number of transpositions required to bring k, m, n in the form 1, 2, 3 is odd, then $e_{kmn} = -1$.
3. If any two or all three values among the index are equal, then $e_{kmn} = 0$.
   Thus:

$$e_{123} = e_{231} = e_{312} = 1$$

$$e_{132} = e_{321} = e_{213} = -1$$

otherwise zero.

3. SCALARS, VECTORS, AND TENSORS

In all fluid dynamic problems we have to manipulate with scalars, vectors, and tensors. Scalars are entitites each with a magnitude but no sense of direction. Vectors are entitites each with a magnitude and a direction. Besides these properties vectors are governed by a law of addition of two vectors and a law of multiplication of a vector by a scalar. Tensors are higher order vectors and have a number of components depending on their orders and also on the dimension of the space in which they occur. Tensors of various orders either appear naturally in a derivation or can be constructed to express a natural phenomenon. In fact, every quantity can be called a tensor of a particular order. Thus a scalar is a tensor of order zero while a vector is a tensor of order one. The most important property of vector and tensor representations lies in the invariant character of physical phenomena. For example, if the velocity vector at a point in a fluid medium is **u**, then it can be represented with reference to either a rectangular Cartesian coordinate system or any other general curvilinear system without changing its magnitude or direction.

Let T be a tensor of order n in a space of dimension N. Then the number of components of T are N^n. Another way of looking at tensors is to represent them as an indefinite product of vectors. For example, consider two arbitrary vectors **a** and **b**. We now define a tensor **T** as:

$$\mathbf{T} = \mathbf{ab}, \qquad (\mathbf{ab} \neq \mathbf{ba}) \tag{6a}$$

where **ab** is called the *dyadic product* of vectors. Substituting the component form (Equation 2) for **a** and **b**, we have:

$$\mathbf{T} = a_m b_n \mathbf{i}_m \mathbf{i}_n \tag{6b}$$

so that the components of **T** are $a_m b_n$ and in a 3-D space there are nine such components. It must, however, be pointed out that the dyadic representation, Equation 6, is a particular case of the second order tensors. Refer to the end of Sect. 9.

Tensors of various orders are usually written in their component forms, e.g., T_{ij}, T^{i}_{mn} etc. Tensors of order two are most efficiently handled by using the dyadic product of base vectors. Thus a second order tensor in rectangular Cartesian coordinates is written as:

$$\mathbf{T} = T_{mn}\mathbf{i}_m\mathbf{i}_n \tag{7a}$$

The transpose of a second order tensor is obtained by interchanging the rows and columns of the matrix $[T_{mn}]$. In dyadic form, the transpose of $\mathbf{T}$ is

$$\mathbf{T}^{\mathrm{T}} = T_{nm}\mathbf{i}_m\mathbf{i}_n = T_{mn}\mathbf{i}_n\mathbf{i}_m \tag{7b}$$

In the same manner the transpose of Equation 6b is

$$\mathbf{T}^{\mathrm{T}} = a_m b_n \mathbf{i}_n \mathbf{i}_m$$

A second order tensor is said to be symmetric when:

$$T_{mn} = T_{nm}$$

which implies that:

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}$$

for symmetric tensors.

A second order tensor is said to be skew-symmetric when:

$$T_{mn} = -T_{nm}$$

which implies that:

$$\mathbf{T} = -\mathbf{T}^{\mathrm{T}}$$

for skew-symmetric tensors.

4. DIFFERENTIAL OPERATIONS ON TENSORS

Various new entities are formed when scalars, vectors, and tensors are differentiated. We first consider them in the context of a rectangular Cartesian frame of reference as follows:

(a) Gradient

A first order differential operator is defined as:

$$\nabla = \mathrm{grad} = \mathbf{i}_k \frac{\partial}{\partial x_k} \tag{8}$$

Let ϕ be a scalar; then:

$$\mathrm{grad}\ \phi = \frac{\partial \phi}{\partial x_k}\mathbf{i}_k \tag{9}$$

is a vector, called the gradient of a scalar.

Let $\mathbf{v} = v_m \mathbf{i}_m$ be a vector, then*

* There is a duality of representation here. Some authors use premultiplication by $\mathbf{i}_k$ in the definition of grad $\mathbf{v}$.

$$\text{grad } \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_k}\,\mathbf{i}_k \tag{10a}$$

$$= \frac{\partial v_m}{\partial x_k}\,\mathbf{i}_m\mathbf{i}_k \tag{10b}$$

which is a second order tensor.

Let $\mathbf{T} = T_{mn}\mathbf{i}_m\mathbf{i}_n$ be a tensor, then:

$$\text{grad } \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k}\,\mathbf{i}_k \tag{11a}$$

$$= \frac{\partial T_{mn}}{\partial x_k}\,\mathbf{i}_m\mathbf{i}_n\mathbf{i}_k \tag{11b}$$

which is a third order tensor.

Here we emphasize that throughout this text we have adopted the nondyadic convention for the gradient of a vector or the divergence of a tensor. As has been discussed in Truesdell and Noll,[1a] the gradient of a vector field **u** is a tensor field grad **u**, which is defined as a linear transformation that assigns to a vector field **v** the vector field (grad **u**). **v**, where

$$(\text{grad } \mathbf{u}) \cdot \mathbf{v} = \lim_{s\to 0}\,[\{\mathbf{u}(\mathbf{r}+s\mathbf{v}) - \mathbf{u}(\mathbf{r})\}/s]$$

Thus,

$$\text{grad } \mathbf{u} = \frac{\partial u_m}{\partial x_n}\mathbf{i}_m\mathbf{i}_n, \quad (\mathbf{u}\cdot\nabla)\mathbf{v} = (\text{grad } \mathbf{v})\cdot\mathbf{u}$$
$$\text{div}\,(\mathbf{ab}) = (\text{grad } \mathbf{a})\cdot\mathbf{b} + \mathbf{a}\,(\text{div}\,\mathbf{b})$$
$$\text{div}\,(\mathbf{T}\cdot\mathbf{u}) = (\text{div}\,\mathbf{T})\cdot\mathbf{u} + \mathbf{T}\colon(\text{grad } \mathbf{u})$$

In fluid dynamics since we are usually interested in the gradient of a vector, we consider a few consequences of Equations 10. The first thing to note is that the components of grad **v**, viz., $\partial v_m/\partial x_k$, form a square matrix. The index m signifies the row number, while the index k signifies the column number. In the same manner the representation of a tensor such as in Equation 7a follows the same rule. For future use we state it in words as follows:

$$\text{Second order tensor} = \text{component}\begin{pmatrix}\text{Base vector}\\ \text{Signifying a row}\end{pmatrix}\begin{pmatrix}\text{Base vector}\\ \text{Signifying a column}\end{pmatrix} \tag{12}$$

The transpose of grad **v** will be denoted as $(\text{grad } \mathbf{v})^T$ and expressed as:

$$(\text{grad } \mathbf{v})^T = \frac{\partial v_m}{\partial x_k}\,\mathbf{i}_k\mathbf{i}_m \tag{13}$$

which is obtained by interchanging the rows and columns of grad **v**.

(b) Divergence

The divergence of a vector **v** is a scalar and obtained as:

$$\text{div } \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_k} \cdot \mathbf{i}_k \tag{14a}$$

$$= \frac{\partial v_m}{\partial x_k} \delta_{mk} = \frac{\partial v_k}{\partial x_k} \tag{14b}$$

Similarly, the divergence of a tensor is

$$\text{div } \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k} \cdot \mathbf{i}_k \tag{15a}$$

If **T** is a second order tensor, then:

$$\text{div } \mathbf{T} = \frac{\partial T_{mn}}{\partial x_k} \mathbf{i}_m \delta_{nk} = \frac{\partial T_{mk}}{\partial x_k} \mathbf{i}_m \tag{15b}$$

Equation 15 is the adopted definition in this book for the divergence of a tensor.

The simplicity in all these operations is because of the constancy of the base vectors $\mathbf{i}_k$.

(c) Curl

The curl of a vector is obtained as:

$$\text{curl } \mathbf{v} = \mathbf{i}_k \times \frac{\partial \mathbf{v}}{\partial x_k} = -\frac{\partial \mathbf{v}}{\partial x_k} \times \mathbf{i}_k \tag{16a}$$

Thus, using the permutation symbol, we have:

$$\text{curl } \mathbf{v} = e_{kmr} \frac{\partial v_m}{\partial x_k} \mathbf{i}_r \tag{16b}$$

Similarly:

$$\text{curl } \mathbf{T} = \mathbf{i}_k \times \frac{\partial \mathbf{T}}{\partial x_k} \tag{16c}$$

5. MULTIPLICATION OF A TENSOR AND A VECTOR

Let **T** be a second order tensor and **v** a vector. We consider the result of scalar multiplications $\mathbf{T} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{T}$.

$$\mathbf{T} \cdot \mathbf{v} = (T_{mn}\mathbf{i}_m\mathbf{i}_n) \cdot (v_k\mathbf{i}_k) \tag{17a}$$

$$= T_{mk}v_k\mathbf{i}_m \tag{17b}$$

where $T_{mk}v_k$ form a vector.:

$$\mathbf{T}^{\mathrm{T}} \cdot \mathbf{v} = (T_{mn}\mathbf{i}_n\mathbf{i}_m) \cdot (v_k\mathbf{i}_k)$$

$$= T_{kn}v_k\mathbf{i}_n \tag{17c}$$

By comparing Equations 17b, c we have the result:

$$\mathbf{T}^T \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{T} \tag{18}$$

Since Equation 18 is a tensor invariant form, it should be valid in every coordinate system.

Let v and w be arbitrary vectors and T a second order tensor. Then based on the results in Equations 17a, c it is easy to see that:

$$\mathbf{w} \cdot (\mathbf{T} \cdot \mathbf{v}) = \mathbf{v} \cdot (\mathbf{T}^T \cdot \mathbf{w}) = (\mathbf{T} \cdot \mathbf{v}) \cdot \mathbf{w} \tag{19}$$

which is also valid in every coordinate system.

If the tensor is grad **u**, then:

$$(\text{grad } \mathbf{u}) \cdot \mathbf{v} = v_k \frac{\partial u_m}{\partial x_k} \mathbf{i}_m \tag{20a}$$

On the other hand, it is easy to verify that:

$$(\mathbf{v} \cdot \text{grad})\mathbf{u} = v_k \frac{\partial u_m}{\partial x_k} \mathbf{i}_m \tag{20b}$$

Thus:

$$(\text{grad } \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \cdot \text{grad})\mathbf{u} \tag{21}$$

If $\mathbf{v} = \mathbf{u}$, then:

$$(\text{grad } \mathbf{u}) \cdot \mathbf{u} = (\mathbf{u} \cdot \text{grad})\mathbf{u} \tag{22}$$

which should also be valid in every coordinate system.

6. SCALAR MULTIPLICATION OF TWO TENSORS

There can be two types of scalar multiplication between a pair of tensors. In the first type, the multiplication of two second order tensors results in a new second order tensor. In the second type, the multiplication of two second order tensors results in a pure scalar.

Let S and T be a pair of second order tensors. Writing both in the form of Equation 7a, the scalar product is given as:

$$\begin{aligned} \mathbf{S} \cdot \mathbf{T} &= (S_{mn}\mathbf{i}_m\mathbf{i}_n) \cdot (T_{pq}\mathbf{i}_p\mathbf{i}_q) \\ &= (S_{mn}T_{pq}\mathbf{i}_m\mathbf{i}_q)\delta_{np} \\ &= S_{mp}T_{pq}\mathbf{i}_m\mathbf{i}_q \end{aligned} \tag{23}$$

It is immediately recognized that the coefficient in Equation 23 is a result of the matrix multiplication of two square matrices.

The other type of tensor multiplication is based on the definition of inner multiplication of two dyads. Here we follow the definition due to Gibbs (see Reference 1 at the end of the MEs), which is

$$(\mathbf{ab}) : (\mathbf{cd}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{24}$$

where **a**, ..., **d** are arbitrary vectors. Thus according to Equation 24:

$$\mathbf{S} : \mathbf{T} = (S_{mn}\mathbf{i}_m\mathbf{i}_n) : (T_{pq}\mathbf{i}_p\mathbf{i}_q)$$
$$= S_{mn}T_{pq}\delta_{mp}\delta_{nq} = S_{mn}T_{mn} \tag{25}$$

The product defined in Equation 24 is called the "inner" or the double dot product of two dyads.

7. A COLLECTION OF USABLE FORMULAE

The following formulae have been stated without proof or derivation. It is expected that a student will verify these formulae with the help of techniques developed so far:

(a) For any three arbitrary vectors **u**, **v**, **w**, the triple scalar product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = e_{ijk}u_i v_j w_k \tag{26}$$

Also,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \tag{27}$$

(b) The triple vector products are*

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) = u_p v_q w_r e_{pqn} e_{nrs} \mathbf{i}_s$$
$$= u_p v_q w_r (\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr})\mathbf{i}_s \tag{28a}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) = u_p v_q w_r e_{pns} e_{qrn} \mathbf{i}_s$$
$$= u_p v_q w_r (\delta_{pr}\delta_{qs} - \delta_{pq}\delta_{sr})\mathbf{i}_s \tag{28b}$$

(c) For four arbitrary vectors **a**, **b**, **c**, **d**:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \tag{29}$$

This is also known as the Lagrange identity.

(d) Using the definition of a dyad and the divergence of a tensor, we have:

$$\operatorname{div}(\mathbf{uv}) = (\operatorname{grad} \mathbf{u}) \cdot \mathbf{v} + (\operatorname{div} \mathbf{v})\mathbf{u} \tag{30}$$

(e) A unit tensor, denoted as **I**, has the value 1 on the main diagonal and zero elsewhere in the matrix representation of its components. A representation of **I**, which is *valid only in rectangular coordinates* is

$$\mathbf{I} = \delta_{mn}\mathbf{i}_m\mathbf{i}_n \tag{31a}$$

$$\mathbf{I} \cdot \mathbf{v} = \mathbf{v}, \qquad \mathbf{I} \cdot \mathbf{T} = \mathbf{T} \quad \text{etc.} \tag{31b}$$

* $e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

If f is a scalar, $\mathbf{u}$ a vector, and $\mathbf{I}$ the unit tensor, then:

$$\operatorname{div}(\mathbf{I}f) = \operatorname{grad} f \tag{32}$$

$$\operatorname{div}(f\mathbf{u}) = f \operatorname{div} \mathbf{u} + \mathbf{u} \cdot (\operatorname{grad} f) \tag{33}$$

$$\operatorname{curl}(f\mathbf{u}) = f \operatorname{curl} \mathbf{u} - \mathbf{u} \times (\operatorname{grad} f) \tag{34}$$

(f) The Laplacian ∇^2 is

$$\nabla^2(\ ) = \operatorname{div}[\operatorname{grad}(\ )] \tag{35}$$

then:

$$\nabla^2\mathbf{u} = \operatorname{div}(\operatorname{grad} \mathbf{u}) = \operatorname{grad}(\operatorname{div} \mathbf{u}) - \operatorname{curl}(\operatorname{curl} \mathbf{u}) \tag{36}$$

(g)
$$(\operatorname{grad} \mathbf{u}) \cdot \mathbf{u} = \operatorname{grad}(\tfrac{1}{2} |\mathbf{u}|^2) + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} \tag{37}$$

(h)
$$\operatorname{div}(\operatorname{grad} \mathbf{u})^T = \operatorname{grad}(\operatorname{div} \mathbf{u}) \tag{38}$$

and thus:

$$\operatorname{div}[(\operatorname{grad} \mathbf{u}) - (\operatorname{grad} \mathbf{u})^T] = -\operatorname{curl}(\operatorname{curl} \mathbf{u}) \tag{39}$$

(i) For two arbitrary vectors:

$$(\operatorname{curl} \mathbf{u}) \times \mathbf{v} = [(\operatorname{grad} \mathbf{u}) - (\operatorname{grad} \mathbf{u})^T] \cdot \mathbf{v} \tag{40}$$

$$(\mathbf{u} \times \operatorname{grad}) \times \mathbf{v} = (\operatorname{grad} \mathbf{v})^T \cdot \mathbf{u} - \mathbf{u}(\operatorname{div} \mathbf{v}) \tag{41a}$$

$$= (\operatorname{grad} \mathbf{v}) \cdot \mathbf{u} - (\operatorname{curl} \mathbf{v}) \times \mathbf{u} - \mathbf{u}(\operatorname{div} \mathbf{v}) \tag{41b}$$

(j)
$$\operatorname{grad}(\mathbf{u} \cdot \mathbf{v}) = (\operatorname{grad} \mathbf{u})^T \cdot \mathbf{v} + (\operatorname{grad} \mathbf{v})^T \cdot \mathbf{u} \tag{42a}$$

$$= (\operatorname{grad} \mathbf{u}) \cdot \mathbf{v} + (\operatorname{grad} \mathbf{v}) \cdot \mathbf{u} + \mathbf{u} \times (\operatorname{curl} \mathbf{v}) + \mathbf{v} \times (\operatorname{curl} \mathbf{u}) \tag{42b}$$

(k1)
$$\operatorname{curl}(\mathbf{u} \times \mathbf{v}) = \operatorname{div}(\mathbf{uv} - \mathbf{vu}) \tag{43a}$$

$$= (\operatorname{grad} \mathbf{u}) \cdot \mathbf{v} - (\operatorname{grad} \mathbf{v}) \cdot \mathbf{u} + \mathbf{u}(\operatorname{div} \mathbf{v}) - \mathbf{v}(\operatorname{div} \mathbf{u}) \tag{43b}$$

(k2)
$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\operatorname{curl} \mathbf{u}) - \mathbf{u} \cdot (\operatorname{curl} \mathbf{v}) \tag{43c}$$

(l) For a vector $\mathbf{u}$ and a tensor $\mathbf{T}$:

$$\mathbf{I} : (\operatorname{grad} \mathbf{u}) = \operatorname{div} \mathbf{u} \tag{44a}$$

$$\operatorname{div}(\mathbf{T} \cdot \mathbf{u}) = (\operatorname{div} \mathbf{T}^T) \cdot \mathbf{u} + \mathbf{T}^T : (\operatorname{grad} \mathbf{u}) \tag{44b}$$

(m) Let $\mathbf{W}$ be a skew-symmetric tensor in a three-dimensional space. With $\mathbf{W}$, a vector $\mathbf{w}$ can be associated, called its dual, such that:

$$\operatorname{div} \mathbf{W} = -\operatorname{curl} \mathbf{w} \tag{45a}$$

$$\operatorname{div} \mathbf{W}^T = \operatorname{curl} \mathbf{w} \tag{45b}$$

If $\mathbf{W}$ is the rotation tensor, then $2\mathbf{w} = \boldsymbol{\omega}$ and $2 \text{ div } \mathbf{W} = -\text{curl } \boldsymbol{\omega}$. For an arbitrary vector $\mathbf{v}$ and the associated vector $\mathbf{w}$:

$$\mathbf{W} \cdot \mathbf{v} = \mathbf{w} \times \mathbf{v} \tag{45c}$$

and:

$$\mathbf{W} : (\text{grad } \mathbf{v}) = \mathbf{w} \cdot (\text{curl } \mathbf{v}) \tag{45d}$$

For an arbitrary second order tensor $\mathbf{T}$ and the associated vector $\mathbf{w}$:

$$\mathbf{w} \times \mathbf{T} = \mathbf{W} \cdot \mathbf{T} \tag{45e}$$

Formulae 27–45 are in vector or tensor invariant forms, and therefore they are applicable in all coordinate systems.

8. TAYLOR'S EXPANSION IN VECTOR FORM

Let $f_m(\mathbf{r})$, $m = 1, 2, 3$, be three differentiable functions of the coordinates. Replacing $\mathbf{r}$ by $\mathbf{r} + d\mathbf{r}$ we form Taylor's expansion for $f_m(\mathbf{r} + d\mathbf{r})$ correct to the first order as:

$$f_m(\mathbf{r} + d\mathbf{r}) = f_m(\mathbf{r}) + (d\mathbf{r} \cdot \text{grad})f_m(\mathbf{r}) + 0(|d\mathbf{r}|^2)$$

where $m = 1, 2, 3$, and:

$$d\mathbf{r} \cdot \text{grad} = dx_k \frac{\partial}{\partial x_k}$$

Multiplying the first, second, and third equation, respectively, by $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and adding, we get:

$$\mathbf{f}(\mathbf{r} + d\mathbf{r}) = \mathbf{f}(\mathbf{r}) + (d\mathbf{r} \cdot \text{grad})\mathbf{f}(\mathbf{r}) + 0(|d\mathbf{r}|^2) \tag{46}$$

From such an elementary analysis the gradient of a vector can be defined by first noting that (cf. Equation 21):

$$(d\mathbf{r} \cdot \text{grad})\mathbf{f} = (\text{grad } \mathbf{f}) \cdot d\mathbf{r}$$

and thus Equation 46 defines grad $\mathbf{f}$ through the equation:

$$(\text{grad } \mathbf{f}) \cdot d\mathbf{r} = \mathbf{f}(\mathbf{r} + d\mathbf{r}) - \mathbf{f}(\mathbf{r})$$

9. PRINCIPAL AXES OF A TENSOR

Let $\mathbf{T}$ be a second order tensor and $\mathbf{v}$ be a vector. The scalar product $\mathbf{T} \cdot \mathbf{v}$ will be a new vector which is neither parallel nor equal in magnitude to $\mathbf{v}$. If, however, the vector $\mathbf{T} \cdot \mathbf{v}$ is parallel to $\mathbf{v}$ but different in magnitude (from that of $\mathbf{v}$), then:

$$\mathbf{T} \cdot \mathbf{v} = \lambda \mathbf{v} \tag{47}$$

where λ is a scalar. The structure of Equation 47 clearly shows that for its satisfaction neither $\mathbf{v}$ nor λ can be arbitrarily selected. In fact, both $\mathbf{v}$ and λ now depend on the latent properties of the tensor **T**, and are called the eigenvector and eigenvalue, respectively. Further since the operation $\mathbf{T} \cdot \mathbf{v}$ is linear, we conclude that if $\mathbf{v}$ is an eigenvector, then so is $\alpha\mathbf{v}$ where α is an arbitrary scalar.

Another form of Equation 47 which leads to a useful component representation is obtained by utilizing the unit tensor, Equation 31b. Thus:

$$\mathbf{T} \cdot \mathbf{v} = \lambda(\mathbf{I} \cdot \mathbf{v})$$

or:

$$(\mathbf{T} - \lambda\mathbf{I}) \cdot \mathbf{v} = 0 \tag{48}$$

In component form, Equation 48 is

$$(T_{mn} - \lambda\delta_{mn})v_n\mathbf{i}_m = 0$$

However, $\mathbf{i}_m$ are independent vectors; therefore:

$$(T_{mn} - \lambda\delta_{mn})v_n = 0 \tag{49}$$

For each index value of m, there results a linear homogeneous algebraic equation. For the purpose of obtaining concrete formulae for future use, we consider both m and n as assuming values from 1 to 3. Thus from Equation 49 we obtain three simultaneous homogeneous equations which yield nontrivial solutions only when:

$$\det(T_{mn} - \lambda\delta_{mn}) = 0 \tag{50}$$

On expanding the determinant in Equation 50, we obtain a cubic equation in λ which can be written as:

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

or:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{51}$$

where I_1, I_2, I_3 are the three *invariants* of the second order tensor **T** and defined as:

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\ &= T_{11} + T_{22} + T_{33} \end{aligned} \tag{52a}$$

$$\begin{aligned} I_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ &= \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \end{aligned} \tag{52b}$$

$$I_3 = \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} \tag{52c}$$

Corresponding to each root* $\lambda_{(r)}$ ($r = 1, 2, 3$) of Equation 51 there will be an eigenvector $\mathbf{v}_{(r)}$ of components $v_{n(r)}$ ($n = 1, 2, 3$) satisfying Equation 49. Thus, for each r, the components $v_{n(r)}$ are given by:

$$\frac{v_{1(r)}}{D_1} = \frac{v_{2(r)}}{D_2} = \frac{v_{3(r)}}{D_3} = \alpha \tag{53}$$

where α is an arbitrary scalar, and:

$$D_1 = \begin{vmatrix} T_{22} - \lambda_{(r)} & T_{23} \\ T_{32} & T_{33} - \lambda_{(r)} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} T_{23} & T_{21} \\ T_{33} - \lambda_{(r)} & T_{31} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} T_{21} & T_{22} - \lambda_{(r)} \\ T_{31} & T_{32} \end{vmatrix}$$

The vector $\mathbf{v}_{(r)}$ associated with $\lambda_{(r)}$ is directed along the *principal direction* of the tensor $\mathbf{T}$. The coordinate axes thus generated are known as the *principal axes*.

Before we proceed further we state the following theorem:

Theorem: *If the tensor* $\mathbf{T}$ *is symmetric then the eigenvalues* $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}$ *are real and the eigenvectors* $\mathbf{v}_{(1)}, \mathbf{v}_{(2)}, \mathbf{v}_{(3)}$ *are mutually orthogonal.*

From this theorem we conclude that the principal directions of a symmetric tensor are mutually orthogonal. (If the tensor is nonsymmetric, then the eigenvectors are orthogonal to the eigenvectors of the transpose of the tensor.)

We confine ourselves only to symmetric tensors and in this case the above theorem provides us the corresponding principal axes of the tensor. Two cases in regard to the eigenvalues are of importance:

1. If $\lambda_1 = \lambda_2$ and λ_3 is distinct, then the vector $\mathbf{v}_{(3)}$ is orthogonal to both $\mathbf{v}_{(1)}$ and $\mathbf{v}_{(2)}$. The vectors $\mathbf{v}_{(1)}$ and $\mathbf{v}_{(2)}$ can be oriented orthogonal to one another in a plane in any desired fashion since $\mathbf{T}$ is isotropic in this plane.
2. If $\lambda_1 = \lambda_2 = \lambda_3$, then any three mutually perpendicular axes can be chosen in a 3-D space since $\mathbf{T}$ is isotropic in this space. In this case any three orthogonal axes are the principal axes. For a dyad $\mathbf{ab}$, $\lambda_1 = \mathbf{a}\cdot\mathbf{b}$, $\lambda_2 = \lambda_3 = 0$.

