Decompositions

Sophie Woodward

2023-06-08

Simulation Study

Denote X as exposure X, Z as unmeasured confounder, and Y as outcome.

Simulation 1: Nested DGM

We simulate the following scenario 100 times. Across $n_1 = 25$ states of 5×5 grids,

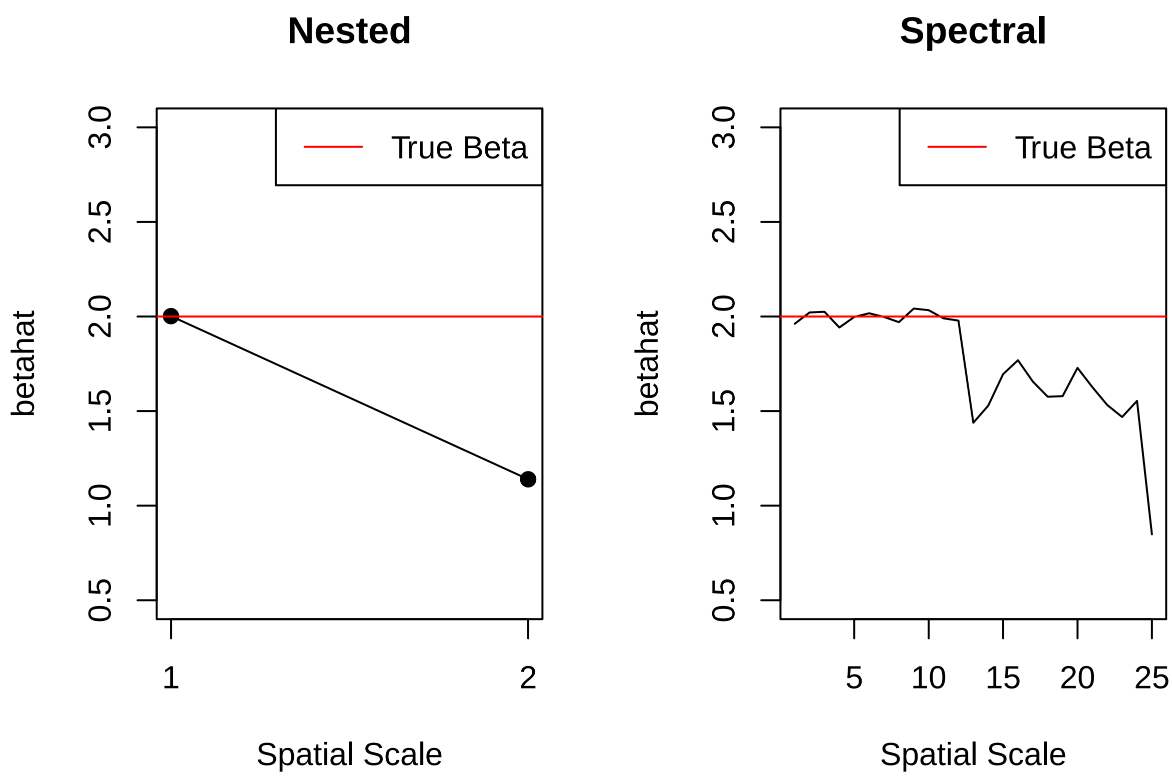
$$X_{1,s} \sim \text{Exp}(1)$$
$$Z_{1,s} \sim 0.9X_2 + \sqrt{1 - 0.9^2}\text{Exp}(1)$$

for $s = 1, \ldots, n_1 = 25$. Across $n_2 = 625$ counties of 1×1 grids, let

$$X_{2,i} \sim \text{Exp}(1)$$
$$Z_{2,i} \sim 0.001X_2 + \sqrt{1 - 0.001^2}\text{Exp}(1)$$

and $X_i = X_{1,s(i)} + X_{2,i}, Z_i = Z_{1,s(i)} + Z_{2,i}$ for $i = 1, \ldots, n_2 = 625$. By construction, X, Z are nearly uncorrelated within the states of 5×5, but correlated across states. (Note that this DGM exactly follows the nested decomposition.) We let $Y_i = 2X_i - Z_i + \epsilon$ where $\epsilon_i \sim \mathcal{N}(0, 1)$ independently across i.

