Math 327 Homework 6

Sathvik Chinta

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1. **7 a**

We want to prove that the function is continuous on the interval $[0, 1]$.

Let x_n be a sequence in $[0, 1]$ such that $x_n \to x_0 \in [0, 1]$. By the sum and product properties of convergent sequences, we have

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \sqrt{x_n} \\ &= \sqrt{x_0} \\ &= f(x_0)\end{aligned}$$

Thus, f is continuous at x_0.

b

We want to prove that the function is uniformly continuous on the interval $[0, 1]$.

Let u_n and v_n be sequences in $[0, 1]$ such that

$$lim_{n\to\infty}[u_n - v_n] = 0$$

We want to prove that

$$lim_{n\to\infty}|f(u_n) - f(v_n)| = 0$$

We shall prove so by arguing the contradition. Suppose that the differences between the two limits is not equal to 0. Then, thre must exist some $\epsilon > 0$ such that

$$|f(u_n) - f(v_n)| \geq \epsilon$$

for all n.

We know, however, that the domain of f is $[0, 1]$. By the Sequential Compactness Theorem, there exists a subsequence u_{n_k} of u_n and a point x_0 in $[0, 1]$ such that

$$lim_{k\to\infty}u_{n_k} = x_0$$

Similarly, we also conclude that there exists a subsequence v_{n_k} of v_n and a point x_0 in $[0, 1]$ such that

$$lim_{k\to\infty}v_{n_k} = x_0$$

Knowing, however, that f is continuous at x_0, we have

$$f(u_{n_k}) = f(x_0) = f(v_{n_k})$$

for all k. Thus, we have

$$|f(u_{n_k}) - f(v_{n_k})| = 0$$

for all k.

This contradicts our assumption that there exists some $\epsilon > 0$ such that

$$|f(u_n) - f(v_n)| \geq \epsilon$$

for all n. Thus, we have proved that the function is uniformly continuous on the interval $[0, 1]$.

c

We want to prove that the function is not Lipschitz. We prove by contradiction
Suppose there $\exists C \in \mathbb{R}$ such that $|f(x) - f(y)| \leq C|x - y|$ for any $x, y \in [0, 1]$.

$$\begin{aligned}
|\sqrt{x} - \sqrt{y}| \leq C|x - y| &= C|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| \\
1 &\leq C|\sqrt{x} + \sqrt{y}| \\
\frac{1}{c} &\leq |\sqrt{x} + \sqrt{y}|
\end{aligned}$$

For any $x, y \in [0, 1]$ where $x \neq y$. However, this cannot be true. For instance, let $y = 0$ and $x = \frac{1}{c+1}$. Since $\frac{1}{c} > \frac{1}{c+1}$, $\frac{1}{c} > \sqrt{\frac{1}{c+1}}$. This is a contradiction to our equation above. Thus, we have proved that the function is not Lipschitz.