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Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory

PART III

Edited by O. A. Ladyzhenskaya

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О. А. ЛАДЫЖЕНСКАЯ

КРАЕВЫЕ ЗАДАЧИ МАТЕМАТИЧЕСКОЙ ФИЗИКИ И СМЕЖНЫЕ
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EDITOR'S NOTE

In the present compendium two papers are devoted to the study of *a priori* estimates of the maxima of moduli of first derivatives for the solutions of various classes of nonuniformly elliptic and nonuniformly parabolic equations. In the paper by O. A. Ladyzhenskaya, an investigation is made of the solvability of an initial boundary problem for nonstationary systems of Navier–Stokes equations in domains where the boundary varies with time. In the paper by V. P. Il'in, relations are obtained between convergences in $W_q^s(\Omega)$ and $W_p^\ell(\Omega)$ in the class of functions of polynomial type. In the paper by N. K. Nikol'skii and B. S. Pavlov, necessary and sufficient conditions are given for a system of characteristic vectors of a contraction operator, close to unitary, to form an absolute basis in Hilbert space. In the paper by A. A. Chervyakova, it is shown that if the Fourier transform of a generalized function is "smeared" along a smooth curve of nonzero curvature, then this function diminishes at infinity as $\tau^{-1/2}$.

O. A. Ladyzhenskaya

CONTENTS

NONLOCAL ESTIMATES OF FIRST DERIVATIVES OF THE SOLUTIONS OF THE INITIAL BOUNDARY PROBLEM FOR NONUNIFORMLY ELLIPTIC AND NONUNIFORMLY PARABOLIC NONDIVERGENT EQUATIONS

N. M. Ivochkina and A. P. Oskolkov

This paper is devoted to a study of nonlocal *a priori* estimates of maxima of moduli of the first derivatives of solutions of Dirichlet's problem, and, correspondingly, the first initial-boundary problem for nonuniformly elliptic and nonuniformly parabolic nondivergent quasi-linear equations. It is closely related to known investigations of O. A. Ladyzhenskaya and N. N. Ural'tseva on quasi-linear elliptic and parabolic equations and systems [1, 2]. A characteristic peculiarity of the paper is the fact that the method, developed by O. A. Ladyzhenskaya and N. N. Ural'tseva, for obtaining *a priori* estimates of maxima of moduli of the first derivatives for solutions of uniformly elliptic and uniformly parabolic quasi-linear equations with divergent principal part, is used here for studying analogous estimates for solutions of nondivergent equations; moreover, the method enables one to investigate specific classes of nonuniformly elliptic and nonuniformly parabolic quasi-linear equations, including those not belonging to S. N. Bernshtein's class (L).

Let Ω be an n-dimensional bounded domain with boundary $\partial\Omega$ of class $C^{(2)}$, $n \geq 2$, $Q_T = \Omega \times [0,T]$, $\partial Q_T = \partial\Omega \times [0,T]$, $T > 0$. We shall say that $u(x,t) \in C^{(2,1)}(Q_T)$ if $u(x,t)$ is twice continuously differentiable with respect to the space variables $x = (x_1, \ldots x_n)$, and continuously differentiable with respect to the time variable t, when $(x,t) \in Q_T$; in addition, we introduce the following notation:

$$M[u; Q_T] \equiv \max_{Q_T} |u(x,t)|, \tag{0.1}$$

$$\omega(u; \delta; Q_T) \equiv \max_{[0,T]} \max_{\substack{x,x' \in \Omega \\ |x-x'| \leq \delta}} |u(x',t) - u(x,t)| \tag{0.2}$$

$$M_{1,x}[u; Q_T] \equiv \max_{Q_T} \sqrt{\sum_{i=1}^{n} u_{x_i}^2(x,t)}. \tag{0.3}$$

Further, we shall say that $A(x,t,u,p) \in \mathfrak{M}^{(l)}(Q_T, M, \infty)$, $l = 0,1,2,\ldots$, $p = (p_i)$, if the function $A(x,t,u,p)$ is l times continuously differentiable with respect to (x,u,p) for $(x,t) \in Q_T$, $|u| \leq M$, $|p| \equiv \sqrt{\sum_{i=1}^{n} p_i^2} < \infty$.

Let us consider in the cylinder Q_T the first initial-boundary problem for the quasi-linear parabolic equation

$$Lu \equiv \frac{\partial u}{\partial t} - a_{ij}(x,t,u,u_x)u_{x_i x_j} + a(x,t,u,u_x) = 0, \tag{0.4}$$

$$u|_{t=0} = f(x), \quad f(x) \in C^{(2)}(\Omega), \tag{0.5}$$

$$u|_{\partial Q_T} = \mathcal{F}(x,t), \quad \mathcal{F}(x,t) \in C^{(2,1)}(\partial Q_T), \tag{0.6}$$

and let us assume that the coefficients $a_{ij}(x,t,u,u_x) \in \mathfrak{M}^{(1)}(Q_T, M[u;Q_T], \infty)$, $i,j = 1,\dots,n$, satisfy the following conditions:

$$\nu(|u|)\lambda(b)|\xi|^2 \leq a_{ij}(x,t,u,p)\xi_i\xi_j, \quad \xi \neq 0, \tag{0.7}$$

$$\sum \left|\frac{\partial a_{ij}}{\partial p_l}\right| \leq \mu(|u|)\theta(b), \tag{0.8}$$

$$\sum\left|\frac{\partial a_{ij}}{\partial u}\right| b^{1/2} + \sum\left|\frac{\partial a_{ij}}{\partial x_l}\right| \leq \begin{cases} \mu(|u|)\pi(b)\dfrac{\lambda^2(b)}{\theta(b)}, & \text{if } \lambda(b) \geq \frac{1}{b}, \\[2ex] \mu(|u|) b^{-1/2}\dfrac{\lambda^{3/2}(b)}{\theta(b)}, & \text{if } \lambda(b) < \frac{1}{b}, \end{cases} \tag{0.9}$$

and that the coefficient $a(x,t,u,p)$ satisfies one of the following conditions:

I. $a(x,t,u,p) \in \mathfrak{M}^{(0)}(Q_T, M[u;Q_T], \infty)$, and the following inequalities are satisfied:

$$|a| \leq \begin{cases} \mu(|u|)\pi(b)\dfrac{\lambda^3(b)}{\theta^2(b)}, & \text{if } \lambda(b) \geq \frac{1}{b}, \\[2ex] \mu(|u|) b^{-1/2}\dfrac{\lambda^{5/2}(b)}{\theta^2(b)}, & \text{if } \lambda(b) < \frac{1}{b}; \end{cases} \tag{0.10}$$

II. $a(x,t,u,p) \in \mathfrak{M}^{(1)}(Q_T, M[u;Q_T], \infty)$, and we have the following inequalities:

1. $$\sum\left|\frac{\partial a}{\partial p_l}\right| + \left|\frac{\partial a}{\partial u}\right| b^{1/2}\frac{\theta^2(b)}{\lambda^2(b)} + \sum\left|\frac{\partial a}{\partial x_l}\right|\frac{\theta^2(b)}{\lambda^2(b)} \leq \mu(|u|)\pi(b) b^{1/2}\lambda(b), \quad \text{if } \lambda(b) \geq \frac{1}{b}; \tag{0.11}$$

2. $$\sum\left|\frac{\partial a}{\partial p_l}\right| + \left|\frac{\partial a}{\partial u}\right| b\,\frac{\theta^2 b}{\lambda^{3/2}(b)} + \sum\left|\frac{\partial a}{\partial x_l}\right| b^{1/2}\frac{\theta^2(b)}{\lambda^{3/2}(b)} \leq \mu(|u|)\lambda^{1/2}(b), \quad \text{if } \lambda(b) < \frac{1}{b}; \tag{0.12}$$

III. $a(x,t,u,p) \in \mathfrak{M}^{(2)}(Q_T, M[u;Q_T], \infty)$, and the following inequalities are satisfied:

$$\left|\frac{\partial a}{\partial u}\right| b^{1/2}\frac{\theta^2(b)}{\lambda^2(b)} + \sum\left|\frac{\partial a}{\partial x_l}\right|\frac{\theta^2(b)}{\lambda^2(b)} + \sum\left|\frac{\partial^2 a}{\partial p_k \partial p_l}\right|\frac{\lambda(b)}{\theta(b)} + \sum\left|\frac{\partial^2 a}{\partial u \partial p_k}\right| + b^{-1/2}\sum\left|\frac{\partial^2 a}{\partial x_l \partial p_k}\right| \leq \begin{cases} \mu(|u|)\pi(b) b^{1/2}\lambda(b), & \text{if } \lambda(b) \geq \frac{1}{b}, \\[2ex] \mu(|u|) b^{-1/2}, & \text{if } \lambda(b) < \frac{1}{b}.^{*} \end{cases} \tag{0.13}$$

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*If $\lambda(b) < \frac{1}{b}$, then $\sum\left|\frac{\partial^2 a}{\partial p_k \partial p_l}\right| \leq \frac{\theta(b)}{\sqrt{\lambda(b)}}$.

In conditions (0.7)-(0.13), $\mathscr{b} \equiv 1+|p^2| \equiv 1+\sum_{i=1}^{n} p_i^2$, $\nu(\tau)$ and $\mu(\tau)$, $0 \leq \tau \leq M[u; Q_T]$ are, respectively, nonincreasing and nondecreasing positive functions, $\lambda(\mathscr{b})$ and $\theta(\mathscr{b})$, $1 \leq \mathscr{b} < \infty$, are strictly positive continuously differentiable functions, monotone for large $\mathscr{b}$, and satisfy the following intrinsic conditions:*

$$\frac{\lambda(\mathscr{b})}{\mathscr{b}^{1/2}} \leq C_\lambda \cdot \theta(\mathscr{b}), \tag{0.14}$$

if $\lambda(\mathscr{b}) \leq \frac{1}{\mathscr{b}}$, and coefficient a satisfies conditions II or III;

$$\left.\begin{aligned} &\frac{\lambda(\mathscr{b})}{\mathscr{b}^{1/2}} \leq C_\lambda \cdot \theta(\mathscr{b}), \\ & \qquad\qquad\qquad \text{if} \quad \lambda(\mathscr{b}) > \frac{1}{\mathscr{b}}; \\ &|\lambda'(\mathscr{b})| \leq C_\lambda \cdot \frac{\theta(\mathscr{b})}{\mathscr{b}^{1/2}}, \end{aligned}\right\} \tag{0.15}$$

$$\left.\begin{aligned} &\frac{\lambda(\mathscr{b})}{\mathscr{b}^{1/2}} \leq C_\lambda \cdot \theta(\mathscr{b}), \\ &-\lambda'(\mathscr{b}) \leq C_\lambda \cdot \frac{\theta(\mathscr{b})}{\mathscr{b}^{1/2}}, \end{aligned}\right\} \tag{0.16}$$

if $\frac{1}{\mathscr{b}} < \lambda(\mathscr{b}) \leq 1$, and coefficient a satisfies condition I; further,

$$\mathscr{T}(\mathscr{b}) \equiv 1,$$

if, for the solution $u(x,t)$, $M[u; Q_T]$ is known *a priori*, along with the modulus of continuity $\omega(u; \delta; Q_T)$, and

$$\mathscr{T}(\mathscr{b}) > 0, \quad \mathscr{T}(\mathscr{b}) \downarrow 0, \quad \mathscr{b} \to \infty,$$

if, for the solution $u(x,t)$, only $M[u; Q_T]$ is known *a priori*.

We shall, in addition, consider classes of equations (0.4), whose coefficients satisfy, besides the conditions (0.7)-(0.8), the following conditions:

$$\frac{\partial a}{\partial u} \geq \nu_1(|u|)\, \Lambda(\mathscr{b}), \quad \nu_1 > 0, \tag{0.17}$$

$$\sum \left| \frac{\partial a}{\partial p_i} \right| \leq \mu(|u|)\, \theta(\mathscr{b}) \sqrt{\mathscr{b} \frac{\Lambda(\mathscr{b})}{\lambda(\mathscr{b})}}\, \mathscr{P}(\mathscr{b}), \tag{0.18}$$

$$\sum \left| \frac{\partial a_{ij}}{\partial u} \right| + \mathscr{b}^{-1/2} \sum \left| \frac{\partial a_{ij}}{\partial x_i} \right| + \mathscr{b}^{-1/2} \sqrt{\frac{\Lambda(\mathscr{b})}{\lambda(\mathscr{b})}} \sum \left| \frac{\partial a}{\partial x_i} \right| \leq \mu(|u|) \sqrt{\frac{\lambda(\mathscr{b})\Lambda(\mathscr{b})}{\mathscr{b}}}\, \mathscr{P}(\mathscr{b}), \tag{0.19}$$

where $\Lambda(\mathscr{b})$ is an arbitrary strictly positive function, $\lambda(\mathscr{b})$, $\theta(\mathscr{b})$ are strictly positive functions, which satisfy inequality (0.14) for arbitrary $\lambda(\mathscr{b})$, and $\mathscr{P}(\mathscr{b}) \downarrow 0$ when $\mathscr{b} \to \infty$.

The first part of our paper is devoted to a study of nonlocal estimates of maxima of moduli of the first derivatives, with respect to the space variables, of the solutions $u(x,t)$ of the first initial-boundary problem (0.4)-(0.6) for the quasi-linear parabolic equation (0.4), the coefficients of which satisfy the conditions (0.7)-(0.16), or the conditions (0.7)-(0.8), (0.17)-(0.19). In obtaining these estimates we shall assume that the solution $u(x,t)$ belongs to class $C^{(2,1)}(Q_T)$, and that for it we have *a priori* knowledge of the following: a) $M[u; Q_T]$ and the modulus of continuity $\omega(u; \delta; Q_T)$, or, correspond-

*Conditions (0.14)-(0.16) are intrinsic in the sense that they are automatically satisfied (with the sign of equality) if the coefficients $a_{ij}(x, t, u, p)$ have orders of increase or decrease which are powers of $|p|$.

ingly, only $M[u;Q_T]$, b) $\max_{\partial Q_T} |\nabla_x u(x,t)| \equiv M_{1,x}[u;\partial Q_T]$; and we shall show that the following theorem is valid.

Theorem 1. Let $u(x,t)$ be a solution of the initial-boundary problem (0.4)-(0.6) of class $C^{(2,1)}(Q_T)$, for which the following are known *a priori* : a) $M[u;Q_T]$ and $\omega(u;\delta;Q_T)$, or, correspondingly, only $M[u;Q_T]$; b) $M_{1,x}[u;\partial Q_T]$, and let the coefficients of equation (0.4) satisfy conditions (0.7)-(0.16) or conditions (0.7)-(0.8), (0.17)-(0.19). Then, the quantity $M_{1,x}[u;Q_T]$ can be estimated only in terms of $M[u;Q_T]$ and $\omega(u;\delta;Q_T)$ (correspondingly, $M[u;Q_T]$, $M_{1,x}[u;\partial Q_T]$), the constants $\nu(M)$ $\mu(M)$, $\nu_1(M)$, $|f|_{C^{(1)}(\Omega)}$, $|\mathcal{F}|_{C^{(2,1)}(\partial Q_T)}$, mes Ω, T and the boundary $\partial\Omega \in C^{(2)}$.

Conditions sufficient for obtaining an estimate of the quantity $M_{1,x}[u;\partial Q_T]$ are given in Lemma 3 (see §1).

Further, let us consider in the domain Ω the Dirichlet problem for the quasi-linear elliptic equation

$$Lu \equiv a_{ij}(x,u,u_x)u_{x_ix_j} - a(x,u,u_x) = 0, \tag{0.20}$$

$$u|_{\partial\Omega} = \mathcal{F}(x), \quad \mathcal{F}(x) \in C^{(2)}(\partial\Omega) \tag{0.21}$$

and let us assume that the coefficients $a_{ij}(x,u,p)$, $i,j=1,\dots,n$, and $a(x,u,p)$ of equation (0.20) satisfy conditions (0.7)-(0.8), (0.18)-(0.19) with $\lambda(b) \equiv 1$.

Let us introduce the notation

$$M[u;\Omega] \equiv \max_{\Omega} |u(x)|, \tag{0.22}$$

$$\omega(u;\delta;\Omega) \equiv \max_{\substack{x,x'\in\Omega \\ |x-x'|\le\delta}} |u(x)-u(x')|, \tag{0.23}$$

$$M_1[u;\Omega] \equiv \max_{\Omega} \sqrt{\sum_{i=1}^{n} u_{x_i}^2(x)} \ . \tag{0.24}$$

The second part of the paper is devoted to a study of nonlocal *a priori* estimates of maxima of moduli of the first derivatives of solutions of the Dirichlet problem (0.20)-(0.21) for equation (0.20), the coefficients of which satisfies conditions (0.7)-(0.16), or conditions (0.7)-(0.8), (0.17-(0.19) with $\lambda(b) \equiv 1$. We shall show that the following theorem is valid.

Theorem 2. Let $u(x)$ be the solution of the Dirichlet problem (0.20)-(0.21) of class $C^{(2)}(\Omega)$, for which it is assumed that the following are known *a priori*: a) $M[u;\Omega]$ and $\omega(u;\delta;\Omega)$, or, correspondingly, only $M[u;\Omega]$, b) $M_1[u;\partial\Omega]$; and let the coefficients of equation (0.20) satisfy conditions (0.7)-(0.16) or (0.7)-(0.8), (0.17)-(0.19) with $\lambda(b) \equiv 1$.* Then, the quantity $M_1[u;\Omega]$ is estimateable only in terms of $M[u;\Omega]$ and $\omega(u;\delta;\Omega)$ (correspondingly, $M[u;\Omega]$), $M_1[u;\partial\Omega]$, the constants $\nu(M)$, $\mu(M)$, $\nu_1(M)$, $|\mathcal{F}|_{C^{(2)}(\partial\Omega)}$, mes Ω , and the boundary $\partial\Omega \in C^{(2)}$.

*If the coefficients of equation (0.20) satisfy conditions (0.17)-(0.19) (with $\lambda(b) \equiv 1$), then it is sufficient to assume that only $M_1[u;\partial\Omega]$ is known *a priori.*

Sufficient conditions for obtaining an estimate of the quantity $M_1[u;\partial\Omega]$ are given in Lemma 4 of of a previous paper [3] by the authors.*

A priori bounds of maxima of moduli of the first derivatives of solutions of the first boundary problem for uniformly elliptic and uniformly parabolic nondivergent equations with differentiable coefficients were obtained for the first time, as we have already remarked above, by O. A. Ladyzhenskaya and N. N. Ural'tseva (see [1], Chap. 6; [2], Chap. 6). They obtained these bounds subject to conditions on the appearance in the coefficients of the equations of arguments (x,u,u_x), close to "limiting," in the sense of belonging to equations of S. N. Bernshtein's class (L). In addition, O. A. Ladyzhenskaya and N. N. Ural'tseva studied some interesting classes of nonuniformly elliptic equations, including some not belonging to S. N. Bernshtein's class (L) (see [1], Chap. 6, and also [4]). Our conditions (0.7)-(0.9), (0.11), with $\lambda(\ell)\equiv\ell^{\frac{m-2}{2}}$, $\theta(\ell)\equiv\ell^{\frac{m-3}{2}}$, $m>1$, i.e., for the case when equations (0.20) (with $m=2$) and (0.4) are, respectively, uniformly elliptic and uniformly parabolic in the sense of O. A. Ladyzhenskaya and N. N. Ural'tseva [1, 2], and with differentiable a (see conditions II), coincide basically with the conditions of O. A. Ladyzhenskaya and N. N. Ural'tseva. In the present paper we shall, first of all, waive the condition of uniform ellipticity of equation (0.20) and uniform parabolicity of equation (0.4) for differentiable a (see conditions II), i.e., we shall consider arbitrary $\lambda(\ell)>0$ and arbitrary $\theta(\ell)>0$, related only by the intrinsic conditions (0.14), (0.15), and, secondly, we shall consider the case of nondifferentiable (see condition I) and twice differentiable (see conditions III) coefficients a. In this regard, it will be found that the conditions for the appearance of the arguments (x,u) in the coefficients a_{ij} and a are determined by the function $\lambda(\ell)$ (in the parabolic case), and by the function $\theta(\ell)$, becoming worse as $\theta(\ell)$ increases, and that the condition for the appearance of the argument u_x in the coefficient a does not depend on $\theta(\ell)$ for differentiable a (see conditions II) and is actually completely arbitrary for twice differentiable a (see conditions III). We note also that the condition for the appearance of the arguments (x,u,u_x) in the coefficients a_{ij} and a of the parabolic equation (0.4), the coefficient $a(x,t,u,u_x)$ of which satisfies conditions II or III, deteriorate as $\lambda(\ell)$ decreases, with $\lambda(\ell)<\frac{1}{6}$, and, in addition, under these same conditions on a and with $\lambda(\ell)<\frac{1}{6}$ it is not necessary, in the "limiting" case (i.e., with $\pi(\ell)\equiv 1$), to assume the modulus of continuity $M_{1,x}[u;Q_T]$ to be known *a priori* in order to obtain estimates of $\omega(u;\delta;Q_T)$. Further, let us emphasize that, depending on the relation between the functions $\lambda(\ell)$ and $\theta(\ell)$, the conditions (0.11)-(0.13) may be satisfied even by equations not belonging to S. N. Bernshtein's class (L) (the parabolic equations (0.4), with $\lambda(\ell)<\frac{1}{6}$), are automatically of this kind).

The conditions (0.9)-(0.17) show that the more nonuniform the equation (0.4) or (0.20) becomes, i.e., the more rapidly the function $\theta(\ell)$, grows in comparison with $\lambda(\ell)$ as $\ell\to\infty$, the more stringent, generally speaking, become the conditions for the appearance of the arguments (x,u) in the coefficients a_{ij} and a. Hence, it is of some interest to study the class of equations for which the coefficients a_{ij} and a, satisfy, apart from the conditions (0.7)-(0.8), the conditions (0.17)-(0.19). In these conditions the appearance of the arguments (x,u) in the coefficients a_{ij} and a does not depend on $\theta(\ell)$ and is determined entirely by the functions $\lambda(\ell)$ and $\Lambda(\ell)$ appearing in conditions (0.7) and (0.17), and the appearance of the argument u_x may be completely arbitrary whenever $\theta(\ell)$ is completely arbitrary. Let us note, incidentally, that if the coefficient a satisfies condition (0.17), then, for the solutions of the boundary problems (0.4)-(0.6) and (0.20)-(0.21) for the equations (0.4) and (0.20), respectively, the quantities $M[u;Q_T]$ and $M[u;\Omega]$ may be estimated in terms of the initial data of the problem (i.e., the coefficients of the equation and the initial and boundary conditions), so that it is not necessary, in this case, to assume *a priori* knowledge of the quantities $M[u;Q_T]$ and $M[u;\Omega]$ in the proof of Theorems 1 and 2.

*See also the dissertation of N. Trudinger [10] and a recent paper by J. Serrin [11].

The results we have obtained, under conditions I, III and the conditions (0.17)-(0.18) on the "free" term $a(x,t,u,p)$, are new, even for uniformly elliptic and uniformly parabolic quasi-linear equations.

As we have already remarked above, the method used to prove Theorems 1 and 2 is similar to the method used by O. A. Ladyzhenskaya and N. N. Ural'tseva ([1], Chap. 4; [2], Chap. 5) to obtain analogous estimates for uniformly elliptic and uniformly parabolic equations with divergent principal part. We shall first obtain, for the solutions of the boundary problems (0.4)-(0.6) and (0.20)-(0.21), sufficiently strong *a priori* integral estimates, in particular, we shall estimate the integral

$$\int_\Omega \psi^{s+1}(\mathscr{b})\,dx, \quad \mathscr{b} \equiv 1+\sum_{i=1}^n u_{x_i}^2, \quad s=0,1,2,\ldots, \tag{0.25}$$

where

$$\psi(\mathscr{b}) \equiv \int_1^{\mathscr{b}} e^{B\int_1^\tau \theta^2(\xi)d\xi}\,d\tau, \quad B \gg 1, \tag{0.26}$$

in the elliptic case, and

$$\psi(\mathscr{b}) \equiv \int_1^{\mathscr{b}} e^{B\int_1^\tau \frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\,d\tau, \quad B \gg 1, \tag{0.27}$$

in the parabolic case. Following this, we shall derive estimates of the quantities $M_1[u;\Omega]$ and $M_{1,x}[u;Q_T]$ for $\lambda(\mathscr{b}) \geqslant \frac{1}{\mathscr{b}}$, as in the case of equations with divergent principal part, using assumptions of a theoretic-functional nature (namely, Lemma 5.2, Chap. 2 of [1] in the elliptic case and Theorem 6.1, Chap. 2 of [2] in the parabolic case), If, in the parabolic case, $\lambda(\mathscr{b}) < \frac{1}{\mathscr{b}}$, then, using the estimates for the integrals (0.25), we shall obtain, for arbitrary $s=1,2,\ldots$, the following inequality:

$$\left(\iint_{Q_T} \psi^{s+1}(\mathscr{b})\,dx\,dt\right)^{\frac{1}{s+1}} \leqslant c(c_1 sT)^{\frac{1}{s+1}} e^{c_1 T}, \tag{0.28}$$

where the constants c and c_1 depend only on those quantities which appear in the formulation of Theorem 1, and do not depend on s. Letting $s \to \infty$, and using the monotonicity of the function $\psi(\mathscr{b})$, we shall obtain an estimate, from inequality (0.28), of the quantity $M_{1,x}[u;Q_T]$ even for $\lambda(\mathscr{b}) < \frac{1}{\mathscr{b}}$.

Theorems 1 and 2, together with the estimates of the Hölder constants for the first derivatives of the solutions of the boundary problems (0.4)-(0.6) and (0.20)-(0.21), obtained by O. A. Ladyzhenskaya and N. N. Ural'tseva ([1], Chap. 6, § 1; [2], Chap. 6, §1), enable us to prove existence theorems for solutions of the boundary problems (0.4)-(0.6) and (0.20)-(0.21), analogous to the existence theorems obtained by O. A. Ladyzhenskaya and N. N. Ural'tseva ([1], Chap. 6, §3; [2], Chap. 6, §4) for solutions of the first boundary problem for uniformly elliptic and uniformly parabolic nondivergent equations. In addition, Theorems 1 and 2 even permit us to obtain unconditional existence theorems in certain cases (for example, if the coefficients of equation (0.4) or (0.20) satisfy conditions (0.7)-(0.8), (0.17)-(0.19), and these conditions are such that the equations belong to S. N. Bernshtein's class (L) and, by the same token, it is possible to obtain estimates of the quantities $M_{1,x}[u;\partial Q_T]$ and $M_1[u;\partial\Omega]$, respectively).

Some particular cases of Theorem 2 were proved in the authors' previous papers [3, 5]. In the two-dimensional case, an estimate of the quantity $M_1[u;\Omega]$ for the nonuniformly elliptic equation (0.20) under more general conditions on the coefficients than the conditions (0.9)-(0.16) was obtained by one of the authors in [6]. An intrinsic *a priori* estimate of the maximum of the modulus of the first derivatives for a class of two dimensional nonuniformly elliptic nondivergent equations with nondifferentiable coefficients was obtained by N. Meyers in [7].

Several words on the arrangement of the paper and its contents are in order. In §1 we give some auxiliary propositions needed in the sequel, a number of them having some independent interest. In §2 we prove Theorem 1, and in §3, Theorem 2. Formulas and constants in each of these paragraphs are numbered independently.

In conclusion, the authors wish to express their sincere thanks to Professors O. A. Ladyzhenskaya and N. N. Ural'tseva for their interest in our work.

§1. Auxiliary Propositions

As before, let Ω be an n-dimensional bounded domain with boundary $\partial\Omega$ of class $C^{(2)}$, $n \geq 2$. Further, let $K_\tau(x_0)$ be a ball of radius τ with center at the point $x_0 \in \bar{\Omega}$, let $S_\tau(x_0)$ be the boundary of the ball $K_\tau(x_0)$, $\Omega_\tau(x_0) \equiv \Omega \cap K_\tau(x_0)$, and let $\zeta(x)$ be a sectional function for the ball $K_{2\tau}(x_0)$, i.e., a continuously differentiable function, defined in the following way:

$$\zeta(x) = \begin{cases} 1, & |x - x_0| \leq \tau, \\ 0 \leq \zeta \leq 1, & \tau \leq |x - x_0| \leq 2\tau, \\ 0, & |x - x_0| \geq 2\tau, \end{cases} \tag{1.1}$$

$$|\nabla\zeta| < \frac{C_{1,1}}{\tau}. \tag{1.2}$$

Finally, let $\{\Omega_\tau^l(x_0^l)\}$, $l = 1, \ldots, N(\tau)$ be a covering of the domain Ω:

$$\Omega \subset \bigcup_{l=1}^{N} \Omega_\tau^l(x_0^l). \tag{1.3}$$

1°. To obtain integral estimates of the solution of the boundary problems (0.4)-(0.6) and (0.20)-(0.21), substantial use is made of the following lemma (cf. [1], Chap. 2, Lemma 4.5, and [3], Lemma 1).

Lemma 1. Let $u(x)$ be an arbitrary function of class $C^{(1)}(\bar{\Omega}) \cap C^{(2)}(\Omega)$, and suppose that for it the following are known *a priori*: $M[u;\Omega]$, $\omega(u;\delta;\Omega)$, and

$$M_1[u;\partial\Omega] \equiv \max_{\partial\Omega} |\nabla u|. \tag{1.4}$$

Further, let $\lambda(\mathscr{b})$ and $\theta(\mathscr{b})$, $\mathscr{b} \geq 1$, be strictly positive continuous, continuously differentiable, monotonic functions for large $\mathscr{b}$, which satisfy the following conditions:

$$\frac{\lambda(\mathscr{b})}{\mathscr{b}^{1/2}} \leq C_\lambda \cdot \theta(\mathscr{b}), \qquad |\lambda'(\mathscr{b})| \leq C_\lambda \cdot \frac{\theta(\mathscr{b})}{\mathscr{b}^{1/2}}, \tag{1.5}$$

let

$$\psi(\mathscr{b}) \equiv \int_1^{\mathscr{b}} e^{B\int_1^{\tau} \frac{\theta^2(\xi)}{\lambda^2(\xi)} d\xi} d\tau, \quad B \equiv \mathrm{const} \gg 1, \tag{1.6}$$

and let $\mathscr{b} \equiv 1 + |\nabla u|^2$. Then, for large $s \geq 0$ and $\varepsilon > 0$, the following inequality holds:

$$\int_\Omega \mathscr{b}\lambda(\mathscr{b}) \frac{\psi^s(\mathscr{b})\psi'^2(\mathscr{b})}{\psi''(\mathscr{b})} dx \leq \varepsilon \left\{ \int_\Omega \psi^s(\mathscr{b})\psi'(\mathscr{b})\lambda(\mathscr{b}) \sum_{i,k} u^2_{x_i x_k} dx + \int_\Omega \lambda(\mathscr{b}) \left(\psi^s(\mathscr{b})\psi''(\mathscr{b}) + \psi^{s-1}(\mathscr{b})\psi'^2(\mathscr{b}) \right) |\nabla\mathscr{b}|^2 dx \right\} + C_{1,2}, \tag{1.7}$$

wherein the constant $C_{1,2}$ depends only on $M[u;\Omega]$, $M_1[u;\partial\Omega]$, s, ε, the constants C_λ and B, and $\operatorname{mes}\partial\Omega$.

Proof. Let $\{\Omega^{\ell}_{\tau}(x^{\ell}_{0})\}$, $x^{\ell}_{0}\in\bar\Omega$, $\ell=1,\dots,N(\tau)$ be an open covering of the domain Ω, in which the number τ, $0<\tau\ll 1$, will be selcted later. In addition, let

$$h(\mathscr{b})\equiv C\lambda(\mathscr{b})\frac{\psi'^{2}(\mathscr{b})}{\psi''(\mathscr{b})}+C^{*}. \tag{1.8}$$

Then, for an arbitrary function $u(x)\in C^{(2)}(\Omega)\cap C^{(1)}(\bar\Omega)$, and for arbitrary $s\geq 0$, the following equation is valid:

$$\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})|\nabla u|^{2}dx=\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})[u(x)-u(x^{\ell}_{0})]_{x_i}u_{x_i}dx=\int\limits_{\partial\Omega\cap S^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}[u(x)-u(x^{\ell}_{0})]\psi^{s}(\mathscr{b})h(\mathscr{b})\frac{\partial u}{\partial n}dS-\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}[u(x)-u(x^{\ell}_{0})]\{\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})\Delta u+$$

$$+\zeta^{2}\psi^{s}(\mathscr{b})h'(\mathscr{b})\mathscr{b}_{x_i}u_{x_i}+s\zeta^{2}\psi^{s-1}(\mathscr{b})\psi'(\mathscr{b})h(\mathscr{b})\mathscr{b}_{x_i}u_{x_i}+2\zeta\zeta_{x_i}\psi^{s}(\mathscr{b})h(\mathscr{b})u_{x_i}\}dx,\quad \ell=1,\dots,N(\tau). \tag{1.9}$$

If we apply Cauchy's inequality to obtain estimates of the volume integrals appearing in the right member of (1.9), namely,

$$pq\leq\varepsilon_1 p^{2}+\frac{1}{4\varepsilon_1}q^{2},\quad \varepsilon_1>0, \tag{1.10}$$

and if we use the fact that $M[u;\Omega]$, $\omega(u;\delta;\Omega)$ and $M_1[u;\partial\Omega]$ are known *a priori*, and if we use the properties of the sectional function $\zeta(x)$, as well as the obvious equation

$$\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})\mathscr{b}\,dx=\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}[\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})|\nabla u|^{2}+\zeta^{2}\psi^{s}(\mathscr{b})h(\mathscr{b})]dx, \tag{1.11}$$

we obtain the following inequality:

$$\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\mathscr{b}\psi^{s}(\mathscr{b})h(\mathscr{b})dx\leq C_{1,3}(s,\varepsilon_1)\omega(u;2\tau;\Omega)\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\{\lambda(\mathscr{b})\psi^{s}(\mathscr{b})\psi'\mathscr{b}\sum_{i,k}u^{2}_{x_ix_k}+\lambda(\mathscr{b})[\psi^{s}(\mathscr{b})\psi''(\mathscr{b})+\psi^{s-1}(\mathscr{b})\psi'^{2}(\mathscr{b})]|\nabla\mathscr{b}|^{2}\}dx+\varepsilon_1\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\mathscr{b}\psi^{s}(\mathscr{b})h(\mathscr{b})dx+$$

$$+\varepsilon_1\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\zeta^{2}\left[\frac{\mathscr{b}\psi^{s}(\mathscr{b})h'^{2}(\mathscr{b})}{\psi''(\mathscr{b})}+\frac{\psi^{s}(\mathscr{b})h^{2}(\mathscr{b})}{\psi'(\mathscr{b})}+\mathscr{b}\psi^{s-1}(\mathscr{b})h^{2}(\mathscr{b})\right]\frac{dx}{\lambda(\mathscr{b})}+C_{1,4}(s,\tau,\varepsilon_1)\int\limits_{\Omega^{\ell}_{2\tau}(x^{\ell}_{0})}\psi^{s}(\mathscr{b})h(\mathscr{b})dx+C_{1,5}(M[u;\Omega],M_1[u;\partial\Omega])\operatorname{mes}(\partial\Omega\cap S^{\ell}_{2\tau}(x^{\ell}_{0})), \tag{1.12}$$

$$\varepsilon_1>0,\quad \ell=1,\dots,N(\tau).$$

We shall show that if the functions $\lambda(\mathscr{b})$ and $\theta(\mathscr{b})$ satisfy condition (1.5), then the following inequality will hold:

$$\frac{1}{\lambda(\mathscr{b})}\left\{\frac{\mathscr{b}\psi^{s}(\mathscr{b})h'^{2}(\mathscr{b})}{\psi''(\mathscr{b})}+\frac{\psi^{s}(\mathscr{b})h^{2}(\mathscr{b})}{\psi'(\mathscr{b})}+\mathscr{b}\psi^{s-1}(\mathscr{b})h^{2}(\mathscr{b})\right\}\leq C_{1,6}(B,C_\lambda)\mathscr{b}\psi^{s}(\mathscr{b})h(\mathscr{b}),\quad s=0,1,2\dots. \tag{1.13}$$

More precisely, let us show that for inequality (1.13) to be satisfied, the function $h(\mathscr{b})$ must be given by inequality (1.8) with constants $C=C(B,C_\lambda)$ and $C^{*}\equiv C^{*}(B,C_\lambda)$.

Proceeding to the proof of inequality (1.13), we observe, first of all, that if the functions $\lambda(\mathscr{b})$ and $\theta(\mathscr{b})$ satisfy condition (1.5), then the following inequality is valid, an inequality which will be used often in the subsequent work:

$$C_{1,7}(B,C_\lambda)\frac{\theta^{2}(\mathscr{b})}{\lambda^{2}(\mathscr{b})}\leq\frac{\psi'(\mathscr{b})}{\psi(\mathscr{b})}\leq C_{1,8}(B,C_\lambda)\frac{\theta^{2}(\mathscr{b})}{\lambda^{2}(\mathscr{b})}. \tag{1.14}$$

Indeed, integrating by parts, we find

$$\psi(\mathscr{b})\equiv\int_1^{\mathscr{b}}e^{B\int_1^{\tau}\frac{\theta^{2}(\xi)}{\lambda^{2}(\xi)}d\xi}d\tau=\frac{1}{B}\int_1^{\mathscr{b}}\frac{\lambda^{2}(\tau)}{\theta^{2}(\tau)}\frac{d}{d\tau}e^{B\int_1^{\tau}\frac{\theta^{2}(\xi)}{\lambda^{2}(\xi)}d\xi}d\tau\equiv\frac{1}{B}\left\{\frac{\lambda^{2}(\mathscr{b})}{\theta^{2}(\mathscr{b})}\psi'(\mathscr{b})-\frac{\lambda^{2}(1)}{\theta^{2}(1)}-\int_1^{\mathscr{b}}\left[\frac{\lambda^{2}(\tau)}{\theta^{2}(\tau)}\right]'e^{B\int_1^{\tau}\frac{\theta^{2}(\xi)}{\lambda^{2}(\xi)}d\xi}d\tau\right\}. \tag{1.15}$$

Further, it is easy to see that if the derivative $\left[\frac{\lambda^2(\tau)}{\theta^2(\tau)}\right]'$ satisfies one of the following conditions:

$$0 \le \left[\frac{\lambda^2(\tau)}{\theta^2(\tau)}\right]' \le C_\lambda^2, \tag{1.16}$$

$$\left[\frac{\lambda^2(\tau)}{\theta^2(\tau)}\right]' = o(1), \tau \to \infty, \tag{1.17}$$

then inequality (1.14) is an immediate consequence of inequality (1.15). On the other hand, if the functions $\lambda(\ell)$ and $\theta(\ell)$ satisfy condition (1.5), then at least one of the relations (1.16), (1.17) holds automatically.

Further, from inequality (1.14) and condition (1.5), it follows that

$$\frac{\psi(\ell)}{\psi'(\ell)} \le C_{1,9}(B,c_\lambda)\ell, \tag{1.18}$$

hence, instead of inequality (1.13), it is sufficient to prove the following inequality:

$$\frac{h'^2(\ell)}{\psi''(\ell)} + \frac{h^2(\ell)}{\psi(\ell)} \le C_{1,10}(B,c_\lambda)\lambda(\ell)n(\ell). \tag{1.19}$$

From condition (1.5), inequality (1.14), and the monotonicity of function $\psi(\ell)$, it follows that the function $h(\ell)$ increases monotonically as ℓ increases, and therefore, $h'(\ell)>0$. Further, from inequality (1.14) and the definition of $\psi(\ell)$, it follows that

$$\frac{\psi''(\ell)}{\psi(\ell)} \equiv \frac{\psi''(\ell)}{\psi'(\ell)}\cdot\frac{\psi'(\ell)}{\psi(\ell)} \le C_{1,11}(B,c_\lambda)\frac{\theta^4(\ell)}{\lambda^4(\ell)}. \tag{1.20}$$

Then, bearing in mind inequality (1.20) and the obvious inequality

$$\frac{h'^2(\ell)}{\psi''(\ell)} + \frac{h^2(\ell)}{\psi(\ell)} \le \left(\frac{h'(\ell)}{\sqrt{\psi''(\ell)}} + \frac{h(\ell)}{\sqrt{\psi(\ell)}}\right)^2, \tag{1.21}$$

it is sufficient to show that the function $h(\ell)$ is, for some $C \equiv C(B,c_\lambda)$ and $C^* \equiv C^*(B,c_\lambda)$, equivalent to the solution of the following differential equation:

$$H'(\ell) + \frac{\theta^2(\ell)}{\lambda^2(\ell)}H(\ell) = \sqrt{\lambda(\ell)\psi''(\ell)}\,H^{1/2}(\ell). \tag{1.22}$$

It is easy to show that the solution of (1.22) has the following form:

$$H^{1/2}(\ell) = \frac{1}{2}e^{-\frac{1}{2}\int_1^\ell\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\left\{\int_1^\ell\sqrt{\lambda(\tau)\psi''(\tau)}\,e^{\frac{1}{2}\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}d\tau + 1\right\}. \tag{1.23}$$

From the definition of $\psi(\ell)$, it follows that

$$\sqrt{\lambda(\tau)\psi''(\tau)} = \frac{\theta(\tau)}{\sqrt{\lambda(\tau)}}e^{\frac{1}{2}\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}; \tag{1.24}$$

therefore

$$H^{1/2}(\ell) = \frac{1}{2}e^{-\frac{1}{2}\int_1^\ell\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\left\{\int_1^\ell\frac{\theta(\tau)}{\sqrt{\lambda(\tau)}}e^{\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}d\tau + 1\right\}. \tag{1.25}$$

Integrating by parts, we find

$$H^{1/2}(b)=\frac{1}{2}e^{-\frac{1}{2}\int_1^b\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\left\{\int\frac{\lambda^{3/2}(\tau)}{\theta(\tau)}\frac{d}{d\tau}e^{\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}d\tau+1\right\}=\frac{1}{2}e^{-\frac{1}{2}\int_1^b\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\left\{\frac{\lambda^{3/2}(\tau)}{\theta(\tau)}e^{\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}\Big|_1^b-\int_1^b\left(\frac{\lambda^{3/2}}{\theta}\right)'e^{\int_1^\tau\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}d\tau+1\right\}. \tag{1.26}$$

From condition (1.5) and the relations (1.16), (1.17), it follows that the derivative $\left(\frac{\lambda^{3/2}}{\theta}\right)'$ satisfies one of the following conditions:

$$0\leq\left(\frac{\lambda^{3/2}(\tau)}{\theta(\tau)}\right)'\leq C_{1,12}(c_\lambda)\frac{\theta(\tau)}{\lambda^{1/2}(\tau)}, \tag{1.27}$$

$$\left(\frac{\lambda^{3/2}(\tau)}{\theta(\tau)}\right)'=o\left(\frac{\theta(\tau)}{\lambda^{1/2}(\tau)}\right),\quad \tau\to\infty, \tag{1.28}$$

and then, from (1.26), noting (1.24), we have

$$H(b)\sim C_{1,13}(B,c_\lambda)\frac{\lambda^3(b)}{\theta^2(b)}e^{\int_1^b\frac{\theta^2(\xi)}{\lambda^2(\xi)}d\xi}+C_{1,14}(B,c_\lambda), \tag{1.29}$$

whence, using the fact that

$$\frac{\lambda^2(b)}{\theta^2(b)}=B\frac{\psi'(b)}{\psi''(b)}, \tag{1.30}$$

we obtain, finally,

$$H(b)\sim C_{1,15}(B,c_\lambda)\lambda(b)\frac{\psi'^2(b)}{\psi''(b)}+C_{1,14}\equiv h(b). \tag{1.31}$$

By the same token, inequality (1.13) is proved.

