Task 15: Interpolation Polynomials

For the function $y = y(x)$, given by its table of values, we will construct interpolation polynomials in the form of Lagrange and Newton. Using them, we will calculate the approximate value of the function at the point $\tilde{x}$.

Variant 14

N	table					$\tilde{x}$	N	table	$\tilde{x}$
14	x	3	4	5	6	3.6			
	y	0	3	1	4				

Lagrange Polynomial

The Lagrange interpolation polynomial for nodes (x_i, y_i) is written as:

$$L(x) = \sum_{i=0}^{n} y_i \ell_i(x),$$

where the basis polynomials $\ell_i(x)$ are defined as:

$$\ell_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

For the given table $(x_0, y_0) = (3, 0)$, $(x_1, y_1) = (4, 3)$, $(x_2, y_2) = (5, 1)$, $(x_3, y_3) = (6, 4)$:

$$L(x) = 0 \cdot \ell_0(x) + 3 \cdot \ell_1(x) + 1 \cdot \ell_2(x) + 4 \cdot \ell_3(x),$$

where

$$
\begin{aligned}
\ell_0(x) &= \frac{(x-4)(x-5)(x-6)}{(3-4)(3-5)(3-6)}, \\
\ell_1(x) &= \frac{(x-3)(x-5)(x-6)}{(4-3)(4-5)(4-6)}, \\
\ell_2(x) &= \frac{(x-3)(x-4)(x-6)}{(5-3)(5-4)(5-6)}, \\
\ell_3(x) &= \frac{(x-3)(x-4)(x-5)}{(6-3)(6-4)(6-5)}.
\end{aligned}
$$

Calculate $L(3.6)$:

$$
\begin{aligned}
\ell_0(3.6) &= \frac{(3.6-4)(3.6-5)(3.6-6)}{(3-4)(3-5)(3-6)}, \\
\ell_1(3.6) &= \frac{(3.6-3)(3.6-5)(3.6-6)}{(4-3)(4-5)(4-6)}, \\
\ell_2(3.6) &= \frac{(3.6-3)(3.6-4)(3.6-6)}{(5-3)(5-4)(5-6)}, \\
\ell_3(3.6) &= \frac{(3.6-3)(3.6-4)(3.6-5)}{(6-3)(6-4)(6-5)}.
\end{aligned}
$$

Newton Polynomial

The Newton interpolation polynomial is given by:

$$
N(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2),
$$

where the coefficients a_i are calculated based on divided differences.
For the given table:

$$
\begin{aligned}
a_0 &= y_0 = 0, \\
a_1 &= \frac{y_1-y_0}{x_1-x_0} = \frac{3-0}{4-3} = 3, \\
a_2 &= \frac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0} = \frac{\frac{1-3}{5-4} - 3}{5-3} = \frac{-2-3}{2} = -2.5, \\
a_3 &= \frac{\frac{\frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1}}{x_3-x_1} - \frac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0}}{x_3-x_0} \\
&= \frac{\frac{\frac{4-1}{6-5} - \frac{1-3}{5-4}}{6-4} - \frac{\frac{1-3}{5-4} - \frac{3-0}{4-3}}{5-3}}{6-3} \\
&= \frac{\frac{3-(-2)}{2} - (-2.5)}{3} \\
&= \frac{2.5}{3} \approx 0.833.
\end{aligned}
$$

Newton Polynomial:

$$N(x) = 0 + 3(x - 3) - 2.5(x - 3)(x - 4) + 0.833(x - 3)(x - 4)(x - 5).$$

Calculate $N(3.6)$:

$$N(3.6) = 3(3.6 - 3) - 2.5(3.6 - 3)(3.6 - 4) + 0.833(3.6 - 3)(3.6 - 4)(3.6 - 5).$$

Laboratory Work: Numerical Methods

Task 20

Calculate the approximate value of the integral $\int_a^b f(x)dx$, using quadrature formulas: a) mid-rectangular with step $h = 0.4$; b) trapezoidal with steps $h = 0.4$ and $h = 0.2$; estimate the error of the latter result using the Runge rule and refine the latter result using Simpson's rule with step $h = 0.4$.

