Math 241 Section 11.1: 3D, Points, Axes, Spheres, Distance
Dr. Justin O. Wyss-Gallifent

1. Goal/Intro: Most of MATH 241 (Multivariable Calculus) takes place in 3D space so we need to understand visually how all this works.

2. In addition to the x and y axis we add an extra axis, the z-axis. We rearrange so that the z-axis is pointing up. The reason for this is that most of our functions are of the form $z = f(x, y)$ and we're used to the dependent variable being vertical like with $y = f(x)$.
   Show: A picture.

3. We won't plot points much but the easiest way to do this is to plot x and y first then go up or down by z. Tick marks on the axes can help. A grid on the xy plane can help too. Perspective can make this a bit confusing at first. It can help to visualize a box in 3D with one corner at the origin and the other at (x, y, z). This works if they're all nonzero. Points are usually denoted by capital letters.

   Example: Plot $P = (2, 3, 5)$, $Q = (-2, 3, -1)$, $R = (0, 0, 2)$, $S = (4, 0, 0)$.
   Show: A picture.

4. Along with the three axis we get the three coordinate planes, those being the xy-plane, the yz-plane and the xz-plane. These divide 3D space into eight octants. The first octant is the one with $x, y, z \geq 0$. **PIC**

   Example: Make one up.

5. In 3D space we have a measurement of distance between $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$. This is denoted $|PQ|$ and is

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

   Example: Make one up.

6. We also get some shapes that we'll encounter frequently:

   (a) The sphere with center (x_0, y_0, z_0) and radius r has equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

   Example: Make one up with picture.

   (b) The (closed) ball with center (x_0, y_0, z_0) and radius r has equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq r^2$$

   Example: Make one up with picture.

Math 241 Section 11.2: Vectors
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1. Definition of a vector:

   Defined a vector as a triple (a, b, c) or pair (a, b) of points. The notation is confusing because it looks like a point. We have notations (a, b, c), $[a\,b\,c]$, $\langle a, b, c\rangle$ and the one we'll use, $a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$. I mentioned that what's really going on is $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ and we're taking combinations.

2. Define $+$, $-$ and scalar multiple for vectors.

3. Visualization and Application:

   Vectors can be visualized as arrows pointing in space. The specific location where the arrow is anchored is not relevant but sometimes we'll anchor our vectors somewhere specific (like the origin, or at an object) if we need to.
   Show: Pictures to illustrate.

   I mentioned how we might anchor a vector at the origin if we're using it to point to an object and we might anchor it at an object if we're using it to point in the direction that the object is moving, for example.

4. Definitions/Properties:

   (a) Zero vector $\mathbf{0}$.

   (b) The vector $\overrightarrow{PQ}$ pointing from P to Q. We get this by "subtracting Q from P").

   (c) If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ then the length (magnitude, norm) is $||\mathbf{v}|| = \sqrt{a^2 + b^2 + c^2}$.

   (d) A unit vector has length 1.

   (e) The unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{v}}{||\mathbf{v}||}$.

   (f) Parallel vectors are vectors which are multiples of one another.

NOTE: Problems like 27 and 28 on the homework can be a bit confusing at first. Basically if you know the angle of the vector you can use this (along with sine and cosine) to find the components.

Math 241 Section 11.3: Dot Product
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1. Defined the dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.
   Example: Make one up.

2. Basic properties:

   (a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

   (b) $\mathbf{a} \cdot (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \pm \mathbf{a} \cdot \mathbf{c}$

   (c) $\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha \mathbf{b})$

3. Advanced properties:

   (a) $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta$ where θ is the angle between them. This follows from the Law of Cosines and is sometimes (physics especially) used as an alternate definition of the dot product.

   (b) $\mathbf{a} \perp \mathbf{b}$ iff $\mathbf{a} \cdot \mathbf{b} = 0$ and how this follows from the previous.

   (c) $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}$

   (d) $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$ and $||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

4. Definition of projection and the formula

$$\text{Pr}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

   Example: Make one up.

Note: Questions like 15-17 in the homework can be confusing. All you're doing is writing the original vector as a sum of two vectors, those two vectors perpendicular to one another.

Math 241 Section 11.4: Cross Product
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1. Define the cross product. First define:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

   and then:

$$\begin{aligned} \mathbf{a} &= a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \\ \mathbf{b} &= b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k} \\ \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \end{aligned}$$

   It's easiest to think of doing this on a case-by-case basis without writing down a messy general formula. They may have seen other definitions and that's fine too.
   Example: Make one up.

2. Basic properties:
   (a) $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c}) = \mathbf{a}\times\mathbf{b}\pm\mathbf{a}\times\mathbf{c}$
   (b) $\alpha(\mathbf{a}\times\mathbf{b}) = (\alpha\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\alpha\mathbf{b})$
   (c) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ (Note the negative in this one!)

   Anti-commutative, distributive over $\pm$ on both ends, associative with scalar multiplication.

3. Advanced properties:
   (a) $||\mathbf{a}\times\mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}||\sin\theta$.
   (b) $\mathbf{a}$ is parallel to $\mathbf{b}$ iff $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.
   (c) The vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ via the right-hand rule. This will be one of the most useful properties of the cross product.

Math 241 Section 11.5: Equations of Lines
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1. Equations of lines are not easy; no sense of slope etc. from which to build an equation. Instead we'll construct lines three different ways, all of which have their own use.

2. Parametric form: If (x_0, y_0, z_0) is a point on the line and $\mathbf{L} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is a direction vector (the direction the line goes) then the *parametric equations* are

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

   form the other points for all possible real numbers t. Emphasized how each point corresponds to a t-value and each t gives a point.
   Example: When (x_0, y_0, z_0) and $\mathbf{L} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ are both explicitly given.

3. Changed this to vector form

$$\mathbf{r} = \mathbf{r}(t) = (x_0 + at)\,\mathbf{i} + (y_0 + bt)\,\mathbf{j} + (z_0 + ct)\,\mathbf{k}$$

   and how this written like a vector but we think of it like a point. In other words we can think of it as a vector which points from the origin to the points on the line. This is actually the primary way we'll see lines later in the course.
   Example: Rewrite the previous.

4. Developed the symmetric forms by solving the parametric forms for t and setting them equal.
   Example: Rewrite the previous.
   Example. Did one where one of a, b, c is 0. In this case the variable with no t is left alone and the other two are solved for t and set equal.
   Example. Did one where two of a, b, c are 0. In this case the two with no t are left alone and the other isn't mentioned because the variable can be anything.

5. Distance formula from point to line. If a line has point P and direction vector $\mathbf{L}$ then the distance from the line to another point Q equals:

$$\text{distance} = \frac{||\overrightarrow{PQ} \times \mathbf{L}||}{||\mathbf{L}||}$$

   Example: Make one up.

Trickier examples:

- Finding the equation of a line when two points are given, since $\mathbf{L}$ must be found first, and either point can be used.

- Finding where a line intersects a sphere, for example, by finding the parametric equations and plugging them into the sphere equation and solving for t.

- Doing a distance from point-to-line problem when the line is given as a confusing symmetric equation since this involves extracting the necessary information from the equation.