10. TRANSFORMATION OF T TO THE PRINCIPAL AXES

Let $\mathbf{T}$ be a symmetric tensor, and we wish to express it in terms of the unit vectors along its principal axes. Denoting by $\mathbf{u}_{(r)}$ the unit vector along the r-th principal axis, i.e.:

$$\mathbf{u}_{(r)} = \mathbf{v}_{(r)}/|\mathbf{v}_{(r)}|$$

then according to Equation 48:

$$(\mathbf{T} - \lambda_{(r)}\mathbf{I}) \cdot \mathbf{u}_{(r)} = 0 \tag{54}$$

In terms of $\mathbf{u}_{(r)}$ the tensor $\mathbf{T}$ is

* We have enclosed r in parentheses to suspend the summation convention on r.

$$\mathbf{T} = S_{pq}\mathbf{u}_p\mathbf{u}_q$$

and using this form in Equation 54 we have:

$$(S_{pq}\mathbf{u}_p\mathbf{u}_q - \lambda_{(r)}\mathbf{I}) \cdot \mathbf{u}_{(r)} = 0$$

or:

$$S_{pr}\mathbf{u}_p = \lambda_{(r)}\mathbf{u}_{(r)} \tag{55}$$

Equation 55 is satisfied only when:

$$S_{pr} = \delta_{pr}\lambda_{(r)}$$

Thus the principal axis transformation of T is

$$\mathbf{T} = \lambda_{(1)}\mathbf{u}_1\mathbf{u}_1 + \lambda_{(2)}\mathbf{u}_2\mathbf{u}_2 + \lambda_{(3)}\mathbf{u}_3\mathbf{u}_3 \tag{56}$$

Example M1.1

Express the symmetric tensor T of components:

$$\begin{array}{lll} T_{11} = 3, & T_{12} = 7, & T_{13} = 9 \\ T_{21} = 7, & T_{22} = 4, & T_{23} = 3 \\ T_{31} = 9, & T_{32} = 3, & T_{33} = 8 \end{array}$$

with respect to its principal axis system.

First of all, from Equation 52:

$$I_1 = 15, \qquad I_2 = -71, \qquad I_3 = -269$$

and then the roots of Equation 51 are

$$\lambda_{(1)} = 18.10138, \qquad \lambda_{(2)} = 2.60447, \qquad \lambda_{(3)} = -5.70585$$

$\underline{r = 1}$

$$D_1 = 133.44340, \qquad D_2 = 97.70966, \qquad D_3 = 147.91242$$

$\underline{r = 2}$

$$D_1 = -1.47038, \qquad D_2 = -10.76871, \qquad D_3 = 8.44023$$

$\underline{r = 3}$

$$D_1 = 124.02692, \qquad D_2 = -68.94095, \qquad D_3 = -66.35265$$

Choosing $\alpha = 1.0/147.91242$, the eigenvectors from Equation 53 are

$$\begin{aligned} \mathbf{v}_{(1)} &= 0.902178\mathbf{i}_1 + 0.660591\mathbf{i}_2 + \mathbf{i}_3 \\ \mathbf{v}_{(2)} &= -0.009941\mathbf{i}_1 - 0.072805\mathbf{i}_2 + 0.057062\mathbf{i}_3 \\ \mathbf{v}_{(3)} &= 0.838516\mathbf{i}_1 - 0.466093\mathbf{i}_2 - 0.448594\mathbf{i}_3 \end{aligned}$$

Thus the unit vectors along the principal axes are

$$\mathbf{u}_1 = 0.601411\mathbf{i}_1 + 0.440364\mathbf{i}_2 + 0.666621\mathbf{i}_3$$
$$\mathbf{u}_2 = -0.106853\mathbf{i}_1 - 0.782557\mathbf{i}_2 + 0.613341\mathbf{i}_3$$
$$\mathbf{u}_3 = 0.791762\mathbf{i}_1 - 0.440105\mathbf{i}_2 - 0.423581\mathbf{i}_3$$

The tensor **T** referred to the principal axes is

$$\mathbf{T} = 18.10138\mathbf{u}_1\mathbf{u}_1 + 2.60447\mathbf{u}_2\mathbf{u}_2 - 5.70585\mathbf{u}_3\mathbf{u}_3$$

11. QUADRATIC FORM AND THE EIGENVALUE PROBLEM

A quadratic form is simply $\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}$ or:

$$A(x) = A_{mn}x_m x_n \tag{57}$$

where, without any loss of generality, the coefficients A_{mn} are symmetric, viz., $A_{mn} = A_{nm}$. In three dimensions the indices m and n take values 1 to 3. It is immediately apparent that the coefficients A_{mn} form the components of a second order tensor or matrix A. Because of the symmetry of A, its eigenvalues will be real and the corresponding eigenvectors will form the principal axis system.

Let the coordinate system referred to the principal axes be denoted by $\bar{x}_m$. When the quadratic form (Equation 57) is referred to the principal axis system, then:

$$A(\bar{x}) = \lambda_{(1)}(\bar{x}_1)^2 + \lambda_{(2)}(\bar{x}_2)^2 + \lambda_{(3)}(\bar{x}_3)^2 \tag{58}$$

From Equation 58 we conclude that when a quadratic form is referred to the principal directions of its coefficient matrix, the quadratic form takes a canonical form in which there appear no cross products. The representation in Equation 58 is unique.

12. REPRESENTATION IN CURVILINEAR COORDINATES

In general curvilinear coordinate x^i ($i = 1, 2, 3$)* the base vectors are not constants either in magnitude or direction. Also, now the placement of subscript and superscript on the components of physical or mathematical entitites has definite meaning. To illustrate these ideas and also the notations clearly, we consider the simplest case of two-dimensional nonrectangular coordinates.

Consider a two-dimensional oblique coordinate system (x^1, x^2) as is shown in Figure M1.2(a).

We start with a result from the theory of linear vector spaces, which states that in an N dimensional space any N noncollinear vectors are linearly independent and then any other vector can be expressed linearly in terms of these chosen vectors. As shown in Figure M1.2(b), we choose linearly independent vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively, along x^1 and x^2. In this space we consider another vector **w** drawn from the origin O. From the tip of **w** draw lines parallel to the axes Ox^1 and Ox^2 to have a parallelogram $OAPB$. Thus a linear representation of **w** in terms of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is

$$\mathbf{w} = \boldsymbol{\lambda}p + \boldsymbol{\mu}q \tag{59}$$

where p and q are called the "parallel projections" of **w** with respect to the basis $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. (Under this basis the physical components of **w** are given by the representation:

$$\mathbf{w} = \mathbf{u}|\boldsymbol{\lambda}|p + \mathbf{v}|\boldsymbol{\mu}|q$$

* The results obtained here are applicable to any index range.

where **u** and **v** are the unit vectors along $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively.

Another method of writing **w** in a linear form is to draw perpendicular lines DP on Ox^2 and CP on Ox^1 as shown in Figure M1.2(c). We now introduce a new coordinate system by drawing lines $O\bar{x}^1$ and $O\bar{x}^2$ are perpendicular to Ox^2 and Ox^1, respectively. Let ψ and χ be the basis of the new coordinate system. With reference to $O\bar{x}^1\bar{x}^2$, the vector **w** is expressed as:

$$\mathbf{w} = \psi R + \chi S \tag{60}$$

Since:

$$\boldsymbol{\lambda} \cdot \chi = 0, \qquad \boldsymbol{\mu} \cdot \psi = 0 \tag{61}$$

writing:

$$r = \mathbf{w} \cdot \boldsymbol{\lambda}, \qquad s = \mathbf{w} \cdot \boldsymbol{\mu}$$

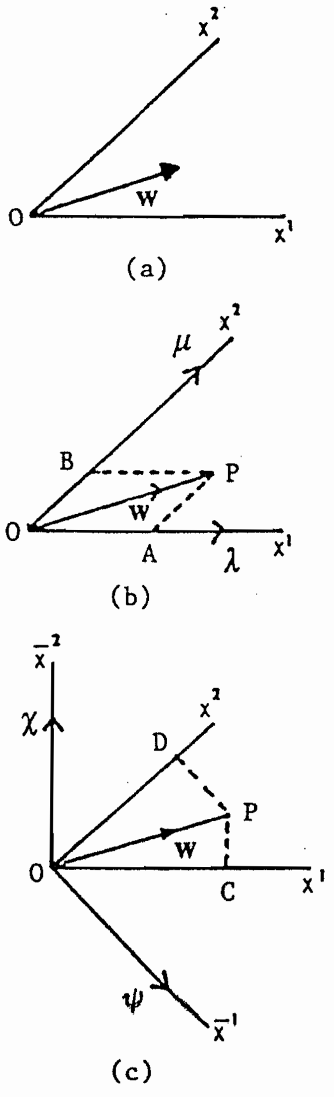


Fig. M1.2 Illustrating the resolution of a vector w with respect to a non-rectangular Cartesian system.

Equation 60 can be rewritten as:

$$\mathbf{w} = \frac{\boldsymbol{\psi}}{\boldsymbol{\psi} \cdot \boldsymbol{\lambda}} r + \frac{\boldsymbol{\chi}}{\boldsymbol{\mu} \cdot \boldsymbol{\chi}} s \tag{62}$$

The quantities r and s are called the "orthogonal projections" of $\mathbf{w}$ with respect to the basis $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Because of two possible linear representations of the same vector $\mathbf{w}$, it is important to have a new system of labeling. It is a standard practice to write:

$$\boldsymbol{\lambda} = \mathbf{a}_1, \qquad \boldsymbol{\mu} = \mathbf{a}_2 \tag{63a}$$

$$\frac{\boldsymbol{\psi}}{\boldsymbol{\psi} \cdot \boldsymbol{\lambda}} = \mathbf{a}^1, \qquad \frac{\boldsymbol{\chi}}{\boldsymbol{\mu} \cdot \boldsymbol{\chi}} = \mathbf{a}^2 \tag{63b}$$

$$p = w^1, \qquad q = w^2 \tag{64a}$$

$$r = w_1, \qquad s = w_2 \tag{64b}$$

Thus, Equations 59 and 62 are written as:

$$\mathbf{w} = w^1\mathbf{a}_1 + w^2\mathbf{a}_2 \tag{65a}$$

$$\mathbf{w} = w_1\mathbf{a}^1 + w_2\mathbf{a}^2 \tag{65b}$$

Also from Equations 63 using Equation 61, we get:

$$\mathbf{a}_1 \cdot \mathbf{a}^1 = 1, \quad \mathbf{a}_2 \cdot \mathbf{a}^2 = 1, \quad \mathbf{a}_1 \cdot \mathbf{a}^2 = 0, \quad \mathbf{a}_2 \cdot \mathbf{a}^1 = 0 \tag{66}$$

Vectors $\mathbf{a}_1$, $\mathbf{a}_2$ are called the covariant base vectors, while $\mathbf{a}^1$, $\mathbf{a}^2$ are called the contravariant base vectors. In the same manner w_1, w_2 are called the covariant and w^1, w^2 are called the contravariant components of the vector $\mathbf{w}$.

Based on the results in Equations 64–66, we now state that for *any* coordinate system there exists a system of natural basis $(\mathbf{a}_i)$ forming a set of independent vectors and a system of reciprocal basis $(\mathbf{a}^i)$ also forming a set of independent vectors, such that:

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta^i_j, \qquad i, j = 1, 2, \ldots, N \tag{67}$$

where δ^i_j is the Kronecker symbol:

$$\delta^i_j = 1 \quad \text{if} \quad i = j$$
$$= 0 \quad \text{if} \quad i \neq j$$

The natural basis $(\mathbf{a}_i)$ is such that each vector, e.g., $\mathbf{a}_j$ is locally tangent to the j-th coordinate. For a three-dimensional coordinate system the base vectors are shown in Figure M1.3.

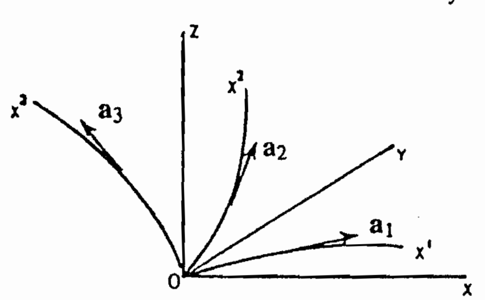


Fig. M1.3 Base vectors for curvilinear coordinates.

The base vectors ($\mathbf{a}_i$) are neither of unit magnitudes nor of constant directions. In case the coordinates become rectangular Cartesian, then $\mathbf{a}_k = \mathbf{i}_k = \mathbf{a}^k$ and the distinction between a basis and its reciprocal basis disappears.

In terms of the basis ($\mathbf{a}^i$) any vector $\mathbf{v}$ can now be represented by using the summation convention (cf. Section 1) as:

$$\mathbf{v} = v^i \mathbf{a}_i \tag{68a}$$

In terms of the reciprocal basis, the same vector is

$$\mathbf{v} = v_i \mathbf{a}^i \tag{68b}$$

In Equations 68a, b, v^i are the contravariant components and v_i are the covariant components.

Similar to the representations in rectangular Cartesian coordinates, a second order tensor $\mathbf{T}$ is now represented as:

$$\mathbf{T} = T^{ij}\mathbf{a}_i\mathbf{a}_j \tag{69a}$$

$$= T_{ij}\mathbf{a}^i\mathbf{a}^j \tag{69b}$$

where T^{ij} and T_{ij} are the contravariant and covariant components of $\mathbf{T}$, respectively. In general coordinates, tensor $\mathbf{T}$ can also be represented in a mixed component form as:

$$\mathbf{T} = T^i{}_j\mathbf{a}_i\mathbf{a}^j = T_j{}^i\mathbf{a}^j\mathbf{a}_i \tag{69c}$$

where superscript index i stands for the contravariant and subscript j stands for the covariant nature of $T^i{}_j$ and $T_j{}^i$.

Fundamental Metric Components

Using the two types of basis vectors (viz., $\mathbf{a}_i$ and $\mathbf{a}^i$), we form the scalars:

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{a}_i \tag{70a}$$

$$g^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j = \mathbf{a}^j \cdot \mathbf{a}^i \tag{70b}$$

which are the fundamental metric components of the space in which the curvilinear coordinates have been introduced. The components g_{ij} and g^{ij} are the covariant and contravariant components, respectively, of a tensor, called the metric tensor. As shown by Equations 70 both components are symmetric.

Denoting the metric tensor as $\mathbf{I}$, we have*:

$$\mathbf{I} = g_{ij}\mathbf{a}^i\mathbf{a}^j \tag{71a}$$

$$= g^{ij}\mathbf{a}_i\mathbf{a}_j \tag{71b}$$

$$= \delta^i_j\mathbf{a}_i\mathbf{a}^j \tag{71c}$$

Thus, with $\mathbf{I}$ there are associated three forms of matrices, which in a three-dimensional space are

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{72a}$$

* Each form in Equations 71 reduces to a unit tensor when referenced to a Cartesian frame.

$$[g^{ij}] = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{72b}$$

$$[\delta^i_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{72c}$$

The structure of **I** given in Equations 72 can now be used to establish some very important formulae involving the metric tensor components. First of all, using Equations 67 and 72c we have:

$$\begin{aligned} \mathbf{I} \cdot \mathbf{I} &= (\delta^i_j \mathbf{a}_i \mathbf{a}^j) \cdot (\delta^p_q \mathbf{a}_p \mathbf{a}^q) \\ &= \delta^i_j \delta^p_q \delta^j_p \mathbf{a}_i \mathbf{a}^q \\ &= \delta^i_q \mathbf{a}_i \mathbf{a}^q \end{aligned}$$

Thus:

$$\mathbf{I} \cdot \mathbf{I} = \mathbf{I} \tag{73}$$

and **I** is a unit or identity tensor. Next substituting Equations 71a, b in Equation 73 and again using Equation 67, we get:

$$\begin{aligned} \mathbf{I} \cdot \mathbf{I} &= (g_{ij} \mathbf{a}^i \mathbf{a}^j) \cdot (g^{mk} \mathbf{a}_m \mathbf{a}_k) \\ &= g_{ij} g^{jk} \mathbf{a}^i \mathbf{a}_k = \mathbf{I} = \delta^k_i \mathbf{a}^i \mathbf{a}_k \end{aligned}$$

Thus:

$$g_{ij} g^{jk} = \delta^k_i \tag{74}$$

In three dimensions Equation 74 represents six algebraic equations connecting covariant and contravariant components of the metric tensor. Let:

$$g = \det(g_{ij}) \tag{75}$$

then solving Equation 74, we obtain:

$$g^{ij} = (g_{rs} g_{mn} - g_{rn} g_{ms})/g \tag{76a}$$

$$g_{ij} = g(g^{rs} g^{mn} - g^{rn} g^{ms}) \tag{76b}$$

where:

$$i = 1 : r = 2, \quad m = 3; \qquad j = 1 : s = 2, \quad n = 3$$
$$i = 2 : r = 3, \quad m = 1; \qquad j = 2 : s = 3, \quad n = 1$$
$$i = 3 : r = 1, \quad m = 2; \qquad j = 3 : s = 1, \quad n = 2 \tag{77}$$

The following formulae of much importance can be established by using Equations 67 and 74:

$$\mathbf{a}_i = g_{ij}\mathbf{a}^j \tag{78a}$$
$$\mathbf{a}^i = g^{ij}\mathbf{a}_j \tag{78b}$$
$$v_i = g_{ik}v^k \tag{78c}$$
$$v^j = g^{jk}v_k \tag{78d}$$

Elemental Displacement Vector

Consider points P and Q in an Euclidean space* whose position vectors from a reference point O are $\mathbf{r}$ and $\mathbf{r} + \Delta\mathbf{r}$ as shown in Figure M1.4.

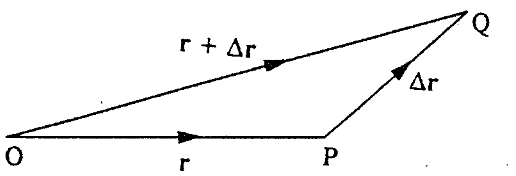


Fig. M1.4 Vector triangle with displacement vector as its one side.

As is obvious, the vector increment $\Delta\mathbf{r}$ is independent of reference point 0 and is also independent of any reference coordinate system which can be used to measure its magnitude and direction. The principal part of $\Delta\mathbf{r}$ is $d\mathbf{r}$, which is the elemental displacement vector. In a coordinate system x^i the coordinates of P and Q will be (x^i) and $(x^i + dx^i)$. Thus using the definition of a differential, we have:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i}\, dx^i \tag{79}$$

Noting that dx^i are the contravariant components of $d\mathbf{r}$, we can also write $d\mathbf{r}$, according to Equation 68a, as:

$$d\mathbf{r} = \mathbf{a}_i\, dx^i \tag{80}$$

and thus:

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial x^i} \tag{81}$$

Equation 81 provides a concrete formula for the determination of the covariant base vectors in an Euclidean space. Thus, for brevity writing:

$$x^1 = \xi, \quad x^2 = \eta, \quad x^3 = \zeta, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$
$$\mathbf{i}_1 = \mathbf{i}, \quad \mathbf{i}_2 = \mathbf{j}, \quad \mathbf{i}_3 = \mathbf{k}$$

* An Euclidean space is a space in which rectangular Cartesian coordinates can be introduced on a global scale.

we have:

$$\left.\begin{aligned} \mathbf{a}_1 &= \mathbf{i}x_\xi + \mathbf{j}y_\xi + \mathbf{k}z_\xi \\ \mathbf{a}_2 &= \mathbf{i}x_\eta + \mathbf{j}y_\eta + \mathbf{k}z_\eta \\ \mathbf{a}_3 &= \mathbf{i}x_\zeta + \mathbf{j}y_\zeta + \mathbf{k}z_\zeta \end{aligned}\right\} \tag{82}$$

where a variable subscript denotes a partial derivative.

The magnitude of $d\mathbf{r}$, denoted as ds, is given by:

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= (\mathbf{a}_i \cdot \mathbf{a}_j)\, dx^i\, dx^j \end{aligned}$$

Using the definition Equation 70a, we have:

$$(ds)^2 = g_{ij}\, dx^i\, dx^j \tag{83}$$

Equation 83 is a formula for measuring infinitesimal lengths between points in all spaces and is a basic postulate (and the only postulate) for a Riemannian space and its geometry.

Differentiation of Base Vectors

From the representations of vectors and tensors in Equations 68 and 69 it is obvious that the operations of grad, curl, and div on vectors and tensors require a knowledge of partial derivatives of the base vectors.

The first result we need in this connection is obtained by using Equation 81, which is

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{r}}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial \mathbf{r}}{\partial x^j}\right)$$

Hence:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \frac{\partial \mathbf{a}_j}{\partial x^i} \tag{84}$$

Now, from the equations:

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \quad g_{jk} = \mathbf{a}_j \cdot \mathbf{a}_k, \quad g_{ik} = \mathbf{a}_i \cdot \mathbf{a}_k$$

we form the derivatives:

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \mathbf{a}_i}{\partial x^k} \cdot \mathbf{a}_j + \mathbf{a}_i \cdot \frac{\partial \mathbf{a}_j}{\partial x^k} \tag{85a}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial \mathbf{a}_j}{\partial x^i} \cdot \mathbf{a}_k + \mathbf{a}_j \cdot \frac{\partial \mathbf{a}_k}{\partial x^i} \tag{85b}$$

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}_k + \mathbf{a}_i \cdot \frac{\partial \mathbf{a}_k}{\partial x^j} \tag{85c}$$

Adding Equation 85b, c, subtracting Equation 85a from it, and using Equation 84 we get:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}_k = [ij, k]$$

which implies:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = [ij, k]\mathbf{a}^k \tag{86}$$

where:

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{87}$$

are called the Christoffel symbols of the first kind.

We now multiply both sides of Equation 86 scalarly by $\mathbf{a}^m$ to have:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}^m = \Gamma^m_{ij}$$

which implies:

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \Gamma^m_{ij}\mathbf{a}_m \tag{88}$$

where:

$$\Gamma^m_{ij} = g^{mk}[ij, k] \tag{89}$$

are called the Christoffel symbols of the second kind.* Fully expanded forms of both Christoffel symbols in three dimensions have been given in Section 13 of this ME.

In a similar fashion, differentiating Equation 67 and using Equation 88 we get:

$$\frac{\partial \mathbf{a}^i}{\partial x^k} \cdot \mathbf{a}_j = -\Gamma^i_{jk}$$

which implies:

$$\frac{\partial \mathbf{a}^i}{\partial x^k} = -\Gamma^i_{rk}\mathbf{a}^r \tag{90}$$

If we set $m = i$ in Equation 89 and perform summation over the repeated indices, then it is easy to show that:

$$\Gamma^i_{ij} = \frac{1}{2g}\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j}(\ell n\sqrt{g}) \tag{91}$$

where $g = \det(g_{ij})$. From Equations 90 and 91, we have the result:

$$\frac{\partial}{\partial x^j}(\sqrt{g}\mathbf{a}^j) = 0 \tag{92}$$

* $[ij, k] = [ji, k]$, $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Gradient of a Vector

In curvilinear coordinates, the grad operator is

$$\text{grad} = \nabla = \mathbf{a}^k \frac{\partial}{\partial x^k} \tag{93}$$

Next we define the gradient of a vector** in the manner of Equation 10a as:

$$\text{grad } \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x^i} \mathbf{a}^i \tag{94}$$

and its transpose is then:

$$(\text{grad } \mathbf{v})^{\mathrm{T}} = \mathbf{a}^i \frac{\partial \mathbf{v}}{\partial x^i} \tag{95}$$

Substituting $\mathbf{v} = v^k \mathbf{a}_k$ in $\partial \mathbf{v}/\partial x^i$ and using Equation 88, we get:

$$\frac{\partial \mathbf{v}}{\partial x^i} = \left(\frac{\partial v^k}{\partial x^i} + v^r \Gamma^k_{ir}\right)\mathbf{a}_k \tag{96}$$

The quantity in parentheses is called the covariant derivative of the contravariant components and is written as:

$$v^k_{,i} = \frac{\partial v^k}{\partial x^i} + v^r \Gamma^k_{ir} \tag{97}$$

Thus Equations 96 and 94 are, respectively:

$$\frac{\partial \mathbf{v}}{\partial x^i} = v^k_{,i} \mathbf{a}_k \tag{98}$$

$$\text{grad } \mathbf{v} = v^k_{,i} \mathbf{a}_k \mathbf{a}^i \tag{99}$$

Similarly by substituting $\mathbf{v} = v_k \mathbf{a}^k$ in $\partial \mathbf{v}/\partial x^i$ and using Equation 90, we get:

$$\frac{\partial \mathbf{v}}{\partial x^i} = v_{k,i} \mathbf{a}^k \tag{100}$$

$$\text{grad } \mathbf{v} = v_{k,i} \mathbf{a}^k \mathbf{a}^i \tag{101}$$

where:

$$v_{k,i} = \frac{\partial v_k}{\partial x^i} - v_r \Gamma^r_{ik} \tag{102}$$

is the covariant derivative of the covariant components. It should be noted that:

$$(\text{grad } \mathbf{v})^{\mathrm{T}} = v^k_{,i} \mathbf{a}^i \mathbf{a}_k \tag{103}$$

** Refer also to Equation M2.14.

$$= v_{k,i}\mathbf{a}^i\mathbf{a}^k \tag{104}$$

The covariant differentiations (Equations 97 and 102) are applicable in every coordinate system. For a scalar ϕ, the gradient is simply:

$$\text{grad } \phi = \frac{\partial \phi}{\partial x^i}\,\mathbf{a}^i = \phi_{,i}\mathbf{a}^i \tag{105a}$$

By using Equation 92, the conservation law form of the gradient is

$$\text{grad } \phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\mathbf{a}^j\phi) \tag{105b}$$

Divergence and Curl of Vectors

The divergence of a vector $\mathbf{v}$ is

$$\text{div } \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x^i}\cdot \mathbf{a}^i$$

Using Equations 67 and 98, we get:

$$\text{div } \mathbf{v} = v^i_{,i} \tag{106}$$

where according to Equations 91 and 97:

$$v^i_{,i} = \frac{\partial v^i}{\partial x^i} + v^j \frac{\partial}{\partial x^j}(\ell n\sqrt{g})$$

Thus:

$$\text{div } \mathbf{v} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}v^i) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\mathbf{v}\cdot\mathbf{a}^i) \tag{107}$$

If covariant components of $\mathbf{v}$ are used, then according to Equation 100:

$$\text{div } \mathbf{v} = g^{ik}v_{i,k} \tag{108}$$

Before giving the expression for the curl of a vector, we state the following easily provable identities:

$$\mathbf{a}_j \times \mathbf{a}_k = \sqrt{g}e_{ijk}\mathbf{a}^i \tag{109a}$$

$$\mathbf{a}^j \times \mathbf{a}^k = \frac{1}{\sqrt{g}}e^{ijk}\mathbf{a}_i \tag{109b}$$

where both e^{ijk} and e_{ijk} are the permutation symbols and superscripted symbols are used only to have a consistent notation for summation on repeated upper and lower indices. From Equations 109 we deduce that:

$$\mathbf{a}^i = \frac{1}{2\sqrt{g}}e^{ijk}(\mathbf{a}_j \times \mathbf{a}_k) \tag{110a}$$

$$\mathbf{a}_i = \frac{\sqrt{g}}{2} e_{ijk}(\mathbf{a}^j \times \mathbf{a}^k) \tag{110b}$$

Thus:

$$\text{curl } \mathbf{v} = \mathbf{a}^i \times \frac{\partial \mathbf{v}}{\partial x^i}$$

Using the previously developed formulae, we get:

$$\text{curl } \mathbf{v} = \frac{1}{\sqrt{g}} e^{ijk} v_{k,j} \mathbf{a}_i \tag{111}$$

Thus the contravariant components of curl **v** are

$$(\text{curl } \mathbf{v})^i = \frac{1}{\sqrt{g}} \left(\frac{\partial v_k}{\partial x^j} - \frac{\partial v_j}{\partial x^k} \right) \tag{112}$$

where i, j, k are to be taken cyclically in the order 1, 2, 3.

