Типовой расчёт номер 20

$f(x) = ln(1+x^2);$
$a = 3.2;$
$b = 4.8$

Решение:

$$I = \int_{3.2}^{4.8} ln(1+x^2)\,dx$$

а) Центр. Прямоуг.: $h = 0.4$, отрезок $[3.2; 4.8]$
$N = 4$

$[x_i]_{i=0}^{4} = [3.2; 3.6; 4.0; 4.4; 4.8]$

$$I = \int_a^b f(x)\,dx = h * \Sigma f(x_{i-1} + \frac{h}{2}) = 0.4(f(3.4) + f(3.8) + f(4.6) + f(4.2)) = 0.4(2.53052 + 2.73696 + 2.92531 + 3.09829) = 4.516431$$

Априорная оценка погрешности: $\left| Y - I \right| \leq \frac{M_2(b-a)}{24}h^2$

$M_2 = \max |f''(x)|$ на $[a, b]$

$$f(x) = ln(1+x^2); f'(x) = \frac{2x}{1+x^2}; f''(x) = \frac{2-2x^2}{(1+x^2)^2}$$

$$f''(4.8) = -0.07627 \Rightarrow \left| Y - I \right| \leq |\frac{-0.7627(4.8-3.2)}{24}0.4^2 = 0.0081355$$

Б) Трапец.: $1)h = 0.4$

Ф-ла:

$$I_{tp}^{0.4} = \int_a^b f(x)\,dx = h(\frac{f(x_0) + f(x_n)}{2} + \Sigma f(x_i) = 0.4(\frac{f(x_0) + f(x_n)}{2} + f(3.6) + f(4.0) + f(4.4))$$

$$= 0.4(\frac{2.41948 + 3.17972}{2} + 2.6362 + 2.83321 + 3.01357) = 4.513032$$

$2)h = 0.2$
$[x_i] = [3.2; 3.4; 3.6; 3.8; 4.0; 4.2; 4.4; 4.6; 4.8]$

Аналогично пункту 1:

$$I_{tp}^{0.2} = \int_a^b f(x)\,dx = h(\frac{f(x_0) + f(x_n)}{2} + \Sigma f(x_i) = 0.2(\frac{f(x_0) + f(x_n)}{2} + f(3.2) + f(4.8) + f(3.4) + f(3.6) + f(3.8) + f(4.0) + f(4.2) + f(4.4) + f(4.6))$$

$$= 0.2(\frac{2.41948 + 3.17972}{2} + 2.53052 + 2.6362 + 2.73696 + 2.83321 = 2.92531 + 3.01357 + 3.09829) = 4.714732$$

Апостериорная оценка погрешности:

$$|Y - I_{tp}^{0.2}| = \frac{I_{tp}^{0.2} - I_{tp}^{0.4}}{2^2 - 1} = \frac{4.714732 - 4.513032}{3} = 0.0672333$$

Уточнения по Рунге:

$$I_{utochn} = I_{tp}^{0.2} = 4.714732 + 0..672333 = 4.78197$$

В) Симпсон:

$$h = 0.4$$

$$[x_i]_{i=0}^{4} = [3.2; 3.6; 4.0; 4.4; 4.8]$$

$$I_c^{0.4} = \frac{h}{6}(f(x_0) + f(x_4) + \Sigma f(\frac{x_{i-1} + x_i}{2}) + 2\Sigma f(x_i) = \frac{0.4}{6}(f(3.2) + f(4.8) + 4(f(3.4) + f(3.8) + f(4.2) + f(4.6)) + 2(f(3.6) + f(4.0) + f(4.4))$$

$$= \frac{0.4}{6}(2.41948 + 3.17972 + 4(2.53052 + 2.73696 + 2.92531 + 3.09829) + 2(2.6362 + 2.83321 + 3.01357)) = 4.5153$$

Типовой расчёт номер 21

A	B	C_0	C_1	C_2	C_3	C_4
0.4	0.9	1	-1	0	4	0

Решение:

$Y = \int_{0.4}^{0.9} (1 - x + 4x^3)\,dx = -0.1945$ - точное значение

$$\left| Y - I_{tp}^{n} \right| \leq \frac{M_2(b-a)^3}{12} h^2 < \epsilon \implies h < \sqrt{\frac{12 \cdot \epsilon}{M_2(b-a)}}$$

