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Stationary Stokes, Oseen and Navier–Stokes Equations with Singular Data

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Abstract

The concept of very weak solution introduced by Giga (Math Z 178:287–329, 1981) for the Stokes equations has hardly been studied in recent years for either the Navier–Stokes equations or the Navier–Stokes type equations. We treat the stationary Stokes, Oseen and Navier–Stokes systems in the case of a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$. Taking up once again the duality method introduced by Lions and Magenes (Problèmes aus limites non-homogènes et applications, vols. 1 & 2, Dunod, Paris, 1968) and Giga (Math Z 178:287–329, 1981) for open sets of class $\mathcal{C}^\infty$ [see also chapter 4 of Necas (Les méthodes directes en théorie des équations elliptiques. (French) Masson et Cie, Éd., Paris; Academia, Éditeurs, Prague, 1967), which considers the Hilbertian case $p = 2$ for general elliptic operators], we give a simpler proof of the existence of a very weak solution for stationary Oseen and Navier–Stokes equations when data are not regular enough, based on density arguments and a functional framework adequate for defining more rigourously the traces of non-regular vector fields. In the stationary Navier–Stokes case, the results will be valid for external forces not necessarily small, which lets us extend the uniqueness class of solutions for these equations. Considering more regular data, regularity results in fractional Sobolev spaces will also be discussed for the three systems. All these results can be extended to other dimensions.

1. Introduction and notation

Let Ω be a bounded open set of $\mathbb{R}^3$ with boundary Γ. In this work, we are interested in some questions concerning the Navier–Stokes equations:

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$$(NS)\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} = h & \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma,\end{cases}$$

where $\boldsymbol{u}$ denotes the velocity and q the pressure and both are unknown, $\boldsymbol{f}$ the external forces, h the compressibility condition and $\boldsymbol{g}$ the boundary condition for the velocity, the three functions being known. The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^3$ are respectively denoted by boldface Roman and special Roman.

In the case of incompressible fluids, $h = 0$, it has been well-known since LERAY [28] (see also [29]) that if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $p \geqq 2$ and for any $i = 0, \ldots, I$,

$$\int_{\Gamma_i} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0, \tag{1}$$

where Γ_i denotes the connected components of the boundary Γ of the open set Ω, then there exists a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (NS). In [37], SERRE proved the existence of a weak solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ for any $\frac{3}{2} < p < 2$ when $h = 0$ and $\boldsymbol{g}$ satisfies the above conditions. More recently, KIM [26] improves Serre's existence and regularity results on weak solutions of (NS) for any $\frac{3}{2} \leqq p < 2$ (including the case $p = \frac{3}{2}$), when the boundary of Ω is connected ($I = 0$), provided h is small in an appropriate norm (due to the compatibility condition between h and $\boldsymbol{g}$, then $\boldsymbol{g}$ is also small in the corresponding appropriate norm).

On the other hand, the notion of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for Stokes or Navier–Stokes equations corresponding to very irregular data has been developed in the last years by GIGA [22] (in a domain Ω of class $\mathcal{C}^\infty$), AMROUCHE AND GIRAULT [5] (in a domain Ω of class $\mathcal{C}^{1,1}$) and more recently by GALDI ET AL. [21], FARWIG AND GALDI [18] (in a domain Ω of class $\mathcal{C}^{2,1}$, see also SCHUMACHER [36]) and KIM [26] (in a domain of class $\mathcal{C}^2$). In this context, the boundary condition is chosen in $\mathbf{L}^p(\Gamma)$ (see BROWN AND SHEN [13], CONCA [15], FABES ET AL. [16], MOUSSAOUI [32], SHEN [38], SAVARÉ [35], MARUSIC-PALOKA [31]) or more generally in $\mathbf{W}^{-1/p,p}(\Gamma)$. For the non-stationary case, the existence, uniqueness and regularity of very weak solutions for the Navier–Stokes equations have been investigated (among other authors) by AMANN [2,3]. In this work, we again take up the method developed in [7], where the existence of very weak solutions for the stationary Stokes equations in a half-space for weighted Sobolev spaces was established.

The purpose of our work is to develop a unified theory of very weak solutions of the Dirichlet problem for Stokes, Oseen and Navier–Stokes equations (and also for the Laplace equation), see Theorems 2 and 4. One important question is to rigorously define the traces of the vector functions which are living in subspaces of $\mathbf{L}^p(\Omega)$ (see Lemmas 12 and 13). We prove the existence and regularity of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ of Stokes and Oseen equations for any $1 < p < \infty$ with arbitrarily large data belonging to some Sobolev spaces of negative order. In the case of Navier–Stokes equations, the existence of a very weak solution is proved for arbitrarily large external forces $\boldsymbol{f}$, but with a smallness condition for both h and $\boldsymbol{g}$. Uniqueness of very weak solutions is also proved for small enough data. Observe that the ideas used in this work are the same as those of Amrouche and Girault appearing for the Stokes problem in [4,5] for a bounded open set, and those of Amrouche et al. [7] for a half-space. However, from a technical point of view, there are very important differences: the functional spaces, all the density lemmas and the nature of the boundary are different. Moreover, we extend the study made for the Stokes equations to the Oseen and Navier–Stokes equations.

The existence of a very weak solution $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$, for arbitrarily large external forces $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $h = 0$ and arbitrarily large boundary conditions $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ and without assuming condition (1), was first proved by Marusic-Paloka in Theorem 5 of [31], with Ω a bounded simply-connected open set of class $\mathcal{C}^{1,1}$. But the proof of such a theorem becomes correct only if either condition (1) or condition (7) holds. The same result was proved by Kim [26] for arbitrarily large external forces $\boldsymbol{f} \in [\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)]'$, for small $h \in [W^{1,3/2}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ and where the boundary of Ω is assumed to be connected ($I = 0$). We remark that the space chosen for the divergence condition h, the dual space of $W^{1,3/2}(\Omega)$, and for the external forces $\boldsymbol{f}$, the dual of $\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)$, are not correct either (see Remark 2). In a closed context, we also consider the case where the data, and then the solutions, belong to fractionary Sobolev spaces $W^{s,p}(\Omega)$ with s a real number, possibly not integer, more precisely (see Theorem 3):

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega),\ h \in W^{\sigma-1,p}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),$$

where σ is a real number such that $\frac{1}{p} < \sigma \leqq 2$. As for the result of the existence of a very weak solution for Stokes and Navier–Stokes equations proved by Galdi et al. [21], they consider a more regular domain, different spaces for the data and impose smallness assumptions for all the data $\boldsymbol{f}$, h and $\boldsymbol{g}$ (see Remarks 1 and 7).

The work is organized as follows: In the remainder of this section, we recall the definitions of some spaces and their respective norms, besides some density results, trace theorems and Sobolev spaces embeddings. In Section 2, we state the main results of this paper related to the very weak solution for the Oseen and Navier–Stokes equations, together with the regularity solutions associated with more regular data. In Section 3, we present a previous study of the existence of a solution for the Laplace problem when irregular Dirichlet and Neumann boundary data are considered. In Section 4, we present some preliminary results, including density lemmas, characterization of dual spaces and a trace result for very weak solutions. Moreover, we recall the Stokes' results related to the very weak solution, treating the cases in which the divergence operator vanishes and the case in which it does not vanish (and is a given function). In Section 5, we extend previous results for the Oseen equation in order to obtain in Section 6 the result for the Navier–Stokes equation using a fixed point technique. The study of the very weak solution in both cases, Stokes and Oseen equations, needs regularity results for their dual problems that we also present in both sections. In Section 6, we first treat the case of small data, and then the general case. In this latest section, there is a difference between the case of small external forces and the case without. For the former, the existence is proved, whereas when arbitrary external forces are considered it is necessary to impose smallness assumptions for the divergence and boundary data. This work is announced in [9] and in [10].

Throughout this work, if we do not state otherwise, Ω will be considered as a Lipschitz open bounded set of $\mathbb{R}^3$. When Ω is connected, we will say Ω is a domain. We will specify the regularity of Ω only when it is to be different from the convention presented above.

In what follows, s is any real number, p denotes a real number such that $1 < p < \infty$ and p' stands for its conjugate: $1/p + 1/p' = 1$. We shall denote by m the integer part of s and by σ its fractional part: $s = m + \sigma$ with $0 \leqq \sigma < 1$. We denote by $W^{s,p}(\mathbb{R}^3)$ the space of all distributions v defined in $\mathbb{R}^3$ such that:

– $D^\alpha v \in L^p(\mathbb{R}^3)$, for all $|\alpha| \leqq m$, when $s = m$ is a nonnegative integer
– $v \in W^{m,p}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{3+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y < \infty,$$

for all $|\alpha| = m$, when $s = m + \sigma$ is nonnegative and is not an integer.

The space $W^{s,p}(\mathbb{R}^3)$ is a reflexive Banach space equipped by the norm:

$$\|v\|_{W^{m,p}(\mathbb{R}^3)} = \left(\sum_{|\alpha|\leqq m} \int_{\mathbb{R}^3} |D^\alpha v(x)|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p}$$

in the first case, and by the norm

$$\|v\|_{W^{s,p}(\mathbb{R}^3)} = \left(\|v\|^p_{W^{m,p}(\mathbb{R}^3)} + \sum_{|\alpha|=m} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{3+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p},$$

in the second case. For $s < 0$, we denote by $W^{s,p}(\mathbb{R}^3)$ the dual space of $W^{-s,p'}(\mathbb{R}^3)$.

In the special case of $p = 2$, we shall use the notation $H^s(\mathbb{R}^3)$ instead of $W^{s,2}(\mathbb{R}^3)$.

Now, we introduce the Sobolev space

$$H^{s,p}(\mathbb{R}^3) = \{v \in L^p(\mathbb{R}^3);\ (I - \Delta)^{s/2} v \in L^p(\mathbb{R}^3)\}.$$

It is known that $H^{s,p}(\mathbb{R}^3) = W^{s,p}(\mathbb{R}^3)$ if s is an integer or if $p = 2$. Furthermore, for any real number s, we have the following embeddings:

$$W^{s,p}(\mathbb{R}^3) \hookrightarrow H^{s,p}(\mathbb{R}^3) \quad \text{if } p \leqq 2 \quad \text{and} \quad H^{s,p}(\mathbb{R}^3) \hookrightarrow W^{s,p}(\mathbb{R}^3) \quad \text{if } p \geqq 2.$$

The definition of the space $W^{s,p}(\Omega)$ is exactly the same as in the case of the whole space. Because $\mathcal{D}(\Omega)$ is not dense in $W^{s,p}(\Omega)$, the dual space of $W^{s,p}(\Omega)$ cannot be identified as a space of distributions in Ω. For this reason, we define $W_0^{s,p}(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$ and we denote by $W^{-s,p'}(\Omega)$ its dual space.

For every $s > 0$, we denote by $W^{s,p}(\overline{\Omega})$ the space of all distributions in Ω which are restrictions of elements of $W^{s,p}(\mathbb{R}^3)$ and by $\widetilde{W}^{s,p}(\Omega)$ the space of functions $u \in W^{s,p}(\overline{\Omega})$ such that the extension $\widetilde{u}$ by zero outside of Ω belongs to $W^{s,p}(\mathbb{R}^3)$.

Recall now some density results ([1,24]):

(i) The space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any real s.
(ii) The space $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^{s,p}(\mathbb{R}^3)$ and in $H^{s,p}(\mathbb{R}^3)$ for any real s.
(iii) The space $\mathcal{D}(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$.
(iv) The space $\mathcal{D}(\Omega)$ is dense in $W^{s,p}(\Omega)$ for all $0 < s \leqq 1/p$, that means that $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$.

The following theorem gives some properties of traces of functions living in $W^{s,\,p}(\Omega)$ ([1,24]).

Theorem 1. *Let Ω be a bounded open set of class $\mathcal{C}^{k,1}$, for some integer $k \geqq 0$. Let s be a real number such that $s \leqq k+1$, $s - 1/p = m + \sigma$, where $m \geqq 0$ is an integer and $0 < \sigma < 1$.*

(i) *The following mapping*

$$\begin{aligned} \gamma_0 : u &\mapsto u_{|\Gamma} \\ W^{s,\,p}(\Omega) &\to W^{s-1/p,\,p}(\Gamma) \end{aligned}$$

is continuous and surjective. When $1/p < s < 1 + 1/p$, we have $Ker(\gamma_0) = W^{s,\,p}_0(\Omega)$.

(ii) *For $m \geqq 1$, the following mapping*

$$\begin{aligned} (\gamma_0, \gamma_1) : u &\mapsto (u_{|\Gamma}, \frac{\partial u}{\partial \boldsymbol{n}}_{|\Gamma}) \\ W^{s,\,p}(\Omega) &\to (W^{s-1/p,\,p}(\Gamma) \times W^{s-1-1/p,\,p}(\Gamma)) \end{aligned}$$

is continuous and surjective. When $1 + 1/p < s < 2 + 1/p$, we have $Ker(\gamma_0, \gamma_1) = W^{s,\,p}_0(\Omega)$.

We recall, also, the following embeddings:

$$W^{s,\,p}(\Omega) \hookrightarrow W^{t,\,q}(\Omega) \quad \text{for } t \leqq s,\ \ p \leqq q \quad \text{such that } s - 3/p = t - 3/q$$

and

$$W^{s,\,p}(\Omega) \hookrightarrow \mathcal{C}^{k,\,\alpha}(\overline{\Omega}) \quad \text{for } k < s - 3/p < k+1, \quad \alpha = s - k - 3/p,$$

where k is a non-negative integer.

2. Main results

We begin by introducing some spaces: First,

$$\mathbf{X}_{r,\,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_0(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W^{1,p}_0(\Omega)\}, \quad 1 < r,\ p < \infty, \tag{2}$$

and we set $\mathbf{X}_{p,\,p}(\Omega) = \mathbf{X}_p(\Omega)$. Their dual spaces, $(\mathbf{X}_{r,\,p}(\Omega))'$ and $(\mathbf{X}_p(\Omega))'$, are characterized by Lemma 9. Second, the solenoidal space:

$$\mathbf{H}_p(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}. \tag{3}$$

And finally, the spaces:

$$\begin{aligned} \mathbf{T}_{p,r}(\Omega) &= \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \ \Delta \boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}, \\ \mathbf{T}_{p,r,\sigma}(\Omega) &= \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \ \nabla\cdot\boldsymbol{v} = 0\}, \end{aligned} \tag{4}$$

endowed with the topology given by the norm:

$$\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta\boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}.$$

Observe that when $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively.

We treat the Stokes, Oseen and Navier–Stokes equations under the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \tag{5}$$

Our main results are as follows.
The first main result in this paper is related to the Oseen equations, that is:

$$(O) \qquad -\Delta\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma$$

where $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ ($s \geqq 3$) is given. We prove the existence and uniqueness of a very weak solution for the Oseen equations in $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ with $1 < p < \infty$.

Theorem 2. (Very weak solution of Oseen equations) *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy the compatibility condition* (5)*,*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))',\ \ h \in L^r(\Omega),\ \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),\ \text{ with } \frac{1}{r} = \frac{1}{p} + \frac{1}{s}$$

and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ with

$$s = 3 \ \text{ if } p > 3/2, \quad s = p' \ \text{ if } p < 3/2, \quad s = 3+\varepsilon \ \text{ if } p = 3/2. \tag{6}$$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ verifying the following estimates:

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leqq C\ \left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right),$$

$$\|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leqq C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)^2\left(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right).$$

(See Theorem 17).

The second main result concerns the regularity of solutions for the Oseen equations. We consider, in particular, the case in which the external forces $\boldsymbol{f}$ and the divergence condition h are not regular, more precisely $\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega)$ and $h \in W^{\sigma-1,p}(\Omega)$ with $\frac{1}{p} < \sigma \leqq 2$. The result is proved by combining Theorems 15, 18 and Remark 15 point (i).

Theorem 3. (Regularity for Oseen equations) *Let σ be a real number such that $\frac{1}{p} < \sigma \leqq 2$. Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the compatibility condition (5) with*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega),\ h \in W^{\sigma-1,p}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Let $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ satisfy (6). Then, the Oseen problem (O) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}}\\ &\quad\leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)} + \|h\|_{W^{\sigma-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Omega)}).\end{aligned}$$

The following theorem gives the existence of very weak solutions for the Navier–Stokes equations in $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ for arbitrarily large $\boldsymbol{f}$ but sufficiently small h and $\boldsymbol{g}$ in a possibly multiply-connected domain (see Theorem 20).

Theorem 4. (Very weak solution of Navier–Stokes equations) *Let $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ satisfy the compatibility condition (5). There exists a constant $\delta > 0$ depending only on Ω such that if*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leqq \delta, \tag{7}$$

then the problem (NS) has a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$.

In the last theorem, we prove weak and strong regularity results on very weak solutions for the Navier–Stokes equations. The result is proved by combining Theorems 21 and 22.

Theorem 5. (Regularity for Navier–Stokes equations) *Let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 4. Then, the following regularity results hold:*

(i) *Suppose that*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))',\ h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and $\max\{r, 3\} \leqq p$. Then $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$.

(ii) *Let $r \geqq 3/2$ and suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega),\ h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma).$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

(iii) *Let $1 < r < \infty$ and suppose that*

$$\boldsymbol{f} \in \mathbf{L}^r(\Omega),\ h \in W^{1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma).$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.

(iv) *Suppose that* $3/2 \leqq p \leqq 3$, $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$ *and*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega), \; h \in W^{\sigma,r}(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),$$

with $\sigma = \frac{3}{p} - 1$, $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ *and* $r \leqq p$. *Then* $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

(v) *Let* σ *be such that* $1/p < \sigma \leqq 1$ *and* $\sigma \geqq 3/p - 1$. *Suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \; h \in W^{\sigma-1,p}(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

Remark 1. (i) Point (i) shows, in particular, that for any $p \geqq 3$, if

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma), \quad \text{with } \frac{3p}{3+p} \leqq r \leqq p,$$

and $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ for any $i = 1, \ldots, I$ and $h = 0$, then Problem (NS) has a solution $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$. In [37], SERRE proves that for any $3/2 < r < 2$ (and then for any $r > 3/2$), if

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma),$$

with $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ for any $i = 0, \ldots, I$ and $h = 0$, then (NS) has a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$. Our point (ii) proves that this result holds if $r = 3/2$ without supposing h or the flux $\boldsymbol{g}$ through Γ_i to be equal to 0, more precisely, it suffices to assume the condition of smallness:

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leqq \delta.$$

(ii) Because of the relation (5), the condition (7) is automatically fulfilled when the norm $\|h\|_{L^{3/2}(\Omega)}$ is sufficiently small and $I = 0$, which means that the boundary Γ is connected, which is the case considered by KIM [26].

(iii) MARUSIC-PALOKA in [31] proves Theorem 4 with $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ (which is included in the dual space $(\mathbf{X}_{3,3/2}(\Omega))'$), $h = 0$ and $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ (which is included in $\mathbf{W}^{-1/3,3}(\Gamma)$) with $\|\boldsymbol{g}\|_{\mathbf{L}^2(\Gamma)}$ small. Moreover, the domain Ω is assumed to be simply-connected. In fact, the solution $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$ is more regular and belongs to $\mathbf{H}^{1/2}(\Omega)$ as pointed out in Remark 16.

(iv) GALDI ET AL. in [21] prove Theorem 4 and Theorem 5 point (i) with $\boldsymbol{f} = \operatorname{div} \mathbb{F}_0$, where $\mathbb{F}_0 \in \mathbb{L}^r(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and $\max\{2r, 3\} \leqq p$. They assume the domain Ω is of class $\mathcal{C}^{2,1}$. Moreover, they suppose $\boldsymbol{f}$, h and $\boldsymbol{g}$ to be sufficiently small with respect to their norms. The smallness condition on the external forces is, in fact, unnecessary, as we prove in Theorem 20. The essential idea consists of decomposing the Navier–Stokes problem (NS) into two, $(NS)_1$ and $(NS)_2$ (see the proof of Theorem 20). The first one is an Oseen problem where the existence of a very weak solution is proved, for a velocity small enough in norm $\mathbf{L}^3(\Omega)$.