Divergence of Second Order Tensors

The divergence of a tensor **T** is defined as:

$$\text{div } \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x^k} \cdot \mathbf{a}^k$$

Using the previously defined expressions for the derivatives of base vectors, we have the following results:

(i)
$$\mathbf{T} = T^{ij}\mathbf{a}_i\mathbf{a}_j$$

$$\text{div } \mathbf{T} = T^{ik}_{\ \ ,k}\mathbf{a}_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g}\, T^{ik}\mathbf{a}_i) \tag{113}$$

(ii)
$$\mathbf{T} = T_{ij}\mathbf{a}^i\mathbf{a}^j$$

$$\text{div } \mathbf{T} = g^{jk} T_{ij,k} \mathbf{a}^i \tag{114}$$

(iii)
$$\mathbf{T} = T^i_{\ j}\mathbf{a}_i\mathbf{a}^j$$

$$\text{div } \mathbf{T} = g^{jk} T^i_{\ j,k} \mathbf{a}_i \tag{115}$$

The covariant derivatives in Equation 113–115 are

$$T^{ij}_{\ \ ,k} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk} T^{mj} + \Gamma^j_{mk} T^{im} \tag{116a}$$

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im} \tag{116b}$$

$$T^i_{\ j,k} = \frac{\partial T^i_{\ j}}{\partial x^k} + \Gamma^i_{mk} T^m_{\ j} - \Gamma^m_{jk} T^i_{\ m} \tag{116c}$$

In this connection it is an easy exercise to show that if one replaces T by g in Equations 116a, b and uses the symmetry property $g_{ij} = g_{ji}$, then:

$$g^{ij}_{.k} = 0, \qquad g_{ij,k} = 0 \tag{117}$$

These equations show that the metric coefficients behave like constants under a covariant differentiation. Using Equations 117, we also get the following results:

$$\frac{\partial g^{ij}}{\partial x^k} = -\Gamma^i_{mk} g^{mj} - \Gamma^j_{mk} g^{im} \tag{118a}$$

and:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^m_{ik} g_{mj} + \Gamma^m_{jk} g_{im} \tag{118b}$$

13. CHRISTOFFEL SYMBOLS IN THREE DIMENSIONS

For brevity, we shall use the notation:

$$x^1 = \xi, \qquad x^2 = \eta, \qquad x^3 = \zeta$$

The metric tensor g_{ij} in three dimensions has six distinct components. The determinant g is then:

$$g = g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} \tag{119}$$

Writing:

$$\begin{aligned}
G_1 &= g_{22}g_{33} - (g_{23})^2 \\
G_2 &= g_{11}g_{33} - (g_{13})^2 \\
G_3 &= g_{11}g_{22} - (g_{12})^2 \\
G_4 &= g_{13}g_{23} - g_{12}g_{33} \\
G_5 &= g_{12}g_{23} - g_{13}g_{22} \\
G_6 &= g_{12}g_{13} - g_{11}g_{23}
\end{aligned} \tag{120}$$

we have on using Equation 76a:

$$\begin{aligned}
&g^{11} = G_1/g, \qquad g^{22} = G_2/g, \qquad g^{33} = G_3/g \\
&g^{12} = G_4/g, \qquad g^{13} = G_5/g, \qquad g^{23} = G_6/g
\end{aligned} \tag{121}$$

First Kind

$$[ij, k] = [ji, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$[11, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial \xi}$$

$$[12, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial \eta}$$

$$[13, 1] = \tfrac{1}{2}\frac{\partial g_{11}}{\partial \zeta}$$

$$[22, 1] = \tfrac{1}{2}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right)$$

$$[23, 1] = \tfrac{1}{2}\left(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{13}}{\partial \eta} - \frac{\partial g_{23}}{\partial \xi}\right)$$

$$[33, 1] = \tfrac{1}{2}\left(2\frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}\right)$$

$$[11, 2] = \tfrac{1}{2}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right)$$

$$[12, 2] = \tfrac{1}{2}\frac{\partial g_{22}}{\partial \xi}$$

$$[13, 2] = \tfrac{1}{2}\left(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{13}}{\partial \eta}\right)$$

$$[22, 2] = \tfrac{1}{2}\frac{\partial g_{22}}{\partial \eta}$$

$$[23, 2] = \tfrac{1}{2}\frac{\partial g_{22}}{\partial \zeta}$$

$$[33, 2] = \tfrac{1}{2}\left(2\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}\right)$$

$$[11, 3] = \tfrac{1}{2}\left(2\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta}\right)$$

$$[12, 3] = \tfrac{1}{2}\left(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta}\right)$$

$$[13, 3] = \tfrac{1}{2}\frac{\partial g_{33}}{\partial \xi}$$

$$[22, 3] = \tfrac{1}{2}\left(2\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta}\right)$$

$$[23, 3] = \tfrac{1}{2}\frac{\partial g_{33}}{\partial \eta}$$

$$[33, 3] = \tfrac{1}{2}\frac{\partial g_{33}}{\partial \zeta} \tag{122a}$$

Second Kind

$$\Gamma^i_{jk} = \Gamma^i_{kj} = g^{i\ell}[jk, \ell]$$

Therefore, using Equations 120 and 121, we have:

$$\Gamma^1_{jk} = \frac{1}{g}\{G_1[jk, 1] + G_4[jk, 2] + G_5[jk, 3]\}$$

$$\Gamma^2_{jk} = \frac{1}{g}\{G_4[jk, 1] + G_2[jk, 2] + G_6[jk, 3]\}$$

$$\Gamma^3_{jk} = \frac{1}{g}\{G_5[jk, 1] + G_6[jk, 2] + G_3[jk, 3]\}$$

$$\Gamma^1_{11} = \frac{1}{2g}\left\{G_1\frac{\partial g_{11}}{\partial \xi} + G_4\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right) + G_5\left(2\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta}\right)\right\}$$

$$\Gamma^1_{12} = \frac{1}{2g}\left\{G_1\frac{\partial g_{11}}{\partial \eta} + G_4\frac{\partial g_{22}}{\partial \xi} + G_5\left(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta}\right)\right\}$$

$$
\begin{aligned}
\Gamma^1_{13} &= \frac{1}{2g}\left\{G_1\frac{\partial g_{11}}{\partial\zeta} + G_4\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{23}}{\partial\xi} - \frac{\partial g_{13}}{\partial\eta}\right) + G_5\frac{\partial g_{33}}{\partial\xi}\right\}\\
\Gamma^1_{22} &= \frac{1}{2g}\left\{G_1\left(2\frac{\partial g_{12}}{\partial\eta} - \frac{\partial g_{22}}{\partial\xi}\right) + G_4\frac{\partial g_{22}}{\partial\eta} + G_5\left(2\frac{\partial g_{23}}{\partial\eta} - \frac{\partial g_{22}}{\partial\zeta}\right)\right\}\\
\Gamma^1_{23} &= \frac{1}{2g}\left\{G_1\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{13}}{\partial\eta} - \frac{\partial g_{23}}{\partial\xi}\right) + G_4\frac{\partial g_{22}}{\partial\zeta} + G_5\frac{\partial g_{33}}{\partial\eta}\right\}\\
\Gamma^1_{33} &= \frac{1}{2g}\left\{G_1\left(2\frac{\partial g_{13}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\xi}\right) + G_4\left(2\frac{\partial g_{23}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\eta}\right) + G_5\frac{\partial g_{33}}{\partial\zeta}\right\}\\
\Gamma^2_{11} &= \frac{1}{2g}\left\{G_4\frac{\partial g_{11}}{\partial\xi} + G_2\left(2\frac{\partial g_{12}}{\partial\xi} - \frac{\partial g_{11}}{\partial\eta}\right) + G_6\left(2\frac{\partial g_{13}}{\partial\xi} - \frac{\partial g_{11}}{\partial\zeta}\right)\right\}\\
\Gamma^2_{12} &= \frac{1}{2g}\left\{G_4\frac{\partial g_{11}}{\partial\eta} + G_2\frac{\partial g_{22}}{\partial\xi} + G_6\left(\frac{\partial g_{13}}{\partial\eta} + \frac{\partial g_{23}}{\partial\xi} - \frac{\partial g_{12}}{\partial\zeta}\right)\right\}\\
\Gamma^2_{13} &= \frac{1}{2g}\left\{G_4\frac{\partial g_{11}}{\partial\zeta} + G_2\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{23}}{\partial\xi} - \frac{\partial g_{13}}{\partial\eta}\right) + G_6\frac{\partial g_{33}}{\partial\xi}\right\}\\
\Gamma^2_{22} &= \frac{1}{2g}\left\{G_4\left(2\frac{\partial g_{12}}{\partial\eta} - \frac{\partial g_{22}}{\partial\xi}\right) + G_2\frac{\partial g_{22}}{\partial\eta} + G_6\left(2\frac{\partial g_{23}}{\partial\eta} - \frac{\partial g_{22}}{\partial\zeta}\right)\right\}\\
\Gamma^2_{23} &= \frac{1}{2g}\left\{G_4\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{13}}{\partial\eta} - \frac{\partial g_{23}}{\partial\xi}\right) + G_2\frac{\partial g_{22}}{\partial\zeta} + G_6\frac{\partial g_{33}}{\partial\eta})\right\}\\
\Gamma^2_{33} &= \frac{1}{2g}\left\{G_4\left(2\frac{\partial g_{13}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\xi}\right) + G_2\left(2\frac{\partial g_{23}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\eta}\right) + G_6\frac{\partial g_{33}}{\partial\zeta}\right\}\\
\Gamma^3_{11} &= \frac{1}{2g}\left\{G_5\frac{\partial g_{11}}{\partial\xi} + G_6\left(2\frac{\partial g_{12}}{\partial\xi} - \frac{\partial g_{11}}{\partial\eta}\right) + G_3\left(2\frac{\partial g_{13}}{\partial\xi} - \frac{\partial g_{11}}{\partial\zeta}\right)\right\}\\
\Gamma^3_{12} &= \frac{1}{2g}\left\{G_5\frac{\partial g_{11}}{\partial\eta} + G_6\frac{\partial g_{22}}{\partial\xi} + G_3\left(\frac{\partial g_{13}}{\partial\eta} + \frac{\partial g_{23}}{\partial\xi} - \frac{\partial g_{12}}{\partial\zeta}\right)\right\}\\
\Gamma^3_{13} &= \frac{1}{2g}\left\{G_5\frac{\partial g_{11}}{\partial\zeta} + G_6\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{23}}{\partial\xi} - \frac{\partial g_{13}}{\partial\eta}\right) + G_3\frac{\partial g_{33}}{\partial\xi}\right\}\\
\Gamma^3_{22} &= \frac{1}{2g}\left\{G_5\left(2\frac{\partial g_{12}}{\partial\eta} - \frac{\partial g_{22}}{\partial\xi}\right) + G_6\frac{\partial g_{22}}{\partial\eta} + G_3\left(2\frac{\partial g_{23}}{\partial\eta} - \frac{\partial g_{22}}{\partial\zeta}\right)\right\}\\
\Gamma^3_{23} &= \frac{1}{2g}\left\{G_5\left(\frac{\partial g_{12}}{\partial\zeta} + \frac{\partial g_{13}}{\partial\eta} - \frac{\partial g_{23}}{\partial\xi}\right) + G_6\frac{\partial g_{22}}{\partial\zeta} + G_3\frac{\partial g_{33}}{\partial\eta}\right\}\\
\Gamma^3_{33} &= \frac{1}{2g}\left\{G_5\left(2\frac{\partial g_{13}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\xi}\right) + G_6\left(2\frac{\partial g_{23}}{\partial\zeta} - \frac{\partial g_{33}}{\partial\eta}\right) + G_3\frac{\partial g_{33}}{\partial\zeta}\right\}
\end{aligned}
\tag{122b}
$$

For a triply orthogonal system:

$$[12, 3] = [13, 2] = [23, 1] = 0$$

$$\Gamma^1_{23} = \Gamma^2_{13} = \Gamma^3_{12} = 0$$

Further, writing for brevity:

$$g_{11} = h_1^2, \qquad g_{22} = h_2^2, \qquad g_{33} = h_3^2$$

the remaining Christoffel symbols of the second kind are

$$\Gamma^1_{11} = \frac{1}{h_1}\frac{\partial h_1}{\partial\xi}, \qquad \Gamma^1_{12} = \frac{1}{h_1}\frac{\partial h_1}{\partial\eta}, \qquad \Gamma^1_{13} = \frac{1}{h_1}\frac{\partial h_1}{\partial\zeta}$$

$$\Gamma^1_{22} = \frac{-h_2}{h_1^2}\frac{\partial h_2}{\partial\xi}, \qquad \Gamma^1_{33} = \frac{-h_3}{h_1^2}\frac{\partial h_3}{\partial\xi}$$

$$\Gamma^2_{11} = \frac{-h_1}{h_2^2}\frac{\partial h_1}{\partial \eta}, \quad \Gamma^2_{12} = \frac{1}{h_2}\frac{\partial h_2}{\partial \xi}$$

$$\Gamma^2_{22} = \frac{1}{h_2}\frac{\partial h_2}{\partial \eta}, \quad \Gamma^2_{23} = \frac{1}{h_2}\frac{\partial h_2}{\partial \zeta}, \quad \Gamma^2_{33} = \frac{-h_3}{h_2^2}\frac{\partial h_3}{\partial \eta}$$

$$\Gamma^3_{11} = \frac{-h_1}{h_3^2}\frac{\partial h_1}{\partial \zeta}, \quad \Gamma^3_{13} = \frac{1}{h_3}\frac{\partial h_3}{\partial \xi}$$

$$\Gamma^3_{22} = \frac{-h_2}{h_3^2}\frac{\partial h_2}{\partial \zeta}, \quad \Gamma^3_{23} = \frac{1}{h_3}\frac{\partial h_3}{\partial \eta}, \quad \Gamma^3_{33} = \frac{1}{h_3}\frac{\partial h_3}{\partial \zeta} \tag{123}$$

Example M1.2

Prove the following formulae:

$$\text{(a)} \qquad \frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}) = -\sqrt{g}\Gamma^j_{km}g^{km} \tag{124}$$

$$\text{(b)} \qquad \operatorname{div}\mathbf{T} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}T^{ik}\mathbf{a}_i) \tag{125}$$

(a) Form the inner sum by setting $k = i$ in Equation 118a and then use Equation 91 to yield the result.
(b) First form the inner sum by setting $j = k$ in Equation 116a and then use Equations 88 and 113 to get the result.

Example M1.3

Show that if the base vectors are functions of time t then:

$$\mathbf{a}_1 \cdot \left(\frac{\partial \mathbf{a}_2}{\partial t} \times \mathbf{a}_3\right) = \frac{\partial \mathbf{a}_2}{\partial t} \cdot (\mathbf{a}_3 \times \mathbf{a}_1) \tag{126}$$

and:

$$\mathbf{a}_1 \cdot \left(\mathbf{a}_2 \times \frac{\partial \mathbf{a}_3}{\partial t}\right) = \frac{\partial \mathbf{a}_3}{\partial t} \cdot (\mathbf{a}_1 \times \mathbf{a}_2) \tag{127}$$

Both identities are simply obvious if one notes that for three arbitrary vectors **a**, **b**, **c**, the following identity is true by taking **a**, **b**, **c** cyclically:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

Example M1.4

If x^i are the curvilinear coordinates in an Euclidean -3 space, show that the contravariant base vectors are given by:

$$\mathbf{a}^m = \operatorname{grad} x^m \tag{128}$$

where:

$$\operatorname{grad} = \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z$$

From Equation 67 we note that:

$$\mathbf{a}^1 \cdot \mathbf{a}_2 = 0, \qquad \mathbf{a}^1 \cdot \mathbf{a}_3 = 0$$

Also grad x^1 is orthogonal to the surface x^1 = constant on which x^2 and x^3, and so also $\mathbf{a}_2$ and $\mathbf{a}_3$ lie. Thus:

$$\mathbf{a}^1 = \text{grad } x^1$$

Similarly, the results of $m = 2$ and 3 can be obtained.

Example M1.5

Show that:

$$\text{div } \mathbf{a}^m = -\Gamma^m_{pq} g^{pq} \tag{129}$$

According to the definition of div for a vector, we have:

$$\text{div } \mathbf{a}^m = \frac{\partial \mathbf{a}^m}{\partial x^r} \cdot \mathbf{a}^r$$

Using Equation 90, we get Equation 129.

14. SOME DERIVATIVE RELATIONS

Let us consider a two-dimensional Euclidean plane in which x and y are rectangular Cartesian coordinates. In this plane let $x^1 = \xi$, $x^2 = \eta$ be a pair of curvilinear coordinates such that:

$$x = x(\xi, \eta), \quad y = y(\xi, \eta); \qquad \xi = \xi(x, y) \quad \eta = \eta(x, y)$$

From Equation 82:

$$\mathbf{a}_1 = \mathbf{i}x_\xi + \mathbf{j}y_\xi, \qquad \mathbf{a}_2 = \mathbf{i}x_\eta + \mathbf{j}y_\eta \tag{130a}$$

From Equations 110a and 128:

$$\mathbf{a}^1 = \text{grad } \xi = \frac{1}{\sqrt{g}} (\mathbf{a}_2 \times \mathbf{k}) \tag{130b}$$

$$\mathbf{a}^2 = \text{grad } \eta = \frac{1}{\sqrt{g}} (\mathbf{k} \times \mathbf{a}_1) \tag{130c}$$

where $\mathbf{k} = \mathbf{a}_3$ is the unit constant vector perpendicular to the plane containing ξ and η. Thus, from Equations 130b, c we have:

$$\xi_x = y_\eta/\sqrt{g}, \quad \xi_y = -x_\eta/\sqrt{g}, \quad \eta_x = -y_\xi/\sqrt{g}, \quad \eta_y = x_\xi/\sqrt{g} \tag{131a}$$

where:

$$\sqrt{g} = x_\xi y_\eta - x_\eta y_\xi \tag{131b}$$

In the same manner, using Equations 78b and 128 and the definitions given in Equation 120, we have the following relations in three dimensions:

$$\frac{\partial \xi}{\partial x_i} = \left(G_1 \frac{\partial x_i}{\partial \xi} + G_4 \frac{\partial x_i}{\partial \eta} + G_5 \frac{\partial x_i}{\partial \zeta}\right) \Big/ g \tag{132a}$$

$$\frac{\partial \eta}{\partial x_i} = \left(G_4 \frac{\partial x_i}{\partial \xi} + G_2 \frac{\partial x_i}{\partial \eta} + G_6 \frac{\partial x_i}{\partial \zeta}\right) \Big/ g \tag{132b}$$

$$\frac{\partial \zeta}{\partial x_i} = \left(G_5 \frac{\partial x_i}{\partial \xi} + G_6 \frac{\partial x_i}{\partial \eta} + G_3 \frac{\partial x_i}{\partial \zeta}\right) \Big/ g \tag{132c}$$

where $i = 1, 2, 3$, and $x_1 = x$, $x_2 = y$, $x_3 = z$.

Normal Derivative of Functions

Let $\phi(x^i) =$ constant be a surface in Euclidean -3 and $F(x^i)$ be a function of position defined on $\phi =$ constant. The unit normal is

$$\mathbf{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{1}{|\text{grad } \phi|} \frac{\partial \phi}{\partial x^i} \mathbf{a}^i \tag{133}$$

Using the formula:

$$\left.\frac{\partial F}{\partial n}\right|_{\phi = \text{constant}} = \mathbf{n} \cdot \text{grad } F$$

we obtain:

$$\left.\frac{\partial F}{\partial n}\right|_{\phi = \text{constant}} = \frac{1}{|\text{grad } \phi|} g^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial F}{\partial x^j} \tag{134}$$

In two dimensions with $x^1 = \xi$, $x^2 = \eta$, we have:

$$\left.\frac{\partial F}{\partial n}\right|_{\xi = \text{constant}} = \frac{1}{\sqrt{g g_{22}}} \left(g_{22} \frac{\partial F}{\partial \xi} - g_{12} \frac{\partial F}{\partial \eta}\right) \tag{135a}$$

$$\left.\frac{\partial F}{\partial n}\right|_{\eta = \text{constant}} = \frac{1}{\sqrt{g g_{11}}} \left(g_{11} \frac{\partial F}{\partial \eta} - g_{12} \frac{\partial F}{\partial \xi}\right) \tag{135b}$$

Physical Components in Curvilinear Coordinates

If $\mathbf{v}$ is a vector, then its physical components with respect to an orthogonal coordinate system are

$$V_i = h_i v^i = \frac{v_i}{h_i} \quad \text{no sum on } i \tag{136a}$$

where v^i are the contravariant and v_i are the covariant components. The subscript i in V must be treated only as a label. Similarly, if $\boldsymbol{\tau}$ is a tensor, then its physical components are

$$T_{ij} = h_i h_j \tau^{ij} = \tau_{ij}/h_i h_j \tag{136b}$$

no sum on i and j. With these definitions the divergence of the tensor $\boldsymbol{\tau}$ has the physical components:

$$(\operatorname{div} \tau)_i = \frac{h_i}{\sqrt{g}} \sum_j \frac{\partial}{\partial x^j} \left(\frac{\sqrt{g} T_{ij}}{h_i h_j} \right) + \sum_j \frac{\partial h_i}{\partial x^j} \left(\frac{T_{ij} + T_{ji}}{h_i h_j} \right) - \sum_j \frac{\partial h_j}{\partial x^i} \frac{T_{jj}}{h_i h_j} \tag{137}$$

Here $i = 1, 2, 3$ and there is *no sum on repeated indices.*

If the coordinates are nonorthogonal, then the physical components of a vector **v** are

$$V_i = \sqrt{g_{ii}} v^i \tag{138a}$$

$$= g^{ij} \sqrt{g_{ii}} v_j \tag{138b}$$

where there is no sum on i; and as before, the subscript i in V should be treated as a label.

For a second order tensor τ and with the adopted convention on multiplication of a tensor and vector as followed throughout the text (e.g., refer to Equation M1.17a), the physical components T_{ij} are

$$T_{ij} = \sqrt{\frac{g_{ii}}{g_{jj}}} g_{kj} \tau^{ik} = \sqrt{\frac{g_{ii}}{g_{jj}}} g_{kj} g^{im} g^{kn} \tau_{mn} \tag{138c}$$

In Equation 138c, there is no sum on i and j and the subscripts i and j in T should be treated as labels. (For components based on the other convention refer to References 1a, 1b, and 1c).

15. Scalar and Double Dot Products of Two Tensors

Using Equation 67, the scalar product of two second order tensor **T** and **S** is

$$\mathbf{T} \cdot \mathbf{S} = T_{ij} S^{jn} \mathbf{a}^i \mathbf{a}_n \tag{139a}$$

$$= g^{ir} T_{ij} S^{jn} \mathbf{a}_r \mathbf{a}_n \tag{139b}$$

etc.

The double dot product is obtained by using Equations 24 and 67 as:

$$\mathbf{T} : \mathbf{S} = T_{ij} S^{ij} \tag{140}$$

etc.

MATHEMATICAL EXPOSITION 2

THEOREMS OF GAUSS, GREEN, AND STOKES

1. GAUSS' THEOREM

Gauss' theorem establishes an equality between an integral on a volume V and an integral on the surface S which bounds the volume V. Let $\mathbf{n}$ be the unit normal vector on S, which is assumed to be positive when drawn outward with respect to the region enclosed by S.

In its simplest form, Gauss' theorem is stated as:

$$\int_V \frac{\partial F}{\partial x_i}\, dv = \int_S F n_i\, dS, \qquad i = 1, 2, 3 \tag{1}$$

where x_i are the rectangular Cartesian coordinates; $n_i = \cos(\mathbf{n}, x_i)$ are the components of $\mathbf{n}$; and $F(x_i)$ is a scalar, or the component of a vector or a tensor. The only requirement of F is that it should be continuously differentiable at least to the first order.

Replacing F by v_i, where v_i is the component of a vector $\mathbf{v}$ along the x_i coordinate, and using the summation convention, we get:

$$\int_V \operatorname{div} \mathbf{v}\, dv = \int_S \mathbf{v} \cdot \mathbf{n}\, dS \tag{2}$$

Similarly, replacing F by T_{ki}, we get:

$$\int_V \operatorname{div} \mathbf{T}\, dv = \int_S \mathbf{T} \cdot \mathbf{n}\, dS \tag{3}$$

Obviously Equation 3 represents three equations.

In Equation 1 replacing F by $e_{kij} v_j$, we obtain:

$$\int_V \operatorname{curl} \mathbf{v}\, dv = \int_S \mathbf{n} \times \mathbf{v}\, dS \tag{4}$$

which also represents three equations. Equations 2–4 are in the vector-tensor invariant form, and therefore each is applicable in every coordinate system. In essence:

$$\operatorname{grad}(\ ) = \operatorname*{Lim}_{V \to 0} \frac{1}{V} \int_S (\ )\mathbf{n}\, dS$$

$$\operatorname{div}(\ ) = \operatorname*{Lim}_{V \to 0} \frac{1}{V} \int_S (\ ) \cdot \mathbf{n}\, dS, \qquad \operatorname{curl}(\ ) = \operatorname*{Lim}_{V \to 0} \frac{1}{V} \int_S \mathbf{n} \times (\ )\, dS$$

2. GREEN'S THEOREM

We substitute the Cartesian forms:

$$\mathbf{v} = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}U, \qquad \mathbf{n} = \mathbf{i}n_1 + \mathbf{j}n_2 + \mathbf{k}n_3$$

in Equation 4 and consider only the coefficient of the unit vector $\mathbf{k}$, which is

$$\int_V \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dv = \int_S (n_1 Q - n_2 P)\, dS$$

Consider now a particular case in which V is a region R in the xy-plane and S is a closed curve C enclosing R. Denoting by s the arc length along C, we get:

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C (n_1 Q - n_2 P)\, ds \tag{5}$$

which is Green's theorem. In the xy-plane considering the counterclockwise traverse on C as positive and $\mathbf{n}$ as the outward drawn normal to C, then according to Equation M3.17 we get:

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C (P\, dx + Q\, dy) \tag{6}$$

Equation 6 is also valid on a curved surface if x and y are replaced by the curvilinear coordinates ξ and η, respectively.