For each of the 100 scenarios, we decompose X, Z, Y at different spatial scales using 1) nested decomposition and 2) spectral decomposition. At each spatial scale ω, we obtain an estimate $\hat{\beta}(\omega)$ of $\beta = 2$ from a linear regression of $Y(\omega)$ on $X(\omega)$. I hypothesize that $\hat{\beta}(\omega)$ is unbiased for high ω (finer spatial scales) since by construction confounding dissipates locally.


Nested
True Beta
betahat
0.5
1.0
1.5
2.0
2.5
3.0
1
2
Spatial Scale
Spectral
True Beta
betahat
0.5
1.0
1.5
2.0
2.5
3.0
5
10
15
20
25
Spatial Scale

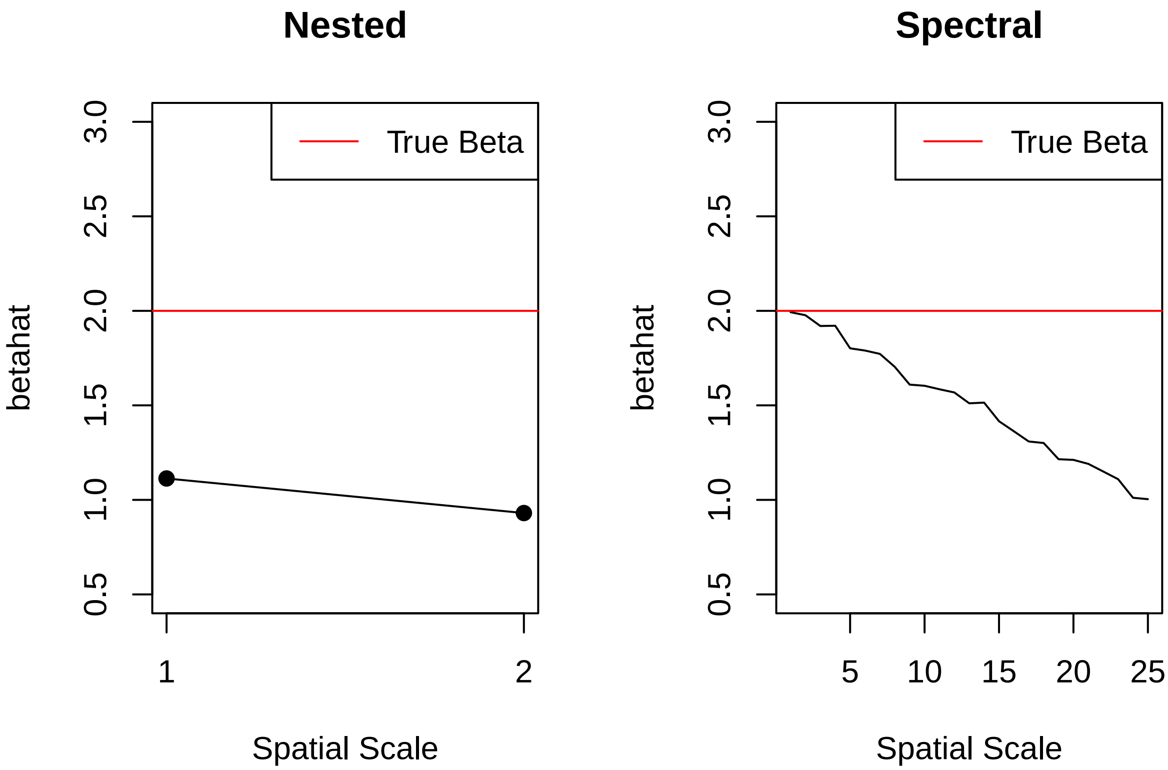
Simulation 2: Spectral DGM

We repeat simulation 1 but now the data-generating model originates from the spectral decomposition rather than the nested. In particular, we use the graph Fourier transform to project X and Z into the spectral domain. In the spectral domain,

