Math 327 Homework 3

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1. For each of the following two sets, find the maximum, minimum, infimum, and supremum if they are defined. Justify your conclusions.

   a. $\{1/n \mid n \text{ in } \mathbb{N}\}$
   maximum:

   The set $\{1/n \mid n \text{ in } \mathbb{N}\}$ consists of values

   $$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}\}$$

   Notice that $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4}, ..., \frac{1}{n}$. Thus, this set is strictly decreasing. Therefore, the maximum of this set is 1.

   minimum:

   There is no minimum for this set. This is because the infimum is not a member of the set.

   infimum:

   For any $\frac{1}{n} \in \mathbb{N}$, $0 < \frac{1}{n}$. If there is an $\epsilon > 0$, we can find an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$ by the Archimedean principle. Thus, the infimum of this set is 0.

   supremum:

   The defintion of the supremum of a set is the least upper bound of the set. In other words, if there is an upper bound b for the set, $b \geq x$ for all x in the set. Since the maximum of this set is 1, the supremum of this set is also 1. Any value less than 1 is not an upper bound for the set since 1 is in the set, while any value greater than 1 is not the least upper bound for the set since 1 is a smaller upper bound.

   b. $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$
   maximum:

   There is no maximum for this set since the supremum is not a member of the set.

   minimum:

   There is no minimum for this set since the infimum is not a member of the set.

   infimum:

   The infimum of this set is $-\sqrt{2}$. This is because $-\sqrt{2} \leq x$ for all x in the set. Any value greater than $-\sqrt{2}$ (which we denote as x) is not a lower bound for the set because there must exist some real number q in the set such that $-\sqrt{2} < q < x$ for all x.

   supremum:

   The supremum of this set is $\sqrt{2}$. This is because $\sqrt{2} \geq x$ for all x in the set. Any value less than $\sqrt{2}$ (which we denote as x) is not an upper bound for the set because there must exist some real number q in the set such that $x < q < \sqrt{2}$ for all x.

2. a. Prove that if n is a natural number, then $2^n > n$.

   We prove by induction. The base case is $n = 1$. Then $2^1 = 2 > 1$. Thus, the base case is true.

   Now, assume that $2^k > k$ for some $k \in \mathbb{N} < n$. We must show that $2^{k+1} > k + 1$.

$$2^k > k$$

multiply both sides by 2

$$2 * 2^k > 2 * k$$
$$2^{k+1} > 2 * k$$

Now, we prove that $2 * k > k + 1$. We prove by induction. The base case is $k = 1$. Then $2 * 1 = 2 > 1 + 1$. Thus, the base case is true.

Now, assume that $2 * g > g + 1$ for all $g \in \mathbb{N} < k$. We must show that $2 * (g + 1) > (g + 1) + 1$. This is equivalent to showing that $2g + 2 > g + 2$.

$$2 * g > g + 1$$

add 2 to both sides

$$2 * g + 2 > g + 3$$

Since $g + 3 > g + 2$ for all $g \in \mathbb{N}$, we have that $2g + 2 > g + 2$ for all $g \in \mathbb{N}$.

Thus, $2 * k > k + 1$. Since $2 * k > k + 1$ for all $k \in \mathbb{N}$, we have that $2^{k+1} > 2 * (k + 1)$ for all $k \in \mathbb{N}$.

Thus, $2^n > n$ for all $n \in \mathbb{N}$.

b. Prove that n is a natural number, then

$$n = 2^{k_0} l_0$$

for some odd natural number l_0 and some nonnegative integer k_0. (Hint: if n is odd, let $k = 0$ and $l = n$; if n is even, let $A = \{k \text{ in } \mathbb{N} \mid n = 2^k l \text{ for some l in } \mathbb{N}.\}$ By (a), $A \subseteq \{1, 2, ..., n\}$. Choose k_0 to be the maximum of A.)

We prove by induction. The base case is $n = 1$. Then $1 = 2^0 1$. Thus, the base case is true.

We assume that this is true for some g in $\mathbb{N} < n$. Thus, $g = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0. We must show that this is true for $g + 1$ as well.

If $g + 1$ is odd, then

$$g + 1 = 2^0 (g + 1)$$

Since 0 is a nonnegative integer and $g + 1$ is odd.

If $g + 1$ is even, then $\frac{g+1}{2}$ is some integer as well. By our hypothesis, $\frac{g+1}{2} = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0. Thus,

$$\frac{g+1}{2} = 2^{k_0} l_0$$
$$g+1 = 2^{k_0+1} l_0$$

Since $k_0 + 1$ is a nonnegative integer and l_0 is an odd natural number, the statement is true for $g+1$ as well.

Thus, $n = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0 for all $n \in \mathbb{N}$.

3. A real number of the form $m/2^n$ where m and n are integers, is called a dyadic rational. Prove that the set of dyadic rationals is dense in $\mathbb{R}$.

   A set is dense in $\mathbb{R}$ if for every $a < b$, there is some x in the target set such that $a < x < b$. So, if we prove that $a < \frac{m}{2^n} < b$, then we have shown that the set of dyadic rationals is dense in $\mathbb{R}$.