3. STOKES' THEOREM

Stokes' theorem establishes an equality between an integral on an *open* surface S and an integral on the curve C bounding the surface S. Refer to Figure M2.1.

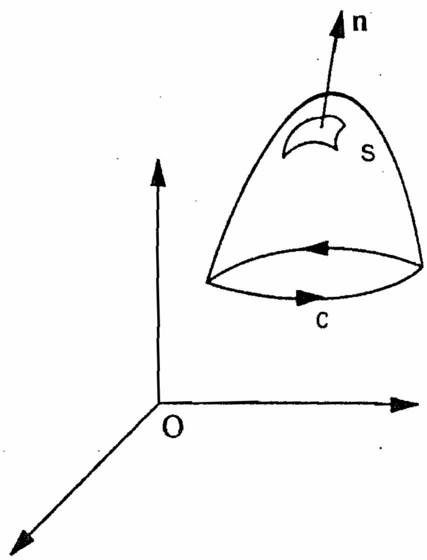


Fig. M2.1 Open surface with the bounding curve C.

The vector-invariant form of Stokes' theorem is

$$\int_S \mathbf{n} \cdot (\text{curl } \mathbf{v})\, dS = \oint_C \mathbf{v} \cdot d\mathbf{r} \tag{7}$$

Thus Equation 7 is valid in all coordinate systems. Another useful form of Equation 7 can be obtained by noting the identity:

$$(\mathbf{n} \times \text{grad}) \cdot \mathbf{v} = \mathbf{n} \cdot (\text{curl } \mathbf{v})$$

Thus:

$$\int_S (\mathbf{n} \times \text{grad}) \cdot \mathbf{v}\, dS = \oint_C \mathbf{v} \cdot d\mathbf{r} \tag{8}$$

With reference to a Cartesian coordinate system, let:

$$\mathbf{v} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

then either Equation 7 or 8 is written as:

$$\int_S \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) n_1 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) n_2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) n_3 \right] dS = \oint_C (u\, dx + v\, dy + w\, dz) \tag{9}$$

From Equation 9, we form three equations according to the following scheme:

1. Replace u by 0, v by w, w by $-v$.
2. Replace u by $-w$, v by 0, w by u.
3. Replace u by v, v by $-u$, w by 0.

Multiply the first, second, and third equation by **i**, **j**, and **k**, respectively, then add. The resulting equation is

$$\int_S (\mathbf{n} \times \text{grad}) \times \mathbf{v}\, dS = \oint_C d\mathbf{r} \times \mathbf{v} \tag{10}$$

Noting the identity:

$$(\mathbf{n} \times \text{grad}) \times \mathbf{v} = (\text{grad } \mathbf{v})^{\mathrm{T}} \cdot \mathbf{n} - \mathbf{n}(\text{div } \mathbf{v})$$

Equation 10 can also be stated as:

$$\int_S [(\text{grad } \mathbf{v})^{\mathrm{T}} \cdot \mathbf{n} - \mathbf{n}(\text{div } \mathbf{v})]\, dS = \oint_C d\mathbf{r} \times \mathbf{v} \tag{11}$$

Another way of writing Equation 1 is

$$\int_V \text{grad } F\, dv = \int_S F\mathbf{n}\, dS \tag{12}$$

from this and from Equation 4, we obtain the respective identities:

$$\int_S \mathbf{n}\, dS = 0, \qquad \int_S \mathbf{r} \times \mathbf{n}\, dS = 0 \tag{13a, b}$$

Both Equations 13a, b are valid for any closed surface. In passing, we note that by applying Equation 13a to an infinitesimal closed surface with area elements $\sqrt{g}\mathbf{a}^1\, dx^2\, dx^3$, $\sqrt{g}\mathbf{a}^2\, dx^1\, dx^3$, $\sqrt{g}\mathbf{a}^3\, dx^1\, dx^2$ of positive normals along x^1, x^2, x^3, respectively, we obtain Equation M1.92. Using the result, we can also write grad **v** as:

$$\text{grad } \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\mathbf{v}\sqrt{g}\mathbf{a}^i) \tag{14}$$

General Statement of the Stokes' Theorem

From the preceding analysis, a general statement of the Stokes' theorem can be written as

$$\int_S (\mathbf{n} \times \text{grad}) * \mathbf{v} dS = \oint_C d\mathbf{r} * \mathbf{v} \tag{15}$$

where the $*$ can be either $\cdot$ or x or a dyadic product.

In the present context it is helpful to note the following expansions in Cartesian coordiantes:

$$(\mathbf{n} \times \text{grad})_i = e_{ijk} n_j \frac{\partial}{\partial x_k}$$

$$\mathbf{n} \times \text{grad} = e_{mjk} \mathbf{i}_m n_j \frac{\partial}{\partial x_k}$$

$$(\mathbf{n} \times \text{grad}) \cdot \mathbf{v} = e_{ijk} n_j \frac{\partial v_i}{\partial x_k}$$

$$\text{curl } \mathbf{v} = e_{mjk} \mathbf{i}_m \frac{\partial v_k}{\partial x_j}$$

MATHEMATICAL EXPOSITION 3

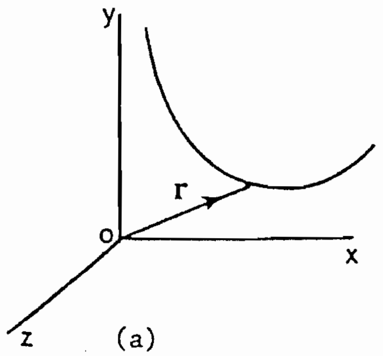
Geometry of Space and Plane Curves

1. BASIC THEORY OF CURVES

In three-dimensional space let the parametric equations of a curve be represented as:

$$\mathbf{r} = \mathbf{r}(t) \tag{1}$$

where $\mathbf{r} = (x^i)$ is the position vector of a position on the curve as shown in Figure M3.1(a) and t is a real parameter which takes values in a certain interval $a \leqq t \leqq b$. It is assumed that the real vector function $\mathbf{r}(t)$ is $p \geqq 1$ times continuously differentiable for every value of t in the specified interval and at least one component of the first derivative $\mathbf{r}'$:



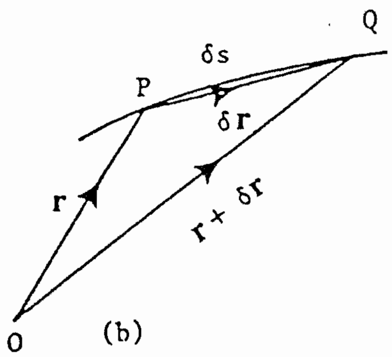


Fig. M3.1 (a) Space curve; (b) Vector displacement $\delta\mathbf{r}$ on the curve.

$$\mathbf{r}' = \frac{d\mathbf{r}}{dt} \tag{2}$$

is different from zero. Note that the parameter t can be expressed in terms of other parameter, say τ, provided that:

$$\frac{d\tau}{dt} \neq 0$$

Tangent Vector

In place of t let us take the arc length s along the curve as a parameter. The direction on the curve along which s assumes increasing values is taken as the positive direction. Let the two nearby points on the curve be denoted as $\mathbf{r}(s)$ and $\mathbf{r}(s + h)$. Then the limit:

$$\begin{aligned} \mathbf{t}(s) &= \lim_{h \to 0} \frac{\mathbf{r}(s + h) - \mathbf{r}(s)}{h} \\ &= \frac{d\mathbf{r}}{ds} = \frac{dx_m}{ds}\mathbf{i}_m = \frac{dx^k}{ds}\mathbf{a}_k \end{aligned} \tag{3}$$

is the *unit tangent vector* at point s on the curve. Note that:

$$|\mathbf{t}| = \frac{|d\mathbf{r}|}{ds} = 1$$

If s is replaced by another parameter t, then:

$$\mathbf{t} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds} = \mathbf{r}'/|\mathbf{r}'| \tag{4}$$

A straight line in the direction of $\mathbf{t}$ from point P on the curve is the tangent line to the curve; see Figure M3.1(b).

Principal Normal

Since:

$$\mathbf{t} \cdot \mathbf{t} = 1$$

by differentiation we get:

$$\mathbf{t} \cdot \frac{d\mathbf{t}}{ds} = 0$$

so that the vector $d\mathbf{t}/ds$ is orthogonal to $\mathbf{t}$ and is called the *curvature vector.* We shall denote it by $\hat{\mathbf{k}}$:

$$\hat{\mathbf{k}} = \frac{d\mathbf{t}}{ds} \tag{5}$$

The *unit principal normal* vector is then defined as:

$$\mathbf{p} = \hat{\mathbf{k}}/|\hat{\mathbf{k}}| \tag{6}$$

The magnitude:

$$k(s) = |\hat{\mathbf{k}}| \tag{7}$$

is the *curvature* of the curve, and:

$$\rho(s) = 1/k(s) \tag{8}$$

is the *radius of curvature.*

The totality of all vectors which are bound at a point of the curve and which are orthogonal to the unit tangent vector at that point lie in a plane. This plane is called the *normal plane*.

The plane spanned by the unit tangent vector and the unit principal normal vector is called the *osculating plane*.

Binormal Vector

A unint vector $\mathbf{b}(s)$ which is orthogonal to both $\mathbf{t}$ and $\mathbf{p}$ is called the binormal vector. Its orientation is fixed by taking $\mathbf{t}, \mathbf{p}, \mathbf{b}$ to form a right-handed system as shown in Figure M3.2, so that:

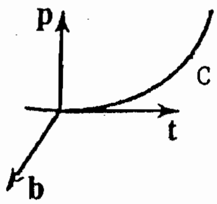


Fig. M3.2 The right-handed system ($\mathbf{t}, \mathbf{p}, \mathbf{b}$).

$$\mathbf{b} = \mathbf{t} \times \mathbf{p} \tag{9}$$

The derivative of $\mathbf{b}$ with respect to s defines a new quantity, called torsion. Thus:

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{p} \tag{10}$$

Serret-Frenet Equations

The following equations, known as the Serret-Frenet equations, are the intrinsic equations of a curve. They are

$$\frac{d\mathbf{t}}{ds} = k\mathbf{p}; \qquad k = \text{curvature}$$

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{p}; \qquad \tau = \text{torsion}$$

$$\frac{d\mathbf{p}}{ds} = \tau\mathbf{b} - k\mathbf{t} \tag{11}$$

Plane Curves

For curves in a plane, the torsion $\tau = 0$ and Equation 11 reduces as follows:

$$\mathbf{b} = \text{constant}$$

$$\frac{d\mathbf{t}}{ds} = k\mathbf{p}$$

$$\frac{d\mathbf{p}}{ds} = -k\mathbf{t} \tag{12}$$

Let there be an arbitrary curve in the xy-plane as shown in Figure M3.3. The unit vector $\mathbf{b}$ is now the constant unit vector $\mathbf{k}$ along the z-axis.

Is has been mentioned earlier that the unit principal normal $\mathbf{p}$ is always directed toward the center of curvature of the curve at that point. Therefore, for general closed curves the direction

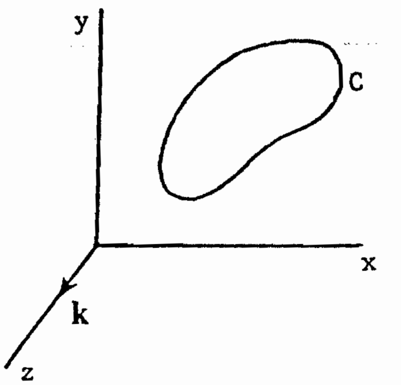


Fig. M3.3 A closed curve in the xy-plane.

of the principal normal can be either directed toward or away from the region enclosed by the curve. To have a uniform sense, we introduce another unit vector which is always directed away from the region bounded by the curve while remaining orthogonal to the unit tangent vector. This vector, denoted as $\mathbf{n}$, is called the unit outward drawn normal to the curve.

As shown in Figure M3.4(a), (b), although angle θ is always positive by convention, angle ϕ can be either positive or negative. However, in either case:

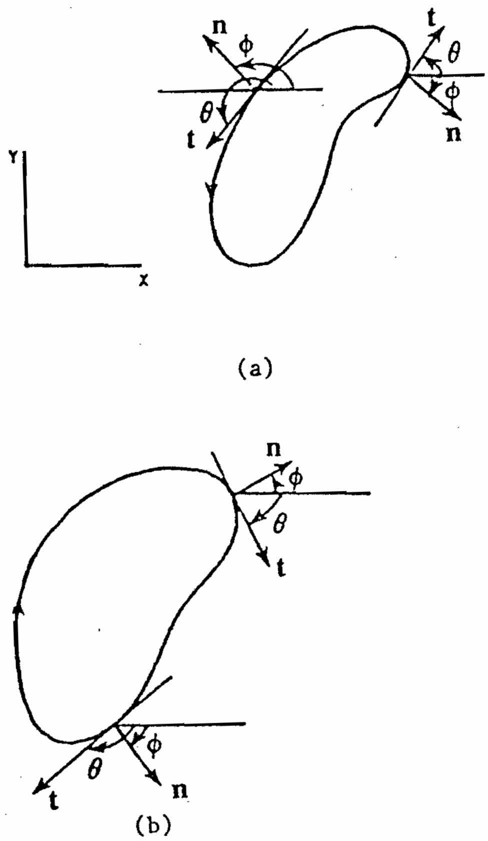


Fig. M3.4 (a) Counter-clockwise traverse on a curve in the xy-plane; (b) Clockwise traverse on a curve in the xy-plane.

$$\theta - \phi = \frac{\pi}{2} \tag{13}$$

For a plane curve, according to Equation 3:

$$\mathbf{t} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds} \tag{14}$$

and:

$$\mathbf{t} \cdot \mathbf{n} = 0 \tag{15}$$

Thus, for a clockwise traverse of the curve:

$$\mathbf{t} \times \mathbf{n} = \mathbf{k}, \quad \mathbf{n} = -\mathbf{i}\frac{dy}{ds} + \mathbf{j}\frac{dx}{ds} \tag{16}$$

and for a counterclockwise traverse of the curve:

$$\mathbf{n} \times \mathbf{t} = \mathbf{k}, \quad \mathbf{n} = \mathbf{i}\frac{dy}{ds} - \mathbf{j}\frac{dx}{ds} \tag{17}$$

MATHEMATICAL EXPOSITION 4

Formulae for Coordinate Transformation

1. INTRODUCTION

In this ME we shall collect those pertinent formulae which are needed in the transformation of scalars, vectors, and tensors under a transformation of coordinates. In this connection it is important to review the basic material on scalars, vectors, and tensors as given in ME.1.

Let x^i and $\bar{x}^i$ be two general coordinate systems with index i varying from 1 to n, $n \geqq 2$. If the components of an entity are known in the x^i system, then we are interested to know the components in the $\bar{x}^i$ system and vice versa. One of the coordinate systems can be rectangular Cartesian, and the other can be a general curvilinear system.

An important consideration which is common to all the discussions that follow in this ME is that there is a functional relation between the two coordinate systems. Thus:

$$\begin{aligned}\bar{x}^i &= f^i(x^1, x^2, \ldots, x^n)\\ &= f^i(x^j), \qquad i = 1, 2, \ldots, n\end{aligned} \tag{1}$$

We assume that the transformation or mapping in Equation 1 is one-to-one and nonsingular. Thus the functions f^i are continuously differentiable and their functional determinants, i.e., the Jacobians are nonvanishing. Thus:

$$\bar{J} = \det\left(\frac{\partial \bar{x}^i}{dx^j}\right), \qquad J = \det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \tag{2}$$

are such that $\bar{J} \neq 0$ and $J \neq 0$ everywhere in the region under consideration. The assumption $\bar{J} \neq 0$ implies that Equation 1 can be inverted to have:

$$x^i = g^i(\bar{x}^j), \qquad i = 1, 2, \ldots, n \tag{3}$$

2. TRANSFORMATION LAW FOR SCALARS

There are two types of scalar quantities. One is called an *absolute scalar* or an *invariant*, while the other is called a *scalar density*.

Let $\phi(x^i)$ be a scalar function of position. If on coordinate transformation the value of ϕ does not change, viz., if:

$$\phi(x^1, x^2, \ldots, x^n) = \bar{\phi}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \tag{4}$$

then ϕ is called an absolute scalar. Equation 4 means that the numerical value of ϕ at x^i and at $\bar{x}^i$, which correspond to the same point, is the same.

There are scalars which on coordinate transformation do not transform like Equation 4. We shall give an example of this type of scalar later. (See Equation 41.)

3. TRANSFORMATION LAWS FOR VECTORS

We first consider the gradient of a scalar f. From Equations M1.105, we find that the components of grad f in the coordinates x^i are given by:

$$\text{grad } f = \frac{\partial f}{\partial x^i} \mathbf{a}^i \tag{5a}$$

Thus the components of grad f are $\partial f/\partial x^i$. On changing the coordinates from x^i to $\bar{x}^i$, the same vector grad f is written as:

$$\text{grad } f = \frac{\partial f}{\partial \bar{x}^i} \bar{\mathbf{a}}^i \tag{5b}$$

Then by the chain rule of differentiation:

$$\frac{\partial f}{\partial x^j} = \frac{\partial f}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^j} \tag{6}$$

and:

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^i} \tag{7}$$

hence, the first partial derivatives of a scalar transform according to the laws stated in Equations 6 and 7.

Next we consider the differential displacement vector $d\mathbf{r}$, which from Equation M1.80 is representable as:

$$d\mathbf{r} = \mathbf{a}_i \, dx^i \tag{8}$$

where:

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial x^i} \tag{9}$$

are the base vectors. Thus dx^i are the components of $d\mathbf{r}$ in the x^i coordinate system.

On changing the coordinates from x^i to $\bar{x}^i$, the same vector $d\mathbf{r}$ is written as:

$$d\mathbf{r} = \bar{\mathbf{a}}_i \, d\bar{x}^i \tag{10}$$

where:

$$\bar{\mathbf{a}}_i = \frac{\partial \mathbf{r}}{\partial \bar{x}^i} \tag{11}$$

are the base vectors in the coordinates $\bar{x}^i$. Obviously:

$$\mathbf{a}_j \, dx_j = \bar{\mathbf{a}}_i \, \mathrm{d}\bar{x}^i \tag{12}$$

By the chain rule of differentiation:

$$dx^j = \frac{\partial x^j}{\partial \bar{x}^i} d\bar{x}^i \tag{13}$$

and:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \tag{14}$$

Thus the differentials of coordinates transform according to the laws stated in Equations 13 and 14.

Based on the above derivations the transformation laws of base vectors can be easily established. Substituting Equations 13 and 14 in Equation 12, we have:

$$\mathbf{a}_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{\mathbf{a}}_i \tag{15}$$

and:

$$\bar{\mathbf{a}}_i = \frac{\partial x^j}{\partial \bar{x}^i} \mathbf{a}_j \tag{16}$$

A study of Equations 6 and 13 or 7 and 14 shows that the transformation laws for components of the gradient of a scalar and of the differential displacement vector are not identical. It is a standard convention to call the components which transform according to Equations 6 and 7 as the *covariant* components (or vectors) and those which transform according to Equations 13 and 14 as the *contravariant* components (or vectors). Using this convention we find from Equations 15 and 16 that the base vectors transform like the covariant components and that is why $\mathbf{a}_i$ are called the covariant base vectors.

Based on the above deductions we state the transformation laws for any vector **A**. The covariant components of a vector **A** denoted as A_i transform according to the laws:[2]

$$A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i \tag{17}$$

and:

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \tag{18}$$

The contravariant components of a vector **A** denoted as A^i transform according to the laws:

$$A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i \tag{19}$$

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \tag{20}$$

It must be noted that the transformation laws for vectors are linear. That is, the vector components in the new system are linear functions of the vector components in the old system.

4. TRANSFORMATION LAWS FOR TENSORS

In Equation M1.69 it was shown that a second order tensor **T** in entity form has the representation:

$$\mathbf{T} = T^{ij}\mathbf{a}_i\mathbf{a}_j \tag{21}$$

in the x^i coordinate system. In the $\bar{x}^i$ coordinate system the same tensor **T** is represented as:

$$\mathbf{T} = \bar{T}^{k\ell}\bar{\mathbf{a}}_k\bar{\mathbf{a}}_\ell \tag{22}$$

Thus:

$$T^{ij}\mathbf{a}_i\mathbf{a}_j = \bar{T}^{k\ell}\bar{\mathbf{a}}_k\bar{\mathbf{a}}_\ell \tag{23}$$

Using the transformation law of base vectors Equation 15 in Equation 23, we get:

$$\bar{T}^{k\ell} = \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^\ell}{\partial x^j}T^{ij} \tag{24}$$

Similarly:

$$T^{ij} = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^\ell}\bar{T}^{k\ell} \tag{25}$$

Equations 24 and 25 are the transformation laws for the contravariant components of a tensor.
For the covariant components we start from the equation:

$$T_{ij}\mathbf{a}^i\mathbf{a}^j = \bar{T}_{k\ell}\bar{\mathbf{a}}^k\bar{\mathbf{a}}^\ell \tag{26}$$

Taking the dot product of Equation 26 with $\bar{a}_p$ and using:

$$\bar{\mathbf{a}}_p = \mathbf{a}_m\frac{\partial x^m}{\partial \bar{x}^p}$$

we get:

$$T_{im}\frac{\partial x^m}{\partial \bar{x}^p}\mathbf{a}^i = \bar{T}_{kp}\mathbf{a}^k \tag{27}$$

Taking another dot product with $\bar{\mathbf{a}}_n$, we get:

$$\bar{T}_{np} = \frac{\partial x^s}{\partial \bar{x}^n}\frac{\partial x^m}{\partial \bar{x}^p}T_{sm} \tag{28}$$

Similarly:

$$T_{sm} = \frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^p}{\partial x^m}\bar{T}_{np} \tag{29}$$

Equations 28 and 29 are the transformation laws for the covariant components of a tensor.
Following the same procedure, the transformation laws for the mixed components of **T** are

$$\bar{T}^k_p = \frac{\partial x^m}{\partial \bar{x}^p}\frac{\partial \bar{x}^k}{\partial x^s} T^s_m \tag{30}$$

$$T^s_m = \frac{\partial \bar{x}^p}{\partial x^m}\frac{\partial x^s}{\partial \bar{x}^k} \bar{T}^k_p \tag{31}$$

The preceding procedure can be used to obtain the transformation formulae for third and higher order tensors.

Referring to Equations M1.71, we know that g_{ij} and g^{ij} are the covariant and contravariant components, respectively, of the metric tensor. Both components are symmetric, and g_{ij} define the length element ds as:

$$(ds)^2 = g_{ij}\, dx^i\, dx^j \tag{32}$$

According to Equations 24 and 28 we have the transformation laws for these components as:

$$\bar{g}^{np} = \frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^p}{\partial x^m} g^{sm} \tag{33}$$

$$\bar{g}_{np} = \frac{\partial x^s}{\partial \bar{x}^n}\frac{\partial x^m}{\partial \bar{x}^p} g_{sm} \tag{34}$$

In Equations M1.97 and M1.102 the covariant derivatives of the components of a vector have been defined. The covariant derivatives of the contravariant components are the mixed components of a second order tensor. Their transformation laws are the following;

$$\bar{A}^i_{,n} = \frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^i}{\partial x^k} A^k_{,j} \tag{35}$$

and:

$$\bar{A}_{i,\ell} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^\ell} A_{j,n} \tag{36}$$

where:

$$\bar{A}^i_{,n} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \bar{\Gamma}^i_{rn}\bar{A}^r \tag{37}$$

$$\bar{A}_{i,\ell} = \frac{\partial \bar{A}_i}{\partial \bar{x}^\ell} - \bar{\Gamma}^r_{i\ell}\bar{A}_r \tag{38}$$

From the transformation laws of the metric components given in Equations 33 and 34 we establish the result that scalars g and $\bar{g}$ are not absolute scalars. For:

$$g = \det(g_{ij}) \tag{39a}$$

and:

$$\bar{g} = \det(\bar{g}_{ij}) \tag{39b}$$

and on substituting Equation 34 in the determinant expansion of Equation 39b, we get

$$\bar{g} = (J)^2 g \tag{40}$$

Similarly:

$$g = (\bar{J})^2 \bar{g} \tag{41}$$

Both J and $\bar{J}$ have been defined in Equation 2.

Thus $\sqrt{g}$ is not an absolute scalar, and its value in some other coordinate system is $\bar{J}\sqrt{\bar{g}}$.

5. TRANSFORMATION LAWS FOR THE CHRISTOFFEL SYMBOLS

From Equations M1.87 and M1.89, the Christoffel symbols[2] of the first and second kinds, respectively, in the coordinate system $\bar{x}^i$ are

$$\overline{[ij,k]} = \frac{1}{2}\left(\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} + \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^i} - \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right) \tag{42}$$

$$\bar{\Gamma}^{\ell}_{ij} = \bar{g}^{k\ell}\,\overline{[ij,k]} \tag{43}$$

If we use Equation 34 in Equation 42 and perform the indicated differentiations, we get:

$$\overline{[\ell m, n]} = [ij,k]\,\frac{\partial x^i}{\partial \bar{x}^\ell}\,\frac{\partial x^j}{\partial \bar{x}^m}\,\frac{\partial x^k}{\partial \bar{x}^n} + g_{ij}\,\frac{\partial^2 x^i}{\partial \bar{x}^\ell\,\partial \bar{x}^m}\,\frac{\partial x^j}{\partial \bar{x}^n} \tag{44}$$

Inner multiplication by $\bar{g}^{np}$ (given in Equation 33) gives:

$$\bar{\Gamma}^p_{\ell m} = \Gamma^r_{ij}\,\frac{\partial \bar{x}^p}{\partial x^r}\,\frac{\partial x^i}{\partial \bar{x}^\ell}\,\frac{\partial x^j}{\partial \bar{x}^m} + \frac{\partial \bar{x}^p}{\partial x^j}\,\frac{\partial^2 x^j}{\partial \bar{x}^\ell\,\partial \bar{x}^m} \tag{45}$$

Equations 44 and 45 are the transformation laws for the Christoffel symbols. Because of the appearance of second derivatives of the coordinates on the right of Equations 44 and 45, the symbols do not transform like the components of a tensor. This shows that the Christoffel symbols are not the components of some tensor.

