Elastic multilayer: Stress analytical solution  
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1. Problem definition and boundary conditions

Multi-layer elastic isotropic material subjected to a uniform temperature delta (, with each layer *i* having a distinct coefficient of thermal expansion . For a battery analogy, this would be a multi-layer elastic isotropic material subjected to a uniform concentration delta (, with each layer *i* having a distinct chemical expansion coefficient . The applied loading is a thermal strain for the baseline problem or concentration-induced strain for the battery analogy:

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| *Geometry and coefficients for each layer. Material coefficients: Young modulus , Poisson ratio: thermal expansion [K-1], chemical expansion [mol-1.m3]. Geometry: layer thickness: e [m].* |
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This problem suits well a test case for battery mechanical modeling as not only loading is battery-relevant (concentration-induced strain), but it also checks material property discontinuities.

Two set of boundary conditions are considered:

* The first one enforces a unique point in the middle of the bottom plane to be fixed (and with no rotation allowed to guarantee solution unicity). In such a case, due to the applied strain incompatibility and the interface continuity, the material will bend and develop a curvature.
* The second one constrains all the bottom plane for the y displacement and constrains also the top plane but for all the rotations (i.e., the top plane is free to expand along the y direction but with a constant thickness everywhere).

The absence of flexion in the second case implies strain (and thus stress) induced by the material flexion are ignored in the analytical solution below.

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| *Boundary conditions (two cases are considered).* |

1. Stress tensor and Hooke’s law

Problem is unidimensional, along y-axis. Layers being isotropic . Layers being arranged in serial as there is no restrain along y-axis (only bottom location y=0 is fixed). Therefore, stress tensor is:

Mechanical behavior follows Hooke’s law (is the identity tensor):

And then:

1. Strain decomposition

Total strain is the sum of applied strain and elastic strain . Elastic strain itself is the sum of uniform elastic strain (when flexion if not considered) and elastic strain due to the flexion of the multilayer :

If the material boundary conditions are set so that multilayer cannot bend (second case), then . Strain induced by the multi-layer flexion is determined by the position of the neutral axis , defined as , and the curvature radius :

According to elastic superposition principle for small deformations, the load effects caused by two or more loadings are the sum of the load effects caused by each loading *separately*. Therefore and can be calculated independently.

1. Elastic strain calculation

We have three independent unknows: , and . We need three independent equations:

* 1. Elastic strain non due to flexion

We apply the 1st principle of static along the x direction (cf. eq. 8), i.e., sum of external forces applies on facets of normal [1,0,0] (similar expression for z [1])

Problem is adimensional: and therefore eq. 10 can be simplified:

Let’s introduce elastic strain (cf. eq. 4):

There is no flexion . Due to interface continuity, , thus ) is constant per layer (since is also constant per layer), therefore:

We can combine equations 13 and 5 to form a system of two equations and inverse it to find and (see first reference). Alternatively, we can avoid such complexity by rewriting the total strain as follows:

Then replacing using strain decomposition:

And replacing for the elastic strain part:

We just make appear the zero term of equation 13. Therefore:

Knowing the total strain, we can deduce the elastic strain (cf. eq. 5):

This approach provides interpretable result: the total deformation is a rule of mixture of the applied strains weighted with the product of the layer Young modulus and the layer thickness. That is the layers which are both stiff and thick control the total strain of the material.

* 1. Elastic strain due to flexion

*Neutral axis* . We re-apply the 1st principle of static, this time with the two elastic strains:

is constant per layer and can then exit the integral. We also replace with its expression (cf. eq. 6):

Left term is 0 (cf. eq. 13). Curvature radius is constant and exit the integral. We multiply by r both sides to remove the term and keep only as unknown:

After calculating the integral, we get:

And we deduce :

*Curvature radius r*. We apply the 2nd principle of static, along direction x:

After replacing stress with strain and effective Young modulus:

After calculating the integral, we get:

And we deduce :

1. Stress calculation

Knowing both elastic strains, the stress are calculated as:

1. Example

Bi-layer with e[m] = [10e-6 20e-6], E[Pa] = [10e9 20e9], = [0.3 0.3], [K-1] = [10e-6 20e-6], [K] = 10.

Chart, line chart

Description automatically generated

References

This problem has been investigated for electrolyzer/fuel cells:

- Detailed derivation of the equations (Usseglio thesis): <https://tel.archives-ouvertes.fr/tel-01223428>, p229-240. Present document is mostly a shorten translation of it.

- J. Laurencin, V. Roche, C. Jaboutian, I. Kieffer, J. Mougin, M.C. Steil, Ni-8YSZ cermet re-oxidation of anode supported solid oxide fuel cell: From kinetics measurements to mechanical damage prediction, International Journal of Hydrogen Energy, 2012, 37 (17), pp. 12557-12573, <https://doi.org/10.1016/j.ijhydene.2012.06.019>

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