Math 327 Homework 1

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1. For each of the following two sets, find the maximum, minimum, infimum, and supremum if they are defined. Justify your conclusions.

   a. $\{1/n \mid n \text{ in } \mathbb{N}\}$
   maximum:

   The set $\{1/n \mid n \text{ in } \mathbb{N}\}$ consists of values

   $$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}\}$$

   Notice that $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4}, ..., \frac{1}{n}$. Thus, this set is strictly decreasing. Therefore, the maximum of this set is 1.

   minimum:

   There is no minimum for this set. This is because for any $1/n$, $1/(n+1)$ is also in the set and smaller in value.

   infimum:

   For any $\frac{1}{n} \in \mathbb{N}$, $0 < \frac{1}{n}$. If there is an $\epsilon > 0$, we can find an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$ by the Archimedean principle. Thus, the infimum of this set is 0.

   supremum:

   The defintion of the supremum of a set is the least upper bound of the set. In other words, if there is an upper bound b for the set, $b \geq x$ for all x in the set. Since the maximum of this set is 1, the supremum of this set is also 1. Any value less than 1 is not an upper bound for the set since 1 is in the set, while any value greater than 1 is not the least upper bound for the set since 1 is a smaller upper bound.

   b. $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$
   maximum:

   The set $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$ contains an infinite number of values. Thus, there is no maximum for this set since for any potential maximum x in the set, there exists some real number $q < \sqrt{2}$ such that $x < q$.

   minimum:

   The set $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$ contains an infinite number of values. Thus, there is no minimum for this set since for any potential minimum x in the set, there exists some real number $q > -\sqrt{2}$ such that $x > q$.

   infimum:

   The infimum of this set is $-\sqrt{2}$. This is because $-\sqrt{2} \leq x$ for all x in the set. Any value greater than $-\sqrt{2}$ (which we denote as x) is not a lower bound for the set because there must exist some real number q in the set such that $-\sqrt{2} < q < x$ for all x.

   supremum:

   The supremum of this set is $\sqrt{2}$. This is because $\sqrt{2} \geq x$ for all x in the set. Any value less than $\sqrt{2}$ (which we denote as x) is not an upper bound for the set because there must exist some real number q in the set such that $x < q < \sqrt{2}$ for all x.

2. a. Prove that if n is a natural number, then $2^n > n$.

We prove by induction. The base case is $n = 1$. Then $2^1 = 2 > 1$. Thus, the base case is true. Now, assume that $2^k > k$ for some $k \in \mathbb{N} < n$. We must show that $2^{k+1} > k + 1$.

$$2^k > k$$

multiply both sides by 2

$$2 * 2^k > 2 * k$$
$$2^{k+1} > 2 * k$$

Now, we prove that $2 * k > k + 1$. We prove by induction. The base case is $k = 1$. Then $2 * 1 = 2 > 1 + 1$. Thus, the base case is true.

Now, assume that $2 * g > g + 1$ for all $g \in \mathbb{N} < k$. We must show that $2 * (g + 1) > (g + 1) + 1$. This is equivalent to showing that $2g + 2 > g + 2$.

$$2 * g > g + 1$$

add 2 to both sides

$$2 * g + 2 > g + 3$$

Since $g + 3 > g + 2$ for all $g \in \mathbb{N}$, we have that $2g + 2 > g + 2$ for all $g \in \mathbb{N}$.

Thus, $2 * k > k + 1$. Since $2 * k > k + 1$ for all $k \in \mathbb{N}$, we have that $2^{k+1} > 2 * (k + 1)$ for all $k \in \mathbb{N}$.

Thus, $2^n > n$ for all $n \in \mathbb{N}$.

b. Prove that n is a natural number, then

$$n = 2^{k_0} l_0$$

for some odd natural number l_0 and some nonnegative integer k_0. (Hint: if n is odd, let $k = 0$ and $l = n$; if n is even, let $A = \{k \text{ in } \mathbb{N} \mid n = 2^k l$ for some l in $\mathbb{N}.\}$ By (a), $A \subseteq \{1, 2, ..., n\}$. Choose k_0 to be the maximum of A.)

We prove by induction. The base case is $n = 1$. Then $1 = 2^0 1$. Thus, the base case is true.

We assume that this is true for some g in $\mathbb{N} < n$. Thus, $g = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0. We must show that this is true for $g + 1$ as well.

If $g + 1$ is odd, then

$$g + 1 = 2^0 (g + 1)$$

Since 0 is a nonnegative integer and $g + 1$ is odd.

If $g+1$ is even, then $\frac{g+1}{2}$ is some integer as well. By our hypothesis, $\frac{g+1}{2} = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0. Thus,

$$\frac{g+1}{2} = 2^{k_0} l_0$$
$$g+1 = 2^{k_0+1} l_0$$

Since $k_0 + 1$ is a nonnegative integer and l_0 is an odd natural number, the statement is true for $g+1$ as well.

Thus, $n = 2^{k_0} l_0$ for some odd natural number l_0 and some nonnegative integer k_0 for all $n \in \mathbb{N}$.

3. A real number of the form $m/2^n$ where m and n are integers, is called a dyadic rational. Prove that the set of dyadic rationals is dense in $\mathbb{R}$.