We can isolate the second derivative term from Equation 45 to have an expression for the second partial derivative of the old coordinates with respect to the new coordinates. On taking the inner multiplication of Equation 45 by $\partial x^r/\partial \bar{x}^p$, we get:

$$\frac{\partial^2 x^r}{\partial \bar{x}^\ell\,\partial \bar{x}^m} = \bar{\Gamma}^p_{\ell m}\,\frac{\partial x^r}{\partial \bar{x}^p} - \Gamma^r_{ij}\,\frac{\partial x^i}{\partial \bar{x}^\ell}\,\frac{\partial x^j}{\partial \bar{x}^m} \tag{46}$$

6. SOME FORMULAE IN CARTESIAN AND CURVILINEAR COORDINATES

Let one coordinate system be a rectangular Cartesian and the other be a general curvilinear system. We denote the Cartesian coordinates as x_i and the general curvilinear coordinates as x^i. We also adopt the convention that a repeated index on the Cartesian coordinates implies a summation. Using the results obtained in the previous section we have the following results (for details refer to Reference 2):

$$g_{ij} = \frac{\partial x_k}{\partial x^i}\,\frac{\partial x_k}{\partial x^j} \tag{47}$$

$$g^{ij} = \frac{\partial x^i}{\partial x_k}\frac{\partial x^j}{\partial x_k} \tag{48}$$

$$\frac{\partial^2 x_r}{\partial x^\ell \partial x^m} = \Gamma^p_{\ell m}\frac{\partial x_r}{\partial x^p} \tag{49}$$

$$\Gamma^r_{ij} = g^{rt}\frac{\partial x_s}{\partial x^t}\frac{\partial^2 x_s}{\partial x^i \partial x^j} \tag{50}$$

$$\frac{\partial^2 x^r}{\partial x_\ell \partial x_m} = -\Gamma^r_{ij}\frac{\partial x^i}{\partial x_\ell}\frac{\partial x^j}{\partial x_m} \tag{51}$$

$$= -\Gamma^r_{ij}g^{ip}g^{jq}\frac{\partial x_\ell}{\partial x^p}\frac{\partial x_m}{\partial x^q} \tag{52}$$

$$= -\Gamma^r_{ij}g^{jq}\frac{\partial x^i}{\partial x_\ell}\frac{\partial x_m}{\partial x^q} \tag{53}$$

Equation 53 is very suitable for obtaining the Laplacian of a curvilinear coordinate. Defining the Laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial x_m \partial x_m} \tag{54}$$

we obtain from Equation 53:

$$\nabla^2 x^r = -g^{ij}\Gamma^r_{ij} \tag{55}$$

Laplacian of an Absolute Scalar

Let ϕ be an absolute scalar. The covariant components of grad ϕ are then:

$$\phi_{,i} = \frac{\partial \phi}{\partial x^i}$$

Thus from Equation 38:

$$(\phi_{,i})_{,j} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\frac{\partial \phi}{\partial x^r} \tag{56}$$

Since $\Gamma^r_{ij} = \Gamma^r_{ji}$, then from Equation 56 we have the result:

$$(\phi_{,i})_{,j} = (\phi_{,j})_{,i} \tag{57}$$

That is, the covariant differentiation of absolute scalars is commutative.

Having obtained the covariant derivative in Equation 56 we use Equation M1.108 to obtain:

$$\nabla^2\phi = \operatorname{div}(\text{grad } \phi) = g^{ij}\left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\frac{\partial \phi}{\partial x^r}\right) \tag{58a}$$

In Equation M1.107 writing $\mathbf{v}$ = grad ϕ and using Equation M1.105b, the conservation law form of the Laplacian is

$$\sqrt{g}\nabla^2\phi = \frac{\partial}{\partial x^i}\left[\mathbf{a}^i \cdot \frac{\partial}{\partial x^j}(\sqrt{g}\mathbf{a}^j\phi)\right] \tag{58b}$$

If in Equation 58a we set $\phi = x^m$, a curvilinear coordinate, then we recover the result in Equation 55. If on the other hand we take $\phi = x_n$, a Cartesian coordinate, then we have:

$$g^{ij}\frac{\partial^2 x_n}{\partial x^i\,\partial x^j} + \frac{\partial x_n}{\partial x^r}\nabla^2 x^r = 0 \tag{59}$$

MATHEMATICAL EXPOSITION 5

Potential Theory

1. INTRODUCTION

Consider the Gauss theorem (Equation M2.1) in an m-dimensional Euclidean space. viz.. $i = 1, 2, 3, \ldots, m$. Writing:

$$F = PQ$$

Equation M2.1 yields:

$$\int_V P \frac{\partial Q}{\partial x_i} dv = -\int_V Q \frac{\partial P}{\partial x_i} dv + \int_S PQ\, n_i\, dS \tag{1}$$

Equation 1 provides the formula for integration by parts in the case of multiple integrals. If either P or Q is zero at S, then:

$$\int_V P \frac{\partial Q}{\partial x_i} dv = -\int_V Q \frac{\partial P}{\partial x_i} dv \tag{2}$$

2. FORMULAE OF GREEN

Using summation convention on repeated incides, consider a linear differential operator defined as:

$$L = -\frac{\partial}{\partial x_i}\left(A_{ik} \frac{\partial}{\partial x_k}\right) + B \tag{3}$$

where A_{ik} are symmetric functions of the coordinates $(x_1, x_2, \ldots, x_m)$, and B is also a function of the coordinates. Then for two continuously differentiable functions u and v, we have the *identity*:

$$\int_V vLu\, dv = -\int_V v \frac{\partial}{\partial x_i}\left(A_{ik} \frac{\partial u}{\partial x_k}\right) dv + \int_V Buv\, dv \tag{4}$$

Applying Equation 1 to the first term on the right-hand side of Equation 4, we get

$$\int_V vLu\, dv = \int_V A_{ik} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_k} dv - \int_S vNu\, dS + \int_V Buv\, dv \tag{5}$$

where:

$$N = A_{ik}\, n_i \frac{\partial}{\partial x_k} \tag{6}$$

is a differential operator. Equation 5 is called *Green's first formula*. Setting $v = u$ in Equation 5, we have:

$$\int_V uLu\,dv = \int_V A_{ik}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\,dv - \int_S uNu\,dS + \int_V Bu^2\,dv \tag{7}$$

Equation 7 is called *Green's second formula*. Interchanging u and v in Equation 5 and subtracting the result from Equation 5 while using the symmetry property of A_{ik}, we get:

$$\int_V (vLu - uLv)\,dv = \int_S (uNv - vNu)\,dS \tag{8}$$

Equation 8 is called *Green's third formula.*

Green's Formulae for Laplace Operator

If $A_{ik} = \delta_{ik}$ and $B = 0$, then $L = -\nabla^2$ where:

$$\nabla^2 = \frac{\partial^2}{\partial x_i\,\partial x_i}$$

is the Laplace operator. In this case Equation 6 yields:

$$N = n_i\frac{\partial}{\partial x_i} = \frac{\partial}{\partial n} \tag{9}$$

which is the normal derivative operator. Green's first, second, and third formulae are, respectively:

$$\int_V v\nabla^2 u\,dv = \int_S v\frac{\partial u}{\partial n}\,dS - \int_V (\text{grad } u)\cdot(\text{grad } v)\,dv \tag{10}$$

$$\int_V u\nabla^2 u\,dv = \int_S u\frac{\partial u}{\partial n}\,dS - \int_V |\text{grad } u|^2\,dv \tag{11}$$

$$\int_V (v\nabla^2 u - u\nabla^2 v)\,dv = \int_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS \tag{12}$$

As a corollary setting $v = 1$ in either Equation 10 or 12 we have:

$$\int_V \nabla^2 u\,dv = \int_S \frac{\partial u}{\partial n}\,dS \tag{13}$$

3. POTENTIAL THEORY

In an m-dimensional space we choose two points denoted as $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$. The distance r between the points is given by:

$$r = \left[\sum_{k=1}^{m}(x_k - \xi_k)^2\right]^{1/2} = |\mathbf{x} - \boldsymbol{\xi}|$$

It can readily be verified by performing differentiation with respect to $\mathbf{x}$ that:

$$\nabla^2\left(\frac{1}{r^{m-2}}\right) = 0 \qquad \text{for} \quad m > 2 \tag{14a}$$

and:

$$\nabla^2\left[\ell n\left(\frac{1}{r}\right)\right] = 0 \qquad \text{for} \quad m = 2 \tag{14b}$$

provided that $r \neq 0$. The functions $1/r^{m-2}$ and $\ell n(1/r)$ for $m > 2$ and $m = 2$, respectively, are called the fundamental solutions of the Laplace equation.

Integral Representation

Let $u(\mathbf{x})$ be continuously differentiable at least up to the second order in a volume V of surface S in an m-dimensional space. In Equation 12 let us set:

$$v = \frac{1}{r^{m-2}}, \qquad r \neq 0$$

and surround point $\mathbf{x}$ by a hypersphere of radius ϵ. The surface area S_ϵ of the hypersphere is

$$S_\epsilon = S_1 \epsilon^{m-1}$$

where:

$$S_1 = 2\pi^{m/2} \Big/ \Gamma\left(\frac{m}{2}\right)$$

where Γ is the Gamma function. Let ξ be a variable point in V with the elements of volume and surface surrounding ξ as $dv(\xi)$ and $dS(\xi)$, respectively. Then Equation 12 becomes:

$$\int_V \frac{1}{r^{m-2}} \nabla_\xi^2 u(\xi)\, dv(\xi) = \int_S \left[\frac{1}{r^{m-2}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right)\right] dS(\xi)$$
$$+ \int_{S_\epsilon} \left[\frac{1}{r^{m-2}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right)\right] dS(\xi)$$

Now:

$$\frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right)\bigg|_{S_\epsilon} = -\frac{\partial}{\partial r}\left(\frac{1}{r^{m-2}}\right)\bigg|_{S_\epsilon} = \frac{m-2}{\epsilon^{m-1}}$$

Using the mean-value theorem of integral calculus, we have:

$$\int_{S_\epsilon} u(\xi) \frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right) dS(\xi) = S_1(m-2)u(\xi^*)$$
$$\int_{S_\epsilon} \frac{1}{r^{m-2}} \frac{\partial u}{\partial n} dS(\xi) = \epsilon S_1 \frac{\partial u}{\partial n}\bigg|_{\xi^*}$$

where ξ^* is some point in S_ϵ. As $\epsilon \to 0$, $\xi^* \to \mathbf{x}$ and we get:

$$u(\mathbf{x}) = \frac{1}{S_1(m-2)} \int_S \left[\frac{1}{r^{m-2}} \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n}\left(\frac{1}{r^{m-2}}\right)\right] dS(\xi)$$
$$- \frac{1}{S_1(m-2)} \int_V \frac{1}{r^{m-2}} \nabla_\xi^2 u(\xi)\, dv(\xi) \tag{15}$$

Equation 15 is the integral representation of a function u, which is continuously differentiable at least up to the second order. It is valid for $m > 2$, and ∇_ξ^2 is the Laplacian in the ξ-coordinates. If $m = 2$, then following the previous procedure we have:

$$u(\mathbf{x}) = \frac{1}{2\pi} \oint_C \left[\ell n\left(\frac{1}{r}\right) \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n} \ell n\left(\frac{1}{r}\right) \right] dS - \frac{1}{2\pi} \int_R \ell n\left(\frac{1}{r}\right) \nabla_\xi^2 u(\xi)\, dR(\xi) \tag{16}$$

where R is a 2-D region bounded by the closed curve C on which the arc length element is dS.
For $m = 3$, Equation 15 is

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_S \left[\frac{1}{r} \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n} \left(\frac{1}{r}\right) \right] dS(\xi) - \frac{1}{4\pi} \int_V \frac{1}{r} \nabla_\xi^2 u(\xi)\, dv(\xi) \tag{17}$$

The Delta Function

For many physical and mathematical problems much progress toward their understanding is achieved by the introduction of a generalized function, also known as the Dirac delta function or impulse function. The delta function, denoted as $\delta(x)$, is not a function in an ordinary sense of a function since it is defined as follows:

$$\begin{aligned} &\text{(i)} && \delta(x) = 0 && \text{if } x \neq 0 \\ &\text{(ii)} && \delta(x) = \infty && \text{if } x = 0 \\ &\text{(iii)} && \int_{-\infty}^{\infty} \delta(x)\, dx = 1 && \end{aligned} \tag{18}$$

No function in the classical sense can satisfy these properties, since ∞ as a function value is not acceptable. Added to the list of properties in Equation 18 is

$$\text{(iv)} \qquad \int_{-\infty}^{\infty} \delta(x)\phi(x)\, dx = \phi(0) \tag{19}$$

where $\phi(x)$ is an arbitrary test function defined at $x = 0$. Because of the property in Equation 19 the delta function is a distribution or generalized function since it assigns a number $\phi(0)$ to an arbitrary function $\phi(x)$. An interesting symbolic operation emerges by combining properties iii and iv:

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\, dx = \phi(0) \int_{-\infty}^{\infty} \delta(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\phi(0)\, dx$$

Thus:

$$\delta(x)\phi(x) = \delta(x)\phi(0) \tag{20}$$

Some other properties are

$$\text{(v)} \qquad \int_{-\infty}^{\infty} \delta(x - \xi)\phi(\xi)\, d\xi = \phi(x) \tag{21}$$

$$\text{(vi)} \qquad \delta(\alpha x) = \frac{1}{|\alpha|} \delta(x) \tag{22}$$

$$\text{(vii)} \qquad \delta(-x) = \delta(x) \tag{23}$$

A sequence $\{f_n(x)\}$ of ordinary functions which is such that for every test function $\phi(x)$ we have:

$$\operatorname*{Lim}_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx$$

then:

$$\delta(x) = \operatorname*{Lim}_{n\to\infty} f_n(x)$$

Some examples of functions which tend to $\delta(x)$ for $n \to \infty$ are

$$\begin{aligned} f_n(x) &= \frac{n}{2} \qquad \text{for} \qquad -\frac{1}{n} < x < \frac{1}{n} \\ &= 0 \qquad \text{otherwise} \\ f_n(x) &= \frac{n}{\pi(1 + n^2x^2)} \\ f_n(x) &= \frac{\sin(2n + 1)\dfrac{\pi x}{L}}{L \sin \dfrac{\pi x}{L}} \end{aligned}$$

Similarly, when $\alpha \to \infty$, then:

$$\operatorname*{Lim}_{\alpha\to\infty} \frac{\sin \alpha x}{\pi x} = \delta(x)$$

Integral Representation of Delta Function

Consider the Fourier transformation pair:

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx \tag{24a}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx}\,dk \tag{24b}$$

Substituting Equation 24a in Equation 24b, we have:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi}\,d\xi \right] e^{ikx}\,dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{ik(x-\xi)}\,dk \right] d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f(\xi) \int_{-\infty}^{\infty} e^{ik(x-\xi)}\,dk \right] d\xi \end{aligned}$$

Writing:

$$\delta(x - \xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)}\, dk \qquad (25)$$

we get:

$$f(x) = \int_{-\infty}^{\infty} \delta(x - \xi) f(\xi)\, d\xi$$

which according to Equation 21 is true. Thus, an integral representation of the delta function is Equation 25, or:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$= \lim_{\alpha \to \infty} \frac{\sin \alpha x}{\pi x} \qquad (26)$$

Delta Function in Higher Dimensions

All the delta function representations in the foregoing were in one dimension. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then:

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\ldots\delta(x_n) \qquad (27)$$

and:

$$\int_{-\infty}^{\infty} \delta(\mathbf{x} - \xi)\phi(\xi)\, d\xi = \phi(\mathbf{x}) \qquad (28)$$

where $d\xi$ is the volume element and $\int_{-\infty}^{\infty}$ is the n-tuple integral. The integral representation is then:

$$\delta(\mathbf{x} - \xi) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{i\alpha(\mathbf{x}-\xi)}\, d\alpha \qquad (29)$$

Delta Function and Fundamental Solution of the Laplace Equation

In three dimensions, let:

$$r = |\mathbf{x} - \xi|$$

then it can immediately be verified that:

$$\nabla^2\left(\frac{1}{r}\right) = 0, \qquad r \neq 0 \qquad (30)$$

where ∇^2 operator can be taken either in $\mathbf{x}$ or ξ variables. The function $1/r$ is called the fundamental solution of the Laplace equation, which is singular at $r = 0$. In place of Equation 30, we can write:

$$\nabla^2\left(-\frac{1}{4\pi r}\right) = \delta(\mathbf{x} - \xi) \qquad (31)$$

The justification of Equation 31 is obvious from the properties of delta function along with the behavior of $1/r$ as $r \to 0$. An equation which is consistent with both Equation 29 at $n = 3$, and Equation 31 is

$$\frac{1}{4\pi r} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{i\alpha(x-\xi)}}{|\alpha|^2}\, d\alpha \tag{32}$$

It can also be verified that:

$$\nabla^4\left(\frac{-r}{8\pi}\right) = \delta(x - \xi) \tag{33a}$$

and:

$$-\frac{r}{8\pi} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{i\alpha(x-\xi)}}{|\alpha|^4}\, d\alpha \tag{33b}$$

The Dirichlet Problem for the Poisson Equation

The Dirichlet problem for the Poisson equation is posed as:

$$\begin{aligned} \nabla^2 u(x) &= -\rho(x), \qquad \text{in } V \\ u(x) &= f(x), \qquad \text{on } S \end{aligned} \tag{34a}$$

The Poisson equation becomes a Laplace equation when $\rho = 0$. Equation 17 is already at hand to provide the solution of the problem in Equation 34a but the term $\partial u/\partial n$ appearing under the surface integral needs to be eliminated. To achieve this elimination we introduce Green's function. Let $h(x, \xi)$ be a solution of the Laplace equation such that:

$$\begin{aligned} \nabla_\xi^2 h &= 0 \qquad \text{in } V \\ h &= \frac{1}{4\pi r} \qquad \text{on } S \end{aligned} \tag{34b}$$

We also replace v by h in Equation 12 and have:

$$\int_V h \nabla_\xi^2 u \, dv(\xi) = \int_S \left(h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n}\right) dS(\xi) \tag{34c}$$

Green's function $G(x; \xi)$ is now defined as:

$$G(x; \xi) = -\frac{1}{4\pi r} + h(x, \xi) \tag{34d}$$

so that according to Equation 34b:

$$G(x; \xi) = 0 \qquad \text{on } S \tag{34e}$$

To obtain the solution of the Dirichlet problem as posed in Equation 34a, we consider Equation 17 with $1/r$ taken from Equation 34d and use Equations 34c, e, thus having:

$$u(x) = \int_S f(\xi) \frac{\partial G}{\partial n}\, dS(\xi) - \int_V G(x; \xi)\rho(\xi)\, dv(\xi) \tag{35a}$$

For the Laplace equation $\rho = 0$, and thus the solution of the Laplace equation under the Dirichlet boundary conditions is

$$u(\mathbf{x}) = \int_S f(\xi) \frac{\partial G}{\partial n} dS(\xi) \tag{35b}$$

Particular Solution of Poisson's Equation
Let:

$$\nabla^2 u(\mathbf{x}) = -\rho(\mathbf{x}) \tag{36}$$

Then a *particular solution* of Equation 36 in three dimensions is

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{\rho(\xi)}{r} dv(\xi) \tag{37}$$

As a verification for the validity of Equation 37, we take the Laplacian with $\mathbf{x}$ on both sides of Equation 37 and use Equations 28 and 31 to obtain Equation 36.

Equation 37 can also be obtained from Equation 17 if domain V extends to infinity in all directions and $u \sim 1/r$ on boundary S at infinity. For proof refer to Reference 3.

4. GENERAL REPRESENTATION OF A VECTOR

Let $\mathbf{u}$ be an arbitrary vector function. We can represent it as:

$$\mathbf{u} = \mathbf{u}_S + \mathbf{u}_I + \mathbf{v}$$

where $\mathbf{u}_S$ is a solenoidal vector, $\mathbf{u}_I$ is irrotational or curl-free, and $\mathbf{v}$ is both solenoidal and curl-free. Thus:

$$\text{div } \mathbf{u}_S = 0, \qquad \text{curl } \mathbf{u}_I = 0$$

$$\text{div } \mathbf{v} = 0, \qquad \text{curl } \mathbf{v} = 0$$

Consequently:

$$\mathbf{u}_S = \text{curl } \mathbf{U}$$

$$\mathbf{u}_I = \text{grad } \phi$$

where $\mathbf{U}$ is an arbitrary vector, ϕ is an arbitrary scalar, and:

$$\mathbf{u} = \text{curl } \mathbf{U} + \text{grad } \phi + \mathbf{v} \tag{38}$$

Let:

$$\text{div } \mathbf{u} = \theta(\mathbf{x}), \qquad \text{curl } \mathbf{u} = \omega(\mathbf{x})$$

be given. Then:

$$\nabla^2 \phi = \theta(\mathbf{x}) \tag{39}$$

and:

$$\operatorname{curl}(\operatorname{curl} \mathbf{U}) = \boldsymbol{\omega}(\mathbf{x})$$

Since:

$$\operatorname{curl}(\operatorname{curl} \mathbf{U}) = \operatorname{grad}(\operatorname{div} \mathbf{U}) - \nabla^2 \mathbf{U}$$

and U is arbitrary, we can without any loss of generality take div $\mathbf{U} = 0$. Thus:

$$\nabla^2 \mathbf{U} = -\boldsymbol{\omega}(\mathbf{x}) \tag{40}$$

According to Equation 37, the particular solutions of Equations 39 and 40 are, respectively:

$$\phi(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{\theta(\boldsymbol{\xi})}{r}\, d\nu(\boldsymbol{\xi}) \tag{41}$$

$$\mathbf{U}(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{\boldsymbol{\omega}(\boldsymbol{\xi})}{r}\, d\nu(\boldsymbol{\xi}) \tag{42}$$

where:

$$r = |\mathbf{x} - \boldsymbol{\xi}|$$

Therefore:

$$\mathbf{u}_S = \operatorname{curl} \mathbf{U} = -\frac{1}{4\pi}\int_V \frac{(\mathbf{x} - \boldsymbol{\xi}) \times \boldsymbol{\omega}(\boldsymbol{\xi})}{r^3}\, d\nu(\boldsymbol{\xi}) \tag{43}$$

$$\mathbf{u}_I = \operatorname{grad} \phi = \frac{1}{4\pi}\int_V \frac{(\mathbf{x} - \boldsymbol{\xi})\,\theta(\boldsymbol{\xi})}{r^3}\, d\nu(\boldsymbol{\xi}) \tag{44}$$

and the complete representation Equation 38 is

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{(\mathbf{x} - \boldsymbol{\xi})\,\theta(\boldsymbol{\xi})}{r^3}\, d\nu(\boldsymbol{\xi}) - \frac{1}{4\pi}\int_V \frac{(\mathbf{x} - \boldsymbol{\xi}) \times \boldsymbol{\omega}(\boldsymbol{\xi})}{r^3}\, d\nu(\boldsymbol{\xi}) + \mathbf{v} \tag{45}$$

5. AN APPLICATION OF GREEN'S FIRST FORMULA

Suppose a linear equation

$$Lu = f \tag{46}$$

with L defined in Equation 3 is to be solved under the mixed boundary conditions

$$u\big|_{S_1} = \hat{u} \tag{47a}$$

$$Nu\big|_{S_2} = \hat{g} \tag{47b}$$

where $S = S_1 + S_2$. Using Equation 46 in Equation 5 with $v\big|_{S_1} = 0$, we get

$$Ib(\upsilon, u) = Il(\upsilon) \tag{48}$$

where Ib is the bilinear functional

$$Ib(\upsilon, u) = \int_V \left(A_{ik} \frac{\partial \upsilon}{\partial x_i} \frac{\partial u}{\partial x_k} + Bu\upsilon \right) d\mathrm{v} \tag{49a}$$

and Il is the linear functional

$$Il(\upsilon) = \int_V \upsilon f d\mathrm{v} + \int_{S_1} \upsilon \hat{g} dS \tag{49b}$$

The generalized Ritz and also the Kantorovich methods of approximate solutions of Equations 46 and 47 (Refer to Reference 16) can be stated in a generalized form using Equation 48. Suppose the nth approximation of u is u_n then, using Equation 5 with $\upsilon|_{S_1} = 0$ and 48, we get

$$\begin{aligned} \int_V \upsilon(Lu_n - f) d\mathrm{v} &= Ib(\upsilon, u_n) - Il(\upsilon) \\ &= 0 \end{aligned}$$

Thus, the fundamental formula for the approximate solution can either be taken as

$$Ib(\upsilon, u_n) = Il(\upsilon) \tag{50a}$$

or

$$\int_V \upsilon(Lu_n - f) d\mathrm{v} = 0 \tag{50b}$$

Suppose

$$u_n = \phi_0 + c_j \phi_j \qquad (sum\ on\ j)$$

and

$$\upsilon = \phi_i$$

where the coordinate function ϕ_0 satisfies 47a, $\phi_i|_{S_1} = 0$, and c_j are constants. Then the generalized Ritz method using Equation 50a is

$$c_j\ Ib(\phi_i, \phi_j) = Il(\phi_i) - Ib(\phi_i, \phi_0), \quad i = 1, 2, \ldots, n$$

Equation 50b is used in the development of Kantorovich method.

MATHEMATICAL EXPOSITION 6

Singularities of the First Order ODEs

1. INTRODUCTION

First order ordinary differential equations (ODEs) sometime appear quite naturally in the formulation of many problems in mechanics. One case in point is the formulation of the 3-D boundary layer separation problem as discussed in Chapter 5, Section 5.14.