That shows that the study of the Oseen problem is very important: Galdi et al. solve the Navier–Stokes problem directly using a fixed point theorem for a Stokes problem, and therefore impose smallness assumptions over all the data. These smallness assumptions over h and $\boldsymbol{g}$ are important in order to measure the nonlinear term, appearing in the second problem $(NS)_2$, and to apply Hopf's Lemma (which allows us to lift the divergence and boundary data conditions). This is the reason we suppose h and $\boldsymbol{g}$ to be small enough in adequate norms.

3. The Laplace equation

3.1. The Laplace equation with Dirichlet conditions

We are interested here in the resolution of the problem

$$(L_D) \qquad -\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = g \ \text{ on } \Gamma,$$

with data in some Sobolev spaces. Before starting our study, we review some results concerning this problem. Recall that one consequence of the Calderon–Zygmund theory of singular integrals and boundary layer potential is that for every $f \in W^{m-2,p}(\Omega)$ and $g \in W^{m-1/p,p}(\Gamma)$, with m a positive integer, the problem (L_D) has a unique solution $u \in W^{m,p}(\Omega)$ when Ω is of class $\mathcal{C}^{r,1}$ with $r = \max\{1, m-1\}$. If $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$, with $s > 1/p$, then $u \in W^{s,p}(\Omega)$ provided that Ω is of class $\mathcal{C}^{r,1}$ with $r = \max\{1, [s]\}$, where $[s]$ is the integer part of s. In [30], Lions and Magenes made a complete study for smooth domains and $p = 2$. Grisvard [24] treats the case where Ω is of class $\mathcal{C}^{r,1}$.

Jerison and Kenig [25] and many other authors study the case where Ω is only a bounded Lipschitz-continuous domain. First, we recall some results for $p = 2$.

- (i) If $f \in H^{-1/2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$ or $f \in L^2(\Omega)$ and $g = 0$, then the unique solution u of (L_D) satisfies $u \in H^{3/2}(\Omega)$.
- (ii) If $f \in H^{-1+s}(\Omega)$, with $-1/2 < s < 1/2$ and $g = 0$, then $u \in H^{1+s}(\Omega)$.
- (iii) If $f = 0$ and $g \in H^{s+1/2}(\Gamma)$, with $-1/2 \leqq s \leqq 1/2$, then $u \in H^{1+s}(\Omega)$.
- (iv) The conclusion in point (i) is not true for $\varepsilon = 0$: There exist a Lipschitz domain Ω and $f \in H^{-1/2}(\Omega)$ such that $u \notin H^{3/2}(\Omega)$.
- (v) The conclusion in point (ii) is not true for $s > 1/2$: There exist a Lipschitz domain Ω and $f \in \mathcal{C}^\infty(\overline{\Omega})$ such that $u \notin H^{1+s}(\Omega)$.

In the case in which p is arbitrary, we have the following result (see Jerison and Kenig [25]).

Theorem 6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geqq 3$. There exists $\varepsilon \in \,]0, 1]$, depending only on the Lipschitz constant of Ω, such that for every $f \in H^{s-2,p}(\Omega)$ and $g = 0$, there is a unique solution $u \in H^{s,p}(\Omega)$ to (L_D),* *provided one of the following holds:*

$$\begin{aligned} p_0 < p < p_0' \quad &\text{and} \quad \frac{1}{p} < s < 1 + \frac{1}{p} \\ 1 < p \leqq p_0 \quad &\text{and} \quad \frac{3}{p} - 1 - \varepsilon < s < 1 + \frac{1}{p} \\ p_0' \leqq p < \infty \quad &\text{and} \quad \frac{1}{p} < s < \frac{3}{p} + \varepsilon \end{aligned}$$

where $1/p_0 = 1/2 + \varepsilon/2$ *and* $1/p_0' = 1/2 - \varepsilon/2$. *Moreover, we have the estimate*

$$\|u\|_{H^{s,p}(\Omega)} \leqq C\|f\|_{H^{s-2,p}(\Omega)}$$

for all $f \in H^{s-2,p}(\Omega)$. *When the domain is* $\mathcal{C}^1$, *the exponent* p_0 *may be taken to be 1. When* $s = 1$, *there is* $p_1 > 3$ *such that if* $p_1' < p < p_1$, *then the inhomogeneous Dirichlet problem has a unique solution* $u \in H^{s,p}(\Omega)$.

As a particular case of the third condition, for any $N \geqq 3$ (and also $N = 2$), there exists a $\mathcal{C}^1$ domain Ω in $\mathbb{R}^N$ and $f \in H^{-1+1/p,p}(\Omega)$ for which the solution u of (L_D) with $g = 0$ does not belong to $H^{1+1/p,p}(\Omega)$ for all $1 < p < \infty$.

As we said before, if Ω is an open set of class $\mathcal{C}^{1,1}$, for each $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Omega)$, the problem (L_D) has a unique solution $u \in W^{s,p}(\Omega)$ assuming $1/p < s \leqq 2$. In this work, we are interested in the search for very weak solutions, that is, solutions belonging to $W^{s,p}(\Omega)$ with $0 \leqq s \leqq 1/p$ and for a regular open set Ω, here of class $\mathcal{C}^{1,1}$. Moreover, we look for optimal conditions for the data f and g in order to obtain such solutions.

With this aim, we introduce the space:

$$M_p(\Omega) = \left\{ v \in L^p(\Omega);\ \Delta v \in W^{-2+1/p,p}(\Omega) \right\},$$

which is a reflexive Banach space for the norm

$$\|v\|_{M_p(\Omega)} = \|v\|_{L^p(\Omega)} + \|\Delta v\|_{W^{-2+1/p,p}(\Omega)}.$$

Lemma 1. *The space* $\mathcal{D}\left(\overline{\Omega}\right)$ *is dense in* $M_p(\Omega)$.

Proof. For every continuous linear form $\ell \in \left(M_p(\Omega)\right)'$, there exists a pair $(f,\ g) \in L^{p'}(\Omega) \times W_0^{2-1/p,p'}(\Omega)$, such that

$$\forall v \in M_p(\Omega), \quad \langle \ell, v \rangle = \int_\Omega f v \,\mathrm{d}x + \langle \Delta v, g \rangle_{W^{-2+1/p,p}(\Omega) \times W_0^{2-1/p,p'}(\Omega)}. \tag{8}$$

Thanks to the Hahn–Banach theorem, it suffices to show that any ℓ which vanishes on $\mathcal{D}(\overline{\Omega})$ is actually zero on $M_p(\Omega)$. Let's suppose that $\ell = 0$ on $\mathcal{D}(\overline{\Omega})$, thus on $\mathcal{D}(\Omega)$. Then we can deduce from (8) that

$$f + \Delta g = 0 \quad \text{in } \Omega,$$

hence we have $\Delta g \in L^{p'}(\Omega)$, and then $g \in W^{2,p'}(\Omega) \cap W_0^{2-1/p,p'}(\Omega)$. Let $\tilde{f} \in L^{p'}(\mathbb{R}^N)$ and $\tilde{g} \in W^{1,p'}(\mathbb{R}^N)$ be respectively the extensions by 0 of f and g to $\mathbb{R}^N$. From (8), we get $\tilde{f} + \Delta\tilde{g} = 0$ in $\mathbb{R}^N$, and thus $\Delta\tilde{g} \in L^{p'}(\mathbb{R}^N)$. Now, according to the properties for Δ in $\mathbb{R}^N$ (see [6]), we can deduce that $\tilde{g} \in W^{2,p'}(\mathbb{R}^N)$. Since $\tilde{g}$ is an extension by 0, it follows that $g \in W_0^{2,p'}(\Omega)$. Then, by density of $\mathcal{D}(\Omega)$ in $W_0^{2,p'}(\Omega)$, there exists a sequence $(\varphi_k)_k \subset \mathcal{D}(\Omega)$ such that $\varphi_k \to g$ in $W^{2,p'}(\Omega)$. Thus, for any $v \in M_p(\Omega)$, we have

$$\begin{aligned}\langle \ell, v\rangle &= -\int_\Omega v\cdot\Delta g\,\mathrm{d}x + \langle \Delta v, g\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\\ &= \lim_{k\to\infty}\left(-\int_\Omega v\cdot\Delta\varphi_k\,\mathrm{d}x + \langle\Delta v,\varphi_k\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\right)\\ &= 0,\end{aligned}$$

that is ℓ is identically zero. □

To study the traces of functions which belong to $M_p(\Omega)$, we have the following lemma.

Lemma 2. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. The linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma_0 : M_p(\Omega) \longrightarrow W^{-1/p,p}(\Gamma).$$

Moreover, we have the Green formula:

$$\begin{aligned}&\forall v \in M_p(\Omega), \quad \forall\varphi \in W^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega),\\ &\int_\Omega v\Delta\varphi\,\mathrm{d}x - \langle\Delta v,\varphi\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\\ &\quad = \left\langle v, \frac{\partial\varphi}{\partial\boldsymbol{n}}\right\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}.\end{aligned}\tag{9}$$

Proof. Let $v \in \mathcal{D}\left(\overline{\Omega}\right)$ and $\varphi \in W^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega)$, then formula (9) obviously holds. For every $\mu \in W^{1/p,p'}(\Gamma)$, there exists $\varphi \in W^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega)$ such that $\frac{\partial\varphi}{\partial\boldsymbol{n}} = \mu$ on Γ, with $\|\varphi\|_{W^{2,p'}(\Omega)} \leqq C\,\|\mu\|_{W^{1/p,p'}(\Gamma)}$. Consequently,

$$\left|\langle v,\mu\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}\right| \leqq C\,\|v\|_{M_p(\Omega)}\,\|\mu\|_{W^{1/p,p'}(\Gamma)}.$$

Thus

$$\|v\|_{W^{-1/p,p}(\Gamma)} \leqq C\,\|v\|_{M_p(\Omega)}.$$

We can deduce that the linear mapping γ is continuous for the norm of $M_p(\Omega)$. Since $\mathcal{D}\left(\overline{\Omega}\right)$ is dense in $M_p(\Omega)$, γ can be extended by continuity to $\gamma \in \mathcal{L}(M_p(\Omega); W^{-1/p,p}(\Gamma))$ and formula (9) holds for all $v \in M_p(\Omega)$ and $\varphi \in W^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega)$. □

We now can solve the Laplace equation with singular boundary conditions.

Theorem 7. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. For any $f \in W^{-2+1/p,p}(\Omega)$ and $g \in W^{-1/p,\,p}(\Gamma)$, the Laplace equation (L_D) has a unique solution $u \in L^p(\Omega)$, with the estimate*

$$\|u\|_{M_p(\Omega)} \leqq C\,(\|f\|_{W^{-2+1/p,p}(\Omega)} + \|g\|_{W^{-1/p,\,p}(\Gamma)}).$$

Proof. Thanks to the Green formula (9), it is easy to verify that $u \in L^p(\Omega)$ is a solution of problem (L_D) equivalent to the variational formulation: Find $u \in L^p(\Omega)$ such that

$$\begin{aligned}
&\forall v \in W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega),\\
&\int_\Omega u\Delta v\,\mathrm{d}x = -\,\langle f, v\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)}\\
&\qquad\qquad + \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)}.
\end{aligned} \tag{10}$$

Indeed, let $u \in L^p(\Omega)$ be a solution to (L_D). Then, the Green formula (9) yields (10). Conversely, let $u \in L^p(\Omega)$ be a solution to (10). Taking v in $\mathcal{D}(\Omega)$, we obtain $-\Delta u = f$ in Ω and $u \in M_p(\Omega)$. Using this last relation and (again) the Green formula (9), we deduce that for all $v \in W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)$,

$$\left\langle u, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)} = \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)}$$

and finally $u = g$ on Γ.

Let us now solve problem (10). We know that for all $F \in L^{p'}(\Omega)$, there exists a unique $v \in W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)$ satisfying $-\Delta v = F$ in Ω, with the estimate

$$\|v\|_{W^{2,\,p'}(\Omega)} \leqq C\|F\|_{L^{p'}(\Omega)}.$$

Then we have

$$\begin{aligned}
&\left|\langle f, v\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,p'}(\Omega)} - \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)}\right|\\
&\leqq C\,\|f\|_{W^{-2+1/p,p}(\Omega)}\|v\|_{W^{2-1/p,p'}(\Omega)} + \|g\|_{W^{-1/p,\,p}(\Gamma)}\|\frac{\partial v}{\partial \boldsymbol{n}}\|_{W^{1/p,\,p'}(\Gamma)}\\
&\leqq C\,\left(\|f\|_{W^{-2+1/p,p}(\Omega)} + \|g\|_{W^{-1/p,\,p}(\Gamma)}\right)\|F\|_{L^{p'}(\Omega)}.
\end{aligned}$$

In other words, we can say that the linear mapping

$$T : F \longmapsto \langle f, v\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,\,p'}(\Omega)} - \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)}$$

is continuous on $L^{p'}(\Omega)$, and according to the Riesz representation theorem there exists a unique $u \in L^p(\Omega)$, such that

$$\forall F \in L^{p'}(\Omega),\ T(F) = \langle u, F\rangle_{L^p(\Omega)\times L^{p'}(\Omega)},$$

that is, u is solution of (L_D). $\square$

Remark 2. In many works, and in particular in [26], we find a similar result with f given in the dual space of $W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)$ (the same remark holds for the dual space of $W^{1,\,p}(\Omega)$). But this assumption is not suitable, because $\mathcal{D}(\Omega)$ not being dense in this last space, its dual is not a subspace of distributions. It is then not sensible to solve Problem (L_D) with such data f as shown in the following counter-example. Indeed, choose $f = \mu_\Gamma$ which is defined by:

$$\forall v \in W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega), \quad \mu_\Gamma(\varphi) = \int_\Gamma \frac{\partial \varphi}{\partial \boldsymbol{n}}.$$

It is clear that f belongs to $[W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)]'$ and then the weak solution u obtained in this work satisfies: $\forall v \in W^{2,\,p'}(\Omega) \cap W_0^{1,\,p'}(\Omega)$,

$$\int_\Omega u \Delta v \,\mathrm{d}x = \int_\Gamma (g-1) \frac{\partial v}{\partial \boldsymbol{n}}.$$

Consequently u verifies:

$$\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = g - 1 \;\text{ on } \Gamma,$$

which means that this solution does not correspond to the solution of Problem (L_D). The origin of this mistake (also present everywhere in the same paper [26]) is due to the fact that when we want to solve a boundary value problem, it is necessary to have an adequate Green formula and corresponding density lemmas.

Corollary 1. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ and σ be a real number such that $0 \leqq \sigma \leqq 1$.*

(i) *We assume that*

$$f \in W^{-2+\sigma/p'+1/p,p}(\Omega) \quad \text{and} \quad g \in W^{\sigma-1/p,p}(\Gamma).$$

Then the solution u given by Theorem 7 belongs to $W^{\sigma,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{\sigma,p}(\Omega)} \leqq C\,(\|f\|_{W^{-2+\sigma/p'+1/p,p}(\Omega)} + \|g\|_{W^{\sigma-1/p,p}(\Gamma)}).$$

(ii) *If moreover*

$$f \in W^{\sigma-1,p}(\Omega) \quad \text{and} \quad g \in W^{\sigma+1/p',p}(\Gamma),$$

then $u \in W^{\sigma+1,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{\sigma+1,p}(\Omega)} \leqq C\,(\|f\|_{W^{\sigma-1,p}(\Omega)} + \|g\|_{W^{\sigma+1/p',p}(\Gamma)}).$$

Proof. First, we observe that if $\sigma = 0$, the conclusion in point (i) holds because of Theorem 7 and the conclusion in point (ii) is satisfied thanks to classical regularity of generalized solutions for Problem (L_D). If $\sigma = 1$, then point (i) holds for the same reason as the second point due to the classical regularity of strong solutions for Problem (L_D). Hence, we can suppose that $0 < \sigma < 1$. In this case, it suffices to use an interpolation argument (see [11,30,39]) and the elliptic regularity problem for the generalized solutions. Note that we have the following interpolation spaces:

$$\begin{aligned}
\Big[W^{1,p}(\Omega), L^p(\Omega)\Big]_{1-\sigma} &= W^{\sigma,p}(\Omega),\\
\Big[W^{1-1/p,p}(\Gamma), W^{-1/p,p}(\Gamma)\Big]_{1-\sigma} &= W^{\sigma-1/p,p}(\Gamma),\\
\Big[W^{-1,p}(\Omega),\ W^{-2+1/p,p}(\Omega)\Big]_{1-\sigma} &\hookleftarrow W^{-2+\sigma/p'+1/p,p}(\Omega)
\end{aligned}$$

and

$$\begin{aligned}
\Big[L^p(\Omega),\ W^{-1,p}(\Omega)\Big]_{1-\sigma} &\hookleftarrow W^{\sigma-1,p}(\Omega),\\
\Big[W^{2-1/p,p}(\Gamma), W^{1-1/p,p}(\Gamma)\Big]_{1-\sigma} &= W^{1/p'+\sigma,p}(\Gamma),\\
\Big[W^{2,p}(\Omega),\ W^{1,p}(\Omega)\Big]_{1-\sigma} &= W^{1+\sigma,p}(\Omega)
\end{aligned}$$

□

Remark 3. (i) The results of the second point are optimal, unlike part (i) which is optimal only when $f = 0$.

(ii) We can reformulate point (ii) as follows. For any $f \in W^{-s,p}(\Omega)$ and $g \in W^{2-s-1/p,p}(\Gamma)$, with $0 \leqq s \leqq 1$, Problem (L_D) has a unique solution $u \in W^{2-s,p}(\Omega)$ satisfying $u = g$ on Γ.

Theorem 8. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$, and let s be a real number such that $\frac{1}{p} < s \leqq 2$. We assume that $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$. Then Problem ($\mathrm{L}_D$) has a unique solution $u \in W^{s,p}(\Omega)$ which satisfies the estimate*

$$\|u\|_{W^{s,p}(\Omega)} \leqq C\,(\|f\|_{W^{s-2,p}(\Omega)} + \|g\|_{W^{s-1/p,p}(\Gamma)}).$$

Proof. The theorem is proved by Corollary 1 point (ii) if $1 \leqq s \leqq 2$. Let be then s a real number such that $\frac{1}{p} < s \leqq 1$. Using Theorem 1, we can suppose $g = 0$. We know that $\mathcal{D}(\Omega)$ is dense in the space of functions of $W^{s,p}(\Omega)$ equal to zero on Γ, which means that

$$W^{s,p}_0(\Omega) = \{v \in W^{s,p}(\Omega);\ v = 0 \text{ on } \Gamma\}.$$

We also have the same relation for the space $W^{2-s,p'}_0(\Omega)$ because $1 \leqq 2 - s < 1 + 1/p'$. Consequently $u \in W^{s,p}_0(\Omega)$ satisfies $-\Delta u = f$ in Ω if and only if

$$\begin{gathered}
\forall v \in W_0^{-s+2,\,p'}(\Omega),\\
\langle u,\ \Delta v\rangle_{W^{s,p}_0(\Omega)\times W^{-s,p'}(\Omega)} = -\langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2\,p'}(\Omega)}
\end{gathered}\tag{11}$$

Let's solve problem (11). By Remark 3 point (ii), we know that for all $F \in W^{-s,p'}(\Omega)$, there exists a unique $v \in W_0^{-s+2,\,p'}(\Omega)$ satisfying $-\Delta v = F$ in Ω, with the estimate

$$\|v\|_{W^{-s+2,\,p'}(\Omega)} \leqq C\|F\|_{W^{-s,p'}(\Omega)}.$$

Then

$$\left|\langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2\,p'}(\Omega)}\right| \leqq C\,\|f\|_{W^{s-2,p}(\Omega)}\|v\|_{W^{-s+2\,p'}(\Omega)}$$
$$\leqq C\,\|f\|_{W^{s-2,p}(\Omega)}\|F\|_{W^{-s,p'}(\Omega)}.$$

In other words, we can say that the linear mapping

$$T\,:\;F\longmapsto \langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2,\,p'}(\Omega)}$$

is continuous on $W^{-s,p'}(\Omega)$, and according to the Riesz representation theorem, there exists a unique $u \in W_0^{s,p}(\Omega)$, such that

$$\forall F \in W^{-s,p'}(\Omega),\;\; T(F) = \langle u, F\rangle_{W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)}\,,$$

that is, u is a solution of (L_D) with $g = 0$. □

Remark 4. (i) When $f \in W^{1/p-2,p}(\Omega)$, we can conjecture that $u \notin W^{1/p,p}(\Omega)$.