N	$f(x)$	a	b
14	$x \arctan x$	4.6	6.2

Mid-rectangular Method

The interval $[4.6, 6.2]$ is divided into subintervals of length $h = 0.4$:

$$\int_{4.6}^{6.2} x \arctan(x)\, dx \approx h \sum_{i=1}^{n} f\left(x_i - \frac{h}{2}\right),$$

where $x_i = 4.6 + i \cdot h$, $i = 1, 2, 3, 4$.

Trapezoidal Method

$$\int_{4.6}^{6.2} x \arctan(x)\, dx \approx \frac{h}{2}\left(f(4.6) + 2\sum_{i=1}^{n-1} f(x_i) + f(6.2)\right),$$

where $x_i = 4.6 + i \cdot h$, $i = 1, 2, 3, 4$.

Trapezoidal Method (continued)

For step $h = 0.2$:

$$\int_{4.6}^{6.2} x \arctan(x)\, dx \approx \frac{h}{2}\left(f(4.6) + 2\sum_{i=1}^{n-1} f(x_i) + f(6.2)\right),$$

where $x_i = 4.6 + i \cdot h$, $i = 1, 2, \ldots, n - 1$.

Error Estimation using the Runge Rule

Let I_1 and I_2 be the approximate values of the integral with step sizes h_1 and h_2, respectively. Then the error estimation using the Runge rule is:

$$R = \frac{I_2 - I_1}{2^p - 1},$$

where p is the order of accuracy of the method. For the trapezoidal method, $p = 2$.

Simpson's Method

Simpson's method with step $h = 0.4$:

$$\int_{4.6}^{6.2} x \arctan(x)\, dx \approx \frac{h}{3} \left(f(4.6) + 4f\left(4.6 + \frac{h}{2}\right) + 2f(5) + 4f\left(5 + \frac{h}{2}\right) + f(6.2) \right).$$

Laboratory Work: Numerical Methods

Task 21

Given the integral of the form $\int_a^b (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4)\, dx$. Using the a priori error estimation of the trapezoidal rule, we will determine the integration step sufficient to achieve the accuracy $\epsilon = 0.01$, and compute the integral with this step. We will calculate the exact value of the integral and confirm the achievement of the specified accuracy.

N	a	b	c_0	c_1	c_2	c_3	c_4
14	-0.3	0.2	3	2	4	3	-2

Integration Step

For the a priori error estimation, we use the formula:

$$h = \sqrt[4]{\frac{12 \cdot \epsilon}{(b-a) \cdot \max|f''(x)|}},$$

where $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$ and $f''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2$.

Step Selection

We determine $\max|f''(x)|$ on the interval $[-0.3, 0.2]$:

$$f''(x) = 2 \cdot 4 + 6 \cdot 3x + 12 \cdot (-2)x^2 = 8 + 18x - 24x^2.$$

Calculate $f''(x)$ at points $x = -0.3$ and $x = 0.2$:

$$f''(-0.3) = 8 + 18 \cdot (-0.3) - 24 \cdot (-0.3)^2 = 8 - 5.4 - 2.16 = 0.44,$$

$$f''(0.2) = 8 + 18 \cdot 0.2 - 24 \cdot 0.2^2 = 8 + 3.6 - 0.96 = 10.64.$$

Thus, $\max|f''(x)| = 10.64$.

Now we find the step:

$$h = \sqrt[4]{\frac{12 \cdot 0.01}{(0.2 + 0.3) \cdot 10.64}} \approx 0.224.$$

Calculating the Integral using the Trapezoidal Method

With step $h \approx 0.224$:

$$\int_{-0.3}^{0.2} (3 + 2x + 4x^2 + 3x^3 - 2x^4)\, dx \approx \frac{h}{2} \left(f(-0.3) + 2 \sum_{i=1}^{n-1} f(x_i) + f(0.2) \right),$$

where $x_i = -0.3 + i \cdot h$, $i = 1, 2, \dots, n - 1$.
The exact value of the integral:

$$\int_{-0.3}^{0.2} (3 + 2x + 4x^2 + 3x^3 - 2x^4)\, dx = \left(3x + x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \right) \Bigg|_{-0.3}^{0.2}.$$

Laboratory Work: Numerical Methods

Task 24

Numerically solve the Cauchy problem for a first-order ordinary differential equation

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0 \end{cases}$$

on the interval $[t_0, t_0+0.8]$ with step $h = 0.2$: a) using Euler's method; b) using the second-order Runge-Kutta method. Estimate the error of both methods using the Runge rule. Find the exact solution of the problem. Plot the graphs of the exact and approximate solutions on the same chart.