Significant qualitative information about the solutions of first-order ODE's can be obtained from a study of their singularities. Below we briefly discuss the types of singularities and the conditions under which they appear. For details refer to Kaplan.[4]

2. SINGULARITIES AND THEIR CLASSIFICATION

We consider the first order ODE:

$$\frac{dz}{dx} = \frac{R(x, z)}{P(x, z)} \tag{1}$$

A point (x_0, z_0) where both P and R are simultaneously zero is called a singular point of Equation 1. Without any loss of generality we can affect a coordinate transformation to have the singular point at the origin (0, 0). In the neighborhood of (0, 0) the analytic functions P and R can be expressed as:

$$P(x, z) = a_1x + c_1z + \dots, \qquad R(x, z) = a_3x + c_3z + \dots$$

Thus, very close to (0, 0), we retain only the first order terms and have the homogeneous form:

$$\frac{dz}{dx} = \frac{a_3x + c_3z}{a_1x + c_1z} \tag{2}$$

where we disregard the trivial case $a_3c_1 - a_1c_3 = 0$.

Introducing a parameter σ, we can write Equation 1 as:

$$\frac{dx}{d\sigma} = P \cong a_1x + c_1z$$

$$\frac{dz}{d\sigma} = R \cong a_3x + c_3z$$

which in vector notation can be written as:

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{A} \cdot \mathbf{x} \tag{3}$$

where:

$$\mathbf{x} = \begin{pmatrix} x \\ z \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_1 & c_1 \\ a_3 & c_3 \end{pmatrix}$$

A trial solution:

$$\mathbf{x} = e^{\lambda\sigma}\mathbf{u}$$

with $\mathbf{u}$ being independent of σ, immediately yields:

$$(\mathbf{A} - \lambda\mathbf{I}) \cdot \mathbf{u} = 0$$

which can be satisfied when:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

The eigenvalues λ_1, λ_2 are

$$\lambda_1 = \tfrac{1}{2}(\nabla + \sqrt{\Delta}), \qquad \lambda_2 = \tfrac{1}{2}(\nabla - \sqrt{\Delta})$$

where:

$$\left.\begin{aligned} \nabla &= a_1 + c_3 \\ J &= a_1c_3 - a_3c_1 \\ \Delta &= \nabla^2 - 4J \end{aligned}\right\} \tag{4}$$

Obviously:

$$\nabla = \lambda_1 + \lambda_2, \qquad J = \lambda_1\lambda_2, \qquad \Delta = (\lambda_1 - \lambda_2)^2 \tag{5}$$

The nature of the singularity depends on the signs of J and Δ as follows:

$\Delta > 0, J < 0$	Saddle point
$\Delta > 0, J > 0$	Nodal point (type 1)
$\Delta = 0$	Nodal point (type 2)
$\Delta < 0, \nabla = 0$	Center
$\Delta < 0, \nabla \neq 0$	Focus

Note that $\Delta < 0$ if the eigenvalues are complex conjugates.

Based on the preceding results we now write the general expressions for ∇, J, and Δ which are to be evaluated at the singular point of Equation 1:

$$\nabla = \frac{\partial P}{\partial x} + \frac{\partial R}{\partial z}$$

$$J = \frac{\partial P}{\partial x}\frac{\partial R}{\partial z} - \frac{\partial P}{\partial z}\frac{\partial R}{\partial x}$$

$$\Delta = \left(\frac{\partial P}{\partial x} + \frac{\partial R}{\partial z}\right)^2 - 4\left(\frac{\partial P}{\partial x}\frac{\partial R}{\partial z} - \frac{\partial P}{\partial z}\frac{\partial R}{\partial x}\right)$$

MATHEMATICAL EXPOSITION 7

Geometry of Surfaces

1. BASIC DEFINITIONS

In ME.3 we have already stated some formulae on the various geometric properties of space curves. These formulae are fundamental to an understanding of the differential-geometric properties of a surface embedded in Euclidean $-R^3$

For a concise representation of the formulae, we shall use the tensor index notation, with the summation convention in effect when one upper and one lower index in a term or in a product are the same, e.g., T^i_i, $a^\alpha b_\alpha$, etc.

We consider a surface embedded in R^3 and choose a general coordinate system x^i ($i = 1, 2, 3$) in such a manner that one coordinate x^ν = constant ($\nu = 1$, or 2, or 3) defines the surface while the remaining two are the coordinates in the surface. All indices associated with the surface will be denoted by Greek letters (except the letter ν). Denoting the partial derivatives by a comma preceding an index, the unit outward drawn normal to the surface x^ν = constant is

$$\mathbf{n}^{(\nu)} = \frac{\mathbf{r}_{,\alpha} \times \mathbf{r}_{,\beta}}{|\mathbf{r}_{,\alpha} \times \mathbf{r}_{,\beta}|} \tag{1}$$

where:

$$\mathbf{r}_{,\alpha} = \frac{\partial \mathbf{r}}{\partial x^\alpha}, \qquad \mathbf{r}_{,\alpha\beta} = \frac{\partial^2 \mathbf{r}}{\partial x^\alpha \, \partial x^\beta} \quad \text{etc.}$$

The values of ν and other Greek indices follow the following scheme:

$\nu = 1$: Greek indices α, β, etc. assume values 2 and 3

$\nu = 2$: Greek indices α, β, etc. assume values 3 and 1

$\nu = 3$: Greek indices α, β, etc. assume values 1 and 2

2. FORMULAE OF GAUSS

The formulae of Gauss express the second derivatives of the position vector at any point of the surface in terms of surface Christoffel symbols and coefficients of the second fundamental form. In tensor notation they are

$$\mathbf{r}_{,\alpha\beta} = Y^\delta_{\alpha\beta}\mathbf{r}_{,\delta} + \mathbf{n}^{(\nu)} b_{\alpha\beta} \tag{2}$$

Here the Greek indices range over two values as stated in the scheme above; $Y^\delta_{\alpha\beta}$ which depend on $g_{\alpha\beta}$ are the surface Christoffel symbols (for space Christoffel symbols refer to ME.1, Section 13), and $b_{\alpha\beta}$ are the coefficients of the second fundamental form. The covariant components $g_{\alpha\beta}$ are the coefficients of the first fundamental form (refer to ME.1), and in the case of surface theory:

$$g_{\alpha\beta} = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta} = \frac{\partial x}{\partial x^\alpha}\frac{\partial x}{\partial x^\beta} + \frac{\partial y}{\partial x^\alpha}\frac{\partial y}{\partial x^\beta} + \frac{\partial z}{\partial x^\alpha}\frac{\partial z}{\partial x^\beta}$$

The contravariant components $g^{\gamma\delta}$ are related with $g_{\alpha\beta}$ according to the equation:

$$g_{\alpha\beta}g^{\alpha\gamma} = \delta^\gamma_\beta$$

It is obvious from Equation 2 that:

$$b_{\alpha\beta} = \mathbf{n}^{(\nu)} \cdot \mathbf{r}_{,\alpha\beta} \tag{3}$$

The surface Christoffel symbols $Y^\delta_{\alpha\beta}$ for the three possible cases, i.e., $x^1 = \xi =$ constant, $x^2 = \eta =$ constant, $x^3 = \zeta =$ constant have been collected below in an expanded form.

Christoffel Symbols Based on Surface Coefficients

(i) Surface ζ = **constant.**

$$Y^1_{11} = \frac{1}{2G_3}\left[g_{22}\frac{\partial g_{11}}{\partial \xi} + g_{12}\left(\frac{\partial g_{11}}{\partial \eta} - 2\frac{\partial g_{12}}{\partial \xi}\right)\right]$$

$$Y^2_{22} = \frac{1}{2G_3}\left[g_{11}\frac{\partial g_{22}}{\partial \eta} + g_{12}\left(\frac{\partial g_{22}}{\partial \xi} - 2\frac{\partial g_{12}}{\partial \eta}\right)\right]$$

$$Y^1_{22} = \frac{1}{2G_3}\left[g_{22}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right) - g_{12}\frac{\partial g_{22}}{\partial \eta}\right]$$

$$Y^2_{11} = \frac{1}{2G_3}\left[g_{11}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right) - g_{12}\frac{\partial g_{11}}{\partial \xi}\right]$$

$$Y^1_{12} = Y^1_{21} = \frac{1}{2G_3}\left(g_{22}\frac{\partial g_{11}}{\partial \eta} - g_{12}\frac{\partial g_{22}}{\partial \xi}\right)$$

$$Y^2_{12} = Y^2_{21} = \frac{1}{2G_3}\left(g_{11}\frac{\partial g_{22}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \eta}\right) \tag{4}$$

(ii) Surface η = **constant.**

$$Y^1_{11} = \frac{1}{2G_2}\left[g_{33}\frac{\partial g_{11}}{\partial \xi} + g_{13}\left(\frac{\partial g_{11}}{\partial \zeta} - 2\frac{\partial g_{13}}{\partial \xi}\right)\right]$$

$$Y^3_{33} = \frac{1}{2G_2}\left[g_{11}\frac{\partial g_{33}}{\partial \zeta} + g_{13}\left(\frac{\partial g_{33}}{\partial \xi} - 2\frac{\partial g_{13}}{\partial \zeta}\right)\right]$$

$$Y^1_{33} = \frac{1}{2G_2}\left[g_{33}\left(2\frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}\right) - g_{13}\frac{\partial g_{33}}{\partial \zeta}\right]$$

$$Y^3_{11} = \frac{1}{2G_2}\left[g_{11}\left(2\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta}\right) - g_{13}\frac{\partial g_{11}}{\partial \xi}\right]$$

$$\Upsilon^1_{13} = \Upsilon^1_{31} = \frac{1}{2G_2}\left(g_{33}\frac{\partial g_{11}}{\partial \zeta} - g_{13}\frac{\partial g_{33}}{\partial \xi}\right)$$

$$\Upsilon^3_{13} = \Upsilon^3_{31} = \frac{1}{2G_2}\left(g_{11}\frac{\partial g_{33}}{\partial \xi} - g_{13}\frac{\partial g_{11}}{\partial \zeta}\right) \tag{5}$$

***(iii) Surface* ξ = constant.**

$$\Upsilon^2_{22} = \frac{1}{2G_1}\left[g_{33}\frac{\partial g_{22}}{\partial \eta} + g_{23}\left(\frac{\partial g_{22}}{\partial \zeta} - 2\frac{\partial g_{23}}{\partial \eta}\right)\right]$$

$$\Upsilon^3_{33} = \frac{1}{2G_1}\left[g_{22}\frac{\partial g_{33}}{\partial \zeta} + g_{23}\left(\frac{\partial g_{33}}{\partial \eta} - 2\frac{\partial g_{23}}{\partial \zeta}\right)\right]$$

$$\Upsilon^2_{33} = \frac{1}{2G_1}\left[g_{33}\left(2\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}\right) - g_{23}\frac{\partial g_{33}}{\partial \zeta}\right]$$

$$\Upsilon^3_{22} = \frac{1}{2G_1}\left[g_{22}\left(2\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta}\right) - g_{23}\frac{\partial g_{22}}{\partial \eta}\right]$$

$$\Upsilon^2_{23} = \Upsilon^2_{32} = \frac{1}{2G_1}\left(g_{33}\frac{\partial g_{22}}{\partial \zeta} - g_{23}\frac{\partial g_{33}}{\partial \eta}\right)$$

$$\Upsilon^3_{23} = \Upsilon^3_{32} = \frac{1}{2G_1}\left(g_{22}\frac{\partial g_{33}}{\partial \eta} - g_{23}\frac{\partial g_{22}}{\partial \zeta}\right) \tag{6}$$

In the preceding formulae, the coefficients G_1, G_2, and G_3 are those which have been defined in Equation M1.120.

For a surface in a 3D space, there are 6 distinct Christoffel symbols. These Christoffel symbols can also be written by using Equation M1.122b by *setting* certain coefficients equal to 1 and 0 according to the following scheme:

For a surface ξ = *const.*, $r = r\,(\eta,\zeta)$.

Setting $g_{11} = 1$, $g_{12} = 0$, $g_{13} = 0$, and using Equation M1.120, we have

$G_1 = g_{22}\, g_{33} - (g_{23})^2$, $G_2 = g_{33}$, $G_3 = g_{22}$,
$G_4 = 0$, $G_5 = 0$, $G_6 = -g_{23}$.
For a surface η = *const.*, $r = r\,(\xi,\zeta)$.

Setting $g_{22} = 1$, $g_{12} = 0$, $g_{23} = 0$, and using Equation M1.120, we have

$G_1 = g_{33}$, $G_2 = g_{11}g_{33} - (g_{13})^2$, $G_3 = g_{11}$,
$G_4 = 0$, $G_5 = -g_{13}$, $G_6 = 0$.
For a surface ζ = *const.*, $r = r\,(\xi,\eta)$.

Setting $g_{33} = 1$, $g_{13} = 0$, $g_{23} = 0$, and using Equation M1.120, we have

$$G_1 = g_{22},\ G_2 = g_{11}\ ,\ G_3 = g_{11}g_{22} - (g_{12})^2\ ,$$
$$G_4 = -g_{12},\ G_5 = 0\ ,\ G_6 = 0\ .$$

Using the above g's and G's in Equation M1.122b, we get the desired Christoffel symbols for a particular surface as have been collected in Equations 4–6 and shown by the symbol Υ. It must, however, be mentioned that the coordinate which is held fixed on a surface, e.g., $\zeta =$ *const.*, shows up only as a constant in the description of a surface. Consequently, in the context of intrinsic surface geometry, the need of setting $g_{33} = 1$, $g_{13} = g_{23} = 0$ does not arise. In the 3D context, g_{33} may not be 1 and g_{13}, g_{23} may not be zero.

3. FORMULAE OF WEINGARTEN

For any surface $\mathbf{n}^{(\nu)}$ is the unit surface normal vector. Thus:

$$\mathbf{n}^{(\nu)} \cdot \mathbf{n}^{(\nu)} = 1$$

and:

$$\mathbf{n}^{(\nu)} \cdot \mathbf{n}^{(\nu)}_{,\alpha} = 0$$

The second equation implies that $\mathbf{n}^{(\nu)}_{,\alpha}$ lies in the plane containing $\mathbf{r}_{,\alpha}$. Thus:

$$\mathbf{n}^{(\nu)}_{,\alpha} = -b_{\alpha\beta} g^{\beta\gamma} \mathbf{r}_{,\gamma} \tag{7}$$

called the formulae of Weingarten.

4. EQUATIONS OF GAUSS

The formulae of Gauss define the coordinates $\mathbf{r}$ of a surface as functions of the surface curvilinear coordinates. The following conditions, known as the compatibility conditions, are satisfied by the formulae of Gauss:

$$(\mathbf{r}_{,\alpha\beta})_{,\gamma} = (\mathbf{r}_{,\alpha\gamma})_{,\beta}$$

Substituting Equation 2 and using the condition that $\mathbf{r}_{,\delta}$ and $\mathbf{n}^{(\nu)}$ are independent, we have after some algebra:

$$\frac{\partial \Upsilon^{\delta}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Upsilon^{\delta}_{\alpha\gamma}}{\partial x^{\beta}} + \Upsilon^{\epsilon}_{\alpha\beta}\Upsilon^{\delta}_{\epsilon\gamma} - \Upsilon^{\epsilon}_{\alpha\gamma}\Upsilon^{\delta}_{\epsilon\beta} = b_{\alpha\beta} b_{\gamma\mu} g^{\mu\delta} - b_{\beta\theta} g^{\theta\delta} b_{\alpha\gamma} \tag{8}$$

Equation 8 (two in number) are the equations of Gauss. From the same analysis the coefficient of $\mathbf{n}^{(\nu)}$ when set equal to zero yields:

$$\frac{\partial b_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial b_{\alpha\gamma}}{\partial x^{\beta}} = \Upsilon^{\epsilon}_{\alpha\gamma} b_{\epsilon\beta} - \Upsilon^{\epsilon}_{\alpha\beta} b_{\epsilon\gamma} \tag{9}$$

Equation 9 is called the Codazzi equations.

5. NORMAL AND GEODESIC CURVATURES

Let C be a curve in a surface. When viewed from a three-dimensional frame of reference, curve C is a space curve. Let $\mathbf{t}$ be the unit tangent vector to C at a point P and $\mathbf{n}^{(\nu)}$ the unit normal to the surface. A plane containing $\mathbf{t}$ and $\mathbf{n}^{(\nu)}$ at a point P of the surface cuts the surface in different curves when rotated about $\mathbf{n}$ as an axis. Each curve is known as a normal section of the surface at point P. Since these curves belong both to the surface and also to the embedding space, a study of curvature properties of these as space curves also reveals the curvature properties of the surfaces in which they lie.

We decompose the curvature vector $\hat{\mathbf{k}}$ at P of C, defined in Equation M3.5 into a vector $\mathbf{k}_n^{(\nu)}$ normal to the surface and a vector $\mathbf{k}_g^{(\nu)}$ tangential to the surface. Thus:

$$\hat{\mathbf{k}} = \mathbf{k}_n^{(\nu)} + \mathbf{k}_g^{(\nu)} \tag{10}$$

The vector $\mathbf{k}_n^{(\nu)}$ is called the normal curvature vector at point P. It is directed either toward or against the direction of $\mathbf{n}^{(\nu)}$, so that:

$$\mathbf{k}_n^{(\nu)} = \mathbf{n}^{(\nu)} k_n^{(\nu)} \tag{11}$$

where $k_n^{(\nu)}$ is called the normal curvature of the normal section of the surface, and is an algebraic quantity.

To find the expression for $k_n^{(\nu)}$, we consider the equation:

$$\mathbf{n} \cdot \mathbf{t} = 0$$

and differentiate it with respect to s, the arc distance along C:

$$\frac{d\mathbf{n}^{(\nu)}}{ds} \cdot \mathbf{t} + \mathbf{n}^{(\nu)} \cdot (\mathbf{n}^{(\nu)} k_n^{(\nu)} + \mathbf{k}_g^{(\nu)}) = 0$$

or:

$$k_n^{(\nu)} = \frac{-d\mathbf{n}^{(\nu)} \cdot d\mathbf{r}}{(ds)^2} = \frac{b_{\alpha\beta}\, dx^\alpha\, dx^\beta}{g_{\alpha\beta}\, dx^\alpha\, dx^\beta} \tag{12}$$

(The above formula comes by differentiating $\mathbf{r}_{,\alpha} \cdot \mathbf{n}^{(\nu)} = \mathbf{r}_{,\beta} \cdot \mathbf{n}^{(\nu)}$ and using Equation 3. As has been shown in various books (e.g., References 2 and 5), the two extermal values of $k_n^{(\nu)}$, denoted as $k_I^{(\nu)}$ and $k_{II}^{(\nu)}$, are called the *principal curvatures of the surface* at point P. It has been shown that:

$$k_I^{(\nu)} + k_{II}^{(\nu)} = g^{\alpha\beta} b_{\alpha\beta} \tag{13}$$

and:

$$k_I^{(\nu)} k_{II}^{(\nu)} = K^{(\nu)} = \frac{\det(b_{\alpha\beta})}{\det(g_{\alpha\beta})} \tag{14}$$

$K^{(\nu)}$ is called the *Gaussian* or total curvature of the surface x^ν = constant at point P.

Corresponding to the two extreme values of $k_n^{(\nu)}$, denoted as $k_I^{(\nu)}$ and $k_{II}^{(\nu)}$, there are two directions called the *principal directions*. Suspending the summation convention on repeated indices, the two principal directions on the surface x^ν = constant are given by:

$$(g_{\alpha\beta} b_{\beta\beta} - g_{\beta\beta} b_{\alpha\beta})(dx^\beta)^2 + (g_{\alpha\alpha} b_{\beta\beta} - g_{\beta\beta} b_{\alpha\alpha})\, dx^\alpha\, dx^\beta + (g_{\alpha\alpha} b_{\alpha\beta} - g_{\alpha\beta} b_{\alpha\alpha})(dx^\alpha)^2 = 0 \tag{15}$$

The curves satisfying Equation 15 are called the *lines of curvature*. If coordinate curves x^α and x^β are themselves the lines of curvature, then by setting $dx^\alpha = 0$ and dx^β arbitrary and $dx^\beta = 0$ and dx^α arbitrary, we find that $g_{\alpha\beta} = 0$, $b_{\alpha\beta} = 0$ where $\alpha \neq \beta$; and thus:

$$k_I^{(\nu)} + k_{II}^{(\nu)} = \frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} + \frac{b_{\beta\beta}}{g_{\beta\beta}}, \qquad k_I^{(\nu)} k_{II}^{(\nu)} = \frac{b_{\alpha\alpha} b_{\beta\beta}}{g_{\alpha\alpha} g_{\beta\beta}} \tag{16}$$

$$k_I^{(\nu)} = \frac{b_{\alpha\alpha}}{g_{\alpha\alpha}}, \qquad k_{II}^{(\nu)} = \frac{b_{\beta\beta}}{g_{\beta\beta}} \tag{17}$$

We now consider the tangential part of $\hat{\mathbf{k}}$, which according to Equation 10 is $\mathbf{k}_g$, and is called the *geodesic curvature vector* of curve C. The magnitude of geodesic curvature denoted as k_g is obtained through the equation:

$$\mathbf{k}_g^{(\nu)} = k_g^{(\nu)} \mathbf{e}$$

where $\mathbf{e}$ is the unit vector tangent to the surface such that both $(\mathbf{t}, \mathbf{e}, \mathbf{n}^{(\nu)})$ and $(\mathbf{e}, \hat{\mathbf{k}}, \mathbf{t})$ form a right-handed system of triads. Thus:

$$k_g = \mathbf{t} \cdot \left(\frac{d\mathbf{t}}{ds} \times \mathbf{n}^{(\nu)} \right)$$

where s is the arc distance along C. Carrying out the algebra, we obtain (refer to Reference 5):

$$k_g^{(\nu)} = \sqrt{G_\nu} \left[A \frac{dx^\alpha}{ds} - B \frac{dx^\beta}{ds} \right] \tag{18a}$$

where:

$$A = \frac{d^2 x^\beta}{ds^2} + Y_{\beta\beta}^{\beta} \left(\frac{dx^\beta}{ds} \right)^2 + 2 Y_{\alpha\beta}^{\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} + Y_{\alpha\alpha}^{\beta} \left(\frac{dx^\alpha}{ds} \right)^2 \tag{18b}$$

$$B = \frac{d^2 x^\alpha}{ds^2} + Y_{\alpha\alpha}^{\alpha} \left(\frac{dx^\alpha}{ds} \right)^2 + 2 Y_{\alpha\beta}^{\alpha} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} + Y_{\beta\beta}^{\alpha} \left(\frac{dx^\beta}{ds} \right)^2 \tag{18c}$$

and in Equations 18b, c there is *no* summation on repeated indices and ν, α, β are cyclic. If C itself is a coordinate curve, either x^α or x^β; then first of all:

$$(ds)_{x^\alpha = \text{constant}} = \sqrt{g_{\beta\beta}}\, dx^\beta$$

$$(ds)_{x^\beta = \text{constant}} = \sqrt{g_{\alpha\alpha}}\, dx^\alpha$$

and consequently:

$$(k_g^{(\nu)})_{x^\alpha = \text{constant}} = -\sqrt{G_\nu}\, Y_{\beta\beta}^{\alpha} / g_{\beta\beta}^{3/2} \tag{19}$$

$$(k_g^{(\nu)})_{x^\beta = \text{constant}} = \sqrt{G_\nu}\, Y_{\alpha\alpha}^{\beta} / g_{\alpha\alpha}^{3/2} \tag{20}$$

are called the geodesic curvatures of coordinate curves x^α = constant and x^β = constant, respectively. As an application, suppose the surface is designated as $\nu = 2$; then $\alpha = 3$ and $\beta = 1$. Writing $x^1 = \xi$, $x^3 = \zeta$, we have:

$$(k_g^{(2)})_{\zeta=\text{constant}} = -\sqrt{G_2}\, Y_{11}^3/g_{11}^{3/2} \tag{21}$$

$$(k_g^{(2)})_{\xi=\text{constant}} = \sqrt{G_2}\, Y_{33}^1/g_{33}^{3/2} \tag{22}$$

If coordinates ζ and ξ are orthogonal, then $g_{13} = 0$; and from the expression in Equation 5, we have:

$$(k_g^{(2)})_{\zeta=\text{constant}} = \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial \zeta} \tag{23}$$

$$(k_g^{(2)})_{\xi=\text{constant}} = -\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial \xi} \tag{24}$$

where $h_1^2 = g_{11}$, $h_3^2 = g_{33}$.

As a matter of definition, if C as an arbitrary curve defined by $x^\alpha = x^\alpha(s)$ and $x^\beta = x^\beta(s)$ in the surface $x^\nu =$ constant, then C is a geodesic curve if $k_g^{(\nu)} = 0$. To obtain the solution $x^\alpha(s)$ and $x^\beta(s)$ one has to solve the two ordinary differential equations $A = 0$ and $B = 0$ under the given initial conditions $x^\alpha(s_0)$, $x^\beta(s_0)$, $(dx^\alpha/ds)_{s=s_0}$, and $(dx^\beta/ds)_{s=s_0}$. If in particular $x^\alpha =$ constant is a geodesic, then from Equation 19 $Y^\alpha_{\beta\beta} = 0$; and similarly if $x^\beta =$ constant is a geodesic, then from Equation 20 $Y^\beta_{\alpha\alpha} = 0$. The same conclusion applies to Equations 23 and 24.

Longitudinal and Transverse Curvatures

In the case of flow past axially symmetric bodies two types of curvature effects are quite substantial. One is due to the longitudinal curving of the body contour which is in a meridian plane, and the other is transverse to it. Both of these effects were considered separately in Chapter 5, Section 5.6 and 5.15, respectively. To formalize both of these effects together in high Reynolds' number viscous flows past axially symmetric bodies, we first consider orthogonal curvilinear coordinates ξ_1, ξ_2, ξ_3 in which the elemental length ds is

$$ds^2 = h_1^2\, d\xi_1^2 + h_2^2\, d\xi_2^2 + h_3^2\, d\xi_3^2$$

We now write as in Chapter 5, Section 5.15:

$$h_1 = h_{1w}(\xi_1)h, \qquad h_2 = 1, \qquad h_3 = r$$

$$h_{1w}\, d\xi_1 = dx, \qquad \xi_2 = y, \qquad \xi_3 = \phi$$

so that:

$$ds^2 = h^2\, dx^2 + dy^2 + r^2\, d\phi^2$$

where x is the curvilinear distance along the body measured from the stagnation point, y is the actual distance normal to the surface, and ϕ is the azimuthal angle. (Refer to Figure M7.1.)