(ii) If $1/p < s < 1$, $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$, then the solution u of (L_D) belongs to $W^{s,p}(\Omega)$. These assumptions are weaker than those of Corollary 1 point (i) because $W^{-2+s/p'+1/p,p}(\Omega) \hookrightarrow W^{s-2,p}(\Omega)$ if $1/p < s < 1$. Moreover, they are optimal.

(iii) If $0 \leqq s \leqq 1/p$, Theorem 8 cannot be applied. Indeed, the trace mapping is not continuous (and not surjective) from $W^{s,p}(\Omega)$ into $W^{s-1/p,p}(\Gamma)$. If $s = 0$ and $g \in W^{-1/p,p}(\Gamma)$, we cannot expect to find a solution u more regular than $L^p(\Omega)$. Theorem 7 shows that it is possible if $f \in W^{-2+1/p,p}(\Omega)$. In the case of $0 < s \leqq 1/p$ and $g \in W^{s-1/p,p}(\Gamma)$, we cannot expect, either, to find a solution u better than $W^{s,p}(\Omega)$. Corollary 1 point (i) shows that it is possible if $f \in W^{-2+s/p'+1/p,p}(\Omega)$, taking into account that $-2+s/p'+1/p > -2+s$.

Remark 5. In the case $p = 2$, we have proved in particular the following results which are naturally better than the case where Ω is considered to be only Lipschitz:

(i) if $f \in H^{-1/2}(\Omega)$ and $g \in H^1(\Gamma)$, then $u \in H^{3/2}(\Omega)$,
(ii) if $f \in H^{-1+s}(\Omega)$, with $-1/2 < s \leqq 1$ and $g = 0$, then $u \in H^{1+s}(\Omega)$,
(iii) if $f = 0$ and $g \in H^{s+1/2}(\Gamma)$, with $-1 \leqq s \leqq 1$ then $u \in H^{1+s}(\Omega)$.

3.2. The Laplace equation with Neumann condition

We introduce the space:

$$K_p(\Omega) = \left\{v \in L^p(\Omega);\;\; \Delta v \in L^p(\Omega)\right\},$$

which is a reflexive Banach space for the norm

$$\|v\|_{K_p(\Omega)} = \|v\|_{L^p(\Omega)} + \|\Delta v\|_{L^p(\Omega)}.$$

We are interested here in the resolution of the problem

$$(L_N) \qquad -\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{n}} = g \;\text{ on } \Gamma,$$

when the boundary condition g is not regular.

As for Lemma 1, we can prove the following lemma:

Lemma 3. *The space $\mathcal{D}\left(\overline{\Omega}\right)$ is dense in $K_p(\Omega)$.*

To study the traces of functions which belong to $K_p(\Omega)$, we have the following lemma:

Lemma 4. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. The linear mapping $\gamma : v \longmapsto (v|_\Gamma, \frac{\partial v}{\partial \boldsymbol{n}}|_\Gamma)$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma : K_p(\Omega) \longrightarrow W^{-1/p,\, p}(\Gamma) \times W^{-1-1/p,\, p}(\Gamma).$$

Moreover, we have the Green formula: $\forall v \in K_p(\Omega), \quad \forall \varphi \in W^{2,\, p'}(\Omega)$,

$$\begin{aligned} &\int_\Omega (v\Delta\varphi \ - \varphi\Delta v)\,\mathrm{d}x \\ &\quad = \left\langle v, \frac{\partial \varphi}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\, p}(\Gamma)\times W^{1/p,\, p'}(\Gamma)} - \left\langle \frac{\partial v}{\partial \boldsymbol{n}}, \varphi\right\rangle_{W^{-1-1/p,\, p}(\Gamma)\times W^{1+1/p,\, p'}(\Gamma)}. \end{aligned} \tag{12}$$

We can now solve the Laplace equation with singular boundary conditions.

Theorem 9. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. For any $f \in L^p(\Omega)$ and $g \in W^{-1-1/p,\, p}(\Gamma)$ satisfying the compatibility condition*

$$\int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\langle g, 1\rangle_{\mathbf{W}^{-1-1/p,p}(\Gamma)\times \mathbf{W}^{1+1/p,p'}(\Gamma)}. \tag{13}$$

the Laplace equation (L_N) has a unique solution $u \in L^p(\Omega)$, with the estimate

$$\|u\|_{K_p(\Omega)} \leqq \ C\ (\|f\|_{L^p(\Omega)} + \|g\|_{W^{-1-1/p,\, p}(\Gamma)}).$$

By an interpolation argument, we can prove the following corollary.

Corollary 2. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ and let σ be a real number such that $0 \leqq \sigma \leqq 1$.*

(i) *Assume that*

$$f \in L^p(\Omega) \quad \text{and} \quad g \in W^{\sigma-1-1,p}(\Gamma)$$

satisfying the compatibility condition (13). Then the solution u given by Theorem 9 belongs to $W^{\sigma,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{\sigma,p}(\Omega)} \leqq C\,(\|f\|_{L^p(\Omega)} + \|g\|_{W^{\sigma-1-1/p,p}(\Gamma)}).$$

(ii) *If moreover $g \in W^{\sigma-1/p,p}(\Gamma)$ then $u \in W^{\sigma+1,p}(\Omega)$ and satisfies the estimate*

$$\|u\|_{W^{\sigma+1,p}(\Omega)} \leqq C\,(\|f\|_{L^p(\Omega)} + \|g\|_{W^{\sigma-1/p,p}(\Gamma)}).$$

4. The Stokes problem

Throughout the rest of this work, if we do not say otherwise, we assume that Ω is a *bounded connected open set of class* $\mathcal{C}^{1,1}$. We focus on the study of the Stokes problem:

$$(S) \qquad -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \ \text{ and } \ \nabla \cdot \boldsymbol{u} = h \ \text{ in } \Omega, \ \boldsymbol{u} = \boldsymbol{g} \ \text{ on } \Gamma,$$

with the compatibility condition (5). Basic results on weak and strong solutions of problem (S) may be summarized by the following theorem (see [5,14]).

Theorem 10. (i) *For every* $\boldsymbol{f}$, h, $\boldsymbol{g}$ *with*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma),$$

and satisfying the compatibility condition (5)*, the Stokes problem* (*S*) *has exactly one solution* $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ *and* $q \in L^p(\Omega)/\mathbb{R}$*. Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \tag{14}$$

(ii) *Moreover, if*

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma),$$

then $\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega)$*,* $q \in W^{1,p}(\Omega)$ *and there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{L}^{p}(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}). \tag{15}$$

Remark 6. If Ω is only a bounded Lipschitz domain, there exists $\varepsilon > 0$ depending only on the Lipschitz constant of Ω such that if $2 \leqq p \leqq 3 + \varepsilon, \boldsymbol{f} = \boldsymbol{0}$, $h = 0$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$, the conclusion of the first part of Theorem 10 holds. The result is also valid under the assumptions $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, for a ε such that $(3+\varepsilon)/(2+\varepsilon) < p < 3 + \varepsilon$ (see [13]).

We are interested here in the case of singular data satisfying precisely the following assumptions:

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \ h \in L^r(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \ \text{with } \frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3} \text{ and } r \leqq p. \tag{16}$$

Recall that the space $(\mathbf{X}_{r',p'}(\Omega))'$ is an intermediate space between $W^{-1,r}(\Omega)$ and $W^{-2,p}(\Omega)$ (see embeddings (19)).

4.1. Preliminary results

We recall now some versions of De Rham's Theorem which we shall use in the sequel. First, we introduce the following spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}, \quad \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) = \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega});\ \nabla \cdot \boldsymbol{\psi} = 0\}.$$

The first result is proved by de Rham [34]:

Lemma 5. (De Rham's Theorem for distributions) *Let Ω be any open subset of $\mathbb{R}^3$ and let $\boldsymbol{f}$ be a distribution of $\boldsymbol{\mathcal{D}}'(\Omega)$ that satisfies:*

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0.$$

Then, there exists a distribution π in $\mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla\pi$.

The second result is proved by [5]:

Lemma 6. (De Rham's Theorem in $\mathbf{W}^{-m,p}(\Omega)$) *Let m be any integer, p any real number with $1 < p < \infty$. Let $\boldsymbol{f} \in \mathbf{W}^{-m,p}(\Omega)$ satisfy:*

$$\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle = 0.$$

Then, there exists $\pi \in W^{-m+1,p}(\Omega)$ such that $\boldsymbol{f} = \nabla\pi$. If, in addition, the set Ω is connected, then π is defined uniquely, up to an additive constant, and there exists a positive constant C, independent of $\boldsymbol{f}$, such that:

$$\inf_{K\in\mathbb{R}} \|\pi + K\|_{W^{-m+1,p}(\Omega)} \leqq C\, \|\boldsymbol{f}\|_{\mathbf{W}^{-m,p}(\Omega)}.$$

Then, we show some density results that will be essential for the proofs below.

Lemma 7. *The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{H}_p(\Omega)$ (defined by (3)).*

Proof. Let $\boldsymbol{\ell}$ be a linear and continuous mapping in $\mathbf{H}_p(\Omega)$ such that $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$. Since $\mathbf{H}_p$ is a subspace of $\mathbf{L}^p(\Omega)$, we can extend $\boldsymbol{\ell}$ to $\boldsymbol{L} \in \mathbf{L}^{p'}(\Omega)$.

We start by supposing Ω to be a connected open set. At this point, we shall make the following assumptions on Ω: Ω is bounded, connected but possibly multiply-connected, with $\cup_{1\leqq i\leqq I}\omega_i$ its wholes, and its boundary Γ is Lipschitz-continuous. We denote by ω_0 the exterior of Ω, by Γ_0 the exterior boundary of Ω and by Γ_i, $1 \leqq i \leqq I$, the other components of Γ. The duality between $\mathbf{W}^{-1/p,p'}(\Gamma_i)$ and $\mathbf{W}^{1/p,p}(\Gamma_i)$, and $\mathbf{W}^{-1/p,p'}(\Gamma_0)$ and $\mathbf{W}^{1/p,p}(\Gamma_0)$, will be denoted by $\langle\cdot,\cdot\rangle_{\Gamma_i}$ and $\langle\cdot,\cdot\rangle_{\Gamma_0}$, respectively. By De Rham's Lemma 6, then there exists a unique $q \in W^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)$ such that $\boldsymbol{L} = \nabla q$ and where

$$L_0^{p'}(\Omega) = \left\{\varphi \in L^{p'}(\Omega);\ \int_\Omega \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\right\}.$$

Moreover,

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}), \qquad \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \langle q, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = 0.$$

We extend $\boldsymbol{L}$ by zero out of Ω and denote the extension by $\widetilde{\boldsymbol{L}}$. Then, for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ such that $\nabla \cdot \boldsymbol{\varphi} = 0$ in $\mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{L}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{L} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} = 0.$$

From this, we deduce that, thanks to De Rham's Lemma 5, there exists $h \in \mathcal{D}'(\mathbb{R}^3)$ verifying $\nabla h \in \mathbf{L}^{p'}(\mathbb{R}^3)$ such that $\widetilde{\boldsymbol{L}} = \nabla h$ (see Lemma 2.1 in [8]). It is clear that $h \in W^{1,p'}_{loc}(\mathbb{R}^3)$. As h is unique up to an additive constant and $\nabla h = 0$ in ω_0, we can choose this constant in such a way that $h = 0$ in ω_0. Therefore, we deduce that:

$$h = 0 \quad \text{in } \omega_0, \qquad h = c_i \quad \text{in each } \omega_i, \qquad h = q + c_0 \quad \text{in } \Omega,$$

and thus:

$$q = -c_0 \quad \text{on } \Gamma_0, \qquad q = c_i - c_0 \quad \text{on } \Gamma_i,\ 1 \leqq i \leqq I.$$

Let $j \in \{1, \ldots, I\}$ be a fixed index, choosing $\boldsymbol{v}_j \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that $\langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k} = \delta_{jk}$ for $1 \leqq k \leqq I$ and $\langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} = -1$, we can deduce that $c_j = 0$. Indeed, for any $j = 1, \ldots, I$, we have:

$$\begin{aligned} \langle \boldsymbol{\ell}, \boldsymbol{v}_j\rangle = \langle \boldsymbol{v}_j \cdot \boldsymbol{n}, q\rangle_\Gamma &= -c_0 \langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \sum_{i=1}^{I} \langle c_i - c_0, \boldsymbol{v}_j \cdot \boldsymbol{n}\rangle_{\Gamma_i} \\ &= c_0 \langle \boldsymbol{v}_j \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \langle c_j - c_0, \boldsymbol{v}_j \cdot \boldsymbol{n}\rangle_{\Gamma_j} = c_j, \end{aligned}$$

so $c_j = 0$ for $j = 1, \ldots, I$. Therefore, $h = q + c_0 \in W^{1,p'}_0(\Omega)$.

When Ω is not connected, we can repeat the procedure above in each connected component of Ω.

In consequence, for every $\boldsymbol{v} \in \mathbf{H}_p(\Omega)$, we have:

$$\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \int_{\Omega} \nabla h \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0.$$

Thus, we deduce that $\boldsymbol{\ell} = \mathbf{0}$ in $\mathbf{H}'_p(\Omega)$. $\square$

The proofs of the following two lemmas are classical, although the functional spaces are changed.

Lemma 8. *The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$ (defined by (2)) and for all $q \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, we have*

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W^{1,p'}_0(\Omega)}. \tag{17}$$

Proof. In order to do the proof, we want to extend the functions in $\mathbf{X}_{r,p}(\Omega)$ to all of the space $\mathbb{R}^N$. Therefore, following a proof made by Amrouche and Girault [4], as a first step we consider that Ω is strictly star-shaped with respect to one of its points, which is taken as the origin. Then, for every $\boldsymbol{v} \in \mathbf{X}_{r,p}(\Omega)$ we take $\widetilde{\boldsymbol{v}}$ its extension by zero to $\mathbb{R}^N$. Thus, $\widetilde{\boldsymbol{v}} \in \mathbf{W}^{1,r}(\mathbb{R}^N)$ and if we define $\nabla \cdot \widetilde{\boldsymbol{v}} = \widetilde{\nabla \cdot \boldsymbol{v}} \in W^{1,p}(\mathbb{R}^N)$, then $\widetilde{\boldsymbol{v}} \in \mathbf{X}_{r,p}(\mathbb{R}^N)$.

Now, we take $\theta < 1$ and define the functions $\widetilde{\boldsymbol{v}}_\theta(\boldsymbol{x}) = \widetilde{\boldsymbol{v}}\left(\dfrac{\boldsymbol{x}}{\theta}\right)$ for any $\boldsymbol{x} \in \mathbb{R}^N$. We observe that $\widetilde{\boldsymbol{v}}_\theta$ has a compact support in Ω, since $\operatorname{supp}\widetilde{\boldsymbol{v}}_\theta \subseteq \theta\overline{\Omega} \subset \Omega$, $\widetilde{\boldsymbol{v}}_\theta \in \mathbf{X}_{r,p}(\mathbb{R}^N)$ and:

$$\lim_{\theta\to 1} \widetilde{\boldsymbol{v}}_\theta = \widetilde{\boldsymbol{v}} \quad \text{in } \mathbf{X}_{r,p}(\mathbb{R}^N).$$

Therefore, for $\varepsilon > 0$ small enough and the mollifiers $\{\rho_\varepsilon\}$, we have that $\rho_\varepsilon \star \widetilde{\boldsymbol{v}}_\theta \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\lim\limits_{\varepsilon\to 0}\lim\limits_{\theta\to 1} \rho_\varepsilon \star \widetilde{\boldsymbol{v}}_\theta = \widetilde{\boldsymbol{v}}$ in $\mathbf{X}_{r,p}(\Omega)$. Consequently, $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$.

In the case where Ω is only Lipschitz, we have to recover Ω by a finite number of star open sets and partitions of unity.

Finally, the relation (17) is a simple consequence of this density. □

The following lemma and its proof are classical.

Lemma 9. *Let $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$. Then, there exist $\mathbb{F}_0 = (f_{ij})_{1\leqq i,j\leqq 3}$ such that $\mathbb{F}_0 \in \mathbb{L}^{r'}(\Omega)$, $f_1 \in W^{-1,p'}(\Omega)$ and satisfying:*

$$\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1. \tag{18}$$

Moreover,

$$\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1 \leqq i, j \leqq 3, \|f_1\|_{W^{-1,p'}(\Omega)}\}.$$

Conversely, if $\boldsymbol{f}$ satisfies (18), then $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$.

Proof. Let $\mathbf{E} = \mathbb{L}^r(\Omega) \times W_0^{1,p}(\Omega)$ be a space with topology given by the norm:

$$\forall \boldsymbol{h} = (\mathbb{H}_0, h_1) \in \mathbf{E}, \quad \|\boldsymbol{h}\|_{\mathbf{E}} = \sum_{1\leqq i,j\leqq 3} \|h_{ij}\|_{L^r(\Omega)} + \|h_1\|_{W_0^{1,p}(\Omega)},$$

where $\mathbb{H}_0 = (h_{ij})_{1\leqq i,j\leqq 3}$. The application $T : \boldsymbol{\varphi} \in \mathbf{X}_{r,p}(\Omega) \mapsto (\nabla\boldsymbol{\varphi}, \nabla\cdot\boldsymbol{\varphi}) \in \mathbf{E}$ is an isometry from $\mathbf{X}_{r,p}(\Omega)$ into $\mathbf{E}$. Thus, if we suppose $\mathbf{G} = T(\mathbf{X}_{r,p}(\Omega))$ with the $\mathbf{E}$-topology, and consider $S = T^{-1} : \mathbf{G} \mapsto \mathbf{X}_{r,p}(\Omega)$, for any $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$ we can define a linear continuous form on $\mathbf{G}$ as follows:

$$\boldsymbol{h} \in \mathbf{G} \mapsto \langle \boldsymbol{f}, S\boldsymbol{h}\rangle_{[\mathbf{X}_{r,p}(\Omega)]'\times\mathbf{X}_{r,p}(\Omega)}.$$

Thanks to the Hahn–Banach Theorem, such application can be extended to a linear continuous form on $\mathbf{E}$, denoted by $\boldsymbol{\Pi}$ such that $\|\boldsymbol{\Pi}\|_{\mathbf{E}'} = \|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'}$.

Moreover, by the characterization of the dual spaces for $\mathbf{W}_0^{1,r}(\Omega)$ and $W_0^{1,p}(\Omega)$, there exist $\mathbb{F}_0 = (f_{ij})_{1\leqq i,j\leqq 3} \in \mathbb{L}^{r'}(\Omega)$ and $f_1 \in W^{-1,p'}(\Omega)$ such that for any $\boldsymbol{h} = (\mathbb{H}_0, h_1) \in \mathbf{E}$,

$$\langle \boldsymbol{\Pi}, \boldsymbol{h}\rangle_{\mathbf{E}'\times\mathbf{E}} = \sum_{i,j=1}^{3} \langle f_{ij}, h_{ij}\rangle_{L^{r'}(\Omega)\times L^r(\Omega)} + \langle f_1, h_1\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)},$$

with $\|\boldsymbol{\Pi}\|_{\mathbf{E}'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1 \leqq i, j \leqq 3, \|f_1\|_{W^{-1,r'}(\Omega)}\}$. In particular, if $\boldsymbol{h} = T\boldsymbol{\varphi} \in \mathbf{G}$, where $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, we have:

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r,p}(\Omega)]' \times \mathbf{X}_{r,p}(\Omega)} = \langle -\nabla \cdot \mathbb{F}_0 - \nabla f_1, \boldsymbol{\varphi}\rangle.$$

To finish, it is easy to verify that the reciprocal holds. □

As a consequence of Lemma 8, we have the following embeddings:

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{19}$$

where the second embedding holds if $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$.