   By Archimedes Principle we know that there must exist some $\frac{1}{n}$ in the interval $a < b$. So, we can re-write this as $\frac{1}{n} < b - a$. By 2(a), we know that $2^n > n$, so the inequaltiy $\frac{1}{2^n} < \frac{1}{n} < b - a$ must hold. We claim that there must be some m such that $a < \frac{m}{2^n} < b$. We prove this by contradiction.

   Assume there is no m such that $a < \frac{m}{2^n} < b$. This will only be possible if the distance between the two numbers is less than $\frac{1}{2^n}$. In other words, $b - a \leq \frac{1}{2^n}$. However, this is a contradiction to what we have proven earlier since $\frac{1}{2^n} < \frac{1}{n} < b - a$. Thus, there must be some m such that $a < \frac{m}{2^n} < b$.

   Thus, the set of dyadic rationals is dense in $\mathbb{R}$.

4. For each of the following statments, determine whether it is true or false and justify your answer

   (a) The set $\mathbb{Z}$ of integers is dense in $\mathbb{R}$

   A set is dense in $\mathbb{R}$ if for every $a < b$, there is some x in the target set such that $a < x < b$. However, if we let $b - a \leq 1$, then there cannot be any $x \in \mathbb{Z}$ in this interval. For instance, if $b = 1$ and $a = 0$, then there is no $x \in \mathbb{Z}$ between 0 and 1. Thus, the set $\mathbb{Z}$ is not dense in $\mathbb{R}$.

   (b) The set of positive real numbers is dense in $\mathbb{R}$

   A set is dense in $\mathbb{R}$ if for every $a < b$, there is some x in the target set such that $a < x < b$. However, if we choose b and a to both be negative, then there cannot be any x in the set of positive real numbers in this interval. For instance, if $b = -1$ and $a = -2$, then there is no $x \in \mathbb{R}^+$ between -2 and -1. Thus, the set of positive real numbers is not dense in $\mathbb{R}$.

   (c) The set of $\mathbb{Q}/\mathbb{N}$ of rational numbers that are not integers is dense in $\mathbb{R}$

   A set is dense in $\mathbb{R}$ if for every $a < b$, there is some x in the target set such that $a < x < b$. This set is dense in $\mathbb{R}$ because, by definition, Q and N are both dense in $\mathbb{R}$. Thus, the set of $\mathbb{Q}/\mathbb{N}$ is dense in $\mathbb{R}$.

5. Suppose that the number a has the property that for every natural number n, $a \leq 1/n$. Prove that $a \leq 0$.

   We prove by contradiction. Let $a > 0$. Let ϵ be some positive real number. Then, there must exist some n such that $0 < \frac{1}{n} < \epsilon$ for all n. However, if $a > 0$, then in order for this inequality to be satisfied, $a > \frac{1}{n}$ to get $0 < \frac{1}{n} < a < \epsilon$. This is a contradiction to the fact that $a \leq 1/n$ for all n.

6. For each of the following statments, determine whether it is true or false and justify your answer

   a. If the sequence $\{a_n{}^2\}$ converges, then the sequence $\{a_n\}$ converges.

   If the sequence $\{a_n{}^2\}$ converges, then it is bounded. Since it is bounded, $\exists x \in \mathbb{R}$ such that $|a_n|^2 \leq x$. Thus, $|a_n| \leq \sqrt{x}$, therefore $\{a_n\}$ is bounded. Since $\{a_n\}$ is bounded, it converges.

   b. If the sequence $\{a_n + b_n\}$ converges, then the sequences $\{a_n\}$ and $\{b_n\}$ also converge.

   Let all elements of $\{a_n\}$ be real numbers $\mathbb{R}$ and all elements of $\{b_n\}$ be negative real numbers $\mathbb{R}^-$, then $\{a_n + b_n\}$ is will cancel out to 0 and thus converge. However, both sets are unbounded, so $\{a_n\}$ and $\{b_n\}$ do not converge

c. If the sequence $\{a_b + b_n\}$ and $\{a_n\}$ converge, then the sequence $\{b_n\}$ also converges.

Let $\{a_b + b_n\}$ converge to some value x and $\{a_n\}$ converge to some value y. Then,

$$\lim_{n\to\infty} ((a_n + b_n) - (b_n)) = \lim_{n\to\infty} a_n$$

So,

$$\begin{aligned}
\lim_{n\to\infty} ((a_n + b_n) - (b_n)) &= \lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} (b_n) \\
x &= y - \lim_{n\to\infty} (b_n) \\
x - y &= - \lim_{n\to\infty} (b_n) \\
\lim_{n\to\infty} (b_n) &= y - x
\end{aligned}$$

So, $\{b_n\}$ converges to $y - x$.

d. If the sequence $\{|a_n|\}$ converges, then the sequence $\{a_n\}$ also converges.

Let $\{a_n\} = \{(-1)^n\}$. Then, $\{a_n\}$ does not converge but $\{|a_n|\}$ does converge ($\{|a_n|\} = 1$ for all n). Thus, this is false.