Math 327 Homework 1

Sathvik Chinta

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1. Let x and y be two positive numbers.

   (i) Use the mathematical induction to show that if $x < y$, then $x^n < y^n$ for all n $\in \mathbb{N}$.

   First, we let $k = 1$. Given that $x < y$, $x^k < y^k = x^1 < y^1 = x < y$ which we are given so it is true.

   Now, we assume this is true for k. We want to show that it is true for $k + 1$.

   $$x^k < y^k$$

   Since $x < y$, we can multiply both sides by x to get

   $$x^{k+1} < y^k x$$

   Knowing that $x < x^k < y^k$, we can substitute x for y^k since the inequality will still hold. Thus, we can write

   $$x^{k+1} < y^k y^k$$
   $$x^{k+1} < y^{k+1}$$

   (ii) Deduce that if $x^n < y^n$ for some $n \in \mathbb{N}$, then $x < y$.

   Assume that $x^n < y^n$ for all n, but $x \geq y$. We then have two cases,

   Case 1: $x = y$.

   If $x = y$. We can thus multiply both sides by x and y respectively (they are both equal, so the order is irrelevant) n times to get $x^n = y^n$ for all n. This is a contradiction to our original statment of $x^n < y^n$, so $x \neq y$.

   Case 2: $x > y$.

   If $x > y$, we can multiply both sides by y n times to get

   $$xy^n > yy^n$$

   Since $x^n < y^n$, we can substitute y^n for x^n since the inequality will still hold. Thus, we can write

   $$xx^n > yy^n$$

   $$x^{n+1} > y^{n+1}$$

   For all n. However, if we plug in $n = n - 1$, we get

   $$x^n < y^n$$

   which is a contradiction to our original statement of $x^n < y^n$, so x cannot be less than y.

   Thus, we have shown that $x < y$.

2. Do problem 17 on page 11 of the textbook [F]

3. Suppose that S is a non-empty set of real numbers that is bounded. Prove that $\inf S \leq \sup S$, and the quality holds if and only if S consists of exactly one number.

4. Do Problem 10 on page 11 of the textbook [F].