Giving a meaning to the trace of a very weak solution of a Stokes, Oseen or Navier–Stokes problem is not trivial. Remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$$

that can also be described (see [5]) as:

$$\mathbf{Y}_{p'}(\Omega) = \left\{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\right\}. \tag{20}$$

Observe that the range space of the normal derivative $\gamma_1 : \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is:

$$\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z} \cdot \boldsymbol{n} = 0\}.$$

We also introduce the space

$$\mathbf{H}_{p,r}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} \in L^r(\Omega)\},$$

which is equipped with the graph norm. The following lemmas will help us to prove a trace result (the space $\mathbf{T}_{p,r}(\Omega)$ is defined by (4)):

Lemma 10. (i) *The space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *is dense in* $\mathbf{T}_{p,r}(\Omega)$.
(ii) *The space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *is dense in* $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$.

Proof. (i) Let $\boldsymbol{\ell} \in (\mathbf{T}_{p,r}(\Omega))'$ be such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = 0$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$. Using Riesz's Representation Lemma, there exists $(\boldsymbol{u}_{\boldsymbol{\ell}}, \boldsymbol{z}_{\boldsymbol{\ell}}) \in \mathbf{L}^{p'}(\Omega) \times \mathbf{X}_{r',p'}(\Omega)$ such that: for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$,

$$\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{u}_{\boldsymbol{\ell}} \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{z}_{\boldsymbol{\ell}}, \Delta \boldsymbol{v}\rangle_{\mathbf{X}_{r',p'}(\Omega) \times [\mathbf{X}_{r',p'}(\Omega)]'}$$

Observe that we can easily extend by zero the functions $\boldsymbol{u}_{\boldsymbol{\ell}}$ and $\boldsymbol{z}_{\boldsymbol{\ell}}$, in such a way that

$$\widetilde{\boldsymbol{u}}_{\boldsymbol{\ell}} \in \mathbf{L}^{p'}(\mathbb{R}^3) \quad \text{and} \quad \widetilde{\boldsymbol{z}}_{\boldsymbol{\ell}} \in \mathbf{X}_{r',p'}(\mathbb{R}^3),$$

that is, $\widetilde{z}_{\ell} \in \mathbf{W}^{1,r'}(\mathbb{R}^3)$ and $\nabla \cdot \widetilde{z}_{\ell} \in W^{1,p'}(\mathbb{R}^3)$. Now, we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ and we observe that $\Delta\boldsymbol{\varphi} \in (\mathbf{X}_{r',p'}(\Omega))'$. Then, we have by definition that:

$$\int_{\mathbb{R}^N} \widetilde{\boldsymbol{u}}_{\ell} \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^N} \widetilde{z}_{\ell} \cdot \Delta\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{u}_{\ell} \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \int_{\Omega} z_{\ell} \cdot \Delta\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle = 0.$$

As $\int_{\mathbb{R}^N} \widetilde{z}_{\ell} \cdot \Delta\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \Delta\widetilde{z}_{\ell}, \boldsymbol{\varphi} \rangle$, therefore:

$$\int_{\mathbb{R}^N} \widetilde{\boldsymbol{u}}_{\ell} \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^N} \widetilde{z}_{\ell} \cdot \Delta\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \widetilde{\boldsymbol{u}}_{\ell} + \Delta\widetilde{z}_{\ell}, \boldsymbol{\varphi} \rangle,$$

and thus $\widetilde{\boldsymbol{u}}_{\ell} + \Delta\widetilde{z}_{\ell} = 0$ in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$. Using that $\widetilde{z}_{\ell} \in \mathbf{W}^{1,r'}(\mathbb{R}^3)$ and $\Delta\widetilde{z}_{\ell} = -\widetilde{\boldsymbol{u}}_{\ell} \in \mathbf{L}^{p'}(\mathbb{R}^3)$, we conclude that $\widetilde{z}_{\ell} \in \mathbf{W}^{2,p'}(\mathbb{R}^3)$ and therefore $z_{\ell} \in \mathbf{W}^{2,p'}(\Omega)$ and $\frac{\partial z_{\ell}}{\partial \boldsymbol{n}} = \mathbf{0}$ on Γ. Thus, $z_{\ell} \in \mathbf{W}_0^{2,p'}(\Omega)$. As $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{W}_0^{2,p'}(\Omega)$, there exists a sequence $\{z_{\ell}^k\}_k \subset \boldsymbol{\mathcal{D}}(\Omega)$ such that $z_{\ell}^k \to z_{\ell}$ in $\mathbf{W}^{2,p'}(\Omega)$, when $k \to +\infty$. In particular, $z_{\ell}^k \to z_{\ell}$ in $\mathbf{X}_{r',p'}(\Omega)$.

To finish, we consider $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and we have to prove that $\boldsymbol{\ell} = \mathbf{0}$. Observe that:

$$\begin{aligned}
\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle &= -\int_{\Omega} \Delta z_{\ell} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \langle \Delta\boldsymbol{v}, z_{\ell} \rangle_{(\mathbf{X}_{r',p'}(\Omega))' \times \mathbf{X}_{r',p'}(\Omega)} \\
&= \lim_{k\to+\infty} \left(-\int_{\Omega} \boldsymbol{v} \cdot \Delta z_{\ell}^k\, \mathrm{d}\boldsymbol{x} + \langle \Delta\boldsymbol{v}, z_{\ell}^k \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} \right) \\
&= \lim_{k\to+\infty} \left(-\int_{\Omega} \boldsymbol{v} \cdot \Delta z_{\ell}^k\, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \boldsymbol{v} \cdot \Delta z_{\ell}^k\, \mathrm{d}\boldsymbol{x} \right) = 0
\end{aligned}$$

Therefore, $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$.
(ii) Let $\boldsymbol{\ell} \in (\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div}; \Omega))'$ be such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = 0$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$. Using the Riesz's Representation Lemma again, there exists $(\boldsymbol{u}_{\ell}, \pi_{\ell}, z_{\ell}) \in \mathbf{L}^{p'}(\Omega) \times L^{r'}(\Omega) \times \mathbf{X}_{r',p'}(\Omega)$ such that: for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div}; \Omega)$,

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = \int_{\Omega} (\boldsymbol{u}_{\ell} \cdot \boldsymbol{v} + \pi_{\ell} \nabla \cdot \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + \langle z_{\ell}, \Delta\boldsymbol{v} \rangle_{\mathbf{X}_{r',p'}(\Omega) \times [\mathbf{X}_{r',p'}(\Omega)]'}$$

As above, we prove first that $-\Delta\widetilde{z}_{\ell} + \nabla\widetilde{\pi}_{\ell} = \widetilde{\boldsymbol{u}}_{\ell} \in \mathbf{L}^{p'}(\mathbb{R}^3)$ and then $z_{\ell} \in \mathbf{W}_0^{2,p'}(\Omega)$ and $\pi_{\ell} \in W_0^{1,p'}(\Omega)$. The rest of the proof is unchanged. □

Lemma 11. *The space* $\boldsymbol{\mathcal{D}}_{\sigma}(\overline{\Omega})$ *is dense in* $\mathbf{T}_{p,r,\sigma}(\Omega)$.

Proof. The proof can be made in a manner similar to that of Lemma 7. □

The following two lemmas prove that the tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belong to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is:

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}. \tag{21}$$

First, we recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_{\tau}$, and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_{\tau} + (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}$.

Lemma 12. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}.$$

Proof. We start with the expression: let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, then

$$\left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} - \langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} \tag{22}$$

which is valid for any $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Observe that $\mathbf{Y}_{p'}(\Omega) \subset \mathbf{X}_{r',p'}(\Omega)$ because $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and the normal trace of the functions of $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$ belongs to the space $\mathbf{Z}_{p'}(\Gamma)$.

Let $\boldsymbol{\mu} \in \mathbf{W}^{1/p,p'}(\Gamma)$. Then, $\boldsymbol{\mu} = \boldsymbol{\mu}_\tau + (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}$. Since Ω is of class $\mathcal{C}^{1,1}$, we know that there exists $\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{0}$ and $\dfrac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} = \boldsymbol{\mu}_\tau$ on Γ and verifying:

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} \leqq C \, \|\boldsymbol{\mu}_\tau\|_{\mathbf{W}^{1/p,p'}(\Gamma)} \leqq C \, \|\boldsymbol{\mu}\|_{\mathbf{W}^{1/p,p'}(\Gamma)}.$$

Moreover, $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Therefore, we can bound the boundary term as follows for such functions $\boldsymbol{\psi}$:

$$\begin{aligned}
\left| \langle \boldsymbol{v}_\tau, \boldsymbol{\mu} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} \right| &= \left| \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} \right| \\
&\leqq \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} \\
&\quad + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} \|\boldsymbol{\psi}\|_{\mathbf{X}_{r',p'}(\Omega)} \\
&\leqq C \, \|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{Y}_{p'}(\Omega)}
\end{aligned}$$

Thus,

$$\|\boldsymbol{v}_\tau\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leqq C \, \|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)}.$$

Therefore, the linear continuous mapping $\boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\mathbf{T}_{p,r}(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$, then we can extend this mapping from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, that is, the tangential trace of functions of $\mathbf{T}_{p,r}(\Omega)$ belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. □

Lemma 13. (i) *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}_{p,r}(\mathrm{div}; \Omega)$.*

(ii) *Let* $1 < p < \infty$ *and* $r > 1$ *be such that* $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$. *The mapping* $\gamma_{\boldsymbol{n}} : \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ *on the space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping, still denoted by* $\gamma_{\boldsymbol{n}}$*, from* $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$ *into* $W^{-1/p,p}(\Gamma)$*, and we have the Green formula: for any* $\boldsymbol{v} \in \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ *and* $\varphi \in W^{1,p'}(\Omega)$*,*

$$\int_\Omega \boldsymbol{v} \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \varphi \, \mathrm{div}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} \, .$$

Proof. The proof of the first point is similar to Lemma 10 and the second point is a consequence of point (i). □

4.2. Large class of solutions for the Stokes equations

We recall the definition and the existence result of very weak solutions for the Stokes problem.

Definition 1. (*Very weak solution for the Stokes problem*) We say that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a *very weak solution* of (S) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned} -\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} &= \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla\pi \, \mathrm{d}\boldsymbol{x} &= -\int_\Omega h \, \pi \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma, \end{aligned} \tag{23}$$

where the dualities on Ω and Γ are defined by:

$$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)}, \ \langle \cdot, \cdot \rangle_\Gamma = \langle \cdot, \cdot \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}. \tag{24}$$

Note that $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ and $\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$, which means that all the brackets and integrals have a sense.

Proposition 1. *Suppose that* $\boldsymbol{f}, \ h, \ \boldsymbol{g}$ *satisfy* (16)*. Then the following two statements are equivalent:*

(i) $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *is a very weak solution of (S),*
(ii) $(\boldsymbol{u}, q)$ *satisfies the system (S) in the sense of distributions.*

Proof. (i) Let $(\boldsymbol{u}, q)$ be a very weak solution to problem (S). It is clear that $-\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u} = h$ in Ω and consequently $\boldsymbol{u}$ belongs to $\mathbf{T}_{p,r}(\Omega)$. Using Lemma 13 point (ii), Lemma 12 and (17), we obtain

$$\begin{aligned} -\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \left\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} \\ -\langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega \, . \end{aligned}$$

Since for any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$,

$$\left\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)},$$

we deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\mathbf{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla \cdot \boldsymbol{u} = h$, we deduce that for any $\pi \in W^{1,p'}(\Omega)$, we have

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \pi \rangle_\Gamma = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma .$$

Consequently $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u} = \boldsymbol{g}$ on Γ.
(ii) The converse is a simple consequence of Lemma 13 point (ii), Lemma 12 and (17). □

Observe that the following result is a variation of Proposition 4.11 in [5], which was made for $\boldsymbol{f} = \mathbf{0}$ and $h = 0$. Here, we focus on the fact that taking $\boldsymbol{f} \neq \mathbf{0}$ and $h \neq 0$ makes over the whole proof appearing there. In the case $r = p$, we have:

Proposition 2. *Let*

$$\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))', \quad h \in L^p(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$$

and satisfying the compatibility condition (5). *Then, the Stokes problem* (S) *has exactly one solution* $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ *and* $q \in W^{-1,p}(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leqq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}. \tag{25}$$

Moreover $\boldsymbol{u} \in \mathbf{T}_p(\Omega)$ *and*

$$\|\boldsymbol{u}\|_{\mathbf{T}_p(\Omega)} \leqq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}.$$

Proof. In [5], the proof of Proposition 2 is made for $\boldsymbol{f} = \mathbf{0}$ and $h = 0$. Below, we focus on the aspects of the proof given in [5] in which $\boldsymbol{f}$ and h take part.
(i) First step: We suppose that $\mathbf{g} \cdot \boldsymbol{n} = 0$ on Γ and $\int_\Omega h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0$.
It remains to consider the following equivalent problem:

Find $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega), \forall \pi \in W^{1,p'}(\Omega)$

$$\begin{aligned} &\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \pi)\, \mathrm{d}\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{w} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ &\quad = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{p'}(\Omega)]' \times \mathbf{X}_{p'}(\Omega)} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma - \int_\Omega h\, \pi\, \mathrm{d}\boldsymbol{x} \end{aligned}$$

being $\mathbf{Y}_{p'}(\Omega)$ the space defined by (20) that verifies the embedding $\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{p'}(\Omega)$. The duality brackets are given in (24).

We can prove (as in [5]) that for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$, we have:

$$\begin{aligned} &\left| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{p'}(\Omega)]' \times \mathbf{X}_{p'}(\Omega)} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma - \int_\Omega h\, \pi\, \mathrm{d}\boldsymbol{x} \right| \\ &\quad \leqq C \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|h\|_{L^p(\Omega)} \right) \left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right) \end{aligned}$$

being $(\boldsymbol{w}, \pi) \in \mathbf{Y}_{p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ the unique solution of the Stokes (dual) problem:

$$-\Delta \boldsymbol{w} + \nabla \pi = \mathbf{F} \ \text{ and } \ \nabla \cdot \boldsymbol{w} = \varphi \ \text{ in } \Omega, \ \boldsymbol{w} = \mathbf{0} \ \text{ on } \Gamma.$$

Note that for any $k \in \mathbb{R}$,

$$\left| \int_\Omega h \, \pi \, \mathrm{d}\boldsymbol{x} \right| = \left| \int_\Omega h \, (\pi + k) \, \mathrm{d}\boldsymbol{x} \right| \leqq \|h\|_{L^p(\Omega)} \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} \tag{26}$$

and

$$\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leqq C \left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right).$$

From this bound, we deduce that the mapping

$$(\mathbf{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma - \int_\Omega h \, \pi \, \mathrm{d}\boldsymbol{x}$$

defines an element of the dual space of $\mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$ with norm bounded by $C(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)})$.

From Riesz' Representation Theorem we deduce that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of (S) satisfying the bound (25).

(ii) Second step: Now, we suppose that $\int_\Omega h(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \langle \mathbf{g} \cdot \boldsymbol{n}, 1 \rangle_\Gamma$ and consider the Neumann problem: Find $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ such that:

$$(N) \quad \Delta \theta = h \ \text{ in } \Omega, \ \ \frac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{g} \cdot \boldsymbol{n} \ \text{ on } \Gamma,$$

which has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ and verifies the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqq C \left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g} \cdot \boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right). \tag{27}$$

Set $\boldsymbol{u}_0 = \nabla \theta$. By step (i), there exists a unique $(z, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem:

$$-\Delta z + \nabla q = \boldsymbol{f} + \nabla h \ \text{ and } \ \nabla \cdot z = 0 \ \text{ in } \Omega, \quad z = \boldsymbol{g} - \boldsymbol{u}_0|_\Gamma \ \text{ on } \Gamma,$$

where the characterization given by Lemma 9 implies that $\nabla h \in (\mathbf{X}_{p'}(\Omega))'$ and $\boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ satisfies the hypothesis of Step (i). Finally, the pair of functions $(\boldsymbol{u}, q) = (z + \boldsymbol{u}_0, q)$ is the required solution. □

More generally, we have:

Theorem 11. *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy* (16) *and* (5). *Then, the Stokes problem* (S) *has exactly one solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\} \end{aligned} \tag{28}$$

Moreover, $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ *and*

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leqq C \left\{\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right\}.$$

In particular, if $\boldsymbol{f} \in \mathbf{W}^{-1,r_0}(\Omega)$ *and* $h \in L^{r_0}(\Omega)$ *with* $r_0 = 3p/(3+p)$*, then* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ *with the corresponding estimates.*

Proof. If we want to use hypotheses $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$ instead of $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$ and $h \in L^r(\Omega)$ instead of $h \in L^p(\Omega)$, appearing in Definition 1 and Proposition 2, then the differences in the proof are linked to:

– Instead of $\langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega$, we have:

$$\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]'\times \mathbf{X}_{r',p'}(\Omega)} \qquad \text{for } \boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega).$$

Observe that $\mathbf{Y}_{p'}(\Omega) \subset \mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$, which is the case defined in Lemma 12. Therefore, the same study can be made, only replacing the bound $\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'}$ by $\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$.

– Now, we solve problem (N) with $h \in L^r(\Omega)$. Problem (N) is equivalent to the problem of finding $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ such that:

$$\forall \varphi \in W^{1,p'}(\Omega), \qquad \int_\Omega \nabla\theta \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, \varphi\rangle_\Gamma - \int_\Omega h\,\varphi\,\mathrm{d}\boldsymbol{x}$$

which is well defined for any $\varphi \in W^{1,p'}(\Omega)$ (observe that $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ if $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$).
The mapping $\ell : \varphi \mapsto \langle \boldsymbol{g}\cdot\boldsymbol{n}, \varphi\rangle_\Gamma - \int_\Omega h\,\varphi\,\mathrm{d}\boldsymbol{x}$ defines an element of the dual $(W^{1,p'}(\Omega)/\mathbb{R})'$ because $\langle \ell, 1\rangle = 0$. Furthermore, the following inf–sup condition is verified:

$$\inf_{\substack{\dot\varphi \in W^{1,p'}(\Omega)/\mathbb{R}\\ \dot\varphi \neq 0}} \sup_{\substack{\dot\theta \in W^{1,p}(\Omega)/\mathbb{R}\\ \dot\theta \neq 0}} \frac{\int_\Omega \nabla\dot\theta\cdot\nabla\dot\varphi\,\mathrm{d}\boldsymbol{x}}{\|\nabla\dot\theta\|_{\mathbf{L}^p(\Omega)}\|\nabla\dot\varphi\|_{\mathbf{L}^{p'}(\Omega)}} > 0.$$

Therefore, the problem (N) has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ and satisfies the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqq C \left(\|\boldsymbol{g}\cdot\boldsymbol{n}\|_{W^{-1/p,p}(\Gamma)} + \|h\|_{L^r(\Omega)}\right)$$

□

Remark 7. Observe that in [21] Theorem 3, the domain considered is of class $\mathcal{C}^{2,1}$ instead of class $\mathcal{C}^{1,1}$, and the divergence term $h \in L^p(\Omega)$ instead of $h \in L^r(\Omega)$. The regularity considered for $\boldsymbol{f}$, taking into account Lemma 9, is the same as we consider here ($\boldsymbol{f}$ is the divergence of a tensor in $\mathbb{L}^r(\Omega)$ because the gradient part can be associated to the pressure). But for the divergence condition h, Galdi et al. consider $h \in L^p(\Omega)$, which is a space smaller than that considered in this work ($h \in L^r(\Omega)$ for $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$). Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$ which has been clearly characterized, contrary to the space $\widehat{\mathbf{W}}^{1,p}(\Omega)$ appearing in [21] which is not characterized, is completely abstract and is obtained as the closure of $\mathbf{W}^{1,p}(\Omega)$ for the norm

$$\|\boldsymbol{u}\|_{\widehat{\mathbf{W}}^{1,p}(\Omega)} = \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|A_r^{-1/2}\mathcal{P}_r\Delta\boldsymbol{u}\|_{\mathbf{L}^r(\Omega)},$$

where A_r is the Stokes operator with domain equal to $\mathbf{W}^{2,p}(\Omega)\cap\mathbf{W}_0^{1,p}(\Omega)\cap\mathbf{L}^p_\sigma(\Omega)$ and $\mathcal{P}_r$ is the Helmholtz projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}^r_\sigma(\Omega)$.