Math 241 Section 11.6: Equations of Planes
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1. Development: Start with a point $P = (x_0, y_0, z_0)$ and a vector $\mathbf{N} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. The plane will pass through P and be perpendicular to $\mathbf{N}$. The vector $\mathbf{N}$ is called the *normal vector* for this reason, the word *normal* often means *perpendicular* in mathematics. Then if $R = (x, y, z)$ is another point on the plane then $\mathbf{N} \cdot \overrightarrow{PR} = 0$ which gets us

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

   We can then simplify this to $ax + by + cz = d$. Note how even in the simplified version the a, b, c are still the coefficients of x, y, z if they're on the same side.
   Example: Make one up.

2. Sketching planes:
   (a) None of $a, b, c = 0$: Find and plot x, y, z intercepts and draw a little triangle. Note that the plane extends from the triangle but it gives a good idea of what's going on.
   Example: Make one up.
   (b) Two of $a, b, c = 0$: Get parallel to either xy, yz or xz-plane. Think of these as shifting these planes.
   Example: Make one up.
   (c) One of $a, b, c = 0$: Draw the corresponding line in the xy, yz or xz-plane and "extend" it to a plane.
   Example: $3x - 6y = 18$ we'd plot the line in the xy-plane and then extend it in the z-direction because z can be anything.
   Example: $2x + 3z = 6$ we'd plot the line in the xz-plane and then extend it in the y-direction because y can be anything.
   Example: $2y + 2z = 8$ we'd plot the line in the yz-plane and then extend it in the x-direction because x can be anything.

3. Distance formula from point to plane. If a plane has point P and normal vector $\mathbf{N}$ and if Q is another point then the distance between Q and the plane is:

$$\text{distance} = \frac{|\mathbf{N} \cdot \overrightarrow{PQ}|}{||\mathbf{N}||}$$

   Example: Make one up.

Note: The following problems can be tricky at first:

- Finding the equation of a plane given other than P and $\mathbf{N}$. For example given three points, or given one point and a line in the plane, or given one point and a line perpendicular to the plane.
- Problems that combine 11.5 and 11.6, for example find the equation of the line formed by the intersection of two planes.

Math 241 Section 12.1: Basics of Vector Valued Functions
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1. Define a VVF as a function where a number (a parameter, typically t) goes in and a vector comes out. Typical notation:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

often with a range of t given and in 2D simply withouth the $\mathbf{k}$ component. We usually treat the vector as a point to describe location of an object at a time t. For example the vector equation of a line is a VVF.

2. To graph these we picture the vectors as anchored at the origin and we plot the endpoint. I did examples similar to the following. Note that we're not going to ask the students to draw many of these but having an idea of what the graphs look like will be extremely helpful for future problems.
Examples: Such as the following...
Example: $\mathbf{r}(t) = (1+2t)\,\mathbf{i} + (3-t)\,\mathbf{j}$ with $0 \leq t \leq 2$ in 2D because it's familiar - a line!
Example: $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}$ with $0 \leq t \leq \pi$ in 2D.
Example: The above with different ranges of t.
Example: $\mathbf{r}(t) = \cos(2t)\,\mathbf{i} + \sin(2t)\,\mathbf{j}$ with $0 \leq t \leq \pi/2$ in 2D to point out how this and the previous example have the same picture.
Example: $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + 2\,\mathbf{k}$ with $0 \leq t \leq \pi$ in 3D.
Example: $\mathbf{r}(t) = (2+\cos t)\,\mathbf{i} + 0\,\mathbf{j} + (3+\sin t)\,\mathbf{k}$ with $0 \leq t \leq \pi$.
Example: $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$ with $t \geq 0$ it's a half-helix spiraling upwards.
Example: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ with $-2 \leq t \leq 1$ in order to draw the function $y = x^2$ between $x = -2$ and $x = 1$.
Example: $\mathbf{r}(t) = t^2\,\mathbf{i} + e^t\sin(t)\,\mathbf{j} + t\cos(t)\,\mathbf{k}$ just to point out that often we have no idea what these look like.

Note: We can use the VVF to determine when and where an object hits something. If an object whose position is described by a VVF then we can know when the object hits a plane (for example) by plugging $x(t)$, $y(t)$ and $z(t)$ into the plane equation and solving for t. We can then figure out where it hit the plane by plugging that t back into the VVF. This works for objects other than planes, too, anything with an equation.

Math 241 Section 12.2: Limits of VVFs
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1. The limit of a VVF is found by taking the limit of the components. This won't be used much but it's good to keep in mind.
   Example: Make one up.

Note: There's no homework on this.

Math 241 Section 12.3: Derivatives and Integrals of VVFs
Dr. Justin O. Wyss-Gallifent

1. Define the derivative of $\mathbf{r}(t)$ with a limit (just to make the point) but said that in practice we just take the derivatives of the components.
   Example: Make one up.

2. Application: If $\mathbf{r}(t)$ gives position of an object then:

   (a) $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity (vector) which is tangent to the curve in the direction of the curve.

   (b) $s(t) = ||\mathbf{v}(t)||$ is the speed.

   (c) $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ is the acceleration vector which indicates how $\mathbf{r}(t)$ is changing.

   Example: $\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j}$ at $t = 1$ with pictures to help explain.

3. Integrals of VVFs.

   (a) Indefinite integrals of VVF: We take the individual integrals and I talked about the $+\mathbf{C}$ at the end rather than giving each components its own constant.
   Example: An application where $\mathbf{a}(t)$, $\mathbf{v}(0)$ and $\mathbf{r}(1)$ are given and we calculate $\mathbf{r}(t)$ by integrating backwards and finding the constants.

   (b) Definite Integrals of VVFs: We just take the individual integrals. We will never do this in practice.

4. Commented on the sum,difference and dot and cross-product rules with derivatives but did not emphasize that much because we don't use them that much.

Math 241 Section 12.4: Curves and Associated Definitions
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1. A space curve (a curve) is the range of a VVF. Really it's just the curve we draw. A curve can be discussed without explicitly giving an $\mathbf{r}(t)$. A choice of $\mathbf{r}(t)$ is defined as a parametrization of the curve and one curve can have many parametrizations.
   Example: A line.
   Example: A circle.

2. Associated Definitions

   (a) Closed:

   - A *closed parametrization* is a parametrization with the property that $\mathbf{r}(a) = \mathbf{r}(b)$ (it starts where it ends) but otherwise does not cross itself infinitely many times.
     Example: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq 2\pi$.
     Example: If t goes any further like twice around then the curve crosses itself infinitely many times and that's a no-no.
   - A curve is *closed* if it has a closed parametrizations.
     Example: The circle again. There are lots of parametrizations of the circle but since there is one that's closed then the curve is closed.

   (b) Smooth:

   - A *smooth parametrization* has $\mathbf{r}'(t) \neq \mathbf{0}$ except it is permitted to be $\mathbf{0}$ at the endpoints, if it has endpoints. A good analogy is an ideal commute where your velocity is (probably) $\mathbf{0}$ at the start and end but you never need to stop on the way.
   - A curve is *smooth* if it has a smooth parametrization.

   (c) Piecewise Smooth:

   - A *piecewise smooth* parametrization is a parametrization in which you can break the t-range into pieces on which the parametrization is smooth. Your commute is probably more like this.
   - A curve is *piecewise smooth* if it has a piecewise smooth parametrization.

