Math 327 Homework 4

Sathvik Chinta

October 28th, 2022

1. **11 a**

   First, we prove that $\frac{\alpha+\beta}{2} \geq \sqrt{\alpha}\beta$.

$$
\begin{aligned}
(\sqrt{\alpha} - \sqrt{\beta})^2 &\geq 0 \\
(\sqrt{\alpha} - \sqrt{\beta})(\sqrt{\alpha} - \sqrt{\beta}) &\geq 0 \\
\alpha - 2\sqrt{\alpha}\sqrt{\beta} + \beta &\geq 0 \\
\alpha + \beta &\geq 2\sqrt{\alpha}\sqrt{\beta} \\
\frac{\alpha+\beta}{2} &\geq \sqrt{\alpha}\sqrt{\beta} \\
\frac{\alpha+\beta}{2} &\geq \sqrt{\alpha\beta}
\end{aligned}
$$

   Now, we are given two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ with initial values $a_1 = a$ and $b_1 = b$. We want to prove that for all $n \geq 2$:

$$a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$$

   We proved already that $\frac{\alpha+\beta}{2} \geq \sqrt{\alpha}\beta$ for all $\alpha, \beta \geq 0$. Thus, we know that $a_{n+1} = \frac{a_n+b_n}{2} \geq b_{n+1} = \sqrt{a_n b_n}$ for all $n \geq 1$. Since $a_n \geq b_n$ for all $n \geq 2$, the arithemtic mean of the two values must be less than or equal to a_n but greater than or equal to b_n. Thus, we have $a_n \geq a_{n+1} \geq b_n$. Since $a_n \geq b_n$ for all $n \geq 2$, the geometric mean of these two numbers must be greater than or equal to b_n but less than or equal to a_n. Thus, we have $a_n \geq b_{n+1} \geq b_n$.

   Combining all that we have proved, we have $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for all $n \geq 2$.

   b We look at the sequence $\{a_n\}$ first. We proved previousy that $a_n \geq a_{n+1}$. Thus, we know the sequence must be monotonically decreasing. We further know that $a_{n+1} \geq b_{n+1}$ for all $n \geq 1$. Thus, we know that the sequence of a_n must be bounded below by the sequence of b_n. Thus, the sequence of a_n must converge.

   Similarly, we can say that the sequence of b_n must be monotonically increasing and bounded above by the sequence of a_n. Thus, the sequence of b_n must converge.

   By the nested inteval theorem, we know that there is exactly one point x that belongs to the interval $[a_n, b_n]$ for all $n \geq 1$, and both sequences converge to x. Thus, we have $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

   Thus, the sequence $\{a_n\}$ and $\{b_n\}$ converge to the same point x, and have the same limit.

2. The solution to the equation $x^2 - x - c = 0$ where $c, x > 0$ is

$$\frac{\sqrt{4c+1}+1}{2}$$

   We want to prove that the recursively defined sequence

$$x_{n+1} = \sqrt{c + x_n}$$

Where $x_1 > 0$ converges monotonically to the same solution.

Let $f(x) = \sqrt{c+x}$. Then, we say that $x_{n+1} = f(x)$ for ease of notation. Notice that if we plug in the solution to our equation, we get $f(\frac{\sqrt{4c+1}+1}{2}) = \frac{\sqrt{4c+1}+1}{2}$.

Let $0 < a_n < \frac{\sqrt{4c+1}+1}{2}$ for any a_n. We can then represent $a_n = \frac{\sqrt{4c+1}+1}{2} - \alpha$ for some $\alpha \in \mathbb{R}$ where $0 < \alpha < \frac{\sqrt{4c+1}+1}{2}$. $f(x_n) = f(\frac{\sqrt{4c+1}+1}{2} - \alpha) = \frac{\sqrt{-2(2\alpha - \sqrt{4c+1} - 2c - 1)}}{2}$. Notice that $\frac{\sqrt{-2(2\alpha - \sqrt{4c+1} - 2c - 1)}}{2} > \frac{\sqrt{4c+1}+1}{2} - \alpha$ for all $\alpha \in \mathbb{R}$ where $0 < \alpha < \frac{\sqrt{4c+1}+1}{2}$. Thus, we know that this sequence is monotonically increasing.