$$X_i^* \sim \text{Exp}(1)$$
$$Z_i^* \sim \rho_i X_2 + \sqrt{1-\rho_i^2}\text{Exp}(1)$$

for $i = 1, \ldots, 625$ where $\rho_i = 0, 1/624, 2/624, \ldots, 1$. So the covariance between X and Z dissipates for smaller i, which corresponds to larger eigenvalues ω of the graph Laplacian, or smaller spatial scales.

Again, for each of the 100 scenarios we decompose X, Z, Y at different spatial scales using 1) nested decomposition and 2) spectral decomposition. At each spatial scale ω, we obtain an estimate $\hat{\beta}(\omega)$ of $\beta = 2$ from a linear regression of $Y(\omega)$ on $X(\omega)$. I hypothesize that $\hat{\beta}(\omega)$ is unbiased for high ω (finer spatial scales) since by construction confounding dissipates locally.



Simulation 3: Local Confounding

We repeat simulations 1 and 2 but now confounding dissipates at larger scales.

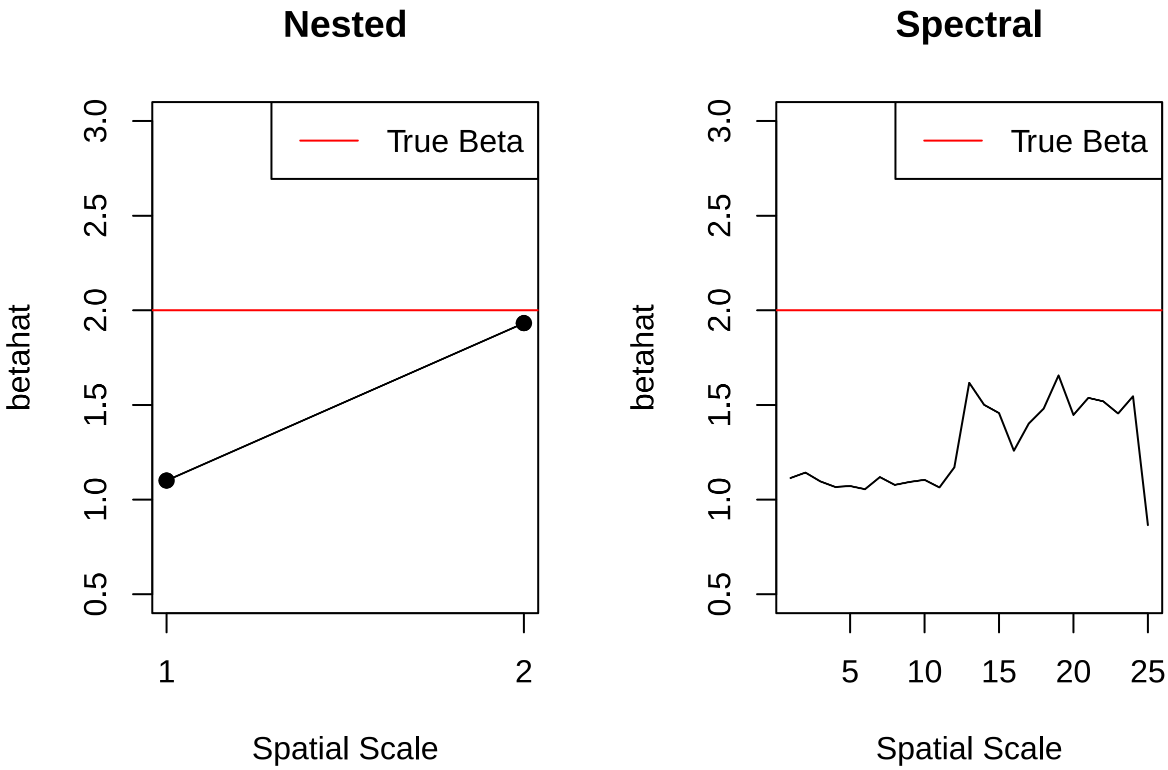
For the nested DGM,

$$X_{1,s} \sim \text{Exp}(1)$$
$$Z_{1,s} \sim 0.001X_2 + \sqrt{1-0.001^2}\text{Exp}(1)$$