   Example: $\mathbf{r}(t) = (2t+1)\,\mathbf{i} + (3-t)\,\mathbf{j} + t\,\mathbf{j}$ for all t. Here $\mathbf{r}'(t) = 2\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k}$ which is never $\mathbf{0}$ hence smooth.
   Example: $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + 2\,\mathbf{k}$ on $[0, 5]$. Here $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 0\,\mathbf{k}$ which is $\mathbf{0}$ at $t = 0$ but that's an endpoint so it's okay.
   Example: $\mathbf{r}(t)$ same as above but on $[-5, 5]$. Here $t = 0$ is no longer an endpoint so it's not smooth. It is, however, piecewise smooth.
   Example: $\mathbf{r}(t) = t^{1/3}\,\mathbf{i}$ on $[-1, 1]$. Here $\mathbf{r}'(t) = \frac{1}{3t^{2/3}}\,\mathbf{i}$ which is undefined at $t = 0$ hence neither smooth nor piecewise smooth.

   (d) Length: If C is piecewise smooth on $[a, b]$ then:

   $$\text{Length of } C = \int_a^b ||\mathbf{r}'(t)||\,dt$$

   Example: An easy one.

Note: Examples like 22,25 in the text are good because they require a sneaky factorization inside the square root before integration.

Math 241 Section 12.5: Tangents and Normals to Curves
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1. Intro:

   I discussed how sometimes we want unit vectors which are tangent to and normal to a curve, what "normal to a curve" might mean. We'll see why soon in this section.

2. The Tangent Vector:

   Defined
   $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$
   This one is usually pretty intuitive. Emphasized that it's length 1 and points in the direction of the curve.

3. The Normal Vector:

   Defined
   $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$
   We do this because $\mathbf{a}(t)$ is not normal (it contains both change in velocity in the direction of motion and perpendicular to it) and so by taking $\mathbf{T}'(t)$ instead, since $||\mathbf{T}|| = 1$, we only capture the change in direction. More formally we can see the result is normal to the curve by showing it's perpendicular to $\mathbf{T}(t)$:
   $$\begin{aligned} \mathbf{T}(t) \cdot \mathbf{T}(t) &= ||\mathbf{T}|| = 1 \\ \frac{d}{dt}\left(\mathbf{T}(t) \cdot \mathbf{T}(t)\right) &= 0 \\ 2\mathbf{T}'(t) \cdot T(t) &= 0 \end{aligned}$$
   Since $\mathbf{T}' \perp \mathbf{T}$ and $\mathbf{N}$ is a multiple of $\mathbf{T}'$ we know $\mathbf{N} \perp \mathbf{T}$.

   Example: Find $\mathbf{T}$ and $\mathbf{N}$ at $(4, 2)$ on the curve $x = y^2$ by parametrizing as $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$ and working it out.

4. Tangential and normal components of acceleration:

   Acceleration breaks down into two components, one in the direction of motion and one perpendicular to it. These turn out to be multiples of $\mathbf{T}$ and $\mathbf{N}$ and in fact:
   $$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$
   where:

   - The tangential component of acceleration is: $a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{||\mathbf{v}||}$
   - The normal component of acceleration is: $a_{\mathbf{N}} = \frac{||\mathbf{v}\times\mathbf{a}||}{||\mathbf{v}||}$

   Example: Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = 1$ for $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + 1/3t^3\,\mathbf{k}$.

Math 241 Section 13.1: Functions of Several Variables
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1. Basic Definitions:
   Defined functions of two and three variables. Mentioned both $z =$ and $f(x, y) =$ notation and both $w =$ and $f(x, y, z)$ notation. Mentioned that every function in two variables is an equation in three, but the reverse is not always the case. Similarly for functions of three variables and equations in four.

2. Examples:
   Drew lots of pictures, including:

   - Paraboloids
   - Cones
   - Spheres
   - Cylinders
   - Planes
   - Ellipsoids
   - Parabolic sheets

   I encouraged them to think of new types, like how could they center a cylinder around something not an axis or how could a paraboloid be made to open around the x-axis.

3. Level Curves and Surfaces:
   Defined level curves and level surfaces and did examples. Pointed out that the graph of a function of one variable is the level surface for a function of two variables and the graph of a function of two variables is the level surface for a function of three variables.

Math 241 Section 13.3: Partial Derivatives
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1. Definition and Notation
   Give a formal definition of the partial derivatives (in terms of limits) and then give the way that we actually do them - treating other variables as constants. We have notation $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.
   Examples: Lots of examples, making sure to hit some that involve quotient rules, chain rules, etc.

2. Methods of Visualization:

   (a) Talked about how $f_x(x, y)$ gives the slope of the tangent line in the x-direction with y fixed and similarly $f_y(x, y)$ gives the slope of the tangent line in the y-direction with x fixed. This is harder to see for more variables like $f_x(x, y, z)$.

   (b) If $f(x, y)$ gives the temperature of the plane at (x, y) then $f_x(x, y)$ give the instantaneous temperature change of an object with respect to distance (for example, degrees Celsius per meter) as it passes through (x, y) in the positive x-direction. Similarly for f_y. This works nicely in 3D too.
   Example: Do one with units and explanation.

3. Higher Derivatives
   Talk about higher derivatives and especially comment on the notation $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$.
   Example: Do one.
   Mentioned that almost always $f_{xy} = f_{yx}$ for example.

Math 241 Section 13.4: The Chain Rule
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1. There's lots of crap in the book for the chain rule. They can ignore most of it.

2. First I pointed out that for example if $z = x^3y + y^2$ and $x = u\sin(v)$ and $y = v\cos(u)$ then really z is a function of u and v and so $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ make sense.

   Likewise if $z = xe^{xy}$ and $x = 2t^3$ and $y = \sqrt{t}$ then really z is a function of t and so $\frac{dz}{dt}$ makes sense.

3. Method for applying the chain rule:
   Step 1: Draw a tree diagram.
   Step 2: On each branch put either a d or a ∂ depending on whether it's a regular derivative (one var) or a partial derivative.
   Step 3: Find all routes from the left side to the variable we're taking the derivative with respect to.
   Step 4: Along each path find the derivatives and multiply.
   Step 5: Add the paths.
   Step 6: Substitute in for the final variable(s).

   Example: Standard.
   Example: One where the starting function ends up being a function of one variable.
   Example: One like $w = t^2 + 1/s$ and $s = t^3 + t$ because $\frac{dw}{dt} = \frac{\partial w}{\partial s}\frac{ds}{dt} + \frac{\partial w}{\partial t}$. This example is useful because it demonstrates that (a) tree branches need not be the same length and (b) $\frac{dw}{dt}$ and $\frac{\partial w}{\partial t}$ are quite different.

4. I mentioned related rates and how the chain rule is useful. The example I did was something like: Sand falls in a conical pile at $2\pi\,\text{in}^3/\text{min}$. The radius increases at 3 in/min. How fast is the height changing when $h = 10$ and $r = 8$?

Math 241 Section 13.5: The Directional Derivative
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1. Recall that f_x means the change in f as x increases (in the $\mathbf{i}$ direction) and likewise for f_y (in the $\mathbf{j}$ direction) and f_z (in the $\mathbf{k}$ direction) and so on. We might ask how f changes if we go in some other direction.

2. Defn: The directional derivative of f in the direction of the unit vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is denoted $D_{\mathbf{u}}f$ and is defined by

$$D_{\mathbf{u}}f = af_x + bf_y + cf_z$$

   Here the $+cf_z$ only appears in the 3D case.

   Note: The phrase "directional derivative in the direction of" is used even when the vector is not a unit vector but you must make it a unit vector before using the formula.

   Examples.

