# Numerical approximations to the derivative - Finite difference

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

In physics problems, we approximate derivatives by finite difference approximations

$$\left.\frac{dy}{dt}\right|_{t+\Delta t/2} \cong \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

where Δt is small but finite. This is called a finite difference approximation to the derivative

If $y(t) = sin(t)$ then the derivative is $\frac{\mathrm{d}y}{dt} = cos(t)$. Write a python function that approximates the derivative of $y(t)$ for $0 \leq t \leq 2\pi$ using the above finite difference approximation. Make plots of your approximate derivative for $\Delta t = 1, \frac{1}{2}, \frac{1}{3}$ and compare with the actual derivative $\frac{dy}{dt} = cos(t)$.

# Differential equations

Scientists use differential equations to describe how things in the physical universe change. For example, if the number of people infected with COVID is $y(t)$ and $\alpha$ is the number of people that, on average, an infected person passes the disease on to, then an equation that describes the rate at which new people become infected can be written as

$$\frac{dy}{dt} = \alpha y$$

Starting with $y(0) = 1$ infected person, the solution is $y(t) = e^{\alpha t}$.
Often the differential equations that represent a physical system are very difficult to solve using elementary functions and physicist resort to numerical approximations. One way to approximate the above differential equation is to use finite difference approximations for the derivative. One possible finite difference equation for the above differential equation is

$$\frac{y_{n+1} - y_n}{\Delta t} = \alpha y_n + O(\Delta t) \ (1^{st} \ order \ accurate)$$
$$y(0) = y_0 = 1$$

where $O(\Delta t)$ means that the error is on the order of (or scales with) $\Delta t$. Rearranging to solve for $y_{n+1}$ gives

$$y_{n+1} = y_n + \alpha y_n \Delta \mathrm{t} = (1 + \alpha \Delta t) y_n \ (n = 0,1,2, \dots)$$

For example, given that $y_0 = 1$, the next value in the sequence is $y_1 = (1 + \alpha\Delta t)y_0 = (1 + \alpha\Delta t)$. The next value is $y_2 = (1 + \alpha\Delta t)y_1 = (1 + \alpha\Delta t)^2$. The ordered pairs $(t_n, y_n)$ for $n = 1,2,3, \dots, N$ constitute a numerical solution to the equation.

A better way to approximate the above differential equation with a finite difference equation is

$$\frac{y_{n+1} - y_n}{\Delta t} = \alpha \frac{1}{2}(y_n + y_{n+1}) + O(\Delta t^2) \ \ (2^{nd} \ order \ accurate)$$

Notice that in writing the RHS as $\frac{1}{2}(y_n + y_{n+1})$ we are approximating $y_{n+\frac{1}{2}} \cong \frac{1}{2}(y_n + y_{n+1})$ so that both the derivative and the RHS are now approximated at $t_n + \frac{1}{2}$. This is known as central differencing and results in an error term that is $O(\Delta t^2)$. This means that the error terms scales with $\Delta t^2$ which is very attractive since reducing $\Delta t$ by a factor of 2 results in a four-fold decrease in the error.

Problem: Write a python program to solve the difference equation with $y_0 = 1$ initial infection using both the first and second-order accurate finite difference approximations. Try $\alpha = 1/day$, $\alpha = 2/day$. What happens after 5 days? What happens when $\alpha < 0$ and what does this physically mean? Using the exact solution, $y(t) = e^{\alpha t}$, make a plot of the finite difference error after 5 days as a function of $\Delta$t ($\alpha = 1$) for both the first and second order accurate finite difference solutions.