Now, we let $b_n > \frac{\sqrt{4c+1}+1}{2}$. Thus, we can represent $b_n = \frac{\sqrt{4c+1}+1}{2} + \beta$ for some $\beta \in \mathbb{R}$. $f(x_n) = f(\frac{\sqrt{4c+1}+1}{2} + \beta) = \frac{\sqrt{2(2\beta + \sqrt{4c+1} + 2c + 1)}}{2}$. Notice that $\frac{\sqrt{2(2\beta + \sqrt{4c+1} + 2c + 1)}}{2} < \frac{\sqrt{4c+1}+1}{2} + \beta$ for all $\beta \in \mathbb{R}$. Thus, we know that this sequence is monotonically decreasing.

So, we can write $a_n < a_{n+1} < \frac{\sqrt{4c+1}+1}{2} < b_{n+1} < b_n$ for all n. Thus, by the nested interval theorem we know that these two sequences must converge to the same point ($\frac{\sqrt{4c+1}+1}{2}$), and that point must be the solution to the equation $x^2 - x - c = 0$.

3. We prove by contradiction. Let $\limsup_{n\to\infty} x_n \neq \sup A$. Thus, the sequence $\{x_n\}$ converges to some value $\alpha \neq \sup A$. Thus, $\alpha < \sup A$ or $\alpha > \sup A$. If $\alpha < \sup A$, then there must be some convergent subsequence of $\{x_n\}$ that converges to $\sup A$ by the definition of A, meaning that $\limsup_{n\to\infty} x_n \neq \alpha$ which is a contradiction. If $\alpha > \sup A$, then there is some convergent subsequence of $\{x_n\}$ that converges to α by the definition of A, meaning that α must be contained in the set A, which would make it the supremum of A, which is a contradiction. Thus, we have proven that $\limsup_{n\to\infty} x_n = \sup A$.

   We can follow a similar proof for the infimum. Let $\liminf_{n\to\infty} x_n \neq \inf A$. Thus, the sequence $\{x_n\}$ converges to some value β which is not the infimum of A. Thus, $\beta < \inf A$ or $\beta > \inf A$. If $\beta < \inf A$, then there is some convergent subsequence of $\{x_n\}$ that converges to β by the definition of A, meaning that β must be contained in the set A, which would make it the infimum of A, which is a contradiction. If $\beta > \inf A$, then there must be some convergent subsequence of $\{x_n\}$ that converges to $\inf A$ by the definition of A, meaning that $\liminf_{n\to\infty} x_n \neq \beta$ which is a contradiction. Thus, we have proven that $\liminf_{n\to\infty} x_n = \inf A$.

   Thus, we have proven that $\limsup_{n\to\infty} x_n = \sup A$ and $\liminf_{n\to\infty} x_n = \inf A$.

4. **i**

   ii

5. **i** Let $f(n) = \sqrt{n+1} - n$. Notice that $\sqrt{n+1} - n > \sqrt{n+2} - n + 1$ for all n. Thus, this sequence is monotonically decreasing. However, there is no bound on the sequence, so we cannot say that it converges.

   ii Let $f(n) = \sqrt{n+1} - \sqrt{n}$. Notice that $\sqrt{n+1} - \sqrt{n} > \sqrt{n+2} - \sqrt{n+1}$ for all n. Thus, this sequence is monotonically decreasing. However, notice that when $n < 0$, our equation is undefined. Thus, the limit of this sequence is 0.

   iii Let $f(n) = \sqrt{4n^2 + n - 1} - 2n$. $f(n) < f(n+1)$ for all $n \geq \sqrt{\sqrt{17} - 18}$. Thus, the sequence is monotonically inscreasing. This sequence will only have a limit if there is an upper bound when $n \to \infty$. Let M be such a limit. Thus, there exists no value for which $f(n) > M$ for all $n \geq \sqrt{\sqrt{17} - 18}$. Thus, we can write

$$\begin{aligned} f(n) &< M \\ \sqrt{4n^2 + n - 1} - 2n &< M \\ n &< \frac{-(m^2+1)}{4m-1} \end{aligned}$$

   The right side is udefined for when $M = \frac{1}{4}$, thus the limit of this sequence is $\frac{1}{4}$.

   iv Let $f(n) = (5^n + 3^n)^{\frac{1}{n}}$.

6.