Indices of Refraction

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I. THEORETICAL BACKGROUND

A. Optical Path Length

The speed of light varies depending on the medium through which light propagates; in a vacuum, this speed is referred to as c, but in all other media this quantity is v. The index of refraction, n, relates these two quantities such that

$$n = \frac{c}{v}. \tag{1}$$

For light that travels through a material of thickness l with index of refraction n, the optical path length τ can be described by

$$\tau = nl. \tag{2}$$

In our experiment, light travels through multiple materials, so the total optical path length is the sum of the individual optical path lengths. Thus,

$$\tau = \sum_{i=1}^{M} n_i l_i \tag{3}$$

computes the total optical path length. Knowing this value, we can improve the approximation made in Lab A-II. Previously, we were told that

$$\lambda N = 2\Delta d, \tag{4}$$

where Δd was the physical path length. We used this value rather than the optical path length, because this approximation made the calculation easier. Substituting the total optical path length for the physical path length leaves

$$\lambda N = 2\Delta\tau. \tag{5}$$

B. Fringe Number Dependence on Angle of Rotation

Next, consider a transparent acrylic sheet with thickness t and index of refraction n. An image showing how an incident beam of light interacts with this sheet appears in Fig. 1 from the lab manual[1]. Here, the sheet has thickness t, and internal physical path length l. The value d denotes a physical length within the sheet.

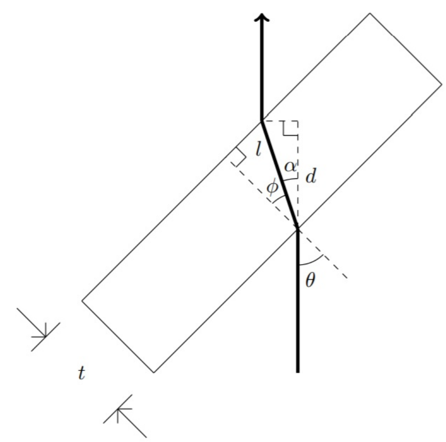

FIG. 1: The acrylic sheet is positioned such that incident light — denoted by the solid black ray — strikes the face of the sheet to form angle θ. This angle was measured from the normal of the surface, so a ray of light perpendicular to the face of the sheet would have $\theta = 0$. This figure depicts the cross-section of the sheet as seen from above in the experimental setup. The bend in the light ray is caused by refraction.

We need to relate l and d to the change in optical path length $\Delta\tau$. Because τ is a function of θ, we can write it as

$$\Delta\tau = \tau(\theta_1) - \tau(\theta_0), \tag{6}$$

with θ_0 being 0°. Assuming the total distance light travels is X, we can replace replace each $\tau(\theta)$ with Eq. 3 to produce

$$\Delta\tau = [nl + n_{air}(X - d)] - [nt + n_{air}(X - t)], \tag{7}$$

where $X - t$ and $X - d$ are the distances light travels through air for θ_1 and θ_0, respectfully. We also made an approximation that n_{air} was equal to 1; applying this and simplifying leaves

$$\Delta\tau = n(l - t) + t - d. \tag{8}$$

Putting l and d in terms of known variables — also knowing that $\alpha + \phi = \theta$ — and using Snell's law, we see that

$$cos(\phi) = \frac{t}{l} \quad \text{and} \quad cos(\alpha) = \frac{d}{l}. \tag{9}$$

Solving for l and d, plugging them into Eq. 8, and performing a Taylor expansion around $\theta = 0$ to approximate the small angle, we have

$$\Delta\tau = \frac{(n-1)t\theta^2}{2n}. \tag{10}$$

Substituting this equation into Eq. 5, we are left with

$$N\lambda = \frac{(n-1)t\theta^2}{n}. \tag{11}$$

Since our linear fitter only accounts for uncertainty in the dependent variable, we switched variables to allow the largest source of uncertainty — which comes from $t\theta^2$ — to affect the fit so as to minimize the loss of uncertainty. Additionally, n in this context is $n_{acrylic}$. After accounting for these factors, our measurement model is

$$\frac{1}{t\theta^2} = \left(1 - \frac{1}{n_{acrylic}}\right)\frac{1}{N\lambda}. \tag{12}$$

C. Fringe Number Dependence on Air Pressure Within a Medium

In the second part of the experiment, the number of fringes varied again due to a changing optical path length. However, the method of manipulating the optical path length was different; the angle of the acrylic sheet was kept constant, and light passed through a vacuum chamber with adjustable pressure.