Further, using inequality (1.13) and putting $\varepsilon_1=\frac{1}{2C_{1,6}}$, we obtain the following inequality from inequality (1.12):

$$\int_{\Omega_\tau^l(x_0^l)}b\psi^s(b)h(b)dx\leq C_{1,16}(s)\omega(u,2\tau;\Omega)\int_{\Omega_{2\tau}^l(x_0^l)}\lambda(b)\Big\{\psi^s(b)\psi'(b)\sum_{i,k}u^2_{x_ix_k}+|\nabla b|^2[\psi^s(b)\psi''(b)+\psi^{s-1}(b)\psi'^2(b)]\Big\}dx+$$

$$+C_{1,17}(s,\tau)\int_{\Omega_{2\tau}^l(x_0^l)}\psi^s(b)h(b)dx+C_{1,18}\operatorname{mes}(\partial\Omega\cap S^l_{2\tau}(x_0^l)),\ l=1,\dots,N(\tau),\ s=0,1,2,\dots \tag{1.32}$$

Summing the inequalities (1.32) with respect to l from 1 to $N(\tau)$, we obtain the following inequality:

$$\int_\Omega b\psi^s(b)h(b)dx\leq C_{1,19}(s)\omega(u;2\tau;\Omega)\int\lambda(b)\Big\{\psi^s(b)\psi'(b)\sum_{i,k}u^2_{x_ix_k}+$$

$$+[\psi^s(b)\psi''(b)+\psi^{s-1}(b)\psi'^2(b)]|\nabla b|^2\Big\}dx+C_{1,20}(s,\tau)\int_\Omega\psi^s(b)h(b)dx+C_{1,21}(M[u;\Omega],M_1[u;\partial\Omega])\operatorname{mes}\partial\Omega,\ s=0,1,2,\dots \tag{1.33}$$

From inequality (1.33), using the inequality

$$\int_\Omega\psi^s(b)h(b)dx\leq\varepsilon_2\int_\Omega b\psi^s(b)h(b)dx+\psi^s\left(\tfrac{1}{\varepsilon_2}\right)h\left(\tfrac{1}{\varepsilon_2}\right)\operatorname{mes}\Omega, \tag{1.34}$$

which is valid for arbitrary $\varepsilon_2>0$, then applying the known isoperimetric inequality

$$\operatorname{mes}\Omega\leq C_{1,22}(n)(\operatorname{mes}\partial\Omega)^{\frac{n}{n-1}}, \tag{1.35}$$

selecting ε_2 and τ small enough so that

$$\frac{C_{1,19}\,\omega(u;2\tau;\Omega)}{C(1-\varepsilon_2 C_{1,20})} \equiv \varepsilon, \tag{1.36}$$

and recalling the definition of the function $h(\mathscr{b})$, we obtain inequality (1.7). With this, the proof of Lemma 1 is complete.

Remark. For $\lambda(\mathscr{b}) = \mathscr{b}^{\frac{m-2}{2}}$, $\theta(\mathscr{b}) = \mathscr{b}^{\frac{m-2}{2}}$, $m>1$, Lemma 1 coincides essentially with Lemma 4.5 in Chap. 2 of [1].

Besides Lemma 1 we shall have need of the following, much simpler, lemma applicable in "limiting" cases, which may be proved basically in the manner of Lemma 1, but without decomposition into small subdomains.

Lemma 2. Let $u(x)$ be an arbitrary function of class $C^{(1)}(\bar\Omega)\cap C^{(2)}(\Omega)$, and, for it, let $M[u;\Omega]$ and $M_1[u;\partial\Omega]$ be known *a priori*. Further, let the functions $\lambda(\mathscr{b})$, $\theta(\mathscr{b})$, and $\psi(\mathscr{b})$ satisfy the same conditions as in Lemma 1, and, as before, let $\mathscr{b}\equiv 1+|\nabla u|^2$. Then, for arbitrary $s\ge 0$, the following inequality holds:

$$\int_\Omega \mathscr{b}\lambda(\mathscr{b})\,\frac{\psi^s(\mathscr{b})\psi'^2(\mathscr{b})}{\psi''(\mathscr{b})}\,dx \le C_{1,23}(s,M[u;\Omega])\Big\{\int_\Omega \psi^s(\mathscr{b})\psi'(\mathscr{b})\lambda(\mathscr{b})\sum_{i,k} u^2_{x_ix_k}\,dx +$$

$$+\int_\Omega \lambda(\mathscr{b})\Big[\psi^s(\mathscr{b})\psi''(\mathscr{b})+\psi^{s-1}(\mathscr{b})\psi'^2(\mathscr{b})\Big]|\nabla\mathscr{b}|^2dx\Big\} + C_{1,24}(M[u;\Omega],M_1[u;\partial\Omega],s,B,C_\lambda,\operatorname{mes}\partial\Omega). \tag{1.37}$$

Remark. It is easy to see that the constant $C_{1,23}$ depends on s in the following way:

$$C_{1,23}\sim s^2,\quad s=1,2,\dots .$$

2°. In the cylinder Q_T, let us consider the first initial-boundary problem (0.4)-(0.6), and let us assume that the coefficients of (0.4) satisfy the following conditions:

$$\nu(|u|)\lambda(\mathscr{b})|\xi|^2 \le a_{ij}(x,t,u,p)\xi_i\xi_j \le \nu_1(|u|)\lambda_1(\mathscr{b})|\xi|^2,\ \xi\ne 0, \tag{1.38}$$

$$|a(x,t,u,p)| \le \mu(|u|)\mathscr{b}\lambda(\mathscr{b}), \tag{1.39}$$

and that the boundary function $\mathscr{F}(x,t)$ in condition (0.6) satisfies the condition

$$\mathscr{F}(x,t)\equiv\mathscr{F}(x),\ 0\le t\le T,\quad \text{if}\quad \lambda(\mathscr{b})<\frac{1}{\mathscr{b}}, \tag{1.40}$$

where $\nu(\tau),\nu_1(\tau)$ and $\mu(\tau)$, $0\le\tau\le M[u;Q_T]$ are, respectively, nonincreasing and nondecreasing strictly positive functions, $\lambda(\mathscr{b})$ and $\lambda_1(\mathscr{b})$, $\lambda(\mathscr{b})\le\lambda_1(\mathscr{b})$ are strictly positive functions, and $\mathscr{b}\equiv 1+|p|^2$.

Lemma 3. Let $u(x,t)$ be the solution of the first initial-boundary problem (0.4)-(0.6), (1.40) for equation (0.4), (1.38), (1.39) of class $C^{(2,1)}(Q_T)$, and, for it, let $M[u;Q_T]$ be known *a priori*. Further, let us assume that one of the following conditions is satisfied:

1. Ω is an arbitrary (bounded) domain, with boundary $\partial\Omega$ of class $C^{(2)}$, and

$$\lambda_1(\mathscr{b})\le C_\lambda\mathscr{b}^{1/2}\lambda(\mathscr{b}); \tag{1.41}$$

2. Ω is an arbitrary convex domain, with boundary $\partial\Omega\in C^{(2)}$, and

$$\lambda_1(\mathscr{b})\le C_\lambda\mathscr{b}\lambda(\mathscr{b}); \tag{1.42}$$

3. Ω is a strictly convex domain, with boundary $\partial\Omega \in C^{(2)}$, and

$$\lambda_1(b) \geqslant \lambda(b). \tag{1.43}$$

Then the quantity

$$\max_{\partial Q_T} \left|\frac{\partial u}{\partial n}(x,t)\right| \tag{1.44}$$

is estimateable solely in terms of $M[u; Q_T]$, the constants $\nu(M), \mu(M), \nu_1(M)$,* C_λ in conditions (1.38), (1.39), (1.41)-(1.42), the norms $|f|_{C^{(1)}(\Omega)}$, $|\mathcal{F}|_{C^{(2,1)}(\partial Q_T)}$, and the boundary $\partial\Omega \in C^{(2)}$.

Indeed, from the proof of Lemma 3.1, Chap. 6 of [2], it follows readily that the estimate of the quantity $\max_{\partial Q_T}\left|\frac{\partial u}{\partial n}\right|$, indicated in Lemma 3, holds automatically for arbitrary $\lambda_1(b) \geq \lambda(b) > 0$, if the domain Ω is convex (not necessarily strictly convex!), the boundary condition (0.6) is homogeneous, i.e., $\mathcal{F}(x,t) \equiv 0$, and the "free term" $a(x,t,u,p)$ satisfies the "limiting" condition (1.39). Further, it is readily seen that if the first two conditions of Lemma 3 are fulfilled, then the domain Ω is either convex or is transformable into a convex domain, and the nonhomogeneous boundary condition (0.6) may be reduced to a homogeneous one, and under these transformations (for both domain and boundary condition), Eq. (0.4), by virtue of conditions (1.40)-(1.42), continues to belong to S. N. Bernshtein's class (L), i.e., the "free term" of the transformed equation will, as before, satisfy condition (1.39); therefore, Lemma 3 is proved for the first and second cases.

On the other hand, it is easy to see, from the proof of Lemma 3.1., Chap. 6 of [2], that in order to obtain an estimate of the quantity (1.44) for solutions $u(x,t)$ of the initial-boundary problem (0.4)-(0.6), with nonhomogeneous boundary condition, it is sufficient to construct two functions $w_1(x,t)$, $w_2(x,t)$ of class $C^{(2,1)}(Q_T)$, satisfying the following conditions:

1. $a_{ij} w_{1x_ix_j}(x,t) \leq 0$, $a_{ij} w_{2x_ix_j}(x,t) \geq 0$, $(x,t) \in Q_T$;

2. $w_1\Big|_{\partial Q_T} = w_2\Big|_{\partial Q_T} \equiv \mathcal{F}(x,t)$;

3. $\max_{Q_T}\left|\frac{\partial w_i}{\partial t}(x,t)\right| < C_{1,25}$, $i = 1,2$ if $\lambda(b) \geq \frac{1}{b}$,

$w_i(x,t) \equiv w_i(x)$, $0 \leq t \leq T$, $i = 1,2$, if $\lambda(b) < \frac{1}{b}$;

4. for the functions $w_i(x,t)$, it is possible to estimate, in terms of the initial data of the problem, enumerated in the formulation of Lemma 3, the quantities $\max_{\partial Q_T}\left|\frac{\partial w_i(x,t)}{\partial n}\right|$, $i = 1,2$.

If now the third condition of Lemma 3 is fulfilled, i.e., the domain Ω is strictly convex, then, using a construction employed by us in the proof of Lemma 4 of [4], it is easy to construct functions $w_i(x,t)$ such that each of the functions $w_i(x,t)$ represents a single-parameter family of convex or, correspondingly, concave cones, coinciding on the boundary ∂Q_T with the function $\mathcal{F}(x,t)$, whose vertices are smoothly truncated by smooth (of class $C^{(2,1)}(Q_T)$) convex (concave) "caps." Moreover, the constant $C_{1,25}$ will depend only on the norm $|\mathcal{F}|_{C^{(2,1)}(\partial Q_T)}$, the boundary $\partial\Omega \in C^{(2)}$, and a measure of the convexity of the domain Ω. Thus, being proved in the third case, the lemma is now completely proven.

*In the third case of the cases being considered, the quantity (1.44) does not depend on the constant $\nu_1(M)$.

3°. Lemma 4. Let $u(x,t)$ be the solution of the initial-boundary problem (0.4)-(0.6), when Eq. (0.4) is parabolic, of class $C^{(2,1)}(Q_T)$, and, for this solution, let $M[u;Q_T]$ and $M_{1,x}[u;\partial Q_T]$ be known *a priori*, and let the boundary $\partial\Omega$ of the domain Ω belong to class $C^{(2)}$. Further, let the functions $\lambda(\mathscr{b})$ and $\Theta(\mathscr{b})$ satisfy the following condition:

$$\frac{\lambda(\mathscr{b})}{\mathscr{b}^{1/2}} \leq C_\lambda \cdot \Theta(\mathscr{b}), \quad \mathscr{b} \geq 1, \tag{1.45}$$

let the function $\psi(\mathscr{b})$ be defined by (1.6), and let $\mathscr{b} \equiv 1 + \sum_{i=1}^{n} u_{x_i}^2(x,t)$. Then, for arbitrary $s = 0, 1, 2, \ldots$, the following inequalities hold:

$$J_{\partial\Omega}^{(1)} \equiv \int_{\partial\Omega} [\psi^{s+1}(\mathscr{b})]' a_{ij} u_{x_k} (u_{x_i x_j} \cos(n, x_k) - u_{x_i x_k} \cos(n, x_j))\, ds \leq (s+1) C_{1,26}(M[u;Q_T], M_{1,x}[u;\partial Q_T], |\mathscr{F}|_{C^{(2,1)}(\partial Q_T)}, \partial\Omega)\, \psi^{s+2}(M_{1,x}[u;\partial Q_T]), \tag{1.46}$$

$$J_{\partial\Omega}^{(2)} \equiv \int_{\partial\Omega} [\psi^{s+1}(\mathscr{b})]' a \frac{\partial u}{\partial n}\, dS \leq (s+1) C_{1,27}(M[u;Q_T], M_{1,x}[u;\partial Q_T])\, \psi^{s+2}(M_{1,x}[u;\partial Q_T]), \tag{1.47}$$

$$J_{\partial Q_t}^{(3)} \equiv \iint_{\partial Q_t} \frac{\partial u}{\partial t} [\psi^{s+1}(\mathscr{b})]' \frac{\partial u}{\partial n}\, dS\, dt \leq (s+1) C_{1,28}(M_{1,x}[u;\partial Q_T], |\mathscr{F}|_{C^{(2,1)}(\partial Q_T)}, T)\, \psi^{s+2}(M_{1,x}[u;\partial Q_T]), \tag{1.48}$$

where, essentially, the constants $C_{1,26} - C_{1,28}$ no longer depend on s.

If $u(x,t) \equiv u(x)$, then the inequalities (1.46) and (1.47) hold (with obvious replacement of $M[u;Q_T]$ by $M[u;\Omega]$, and $M_{1,x}[u;\partial Q_T]$ by $M_1[u;\partial\Omega]$) for solutions $u(x)$ of class $C^{(2)}(\Omega)$ of the Dirichlet problem (0.20)-(0.21) for the elliptic equation (0.20).

Inequality (1.46) can be proved essentially as was done for analogous inequalities for nonuniformly elliptic equations with divergent principal part ([8], §2; [3], Lemma 5). That is, proceeding at first by steps analogous to those used by O. A. Ladyzhenskaya in estimating surface integrals in the proof of the second fundamental inequality for linear elliptic equations (see, e.g., [1], Chap. 3, §8), and integrating by parts over the curvilinear surface, one can transform the integral $J_{\partial\Omega}^{(1)}$ so that it will not involve second derivatives with differentiation along the normal (for more detail, see [8], §2). Further, the second derivatives with respect to the spatial tangential variables can be estimated in terms of the norm $|\mathscr{F}|_{C^{(2)}(\partial Q_T)}$ and the maxima of the moduli of the first and second derivatives, given in local coordinates of the surface $\partial\Omega$. Then, using the fact that, in view of condition (1.45), the function $\psi(\mathscr{b})$ increases monotonically as $\mathscr{b}$ increases, and hence satisfies the obvious inequality

$$\psi'(\mathscr{b}) \leq \psi^2(\mathscr{b}), \tag{1.49}$$

we obtain the following inequality for the integral $J_{\partial\Omega}^{(1)}$:

$$J_{\partial\Omega}^{(1)} \leq (s+1) C_{1,29}(M[u;Q_T], |\mathscr{F}|_{C^{(2,1)}(\partial Q_T)}, \partial\Omega)\, \psi^{s+2}(M_{1,x}[u;\partial Q_T]) \int_{\partial\Omega} \Psi(\mathscr{b}, \Theta(\mathscr{b}))\, ds \equiv (s+1) C_{1,26}\, \psi^{s+2}(M_{1,x}[u;\partial Q_T]), \quad 0 \leq t \leq T. \tag{1.50}$$

Further, the inequalities (1.47) and (1.48), in view of inequality (1.49), follow at once from the definition of $J_{\partial\Omega}^{(2)}$ and $J_{\partial Q_t}^{(3)}$, respectively. Thus, Lemma 4 has been demonstrated.

§2. Proof of Theorem 1

In this section we shall prove Theorem 1, relying on Lemmas 1-4, which were proved in §1. The proof will be carried out in the following sequence: we consider first the case when the coefficients $a_{ij}(x,t,u,u_x)$ of Eq. (0.4) satisfy conditions (0.7)-(0.9), and the coefficient $a(x,t,u,u_x)$ satisfies, in turn, conditions II, III, I, wherein, in each of the three cases, we shall distinguish the subcases $\lambda(\mathscr{b}) \geq \frac{1}{6}$ and $\lambda(\mathscr{b}) < \frac{1}{6}$, and, in addition, for $\lambda(\mathscr{b}) \geq \frac{1}{6}$, we shall assume in each of these subcases that, for the solution $u(x,t)$, either $M[u;Q_T]$ and $\omega(u;\delta;Q_T)$ or only $M[u;Q_T]$ are known *a priori* following which we shall consider the case when the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.8), (0.17)-(0.19). The function

$$\psi(\mathscr{b}) \equiv \int_1^{\mathscr{b}} e^{B\int_1^{\tau} \frac{\theta^2(\xi)}{\lambda^2(\xi)} d\xi} d\tau, \quad \mathscr{b} \geq 1, \tag{2.1}$$

will play an essential role in the proof of Theorem 1, where the constant $B, B \gg 1$ will be selected later.*

1°. Thus, let us assume, at first, that the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.9), (0.11), i.e., $\lambda(\mathscr{b}) \geq \frac{1}{6}$, and assume that for the solution $u(x,t)$, $M[u;Q_T]$ and the modulus of continuity $\omega(u;\delta;Q_T)$ are known *a priori*. Let us multiply both sides of the equation by the function $\{[\psi^{s+1}(\mathscr{b})]' u_{x_k}\}_{x_k}$, $s=0,1,2,\dots$, $\mathscr{b}=1+\sum_{i=1}^{n} u_{x_i}^2$, wherein the exponent B in the expression for the function $\psi(\mathscr{b})$ will be selected later, and let us then integrate the result over the cylinder Q_t, $t \leq T$. Integrating the variable term and the term with coefficient a by parts once, and the terms with coefficients a_{ij}, twice, we obtain the following equation:

$$\frac{1}{2}\int_{\Omega_t} \psi^{s+1}(\mathscr{b})\,dx + \iint_{Q_t} \Big\{[\psi^{s+1}(\mathscr{b})]' a_{ij} u_{x_k x_i} u_{x_k x_j} + \frac{1}{2}[\psi^{s+1}(\mathscr{b})]'' a_{ij} \mathscr{b}_{x_i} \mathscr{b}_{x_j}\Big\}\,dx\,dt + \iint_{Q_t} [\psi^{s+1}(\mathscr{b})]' \Big\{\frac{da_{ij}}{dx_j} u_{x_k} u_{x_k x_i} - \frac{da_{ij}}{dx_k} u_{x_k} u_{x_i x_j}\Big\}\,dx\,dt - \frac{1}{2}\int_{\Omega_0} \psi^{s+1}(\mathscr{b})\,dx -$$

$$-\iint_{\partial Q_t} \frac{\partial u}{\partial t}[\psi^{s+1}(\mathscr{b})]' \frac{\partial u}{\partial n}\,dS\,dt + \iint_{Q_t} [\psi^{s+1}(\mathscr{b})]' \Big\{\frac{\partial a}{\partial p_\ell} u_{x_k} u_{x_k x_\ell} + \frac{\partial a}{\partial u} u_{x_k}^2 + \frac{\partial a}{\partial x_k} u_{x_k}\Big\}\,dx\,dt - \iint_{\partial Q_t} [\psi^{s+1}(\mathscr{b})]' a \frac{\partial u}{\partial n}\,dS\,dt +$$

$$+\iint_{\partial Q_t} [\psi^{s+1}(\mathscr{b})]' a_{ij} u_{x_k} (u_{x_i x_j} \cos(n,x_k) - u_{x_i x_k} \cos(n,x_j))\,dS\,dt = 0. \tag{2.2}$$

If we estimate the first volume integral and the volume integral containing $\frac{\partial a_{ij}}{\partial p_\ell}$ by using conditions (0.7), (0.8), if we use the fact that $u_{x_k} u_{x_i x_k} = \frac{1}{2}\mathscr{b}_{x_i}$, applying the Cauchy inequality (1.10), and use the monotonicity of the function $\psi(\mathscr{b})$, estimating the surface integrals over ∂Q_t with the aid of Lemma 4, and if we use the initial condition (0.6) as an estimate of the integral over the section Ω_0, we then obtain the following inequality from Eq. (2.2):

$$\int_{\Omega_t} \psi^{s+1}(\mathscr{b})\,dx + (s+1)\iint_{Q_t} \psi^{s}(\mathscr{b})\psi'(\mathscr{b})\lambda(\mathscr{b})\sum_{i,k} u_{x_i x_k}^2\,dx\,dt + \iint_{Q_t} [(s+1)\psi^{s}(\mathscr{b})\psi''(\mathscr{b}) + s(s+1)\psi^{s-1}(\mathscr{b})\psi'^2(\mathscr{b})]\lambda(\mathscr{b})\,|\nabla \mathscr{b}|^2\,dx\,dt \leq$$

$$\leq C_{2,1}(\nu,\Omega)\psi^{s+1}(M_1^2[f;\Omega_0]) + C_{2,2}(\nu,T,C_{1,26}-C_{1,28})(s+1)\psi^{s+2}(M_{1x}^2[u;\partial Q_T]) + \varepsilon(s+1)\iint_{Q_t} \lambda(\mathscr{b})\psi^{s+1}(\mathscr{b})[\psi(\mathscr{b})\sum_{i,k} u_{x_i x_k}^2 + \psi''(\mathscr{b})|\nabla \mathscr{b}|^2]\,dx\,dt +$$

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*The function $\Psi(\mathscr{b})$ satisfies the following differential equation:

$$\frac{\psi''(\mathscr{b})}{\psi'(\mathscr{b})} = B\,\frac{\theta^2(\mathscr{b})}{\lambda^2(\mathscr{b})}, \quad \mathscr{b} \geq 1,$$

whose solution, strictly speaking, gives rise to the function (see the inequalities (2.4), obtained from inequality (2.3)).

$$+(s+1)C_{2,3}(\nu,\mu,\varepsilon)\iint\limits_{Q_t}\frac{\psi^s(\theta)\psi'^2(\theta)}{\lambda(\theta)\psi''(\theta)}\theta^2(\theta)\sum_{i,k}u^2_{x_ix_k}\,dx\,dt+(s+1)\,C_{2,4}(\nu,\varepsilon)\iint\limits_{Q_t}\frac{\psi^s(\theta)\psi'^2(\theta)}{\lambda(\theta)\psi''(\theta)}\sum\left|\frac{\partial a}{\partial p_l}\right|^2dx\,dt+$$

$$+(s+1)\,C_{2,5}(\nu,\varepsilon)\iint\limits_{Q_t}\left\{\frac{\psi^s(\theta)\psi'(\theta)\theta^2}{\lambda(\theta)}\sum\left|\frac{\partial a_{ij}}{\partial u}\right|^2+\frac{\psi^s(\theta)\psi'(\theta)\theta}{\lambda(\theta)}\sum\left|\frac{\partial a_{ij}}{\partial x_l}\right|^2+\psi^s(\theta)\psi'(\theta)\theta\left|\frac{\partial a}{\partial u}\right|+\psi^s(\theta)\psi'(\theta)\theta^{1/2}\sum\left|\frac{\partial a}{\partial x_j}\right|\right\}dx\,dt,\ \varepsilon>0. \tag{2.3}$$

If in inequality (2.3) we put $\varepsilon=\frac{1}{2}$, choose $B\equiv 2C_{2,3}$, and use the first of conditions (0.9) and condition (0.11) to estimate the last volume integral, we obtain the following inequality from inequality (2.3):

$$\int\limits_{\Omega_t}\psi^{s+1}(\theta)dx+(s+1)\iint\limits_{Q_t}\psi^s(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k}u^2_{x_ix_k}\,dx\,dt+\iint\limits_{Q_t}\left\{(s+1)\psi^s(\theta)\psi''(\theta)+s(s+1)\psi^{s-1}(\theta)\psi'^2(\theta)\right\}\lambda(\theta)|\nabla\theta|^2dx\,dt\leq$$

$$\leq C_{2,6}(\nu,\mu,\Omega,T,s,C_{1,26}-C_{1,28},M_1[f;\Omega_0],M_{1,x}[u;\partial Q_T])+C_{2,7}(\nu,\mu)(s+1)\iint\limits_{Q_t}\theta\lambda(\theta)\frac{\psi^s(\theta)\psi'^2(\theta)}{\psi''(\theta)}dx\,dt,\quad s=0,1,2,\ldots. \tag{2.4}$$

If, to estimate the integral appearing in the right member of inequality (2.4), we apply Lemma 1, and if we select ε in inequality (1.7) to satisfy the condition $\varepsilon C_{2,7}=\frac{1}{2}$, we will obtain, from inequality (2.4), for arbitrary $s=0,1,2,\ldots$, the following fundamental inequality:

$$\int\limits_{\Omega_t}\psi^{s+1}(\theta)dx+(s+1)\iint\limits_{Q_t}\psi^s(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k}u^2_{x_ix_k}\,dx\,dt+(s+1)\iint\limits_{Q_t}\psi^s(\theta)\psi''(\theta)\lambda(\theta)|\nabla\theta|^2dx\,dt\leq 2C_{2,6}+C_{2,8}(C_{1,2},C_{2,7})\equiv C_{2,9}, \tag{2.5}$$
$$s=0,1,2,\ldots,\qquad 0<t\leq T.$$

Further, let us multiply Eq. (0.4) by the function $\{\psi'(\theta)\max\{\psi(\theta)-\psi(\kappa);0\}u_{x_l}\}_{x_l}$, wherein we shall consider that

$$\kappa\geq\max\left\{1+M_1^2[f,\Omega_0];\ 1+M^2_{1,x}[u;\partial Q_T]\right\}, \tag{2.6}$$

and let us then integrate the result over the cylinder Q_t, $0<t\leq T$. Proceeding as before, integrating the variable term and the term with coefficient a by parts once, and the terms with coefficients a_{ij} twice, using conditions (0.7)-(0.9), (0.11) to estimate the volume integrals, applying the Cauchy inequality (1.10), using the notation

$$A_\kappa(t)\equiv\left\{x\in\Omega_t:\ \theta(x,t)>\kappa\right\},\ 0<t\leq T, \tag{2.7}$$

we shall obtain, upon neglecting the two automatically nonnegative terms on the left side of the inequality (cf. inequality (2.4)):

$$\int\limits_{A_\kappa(t)}[\psi(\theta)-\psi(\kappa)]^2dx+\int_0^t\int\limits_{A_\kappa(t)}\lambda(\theta)\psi'^2(\theta)|\nabla\theta|^2dx\,dt\leq C_{2,10}(\nu,\mu)\int_0^t\int\limits_{A_\kappa(t)}\theta\lambda(\theta)\psi(\theta)\frac{\psi'^2(\theta)}{\psi''(\theta)}dx\,dt,\quad 0<t\leq T. \tag{2.8}$$

Let $\frac{1}{\theta}\leq\lambda(\theta)<1$. Then, using the fact that in this case, according to the condition $\lambda(\theta)\downarrow 0$, $\theta\to\infty$, introducing the new function

$$\tilde{\psi}(\theta)\equiv\sqrt{\lambda(\theta)}\,\psi(\theta), \tag{2.9}$$

taking account of the easily verifiable inequality, for $\frac{1}{\theta}\leq\lambda(\theta)<1$,

$$\tilde{\psi}'^2(\theta)\leq C_{2,11}\lambda(\theta)\psi'^2(\theta), \tag{2.10}$$

using the fact that $\tilde{\varphi}'^2(b)|\nabla b|^2 \equiv |\nabla\tilde{\varphi}(b)|^2$, and using inequality (1.14), we shall obtain, from inequality (2.8), the following inequality:

$$\max_{t\in[0,T]}\int_{A_k(t)}[\tilde{\varphi}(b)-\tilde{\varphi}(k)]^2dx+\int_0^T\int_{A_k(t)}|\nabla\tilde{\varphi}(b)|^2dx\,dt \le C_{2,12}(B,C_{2,10})\int_0^T\int_{A_k(t)}b\,\varphi^2(b)\,dx\,dt. \tag{2.11}$$

From the definition of $\varphi(b)$ and condition (0.14), it follows that

$$b \le C_{2,13}(B)\,\varphi(b). \tag{2.12}$$

Then, applying Hölder's inequality and using our fundamental inequality (2.5), we will obtain, for arbitrary, suitably small $\delta>0$, the following inequality:

$$\int_{A_k(t)} b\,\varphi^2(b)\,dx \le C_{2,14}(C_{2,9},B,\delta)\,\mathrm{mes}^{1-\delta}A_k(t). \tag{2.13}$$

It is easy to show that if $\frac{1}{b}\le\lambda(b)<1$, then $\tilde{\varphi}(b)\uparrow\infty$, $b\to\infty$, and that, then, obviously, $A_k(t)\equiv A_{\tilde{\varphi}(k)}(t)$. Therefore, from inequality (2.11), with the aid of inequality (2.13), we obtain the following inequality:

$$\max_{t\in[0,T]}\int_{A_{\tilde{\varphi}(k)}(t)}[\tilde{\varphi}(b)-\tilde{\varphi}(k)]^2dx+\int_0^T\int_{A_{\tilde{\varphi}(k)}(t)}|\nabla\tilde{\varphi}(b)|^2dx\,dt \le C_{2,15}(C_{2,12},C_{2,14})\int_0^T \mathrm{mes}^{1-\delta}A_{\tilde{\varphi}(k)}(t)\,dt\,,\quad 0<\delta\ll 1. \tag{2.14}$$

Thus, from inequality (2.14), in accord with Theorem 6.1 in Chap. 2 of [2], we obtain an estimate of $\max\tilde{\varphi}(b)$, and, hence, in view of the monotonicity of $\tilde{\varphi}(b)$, also an estimate of $M^{Q_T}_{1,x}[u;Q_T]$ in terms of quantities given in the formulation of Theorem 1.

Now, let $\lambda(b)\ge 1$. From the definition of the function $\varphi(b)$ and condition (0.15), it follows readily that in this case

$$\lambda(b) \le \varphi'(b). \tag{2.15}$$

We stress the fact that the second of the conditions (0.15) is introduced in order that the function $\lambda(b)$ and $\varphi(b)$ may be related by means of the inequality (2.15).

Using inequalities (2.15) and (1.14) and the fact that

$$\varphi'^2(b)\,|\nabla b|^2 \equiv |\nabla\varphi(b)|^2, \tag{2.16}$$

we obtain, from inequality (2.8), the inequality

$$\int_{A_k(t)}[\varphi(b)-\varphi(k)]^2dx+\int_0^t\int_{A_k(t)}|\nabla\varphi(b)|^2dx\,dt \le C_{2,16}(B,C_{2,10})\int_0^t\int_{A_k(t)}\varphi^5(b)\,dx\,dt,\qquad 0<t\le T. \tag{2.17}$$

Further, applying Hölder's inequality and using our fundamental inequality (2.5), we obtain, for arbitrary suitably small $\delta>0$, the inequality

$$\int_{A_k(t)}\varphi^4(b)\,dx \le C_{2,17}(C_{2,9},\delta)\,\mathrm{mes}^{1-\delta}A_k(t). \tag{2.18}$$

It is obvious that $\varphi(b)\uparrow\infty$, $b\to\infty$, and hence that $A_k(t)\equiv A_{\varphi(k)}(t)$. Therefore, from inequality (2.17), we obtain, with the aid of inequality (2.18), the inequality

$$\max_{t\in[0,T]}\int_{A_{\varphi(k)}(t)}[\varphi(b)-\varphi(k)]^2dx+\int_0^T\int_{A_{\varphi(k)}(t)}|\nabla\varphi(b)|^2dx\,dt \le C_{2,18}(C_{2,16},C_{2,17})\int_0^T\mathrm{mes}^{1-\delta}A_{\varphi(k)}(t)\,dt,\quad 0<\delta\ll 1. \tag{2.19}$$

As before, we obtain, from inequality (2.19), in accord with Theorem 6.1 of Chap. 2 of [2], an estimate of $\max_{Q_T} \psi(\theta)$, and, hence, in view of the monotonicity of $\psi(\theta)$, also an estimate of the quantity $M_{1,x}[u; Q_T]$ in terms of quantities given in the formulation of Theorem 1.

Let us assume now that the coefficients of Eq. (0.4) satisfy, as before, conditions (0.7)-(0.9), (0.11), i.e., $\lambda(\theta) \geq \frac{1}{\theta}$, but that only $u(x,t)$ is known *a priori* for the solution $M[u; Q_T]$. Then, instead of inequality (2.4), we obtain the following inequality:

$$\int_{\Omega_t} \psi^{s+1}(\theta)\,dx + (s+1)\iint_{Q_t} \psi^{s}(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k} u^2_{x_i x_k}\,dx\,dt + \iint_{Q_t}\left\{(s+1)\psi^{s}(\theta)\psi''(\theta) + s(s+1)\psi^{s-1}(\theta)\psi'^2(\theta)\right\}\lambda(\theta)|\nabla\theta|^2\,dx\,dt \leq$$
$$\leq C_{2,19} + (s+1)\,C_{2,20}\iint_{Q_t} \pi(\theta)\theta\lambda(\theta)\frac{\psi^{s}(\theta)\psi'^2(\theta)}{\psi''(\theta)}\,dx\,dt, \quad s = 0,1,2,\ldots, \tag{2.20}$$

wherein $\pi(\theta) \downarrow 0$, $\theta \to \infty$, and the constants $C_{2,19}$ and $C_{2,20}$ depend, respectively, on the same quantities as do the constants $C_{2,6}$ and $C_{2,7}$ of inequality (2.4).

Further, let the number $\varepsilon > 0$ satisfy the following condition:

$$\varepsilon\, C_{1,25}\, C_{2,20} = \frac{1}{2}, \tag{2.21}$$

where $C_{1,25}$ is the constant in inequality (1.37). Since the function $\pi(\theta)$ decreases monotonically as θ increases, then, corresponding to the ε chosen, one can find a $\tilde{\theta} \equiv \tilde{\theta}(\varepsilon)$, such that for $\theta \geq \tilde{\theta}$, the following inequality will hold:

$$\pi^2(\theta) < \varepsilon. \tag{2.22}$$

Using the obvious inequality

$$\iint_{Q_t} \pi^2(\theta)\theta\lambda(\theta)\frac{\psi^{s}(\theta)\psi'^2(\theta)}{\psi''(\theta)}\,dx\,dt \leq \varepsilon\iint_{Q_t}\theta\lambda(\theta)\frac{\psi^{s}(\theta)\psi'^2(\theta)}{\psi''(\theta)}\,dx\,dt + \tilde{\theta}\lambda(\tilde{\theta})\frac{\psi^{s}(\tilde{\theta})\psi'^2(\tilde{\theta})}{\psi''(\tilde{\theta})}\operatorname{mes} Q_T, \tag{2.23}$$
$$0 < t \leq T,$$

and applying Lemma 2 in order to estimate the integral in the right member of inequality (2.23), we obtain, from inequality (2.20), the following inequality, analogous to inequality (2.5):

$$\int_{\Omega_t} \psi^{s+1}(\theta)\,dx + (s+1)\iint_{Q_t}\psi^{s}(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k} u^2_{x_i x_k}\,dx\,dt + (s+1)\iint_{Q_t}\psi^{s}(\theta)\psi''(\theta)\lambda(\theta)|\nabla\theta|^2\,dx\,dt \leq C_{2,21}(C_{2,19}, C_{2,20}, C_{1,25}, Q_T), \tag{2.24}$$
$$s = 0,1,2,\ldots, \quad 0 < t \leq T.$$

After this, one obtains an estimate of the quantity $M_{1,x}[u; Q_T]$ in exactly the same way as in the case considered above.

2°. Let the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.9), (0.12), i.e., $\lambda(\theta) < \frac{1}{\theta}$, and let only $u(x,t)$ be known *a priori* for the solution $M[u; Q_T]$. Then, putting, as before, $\varepsilon = \frac{1}{2}$ in inequality (2.3), and $B = 2C_{2,3}$, and using the second of conditions (0.9) and condition (0.12) to estimate the last volume integral, we obtain, from inequality (2.3), the following inequality:

$$\int_{\Omega_t} \psi^{s+1}(\theta)\,dx + (s+1)\iint_{Q_t}\psi^{s+1}(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k} u^2_{x_i x_k}\,dx\,dt + \iint_{Q_t}\left\{(s+1)\psi^{s}(\theta)\psi''(\theta) + s(s+1)\psi^{s-1}(\theta)\psi'^2(\theta)\right\}\lambda(\theta)|\nabla\theta|^2\,dx\,dt \leq$$
$$\leq C_{2,1}\psi^{s+1}(M_1^2[f; \Omega_0]) + (s+1)C_{2,2}\psi^{s+2}(M_{1,x}[u; \partial Q_T]) + (s+1)C_{2,25}(\nu,\mu)\iint_{Q_t}\psi^{s+1}(\theta)\,dx\,dt, \quad s = 0,1,2,\ldots, \quad 0 < t \leq T. \tag{2.25}$$

Let us put

$$y_s(t) \equiv \iint_{Q_t} \psi^{s+1}(b)\,dx\,dt, \; s=0,1,2,\ldots, \; 0<t\le T. \tag{2.26}$$

Then, obviously,

$$\frac{dy_s(t)}{dt} \equiv \int_{\Omega_t} \psi^{s+1}(b)\,dx\,. \tag{2.27}$$

Using (2.26) and (2.27), and neglecting the automatically positive terms on the left side of inequality (2.25), we obtain, from inequality (2.25), the following differential inequality:

$$\frac{dy_s(t)}{dt} \le C_{2,26}(s) + (s+1)C_{2,25}\,y_s(t), \tag{2.28}$$
$$s=0,1,2,\ldots, \quad 0<t\le T,$$

where, for brevity, we have put

$$C_{2,26}(s) \equiv C_{2,1}\,\psi^{s+1}(M_1^2[f;\Omega_0])+(s+1)C_{2,2}\,\psi^{s+2}(M_{1,x}^2[u;\partial Q_T]). \tag{2.29}$$

Upon integrating the differential inequality (2.28) and using the fact that $y_s(0)=0$, $s=0,1,2,\ldots$, we obtain the estimate (cf. [2], Chap. 2, §5):

$$y_s(t) \le C_{2,26}(s)\,e^{(s+1)\cdot C_{2,25}\cdot t}\int_0^t e^{-(s+1)C_{2,25}\cdot\tau}\,d\tau \le C_{2,26}(s)\,e^{(s+1)C_{2,25}T}\cdot T\,, \qquad 0<t\le T\,. \tag{2.30}$$

Then, from inequalities (2.30) and (2.26), we obtain, finally, the following inequality:

$$\iint_{Q_T} \psi^{s+1}(b)\,dx\,dt \le C_{2,26}(s)\,T e^{(s+1)TC_{2,25}}, \; s=0,1,2,\ldots. \tag{2.31}$$

It is well known that

$$\left(\iint_{Q_T} \psi^{s+1}(b)\,dx\,dt\right)^{\frac{1}{s+1}} \xrightarrow[s\to\infty]{} \max_{Q_T} \psi(b). \tag{2.32}$$

Therefore, upon taking the $(s+1)$-th root of both sides of inequality (2.31), passing to the limit as $s\to\infty$, using the obvious relations

$$(\alpha+\beta s)^{1/s} \to 1, \; s\to\infty, \; \alpha\ge 0, \; \beta>0, \tag{2.33}$$

$$(\alpha+\beta s\gamma^s)^{1/s} \to 1, \; s\to\infty, \; \alpha,\beta>0, \; 0<\gamma<1, \tag{2.34}$$

and recalling the definition of the constant $C_{2,26}(s)$, we obtain, from inequality (2.31), the following inequality:

$$\max_{Q_T} \psi(b) \le e^{TC_{2,25}} \max\left\{\psi(M_1^2[f;\Omega_0]), \psi(M_{1,x}^2[u;\partial Q_T])\right\}. \tag{2.35}$$

Further, it is easy to see that since the functions $\lambda(b)$ and $\theta(b)$ satisfy, for $\lambda(b)<\frac{1}{6}$, the condition (0.14), the function $\psi(b)$ increases monotonically as b increases, even for $\lambda(b)<\frac{1}{6}$, and, then, from inequality (2.35), we obtain also an estimate of the quantity $M_{1,x}[u;Q_T]$ in terms of quantities appearing in the formulation of Theorem 1. Thus, we have proved Theorem 1 subject to conditions II on the coefficient $a(x,t,u,u_x)$.

3°. Let the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.9), (0.12) for $\lambda(\theta) \geq \frac{1}{6}$, and assume that, for the solution $u(x,t)$, there is known *a priori* either $M[u;Q_T]$ and $\omega(u;\delta;Q_T)$, or only $M[u;Q_T]$. Repeating the procedure which led to Eq. (2.2), and using the fact that now the coefficient $a(x,t,u,u_x)$ is twice continuously differentiable, let us transform the integral $\iint_{Q_t}[\psi^{s+1}(\theta)]'\frac{\partial a}{\partial p_l}u_{x_k}u_{x_kx_l}\,dx\,dt$ as follows, using an integration by parts:

$$\iint_{Q_t}[\psi^{s+1}(\theta)]'\frac{\partial a}{\partial p_l}u_{x_k}u_{x_kx_l}\,dx\,dt=\frac{1}{2}\iint_{Q_t}\frac{\partial a}{\partial p_l}\frac{d}{dx_l}[\psi^{s+1}(\theta)]\,dx\,dt=$$

$$=\frac{1}{2}\iint_{\partial Q_t}\psi^{s+1}(\theta)\frac{\partial a}{\partial p_l}\cos(n,x_l)\,dS\,dt-\frac{1}{2}\iint_{Q_t}\psi^{s+1}(\theta)\left\{\frac{\partial^2 a}{\partial p_l\partial p_m}u_{x_lx_m}+\frac{\partial^2 a}{\partial u\partial p_l}u_{x_l}+\frac{\partial^2 a}{\partial x_l\partial p_l}\right\}dx\,dt,\qquad 0<t\leq T. \tag{2.36}$$

An inequality analogous to inequalities (2.4) and (2.20) may be obtain from (2.2) by taking into account (2.36), using the conditions (0.7)-(0.9), (0.12), $\lambda(\theta)\geq\frac{1}{6}$, to estimate the volume integrals, applying Cauchy's inequality, using the monotonicity of the function $\psi(\theta)$, estimating the surface integrals over ∂Q_t with the aid of Lemma 4, using the initial condition (0.6) to estimate the integral over the lower base of Ω_0, and selecting ε and B in the same way they were selected when inequality (2.4) was obtained from inequality (2.3). The result is

$$J_s(t)\equiv\int_{\Omega_t}\psi^{s+1}(\theta)\,dx+(s+1)\iint_{Q_t}\psi^s(\theta)\psi'(\theta)\lambda(\theta)\sum_{i,k}u^2_{x_ix_k}\,dx\,dt+\iint_{Q_t}\left\{(s+1)\psi^s(\theta)\psi''(\theta)+s(s+1)\psi^{s-1}(\theta)\psi'^2(\theta)\right\}\lambda(\theta)|\nabla\theta|^2\,dx\,dt\leq$$

$$\leq C_{2,28}(\nu,T,M_{1,x}[u;\partial Q_T],|\mathcal{F}|_{C^{(2,1)}(\partial Q_T)})(s+1)\psi^{s+2}(M^2_{1,x}[u;\partial Q_T])+C_{2,27}(\nu,\Omega)\psi^{s+1}(M^2_1[f;\Omega_0])+C_{2,29}(\nu,\mu)(s+1)\iint_{Q_t}\mathfrak{F}^2(\theta)\theta\lambda(\theta)\frac{\psi^s(\theta)\psi'^2(\theta)}{\psi''(\theta)}\,dx\,dt,$$

$$s=0,1,2,\ldots,\quad 0<t\leq T. \tag{2.37}$$

Following this, an estimate of the quantity $M_{1,x}[u;Q_T]$ may be obtained from inequality (2.37) in the same way it was obtained above from inequalities (2.4) and (2.20).