for each 5×5 state, $s = 1, \dots, n_1 = 25$. Across $n_2 = 625$ counties of 1×1 grids, let

$$X_{2,i} \sim \text{Exp}(1)$$
$$Z_{2,i} \sim 0.9X_2 + \sqrt{1-0.9^2}\text{Exp}(1)$$

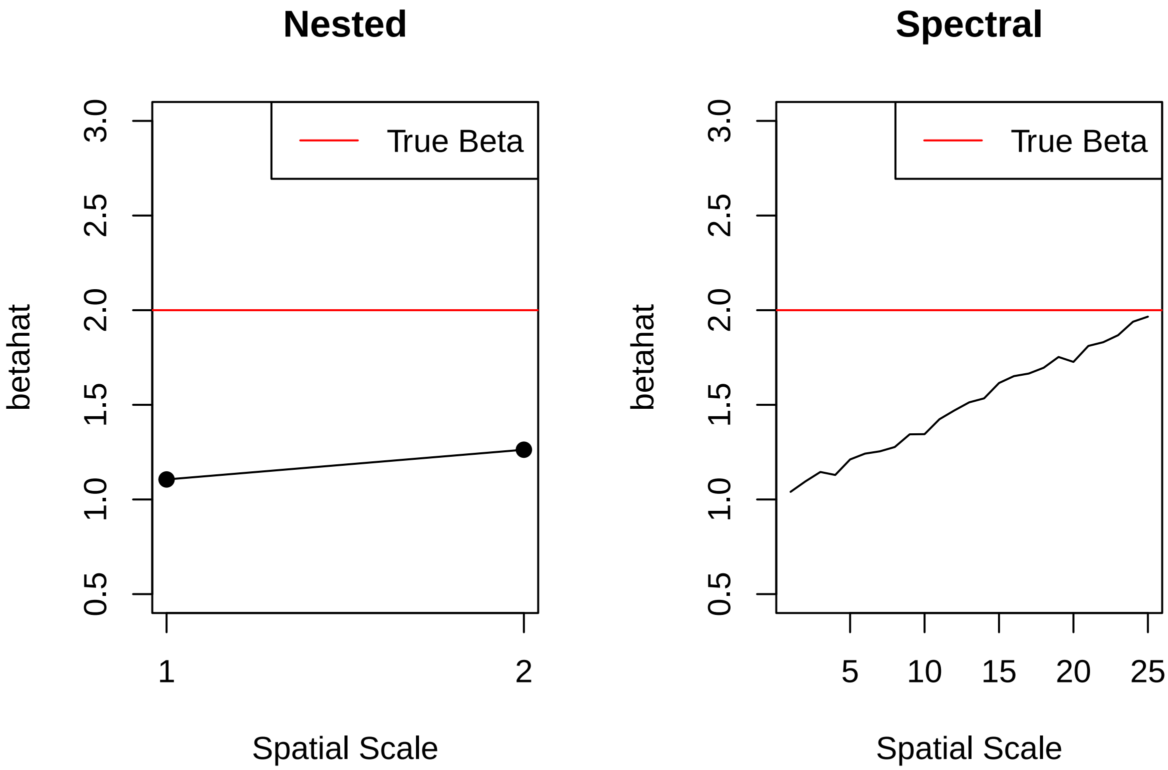
and $X_i = X_{1,s(i)} + X_{2,i}, Z_i = Z_{1,s(i)} + Z_{2,i}$ for $i = 1, \dots, n_2 = 625$. By construction, X, Z are nearly uncorrelated across the states of 5×5, but correlated within states.



