Math 327 Homework 6

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1. **7 a**

   We want to prove that the function is continuous on the interval $[0, 1]$.

   Let x_n be a sequence in $(0, 1]$ such that $x_n \to x_0 \in [0, 1]$. By the sum and product properties of convergent sequences, we have

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \sqrt{x_n} \\ &= \sqrt{x_0} \\ &= f(x_0)\end{aligned}$$

   When $x_0 = 0$,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \sqrt{x_0} \\ &= 0 \\ &= f(x_0)\end{aligned}$$

   Thus, f is continuous at x_0, thus f is continous on the interval $[0, 1]$

   b

   We want to prove that the function is uniformly continuous on the interval $[0, 1]$.

   Let u_n and v_n be sequences in $[0, 1]$ such that

$$lim_{n\to\infty}[u_n - v_n] = 0$$

   We want to prove that

$$lim_{n\to\infty}|f(u_n) - f(v_n)| = 0$$

   We shall prove so by arguing the contradition. Suppose that the differences between the two limits is not equal to 0. Then, thre must exist some $\epsilon > 0$ such that

$$|f(u_n) - f(v_n)| \geq \epsilon$$

   for all n.

   We know, however, that the domain of f is $[0, 1]$. By the Sequential Compactness Theorem, there exists a subsequence u_{n_k} of u_n and a point x_0 in $[0, 1]$ such that

$$lim_{k\to\infty}u_{n_k} = x_0$$

Similarly, we also conclude that there exists a subsequence v_{n_k} of v_n and a point x_0 in $[0,1]$ such that

$$lim_{k\to\infty} v_{n_k} = x_0$$

Knowing, however, that f is continuous at x_0, we have

$$f(u_{n_k}) = f(x_0) = f(v_{n_k})$$

for all k. Thus, we have

$$|f(u_{n_k}) - f(v_{n_k})| = 0$$

for all k.

This contradicts our assumption that there exists some $\epsilon > 0$ such that

$$|f(u_n) - f(v_n)| \geq \epsilon$$

for all n. Thus, we have proved that the function is uniformly continuous on the interval $[0,1]$.

c

We want to prove that the function is not Lipschitz. We prove by contradiction
Suppose there $\exists C \in \mathbb{R}$ such that $|f(x) - f(y)| \leq C|x-y|$ for any $x, y \in [0,1]$.

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &\leq C|x-y| = C|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| \\ 1 &\leq C|\sqrt{x} + \sqrt{y}| \\ \frac{1}{c} &\leq |\sqrt{x} + \sqrt{y}| \end{aligned}$$

For any $x, y \in [0,1]$ where $x \neq y$. However, this cannot be true. For instance, let $y = 0$ and $x = \frac{1}{c+1}$. Since $\frac{1}{c} > \frac{1}{c+1}$, $\frac{1}{c} > \sqrt{\frac{1}{c+1}}$. This is a contradiction to our equation above. Thus, we have proved that the function is not Lipschitz.

2. **10**
We prove both directions using contradiction. First, we assume that f is uniformly continuous, but it does not satisfy the $\epsilon - \delta$ criterion. Then, there exists some $\epsilon > 0$ such that for any $\delta > 0$, there exists some $u, v \in D$ such that $|u - v| < \delta$ but $|f(u) - f(v)| \geq \epsilon$.

Let $\delta = \frac{1}{n}$. Then, there exists sequences $u_n, v_n \in D$ such that $|u_n - v_n| < \frac{1}{n}$ and $|f(u_n) - f(v_n)| \geq \epsilon$.

Taking the limit as $n \to \infty$, we have

$$lim_{n\to\infty} |u_n - v_n| = 0$$

Which implies

$$lim_{n\to\infty} |f(u_n) - f(v_n)| = 0$$

As well since f is uniformly continuous. However, this is a contradiction to our assumption that $|f(u_n) - f(v_n)| \geq \epsilon$!

Next, we prove the other way. Let the $\epsilon - \delta$ criterion be met, but f is not uniformly continuous. Then, there exists some $\epsilon > 0$ such that for any $\delta > 0$, there exists some $u, v \in D$ such that $|u - v| < \delta$ and $|f(u) - f(v)| < \epsilon$.