If the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.9), (0.12) for $\lambda(\theta)<\frac{1}{6}$, and if $M[u;Q_T]$ is known *a priori* for the solution $u(x,t)$, then, instead of the inequality (2.37), we obtain the following inequality, analogous to inequality (2.25):

$$J_s(t)\leq(s+1)C_{2,30}(\nu,\mu)\iint_{Q_t}\psi^{s+1}(\theta)\,dx\,dt+C_{2,27}\psi^{s+1}(M^2_1[f;\Omega_0])+(s+1)C_{2,28}\psi^{s+2}(M^2_{1,x}[u;\partial Q_T]), \tag{2.38}$$

$$s=0,1,2,\ldots,\quad 0<t\leq T,$$

and one then obtain from this inequality an estimate of $M_{1,x}[u;Q_T]$, in the same way as was done above for inequality (2.25). Thus, Theorem 1 has been proved also for conditions III on the coefficient $a(x,t,u,u_x)$.

4°. Let the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.10), and assume that, for the solution $u(x,t)$, either $M[u;Q_T]$ and $\omega(u;\delta;Q_T)$ or, correspondingly, only $M[u;Q_T]$ are known *a priori.* Repeating the procedure which led us to Eq. (2.2), but not integrating by parts the term with coefficient $a(x,t,u,u_x)$, we obtain, in place of (2.2),

$$\frac{1}{2}\int_{\Omega_t}\psi^{s+1}(\theta)\,dx+\iint_{Q_t}\left\{[\psi^{s+1}(\theta)]'a_{ij}u_{x_kx_i}u_{x_kx_j}+\frac{1}{2}[\psi^{s+1}(\theta)]''a_{ij}\theta_{x_i}\theta_{x_j}\right.+\iint_{Q_t}[\psi^{s+1}(\theta)]'\left\{\frac{da_{ij}}{dx_j}u_{x_k}u_{x_kx_i}-\frac{da_{ij}}{dx_k}u_{x_k}u_{x_ix_j}\right\}dx\,dt-\frac{1}{2}\int_{\Omega_0}\psi^{s+1}(\theta)\,dx+$$

$$+\iint_{\partial Q_t}[\psi^{s+1}(\theta)]'a_{ij}u_{x_k}(u_{x_ix_j}\cos(n,x_k)-u_{x_ix_k}\cos(n,x_j))\,dS\,dt-\iint_{Q_t}a\left\{[\psi^{s+1}(\theta)]'u_{x_k}\right\}_{x_k}dx\,dt=0,\quad s=0,1,2,\ldots,\quad 0<t\leq T. \tag{2.39}$$

and from (2.39), repeating the same estimates as was done in connection with the derivation of inequalities (2.5), (2.24), and (2.31), but using, naturally, the conditions (0.7)-(0.10) to estimate the volume integrals, we obtain, for $\lambda(\ell) \geq \frac{1}{\ell}$, depending on the *a priori* assumptions on the solution $u(x)$, inequalities (2.5) and (2.24), but for $\lambda(\ell) < \frac{1}{\ell}$, where now the "parabolic part" of Eq. (0.4) begins to play the principal role; then, instead of inequality (2.31), we obtain the following, essentially different, inequality:

$$\iint_{Q_T} \psi^{s+1}(\ell)\,dx\,dt \leq C_{2,26}(s)\,T\,e^{s(s+1)TC_{2,25}}, \quad s=0,1,2,\dots \tag{2.31'}$$

Further, an estimate of the quantity $M_{1,x}[u; Q_T]$, for $\lambda(\ell) > 1$, may be obtained in exactly the same way as in section 1°. If, however, $\lambda(\ell) \leq 1$, then, as we did in section 1° for $\frac{1}{\ell} \leq \lambda(\ell) \leq 1$, we introduce the function

$$\tilde{\psi}(\ell) \equiv \sqrt{\lambda(\ell)}\,\psi(\ell)$$

(see Eq. (2.9)). From conditions (0.16), it follows readily that the function $\tilde{\psi}(\ell)$, for arbitrary $\lambda(\ell) \leq 1$, increases monotonically as ℓ increases and satisfies inequalities (2.10) and (2.12).* After this, one obtains an estimate of the quantity $M_{1,x}[u; Q_T]$, for arbitrary $\lambda(\ell) \leq 1$, in exactly the same way as was done in section 1° for $\frac{1}{\ell} \leq \lambda(\ell) \leq 1$. Thus, we have also proved Theorem 1 when coefficient $a(x,t,u,u_x)$ is subject to condition I.

Let us assume, finally, that the coefficients of Eq. (0.4) satisfy conditions (0.7)-(0.8), (0.17)-(0.19), and assume that only $M[u; Q_T]$ is known *a priori* for the solution $u(x,t)$. Then, from Eq. (2.2), using as an estimate of the volume integrals the conditions (0.7)-(0.8), (0.17)-(0.19), applying Cauchy's inequality, using the monotonicity of the function $\psi(\ell)$, estimating the surface integrals over ∂Q_t with the aid of Lemma 4, using as an estimate of the integral over Ω_0 the initial condition (0.6), and selecting ε and B as was done when inequality (2.4) was obtained from inequality (2.3), we obtain the following inequality:

$$\begin{aligned} I_s(t) \equiv {} & \int_{\Omega_t} \psi^{s+1}(\ell)\,dx + (s+1)\iint_{Q_t} \psi^{s}(\ell)\psi'(\ell)\lambda(\ell)\sum_{i,k} u^2_{x_i x_k}\,dx\,dt + \iint_{Q_t}\left\{(s+1)\psi^{s}(\ell)\psi''(\ell) + s(s+1)\psi^{s-1}(\ell)\psi'^2(\ell)\right\}\lambda(\ell)|\nabla \ell|^2\,dx\,dt + \\ & + (s+1)\iint_{Q_t} \ell\,\Lambda(\ell)\psi^{s}(\ell)\psi'(\ell)\,dx\,dt \leq C_{2,32}(\nu,\nu_1,T,M_{1,x}[u;\partial Q_T],|\mathcal{F}|_{C^{(2,1)}(\partial Q_T)})(s+1)\psi^{s+2}(M^2_{1,x}[u;\partial Q_T]) + \\ & + (s+1)C_{2,33}(\nu,\mu,\nu_1)\iint_{Q_t} \mathcal{P}^2(\ell)\,\ell\,\Lambda(\ell)\psi^{s}(\ell)\psi'(\ell)\,dx\,dt + C_{2,31}(\nu,\nu_1,\Omega)\,\psi^{s+1}(M_1^2[f;\Omega_0]), \quad s=0,1,2,\dots \end{aligned} \tag{2.40}$$

Let the number $\varepsilon > 0$ satisfy a condition like

$$\varepsilon C_{2,33} = \frac{1}{2}. \tag{2.41}$$

Since, by hypothesis, the function $\mathcal{P}(\ell)$ decreases monotonically as ℓ increases, then, corresponding to the ε chosen, one can find a $\tilde{\ell}(\varepsilon)$, such that, for $\ell \geq \tilde{\ell}$, the following inequality holds:

$$\mathcal{P}^2(\ell) < \varepsilon. \tag{2.42}$$

*In particular, in order that the function $\tilde{\psi}(\ell)$ might possess the properties enumerated, the second of the conditions (0.16) was introduced.

Using the obvious inequality

$$\iint_{Q_t} \mathcal{P}^2(\mathscr{b})\mathscr{b}\Lambda(\mathscr{b})\psi^s(\mathscr{b})\psi'(\mathscr{b})\,dx\,dt \le \varepsilon\iint_{Q_t} \mathscr{b}\Lambda(\mathscr{b})\psi^s(\mathscr{b})\psi'(\mathscr{b})\,dx\,dt + \tilde{\mathscr{b}}\Lambda(\tilde{\mathscr{b}})\psi^s(\tilde{\mathscr{b}})\psi'(\tilde{\mathscr{b}})\,\mathrm{mes}\,Q_T, \qquad (2.43)$$

$$0 < t \le T,$$

and using the fact that

$$\psi'(\mathscr{b}) \le C_{2,34}(C_\lambda, B)\psi^2(\mathscr{b}), \qquad (2.44)$$

we obtain, from inequality (2.40), the following inequality:

$$\bar{I}_s(t) \le 2C_{2,31}\psi^{s+1}(M_1^2[f;\Omega_0]) + (s+1)\Big\{2C_{2,32}\psi^{s+2}(M_{1,x}^2[u;\partial Q_t]) + C_{2,35}(C_{2,33}, C_{2,34}, Q_T)\tilde{\mathscr{b}}\Lambda(\tilde{\mathscr{b}})\psi^{s+2}(\tilde{\mathscr{b}})\Big\}, \qquad (2.45)$$

$$s = 0, 1, 2, \ldots, \quad 0 < t \le T,$$

and, from inequality (2.45), we obtain the following fundamental inequality:

$$\iint_{Q_T} \psi^{s+1}(\mathscr{b})\,dx\,dt \le 2TC_{2,31}\psi^{s+1}(M_1^2[f;\Omega_0]) + (s+1)\Big\{2TC_{2,32}\psi^{s+2}(M_{1,x}^2[u;\partial Q_T]) + TC_{2,36}\tilde{\mathscr{b}}\Lambda(\tilde{\mathscr{b}})\psi^{s+2}(\tilde{\mathscr{b}})\Big\}, \qquad (2.46)$$

$$s = 0, 1, 2, \ldots .$$

Taking the $(s+1)$-th root of both sides of inequality (2.46), letting $s \to \infty$, and using relations (2.32)-(2.34), we obtain the following inequality:

$$\max_{Q_T} \psi(\mathscr{b}) \le \max\Big\{\psi(M_1^2[f;\Omega_0]);\ \psi(M_{1,x}^2[u;\partial Q_T];\ \psi(\tilde{\mathscr{b}})\Big\}, \qquad (2.47)$$

where the constant $\tilde{\mathscr{b}}$ is determined by conditions (2.41)-(2.42), i.e., in the final computation the quantity in the right member of inequality (2.47) may be determined in terms of known quantities, which were given in the formulation of Theorem 1. Further, since in the given case, the functions $\lambda(\mathscr{b})$ and $\theta(\mathscr{b})$, for arbitrary $\lambda(\mathscr{b})$, satisfy condition (0.14), the function $\psi(\mathscr{b})$ will increase monotonically as $\mathscr{b}$ increases, and an estimate of the quantity $M_{1,x}[u;Q_T]$ may then be obtained from inequality (2.47) in terms of quantities given in the formulation of Theorem 1. Thus, Theorem 1 has also been proved subject to the conditions (0.7)-(0.8), (0.17)-(0.19) on the coefficients of Eq. (0.4), i.e., it has now been proved in full.

§3. Proof of Theorem 2

The present section is devoted to a proof of Theorem 2. The proof of Theorem 2 is basically analogous to the proof of Theorem 1, but simpler on account of the fact that in nondivergent elliptic equations one can always construct a norm of the coefficients, and consider that $\lambda(\mathscr{b}) \equiv 1$. Just as in the proof of Theorem 1, we shall consider in detail the case when coefficient $a(x,u,u_x)$ satisfies condition II, after which we shall point out briefly those changes which arise when coefficient $a(x,u,u_x)$ satisfies other conditions. As was previously the case, an essential role will be played by the function

$$\psi(\mathscr{b}) \equiv \int_1^{\mathscr{b}} e^{B\int_1^{\tau}\theta^2(\xi)\,d\xi}\,d\tau, \quad \mathscr{b} \ge 1, \qquad (3.1)$$

where again the constant B, $B \gg 1$, will be selected later.

1°. Thus, to begin, let us assume that the coefficients of Eq. (0.20) satisfy conditions (0.7)-(0.9), (0.11), and that, for the solution $u(x)$, $M[u;\Omega]$ and the modulus of continuity $\omega(u;\delta;\Omega)$ are known *a priori*. As before, let us multiply both sides of Eq. (0.20) by the function $\{[\psi^{s+1}(\mathscr{b})]'u_{x_k}\}_{x_k}$, $s = 0, 1, 2, \ldots$, $\mathscr{b} \equiv 1 + |\nabla u|^2$, where the exponent B in the expression for the function $\psi(\mathscr{b})$ will be selected later, and let us then integrate the result over the domain Ω. Integrating terms with coefficients a_{ij} by parts twice, and the term with coefficient a once, we obtain the following equation (cf. Eq. (2.2)):

$$\iint_\Omega \Big\{[\psi^{s+1}(v)]' a_{ij} u_{x_k x_i} u_{x_k x_j} + \frac{1}{2}[\psi^{s+1}(v)]'' a_{ij} v_{x_i} v_{x_j}\Big\} dx + \int_\Omega [\psi^{s+1}(v)]' \Big\{\frac{da_{ij}}{dx_j} u_{x_k} u_{x_k x_i} - \frac{da_{ij}}{dx_k} u_{x_k} u_{x_i x_j} + \frac{\partial a}{\partial p_l} u_{x_k} u_{x_k x_l} +$$

$$+ \frac{\partial a}{\partial u} u_{x_k}^2 + \frac{\partial a}{\partial x_l} u_{x_k}\Big\} dx - \int_\Omega [\psi^{s+1}(v)]' a \frac{\partial u}{\partial n} dS + \int_{\partial\Omega} [\psi^{s+1}(v)]' a_{ij} u_{x_k} [u_{x_i x_j} \cos(n, x_k) - u_{x_i x_k} \cos(n, x_j)] dS = 0, \; s = 0, 1, 2, \ldots \quad (3.2)$$

Using as an estimate of the volume integrals the conditions (0.7)-(0.9), (0.11), applying the Cauchy inequality (1.10), and estimating the surface integrals with the aid of Lemma 4, we obtain from Eq. (3.2), the following inequality:

$$J_s \equiv \int_\Omega \psi^s(v)\psi'(v) \sum_{i,k} u_{x_i x_k}^2 dx + \int_\Omega [\psi^s(v)\psi''(v) + s\psi^{s-1}(v)\psi'^2(v)]|\nabla v|^2 dx \leq$$

$$\leq C_{3,1}(\nu, s, M_1[u; \partial\Omega]) + \varepsilon \int_\Omega \psi^s(v)[\psi'(v) \sum_{i,k} u_{x_i x_k}^2 + \psi''(v)|\nabla v|^2] dx + C_{3,2}(\nu, \mu, \varepsilon) \int \frac{\psi^s(v)\psi'^2(v)}{\psi''(v)} \theta^2(v) \sum_{i,k} u_{x_i x_k}^2 dx + C_{3,3}(\nu, \mu, \varepsilon) \int_\Omega v \frac{\psi^s(v)\psi'^2(v)}{\psi''(v)} dx, \quad (3.3)$$

$$\varepsilon > 0, \quad s = 0, 1, 2, \ldots,$$

and, from inequality (3.3), we obtain, upon putting $\varepsilon = \frac{1}{2}$ and $B = 2C_{3,2}(\nu, \mu, \frac{1}{2})$, the inequality (cf. inequality 2.4)):

$$J_s \leq 2C_{3,1} + 2C_{3,3} \int_\Omega v \frac{\psi^s(v)\psi'^2(v)}{\psi''(v)} dx, \quad s = 0, 1, 2, \ldots. \quad (3.4)$$

If, to estimate the integral appearing on the right side of inequality (3.4), we apply Lemma 1, and if we select ε in inequality (1.7) to satisfy the condition $2\varepsilon C_{3,3} = \frac{1}{2}$, we obtain, from inequality (3.4), for arbitrary $s = 0, 1, 2, \ldots$, the following inequality:

$$J_s \leq 4C_{3,1} + C_{3,4}(C_{1,2}, C_{3,3}) \equiv C_{3,5}. \quad (3.5)$$

After this, if we again use Lemma 1 (or Lemma 2) and the inequality (3.5), just derived, and apply inequality (1.14), we obtain, for arbitrary $s = 0, 1, 2, \ldots$, the following fundamental inequality:

$$\int_\Omega \psi^{s+1}(v) dx \leq C_{3,6}(C_{3,1}, C_{3,5}, C_{1,2}, B), \quad s = 0, 1, 2, \ldots. \quad (3.6)$$

Further, let us multiply Eq. (0.20) by the function $\{\psi'(v) \max[\psi(v) - \psi(k); 0] u_{x_l}\}_{x_l}$, where we shall consider that

$$k \geq 1 + M_1^2[u; \partial\Omega], \quad (3.7)$$

and let us then integrate the result over the domain Ω. Integrating the terms with coefficients a_{ij} by parts twice, and the term with coefficient a once, using the conditions (0.7)-(0.9), (0.11) to estimate the volume integrals, applying the Cauchy inequality, using the notation

$$A_k \equiv \{x \in \Omega : v(x) > k\} \quad (3.8)$$

and neglecting on the left side of the inequality the two automatically nonnegative terms, we obtain the following inequality (cf. inequality (2.8)):

$$\int_{A_k} \psi'^2(v)|\nabla v|^2 dx \leq C_{3,7}(\nu, \mu) \int_{A_k} v\,\psi(v) \frac{\psi'^2(v)}{\psi''(v)} dx, \quad (3.9)$$

and, from inequality (3.9), using the fact that $\psi'^2(v)|\nabla v|^2 \equiv |\nabla\psi(v)|^2$, using inequality (1.14) and our fundamental inequality (3.6), applying Hölder's inequality and using the fact that by virtue of the monotonicity of the function $\psi(\theta)$, $A_k \equiv A_{\psi(k)}$, we obtain, for arbitrary, suitably small $\delta > 0$, the inequality

$$\int_{A_{\psi(k)}} |\nabla\psi|^2 dx \leq C_{3,8}(B, C_{3,7}) \int_{A_{\psi(k)}} \psi^4(\theta)\, dx \leq C_{3,9}(C_{3,6}, C_{3,8}, \delta)\, \mathrm{mes}^{1-\delta} A_{\psi(k)}. \tag{3.10}$$

An estimate for $\max_\Omega \psi(\theta)$ then follows from inequality (3.10) and Lemma 5.2 of Chap. 2 in [1], and therefore, in view of the monotonicity of $\psi(\theta)$, an estimate also follows for $M_1[u;\Omega]$ in terms of quantities given in the formulation of Theorem 2.

If, for the solution $u(x)$, only $M[u;\Omega]$, is known *a priori*, then, instead of inequality (3:4), we obtain, correspondingly, the inequality

$$J_s \leq 2C_{3,1} + 2C_{3,3} \int_\Omega \pi^2(\theta)\theta \frac{\psi^s(\theta)\psi'^2(\theta)}{\psi''(\theta)} dx, \quad s = 0, 1, 2, \ldots, \tag{3.11}$$

where $\pi(\theta) \downarrow 0$, $\theta \to \infty$, and, from inequality (3.11), choosing $\varepsilon > 0$ to satisfy the condition

$$2\varepsilon\, C_{1,23} C_{3,3} = \frac{1}{2}, \tag{3.12}$$

where $C_{1,23}$ is the constant of Lemma 2, using an inequality of the type of inequality (2.23) and applying Lemma 2, we obtain the following inequality, analogous to inequality (3.5):

$$J_s \leq 4C_{3,1} + C_{3,10}(C_{1,24}, C_{3,3}) \equiv C_{3,11}, \quad s = 0, 1, 2, \ldots. \tag{3.13}$$

Following this, an estimate of $M_1[u;\Omega]$ may be obtained in exactly the same way as in the case above. Thus, we have proved Theorem 2 subject to the condition II on the coefficient $a(x, u, u_x)$.

2°. Let the coefficients of Eq. (0.20) satisfy conditions (0.7)-(0.9), (0.12) and, for the solution $u(x)$, assume that either $M[u;\Omega]$ and $\omega(u;\delta;\Omega)$ or only $M[u;\Omega]$ are known *a priori.* We repeat the procedure which led us to Eq. (3.2) and, using the fact that now the coefficient $a(x, u, u_x)$ is twice continuously differentiable, we transform the integral $\int_\Omega [\psi^{s+1}(\theta)]' \frac{\partial a}{\partial p_l} u_{x_k} u_{x_k x_l} dx$ by an integration by parts, as follows (cf. (2.36)):

$$\int_\Omega [\psi^{s+1}(\theta)]' \frac{\partial a}{\partial p_l} u_{x_k} u_{x_k x_l} dx = \frac{1}{2} \int_\Omega \frac{\partial a}{\partial p_l} \frac{d}{dx_l} [\psi^{s+1}(\theta)]\, dx =$$

$$= \frac{1}{2} \int_{\partial\Omega} \psi^{s+1}(\theta) \frac{\partial a}{\partial p_l} \cos(n, x_l)\, dS - \frac{1}{2} \int_\Omega \psi^{s+1}(\theta) \left\{ \frac{\partial^2 a}{\partial p_l \partial p_m} u_{x_l x_m} + \frac{\partial^2 a}{\partial u \partial p_l} u_{x_l} + \frac{\partial^2 a}{\partial x_l \partial p_l} \right\} dx, \quad s = 0, 1, 2, \ldots. \tag{3.14}$$

From Eq. (3.2), taking into account Eq. (3.14), using the conditions (0.7)-(0.9), (0.12) to estimate the volume integrals, applying Cauchy's inequality, estimating the surface integrals over $\partial\Omega$ with the aid of Lemma 4 and selecting ε and B just as was done when inequality (3.4) was obtained from inequality (3.3), we obtain the following inequality, analogous to inequalities (3.4) and (3.11):

$$J_s \leq C_{3,12} + C_{3,13} \int_\Omega \pi^2(\theta)\theta \frac{\psi^s(\theta)\psi'^2(\theta)}{\psi''(\theta)} dx, \quad s = 0, 1, 2, \ldots. \tag{3.15}$$

Following this, an estimate of quantity $M_1[u;\Omega]$ may be obtained exactly as in section 1°. Thus, Theorem 2 has been proved also subject to condition III on $a(x, u, u_x)$.

3°. Let the coefficients of Eq. (0.20) satisfy conditions (0.7)-(0.10), and assume that, for the solution $u(x)$, either $M[u;\Omega]$ and $\omega(u;\delta;\Omega)$ or, correspondingly, only $M[u;\Omega]$ are known *a priori.* Repeating the procedure which led us to Eq. (3.2), but not integrating by parts the term with $a(x, u, u_x)$, we obtain, instead of Eq. (3.2), the equation (cf. Eq. (2.39)):

$$\int_\Omega \left\{ [\psi^{s+1}(\theta)]' a_{ij} u_{x_k x_i} u_{x_k x_j} + \frac{1}{2} [\psi^{s+1}(\theta)]'' a_{ij} \theta_{x_i} \theta_{x_j} \right\} dx + \int_\Omega [\psi^{s+1}(\theta)]' \left\{ \frac{da_{ij}}{dx_j} u_{x_k} u_{x_k x_i} - \frac{da_{ij}}{dx_k} u_{x_k} u_{x_i x_j} \right\} dx - \int_\Omega a \{ [\psi^{s+1}(\theta)]' u_{x_k} \}_{x_k} dx +$$

$$+ \int_{\partial\Omega} [\psi^{s+1}(\theta)]' a_{ij} u_{x_k} [u_{x_i x_j} \cos(n, x_k) - u_{x_i x_k} \cos(n, x_j)]\, dS = 0, \quad s = 0, 1, 2, \ldots, \tag{3.16}$$

and, from Eq. (3.16), repeating the same estimates involved in obtaining inequalities (3.4), (3.11), and (3.15), but using, naturally, the conditions (0.7)-(0.10) to estimate the volume integrals, we obtain an inequality analogous to inequality (3.15). Following this, an estimate of the quantity $M_1[u;\Omega]$ may be obtained just as was done in sections 1°, 2°. Thus, we have also proved Theorem 2, subject to condition I on the coefficient $a(x,u,u_x)$.

4°. Finally, let the coefficients of Eq. (0.20) satisfy conditions (0.7)-(0.8), (0.17)-(0.19) and assume that, for the solution $u(x)$, only $M[u;\Omega]$ is known *a priori*. Then, from Eq. (3.2), using the conditions (0.7)-(0.8), (0.17)-(0.19) to estimate the volume integrals, applying Cauchy's inequality, estimating the surface integrals over $\partial\Omega$ with the aid of Lemma 4 and selecting ε and B exactly as was done in obtaining inequality (3.4) from inequality (3.3), we obtain the following inequality (cf. inequality (2.40)):

$$I_s \equiv J_s + \int_\Omega \mathscr{E}\Lambda(\mathscr{E})\psi^s(\mathscr{E})\psi'(\mathscr{E})\,dx \le C_{3,14}(\nu,\nu_1,s,M_1[u;\partial\Omega]) + C_{3,15}(\nu,\nu_1,\mu)\int_\Omega \mathscr{P}^2(\mathscr{E})\mathscr{E}\Lambda(\mathscr{E})\psi^s(\mathscr{E})\psi'(\mathscr{E})\,dx, \qquad s=0,1,2,\ldots, \tag{3.17}$$

and, from inequality (3.17), taking $\varepsilon>0$ to satisfy the condition

$$\varepsilon C_{3,15} = \frac{1}{2}, \tag{3.18}$$

the number $\tilde{\mathscr{E}}(\varepsilon)$ to satisfy the condition

$$\mathscr{P}^2(\tilde{\mathscr{E}}) < \varepsilon \tag{3.19}$$

and using an inequality of the type of inequality (2.43) and inequality (2.44), we obtain the following inequality:

$$I_s \le 2C_{3,14} + C_{3,16}(C_{3,15},B,C_\lambda,\Omega)\tilde{\mathscr{E}}\Lambda(\tilde{\mathscr{E}})\psi^{s+2}(\tilde{\mathscr{E}}), \quad s=0,1,2,\ldots. \tag{3.20}$$

Further, using inequality (3.20) and Lemma 2, we obtain, as before, for arbitrary $s = 0, 1, 2, \ldots$, the following inequality:

$$\int_\Omega \psi^{s+1}(\mathscr{E})\,dx \le C_{3,17}, \qquad s=0,1,2,\ldots, \tag{3.21}$$

where the constant $C_{3,17}$ depends only on those quantities mentioned in the formulation of Theorem 2. Following this, an estimate of the quantity $M_1[u;\Omega]$ may be obtained exactly as was done in sections 1°, 3°. Thus, we have also proved Theorem 2, subject to the conditions (0.7)-(0.8), (0.17)-(0.19) on the coefficients of Eq. (0.20), i.e., the proof of the theorem is now complete.

Remark. In our theses [5], several different variants of the proof of Theorem 2 are presented. In particular, we begin by multiplying Eq. (0.20) by the function $\{\mathscr{E}^s\varphi(\mathscr{E})u_{x_i}\}_{x_i}$, where $s=0,1,2,\ldots,a$, and

$$\varphi(\mathscr{E}) \equiv e^{B\int_1^{\mathscr{E}}\theta^2(\tau)d\tau}, \quad B \gg 1, \tag{3.22}$$

and we obtain estimates of the integrals

$$\int_\Omega \mathscr{E}^s\varphi(\mathscr{E})\sum_{i,k}u^2_{x_ix_k}\,dx + \int_\Omega \mathscr{E}^s\varphi'(\mathscr{E})|\nabla\mathscr{E}|^2dx + \int_\Omega\Big(\int_1^{\mathscr{E}}\sqrt{\varphi'(\tau)}\,d\tau\Big)^2\mathscr{E}^s\,dx, \qquad s=0,1,2,\ldots. \tag{3.23}$$

We then multiply Eq. (0.20) by the function $\{\max[\varphi(\mathscr{E})-\varphi(k);0]u_{x_i}\}_{x_i}$, where $k \ge 1+M_1^2[u;\partial\Omega]$, and we then obtain the following inequality:

$$\int_{A_k}\varphi'(\mathscr{E})|\nabla\mathscr{E}|^2dx \le C_{3,18}\int_{A_k}\Big(\int_1^{\mathscr{E}}\sqrt{\varphi'(\tau)}\,d\tau\Big)^2\mathscr{E}\,dx, \tag{3.24}$$

following which, through use of the already available Lemma 3 of [3], the situation reduces to Lemma 5.1, Chap. 2 of [1], from which an estimate of the quantity $M_1[u;\Omega]$ follows. While it is true that in [5] we have considered only Eq. (0.20) for which the coefficient $a(x,u,u_x)$ satisfies condition II, the situations wherein the coefficient satisfies all the other conditions considered in this paper are handled entirely analogously. Details of this variation in the proof of Theorem 2 are presented in N. M. Ivochkina's candidate's dissertation [9].

LITERATURE CITED

1. O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasi-linear Equations of Elliptic Type, Mathematics in Science and Engineering, Vol. 46, Academic Press, New York (1968).
2. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, Linear and Quasi-linear Equations of Parabolic Type, "Nauka," Moscow (1967).
3. N. M. Ivochkina and A. P. Oskolkov, "Nonlocal estimates of first derivatives of solutions of Dirichlet's problem for nonuniformly elliptic quasi-linear equations," Seminars in Mathematics, Vol. 5, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part I, Consultants Bureau, New York (1969), pp. 12-34.
4. O. A. Ladyzhenskaya and N. N. Ural'tseva, "Certain classes of nonuniformly elliptic equations," Seminars in Mathematics, Vol. 5, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part I, Consultants Bureau, New York (1969), pp. 67-69.
5. N. M. Ivochkina and A. P. Oskolkov, "On estimates of the first derivatives for nonuniformly elliptic nondivergent equations," Problems of Applied Mathematics and Geometrical Modeling, Leningrad Institute of Engineering and Construction, Leningrad (1968).
6. N. M. Ivochkina, "The Dirichlet problem for two-dimensional quasi-linear second-order elliptic equations," Problems of Mathematical Analysis, Vol. 2, Leningrad State University, Leningrad (1968) [English translation: Consultants Bureau, New York (1971)].
7. N. Meyers, "On a class of nonuniformly elliptic quasi-linear equations in the plane," Arch. Rat. Mech. Anal., Vol. 12, No. 5, pp. 367-391 (1963).
8. T. B. Solomyak, "Dirichlet problem for nonuniformly elliptic quasi-linear equations," Problems of Mathematical Analysis, Vol. 1, Boundary Value Problems and Integral Equations, Consultants Bureau, New York (1968), pp. 81-101.
9. N. M. Ivochkina, The First Boundary Problem for Nonuniformly Elliptic and Nonuniformly Parabolic Quasi-linear Equations, Candidate's Dissertation, Leningrad Branch of the Academy of Sciences of the USSR (1968).
10. N. S. Trudinger, "Quasi-linear elliptic partial differential equations in n variables," Department of Mathematics, Stanford University, Stanford (July, 1966).
11. J. Serrin, "The Dirichlet problem for quasi-linear equations with many independent variables," Proc. Nat. Acad. Sci. USA, Vol. 58, No. 5 (1967).

SOME REMARKS ON THE CONVERGENCE OF SEQUENCES OF FUNCTIONS OF POLYNOMIAL TYPE IN $W_p^1(G)$ SPACES

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In [1],* based on multiplicative inequalities, corresponding to theorems of imbedding, some results were obtained concerning the convergence of sequences of functions in $W_p^l(G)$ spaces. For the particular case when the functions of the sequence considered are functions of polynomial type, i.e., functions for which inequalities of Markov or Bernshtein type are valid, and the convergence is considered in $C^s(G)$ space, the results obtained may be defined concretely. As consequences of these results, some theorems were obtained on the convergence of the method of Ritz, applied to boundary problems for elliptic equations of the second and forth orders.

In this note we generalize the results of [1]. Basic to ur discussion here is a theorem which generalizes a corresponding theorem of [1], in which, based on a known rate of decrease of $\|f - g_i\|_{W_p^l(G)}$, where $g_i\ (i=1,\ldots)$ are functions of polynomial type in the domain G, a conclusion is made concerning the order of convergence of $\{g_i\}_1^\infty$ to f in the norm of $W_q^s(G)$, where $p \leq q \leq \infty$, and s, generally speaking, is an arbitrary positive integer. In the proof of the theorem inequalities for functions of polynomial type are applied, which give an estimate of their norms in $W_q^s(G)$ in terms of the norm in $W_p^l(G)$. Although inequalities of this kind for several classes of domains are well known, we shall establish them by applying a method based on the use of parametric inequalities for functions belonging to $W_p^l(G)$ spaces.

From the theorem which has been proved and theorems of the constructive theory of functions, which enable one to estimate the order of decrease of $\|f - g_i\|_{W_p^l(G)}$, several concrete results follow concerning the order of convergence of variational processes in $W_q^s(G)$ spaces.

Let us note that in the case of ordinary differential equations, the problem concerning the order of convergence of derivatives of arbitrary order in the Ritz–Galerkin method was investigated by I. K. Daugavet [3], and, in the case of an arbitrary number of independent variables, by S. G. Mikhlin†; however, the methods used by these authors, and partly, also their results, are different from ours.

I. In this section G will denote a domain of the n-dimensional Euclidean space E^n, satisfying the condition: for each point $x=(x_1,\ldots,x_n)\in G$, there exists an n-dimensional spherical sector of constant radius and form, with vertex at x and lying entirely in G. Let H be the magnitude of the largest al-

*Reference [1] is a somewhat revised and supplemented presentation of the results in [2].
†The author was able to acquaint himself with S. G. Mikhlin's results in the original manuscript.

lowable radius of the sector reaching each point x of the domain G. Then, if G satisfies the condition stated, we shall write $G \in A(H)$.

Let the function $f(x)$ be defined in the domain G and let it possess there generalized derivatives, in the sense of S. L. Sobolev, of all orders to be considered below. Let us assume also that all the norms of the function f considered below possess a sense.

Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a nonnegative integer vector ($\alpha_i \geqslant 0$ are integers), $|\bar{\alpha}| = \alpha_1 + \dots + \alpha_n$. Let us put

$$\mathfrak{D}^{\bar{\alpha}} f(x) = \frac{\partial^{|\bar{\alpha}|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \mathfrak{D}^{\ell} f(x) = \sqrt{\sum_{|\bar{\alpha}| = \ell} |\mathfrak{D}^{\bar{\alpha}} f(x)|^2},$$

where ℓ is a nonnegative integer.

Let $p > 1$,*

$$\|f\|_{W_p^{\ell}(G)} = \|f\|_{L_p(G)} + \|\mathfrak{D}^{\ell} f\|_{L_p(G)}$$

$$(\text{for } \ell = 0 \quad \|f\|_{W_p^{\ell}(G)} = \|f\|_{L_p(G)}),$$

where

$$\|f\|_{L_p(G)} = \left[\int_G |f(x)|^p dx \right]^{\frac{1}{p}}.$$

The principal aim of this section is to obtain estimates of a special type for $\|\mathfrak{D}^s f\|_{L_q(G)}$, in terms of $\|f\|_{W_p^{\ell}(G)}$, where s is an arbitrary nonnegative integer, and the number q satisfies the inequalities: $p \leqslant q \leqslant \infty$.

We shall distinguish the following cases, depending on the relations among the parameters ℓ, s, p, q, n, and (where n is the dimension of the domain G):

1. $\ell - s - \frac{n}{p} + \frac{n}{q} \geqslant 0$, where we exclude the case in which the following equations are simultaneously satisfied: $\ell - s - \frac{n}{p} + \frac{n}{q} = 0$ and $q = \infty$;
2. $\ell - s - \frac{n}{p} = 0$, $q = \infty$;
3. $\ell - s - \frac{n}{p} + \frac{n}{q} < 0$.

1) Let $\ell - s - \frac{n}{p} + \frac{n}{q} \geqslant 0$, where if $\ell - s - \frac{n}{p} + \frac{n}{q} = 0$, $q \neq \infty$. In this case, based on Theorem 2, p. 76 of [1], the following estimate holds:

$$\|\mathfrak{D}^s f\|_{L_q(G)} \leqslant C \|f\|_{L_p(G)}^{\alpha} \cdot \|f\|_{W_p^{\ell}(G)}^{1-\alpha} \leqslant C \|f\|_{W_p^{\ell}(G)}, \tag{1}$$

where $\alpha = \frac{1}{\ell}(\ell - s - \frac{n}{p} + \frac{n}{q})$, and C is a constant, which depends on the form of the domain G but not on f.

2) Let $\ell - s - \frac{n}{p} = 0$, $q = \infty$. Let τ be an arbitrary positive number, $\tau > p$. By inequality (16"), p. 71 of [1], the following estimate holds for an arbitrary point $x \in G$:

$$|\mathfrak{D}^s f(x)| \leqslant C_1 h^{-\frac{n}{\tau}} \|\mathfrak{D}^s f\|_{L_\tau(G \cap \omega_h(x))} + C_2 h \|\mathfrak{D}^{s+1} f\|_{L_\infty(G)} \tag{2}$$

where h is an arbitrary positive number, satisfying the inequalities: $0 < h \leqslant H$, where H is a

*We make the assumption $p > 1$ for simplicity. Of the estimates introduced below, the fundamental ones can be obtained, with some complication in the arguments, even for $p = 1$.

parameter appearing in the definition of the class $A(H)$, to which the domain G belongs; $\omega_h(x)$ is an n-dimensional ball of radius h, with center at x, and C_1 and C_2 are constants, independent of f, h, and τ.

Further, by inequality (25), p. 73 of [1], we have

$$\|\mathfrak{D}^s f\|_{L_\tau(G\cap\omega_h(x))} \leq C_3 h^{\frac{n}{\tau}} \lambda^{-\frac{n}{p}} \|\mathfrak{D}^s f\|_{L_p(G)} + C_4 (2\tau)^{\frac{1}{p'}} h^{\frac{n}{2\tau}} \|\mathfrak{D}^\ell f\|_{L_p(G)}, \tag{3}$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$, λ is a fixed number, $0<\lambda\leq H$, and C_3, C_4 are constants, independent of h, τ, λ, and f.

Substituting Eq. (3) into Eq. (2), and noting here that

$$\|\mathfrak{D}^s f\|_{L_p(G)} \leq C_5 \|f\|_{W_p^\ell(G)},$$

since $s\leq \ell$, we obtain

$$\|\mathfrak{D}^s f\|_{L_\infty(G)} \leq C_6 \|f\|_{W_p^\ell(G)} + C_7 (2\tau)^{\frac{1}{p'}} h^{-\frac{n}{2\tau}} \|f\|_{W_p^\ell(G)} + C_2 h \|\mathfrak{D}^{s+1} f\|_{L_\infty(G)}, \tag{4}$$

where C_6, C_7, and C_2 are constants, independent of f, h, and τ, and h and τ are arbitrary numbers, satisfying the inequalities: $0<h\leq H$, $p\leq\tau<\infty$.

3) Finally, let $\ell-s-\frac{n}{p}+\frac{n}{q}<0$. Let us consider two subcases: a) $s\leq\ell-1$, b) $s\geq\ell$.

First, let $s\leq\ell-1$. Since $\ell-s-\frac{n}{p}+\frac{n}{q}<0$, then there exists an τ, $p<\tau<q\leq\infty$, such that $\ell-s-\frac{n}{p}+\frac{n}{\tau}=0$, and, by virtue of an imbedding theorem (see [4]),

$$\|\mathfrak{D}^s f\|_{L_\tau(G)} \leq C_9 \|f\|_{W_p^\ell(G)}, \tag{5}$$

where C_9 does not depend on f.

By inequality (24), p. 73 of [1], and inequality (5), we then obtain

$$\|\mathfrak{D}^s f\|_{L_q(G)} \leq C_{10} h^{-\frac{n}{\tau}+\frac{n}{q}} \|\mathfrak{D}^s f\|_{L_\tau(G)} + C_{11} h \|\mathfrak{D}^{s+1} f\|_{L_q(G)} \leq C_{12} h^{\ell-s-\frac{n}{p}+\frac{n}{q}} \|f\|_{W_p^\ell(G)} + C_{11} h \|\mathfrak{D}^{s+1} f\|_{L_q(G)}, \tag{6}$$

where h is an arbitrary positive number, $0<h\leq H$, and C_{11}, C_{12} are constants, independent of f and h.

Now let $s\geq\ell$. Then, by inequality (37), p. 77 of [1], we obtain

$$\|\mathfrak{D}^s f\|_{L_q(G)} \leq C_{13} h^{\ell-s-\frac{n}{p}+\frac{n}{q}} \|\mathfrak{D}^\ell f\|_{L_p(G)} + C_{14} h \|\mathfrak{D}^{s+1} f\|_{L_q(G)} \leq C_{13} h^{\ell-s-\frac{n}{p}+\frac{n}{q}} \|f\|_{W_p^\ell(G)} + C_{14} h \|\mathfrak{D}^{s+1} f\|_{L_q(G)}, \tag{7}$$

where $0<h\leq H$, and C_{13}, C_{14} do not depend on f and h. Comparing inequalities (6) and (7), we see that for $\ell-s-\frac{n}{p}+\frac{n}{q}<0$, one may consider that the estimate (7) is always valid.

In the following section we shall apply the estimates (1), (4), and (7), corresponding to the conditions 1), 2), and 3), to find necessary inequalities for functions of much narrower classes.

II. Let $\nu>0$ be an integer, i, let an arbitrary positive number, $\sigma(\xi)$, be a monotone increasing function positive for $\xi>0$, satisfying the condition:

$$\sigma(m\xi)\leq C(m)\sigma(\xi), \tag{8}$$

where $m>1$ is an arbitrary positive number, and $C(m)$ is a constant, which does not depend on ξ.

We shall say that the function $g(x)$ is a function of class $\mathcal{P}(i;\sigma;\nu;G)$ in the domain G if, for an arbitrary nonnegative integral vector $\bar{\alpha}$, $|\bar{\alpha}|=s$, $0\leq s\leq\nu-1$, and an arbitrary number q, $1\leq q\leq\infty$, the following inequalities hold:

$$\|\mathfrak{D}^{\bar{\alpha}+\bar{\omega}_k} g\|_{L_q(G)} \leq C\sigma(i) \|\mathfrak{D}^{\bar{\alpha}} g\|_{L_q(G)} \quad (k=1,\dots,n), \tag{9}$$

where $\vec{\omega}_k = (0,\dots,0,\overset{k}{1},0,\dots 0)$; $\tilde{C}$ is a constant, depending on the properties of the domain G and being independent of ι and g; σ is a function satisfying the conditions stated above (in general, it, too, depends on G).

For a certain class of domains G, for example for domains satisfying the conditions given in a remark below (as the simplest example of such a domain, we have a rectangular parallelepiped with edges parallel to the coordinate axes), the functions of class $\mathcal{P}(\iota;\sigma;\nu;G)$ for arbitrary ν and for $\sigma(\iota)=\iota^2$ are algebraic and trigonometric polynomials of degree ι in each variable.

