Volume of an n-ball using spherical coordinates

Josh O'Connor, Nathan Phillips, Luke Hui, Xheonray Jhoey Arnado, Carson Crusinberry

December 19th 2024

Let $S(n)$ be a sphere of radius R in $\mathbb{R}^n$ centered at the origin. How can we calculate the hyper volume of $S(n)$?

We know the case for calculating the volume of a sphere in $\mathbb{R}^3$. We evaluate the triple integral

$$\int\int\int_S dV.$$

After applying a spherical coordinate transformation, we get

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} \rho^2 \sin(\theta)\, d\rho\, d\phi\, d\theta$$

It follows from this notion of volume in $\mathbb{R}^3$ that to compute the volume of $S(n)$, we will need to evaluate

$$\int\int\cdots\int_S d^nV$$

To calculate this volume, we have three tasks:

1. Set up spherical coordinates for n dimensions
2. Compute the Jacobian of this coordinate transformation
3. Evaluate the resulting integral

n-dimensional Spherical Coordinates

For n-dimensional spherical coordinates, we have r, the radial coordinate, $n-2$ angles $\phi_1, \phi_2, \ldots, \phi_{n-2}$, which range between $[0, \pi]$, and ϕ_{n-1}, which ranges between $[0, 2\pi]$ and is called the azimuthal angle.

With this we can set up the transformation between cartesian coordinates $(x_1 \ldots x_n)$ to

$$\begin{gathered}
x_1 = r\cos(\phi_1)\\
x_2 = r\sin(\phi_1)\cos(\phi_2)\\
x_3 = r\sin(\phi_1)\sin(\phi_2)cos(\phi_3)\\
.\\
.\\
.\\
x_{n-1} = r\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1})\\
x_n = r\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1})
\end{gathered}$$

Once we have this, we can try to compute the Jacobian of our n dimensional spherical coordinates.

Jacobian of n-dimensional spherical coordinate transformation

$$J_n = \begin{bmatrix} c_1 & -rs_1c_2 & 0 & \cdots & 0 \\ s_1c_2 & rc_1c_2 & -rs_2c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_1\cdots s_{n-2}c_{n-1} & rc_1\cdots c_{n-2} & \cdots & \cdots & -rs_{n-2}s_{n-1} \\ s_1\cdots s_{n-2}c_{n-1} & rs_1\cdots s_{n-2} & \cdots & \cdots & rs_{n-2}\cdots s_{n-1}c_{n-1} \end{bmatrix}$$

The determinant using a technique called Laplace expansion of which we are not familiar. This takes us to the following equation:

$$|J_n| = (r\sin(\phi_1)\ldots\sin(\phi_{n-2})\cdot|J_{n-1}|$$

This is a recursive relation of the Jacobian. Let's investigate this a little further. One can easiy see that for $n = 2$, our Jacobian $|J_2| = r$.

From our recursive relation, we know

$$|J_3| = r\sin(\phi_1)|J_2| = r^2\sin(\phi_1)$$

Let's use this to compute J_4.

$$|J_4| = r\sin(\phi_1)\sin(\phi_2)|J_3| = r^3\sin^2(\phi_1)\sin(\phi_2)$$

Now for J_5:

$$|J_4| = r\sin(\phi_1)\sin(\phi_2)\sin(\phi_2)|J_4| = r^4\sin^3(\phi_1)\sin^2(\phi_2)\sin(\phi_3)$$

We can clearly see a pattern arising.

$$|J_n| = r^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\ldots\sin(\phi_{n-2})$$

We know the volume element in our integral d^nV equal to $|J_n|$ multiplied by the differtials for each quantity we are integrating, so we have

$$d^nV = |J_n|\,dr\,d\phi_1\,d\phi_2\ldots\,d\phi_{n-1} = r^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\ldots\sin(\phi_{n-2})\,dr\,d\phi_1\,d\phi_2\ldots\,d\phi_{n-1}$$

This lets us set up our integral as

$$\int_0^R\int_0^\pi\ldots\int_0^{2\pi} r^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\ldots\sin(\phi_{n-2})\,d\phi_{n-1}\ldots\,d\phi_1\,dr$$

Since these are all separate quantities, lets write this as a product of integrals.

$$V_n(R) = \int_0^R r^{n-1}\,dr\int_0^\pi\sin^{n-2}(\phi_1)d\phi_1\ldots\int_0^{2\pi}d\phi_{n-1}$$

Tangent on gamma and beta functions

To complete the next step of the proof, we must understand how the gamma and beta function work.

The gamma function

The gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

which can also be understood as $\Gamma(n) = (n-1)!$. The purpose of the gamma function is to extend the factorial function to non-integer numbers. This becomes necessary when applying the formula for the volume of an n-ball when "n" is an odd number. We also list some important properties of the gamma function below which will come in handy later.