Corollary 3. *Let f, h, $\boldsymbol{g}$ satisfy (5) and $\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1$ with $\mathbb{F}_0 \in \mathbb{L}^r(\Omega)$, $f_1 \in W^{-1,p}(\Omega)$, $h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$. Then the solution $\boldsymbol{u}$ given by Theorem 11 belongs to $\mathbf{W}^{1,r}(\Omega)$. If moreover $f_1 \in L^r(\Omega)$, then the solution q given by Theorem 11 belongs to $L^r(\Omega)$. In both cases, we have the corresponding estimates.*

Proof. Let $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ be the solution given by Theorem 11. Then

$$-\Delta\boldsymbol{u} + \nabla(q - f_1) = \nabla\cdot\mathbb{F}_0 \ \text{ and } \ \nabla\cdot\boldsymbol{u} = h \ \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \ \text{ on } \Gamma.$$

By Theorem 10 point (i), we deduce that $(\boldsymbol{u}, q - f_1) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$. Note that if $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$, we have the following embeddings:

$$W^{1,r}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{and} \quad W^{1-1/r,r}(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma).$$

□

Remark 8. It is clear that

$$\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{T}_{p,r}(\Omega) \quad \text{when} \quad \frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3},$$

that is, $\mathbf{T}_{p,r}(\Omega)$ is an intermediate space between $\mathbf{W}^{1,r}(\Omega)$ and $\mathbf{L}^p(\Omega)$.

Remark 9. (i) First, we have as a consequence of Proposition 2 the following Helmholtz decomposition: for any $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$, there exist $\boldsymbol{\psi} \in \mathbf{W}^{-1,p}(\Omega)$ and $q \in W^{-1,p}(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\psi} + \nabla q, \ \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \Omega.$$

(ii) In the same way, suppose that $\boldsymbol{f} = \nabla\cdot\mathbb{F}$ with $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ verifying the compatibility condition (5). Then, the solution $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ given by Theorem 11 satisfies $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ with the appropriate estimate.

Corollary 4. *Let h and $\boldsymbol{g}$ satisfy:*

$$h \in L^r(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \quad \int_\Omega h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_\Gamma,$$

with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and $r \leqq p$. Then, there exists at least one solution $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ verifying

$$\nabla\cdot\boldsymbol{u} = h \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma.$$

Moreover, there exists a constant $C = C(\Omega, p, r)$ such that:

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leqq C \left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right).$$

The following corollary gives Stokes solutions $(\boldsymbol{u}, q)$ in fractionary Sobolev spaces of type $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$, with $0 < \sigma < 2$.

Corollary 5. *Let s be a real number such that $0 \leqq s \leqq 1$.*

(i) *Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfy the compatibility condition (5) with*

$$\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega), \quad f_1 \in W^{s-1,p}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma), \quad h \in W^{s,r}(\Omega),$$

with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and $r \leqq p$. Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u}, q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C\,(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)})\end{aligned}$$

(ii) *Assume that*

$$\boldsymbol{f} \in \mathbf{W}^{s-1,p}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{s+1-1/p,p}(\Gamma), \quad h \in W^{s,p}(\Omega),$$

with the compatibility condition (5). Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u}, q) \in \mathbf{W}^{s+1,p}(\Omega) \times W^{s,p}(\Omega)/\mathbb{R}$ with

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{s+1,p}(\Omega)} + \|q\|_{W^{s,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{s-1,p}(\Omega)} + \|h\|_{W^{s,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s+1-1/p,p}(\Gamma)})\end{aligned}$$

Proof. It suffices to use an interpolation argument. Indeed, if

$$\begin{aligned}&\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1 \text{ with } \mathbb{F}_0 \in \mathbb{L}^r(\Omega), \\ &f_1 \in W^{-1,p}(\Omega), \quad h \in L^r(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),\end{aligned}$$

by Theorem 11 there exists a unique solution $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ of (S) satisfying the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C\,(\|\mathbb{F}_0\|_{\mathbb{L}^r(\Omega)} + \|f_1\|_{W^{-1,p}(\Omega)} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}).\end{aligned}$$

If now

$$\begin{aligned}&\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1 \text{ with } \mathbb{F}_0 \in \mathbb{W}^{1,r}(\Omega), \\ &f_1 \in L^p(\Omega), \quad h \in W^{1,r}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma),\end{aligned}$$

by Theorem 10 point (i) there exists a unique solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ of (S) satisfying the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \\ &\quad \leqq C\,(\|\mathbb{F}_0\|_{\mathbb{W}^{1,r}(\Omega)} + \|f_1\|_{L^p(\Omega)} + \|h\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}).\end{aligned}$$

Note that $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$. The result is then a consequence of the following interpolation spaces:

$$[W^{1,r}(\Omega),\ L^r(\Omega)]_{1-s} = W^{s,r}(\Omega), \quad [L^p(\Omega),\ W^{-1,p}(\Omega)]_{1-s} \hookleftarrow W^{s-1,p}(\Omega)$$

and

$$[W^{1-1/p,p}(\Gamma),\ W^{-1/p,p}(\Gamma)]_{1-s} = W^{s-1/p,p}(\Gamma),$$
$$[W^{1,p}(\Omega),\ L^p(\Omega)]_{1-s} = W^{s,p}(\Omega).$$

The proof of point (ii) is similarly obtained by using Theorem 10 point (ii) and an interpolation argument. □

Remark 10. We can reformulate point (ii) as follows. For any

$$\boldsymbol{f} \in \mathbf{W}^{-s,p'}(\Omega),\ h \in W^{-s+1,p'}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{2-s-1/p',p'}(\Gamma),$$

with $0 \leqq s \leqq 1$, then problem (S) has a unique solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{2-s,p'}(\Omega) \times W^{1-s,p'}(\Omega)/\mathbb{R}$.

The following theorem gives solutions for external forces $\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega)$ and divergence condition $h \in W^{s-1,p}(\Omega)$ with $1/p < s < 2$. If $p = 2$, we can obtain solutions in $\mathbf{H}^{1/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$, $0 < \varepsilon \leqq 3/2$.

Theorem 12. *Let s be a real number such that $\frac{1}{p} < s \leqq 2$. Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the compatibility condition (5) with*

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega),\ h \in W^{s-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma).$$

Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}} + \|q\|_{W^{s-1,p}/\mathbb{R}} \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}) \tag{29}$$

Proof. The theorem is proved by Corollary 5 point (ii) if $1 \leqq s \leqq 2$. Using Theorem 1, we can suppose $\boldsymbol{g} = \mathbf{0}$. Let s, then, be a real number such that $\frac{1}{p} < s < 1$. It remains to consider the following equivalent problem:

Find $(\boldsymbol{u},\ q) \in \mathbf{W}_0^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{W}_0^{-s+2,p'}(\Omega)$, $\forall \pi \in W^{-s+1,p'}(\Omega)$

$$\langle \boldsymbol{u},\ -\Delta \boldsymbol{w} + \nabla \pi \rangle_{\mathbf{W}_0^{s,p}(\Omega) \times \mathbf{W}^{-s,p'}(\Omega)} - \langle q,\ \nabla \cdot \boldsymbol{w} \rangle_{W^{s-1,p}(\Omega) \times W_0^{-s+1,p'}(\Omega)}$$
$$= \langle \boldsymbol{f},\ \boldsymbol{w} \rangle_{\mathbf{W}^{s-2,p}(\Omega) \times \mathbf{W}_0^{-s+2,p'}(\Omega)} - \langle h,\ \pi \rangle_{W^{s-1,p}(\Omega) \times W_0^{-s+1,p'}(\Omega)}.$$

Note that $W_0^{-s+1,p'}(\Omega) = W^{-s+1,p'}(\Omega)$ because $-s + 1 < 1/p'$.

As in the proof of Proposition 2, for any $\mathbf{H} \in \mathbf{W}^{-s,p'}(\Omega)$ and $\varphi \in W^{-s+1,p'}(\Omega)$, we have:

$$
\begin{aligned}
&\left| -\langle \boldsymbol{f},\ \boldsymbol{w}\rangle_{\mathbf{W}^{s-2,p}(\Omega)\times \mathbf{W}_0^{-s+2,p'}(\Omega)} + \langle h,\ \pi\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)} \right| \\
&\quad \leqq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}(\Omega)}\right)\left(\|\mathbf{H}\|_{\mathbf{W}^{-s,p'}(\Omega)} + \|\varphi\|_{W^{-s+1,p'}(\Omega)}\right)
\end{aligned}
$$

being $(\boldsymbol{w}, \pi) \in \mathbf{W}_0^{-s+2,p'}(\Omega) \times W_0^{-s+1,p'}(\Omega)/\mathbb{R}$, the unique solution of the Stokes problem given by Corollary 5 point (ii) (see also Remark 10):

$$
-\Delta \boldsymbol{w} + \nabla \pi = \mathbf{H} \ \text{ and } \ \nabla \cdot \boldsymbol{w} = \varphi \ \text{ in } \Omega, \ \boldsymbol{w} = \mathbf{0} \ \text{ on } \Gamma.
$$

Note that $\mathbb{R} \subset W^{-s+1,p'}(\Omega)$ and for any $k \in \mathbb{R}$,

$$
\begin{aligned}
\left|\langle h,\ \pi\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)}\right| &= \left|\langle h,\ \pi + k\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)}\right| \\
&\leqq C\|h\|_{W^{s-1,p}(\Omega)}\|\pi\|_{W^{-s+1,p'}(\Omega)/\mathbb{R}}
\end{aligned}
$$

and

$$
\|\boldsymbol{w}\|_{\mathbf{W}_0^{-s+2,p'}(\Omega)} + \|\pi\|_{W^{-s+1,p'}(\Omega)/\mathbb{R}} \leqq C\left(\|\mathbf{H}\|_{\mathbf{W}^{-s,p'}(\Omega)} + \|\varphi\|_{W^{-s+1,p'}(\Omega)}\right).
$$

From this bound, we deduce that the mapping

$$
(\mathbf{H},\ \varphi) \to -\langle \boldsymbol{f},\ \boldsymbol{w}\rangle_{\mathbf{W}^{s-2,p}(\Omega)\times \mathbf{W}_0^{-s+2,p'}(\Omega)} + \langle h,\ \pi\rangle_{W^{s-1,p}(\Omega)\times W_0^{-s+1,p'}(\Omega)}
$$

defines an element of the dual space of $\mathbf{W}^{-s,p'}(\Omega) \times W^{-s+1,p'}(\Omega)$ with norm bounded by $C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}(\Omega)}\right)$. From Riesz' Representation Theorem we deduce that there exists a unique $(\boldsymbol{u},\ q) \in \mathbf{W}_0^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ solution of (S) and satisfying the bound (29). □

Remark 11. (i) Remark 4 point (ii) and (iii) holds.

(ii) If $n = 2$, Ω is a convex polygon, with $\Gamma = \cup\Gamma_i$, Γ_i linear segments, $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in H^s(\Gamma_i)$, for $i = 1, \ldots, I_0$ and $-1/2 < s < 1/2$, then $\boldsymbol{u} \in \mathbf{H}^r(\Omega)$ for any $r < s + 1/2$ and $q \in H^{s-1/2}(\Omega)$ (see [32]).

(iii) When Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geqq 3$, $\boldsymbol{f} = \mathbf{0}$, $h = 0$, $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ (respectively $\boldsymbol{g} \in \mathbf{W}^{1,2}(\Gamma)$) , with $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$, then $\boldsymbol{u} \in \mathbf{H}^{1/2}(\Omega)$ (respectively $\boldsymbol{u} \in \mathbf{H}^{3/2}(\Omega)$ and $q \in H^{-1/2}(\Omega)$ (respectively $q \in H^{1/2}(\Omega)$) (see Fabes et al. [16]). If $\boldsymbol{g} \in \mathbf{L}^p(\Gamma)$, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $2 - \varepsilon \leqq p \leqq 2 + \varepsilon$, then $\boldsymbol{u} \in \mathbf{W}^{1-1/p}(\Omega)$ and $q \in W^{-1/p}(\Omega)$. For a similar result, when $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ and Ω is a simply connected domain of $\mathbb{R}^2$, we can refer the reader to [12].

(iv) When Ω is only a bounded Lipschitz domain, with connected boundary, the same result has been proved in [38] with $\boldsymbol{f} = \mathbf{0}$ and $h = 0$ for any $p \geqq 2$.

5. The Oseen problem

We want to study the existence of generalized, strong and very weak solutions for problem (O) presented in Section 2.

5.1. Existence of solution in $\mathbf{H}^1(\Omega) \times L^2(\Omega)$

First, we will study the existence of a solution for (O):

Theorem 13. (Existence of a solution for (O)) *Let Ω be a Lipschitz bounded domain. Let*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$$

verify the compatibility condition (5) *for* $p = 2$*. Then, problem* (O) *has a unique solution* $(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$*. Moreover, there exist some constants* $C_1 > 0$ *and* $C_2 > 0$ *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leqq C_1 \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right), \tag{30}$$

$$\|q\|_{L^2(\Omega)/\mathbb{R}} \leqq C_2 \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right) \tag{31}$$

where $C_1 = C(\Omega)$ *and* $C_2 = C_1 \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$.

Proof. In order to prove the existence of a solution, first (using Lemma 3.3 in [5], for instance) we lift the boundary and the divergence data. Then, there exists $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \boldsymbol{u}_0 = h$ in Ω, $\boldsymbol{u}_0 = \boldsymbol{g}$ on Γ and:

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leqq C \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right). \tag{32}$$

Therefore, it remains to find $(\boldsymbol{z}, q) = (\boldsymbol{u} - \boldsymbol{u}_0, q)$ in $\mathbf{H}^1_0(\Omega) \times L^2(\Omega)$ such that:

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla q = \widetilde{\boldsymbol{f}} \ \text{ and } \ \nabla \cdot \boldsymbol{z} = 0 \ \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{0} \ \text{ on } \Gamma.$$

being $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 - (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_0$. Observe that $\widetilde{\boldsymbol{f}} \in \mathbf{H}^{-1}(\Omega)$. Since the space $\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}$ is dense in the space $\mathbf{V} = \{\boldsymbol{z} \in \mathbf{H}^1_0(\Omega);\ \nabla \cdot \boldsymbol{z} = 0\}$, the previous problem is equivalent to finding $\boldsymbol{z} \in \mathbf{V}$ such that:

$$\forall \boldsymbol{\varphi} \in \mathbf{V}, \qquad \int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)},$$

where b is a trilinear antisymmetric form with respect to the last two variables, well-defined for $\boldsymbol{v} \in \mathbf{L}^3(\Omega), \boldsymbol{z}, \boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega)$. (We can recover the pressure π thanks to the De Rham's Lemma 6). By Lax-Milgram's Theorem we can deduce the existence of a unique $\boldsymbol{z} \in \mathbf{H}^1_0(\Omega)$ verifying:

$$\begin{aligned}
\|\boldsymbol{z}\|_{\mathbf{H}^1(\Omega)} &\leqq C(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}} + \|\Delta \boldsymbol{u}_0\|_{\mathbf{H}^{-1}(\Omega)} + \|\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}_0)\|_{\mathbf{H}^{-1}(\Omega)}) \\
&\leqq C(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{v} \otimes \boldsymbol{u}_0\|_{\mathbb{L}^2(\Omega)}) \\
&\leqq C\left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)}\right) \\
&\leqq C\left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right),
\end{aligned}$$

which added to estimate (32) makes (30).

Now, $-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \widetilde{\boldsymbol{f}} \in \mathbf{H}^{-1}(\Omega)$ and:

$$\forall \boldsymbol{\varphi} \in \mathbf{V}, \qquad \langle -\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)} = 0.$$

Thanks to De Rham's Lemma 6, there exists a unique $q \in L^2(\Omega)/\mathbb{R}$ such that:

$$-\Delta z + \boldsymbol{v} \cdot \nabla z + \nabla q = \widetilde{\boldsymbol{f}}$$

with $\|q\|_{L^2(\Omega)/\mathbb{R}} \leqq C \|\nabla q\|_{\mathbf{H}^{-1}(\Omega)}$. Finally, estimate (31) follows from the previous equation and the estimate for z. □

As a consequence of Theorem 13, Theorem 10 and the inequality

$$\|\boldsymbol{v} \cdot \nabla \boldsymbol{u}\|_{\mathbf{L}^{6/5}(\Omega)} \leqq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\nabla \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)},$$

we can deduce the following result:

Corollary 6. *Let*

$$\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in W^{1,6/5}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{7/6,6/5}(\Gamma)$$

verify the compatibility condition (5)*. Then, the solution* $(\boldsymbol{u}, q)$ *given by Theorem* 13 *belongs to* $\mathbf{W}^{2,6/5}(\Omega) \times W^{1,6/5}(\Omega)$ *and verifies the following estimate:*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{2,6/5}(\Omega)} + \|q\|_{W^{1,6/5}(\Omega)/\mathbb{R}} \leqq\ & C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ & \times \left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\right. \\ & \left.\times \left(\|h\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{7/6,6/5}(\Gamma)}\right)\right)\end{aligned}$$

5.2. Generalized and strong solutions

In this subsection, we are interested in the study of generalized solutions and strong solutions of the system (O). Let us first consider $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ a generalized solution of (O). If $p < 3$, then $\boldsymbol{v} \otimes \boldsymbol{u}$ belongs to $\mathbf{L}^p(\Omega)$ and then div $(\boldsymbol{v} \otimes \boldsymbol{u})$ belongs to $\mathbf{W}^{-1,p}(\Omega)$. If $p \geqq 3$, $\boldsymbol{v} \otimes \boldsymbol{u} \notin \mathbf{L}^p(\Omega)$, but $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^r(\Omega)$, with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$ and $\mathbf{L}^r(\Omega) \hookrightarrow \mathbf{W}^{-1,p}(\Omega)$. This means that all terms appearing in system (O) belong to $\mathbf{W}^{-1,p}(\Omega)$. In the case of the strong solutions, the situation is different. In fact, when $p < 3$, because of the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p*}(\Omega)$, the term $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^p(\Omega)$. But this is no longer the case when $p \geqq 3$ and $\boldsymbol{v}$ belongs only to $\mathbf{H}_3(\Omega)$. The next proposition gives good conditions for ensuring the existence of strong solutions.

Theorem 14. (Strong solutions) *Let* $p \geqq \frac{6}{5}$,

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_s(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma),$$

with

$$s = 3 \quad \text{if } p < 3, \quad s = p \quad \text{if } p > 3, \quad s = 3 + \varepsilon \quad \text{if } p = 3, \tag{33}$$

for some arbitrary $\varepsilon > 0$*, and satisfying the compatibility condition:*

$$\int_\Omega h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, \mathrm{d}\sigma.$$

Then, the unique solution of (O) given by Theorem 13 verifies $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. *Moreover, there exists a constant* $C > 0$ *such that:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\right) \end{aligned} \tag{34}$$

Proof. First, by Corollary 6, we can suppose $p \geqq 6/5$ and then we have the following embeddings:

$$\mathbf{L}^p(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega), \quad W^{1,p}(\Omega) \hookrightarrow L^2(\Omega), \quad \text{and} \quad \mathbf{W}^{2-1/p,p}(\Gamma) \hookrightarrow \mathbf{H}^{1/2}(\Gamma).$$

Thanks to the regularity of $\boldsymbol{f}$, by Theorem 13 there exists a unique solution $(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ verifying the following estimates:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leqq C\left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right) \tag{35}$$

and

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} + \|q\|_{L^2(\Omega)/\mathbb{R}} &\leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\right). \end{aligned} \tag{36}$$

Observe that a priori the regularity for the Oseen problem cannot be deduced from the Stokes problem. This follows from the fact that $\boldsymbol{v} \cdot \nabla \boldsymbol{u} = \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) \in \mathbf{H}^{-1}(\Omega)$ (see the proof of Theorem 13).