Let G be a bounded domain, with boundary Γ. Further, let $\omega(x)$ be a sufficiently smooth function, defined in an open domain containing $\bar{G}=G+\Gamma$ and satisfying the conditions: 1) $\omega(x)=0$ on Γ, 2) $\omega(x)>0$ in G, 3) $|\operatorname{grad}\omega/_{\Gamma}>0$. Then the functions of class $\mathcal{P}(\iota;\sigma;\ell;G)$, where $\sigma(\iota)=\iota^2$, are functions of the form

$$g(x)=\omega^{\ell}(x)P_{\iota}(x), \tag{10}$$

where the $P_\iota(x)$- are algebraic polynomials of degree not exceeding ι in each of the variables $x_1,\dots,x_n$, and ℓ is a natural number. This result was proved in [5] for $\ell=1$.

The functions of class $\mathcal{P}(\iota\ \sigma;\nu;E^n)(G=E^n)$ are, for arbitrary ν, entire functions of degree ι in each of the variables, with summable q-th powers over E^n, wherein $\sigma(\iota)=\iota$. An analogous statement is true for trigonometric polynomials, taken over the period $\Pi=G$.*

Let us assume now that $g(x)$ is a function of class $\mathcal{P}(\iota;\sigma;s+1;G)$, and that $g(x)\in W_p^{\ell}(G)$, $G\in A(H)$ For $g(x)$, we note the following consequences of the inequalities (1), (4), and (7).

1) If $\ell-s-\frac{n}{p}+\frac{n}{q}\geqslant 0$ (for $\ell-s-\frac{n}{p}+\frac{n}{q}=0, q\neq\infty$), then, by inequality (1),

$$\|\mathcal{D}^s g\|_{L_q(G)}\leq C\|g\|_{W_p^{\ell}(G)}. \tag{11}$$

2) If $\ell-s-\frac{n}{p}=0$ and $q=\infty$, then, by inequality (4),

$$\|\mathcal{D}^s g\|_{L_\infty(G)}\leq C_6\|g\|_{W_p^{\ell}(G)}+C_7(2\tau)^{\frac{1}{p'}}h^{-\frac{n}{2\tau}}\|g\|_{W_p^{\ell}(G)}+C_2 h\|\mathcal{D}^{s+1}g\|_{L_\infty(G)}, \tag{4'}$$

where $0<h\leq H$, $p\leq\tau<\infty$.

In accord with inequalities (9),

$$\|\mathcal{D}^{s+1}g\|_{L_\infty(G)}\leq\tilde{C}\sigma(\iota)\|\mathcal{D}^s g\|_{L_\infty(G)}, \tag{12}$$

where $\tilde{C}$ does not depend on ι and g.

Let us estimate the last term in inequality (4') on the basis of inequality (12), and let us put

$$h=\frac{1}{\kappa C_2\tilde{C}\sigma(\iota)},$$

where $\kappa\geqslant 2$ is chosen so that $h\leq H$. Then, taking the term containing $\|\mathcal{D}^s g\|_{L_\infty(G)}$ to the left side of the inequality and noting that its coefficient $\leq\frac{1}{2}$, we obtain, after simple transformations,

$$\|\mathcal{D}^s g\|_{L_\infty(G)}\leq C_{15}\|g\|_{W_p^{\ell}(G)}+C_{16}(2\tau)^{\frac{1}{p'}}[\sigma(\iota)]^{\frac{n}{2\tau}}\|g\|_{W_p^{\ell}(G)}, \tag{13}$$

where C_{15} and C_{16} do not depend on τ, ι, and g.

*In the terminology used above, these are functions of polynomial type, relating, namely, to functions of the classes $\mathcal{P}(\iota;\sigma;\nu;G)$.

If $\ln[\sigma(i)]^n \geqslant 2p$, then, putting

$$2\tau = \ln[\sigma(i)]^n,$$

we obtain

$$\|\mathcal{D}^s g\|_{L_\infty(G)} \leq C_{17}\|g\|_{W_p^\ell(G)}[\ln\sigma(i)]^{\frac{1}{p}} \tag{14}$$

If, however, $\ln[\sigma(i)]^n \leqslant 2p$, then, putting $2\tau = 2p$ in inequality (13), we obtain

$$\|\mathcal{D}^s g\|_{L_\infty(G)} \leq C_{17}\|g\|_{W_p^\ell(G)}.$$

For $\sigma(i) \geqslant 2$, we can always consider that inequality (14) holds, where C_{17} does not depend on i and g.

3) If $\ell - s - \frac{n}{p} + \frac{n}{q} < 0$, then, by inequality (7),

$$\|\mathcal{D}^s g\|_{L_q(G)} \leq C_{13} h^{\ell-s-\frac{n}{p}+\frac{n}{q}}\|g\|_{W_p^\ell(G)} + C_{14} h\|\mathcal{D}^{s+1} g\|_{L_q(G)}.$$

If, as before, we estimate $\|\mathcal{D}^{s+1} g\|_{L_q(G)}$ by application of inequality (12), and if we select h in a corresponding way, we obtain

$$\|\mathcal{D}^s g\|_{L_q(G)} \leq C_{19}\|g\|_{W_p^\ell(G)}[\sigma(i)]^{-(\ell-s-\frac{n}{p}+\frac{n}{q})} \tag{15}$$

where the constant C_{19} does not depend on i and g.

Remark. Let $\ell = s = 0$, $1 \leq p < q \leq \infty$. Then case 3) applies ($\ell - s - \frac{n}{p} + \frac{n}{q} < 0$), and, from inequality (15), we obtain the well-known inequality

$$\|g\|_{L_q(G)} \leq C\|g\|_{L_p(G)}[\sigma(i)]^{\frac{n}{p}-\frac{n}{q}}. \tag{16}$$

If G^m is the cross section of domain G by a hyperplane of dimension m, $1 \leq m < n$, then we obtain, in an analogous way, the following inequality:

$$\|g\|_{L_q(G^m)} \leq C\|g\|_{L_p(G)}[\sigma(i)]^{\frac{n}{p}-\frac{m}{q}}. \tag{17}$$

Inequalities (16) and (17) (as well as inequalities (14) and (15)) may be made more precise by placing more stringent limitations on the domain G.

Let us assume that the domain G satisfies the following condition: for each point $x \in G$, there exists a rectangular parallelepiped of fixed dimensions, which has its edges parallel to the coordinate axes and has one vertex at x and lies entirely in G. Let H_k $(k=1,\dots,n)$ be the largest allowable lengths of the edges of the parallelepiped, which reaches each point x of the domain G.

Let us denote by e an arbitrary subset (which may be the null set) of the set $e^n = \{1,\dots,n\}$ of natural numbers; let us put $e' = e^n \setminus e$, $\bar\omega^e = (\omega_1^e,\dots,\omega_n^e)$, where $\omega_j^e = 1$, if $j \in e$, and $\omega_j^e = 0$, if $j \in e'$,

$$\mathcal{D}^{\bar\omega^e} g(x) = \frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_k}} g(x), \qquad \{j_1,\dots,j_k\} = e.$$

It is not difficult to prove the following inequality by the method of integral representations of differentiable functions:

$$\|g\|_{L_q(G)} \leq \sum_{e \subseteq e^n} C_e \|\mathcal{D}^{\bar\omega^e} g\|_{L_p(G)} \Big(\prod_{k \in e} h_k^{1-\frac{1}{p}+\frac{1}{q}}\Big)\Big(\prod_{k \in e'} h_k^{-\frac{1}{p}+\frac{1}{q}}\Big), \tag{18}$$

where $1 \leq p < q \leq \infty$, h_k are arbitrary numbers satisfying the conditions $0 < h_k \leq H_k$ $(k=1,\dots,n)$, and the C_e are constants independent of g and h_k.

Let us assume also that in the domain G the inequalities (9) are valid for the function $g(x)$ in the form

$$\|\mathfrak{D}^{\bar{\alpha}+\bar{\omega}_k} g\|_{L_p(G)} \leq \bar{C}\sigma(i_k)\|\mathfrak{D}^{\bar{\alpha}} g\|_{L_p(G)} \quad (k=1,\dots,n), \tag{9'}$$

i.e., $g(x)$ is a function of polynomial type of degree i_k in the variable x_k $(k=1,\dots,n)$.

Then, by inequality (9'), we may write inequality (18) in the form:

$$\|g\|_{L_q(G)} \leq \|g\|_{L_p(G)} \sum_{e \in \mathcal{E}^n} \tilde{C}_e \Big(\prod_{k \in e} h_k^{1-\frac{1}{p}+\frac{1}{q}} \cdot \sigma(i_k)\Big)\Big(\prod_{k \in e'} h_k^{-\frac{1}{p}+\frac{1}{q}}\Big).$$

Putting

$$h_k = \frac{1}{M\sigma(i_k)} \quad (k=1,\dots,n),$$

where $M>0$ is selected so that $h_k \leq H_k$ $(k=1,\dots,n)$, we obtain

$$\|g\|_{L_q(G)} \leq C\|g\|_{L_p(G)}\Big(\prod_{k=1}^{n}\sigma(i_k)\Big)^{\frac{1}{p}-\frac{1}{q}} \tag{19}$$

where C does not depend on g and i_k $(k=1,\dots,n)$.

Further, let m be a natural number, $1 \leq m < n$, $e^{n-m} = \{m+1,\dots,n\}$, e be an arbitrary subset of e^{n-m}, $e' = e^{n-m} \setminus e$, G^m be the intersection of the domain G with the hyperplane $x_{m+1} = \mathrm{const}, \dots, x_n = \mathrm{const}$. One can then establish the inequality

$$\|g\|_{L_p(G^m)} \leq \sum_{e \in \mathcal{E}^{n-m}} C_e \|\mathfrak{D}^{\bar{\omega}^e} g\|_{L_p(G)} \Big(\prod_{k \in e} h_k^{1-\frac{1}{p}}\Big)\Big(\prod_{k \in e'} h_k^{-\frac{1}{p}}\Big).$$

From this, with the aid of inequality (9'), we obtain

$$\|g\|_{L_p(G^m)} \leq C\|g\|_{L_p(G)} \cdot \Big(\prod_{k=m+1}^{n}\sigma(i_k)\Big)^{\frac{1}{p}}. \tag{20}$$

For entire functions of finite degree, for $G = E^n$, and for trigonometric polynomials, taken over a period, inequalities (19) and (20), with the indicated values of the constants C, were first obtained by S. M. Nikol'skii [6] by another method. Other cases for which inequalities (19) and (20) are valid, essentially those for functions of a single variable, were considered later by a number of other authors.

In what follows we shall consider sequences $\{g_i\}_{i=1}^{\infty}$ of functions such that $g_i \in \mathfrak{P}(i;\sigma;\nu;G)$ $(i=1,2,\dots)$, and, if $i>k$, such that $g_i - g_k \in \mathfrak{P}(i;\sigma;\nu;G)$. Then, it is obvious that for the difference $g_i - g_k$, the inequalities (11), (14), and (15) will be valid, respectively, by replacing in them g_i by $g_i - g_k$.

To simplify the writing in what follows, let us put

$$E_i = E_i^{l,p} = \|f - g_i\|_{W_p^l(G)}, \quad E_{ik} = \|g_i - g_k\|_{W_p^l(G)}.$$

Theorem. Let $G \in A(H)$, $f \in W_p^l(G)$, and let $\{g_i\}_1^{\infty}$ be a sequence of functions such that $g_i \in W_p^l(G) \cap \mathfrak{P}(i;\sigma;s+1;G)$ $(i=1,2,\dots)$; also, let s be a natural number, let q be a positive number for which $p \leq q \leq \infty$.

Further, let the E_i decrease monotonically (as $i \to \infty$) and

1) $\lim_{i\to\infty} E_i = 0$, if $l - s - \frac{n}{p} + \frac{n}{q} > 0$ (for $l - s - \frac{n}{p} + \frac{n}{q} = 0$ $q \neq \infty$); (21)

2) The series $\sum_{m=0}^{\infty} E_{2^m} |\ln \sigma(2^m)|^{\frac{1}{p'}}$ converges if $l - s - \frac{n}{p} = 0$ and $q = \infty$; (22)

3) The series $\sum_{m=0}^{\infty} E_{2^m} [\sigma(2^m)]^{-(l-s-\frac{n}{p}+\frac{n}{q})}$ converges if $l - s - \frac{n}{p} + \frac{n}{q} < 0$. (23)

Then, in the domain G, the function $f(x)$ has all the generalized derivatives $\mathcal{D}^{\bar\alpha} f$ of order $|\bar\alpha| = s$, and the sequence $\{\mathcal{D}^{\bar\alpha} g_i\}_1^{\infty}$ converges to $\mathcal{D}^{\bar\alpha} f$ in $L_q(G)$; moreover,

1) $\|\mathcal{D}^{\bar\alpha} f - \mathcal{D}^{\bar\alpha} g_i\|_{L_q(G)} \leq C_{20} E_i$, for condition (21); (24)

2) $\|\mathcal{D}^{\bar\alpha} f - \mathcal{D}^{\bar\alpha} g_i\|_{L_\infty(G)} \leq C_{21} E_i |\ln \sigma(i)|^{\frac{1}{p'}} + C_{22} \sum_{m=\tau}^{\infty} E_{2^m} |\ln \sigma(2^m)|^{\frac{1}{p'}}$, for condition (22); (25)

3) $\|\mathcal{D}^{\bar\alpha} f - \mathcal{D}^{\bar\alpha} g_i\|_{L_q(G)} \leq C_{23} E_i [\sigma(i)]^{-(l-s-\frac{n}{p}+\frac{n}{q})} + C_{24} \sum_{m=\tau}^{\infty} E_{2^m} [\sigma(2^m)]^{-(l-s-\frac{n}{p}+\frac{n}{q})}$, for condition (23); (26)

where the C_j are constants, independent of f and g_i, and the integer τ if such that $2^{\tau-1} < i \leq 2^{\tau}$.

<u>Proof.</u> The case $l - s - \frac{n}{p} + \frac{n}{q} \geq 0$ need not be considered since inequality (24) is a direct consequence of inequality (1). The remaining two cases are considered identical. We examine, therefore, the proof of the assertion of the theorem in the case $l - s - \frac{n}{p} + \frac{n}{q} < 0$.

We note, first of all, that

$$E_{ik} \leq E_i + E_k \leq 2E_k \quad , \text{ if } \quad i > k, \tag{27}$$

since the E_i, by hypothesis, decrease monotonically.

Let us show that the sequence $\{\mathcal{D}^{\bar\alpha} g_{2^m}\}_{m=1}^{\infty}$, $|\bar\alpha| = s$ converges in $L_q(G)$ to some function φ. For this, we establish the convergence of the series

$$\sum_{m=1}^{\infty} \|\mathcal{D}^{\bar\alpha}(g_{2^{m+1}} - g_{2^m})\|_{L_q(G)} . \tag{28}$$

Since $g_{2^{m+1}} - g_{2^m}$ is a function of class $\mathcal{P}(2^{m+1}; \sigma; s+1; G)$, then, on the basis of inequalities (15), (27), and (8), we shall have

$$\|\mathcal{D}^{\bar\alpha}(g_{2^{m+1}} - g_{2^m})\|_{L_q(G)} \leq C_{25} E_{2^m} [\sigma(2^{m+1})]^{-(l-s-\frac{n}{p}+\frac{n}{q})} \leq C_{26} E_{2^m} [\sigma(2^m)]^{-(l-s-\frac{n}{p}+\frac{n}{q})}, \tag{29}$$

where C_{26} is independent of m.

Convergence of the series (28) follows from inequality (29) and the convergence of the series (23). Hence, there exists a function $\varphi \in L_q(G)$ such that

$$\lim_{m\to\infty} \|\mathcal{D}^{\bar\alpha} g_{2^m} - \varphi\|_{L_q(G)} = 0$$

Since, in addition,

$$\lim_{m\to\infty} \|g_{2^m} - f\|_{L_p(G)} = 0 \qquad (E_i \to 0 \text{ as } i \to \infty),$$

then, $\mathcal{D}^{\bar\alpha} f$ and $\mathcal{D}^{\bar\alpha} f = \varphi \in L_q(G)$ exist.

Let us now estimate $\|\mathcal{D}^{\bar{\alpha}} f - \mathcal{D}^{\bar{\alpha}} g_i\|_{L_q(G)}$ for arbitrary natural i. Let us choose the integer τ so that $2^{\tau-1} < i \leq 2^{\tau}$. Then, using the inequality:

$$\|\mathcal{D}^{\bar{\alpha}}(g_{2^\tau} - g_i)\|_{L_q(G)} \leq C_{27} E_i [\sigma(i)]^{-(l-s-\frac{n}{p}+\frac{n}{q})} (i = 2^{\tau-1}, \ldots, 2^{\tau}),$$

obtained in the same way as inequality (29), we obtain

$$\|\mathcal{D}^{\bar{\alpha}}(f - g_i)\|_{L_q(G)} \leq \|\mathcal{D}^{\bar{\alpha}}(g_{2^\tau} - g_i)\|_{L_q(G)} + \|\mathcal{D}^{\bar{\alpha}}(f - g_{2^\tau})\|_{L_q(G)} \leq$$

$$\leq \|\mathcal{D}^{\bar{\alpha}}(g_{2^\tau} - g_i)\|_{L_q(G)} + \sum_{m=\tau}^{\infty} \|\mathcal{D}^{\bar{\alpha}}(g_{2^{m+1}} - g_{2^m})\|_{L_q(G)} \leq C_{28} E_i [\sigma(i)]^{-(l-s-\frac{n}{p}+\frac{n}{q})} + C_{29} \sum_{m=\tau}^{\infty} E_{2^m} [\sigma(2^m)]^{-(l-s-\frac{n}{p}+\frac{n}{q})}.$$

This completes the proof of the theorem.

III. The theorem just proven enables us, knowing an estimate for the rate of decrease of $E_i^{l,p} = \|f - g\|_{W_p^l(G)}$, to make a statement concerning the rate of convergence of the derivatives of arbitrary order s of the function g_i of classes $\mathcal{P}(i; \sigma; \nu; G)$ to the corresponding derivative of the function f in the norm of $L_q(G)$. In certain cases, an estimate for $E_i^{l,p}$ can be made on the basis of theorems of the constructive theory of functions.

We shall apply the theorem just proven to estimate the rapidity of convergence of variational processes.

Let us assume we are seeking a solution of the first boundary problem for a linear self-adjoint differential equation of elliptic type of order $2l$ in an n-dimensional bounded domain G:

$$\mathcal{L}u = f \tag{30}$$

$$\mathcal{D}^{\bar{\alpha}} u/_{\Gamma} = 0 \qquad |\bar{\alpha}| = 0, 1, \ldots, l-1.$$

Let the solution u be sought as the limit of a minimizing sequence $\{u_i\}_1^\infty$ for a corresponding problem of the quadratic functional $\mathcal{F}(u)$. Let us assume that the minimizing sequence is formulated according to the method of Ritz and that the functions u_i are functions of the corresponding classes $\mathcal{P}(i; \sigma; \nu; G)$ of the form (10) (for certain particular types of domains G, the functions u_i turn out to be simple algebraic polynomials).

Let $\tilde{u}_i$ be an arbitrary function of the form given in Eq. (10) ($\tilde{u}_i = \omega^l(x) Q_i(x)$, where $Q_i(x)$ is an arbitrary polynomial of degree not higher than i in each of the variables $x_1, \ldots, x_n$).

Then, as is well known, the following chain of inequalities is valid:

$$E_i^{l,2} = \|u - u_i\|_{W_2^l(G)} \leq C_1(\mathcal{F}(u_i) - \mathcal{F}(u)) \leq C_1(\mathcal{F}(\tilde{u}_i) - \mathcal{F}(u)) \leq C_2 \|u - \tilde{u}_i\|_{W_2^l(G)}, \tag{31}$$

where C_2 is independent of u and $\tilde{u}_i$.

The possibility of obtaining an estimate for $E_i^{l,2}$ is based on the inequality (31) and theorems giving an estimate for $\|u - \tilde{u}_i\|_{W_2^l(G)}$. An estimate for $\|u - \tilde{u}_i\|_{W_2^l(G)}$ may be obtained by using a theorem of I. Yu. Kharrik [7].

Let the function u, which is a solution of problem (30), be κ times continuously differentiable in the domain $\bar{G}$, and let $\omega(x)$ be $\kappa+1$ times continuously differentiable. Then, as shown in [7], there exists a function $\tilde{u}_i$, introduced above, of a form such that

$$\|u - \tilde{u}_i\|_{C^l(\bar{G})} = O\left(\frac{\omega^{(\kappa)}(u; \frac{1}{i})}{i^{\kappa-l}}\right), \quad \kappa > l, \tag{32}$$

where $\omega^{(\kappa)}(u; \frac{1}{i})$ is the maximum modulus of continuity of the derivatives of order κ of the function u.

Under the conditions stated, it follows from the relations (31) and (32) that

$$E_i^{l,2} \le C \frac{\omega^{(k)}(u;\frac{1}{i})}{i^{k-l}}.$$

Applying this estimate to the theorem proved above, with $p=2$, $\sigma(i)=i^2$, we obtain the inequalities:

1) $\|\mathcal{D}^s u - \mathcal{D}^s u_i\|_{L_q(G)} \le C_1 i^{-k+l}\omega^{(k)}(u;\frac{1}{i})$, if $l-s-\frac{n}{2}+\frac{n}{q}>0$ (for $l-s-\frac{n}{2}+\frac{n}{q}=0$ $q\neq\infty$), $k>l$;

2) $\|\mathcal{D}^s u - \mathcal{D}^s u_i\|_{C(G)} \le C_2 i^{-k+l}\omega^{(k)}(u;\frac{1}{i})\sqrt{\ln i}$, if $l-s-\frac{n}{2}=0$, $k>l$;

3) $\|\mathcal{D}^s u - \mathcal{D}^s u_i\|_{L_q(G)} \le C_3 i^{-k+l-2[l-s-\frac{n}{2}+\frac{n}{q}]}\cdot\omega^{(k)}(u;\frac{1}{i})$,

if $l-s-\frac{n}{2}+\frac{n}{q}<0$ and $k>l-2[l-s-\frac{n}{2}+\frac{n}{q}]$, where C_1, C_2, and C_3 do not depend on i.

LITERATURE CITED

1. V. P. Il'in, "Some inequalities in functional spaces and their application to the study of convergence of variational processes," Trudy Matem. Inst. im. V. A. Steklova, Akad. Nauk SSSR, Vol. 53, pp. 64-127 (1959).
2. V. P. Il'in, Candidate's Dissertation, Leningrad State University (1951).
3. I. K. Daugavet, "On the rapidity of convergence of the method of Galerkin for ordinary differential equations," Izvestiya VUZov, Matematika, No. 5, pp. 158-165 (1958).
4. S. L. Sobolev, "On a theorem of functional analysis," Matem Sb., Vol. 4(46):3, pp. 471-497 (1938).
5. I. Yu. Kharrik, "On an analogue of an inequality of Markov," Doklady Akad. Nauk SSSR, Vol. 106, No. 2, pp. 203-206 (1956).
6. S. M. Nikol'skii, "Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of many variables," Trudy Matem. Inst. im. V. A. Steklova, Akad. Nauk SSSR, Vol. 38, pp. 244-278 (1951).
7. I. Yu. Kharrik, "On the approximation of functions, which, together with their partial derivatives, vanish on the boundary of a domain, by functions of a particular form," Sibirsk. Matem. Zh., Vol. 4, No. 2, pp. 408-425 (1963).

INITIAL-BOUNDARY PROBLEM FOR NAVIER-STOKES EQUATIONS IN DOMAINS WITH TIME-VARYING BOUNDARIES

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Let us consider the system

$$\left.\begin{aligned} &v_t - \nu\Delta v + \sum_{i=1}^{3} v_i v_{x_i} = -\nabla p + f(x,t) \\ &\operatorname{div} v = 0 \end{aligned}\right\} \tag{1}$$

in the bounded domain $Q^T = \{(x,t): t\in(0,T), x\in\Omega_t\}$ of the space $E_4 \{(x,t): t\in(-\infty,\infty), x=(x_1,x_2,x_3)\in E_3\}$, and let us assume that the boundary S_t of the domain Ω_t belongs to C^2 for all $t\in[0,T]$ (where the "norms" of S_t in C^2 are uniformly bounded) and changes with time at a finite rate. With the system (1) we shall associate the initial and boundary conditions

$$v\big|_{t=0} = v_0(x),\ x\in\Omega_0,\quad v\big|_{S^T_{\delta\text{-nbhd}}} = \psi(s,t). \tag{2}$$

In view of the equation $\operatorname{div} v=0$, it is necessary to place on v_0 and ψ the restrictions $\operatorname{div} v_0 = 0$ and $\int_{S_t}\psi(s,t)\cdot n\,ds = 0$, $\forall t\in[0,T]$. The problem of determining the speed $v=(v_1(x,t), v_2(x,t), v_3(x,t))$ and the pressure $p=p(x,t)$ leads, upon introducing new unknown functions $u(x,t)=v(x,t)-\Psi(x,t)$, to the problem

$$\left.\begin{aligned} &u_t - \nu\Delta u + (u_i+\Psi_i)u_{x_i} + u_i\Psi_{x_i} = -\nabla p + \mathcal{F} \\ &\operatorname{div} u = 0,\ u\big|_{t=0} = u_0(x),\ u\big|_{S^T_{\delta\text{-nbhd}}} = 0, \end{aligned}\right\} \tag{3}$$

where $\mathcal{F} = f - \Psi_t + \nu\Delta\Psi - \Psi_i\Psi_{x_i}$, $u_0(x) = v_0(x) - \Psi(x,0)$ and $\Psi(x,t)$ is a sufficiently smooth vector function satisfying the conditions

$$\operatorname{div}\Psi(x,t) = 0 \text{ and } \Psi(x,t)\big|_{x=s\in S_t} = \psi(s,t) \text{ for } \forall t\in[0,T].$$

We shall prove that problem (3) is uniquely solvable in $W_2^{2,1}(Q^T)$ for arbitrary T if $\psi\equiv 0$, and that $u_0(x)$ and $f(x,t)$ are, in a definite sense, small, even for small T, if Ψ, u_0, and f only possess a certain smoothness. Approximate solutions will be found by the method of Rothe, and their convergence to the solution of problem (3) will be established with the aid of two *a priori* estimates.

§1. Let us divide Q^T by the planes $t = t_\kappa = \kappa\Delta t \equiv \kappa h$ into layers, and let us denote by Ω_κ the cross section of Q^T made by the plane $t=t_\kappa$, and by S_κ the boundary Ω_κ.* The approximate solutions u_h will be determined consecutively on the cross sections Ω_κ, $\kappa=1,2,\ldots[\frac{T}{h}]$, as solutions of the following linear stationary problems:

$$\left.\begin{aligned} &u_{\bar t}(\kappa) - \nu\Delta u(\kappa) + \left[u_i(\kappa-1)+\Psi_i(\kappa)\right]u_{x_i}(\kappa) + \\ &+ u_i(\kappa)\Psi_{x_i}(\kappa) = -\nabla p(\kappa) + \mathfrak{F}_h(\kappa), \\ &\operatorname{div} u(\kappa)=0, \quad u\big|_{S_\kappa}=0, \quad u(0)=u_{0h}, \end{aligned}\right\} \tag{4}$$

in which $u(\kappa) \equiv u(x, t_\kappa)$, $x \in \bar\Omega_\kappa$, $u_{\bar t}(\kappa) = \frac{1}{h}[u(\kappa) - u(\kappa-1)]$, $\mathfrak{F}_h(\kappa) = \frac{1}{h}\int_{(\kappa-1)h}^{\kappa h} \mathfrak{F}(x,\tau)d\tau$, $u_{0h}(x)$ is a solenoidal vector in $W_2^1(E_3)$, which vanishes outside of $\Omega_0 \cap \Omega_1$ (we shall require below that $u_{0h}(x)$ converge strongly in the norm of $W_2^1(E_3)$ to $u_0(x)$ as $h \to 0$). In addition, $u_{\bar t}(x, t_\kappa)$, $x \in \bar\Omega_\kappa$, in Eq. (4), is considered to be equal to $\frac{1}{h}u(x,t_\kappa)$ if the point $(x, t_{\kappa-1}) \bar\in \bar\Omega_{\kappa-1}$. On each cross section Ω_κ, we have, by the same token, a linear stationary problem. It will be uniquely solvable in $W_2^2(\Omega_\kappa)$ (more precisely, $u(\kappa) \in W_2^2(\Omega_\kappa)$, $\nabla p(\kappa) \in L_2(\Omega_\kappa)$) providing only that u_0, $\mathfrak{F}$, and Ψ possess a certain regularity. Namely, we shall require that

$$\left.\begin{aligned} &u_0(x) \in \overset{\circ}{W}{}_2^1(\Omega_0), \ \operatorname{div} u_0 = 0, \\ &\Psi(x,t) \in W_2^{2,1}(Q^T); \quad f(x,t) \in L_2(Q^T), \end{aligned}\right\} \tag{5}$$

from which it follows, as is known, that

$$\left.\begin{aligned} &\|\mathfrak{F}\|_{2,Q^T} \equiv \Big(\int_{Q^T} |\mathfrak{F}|^2 dx\,dt\Big)^{1/2} \le c, \\ &\|\Psi_x\|_{2,\Omega_t} \equiv \Big(\int_{\Omega_t} |\Psi_x|^2 dx\Big)^{1/2} \le c, \end{aligned}\right\} \tag{5'}$$

where

$$|\Psi| = \sqrt{\sum_{i=1} \Psi_i^2}\,, \quad |\Psi_x| = \sqrt{\sum_{i,\kappa=1} \Psi_{ix_\kappa}^2}\,.$$

Subject to satisfaction of these conditions, problem (4) has, for all $\kappa=1,2,\ldots\left[\frac{T}{h}\right]$ and for h not exceeding some h_0, a unique solution $u(\kappa) \in W_2^2(\Omega_\kappa)$, $\nabla p(\kappa) \in L_2(\Omega_\kappa)$, for which the following estimate is valid:

$$\|u(\kappa)\|_{W_2^2(\Omega_\kappa)} + \|\nabla p(\kappa)\|_{2,\Omega_\kappa} \le c(\kappa) \tag{6}$$

with constant $c(\kappa)$ depending on h, $\mathfrak{F}$, Ψ, $u(\kappa-1)$, and S_κ. This is proved in the manner employed in [1] for a Navier–Stokes system, linearized according to Stokes (see Chap. 6, §11, and Chap. 3, §5). The presence of linear terms of lower order entails no difficulties. From §5, Chap. 3 of [1], we shall need an estimate of the form (6) (estimate of V. A. Solonnikov) for the Stokes linearization, namely,

$$\|v(x)\|_{W_2^2(\Omega_\kappa)} \le \beta_1 \|P^\kappa \Delta v(x)\|_{2,\Omega_\kappa}, \tag{7}$$

valid for an arbitrary solenoidal vector $v(x)$ of $W_2^2(\Omega_\kappa)$, which vanishes on S_κ. We designate the set of such $v(x)$ by $J_{2,0}^2(\Omega_\kappa)$. The constant β_1 in inequality (7) depends on the "norm" of S_κ in C^2. The assumption we make concerning the smoothness of the boundaries S_t, $t \in [0, T]$, consists in taking the constant β_1 in inequality (7) to be common for all the S_t, $t \in [0,T]$. The symbol P^κ indicates, in the space of vector functions of $L_2(\Omega_\kappa)$, the operation of orthogonal projection onto $\overset{\circ}{J}(\Omega_\kappa) = L_2(\Omega_\kappa) \ominus G(\Omega_\kappa)$, where $G(\Omega_\kappa)$ is the set of all vector functions of the form $\nabla\varphi(x)$, $\varphi(x) \in W_2^1(\Omega_\kappa)$.

*The notations Ω_κ and S_κ will also be used for their orthogonal projections from E_4 onto the hyperplane $t=0$, which is the space E_3 for the variation of $x = (x_1, x_2, x_3)$.

Thus, we shall determine on each Ω_K, $K=1,2,\dots\left[\frac{T}{h}\right]$, a function $u(K)\in \overset{\vee}{J}{}^{2}_{2,0}(\Omega_K)$, taking

$$u_0(x)\in \overset{\vee}{J}{}^{1}_{2,0}(\Omega_0) \tag{8}$$

(i.e., $u_0(x)\in \overset{0}{W}{}^{1}_{2}(\Omega_0)$ and $\operatorname{div} u_0=0$). We shall extend each $u(K)$, $K=0,1,\dots$ to be zero outside of $\bar{\Omega}_K$, keeping, after this extension, the previous notation $u(K)$. We note that such $u(K)$ will belong to $\overset{\vee}{J}{}^{1}_{2,0}(E_3)$, but not to $\overset{\vee}{J}{}^{2}_{2,0}(E_3)$.

§2. Let us now obtain estimates for $u(K)$, which are independent of h. To do this, we multiply Eq. (4) scalarly by $u(K)$, integrate over Ω_K, and, after elementary transformations, put the result in the form

$$\int_{\Omega_K}\left[u_t(K)u(K)+\nu u_x^2(K)+u_i(K)\Psi_{x_i}(K)u(K)\right]dx=\int_{\Omega_K}\mathcal{F}_h(K)u(K)\,dx. \tag{9}$$

Here, and in what follows, we shall have need of the following equation:

$$[u(K)-u(K-1)]u(K)=\frac{1}{2}\left[u^2(K)-u^2(K-1)\right]+\frac{1}{2}\left[u(K)-u(K-1)\right]^2. \tag{10}$$

Let us multiply Eq. (9) through by h and sum with respect to K from 1 to $m\le\left[\frac{T}{h}\right]$. By virtue of Eq. (10) and our definition of $u_t(K)$, we will have

$$h\sum_{K=1}^{m}\int_{\Omega_K}u_t(K)u(K)\,dx\ge\frac{1}{2}\int_{\Omega_m}u^2(m)\,dx-\frac{1}{2}\int_{\Omega_0}u^2(0)\,dx \tag{11}$$

and

$$\frac{1}{2}\int_{\Omega_m}u^2(m)\,dx-\frac{1}{2}\int_{\Omega_0}u^2(0)\,dx+h\sum_{K=1}^{m}\nu\int_{\Omega_K}u_x^2(K)\,dx\le h\sum_{K=1}^{m}\int_{\Omega_K}\left[-u_i(K)\Psi_{x_i}(K)u(K)+\mathcal{F}_h(K)u(K)\right]dx. \tag{12}$$

Let us estimate the right side through use of the inequality of Cauchy and Jung and inequality (3), §1, Chap. 1 of [1], thus:

$$\left|\int_{\Omega_K}u_i(K)\Psi_{x_i}(K)u(K)\,dx\right|\le\|\Psi_x\|_{2,\Omega_K}\|u\|^2_{4,\Omega_K}\le 2\|\Psi_x\|_{2,\Omega_K}\|u\|^{1/2}_{2,\Omega_K}\|u_x\|^{3/2}_{2,\Omega_K}\le c\left(\frac{3}{2}\varepsilon^{4/3}\|u_x\|^2_{2,\Omega_K}+\frac{1}{2\varepsilon^4}\|u\|^2_{2,\Omega_K}\right)^{*} \tag{13}$$

and

$$\left|\int_{\Omega_K}\mathcal{F}_h(K)u(K)\,dx\right|\le\frac{1}{2}\|\rho^K\mathcal{F}_h\|^2_{2,\Omega_K}+\frac{1}{2}\|u\|^2_{2,\Omega_K}.$$

In inequality (13) let us take ε such that $\frac{3}{2}c\,\varepsilon^{4/3}=\frac{\nu}{2}$. From inequality (12), by virtue of the last two inequalities, it follows that

$$\int_{\Omega_m}u^2\,dx-\int_{\Omega_0}u^2\,dx+h\sum_{K=1}^{m}\nu\int_{\Omega_K}u_x^2\,dx\le h\sum_{K=1}^{m}\int_{\Omega_K}\left[c_1u^2+(\rho^K\mathcal{F}_h)^2\right]dx, \tag{14}$$

where $c_1=1+\frac{3^3c^4}{\nu^3}$. From inequality (14) we can derive, by a known method, the estimate

$$\int_{\Omega_m}u^2\,dx+h\sum_{K=1}^{m}\nu\int_{\Omega_K}u_x^2\,dx\le c_2\left[\int_{\Omega_0}u^2\,dx+h\sum_{K=1}^{m}\int_{\Omega_K}(\rho^K\mathcal{F}_h)^2\,dx\right] \tag{15}$$

for $m=1,2,\dots\left[\frac{T}{h}\right]$, $h\le\frac{1}{2c_1}$, where the constant c_2 depends only on T and c_1.

$^{*}\ \|u\|_{q,\Omega}=\left(\int_{\Omega}|u|^q\,dx\right)^{1/q}$

We pass now to a derivation of the second estimate. For this we multiply Eq. (4) scalarly by $-\rho^{\kappa}\Delta u(\kappa)$ and integrate over Ω_κ:

$$-\int_{\Omega_\kappa} u_{\bar t}(\kappa)\rho^{\kappa}\Delta u(\kappa)dx+\nu\int_{\Omega_\kappa}(\rho^{\kappa}\Delta u(\kappa))^2dx=\int_{\Omega_\kappa}\left\{[u_i(\kappa-1)+\Psi_i(\kappa)]u_{x_i}(\kappa)+u_i(\kappa)\Psi_{x_i}(\kappa)\right\}\rho^{\kappa}\Delta u(\kappa)dx-\int_{\Omega_\kappa}\mathcal{F}_h(\kappa)\rho^{\kappa}\Delta u(\kappa)dx.$$

Let us estimate, first of all, the terms appearing in the right member of Eq. (16). In this regard, we use, besides the inequality (7), the inequalities of Hõlder and the assumptions (5), the known inequalities

$$\|u\|_{6,\Omega}\le\beta_2\,\|u_x\|_{2,\Omega}\qquad\text{for }\ \forall u\in \overset{\circ}{W}{}_2^1(\Omega), \tag{17}$$

$$\|u\|_{6,\Omega}\le\beta_3\,\|u\|_{W_2^1(\Omega)}\quad\text{for }\ \forall u\in W_2^1(\Omega), \tag{18}$$

$$\|u\|_{q,\Omega}\le\varepsilon\,\|u_x\|_{2,\Omega}+c_{\varepsilon,q}\|u\|_{2,\Omega}, \tag{19}$$

for $\forall u\in W_2^1(\Omega)$, with arbitrary, $q<6$, and

$$\max_{\Omega}|u|\le\varepsilon\,\|u_{xx}\|_{2,\Omega}+c_\varepsilon\,\|u_x\|_{2,\Omega} \tag{20}$$

for $\forall u\in W_2^2(\Omega)$ and $u|_S=0$, with $\varepsilon>0$ arbitrary.* Namely,

$$\left|\int_{\Omega_\kappa}\mathcal{F}_h(\kappa)\rho^{\kappa}\Delta u(\kappa)dx\right|\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|\rho\mathcal{F}_h\|_{2,\Omega_\kappa}\le\frac{\nu}{8}\|\rho\Delta u\|^2_{2,\Omega_\kappa}+\frac{2}{\nu}\|\rho\mathcal{F}_h\|^2_{2,\Omega_\kappa};$$

$$\left|\int_{\Omega_\kappa}\Psi_i(\kappa)u_{x_i}(\kappa)\rho^{\kappa}\Delta(\kappa)dx\right|\le\|\rho\Delta u\|_{2,\Omega_\kappa}\,\||\Psi|\cdot|u_x|\|_{2,\Omega_\kappa}\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|\Psi\|_{6,\Omega_\kappa}\|u_x\|_{3,\Omega_\kappa}\le$$

$$\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|\Psi\|_{6,\Omega_\kappa}(\varepsilon\|u_{xx}\|_{2,\Omega_\kappa}+c_{\varepsilon,3}\|u_x\|_{2,\Omega_\kappa})\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|\Psi\|_{6,\Omega_\kappa}(\varepsilon\beta_1\|\rho\Delta u\|_{2,\Omega_\kappa}+c_{\varepsilon,3}\|u_x\|_{2,\Omega_\kappa})\le$$

$$\le\|\Psi\|_{6,\Omega_\kappa}\left(\frac{3}{2}\varepsilon\beta_1\|\rho\Delta u\|^2_{2,\Omega_\kappa}+\frac{c^2_{\varepsilon,3}}{2\varepsilon\beta_1}\|u_x\|^2_{2,\Omega_\kappa}\right);$$

$$\left|\int_{\Omega_\kappa}u_i(\kappa)\Psi_{x_i}(\kappa)\rho^{\kappa}\Delta u(\kappa)dx\right|\le\|\rho\Delta u\|_{2,\Omega_\kappa}\,\||u|\cdot|\Psi_x|\|_{2,\Omega_\kappa}\le\|\rho\Delta u\|_{2,\Omega_\kappa}\cdot\max_{\Omega_\kappa}|u(\kappa)|\cdot\|\Psi_x\|_{2,\Omega_\kappa}\le$$

$$\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|\Psi_x\|_{2,\Omega_\kappa}(\varepsilon\|u_{xx}\|_{2,\Omega_\kappa}+c_\varepsilon\|u_x\|_{2,\Omega_\kappa})\le\|\Psi_x\|_{2,\Omega_\kappa}\left(\frac{3}{2}\varepsilon\beta_1\|\rho\Delta u\|^2_{2,\Omega_\kappa}+\frac{c_\varepsilon^2}{2\varepsilon\beta_1}\|u_x\|^2_{2,\Omega_\kappa}\right).$$

The remaining terms on the right side of Eq. (16) can be estimated in the manner employed for the problem with a fixed boundary, where estimates, analogous to those derived here, were obtained (see Chap. 6 of the second English edition of [1]):

$$\left|\int_{\Omega_\kappa}u_i(\kappa-1)u_{x_i}(\kappa)\rho^{\kappa}\Delta u(\kappa)dx\right|\le\|\rho\Delta u\|_{2,\Omega_\kappa}\,\||u(\kappa-1)|\cdot|u_x(\kappa)|\|_{2,\Omega_\kappa}\le$$

$$\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|u(\kappa-1)\|_{6,\Omega_\kappa}\left(\int_{\Omega_\kappa}|u_x|^{3/2}\cdot|u_x|^{3/2}dx\right)^{1/3}\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|u(\kappa-1)\|_{6,\Omega_\kappa}\|u_x\|^{1/2}_{2,\Omega_\kappa}\|u_x\|^{1/2}_{6,\Omega_\kappa}\le$$

$$\le\|\rho\Delta u\|_{2,\Omega_\kappa}\|u\|_{6,\Omega_{\kappa-1}}\|u_x\|^{1/2}_{2,\Omega_\kappa}\beta_3^{1/2}\|u_x\|^{1/2}_{W_2^1(\Omega_\kappa)}\le\sqrt{\beta_1\beta_3}\,\|\rho\Delta u\|^{3/2}_{2,\Omega_\kappa}\|u\|_{6,\Omega_{\kappa-1}}\|u_x\|^{1/2}_{2,\Omega_\kappa}\le$$

$$\le\beta_2\sqrt{\beta_1\beta_3}\,\|\rho\Delta u\|^{3/2}_{2,\Omega_\kappa}\|u_x\|_{2,\Omega_{\kappa-1}}\|u_x\|^{1/2}_{2,\Omega_\kappa}\le\beta_2\sqrt{\beta_1\beta_3}\left[\frac{3}{4}\varepsilon^{4/3}\|\rho\Delta u\|^2_{2,\Omega_\kappa}+\frac{1}{4\varepsilon^4}\|u_x\|^4_{2,\Omega_{\kappa-1}}\|u_x\|^2_{2,\Omega_\kappa}\right]. \tag{21}$$

*The constants in inequalities (17)-(20) depend on S. With our assumptions on S_t, $t\in[0,T]$, they may be taken as common for all the S_t, $t\in[0,T]$.