$f(x) = 1 - x + 4x^3$

$f'(x) = -1 + 12x^2$

$f''(x) = 24x$

$M_2 = \max |f''(x)| = f''(0.9) = 21.6$

$$h < \sqrt{\frac{12 \cdot 0.001}{21.6 \cdot (0.9 - 0.4)}} \approx 0.105409$$

$h = 0.1$

$$N = \frac{b-a}{h} = \frac{0.5}{0.1} = 5$$

$x_0 = 0.4, \quad x_1 = 0.5, \quad x_2 = 0.6, \quad x_3 = 0.7, \quad x_4 = 0.8, \quad x_5 = 0.9$

$$I_{tp}^{0.1} = \frac{f(0.2) + f(0.9)}{2} + f(0.5) + f(0.6) + f(0.7) + f(0.8) = -0.188$$

$|Y - I_{tp}^{n}| \leq \epsilon$

$|-0.1945 + 0.188| = 0.0065$ - нужная точность достигнута.

Типовой расчёт номер 23

$f(x) = ln(1 + x^2); a = 3.2; b = 4.8.$

Решение:

$$x_0 = \frac{3.2 + 4.8}{2} = 4$$

$$f'(x_0) = \frac{f(x_0 + h) + f(x_0 - h)}{2h}$$

$$f'(x_1) = \frac{f(x_0) + f(x_0 - h)}{h}$$

$$f'(4) = \frac{f(4.1) + f(3.9)}{2 * 0.1} = 0.470659$$

$$f_A(x_0) = \frac{f(4) - f(3.9)}{0.1} = 0.47585$$

Вычислим точное значение производной:

$$f'(x) = \frac{2x}{1 + x^2}$$

$$f(x) = 0.470588$$

Абсолютная погрешность:

$$\Delta = |0.470588 - 0.470659| = 0.000071$$

$$\Delta_1 = |0.470588 - 0.47585| = 0.005625$$

Центральная разностная производная даёт более точный результат:

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

$$f''(4) = \frac{f(4.1) - 2f(4) + f(3.9)}{0.1^2} = -0.103826$$

Точное значение:

$$f''(x) = \frac{2 - 2x^2}{(1 + x^2)^2}$$

$$f''(4) = \frac{2 - 2 * 4^2}{(1 + 4^2)^2} = -0.103806$$

Абсолютная погрешность:

$$\Delta_2 = |-0.103806 + 0.103826| = 0.00002$$

Типовой расчёт номер 24

$$\begin{cases} y' = f(t, y) \\ y(2) = 0 \end{cases}$$

$$f(t, y) = \frac{y}{t+1} + t + 1, \quad t_0 = 1, \quad y_0 = 0$$

Решение:

$$\begin{cases} y' = \frac{y}{t+1} + t + 1 \\ y(t_0) = 0, \quad t \in [1{,}1.8] \end{cases}$$

Найдём точное решение задачи:

$$y' = \frac{y}{t+1} + t + 1$$

$$y = a(t), \quad y = b(t)$$

Подстановка:

$$y = uv$$

$$y' = uv' + u'v$$

$$uv' + u'v - \frac{uv}{t+1} = t + 1$$

$$u'v + u(v' - \frac{u}{t+1}) = t + 1$$

1) $u' - \frac{u}{t+1} = 0$

$$\frac{du}{dt} = \frac{u}{t+1}$$

$$\int \frac{du}{u} = \int \frac{dt}{t+1}$$

$$\ln u = \ln(t+1)$$

$$u = t + 1$$

2) $(t+1)u' = t + 1 \; |:(t+1) \qquad t = -1$

$$\frac{du}{dt} = 1$$

$$du = dt$$

$u = t + c$
$u = t + c, \quad t = -1$

Обратная замена:
$y = (t + c)(t + 1) = t^2 + ct + t + c$ - общее решение

$y(1) = 0 \implies 0 = 2 + c + 2 \implies c = -1$
$\implies y = t^2 - 1(t + 1) + t$
$y = t^2 - t - 1 + t$
$y = t^2 - 1$ - частное решение(точное)

$t \in [1,1.8]$ с шагом $h = 0.2$, имеем: $t_0 = 1, t_1 = 1.2, t_2 = 1.4, t_3 = 1.6, t_4 = 1.8$

T	Y
1	0
1.2	0.44
1.4	0.96
1.6	1.56
1.8	2.24

Метод Эйлера:

$\dfrac{y_{n+1} - y_n}{h} = f(t_n, y_n) \qquad f(t_n, y_n) = \dfrac{y_n}{t_n + 1} + t_n + 1$