Math 241 Section 13.6: The Gradient
Dr. Justin O. Wyss-Gallifent

1. Definition
   The gradient of f is denoted either Grad$f$ or ∇f (note that ∇ is pronounced "nabla" and comes from the Hellenistic Greek word $\nu\alpha\beta\lambda\alpha$ for a Phoenician harp) and is defined by:

   $$\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$$

   Note that this is a vector and dhe $+f_z\,\mathbf{k}$ only appears in the 3D case.
   Examples.

2. Basic properties

   (a) Observe that since $||\mathbf{u}|| = 1$ we have

   $$D_{\mathbf{u}}f = \mathbf{u}\cdot\nabla f = ||\mathbf{u}||||\nabla f||\cos\theta = ||\nabla f||\cos\theta$$

   Where θ is the angle between $\mathbf{u}$ and ∇f.
   It follows that $D_{\mathbf{u}}f$ is largest when $\theta = 0$ in which case $\mathbf{u}$ points in the same direction as ∇f and $D_{\mathbf{u}}f$ equals $||\nabla f||$.

   (b) First this means that ∇f points in the direction of maximum instantaneous increase of f.

   (c) Second this means that the largest possible $D_{\mathbf{u}}f$ is in fact $||\nabla f||$.

   (d) To put (b) and (c) together: Different $\mathbf{u}$ give different values for $D_{\mathbf{u}}f$. The largest value is when $\mathbf{u} = \nabla f$ and that largest value is $||\nabla f||$.

   Example. If the temp at (x, y) is $f(x, y) = x^2y$ and a bug is at $(1, 2)$ in which direction does it detect the greatest increase in temperature and what is that increase?

3. Normal/Perpendicular properties

   (a) $\nabla f(x, y)$ is normal to the level curve of $f(x, y)$ at (x, y).
   Example: Find a vector $\perp$ to $y = x^2$ at $(3, 9)$.
   Solution: Set $f(x, y) = y - x^2$ then $\nabla f = -2x\,\mathbf{i} + 1\,\mathbf{j}$ and so $\nabla f(3, 9) = -18\,\mathbf{i} + 1\,\mathbf{j}$ works.

   (b) $\nabla f(x, y, z)$ is normal to the level surface of $f(x, y, z)$ at (x, y, z).
   Example.

Math 241 Section 13.7: Tangent Plane Approximation
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1. First recall how if $f(x)$ is a function and x_0 is an x-value we can draw the tangent line at $(x_0, f(x_0))$. Using point-slope form this line is $y - y_0 = f'(x_0)(x - x_0)$ or $y = y_0 + f'(x_0)(x - x_0)$. If x is close to x_0 then we have:

$$f(x) \approx y_0 + f'(x_0)(x - x_0)$$

While this is a weak approximation it's the beginning of the Taylor Series which is extremely useful:

$$f(x) = y_0 + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + ...$$

2. In the case of $f(x, y)$ we can do the same thing. We take the point (x_0, y_0) and construct the tangent plane. How do we do this? Well, the tangent plane passes through $(x_0, y_0, f(x_0, y_0))$ and has normal vector perpendicular to the graph of $f(x, y)$. Well, the graph of $z = f(x, y)$ is the level surface for the function of three variables $g(x, y, z) = f(x, y) - z$ and so we can obtain a perpendicular vector using the gradient and use this for $\mathbf{n}$:

$$\begin{aligned} \nabla g(x, y, z) &= f_x\,\mathbf{i} + f_y\,\mathbf{j} - 1\,\mathbf{k} \\ \nabla g(x_0, y_0, f(x_0, y_0)) &= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{n} &= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k} \end{aligned}$$

Therefore the tangent plane has equation:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - 1(z - f(x_0, y_0)) = 0$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

And similiarly now if (x, y) is close to (x_0, y_0) then:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(Note: The book rewrites this in a slightly different way which tends to be more confusing.)

This can then be used to approximate functions and is the first step in a 2D version of the Taylor Series (which we won't do).

Example: To approximate $\sqrt{3.02^2 + 6.95}$ we note this is close to $\sqrt{3^2 + 7}$ so we set $f(x, y) = \sqrt{x^2 + y}$ and use the approximation.

3. This can be expanded to 3D. For $f(x, y, z)$ if we have a point (x_0, y_0, z_0) then if (x, y, z) is close to (x_0, y_0, z_0) then

$$f(x, y, z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

Math 241 Section 13.8: Maxima and Minima
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1. Described (informally with a picture) maxima, minima, relative maxima and minima.
   Definition: For a function f a critical point is a point where the function is defined and where either all the partial derivatives are zero or at least one partial derivative is undefined.

2. Finding relative max and min. Procedure:

   (a) Find critical points.

   (b) Find the discriminant $D(x,y) = f_x x f_y y - f_{xy}^2$

   (c) For each CP:

   - If $D(x_0,y_0) > 0$ then check $f_x x(x_0,y_0)$. If $+$ rel min. If $-$ rel max.
   - If $D(x_0,y_0) < 0$ then saddle point.

   Example: $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

3. Finding absolute extrema for $f(x,y)$ on a closed bounded region. Procedure:

   (a) Find all critical points inside the region. Take f at these point.

   (b) Find the max and min of $f(x,y)$ on the edge. How this is done depends on f and the shape of the edge but the basic idea is to use the edge equation to change f into an equation involving one variable and then work it out intuitively.

   (c) Take the largest value and smallest value from Steps 1 and 2.

   Example: $f(x,y) = 2x^2 - 3y^2$ on $x^2 + y^2 \leq 4$.

   Example: $f(x,y) = x^2 + y^2$ on the triangle with corners $(0,0), (5,0), (0,3)$.

Math 241 Section 13.9: Lagrange Multipliers
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1. Goal: To find the max and/or min of $f(x,y)$ (the objective function) subject to $g(x,y) = 0$ (the constraint equation).

2. Fact: If $f(x,y)$ attains such a max and/or min then the max/min occur where $\nabla f = \lambda \nabla g$. The reason for this is not screamingly obvious and we will not discuss it in detail. The basic idea is that it happens when the level curves are parallel, which means the gradients must be parallel.

3. Method:

   (a) Identify objective $f(x,y)$
   Identify constraint g(x,y)=0

   (b) Solve for all (x,y) satisfying the system:

   $$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= 0 \end{aligned}$$

   Note: You may also find λ while doing this. That's fine, but you don't ever need λ.

   (c) Plug each resulting (x,y) into f and identify the largest and/or smallest.

   Example: Find max and min of $f(x,y) = x + 3y$ with $x^2 + y^2 = 9$.
   Example: Find min of $f(x,y) = xy$ with $y = 2x + 4$.
   Example: Find max and min of $f(x,y) = x + xy$ with $x^2 + y^2 = 1$.

Math 241 Section 14.1: Double Integrals
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1. Reminded them that for $f(x)$ defined on $I = [a, b]$ the single integral is (signed) area under a curve. Drew pictures.

2. For $f(x, y)$ defined on R in xy-plane define the double integral
$$\int\int_R f(x, y)\, dA$$
as the (signed) volume. Picture to clarify.

3. Switch gears and do examples of iterated double integrals, basically just understanding the notation and process, ignoring relevance.

4. The point now is to rewrite $\int\int_R f(x, y)\, dA$ as an iterated double integral. This process with depend entirely on the shape of R. For now just rectangular parametrizations.

   (a) Definition: R is vertically simple if R is:

   - Between two constant x-values $x = a$ and $x = b$.
   - Between functions $y = B(x)$ and $y = T(x)$.