Let us use all the inequalities just obtained in estimating the right side of Eq. (16):

$$-\int_{\Omega_K} u_{\bar t}(K)\rho^K \Delta u(K)\,dx + \nu\|\rho\Delta u\|^2_{2,\Omega_K} \leq \|\rho\Delta u\|^2_{2,\Omega_K}\Big(\frac{\nu}{8} + \frac{3\varepsilon\beta_1}{2}\|\psi\|_{6,\Omega_K} + \frac{3\varepsilon\beta_1}{2}\|\psi_x\|_{2,\Omega_K} +$$

$$+\beta_2\sqrt{\beta_1\beta_3}\,\frac{3}{4}\varepsilon^{4/3}\Big) + \frac{\beta_2\sqrt{\beta_1\beta_3}}{4\varepsilon^4}\|u_x\|^4_{2,\Omega_{K-1}}\|u_x\|^2_{2,\Omega_K} + \|u_x\|^2_{2,\Omega_K}\Big(\frac{C^2_{\varepsilon,3}}{2\varepsilon\beta_1}\|\psi\|_{6,\Omega_K} + \frac{C^2_\varepsilon}{2\varepsilon\beta_1}\|\psi_x\|_{2,\Omega_K}\Big) + \frac{2}{\nu}\|\rho\mathcal{F}_h\|^2_{2,\Omega_K}. \tag{22}$$

We take up now the study of the first term on the left side of inequality (22). For this we observe that if the vector $v \in L_2(\Omega)$, and if $v = \rho v \oplus \nabla\psi$ is its orthogonal decomposition, then, multiplying this equation scalarly by $\nabla\Phi$ and integrating, we obtain

$$\int_\Omega v\nabla\Phi\,dx = \int_\Omega \psi_x\Phi_x\,dx \tag{23}$$

for arbitrary scalar function Φ of $W_2^1(\Omega)$.

Let us denote by $\hat{W}_2^1(\Omega)$, the Hilbert space consisting of the scalar functions $\Phi(x) \in W_2^1(\Omega)$, subject to the condition $\int_\Omega \Phi\,dx = 0$. The scalar product in this space is taken as

$$[\psi,\Phi] = \int_\Omega \psi_x\Phi_x\,dx.$$

The identity (23) uniquely defines ψ in $\hat{W}_2^1(\Omega)$ with respect to v of $L_2(\Omega)$. Indeed, Eq. (23) may be written in the form

$$[\psi,\Phi] = \int_\Omega v\nabla\Phi\,dx. \tag{$\widetilde{23}$}$$

The right side of ($\widetilde{23}$), as one readily sees, defines a linear functional over Φ in $\hat{W}_2^1(\Omega)$. By virtue of a theorem of Riesz, it may be represented in the form of a scalar product $[A(v),\Phi]$, where $A(v)$ is uniquely defined with respect to v. This element $A(v)$ of $\hat{W}_2^1(\Omega)$ is, obviously, the desired solution ψ of $\hat{W}_2^1(\Omega)$.

The first term of inequality (22) may be written as

$$-\int_{\Omega_K} u_{\bar t}(K)\rho^K\Delta u(K)\,dx = -\int_{\Omega_K}\rho^K u_{\bar t}(K)\Delta u(K)\,dx = -\int_{\Omega_K} u_{\bar t}(K)\Delta u(K)\,dx - \int_{\Omega_K}\frac{u(K-1)-\rho^K u(K-1)}{\Delta t}\Delta u(K)\,dx =$$

$$= \int_{\Omega_K} u_{x\bar t}(K)u_x(K)\,dx - \int_{S_K} u_{\bar t}(K)\frac{\partial u(K)}{\partial n}\,dS - \int_{\Omega_K}\frac{u(K-1)-\rho^K u(K-1)}{\Delta t}\Delta u(K)\,dx, \tag{24}$$

taking into account that $\rho^K u(K) = u(K)$. However,

$$j_1 \equiv \Big|\int_{S_K} u_{\bar t}(K)\frac{\partial u(K)}{\partial n}\,ds\Big| = \Big|\int_{S_K}\frac{u(K-1)}{\Delta t}\frac{\partial u(K)}{\partial n}\,ds\Big| \leq \frac{1}{\Delta t}\|u(K-1)\|_{2,S_K}\|u_x(K)\|_{2,S_K}. \tag{25}$$

It will be shown below that

$$\|u(K-1)\|_{2,S_K} \leq c_1\Delta t\big(\varepsilon\|u_{xx}(K-1)\|_{2,\Omega_{K-1}} + \tfrac{1}{\varepsilon}\|u_x(K-1)\|_{2,\Omega_{K-1}}\big) \tag{26}$$

with arbitrary $\varepsilon \in (0,1]$. In addition it is known that for an arbitrary function u of $W_2^2(\Omega)$,

$$\|u_x\|_{2,S} \leq c\big(\varepsilon\|u_{xx}\|_{2,\Omega} + \tfrac{1}{\varepsilon}\|u_x\|_{2,\Omega}\big) \tag{27}$$

with arbitrary $\varepsilon \in (0,1]$. (see Eq. (51') below). By virtue of inequalities (26), and (27), we obtain, from inequality (25),

$$j_1 \leq cc_1\big(\varepsilon\|u_{xx}\|_{2,\Omega_{K-1}} + \tfrac{1}{\varepsilon}\|u_x\|_{2,\Omega_{K-1}}\big)\big(\varepsilon\|u_{xx}\|_{2,\Omega_K} + \tfrac{1}{\varepsilon}\|u_x\|_{2,\Omega_K}\big). \tag{28}$$

The element $u(\kappa-1)-p^{\kappa}u(\kappa-1)$ has the form $\nabla\varphi$, and for it, by virtue of Eq. ($\widetilde{23}$), we have

$$\int_{\Omega_\kappa} \varphi_x \Phi_x \, dx = \int_{\Omega_\kappa} u(\kappa-1)\nabla\Phi \, dx \quad \text{for } \forall \Phi \in \hat{W}_2^1(\Omega_\kappa) \tag{29}$$

Let us transform the right side of Eq. (29), recalling that $u(\kappa-1)\in J^1_{2,0}(E_3)$,

$$\int_{\Omega_\kappa} \varphi_x \Phi_x \, dx = -\int_{\Omega_\kappa} \Phi \operatorname{div} u(\kappa-1)\, dx + \int_{S_\kappa} u(\kappa-1)\cdot n \, \Phi \, ds = \int_{S_\kappa} u_n(\kappa-1)\Phi \, ds.$$

It follows from this that

$$\left| \int_{\Omega_\kappa} \varphi_x \Phi_x \, dx \right| \le \| u_n(\kappa-1)\|_{2,S_\kappa} \|\Phi\|_{2,S_\kappa} \le c_2 \|u(\kappa-1)\|_{2,S_\kappa} \|\Phi_x\|_{2,\Omega_\kappa},$$

and, in view of the arbitrariness of Φ,

$$\|\varphi_x\|_{2,\Omega_\kappa} = \|u(\kappa-1)-p^{\kappa}u(\kappa-1)\|_{2,\Omega_\kappa} \le c_2 \|u(\kappa-1)\|_{2,S_\kappa}. \tag{30}$$

Thanks to relations (30) and (26), we have

$$\left| \int_{\Omega_\kappa} \frac{u(\kappa-1)-p^{\kappa}u(\kappa-1)}{\Delta t} \Delta u(\kappa)\, dx \right| \le c_1 c_2 \|\Delta u\|_{2,\Omega_\kappa} \left(\varepsilon \|u_{xx}\|_{2,\Omega_{\kappa-1}} + \frac{1}{\varepsilon}\|u_x\|_{2,\Omega_{\kappa-1}}\right) \tag{31}$$

and, hence, we obtain from Eq. (24), by virtue of inequalities (28), (31), and (7),

$$-\int_{\Omega_\kappa} u_{\bar t}(\kappa) p^{\kappa} \Delta u(\kappa)\, dx \ge \int_{\Omega_\kappa} u_{x\bar t}(\kappa) u_x(\kappa)\, dx - c_3\left(\varepsilon \|p\Delta u\|^2_{2,\Omega_\kappa} + \varepsilon \|p\Delta u\|^2_{2,\Omega_{\kappa-1}} + \frac{1}{\varepsilon}\|u_x\|^2_{2,\Omega_\kappa} + \frac{1}{\varepsilon}\|u_x\|^2_{2,\Omega_{\kappa-1}}\right). \tag{32}$$

Let us replace the first term in inequality (22) by the smaller quantity in inequality (32) and then sum the resulting inequalities with respect to κ from $\kappa=2$ to $\kappa=m$; we then collect similar terms, multiply through by h, take account of condition (5), (8), and the estimate (15), and choose all the ε so small that the coefficients for $\|p\Delta u\|^2_{2,\Omega_\kappa}$ and $\|p\Delta u\|^2_{2,\Omega_{\kappa-1}}$ do not exceed $\frac{\nu}{4}$. As a result of this we arrive at the inequality

$$h\sum_{\kappa=2}^{m}\left(\int_{\Omega_\kappa} u_{x\bar t}(\kappa)u_x(\kappa)\,dx + \frac{\nu}{2}\|p\Delta u\|^2_{2,\Omega_\kappa}\right) \le c + ch\sum_{\kappa=2}^{m}\|u_x\|^4_{2,\Omega_{\kappa-1}}\|u_x\|^2_{2,\Omega_\kappa} + \frac{\nu}{4}h\|p\Delta u\|^2_{2,\Omega_1}. \tag{33}$$

To this inequality let us add inequality (22) with $\kappa=1$, multiplied through by h, wherein we take account of the fact that $u(0)|_{S_1} = u_{0h}|_{S_1} = 0$ for $\kappa=1$; we then obtain, instead of Eq. (24), the equation

$$-\int_{\Omega_1} u_{\bar t}(1)p^1\Delta u(1)\,dx = \int_{\Omega_1} u_{x\bar t}(1)u_x(1)\,dx. \tag{34}$$

This gives

$$h\sum_{\kappa=1}^{m}\left(\int_{\Omega_\kappa} u_{x\bar t}(\kappa)u_x(\kappa)\,dx + \frac{\nu}{2}\|p\Delta u\|^2_{2,\Omega_\kappa}\right) \le c_1 + c_2 h\sum_{\kappa=1}^{m}\|u_x\|^4_{2,\Omega_{\kappa-1}}\|u_x\|^2_{2,\Omega_\kappa}. \tag{35}$$

For the first term of the left member of inequality (35), by virtue of Eq. (10) and the fact that $u(x,t_\kappa)=0$ for $x \bar{\in} \Omega_\kappa$, inequality (11) holds, with the function u_x replacing u, namely,

$$h\sum_{\kappa=1}^{m}\int_{\Omega_\kappa} u_{x\bar t}(\kappa)u_x(\kappa)\,dx \ge \frac{1}{2}\int_{\Omega_m} u_x^2\,dx - \frac{1}{2}\int_{\Omega_0}(u_{0hx})^2\,dx. \tag{36}$$

Hence, it follows from inequality (35) that

$$\|u_x\|^2_{2,\Omega_m} + \nu h \sum_{k=1}^{m} \|p\Delta u\|^2_{2,\Omega_k} \leq c_3 + c_4 h \sum_{k=1}^{m} \|u_x\|^4_{2,\Omega_{k-1}} \|u_x\|^2_{2,\Omega_k}, \tag{37}$$

for $m = 1, 2, .., [\frac{T}{h}]$, $(c_4 = 2c_2)$.

From these inequalities one may conclude that on some interval $[0, T_1]$, the quantity on the left side of inequality (37) does not exceed some constant c for $m \leq m_0 = [\frac{T_1}{h}]$, where T_1 and c are defined by c_3, c_4, and $\|u_x\|_{2,\Omega_0}$. Indeed, upon discarding, at first, the second term on the left side of inequality (37) and denoting the right side of this inequality by $z(m)$ we obtain

$$\|u_x\|^2_{2,\Omega_m} \leq z(m), \quad m = 1, 2, \dots m_0, \tag{38}$$

and

$$\frac{1}{h}\left[z(m) - z(m-1)\right] = c_4 \|u_x\|^4_{2,\Omega_{m-1}} \|u_x\|^2_{2,\Omega_m} \leq c_4 z^2(m-1) z(m), \tag{39}$$

$$m = 2, \dots m_0$$

Let us calculate the m for which the inequality $z(m) \leq M z(1)$ is satisfied, where M is a number which will be selected below $(M > 1)$. Let $z(k) \leq M z(1)$ for $k \leq m$. Then, from Eq. (39) we obtain

$$z(k) \leq \frac{z(k-1)}{1 - c_4 h z^2(k-1)} \leq \frac{z(k-1)}{1 - c_4 h M^2 z^2(1)} \leq \left[1 + 2c_4 h M^2 z^2(1)\right] z(k-1), \tag{40}$$

assuming that h satisfies the condition

$$c_4 h M^2 z^2(1) \leq \frac{1}{2} \tag{41}$$

Further, it follows from inequality (40) that

$$z(k) \leq (1 + 2c_4 h M^2 z^2(1))^{k-1} z(1) \leq \exp\{2c_4 M^2 z^2(1)(k-1)h\} z(1). \tag{42}$$

This inequality holds as long as $z(k) \leq M z(1)$, i.e., for k satisfying the condition

$$\exp\{2c_4 M^2 z^2(1)(k-1)h\} \leq M. \tag{43}$$

It is an easy calculation to show that for $M = \sqrt{e}$ the quantity $(k-1)h$, satisfying relation (43) with the equality sign, has its largest possible value. Therefore, let us take $M = \sqrt{e}$ and denote the number $(4c_4 e z^2(1))^{-1}$ by T^h. When $k \leq 1 + [\frac{T^h}{h}]$, inequality (43) will hold with $M = \sqrt{e}$, and hence also the estimate $z(k) \leq \sqrt{e}\, z(1)$. Returning to inequality (37), we find that

$$\|u_x\|^2_{2,\Omega_m} + \nu h \sum_{k=1}^{m} \|p\Delta u\|^2_{2,\Omega_k} \leq \sqrt{e}\, z(1) \tag{44}$$

for $m \leq 1 + [\frac{T^h}{h}] \equiv m^h$. The quantity $z(1)$ can be estimated in terms of quantities, known to us from inequality (37), for $m = 1$, namely

$$\|u_x\|^2_{2,\Omega_1} \leq z(1) = c_3 + c_4 h \|u_x\|^4_{2,\Omega_0} \|u_x\|^2_{2,\Omega_1},$$

and therefore

$$\|u_x\|^2_{2,\Omega_1} \leq 2c_3$$

and

$$z(1) \leq 2c_3 \tag{45}$$

for h satisfying the condition

$$h \leq \frac{1}{2c_4 \|u_x\|^4_{2,\Omega_0}}. \tag{46}$$

The condition (41), which was placed on h above, will be satisfied if we require that h satisfy, besides inequality (46), the inequality

$$h \leq (8 c_4 c_3^2 e)^{-1}. \tag{47}$$

Let us recall, finally, that in the proof of inequality (15) (which was used in the present derivation), we assumed that $h \leq (2c_1)^{-1}$, $c_1 = 1 + (\frac{3}{\nu})^3 c^4$, where the " c " was taken from relation (5).

Inequalities (44) and (45) yield, when account is taken of inequality (7), the estimate

$$\| u_x \|^2_{2,\Omega_m} + h \sum_{k=1}^{m} \| u \|^2_{W_2^2(\Omega_k)} \leq c_5 \tag{48}$$

for $m = 1, 2, \ldots, m_1 = [\frac{T_1}{h}]$, where $T_1 = (16 c_3^2 c_4 e)^{-1}$. From inequality (48) and the system (4), we obtain an estimate for $u_{\bar{t}}$. By virtue of system (4),

$$u_{\bar{t}}(k) = \rho^k u_{\bar{t}}(k) + \frac{\rho^k u(k-1) - u(k-1)}{\Delta t} = \nu \rho^k \Delta u(k) - \rho^k [(u_i(k-1) + \varphi_i(k)) u_{x_i}(k) + u_i(k) \varphi_{x_i}(k) - \mathcal{F}_h] + \frac{\rho^k u(k-1) - u(k-1)}{\Delta t}.$$

From this relation, the estimates (26) and (30), Eq. (34), the estimates of the terms in the right member of Eq. (16), the data following formula (20), and the previously established estimate (48), we obtain

$$h \sum_{k=1}^{m} \| u_{\bar{t}}(k) \|^2_{2,\Omega_k} \leq c_6, \quad m = 1, 2, \ldots m_1.$$

This inequality, along with inequality (48), gives us a second fundamental estimate:

$$\| u_x \|^2_{2,\Omega_m} + h \sum_{k=1}^{m} \left(\| u_{\bar{t}} \|^2_{2,\Omega_k} + \| u \|^2_{W_2^2(\Omega_k)} \right) \leq c_7, \tag{49}$$

for $m = 1, 2, \ldots m_1 = [\frac{T_1}{h}]$, where $T_1 = (16 c_3^2 c_4 e)^{-1}$. The constant c_7 is common for all sufficiently small h (h satisfying the requirements (46) and (47)).

To complete the proof of estimate (49), it remains for us to verify the validity of inequality (26). We shall prove this inequality for an arbitrary scalar function $u(x)$, using the fact that it belongs to $W_2^2(\Omega_{k-1}) \cap \overset{\circ}{W}{}_2^1(\Omega_{k-1})$ and vanishes outside of Ω_{k-1}. As for the variation of the boundary S_t, we shall assume that it takes place with a speed not exceeding some number β_0, more precisely, we shall assume that for all $\Delta t \leq h_0$ and for all $t \in [0,T]$: 1) the set of points $\bar{\Omega}_t + \Pi_{t,\Delta t}$ contains $\bar{\Omega}_{t-\Delta t}$, where $\Pi_{t,\Delta t}$ is the strip $[S_t, S_t + \beta_0 \Delta t\, n_t]$ of width $\beta_0 \Delta t$, which adjoints S_t (i.e., the set of points x, placed exterior to S_t along the normals n_t and at a distance from S_t along these normals not exceeding $\beta_0 \Delta t$), and 2) the strip adjointing $S_{t-\Delta t}$, namely, $\tilde{\Pi}_{t-\Delta t, \Delta t} = [S_{t-\Delta t} - \beta_0 \Delta t\, n_{t-\Delta t}, S_{t-\Delta t}]$, contains $\bar{\Omega}_{t-\Delta t} \cap \Pi_{t,\Delta t}$ ($\tilde{\Pi}_{t-\Delta t, \Delta t}$ consists of points x, placed on interior normals to $S_{t-\Delta t}$ at a distance not exceeding $\beta_0 \Delta t$). From conditions 1) and 2), it is evident that S_t can, as t increases, "broaden out" with arbitrary speed, but "narrow down" with a finite speed.

Thus, it is necessary for us to estimate $\int_{S_t} u^2(x)\, ds$, knowing that $u \in W_2^2(\Omega_{t-\Delta t}) \cap \overset{\circ}{W}{}_2^1(\Omega_{t-\Delta t})$, and that $u(x) \equiv 0$ for $x \bar{\in} \Omega_{t-\Delta t}$. By virtue of condition 1), the function $u(x)$ is equal to zero on the outside boundary of $\Pi_{t,\Delta t}$, therefore, as is well known,

$$\int_{S_t} u^2 ds \leq c\beta_0 \Delta t \int_{\Pi_{t,\Delta t}} u_x^2\, dx = c\beta_0 \Delta t \int_{\Pi_{t,\Delta t} \cap \Omega_{t-\Delta t}} u_x^2\, dx.$$

If we take into account condition 2), we may conclude from this that

$$\int_{S_t} u^2 ds \leq c\beta_0 \Delta t \int_{\tilde{\Pi}_{t-\Delta t,\Delta t}} u_x^2\, dx \leq c_1 \beta_0 \Delta t \int_0^{\beta_0 \Delta t} \int_{S^\tau} u_x^2\, ds\, d\tau, \tag{50}$$

where S^τ denotes the family of surfaces $S_{t-\Delta t} - \tau n_{t-\Delta t}$ (i.e., S^τ is the set of endpoints of interior normals to $S_{t-\Delta t}$ of length τ). The quantity Δt is here regarded so small that S^τ, $0 \le \tau \le \beta_0 \Delta t$ constitutes a regular family of coordinate surfaces. Further, it is known that in the domain Ω, bounded by a smooth surface S, the inequality

$$\int_S |v(x)|\,ds \le c \int_\Omega (|v| + |v_x|)\,dx \tag{51}$$

is valid for arbitrary $v(x) \in W_1^1(\Omega)$, where the constant c is defined only by the "norm" of S in C^1.

Applying this inequality to $v = u_x^2$, we obtain

$$\int_{S^\tau} u_x^2\,dx \le c \int_{\Omega^\tau} \{u_x^2 + [\sum_{i=1}^{3} (2u_{x_k} u_{x_k x_i})^2]^{1/2}\}\,dx \le c \int_{\Omega^\tau} [u_x^2 + 2\sum_{i,k=1}^{3} |u_{x_k} u_{x_k x_i}|]\,dx \le c \int_{\Omega_{t-\Delta t}} [\varepsilon u_{xx}^2 + (1 + \tfrac{3}{\varepsilon}) u_x^2]\,dx \tag{51'}$$

with $\varepsilon > 0$ arbitrary. Substituting this estimate into inequality (50), we arrive, after elementary evaluations, at the desired inequality:

$$\int_{S_t} u^2 ds \le c_2 (\beta_0 \Delta t)^2 \int_{\Omega_{t-\Delta t}} (\varepsilon u_{xx}^2 + \tfrac{1}{\varepsilon} u_x^2)\,dx,\ \varepsilon \in (0,1].$$

This concludes the derivation of the estimate (49).

§3. Let us now let h tend to zero. The functions $u_{0h}(x)$ will then converge to $u_0(x)$ in $W_2^1(E_3)$, and $\mathfrak{F}_h(x,t)$ will converge to $\mathfrak{F}(x,t)$ in $L_2(Q^T)$. Let us denote by u_h the vector function which we defined earlier on the cross sections Ω_k, $k=0,1,\dots[\frac{T}{h}] \equiv m_0$ as the solution of problems (4) with $\Delta t = h$, where $u_h(x,t_k)$ is considered to be extended to zero exterior to Ω_k. We enclose Q^T in the cylinder $Q_T = \Omega \times (0,T)$ and construct, with respect to u_h, continuous functions $u_h'(x,t)$ in $Q_{m_0 h} = \Omega \times (0, m_0 h)$ which coincide with u_h on the cross-sections $t = kt$, $k = 0,1,\dots m_0$, and are linear in t for $t \in ((k-1)h, kh$. One readily sees that such functions belong to $\mathfrak{J}_{2,0}^{1,0}(Q_{m_0 h})$, i.e. they are solenoidal, they vanish on the lateral surface of $Q_{m_0 h}$ and have the finite norm

$$\|u_h'\|_{\mathfrak{J}_{2,0}^{1,0}(Q_{m_0 h})} \equiv \left\{\int\int_{Q_{m_0 h}} [(u_h')^2 + (u_{hx}')^2]\,dx\,dt\right\}^{1/2},$$

where, by virtue of inequality (15), the quantities $\|u_h'\|_{\mathfrak{J}_{2,0}^{1,0}(Q_{m_0 h})}$ are uniformly bounded. Therefore, from $\{u_h'\}$ one may extract a subsequence, which converges weakly in $\mathfrak{J}_{2,0}^{1,0}(Q_{T-\varepsilon})$ (ε is arbitrarily small) to some element $u(x,t)$ of $\mathfrak{J}_{2,0}^{1,0}(Q_T)$. Since at an arbitrary point $(x,t) \in \bar{Q}_T \setminus \bar{Q}^T$, all the functions u_h', beginning with h sufficiently small, are equal to zero, then, in them, u is also equal to zero, and therefore $u(x,t) \in \mathfrak{J}_{2,0}^{1,0}(Q_T)$.

Further, $u_h(k) \in W_{2,0}^2(\Omega_k)$, therefore, for an arbitrary domain Q', at a positive distance from the lateral surface and upper base of Q^T, all the functions u_h', starting with some h, belong to $W_2^{2,1}(Q')$, and their norms

$$\|u_h'\|_{W_2^{2,1}(Q')} \equiv \left\{\int\int_{Q'} [|u_h'|^2 + |\tfrac{\partial u_h'}{\partial t}|^2 + |u_{hxx}'|^2]\,dx\,dt\right\}^{1/2}$$

are bounded by a constant c, depending neither on h nor on δ. In view of this, we shall also have $\|u\|_{W_2^{2,1}(Q')} \le c$ for the limit function, and, since δ is arbitrary, also $\|u\|_{W_2^{2,1}(Q^{T'})} \le c$.

It remains to show that the function u satisfies the system (3). This is done just as in [2], where a proof was given for the convergence of the solutions of difference equations to the solution of an initial-boundary problem for hyperbolic equations (see §6, Chap. 1, and §§1-2, Chap. 3).

Namely, along with the interpolation u'_h, let us introduce the interpolation $\tilde{u}_h$ for u_h. On the layers $t=\kappa h$, the function $\tilde{u}_h(x,t)$, is equal to $u_h(\kappa)=u_h(x,t_\kappa)$, and, for $t\in((\kappa-1)h,\kappa h]$, it is equal to $u_h(\kappa)$. It is clear that $(\widetilde{u_{h\bar{t}}})(x,t)=\frac{\partial u_h(x,t)}{\partial t}$ for $t\in((\kappa-1)h,\kappa h)$ and $(\widetilde{u_{hxx}})=(\tilde{u}_h)_{xx}$.

Let us take a smooth solenoidal vector function $\Phi(x,t)$, which is equal to zero close to the lateral surface and upper base of the domain Q^{T_1}. Let us multiply the first of equations (4) scalarly by $h\,\Phi(x,t)$, integrate the resulting equation over Ω_κ, sum with respect to κ from 1 to m_1, and write the result as follows:

$$0=h\sum_{\kappa=1}^{m_1}\int_{\Omega_\kappa}[u_{h\bar{t}}(\kappa)-\nu\Delta u_h(\kappa)+(u_{hi}(\kappa-1)+\Psi_i(\kappa))\,u_{hx_i}(\kappa)+u_{hi}(\kappa)\Psi_{x_i}(\kappa)+\nabla p_h(\kappa)-\mathfrak{F}_h(\kappa)]\,\Phi(\kappa)\,dx=$$

$$=\int_{Q^{T_1}}[\frac{\partial u'_h(x,t)}{\partial t}-\nu\Delta\tilde{u}_h(x,t)+(\tilde{u}_{hi}(x,t-h)+\tilde{\Psi}_{hi}(x,t))\,\tilde{u}_{hx_i}(x,t)+\tilde{u}_{hi}(x,t)\tilde{\Psi}_{hx_i}(x,t)-\tilde{\mathfrak{F}}_h(x,t)]\,\tilde{\Phi}_h(x,t)\,dx\,dt, \tag{52}$$

wherein h is taken to be sufficiently small. As was shown in §6, Chap. 1 of [2], the completions of u'_h and $\tilde{u}_h$ have, in view of the estimates (15) and (49), one and the same limit u (weak in $L_2(Q^{T_1})$), and the functions $\frac{\partial u'_h}{\partial t}$, $\tilde{u}_{hx}$ and $\tilde{u}_{hxx}$, have as their limits (weak in $L_2(Q')$) the derivatives $\frac{\partial u}{\partial t}$, u_x and u_{xx}, respectively. In addition, $\tilde{u}_h(x,t)$ and $\tilde{u}_h(x,t-h)$ converge to $u(x,t)$ strongly in $L_2(Q')$ for an arbitrary domain Q' of the form described above. The functions $\tilde{\Psi}_h$ and $\tilde{\mathfrak{F}}_h$ converge strongly to Ψ and $\mathfrak{F}$ in $L_2(Q')$, the functions $\tilde{\Psi}_{hx}$ converge strongly to Ψ_x in $L_2(Q')$, and $\tilde{\Phi}_h$ converges to Φ uniformly in $\bar{Q}^{T_1}$. Thanks to this, the limiting form of Eq. (52) will be

$$\int_{Q^{T_1}}[\frac{\partial u}{\partial t}-\nu\Delta u+(u_i+\Psi_i)\,u_{x_i}+u_i\Psi_{x_i}-\mathfrak{F}]\,\Phi\,dx\,dt=0,$$

whence, in view of the sufficient arbitrariness of Φ and the fact that u belongs to $W_2^{2,1}(Q^{T_1})$, it follows that u satisfies Eqs. (3).

Thus, we have proved the following theorem:

Theorem 1. Problem (3) is uniquely solvable in $W_2^{2,1}(Q^{T_1})$, where T_1 is a positive number defined by $\|u_0(x)\|_{W_2^1(\Omega_0)}$, $\|p\mathfrak{F}\|_{L_2(Q^T)}$, and $\|\Psi\|_{W_2^{2,1}(Q^T)}$. The norm of the solution u, namely, $\|u\|_{W_2^{2,1}(Q^{T_1})}$, is also determined by these same quantities. In this connection, the domain $\bar{Q}^T=\{x\in\bar{\Omega}_t,\ t\in[0,T]\}$ must have a piecewise smooth boundary in E_{n+1}, and the boundaries S_t of the domain Ω_t in E_3 must have uniformly bounded "norms" in C^2 for all $t\in[0\,T]$ and must satisfy conditions 1) and 2), given on page 42. The constants, which appear in the assumptions concerning the boundaries, influence the values of T_1 and $\|u\|_{W_2^{2,1}(Q^{T_1})}$.

The solvability of problem (1), (2), in $W_2^{2,1}(Q^{T_1})$ follows from this theorem, with the same assumptions of this theorem (we recall that $\operatorname{div}u_0=0$ and $\operatorname{div}\Psi(x,t)=0$). Uniqueness of the solutions of problems (1), (2), and (3) in the class $W_2^{2,1}(Q^{T_1})$ is proved as it was for the right cylinder $Q_T=\Omega\times[0,T]$ (see Chap. 6 of [1]). A theorem similar to Theorem 1 was established for the right cylinder Q_T by Prodi [3] and by us [1] independently, and later by Shinbrot and Kaniel [4].

If $p\mathfrak{f}(x,t)\equiv 0$ and $\Psi\equiv 0$ (or if their norms are sufficiently small), then, for sufficiently small $\|u_0(x)\|_{W_2^1(\Omega_0)}$, the time interval T_1, over which a solution belonging to $W_2^{2,1}(Q^{T_1})$ exists, will be equal to ∞. In fact, in this case, the term c_1 in the right member of inequality (35) will be zero, and, therefore, instead of inequality (37), we will have

$$\|u_x\|^2_{2,\Omega_m}+h\sum_{\kappa=1}^{m}\nu\|p\Delta u\|^2_{2,\Omega_\kappa}\le\|u_x\|^2_{2,\Omega_0}+c_4 h\sum_{\kappa=1}^{m}\|u_x\|^4_{2,\Omega_{\kappa-1}}\|u_x\|^2_{2,\Omega_\kappa}. \tag{53}$$

By virtue of inequality (7), or more precisely,

$$\|u_x\|^2_{2,\Omega_k} \le \beta_1^2 \|p\Delta u\|^2_{2,\Omega_k}, \tag{54}$$

we may conclude from inequality (53) that for all m,

$$\|u_x\|^2_{2,\Omega_m} \le \|u_x\|^2_{2,\Omega_0}, \tag{55}$$

providing

$$\nu - c_4\beta_1^2 \|u_x\|^4_{2,\Omega_0} > 0. \tag{56}$$

The estimate (55), together with inequality (37) and system (4), guarantees inequality (49), for arbitrary m. For problem (1), (2), solutions v belonging to $W_2^{2,1}(Q^T)$ are stable with respect to a change in f and ψ. We have, thus, the following theorem:

Theorem 2. Suppose that for $f^0 \in L_2(Q^T)$ and $\psi^0 \in W_2^{2,1}(Q^T)$, there exists a solution $v^0 \in W_2^{2,1}(Q^T)$. Then for all $\tilde f$ and $\tilde\psi$ sufficiently close to f^0 and ψ in the $L_2(Q^T)$ and $W_2^{2,1}(Q^T)$ norms, respectively, there exists a solution v belonging to $W_2^{2,1}(Q^T)$ (with the same T!), which differs little from v^0 in the $W_2^{2,1}(Q_T)$ norm. The assumptions concerning S_t, $t \in [0,T]$, are the same as those in Theorem 1.

Indeed, the function $u = v - (\tilde\psi + v^0 - \psi^0)$ is a solution of the problem of type (3), and, as is evident from the proof of Theorem 2, it exists in $W_2^{2,1}(Q^T)$, and its norm in $W_2^{2,1}(Q^T)$ is small providing only that $\|f - f^0\|_{L_2(Q^T)}$ and $\|\psi - \psi^0\|_{W_2^{2,1}(Q^T)}$ are sufficiently small.

We note that other generalized solutions, those studied in [1], also possess this kind of stability.

In the case of plane-parallel flows, we have solvability in $W_2^{2,1}(Q^T)$ for an arbitrary time interval T. The proof proceeds as in Theorem 1, except that the principal nonlinear term in Eq. (16), instead of inequality (21), is treated differently:

$$\begin{aligned} \left|\int_\Omega u_i(k-1)u_{x_i}(k)p^k \Delta u(k)dx\right| &\le \|p\Delta u\|_{2,\Omega_k} \cdot \|u\|_{4,\Omega_{k-1}} \cdot \|u_x\|_{4,\Omega_k} \le \\ &\le c\|p\Delta u\|_{2,\Omega_k} \|u\|^{1/2}_{2,\Omega_{k-1}} \|u_x\|^{1/2}_{2,\Omega_{k-1}} \|u_x\|^{1/2}_{2,\Omega_k} \|u_{xx}\|^{1/2}_{2,\Omega_k} \cdot c_1 \|p\Delta u\|^{3/2}_{2,\Omega_k} \|u_x\|^{1/2}_{2,\Omega_{k-1}} \|u_x\|^{1/2}_{2,\Omega_k} \le \\ &\le c_1\left(\tfrac{3}{4}\varepsilon^{4/3}\|p\Delta u\|^2_{2,\Omega_k} + \tfrac{1}{4\varepsilon^4}\|u_x\|^2_{2,\Omega_{k-1}}\|u_x\|^2_{2,\Omega_k}\right), \forall\varepsilon>0. \end{aligned} \tag{$\widetilde{21}$}$$

Here we have used inequality (1), §1, Chap. 1 of [1] and inequalities (7) and (15) of this paper. Using inequality ($\widetilde{21}$), instead of inequality (37), we obtain

$$\|u_x\|^2_{2,\Omega_m} + \nu h\sum_{k=1}^m \|p\Delta u\|^2_{2,\Omega_k} \le \tilde c_3 + \tilde c_4 h\sum_{k=1}^m \|u_x\|^2_{2,\Omega_{k-1}} \cdot \|u_x\|^2_{2,\Omega_k}. \tag{$\widetilde{37}$}$$

Denoting the right member of this inequality by $z(m)$, we obtain, from inequality ($\widetilde{37}$),

$$\|u_x\|^2_{2,\Omega_m} \le z(m)$$

and

$$\frac{1}{h}[z(m) - z(m-1)] = \tilde c_4 \|u_x\|^2_{2,\Omega_{m-1}} \cdot \|u_x\|^2_{2,\Omega_m} \le \tilde c_4\, z(m-1)\|u_x\|^2_{2,\Omega_m},$$

whence

$$z(m) \le (1+\tilde c_4 h\|u_x\|^2_{2,\Omega_m})z(m-1) \le (1+\tilde c_4 h\|u_x\|^2_{2,\Omega_m})\dots(1+\tilde c_4 h\|u_x\|^2_{2,\Omega_1})z(0) \le e^{1+\tilde c_4 h\sum_{k=1}^m \|u_x\|^2_{2,\Omega_k}} z(0),$$

i.e. (noting the estimate (15)), $z(m)$, and therefore also the left member of inequality (37), are uniformly bounded for all $m = 0,\dots,[\frac{T}{h}]$. Thus, we have

Theorem 3. In the case of plane-parallel flows, problems (1), (2), and (3) are uniquely solvable in $W_2^{2,1}(Q^T)$ for arbitrary T, if f, ψ, and $S_t, t \in [0,T]$ satisfy the same conditions as in Theorem 1.

LITERATURE CITED

1. O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, 2nd ed., Gordon and Breach, New York (1968).
2. O. A. Ladyzhenskaya, A Mixed Problem for Hyperbolic Equations [in Russian], Moscow (1953).
3. G. Prodi, Résultats récents et problèmes anciens dans la théorie des équations de Navier–Stokes, Institut de Mathématiques, Trieste (1962).
4. M. Shinbrot and S. Kaniel, "The initial value problem for the Navier–Stokes equations," Archive for Rational Mechanics and Analysis, Vol. 21, No. 4, pp. 270-285 (1966).

ON SOME CLASSES OF NONUNIFORMLY ELLIPTIC EQUATIONS*

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We have called the quasi-linear equation

$$a_{ij}(x,u,u_x)u_{x_ix_j}+a(x,u,u_x)=0 \tag{1}$$

uniformly elliptic if

$$\nu(|u|)\xi^2\le a_{ij}(x,u,p)\xi_i\xi_j\le\mu(|u|)\xi^2,$$

where $\nu(\tau)$ and $\mu(\tau)$ are continuous positive functions for $\tau\ge 0$. Such equations have been, in particular, the fundamental object of investigations concerning their solvability for the case of boundary problems "in the large" and the matter of obtaining various *a priori* estimates for all their possible solutions (see [2]). One of the principal *a priori* estimates is the estimate $\max_\Omega |u_x|$. The methods we have given for obtaining this estimate (see §§3,4, Chap. 4, and §2, Chap. 6 of [2]) have also been applied, with corresponding modifications, to some classes of nonuniformly elliptic equations. A series of such classes was singled out in papers [3]-[7]. We point out here still another class of such equations.

Let us consider an equation of the form

$$a_{ij}(u_x)u_{x_ix_j}+a(x,u,u_x)=0 \tag{2}$$

and let us assume that the functions $a_{ij}(p)$ have the form

$$a_{ij}(p)=\overset{\circ}{a}_{ij}+b_{ij}(p)\equiv\overset{\circ}{a}_{ij}[1+f(p)]+f_i(p)p_j+\tilde f_j(p)p_i+\tilde a_{ij}(p),$$

where the $b_{ij}(p)$ are homogeneous functions of p of order $m>0$, $b_{ij}(p)p_ip_j=0$, and the $\overset{\circ}{a}_{ij}$ are constants, which determine a positive-definite form, i.e.,

$$\nu\xi^2\le\overset{\circ}{a}_{ij}\xi_i\xi_j\le\mu\xi^2,\quad \nu,\ \mu=\mathrm{Const}>0\,.$$

The form $a_{ij}(p)\xi_i\xi_j\ge c_1\xi^2$, $c_1=\mathrm{Const}>0$, and for $\tilde a_{ij}(p)$, the following estimate holds:

$$\sqrt{\sum_{i,j=1}^n\tilde a_{ij}^2(p)}\le c_2(1+f(p)).$$

In addition, $0\le f(p)\to\infty$ as $|p|\to\infty$, and all the functions we have introduced will be assumed to be continuously differentiable functions of their arguments.

Let $u(x)$ be a thrice differentiable solution of Eq. (2) in the domain Ω, and let us assume that the estimates $\max_\Omega|u(x)|$ and $\max_S|u_x(x)|$ are known on the boundary S of the domain Ω: $\max_\Omega|u(x)|\le M$, $\max_S|u_x(x)|\le M$. We obtain now the estimate $\max_\Omega|u_x(x)|$. For this we introduce in place of u a new

*The results given here appeared in [1] in abbreviated form.

function v by means of the equation: $u = \varphi(v)$. The function $\varphi(v)$ will be determined below as the solution of an ordinary differential equation of the third order, with a positive derivative $\varphi'(v)$, and with $\varphi''(v) < 0$. Let us substitute $u = \varphi(v)$ in Eq. (2). By virtue of the relations: $u_{x_i} = \varphi' v_{x_i}$, $u_{x_i x_j} = \varphi' v_{x_i x_j} + \varphi'' v_{x_i} v_{x_j}$, we have, from Eq. (2),

$$a_{ij}(u_x) v_{x_i x_j} + \frac{\varphi''}{\varphi'} a_{ij}(u_x) v_{x_i} v_{x_j} + \frac{a(x,u,u_x)}{\varphi'} \equiv \mathcal{L}(u,v) = 0. \tag{3}$$

Let us put $v_x^2 = w$, and let us estimate $w(x)$ at the point where it is a maximum in Ω. If this maximum is attained on S, then the estimate follows from the estimate $\max_S |u_x|$, already known to us. If, however, $\max_\Omega w$ is attained at some interior point $x_0 \in \Omega$, then at this point,

$$w_{x_k} = 2 v_{x_l} v_{x_l x_k} = 0, \quad k = 1, \ldots n, \tag{4}$$

and, also, $a_{ij} w_{x_i x_j} \leq 0$, and therefore,

$$a_{ij} v_{x_k x_i x_j} v_{x_k} \leq -a_{ij} v_{x_k x_i} v_{x_k x_j}. \tag{5}$$

At the point x_0 let us consider the relation

$$v_{x_k} \frac{d\mathcal{L}(u,v)}{dx_k} = 0.$$

It follows, from this relation, upon taking into account Eqs. (4) and (5), that

$$a_{ij} v_{x_k x_i} v_{x_k x_j} - \frac{\partial a_{ij}}{\partial u_{x_l}} u_{x_l}\left(v_{x_i x_j} + \frac{\varphi''}{\varphi'} v_{x_i} v_{x_j}\right)\frac{\varphi''}{\varphi'} v_x^2 - \left(\frac{\varphi''}{\varphi'}\right)' a_{ij} v_{x_i} v_{x_j} v_x^2 + a\frac{\varphi''}{\varphi'^2} v_x^2 -$$
$$- \frac{\partial a}{\partial u_{x_l}} u_{x_l} \frac{\varphi''}{\varphi'^2} v_x^2 - \frac{\partial a}{\partial x_k} u_{x_k} \frac{1}{\varphi'^2} - \frac{\partial a}{\partial u} v_x^2 \leq 0. \tag{6}$$

Let us investigate the term $j \equiv -\frac{\partial a_{ij}}{\partial u_{x_l}} u_{x_l}\left(v_{x_i x_j} + \frac{\varphi''}{\varphi'} v_{x_i} v_{x_j}\right)\frac{\varphi''}{\varphi'} v_x^2$, using Eqs. (3), (4), and the assumptions we have made concerning the a_{ij}:

$$j = -m b_{ij}\left(v_{x_i x_j} + \frac{\varphi''}{\varphi'} v_{x_i} v_{x_j}\right)\frac{\varphi''}{\varphi'} v_x^2 = -m(\overset{\circ}{a}_{ij}\,\xi + \tilde{a}_{ij}) v_{x_i x_j} \frac{\varphi''}{\varphi'} v_x^2.$$

Further, we note that at the point x_0, where v_x^2 is a maximum, we have by virtue of Eq. (4)

$$a_{ij} v_{x_i x_j} = \overset{\circ}{a}_{ij}(1+\xi) v_{x_i x_j} + \tilde{a}_{ij} v_{x_i x_j},$$

and by virtue of Eq. (3)

$$a_{ij} v_{x_i x_j} = -\frac{\varphi''}{\varphi'} a_{ij} v_{x_i} v_{x_j} - \frac{a}{\varphi'} = -\frac{\varphi''}{\varphi'} \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} - \frac{a}{\varphi'},$$

and, therefore,

$$\overset{\circ}{a}_{ij}(1+\xi) v_{x_i x_j} = -\frac{\varphi''}{\varphi'} \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} - \frac{a}{\varphi'} - \tilde{a}_{ij} v_{x_i x_j}.$$

But then,

$$j = m\left(\frac{\varphi''}{\varphi'}\right)^2 \frac{\xi}{1+\xi} \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} v_x^2 + m\left(\frac{a}{\varphi'} + \tilde{a}_{ij} v_{x_i x_j}\right)\frac{\xi}{1+\xi} \frac{\varphi''}{\varphi'} v_x^2 - m\tilde{a}_{ij} v_{x_i x_j} \frac{\varphi''}{\varphi'} v_x^2 \equiv$$
$$\equiv m\left(\frac{\varphi''}{\varphi'}\right)^2 \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} v_x^2 + ma\frac{\xi}{1+\xi} \frac{\varphi''}{\varphi'^2} v_x^2 + j_1,$$

where, for j_1, we have the valid estimate

$$|j_1| \leq \frac{m\mu}{1+\xi}\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 + c_1 v_{xx}^2 + \frac{m^2 c_2^2}{4c_1}\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 \leq c_1 v_{xx}^2 + \left(\frac{m\mu}{1+\xi} + \frac{m^2 c_2^2}{4c_1}\right)\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4.$$

Thanks to these relations and the properties of the a_{ij}, we obtain the following inequality from inequality (6):

$$\left[-\left(\frac{\varphi''}{\varphi'}\right)' + m\left(\frac{\varphi''}{\varphi'}\right)^2\right] \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} v_x^2 + \left(1+\frac{mf}{1+f}\right)\cdot a\frac{\varphi''}{\varphi'^2} v_x^2 - \frac{\partial a}{\partial u_{x_\ell}} u_{x_\ell} \frac{\varphi''}{\varphi'^2} v_x^2 - \frac{\partial a}{\partial x_k} u_{x_k} \frac{1}{\varphi'^3} - \frac{\partial a}{\partial u} v_x^2 \leq$$

$$\leq \left(\frac{m\mu}{1+f} + \frac{m^2 c_2^2}{4c_1}\right)\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 \equiv \left(\frac{m\mu}{1+f} + c_3\right)\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4. \tag{7}$$

We formulate now our assumptions relative to the function $a(x,u,p)$. We shall assume that for $x \in \bar{\Omega}$, $|u| \leq M$

$$-a(x,u,p)\left(1+\frac{mf(p)}{1+f(p)}\right) + p_\ell \frac{\partial a(x,u,p)}{\partial p_\ell} \geq -c_4 p^2 - c_5,$$
$$-\frac{\partial a(x,u,p)}{\partial x_k} p_k \geq -c_6 |p|^4 - c_7, \quad -\frac{\partial a(x,u,p)}{\partial u} \geq -c_8 p^2 - c_9. \tag{8}$$

Then, from inequality (7), we have

$$\left[-\left(\frac{\varphi''}{\varphi'}\right)' + m\left(\frac{\varphi''}{\varphi'}\right)^2\right] \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} v_x^2 \leq \left[c_3 + \frac{m\mu}{1+f(u_x)}\right]\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 + (c_4 \varphi'^2 v_x^2 + c_5)\frac{|\varphi''|}{\varphi'^2} v_x^2 + (c_6 \varphi'^4 |v_x|^4 + c_7)\frac{1}{\varphi'^2} + (c_8 \varphi'^2 v_x^2 + c_9) v_x^2 =$$

$$= \left[c_3\left(\frac{\varphi''}{\varphi'}\right)^2 + c_4 |\varphi''| + (c_6 + c_8)\varphi'^2\right]|v_x|^4 + \frac{m\mu}{1+f(u_x)}\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 + \left(c_5 \frac{|\varphi''|}{\varphi'^2} + c_9\right) v_x^2 + \frac{c_7}{\varphi'^2}. \tag{9}$$

The additional results we desire will be carried out under the assumption that

$$(m-1)\nu \geq c_3. \tag{10}$$

In particular, if $\bar{a}_{ij} \equiv 0$, then $c_3 = 0$, and our our condition then reduces to the requirement that $m \geq 1$.