$y_{n+1} = y_n + hf(t_n, y_n) \qquad y_0 = 0 \quad t_0 = 1$

$1) y_1 = y_0 + hf(t_0, y_0) = [y_0 = 0; t_0 = 1] = 0 + 0.2(\dfrac{0}{1+1} + 1 + 1) = 0.4; t = 1.2$
$2) y_2 = y_1 + hf(t_1, y_1) = [y_1 = 0.4; t_1 = 1.2] = 0.4 + 0.2(\dfrac{0.4}{1.2+1} + 1.2 + 1) = 0.8763; t = 1.4$
$2) y_3 = y_2 + hf(t_2, y_2) = [y_2 = 0.8763; t_2 = 1.4] = 0.8763 + 0.2(\dfrac{0.8763}{1.4+1} + 1.4 + 1) = 1; t = 1.6$
$4) y_4 = 2.05927; t = 1.8$

Для оценки погрешности вычислим значение с шагом $2h = 0.4$

$1) y_1^{2h} = y_0 + 2hf(t_0, y_0) = 0 + 0.4(\dfrac{0}{1+1} + 1 + 1) = 0.8$

$2) y_1^{2h} = 1.893$

i	t_i	y_i^h	y_i^{2h}
0	1	0	0
1	1.2	0.4	
2	1.4	0.876	0.8
3	1.6	1.429	
4	1.8	2.05927	1.893

Расчёт оценки погрешности метода по правилу Рунге

$R_1 = |y_1^h - y_1^{2h}| = |0.876 - 0.8| = 0.076$

$R_2 = |y^h - y_2^{2h}| = |2.05927 - 1.893| = 0.16627$

Б) Метод Рунге-Кутты(2-го порядка):

$t_0 = 1; t_1 = 1.2; t_2 = 1.4; t_3 = 1.6; t_4 = 1.8$

$y_0 = 0$

$$\begin{cases} y_1 = y_0 + hf(t_0, y_0) = 0 + 0.2(\frac{0}{1+1} + 1 + 1) = 0.4 \\ y_1 = y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_1)) = 0 + \frac{0.2}{2}((\frac{0}{1+1} + 1 + 1) + (\frac{0.4}{1.2+1} + 1.2 + 1)) = 0.4381 \end{cases}$$

$t = 1.2$

$$\begin{cases} 0.4381 + 0.2(\frac{0.4381}{1.2+1} + 1 + 1) = 0.917927 \\ 0.4381 + \frac{0.2}{2}((\frac{0.4321}{1.2+1} + 1.2 + 1) + (\frac{0.917927}{1.4+1} + 1.4 + 1)) = 0.956261 \end{cases}$$

$t = 1.4$

$$\begin{cases} 0.956261 + 0.2(\frac{0.956261}{1.4+1} + 1.4 + 1) = 1.51595 \\ 0.956261 + \frac{0.2}{2}((\frac{0.956261}{1.4+1} + 1.4 + 1) + (\frac{1.51595}{1.6+1} + 1.6 + 1)) = 1.55441 \end{cases}$$

$t = 1.6$

$$\begin{cases} 1.55441 + 0.2(\frac{1.55441}{1.6+1} + 1.6 + 1) = 2.19398 \\ 1.55441 + \frac{0.2}{2}((\frac{1.55441}{1.6+1} + 1.6 + 1) + (\frac{2.19398}{1.8+1} + 1.8 + 1)) = 2.23255 \end{cases}$$

$t = 1.8$

$$\begin{cases} 0.4 + 0.4(\frac{0}{1+1} + 1 + 1) = 0.8 \\ 0 + \frac{0.4}{2}((\frac{0}{1+1} + 1 + 1) + (\frac{0.8}{1.4+1} + 1.4 + 1)) = 0.946 \end{cases}$$

