

Math 327 Homework 1

Sathvik Chinta

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1. Let x and y be two positive numbers.

(i) Use the mathematical induction to show that if $x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$.

First, we let $k = 1$. Given that $x < y$, $x^k < y^k = x^1 < y^1 = x < y$ which we are given so it is true.

Now, we assume this is true for k . We want to show that it is true for $k + 1$.

$$x^k < y^k$$

Since $x < y$, we can multiply both sides by x to get

$$x^{k+1} < y^k x$$

Knowing that $x < y$, we can substitute x for y since the inequality will still hold. Thus, we can write

$$\begin{aligned} x^{k+1} &< y^k y \\ x^{k+1} &< y^{k+1} \end{aligned}$$

(ii) Deduce that if $x^n < y^n$ for some $n \in \mathbb{N}$, then $x < y$.

Assume that $x^n < y^n$ for all n , but $x \geq y$. We then have two cases,

Case 1: $x = y$.

If $x = y$. We can thus multiply both sides by x and y respectively (they are both equal, so the order is irrelevant) n times to get $x^n = y^n$ for all n . This is a contradiction to our original statement of $x^n < y^n$, so $x \neq y$.

Case 2: $x > y$.

If $x > y$, we can multiply both sides by y n times to get

$$xy^n > yy^n$$

Since $x^n < y^n$, we can substitute y^n for x^n since the inequality will still hold. Thus, we can write

$$xx^n > yy^n$$

$$x^{n+1} > y^{n+1}$$

For all n . However, if we plug in $n = n - 1$, we get

$$x^n < y^n$$

which is a contradiction to our original statement of $x^n < y^n$, so x cannot be less than y .

Thus, we have shown that $x < y$.

2. Do problem 17 on page 11 of the textbook [F]. Define

$$S = \{x \mid x \text{ in } \mathbb{R}, x \geq 0, x^2 < c\}$$

a. Show that $c + 1$ is an upper bound for S and therefore, by the Completeness Axiom, S has a least upper bound that we denote by b .

We know that every element of the set S must be greater than or equal to 0. Furthermore, we know that every element in x^2 must be less than c . First, we prove that for all $x \geq 0$, $x \leq x^2$. When $x = 0$, we have

$$0 \leq 0^2 = 0$$

When $x > 0$ (equivalent to $x \geq 1$), we can substitute x^2 for xx to get

$$x \leq xx$$

Dividing both sides by x gives us

$$1 \leq x$$

Which is the exact solution set we wanted. Thus, we have shown that for all $x \geq 0$, $x \leq x^2$.

Now, we can say that $x \leq x^2 < c$. For all integers c , $c + 1 > c$. Thus, we can write

$$x \leq x^2 < c \leq c + 1$$

Thus, we know that $x < c + 1$ for all x in S , so $c + 1$ is an upper bound for S .

By the completeness axiom, if S has some upper bound, then there must be some least upper bound for the set S . Thus, there must exist some b such that b is an upper bound for S and b is the least upper bound for S .

b. Show that if $b^2 > c$, then we can choose a suitably small positive number r such that $b - r$ is also an upper bound for S , thus contradicting the choice of b as the *least* upper bound for S .

We know that $x^2 < c$ for all x in S . If we take the square root from both sides, we find that the theoretical solution set is $x \in \mathbb{R}$ such that $x \geq 0$ and $x < \sqrt{c}$. Knowing that $b^2 > c$, we can take the square root from both sides of the inequality to get $b > \sqrt{c}$. Thus, we know that $b > \sqrt{c}$. We can then write $x < \sqrt{c} < b$. If we take the difference between both sides, we get $x < b - \sqrt{c}$. Thus, we know that $b - \sqrt{c}$ must also be an upper bound for S . Since c is positive, we know that $b - \sqrt{c} < b$. Thus, b cannot be the *least* upper bound of S .

c. Show that if $b^2 < c$, then we can choose a suitably small positive number r such that $b + r$ belongs to S , thus contradicting the choice of b as *an* upper bound for S .

We assume that $b^2 < c$ and show that there must exist some r such that $b + r$ belongs to S . We can expand $(b + r)^2$ as

$$(b + r)^2 = b^2 + 2br + r^2$$

Let $r \leq b$, meaning that $r^2 \leq br$. Thus, we can write

$$(b + r)^2 = b^2 + 2br + r^2 \leq b^2 + 3br$$

Now, we assumed $r \leq b$. We can further impose write $r < \frac{c-b^2}{3b}$ since $b^2 < c$ and $b < \sqrt{c}$.

Thus, we get

$$(b + r)^2 = b^2 + 2br + r^2 \leq b^2 + 3br < b^2 + 3b\left(\frac{c-b^2}{3b}\right) = b^2 + c - b^2 = c$$

Thus we have shown that $b + r$ belongs to S , meaning that $b + r < b$ which is a contradiction! So b cannot be an upper bound for S .

d. Use parts (b) and (c) and the Positivity Axioms for \mathbb{R} to conclude that $b^2 = c$.

