Math 327 Homework 1

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1. Let x and y be two positive numbers.

   (i) Use the mathematical induction to show that if $x < y$, then $x^n < y^n$ for all n $\in \mathbb{N}$.

   First, we let $k = 1$. Given that $x < y$, $x^k < y^k = x^1 < y^1 = x < y$ which we are given so it is true.

   Now, we assume this is true for k. We want to show that it is true for $k + 1$.

   $$x^k < y^k$$

   Since $x < y$, we can multiply both sides by x to get

   $$x^{k+1} < y^k x$$

   Knowing that $x < y$, we can substitute x for y since the inequality will still hold. Thus, we can write

   $$x^{k+1} < y^k y$$
   $$x^{k+1} < y^{k+1}$$

   (ii) Deduce that if $x^n < y^n$ for some $n \in \mathbb{N}$, then $x < y$.

   Assume that $x^n < y^n$ for all n, but $x \geq y$. We then have two cases,

   Case 1: $x = y$.

   If $x = y$. We can thus multiply both sides by x and y respectively (they are both equal, so the order is irrelevant) n times to get $x^n = y^n$ for all n. This is a contradiction to our original statment of $x^n < y^n$, so $x \neq y$.

   Case 2: $x > y$.

   If $x > y$, we can multiply both sides by y n times to get

   $$xy^n > yy^n$$

   Since $x^n < y^n$, we can substitute y^n for x^n since the inequality will still hold. Thus, we can write

   $$xx^n > yy^n$$

   $$x^{n+1} > y^{n+1}$$

   For all n. However, if we plug in $n = n - 1$, we get

   $$x^n < y^n$$

   which is a contradiction to our original statement of $x^n < y^n$, so x cannot be less than y.

   Thus, we have shown that $x < y$.

2. Do problem 17 on page 11 of the textbook [F]. Define

$$S = \{x \mid x \text{ in } \mathbb{R}, x \geq 0, x^2 < c\}$$

a. Show that $c + 1$ is an upper bound for S and therefore, by the Completeness Axiom, S has a least upper bound that we denote by b.

We know that every element of the set S must be greater than or equal to 0. Furthermore, we know that every element in x^2 must be less than c. First, we prove that for all $x \geq 0$, $x \leq x^2$. When $x = 0$, we have

$$0 \leq 0^2 = 0$$

When $x > 0$ (equivalent to $x \geq 1$), we can substitute x^2 for xx to get

$$x \leq xx$$

Dividing both sides by x gives us

$$1 \leq x$$

Which is the exact solution set we wanted. Thus, we have shown that for all $x \geq 0$, $x \leq x^2$.

Now, we can say that $x \leq x^2 < c$. For all integers c, $c + 1 > c$. Thus, we can write

$$x \leq x^2 < c \leq c + 1$$

Thus, we know that $x < c + 1$ for all x in S, so $c + 1$ is an upper bound for S.

By the completeness axiom, if S has some upper bound, then there must be some least upper bound for the set S. Thus, there must exist some b such that b is an upper bound for S and b is the least upper bound for S.

b. Show that if $b^2 > c$, then we can choose a suitably small positive number r such that $b - r$ is also an upper bound for S, thus contradicting the choice of b as the *least* upper bound for S.

We know that $x^2 < c$ for all x in S. If we take the square root from both sides, we find that the theoretical solution set is $x \in \mathbb{R}$ such that $x \geq 0$ and $x < \sqrt{c}$. Knowing that $b^2 > c$, we can take the square root from both sides of the inequality to get $b > \sqrt{c}$. Thus, we know that $b > \sqrt{c}$. We can then write $x < \sqrt{c} < b$. If we take the difference between both sides, we get $x < b - \sqrt{c}$. Thus, we know that $b - \sqrt{c}$ must also be an upper bound for S. Since c is positive, we know that $b - \sqrt{c} < b$. Thus, b cannot be the *least* upper bound of S.

c. Show that if $b^2 < c$, then we can choose a suitably small positive number r such that $b + r$ belongs to S, thus contradicting the choice of b as *an* upper bound for S.

We assume that $b^2 < c$ and show that there must exist some r such that $b + r$ belongs to S. We can expand $(b + r)^2$ as

$$(b + r)^2 = b^2 + 2br + r$$

Let $r \leq b$, meaning that $r^2 \leq br$. Thus, we can write

$$(b + r)^2 = b^2 + 2br + r^2 \leq b^2 + 3br$$

Now, we asssumed $r \leq b$. We can further impose write $r < \frac{c-b^2}{3b}$ since $b^2 < c$ and $b < \sqrt{c}$.

Thus, we get

$$(b + r)^2 = b^2 + 2br + r^2 \leq b^2 + 3br < b^2 + 3b(\frac{c - b^2}{3b}) = b^2 + c - b^2 = c$$

Thus we have shown that $b + r$ belongs to S, meaning that $b + r < b$ which is a contradiction! So b cannot be an upper bound for S.

d. Use parts (b) and (c) and the Positivity Axioms for $\mathbb{R}$ to conclude that $b^2 = c$.