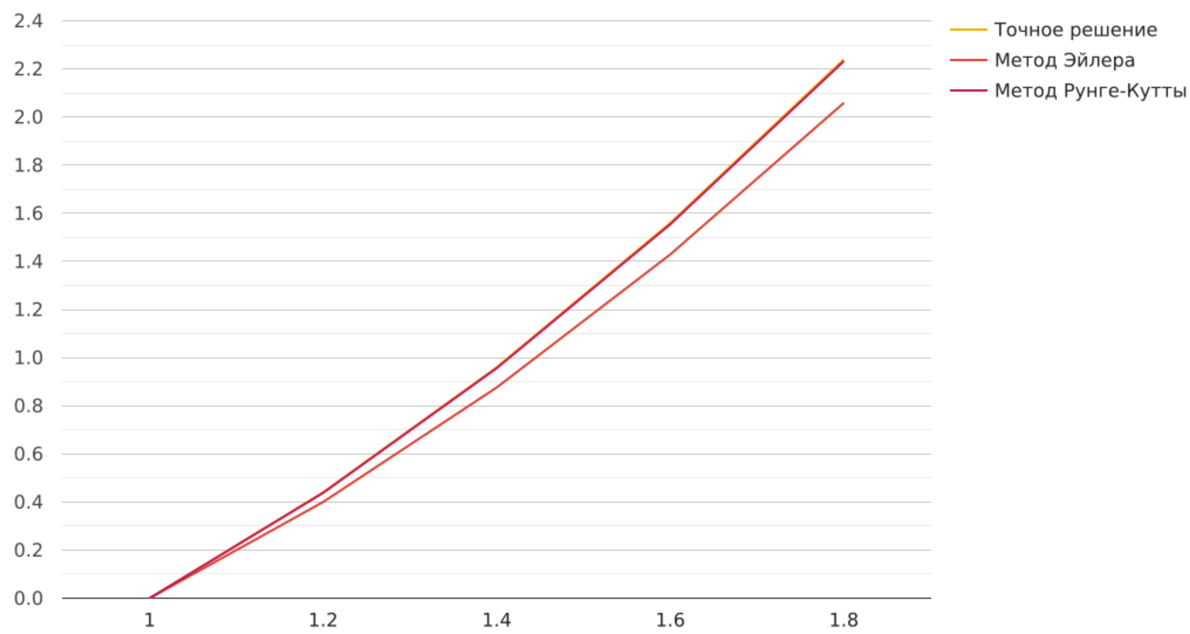
$$\begin{cases} 0.946 + 0.4(\frac{0.946}{1.4+1} + 1.4 + 1) = 2.0636 \\ 0.946 + \frac{0.4}{2}((\frac{0.946}{1.4+1} + 1.4 + 1) + (\frac{2.0636}{1.8+1} + 1.8 + 1)) = 2.21223 \end{cases}$$

i	t_i	y_i^h	y_i^{2h}
0	1	0	0
1	1.2	0.4381	
2	1.4	0.9563	0.946
3	1.6	1.5544	
4	1.8	2.23255	2.21223

Оценка погрешности:

$$R = |\frac{y_i^h - y_i^{2h}}{2^p - 1}|$$
$$p = 2$$

$$R_1 = |\frac{0.946 - 0.9563}{3}| = 0.00343$$
$$R_2 = |\frac{2.23255 - 2.21223}{3}| = 0.006773$$
$$R_0 = |\frac{y_0^h - y_0^{2h}}{3}| = 0$$



t	Точное решение	Метод Эйлера	Метод Рунге-Кутты
1	0	0	0
1.2	0.44	0.4	0.4381
1.4	0.96	0.876	0.9563
1.6	1.56	1.429	1.55441
1.8	2.24	2.059	2.23255

Типовой расчёт номер 27

$q(x) = 6$

$f(x) = 6(1 - x + x^3)$
$y_0 = 1$
$y_1 = 2$

Решение:
$$\begin{cases} -y'' + 6y = 6(1 - x + x^3) \\ y(0) = 1; y(1) = 2 \end{cases}$$

$$-y_{i+1} + (2 + h^2 q_i)y_i - y_{i-1} = h^2 f_i$$

$$-y_{i+1} + (2 + h^2 6)y_i - y_{i-1} = h^2 * 6(1 - x_i + x_i^3)$$

Для шага $h_1 = \dfrac{1}{3}$:

$$[x_i] = [0; \frac{1}{3}; \frac{2}{3}; 1]$$

Из условия:
$y(0) = y(x_0) = y_0 = 1$
$y(1) = y(x_3) = y_3 = 2$

$$\begin{cases} -y_2 + (2 + 6h^2)y_1 - y_0 = 6h^2(1 - x_1 + x_1^3); i = 1 \\ -y_3 + (2 + 6h^2)y_2 - y_1 = 6h^2(1 - x_2 + x_2^3); i = 2 \end{cases}$$

$$\begin{cases} -y_2 + (2 + 6(\frac{1}{9}))y_1 - 1 = 6 * (\frac{1}{9})(1 - \frac{1}{3} + \frac{1}{3}^3) \\ -2 + (2 + 6(\frac{1}{9})y_2 - y_1 = 6(\frac{1}{9})(1 - \frac{2}{3} + \frac{2}{3}^3) \end{cases}$$

$$\begin{cases} -y_2 + (\frac{8}{3})y_1 - 1 = \frac{38}{81} \\ -2 + (\frac{8}{3})y_2 - y_1 = \frac{34}{81} \end{cases}$$
$$\begin{cases} y_1 = \frac{28}{27} \\ y_2 = \frac{35}{27} \end{cases}$$