Understanding pressure dependence of the index of refraction starts at the microscopic level. A gas filled with atoms will have an electric field $\vec{E}$ present that polarizes said atoms. This means that each atom gains a dipole moment $\vec{p}$, pointing in the direction of the field. For fields that are weak enough, there is a linear relationship between the strength of the field and the magnitude of the dipole moment. This relationship can be mathematically realized using the constant α, such that

$$\vec{p} = \alpha\vec{E}, \tag{13}$$

where α is the atomic polarizability. Additionally, we can relate the polarization $\vec{P}$ (dipole moment per unit volume of gas) for linear dielectric materials by

$$\vec{P} = (\varepsilon_r - 1)\varepsilon_0\vec{E}, \tag{14}$$

where ε_0 is the permittivity of free space, and ε_r is the relative permittivity of the gas. Using the relation $\vec{P} = \vec{p}/V$, with V being the volume of gas, we arrive at

$$\frac{1}{V}\frac{\alpha}{\varepsilon_0} = \mathcal{N}\frac{\alpha}{\varepsilon_0} = (\varepsilon_r - 1). \tag{15}$$

Here, $\mathcal{N}$ is the number density of atoms in the gas, or $1/V$. Next, we will incorporate the formulae for expressing the speed of light. Through a vacuum, $c = 1/\sqrt{\varepsilon_0\mu_0}$; through other media, $v = 1/\sqrt{\varepsilon\mu}$. Combining these equations with Eq. 1, noting that $\varepsilon_r = \varepsilon/\varepsilon_0$, and assuming $\mu/\mu_0 = 1$, we are left with

$$n = \sqrt{\varepsilon_r} \tag{16}$$

after solving for n.

Squaring Eq. 16 and subtracting 1, we have $n^2 - 1 = \varepsilon_r - 1$, which appears in Eq. 15. Applying the ideal gas law $PV = n_gRT$ yields

$$n^2 - 1 = (\varepsilon_r - 1) = \frac{\alpha}{V\varepsilon_0} = \frac{P\alpha}{n_gRT\varepsilon_0}, \tag{17}$$

where n_g is the number of moles of gas, P is pressure, and T is temperature. Further simplifying leaves

$$n^2 - 1 = \frac{P\alpha}{T\varepsilon_0k_B}, \tag{18}$$

where k_B is the Boltzmann constant. This equation seems to indicate a nonlinear relationship between n and P; however, our experiment found a linear relationship between the two variables. Eq. 18 can be modified to predict this relationship. Taylor expanding $n^2 - 1$ around $n = 1$ produces a series of terms with varying degrees to n. The terms with degrees greater than 1 can be ignored, because these higher order terms are small enough that their effects on the sum are negligible. Completing the expansion and solving for n gives

$$n = \frac{P\alpha}{2T\varepsilon_0k_B} + 1. \tag{19}$$

Taking the partial derivative of n with respect to P, we have

$$\frac{\partial n}{\partial P} = \frac{\alpha}{2T\varepsilon_0k_B} = \beta. \tag{20}$$

The constant β will assist in future calculations; this value is constant in our experiment, because pressure was the only value that varied. It did so linearly with respect to n. Thus, taking the partial derivative of n with respect to P yields a value that should remain constant for other parts of this experiment. Using this constant and knowing that n in this context is n_{air}, we can simplify Eq. 19 to

$$n_{air} = \beta P + 1. \tag{21}$$

Using Eq. 2 and Eq. 5, We are now able to describe the relationship between the number of fringes and the change in pressure. From Eq. 21, we can extrapolate that

$$\Delta n_{air} = \beta\Delta P. \tag{22}$$

Similarly, Eq. 2 can be expressed as

$$\Delta\tau = \Delta n_{air}l. \tag{23}$$

Substituting Eq. 22 into Eq. 23 produces $\Delta\tau = \beta\Delta Pl$, which can then be plugged into Eq. 5 to end up with