Thus, let $(m-1)\nu \geq c_3$. Then, from inequality (9), it follows that

$$\left[-\left(\frac{\varphi''}{\varphi'}\right)' + \left(\frac{\varphi''}{\varphi'}\right)^2\right] \overset{\circ}{a}_{ij} v_{x_i} v_{x_j} v_x^2 \leq \left[c_4 |\varphi''| + (c_6 + c_8)\varphi'^2\right]|v_x|^4 + \frac{m\mu}{1+f(u_x)}\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 + \left(c_5 \frac{|\varphi''|}{\varphi'^2} + c_9\right) v_x^2 + \frac{c_7}{\varphi'^2}. \tag{11}$$

We shall determine $\varphi(v)$ for $v \geq 0$ as a solution of the equation

$$-\left(\frac{\varphi''}{\varphi'}\right)' + \left(\frac{\varphi''}{\varphi'}\right)^2 = c\varphi'^2 + c|\varphi''|, \tag{12}$$

where

$$c = \frac{1}{\nu}\left[\max\{c_4, c_6 + c_8\} + \nu_2\right], \quad \nu_2 > 0,$$

possessing the properties

$$\varphi'(v) > 0, \; \varphi'' < 0, \; \varphi(v) \to \infty \quad \text{for } v \to \infty.$$

We shall show below that such a $\varphi(v)$ exists. Now, however, we conclude the derivation of an estimate for $v_x^2(x_0)$. For such a $\varphi(v)$, we obtain, from inequality (11),

$$\nu_2(\varphi'^2 + |\varphi''|)|v_x|^4 \leq \frac{m\mu}{1+f(u_x)}\left(\frac{\varphi''}{\varphi'}\right)^2 |v_x|^4 + \left(c_5 \frac{|\varphi''|}{\varphi'^2} + c_9\right) v_x^2 + \frac{c_7}{\varphi'^2}. \tag{13}$$

Since, as u varies over the interval $[-M, M]$, v varies over some finite interval $[v_1, v_2]$, then on this interval: $0 < \nu_3 \leq \varphi'(v) \leq \mu_3$, $0 \leq |\varphi''(v)| \leq \mu_3$.

By virtue of this and the relations $u_{x_i} = \varphi'(v) v_{x_i}$, $|v_x|$ and $|u_x|$ tend to ∞ uniformly, so that $f(u_x) \to \infty$ as $|v_x| \to \infty$. If $\left.\frac{m\mu}{1+f(u_x)}\right|_{x=x_0} \geq \frac{\nu_2 \nu_3^4}{2\mu_3^2}$, then, as a result, we may obtain an upper bound for $|u_x(x_0)|$, and, therefore, also for $|v_x(x_0)|$. If, however, $\left.\frac{m\mu}{1+f(u_x)}\right|_{x=x_0} < \frac{\nu_2 \nu_3^4}{2\mu_3^2}$, then inequality

(13) yields

$$\frac{\nu_2 \nu_3^2}{2} |v_x|^4 \leq \left(c_5 \frac{|\varphi''|}{\varphi'^2} + c_9\right) v_x^2 + \frac{c_7}{\varphi'^2} \leq \left(c_5 \frac{\mu_3}{\nu_3^2} + c_9\right) v_x^2 + \frac{c_7}{\nu_3^2} ,$$

whence

$$v_x^2 (x_0) \leq \frac{A}{2} + \sqrt{\frac{A^2}{4} + B},$$

where

$$A = \frac{2}{\nu_2 \nu_3^2} \left(c_5 \frac{\mu_3}{\nu_3^2} + c_9\right), \quad B = \frac{2c_7}{\nu_2 \nu_3^4} .$$

Thus, having obtained an upper bound for $v_x^2(x_0)$, we are given, by the same token, an estimate for $\max_\Omega u_x^2 \ (x)$.

To complete the proof, we need to show the existence of a solution $\varphi(v)$ of Eq. (12), having the properties stated earlier.

Let us introduce, instead of $\varphi(v)$, a new function $\eta(v)$ by means of the equation $\varphi'(v) = e^{\eta(v)}$. Substituting into Eq. (12), we obtain

$$-\eta'' + (\eta')^2 = c e^{2\eta} + c e^{\eta} |\eta'|. \tag{14}$$

Let us find a solution of this equation for which $\eta' < 0$.

Let us consider η as the argument, and $\zeta = \eta'$ as the unknown function. Then, from Eq. (14), we have

$$-\zeta \frac{d\zeta}{d\eta} + \zeta^2 = c e^{2\eta} - c e^{\eta} \zeta. \tag{15}$$

For $\alpha = -\zeta e^{-\eta}$, we obtain, from Eq. (15): $\alpha \frac{d\alpha}{d\eta} = -c(1 + \alpha)$, wherein we are only interested in the domain where $\alpha > 0$. One of the solutions of this equation is given by the equation

$$\alpha - \ln(1 + \alpha) = -c\eta. \tag{16}$$

The function $y = y(\alpha) = \alpha - \ln(1 + \alpha)$ is monotonically increasing for $\alpha > 0$, and, for large α, it "behaves" like α. Let $z(y)$ be the inverse of this function, so that $\alpha = z(y)$. It is defined for all $y \geq 0$, increases monotonically as $y \to \infty$, and "acts" like y for large y (in other words, it has the line $\alpha = y$ as its asymptote). Upon introducing the function $\alpha = y$, it follows from Eq. (16) that

$$\alpha = z(-c\eta) \quad \text{for} \quad \eta \in (-\infty, 0).$$

This, together with the equation $\alpha = -\zeta e^{-\eta} = -\frac{d\eta}{dv} e^{-\eta}$, yields the equation

$$\frac{d\eta}{dv} e^{-\eta} = -z(-c\eta) \tag{17}$$

which determines the relation between η and v. Namely, from Eq. (17), we have

$$v = \int_\eta^0 \frac{d\eta}{e^\eta z(-c\eta)} .$$

As η varies from $-\infty$ to 0, the quantity v varies from $+\infty$ to 0. As v varies from 0 to $+\infty$, the function $\varphi'(v) = e^{\eta(v)}$ varies from 1 to 0. As the function $u = \varphi(v)$, let us take the function

$$u = \varphi(v) = -M + \int_0^v e^{\eta(v)} dv. \tag{18}$$

Let us verify that $\int_0^\infty e^{\eta(v)} dv = \infty$. Indeed, by virtue of Eq. (17),

$$\int_0^{\infty} e^{z(v)}\,dv = \int_0^{-\infty} e^{z(v)}\frac{dv}{d\eta}\,d\eta = -\int_0^{-\infty}\frac{d\eta}{z(-c\eta)} = \frac{1}{c}\int_0^{\infty}\frac{dy}{z(y)},$$

and the last integral diverges since $z(y)$, for large y, behaves like y. Thanks to this divergence, the values of the function $\varphi(v)$ fill out the entire interval $[-M, M]$ (for arbitrary $M = \max_{\Omega}|u(x)|$) as v varies from $v_1 = 0$ to some $v_2 < \infty$. Thus, we have found the desired function $\varphi(v)$, and, therewidth, have proved the following theorem:

Theorem 1. Let $u(x)$ be a thrice differentiable solution of Eq. (2) with $\max_{\Omega}|u(x)| \le M$ and $\max_{S}|u_x(x)| \le M_1$. Let $a_{ij}(p)$ and $a(x,u,p)$ be continuously differentiable functions of their arguments, where $a(x,u,p)$ satisfies inequalities (8), and where $a_{ij}(p)$ satisfy the conditions enunciated immediately after Eq. (2), in which we must have

$$(m-1)\nu \ge c_3 = \frac{m^2 c_2^2}{4c_1}. \tag{10}$$

Then, $\max_{\Omega}|u_x(x)|$ does not exceed a constant, which depends only on ν, μ, m, M, M_1, $C_1 - C_9$ and $f(p)$.

Remark 1. As an example of an equation to which Theorem 1 is applicable, we can take an equation giving the mean curvature of the surface $u = u(x)$, namely,

$$\left[(1+u_x^2)\delta_i^j - u_{x_i}u_{x_j}\right]u_{x_ix_j} = n\mathcal{H}\cdot(1+u_x^2)^{3/2}, \tag{19}$$

or, equivalently,

$$\frac{d}{dx_i}\left(\frac{u_{x_i}}{\sqrt{1+u_x^2}}\right) = n\mathcal{H}. \tag{19'}$$

For $\mathcal{H} = \text{const}$, we have the known inequality

$$\max_{\Omega}|u_x| \le \max_{S}|u_x|, \tag{20}$$

a simple derivation of which, for generalized solutions of Eq. (19) belonging to $W_2^2(\Omega)$, coincides with the proof of inequality (20) for solutions of the nonuniformly elliptic equation (8.38) of Chap. 4, [2], p. 355. For $\mathcal{H}$, depending on x, u, and u_x, but satisfying inequalities (8) with $a(x,u,p) = -n\mathcal{H}(x,u,p)\cdot(1+p^2)^{3/2}$, Theorem 1 guarantees the possibility of the estimate $\max_{\Omega}|u_x|$ in terms of $\max_{\Omega}|u|$ and $\max_{S}|u_x|$. For Eq. (19), $\tilde{a}_{ij} \equiv 0$ and $m = 2$.

The possibility of obtaining such an estimate for Eqs. (11), for which $\mathcal{H} = \mathcal{H}(x)$ and $\mathcal{H} = \mathcal{H}(x,u)$, was noted by I. Ya. Bakel'man in his paper at the Second All-Union Symposium on Global Geometry in May, 1967 (see [8]).

Remark 2. If condition (10) is not satisfied, then, in place of Eq. (12) for determining $\varphi(v)$, we will have the equation

$$-\left(\frac{\varphi''}{\varphi'}\right)' + c'\left(\frac{\varphi''}{\varphi'}\right)^2 = c\varphi'^2 + c'|\varphi''| \tag{21}$$

with $c' < 1$, $c, c' > 0$. To this equation, we append the two conditions: $\varphi' > 0$ and $u_2 - u_1 = \int_{v_1}^{v_2}\varphi'(v)\,dv = 2M$, where $u_k = \varphi(v_k)$, $k = 1, 2$. We were led to a problem of this kind with $c' < 0$, in connection with obtaining the estimate $\max_{\Omega}|u_x(x)|$ for the general quasi-linear equation

$$a_{ij}(x,u,u_x)u_{x_ix_j} + a(x,u,u_x) = 0 \tag{22}$$

for the case where the latter is uniformly elliptic (see p. 403 of [2]). We shall show that the problem (21) does not necessarily have a solution for large c and M, if $c' < 1$. As above, we introduce a function η with the aid of the equation $\varphi'(v) = e^{\eta(v)}$. In terms of η, Eq. (21) becomes

$$-\eta'' + c'(\eta')^2 = ce^{2\eta} + c''e^{\eta}|\eta'|. \tag{23}$$

Let us consider η as the argument, and $\varsigma = \eta'$ as the unknown function. From Eq. (23), we then have

$$-\varsigma\frac{d\varsigma}{d\eta} + c'\varsigma^2 = ce^{2\eta} + c''e^{\eta}|\varsigma|,$$

whence, for $\alpha = -\varsigma e^{-c'\eta}$, it follows that

$$\alpha\frac{d\alpha}{d\eta} = -ce^{2(1-c')\eta} - c''e^{(1-c')\eta}|\alpha|,$$

and, therefore, also,

$$\alpha\frac{d\alpha}{d\hat\eta} = -\hat c\hat\eta - \hat c''|\alpha|, \text{ where } \hat\eta = e^{(1-c')\eta}, \ \hat c = \frac{c}{1-c'}, \ \hat c'' = \frac{c''}{1-c'}.$$

Let us introduce, in place of α, a new unknown function $\beta = \frac{\alpha^2}{2\hat\eta^2}$. It will satisfy the equation $\hat\eta\frac{d\beta}{d\hat\eta} = -(2\beta + \hat c''\sqrt{2\beta} + \hat c)$, whence, we obtain

$$\int\frac{d\beta}{2\beta + \hat c''\sqrt{2\beta} + \hat c} = -\ln\hat\eta + c'''. \tag{24}$$

This relation enables us to determine $\varphi(v)$ with $\varphi'(v) > 0$. However, for c and M large, no φ can be found for which $\int_{v_1}^{v_2}\varphi'(v)dv = 2M$, since for $\hat c'', \hat c > 0$,

$$\int_{v_1}^{v_2}\varphi'(v)dv = \left|\int_{\eta_1}^{\eta_2} e^{\eta}\frac{dv}{d\eta}d\eta\right| = \left|\int_{\eta_1}^{\eta_2} e^{\eta}\frac{d\eta}{\varsigma}\right| = \left|\int_{\eta_1}^{\eta_2} e^{(1-c')\eta}\frac{d\eta}{\alpha}\right| = \frac{1}{1-c'}\left|\int_{\hat\eta_1}^{\hat\eta_2}\frac{d\hat\eta}{\sqrt{2\beta}\,\hat\eta}\right| \le$$

$$\le \frac{1}{1-c'}\left|\int_0^{\infty}\frac{d\beta}{\sqrt{2\beta}\,(2\beta + \hat c''\sqrt{2\beta} + \hat c)}\right| \equiv \tilde c < \infty$$

and $\tilde c$ may turn out to be less than $2M$.

It is evident from these considerations that if we wish to have a solution of problem (21) for $c' < 1$, it is necessary to require fulfillment of the inequality $\tilde c > 2M$.

Remark 3. From the reasoning given on pp. 394-395 in [2], it is evident that a valid assertion, somewhat stronger than that formulated in Lemma 2.1, can be made. In particular, if Ω is a convex domain and $u|_S = 0$, then $\max_S |u_x|$ may be estimated from above in terms of $M = \max_\Omega |u|$ and $\mu(M)/\nu(M)$, where $\nu(M)$ and $\mu(M)$ are taken from the inequalities $a_{ij}(x,u,p)\xi_i\xi_j \ge \nu(|u|)\xi^2$ and $|a(x,u,p)| \le \mu(|u|)(1+p^2)$. If, however, $u|_S = \psi(s)$, then, for the convex domain Ω, an estimate of $\max_S |u_x|$ depends also on $\mu_1(M) = \sup_{\substack{x\in\Omega\\|u|\le M}}\frac{a_{ij}(x,u,p)\xi_i\xi_j}{(1+p^2)\xi^2}$ and $\|\psi\|_{2,0,S}$, and, for an arbitrary domain for which S belongs to O_2, the estimate will depend on $\mu_2(M) = \sup_{\substack{x\in\Omega\\|u|\le M}}\frac{a_{ij}(x,u,p)\xi_i\xi_j}{(1+|p|)\xi^2}$, $\|\psi\|_{2,0,S}$ and on the norm of S in O_2.

Remark 4. It is known that for a solution $u(x)$ of $C_3(\Omega) \cap C_1(\bar\Omega)$ of the equation

$$a_{ij}(u_x)u_{x_ix_j} + a(u,u_x) = 0, \ \frac{\partial a}{\partial u} \le 0 \tag{25}$$

the $\max_{\Omega} |u_{x_k}|$, $k=1,\ldots n$ are assumed at boundary points of Ω, if $a_{ij}(p)\xi_i\xi_j>0$ for $|\xi|\neq 0$ and the functions $a_{ij}(p)$ and $a(u,p)$ are continuously differentiable, since $u_k \equiv u_{x_k}$ satisfies an equation of the form

$$a_{ij}(u_x)u_{kx_ix_j}+b_j u_{kx_j}+cu_k=0, \quad c\leq 0.$$

In this equation $b_j=\frac{\partial a_{im}}{\partial u_{x_j}}u_{x_ix_m}+\frac{\partial a}{\partial u_{x_j}}$, $c=\frac{\partial a}{\partial u}$. We shall show, in fact, that this is true even for solutions $u(x)\in C_2(\bar{\Omega})$. To do this, we multiply Eq. (25) by $\Phi_{x_k}\in \overset{\circ}{C}_\infty(\Omega)$, integrate over Ω, and then transform the resulting equation, integrating by parts twice, to the following:

$$\int (a_{ij}u_{kx_j}\Phi_{x_i}+[d_j u_{kx_j}-cu_k]\Phi)\,dx=0. \tag{26}$$

In this equation $d_j=\frac{\partial a_{ij}}{\partial u_{x_m}}u_{x_ix_m}-\frac{\partial a_{im}}{\partial u_{x_j}}u_{x_ix_m}-\frac{\partial a}{\partial u_{x_j}}$. It is easy to see that Eq. (26) holds for arbitrary $\Phi\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ and $u(x)\in C_2(\bar{\Omega})$. The identity (26) shows that u_{x_k} may be considered as a generalized solution v of $W_2^{(1)}(\Omega)$ (and even of $C_1(\bar{\Omega})$) of the equation

$$\frac{\partial}{\partial x_i}(a_{ij}v_{x_j})-d_j v_{x_j}+cv=0$$

with bounded functions a_{ij}, d_j, and c, $c\leq 0$. However, for such equations it has been shown that $\operatorname{vrai\,max}_{\Omega} v \leq \max\{\operatorname{vrai\,max}_{S} v; 0\}$ and

$$\operatorname{vrai\,min}_{\Omega} v \geq \min\{\operatorname{vrai\,min}_{S} v; 0\}, \text{ i.e., } \max_{\Omega} u_{x_k}\leq \max\{\max_{S} u_{x_k}; 0\} \text{ and } \min_{\Omega} u_{x_k}\geq \min\{\min_{S} u_{x_k}; 0\}.$$

Approximating Ω by domains Ω_m, $m=1,2,\ldots$, lying strictly within Ω, we see that these inequalities hold even for solutions $u(x)\in C_2(\Omega)\cap C_1(\bar{\Omega})$.

LITERATURE CITED

1. O. A. Ladyzhenskaya and N. N. Ural'tseva, "Certain classes of nonuniformly elliptic equations," Seminars in Mathematics, Vol. 5, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part I, Consultants Bureau, New York (1969), pp. 67-69.
2. O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasi-linear Elliptic Equations, Mathematics in Science and Engineering, Vol. 46, Academic Press, New York (1968).
3. A. P. Oskolkov, "On some estimates for nonuniformly elliptic equations and systems," Trudy Matem. inst. im V. A. Steklova, Vol. 92, pp. 203-232 (1966).
4. A. P. Oskolkov, "*A priori* estimates of the first derivatives of solutions of Dirichlet's problem for nonuniformly elliptic quasi-linear equations," Trudy Matem. inst. im. V. A. Steklova, Vol. 102 (1967).
5. N. M. Ivochkina, "Dirichlet's problem for two-dimensional quasi-linear second-order elliptic equations," in Problems of Mathematical Analysis, Collected Papers [in Russian], No. 2, Leningrad State Univ., Leningrad (1968) [English translation: Consultants Bureau, New York (in preparation).]
6. N. M. Ivochkina and A. P. Oskolkov, "Nonlocal estimates of first derivatives of solutions of Dirichlet's problem for nonuniformly elliptic quasi-linear equations," Seminars in Mathematics, Vol. 5, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part I, Consultants Bureau, New York (1969), pp. 12-35.
7. N. M. Ivochkina and A. P. Oskolkov, "Nonlocal estimates of first derivatives of the solutions of the initial-boundary problem for nonuniformly elliptic and nonuniformly parabolic nondivergent equations," this volume, p. 1.
8. I. Ya. Bakel'man, "The construction of surfaces with given mean curvature and associated problems of the theory of quasi-linear equations," Reports at the Second All-Union Symposium on Global Geometry [in Russian], Petrozavodsk (1967), pp. 9-10.

EXPANSIONS IN CHARACTERISTIC VECTORS OF NONUNITARY OPERATORS AND THE CHARACTERISTIC FUNCTION

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In this paper we shall establish necessary and sufficient conditions for a system of characteristic vectors of a nonunitary operator in Hilbert space to form an unconditional basis (i.e., a basis similar to an orthonormal basis). These conditions will be formulated in terms of the characteristic function of the operator and its functional model, as developed by B. S. Nagy and C. Foias [1]. Therefore, we shall consider only operators of contraction, which are in some (sufficiently weak) sense close to unitary, i.e., operators which are well-placed in the scheme of Nagy and Foias.

The content of this paper has appeared in abbreviated form in [2]. Some applications, along with a detailed study of operators with finite-dimensional deficiency subspaces, will be given in a separate paper.

§0. Introduction

0.1. Let E be a (separable) Hilbert space, R, the Banach algebra of all (linear and continuous) operators from E to E, and let $\mathfrak{S}_1$ ($\mathfrak{S}_2$) be the ideal of all nuclear (or Hilbert–Schmidt, respectively) operators of R. As usual, we shall denote by $H^p, 1 \le p \le \infty$, (see [3], [4]), the known classes of Hardy of all functions f, analytic in the disk $\mathfrak{D} = \{z : |z| < 1\}$ of the complex plane, for which

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty,$$

$$\left(\|f\|_\infty = \sup_{z \in \mathfrak{D}} |f(z)| < \infty, \quad \text{if} \quad p = \infty\right).$$

Further, let us consider the abstract analogues of the Hardy spaces H^p. Namely, if X is a Banach space, then by $H^p(X), 1 \le p \le \infty$, we shall mean the set of all functions f, regular in $\mathfrak{D}$, with values in the space X, for which

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|_X^p \, dt < \infty,$$

$$\left(\|f\|_\infty = \sup_{z \in \mathfrak{D}} \|f(z)\|_X < \infty, \quad \text{if} \quad p = \infty\right).$$

The only cases of importance to us will turn out to be those for which $X = E$, R, $\mathfrak{S}_1$, or $\mathfrak{S}_2$. Properties of the abstract analogues of the Hardy classes H^p have been discussed in detail in, for example, [1] and [3]. We shall use these properties in the sequel with no special reservations.

0.2. We shall speak of the function Θ, $\Theta \in H^{\infty}(\mathfrak{R})$, as an inner function (see [1], [3]) if its angular boundary values $\Theta(e^{it})$ are unitary operators in E for almost all values of t, $t \in [0, 2\pi]$.

Let us consider in the space $H^2(E)$ the shift operator S,

$$(Sf)(z) = zf(z), \quad |z|<1, \quad f \in H^2(E),$$

and let S^* be the operator conjugate to S. According to a theorem of A. Beurling, P. Lax, and P. Halmos (see, for example, [3], [4], and [1]), an arbitrary invariant subspace M of the operator S has the form

$$M = \Theta H^2(E),$$

where $\Theta H^2(E) \stackrel{\text{def}}{=} \{g, g(z) = \Theta(z)f(z), |z|<1, f \in H^2(E)\}$ and Θ is a function in $H^{\infty}(\mathfrak{R})$, whose values, almost everywhere on the unit circle, are partially-isometric operators in E with a constant null subspace. In the sequel we shall consider only subspaces M, which are generated by characteristic (pure) [1] inner functions Θ, i.e., those for which $\|\Theta(0)e\| < \|e\|, e \in E$. Having selected one of those (inner characteristic) functions Θ, we note that then the subspace $K = H^2(E) \ominus \Theta H^2(E)$ is invariant for the operator S^*, and it makes sense to talk about the restriction of the operator S^* to K,

$$T = S^* \big| K, \quad K = H^2(E) \ominus \Theta H(E). \tag{1}$$

This (model) operator T will, in fact, be the principal object of study in this paper. It is known, from the work of B. S. Nagy and C. Foias [1], that an arbitrary linear operator A in a Hilbert space $\mathcal{H}$, for which

$$\|A\| \le 1, \quad \lim_{n\to\infty} A^n = \lim A^{*n} = 0, \tag{2}$$

in the sense of strong operator convergence, is unitarily equivalent to an operator T of the form (1), where the inner function Θ is found to be (to within a constant unitary factor) the characteristic operator-function Θ_{A^*} for the contraction† A^*, i.e., (see the definition in [1])

$$\Theta(z) = \Theta_{A^*}(z) \stackrel{\text{def}}{=} (-A^* + zD_A(1-zA)^{-1}D_{A^*})\big|\mathfrak{D}_{A^*}, \quad |z|<1,$$

where $D_A = (I - A^*A)^{1/2}$, $D_{A^*} = (I - AA^*)^{1/2}$ are deficiency operators, and $\mathfrak{D}_A = \overline{D_A \mathcal{H}}$, $\mathfrak{D}_{A^*} = \overline{D_{A^*}\mathcal{H}}$ are deficiency subspaces of the operator A^*. In particular, the conditions (2) are satisfied if A is a completely nonunitary‡ contraction, which is annihilated by some (scalar) function h, $h \in H^{\infty}$,

$$h(A) = 0.$$

Contractions of this kind are spoken of as contractions of class C_0 (see [1]).

Lastly, we shall agree to call an operator T a basis operator, if the system of its characteristic vectors forms a Riesz basis (see [5]), (i.e., a basis equivalent§ to an orthonormal, or to an unconditional basis), in the closure of its linear hull.

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†An operator A is said to be a contraction if $\|A\| \le 1$.

‡A contraction A is said to be completely nonunitary if it is nonunitary on any one of its invariant subspaces.

§One says (see, e.g., [5]) that a system of vectors $\{x_n\}_{n=1}^{\infty}$ forms a basis of a Hilbert space $\mathcal{H}$, equivalent to an orthonormal basis, if there is a linear isomorphism V of the space $\mathcal{H}$, such that $\{Vx_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$.

0.3. The operator T, considered in Eq. (1), will (unless mention is made to the contrary) be assumed always to satisfy conditions (i) and (ii):

(i) The part of the spectrum $\sigma(T)$ of the operator T which lies in the unit disk $\mathfrak{D}$ consists of normal† characteristic values, and all the poles of the resolvent of the operator T are simple.

(ii) The system of characteristic vectors of the operator T corresponding to characteristic numbers lying inside the unit disk $\mathfrak{D}$ is complete in K.

It may be noted that an abstract contraction operator A, which, together with its conjugate A^*, satisfies conditions (i) and (ii), may be represented in the form (1). In terms of the characteristic function Θ, condition (i) may be rewritten in the following way (see Lemma 2.1; cf. also [16], [17]): if $z_k \in \sigma(T^*) \cap \mathfrak{D}$, then

$$\Theta(z) = \Theta_k(z)\, B_k(z), \quad |z|<1, \tag{3}$$

where Θ_k, B_k are inner functions,

$$B_k(z) = \frac{z_k - z}{1-\bar{z}_k z} \cdot \frac{\bar{z}_k}{|z_k|}\, \pi_k + (I - \pi_k), \quad |z|<1,$$

π_k-is an orthoprojector in E on the subspace $\operatorname{Ker}\Theta(z_k)$ of zeros of the operator $\Theta(z_k)$, and $\Theta_k(z_k)$ is an isomorphism in E.

Condition (ii) may also be expressed in terms of the characteristic function of the operator T (see Theorem 1.11).

0.4. Before proceeding to a study of the results, let us recall (see [1], [2]) that, for operators of the class C_0, the property (ii) of completeness of the system of characteristic (or root) subspaces is a purely spectral property. In fact, this property is satisfied for an operator $A, A \in C_0$, when, and only when, the annihilating function h ($h(A) = \mathbb{O}$, $h \in H^\infty$) contains no singular factors [4].

It follows from Theorems 1.3, 1.9, and 1.15 (see §1) of the present work that the property of being a basis operator is no longer a spectral property, but depends (even in the model case considered) on the geometry of the system of characteristic subspaces. In this regard, it may be noted that until now, the known sufficient conditions for a system of characteristic (or root) vectors to form a basis (see [6-9]) have utilized only the spectral characteristis of the operator T.

§1. Statements of the Theorems

1.1. Let T be an operator of the form (1), satisfying conditions (i) and (ii), and let $\{\bar{z}_k\}_{k=1}^{\infty} = \sigma(T) \cap \mathfrak{D}$ be the sequence of its characteristic numbers.

1.2. Theorem. Under the conditions 1.1, the functions‡

$$\varphi_k(z) = \frac{\Theta_k^{*-1}(z)\, e}{1-\bar{z}_k z}\,(1-|z_k|^2)^{\frac{1}{2}}, \quad e \in \pi_k E, \ \|e\|=1, \tag{4}$$

form a system of characteristic functions (c.f.) of the operator T, where π_k is an orthoprojector from E onto $\operatorname{Ker}\Theta(z_k)$.

The bi-orthogonal system $\{\psi_k\}_{k=1}^{\infty}$, consisting of c.f. of the operator T^*, has the form

$$\psi_k(z) = \Theta_k(z)\frac{e}{1-\bar{z}_k z}\,(1-|z_k|^2)^{\frac{1}{2}}, \quad e \in \pi_k E, \ \|e\|=1.$$

†A characteristic value λ is said to be normal if it is an isolated point of the spectrum, and is a pole of the resolvent.

‡Speaking of system (4), we have in mind an arbitrary one of the systems $\{\varphi_{ki}\}$, obtained from system (4) by choosing an orthonormal basis $e_i^{(k)}$ in $\pi_k E$.

1.3. Theorem. I. Under the assumptions 1.1, the following conditions are equivalent:

1) T is a basis operator.

2) $\sum_k (1-|z_k|^2)\|\pi_k \Theta_k^{-1}(z_k) g(z_k)\pi_k\|_{\mathfrak{S}_1} < \infty$ for an arbitrary function g, $g \in H_0^1(\mathfrak{S}_1)$,

$$H_0^1(\mathfrak{S}_1) \overset{\text{def}}{=} \{g : g \in H^1(\mathfrak{S}_1),\ g(0)=0\}.$$

II. If, in addition, the system $\{\psi_k\}_{k=1}^{\infty}$ is complete in K, then conditions 1) and 2) are equivalent to the following assertions:

3) $\sum_k (1-|z_k|^2)\|\pi_k f(z_k)\|_E^2 < \infty$,

for an arbitrary function f, $f \in H_0^2(E)$, $H_0^2(E) \overset{\text{def}}{=} \{f : f \in H^2(E),\ f(0)=0\}$;

$$\sum_k (1-|z_k|^2)\|\pi_k \Theta_k^{-1}(z_k) f(z_k)\|_E^2 < \infty,$$

for an arbitrary function f, $f \in H^2(E)$.

4) $\sup_k \|\pi_k \Theta_k^{-1}(z_k)\Delta_k\|_R < \infty$ and

$$\sum_{k,n} \frac{(1-|z_k|^2)^{1/2}(1-|z_n|^2)^{1/2}}{1-\bar{z}_n z_k}(\ell_n, \ell_k) \le C \sum_k \|\ell_k\|^2$$

for an arbitrary sequence $\{\ell_n\}_{n=1}^{\infty}$, $\ell_n \in \pi_n E$, $n \ge 1$;

$$\sum_{k,n} \frac{(1-|z_k|^2)^{1/2}(1-|z_n|^2)^{1/2}}{1-\bar{z}_n z_k}(\ell_k, \ell_n) \le C \sum_k \|\ell_k\|^2$$

for an arbitrary sequence $\{\ell_n\}_{n=1}^{\infty}$, $\ell_n \in \Delta_n E$, $n \ge 1$.

1.4. Remark. If $T \in C_0$ and conditions 1.1 are satisfied, then $\{\psi_k\}_1^{\infty}$ is complete in the system K [1]. Then, part II of Theorem 1.3 gives criteria for T to be a basis operator, without additional assumptions. In general, to guarantee that T will be a basis operator, based on the assertions 3) and 4), one needs to add to the conditions 1.1 the requirement that the system $\{\psi_k\}$, bi-orthogonal to $\{\varphi_k\}$, be complete. It remains to clarify whether the requirement of completeness of $\{\psi_k\}$ is an independent condition or whether it is a consequence of the requirements 1.1.

1.5. Let F be a function from $H^{\infty}(R)$. Further, let

$$\Phi_F(\xi) = \Theta^*(\xi) F(\xi) \Theta(\xi), \quad |\xi| = 1.$$

Let $\mathfrak{F}$ be a subspace of $H^{\infty}(R)$, consisting of all functions F such that

$$\Phi_F \in H^{\infty}(R).$$

Now let $f \in K = H^2(E) \ominus \Theta H^2(E)$, and let P be an orthoprojector on K. If $F \in \mathfrak{F}$ and $(F \cdot f)(z) \equiv F(z) f(z)$, $(|z| < 1)$, let $F(T^*)$ be an operator in K, operating according to the rule

$$F(T^*) f = P(F \cdot f).$$

Finally, let us put

$$\mathfrak{F}(T^*) = \{F(T^*) : F \in \mathfrak{F}\},$$

and let us provide $\mathfrak{F}(T^*)$ with an operator norm.

1.6. Theorem. $\mathcal{F}(T^*)$ is isometrically isomorphic to the space $\mathcal{F}/\Theta H^\infty(R)$.

1.7. Remark. Theorem 1.6 is contained, essentially, in the paper by D. Sarason [10] for the case $\Theta = \varphi I$, $\varphi \in H^\infty$ (scalar inner function), since then $\mathcal{F} = H^\infty(R)$.

1.8. Let $\ell^2(\Delta_k)$ be the set of all sequences $x = \{x_k\}_{k=1}^\infty$, $x_k \in \Delta_k E$, $k \geq 1$, such that $\sum \|x_k\|^2 < \infty$. Further, let $\mathfrak{J}$ be the contraction operator

$$\mathfrak{J} f = \{\Delta_k f(z_k)(1-|z_k|^2)^{\frac{1}{2}}\}_{k=1}^\infty, \quad f \in H^2(E),$$

corresponding to the interpolational problem

$$\Delta_k f(z_k)(1-|z_k|^2)^{\frac{1}{2}} = w_k, \tag{5}$$

$$w_k \in \Delta_k E, \quad k \geq 1, \quad f \in H^2(E).$$

In the well-known papers of H. Shapiro and A. Shields [11], L. Carleson [12], and D. Newman [13], this problem has been studied in detail for the case $\dim E = 1$ (in this case, always, $\Delta_k = I_E$). In particular, it was established in these papers that satisfaction of the conditions

$$\inf_k \left| \prod_{\substack{n \\ n \neq k}} \frac{z_k - z_n}{1 - \bar{z}_n z_k} \cdot \frac{\bar{z}_n}{|z_n|} \right| = \delta > 0, \quad (C)$$

$$\sum_k (1-|z_k|^2)\ |f(z_k)|^2 < \infty, \quad f \in H^2, \quad (\mathcal{N})$$

is equivalent to the equation $\mathfrak{J} H^2 = \ell^2$. In fact, Carleson [12] has shown that (for the case $\dim E = 1$) condition (C) already implies condition $(\mathcal{N})$. We shall show below that, for the condition $\dim E > 1$, problem (5) is solvable in the class $\ell^2(\Delta_k)$, if certain conditions, analogous to the conditions (C) and $(\mathcal{N})$, are satisfied. However, in the general case, one can no longer be restricted by only the one condition of type (C) (see, e.g., section 1.15).

On the other hand, it turns out to be the case that the possibility of interpolation in $H^2(E)$ of arbitrary sequences from $\ell^2(\Delta_k)$ implies the basis property of the operators T and T^* considered above. This fact is a consequence of theorems of N. K. Bari [5] concerning spaces of the coefficients in expansions according to Riesz bases.

1.9. Theorem. If an arbitrary system $\{\varphi_k\}_{k=1}^\infty$ is complete in K, and if it is uniformly minimal,† then the following statements are equivalent:

1) $\mathfrak{J} H^2(E) = \ell^2(\Delta_k)$;

2) $\{\varphi_k\}_{k=1}^\infty$ $(\{\psi_k\}_{k=1}^\infty)$ is a Riesz basis in K;

3) a) $\sum (1-|z_k|^2)\|\pi_k f(z_k)\|_E^2 < \infty$, $f \in H_0^2(E)$,

b) $\sum (1-|z_k|^2)\|\Delta_k f(z_k)\|_E^2 < \infty$, $f \in H^2(E)$.

1.10. Corollary. [9]. If condition (C) is satisfied, the system $\{\varphi_k\}_{k=1}^\infty$ $(\{\psi_k\}_{k=1}^\infty)$ is a Riesz basis in K.

We remark, in passing, that in [9] there is proved a much more general sufficient condition for the basis property than that of Corollary 1.10, namely, a condition connected with the transition to bases from the root subspaces of the operators T and T^*.

†A system of vectors $\{x_n\}_{n=1}^\infty$, $\|x_n\|=1$, $n \geq 1$ is said to be uniformly minimal if $\inf_n \rho(x_n, \mathfrak{X}(x_k)_{k \neq n}) > 0$, where $\mathfrak{X}(x_i)$ is the closed linear hull of the vectors x_i, and $\rho(x, \mathfrak{X})$ is the distance of the point x from the subspace $\mathfrak{X}$.

1.11. Theorem. The system (4), $\{\varphi_k\}_{k=1}^{\infty}$, is complete in K if and only if $\mathcal{J}f=0$, $f\in H^2(E)$ implies that $f=\Theta g$, and for some g, $g\in H^2(E)$.

1.12. Theorem. Under the conditions 1.1 on the operator T, the following statements are equivalent:

1) An arbitrary one of the systems $\{\varphi_k\}_{k=1}^{\infty}$ (or† $\{\psi_k\}_{k=1}$) is uniformly minimal;

2) $\sup\limits_k \|\pi_k \Theta_k^{-1}(z_k)\Delta_k\|_R < \infty.$ (6)

1.13. Remark. If $\dim E=1$, condition (6) coincides with condition (C). In the general case, condition (C) always implies condition (6).

1.14. Remark. Statements 1) and 2) or Theorem 1.12 coincide with the condition of uniform minimality of the system of characteristic subspaces $\{P_n K\}_{n=1}^{\infty}$ of the operator T^*. Uniform minimality of at least one of the systems (4) implies that there exist orthonormal bases $\{e_i^{(k)}\}_i$ in $\pi_k E$, such that

$$\sup_{k,i} \|\Theta_k^{*-1}(z_k)e_i^{(k)}\| < \infty.$$

This condition is equivalent to condition (6), for example, in the case $\sup\limits_k \dim \pi_k E < \infty$.

1.15. Example. An operator T exists for which conditions 1.1 and inequality (6) are satisfied, but for which statements 3a) or 3b) of Theorem 1.9 no longer hold (for the formulation of T, see §10).

Such an operator T is not a basis operator, and the corresponding equation $\mathcal{J}H^2(E)=\ell^2(\Delta_k)$ for it is not satisfied (see Theorem 1.9 and the statement of the problem concerning interpolation in 1.8).

§2. Lemmas

2.1. Lemma. Let Θ be an inner function of $H^\infty(R)$, and let λ, $|\lambda|<1$, be a normal characteristic number (c.n.) of the operator T^* in Eq. (1). Then, an expansion of the form (3) is valid:

$$\Theta=\Theta_\lambda\cdot B_\lambda,$$

where $\Theta_\lambda, B_\lambda$ are inner functions, $\Theta_\lambda(\lambda)$ is an isomorphism in E, and

$$B_\lambda(z)=\left(\frac{\lambda-z}{1-\bar\lambda z}\cdot\frac{\bar\lambda}{|\lambda|}\right)^m\pi_\lambda+(I-\pi_\lambda),\quad |z|<1$$

π_λ is an orthoprojector in E onto $\operatorname{Ker}\Theta(\lambda)$, and m is a natural number.