We know by the Completeness Axiom that S must contain a least upper bound b . From part (b), we know that $b^2 > c$ is a contradiction. Thus, $b^2 \leq 0$. From part (c), we know that $b^2 < c$ is a contradiction. Thus, $b^2 \geq c$.

If $b^2 \leq c$, then $c - b^2$ is either positive or 0 by the Positivity Axioms.

If $b^2 \geq c$, then $-(c - b^2)$ is either positive or 0 by the Positivity Axioms.

The only possibility, therefore, is that $c - b^2 = 0$ or $b^2 = c$.

3. Suppose that S is a non-empty set of real numbers that is bounded. Prove that $\inf S \leq \sup S$, and the equality holds if and only if S consists of exactly one number.

We must prove this both ways. We start with if $\inf S \leq \sup S$, then S consists of exactly one number.

We will prove by contradiction. Assume that S has more than one number in it. Then, there exists x and y in S such that $x \neq y$. Thus, either $x < y$ or $x > y$. Assume that $x < y$. Since S is bounded, there exists a lower bound $\inf S$ such that every element of S is greater than or equal to it. Thus, we can write $\inf S \leq x < y$. Similarly, there exists an upper bound $\sup S$ such that every element of S is less than or equal to it. Thus, we can write $\inf S \leq x < y \leq \sup S$. However, since the inequality between x and y is strict, $\inf S < \sup S$ is also strict. This is a contradiction to our original statement of $\inf S \leq \sup S$ since the two values can never be equivalent, so S must consist of exactly one number.

Now, we prove that if S consists of exactly one number, then $\inf S \leq \sup S$.

Since S consists of exactly one number, we can write $S = \{x\}$. By definition, $\inf S = x$ and $\sup S = x$. Thus, we can write $\inf S = x = \sup S$. This is contained in $\inf S \leq \sup S$ since $\inf S \leq \sup S$ is an equivalence relation. Thus, we have shown that if S consists of exactly one number, then $\inf S \leq \sup S$.

Thus $\inf S \leq \sup S$, and the equality holds if and only if S consists of exactly one number.

4. Do Problem 10 on page 11 of the textbook [F].

Use Exercise 9 to prove that the rational numbers satisfy the Field Axiom

Assume that we already have proven exercise 9 and 8. Thus, we know that if n and m are natural numbers such that $n > m$, then $n - m$ is also a natural number. We also know that the sum, difference, and product of two natural numbers is also a natural number.

We first prove the Associative Axiom:

Given $a, b, c \in \mathbb{Q}$, we know that $a + b$ must be a natural number by exercise 8. Similarly, we also know that $b + c$ must be a natural number by exercise 8. Thus, we can write $a + b + c = (a + b) + c$. We can also write $a + b + c = a + (b + c)$. For multiplication, we can write $a(b(c)) = (ab)c$ by exercise 9. We can also write $a(b(c)) = a(bc)$.

We now prove the Commutative Axiom:

Given $a, b \in \mathbb{Q}$, we know that $a + b$ must be a natural number by exercise 8. Similarly, we also know that $b + a$ must be a natural number by exercise 8. Thus, we can write $a + b = b + a$. We can also write $ab = ba$ by exercise 9.

We now prove the Identity Axiom:

Given $a \in \mathbb{Q}$, we know that $a + 0$ must be a natural number by exercise 8. Similarly, we also know that $0 + a$ must be a natural number by exercise 8. Thus, we can write $a + 0 = a$. We can also write $0 + a = a$ by exercise 8. For multiplication, we can write $a(1) = a$ by exercise 9. We can also write $1a = a$ by exercise 9.

We now prove the Inverse Axiom:

Given $a \in \mathbb{Q}$, we know that $a + (-a)$ must be a natural number by exercise 8. Similarly, we also know that $(-a) + a$ must be a natural number by exercise 8. Thus, we can write $a + (-a) = 0$. We can also write $(-a) + a = 0$ by exercise 8. For multiplication, we can write $a(1/a) = 1$ by exercise 9. We can also write $(1/a)a = 1$ by exercise 9.

We now prove the Distributive Axiom:

Given $a, b, c \in \mathbb{Q}$, we know that $a + (bc)$ must be a natural number by exercise 8. Similarly, we also know that $(bc) + a$ must be a natural number by exercise 8. Thus, we can write $a + (bc) = (a + b)c$. We can also write $(bc) + a = (b + c)a$ by exercise 8. For multiplication, we can write $a(bc) = (ab)c$ by exercise 9. We can also write $a(bc) = a(bc)$ by exercise 9.

Using the properties of the rational numbers, as well as the properties of exercise 8 and 9, we have proven that the rational numbers satisfy the Field Axioms.