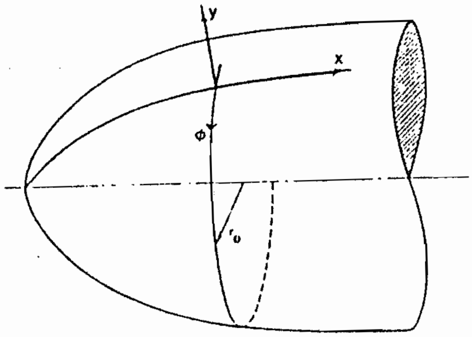


Fig. M7.1 Coordinates on an axially symmetric body.

For thin shear layers it suffices to take:

$$h = 1 + yk(x)$$

and:

$$r = r_0(x) + y \cos \beta(x)$$

where $1/k(x)$ and r_0 are the longitudinal and the transverse radii of curvature of the body, respectively, and β is the angle between the x-coordinate and the axis of revolution.

6. GRID GENERATION IN SURFACES

The preceding simple geometric concepts can be used to formulate a set of equations which can be used for grid generation in arbitrary surfaces. For details on this aspect refer to References 6 and 6a.

The Beltramian of a scalar ϕ is given by:

$$\Delta_2^{(\nu)}\phi = g^{\alpha\beta}(\phi_{.\alpha\beta} - Y^{\delta}_{\alpha\beta}\phi_{.\delta}) \tag{25}$$

If ϕ is taken as one of the surface coordinate curves, say $\phi = x^{\delta}$, then from Equation 25:

$$\Delta_2^{(\nu)}x^{\delta} = -g^{\alpha\beta}Y^{\delta}_{\alpha\beta} \tag{26}$$

The Beltramian in surface theory plays the same role as the Laplacian does in Euclidean space. In this connection it is worthwile to note the following result. Inner multiplication of Equation 2 by $G_\nu g^{\alpha\beta}$ yields:

$$G_\nu g^{\alpha\beta}\mathbf{r}_{.\alpha\beta} = G_\nu g^{\alpha\beta}(Y^{\delta}_{\alpha\beta}\mathbf{r}_{.\delta} + \mathbf{n}^{(\nu)}b_{\alpha\beta}) \tag{27}$$

where G_ν pertains to the surface x^ν = constant and can be either G_1, G_2, or G_3 (refer to Equation M1.120). Equation 27 can be written as:

$$D\mathbf{r} + G_\nu(\Delta_2^{(\nu)}x^{\delta})\mathbf{r}_{.\delta} = \mathbf{n}^{(\nu)}R \tag{28}$$

$$D = G_\nu g^{\alpha\beta}\partial_{\alpha\beta}, \qquad \mathbf{n}^{(\nu)} = \text{unit surface normal}$$

$$G_\nu = g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2, \qquad \nu,\ \alpha,\ \beta \text{ cyclic}$$

$$R = G_\nu g^{\alpha\beta}b_{\alpha\beta} = (k_I^{(\nu)} + k_{II}^{(\nu)})G_\nu$$

Refer to References 6 for the use of Equation 28.

MATHEMATICAL EXPOSTION 8

Finite Difference Approximation Applied to PDEs

1. INTRODUCTION

The subject of computational fluid dynamics (CFD) requires a thorough understanding of a variety of computational techniques. A number of books on computational methods (e.g., References 7, 8, etc.) and on CFD itself (e.g., References 9–11) should be consulted.

2. CALCULUS OF FINITE DIFFERENCE

The foundation of the calculus of finite difference is Taylor's expansion of an analytic function. Let $f(x)$ be an analytic function of x, and h be a small but finite increment in x. To develop the finite difference approximations for the various derivatives, $f'(x)$, $f''(x)$, etc. we need the following Taylor expansions:

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \ldots$$

$$f(x - h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \ldots$$

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4h^3}{3}f'''(x) + \ldots$$

$$f(x - 2h) = f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4h^3}{3}f'''(x) + \ldots$$

In place of x, we can write $x_j = x_0 + jh$, where j is an integer which assumes the values 0, 1, 2, ... For brevity the values of the function and its various derivatives will be denoted by:

$$f_j = f(x_j), \qquad f'_j = f'(x_j), \qquad f''_j = f''(x_j) \quad \text{etc.}$$

and then the various Taylor expansions as noted above are written as:

$$f_{j+1} = f_j + hf'_j + \frac{h^2}{2}f''_j + \frac{h^3}{6}f'''_j + \ldots \tag{1a}$$

$$f_{j-1} = f_j - hf'_j + \frac{h^2}{2}f''_j - \frac{h^3}{6}f'''_j + \ldots \tag{1b}$$

$$f_{j+2} = f_j + 2hf'_j + 2h^2f''_j + \frac{4h^3}{3}f'''_j + \ldots \tag{1c}$$

$$f_{j-2} = f_j - 2hf'_j + 2h^2f''_j - \frac{4h^3}{3}f'''_j + \ldots \tag{1d}$$

The differences:

$$\Delta f_j = f_{j+1} - f_j$$
$$\nabla f_j = f_j - f_{j-1}$$

define two fundamental difference operators and are called the forward and backward differences, respectively. Using the operator property, we have:

$$\Delta^2 f_j = \Delta(\Delta f_j) = f_{j+2} - 2f_{j+1} + f_j$$
$$\nabla^2 f_j = \nabla(\nabla f_j) = f_j - 2f_{j-1} + f_{j-2}$$

Similarly:

$$\Delta^n f_j = \Delta^{n-1}(\Delta f_j), \qquad \nabla^n f_j = \nabla^{n-1}(\nabla f_j)$$

From Equations 1a, b the first derivative approximations correct to order one, i.e., $0(h)$, are

$$f'_j = (f_{j+1} - f_j)/h - \frac{h}{2}f''_j - \frac{h^2}{6}f'''_j - \ldots$$
$$= \Delta f_j/h + 0(h) \tag{2a}$$

$$f'_j = (f_j - f_{j-1})/h + \frac{h}{2}f''_j - \frac{h^2}{6}f'''_j + \ldots$$
$$= \nabla f_j/h + 0(h) \tag{2b}$$

To find the second derivative approximation in terms of the forward difference operator and which is correct to order one, multiply Equation 1a by 2 and subtract it from Equation 1c, thus having:

$$f''_j = (f_{j+2} - 2f_{j+1} + f_j)/h^2 - hf'''_j - \ldots$$
$$= \Delta^2 f_j/h^2 + 0(h) \tag{3a}$$

Similarly:

$$f''_j = (f_j - 2f_{j-1} + f_{j-2})/h^2 - hf'''_j + \ldots$$
$$= \nabla^2 f_j/h^2 + 0(h) \tag{3b}$$

To obtain the third order derivative correct to order one consider the Taylor expansion of f_{j+3} and substitute the expressions for f'_j and f''_j from the first equations in Equations 2a and 3a, respectively, thus having:

$$f'''_j = (f_{j+3} - 3f_{j+2} + 3f_{j+1} - f_j)/h^3 + 0(h) \tag{4a}$$

Similarly:

$$f'''_j = (f_j - 3f_{j-1} + 3f_{j-2} - f_{j-3})/h^3 + 0(h) \tag{4b}$$

The central difference operator is defined as:

$$\delta f_j = f_{j+1/2} - f_{j-1/2} \tag{5a}$$

$$\delta f_{j+1/2} = f_{j+1} - f_j \tag{5b}$$

where:

$$f_{j+1/2} = (f_{j+1} + f_j)/2$$

$$f_{j-1/2} = (f_j + f_{j-1})/2$$

Thus the second order correct expressions for the derivatives are

$$\begin{aligned} f'_j &= (f_{j+1} - f_{j-1})/2h - \frac{h^2}{6} f'''_j \\ &= \delta f_j/h + 0(h^2) \end{aligned} \tag{6}$$

$$\begin{aligned} f''_j &= (f_{j+1} - 2f_j + f_{j-1})/h^2 - \frac{h^2}{12} f''''_j \\ &= \delta^2 f_j/h^2 + 0(h^2) \end{aligned} \tag{7}$$

Some high order accurate expressions for the first derivative which are one-sided (noncentral) can be obtained from the preceding formulae. They are

$$f'_j = (-f_{j+2} + 4f_{j+1} - 3f_j)/2h + 0(h^2) \tag{8}$$

$$f'_j = (3f_j - 4f_{j-1} + f_{j-2})/2h + 0(h^2) \tag{9}$$

$$f'_j = (2f_{j+3} - 9f_{j+2} + 18f_{j+1} - 11f_j)/6h + 0(h^3) \tag{10}$$

$$f'_j = (11f_j - 18f_{j-1} + 9f_{j-2} - 2f_{j-3})/6h + 0(h^3) \tag{11}$$

Methods of Interpolation

Let the function values $f_0 = f(x_0)$ and $f_1 = f(x_1)$ be known. A linear function $g(x)$ which approximates $f(x)$ in the interval $x_0 \leqq x \leqq x_1$ is

$$g(x) = \frac{x_1 - x}{x_1 - x_0} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \tag{12}$$

which is the simplest interpolation formula for $f(x)$.

In general, if the function values:

$$f_0 = f(x_0), \quad f_1 = f(x_1), \quad \ldots, \quad f_n = f(x_n) \tag{13a}$$

are known, then an n-th degree polynomial:

$$g(x) = \sum_{k=0}^{n} a_k x^k \tag{13b}$$

can be used to satisfy the $(n + 1)$ values given in Equation 13a by $g(x)$ to determine the coefficients a_k in Equation 13b. This process yields:

$$g(x) = \sum_{j=0}^{n} P_j(x) f_j \tag{14a}$$

where $f_j = f(x_j)$, and:

$$P_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} (x - x_i) \Big/ \prod_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i) \tag{14b}$$

Equation 14a is called the Lagrange interpolation formula in the range $x_0 \leqq x \leqq x_n$; and $P_j(x)$ defined in Equation 14b are the corresponding *Lagrange polynomials*, each of order n. Note that symbol II stands for the product of the terms.

In computational problems it is preferable to use a piecewise polynomial of low order repeatedly in each interval $x_j \leqq x \leqq x_{j+1}$. Thus, from Equation 12:

$$g_j(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j} f_j + \frac{x - x_j}{x_{j+1} - x_j} f_{j+1} \tag{15}$$

in the interval $x_j \leqq x \leqq x_{j+1}$. If besides the function values the derivatives of the first order are also provided in each interval, then a piecewise cubic can be fitted in each interval.

Cubic Spline Functions

The mathematical analogue of a French curve or a flexible elastic string, which passes in a piecewise manner through a set of given points, is a *cubic spline function*. Such a desired smooth curve can be obtained by using cubic polynomials in all the intervals with the added properties that at the junction of two intervals the first and second derivatives of the function match.

Let the function values $f_j = f(x_j)$ specified at the $(n + 1)$ points $x_0, x_1, \ldots, x_n$ in the range $x_0 \leqq x \leqq x_n$. We consider a set of n cubics $\{g_j(x)\}$, where $g_j(x)$ is defined in the range $x_j \leqq x \leqq x_{j+1}$. Let $g(x)$ be the approximating function for the entire range. As discussed above, we now match the first and the second derivatives of the cubics at the points x_j, $j = 1, 2, \ldots, n - 1$ so that:

$$g_j'(x_j) = g_{j-1}'(x_j) = g'(x_j) \tag{16a}$$

$$g_j''(x_j) = g_{j-1}''(x_j) = g''(x_j) \tag{16b}$$

Since the second derivative of a cubic is linear, the second derivatives can be obtained by a linear interpolation in each interval. Thus according to Equation 15, we have:

$$g_j''(x) = \frac{x_{j+1} - x}{\Delta x_j} g''(x_j) + \frac{x - x_j}{\Delta x_j} g''(x_{j+1}) \tag{17}$$

where $\Delta x_j = x_{j+1} - x_j$. Integrating Equation 17 two times and evaluating the two constants of integration by the conditions:

$$g_j(x_j) = f(x_j) = f_j$$

$$g_j(x_{j-1}) = f(x_{j-1}) = f_{j-1}$$

we obtain the desired cubic on the range $x_j \leqq x < x_{j+1}$ as:

$$g_j(x) = \frac{g''(x_j)}{6}\left[\frac{(x_{j+1} - x)^3}{(\Delta x_j)^2} - (x_{j+1} - x)\right]\Delta x_j$$

$$+ \frac{g''(x_{j+1})}{6}\left[\frac{(x - x_j)^3}{(\Delta x_j)^2} - (x - x_j)\right]\Delta x_j + \frac{f_j}{\Delta x_j}(x_{j+1} - x) + \frac{f_{j+1}}{\Delta x_j}(x - x_j) \tag{18}$$

We now impose the conditions of smoothness or matching of the derivatives stated Equations in 16a, b. The condition in Equation 16a yields the equation:

$$g''(x_{j+1}) + \frac{2(x_{j+1} - x_{j-1})}{\Delta x_j} g''(x_j) + \frac{\Delta x_{j-1}}{\Delta x_j} g''(x_{j-1})$$

$$= \frac{6}{\Delta x_j}\left[\frac{f_{j+1} - f_j}{\Delta x_j} - \frac{f_j - f_{j-1}}{\Delta x_{j-1}}\right] \tag{19}$$

while the condition Equation 16b is already in effect starting from Equation 17.

Now the $n + 1$ function values:

$$f_0, f_1, f_2, \ldots, f_n$$

have been given in the interval (x_0, x_n). If in addition, either $f''(x_0)$ and $f''(x_n)$ are known or we impose any one of the following conditions:

(a) $\quad f''_0 = 0, \quad f''_n = 0$

(b) $\quad f''_0 = f''_1, \quad f''_n = f''_{n-1}$

(c) $\quad f''_0 = \lambda f''_1, \quad f''_n = \mu f''_{n-1}$

where λ and μ are adjustable parameters; then for:

$$j = 1, 2, 3, \ldots, (n - 1)$$

Equations 19 form a system of $(n - 1)$ simultaneous equations for the determination of all the second derivatives. The set of Equation 19 yield a tridiagonal matrix which can be solved quite easily on a computer.

3. ITERATIVE ROOT-FINDING

To find the roots of the equation:

$$f(x) = 0$$

a method known as Newton's method or Newton-Raphson's method is quite suitable. The fundamental basis of the method is again Taylor's expansion. Let the initial guess for the root be $x = x_0$. Then the truncated Taylor series for a neighboring point $x = x_1$ is

$$f(x_1) \cong f(x_0) + (x_1 - x_0)f'(x)$$

Setting $f(x_1) = 0$, we get:

$$x_1 - x_0 = \delta_1 = \frac{-f(x_0)}{f'(x_0)} \tag{20a}$$

Repeated application of Equation 20a yields the iterative procedure:

$$x_{n+1} - x_n = \delta_{n+1} = \frac{-f(x_n)}{f'(x_n)} \tag{20b}$$

which is Newton's method. It has been found by numerical experimentation that in some cases for a success of this method a graphical evaluation of the first guess is necessary, particularly, when the function is periodic or has closely spaced multiple values. In some cases the derivative $f'(x)$ is either difficult to obtain or has to be obtained numerically. In such cases we write:

$$\begin{aligned} f'(x_n) &= [f(x_n) - f(x_{n-1})]/(x_n - x_{n-1}) \\ &= [f(x_n) - f(x_{n-1})]/\delta_n \end{aligned}$$

Thus Equation 20b becomes:

$$x_{n+1} - x_n = \delta_{n+1} = \frac{-f(x_n)\delta_n}{f(x_n) - f(x_{n-1})} \tag{21}$$

which is also known as the *secant method*. For starting the iterative procedure in Equation 21, one needs two initial guesses.

For multiple roots a modified method due to Ralston[12] has been found to be quite effective. Let:

$$g(x) = \frac{f(x)}{f'(x)}$$

Then the roots of $f(x)$ are the zeros of $g(x)$. Applying Newton's method to $g(x)$ we have:

$$x_{n+1} - x_n = \delta_{n+1} = \frac{-g(x_n)}{g'(x_n)}$$

where:

$$g'(x) = 1 - \frac{f(x)f''(x)}{[f'(x)]^2}$$

In the numerical "Shooting Method" of ordinary differential equations, the Newton's method as given in Equation 21 plays an important role. Suppose a boundary-value-problem is stated as

$$u'' = f(x,u,u')$$

with $u(a) = r$ and $u'(b) = s$, where $a \le x \le b$.

It is simpler to write the given differential equation as a system of first order differential equations as

$$\begin{aligned} u' &= g;\ u(a) = r \\ g' &= f(x,u,g);\ g(a) = \alpha \end{aligned}$$

The correct value of α is obtained by numerically solving the equation

$$g(b;\ \alpha) - s = 0$$

by using Equation 21.

4. NUMERICAL INTEGRATION

The simplest formula for the numerical evaluation of integrals is obtained by integrating Equation 12 between limits x_0 and x_1. Thus:

$$\int_{x_0}^{x_1} f(x)\ ds = \frac{h}{2}(f_0 + f_1) \tag{22}$$

where:

$$h = x_1 - x_0$$

The formula in Equation 22 is called the *trapezoidal rule*. If h is sufficiently small, the trapezoidal rule can be used in a piecewise manner to obtain the value of an integral on a computer. Let the limits of integration be x_0 and x_n. Then the repeated use of Equation 22 yields:

$$\int_{x_0}^{x_n} f(x)\ ds = \frac{h}{2}[f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n] \tag{23}$$

A more accurate formula can be obtained by integrating a second degree polynomial between limits x_0 and x_2, where function values at x_0, x_1, x_2 are f_0, f_1, f_2, respectively, and:

$$x_1 - x_0 = x_2 - x_1 = h$$

Thus:

$$\int_{x_0}^{x_2} f(x)\ dx = \frac{h}{3}(f_0 + 4f_1 + f_2), \qquad x_2 = x_0 + 2h \tag{24}$$

and is called *Simpson's rule*. In the same manner if the complete range from a to b is divided into equal intervals h through the equation:

$$h = \frac{b - a}{n}$$

where n is an even integer, then a repeated application of Simpson's rule yields:

$$\int_a^b f(x)\ dx = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 4f_{n-1} + f_n) \tag{25}$$

where:

$$b = a + nh, \qquad (n \text{ even})$$

For three intervals:

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$$

the integration of a third degree polynomial between limits x_0 and x_3 yields:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \tag{26}$$

5. FINITE DIFFERENCE APPROXIMATIONS OF PARTIAL DERIVATIVES

For future reference we quote the finite difference approximations of the partial derivatives. Denoting by integers j and k the positions along the coordinates x and y, respectively, we write the finite difference approximations of:

$$\frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x\,\partial y}$$

First Derivatives

(a) Forward difference; correct to first order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (f_{j+1,k} - f_{j,k})/\Delta x$$

(b) Backward difference; correct to first order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (f_{j,k} - f_{j-1,k})/\Delta x$$

(c) Central difference; correct to second order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (f_{j+1,k} - f_{j-1,k})/2\Delta x$$

(d) One-sided forward difference; correct to second order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (-f_{j+2,k} + 4f_{j+1,k} - 3f_{j,k})/2\Delta x$$

(e) One-sided backward difference; correct to second order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (f_{j-2,k} - 4f_{j-1,k} + 3f_{j,k})/2\Delta x$$

(f) One-sided backward difference; correct to fourth order:

$$\left(\frac{\partial f}{\partial x}\right)_{j,k} = (-25f_{j,k} + 48f_{j+1,k} - 36f_{j+2,k} + 16f_{j+3,k} - 3f_{j+4,k})/12\Delta x$$

Second Derivatives

(g) Central difference; correct to second order:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{j,k} = (f_{j+1,k} - 2f_{j,k} + f_{j-1,k})/(\Delta x)^2$$

(h) Forward difference; correct to first order:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{j,k} = (f_{j+2,k} - 2f_{j+1,k} + f_{j,k})/(\Delta x)^2$$

(i) Backward difference; correct to first order:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{j,k} = (f_{j,k} - 2f_{j-1,k} + f_{j-2,k})/(\Delta x)^2$$

(j) Central differences in both x and y; correct to second order:

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{j,k} = (f_{j+1,k+1} - f_{j+1,k-1} - f_{j-1,k+1} + f_{j-1,k-1})/(4\Delta x\, \Delta y)$$

(k) Forward difference in x and central in y; correct to second order:

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{j,k} = (3f_{j,k-1} - 4f_{j+1,k-1} + f_{j+2,k-1} - 3f_{j,k+1} + 4f_{j+1,k+1} - f_{j+2,k+1})/(4\Delta x\, \Delta y)$$

(l) Central difference in x and forward in y; correct to second order:

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{j,k} = (3f_{j-1,k} - 4f_{j-1,k+1} + f_{j-1,k+2} - 3f_{j+1,k} + 4f_{j+1,k+1} - f_{j+1,k+2})/(4\Delta x\, \Delta y)$$

6. FINITE DIFFERENCE APPROXIMATION OF PARABOLIC PDEs

Let us consider a model quasilinear parabolic equation:

$$\frac{\partial \phi}{\partial t} = f \tag{27}$$

where f is a function of ϕ and its partial derivatives of the first and second orders. Without denoting the dependence of quantities on the spatial variables, we can discritize Equation 27 in the following four forms:

1. Forward differencing (Euler explicit):

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = f^n \tag{28a}$$

2. Backward differencing (Euler implicit):

$$\frac{\phi^n - \phi^{n-1}}{\Delta t} = f^n \tag{28b}$$

3. Second order Adams-Bashforth:

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}(3f^n - f^{n-1}) \tag{28c}$$

4. Modified central differencing (trapezoidal):

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{1}{2}(f^n + f^{n+1}) \tag{28d}$$

For the purpose of understanding the problem of stability we first consider a simple one-dimensional linear parabolic equation and follow the analysis of Fromm.[12a] The equation to be considered is

$$\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x} + \nu \frac{\partial^2 \phi}{\partial x^2} \tag{29}$$

where u is a constant. Using the forward differencing (Equation 28a and central differences for the spatial derivatives, Equation 29 is discritized as follows:

$$\phi_k^{n+1} - \phi_k^n = -\frac{1}{2}c(\phi_{k+1}^n - \phi_{k-1}^n) + d(\phi_{k+1}^n - 2\phi_k^n + \phi_{k-1}^n) \tag{30}$$

where:

$$c = \frac{u\,\Delta t}{\Delta x}, \qquad d = \frac{\nu\,\Delta t}{(\Delta x)^2} \tag{31}$$

Let $\overline{\phi}_k^n$ be the *exact* solution of Equation 30 and δ_k^n be the error due to round-off, etc. Then we write:

$$\phi_k^n = \overline{\phi}_k^n + \delta_k^n$$

which on substitution in Equation 30 yields the same equation, i.e., Equation 30 with ϕ replaced by δ. Following Von Neumann (refer to Reference 10) we consider a Fourier expansion for δ_k^n as:

$$\delta_k^n = \sum_{m=-\infty}^{\infty} \alpha_m A_{(m)}^{(n)} e^{imk\Delta x} \tag{32}$$

where α_m are the Fourier coefficients, $A^{(n)}$ as functions of m are the amplitudes, and i is the imaginary unit:

$$i = \sqrt{-1}$$

Substituting Equation 32 in the equation for δ, we obtain:

$$\frac{A^{n+1}}{A_n} = 1 - ic \sin\theta + 2d(\cos\theta - 1) \tag{33}$$

where:

$$\theta = m\,\Delta x$$

Thus no harmonic will be amplified if for any m:

$$\left|\frac{A^{n+1}}{A^n}\right| \leq 1$$

which should be the condition of stability. Thus from Equation 33 the stability condition is given by:

$$G(\cos\theta) = [1 - 2d(1 - \cos\theta)]^2 + c^2(1 - \cos^2\theta) \leqq 1 \tag{34}$$

Since the inequality in Equation 34 must remain true for all values of θ, the choice $\theta = \pi$ shows that:

$$-1 \leqq 1 - 4d \leqq 1$$

or:

$$1 - 4d \geqq -1$$

thus:

$$d \leqq 1/2 \tag{35a}$$

and then the choice $\theta = \pi/2$ shows that:

$$c \leqq 1 \tag{35b}$$

To investigate the inequality in Equation 34 further, we find on differentiating $G(\cos\theta)$ with respect to $\cos\theta$ that the extremum of G is attained at:

$$\cos\theta = \frac{2d(1 - 2d)}{c^2 - 4d^2}$$

and using this value in the expression for G gives the absurd condition:

$$(c^2 - 2d)^2 \leqq 0$$

Further, the maximum of $G(\cos\theta)$ is attained when:

$$G''(\cos\theta) = 8d^2 - 2c^2 < 0$$

and the minimum is attained when:

$$G''(\cos\theta) = 8d^2 - 2c^2 > 0$$

Obviously, the condition of maximum:

$$c > 2d$$

can never be satisfied. We, therefore, take the condition:

$$c \leqq 2d$$

or:

$$\frac{u\,\Delta x}{\nu} \leqq 2 \tag{36}$$

which is a necessary condition for Equation 34 to remain satisfied.

We, therefore, reach the conclusion that the forward time differencing (explicit scheme) can remain stable if:

$$d \leqq \tfrac{1}{2}, \qquad c \leqq 1, \qquad \frac{u\,\Delta x}{\nu} \leqq 2$$

The quantities d, c, and $u\,\Delta x/\nu$ (Equation 31) are also known as the *diffusion parameter*, *Courant number*, and *mesh Reynolds' number*, respectively.

Following the same approach it can be shown that for a two-dimensional parabolic equation the stability conditions are given by the inequalities:

$$\frac{\nu\,\Delta t}{(\Delta x)^2} + \frac{\nu\,\Delta t}{(\Delta y)^2} \leqq \frac{1}{2} \tag{37a}$$

and:

$$\frac{u\,\Delta t}{\Delta x} + \frac{v\,\Delta t}{\Delta y} \leqq 1 \tag{37b}$$

Stable Schemes for Parabolic Equations

Notice from Equation 34 that the preceding explicit scheme is completely unstable for the purely advection equation, viz., Equation 29 with $\nu = 0$. Performing the stability analysis with the central time differencing Equation 28c in Equation 29 shows that the scheme is *unconditionally unstable*. However, it can be shown that if the backward time differencing (Equation 28b) or if the modified central differencing (Equation 28d) is used, then the scheme is *unconditionally stable*. It has also been confirmed by numerical experimentations that the *upwind differencing* of advection terms enhances the stability of difference approximations. The upwind differencing of $u\;\partial\phi/\partial x$ is defined as follows:

$$u\frac{\partial\phi}{\partial x} = \frac{u\phi_k^n - u\phi_{k-1}^n}{\Delta x}, \qquad \text{if} \quad u > 0$$

$$= \frac{u\phi_{k+1}^n - u\phi_k^n}{\Delta x}, \qquad \text{if} \quad u < 0$$

Besides the backward time differencing, the following three unconditionally stable schemes are quite efficient and accurate.