   Picture to clarify.
   Then:
   $$\int\int_R f(x, y)\, dA = \int_a^b \int_{B(x)}^{T(x)} f(x, y)\, dy\, dx$$
   Example: $\int\int_R xy\, dA$ where R is the region between $y = \frac{1}{2}x$ and $y = x$ and to the left of $x = 6$.

   (b) Definition: R is horizontally simple if R is:

   - Between two constant y-values $y = c$ and $y = d$.
   - Between functions $x = L(y)$ and $x = R(y)$.

   Picture to clarify.
   Then:
   $$\int\int_R f(x, y)\, dA = \int_c^d \int_{L(y)}^{R(y)} f(x, y)\, dx\, dy$$
   Example: $\int\int_R x + y\, dA$ where R is the region between $x = y^2$ and $y = \frac{1}{5}x$.

   (c) Definition: R is simple if it's both, and can then be done either way.

Math 241 Section 14.2: Double Integrals in Polar
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1. Intro: Sometimes R can be easier to describe using polar. A region is described in polar as being between two angles $\theta = \alpha$ and $\theta = \beta$ and between two functions $r = N(\theta)$ and $r = F(\theta)$ where N is nearer to the origin and F is further from the origin.

2. Polar functions to rememeber:

   - Circles: $r = a$, $r = a\cos\theta$, $r = a\sin\theta$
   - Cardioids: $r = a + a\cos\theta$, $r = a + a\sin\theta$
   - Converting rectangular: For example $x = 2$ becomes $r\cos\theta = 2$ or $r = 2\sec\theta$.

3. If R is described in polar then:

$$\int\int_R f(x,y)\,dA = \int_\alpha^\beta \int_{N(\theta)}^{F(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

   Note 1: the $f(r\cos\theta, r\sin\theta)$ means convert f into polar.

   Note 2: Do not forget the additional r. We'll have a good explanation of this later.

   Example: $\int\int_R x dA$ where R is the semicircle $x^2 + y^2 \leq 9$ with $x \geq 0$.

   Example: $\int\int_R 1 dA$ where R is the region inside $r = 1 + \cos\theta$ and outside $r = 1$.

4. Note about 14.1 and 14.2: Sometimes an iterated integral has been set up one way (VS, HS, P) and is easier another way. In this case we might rewrite it.

   Example: The integral

$$\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

   is particular icky. The region R however is simply the pie-slice in the first quadrant inside $r = 2$ and between $\theta = \pi/3$ and $\theta = \pi/2$. Therefore we can rewrite it as

$$\int_{\pi/3}^{\pi/2} \int_0^2 r^2\, dr\, d\theta$$

   which is much more manageable.

Math 241 Section 14.4: Triple Integrals in Rectangular
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1. Introduction: If D is a solid and $f(x,y,z)$ is the density around (x,y,z) then $\int\int\int_D f(x,y,z)\,dV$ represents the mass of D. This idea applies to any density sort of thing, like electrical charge density, for example. The question is how to evaluate this.

2. This depends on how D is described. For 14.4 we assume D is between $F_1(x,y)$ and $F_2(x,y)$ and above the region R in the xy-plane where R is either vertically or horizontally simple. If D is described as such then:

   (a) If R is VS then

   $$\int\int\int_D f(x,y,z)\,dV = \int_a^b \int_{B(x)}^{T(x)} \int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

   (b) If R is HS then

   $$\int\int\int_D f(x,y,z)\,dV = \int_c^d \int_{L(y)}^{R(y)} \int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

   Example: Find the mass of D where is between $z = x^2 + y^2$ and $z = 1 + x^2 + y^2$ and above R the triangle in the xy-plane with corners $(0,0), (0,1), (1,0)$ and the density is $f(x,y,z) = xz$.

3. I then commented that volume is $\int\int\int_D 1\,dV$ and why.

   Example: Find the volume of D the wedge under $x + 2y + z = 6$ and in the first octant.

Math 241 Section 14.5: Triple Integrals in Polar
Dr. Justin O. Wyss-Gallifent

1. Introduction: Cylindrical is like polar plus z however many surfaces can look strange in cylindrical. Examples:

   (a) $z = x^2 + y^2$ becomes $z = r^2$.

   (b) $z = \sqrt{x^2 + y^2}$ becomes $z = r$.

   (c) $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$ or the top half $z = \sqrt{9 - r^2}$.

   (d) $z = 2 - x - y$ becomes $z = 2 - r\cos\theta - r\sin\theta$.

   (e) $r = \sin\theta$ becomes a cylinder, as does $r = \cos\theta$ and $r = 3$.

2. The method: If R is parametrized in polar then:

$$\int\int\int_D f(x,y,z)dV = \int_\alpha^\beta \int_{N(\theta)}^{F(\theta)} \int_{\text{Floor}(r,\theta)}^{\text{Ceiling}(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

   As with triple integrals in rectangular the first two integrals take care of R. The top and bottom functions must be rewritten in terms of r and θ and the integrand must be rewritten too.

   Example: The mass of the ice-cream cone inside $z = \sqrt{(3x^2 + 3y^2)}$ and inside $x^2 + y^2 + z^2 = 4$. It's often tricky to identify R and even the top and bottom functions are often confusing. Here I used $f(x,y,z) = z^2$ for the density.

   Example: The volume of the solid inside $r = \sin(\theta)$, below $z = 9 - x^2 - y^2$ and above the xy-plane

Math 241 Section 14.6: Triple Integrals in Spherical
Dr. Justin O. Wyss-Gallifent

1. Introduction: Introduce spherical coordinates. We locate a point using three variables:

   θ is the familiar one, the angle from the positive x-axis.
   ϕ is the angle measured from the positive z-axis downwards, so $0 \leq \phi \leq \pi$.
   ρ is the distance from the origin.

   Graph some points (very few) and then show a picture which illustrates how to change these to x, y, z which we'll need for substitution:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \\ x^2 + y^2 + z^2 &= \rho^2 \end{aligned}$$

2. Classic examples of equations in spherical:
   (a) Sphere $\rho = 3$
   (b) Cone $\phi = \pi/4$
   (c) The plane $z = 2$ becomes $\rho\cos\phi = 2$ or $\rho = 2\sec\phi$.

3. Describing objects in spherical: A solid D will be described as having:

$$\begin{gathered} \alpha \leq \theta \leq \beta \\ \gamma \leq \phi \leq \delta \\ near(\theta, \phi) \leq \rho \leq far(\theta, \phi) \end{gathered}$$

4. Integration: If D is described in spherical then:

$$\int\int\int_D f(x, y, z)\, dV = \int_\alpha^\beta \int_\gamma^\delta \int_{near(\theta,\phi)}^{far(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

   NOTE 1: The $\rho^2 \sin\phi$ is like the polar/cylindrical r. We just have to remember for now to put it there.

   NOTE 2: The book allows the middle iterated integral to have limits which are functions of theta - see the $h_1(\theta)$ and $h_2(\theta)$ in Thm 14.11 on p.947. In practice these are always constants so I just gave those limits as γ and δ.

   Example: The solid between the spheres $\rho = 1$ and $\rho = 2$.

   Example: If we just take the top half of the previous.

   Example: If we just take the part of the previous having $x \leq 0$, $y \geq 0$ and $z \geq 0$. This one is nice to draw.

   Example: The volume of the ice-cream cone inside $\phi = \pi/6$ and and inside $\rho = 5$.

