Math 327 Homework 1

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1. Recall that in the textbook, a function f : A → B is said to be invertible f is one-to-one and onto. (This is also called bijection in some literature.) Show that a function f : A → B is invertible if and only if there is a function g : B → A such that g ∘ f = identity function on A and f ∘ g = identity function on B.

   We first prove that if there exists a function B → A such that g ∘ f = identity function on A and f ∘ g = identity function on B, then f is invertible.

   A function F is called the identity function given that $F(x) = x$. We are given that $f(g(x)) = x$ and $g(f(x)) = x$. We want to prove that f is both one-to-one and onto.

   Let $f(g(x)) = f(g(y))$. Given that this is the identity function, we know that $f(g(x)) = x$ and $f(g(y)) = y$. As such, $x = y$. So, $f(g(x))$ is one-to-one.

   We know that $f(g(x)) = x$. Let $y \in B$. We want to show that for some $x \in A$, $f(g(x)) = y$. Since $f(g(x))$ is the identity function, we know that $f(g(y)) = y$. As such, when $x = y$, $f(g(x)) = y$. So, $f(g(x))$ is onto.

   With similar lines of reasoning as above, we know that $g(f(x))$ is both one-to-one and onto as well.

   Let $a \in B$. Since f ∘ g is the identity function, there exists $b \in B$ such that $f(g(b)) = a$. So, if $x = g(b) \in B$, then $f(x) = a$. As such, f is onto. By a similar line of reasoning, g is also onto.

   Let $x \in A$ and $y \in A$. Let $f(g(x)) = f(g(y))$. Since

2. Let f : A → B and g : B → C be functions.
   (a) Prove that f and g are injective, then so is g ∘ f.

   Let $g(f(x)) = g(f(y))$. Since g is injective, we know that $f(x) = f(y)$. Since f is injective, we know that $x = y$. So, $g(f(x))$ is injective.

   (b) Suppose g ∘ f is surjective. Does it follow that f is surjective?

   Let $a \in B$. Since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = a$. However, we do not know if $f(x) = a$. As such, f is not surjective.

   (c) Suppose g ∘ f is surjective. Does it follow that g is surjective?

   Let $a \in C$. Since g ∘ f is surjective, there exists $b \in B$ such that $g(f(b)) = a$. So, if $x = f(b) \in B$, then $g(x) = a$. As such, g is surjective.

3. Let f: $A \to$ B. For a set $E \subset A$, the set $F(E) \subset B$ is defined by $F(E) = \{f(a) : a \in E\}$, and is called the image of E. For $F \subset B$, the set $f^{-1}(F) \subset A$ is defined by $f^{-1}(F) = \{a \in A : f(a) \in F\}$ and is called the inverse image of F.
   (i) Let $E \subset A$. Prove that $E \subset f^{-1}(f(E))$. Give an example to show that the inclusion may be strict. What happens when f is injective?

   (ii) Let Λ be a set and that each $\lambda \in \Lambda$, let E_λ be a subset of A. Prove that

   $$f(\cup_{\lambda\in\Lambda}E_\lambda) = \cup_{\lambda\in\Lambda}f(E_\lambda) \text{ and } f(\cap_{\lambda\in\Lambda}E_\lambda) \subset \cup_{\lambda\in\Lambda}f(E_\lambda)$$

   Give an example in which the last inclusion is proper. What happens when f is injective?