Math 327 Homework 7

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1. **10**

   We want to prove that if a function $f : (a, b) \in \mathbb{R}$ is uniformly continuous, then it is bounded.

   By the $\epsilon - \delta$ definition of continuity, a function is continuous in the domain D provided that for every positive number ϵ there is a positive number δ for which

   $$|f(x) - f(y)| < \epsilon \quad \text{if} \quad |x - y| < \delta$$

   Let f be a function on (a, b) that is uniformly continuous.

   We prove by contradiction. Let f be a function on (a, b) that is uniformly continuous but not bounded.

   Then, for all $M > 0$, there exists $x \in (a, b)$ such that $|f(x)| > M$.

   Then, $f(x) > M$ or $f(x) < -M$. Then, let $M = \epsilon + f(y)$ where $|x - y| < \delta$.
   Then, $f(x) > \epsilon + f(y)$ or $f(x) < -\epsilon - f(y)$. This is equivalent to saying that $f(x) - f(y) > \epsilon$ or $f(x) - f(y) < -\epsilon$. In other words:

   $$|f(x) - f(y)| > \epsilon$$

   However, this is a contradiction to the fact that f is uniformly continuous.

2. **7**
   We are given that $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) > 0$ and $f(1) = 0$. We want to prove that there is a number $x_0 \in (0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for $0 \leq x < x_0$.

   We prove by contradiction. Let $f : [0, 1] \to \mathbb{R}$ be continuous, $f(0) > 0$, $f(1) = 0$, and $x_0 \in (0, 1]$. We say that there is no value x_0 for which $f(x_0) = 0$ and $f(x) > 0$ for $0 \leq x < x_0$. Let $x_0 = 1$ and $x = 0$. Then, $0 \leq x < x_0$ with $x_0 \in (0, 1]$. However, $f(x_0) = 0$ and $f(x) > 0$ from our assumptions. This is a contradiction. Thus, there is a value $x_0 \in (0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for $0 \leq x < x_0$.

3. **4**
   We are given that $f : [-1, 1] \to \mathbb{R}$ is continuous, $f(-1) > -1$ and $f(1) < 1$. We want to prove that f has a fixed point where a fixed point is the point at which the line $y = x$ intersects the graph of f.

   Let's define another function $g : [-1, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then, g is continuous since f is continuous and x is continous. We also have that $g(-1) = f(-1) - (-1) > 0$ and $g(1) = f(1) - 1 < 0$.

   In other words

   $$g(1) < 0 < g(-1)$$

By the Intermediate Value Theorem, there is a number $x_0 \in (-1, 1)$ such that $g(x_0) = 0$. Then, $f(x_0) = x_0$ since $g(x) = f(x) - x$.

Thus, f has a fixed point.

4. **1**

   a
   False. Let a function be defined as $f(x) = \frac{1}{x-0.5}$. Then, this function does not have a maximum on the interval [0, 1]

   b
   Yes, if a functionis continous on a closed interval, it must have a minimum. If it were to not have a minimum, then it cannot be continous since it would approach infinty

   c
   No, if the function has it's maximum at x = 1, then the the interval (0, 1) does not contain the maximum.

   d
   No, the function itself is unbounded so its image cannot be bounded.

   e
   Yes, if the image is bounded below there must be some $x \in (0, 1)$ such that $f(x) =$ minimum.

   2

   a
   x = 1 is the maximizer for this function.

   b
   x = 0 is the maximizer for this function

   c
   x = -1 is the maximizer for this function

5. **1**

   a
   No, let $f(x) = 1$ be our function. Then, $f(\mathbb{R}) = 1$.

   b
   No, if the function is unbounded then its image cannot be an interval

   c
   No, if the function is unbounded then its image cannot be an interval

   d
   Yes, since $f(0) < f(x) < f(1)$ for all $0 < x < 1$, the image is the interval [f(0), f(1)]

6. **3**

   Let $f(x) = \frac{1}{\sqrt{x+x^2}} + x^2 - 2x$. Notice that $f(1) < 0$ and $f(2) > 0$. Thus, there must exist some $x \in (1, 2)$ for which $f(x) = 0$ by the Intermediate Value Theorem.