In order to obtain the strong solution in $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, first we apply Lemma 7 to function $\boldsymbol{v}$, and we take for any $\lambda > 0$ $\boldsymbol{v}_\lambda$ as the velocity of the convection term, where $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ is such that $\nabla \cdot \boldsymbol{v}_\lambda = 0$ and $\|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \leqq \lambda$. Therefore, we search for $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ solution of the problem:

$$(O_\lambda) \begin{cases} -\Delta \boldsymbol{u}_\lambda + \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla q_\lambda = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_\lambda = h & \text{in } \Omega, \\ \boldsymbol{u}_\lambda = \boldsymbol{g} & \text{on } \Gamma. \end{cases}$$

Remember that, from above, we can obtain a unique solution $(\boldsymbol{u}_\lambda, q_\lambda)$ bounded in $\mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ independent of λ. Then, we again obtain estimates (35) and (36). As $\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda \in \mathbf{L}^2(\Omega)$, if $\boldsymbol{f}$ and h are regular enough, then using the Stokes regularity we deduce that $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ if $2 \leqq p$ and $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ if $6/5 < p \leqq 2$. A bootstrap argument, moreover, shows that $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ if $2 < p$.

Thus, we focus on obtaining a strong estimate for $(\boldsymbol{u}_\lambda, q_\lambda)$. Let $\varepsilon > 0$ with $0 < \lambda < \varepsilon/2$. We consider:

$$\boldsymbol{v}_\lambda = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}, \quad \text{and} \quad \boldsymbol{v}_{\lambda,2}^\varepsilon = \boldsymbol{v}_\lambda - \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}, \tag{37}$$

$\widetilde{\boldsymbol{v}}$ being the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and $\rho_{\varepsilon/2}$ the classical mollifier. By regularity estimates for the Stokes problem, we have

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} &\leqq C\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)}\right. \\ &\left. + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}\right) \end{aligned} \tag{38}$$

Now, we use the decomposition (37) in order to bound the term $\|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}$. We observe first that

$$\|\boldsymbol{v}^{\varepsilon}_{\lambda,2}\|_{\mathbf{L}^s(\Omega)} \leqq \|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{v} - \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}\|_{\mathbf{L}^s(\Omega)} \leqq \lambda + \varepsilon/2 < \varepsilon.$$

Recall that

$$W^{2,p}(\Omega) \hookrightarrow W^{1,k}(\Omega) \tag{39}$$

for any $k \in [1, p*]$, with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$, if $p < 3$, for any $k \geqq 1$ if $p = 3$ and for any $k \in [1, \infty]$ if $p > 3$. Moreover, the embedding

$$W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \tag{40}$$

is compact for any $q \in [1, p*[$ if $p < 3$, for any $q \in [1, \infty[$ if $p = 3$ and for $q \in [1, \infty]$ if $p > 3$. Then, using the Hölder inequality and the Sobolev embedding, we obtain

$$\|\boldsymbol{v}^{\varepsilon}_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}^{\varepsilon}_{\lambda,2}\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^k(\Omega)} \leqq C\,\varepsilon \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} \tag{41}$$

where $\frac{1}{k} = \frac{1}{p} - \frac{1}{s}$, which is well defined because of the definition of the real number s. For the second estimate, we consider two cases.
(i) Case $p \leqq 2$. Let $r \in]3, \infty]$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$ and $t \geqq 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ satisfying:

$$\begin{aligned}\|\boldsymbol{v}^{\varepsilon}_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leqq \|\boldsymbol{v}^{\varepsilon}_1\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)} \\ &\leqq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Using the estimate (35), we have

$$\begin{aligned}\|\boldsymbol{v}^{\varepsilon}_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq C_\varepsilon \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} &+ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \\ &\times (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)})\Big).\end{aligned} \tag{42}$$

From (42) and (41), we deduce that

$$\begin{aligned}&\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqq C\, \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \\ &\quad \times \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\, (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)})\Big)\end{aligned} \tag{43}$$

(ii) Case $p > 2$. First, we choose the exponent q given in (40) such that $q > 2$. For any ε', we known that there exists $C_{\varepsilon'} > 0$ such that

$$\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leqq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}.$$

Let first consider $p < 3$ and choose $q < p*$ and close of $p*$. Then, there exist $r > 3$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $t > 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ satisfying:

$$\begin{aligned}\|\boldsymbol{v}^{\varepsilon}_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leqq \|\boldsymbol{v}^{\varepsilon}_1\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \\ &\leqq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)}.\end{aligned}$$

If $p \geqq 3$,

$$\begin{aligned} \|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leqq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \\ &\leqq \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \|\rho_{\varepsilon/2}\|_{L^1(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)}, \end{aligned}$$

where we choose $q = \infty$ if $p > 3$ and q large enough, if $p = 3$. In the both cases, in order to control the first term on the right-hand side of (38) with the term on the left-hand side, we fix ε and ε' small enough to obtain

$$\begin{aligned} &\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C \Big\{ \|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} \\ &\qquad + C_{\varepsilon'} \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\Omega)} \\ &\qquad \times \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big) \left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} \right) \Big) \Big\} \end{aligned} \tag{44}$$

Thus, we deduce that $(\boldsymbol{u}_\lambda, q_\lambda)$ satisfies (43), where we replace $\|\boldsymbol{v}\|_{\mathbf{L}^3}$ by $\|\boldsymbol{v}\|_{\mathbf{L}^s}$.

The estimate (43) is uniform in λ, and therefore we can extract subsequences, which we still call $\{\boldsymbol{u}_\lambda\}_\lambda$ and $\{q_\lambda\}_\lambda$, such that if $\lambda \to 0$,

$$\boldsymbol{u}_\lambda \longrightarrow \boldsymbol{u} \quad \text{weakly in } \mathbf{W}^{2,p}(\Omega),$$

and for the pressure, there exists a sequence of real numbers k_λ such that

$$q_\lambda + k_\lambda \to q \quad \text{weakly in } W^{1,p}(\Omega).$$

It is easy to verify that $(\boldsymbol{u}, q)$ is a solution of (O) satisfying estimate (34) and that this solution is unique. □

Remark 12. (i) What happens if $1 < p < 6/5$? We shall try an answer later.

(ii) Observe that the value of p for the regularity cannot be equal to 3 because of $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and thus $\boldsymbol{v} \cdot \nabla \boldsymbol{u}$ cannot be better than $\mathbf{L}^{3-\varepsilon}(\Omega)$ ($\varepsilon > 0$). If we want to reach the case $p = 3$ (respectively $p > 3$), we must suppose $\boldsymbol{v} \in \mathbf{L}^{3+\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$ (respectively $\boldsymbol{v} \in \mathbf{L}^p(\Omega)$).

Theorem 15. *Let*

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

verify the compatibility condition (5). *Then, the problem* (O) *has a unique solution* $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. *Moreover, there exists some constant* $C > 0$ *such that:*

(i) *if* $p \geqq 2$, *then*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \\ &\quad \leqq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \right) \end{aligned} \tag{45}$$

(ii) *if* $p < 2$, *then*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} &\leqq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \\ &\quad \times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \right) \end{aligned} \tag{46}$$

Proof. (i) First case: $p \geqq 2$. Let $(\boldsymbol{u}_0, q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ be the solution of:

$$-\Delta \boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{f} \;\text{ and }\; \nabla \cdot \boldsymbol{u}_0 = h \;\text{ in } \Omega, \quad \boldsymbol{u}_0 = \boldsymbol{g} \;\text{ on } \Gamma.$$

verifying the estimate:

$$\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|q_0\|_{L^p(\Omega)/\mathbb{R}} \leqq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right) \tag{47}$$

and $(\boldsymbol{z}, \theta) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$ verifying:

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \;\text{ and }\; \nabla \cdot \boldsymbol{z} = 0 \;\text{ in } \Omega, \quad \boldsymbol{z} = \mathbf{0} \;\text{ on } \Gamma,$$

with $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$ and satisfying the estimate

$$\begin{aligned}
&\|\boldsymbol{z}\|_{\mathbf{W}^{2,t}(\Omega)} + \|\theta\|_{W^{1,t}(\Omega)/\mathbb{R}} \\
&\quad \leqq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0\|_{\mathbf{L}^t(\Omega)} \\
&\quad \leqq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \\
&\qquad \times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right).
\end{aligned} \tag{48}$$

Here, we have applied Theorem 14 because of $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \in \mathbf{L}^t(\Omega)$. Observe that $\frac{6}{5} \leqq t < 3$, if and only if $p \geqq 2$.

Thanks to the embedding $\mathbf{W}^{2,t}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$, the pair $(\boldsymbol{u}, q) = (\boldsymbol{z} + \boldsymbol{u}_0, \theta + q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies the problem (O). Estimate (45) follows from (47) and (48).

(ii) Second case: $p < 2$. Here we use a duality argument.

1. First, we suppose $h = 0$ and $\boldsymbol{g} = \mathbf{0}$.

The problem (O) is equivalent to the problem:

Find $(\boldsymbol{u}, q) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ such that: $\forall (\boldsymbol{w}, \pi) \in \mathbf{W}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)$

$$\begin{aligned}
&\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla \pi \rangle_{\mathbf{W}_0^{1,p}(\Omega) \times \mathbf{W}^{-1,p'}(\Omega)} - \langle q, \nabla \cdot \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}.
\end{aligned}$$

Thanks to case (i), as $p' > 2$, for any pair $(\mathbf{F}, \varphi) \in \mathbf{W}^{-1,p'}(\Omega) \times L_0^{p'}(\Omega)$, there exists a unique $(\boldsymbol{w}, \pi) \in \mathbf{W}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ such that

$$-\Delta \boldsymbol{w} - \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla \pi = \mathbf{F} \;\text{ and }\; \nabla \cdot \boldsymbol{w} = \varphi \;\text{ in } \Omega, \;\; \boldsymbol{w} = \mathbf{0} \;\text{ on } \Gamma$$

and satisfying the estimate:

$$\|\boldsymbol{w}\|_{\mathbf{W}^{1,p'}(\Omega)} + \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} \leqq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \left(\|\mathbf{F}\|_{\mathbf{W}^{-1,p'}(\Omega)} + \|\varphi\|_{L^{p'}(\Omega)}\right). \tag{49}$$

Furthermore, we have

$$\begin{aligned}
&\left|\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbf{W}^{-1,p}(\Omega) \times \mathbf{W}_0^{1,p'}(\Omega)}\right| \\
&\quad \leqq C \|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \left(\|\mathbf{F}\|_{\mathbf{W}^{-1,p'}(\Omega)} + \|\varphi\|_{L^{p'}(\Omega)}\right)
\end{aligned}$$

In other words, the mapping $(\mathbf{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\mathbf{W}^{-1,p}\times \mathbf{W}_0^{1,p'}}$ defines an element of the dual space of $\mathbf{W}^{-1,p'}(\Omega) \times L_0^{p'}\Omega)$. From Riesz' Representation Theorem, we deduce that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of (O) satisfying

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)}.$$

2. We suppose now $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ verifying the compatibility condition (5). There exists $\boldsymbol{z} \in \mathbf{W}^{1,p}(\Omega)$ such that

$$\nabla \cdot \boldsymbol{z} = h \ \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g} \ \text{ on } \Gamma,$$

with the corresponding estimate.

We know that there exist $(\boldsymbol{u}_0, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ unique solutions to

$$-\Delta \boldsymbol{u}_0 + \boldsymbol{v}\cdot\nabla \boldsymbol{u}_0 + \nabla q = \boldsymbol{f} + \Delta \boldsymbol{z} - \boldsymbol{v}\cdot\nabla\boldsymbol{z} \ \text{ and } \ \nabla\cdot \boldsymbol{u}_0 = 0 \ \text{ in } \Omega, \quad \boldsymbol{u}_0 = \boldsymbol{0} \ \text{ on } \Gamma$$

and

$$\begin{aligned}
&\|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \\
&\quad \leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \|\boldsymbol{f} + \Delta\boldsymbol{z} - \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{z})\|_{\mathbf{W}^{-1,p}(\Omega)} \\
&\quad \leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 (\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} \\
&\qquad + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}\otimes\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)}).
\end{aligned} \tag{50}$$

Using the Hölder inequality and Sobolev embedding, we have

$$\|\boldsymbol{v}\otimes\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}\|_{\mathbf{L}^3}\|\boldsymbol{z}\|_{\mathbf{L}^{p*}(\Omega)} \leqq C\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}).$$

The estimate (46) is then a consequence of the two above estimates. □

Remark 13. (i) Observe that $\boldsymbol{v}\cdot\nabla\boldsymbol{u} \in \mathbf{L}^t(\Omega) \hookrightarrow \mathbf{W}^{-1,p}(\Omega)$, with $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$. This means that all terms in the left-hand side of the system (O) belong to the space $\mathbf{W}^{-1,p}(\Omega)$.
(ii) Estimates (45) and (46) are not optimal.

Proposition 3. *Under the assumptions of Theorem 15 and supposing that* $\frac{6}{5} \leqq p \leqq 6$*, the solution* $(\boldsymbol{u}, q)$ *satisfies the estimate:*

$$\begin{aligned}
&\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leqq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\
&\quad \times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right)\right)
\end{aligned} \tag{51}$$

Moreover, if $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ *on* Γ*, then the estimate* (51) *holds for any* $1 < p < \infty$.

Proof. The case $p = 2$ is treated in Theorem 13. We will quickly repeat the reasoning given in Theorem 14. Let $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ be the solution of the problem (O_λ) given by Theorem 15.

(i) First case: $2 < p < 3$. We have

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|q_\lambda\|_{L^p(\Omega)/\mathbb{R}} \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}) \tag{52}$$

and

$$\|\boldsymbol{v}^\varepsilon_{\lambda,2}\|_{\mathbf{L}^3(\Omega)} \leqq \|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} + \|\boldsymbol{v} - \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}\|_{\mathbf{L}^3(\Omega)} \leqq \lambda + \varepsilon/2 \leqq \varepsilon.$$

Then, using the Hölder inequality and the Sobolev embedding, we obtain

$$\|\boldsymbol{v}^\varepsilon_{\lambda,2} \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}^\varepsilon_{\lambda,2}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^{p*}(\Omega)} \leqq C\,\varepsilon \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}. \tag{53}$$

For the second estimate, we have:

$$\begin{aligned}\|\boldsymbol{v}^\varepsilon_1 \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leqq \|\boldsymbol{v}^\varepsilon_1\|_{\mathbf{L}^r(\Omega)} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)}\\ &\leqq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)},\end{aligned}$$

where $t \in]1, \frac{6}{5}[$ is such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ and $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. We choose $r \in]3, \frac{6p}{6-p}[$ for that $6 < q < p*$. Then, for any ε', we known that there exists $C_{\varepsilon'} > 0$ such that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leqq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} \;+\; C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)}. \tag{54}$$

From (53), we deduce that

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|q_\lambda\|_{L^p(\Omega)/\mathbb{R}} &\leqq C\,\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\,(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)})\right)\end{aligned} \tag{55}$$

(ii) Case $3 \leqq p \leqq 6$. We have now, by Stokes regularity, the estimate:

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|q_\lambda\|_{L^p(\Omega)/\mathbb{R}} \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}) \tag{56}$$

and

$$\|\boldsymbol{v}^\varepsilon_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)} \leqq C \|\boldsymbol{v}^\varepsilon_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^r(\Omega)} \leqq C\,\varepsilon \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}, \tag{57}$$

where $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$. For the second estimate, we have:

$$\begin{aligned}\|\boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)} &\leqq C \|\boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^r(\Omega)}\\ &\leqq C \|\boldsymbol{v}^\varepsilon_1\|_{\mathbf{L}^k(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}\\ &\leqq C \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)},\end{aligned} \tag{58}$$

where $k = \frac{6p}{6-p}$ and $t = \frac{2p}{p+2} \in [\frac{6}{5}, \frac{3}{2}[$.
(iii) Case $\frac{6}{5} \leqq p < 2$. As in the proof of Theorem 15, we use a duality argument.
(iv) Case $p > 6$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ. The functions $\widetilde{\boldsymbol{v}}$ ($\widetilde{\boldsymbol{v}}$ being the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$) and $\boldsymbol{v}^\varepsilon_1$ are then divergence free. As in Case (ii), the estimate (57) is satisfied. For the estimate (58), we observe that

$$\|\boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)} = \|\mathrm{div}(\boldsymbol{v}^\varepsilon_1 \otimes \boldsymbol{u}_\lambda)\|_{\mathbf{W}^{-1,p}(\Omega)} \leqq \|(\boldsymbol{v}^\varepsilon_1 \otimes \boldsymbol{u}_\lambda)\|_{\mathbf{L}^p(\Omega)}.$$

But

$$\|\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)}$$
$$\leqq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)},$$

with $1+\frac{1}{p}=\frac{1}{3}+\frac{1}{t}$ and $t>\frac{6}{5}$. To finish, the estimate (55) is a consequence of the fact that for any ε', we known that there exists $C_{\varepsilon'}>0$ such that

$$\|\boldsymbol{u}_\lambda\|_{L^\infty(\Omega)} \leqq \varepsilon'\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + C_{\varepsilon'}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)}. \tag{59}$$

(v) Case $p<6/5$ and $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ. Again, we use a duality argument. □

Remark 14. If we suppose that $\boldsymbol{v}\in\mathbf{H}_p(\Omega)$, then estimate (51), where we replace the norm $\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}$ by $\|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}$, holds when $p>6$ (and then also, by duality argument, when $p<6/5$ and $\boldsymbol{v}\in\mathbf{H}_{p'}(\Omega)$). Indeed, we rewrite estimate (53):

$$\|\boldsymbol{v}_{\lambda,2}^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{\mathbf{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)} \leqq C\,\varepsilon\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)}. \tag{60}$$

For the second estimate, we have:

$$\|\boldsymbol{v}_1^\varepsilon \otimes \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leqq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)}$$
$$\leqq \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)}.$$

To finish, we again use estimate (59).

Corollary 7. *Let* $1<p<6/5$ *and let*

$$\boldsymbol{f}\in\mathbf{L}^p(\Omega),\quad \boldsymbol{v}\in\mathbf{H}_3(\Omega),\quad h\in W^{1,p}(\Omega)\quad\text{and}\ \boldsymbol{g}\in\mathbf{W}^{2-1/p,p}(\Gamma)$$

verify the compatibility condition (5). *Then, the solution given by Theorem* 15 *satisfies* $(\boldsymbol{u},q)\in\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ *and the estimate*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqq C\left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)$$
$$\times\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1+\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)}+\|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\right) \tag{61}$$

holds.

Proof. Let $1<p<6/5$ and $(\boldsymbol{u},q)\in\mathbf{W}^{1,p}(\Omega)\times L^p(\Omega)$ be the solution given by Theorem 15. But

$$\mathbf{L}^p(\Omega)\hookrightarrow\mathbf{W}^{-1,r}(\Omega),\quad W^{1,p}(\Omega)\hookrightarrow\mathbf{L}^r(\Omega),\quad \mathbf{W}^{2-1/p,p}(\Gamma)\hookrightarrow\mathbf{W}^{1-1/r,}(\Gamma)$$

where $r\in]\frac{3}{2},2[$ satisfies $\frac{1}{r}=\frac{1}{p}-\frac{1}{3}$. From Theorem 15, we deduce that $(\boldsymbol{u},q)\in\mathbf{W}^{1,r}(\Omega)\times L^r(\Omega)$ and then $\boldsymbol{v}\cdot\nabla\boldsymbol{u}\in\mathbf{L}^p(\Omega)$. The Stokes regularity allows us to conclude that $(\boldsymbol{u},q)\in\mathbf{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$. To obtain the estimate (61), we proceed as in the proof of Theorem 14. The only difference consists of obtaining estimate (42). Let $r\in]2,3[$ such that $\frac{1}{p}=\frac{1}{r}+\frac{1}{2}$ and $t\in]3,6[$ such that $1+\frac{1}{r}=\frac{1}{3}+\frac{1}{t}$. Then, we obtain in exactly the same way estimate (42). Passing to the limit on λ for the solutions $(\boldsymbol{u}_\lambda,q_\lambda)$, we obtain the estimate on $(\boldsymbol{u},q)$. □

We can summarize Theorem 14 and Corollary 7 by the following theorem:

Theorem 16. *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ be such that*

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

verify the compatibility condition (5) *and* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with s defined by* (33)*. Then, the solution given by Theorem 15 satisfies* $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *and satisfies estimate* (34)*.*

5.3. Very weak solutions

The concepts of weak and strong solutions are known for the Oseen equations. Here, we want to define and prove the existence of a very weak solution. In this way, before describing the technique, we must give a meaning to the singular data for an Oseen problem, in the same way that it was done for the Stokes problem in [5].