   Example: If we chop off the bottom at $z = 1$.

Math 241 Section 14.8: Change of Variables
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1. Intro Part I: For double integrals, sometimes VS, HS and polar are insufficient for parallelograms, ellipses and other quirky shapes.

2. Intro Part II: Historical note - consider the integral $\int_0^1 \sqrt{1-x^2}\,dx$. We put $x = \sin u$ and get $\int_0^{\pi/2} \sqrt{1-\sin^2 u}\cos u\,du$ and so three things have changed:

   The interval changes, the integrand changes, and the dx is replaced by $\cos u du$.

3. A change of vars is basically a 2D version of this. Our goal is to do this for a double integral. The change of variables formula works as follows: If R is a not-so-nice region and we want to evaluate $\int\int_R f(x,y)\,dA$ and if we can do a substitution $x = g(u,v)$ and $y = h(u,v)$ as functions of u, v which changes R (in the xy-plane) into S (in the uv-plane) then:

   $$\int\int_R f(x,y)\,dA = \int\int_S f(g(u,v),h(u,v))\,|J(x,y)|\,dA \text{ where } J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

   Notes:

   - Often the change of variables is given by $u =$ and $v =$. In this case we need to solve for $x =$ and $y =$ only if we need them for the integrand.
   - We can often use: $J(x,y) = 1 \div J(u,v)$ where $J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

4. Three classic examples.

   Example: Find $\int\int_R x dA$ where R is the parallelogram bounded by $y = 1-x$, $y = 3-x$, $y = \frac{1}{2}x+1$, and $y = \frac{1}{2}x+4$. We rewrite the edges as $x+y = 1$, $x+y = 3$, $y - \frac{1}{2}x = 1$, and $y - \frac{1}{2}x = 4$ and substitute $u = x+y$ and $v = y - \frac{1}{2}x$. Then S is a rectangle. Since the integrand is just x we need to solve for x. We might as well find y too and then find $J(x,y)$ directly.
   Note: If the integrand had been something like $x+y$ then we know it becomes u and we could have used the alternate method for $J(x,y)$.

   Example: Find $\int\int_R y^2 dA$ where R is the ellipse $x^2/9+y^2/4 = 1$. We rewrite as $(x/3)^2+(y/2)^2 = 1$ and substitute $u = x/3$ and $v = y/2$. Then S is a circle and then we go to polar.

   Example: Find $\int\int_R \frac{x}{y} dA$ where R is the region bounded by $y = x$, $y = 2x$, $y = \frac{1}{x}$, and $y = \frac{2}{x}$. We rewrite the edges as $\frac{y}{x} = 1$, $\frac{y}{x} = 3$, $xy = 1$ and $xy = 2$ and then substitute $u = \frac{y}{x}$ and $v = xy$. Then S is a rectangle. Notice here that the integrand is y/x which is u so we never need to solve for x and y directly so we can use the alternate method for $J(x,y)$.

Math 241 Section 14.9: Parametrization of Surfaces
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1. First a reminder about how we parametrized a curve $\mathbf{r}(t)$. That is, for various t we think of $\mathbf{r}(t)$ as a point on the curve and for all t we get all points.

   The idea for a surface is that given a surface Σ we want to give a parametrization $\mathbf{r}(u, v)$ for a range of u and $v$so that as those two variables run over their ranges we get all the points on the surface.

   Typically we don't use u and v (in fact we never do) but those are stand-ins for more common variables like x, y, z, r, θ, ϕ, and ρ.

2. Examples. Emphasize why we make choices of the two variables the way we do. This can often be confusing at first. Emphasize that the choice of variables is often based on the restriction rather than the surface.

   Example: A small rectangle at $z = 2$ with $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$:
   $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$ with $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$.

   Example: If we fix $y = 3$ instead then we can get something like:
   $\mathbf{r}(x, z) = x\,\mathbf{i} + 3\,\mathbf{j} + z\,\mathbf{k}$ with $0 \leq x \leq 1$ and $0 \leq z \leq 5$.

   Example: The disk of radius 2 at $z = 3$ centered on the z-axis:
   $\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 3\,\mathbf{k}$ with $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$.

   Example: Change the previous one so x is fixed:
   $\mathbf{r}(y, z) = ...$

   Example: The part of $z = 16 - x^2 - y^2$ above a rectangle in the xy-plane.
   $\mathbf{r}(x, y) = ...$

   Example: The cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 5$:
   $r(z, \theta) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$ with $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 5$.

   Example: The hemisphere $x^2 + y^2 + z^2 = 9$ above the xy-plane.
   $\mathbf{r}(\theta, \phi) = ...$

NOTE: There is a document on canvas with a bunch of extra examples (the book is sorely lacking) as well as solutions.
NOTE: All the examples I did today have constant bounds on the two variables. Soon enough I'll introduce some which have functional bounds on one of them.

Math 241 Section 15.1: Vector Fields
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1. Definition: A *vector field* is a function which assigns a vector to each point. This can either be 2D:

   Example: $\mathbf{F}(x,y) = xy\,\mathbf{i} + 2x^2\,\mathbf{j}$

   Or it can be 3D:

   Example: $\mathbf{F}(x,y,z) = ye^z\,\mathbf{i} + xz\,\mathbf{j} + y^2\,\mathbf{k}$

   We can sketch a VF by simply plotting a vector at each point for a nice selection of points.
   Example: $\mathbf{F}(x,y) = \frac{1}{5}y\,\mathbf{i} - \frac{1}{5}x\,\mathbf{j}$ is good to draw.

   Vector fields can be thought of as representing any sort of force field, fluid flow, etc.

2. Associated definitions. For a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ define:

   (a) Divergence: $\nabla \cdot \mathbf{F} = M_x + N_y + P_z$. This is a scalar and measures net gain/loss of fluid at a point. A source is a point with positive divergence and a sink is a point with negative divergence.
   Example: Anything.

   (b) Curl: $\nabla \times \mathbf{F}$ which is defined by taking the cross-product $\left(\frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\right) \times \mathbf{F}$ and doing the corresponding partials. Loosely speaking the curl measures the axis of rotation of the fluid at a point.
   Example: Anything.

3. Conservative vector fields.

   (a) We've seen VF before. Recall the gradient $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. This gives us a VF.
   Example: If $f(x,y,z) = xy^2z$ then $\nabla f = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$.
   Definition: A vector field $\mathbf{F}$ is conservative if there is some f so that $\mathbf{F} = \nabla f$. This f is called a potential function.
   Example: $\mathbf{F}(x,y,z) = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$ is conservative.

   (b) Theorems:

   - If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = \mathbf{0}$.
   - If $\nabla \times \mathbf{F} \neq \mathbf{0}$ then $\mathbf{F}$ is not conservative.
     Example: $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + y\,\mathbf{j} + xz\,\mathbf{k}$ is not because $\nabla \times \mathbf{F} \neq \mathbf{0}$.
   - If $\nabla \times \mathbf{F} = \mathbf{0}$ and if $\mathbf{F}$ is defined at all (x,y,z) then $\mathbf{F}$ is conservative.