Then, there exist some N such that for any $n \geq n$,there are sequences $u_n, v_n \in D$ such that $|u_n - v_n| < \delta$ and $|f(u_n) - f(v_n)| < \epsilon$.

Furthermore, we let $\lim_{n\to\infty} |u_n - v_n| = 0$.

Thus, knowing that both sequences converge to the same point, the image of our sequences can be written as $|(f(u_n) - f(v_n)) - 0| < \epsilon$. Thus,

$$lim_{n\to\infty}|f(u_n) - f(v_n)| = 0$$

.

3. **14 a**

We define $m = f(1)$. We want to prove that given the property that f has the property

$$f(u + v) = f(u) + f(v)$$

for all $u, v \in \mathbb{R}$, we have

$$f(x) = mx$$

for all $x \in \mathbb{R}$.

We prove by induction. Let's first consider all Natural Numbers. We know that $f(1) = m$. We have the base case as

$$f(1) = m \times 1 = m$$

Assume this property holds for all $x = k$. Thus,

$$f(k) = mk$$

We want to prove that this property holds for $x = k + 1$. We have

$$\begin{aligned} f(k+1) &= f(k) + f(1) \\ &= mk + m \\ &= (k+1)m \end{aligned}$$

Thus, we have proved that the property holds for all $x \in \mathbb{N}$.

We now consider $f(0)$. We can re-write this as

$$\begin{aligned} f(0) &= f(1-1) \\ &= f(1) - f(1) \\ &= m - m \\ &= 0 \end{aligned}$$

We now consider negative x values. We have

$$
\begin{aligned}
f(-x) &= f(1-1-x) \\
&= (f(1)-f(1)) - f(x) \\
&= (m-m) - f(x) \\
&= 0 - f(x) \\
&= -f(x) \\
&= -mx
\end{aligned}
$$

Now. we wish to prove this for any rational number that can be expressed $\frac{p}{q}$. Let q be positive. So, we wish to prove

$$
f(\frac{p}{q}) = m\frac{p}{q}
$$

$$
\begin{aligned}
mp &= f(p) \\
mp &= f(q\frac{p}{q}) \\
mp &= f(\frac{p}{q} + \frac{p}{q} + ... + \frac{p}{q}) \text{ where there are q number of } \frac{p}{q} \text{ terms} \\
mp &= f(\frac{p}{q}) + f(\frac{p}{q}) + ... + f(\frac{p}{q}) \\
mp &= qf(\frac{p}{q}) \\
m\frac{p}{q} &= f(\frac{p}{q})
\end{aligned}
$$

Thus, we have proved that $f(x) = mx$ for all rational numbers x.

b

We wish to prove that $f(x) = mx$ for all x. We are given that f is continuous and that $f(x) = mx$ for all rational numbers x.

If f is continuous, then there exists some sequence x_n such that if $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x_0$.

Thus, the $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Thus, we have proved that $f(x) = mx$ for all x.

4. **4**

   We want to prove the function does not satisfy the $\epsilon - \delta$ criterion at the point $x_0 = \frac{3}{4}$. Indeed, take $\epsilon = 1.875$. There is no positive number δ having the property such that

   $$
   -1.875 < f(x) < 1.875
   $$

   Around the point

   $$
   \frac{3}{4}
   $$

   . This, the function does not satisfy the $\epsilon - \delta$ criterion.

5. **1 a**

   No, let f be a function that is defined at every point except $x = 0$ and let g be defined only at $x = 0$. Then, nether function is continuous at every point, but their sum is continuous at every point.

b

Yes, if the function squared is continous at every point, then the function is continuous at every point becuase every number must have a positive square root.

c

No, if g and $f + g$ are continuous, that does not necessarily mean that f is continuous. Let f be defined as a piecewise function with a break at $x = 0$. Then, f is not continuous at $x = 0$, but $f + g$ is continuous at $x = 0$ since the g component would "fill in" the gap.

d

Yes, since $\mathbb{N}$ is dense in $\mathbb{R}$, we can find a sequence of rational numbers that converges to x_0. Thus, the limit of $f(x_n)$ is $f(x_0)$.

6. **6**

   For this function to be continous at a point, $x^2 = -x^2$ at that point. Thus, $x = 0$. However, there is no irrational number whose square is 0. Thus, the function is not continuous at any point.