These are the tools we shall use in the definition and existence proof that we present in the sequel.

Definition 2. (*Very weak solution for the Oseen problem*) Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (16) and (5) and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ for s as in (64). We say that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a *very weak solution* of (O) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned}
&\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma, \qquad (62)\\
&\int_\Omega \boldsymbol{u} \cdot \nabla \pi \, \mathrm{d}\boldsymbol{x} = -\int_\Omega h\, \pi \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma,
\end{aligned}$$

where the dualities on Ω and Γ are defined by (24).

As for the Stokes problem, the previous duality has sense. Moreover, note that $\mathbf{W}^{1,p'}(\Omega) \hookrightarrow \mathbf{L}^{p'*}(\Omega)$ and then the integral $\int_\Omega \boldsymbol{u} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}$ is well defined.

Theorem 17. (Very weak solution) *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy* (5)*,*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \; h \in L^r(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \; \text{with } \frac{1}{r} = \frac{1}{p} + \frac{1}{s} \qquad (63)$$

and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with*

$$s = 3 \;\text{ if } p > 3/2, \quad s = p' \;\text{ if } p < 3/2, \quad s = 3 + \varepsilon \;\text{ if } p = 3/2. \qquad (64)$$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ verifying the following estimates:

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leqq C\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\big), \qquad (65)$$

$$\|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leqq C\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)^2\big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\big). \qquad (66)$$

Proof. First, we shall prove that if the pair $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfies the first two equations of (O), then $\boldsymbol{u}$ belongs to $\mathbf{T}_{p,r}(\Omega)$ and thus the boundary condition $\boldsymbol{u} = \boldsymbol{g}$ on Γ makes sense. Hence, if a pair $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ satisfies the first two equations of (O), because of $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ with $\nabla \cdot \boldsymbol{v} = 0$ and thanks (again) to Lemma 9, then $\Delta \boldsymbol{u} = \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla q - \boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$. Therefore, $\boldsymbol{u} \in \mathbf{T}_{p,r,\sigma}(\Omega)$ and its tangential trace belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. Moreover, as $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $\nabla \cdot \boldsymbol{u} \in L^r(\Omega)$, then $\boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma \in W^{-1/p,p}(\Gamma)$, and the whole trace $\boldsymbol{u}|_\Gamma \in \mathbf{W}^{-1/p,p}(\Gamma)$ can be identified with $\boldsymbol{u}|_\Gamma = \boldsymbol{g}$.

It suffices to consider the case in which $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = 0$ and $\int_\Omega h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0$; the general case is similar to the proof given in the end of Proposition 2.

We prove, then, that problem (O) is equivalent to the variational formulation: Find $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega)$, $\forall \pi \in W^{1,p'}(\Omega)$

$$\begin{aligned}
&\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \pi)\, \mathrm{d}\boldsymbol{x} - \langle q,\, \nabla \cdot \boldsymbol{w} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \pi\, \mathrm{d}\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma .
\end{aligned} \tag{67}$$

Now, for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times [W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)]$, using (26) we have:

$$\begin{aligned}
&\left| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \pi\, \mathrm{d}\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \right| \\
&\leqq C \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big) \Big(\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\pi\|_{W^{1,p'}} \Big) \\
&\leqq C\ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big) \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big) \\
&\quad \times \Big(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)\ \|\varphi\|_{W^{1,p'}(\Omega)} \Big)
\end{aligned}$$

That is, the mapping

$$(\mathbf{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \int_\Omega h\, \pi\, \mathrm{d}\boldsymbol{x} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \tag{68}$$

defines an element $(\boldsymbol{u}, q)$ of the dual space of $\mathbf{L}^{p'}(\Omega) \times [\mathbf{W}_0^{1,p'} \cap \mathbf{L}_0^p(\Omega)]$, which is equal to $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$, with

$$\begin{aligned}
&\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)})^{-1} \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leqq C\ \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big) \\
&\quad \times \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big) .
\end{aligned} \tag{69}$$

Therefore, using Riesz's Lemma, there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ verifying (67) and estimate (69). $\square$

As for Corollary 5, we can prove the following result.

Corollary 8. (i) *Let σ be a real number such that* $0 < \sigma < 1$. *Let* $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, *$h$ and $\boldsymbol{g}$ satisfy the compatibility condition (5) with*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),\ h \in W^{\sigma,r}(\Omega),$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$ *and* $r \leqq p$. *Let* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with*

$$s = 3 \quad \text{if } p > 3/2, \quad s = p' \quad \text{if } p < 3/2, \quad s = 3 + \varepsilon \quad \text{if } p = 3/2.$$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, q)$ *belonging to* $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ *and satisfying the estimate*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} &\leqq C\,(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}) \\ &\times \big(\|\mathbb{F}_0\|_{\mathbf{W}^{\sigma,r}(\Omega)} + \|f_1\|_{W^{\sigma-1,p}(\Omega)} \\ &+ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)})(\|h\|_{W^{\sigma,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Gamma)})\big).\end{aligned}$$

(ii) *If, moreover,*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma+1,r}(\Omega), \quad f_1 \in W^{\sigma,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{\sigma+1-1/p,p}(\Gamma), \ h \in W^{\sigma+1,r}(\Omega),$$

with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{s}$ *and* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$, *where*

$$s = 3 \quad \text{if } p < 3, \quad s = p \quad \text{if } p > 3, \quad s = 3 + \varepsilon \quad \text{if } p = 3,$$

then $(\boldsymbol{u}, q) \in \mathbf{W}_0^{\sigma+1,p}(\Omega) \times W^{\sigma,p}(\Omega)$ *and*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma+1,p}(\Omega)} + \|q\|_{W^{\sigma,p}(\Omega)/\mathbb{R}} &\leqq C\,(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}) \\ &\times \big(\|\mathbb{F}_0\|_{\mathbf{W}^{\sigma+1,r}(\Omega)} + \|f_1\|_{W^{\sigma,p}(\Omega)} \\ &+ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)})(\|h\|_{W^{\sigma+1,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma+1-1/p,p}(\Gamma)})\big).\end{aligned}$$

Theorem 18. *Let* σ *be a real number such that* $\frac{1}{p} < \sigma < 1$. *Let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfy the compatibility condition* (5) *with*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \ h \in W^{\sigma-1,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Let $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with*

$$s = 3 \quad \text{if } p > 3/2, \quad s = p' \quad \text{if } p < 3/2, \quad s = 3 + \varepsilon \quad \text{if } p = 3/2.$$

Then, the Oseen problem (O) has exactly one solution $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ *satisfying the estimate*

$$\begin{aligned}&\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} \\ &\quad \leqq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)} + \|h\|_{W^{\sigma-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Gamma)}).\end{aligned}$$

Proof. The proof is exactly the same as in Theorem 12 and it suffices to study the new term containing the function $\boldsymbol{v}$. Moreover, as in Theorem 12, we can suppose $\boldsymbol{g} = \mathbf{0}$. We begin by proving that for any $\boldsymbol{u} \in \mathbf{W}^{\sigma,p}(\Omega)$, we have $\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}) \in \mathbf{W}^{\sigma-2,p}(\Omega)$. Indeed, let $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$. Suppose first that $p' < \frac{3}{1-\sigma}$ and $p < \frac{3}{\sigma}$. Then, by Sobolev embeddings

$$\begin{aligned}|\langle \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}), \ \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}| &= \left|\int_\Omega \boldsymbol{u} \otimes \boldsymbol{v} \ : \ \nabla \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}\right| \\ &\leqq \|\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^k(\Omega)} \\ &\leqq C \|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{2-\sigma,p'}(\Omega)},\end{aligned}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\sigma}{3}$ and $\frac{1}{k} = \frac{1}{p'} - \frac{1-\sigma}{3}$. Observe that $\frac{1}{k} + \frac{1}{q} + \frac{1}{3} = 1$. Using the density of $\mathcal{D}(\Omega)$ in $\mathbf{W}^{2-\sigma,p'}(\Omega)$, we deduce the announced property. Furthermore, the same property holds when $p' \leqq \frac{3}{1-\sigma}$ or $p \leqq \frac{3}{\sigma}$. Moreover, for all $\boldsymbol{w} \in \mathbf{W}_0^{2-\sigma,p'}(\Omega)$, we have:

$$\langle \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}),\ \boldsymbol{w}\rangle_{\mathbf{W}^{\sigma-2,p}(\Omega)\times\mathbf{W}_0^{2-\sigma,p'}(\Omega)} = \langle \boldsymbol{u},\ \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w})\rangle_{\mathbf{W}^{\sigma,p}(\Omega)\times\mathbf{W}_0^{-\sigma,p'}(\Omega)},$$

and for all $\pi \in W^{-\sigma+1,p'}(\Omega)$, we have

$$\langle \boldsymbol{u},\ \nabla\pi\rangle_{W_0^{\sigma,p}(\Omega)\times W^{-\sigma,p'}(\Omega)} = -\langle \operatorname{div} \boldsymbol{u},\ \pi\rangle_{W^{\sigma-1,p}(\Omega)\times W_0^{-\sigma+1,p'}(\Omega)}.$$

Observe that $W^{-\sigma+1,p'}(\Omega) = W_0^{-\sigma+1,p'}(\Omega)$ because $1-\sigma < \frac{1}{p'}$.

As in the proof of Proposition 2, for any $\mathbf{H} \in \mathbf{W}^{-\sigma,p'}(\Omega)$ and $\varphi \in W^{-\sigma+1,p'}(\Omega)$, we have:

$$\begin{aligned}&\left|-\langle \boldsymbol{f},\ \boldsymbol{w}\rangle_{\mathbf{W}^{\sigma-2,p}(\Omega)\times\mathbf{W}_0^{-\sigma+2,p'}(\Omega)} + \langle h,\ \pi\rangle_{W^{\sigma-1,p}(\Omega)\times W_0^{-\sigma+1,p'}(\Omega)}\right|\\ &\quad\leqq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)} + \|h\|_{W^{\sigma-1,p}(\Omega)}\right)\left(\|\mathbf{H}\|_{\mathbf{W}^{-\sigma,p'}(\Omega)} + \|\varphi\|_{W^{-\sigma+1,p'}(\Omega)}\right)\end{aligned}$$

with $(\boldsymbol{w}, \pi) \in \mathbf{W}_0^{-\sigma+2,p'}(\Omega) \times W^{-\sigma+1,p'}(\Omega)/\mathbb{R}$ being the unique solution of the Oseen problem given by Corollary 8 point (ii):

$$-\Delta\boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla\pi = \mathbf{H} \ \text{ and } \ \nabla\cdot\boldsymbol{w} = \varphi \ \text{ in } \Omega,\ \ \boldsymbol{w} = \mathbf{0} \ \text{ on } \Gamma.$$

The rest of the proof is completely similar. □

Remark 15. (i) Note that the previous theorem is also valid if $1 < \sigma \leqq 2$.

(ii) When $\boldsymbol{f} \in \mathbf{W}^{1/p-2,p}(\Omega)$, we can conjecture that $\boldsymbol{u} \notin \mathbf{W}^{1/p,p}(\Omega)$.

(iii) If $1/p < \sigma < 1$, $\boldsymbol{f} \in W^{\sigma-2,p}(\Omega)$, $\boldsymbol{g} \in W^{\sigma-1/p,p}(\Gamma)$, then the solution $(\boldsymbol{u}, q)$ of (O) belongs to $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$. These assumptions are weaker than those of Corollary 8 point (i). Moreover, they are optimal.

(iv) If $0 \leqq \sigma \leqq 1/p$, Theorem 18 cannot be applied. Indeed, the trace mapping is not continuous (and not surjective) from $\mathbf{W}^{\sigma,p}(\Omega)$ into $\mathbf{W}^{\sigma-1/p,p}(\Gamma)$. If we wish to solve Problem (O) with boundary condition $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$, it is necessary to suppose that $\boldsymbol{f}$ and h are more regular, precisely, we must assume $\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1$ with $\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega)$, $f_1 \in W^{\sigma-1,p}(\Omega)$, and $h \in W^{\sigma,r}(\Omega)$, where $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ and $r \leqq p$. The solution is then obtained by Corollary 8 point (i).

6. The Navier–Stokes problem

As a consequence of the previous study, we give the definition of a very weak solution for the Navier–Stokes equations and we look for a result showing the existence of a very weak solution:

Definition 3. (*Very weak solution for the Navier–Stokes problem*) Let $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$, $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ satisfy the compatibility condition (5). We say that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a *very weak solution* of (NS) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned} &\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ &\quad = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma, \\ &\int_\Omega \boldsymbol{u} \cdot \nabla \pi \, \mathrm{d}\boldsymbol{x} = -\int_\Omega h\, \pi \, \mathrm{d}\boldsymbol{x} + \langle (\boldsymbol{g} \cdot \boldsymbol{n}), \pi \rangle_\Gamma, \end{aligned} \tag{70}$$

where the dualities on Ω and Γ are defined in (24).

In the stationary Navier–Stokes equations, the data h and $\boldsymbol{g}$ play a special role, making possible (or not) the existence of a very weak solution. If h and $\boldsymbol{g}$ are small enough, then the result is true. Until we know more, we think that it is not possible to eliminate this last condition.

Therefore, we separate the two results: first, we prove the result for small external forces and then we generalized (when possible) to the general Navier–Stokes case, always supposing that h and $\boldsymbol{g}$ are small enough in their respective norms. Each step will be carried out in the subsections below.

6.1. The case of small external forces

In the search for a proof of the existence of very weak solutions for the incompressible Navier–Stokes equations, and following the scheme used by Marusič-Paloka [31], first, we prove the result for small enough data, and secondly, we generalize to the case of any data.

Theorem 19. (Very weak solution for Navier–Stokes, small data case) *Let* $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ *verify* (5).

(i) *There exists a constant* $\alpha_1 > 0$ *such that, if*

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leqq \alpha_1, \tag{71}$$

then, there exists a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *to problem* (NS) *verifying the following estimates:*

$$\| \boldsymbol{u} \|_{\mathbf{L}^3(\Omega)} \leqq C \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right) \tag{72}$$

$$\begin{aligned} \| q \|_{W^{-1,3}(\Omega)/\mathbb{R}} \leqq C_1 \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega))]'} + 2(1 + C_2) C \\ \times \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Omega)} \right) \end{aligned} \tag{73}$$

where $C > 0$ *is the constant given by* (65), $\alpha_1 = \min \left\{ (2C)^{-1}, (2C^2)^{-1} \right\}$, C_1 *and* C_2 *are constants of Sobolev embeddings.*

(ii) *Moreover, there exists a constant* $\alpha_2 \in]0, \alpha_1]$ *such that this solution is unique, up to an additive constant for* q, *if*

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leqq \alpha_2. \tag{74}$$

Proof. (i) *Existence*. We begin the proof of the existence of a very weak solution. We want to apply Banach's fixed point theorem, so we define a space over which we shall define an invariant operator.

The idea is to apply this fixed point over the Oseen equations, written in an adequate manner. We are searching for a fixed point for the application T,

$$\begin{cases} T : \mathbf{H}_3(\Omega) & \to \mathbf{H}_3(\Omega) \\ \quad \boldsymbol{v} & \mapsto T\boldsymbol{v} = \boldsymbol{u} \end{cases} \tag{75}$$

where given $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $T\boldsymbol{v} = \boldsymbol{u}$ is the unique solution of (O) given by Theorem 17.

In order to apply the fixed point result, we have to define a neighborhood $\mathbf{B}_r$, in the form:

$$\mathbf{B}_r = \{\boldsymbol{v} \in \mathbf{H}_3(\Omega);\ \ \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \leqq r\}. \tag{76}$$

If we choose a contraction method (in order to prove the existence of the fixed point), we must prove that there exists $\theta \in]0, 1[$ such that

$$\|T\boldsymbol{v}_1 - T\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \leqq \theta \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}. \tag{77}$$

In order to estimate $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}$, we observe that for each $i = 1, 2$, we have

$$\begin{aligned} -\Delta \boldsymbol{u}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{u}_i + \nabla q_i &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_i &= h \quad \text{in } \Omega, \\ \boldsymbol{u}_i &= \boldsymbol{g} \quad \text{on } \Gamma, \end{aligned}$$

with the estimates

$$\begin{aligned} \|\boldsymbol{u}_i\|_{\mathbf{L}^3(\Omega)} \leqq C \left(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)}\right) \\ \times \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}\right), \end{aligned} \tag{78}$$

where $C > 0$ is the constant given by (65). However, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to reason differently. We start with the problem verified by $(\boldsymbol{u}, q) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, q_1 - q_2)$, which is the following one:

$$-\Delta \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_2 \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}_1 = T\boldsymbol{v}_1$, $\boldsymbol{u}_2 = T\boldsymbol{v}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Using the very weak estimates (65) made for the Oseen problem successively for $\boldsymbol{u}$ and for $\boldsymbol{u}_2$, we obtain that:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} &\leqq C \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \|(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_2\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\ &\leqq C \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \\ &\leqq C^2 \beta \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \left(1 + \|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}, \end{aligned}$$

where $\beta = \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}$. Thus, we can (for instance) obtain estimate (77) if we consider $C^2\,\beta\,(1+r)^2 < 1$ to be verified, for example, taking:

$$r = \left(2\,C^2\,\beta\right)^{-1/2} - 1 \qquad \text{with} \qquad \beta < (2\,C^2)^{-1}. \tag{79}$$

Therefore, if (79) is verified and again using estimate (65), then the fixed point $\bar{\boldsymbol{u}} \in \mathbf{L}^3(\Omega)$ verifies the estimate:

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \leqq C\beta\, \left(1 + \|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\right).$$

Choose also β such that $\beta < (2\,C)^{-1}$. Thus,

$$\|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)} \leqq C\beta(1 - C\beta)^{-1} \leqq 2\,C\,\beta < 1.$$

Setting $\alpha_1 = \min\left\{(2C)^{-1}, (2C^2)^{-1}\right\}$, then the estimate (72) is satisfied. For the estimate of the associated pressure, we deduce from the equations $\nabla \bar{q} = \Delta \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} + \boldsymbol{f}$ and (72) that:

$$\begin{aligned}
\|\dot{\bar{q}}\|_{W^{-1,3}(\Omega)/\mathbb{R}} &\leqq \|\nabla \bar{q}\|_{W^{-2,3}(\Omega)} \\
&\leqq \|\Delta \bar{\boldsymbol{u}}\|_{\mathbf{W}^{-2,3}(\Omega)} + C_2 \|\bar{\boldsymbol{u}}\|^2_{\mathbf{L}^3(\Omega)} + C_1 \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\
&\leqq \|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\, \left(1 + C_2 \|\bar{\boldsymbol{u}}\|_{\mathbf{L}^3(\Omega)}\right) + C_1 \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\
&\leqq C_1 \|\boldsymbol{f}\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + 2(1 + C_2) C\,\beta,
\end{aligned}$$

where C_1 is the continuity constant of the Sobolev embedding $[\mathbf{X}_{3,3/2}(\Omega)]' \hookrightarrow \mathbf{W}^{-2,3}(\Omega)$ and C_2 is the continuity constant of the Sobolev embedding $\mathbf{W}^{1,3/2}_0(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$, which is (73), and the proof of existence is completed.
(ii) *Uniqueness*. We shall next prove uniqueness. Let us denote by $(\boldsymbol{u}_1, q_1)$ the solution obtained in step (i) and by $(\boldsymbol{u}_2, q_2)$ any other very weak solution corresponding to the same data. Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $q = q_1 - q_2$. We find that

$$-\Delta \boldsymbol{u} + \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma.$$