   (c) Finding a potential function: If $\mathbf{F}$ is conservative then it will be useful to find a corresponding potential function f. This can often be guessed but it's useful to have a process. Here's an example with a process:
   Example: $\mathbf{F}(x,y,z) = 2xy\,\mathbf{i} + \left[x^2 + z\right]\,\mathbf{j} + [y + 2z]\,\mathbf{k}$
   Process: We want $f(x,y,z)$ with $f_x(x,y,z) = 2xy$, $f_y(x,y,z) = x^2+z$ and $f_z(x,y,z) = y+2z$.
   Step 1: Since $f_x(x,y,z) = 2xy$ we have $f(x,y,z) = x^2y + g(y,z)$.
   Step 2: From here $f_y(x,y,z) = x^2+g_y(y,z)$ but this must equal x^2+z so $x^2+g_y(y,z) = x^2+z$ so $g_y(y,z) = z$ so $g(y,z) = yz + h(z)$ and so $f(x,y,x) = x^2y + yz + h(z)$.
   Step 3: From here $f_z(x,y,z) = y + h'(z)$ but this must equal $y + 2z$ so $y + h'(z) = y + 2z$ so $h'(z) = 2z$ so $h(z) = z^2 + C$ and so $f(x,y,z) = x^2y + yz + z^2 + C$.
   Since C can be any constant we typically use $C = 0$.

Math 241 Section 15.2: Line Integrals of Functions and VFs
Dr. Justin O. Wyss-Gallifent

1. Line Integrals of Functions:

   (a) Situation: If C is a curve representing a wire and if $f(x,y,z)$ is the density at (x,y,z) then we can ask the mass. This is the line integral of $f(x,y,z)$ over C denoted $\int_C f(x,y,z)\,ds$.

   (b) Evaluation (Only Way): We parametrize C as $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \leq t \leq b$ and then
   $$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))||\mathbf{r}'(t)||\,dt$$

   Example: C is the semicircle $x^2 + y^2 = 4$ with $y \geq 0$ and density $f(x,y) = y$.

2. Line Integrals of Vector Fields:

   (a) Situation: If C is a curve representing the path of an object through a vector field (force field) $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ we can ask the work done by F on the object. This is the line integral of $\mathbf{F}(x,y,z)$ over C denoted either $\int_C \mathbf{F} \cdot d\mathbf{r}$ or $\int_C M\,dx + N\,dy + P\,dz$.

   (b) Evaluation (Way 1 of 4): We parametrize C as $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \leq t \leq b$ and then
   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t),z(t)) \cdot \mathbf{r}'(t)\,dt$$

   Example:
   Example:

   (c) Note 1: Direction of C matters, changing the direction negates the integral, and why.

   (d) Note 2: Notation can be confusing. Observe the difference between $\int_C x\,dx$, $\int_C x\,ds$, $\int_a^b x\,dx$.

Math 241 Section 15.3: The Fundamental Theorem of Line Integrals
Dr. Justin O. Wyss-Gallifent

1. Reminder about how the Fundamental Theorem of Calculus works. It sometimes helps students see the analogy. Plus this analogy arises later in other theorems.

2. FTOLI: If $\mathbf{F}$ is conservative with potential function f then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{endpoint of } C) - f(\text{startpoint of } C)$$

   Example: Draw a really awful curve in 2D but make the endpoints clear.

   Example: Give $\mathbf{r}(t)$ so we have to find the endpoints via $\mathbf{r}$ in that case.

3. Notes:

   (a) $\mathbf{F}$ MUST BE CONSERVATIVE!!!

   (b) If $\mathbf{F}$ is conservative and C is closed then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

   (c) These problems can also appear with $\int_C M\,dx + N\,dy + P\,dz$ notation.

   (d) If $\mathbf{F}$ is conservative then we say the integral is independent of path.

Math 241 Section 15.4: Green's Theorem
Dr. Justin O. Wyss-Gallifent

1. Green's Thm is a 2D theorem which relates the line integral of a vector field around a closed curve to the double integral over the region contained within that curve. Think: If R is a region then it has an edge C and Green's Theorem relates two integrals.

   Theorem: If R is a region in the xy-plane and C is the boundary, oriented counterclockwise, then

   $$\int_C M\,dx + N\,dy = \int\int_R N_x - M_y\,dA$$

2. Examples and Notes:

   (a) Example: With C the edge of a triangle.

   (b) Example: With C the edge of a quarter-disk in which the orientation is clockwise so we must negate.

   (c) Note: C must be closed.

   (d) Note: The left side is the same as $\int_C (M\,\mathbf{i} + N\,\mathbf{j}) \cdot d\mathbf{r}$ so keep an eye on that!

Math 241 Section 15.5: Surface Integrals of Functions
Dr. Justin O. Wyss-Gallifent

1. Intro: Suppose Σ is a surface and $f(x, y, z)$ is defined at each point in Σ. If we chop Σ into little rectangles and take the area of each multiplied by f at some point in the rectangle, then let the size of the rectangles to go zero, the result is $\int\int_\Sigma f\,dS$. Some uses:

   - If $f(x, y, z) = 1$ we get the surface area of Σ.
   - If $f(x, y, z)$ is the density (mass or electrical charge or whatever) at any point is $f(x, y, z)$ then we get the total (mass,charge, whatever).

2. Method of evaluation; Only one way:

   Parametrize Σ as $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$ for u, v in the region R in the uv-plane (R is usually described simply by inequalities on u and v) and then:

   $$\int\int_\Sigma f(x, y, z)\,dS = \int\int_R f(x(u, v), y(u, v), z(u, v))||\mathbf{r}_u \times \mathbf{r}_v||\,dA$$

   Note: We're used to using R to denote a region in the xy-plane but that's not really the case here. Instead R is a region in the uv-plane for whatever variables we're using, but it's easier to think of R as a set of (u, v) usually described by a pair of inequalities.

3. Examples

   Example: $f(x, y, z) = xyz$ for Σ the part of $z = 9 - x^2 - y^2$ above the rectangle with $0 \le x \le 1$ and $0 \le y \le 2$. Here we use $\mathbf{r}(x, y)$.

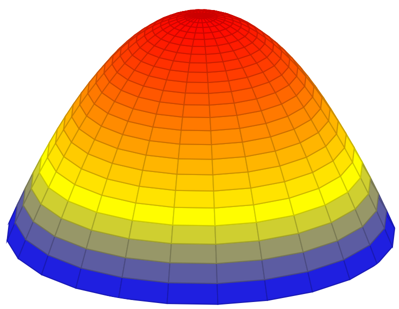
   Example: $f(x, y, z) = xyz$ for Σ the part of the cylinder $x^2 + y^2 = 4$ between $z = 1, 5$. Here we use $\mathbf{r}(z, \theta)$.

   Example: $f(x, y, z) = x^2 z$ for Σ the part of $z = 7 - x$ inside $r = 2\sin(\theta)$. Here we use $\mathbf{r}(r, \theta)$. This is often a bit confusing because we have $0 \le \theta \le \pi$ (generally okay) and $0 \le r \le 2\sin\theta$ (often confusing).

   NOTE: An important point - I really want to be sure that this is very step-by-step, meaning we go from an integral over Σ to an integral over R to an iterated integral and that we don't skip the middle step.

Math 241 Section 15.6: Surface Integrals of Vector Fields
Dr. Justin O. Wyss-Gallifent

1. An oriented surface is a surface with a chosen direction through the surface. More specifically it's a continuous choice of unit normal vectors at each point on the surface.



