Random Variables: Joint, Conditional, and Marginal Distributions

The latest series of collectible Lego Minifigures contains 3 different Minifigure prizes (labeled 1, 2, 3). Each package contains a single unknown prize. Suppose we only buy 3 packages and we consider as our sample space outcome the results of just these 3 packages (prize in package 1, prize in package 2, prize in package 3). For example, 323 (or (3, 2, 3)) represents prize 3 in the first package, prize 2 in the second package, prize 3 in the third package. Let X be the number of distinct prizes obtained in these 3 packages. Let Y be the number of these 3 packages that contain prize 1. Suppose that each package is equally likely to contain any of the 3 prizes, regardless of the contents of other packages; let P denote the corresponding probability measure.

It can be shown that the joint distribution of X and Y can be represented by the following table.

$\mathrm{P}(X = x, Y = y)$

	y			
x	0	1	2	3
1	2/27	0	0	1/27
2	6/27	6/27	6/27	0
3	0	6/27	0	0

1. Briefly explain why there are 27 possible outcomes.
2. Show that $\mathrm{P}(X = 1, Y = 0) = 2/27$ by listing the outcomes that comprise the event $\{X = 1, Y = 0\}$.
3. Show that $\mathrm{P}(X = 1, Y = 3) = 1/27$ by listing the outcomes that comprise the event $\{X = 1, Y = 3\}$.
4. Show that $\mathrm{P}(X = 2, Y = 0) = 6/27$ by listing the outcomes that comprise the event $\{X = 2, Y = 0\}$.
5. Make a table representing the marginal distribution of X and compute $\mathrm{E}(X)$.
6. Make a table representing the marginal distribution of Y and compute $\mathrm{E}(Y)$.
7. Find the conditional distribution of Y given $X = x$ for each possible value of x.
8. Make a table representing the distribution of $\mathrm{E}(Y|X)$.
9. Find the conditional distribution of X given $Y = y$ for each possible value of y.
10. Make a table representing the distribution of $\mathrm{E}(X|Y)$.
11. Describe three methods for how you could use physical objects (e.g., cards, dice, spinners) to simulate an (X, Y) pair with the joint distribution given by the table above.
    a. Method 1: simulate outcomes from the probability space (i.e., prizes in the packages)
    b. Method 2: simulate an (X, Y) pair directly from the joint distribution (without simulating outcomes from the probability space)
    c. Method 3: simulate an (X, Y) pair by first simulating X from directly from its marginal distribution (without simulating outcomes from the probability space).

12. Suppose you have simulated many (X, Y) pairs. Explain how you could use the simulation results to approximate each of the following. You should not do any of the calculations; rather, explain in words how you would use the simulation results and simple operations like counting and averaging.
    a. $\mathrm{P}(X = 3)$
    b. the marginal distribution of X
    c. $\mathrm{E}(X)$
    d. $\mathrm{Var}(X)$
    e. $\mathrm{P}(X = 2, Y = 1)$
    f. $\mathrm{E}(XY)$
    g. $\mathrm{Cov}(X, Y)$
    h. $\mathrm{P}(X = 1|Y = 0)$
    i. the conditional distribution of X given $Y = 0$
    j. $\mathrm{E}(X|Y = 0)$
    k. $\mathrm{P}(Y = 0|X = 1)$
    l. the conditional distribution of Y given $X = 1$
    m. $\mathrm{E}(Y|X = 1)$