

# Math 327 Homework 4

Sathvik Chinta

October 28th, 2022

1. Using only the Archimedean property of  $\mathbb{R}$ , give a direct  $\epsilon - N$  verification of the convergence of the following sequences

a.  $\frac{2}{\sqrt{n}} + \frac{1}{n} + 3$

For any  $\epsilon > 0$ , by the Archimedean property there exists some  $N \in \mathbb{N}$  with  $N > \frac{\sqrt{3}}{\epsilon}$ . Then, for any  $n \geq N$

$$n \leq \frac{3}{n^2} \leq \frac{3}{N^2} < \epsilon^2$$

So  $|a_n - 3| < \epsilon$  where  $a_n$  is the sequence.

So, the sequence converges to 3.

b.  $\frac{n^2}{n^2+n}$

Given  $\epsilon > 0$ , by the Archimedean property there exists some  $N \in \mathbb{N}$  with  $N < \epsilon$ . Then, for any  $n \geq N$

$$n \geq \frac{1}{n} \geq \frac{1}{N} > \frac{1}{\epsilon}$$

So  $|a_n - 1| < \epsilon$  where  $a_n$  is the sequence.

So, the sequence converges to 1.

2. Prove that

$$\lim_{x \rightarrow -\infty} n^{\frac{1}{n}} = 1$$

First, we define

$$\alpha_n = n^{\frac{1}{n}} - 1$$

This implies

$$n^{\frac{1}{n}} = 1 + \alpha_n$$

Raising both sides to the  $n$ th power, we get

$$n = (1 + \alpha_n)^n$$

Doing binomial expansion to the right hand side, we get

$$n = 1 + n\alpha_n + \frac{n(n-1)}{2}\alpha_n^2 + \frac{n(n-1)(n-2)}{6}\alpha_n^3 + \dots$$

Notice that  $n \geq \frac{n(n-1)}{2}\alpha_n^2$ . Dividing both sides by  $n$ , we get

$$1 \leq \frac{n-1}{2}\alpha_n^2. \text{ Re-arranging the terms, we have}$$

$$\alpha_n^2 \leq \frac{2}{n-1}.$$

For any  $\epsilon > 0$ , by the Archimedean property there exists some  $N \in \mathbb{N}$  with  $N > \frac{2}{\epsilon^2} + 1$ . Then, for any  $n \geq N$

$$\alpha_n^2 \leq \frac{2}{n-1} \leq \frac{2}{N-1} < \epsilon^2$$

So,  $|\alpha_n - 0| < \epsilon$  and thus the limit of  $\alpha_n$  as  $n \rightarrow \infty$  is 0.

Since we defined  $\alpha_n$  as  $n^{\frac{1}{n}} - 1$ , we know that the limit of  $n^{\frac{1}{n}}$  as  $n \rightarrow \infty$  is 1.

3. Suppose that the sequence  $\{a_n\}$  converges to  $a$  and that  $|a| < 1$ . Prove that the sequence  $\{(a_n)^n\}$  converges to 0.

Given that  $|a| < 1$ , we know that there must exist some  $\epsilon \in \mathbb{R}$  such that  $\epsilon = \frac{1-|a|}{2}$ .

Thus, there exists some  $N \in \mathbb{N}$  so that  $|a_n - a| < \epsilon$  for all  $n \geq N$ . This implies that  $|a_n| < |a + \epsilon|$  for all  $n \geq N$ . This, in turn, implies  $|a_n|^n < |a + \epsilon|^n$ .

By proposition 2.28 in the textbook, since  $|a + \epsilon| < 1$ ,  $|a + \epsilon|^n$  converges to 0. Since  $|a_n|^n < |a + \epsilon|^n$  for all  $n \geq N$ , we know that  $(a_n)^n$  converges to 0 as well.

4. We define  $D$  as a set of sequences. If we write

$n_0 n_1 n_2 \dots$  for all  $n_k \in D$ , then we get  $n_0.012345678012345678 \dots$ . Since every element of  $D$  is encompassed in this decimal, it gives an order on  $D$ . If we define  $j$  as the smallest non-negative integer such that  $n_j \neq m_j$ , we can multiply both  $n_0.n_1 n_2 n_3 \dots$  and  $m_0.m_1 m_2 m_3 \dots$  by  $10^j$  to get  $n_0 n_1 n_2 \dots n_j . n_{j+1} \dots$  and  $m_0 m_1 m_2 \dots m_j . m_{j+1} \dots$ . Since  $n_j \neq m_j$ . Using this definition, the  $\prec$  operator behaves the same as the inequality operator, thus making  $D$  a linear order and making it so that if  $x, y \in D$ , then only one of the following statements is true:  $x \prec y$ ,  $x \succ y$ , or  $x = y$ . Furthermore, if  $x, y, z \in D$  and  $x \prec y$  and  $y \prec z$ , then  $x \prec z$ .

5. Properties

i. If  $n_k > 9$ , then  $\frac{n_1}{10} \geq 1$ , thus in the boundary  $(x_0, x_1)$  there must exist some integer  $> x$ , which is a contradiction to our initial conditions.

ii.

iii. Since  $n_k \in \{0, 1, 2, \dots, 9\}$  for all  $k \in \mathbb{N}$ , and  $k \geq 0$ ,  $\frac{n_k}{10^k} > 0$ , so  $x_{k-1} \leq x_k$  for all  $k \in \mathbb{N}$ .

iv.

v. Since the sequence is monotonically increasing, if  $x \preceq y$  in  $\mathbb{R}$ , this implies the choice of the initial  $n_0$  is smaller. Thus,  $x < y$  implies  $D(x) \prec D(y)$  and vice versa.

6. i. For each of the following statements, determine whether it is true or false and justify your answer

a. Every bounded sequence converges

False. Let the sequence consist of entirely 0s and 1s, each alternating in occurrence. In this case, bounds are known but the sequence does not converge.

b. A convergent sequence of positive numbers has a positive limit

True, a negative limit would imply that the sequence is bounded below by a negative number and that the sequence is monotonically decreasing. If the set only contains positive numbers, 0 will always be a lower bound and thus the lowest possible value for a limit is 0.

c. The sequence  $\{n^2 + 1\}$  converges

No, the sequence will tend towards infinity as  $n \rightarrow \infty$ .

d. A convergent sequence of rational numbers has a rational limit

False, a convergent sequence of rational numbers can have an irrational limit. For example, the sequence  $(1 + \frac{1}{n})^n$  converges to  $e$ .

e. The limit of a convergent sequence in the interval  $(a, b)$  also belongs to  $(a, b)$

True, if a sequence is in an interval and it converges to a number in that interval, then the limit must also be in that interval.

ii. Show that the set of irrational numbers fails to be closed

We let  $x$  to be a rational number. We let  $a_n$  be the sequence of irrational numbers such that  $a_n$  converges to  $x$ . Since the set of irrational numbers cannot contain  $x$ , but there exists a sequence within the set of irrational numbers that converges to  $x$ , the set of irrational numbers cannot be closed.