1. Functional equation:
$$z\Gamma(z) = \Gamma(z+1)$$
2. Special values:
$$\Gamma(1) = 1$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

The beta function

The beta function is then defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

The function is special because it can also be defined in terms of the gamma function. In this way it is defined as,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

We can prove this relationship by showing that $\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)$.

First write $\Gamma(x)\Gamma(y)$ as integrals:

$$\Gamma(x)\Gamma(y) = \int_0^\infty u^{x-1}e^{-u}\, du \cdot \int_0^\infty v^{y-1}e^{-v}\, dv$$

We can re-write this as:

$$\int_0^\infty \int_0^\infty u^{x-1}v^{y-1}e^{-(u+v)}\, du\, dv$$

Now change variables using $G(s,t)$ where $u = st$ and $v = s(1-t)$ and where the $|\mathrm{Jac}(G)| = s$. The bounds for this would be $0 \le s \le \infty$, $0 \le t \le 1$.

Write new integral as:

$$\int_0^1 \int_0^\infty (st)^{x-1}s^{y-1}(1-t)^{y-1}e^{-st-s+st}s\, ds\, dt$$

This can be split up into:

$$\int_0^1 (1-t)^{y-1}\, dt \cdot \int_0^\infty s^{x+y-1}e^{-s}\, ds$$

Which we can see is $B(x,y)\Gamma(x+y)$, proving that this is a valid definition of the Beta function.

Rewriting our integral in terms of the Beta function

Picking up where we left off,

$$V_n(R) = \int_0^R r^{n-1}dr \cdot \int_0^\pi \sin(\phi_1)^{n-2}d\phi_1 \cdots \int_0^{2\pi} d\rho_{n-1}$$

We see that $\int_0^R r^{n-1}dr = \frac{R^n}{n}$, and by the symmetry of sine, $\int_0^\pi \sin(\phi_1)^{n-2}d\phi_1 = 2\int_0^{\pi/2} \sin(\phi_1)^{n-2}d\phi_1$, and also that $\int_0^{2\pi} d\rho_{n-1} = 2\pi$ Leaving us with,

$$V_n(R) = \frac{R^n}{n} \cdot 2\int_0^{\pi/2} \sin(\phi_1)^{n-2}d\phi_1 \cdots 2\pi$$

We will be using the angular integral I_k to reveal the beta function $B(x, y)$ in $V_n(R)$

$$I_k = 2\int_0^{\pi/2} \sin^k(\phi)\, d\phi \text{ (where k varies from n-2 down to 1)}$$

I_k is basically reformatted $\int_0^\pi \sin(\phi_1)^{n-2}d\phi_1$ from the $V_n(R)$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

$$\sin^2(\phi) = t$$
$$\cos^2(\phi) = 1 - t$$
$$dt = 2\sin\phi\cos\phi\, d\phi$$

Converting $B(x, y)$ into terms of sine:

$$B(x, y) = \int_0^{\pi/2} (\sin(\phi)^{2x-2}\cos(\phi)^{2x-2})(2\sin(\phi)\cos(\phi))\, d\phi$$
$$\rightarrow B(x, y) = 2\int_0^{\pi/2} (\sin(\phi)^{2x-1}\cos(\phi)^{2x-1})\, d\phi$$

Notice that when $y = \frac{1}{2}$, $\cos(\phi)^{2y-1} = 1$. So:

$$B(x, \frac{1}{2}) = 2\int_0^{\pi/2} (\sin(\phi)^{2x-1})\, d\phi$$
$$B(x, \frac{1}{2}) = I_k$$

Recall k of $I_k = n - 2$. Knowing this, we can find x of $B(x, \frac{1}{2})$ from $2x - 1 = k = n - 2$. Thus, $x = \frac{n-1}{2}$ which gives us $B(\frac{n-1}{2}, \frac{1}{2})$

We then substitute this Beta function $B(\frac{n-1}{2}, \frac{1}{2})$ back into $V_n(R)$ for $2\int_0^{\pi/2} \sin(\phi_1)^{n-2}d\phi_1$:

$$V_n(R) = \frac{R^n}{n} \cdot B(\frac{n-1}{2}, \frac{1}{2}) \cdots 2\pi$$

Thus, the Beta Function has been revealed.

The beta functions can be rewritten in terms of gamma functions, which we expand to a few terms for the sake of visuzalizing the next step:

$$V_n(R) = \frac{R^n}{n} \cdot \frac{\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n}{2}-1)\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}-\frac{1}{2})} \cdot \frac{\Gamma(\frac{n}{2}-\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}-1)} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi$$

Most of the Gamma terms in the series expansion cancel, and using the properties of the Gamma function, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$ we can see that,

In the numerator we are left with (n-2) copies of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and a factor of 2π

In the denomenator we are left just with $\Gamma(\frac{n}{2})$

In total, due to term cancellation in the series expansion, we are left with

$$V_n(R) = \frac{R^n}{n} \cdot \frac{2\pi \cdot (\sqrt{\pi})^{n-2}}{\Gamma(\frac{n}{2})} = \frac{2R^n(\sqrt{\pi})^n}{n\Gamma(\frac{n}{2})}$$

as the intermediate terms cancel out. Then applying the functional equation $z\Gamma(z) = \Gamma(z+1)$, this leads to:

$$V_n(R) = \frac{2\pi^{n/2}R^n}{n\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}R^n}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}R^n}{\Gamma(\frac{n}{2}+1)}$$

Showing general formulas for even and odd n

As we apply this formula towards solving volumes in n dimensions, you will notice that the function behaves differently for even and odd n. To show this behavior, we find two separate functions derived from this general expression, which can be used for an even or odd n.