N	$f(t,y)$	t_0	y_0
14	$y\sin t - 2\sin t\cos t$	0	3

Euler's Method

Euler's method for solving the Cauchy problem is written as:

$$y_{n+1} = y_n + hf(t_n, y_n),$$

where $t_n = t_0 + nh$, $n = 0, 1, \ldots.$

We apply Euler's method to the problem:

$$\begin{cases} y' = y \sin t - 2 \sin t \cos t, \\ y(0) = 3, \end{cases}$$

on the interval $[0, 0.8]$ with step $h = 0.2$.

Second-Order Runge-Kutta Method

The second-order Runge-Kutta method for solving the Cauchy problem is written as:

$$\begin{aligned} k_1 &= h f(t_n, y_n), \\ k_2 &= h f(t_n + h, y_n + k_1), \\ y_{n+1} &= y_n + \frac{k_1 + k_2}{2}, \end{aligned}$$

where $t_n = t_0 + nh$, $n = 0, 1, \ldots.$

We apply the second-order Runge-Kutta method to the problem:

$$\begin{cases} y' = y \sin t - 2 \sin t \cos t, \\ y(0) = 3, \end{cases}$$

on the interval $[0, 0.8]$ with step $h = 0.2$.

Error Estimation using the Runge Rule

Let $y_h(t)$ and $y_{h/2}(t)$ be the numerical solutions of the Cauchy problem with steps h and $h/2$, respectively. Then the error estimation using the Runge rule is:

$$R = \frac{y_{h/2}(t) - y_h(t)}{2^p - 1},$$

where p is the order of accuracy of the method. For Euler's method, $p = 1$, for the second-order Runge-Kutta method, $p = 2$.

Exact Solution of the Problem

We find the exact solution of the Cauchy problem:

$$\begin{cases} y' = y \sin t - 2 \sin t \cos t, \\ y(0) = 3. \end{cases}$$

We solve this differential equation analytically or using symbolic computation.

Plotting the Graphs

We plot the graphs of the exact and approximate solutions obtained by Euler’s method and the second-order Runge-Kutta method on the same chart.

Laboratory Work: Numerical Methods

Task 27

Using the finite difference method, we will find the solution to the boundary value problem

$$\begin{cases} -y'' + q(x)y = f(x), \\ y(0) = y_0, \\ y(1) = y_1, \end{cases}$$

with steps $h_1 = \frac{1}{3}$ and $h_2 = \frac{1}{6}$, and estimate the error using the Runge rule. We will plot the graphs of the obtained approximate solutions on the same chart.

N	$q(x)$	$f(x)$	y_0	y_1
14	e^2	e^x	2	$1 + e$

Finite Difference Method

The interval $[0,1]$ is divided into N equal parts with step h. At the grid points $x_i = ih$, $i = 0, 1, \ldots, N$, we apply the finite difference method for the approximate solution of the boundary value problem.

The finite difference scheme for the problem is:

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + q(x_i) y_i = f(x_i),$$

where $i = 1, 2, \ldots, N-1$.

We assemble the system of linear equations and solve it using the Gaussian elimination method or by using library functions.

Error Estimation using the Runge Rule

Let $y_h(x)$ and $y_{h/2}(x)$ be the numerical solutions of the boundary value problem with steps h and $h/2$, respectively. Then the error estimation using the Runge rule is:

$$R = \frac{y_{h/2}(x) - y_h(x)}{2^p - 1},$$

where p is the order of accuracy of the method. For the finite difference method, $p = 2$.

Plotting the Graphs

We plot the graphs of the approximate solutions obtained by the finite difference method with steps h_1 and h_2 on the same chart.