For the spectral DGM,

$$X_i^* \sim \text{Exp}(1)$$
$$Z_i^* \sim \rho_i X_2 + \sqrt{1-\rho_i^2}\text{Exp}(1)$$

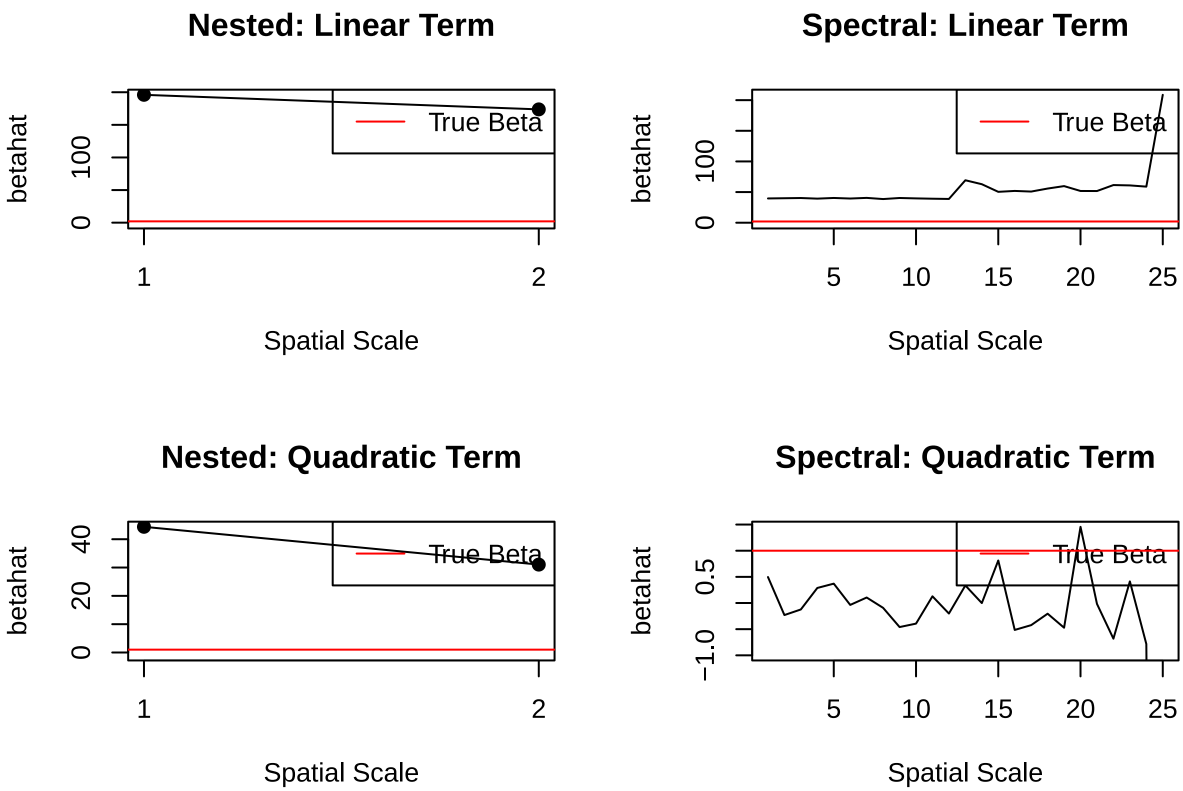
for $i = 1, \ldots, 625$ where $\rho_i = 1, 623/624, 622/624, \ldots, 0$.



Simulation 4: Nonlinear Outcome Model

We repeat simulation 1 and 2 but the outcome model now includes a quadratic term of X. In particular, $Y_i = 2X_i + X_i^2 - Z_i + \epsilon$.

Nested DGM



Spectral DGM