We know by the Completeness Axiom that S must contain a least upper bound b. From part (b), we know that $b^2 > c$ is a contradiction. Thus, $b^2 \leq 0$. From part (c), we know that $b^2 < c$ is a contradiction. Thus, $b^2 \geq c$.

If $b^2 \leq c$, then $c - b^2$ is either positive or 0 by the Positivity Axioms.

If $b^2 \geq c$, then $-(c - b^2)$ is either positive or 0 by the Positivity Axioms.

The only possibility, therefore, is that $c - b^2 = 0$ or $b^2 = c$.

3. Suppose that S is a non-empty set of real numbers that is bounded. Prove that inf $S \leq$ sup S, and the quality holds if and only if S consists of exactly one number.

We must prove this both ways. We start with if inf $S \leq$ sup S, then S consists of exactly one number.

We will prove by contradiciton. Assume that S has more than one number in it. Then, there exists x and y in S such that $x \neq y$. Thus, either $x < y$ or $x > y$. Assume that $x < y$. Since S is bounded, there exists a lower bound inf S such that every element of S is greater than or equal to it. Thus, we can write inf $S \leq x < y$. Similarly, there exists an upper bound sup S such that every element of S is less than or equal to it. Thus, we can write inf $S \leq x < y \leq$ sup S. However, since the inequality between x and y is strict, inf $S <$ sup S is also strict. This is a contradiction to our original statement of inf $S \leq$ sup S since the two values can never be equivalent, so S must consist of exactly one number.

Now, we prove that if S consists of exactly one number, then inf $S \leq$ sup S.

Since S consists of exactly one number, we can write $S = \{x\}$. By definition, inf $S = x$ and sup $S = x$. Thus, we can write inf $S = x =$ sup S. This is contained in inf $S \leq$ sup S since inf $S \leq$ sup S is an equivalence relation. Thus, we have shown that if S consists of exactly one number, then inf $S \leq$ sup S.

Thus inf $S \leq$ sup S, and the quality holds if and only if S consists of exactly one number.

4. Do Problem 10 on page 11 of the textbook [F].

Use Exercise 9 to prove that the rational numbers satisfy the Field Axiom

Assume that we already have proven exercise 9 and 8. Thus, we know that if n and m are natural numbers such that $n > m$, then $n - m$ is also a natural number. We also kniw that the sum, difference, and product of two natural numbers is also a natural number.

We first prove the Associative Axiom:

Given $a, b, c \in \mathbb{Q}$, we know that $a+b$ must be a natural number by excercise 8. Similarly, we also know that $b+c$ must be a natural number by exercise 8. Thus, we can write $a+b+c = (a+b)+c$. We can also write $a + b + c = a + (b + c)$. For multiplication, we can write $a(b(c)) = (ab)c$ by excercise 9. We can also write $a(b(c)) = a(bc)$.

We now prove the Commutative Axiom:

Given $a, b \in \mathbb{Q}$, we know that $a + b$ must be a natural number by exercise 8. Similarly, we also know that $b + a$ must be a natural number by exercise 8. Thus, we can write $a + b = b + a$. We can also write $ab = ba$ by excercise 9.

We now prove the Identity Axiom:

Given $a \in \mathbb{Q}$, we know that $a + 0$ must be a natural number by exercise 8. Similarly, we also know that $0 + a$ must be a natural number by exercise 8. Thus, we can write $a + 0 = a$. We can also write $0 + a = a$ by excercise 8. For multiplication, we can write $a(1) = a$ by excercise 9. We can also write $1a = a$ by excercise 9.

We now prove the Inverse Axiom:

Given $a \in \mathbb{Q}$, we know that $a + (-a)$ must be a natural number by exercise 8. Similarly, we also know that $(-a) + a$ must be a natural number by exercise 8. Thus, we can write $a + (-a) = 0$. We can also write $(-a) + a = 0$ by excercise 8. For multiplication, we can write $a(1/a) = 1$ by excercise 9. We can also write $(1/a)a = 1$ by excercise 9.

We now prove the Distributive Axiom:

Given $a, b, c \in \mathbb{Q}$, we know that $a + (bc)$ must be a natural number by exercise 8. Similarly, we also know that $(bc) + a$ must be a natural number by exercise 8. Thus, we can write $a + (bc) = (a + b)c$. We can also write $(bc) + a = (b + c)a$ by excercise 8. For multiplication, we can write $a(bc) = (ab)c$ by excercise 9. We can also write $a(bc) = a(bc)$ by excercise 9.

Using the properties of the rational numbers, as well as the properties of excercise 8 and 9, we have proven that the rational numbers satisfy the Field Axioms.