Continuous Time Markov Chains: Transition Probabilities and Kolmogorov Equations

1. The weather in a certain city can be in one of 3 states: sunny (1), cloudy (2), or rainy (3). Suppose the weather evolves over time according to a continuous time Markov chain with the following transition rate matrix. Rates are all per day (24 hours). (Diagonals left blank on purpose.)

$$\mathbf{Q} = \begin{bmatrix} & 0.25 & 0 \\ 0.8 & & 0.4 \\ 2.0 & 1.5 & \end{bmatrix}$$

a. Given that it is cloudy now, find the probability that it is rainy next.
b. Given that it is rainy now, *approximate* the probability that it is sunny 30 minutes from now. Justify your approximation without using software or solving any equations.
c. Given that it is sunny now, use software to compute the probability for each type of weather at this time in 2 days.
d. Given that it is cloudy now, use software to compute the probability for each type of weather at this time in 2 days.

2. (Yule process.) Every individual in a population gives birth to a new individual independently at Exponential rate λ. Let X_t denote the number of individuals in the population at time t, assuming no deaths. Assume that $X_0 = 1$; we are interested in the distribution of X_t. That is, we want to find $p_t(1, j) = \mathrm{P}(X_t = j | X_0 = 1)$ for $j = 1, 2, \ldots$.

a. Write out the Kolmogorov forward equations for $p_t'(1, j)$.
b. Check that

$$(1 - e^{-\lambda t})^{j-1} e^{-\lambda t}, \quad j = 1, 2, 3, \ldots$$

is the solution to the Kolmogorov forward equations
c. Identify by name the distribution of X_t (given $X_0 = 1$). Be sure to identify relevant parameters.
d. Provide an intuitive explanation for the previous result.
e. As a concrete example, make a table of the distribution of X_t when $\lambda = 0.1$ and $t = 5$.