Math 327 Homework 4

Sathvik Chinta

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1. Using only the Archimedean property of $\mathbb{R}$, give a direct $\epsilon - N$ verifictation of the convergence of the following sequences

   a. $\frac{2}{\sqrt{n}} + \frac{1}{n} + 3$

   For any $\epsilon > 0$, by the Archimedean property there exists some $N \in \mathbb{N}$ with $N > \frac{\sqrt{3}}{\epsilon}$. Then, for any $n \geq N$

   $$n \leq \frac{3}{n^2} \leq \frac{3}{N^2} < \epsilon^2$$

   So $|a_n - 3| < \epsilon$ where a_n is the sequence.

   So, the sequence converges to 3.

   b. $\frac{n^2}{n^2+n}$

   Given $\epsilon > 0$, by the Archimedean property there exists some $N \in \mathbb{N}$ with $N < \epsilon$. Then, for any $n \geq N$

   $$n \geq \frac{1}{n} \geq \frac{1}{N} > \frac{1}{\epsilon}$$

   So $|a_n - 1| < \epsilon$ where a_n is the sequence.

   So, the sequence converges to 1.

2. Prove that

   $$\lim_{x \to -\infty} n^{\frac{1}{n}} = 1$$

   First, we define

   $$\alpha_n = n^{\frac{1}{n}} - 1$$

   This implies
   $n^{\frac{1}{n}} = 1 + \alpha_n$
   Raising both sides to the nth power, we get
   $n = (1 + \alpha_n)^n$
   Doing binomaial expansion to the right hand side, we get
   $n = 1 + n\alpha_n + \frac{n(n-1)}{2}\alpha_n^2 + \frac{n(n-1)(n-2)}{6}\alpha_n^3 + \cdots$
   Notice that $n \geq \frac{n(n-1)}{2}\alpha_n^2$. Dividing both sides by n, we get
   $1 \leq \frac{n-1}{2}\alpha_n^2$. Re-arranging the terms, we have
   $\alpha_n^2 \leq \frac{2}{n-1}$.
   For any $\epsilon > 0$, by the Archimedean property there exists some $N \in \mathbb{N}$ with $N > \frac{2}{\epsilon^2} + 1$. Then, for any $n \geq N$

   $$\alpha_n^2 \leq \frac{2}{n-1} \leq \frac{2}{N-1} < \epsilon^2$$

So, $|\alpha_n - 0| < \epsilon$ and thus the limit of α_n as $n \to \infty$ is 0.

Since we defined α_n as $n^{\frac{1}{n}} - 1$, we know that the limit of $n^{\frac{1}{n}}$ as $n \to \infty$ is 1.

3. Suppose that the sequence $\{a_n\}$ converges to a and that $|a| < 1$. Prove that the sequence $\{(a_n)^n\}$ converges to 0.

   Given that $|a| < 1$, we know that there must exist some $\epsilon \in \mathbb{R}$ such that $\epsilon = \frac{1-|a|}{2}$.
   Thus, there exists some $N \in \mathbb{N}$ so that $|a_n - a| < \epsilon$ for all $n \geq N$. This implies that $|a_n| < |a+\epsilon|$ for all $n \geq N$. This, in turn, implies $|a_n|^n < |a+\epsilon|^n$.

   By proposition 2.28 in the textbook, since $|a+\epsilon| < 1$, $|a+\epsilon|^n$ converges to 0. Since $|a_n|^n < |a+\epsilon|^n$ for all $n \geq N$, we know that $(a_n)^n$ converges to 0 as well.