Homework 2: Bayesian Decision Theory

(DHS) Chapter 2, p65: **2, 6, 37,** and the following.

Problem 4

Two one-dimensional distributions are uniform in [0,2] for ω_1 and [1,4] for ω_2, and P_1=P_2=0.5.

1. Find the Bayes boundary for minimum error, and compute the probability of error.
2. Find the Bayes decision boundary if $\lambda_{11}= \lambda_{22}=0$ and $\lambda_{12}= 2\lambda_{21}$.

Problem 5

For a two-class recognition problem with salmon (ω = 1) and sea bass (ω = 2), suppose we have two features **x** = (x1, x2) and the two class-conditional densities, $p(x|\omega = 1)$ and $p(x| \omega = 2)$, are 2D Gaussian distributions centered at points (4, 16) and (16, 4) respectively with the same covariance matrix $\mathbf{\Sigma} = 4\mathbf{I}$ (with **I** is the identity matrix). Suppose the priors are $P(\omega = 1) = 0.6$ and $P(\omega = 2) = 0.4$.

1. Suppose we use a Bayes decision rule, write the two discriminant functions $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$.
2. Derive the equation for the decision boundary $g_1(\mathbf{x}) = g_2(\mathbf{x})$. Draw the boundary on the feature space (the 2D plane).

Section 2.2

2. Suppose two equally probable one-dimensional densities are of the form $p(x|\omega_i) \propto e^{-|x-a_i|/b_i}$ for $i = 1, 2$ and $0 < b_i$.

(a) Write an analytic expression for each density, that is, normalize each function for arbitrary a_i and positive b_i.

(b) Calculate the likelihood ratio as a function of your four variables.

(c) Sketch a graph of the likelihood ratio $p(x|\omega_1)/p(x|\omega_2)$ for the case $a_1 = 0$, $b_1 = 1$, $a_2 = 1$ and $b_2 = 2$.

6. Consider the Neyman-Pearson criterion for two univariate normal distributions: $p(x|\omega_i) \sim N(\mu_i, \sigma_i^2)$ and $P(\omega_i) = 1/2$ for $i = 1, 2$. Assume a zero-one error loss, and for convenience let $\mu_2 > \mu_1$.

(a) Suppose the maximum acceptable error rate for classifying a pattern that is actually in ω_1 as if it were in ω_2 is E_1. Determine the single-point decision boundary in terms of the variables given.

(b) For this boundary, what is the error rate for classifying ω_2 as ω_1?

(c) What is the overall error rate under zero-one loss?

(d) Apply your results to the specific case $p(x|\omega_1) \sim N(-1, 1)$ and $p(x|\omega_2) \sim N(1, 1)$ and $E_1 = 0.05$.

(e) Compare your result to the Bayes error rate (i.e., without the Neyman-Pearson conditions).

(c) Integrate explicitly the term ...

37. Consider a two-category classification problem in two dimensions with

$$p(\mathbf{x}|\omega_1) \sim N(\mathbf{0}, \mathbf{I}),\ p(\mathbf{x}|\omega_2) \sim N\left(\begin{pmatrix}1\\1\end{pmatrix}, \mathbf{I}\right), \text{ and } P(\omega_1) = P(\omega_2) = 1/2.$$

(a) Calculate the Bayes decision boundary.

(b) Calculate the Bhattacharyya error bound.

(c) Repeat the above for the same prior probabilities, but

$$p(\mathbf{x}|\omega_1) \sim N\left(\mathbf{0}, \begin{pmatrix}2 & .5\\ .5 & 2\end{pmatrix}\right) \text{ and } p(\mathbf{x}|\omega_2) \sim N\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}5 & 4\\ 4 & 5\end{pmatrix}\right).$$