Proof. Let K_λ be a root subspace of the operator T^*, corresponding to the c.n. λ. Then, $T^*K_\lambda\subset K_\lambda$, and, in accord with a theorem of B. S. Nagy and C. Foias [1] (p. 277), to the subspace K_λ there corresponds a regular (see [1], p. 277, et seq.) factorization

$$\Theta=\Theta_\lambda\cdot B_\lambda,$$

wherein the characteristic part (see [1], p. 178) of the inner function B_λ coincides with the characteristic function (c.f.) of the operator $T^*|K_\lambda$, and Θ_λ is the c.f. of the operator $T_\lambda^*=(I-\tilde P_\lambda)T^*|K_\lambda^{\perp}$, where $\tilde P_\lambda$ represents the orthogonal projector onto the subspace K_λ. Since the factorization $\Theta=\Theta_\lambda\cdot B_\lambda$ is regular, it follows that ([1], pp. 284, 286) Θ_λ and B_λ are inner functions. In addition, $\lambda\notin\sigma(T_\lambda^*)$, and consequently ([1], p. 247), $\Theta_\lambda(\lambda)$ is the inverse operator in E. Since the characteristic function of the oper-

†Subject to the condition of completeness of the system $\{\psi_k\}_{k=1}^{\infty}$.

ator $T^*|K_\lambda$ is b_λ^m, $b_\lambda(z)=(\lambda-z)(1-\bar\lambda z)^{-1}\bar\lambda|\lambda|^{-1}|z|<1$, where m is a natural number, the function B_λ has the form given in the statement of Lemma 2.1.

2.2. Lemma. If P is an orthoprojector from $H^2(E)$ onto $K=H^2(E)\ominus\Theta H^2(E)$, Q is an orthoprojector from $L^2(E)$ onto $H^2(E)$ and Θ is an inner function, $\Theta\in H^\infty(\mathcal{R})$, then $P=I-\Theta Q\Theta^*$, i.e.,

$$Pf=f-\Theta Q(\Theta^* f),\quad f\in H^2(E).$$

Proof. Let $P_0=I-P$ be the orthoprojector onto the subspace $\Theta H^2(E)$. If $h\in H^2(E)$, $\varphi\in H^2(E)$, then

$$\langle h,\Theta\varphi\rangle=\langle P_0 h,\Theta\varphi\rangle=\langle\Theta h_0,\Theta\varphi\rangle=\langle h_0,\varphi\rangle,$$

since Θ is an inner function. On the other hand, $\langle h,\Theta\varphi\rangle=\langle\Theta^* h,\varphi\rangle$. Consequently, $h_0=Q\Theta^* h$, i.e., $P_0 h=\Theta Q(\Theta^* h)$. This completes the proof.

2.3. Lemma. Let T be an operator of the form (1), and assume that $\lambda\notin\sigma(T^*)$. Then the resolvent $R(\lambda,T^*)$ of the operator T^* has the form

$$R(\lambda,T^*)=-P\,\mathcal{U}_\lambda\Theta(\lambda)^{-1},\tag{7}$$

where $\mathcal{U}_\lambda\in H^\infty(\mathcal{R})$, $\mathcal{U}_\lambda(z)=\frac{\Theta(z)-\Theta(\lambda)}{z-\lambda}$, $|z|<1$.

Proof. Since $\lambda\notin\sigma(T^*)$, then $\Theta(\lambda)$ is the inverse operator, [1]. To validate Eq. (7), it is sufficient to show that

$$-(T^*-\lambda)\,P\,\mathcal{U}_\lambda\Theta^{-1}(\lambda)=I.$$

We have $-(T^*-\lambda)\,P\,\mathcal{U}_\lambda\Theta^{-1}(\lambda)=-P(S-\lambda)P\,\mathcal{U}_\lambda\Theta^{-1}(\lambda)=-P(S-\lambda)\mathcal{U}_\lambda\Theta^{-1}(\lambda)$, since $S^*K\subset K$, and, consequently, $PS^*P=S^*P$, i.e., $PSP=PS$. Extending the equation, we have $-P(S-\lambda)\mathcal{U}_\lambda\Theta(\lambda)^{-1}=-P(\Theta-\Theta(\lambda))\Theta(\lambda)^{-1}=$ $=P\Theta(\lambda)\Theta(\lambda)^{-1}=I_K$, which is what we wanted to show.

2.4. Lemma. Let T be an operator of the form (1), satisfying condition (i) of section 0.3, and let λ, $|\lambda|<1$, be a simple pole of the resolvent of operator T^*. Let P_λ be the spectral projector onto the characteristic subspace $K_\lambda=\operatorname{Ker}(T^*-\lambda)$. Then,

$$P_\lambda=-(1-|\lambda|^2)(I-\bar\lambda T^*)^{-1}P\,\Theta_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)=-(1-|\lambda|^2)P\,\Theta_\lambda E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)\overset{\text{def}}{=}PF_\lambda,$$

where $E_\lambda(z)=(1-\bar\lambda z)^{-1}$, $|z|<1$, and $\Theta_\lambda,\pi_\lambda$ are defined in Lemma 2.1.

Proof. Since λ is an isolated point of the spectrum of the operator T^*, then

$$P_\lambda=-\frac{1}{2\pi i}\int_{|\xi-\lambda|=\varepsilon}R(\xi,T^*)\,d\xi,$$

for small δ, $\varepsilon>0$, and use can be made of Lemma 2.3. By this lemma:

$$P_\lambda=\frac{1}{2\pi i}\int_{|\xi-\lambda|=\varepsilon}P\,\mathcal{U}_\xi\Theta^{-1}(\xi)\,d\xi=P\frac{1}{2\pi i}\int_{|\xi-\lambda|=\varepsilon}\mathcal{U}_\xi\Theta^{-1}(\xi)\,d\xi\overset{\text{def}}{=}PF_\lambda,\ F_\lambda\in H^\infty(R).$$

Let us evaluate the function F_λ:

$$F_\lambda(z)=\frac{1}{2\pi i}\int_{|\xi-\lambda|=\varepsilon}\mathcal{U}_\xi(z)\Theta^{-1}(\xi)\,d\xi=\frac{1}{2\pi i}\int\frac{\Theta(z)-\Theta(\xi)}{z-\xi}\,\Theta^{-1}(\xi)\,d\xi=\frac{1}{2\pi i}\int\frac{\Theta(z)-\Theta(\xi)}{z-\xi}\,B_\lambda^{-1}(\xi)\,\Theta_\lambda^{-1}(\xi)\,d\xi,$$

where B_λ and Θ_λ are functions from Lemma 2.1. Continuing the equation, we obtain

$$F_\lambda(z)=\frac{1}{2\pi i}\int\frac{\Theta(z)-\Theta(\xi)}{z-\xi}\left(\frac{1-\bar\lambda\xi}{\lambda-\xi}\cdot\frac{|\lambda|}{\bar\lambda}\pi_\lambda+(I-\pi_\lambda)\right)\Theta_\lambda^{-1}(\xi)\,d\xi=\frac{\Theta(z)-\Theta(\lambda)}{z-\lambda}\cdot\frac{|\lambda|}{\bar\lambda}(1-|\lambda|^2)\pi_\lambda\Theta_\lambda^{-1}(\lambda)=$$
$$=\frac{\Theta(z)}{z-\lambda}\cdot\frac{|\lambda|}{\bar\lambda}(1-|\lambda|^2)\pi_\lambda\Theta_\lambda^{-1}(\lambda)=\Theta_\lambda(z)\frac{B_\lambda(z)}{z-\lambda}\cdot\frac{|\lambda|}{\bar\lambda}(1-|\lambda|^2)\pi_\lambda\Theta_\lambda^{-1}(\lambda)=$$
$$=-\Theta_\lambda(z)\frac{1}{1-\bar\lambda z}(1-|\lambda|^2)\pi_\lambda\Theta_\lambda^{1}(\lambda).$$

Thus, the proof is complete since we have already remarked that $PSP = PS$ and $\Theta_\lambda S = S\Theta_\lambda$.

2.5. Lemma. Let $F\in H^\infty(R)$, and let $F(T^*)$ be the operator in $K=H^2(E)\ominus\Theta H^2(E)$, defined in section 1.5: $F(T^*)f=P(F\cdot f)$, $f\in K$.

If Θ is a characteristic inner function, then

$$\{F : F\in H^\infty(R),\ F(T^*)=0\}=\Theta H^\infty(R).$$

Proof. Let $F\in H^\infty(R)$, $F(T^*)=0$. Then, $F(T^*)P$ is the zero operator in $H^2(E)$ and, consequently, $FK\subset\Theta H^2(E)$, i.e., $\Theta^* FK\subset H^2(E)$. Hence, if M, the smallest S, is an invariant subspace in $H^2(E)$ containing K, then $\Theta^* FM\subset H^2(E)$. It is obvious also that $S^*M\subset M$, and, by the Lax-Halmos theorem [4, 3, 1], that $M=qH^2(E)$, where $q(\zeta)\equiv q_0$, $|\zeta|=1$, and q_0 is an orthoprojector in E. If $q_0\neq I$, then it follows from $K\subset M$, that $\Theta(\zeta)(I-q_0)=I-q_0$. This latter equation contradicts the assumption that Θ is a characteristic inner function (see [1], p. 178). Thus, $q_0=I$, and $\Theta^* FH^2(E)\subset H^2(E)$, i.e., $\Theta^* F\in H^\infty(R)$, $F\in\Theta H^\infty(R)$. With this we have derived one of the required inclusions. The second inclusion is obvious, so that the lemma has now been completely proved.

2.6. Lemma. Let $\mathcal{H}$ be a Hilbert space, and let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a family of pairwise disjunction projectors in $\mathcal{H}$ ($P_\lambda P_\mu=0$, $\lambda\neq\mu$), such that $\{P_\lambda\mathcal{H}\}_{\lambda\in\Lambda}$ is complete in $\mathcal{H}$. If $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a bounded family of operators, where Q_λ is a linear operator in $P_\lambda\mathcal{H}$, $\lambda\in\Lambda$, then it follows from $\sup_\sigma\|\sum_{\lambda\in\sigma}P_\lambda\|<\infty$ that $\sup_\sigma\|\sum_{\lambda\in\sigma}Q_\lambda P_\lambda\|<\infty$.

Here, the supremum is taken over all finite subsets of the set Λ.

Proof. From the conditions of lemma it follows that $\sum P_\lambda g$ converges unconditionally to g, for arbitrary g, $g\in\mathcal{H}$. By a theorem of E. Lorch on unconditional bases [5, p. 381], this convergence is equivalent to the convergence $\sum_\lambda\|P_\lambda g\|^2<\infty$, $g\in\mathcal{H}$. But then, it is also true that $\sum_\lambda\|Q_\lambda P_\lambda g\|^2\leq\sup_\lambda\|Q_\lambda\|\cdot\sum_\lambda\|P_\lambda g\|^2<\infty$. Since $Q_\lambda P_\lambda g\in P_\lambda\mathcal{H}$, it follows, by the same theorem of Lorch, that the series $\sum Q_\lambda P_\lambda g$ converges unconditionally, $g\in\mathcal{H}$, and, consequently, $\sup_\sigma\|\sum_{\lambda\in\sigma}Q_\lambda P_\lambda\|<\infty$.

2.7. Remark. Lemma 2.6 may be considered as a generalization of a proposition of V. Orlicz concerning unconditionally converging series (see [5], p. 379).

2.8. Lemma. 1) $L^1(\mathfrak{r}_1)^*=L(R)$, where the duality is brought about by the form

$$\langle f,g\rangle=\frac{1}{2\pi i}\int_{|z|=1}\operatorname{Sp} f(z)g(z)\frac{dz}{z},$$
$$f\in L^1(\mathfrak{r}_1),\quad g\in L^\infty(R);$$

2) $H^\infty(R)^\perp=H_0^1(\mathfrak{r}_1)$;

3) $(\Theta H^\infty(R))^\perp=H_0^1(\mathfrak{r}_1)\Theta^*$.

Here $M^{\perp}$ is the annihilator of $M, M = \{ f : f \in X, < f, x > = 0, x \in M \}$, if $M \subset X^*$, and $H_0^1(\Gamma_1)$ is the subspace of $H^1(\Gamma^1)$, consisting of all functions f such that $f(0) = 0$.

Proof. The assertions 1) and 2) are a consequence of general theorems on duality of spaces of the type $L^p(X)$, where X is a Banach space (see, e.g., [14] or [10]). Let us verify the last equation: $f \in (\Theta H^\infty(\mathfrak{R}))^{\perp}$ implies that $\int Sp\, \Theta h f = 0$ for all $h, h \in H^\infty(\mathfrak{R})$, i.e., $\int Sp\, h f \Theta = 0$, i.e., $f\Theta \in H_0^1(\Gamma_1)$, i.e., $f \in H_0^1(\Gamma_1)\Theta^*$, which is what we wished to show.

2.9. Lemma. Let $\Theta, P, \{\Delta_k\}_{k=1}^\infty$ and $\{z_k\}_1^\infty = \sigma(T^*) \cap \mathfrak{D}$ have the same meaning as §1. Then,

$$\Delta_k f(z_k) = \Delta_k (Pf)(z_k), \quad f \in H^2(E), \quad k \geq 1.$$

Proof.

$$\Delta_k (f(z_k) - (Pf)(z_k)) = \Delta_k (f(z_k) - f(z_k) + \Theta(z_k) \ (Q\Theta^* f)(z_k)) = \Delta_k \Theta(z_k)(Q\Theta^* f)(z_k) = 0, \quad k \geq 1.$$

Here we have used Lemma 2.2 and the fact that $\Theta(z_k) E$ is orthogonal to $\operatorname{Ker} \Theta^*(z_k)$.

2.10. Lemma. Let $\Theta_k, \pi_k, \Delta_k (k \geq 1)$ have the same meaning as in 1. Then,

1) $\pi_k \Theta_k^{-1}(z_k)\Delta_k = \pi_k \Theta_k^{-1}(z_k)$;

2) $\Theta_k^*(z_k)\Delta_k E = \pi_k E$.

Proof. The first equation is equivalent to the statement that $\Theta_k^{*-1}(z_k)\pi_k = \Delta_k \Theta_k^{*-1}(z_k)\pi_k$, i.e., that $\Theta_k^{*-1}(z_k)\pi_k E \subset \Delta_k E$. Since Δ_k is an orthoprojector on $\operatorname{Ker} \Theta^*(z_k)$, it is sufficient to verify that $\Theta^*(z_k)\Theta_k^{*-1}(z_k)\pi_k = 0$. We have $\Theta^*(z_k)\Theta_k^{*-1}(z_k)\pi_k = B_k^*(z_k)\Theta_k^*(z_k)\Theta_k^{*-1}(z_k)\pi_k = (1-\pi_k)\pi_k = 0$. In analogous fashion, the second equation implies that $\Delta_k E = \{ e : \Theta_k^*(z_k) e \in \pi_k E \}$, i.e., $\Delta_k E = \{ e : (1-\pi_k)\Theta_k^*(z_k) e = 0 \}$ or $\Delta_k E = \operatorname{Ker} \Theta^*(z_k)$.

The validity of the last equality is obvious.

2.11. Remark. Lemma 2.10 shows, in particular, that $\dim \pi_k E = \dim \Delta_k E$, $k \geq 1$ (normal solvability of the operators T and T^*; see, e.g., [14]). This important property turns out to be, apparently, a consequence of the fact that the characteristic function Θ is an inner function.

§3. Proof of Theorem 1.2

Let φ be a characteristic function of the operator $T = S^*|K$ of the form (1), corresponding to the characteristic number $\bar{z}_k$, $|z_k| < 1$. We then have $S^*\varphi = \bar{z}_k \varphi$, i.e., $\frac{\varphi(z) - \varphi(0)}{z} = \bar{z}_k \varphi(z)$, $|z| < 1$. Hence, $\varphi(z) = \frac{\varphi(0)}{1 - \bar{z}_k z}$, $|z| < 1$, $\varphi(0) \in E$. Let us rewrite the requirement $\varphi \in K$ in the form $\varphi \perp \Theta H^2(E)$, i.e.,

$$0 = < \varphi, \Theta f > = \frac{1}{2\pi i} \int\limits_{|z|=1} (\Theta(z) f(z), \varphi(z))_E \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{|z|=1} (\Theta(z) f(z), \varphi(0)) \frac{1}{z - z_k} dz = (\Theta(z_k) f(z_k), \varphi(0)),$$

where f is an arbitrary function from $H^2(E)$.

Thus, $\varphi \in K$ if and only if $\Theta^*(z_k)\varphi(0) = 0$, i.e., $\varphi(0) \in \Delta_k E$, Δ_k is an orthoprojector on $\operatorname{Ker} \Theta^*(z_k)$ i.e., $\varphi(0) \in \Theta_k^{*-1}(z_k)\pi_k E$ (see Lemma 2.10).

Since the system of characteristic functions is assumed to be complete in the space K, then, to complete the proof of the theorem, it is sufficient to verify that the systems $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ (see the statement of Theorem 1.2) are bi-orthogonal, and that $\psi_n \in K$, $n \geq 1$.

We have

$$< \Theta f, \psi_n > = \frac{1}{2\pi i} \int\limits_{|z|=1} (\Theta(z) f(z), \psi_n(z))_E \frac{dz}{z} = \alpha_n \frac{1}{2\pi i} \int\limits_{|z|=1} \left(\Theta(z) f(z), \frac{\Theta_n(z) e}{1 - \bar{z}_n z}\right)_E \frac{dz}{z} = \alpha_n \cdot \frac{1}{2\pi i} \times$$

$$\times \int_{|z|=1} (B_n(z) f(z), e) \frac{dz}{z - z_n} = \alpha_n (f(z_n), (1-\pi_n)e) = 0,$$

since $e \in \pi_n E$. In these equations, α_n is some constant depending on ψ_n, and f is an arbitrary function from $H^2(E)$. Thus, $\psi_n \in K$, $n \geq 1$. Now, let us establish the bi-orthogonality of the systems $\{\psi_n\}$ and $\{\varphi_n\}$:

$$\langle \psi_n, \varphi_k \rangle = \frac{1}{2\pi i} \int_{|z|=1} (\psi_n(z), \varphi_k(z))_E \frac{dz}{z} = \frac{1}{2\pi i} \int_{|z|=1} \left(\frac{\Theta_n(z) e_1}{1 - \bar{z}_n z}, \frac{\Theta_k^{*-1}(z_k) e_2}{1 - \bar{z}_k z} \right) (1-|z_n|^2)^{1/2} (1-|z_k|^2)^{1/2} \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{(\Theta_n(z) e_1, \Theta_k^{*-1}(z_k) e_2)}{(1-\bar{z}_n z)\ (z - z_k)} (1-|z_n|^2)^{1/2} (1-|z_k|^2)^{1/2} dz = \frac{(\Theta_n(z_k) e_1, \Theta_k^{*-1}(z_k) e_2)}{1 - \bar{z}_n z_k} (1-|z_n|^2)^{\frac{1}{2}} (1-|z_k|^2)^{\frac{1}{2}}.$$

In these equations, $e_1 \in \pi_n E$, $e_2 \in \pi_k E$. If $n \neq k$, then

$$(\Theta_n(z_k) e_1, \Theta_k^{*-1}(z_k) e_2) = (\Theta_n(z_k) B_n(z_k) B_n^{-1}(z_k) e_1, \Theta_k^{*-1}(z_k) e_2) = (B_n^{-1}(z_k) e_1, \Theta^*(z_k)\, \Theta_k^{*-1}(z_k) e_2) = 0,$$

since, by virtue of Lemma 2.10, $\Theta_k^{*-1}(z_k) e_2 \in \Delta_k E$.

If $n = k$, but $e_1 \neq e_2$, then (see the footnote given with the statement of Theorem 1.2) e_1 and e_2 are orthogonal, and, once again, $\langle \psi_n, \varphi_k \rangle = 0$. If, however, $n = k$, $e_1 = e_2$, it is then easy to show that $\langle \psi_n, \varphi_n \rangle = 1$. Thus, the theorem is proved.

§4. Proof of Theorem 1.3. Part II

The equivalence of the statement 1), 3), 4) of Theorem 1.3 is established through use of the theorems of N. K. Bari [5] on Riesz bases. Statement 2) is reduced to statement 1) with the aid of the previously used theorem of Lorch, [5].

1) $\Longleftrightarrow$ 3). In order to demonstrate these inclusions, it is sufficient to note that the conditions of N. K. Bari's theorem on bases, equivalent to orthonormalized bases ([5], p. 374), are satisfied, and that the inequalities

$$\sum_k |\langle f, \varphi_k \rangle|^2 < \infty, \quad \sum_k |\langle f, \psi_k \rangle|^2 < \infty, \tag{8}$$

reduce to statement 3) of Theorem 1.3. In fact,

$$\langle f, \varphi_k \rangle = \frac{1}{2\pi i} \int_{|z|=1} \frac{(f(z), \Theta_k^{*-1}(z_k) e) dz}{1 - z_k \bar{z}}\frac{}{z} (1-|z_k|^2)^{1/2} = (1-|z_k|^2)^{1/2} (f(z_k), \Theta_k^{*-1}(z_k) e), \quad e \in \pi_k E$$

from whence it follows that

$$\sum_k |\langle f, \varphi_k \rangle|^2 = \sum_{k,i} |(f(z_k), \Theta_k^{*-1}(z_k) \ell_k^{(i)})|^2 (1-|z_k|^2) =$$

$$= \sum_{k,i} |(\Theta_k(z_k)^{-1} f(z_k), \ell_k^{(i)})|^2 (1-|z_k|^2) = \sum_k \|\pi_k \Theta_k(z_k)^{-1} f(z_k)\|_E^2 (1-|z_k|^2), \quad f \in H^2(E).$$

Further, if $f \in K$, then

$$\langle f, \psi_k \rangle = \frac{1}{2\pi i} \int_{|z|=1} \frac{(f(z), \Theta_k(z) \ell)}{1 - z_k \bar{z}} (1-|z_k|^2)^{1/2} \frac{dz}{z} = (1-|z_k|^2)^{1/2} \frac{1}{2\pi i} \int_{|z|=1} \frac{(B_k^*(z) \Theta_k^*(z) f(z), B_k^*(z) \ell)}{z - z_k} dz =$$

$$= -(1-|z_k|^2)^{1/2} \frac{1}{2\pi i} \int_{|z|=1} \frac{(\Theta^*(z) f(z), e)}{1 - \bar{z}_k z} dz \cdot \frac{z_k}{|z_k|} = (1-|z_k|^2)^{1/2} \frac{1}{|z_k|} ((\Theta^* f)(\tfrac{1}{\bar{z}_k}), e),$$

since $\Theta^* f \perp H^2(E)$ for $f \in K$. Since† $K = \{f : f \perp H^2_-(E),\ f \perp \Theta H^2(E)\}$, then $\Theta^* K = H^2_-(E) \ominus \Theta^* H^2_-(E) \overset{\text{def}}{=} K_-$. Hence, the second of the conditions (8) is equivalent to convergence of the series‡

$$\sum_{k,i} (1-|z_k|^2)\,|(g(\tfrac{1}{\bar z_k}), \ell_k^{(i)})|^2 = \sum_k (1-|z_k|^2)\,\|\pi_k g(\tfrac{1}{\bar z_k})\|_E^2 < \infty, \quad g \in K_-.$$

But if $g \in \Theta^* H^2_-(E)$, $g = \Theta^* f$, $f \in H^2_-(E)$, then $\pi_k g(\frac{1}{\bar z_k}) = \pi_k \Theta^*(z_k) f(\frac{1}{\bar z_k}) = \pi_k (1-\pi_k)\Theta_k^*(z_k) f(\frac{1}{\bar z_k}) = \mathbb{0}$, $k = 1, 2, \dots$, $g(z) = \Theta^*(\frac{1}{\bar z}) f(z)$ since, $|z| > 1$. Consequently,

$$\sum_k (1-|z_k|^2)\,\|\pi_k g(\tfrac{1}{\bar z_k})\|_E^2 < \infty, \quad g \in H^2_-(E).$$

Since the correspondence $f(z) = g(\frac{1}{\bar z})$, $|z| < 1$, $g \in H^2_-(E)$ effects an isometry of $H^2_-(E)$ onto $H^2_0(E) = \{f : f \in H^2(E),\ f(0) = 0\}$, the second of the conditions (8), like the first, reduces § to the corresponding statement of Theorem 1.3.

1) $\Longleftrightarrow$ 4). Actually, by a Theorem of N. K. Bari (see [5], p. 374), the systems $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ form Riesz bases in K if and only if they are complete in K, and if the Gram matrices $\{\langle \varphi_i, \varphi_j \rangle\}_{ij}$, $\{\langle \psi_i, \psi_j \rangle\}_{ij}$ generate bounded operators in ℓ^2. It remains to find these matrices:

$$\langle \psi_k, \psi_n \rangle (1-|z_k|^2)^{-1/2} (1-|z_n|^2)^{-1/2} = \frac{1}{2\pi i} \int_{|z|=1} \frac{(\Theta_k(z)\ell_1, \Theta_n(z)\ell_2)}{(1-\bar z_k z)(1-z_n \bar z)} \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{(\Theta(z) B_k(z)^* \ell_1, \Theta_n(z)\ell_2)}{(1-\bar z_k z)(z-z_n)}\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{(B_n(z)\ell_1, \ell_2)}{(z_k - z)(z-z_n)}\, dz \cdot \frac{|z_k|}{\bar z_k} =$$

$$= -\frac{1}{2\pi i} \int_{|z|=1} \frac{(\ell_1, \ell_2)}{(z_k - z)(1-\bar z_n z)} \cdot \frac{\bar z_n}{|z_n|} \cdot \frac{|z_k|}{\bar z_k}\, dz = \frac{(\ell_1, \ell_2)}{1-\bar z_n z_k} \cdot \frac{|z_k|}{\bar z_k} \cdot \frac{\bar z_n}{|z_n|},$$

where $\ell_1 \in \pi_k E$, $\ell_2 \in \pi_n E$ (see the footnote to the statement of Theorem 1.2). The second matrix is more readily calculable:

$$\langle \varphi_k, \varphi_n \rangle \cdot (1-|z_k|^2)^{-1/2} (1-|z_n|^2)^{-1/2} = \frac{1}{2\pi i} \int_{|z|=1} \frac{(\Theta_k^{*-1}(z_k)\ell_1, \Theta_n^{*-1}(z_n)\ell_2)}{(1-\bar z_k z)(1-z_n \bar z)} \frac{dz}{z} = \frac{(\Theta^{*-1}(z_k)\ell_1, \Theta_n^{*-1}(z_n)\ell_2)}{1-\bar z_k z_n}.$$

If we now choose orthonormalized bases $\{\ell_k^{(i)}\}_i$ in $\pi_k E$, $k \geq 1$ (see the footnote to Theorem 1.2), then the condition of boundedness of the Gram matrices may be rewritten as follows:

$$\sum_{k,n,i,j} (1-|z_k|^2)^{1/2} (1-|z_n|^2)^{1/2} \frac{(\ell_k^{(i)}, \ell_n^{(j)})}{1-\bar z_n z_k} c_k^{(i)} \bar c_n^{(j)} = \sum_{k,n} (1-|z_k|^2)^{1/2} (1-|z_n|^2)^{1/2} \frac{(x_k, x_n)}{1-\bar z_n z_k} \leq$$

$$\leq C \sum_{k,i} |c_k^{(i)}|^2 = C \sum_k \|x_k\|^2, \quad C = \mathrm{Const}, \quad x_k = \sum_i c_k^{(i)} \ell_k^{(i)} \in \pi_k E, \quad k \geq 1 \tag{9}$$

and, analogously,

$$\sum_{k,n} \frac{(1-|z_k|^2)^{1/2} (1-|z_n|^2)^{1/2}}{1-\bar z_k z_n} (\Theta_k^{*-1}(z_k)x_k, \Theta_n^{*-1}(z_n)x_n) \leq C \sum_k \|x_k\|^2, \quad x_k \in \pi_k E, \quad k \geq 1. \tag{10}$$

---

†Here, $H^2_-(E) = L^2(E) \ominus H^2(E)$.

‡See the footnote to Theorem 1.2.

§Here, use is also being made of the fact that the correspondence $f \to \hat f$, $f(z) = \sum_k f_k(z) a_k$, $\hat f(z) = \sum_k \overline{f_k(\bar z)}\, a_k$ is an isometric mapping of $H^2_0(E)$ onto $H^2_0(E)$. Here, $\{a_k\}_1^\infty$ is an orthonormalized basis in E, $f_k \in H^2_0$, $k \geq 1$.

We note now that, as a result of the last inequality

$$\sup_{k} \| \Theta_k^{*-1}(z_k)\pi_k \| \leq C. \tag{11}$$

If $\{\varphi_k\}_{k=1}^{\infty}$ is a Riesz basis in K, then statement 4) of Theorem 1.3 is valid, this being a consequence of inequalities (11), (9), and (10) plus the supplementary inequality $\|x_k\| \leq \|\Theta_k^{*-1}(z_k)x_k\|, k \geq 1$, (see also Lemma 2.10).

Conversely, if the conditions 4) of Theorem 1.3 are satisfied, we then obtain inequalities (9) and (11), and, along with them, also inequality (10), since

$$\|\Theta_k^{*-1}(z_k)x_k\| \leq \|\Theta_k^{*-1}(z_k)\pi_k\| \cdot \|x_k\| \leq C\|x_k\|,$$

if $x_k \in \pi_k E$ (see also Lemma 2.10, in accord with which $\ell_n \in \Delta_n E$ implies that $\ell_n = \Theta_n^{*-1}(z_n)x_n$, $x_n \in \pi_n E$).

The second part of Theorem 1.3 has now been proved.

§5. Proof of Theorem 1.6

5.1. It follows from Lemma 2.8 that $H^\infty(\mathcal{R}) = \left(L^1(\mathcal{T}_1) / H_0^1(\mathcal{T}_1) \right)^*$ and $H^\infty(\mathcal{R}) / \Theta H^\infty(\mathcal{R}) = \left(H_0^1(\mathcal{T}_1)\Theta^* / H_0^1(\mathcal{T}_1) \right)^*$, in the sense of an isometric isomorphism.

5.2. Let $K_2 = H^2(\mathcal{T}_2) \ominus \Theta H^2(\mathcal{T}_2)$ and $\mathcal{F}(T_2^*) = \{F(T_2^*) : F \in \mathcal{F}\}$, where $\mathcal{F}$ is the subspace of $H^\infty(\mathcal{R})$ from Theorem 1.6, and $F(T_2^*)$ is an operator in K_2, operating according to the rule:

$$F(T_2^*)f = P_2(Ff), \; f \in K_2,$$

where P_2 is an orthoprojector from $H^2(\mathcal{T}_2)$ onto K_2.

Let us provide $\mathcal{F}(T_2^*)$ with an operator norm and show, to begin with, that $\mathcal{F}(T_2^*)$ is isometric to $\mathcal{F}/\Theta H^\infty(\mathcal{R})$. The proof will repeat, essentially, the corresponding calculations in Sarason's paper [10], which contains the case involving the scalar inner function $\Theta = \varphi \cdot I, \varphi$. For completeness of the presentation, we include this proof.

5.3. If $A \in \mathcal{F}(T_2^*)$ and $A = F(T_2^*)$, $F \in \mathcal{F}$, then it is obvious, also, that $(F + \Theta\Psi)(T_2^*) = A$, if $\Psi \in H^\infty(\mathcal{R})$, and $\|A\| \leq \|F + \Theta\Psi\|_\infty$. Consequently,

$$\|A\| \leq \inf_{\Psi \in H^\infty(\mathcal{R})} \|F + \Theta\Psi\|_\infty = \|F\|_{\mathcal{F}/\Theta H^\infty(\mathcal{R})}.$$

In order to prove the inverse inequality, let us establish an auxiliary statement.

5.4. Let $f \in H_0^1(\mathcal{T}_1)$, $\|f\|_1 = 1$, $F \in \mathcal{F}$. Then g_1, g_2 ($g_i \in K_2$) exists, such that

$$\|g_i\| \leq 1, \; i = 1, 2, \text{ and } \langle f\Theta^*, F \rangle = \langle F(T_2^*)g_1, g_2 \rangle.$$

P r o o f. Using Sarason's theorem [10] on factorization, let us represent the function f in the form $f = f_1 \cdot f_2$, where

$$f_1 \in H^2(\mathcal{T}_2), \, f_2 \in H_0^2(\mathcal{T}_2), \text{ and } \|f_1\|_2^2 = \|f_2\|_2^2 = \|f\|_1 = 1.$$

Then, †

$$\langle f\Theta^*, F \rangle = \int \operatorname{Sp} F f \Theta^* = \int \operatorname{Sp} F f_1 f_2 \Theta^* = \int \operatorname{Sp} F f_1 (\Theta f_2^*)^*$$
$$= \int (F f_1, \Theta f_2^*)_{\mathcal{T}_2} = \langle F f_1, \Theta f_2^* \rangle_{L^2(\mathcal{T}_2)}.$$

†For brevity, we omit the symbol of Lebesgue measure, the measure with respect to which the integration is made.

We observe now that $f_2^* \in H^2(\Upsilon_2)^\perp = H^2_-(\Upsilon_2)$ and $\Theta f_2^* \in \Theta H^2(\Upsilon_2)^\perp = (\Theta H^2(\Upsilon_2))^\perp$, since Θ - is an inner function. Since $(\Theta H^2(\Upsilon_2))^\perp = H^2_-(\Upsilon_2) \oplus K_2$, then

$$\langle F f_1, \Theta f_2^* \rangle = \langle F f_1, P_2 \Theta f_2^* \rangle .$$

Let $P_2 \Theta f_2^* = g_2$. Since $f_1 - P_2 f_1 \in \Theta H^2(\Upsilon_2)$, then $f_1 - P_2 f_1 = \Theta \varphi$ and $F(f_1 - P_2 f_1) = F\Theta\varphi = \Theta\Psi\varphi \in \Theta H^2(\Upsilon_2)$ for some Ψ, $\Psi \in H^\infty(\mathcal{R})$. We have used here the fact that $F \in \mathfrak{F}$. Putting $g_1 = P_2 f_1$, we obtain

$$\langle f\Theta^*, F \rangle = \langle F f_1, g_2 \rangle = \langle P_2 F g_1, g_2 \rangle = \langle F(T_2^*) g_1, g_2 \rangle .$$

This proves statement 5.4.

5.5. Let us continue the proof of the required isometry. Let $F \in \mathfrak{F}$ and $\| F \|_{\mathfrak{F}/\Theta H^\infty(\mathcal{R})} = 1$. From the duality of section 5.1, we find, for a given arbitrary ε, $\varepsilon > 0$, a function f, $f \in H^1_0(\Upsilon_1)$, $\|f\|_1 = 1$, such that $\langle f\Theta^*, F \rangle \geq 1 - \varepsilon$. Using the statement of section 5.4, let us rewrite the last inequality in the form $\langle F(T_2^*) g_1, g_2 \rangle \geq 1 - \varepsilon$, $\|g_i\| \leq 1$, $g_i \in K_2$. It follows from this that $\|F(T_2^*)\| \geq 1 - \varepsilon$, and since $\varepsilon > 0$ is arbitrary, that $\|F(T_2^*)\| \geq 1$, i.e., that $\|F(T_2^*)\| = \|F\|_{\mathfrak{F}/\Theta H^\infty(\mathcal{R})}$.

5.6. The proof of Theorem 1.6 will be complete if we show that $\mathfrak{F}(T_2^*)$ is isometric to $\mathfrak{F}(T^*)$. In view of what has already been shown, it will be sufficient for this to show that $\|F(T_2^*)\| \leq \|F(T^*)\|$, $F \in \mathfrak{F}$. Validity of the last inequality follows immediately from the following simple proposition.

5.7. Let $\{\mathscr{b}_n\}_{n=1}^\infty$ be an orthonormalized basis in E, and let $e \in E$. Let us denote by ge, $g \in H^2(\Upsilon_2)$ a function f, $f \in H^2(E)$, defined by the equation $f(z) = g(z)e$, $|z| < 1$. Then the following statements are valid:

A. $\|g\|^2_{H^2(\Upsilon_2)} = \sum_{n=1}^\infty \|g\mathscr{b}_n\|^2_{H^2(E)}$, $g \in H^2(\Upsilon_2)$.

B. $(P_2 Fg)e = P(Fge)$, $F \in H^\infty(\mathcal{R})$,
$g \in H^2(\Upsilon_2)$, $e \in E$.

Proof. A. This is obvious.

B. Let $g \in H^2(\Upsilon_2)$, $F \in H^\infty(\mathcal{R})$ and $e \in E$. Then,

$$(Fg)e = (P_2(Fg))e + ((I - P_2)Fg)e. \tag{12}$$

On the other hand,

$$Fge = P(Fge) + (I - P)(Fge). \tag{13}$$

Since the left members of Eqs. (12) and (13) are equal, then to prove statement B it is sufficient to convince ourselves that $(P_2 Fg)e \in K$ and $((I - P_2)Fg)e \perp K$. The last relation is obvious since $(I - P_2)Fg = \Theta\varphi$, $\varphi \in H^2(\Upsilon_2)$. If, however, $\Psi \perp \Theta H^2(\Upsilon_2)$ (e.g., $\Psi = P_2(Fg)$), then $\langle \Psi e, \Theta f \rangle_{H^2(E)} = \int (\Psi e, \Theta f)_E = \int Sp\, \Psi ((\cdot, \Theta f) e) = \langle \Psi, (\cdot, e)\Theta f \rangle_{H^2(\Upsilon_2)} = 0$ for $f \in H^2(E)$. Therefore, $\Psi e \perp \Theta H^2(E)$.

Thus, the proof of proposition 5.7 and, with it, the proof of all of Theorem 1.6 is now complete.

§6. Proof of Theorem 1.3, Part I

6.1. For the conditions 1.1, the basis property of operator T is equivalent to the fact that

$$\sup_\sigma \Big\| \sum_{\lambda \in \sigma} P_\lambda \Big\| < \infty , \tag{14}$$

where σ ranges over all the finite subsets of the discrete spectrum $\sigma(T^*) \cap \mathfrak{D}$, P_λ, $\lambda \in \sigma(T^*) \cap \mathfrak{D}$, and is the spectral projector on the characteristic subspace of the operator T^*. By Lemma 2.6, condition

(14) is equivalent to the inequality

$$\sup_{\|Q_\lambda\|\le 1} \sup_{\sigma} \Big\|\sum_{\lambda\in\sigma} Q_\lambda P_\lambda\Big\| < \infty, \tag{15}$$

where $\{Q_\lambda\}$ is the arbitrary bounded family of operators of Lemma 2.6 (this lemma is applicable, since from inequality (14) it follows that the operator T is a basis operator, and, hence, the bi-orthogonal system $\{\psi_k\}_1^\infty$ is complete in K, i.e., $\{P_\lambda K\}$ is complete in K). Let us now evaluate the number in the left member of inequality (15).

6.2. Selection of Q. Let A_λ, $\lambda\in\sigma(T^*)\cap\mathfrak{D}$, be a bounded operator, operating† in $\pi_\lambda E$. Let us then put

$$Q_\lambda = \Theta_\lambda A_\lambda \Theta_\lambda^{-1},$$

i.e.,

$$(Q_\lambda f)(z) = \Theta_\lambda(z) A_\lambda \Theta_\lambda^{-1}(z) f(z), \quad |z|<1, \; f\in P_\lambda K.$$

We shall show that $Q_\lambda P_\lambda K \subset P_\lambda K$, and that $Q_\lambda P_\lambda f = P Q_\lambda F_\lambda f$, $f\in K$, and F_λ is the function of Lemma 2.4.

Let, at first, $\psi\in P_\lambda K$; then we have (Theorem 1.2)

$$\psi(z) = \Theta_\lambda(z)\frac{e}{1-\bar\lambda z}, \quad |z|<1, \; e\in\pi_\lambda E.$$

Consequently, $(Q_\lambda\psi)(z) = \Theta_\lambda(z)A_\lambda\Theta_\lambda(z)^{-1}\Theta_\lambda(z)\frac{e}{1-\bar\lambda z} = \Theta_\lambda(z)\frac{A_\lambda e}{1-\bar\lambda z}$, $|z|<1$, and $Q_\lambda\psi\in P_\lambda K$.

Now let $f\in K$. We have ‡

$$Q_\lambda P_\lambda f = Q_\lambda P(F_\lambda f) = Q_\lambda(\Theta_\lambda E_\lambda \pi_\lambda \Theta_\lambda^{-1}(\lambda) f) - Q_\lambda \Theta Q(\Theta^*(\Theta_\lambda E_\lambda \pi_\lambda \Theta_\lambda^{-1}(\lambda) f)) \tag{16}$$

Let us calculate the second term of this expression (in fact, we even find the first), considering $|z|=1$:

$$Q_\lambda\Theta Q(\Theta^*(\Theta_\lambda E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f))(z) = \Theta_\lambda(z)A_\lambda B_\lambda(z) Q(B_\lambda^* E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f)(z) =$$

$$= \Theta_\lambda(z)A_\lambda\pi_\lambda B_\lambda(z)Q(B_\lambda^* E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f)(z) = \Theta_\lambda(z)B_\lambda(z)A_\lambda\pi_\lambda Q(B_\lambda^* E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f)(z) =$$

$$= \Theta(z)Q(A_\lambda B_\lambda^* E_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f)(z) = \Theta(z)Q(\Theta^*(z)\Theta_\lambda(z)E_\lambda A_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)f)(z).$$

Returning to Eq. (16), we obtain:

$$Q_\lambda P_\lambda f = (1-\Theta Q\Theta^*)(\Theta_\lambda E_\lambda A_\lambda \pi_\lambda \Theta_\lambda^{-1}(\lambda) f) = P Q_\lambda F_\lambda f.$$

6.3. Let us show now that $Q_\lambda F_\lambda \in \mathfrak{F}$. For this it is necessary to verify that $\Theta^* Q_\lambda F_\lambda \Theta \in H^\infty(\mathfrak{R})$. We have $(|z|=1)$:

$$\Theta^*(z)Q_\lambda(z)F_\lambda(z)\Theta(z) = \Theta^*(z)\Theta_\lambda(z)A_\lambda\Theta_\lambda^*(z)\Theta_\lambda(z)E_\lambda(z)\pi_\lambda\Theta_\lambda^{-1}(\lambda)\Theta(z)(|\lambda|^2-1) =$$

$$= B_\lambda^*(z)A_\lambda E_\lambda(z)\pi_\lambda\Theta_\lambda^{-1}(\lambda)\Theta(z)(|\lambda|^2-1) = -\frac{|\lambda|}{\lambda}\frac{1-|\lambda|^2}{\lambda-z}A_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)\Theta_\lambda(z)B_\lambda(z) \equiv \frac{1-|\lambda|^2}{\lambda-z}G(z).$$

To complete the proof of section 6.3, it is sufficient to note that the analytic continuation of the function G in the disk $\mathfrak{D}$ vanishes at the point $z=\lambda$:

$$G(\lambda) = -\frac{|\lambda|}{\lambda}A_\lambda\pi_\lambda\Theta_\lambda^{-1}(\lambda)\Theta_\lambda(\lambda)B_\lambda(\lambda) = \frac{|\lambda|}{\lambda}A_\lambda\pi_\lambda(1-\pi_\lambda) = 0.$$

---

†If $\lambda = z_k \in\sigma(T^*)\cap\mathfrak{D}$, we then let $\pi_\lambda = \pi_k$, $\Delta_\lambda = \Delta_k$, $\Theta_\lambda = \Theta_k$, $B_\lambda = B_k$, etc.

‡ $E_\lambda(z) = (1-\bar\lambda z)^{-1}$, $|z|<1$, is the function of Lemma 2.4.

6.4. Evaluation of the Left Member of Formula (15). Since $\tilde{P}_\sigma = \sum_{\lambda\in\sigma} Q_\lambda P_\lambda = P\sum_{\lambda\in\sigma} Q_\lambda F_\lambda$ and $Q_\lambda F_\lambda \in \mathfrak{F}$, use can be made of Theorem 1.6. We find then that

$$\|\tilde{P}_\sigma\| = \inf_{\Psi\in H^\infty(R)} \|\sum_{\lambda\in\sigma} Q_\lambda F_\lambda + \Theta\Psi\|_\infty = \inf \|\Theta^* \sum_{\lambda\in\sigma} Q_\lambda F_\lambda + \Psi\|_\infty = \|\tilde{F}_\sigma\|_{H_0^1(\mathfrak{T}_1)^*}$$

by virtue of Lemma 2.8. Here $\tilde{F}_\sigma = \Theta^* \sum_{\lambda\in\sigma} Q_\lambda F_\lambda$.