For odd and even n dimension spheres, the gamma function $\Gamma(\frac{n}{2}+1)$ will behave differently if the term $(\frac{n}{2}+1)$ results in a integer or half-integer value.

For all even n dimensions, $(\frac{n}{2}+1)$ will result in positive integer arguments for the gamma function. We can utilize the fact that positive integer arguments for the gamma function are equal to a factorial.

$$\Gamma(n) = (n-1)!$$

The $\Gamma(\frac{n}{2}+1)$ term for the formula of a volume of a sphere in even n dimensions can be rewritten as (n/2)!. This gives us the following formula for the volume of even n spheres:

$$V_n(R) = \frac{\pi^{n/2}R^n}{(\frac{n}{2})!}$$

For all odd n dimensions, $(\frac{n}{2}+1)$ will result in positive half-integer arguments for the gamma function. This difference in half-integer and integer arguments for the gamma function is what allows us to separate the formula to find the volumes of the even and odd n dimensions balls. In order to simplify $\Gamma(\frac{n}{2}+1)$ for odd n spheres. We will first transform the term $(\frac{n}{2}+1)$ into something that allows us to use the half-integer argument of the gamma function:

$$\Gamma(\frac{n}{2}) = \sqrt{\pi} \cdot \frac{(n-2)!!}{2^{\frac{n-1}{2}}}$$

First, if n is odd, we can do the following:

$$\frac{n}{2} + 1 = \frac{n+1}{2} + \frac{3}{2}$$

We can then replace $\frac{n+1}{2}$ as variable k to get:

$$k + \frac{3}{2}$$

substitute $k + \frac{3}{2}$ into the gamma function to get

$$\Gamma(k + \frac{3}{2})$$

This gamma function is equilavent to a factorial

$$\Gamma(k + \frac{3}{2}) = (k + \frac{1}{2})!$$

The factorial can be expanded into the following function

$$(k + \frac{1}{2})! = \frac{\sqrt{\pi}(2k+1)!!}{2^k + 1}$$

When we substitute k for $\frac{n+1}{2}$, we get the following

$$\frac{\sqrt{\pi}(n)!!}{2^{\frac{n+1}{2}}}$$

Note: (n)!! denotes a double factorial.

With this, we can substitute the following term for the gamma function to get the following formula for the volume of odd n spheres.

$$V_n(R) = \frac{\pi^{\frac{n}{2}} R^n}{\frac{\sqrt{\pi}(n!!)}{2^{\frac{n+1}{2}}}} = \frac{\pi^n R^n 2^{\frac{n+1}{2}}}{n!!}$$

Conclusion

After completing the proof spherically, we found several interesting properties. First, even and odd functions are accounted for in one function through the gamma function, which showcases distinct positive integer outputs that rely on whether the n-dimension is even or odd. The telescoping nature of $V_n(R)$ using $B(x, y)$ for the remaining integrals is revealed when some of the values in the numerator and denominator cancel out, leaving us with $\frac{\pi^{\frac{n}{2}} \cdot R^n}{\Gamma(\frac{n}{2}+1)}$, which really simplifies an otherwise very complex equation. Another interesting finding is that when n-dimensions goes towards infinity, the value for $V_n(R)$ gets smaller, as the factorials in the denominator generated by the gamma function increase way faster than the exponential values in the numerator."

References

N-sphere, December 2024. URL `https://en.wikipedia.org/wiki/N-sphere`.

Guswen. Volume of an n-ball, October 2024. URL `https://en.wikipedia.org/wiki/Volume_of_an_n-ball#Direct_integration_in_spherical_coordinates`.

Jeremy Orloff. 14.2: Definition and properties of the gamma function, May 2023. URL `https://math.libretexts.org/Bookshelves/Analysis/Complex_Variables_with_Applications_(Orloff)/14%3A_Analytic_Continuation_and_the_Gamma_Function/14.02%3A_Definition_and_properties_of_the_Gamma_function`.

Physics and Math Lectures. Intro to the beta function, September 2020. URL `https://www.youtube.com/watch?v=E9sc1FnJF9k`.

Kyle Siegrist. 5.17: The beta distribution, April 2022. URL `https://stats.libretexts.org/Bookshelves/Probability_Theory/Probability_Mathematical_Statistics_and_Stochastic_Processes_(Siegrist)/05%3A_Special_Distributions/5.17%3A_The_Beta_Distribution`.