We consider the two-dimensional model equation:

$$\frac{\partial\phi}{\partial t} = f$$

where:

$$f = -u\frac{\partial\phi}{\partial x} - v\frac{\partial\phi}{\partial y} + \nu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

Extension of the following schemes to three dimesions is straightforward.

(i) Crank-Nicolson's Method[12b]. This method is that of modified central differencing (Equation 28d). If the spatial derivatives are approximated by central differences, then the scheme is accurate to the second order in Δt, Δx, and Δy.

(ii) Alternating Direction Implicit (ADI) Method[12c]. The ADI scheme is a two-step scheme which is stated as follows:

Step 1

$$\frac{\phi^{n+1/2} - \phi^n}{\Delta t/2} = -u\left(\frac{\partial \phi}{\partial x}\right)^{n+1/2} - v\left(\frac{\partial \phi}{\partial y}\right)^n + \nu\left(\frac{\partial^2 \phi}{\partial x^2}\right)^{n+1/2} + \nu\left(\frac{\partial^2 \phi}{\partial y^2}\right)^n \tag{39a}$$

Step 2

$$\frac{\phi^{n+1} - \phi^{n+1/2}}{\Delta t/2} = -u\left(\frac{\partial \phi}{\partial x}\right)^{n+1/2} - v\left(\frac{\partial \phi}{\partial y}\right)^{n+1} + \nu\left(\frac{\partial^2 \phi}{\partial x^2}\right)^{n+1/2} + \nu\left(\frac{\partial^2 \phi}{\partial y^2}\right)^{n+1} \tag{39b}$$

All the spatial derivatives are to be approximated by the central difference expressions.

(iii) Leapfrog DuFort-Frankel Method[12d]. The simple leapfrog method applied to one-dimensional model Equation 29 is

$$\frac{\phi_k^{n+1} - \phi_k^{n-1}}{2\Delta t} = -u\,\frac{\phi_{k+1}^n - \phi_{k-1}^n}{2\Delta x} + \nu\,\frac{\phi_{k+1}^{n-1} - 2\phi_n^{n-1} + \phi_{k-1}^{n-1}}{(\Delta x)^2} \tag{40}$$

which is stable only when the step sizes Δt and Δx are such that the least of c and d is always satisfied, where:

$$c \leqq 1 \qquad \text{and} \qquad d \leqq \tfrac{1}{2}$$

The leapfrog method coupled with the DuFort-Frankel method for linear heat equation takes the form:

$$\frac{\phi_k^{n+1} - \phi_k^{n-1}}{2\Delta t} = -u\,\frac{\phi_{k+1}^n - \phi_{k-1}^n}{2\Delta x} + \nu\,\frac{\phi_{k+1}^n - \phi_k^{n+1} - \phi_k^{n-1} - \phi_{k-1}^n}{(\Delta x)^2}$$

which is unconditionally stable.

7. FINITE DIFFERENCE APPROXIMATION OF ELLIPTIC EQUATIONS

As a model of an elliptic equation which suffices our present purpose, we take Poisson's equation:

$$\nabla^2 \psi = -f \tag{42}$$

General boundary conditions for Equation 42 are represented by a mixed equation of the form:

$$\frac{\partial \psi}{\partial n} + p(s)\psi = q(s) \tag{43}$$

where s represents the boundary surface coordinates and $\partial/\partial n$ denotes differentiation in the direction of the external normal to the boundary. If $p = 0$, then:

$$\frac{\partial \psi}{\partial n} = q(s) \tag{44a}$$

is called the *Neumann boundary condition.* If $p \rightarrow \infty$ and:

$$\lim_{p \to \infty} \frac{q}{p} = \lambda(s)$$

then:

$$\psi = \lambda(s) \tag{44b}$$

is called the *Dirichlet boundary condition.*

For simplicity we consider a two-dimensional rectangular Cartesian plane-xy in which we first consider a mesh system formed by squares as shown in Figure M8.1 that follows.

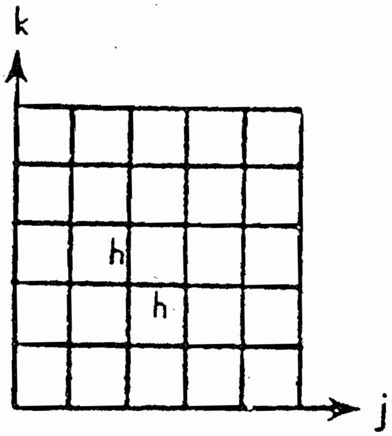


Fig. M8.1 Uniform mesh system in the xy-plane.

Let us consider the grid point (j, k) and the four adjacent grid points $(j - 1, k)$, $(j + 1, k)$, $(j, k - 1)$, and $(j, k + 1)$. Using the central difference approximations for the second derivatives in Equation 42, we get:

$$\psi_{j-1,k} + \psi_{j+1,k} + \psi_{j,k-1} + \psi_{j,k+1} - 4\psi_{j,k} = -h^2 f_{j,k} \tag{45}$$

The difference expression (Equation 45) is called the five-point difference equation for the Laplace/Poisson equation.

For a nonuniform grid system, we can simply proceed from Equation 42 by first writing it as:

$$\operatorname{div}(\operatorname{grad} \psi) = -f$$

Integrating over a region Ω with surface S and using Gauss' divergence theorem, we get:

$$\int_S \frac{\partial \psi}{\partial n} \, dS = -\int_\Omega f \, dv \tag{46}$$

For simplicity we again consider a two-dimensional region with the mesh system of unequal steps as shown in Figure M8.2.

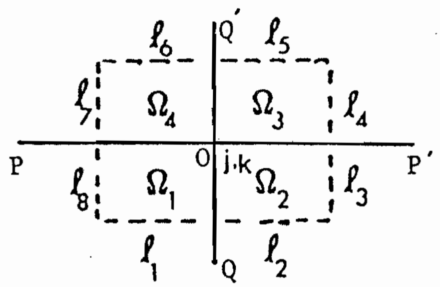


Fig. M8.2 Non-uniform mesh system in the xy-plane.

Let:

$$OP = h_1, \quad OP' = h_1', \quad OQ = h_2, \quad OQ' = h_2'$$

We now erect four rectangular boxes around point (j, k) by drawing lines passing through the midpoints between (j, k) and the four neighboring points as shown in Figure M8.2 by the dashed lines. The measures of the regions are then Ω_1, Ω_2, Ω_3, and Ω_4 where:

$$\Omega_1 = \frac{h_1 h_2}{4}, \quad \Omega_2 = \frac{h_1' h_2}{4}, \quad \Omega_3 = \frac{h_1' h_2'}{4}, \quad \Omega_4 = \frac{h_1 h_2'}{4}$$

The normal derivatives on the faces are as follows:

$$\text{On } \ell_1 \ell_2 : \frac{\partial \psi}{\partial n} = \frac{\psi_{j,k-1} - \psi_{j,k}}{h_2}$$

$$\text{on } \ell_3 \ell_4 : \frac{\partial \psi}{\partial n} = \frac{\psi_{j+1,k} - \psi_{j,k}}{h_1'}$$

$$\text{on } \ell_5 \ell_6 : \frac{\partial \psi}{\partial n} = \frac{\psi_{j,k+1} - \psi_{j,k}}{h_2'}$$

$$\text{on } \ell_7 \ell_8 : \frac{\partial \psi}{\partial n} = \frac{\psi_{j-1,k} - \psi_{j,k}}{h_1}$$

The integrals in Equation 46 are now approximated by the integral meanvalue theorem. Thus:

$$\int_S \frac{\partial \psi}{\partial n} dS \cong \frac{\psi_{j,k-1} - \psi_{j,k}}{h_2} \left(\frac{h_1}{2} + \frac{h_1'}{2} \right) + \ldots$$

and:

$$\int_\Omega f\, dv = f_1 \Omega_1 + f_2 \Omega_2 + f_3 \Omega_3 + f_4 \Omega_4 = F$$

Substituting these expressions in Equation 46, we obtain:

$$A_{jk} \psi_{j-1,k} + B_{jk} \psi_{j+1,k} + C_{jk} \psi_{j,k-1} + D_{jk} \psi_{j,k+1} + E_{jk} \psi_{j,k} = -F_{jk} \tag{47}$$

where:

$$A_{jk} = \frac{h_2 + h_2'}{2h_1}$$

$$B_{jk} = \frac{h_2 + h_2'}{2h_1'}$$

$$C_{jk} = \frac{h_2 + h_1'}{2h_2}$$

$$D_{jk} = \frac{h_1 + h_1'}{2h_2}$$

$$E_{jk} = -(A_{jk} + B_{jk} + C_{jk} + D_{jk})$$

$$F_{jk} = f_1\Omega_1 + f_2\Omega_2 + f_3\Omega_3 + f_4\Omega_4 \tag{48}$$

Note that for equal mesh lengths, i.e., $h_1 = h_1' = h_2 = h_2' = h$, Equation 47 reduces to Equation 45.

Equation 47 can also be considered as being the result of applying difference approximation to any elliptic PDE in two dimensions. The elliptic equations can be either linear, quasilinear, or nonlinear in the xy-plane or in any other 2-D space in which only curvilinear coordinates can be introduced. In the case of quasi- or nonlinear equations the coefficients are considered to be from the previous iteration. Thus, in the following development we have considered Equation 47 in the spirit noted above in which coefficients A, B, etc. are not necessarily those given in Equation 48.

When Equation 47 is written for all grid points, i.e., for all values of j and k, the system of equations can be written in the matrix notation:

$$\mathbf{A} \cdot \boldsymbol{\psi} = -\mathbf{F} \tag{49}$$

and the whole problem reduces to the solution of these equations. One alternative for solution, which is quite suitable for digital computers, is to use iterative methods of solution.[13] Fundamental to the iterative methods is Jacobi's method. Below we discuss various iterative methods.

(i) Point- and Line-Jacobi Method. Let the integer indices j and k vary from $j = 1, 2, \dots J$, and $k = 1, 2, \dots, K$. The point-Jacobi method is then given by:

$$\psi_{j,k}^{(p)} = -\frac{1}{E}\left[F_{j,k} + A\psi_{j-1,k}^{(p-1)} + B\psi_{j+1,k}^{(p-1)} + C\psi_{j,k-1}^{(p-1)} + D\psi_{j,k+1}^{(p-1)}\right] \tag{50}$$

where p is the iteration counter. The initial values $\psi^{(0)}$ are chosen arbitrarily to start the iterative procedure.

Sometimes it is convenient to choose the $(k - 1)$th and $(k + 1)$th columns arbitrarily to start an iterative procedure for the k-th column. In this case the line with $j = 1, 2, \dots J$ is obtained by solving the system of equations:

$$A\psi_{j-1,k}^{(p)} + B\psi_{j+1,k}^{(p)} + E\psi_{j,k}^{(p)} = -F_{j,k} - C\psi_{j,k-1}^{(p-1)} - D\psi_{j,k+1}^{(p-1)} \tag{51}$$

which yields a tridiagonal matrix form.

(ii) Gauss-Seidel Iterative Method. In the Gauss-Seidel method the values at $(j - 1, k)$ and $(j, k - 1)$ are replaced by the most current values available in the iterative cycle. Thus for the p-th cycle the point-Jacobi method is replaced by:

$$\psi_{j,k}^{(p)} = -\frac{1}{E}\left[F_{j,k} + A\psi_{j-1,k}^{(p)} + C\psi_{j,k-1}^{(p)} + B\psi_{j+1,k}^{(p-1)} + D\psi_{j,k+1}^{(p-1)}\right] \tag{52}$$

The line-Gauss-Seidel method is to solve the system of equations:

$$A\psi_{j-1,k}^{(p)} + B\psi_{j+1,k}^{(p)} + E\psi_{j,k}^{(p)} = -F_{j,k} - C\psi_{j,k-1}^{(p)} - D\psi_{j,k+1}^{(p-1)} \tag{53}$$

(iii) Successive-Over-Relaxation (SOR). When the regular Gauss-Seidel method is modified as:

$$\psi_{j,k}^{(p)} = (1-\omega)\psi_{j,k}^{(p-1)} - \frac{\omega}{E}\left[F_{j,k} + A\psi_{j-1,k}^{(p)} + C\psi_{j,k-1}^{(p)} + B\psi_{j+1,k}^{(p-1)} + D\psi_{j,k+1}^{(p-1)}\right] \tag{54}$$

then it is called a point-SOR scheme. Here $1 < \omega < 2$.

The line-SOR method is written as:

$$\psi_{j,k}^{(p)} = \omega\hat{\psi}_{j,k}^{(p)} + (1-\omega)\psi_{j,k}^{(p-1)} \tag{55a}$$

where $\hat{\psi}_{j,k}^{(p)}$ are determined by the tridiagonal system:

$$A\hat{\psi}_{j-1,k}^{(p)} + B\hat{\psi}_{j+1,k}^{(p)} + E\hat{\psi}_{j,k}^{(p)} = -F_{j,k} - C\psi_{j,k-1}^{(p)} - D\psi_{j,k+1}^{(p-1)} \tag{55b}$$

where $j = 1, 2, \ldots, J$.

MATHEMATICAL EXPOSITION 9

Frame Invariancy

1. INTRODUCTION

The purpose of ME. 1 was essentially to introduce the idea that the entities called vectors and tensors are coordinate invariants. Thus a tensor **T** is represented as $T^{ij}\mathbf{a}_i\mathbf{a}_j$ in a coordinate system x^i of base vectors $\mathbf{a}_i$, and the *same* tensor is represented as $\bar{T}^{ij}\bar{\mathbf{a}}_i\bar{\mathbf{a}}_j$ in another coordinate system $\bar{x}^i$ of base vectors $\bar{a}_i$. Rules and principles to achieve the transformation from one coordinate system to the other have been laid out in ME.1 and used at various places in the text.

The other important principle in mechanics is that of frame invariancy, also called the observer invariancy. A check of frame invariancy on the physical quantities guides one in the direction of making some necessary alterations so as to comply with the basic principles of mechanics, e.g., with Newton's second law of motion. For example, although both velocity and acceleration vectors are coordinate invariants, they are not frame invariants. This is why alterations in both the velocity and acceleration vectors had to be made in the case of a rotating reference frame. Refer to Equations 3.130 and 3.140. An important use of frame-invariancy is in the selection of constitutive equations connecting the stress and the rate-of-strain in fluid dynamics. For a thorough analysis the reader is referred to Reference 14. In this ME we shall state the basic principles of frame invariancy starting from the basic properties of orthogonal tensors.

2. ORTHOGONAL TENSOR

A linear transformation is called an orthogonal tensor if the column vectors of the resulting matrix are orthonormal. If we denote the orthogonal tensor by **Q** and its transpose by $\mathbf{Q}^T$, then the property noted above implies that:

$$\mathbf{Q}^{-1} = \mathbf{Q}^T$$

and:

$$\mathbf{Q}\cdot\mathbf{Q}^T = \mathbf{Q}^T\cdot\mathbf{Q} = \mathbf{I} \tag{1}$$

Further, for any two vectors **u** and **v**:

$$|\mathbf{Q}\cdot\mathbf{u}| = |\mathbf{u}|, \qquad (\mathbf{Q}\cdot\mathbf{u})\cdot(\mathbf{Q}\cdot\mathbf{v}) = \mathbf{u}\cdot\mathbf{v} \tag{2}$$

Using the rules of tensor multiplication given in ME.1, we state Equation 1 in component form as:

$$Q_{mn}Q_{pn} = \delta_{mp} \tag{3a}$$

$$Q_{ij}Q_{ik} = \delta_{jk} \tag{3b}$$

The determinant of **Q**, using Equation 1, is

$$\det(\mathbf{Q}) = \pm 1$$

where *a* plus sign is for arbitrary rotation and *a* minus sign is for reflection.

The orthogonal tensor which represents a counterclockwise rotation through an angle θ about an axis having direction cosines ℓ, and passing through the origin[2] has components:

$$Q_{ik} = \delta_{ik} \cos\theta + (1-\cos\theta)\ell_i \ell_k - e_{kji}\ell_j \sin\theta; \quad \bar{x}_i = Q_{ik}x_k$$

Examples include:

1. Counterclockwise rotation of a rigid body about the x_3-axis through 90°. In this case:

$$[Q] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \det(Q) = +1$$

2. Reflection with respect to the x_1x_3-plane.
   In this case:

$$[Q] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \det(Q) = -1$$

Time Differentiation

If Q is a function of time, then from Equation 1:

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\mathbf{Q}}^T = 0$$

$$\dot{\mathbf{Q}}^T \cdot \mathbf{Q} + \mathbf{Q}^T \cdot \dot{\mathbf{Q}} = 0$$

or:

$$\dot{Q}_{mn} Q_{pn} = -Q_{mn} \dot{Q}_{pn} \tag{4a}$$

$$\dot{Q}_{ij} Q_{ik} = -Q_{ij} \dot{Q}_{ik} \tag{4b}$$

The transpose of an orthogonal tensor is also an orthogonal tensor. *In the ensuing analysis, we have worked with Q^T but have denoted it as Q.* To compare the results with the work of others, one only has to change Q by Q^T and Q_{pm} by Q_{mp} in all the formulae given below.

Change of Basis

We consider two rectangular Cartesian frames having bases ($\mathbf{i}_p$) and ($\mathbf{i}_p^*$). The basis vector sets are related through the orthogonal tensor **Q** (linear transformation) as:

$$\mathbf{i}_p^* = Q_{pm}\mathbf{i}_m = Q^T \cdot \mathbf{i}_p \tag{5a}$$

$$\mathbf{i}_p = Q_{mp}\mathbf{i}_m^* = Q \cdot \mathbf{i}_p^* \tag{5b}$$

which are a consequence of the definition

$$Q_{mn} = \mathbf{i}_m^* \cdot \mathbf{i}_n$$

Let **a** be a vector. Then since:

$$\mathbf{a} = a_r \mathbf{i}_r = a_s^* \mathbf{i}_s^*$$

we find on using Equations 5a, b that:

$$a_s^* = Q_{sr} a_r \tag{6a}$$

$$a_s = Q_{rs} a_r^* \tag{6b}$$

We can also interpret Equation 6a as the components of a *new* vector $\mathbf{a}^*$ when referred to the original basis,** i.e.:

$$\mathbf{a}^* = a_s^* \mathbf{i}_s = Q_{sr} a_r \mathbf{i}_s = \mathbf{Q} \cdot \mathbf{a} \tag{7}$$

Let $\mathbf{T}$ be a second order tensor. Then:

$$\mathbf{T} = T_{mn} \mathbf{i}_m \mathbf{i}_n = T_{pq}^* \mathbf{i}_p^* \mathbf{i}_q^*$$

Using Equations 5a, b, we get:

$$T_{pq}^* = Q_{pm} Q_{qn} T_{mn} \tag{8a}$$

$$T_{pq} = Q_{mp} Q_{nq} T_{mn}^* \tag{8b}$$

If we again interpret T_{mn}^* as the components of a new tensor $\mathbf{T}^*$ as referred to the original basis, i.e.:

$$\mathbf{T}^* = T_{mn}^* \mathbf{i}_m \mathbf{i}_n$$

then on substituting Equations 8, we find that:

$$\mathbf{T}^* = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^{\mathrm{T}} \tag{9a}$$

$$\mathbf{T} = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{T}^* \cdot \mathbf{Q} \tag{9b}$$

A general representation of the results in Equations 6a and 8a is

$$A_{j_1 j_2 \cdots j_n}^* = Q_{j_1 k_1} Q_{j_2 k_2} \cdots Q_{j_n k_n} A_{k_1 k_2 \cdots k_n} \tag{10}$$

Tensors which transform according to Equation 10 are called Cartesian tensors. Such tensors are said to be frame indifferent. A further subclass is that of isotropic tensors. Isotropic tensors transform into themselves under $\mathbf{Q}$. Examples are the δ_{ij} and e_{ijk}.

3. ARBITRARY RECTANGULAR FRAMES OF REFERENCE

We now consider a reference frame (*) which is in a state of arbitrary time-dependent rotation and translation with respect to a fixed inertial reference frame. The positions of a particle with respect to the *-frame and the fixed frame are given by the rectangular Cartesian coordinates (x_k^*) and (x_k), respectively. If we consider the components x_k^* as the components of a new vector $\mathbf{x}^*$ referred to the fixed reference frame, i.e.:

$$\mathbf{x}^* = x_k^* \mathbf{i}_k$$

** Note the difference between a_s^* as components and $\mathbf{a}^*$ as an entity.

then a general relationship between $\mathbf{x}^*$ and $\mathbf{x}$ is obtained as:

$$\mathbf{x}^* = \mathbf{Q}(t) \cdot \mathbf{x} - \mathbf{b}(t), \qquad t^* = t \tag{11a}$$

which in component form is

$$x_k^* = Q_{km} x_m - b_k \tag{11b}$$

Multiplying both sides of Equation 11b by Q_{ks} and using Equation 3b, we get:

$$x_s = Q_{ks} x_k^* + Q_{ks} b_k \tag{11c}$$

Time differentiation of Equation 11a yields the velocity. Thus:

$$\mathbf{u}^* = \mathbf{Q} \cdot \mathbf{u} + \dot{\mathbf{Q}} \cdot \mathbf{x} - \dot{\mathbf{b}} \tag{12a}$$

or:

$$u_k^* = Q_{ks} u_s + \dot{Q}_{ks} x_s - \dot{b}_k \tag{12b}$$

Multiplying both sides by Q_{km} and using Equations 3b and 4b we get:

$$u_m = Q_{km} u_k^* + \dot{Q}_{km} Q_{ks} x_s + Q_{km} \dot{b}_k \tag{12c}$$

$$= Q_{km} u_k^* + \dot{Q}_{km} x_k^* + \dot{Q}_{km} b_k + Q_{km} \dot{b}_k \tag{12d}$$

A comparison of Equations 7 and 12 shows that the velocity vector is *not* frame invariant. Particular cases of Equation 11 are

1. If Q is a constant with det(Q) = +1, then the *-frame is in arbitrary translational acceleration.
2. If Q is a constant and $\mathbf{b} = \mathbf{c}t + \mathbf{d}$, $\mathbf{c}$ and $\mathbf{d}$ being constant vectors, then the *-frame is a Galilean reference frame.
3. If $\mathbf{Q} = \mathbf{I}$ and $\mathbf{b} = u_\infty \mathbf{i}_1$, then the *-frame is an inertial Galilean reference frame.

4. CHECK FOR FRAME INVARIANCY

The preceding formulae, i.e., Equations 11 and 12, are used to check the frame invariancy of the physical quantities and also of the differential equations. To get acquainted with the process of establishing the frame invariancy the following exercises must be solved. Refer also to Reference 15.

For example, prove that the substantive operator is frame invariant. Noting that x_k, t are independent variables, we have:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$$

$$= \frac{\partial}{\partial t^*} + \frac{\partial x_k^*}{\partial t} \frac{\partial}{\partial x_k^*} + Q_{kp} u_p \frac{\partial}{\partial x_k^*}$$

$$= \frac{\partial}{\partial t^*} + (Q_{kp}u_p + \dot{Q}_{kp}x_p - \dot{b}_k)\frac{\partial}{\partial x_k^*}$$

$$= \frac{\partial}{\partial t^*} + u_k^* \frac{\partial}{\partial x_k^*}$$

$$= \frac{D^*}{Dt^*}$$

Hence, it is frame invariant.

Example 1 — Perturb Equation 12b and show that:

$$\bar{\tau}_{ij} = Q_{ki}Q_{mj}\overline{\tau_{km}^*}$$

and hence the Reynolds stress is frame invariant. (Refer to Equation 8b for the condition of frame invariancy.)

Example 2 — Show that the turbulence energy is frame invariant, i.e.:

$$K = K^*$$

Example 3 — For a symmetric tensor T_{ik} show that:

$$Q_{ij}\dot{Q}_{kj}T_{ik} = 0$$

Example 4 — By establishing that the components of the tensor grad **u** transform as:

$$\frac{\partial u_i}{\partial x_k} = Q_{si}Q_{nk}\frac{\partial u_s^*}{\partial x_n^*} - Q_{si}\dot{Q}_{sk}$$

convince yourself that although the tensor grad **u** is not frame invariant, the rate-of-strain tensor **D** is frame invariant.

Example 5 — Let u_i be a component of the fluid velocity vector. Use the chain rule and the results in Equations 11 and 12 to establish that:

$$\frac{Du_i}{Dt} = Q_{ki}\left(\frac{D^*u_k^*}{Dt^*} + \ddot{b}_k\right) + 2\dot{Q}_{ki}(u_k^* + \dot{b}_k) + \ddot{Q}_{ki}(x_k^* + b_k)$$

Thus the acceleration vector is not invariant to general motion of the frame. However, if $\mathbf{Q}$ = constant and $\ddot{\mathbf{b}} = 0$, then it is a Galilean invariant. From this result we conclude that the equations of motion are only Galilean invariant.

5. USE OF Q

The word "frame" used in the preceding development is a "frame of reference," with respect to which events are observed. Frame of reference should not be regarded as a synonym for a coordinate system.

A basic question of mechanics is the following: how does a material transmit forces from one part of the medium to another? In answer to this question one may state that certain relationships between the kinematic and mechanical quantities are required. These relationships are called the "constitutive equations." However, the formation of these relationships entails the development of an important principle, called the "principle of consequence." The principle of consequence requires that the proposed relationships should be invariant under a change of reference frame. That is, the manner in which a material medium distributes forces must be independent of the frame of reference. Once the principle of consequence is satisfied, then the invariancy property is called the principle of "material frame indifference," or, "invariance," or, "objectivity." Following are the principles of material frame indifference when $\mathbf{Q}$ is not considered as its transpose:

(i) Scalars remain unchanged under the change of frame.

(ii) A vector $\mathbf{u}$ is transformed into $\mathbf{u}^*$, except when $\mathbf{u}$ is a velocity vector, as

$$\mathbf{u}^* = \mathbf{Q}^T \cdot \mathbf{u}$$

(iii) A tensor $\mathbf{T}$ is transformed into $\mathbf{T}^*$ as

$$\mathbf{T}^* = \mathbf{Q}^T \cdot \mathbf{T} \cdot \mathbf{Q}$$

(Refer to Section 2.12)

Two interpretations are in order here:

(a) $\mathbf{u}$ and $\mathbf{u}^*$, or, $\mathbf{T}$ and $\mathbf{T}^*$ may be considered as the same entities but viewed from different frames.

(b) $\mathbf{u}$ and $\mathbf{u}^*$, or, $\mathbf{T}$ and $\mathbf{T}^*$ may be considered as different entities but viewed in the same frame.

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