Thus,

$$\|\tilde{P}_\sigma\| = \|\tilde{F}_\sigma\|_{H_0^1(\mathfrak{T}_1)^*} = \sup_{\|f\|_{H_0^1(\mathfrak{T}_1)}\le 1} |\langle \tilde{F}_\sigma, f\rangle| = \sup \left|\frac{1}{2\pi i}\int_{|z|=1} \operatorname{Sp} \tilde{F}_\sigma(z) f(z)\frac{dz}{z}\right| =$$

$$= \sup\left|\frac{1}{2\pi i}\operatorname{Sp}\int_{|z|=1}\Theta^*(z)\Big(\sum_{\lambda\in\sigma}(1-|\lambda|^2)\Theta_\lambda(z)\frac{A_\lambda\mathfrak{T}_\lambda}{1-\bar\lambda z}\Theta_\lambda^{-1}(\lambda)\Big)f(z)\frac{dz}{z}\right| =$$

$$= \sup\left|\operatorname{Sp}\sum_{\lambda\in\sigma}(1-|\lambda|^2)\frac{1}{2\pi i}\int_{|z|=1}\frac{B_\lambda^*(z)A_\lambda\mathfrak{T}_\lambda}{1-\bar\lambda z}\Theta_\lambda^{-1}(\lambda)f(z)\frac{dz}{z}\right| = \sup\left|\operatorname{Sp}\sum_{\lambda\in\sigma}(1-|\lambda|^2)\frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)A_\lambda\mathfrak{T}_\lambda}{\lambda-z}\frac{dz}{z}\Theta_\lambda^{-1}(\lambda)\cdot\frac{|\lambda|}{\bar\lambda}\right| =$$

$$= \sup\left|\operatorname{Sp}\sum_{\lambda\in\sigma}(1-|\lambda|^2)\frac{|\lambda|}{|\lambda|^2}f(\lambda)A_\lambda\mathfrak{T}_\lambda\Theta_\lambda^{-1}(\lambda)\right| = \sup_{\|f\|_{H_0^1(\mathfrak{T}_1)}\le 1}\left|\sum_{\lambda\in\sigma}(1-|\lambda|^2)\frac{1}{|\lambda|}\cdot\operatorname{Sp}(f(\lambda)A_\lambda\mathfrak{T}_\lambda\Theta_\lambda^{-1}(\lambda))\right|.^\dagger$$

To complete the evaluation, it is sufficient to pass to the supremum over all A_λ, $\|A_\lambda\|\le 1$ (since $\|Q_\lambda\| = \|A_\lambda\|$):

$$\sup_\sigma \sup_{\|f\|_1\le 1}\sup_{\|A_\lambda\|\le 1}\left|\sum_{\lambda\in\sigma}(1-|\lambda|^2)\frac{1}{|\lambda|}\operatorname{Sp}(f(\lambda)A_\lambda\mathfrak{T}_\lambda\Theta_\lambda^{-1}(\lambda))\right| = \sup_{\substack{\|f\|_1\le 1,\\ f\in H_0^1(\mathfrak{T}_1)}}\sum_\lambda(1-|\lambda|^2)\frac{1}{|\lambda|}\|\mathfrak{T}_\lambda\Theta_\lambda^{-1}(\lambda)f(\lambda)\mathfrak{T}_\lambda\|_{\mathfrak{T}_1} < \infty.$$

The last inequality constitutes the content of assertion 2) of Theorem 1.3. This completes the proof.

§7. Proof of Theorem 1.11

By virtue of Theorem 1.2, the condition $Jf = 0$, $f\in H^2(E)$, implies that

$$\langle f, \varphi_k\rangle = \frac{1}{2\pi i}\int_{|z|=1}\frac{(f(z), e)}{1-z_k\bar z}\frac{dz}{z} = (f(z_k), e) = 0,\quad e\in\Delta_k E,\ k\ge 1.$$

On the other hand, completeness of the system $\{\varphi_k\}_1^\infty$ in the space K implies that, from $\langle f,\varphi_k\rangle = 0$, $k\ge 1$, we have $f\perp K$, i.e., $f\in\Theta\cdot H^2(E)$. With these two remarks, the proof of Theorem 1.11 is complete.

§8. Proof of Theorem 1.12

The uniform minimality of the bi-orthogonal systems $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ implies (in the case $\|\psi_k\| = 1$, $k\ge 1$) that

$$\sup_k \|\varphi_k\| < \infty$$

From formula (4) we obtain $\sup_{k,i}\|\Theta_k(z_k)^{*-1}e_k^{(i)}\| < \infty$, $e_k^{(i)}\in\mathfrak{T}_k E$, $k\ge 1$, $\|e_k^{(i)}\| = 1$. Consequently, with the satisfaction of condition (6), an arbitrary one of the systems (4) is uniformly minimal.

Conversely, if condition (6) is not satisfied, then $\sup_k\|\Theta_k^{*-1}(z_k)e_k\| = \infty$ for some sequence $\{e_k\}_{k=1}^\infty$, $\|e_k\| = 1$, $e_k\in\mathfrak{T}_k E$. Thus, the corresponding system $\{\varphi_k\}_{k=1}^\infty$, for which the vectors e_k, $k\ge 1$ are included in the bases of the subspaces $\mathfrak{T}_k E$, will not be uniformly minimal. This completes the proof of the theorem.

†We have here used the fact that $f(0) = 0$.

§9. Proof of Theorem 1.9

9.1. Using Lemma 2.9, we note first that if the function f, $f \in H^2(E)$ solves the interpolation problem (5) of §1, then the function Pf, $Pf \in K$ also possesses this property. If g, $g \in H^2(E)$ is a second solution of this same problem, then $Pf = Pg$ (Theorem 1.11). Consequently, if problem (5) (for given $w_\kappa, \kappa \geq 1$) is solvable, then there exists a unique solution f of smallest norm, with $f \in K$.

9.2. It is obvious that a necessary and sufficient condition for the inclusion $J H^2(E) \subset \ell^2(\Delta_\kappa)$ is the condition

$$\sum_\kappa (1-|z_\kappa|^2)\|\Delta_\kappa f(z_\kappa)\|^2 < \infty, \quad f \in H^2(E),$$

i.e., condition 3b) of Theorem 1.9.

9.3. Let us now find the condition equivalent to the inclusion $JH^2(E) \supset \ell^2(\Delta_\kappa)$.

Let $JH^2(E) \supset \ell^2(\Delta_\kappa)$. Let us put

$$G_\kappa = \Theta_\kappa \Theta_\kappa^{-1}(z_\kappa)\Delta_\kappa, \quad \kappa \geq 1.$$

Then $\Delta_i G_\kappa(z_i) = 0$, if $i \neq \kappa$. Actually,

$$G_\kappa(z_i)E = \Theta_\kappa(z_i)\Theta_\kappa^{-1}(z_\kappa)\Delta_\kappa E = \Theta(z_i)B_\kappa(z_i)^{-1}\Theta_\kappa(z_\kappa)^{-1}\Delta_\kappa E \subset \Theta(z_i)E \subset (I-\Delta_i)E.$$

But if $i=\kappa$, then $\Delta_\kappa G(z_\kappa) = \Delta_\kappa$.

By virtue of what was said in section 9.1, there exists a linear operator A from $\ell^2(\Delta_\kappa)$ into K, such that $JAw = w$, $w \in \ell^2(\Delta_\kappa)$. From the closed graph theorem, it follows immediately that A is a continuous operator. Consequently, under our assumptions,

$$\sup_{\|w\| \leq 1} \|Aw\|_K < \infty, \tag{17}$$

where the supremum is taken over all "finite" sequences w of $\ell^2(\Delta_\kappa)$, i.e., $w \in \ell^2(\Delta_\kappa)$, $w = \{w_\kappa\}_{\kappa=1}^\infty$, $w_\kappa = 0$ for $\kappa > n = n(w)$, $\|w\|_{\ell^2(\Delta_\kappa)} \leq 1$.

It is obvious that the converse is also true: if condition (17) is satisfied, then $JH^2(E) \supset \ell^2(\Delta_\kappa)$.

Now let $w = \{w_\kappa\}_1^\infty$, $w \in \ell^2(\Delta_\kappa)$, be a "finite" sequence, with $\|w\| \leq 1$. Let us put

$$f = \sum_\kappa G_\kappa w_\kappa (1-|z_\kappa|^2)^{-1/2}.$$

Then, $Aw = Pf$ and

$$\|Aw\|_2 = \|Pf\|_2 = \inf_{g \in H^2(E)} \|f + \Theta g\|_2 = \inf_g |\Theta^* f + g|_2 = \sup_{h \in H^2_-(E), \|h\| \leq 1} |\langle \Theta^* f, h \rangle|,$$

since $L^2(E)/H^2(E)$ is isometric to $H^2_-(E)$. Here, $H^2_-(E) = L^2(E) \ominus H^2(E)$.

Continuing the evaluation, we obtain

$$\|Aw\|_2 = \sup_{h \in H^2_-(E), \|h\| \leq 1} |\langle \Theta^* f, h \rangle| = \sup_h \left|\sum_\kappa \langle \Theta^* G_\kappa w_\kappa, h \rangle (1-|z_\kappa|^2)^{-1/2}\right| = \sup_h \left|\sum_\kappa \langle B_\kappa^* \Theta_\kappa^* \Theta_\kappa \Theta_\kappa^{-1}(z_\kappa) w_\kappa, h \rangle (1-|z_\kappa|^2)^{-1/2}\right| =$$

$$= \sup_h \left|\sum_\kappa (1-|z_\kappa|^2)^{-1/2} \frac{1}{2\pi i} \int_{|z|=1} \frac{1-\bar{z}_\kappa z}{z_\kappa - z} \cdot \frac{|z_\kappa|}{z_\kappa} (\pi_\kappa \Theta_\kappa^{-1}(z_\kappa) w_\kappa, h(z)) \frac{dz}{z}\right| = \sup_h \left|\sum_\kappa (1-|z_\kappa|^2)^{1/2} \frac{1}{|z_\kappa|} \left(\pi_\kappa \Theta_\kappa^{-1}(z_\kappa) w_\kappa, h\left(\frac{1}{\bar{z}_\kappa}\right)\right)\right|. \dagger$$

---

†Here we have used the fact that $\int_{|z|=1} h(z) \frac{dz}{z} = 0$.

Evaluating the supremum in formula (17), we find now that

$$\sup_{\|w\|\leq 1} \|Aw\|_2 = \sup_h \sup_w \Big|\sum_k (1-|z_k|^2)^{1/2}\frac{1}{|z_k|}(w_k, \Theta_k^{*-1}(z_k)\pi_k h(\tfrac{1}{\bar z_k}))\Big| =$$

$$= \sup_{\|h\|_{H^2_-(E)}\leq 1} \sum_k (1-|z_k|^2)\frac{1}{|z_k|^2}\|\Delta_k \Theta_k^{*-1}(z_k)\pi_k h(\tfrac{1}{\bar z_k})\|_E^2 < \infty \tag{18}$$

Thus, conditions (17) and (18) are equivalent. Let us show now that Eq. (18) reduces to the two inequalities

$$\left.\begin{aligned} &\sup_k \|\pi_k \Theta_k^{-1}(z_k)\Delta_k\| < \infty, \\ &\sum_k (1-|z_k|^2)\|\pi_k h(\bar z_k)\|_E^2 < \infty, \; h \in H_0^2(E). \end{aligned}\right\} \tag{19}$$

The first of the inequalities (19) follows from Eq. (18) for a corresponding choice of the functions h, while the second is obtained from the equation $\Delta_k \Theta_k^{*-1}(z_k)\pi_k = \Theta_k^{*-1}(z_k)\pi_k$ (Lemma 2.10) and the inequality $\|\Theta_k^*(z_k)\| \leq 1$. The reverse inclusion ((19) $\Rightarrow$ (18)) is obvious.

9.4. By combining the results of section 9.2 with the results of section 9.3 (the equivalence of inequalities (17) and (19)), we find that the conditions 1) and 3) of Theorem 1.9 are equivalent.

9.5. Let us establish now the equivalence of the conditions 1) and 2) of Theorem 1.9. To do this, let us select at first, from each of the subspaces $\pi_k E$, according to an orthonormalized basis $\{e_k^{(i)}\}_i$, and let us denote the corresponding system (4) of Theorem 1.2 by $\{\varphi_k^{(i)}\}$, and the system biorthogonal to it by $\{\psi_k^{(i)}\}$. By virtue of Theorem 1.12 and Lemma 2.10, the systems $\{g_k^{(i)} = \Theta_k^{*-1}(z_k)e_k^{(i)}\}_i$ are Riesz bases in the subspaces $\Delta_k E$, $k \geq 1$. In this connection, the constants characterizing the deviation of such a basis from an orthonormal basis are uniformly bounded:

$$\sum_i |c_i|^2 \leq \|\sum_i c_i g_k^{(i)}\|^2 \leq \alpha^2 \sum_i |c_i|^2, \; k \geq 1,$$

where $\alpha = \sup_k \|\Theta_k^{*-1}(z_k)\pi_k\|$

Consequently, assignment of the element $w = \{w_k\}_{k=1}^\infty$, $w \in \ell^2(\Delta_k)$, is equivalent to assignment of a family $\{c_k^{(i)}\}_{k,i}$, such that $\sum_{k,i} |c_k^{(i)}|^2 < \infty$ ($c_k^{(i)} = (w_k, g_k^{(i)})$). Therefore, the interpolation problem (5) can be reformulated as follows:

$$(1-|z_k|^2)^{1/2}(f(z_k), g_k^{(i)}) = c_k^{(i)}, k \geq 1, \; f \in H^2(E).$$

The equation $JH^2(E) = \ell^2(\Delta_k)$ of Theorem 1.9 is converted thereby into the equation $J_g H^2(E) = \ell^2$, where J_g denotes the operator $J_g f = \{(f(z_k), g_k^{(i)})(1-|z_k|^2)^{1/2}\}_{k,i}$, $f \in H^2(E)$.

We note, further, that

$$(\psi_k^{(i)}(z_n), g_n^{(j)})(1-|z_n|^2)^{1/2} = \langle \psi_k^{(i)}, \varphi_n^{(j)} \rangle = \begin{cases} 1, & k = n, i = j, \\ 0, & \text{in the remaining cases.} \end{cases}$$

Let us now define an operator A_g by the equation

$$A_g\{c_k^{(i)}\} = \sum_{k,i} c_k^{(i)}\psi_k^{(i)},$$

for all families $\{c_k^{(i)}\}_{k,i}$, with a finite number of terms different from zero. From all that has been said in this section 9.5 (see also section 9.1 and the clarification of inequality (17)), it follows that condition 1) of Theorem 1.9 ($J_g H^2(E) = \ell^2$) is equivalent to the assertion that the operator A_g is extendable to an isomorphism of the spaces ℓ^2 and K. By a theorem of N. K. Bari ([5], p. 375, item 3), this latter assertion implies that $\{\psi_k^{(i)}\}$ is a Riesz basis in the space K. This completes the proof.

§10. Example 1.15

Let E be a Hilbert space, $\dim E=\infty$, and let $\{e_k\}_{k=0}^{\infty}$ be an orthonormalized basis in E. Let us put $g_k=e_0+e_k$, $k\geq 1$, and $\varphi_k(z)=\frac{g_k}{1-\bar{z}_k z}\frac{1}{\sqrt{2}}(1-|z_k|^2)^{1/2}$, $|z|<1$, $k\geq 1$. Let $K=\mathcal{L}(\varphi_k)_{k\geq 1}$ be the closed linear hull of the system $\{\varphi_k\}_{k=1}^{\infty}$ in $H^2(E)$. If $f\in H^2$, then let f_e, $e\in E$, denote the function $f_e(z)=f(z)e$, $|z|<1$.

We have $(f e_0(z_k), g_k)=f(z_k)$, $k\geq 1$. Consequently, the example we seek will have been constructed if we can find a sequence $\{z_k\}_{k=1}^{\infty}$, and a function f, $f\in H^2(E)$, such that

$$\sum_k (1-|z_k|)<\infty, \tag{20}$$

$$\sum_k (1-|z_k|^2)|f(z_k)|^2=\infty. \tag{21}$$

Here, $T^*=S^*|K$, and the conditions 1.1 follow from the fact that $T\in C_0$ ($B(T)=\mathbb{O}$, if B is a Blaschke product with zeros $\{z_k\}_{k=1}^{\infty}$).

To find $\{z_k\}_{k=1}^{\infty}$ and f, $f\in H^2(E)$, with the conditions (20) and (21), one can, for example, employ a theorem of L. Carleson, [15].

It remains then to prove the uniform minimality of the system $\{\varphi_k\}_{k=1}^{\infty}$. We have

$$\Big\|\varphi_k-\sum_{n\neq k}C_n\varphi_n\Big\|_{H^2(E)}\geq\Big\|\varphi_k(z_k)-\sum_{n\neq k}C_n\varphi_n(z_k)\Big\|_E(1-|z_k|^2)^{1/2}=$$

$$=\Big\|\frac{1}{\sqrt{2}}g_k-\sum_{n\neq k}C_n(1-|z_k|^2)^{1/2}\frac{1}{\sqrt{2}}\frac{(1-|z_n|^2)^{1/2}}{1-\bar{z}_n z_k}g_n\Big\|_E\geq\Big\|\frac{1}{\sqrt{2}}e_k-\sum_{n\neq k}C_n(1-|z_k|^2)^{1/2}\frac{1}{\sqrt{2}}\frac{(1-|z_n|^2)^{1/2}}{1-\bar{z}_n z_k}e_n\Big\|\geq\frac{1}{\sqrt{2}},$$

for all k, $k\geq 1$.

LITERATURE CITED

1. B. S. Nagy and C. Foias, Analysé harmonique des operateurs de l'éspace de Hilbert, Budapest (1967).
2. N. K. Nikol'skii and B. S. Pavlov, "Bases of characteristic vectors, the characteristic function, and problems of interpolation in the Hardy space H²," Dokl. Akad. Nauk SSSR (in press).
3. H. Helson, Lectures on Invariant Subspaces, Academic Press, New York (1964).
4. K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, N. J. (1962).
5. I. Ts. Gokhberg and M. G. Krein, Introduction to the Theory of Linear Non-self-adjoint Operators in Hilbert Space [in Russian], "Nauka," Moscow (1965).
6. B. R. Mukminov, "On the expansion in terms of characteristic functions of dissipative kernels," Dokl. Akad. Nauk SSSR, Vol. 99, No. 4, pp. 499-502 (1954).
7. I. M. Glazman, "On the expandibility with respect to a system of characteristic elements of dissipative operators," Uspekh. Matem. Nauk, Vol. 13, No. 3(81), pp. 179-181 (1958).
8. A. S. Markus, "On the basis of the root vectors of a dissipative operator," Dokl. Akad. Nauk SSSR, Vol. 132, No. 3, (1960).
9. V. E. Katsnel'son, "On the conditions for the basis property of the system of root vectors of some classes of operators," Funktsional. Analiz i Ego Prilozhen., Vol. 1, No. 2 (1967).
10. D. Sarason, "Generalized interpolation in H∞," Trans. Am. Math. Soc., Vol. 127, No. 2, pp. 179-203 (1967).
11. H. S. Shapiro and A. L. Shields, "On some interpolation problems for analytic functions," Am. J. Math., Vol. 83, No. 3, pp. 513-532 (1961).
12. L. Carleson, "An interpolation problem for bounded analytic functions," Am. J. Math., Vol. 80, No. 4, pp. 921-930 (1958).

13. D. Newman, "Interpolation in H^∞," Trans. Am. Math. Soc., Vol. 92, No. 3, pp. 501-507 (1959).
14. J. T. Schwartz and N. Dunford, Linear Operators, Vol. 1, Interscience, New York (1958).
15. L. Carleson, "Interpolations by bounded analytic functions and the corona problem," Annals of Math., Vol. 76, No. 3, pp. 547-559 (1962).
16. V. P. Potapov, "Multiplicative structure of J-Nonexpanding matrix-functions," Trudy Mosk. Matem. Obshch., Vol. 4, pp. 125-236 (1955).
17. Yu. P. Ginzburg, "A maximum principle for J-nonexpanding operator-functions and some of its consequences," Izv. VUZov, Matem., No. 1, pp. 42-53 (1963).

ON THE ORDER OF DECREASE AT INFINITY OF A FUNCTION WHOSE FOURIER TRANSFORM IS LOCALIZED ON A CURVE

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Let the Fourier transform $\hat{u}(\alpha,\beta)$ of a function $u(x,y)$ be given on the plane $O\alpha\beta$. For smooth functions $u(x,y)$ it is known that the order of decrease at infinity is related to and depends on the smoothness of the function $\hat{u}(\alpha,\beta)$: the smoother the function, the higher is the order. Let a function $u(x,y)$ be given by its Fourier transform, the latter, generally speaking, being a generalized function. For the definition of the direct and inverse Fourier transforms of a generalized function, the reader should consult [1]. If the Fourier transform $\hat{u}(\alpha,\beta)$ is concentrated at a single point (h_1, h_2), then the function $u(x,y)=(2\pi)^{-2}\exp[-i(xh_1+yh_2)]$ is merely bounded on the $0xy$-plane. We shall answer the question as to the behavior at infinity (as $r=\sqrt{x^2+y^2}\to\infty$) of the function $u(x,y)$, given that its Fourier transform is localized on a curve Γ.

Let Γ be a closed, smooth curve, homeomorphic to a circle, and located in the $0\alpha\beta$-plane. Let us introduce on the curve a parameter s, where $0\le s\le 2\pi$; points, which correspond to the parameter values $s=0$ and $s=2\pi$, coincide.

On an oriented plane, it is possible to assign a sign to the curvature $\mathcal{K}$ of a plane curve, where the direction of traversing the curve Γ is prescribed. Let us assume that $\mathcal{K}$ is positive at each point of Γ. In this case, as one traverses the boundary in a positive direction (alway keeping the interior domain to one's left), the tangent vector rotates counterclockwise, this being one of the definitions of a convex domain (see [2]). Thus, the interior domain bounded by the curve Γ is convex.

On the curve Γ, let a smooth vector field transverse to it be given. Let us extend it, maintaining its smoothness, onto some neighborhood of Γ. Through each point (α,β) of this neighborhood, one can draw an integral curve. We shall denote the distance along the integral curve from the point considered to the point of its intersection with Γ by w. We shall speak of the set on which $w<\sigma$ as the σ-neighborhood of Γ, and we shall denote this set by Π.

The position of each point of the domain Π may be determined by numbers s, w, which are curvilinear coordinates of the point. In this case, for rectangular coordinates α, β, we shall have

$$\begin{cases}\alpha=\alpha(s,w),\\ \beta=\beta(s,w),\end{cases}$$

where $\alpha(s,w)$ and $\beta(s,w)$ have continuous derivatives of the second order, and the nonvanishing Jacobian $J(s,w)=\frac{D(\alpha,\beta)}{D(s,w)}$ is a smooth function.

Let us consider a function u whose Fourier transform is given in terms of the coordinates (s,w) in the domain Π by the formula

$$\hat{u}(s,w)=g(s)\,\delta(w), \tag{1}$$

where $g(s)$ is an arbitrary continuously differentiable function, with $g(0)=g(2\pi)$. Outside of Π, we define $\hat{u}$ to be zero.

To each smooth function $g(s)$, $0 \leq s \leq 2\pi$ and to each smooth vector field transverse to Γ, there corresponds a function $\hat{u}(s,w)$.

If in the formula

$$u(x,y)=(2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{u}(\alpha,\beta)\exp[-i(\alpha x+\beta y)]\,d\alpha\,d\beta$$

defining the function $u(x,y)$ from its Fourier transform $\hat{u}(\alpha,\beta)$, we transform to new variables of integration s and w, we obtain

$$u(x,y)=(2\pi)^{-2}\iint_{\Pi}\hat{u}(s,w)J(s,w)\exp\{-i[\alpha(s,w)x+\beta y]\}\,ds\,dw. \tag{2}$$

With the assignment (1) for $\hat{u}(s,w)$, it is easy to show that

$$u(x,y)=(2\pi)^{-2}\int_{0}^{2\pi}g(s)\,J(s,0)\exp\{-i[\alpha(s,0)x+\beta(s,0)y]\}\,ds. \tag{3}$$

Let us transform the expression (3) by making a translation of the $0\alpha\beta$-system of coordinates to a new system with origin at a point lying inside the curve (if the $0\alpha\beta$-system already has this property, the transformation is unnecessary). We designate the new system of coordinates by $0_1\alpha_1\beta_1$. For points lying on Γ, we shall have

$$\begin{cases}\alpha(s)=\alpha_1(s)+a,\\ \beta(s)=\beta_1(s)+b,\end{cases} \tag{4}$$

where a and b are the coordinates of the new origin 0_1 with respect to the old system $0\alpha\beta$.

Since the domain bounded by Γ is convex, the curve may be represented by an equation of the form $\rho=\rho(\psi)\,(0\leq\psi\leq 2\pi)$ in a polar coordinate system, with the pole at 0_1 and the polar axis along the α-axis. Using Eqs. (4), transforming to polar coordinates r, φ (in the $0xy$-plane) and to ρ, ψ (in the $0_1\alpha_1\beta_1$-plane), we obtain the integral (3) in the form

$$u(r\cos\varphi, r\sin\varphi)=(2\pi)^{-2}\exp[-ir(a\cos\varphi+b\sin\varphi)]\int_{0}^{2\pi}h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]\,d\psi, \tag{5}$$

where

$$h(\psi)=g(\psi)\,J(\psi,0),\qquad \rho=\sqrt{\alpha_1^2(s,0)+\beta_1^2(s,0)},$$
$$\cos\psi=\frac{\alpha_1(s,0)}{\sqrt{\alpha_1^2(s,0)+\beta_1^2(s,0)}},\quad \sin\psi=\frac{\beta_1(s,0)}{\sqrt{\alpha_1^2(s,0)+\beta_1^2(s,0)}}. \tag{6}$$

Noting that the selection of the parameter s on Γ is somewhat arbitrary, we put $s=\psi$.

<u>Remark.</u> As is evident from the formulas obtained, one and the same function $u(x,y)$ may be obtained for various assignments of $\hat{u}$ in accord with formula (1), if different coordinates (s,w) are employed.

To study its behavior at infinity (as $r\to\infty$), it is convenient to represent the function (5) in the form

$$u(r\cos\varphi, r\sin\varphi)=\exp[-ir(a\cos\varphi+b\sin\varphi)]\int_{0}^{2\pi}\frac{h(\psi)\{\exp[-ir\rho\cos(\psi-\varphi)]\}'\,d\psi}{-ir[\rho'(\psi)\cos(\psi-\varphi)-\rho(\psi)\sin(\psi-\varphi)]}. \tag{7}$$

We shall show, under our assumptions on the curve, that all the roots of the denominator

$$\Phi(\psi)=\rho'(\psi)\cos(\psi-\varphi)-\rho(\psi)\sin(\psi-\varphi)$$

in the integrand of Eq. (7), are simple and finite in number.

Let us assume to the contrary that ψ_1 is a root of the denominator $\Phi(\psi)$ of multiplicity two. Then,

$$\begin{cases}\rho'(\psi_1)\cos(\psi_1-\varphi)-\rho(\psi_1)\sin(\psi_1-\varphi)=0\ ,\\ [\rho''(\psi_1)-\rho(\psi_1)]\cos(\psi_1-\varphi)-2\rho'(\psi_1)\sin(\psi_1-\varphi)=0,\end{cases}$$

or

$$[\rho'(\psi_1)\cos\psi_1-\rho(\psi_1)\sin\psi_1]\cos\varphi+[\rho'(\psi_1)\sin\psi_1+\rho(\psi_1)\cos\psi_1]\sin\varphi=0,$$

$$\{[\rho''(\psi_1)-\rho(\psi_1)]\cos\psi_1-2\rho'(\psi_1)\sin\psi_1]\}\cos\varphi+\{[\rho''(\psi_1)-\rho(\psi_1)]\sin\psi_1+2\rho'(\psi_1)\cos\psi_1\}\sin\varphi=0.$$

This set of equations may be regarded as a homogeneous system in $\cos\varphi$ and $\sin\varphi$; since $\cos^2\varphi+\sin^2\varphi=1$, this system has a nontrivial solution. A homogeneous system has a nontrivial solution only if its determinant vanishes, i.e., for the case in question, we must have

$$\begin{vmatrix}\rho'(\psi_1)\cos\psi_1-\rho(\psi_1)\sin\psi_1 & \rho'(\psi_1)\sin\psi_1+\rho(\psi_1)\cos\psi_1\\ [\rho''(\psi_1)-\rho(\psi_1)]\cos\psi_1-2\rho'(\psi_1)\sin\psi_1 & [\rho''(\psi_1)-\rho(\psi_1)]\sin\psi_1+2\rho'(\psi_1)\cos\psi_1\end{vmatrix}=0$$

or $\rho^2+2\rho'^2-\rho\rho''=0$, but this contradicts the fact that, for the curve being considered, the curvature is positive. Thus, a root of the denominator $\Phi(\psi)$ must be a simple root.

Moreover, there exists a constant A, which depends on the curve, such that $\left|\frac{d\Phi}{d\psi}\right|>A$ at all points for which $\Phi(\psi)=0$. As a result, we can estimate the number of roots of $\Phi(\psi)$ on Γ: an N exists such that, for arbitrary φ, $\Phi(\psi)$ has no more than N simple roots.

Let us now fix φ and assume that there are exactly N roots, each being included in an interval (γ_{k-1},γ_k), $k=1,2,\dots,N$. Here, $\gamma_0\equiv\gamma_N\ (\mathrm{mod}\,2\pi)$. Taking these considerations into account, we find that the formulas (5) and (7) transform to the form

$$u(r\cos\varphi,\ r\sin\varphi)=\frac{1}{(2\pi)^2}\exp[-ir(a\cos\varphi+b\sin\varphi)]\sum_{k=1}^{N}\int_{\gamma_{k-1}}^{\gamma_k}h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]d\psi, \tag{8}$$

$$u(r\cos\varphi,\ r\sin\varphi)=\frac{1}{(2\pi)^2}\exp[-ir(a\cos\varphi+b\sin\varphi)]\sum_{k=1}^{N}\int_{\gamma_{k-1}}^{\gamma_k}\frac{h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]}{-ir\Phi(\psi)}d\psi. \tag{9}$$

The function (8) has the same order at infinity as each of the terms appearing in the sum. Let us subdivide the range of integration (γ_{k-1},γ_k) into the three intervals $(\gamma_{k-1},\psi_k-\varepsilon)$, $(\psi_k-\varepsilon,\psi_k+\varepsilon)$, $(\psi_k+\varepsilon,\gamma_k)$, where we have selected a sufficiently small ε-neighborhood of the root ψ_k of the denominator $\Phi(\psi)$. Integrating by parts, we have

$$\int_{\gamma_{k-1}}^{\gamma_k}h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]d\psi=\int_{\psi_k-\varepsilon}^{\psi_k+\varepsilon}h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]d\psi+\frac{1}{ir}\left\{\frac{h(\gamma_{k-1})}{\Phi(\gamma_{k-1})}\exp[-ir\rho(\gamma_{k-1})\cos(\gamma_{k-1}-\varphi)]-\frac{h(\gamma_k)}{\Phi(\gamma_k)}\exp[-ir\rho(\gamma_k)\cos(\gamma_k-\varphi)]\right\}+$$

$$+\frac{h(\psi_k+\varepsilon)}{ir\Phi(\psi_k+\varepsilon)}\exp[-ir\rho(\psi_k+\varepsilon)\cos(\psi_k+\varepsilon-\varphi)]-\frac{h(\psi_k-\varepsilon)}{ir\Phi(\psi_k-\varepsilon)}\exp[-ir\rho(\psi_k-\varepsilon)\cos(\psi_k-\varepsilon-\varphi)]+$$

$$+\frac{1}{ir}\left(\int_{\gamma_{k-1}}^{\psi_k-\varepsilon}+\int_{\psi_k+\varepsilon}^{\gamma_k}\right)\frac{h'(\psi)}{\Phi(\psi)}\exp[-ir\rho(\psi)\cos(\psi-\varphi)]d\psi-\frac{1}{ir}\left(\int_{\gamma_{k-1}}^{\psi_k-\varepsilon}+\int_{\psi_k+\varepsilon}^{\gamma_k}\right)\frac{h(\psi)}{\Phi^2(\psi)}\exp[-ir\rho(\psi)\cos(\psi-\varphi)]d\psi. \tag{10}$$

Let us estimate each term in the right member of Eq. (10), except for the terms already integrated and evaluated at the points γ_{k-1} and γ_k, which disappear if we carry through in detail the integration by parts for each term of the sum (8). Let us denote the remaining terms in the sum (10) by J_1, J_2, J_3, J_4, J_5, the order corresponding to that in which they are written. In estimating the terms J_1-J_5, we shall find it convenient to adopt the following notation:

$$|h|_1 = \max_{0\le\psi\le 2\pi} |h(\psi)| + \max_{0\le\psi\le 2\pi} |h'(\psi)|. \tag{11}$$

It is obvious that

$$|J_1| = \left|\int_{\psi_k-\varepsilon}^{\psi_k+\varepsilon} h(\psi)\exp[-ir\rho\cos(\psi-\varphi)]d\psi\right| \le c_1|h|_1\varepsilon. \tag{12}$$

To estimate the remaining terms, we shall employ a representation of the function $\Phi(\psi)$ in a neighborhood of the zero ψ_k:

$$\Phi(\psi) = A_1(\psi-\psi_k) + O((\psi-\psi_k)^2),$$

where $A_1 = [(\rho''-\rho)\cos(\psi-\varphi) - 2\rho'\sin(\psi-\varphi)]_{\psi=\psi_k} \ne 0$.

If we take note of the representation $\Phi^{-1}(\psi) = A_1^{-1}(\psi-\psi_k)^{-1} + B$, where B is a bounded function of ψ, it is possible to show that

$$|J_4| \le c_4|h|_1|\ln\varepsilon| r^{-1}, \tag{13}$$

$$|J_5| \le c_5|h|_1(r\varepsilon)^{-1}. \tag{14}$$

Using $\Phi^{-1}(\psi_k+\varepsilon) = A_1^{-1}\varepsilon^{-1} + B$, we have

$$|J_2| \le c_2|h|_1(r\varepsilon)^{-1}. \tag{15}$$

Combining the estimates (13)-(15) with an analogous estimate of J_3 and the estimate (12), we obtain

$$|u(r\cos\varphi, r\sin\varphi)| \le |h|_1\left(c_1\varepsilon + \frac{D}{r\varepsilon}\right). \tag{16}$$

Minimizing the right member of inequality (16) with respect to ε, we obtain, uniformly with respect to φ, the estimate

$$|u(x,y)| \le \frac{c|h|_1}{\sqrt{r}}. \tag{17}$$

Let us show that the estimate (17) is valid even when Γ is a smooth, nonclosed, planar curve of positive curvature, which is homeomorphic to an interval. As the parameter s, we take the arc length of Γ, reckoning it from one of the ends of the curve being considered, where $0 \le s \le L$. Considering (as was done in the case involving the closed curve) a smooth vector field transverse to the curve, and extending this field onto some σ-neighborhood of Γ, upon assigning in this neighborhood a Fourier transform in accordance with formula (1), where in this formula $g(s)$ is an arbitrary smooth function which vanishes, along with its derivatives, in a neighborhood of the points $s=0$ and $s=L$, we obtain the result

$$u(x,y) = (2\pi)^{-2}\int_0^L g(s)J(s,0)\exp\{i[\alpha(s,0)x + \beta(s,0)y]\}ds, \tag{18}$$

where the equations $\alpha=\alpha(s)$, $\beta=\beta(s)$ define the curve Γ, and $J(s,0)$ is the value of the Jacobian of the transformation to the new variables s, w for $w=0$.

As we did earlier, let us transform the expression (18). In the present case, however, the curve, generally speaking, cannot be defined by an equation $\rho=\rho(\psi)$ in any polar coordinate system on the

$O\alpha\beta$-plane. Nevertheless, it is not hard to see that an arbitrary part of the curve, of sufficiently small length δ, can be represented by an equation $\rho=\rho(\varphi)$ in a local polar coordinate system, in which the pole is placed at a point on the evolute of the curve Γ, corresponding to the mean of the part of the curve considered, and the polar axis is directed parallel to the α-axis. (The quantity δ depends only on the maximum value of $\mathcal{K}$).

Let us take a finite covering by one-dimensional open (with respect to Γ) curvilinear intervals of length not exceeding δ, and let us construct, subject to this covering, a smooth resolution of the identity. If $\{e_k\}_{k=1}^{m}$ is such a partitioning, then $g(s)$ is representable in the form

$$g(s)=\sum_{k=1}^{m} g_k(s), \quad g_k=g(s)e_k(s),$$

and, correspondingly,

$$\hat{u}(s,w)=\sum_{k=1}^{m}\hat{u}_k(s,w), \quad \hat{u}_k(s,w)=g_k(s)\delta(w). \tag{19}$$

In this coordinate representation, there is, corresponding to formula (19), a corresponding representation of $u(x,y)$:

$$u(x,y)=\sum_{k=1}^{m} u_k(x,y).$$

If we can show that $u_k(x,y)$ is of order $(\sqrt{\tau})^{-1}$ at infinity, the same result will apply to $u(x,y)$. However, since the part of Γ, on which the support of $\hat{u}_k(s,w)$ is concentrated, permits the representation $\rho=\rho(\varphi)$, the estimate of $u_k(x,y)$ will not differ essentially from that for the case of the closed curve.

Using Eq. (18) and passing to the local polar coordinates, we can transform $u_k(x,y)$ to the form

$$u_k(\tau\cos\varphi, \tau\sin\varphi)=\frac{1}{(2\pi)^2}\exp[-i\tau(a_k\cos\varphi+b_k\sin\varphi)]\int_{\alpha_k}^{\beta_k} h_k(\psi)\exp[-i\tau\rho\cos(\psi-\varphi)]d\psi, \tag{20}$$

where a_k, b_k are the coordinates of the pole of the local polar system of coordinates in the $O\alpha\beta$-system of coordinates, and $h_k(\psi)=g_k(\psi)J(\psi,0)$.

Reasoning as we did for the case of the closed curve, we obtain the estimate

$$|u_k(x,y)|\leq\frac{c_k|h|_1}{\sqrt{\tau}}.$$

uniformly with respect to φ.

We have, finally,

$$|u(x,y)|\leq\frac{c|h(s)|_1}{\sqrt{\tau}}, \tag{21}$$

where

$$h(s)=\sum_{k=1}^{m} h_k(s), \quad h_k=g(s)J(s,0)e_k(s). \tag{22}$$

The result obtained may be formulated as follows.

Theorem. Let Γ be a smooth curve, homeomorphic to a circle or line segment, whose curvature is everywhere positive. Let the Fourier transform of the function $u(x,y)$ be a generalized function of the form (1), the support for which is Γ. Then, as $\tau\to\infty$, $u(x,y)$ permits the estimate

$$|u(x,y)|\leq\frac{c|h(s)|_1}{\sqrt{\tau}},$$

where $h(s)=g(s)J(s,0)$.

Remark. As we did earlier, let us consider a smooth, planar curve Γ, homeomorphic to a circle or line segment, which is represented by an equation $\rho=\rho(\varphi)$. Let us assume that the curvature is zero at only one point of the curve, the point being a point of inflection. Let the slope of the curve at this point be different from zero. It is then not difficult to show that as $\tau \to \infty$, the following estimate holds:

$$|u(x,y)| \leqslant \frac{c|h|_1}{\sqrt[3]{\tau}}. \tag{23}$$

Indeed, in this case,

$$u(\tau\cos\varphi, \tau\sin\varphi) = \frac{1}{(2\pi)^2}\int_0^{2\pi} h(\psi)\exp[-i\tau\rho\cos(\psi-\varphi)]\,d\psi, \tag{24}$$

or

$$u(\tau\cos\varphi, \tau\sin\varphi) = \frac{1}{(2\pi)^2}\int \frac{h(\psi)}{-i\tau\Phi(\psi)}\{\exp[-i\tau\rho\cos(\psi-\varphi)]\}'\,d\psi. \tag{25}$$

Assuming the curvature to be zero at the point ψ_1, we obtain, based on what was proven earlier, the result that all possible roots of the denominator $\Phi(\psi)$ of the integrand in Eq. (25), except for ψ_1, are simple, and that ψ_1 is a two-fold root.

If we divide the interval of integration in formulas (24) and (25) into the two subintervals $(0,\gamma)$ and $(\gamma, 2\pi)$, the first of these containing the root ψ_1, and the other all the remaining roots, we obtain

$$u(\tau\cos\varphi, \tau\sin\varphi) = \frac{1}{(2\pi)^2}\int_0^{\gamma} h(\psi)\exp[-i\tau\rho\cos(\psi-\varphi)]\,d\psi + \frac{1}{(2\pi)^2}\int_{\gamma}^{2\pi} h(\psi)\exp[-i\tau\rho\cos(\psi-\varphi)]\,d\psi, \tag{26}$$

$$u(\tau\cos\varphi, \tau\sin\varphi) = \frac{1}{(2\pi)^2}\int_0^{\gamma} \frac{h(\psi)}{-i\tau\Phi(\psi)}\{\exp[-i\tau\rho\cos(\psi-\varphi)]\}'\,d\psi + \frac{1}{(2\pi)^2}\int_{\gamma}^{2\pi} \frac{h(\psi)}{-i\tau\Phi(\psi)}\{\exp[-i\tau\rho\cos(\psi-\varphi)]\}'\,d\psi. \tag{27}$$

It is sufficient to determine the order at infinity of the first term in the right member of Eq. (26), since the order of the second term is known and is equal to $(\sqrt{\tau})^{-1}$. Using a representation of the form

$$\Phi(\psi) = A_2(\psi-\psi_1)^2 + O((\psi-\psi_1)^3),$$

where

$$A_2 = [(\rho'''-3\rho')\cos(\psi-\varphi) - (3\rho''-\rho)\sin(\psi-\varphi)]_{\psi=\psi_1} \neq 0,$$

it is easy to establish, using the same scheme as in the case of the closed curve, the following estimates:

$$|J_1| \leqslant C_1|h|_1\varepsilon, \tag{28}$$

$$|J_2| \leqslant C_2|h|_1\tau^{-1}\varepsilon^{-2}, \tag{29}$$

$$|J_4| \leqslant C_4|h|_1\tau^{-1}\varepsilon^{-1}, \tag{30}$$

$$|J_5| \leqslant C_5|h|_1\tau^{-1}\varepsilon^{-3}. \tag{31}$$

It follows from the estimates (28)-(31) that the first term of the function (26) does not exceed

$$\left(C_1\varepsilon + \frac{\mathcal{D}}{\tau\varepsilon^2}\right)|h|_1.$$

We then obtain the estimate (23) upon minimizing the last expression and taking into account the order at infinity of the second term of the function (26).

As estimate of the type (23) is also valid for the case in which the curve has a finite number of points of inflection.

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LITERATURE CITED

1. I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. I: Properties and Operations, Academic Press, New York (1964).
2. G. M. Goluzin, Geometric Theory of Functions of a Complex Variable, Am. Math. Soc. Publ., New York (1969).