The Permittivity of Free Space

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In this lab, I examined the behavior of the time constant τ for an RC circuit when the capacitance was manipulated. I proved that voltage within the circuit follows an exponential decay curve when the capacitor discharges. Additionally, I found the value of ϵ_0, the permittivity of free space, to be $(8.69 \pm 0.47) \times 10^{-12}$ F/m, which agreed with the nominal value of 8.85×10^{-12} F/m.

I. INTRODUCTION

Plate capacitors are able to be charged and discharged, and the speed at which this occurs depends on a few properties of the circuit they inhabit. One of these properties is the type of material between the plates; for this experiment, this material was the air around us, or free space. Measuring how the capacitor behaves due to various adjustments allows us to calculate how well the capacitor can store charge for the given material between the plates. This value is known as ϵ_0, the permittivity of free space. To accomplish this, I fit curves to curves describing the voltage drop across the capacitor in a circuit with a capacitor and resistor in series, and plotted extrapolated values to find ϵ_0. This permittivity value allows us to determine the permittivity of other materials that may be placed between the plates, known as dielectrics.

II. THEORETICAL BACKGROUND

A. Plate Capacitor

First, we need to derive the equation for the capacitance C of a parallel plate capacitor. Starting with Gauss' law, we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{enc}}{\epsilon_0}, \tag{1}$$

where the surface area integral on the left hand side determines electric flux through a surface A, and Q_{enc} is the charge enclosed on that surface. Evaluating the left hand side for a single capacitor plate yields

$$EA = \frac{Q_{enc}}{\epsilon_0}. \tag{2}$$

Next, one can substitute σA for Q_{enc}. Here, σ is the area charge density. Additionally, it is necessary to multiply the area on the left hand side by 2, because there are two plates. Doing so and solving for E results in

$$E = \frac{\sigma}{2\epsilon_0}. \tag{3}$$

This describes the electric field for a single plate. Because there are two plates — one of positive charge, and one of negative charge — the resulting electric field from each should point in the same direction between the plates, and opposite directions outside of the plates. Thus, they add together in the area between the plates, and should effectively cancel outside the plates. We want the E-field between the plates, so multiplying E by 2 to account for this leaves

$$E = \frac{\sigma}{\epsilon_0}. \tag{4}$$

Next, consider the equation relating E to voltage V:

$$V = -\int E \cdot \mathrm{d}\vec{s}. \tag{5}$$

Substituting Eq. 4 into Eq. 5, substituting Q/A for σ, and evaluating the integral gives

$$V = \frac{Qd}{\epsilon_0 A}. \tag{6}$$

From here, one may use the relation that capacitance $C = Q/V$ to arrive at the expression

$$C_{plate} = \epsilon_0 \frac{A}{d}. \tag{7}$$

B. Time Constant

Starting with the equation describing the time constant τ, we have

$$\tau = RC_{total} \tag{8}$$

Here, R and C_{total} are the total resistance and total capacitance of the circuit, respectively. The value C_{total} is equal the sum of C_{plate} (described by Eq. 7) and C_{stray}, the stray capacitance. Making these substitutions into Eq. 8 produces

$$\tau = \frac{\epsilon_0 AR}{d} + RC_{stray}. \tag{9}$$

It will be helpful later on to have the full equation for C_{total}, so to find this, simply divide Eq. 9 by R to get

$$C_{total} = \epsilon_0 \frac{A}{d} + C_{stray}. \tag{10}$$

It is important to note that V was the value measured in this experiment, and C_{total} was found using the following procedure.

C. Voltage Curve

Because we recorded voltage $V(t)$ as a function of time, we need a way to relate this function to capacitance. Starting with

$$V(t) = V_0 e^{-t/\tau}, \tag{11}$$

we can relate $V(t)$ to τ. By fitting an exponential curve of the form $y = ae^{bt}$ to the data for $V(t)$, we can find τ by dividing -1 by b. From there, C is found using the relation $\tau = RC$. This is our measured capacitance C_{total}.

III. METHODS

A. Procedure

In this experiment, a function generator was placed in series with a resistor and another setup, which comprised an adjustable spacing parallel plate capacitor in parallel with a digital oscilloscope capable of saving data to a USB flash drive. This created an RC circuit, and the setup is depicted in Fig. 1, borrowed from the lab manual [1]. The plate capacitor had a knob that allowed us to adjust the distance between the plates. Varying the distance changed C_{plate}, in turn affecting our measured C_{total}.

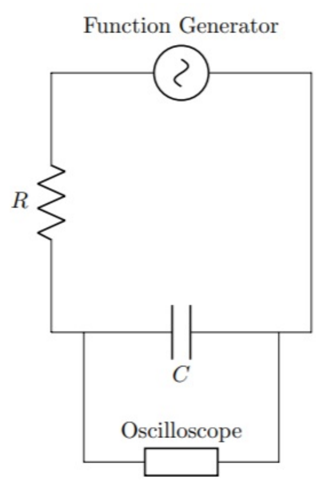

FIG. 1: RC circuit diagram using the oscilloscope to measure the voltage across the capacitor as a function of time.

The function generator needed to be adjusted to certain settings. It was set to output a square wave with "duty cycle" turned off. A frequency for the square wave that would properly display the decay curve was estimated by multiplying the time constant by 5. This provided enough time for each cycle that the full curve was able to be seen, and not cut off or squished into a near-linear form. We estimated a time constant by measuring R and C before the experiment started.

Data for each capacitor plate distance was recorded onto a flash drive that was plugged into the oscilloscope, and then transferred to our computers for analysis.