По итогу:
$$\begin{cases} y_0 = 1 \\ y_1 = \frac{28}{27} = 1.03703704 \\ y_2 = \frac{35}{27} = 1.2962963 \\ y_3 = 2 \end{cases}$$

Для шага $h_2 = \dfrac{1}{6}$:

$$[x_i] = [0; \frac{1}{6}; \frac{1}{3}; \frac{1}{2}; \frac{2}{3}; \frac{5}{6}; 1]$$

Из условия:

$y(0) = y(x_0) = y_0 = 1$

$y(1) = y(x_6) = y_6 = 2$

$$\begin{cases} -y_2 + (2 + 6 * \frac{1}{36})y_1 - y_0 = 6 * \frac{1}{36}(1 - x_1 + x_1^3); i = 1 \\ -y_3 + (2 + 6 * \frac{1}{36})y_2 - y_1 = 6 * \frac{1}{36}(1 - x_2 + x_2^3); i = 2 \\ -y_4 + (2 + 6 * \frac{1}{36})y_3 - y_2 = 6 * \frac{1}{36}(1 - x_3 + x_3^3); i = 3 \\ -y_5 + (2 + 6 * \frac{1}{36}^2)y_4 - y_3 = 6 * \frac{1}{36}(1 - x_4 + x_4^3); i = 4 \\ -y_6 + (2 + 6 * \frac{1}{36})y_5 - y_4 = 6 * \frac{1}{36}(1 - x_5 + x_5^3); i = 5 \end{cases}$$

$$\begin{cases} -y_2 + (2 + 6 * \frac{1}{36})y_1 - y_0 = 6 * \frac{1}{36}(1 - 0 + 0); i = 1 \\ -y_3 + (2 + 6 * \frac{1}{36})y_2 - y_1 = 6 * \frac{1}{36}(1 - \frac{1}{6} + \frac{1}{6}^3); i = 2 \\ -y_4 + (2 + 6 * \frac{1}{36})y_3 - y_2 = 6 * \frac{1}{36}(1 - \frac{1}{3} + \frac{1}{3}^3); i = 3 \\ -y_5 + (2 + 6 * \frac{1}{36})y_4 - y_3 = 6 * \frac{1}{36}(1 - \frac{1}{2} + \frac{1}{2}^3); i = 4 \\ -2 + (2 + 6 * \frac{1}{36})y_5 - y_4 = 6 * \frac{1}{36}(1 - \frac{5}{6} + \frac{5}{6}^3); i = 5 \end{cases}$$

$$\begin{cases} -y_2 + (\frac{13}{6})y_1 - 1 = \frac{1}{6}; \\ -y_3 + (\frac{13}{6})y_2 - y_1 = \frac{181}{1296}; \\ -y_4 + (\frac{13}{6})y_3 - y_2 = \frac{19}{162}; \\ -y_5 + (\frac{13}{6})y_4 - y_3 = \frac{5}{48}; \\ -2 + (\frac{13}{6})y_5 - y_4 = \frac{161}{1296}; \end{cases}$$

$$\begin{cases} -y_2 + (\frac{13}{6})y_1 - 1 = \frac{1}{6}; \\ -y_3 + (\frac{13}{6})y_2 - y_1 = \frac{181}{1296}; \\ -y_4 + (\frac{13}{6})y_3 - y_2 = \frac{19}{162}; \\ -y_5 + (\frac{13}{6})y_4 - y_3 = \frac{5}{48}; \\ -2 + (\frac{13}{6})y_5 - y_4 = \frac{161}{1296}; \end{cases}$$

$$\begin{cases} y_1 = 1.02697; \\ y_2 = 1.05843; \\ y_3 = 1.12664; \\ y_4 = 1.29681; \\ y_5 = 1.57894; \end{cases}$$

По итогу:

$$\begin{cases} y_0 = 1; \\ y_1 = 1.02697; \\ y_2 = 1.05843; \\ y_3 = 1.12664; \\ y_4 = 1.29681; \\ y_5 = 1.57894; \\ y_6 = 2; \end{cases}$$

Оценка погрешности по Рунге:

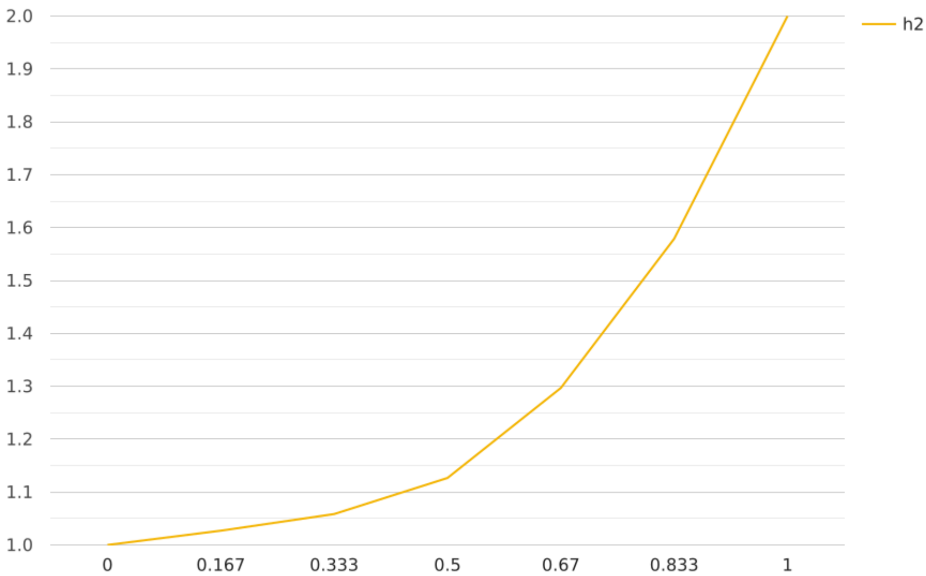
$$R = |\frac{y_i^h - y_i^{\frac{h}{2}}}{2^p - 1}|$$

$$R_0 = |\frac{y_0^{\frac{1}{3}} - y_0^{\frac{1}{6}}}{3}| = 0$$

$$R_1 = |\frac{y_1^{\frac{1}{3}} - y_1^{\frac{1}{6}}}{3}| = |\frac{1.03703 - 1.05843}{3}| = 0.0071333$$

$$R_2 = |\frac{y_2^{\frac{1}{3}} - y_4^{\frac{1}{6}}}{3}| = |\frac{1.2962963 - 1.29681}{3}| = 0.0005137$$

$$R_0 = |\frac{y_3^{\frac{1}{3}} - y_6^{\frac{1}{6}}}{3}| = 0$$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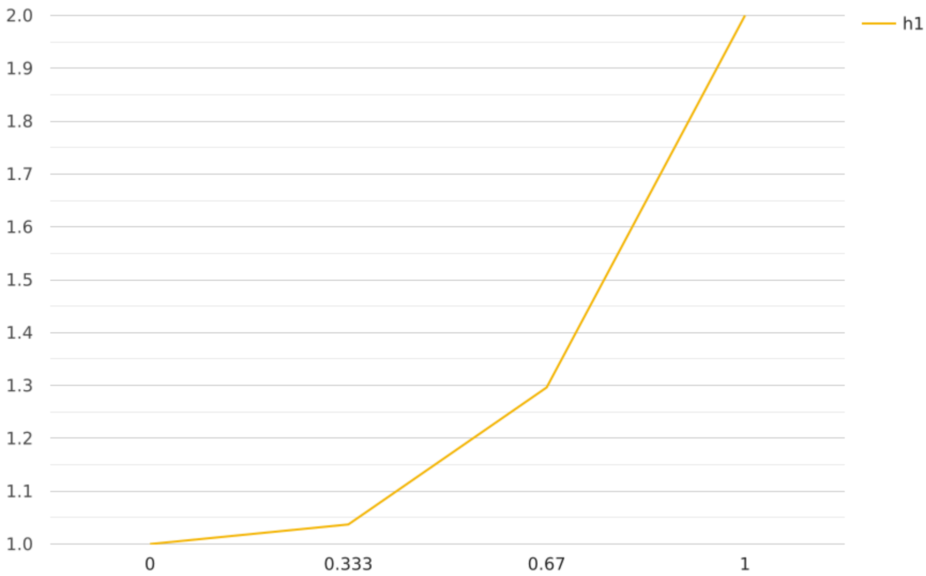
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0
0.167
0.333
0.5
0.67
0.833
1
h2


2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0
0.333
0.67
1
h1