$$\lambda N = 2\beta\Delta Pl. \tag{24}$$

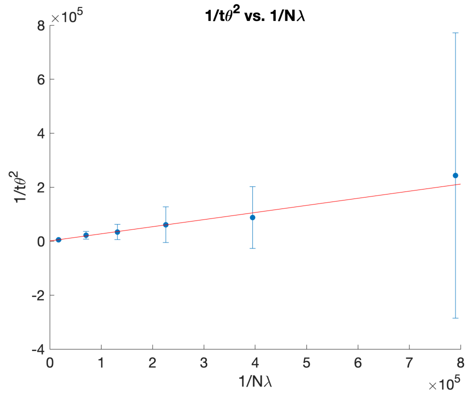

FIG. 2: Plot of $1/t\theta^2$ vs. $1/N\lambda$, with a slope of $(1 - 1/n_{acrylic}) = 0.3 \pm 0.1$.

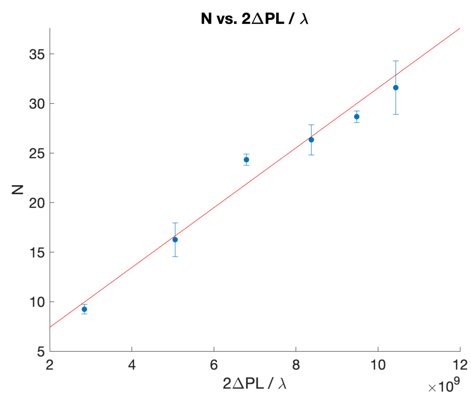

FIG. 3: Plot of N vs. $2\Delta PL/\lambda$, with a slope of $\beta = (3.0 \pm 0.2) \times 10^{-6}$ Pa^{-1}.

II. RESULTS

Fig. 3 has a slope of $\beta = (3.0 \pm 0.2) \times 10^{-9}$ Pa^{-1}. When plugged into Eq. 21, $n_{air} = 1.00030 \pm 0.00002$.

Fitted from Eq. 12, Fig. 2 has a slope of $(1 - 1/n_{acrylic}) = 0.3 \pm 0.1$. This yields yields $n_{acrylic} = 1.4 \pm 0.3$.

III. DISCUSSION AND CONCLUSIONS

Our value of $n_{acrylic} = 1.4 \pm 0.3$ from Fig. 2 agrees with the nominal index of refraction for acrylic, which is $n_{acrylic} = 1.495$. From Fig. 3 we found $\beta = 3.0 \pm 0.2$ Pa^{-1}. This, when plugged into Eq. 21, yields $n_{air} = 1.00030 \pm 0.00002$, which also agrees with the accepted value of $n_{air} = 1.0003$.

To fit the data, we used a weighted linear fitter which only accounts for uncertainty in the dependent variable. For Fig. 3, this was suitable because our uncertainty in N was not significantly less than the uncertainty in $2\Delta PL/\lambda$. As shown in the methods section, L and λ were given values with fairly small uncertainties, and ΔP similarly had a small variance. For Fig. 2, however, the largest source of our uncertainty was originally the independent variable, $t\theta^2$. Therefore, we rearranged our linear model, as described in Section I B, to switch the variables. This produced a weighted linear fit that more accurately represents the data.

The assumption made in Lab A-II, that $\Delta\tau = \Delta d$, is a valid one. Since the index of refraction of air is approximately 1, $\tau = nl \approx l$. Therefore, changing l by Δd would change τ by approximately $\Delta\tau$.

A typical Snell's Law experiment could not work in this situation to measure the indices of refraction of acrylic and air. The acrylic used is simply too thin to be able to accurately measure the angle of refraction inside of it, and the difference in path length through the vacuum tube would be too small to measure with any precision.

IV. ACKNOWLEDGMENTS

The experiment was performed by C.S. and N.M., and C.S. recorded the data. C.S. wrote both the Results and the Discussions and Conclusions sections, and N.M. wrote the Theoretical Background section. LinFit.m code was provided by Ben Levy.

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[1] Lab B-I Manual `https://sakai.unc.edu/access/content/group/41081125-b301-4ad9-8173-970601749ad6/Lab\%20Manuals/Lab\%20B-I\%3A\%20Indices\%20of\%20Refraction/Lab\%20B-I\%20Indices\%20of\%20Refraction.pdf`