As $\boldsymbol{u} \cdot \nabla \boldsymbol{u}_1$ belongs to $\mathbf{W}^{-1,3/2}(\Omega)$, using the uniqueness argument and Proposition 3, the function $\boldsymbol{u}$ belongs to $\mathbf{W}^{1,3/2}(\Omega)$ and we have the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \leqq C_1\, \|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)} \left(1 + \|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\right),$$

where $C_1 > 0$ is given by (51). Thanks to Theorem 17, we also have

$$\|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \leqq C(1 + \|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)})(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}),$$

where $C > 0$ is the constant given in (65). We then deduce

$$\begin{aligned}
\|\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} &\leqq \frac{C(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)})}{1 - C(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)})} \\
&\leqq 2\,\beta C,
\end{aligned}$$

provided that $\beta \leqq \alpha_1$. Finally, using the embedding $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, we obtain the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \leqq 2CC_1C_2\beta(1+2C\beta)\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)},$$

where C_2 is the continuity constant of the above embedding. Consequently

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3/2}(\Omega)} \leqq 0,$$

provided that

$$\beta < \frac{-C_1C_2 + \sqrt{C_1C_2(4+C_1C_2)}}{4CC_1C_2}.$$

We deduce that $\boldsymbol{u} = \mathbf{0}$ and the proof of uniqueness is completed. $\square$

Corollary 9. *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy* (5), (71) *and*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \ h \in L^r(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \ \text{with } \frac{1}{r} \leqq \frac{1}{p} + \frac{1}{s}, \tag{80}$$

where $\max\{r, 3\} \leqq p$ *and s is defined by* (64). *Then, the solution $(\boldsymbol{u}, q)$ given by Theorem* 19 *point (i) belongs to $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$. Moreover, if $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the condition* (74)*, then this solution is unique, up to a constant for q.*

Proof. First, we observe that the assumptions (80) imply that the assumptions of Theorem 19 are verified. Then let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 19 and satisfying the estimate

$$\| \boldsymbol{u} \|_{\mathbf{L}^3(\Omega)} \leqq C \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \right).$$

Observe, then, that $(\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow (\mathbf{X}_{r_0',p'}(\Omega))'$ and $L^r(\Omega) \hookrightarrow L^{r_0}(\Omega)$ where $1/r_0 = 1/p + 1/3$. Using Theorem 17 , there exist a unique $(\boldsymbol{w}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying $-\Delta \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \pi = \boldsymbol{f} = -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q$, div $\boldsymbol{w} = h$ in Ω and $\boldsymbol{w} = \boldsymbol{g}$ on Γ. Setting $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $\theta = \pi - q$, that means that

$$-\Delta \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \nabla \theta = \mathbf{0}, \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

and thanks to Theorem 17 and the uniqueness argument, we deduce that $\boldsymbol{z} = \mathbf{0}$, $\nabla \pi = \nabla q$ and then $\boldsymbol{w} = \boldsymbol{u}$. The uniqueness of $(\boldsymbol{u}, q)$, up to a constant for q, is immediate. $\square$

6.2. The case of arbitrary external forces

Lemma 14. *Let Ω be a Lipschitz bounded open set. Then, the space $\mathcal{D}(\Omega)$ is dense in $(\mathbf{X}_{r,p}(\Omega))'$.*

Proof. From Lemma 9, we have the characterization for the functions belonging to the space $(\mathbf{X}_{r,p}(\Omega))'$:

$$\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))' \quad \Leftrightarrow \quad \boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1,$$

where $\mathbb{F}_0 = (f_{ij})_{1 \leqq i,j \leqq 3} \in \mathbb{L}^{r'}(\Omega)$ and $f_1 \in W^{-1,p'}(\Omega)$, being

$$\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1 \leqq i, j \leqq 3, \ \|f_1\|_{W^{-1,p'}(\Omega)}\}.$$

As $\mathcal{D}(\Omega)$ is dense in $L^r(\Omega)$ and $W^{-1,r}(\Omega)$, we can deduce that for any $\varepsilon > 0$ there exist $\mathbb{F}_0^\varepsilon \in (\mathcal{D}(\Omega))^{3\times 3}$ and $f_1^\varepsilon \in \mathcal{D}(\Omega)$ such that:

$$\max_{1 \leqq i,j \leqq 3} \|f_{ij}^\varepsilon - f_{ij}\|_{L^{r'}(\Omega)} \leqq \varepsilon, \qquad \|f_1^\varepsilon - f_1\|_{W^{-1,p'}(\Omega)} \leqq \varepsilon$$

Then considering $\boldsymbol{f}_\varepsilon = \nabla \cdot \mathbb{F}_0^\varepsilon + \nabla f_1^\varepsilon \in \mathcal{D}(\Omega)^3$, we have that:

$$\begin{aligned}\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{r,p}(\Omega)]'} &= \max\{\|f_{ij}^\varepsilon - f_{ij}\|_{L^{r'}(\Omega)}, 1 \leqq i, j \leqq 3, \ \|f_1^\varepsilon - f_1\|_{W^{-1,p'}(\Omega)}\} \\ &\leqq \varepsilon\end{aligned}$$

□

Lemma 15. *Let Ω be a Lipschitz bounded open set. Let $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ be such that the condition (5) holds. For every $\varepsilon > 0$, there exist sequences $(h_\varepsilon) \subset \mathcal{D}(\overline{\Omega})$ and $(\boldsymbol{g}_\varepsilon) \subset \boldsymbol{\mathcal{C}}^\infty(\Gamma)$ such that*

$$\int_\Omega h_\varepsilon(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_\Gamma \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n} \, \mathrm{d}\sigma \tag{81}$$

and verifying

$$\|h - h_\varepsilon\|_{L^r(\Omega)} \leqq \varepsilon \quad \text{and} \quad \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leqq \varepsilon \tag{82}$$

$$\|h_\varepsilon\|_{L^r(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leqq 2\left(\|h\|_{L^r(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\right). \tag{83}$$

Proof. Let $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ such that the condition (5) holds. By Theorem 11, we know that there exists $\boldsymbol{v} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)$ such that:

$$-\Delta \boldsymbol{v} + \nabla q = \mathbf{0} \quad \text{and} \quad \nabla \cdot \boldsymbol{v} = h \quad \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{g} \quad \text{on } \Gamma.$$

Then, we can deduce that $\Delta \boldsymbol{v} \in (\mathbf{X}_{p'}(\Omega))'$ and, therefore, $\boldsymbol{v} \in \mathbf{T}_p(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div}; \Omega)$. By Lemma 10 point (ii), there exists $(\boldsymbol{v}_\varepsilon) \subset \boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \ \text{ in } \mathbf{L}^p(\Omega), \ \Delta \boldsymbol{v}_\varepsilon \to \Delta \boldsymbol{v} \ \text{ in } (\mathbf{X}_{p'}(\Omega))' \quad \text{and} \quad \nabla \cdot \boldsymbol{v}_\varepsilon \to \nabla \cdot \boldsymbol{v} \quad \text{in } L^r(\Omega).$$

Considering $\boldsymbol{g}_\varepsilon = \boldsymbol{v}_\varepsilon|_\Gamma$ and $h_\varepsilon = \nabla \cdot \boldsymbol{v}_\varepsilon$, then we can deduce, thanks to Lemma 12 and Lemma 13 point (ii), that:

$$h_\varepsilon \to h \quad \text{in} \quad L^r(\Omega) \quad \text{and} \quad \boldsymbol{g}_\varepsilon \to \boldsymbol{g} \quad \text{in} \quad \mathbf{W}^{-1/p,p}(\Gamma)$$

and then we obtain the estimates (82) and (83). □

Theorem 20. (Very weak solution of Navier–Stokes, arbitrary forces) *Let $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ satisfy the compatibility condition* (5). *There exists a constant $\delta > 0$ depending only on Ω such that if:*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leqq \delta, \tag{84}$$

then the problem (NS) has a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$.

Proof. We decompose the problem into two parts. First, to find $(\boldsymbol{v}_\varepsilon, q_\varepsilon^1)$ solution of the problem:

$$(NS_1)\begin{cases} -\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla q_\varepsilon^1 = \boldsymbol{f} - \boldsymbol{f}_\varepsilon & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v}_\varepsilon = h - h_\varepsilon & \text{in } \Omega, \\ \boldsymbol{v}_\varepsilon = \boldsymbol{g} - \boldsymbol{g}_\varepsilon & \text{on } \Gamma, \end{cases}$$

and then to find $(\boldsymbol{z}_\varepsilon, q_\varepsilon^2)$, solution of the problem:

$$(NS_2)\begin{cases} -\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla q_\varepsilon^2 = \boldsymbol{f}_\varepsilon & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{z}_\varepsilon = h_\varepsilon & \text{in } \Omega, \\ \boldsymbol{z}_\varepsilon = \boldsymbol{g}_\varepsilon & \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$, $h_\varepsilon \in L^2(\Omega)$ and $\boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$ satisfy

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leqq \varepsilon$$

and

$$\|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_j}| \leqq 2\delta$$

(see Lemmas 14 and 15). The pair $(\boldsymbol{u}, q) = (\boldsymbol{v}_\varepsilon + \boldsymbol{z}_\varepsilon, q_\varepsilon^1 + q_\varepsilon^2)$ is then a solution to problem (NS).

Given $\boldsymbol{f} - \boldsymbol{f}_\varepsilon \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h - h_\varepsilon \in L^{3/2}(\Omega)$ and $\boldsymbol{g} - \boldsymbol{g}_\varepsilon \in \mathbf{W}^{-1/3,3}(\Gamma)$ such that $\langle (\boldsymbol{g} - \boldsymbol{g}_\varepsilon) \cdot \boldsymbol{n}, 1\rangle_\Gamma = \int_\Omega (h - h_\varepsilon)\, \mathrm{d}\boldsymbol{x}$, the previous study given in Theorem 19 leads us to conclude that, if ε is small enough, then there exists $(\boldsymbol{v}_\varepsilon, q_\varepsilon^1) \in \mathbf{L}^3(\Omega) \times W^{-1/3,3}(\Omega)$ solution to (NS_1) with

$$\begin{aligned} \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} \leqq C \big(& \|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} \\ & + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \big) := \delta(\varepsilon) \end{aligned} \tag{85}$$

Later, we focus on the study of problem (NS_2). First, using Hopf's Lemma (see [20], Remark VIII.4.4, for instance) we lift $h_\varepsilon \in L^2(\Omega)$ and the boundary data $\boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$ (observe that (81) is verified): for any $\alpha > 0$, there exists $\boldsymbol{y}_\varepsilon \in \mathbf{H}^1(\Omega)$, depending on α, such that:

$$\nabla \cdot \boldsymbol{y}_\varepsilon = h_\varepsilon \text{ in } \Omega, \quad \boldsymbol{y}_\varepsilon = \boldsymbol{g}_\varepsilon \text{ on } \Gamma \quad \text{and for any } \boldsymbol{w} \in \mathbf{H}_0^1(\Omega) \text{ with } \nabla \cdot \boldsymbol{w} = 0,$$

$$\begin{aligned} \left| \int_\Omega (\boldsymbol{w} \cdot \nabla) \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}\, \mathrm{d}\boldsymbol{x} \right| &\leqq \left(\alpha + \|h_\varepsilon\|_{L^{3/2}(\Omega)} + C \sum_{i=1}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \right) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)} \\ &\leqq (\alpha + 2C_1\delta) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)}, \end{aligned}$$

where $C_1 > 0$ depends only on Ω. Therefore, the study of problem (NS_2) becomes the study of:

$$(\widetilde{NS_2}) \begin{cases} -\Delta \boldsymbol{w}_\varepsilon + (\boldsymbol{v}_\varepsilon + \boldsymbol{w}_\varepsilon + \boldsymbol{y}_\varepsilon) \cdot \nabla \boldsymbol{w}_\varepsilon + \nabla q_\varepsilon^2 & \\ \qquad + \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon + \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon = \mathbf{F}_\varepsilon & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{w}_\varepsilon = 0 & \text{in } \Omega, \\ \boldsymbol{w}_\varepsilon = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{w}_\varepsilon = \boldsymbol{z}_\varepsilon - \boldsymbol{y}_\varepsilon$ and $\mathbf{F}_\varepsilon = \boldsymbol{f}_\varepsilon + \Delta \boldsymbol{y}_\varepsilon - \boldsymbol{y}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon - \boldsymbol{y}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon - \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$. Note that $\boldsymbol{y}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon = \nabla \cdot (\boldsymbol{y}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - (\nabla \cdot \boldsymbol{y}_\varepsilon)\, \boldsymbol{v}_\varepsilon$ and, since $\boldsymbol{y}_\varepsilon \in \mathbf{L}^6(\Omega)$, then $\boldsymbol{y}_\varepsilon \otimes \boldsymbol{v}_\varepsilon \in \mathbb{L}^2(\Omega)$ and $h_\varepsilon\, \boldsymbol{v}_\varepsilon \in \mathbf{L}^{6/5}(\Omega)$.
Taking $\boldsymbol{w}_\varepsilon$ as a test function in $(\widetilde{NS_2})$, we obtain:

$$\begin{aligned} &\|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}_\varepsilon \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon \,\mathrm{d}\boldsymbol{x} - \int_\Omega h\, |\boldsymbol{w}_\varepsilon|^2 \,\mathrm{d}\boldsymbol{x} \\ &\quad = \langle \mathbf{F}_\varepsilon, \boldsymbol{w}_\varepsilon \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)} \end{aligned}$$

The bounds of every term can be given by:

$$\left| \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}_\varepsilon \,\mathrm{d}\boldsymbol{x} \right| \leqq (\alpha + 2C_1\delta)\, \|\boldsymbol{w}_\varepsilon\|^2_{\mathbf{H}^1(\Omega)}$$

where α is chosen small enough and

$$\begin{aligned} \left| \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon \cdot \boldsymbol{w}_\varepsilon \,\mathrm{d}\boldsymbol{x} \right| &= \left| - \int_\Omega \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{w}_\varepsilon \cdot \boldsymbol{v}_\varepsilon \,\mathrm{d}\boldsymbol{x} \right| \\ &\leqq \|\boldsymbol{w}_\varepsilon\|_{\mathbf{L}^6(\Omega)} \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)} \|\nabla \boldsymbol{w}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \\ &\leqq C_2\, \|\boldsymbol{v}_\varepsilon\|_{\mathbf{L}^3(\Omega)}\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \leqq C_2\, \delta(\varepsilon)\, \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \end{aligned}$$

being $\delta(\varepsilon)$ given by (85) and C_2 the constant of continuity of the Sobolev embedding $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$. Moreover,

$$\left| \int_\Omega h |\boldsymbol{w}_\varepsilon|^2 \,\mathrm{d}\boldsymbol{x} \right| \leqq \|h\|_{L^{3/2}(\Omega)} \|\boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^6(\Omega)} \leqq C_2 \|h\|_{L^{3/2}(\Omega)} \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}.$$

We choose ε, α and $\|h\|_{L^{3/2}(\Omega)}$ such that $\alpha + 2C_1\delta + C_2\delta_\varepsilon + C_2\|h\|_{L^{3/2}(\Omega)} \leqq 1/2$. Now, the classical theory for the problem (NS_2) and an approach analogous to Theorem 19 for the (NS_1), imply the existence of a solution $(\boldsymbol{w}_\varepsilon, q_\varepsilon^2) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ and that concludes the proof. □

Theorem 21. *Let* $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *be the solution given by Theorem 20. Then, the following regularity results hold:*

(i) *Suppose that*

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \;\; h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

with $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ *and* $\max\{r, 3\} \leqq p$. *Then* $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$.

(ii) *Let* $r \geqq 3/2$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega), \;\; h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma). \tag{86}$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

(iii) *Let* $1 < r < \infty$ *and suppose that*

$$\boldsymbol{f} \in \mathbf{L}^r(\Omega), \;\; h \in W^{1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma). \tag{87}$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.

Proof. First, we remark that under the assumptions in (i), (ii) and (iii), we have that $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$.
(i) Let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 20. Using Theorem 17, there exist a unique $(\boldsymbol{w}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying $-\Delta \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \nabla \pi = \boldsymbol{f} = -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q$, $\operatorname{div} \boldsymbol{w} = h$ in Ω and $\boldsymbol{w} = \boldsymbol{g}$ on Γ. Setting $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{u}$ and $\theta = \pi - q$, this means that

$$-\Delta \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \nabla \theta = \mathbf{0}, \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

and thanks to Theorem 17 and the uniqueness argument, we deduce that $\boldsymbol{z} = \nabla \theta = \mathbf{0}$ and then $\boldsymbol{w} = \boldsymbol{u}$ and $\pi = q + c$, with c constant. Point (i) is proved.
(ii) Let $r \geqq 3/2$ and $\boldsymbol{f}, h, \boldsymbol{g}$ satisfying (86). Let $p \geqq 3$ defined by $1/p = 1/r - 1/3$. Then $\mathbf{W}^{1-1/r,r}(\Gamma) \hookrightarrow \mathbf{W}^{-1/p,p}(\Gamma)$ and $\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))'$. If $r \leqq 3$, by point (i), we deduce that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ and then $\boldsymbol{u} \otimes \boldsymbol{u} \in \mathbf{L}^r(\Omega)$. But $-\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} - \operatorname{div} (\boldsymbol{u} \otimes \boldsymbol{u}) \in \mathbf{W}^{-1,r}(\Omega)$ and by Stokes regularity, we obtain that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$. If now $r > 3$, we know that $u \in \mathbf{W}^{1,3}(\Omega)$ and thanks to Sobolev embeddings, $\boldsymbol{u} \otimes \boldsymbol{u} \in \mathbf{L}^r(\Omega)$ and again, as above, we deduce that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.
(iii) Let $1 < r < \infty$ and $\boldsymbol{f}, h, \boldsymbol{g}$ satisfy (87). We observe first that $\mathbf{L}^r(\Omega) \hookrightarrow \mathbf{W}^{-1,3/2}(\Omega)$, $W^{1,r}(\Omega) \hookrightarrow L^{3/2}(\Omega)$ and $\mathbf{W}^{2-1/r,r}(\Gamma) \hookrightarrow \mathbf{W}^{1/3,3/2}(\Gamma)$ and then by step ii), we obtain that $(\boldsymbol{u}, q) \in \mathbf{W}^{1,3/2}(\Omega) \times L^{3/2}(\Omega)$. If $r < 3$, we deduce, thanks to Theorem 16, that $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$. If now $r \geqq 3$, then $\boldsymbol{u} \in \mathbf{L}^\infty(\Omega)$ and, again using Theorem 16, we obtain the same conclusion. □

Theorem 22. *Let* $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *be the solution given by Theorem 20. Then, the following regularity results hold:*

(i) *Suppose that* $3/2 \leqq p \leqq 3$ *and let*

$$\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega), \quad f_1 \in W^{\sigma-1,p}(\Omega), \;\; h \in W^{\sigma,r}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),$$

with $\sigma = \frac{3}{p} - 1$, $\frac{1}{r} \leqq \frac{1}{p} + \frac{1}{3}$ *and* $r \leqq p$. *Then* $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

(ii) *Let* σ *be such that* $1/p < \sigma \leqq 1$ *and* $\sigma \geqq 3/p - 1$. *Suppose that*

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \;\; h \in W^{\sigma-1,p}(\Omega), \;\; \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Then $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

Proof. (i) Note that $\mathbf{W}^{\sigma,r}(\Omega) \hookrightarrow \mathbf{L}^{3/2}(\Omega)$ because $\frac{1}{r} - \frac{\sigma}{3} = \frac{1}{r} - \frac{1}{p} + \frac{1}{3} \leqq \frac{2}{3}$. We also have the embedding $\mathbf{W}^{\sigma-1/p,p}(\Gamma) \hookrightarrow \mathbf{L}^2(\Gamma) \hookrightarrow \mathbf{W}^{-1/3,3}(\Gamma)$ because $1/p - (\sigma - 1/p)/2 = 1/2$ and $2/3 - 1/6 = 1/2$. As $3/2 \leqq p \leqq 3$, then $0 \leqq \sigma \leqq 1$ and we apply Corollary 8 point (i).
(ii) When $\sigma = 1$, the property is a consequence of Theorem 15. If $\sigma < 1$, then $p > \frac{3}{2}$ and Theorem 18 implies that $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$. □

Remark 16. In particular, when $p = 2$ and $r = 6/5$, if

$$\boldsymbol{f} \in \mathbf{W}^{-1/2,6/5}(\Omega),\ h \in W^{1/2,6/5}(\Omega),\ \boldsymbol{g} \in \mathbf{L}^2(\Gamma),$$

then the solution given by the previous theorem satisfies $(\boldsymbol{u}, q) \in \mathbf{H}^{1/2}(\Omega) \times H^{-1/2}(\Omega)$.

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