B. Uncertainty Analysis

My calculated values of C_{total} had only one source of uncertainty. They were found using τ and R, and τ had no uncertainty. The resistor had a resistance of $(100 \pm 5)\text{K}\Omega$, denoted by its colored bands. This was a manufacturer-specified uncertainty, so I used a uniform continuous distribution.

My values of A/d had two sources of uncertainty: uncertainty in A, and uncertainty in d. Both measurements were limited by the precision of the rulers they were measured with, so I selected uniform continuous distributions for each. A was 0.0400 ± 0.002 m^2, and d had an uncertainty of 0.0005 m.

Uncertainty in both C and A/d was then calculated for each data point using the formula from the Guide to expressing Uncertainty in Measurement.

IV. RESULTS

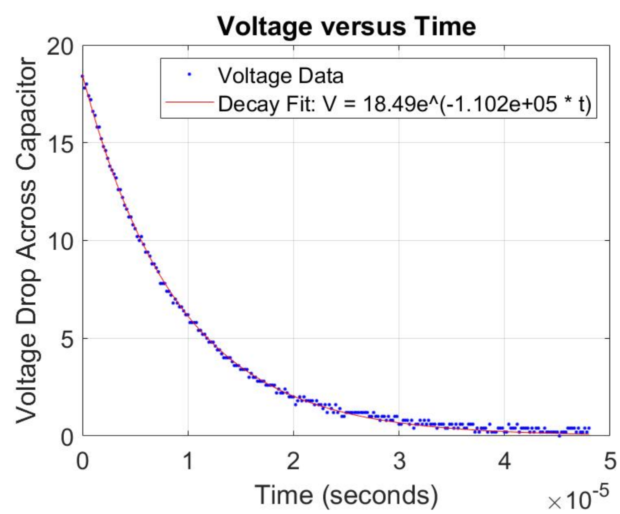

FIG. 2: This plot displays the fitted data for the eighth trial, which had a separation distance of 0.0117 m.

I fit the data in Fig. 3 to Eq. 10, such that the slope was $\epsilon_0 = (8.69 \pm 0.47) \times 10^{-12}$ F/m and the y-intercept was $C_{stray} = (5.89 \pm 0.25) \times 10^{-11}$ F.

V. DISCUSSION AND CONCLUSIONS

A. Decay Curve

I confirmed that V follows the behavior of Eq. 5, as seen in Fig. 2. The collected data was for the entire

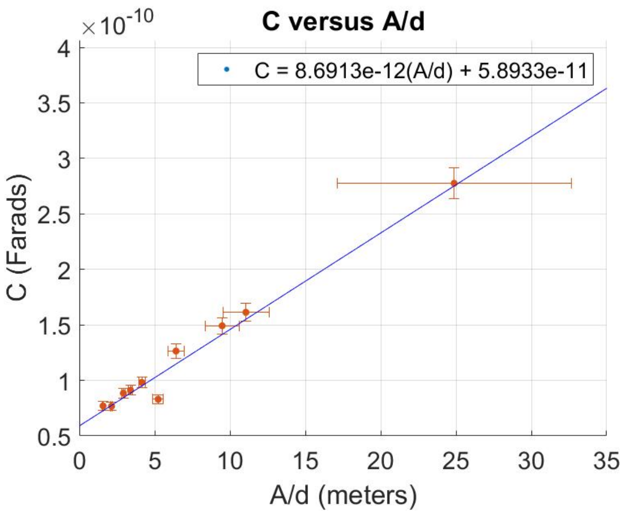

FIG. 3: This plot shows the linear relationship between C and A/d.

curve, not just the decay region, so I had to trim it so that it only included the decay. The recorded data also went into negative values for V as it oscillated between 10 and -10. Additionally, the data did not start at $t = 0$. To fix these problems, I added a constant of 10 to all values of V, and subtracted a constant from all values of t. This constant was equal to t at the first recorded value of V that I used. Doing so allowed me to fit the exponential decay curve to the data, as an exponential decay curve requires that all y-values be positive, and that the maximum y-value occurs at $t = 0$.

B. Value of ϵ_0

The value for the permittivity of free space that I found agreed with the nominal value. I anticipated stray capacitance being the largest source of error, but fortunately it was small enough that my ϵ_0 still fell within an acceptable range. I found C_{stray} to be 5.89×10^{-11} F/m, which was slightly less than half of the average value for C_{total}.

The next source of error I considered was the accuracy of the curves I fit to each measurement. Even with the excess parts of the V curve trimmed, the fitted curve did not fully agree with the plotted data. It generally matched it, but the fit was either too sharp, or not sharp enough. However, removing values on the end of the decay data improved the fit. This was due to the fact that the tail of the decay curve extended out a little ways, so this area where the slope was significantly lower reduced the accuracy of the fitted curve. Thus, I had to sacrifice some of my data to improve the quality of the fit. Generally, more data allows you to draw better conclusions, and it is advisable not to force the data to conform to your expectations too much. In this case though, discretion was necessary, and it yielded favorable — and accurate — results.

VI. ACKNOWLEDGMENTS

The experiment was performed by C.S. and N.M., and C.S. recorded the data. All sections of this report were written by N.M. Data was fit using the york_fit.m MatLab function provided by Sean Washburn.

[1] Lab B-IV Manual `https://sakai.unc.edu/access/content/group/41081125-b301-4ad9-8173-970601749ad6/Lab\%20Manuals/Lab\%20B-IV\%3A\%20Permittivity\%20of\%20Free\%20Space/Lab\%20B-IV\%20Permittivity\%20of\%20Free\%20Space.pdf`