2. If Σ is an oriented surface immersed in a fluid with fluid flow $\mathbf{F}$ then the rate at which $\mathbf{F}$ flows through Σ in the direction of the orientation is given by the surface integral of the vector field, denoted
$$\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS$$

3. The most basic method of evaluation: Parametrize Σ using $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ with u,v constrained by inequalities described by R. Then:
$$\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \pm\int\int_R \mathbf{F}(x(u,v),y(u,v),z(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$
In which the $\pm$ is determined as follows:
   - Use $+$ if the vectors $\mathbf{r}_u\times\mathbf{r}_v$ match the direction of the orientation of Σ.
   - Use $-$ if they are opposite.

4. Examples:

   Example: Σ is the part of $x+2y+2z=6$ in the first octant oriented downwards and with $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + y\,\mathbf{j} + xz\,\mathbf{k}$.

   Example: Σ is the part of $x^2+y^2=9$ with $0\le z\le 5$ oriented outwards and with $\mathbf{F}(x,y,z) = z\,\mathbf{i} + xy\,\mathbf{j} + y^2\,\mathbf{k}$.

Math 241 Section 15.7: Stokes' Theorem
Dr. Justin O. Wyss-Gallifent

1. Introduction: Stokes' Theorem can be thought of as a version of Green's Theorem which applies to surface in 3D space, although it would be more accurate to say that Green's Theorem is a version of Stokes' Theorem when the surface is in the xy-plane.

2. Induced Orientations: If C is the edge of Σ and if C has an orientation (direction) then C induces an orientation on Σ using the right-hand rule; if your fingers follow C's orientation then your thumb points in an orientation for Σ. We say Σ has the induced orientation from R.

   This also works in reverse but we will not need it this way.

3. Stokes' Theorem: Suppose C is the edge of Σ and C is oriented. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

   where Σ has the induced orientation from R.

   Note: Stokes' Theorem leads to a multiple-step procedure:

$$\int_C \underset{\text{Stokes'}}{\longrightarrow} \int\int_\Sigma \underset{\text{Param } \Sigma}{\longrightarrow} \pm\int\int_R \underset{\text{Use Ineq}}{\longrightarrow} \int_*^* \int_*^* \underset{\text{Grunt}}{\longrightarrow} \text{Value}$$

   So be clear about showing all necessary steps.

   Example: Suppose C is the curve on the surface $z = 9 - x^2$ lying above the triangle with vertices $(0,0,0)$, $(2,0,0)$, $(0,1,0)$ and having counterclockwise orientation when viewed from above. Evaluate $\int_C 2y\,dx + xz\,dy + z\,dz$.

   Example: Suppose C is the intersection of $x^2 + z^2 = 16$ with $y = 9 - x^2$ having clockwise orientation when viewed from the positive y-axis. Evaluate $\int_C (x^2\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}) \cdot d\mathbf{r}$.

Math 241 Section 15.8: The Divergence Theorem (Gauss' Theorem)
Dr. Justin O. Wyss-Gallifent

1. The Divergence Theorem: If Σ is a surface which completely surrounds a solid D and if Σ has outwards orientation then:

$$\int\int_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_{D} \nabla \cdot \mathbf{F}\, dV$$

   Two immediate notes:

   - Σ is the surface but D is the solid inside it.
   - How we do $\int\int\int_D \nabla \cdot \mathbf{F}\, dV$ depends on D. Could be rectangular, cylindrical or spherical.

   Example: Suppose Σ is the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$ with the caps at the ends, oriented outwards, evaluate $\int\int_{\Sigma}(x^2\,\mathbf{i} + xz\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n}\, dS$.

2. Notes

   (a) If Σ is oriented inwards we can negate.

   (b) If $\nabla \cdot \mathbf{F}$ is a constant then the result is that constant times the volume of D. This is only useful if the volume can be conveniently calculated.

   Example of (a) and (b): Suppose Σ is the top hemisphere part of $x^2 + y^2 + z^2 = 9$ along with the base, oriented inwards, evaluate $\int\int_{\Sigma}(2x\,\mathbf{i} + 5y\,\mathbf{j} + 7z\,\mathbf{k}) \cdot \mathbf{n}\, dS$.

   (c) Σ must completely surround D to do this. In that sense it's like a 3D version of Green's Theorem.

   Example of (c): In the previous problem if we hadn't included the base then the Divergence Theorem would not apply.

   (d) This Theorem makes sense. The surface integral is measuring the fluid flow across the surface while the triple integral is measuring the "total divergence" which can be thought of as the "total sink/source" nature of the vector field inside the object. It makes sense that these are equal because the fluid can only appear (source) or disappear (sink) equal to how it crosses the boundary.

Math 241 Chapter 15 Integral Study Guide

Important 1: Curves are always parametrized by $\bar{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ for $a \le t \le b$ Note that some components might be 0 and the $\hat{k}$ component will definitely be 0 in 2D.
Important 2: Surfaces are always parametrized by $\bar{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$ for u, v restricted by R. Note that often it's not u, v but x, y or z, θ or ...
Important 3: Keep in mind that $\int_C 1\ ds$ is length of C, $\iint_\Sigma 1\ dS$ is surface area of Σ, $\iint_R 1\ dA$ is area of R and $\iiint_D 1\ dV$ is volume of D.

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1. Line integral of a function. One choice:

$$\int_C f\ ds = \int_a^b f(x(t), y(t), z(t))||\bar{r}'(t)||\ dt$$

2. Line integral of a vector field. Lots of possibilities. This one has the most options.

   (a) Most basic. Could be so ugly as to be nonintegrable:

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(x(t), y(t), z(t)) \cdot \bar{r}'(t)\ dt$$

   (b) If $\bar{F}$ is conservative and f is a potential function then:

$$\int_C \bar{F} \cdot d\bar{r} = f(\text{endpt of C}) - f(\text{startpt of C})$$

   (c) If $\bar{F}$ is conservative and C is closed then:

$$\int_C \bar{F} \cdot d\bar{r} = 0$$

   (d) If C is the edge of Σ with induced orientation then Stokes's Theorem gives:

$$\int_C \bar{F} \cdot d\bar{r} = \iint_\Sigma (\nabla \times \bar{F}) \cdot \bar{n}\ dS \quad \text{Then go to 4(a)}$$

   (e) Note that there is alternate notation for this:

$$\int_C M dx + N dy + P dz \quad \text{means} \quad \int_C (M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}) \cdot d\bar{r}$$

   (f) If in 2D and C is the edge of R with the cows on the left then Green's Theorem gives:

$$\int_C (M\,\hat{\imath} + N\,\hat{\jmath}) \cdot d\bar{r} \ \text{ or } \int_C M dx + N dy = \iint_R N_x - M_y\ dA$$

3. Surface integral of a function. One choice:

$$\iint\limits_\Sigma f\ dS = \iint\limits_R f(x(u,v), y(u,v), z(u,v))||\bar{r}_u \times \bar{r}_v||\ dA$$

4. Surface integral of a vector field. Two possibilities.

   (a) Always unless the Divergence Theorem applies:

$$\iint\limits_\Sigma \bar{F} \cdot \bar{n}\ dS = \pm \iint\limits_R \bar{F}(x(u,v), y(u,v), z(u,v)) \cdot [\bar{r}_u \times \bar{r}_v]\ dA$$

   (b) If Σ is the boundary surface of a solid object D with outward orientation then we can use the Divergence Theorem (Gauss's Theorem) giving:

$$\iint\limits_\Sigma \bar{F} \cdot \bar{n}\ dS = \iiint\limits_D \nabla \cdot \bar{F}\ dV$$

Math 241 Parametrization of Surfaces

First make sure that you understand what a parametrization of a surface Σ actually means. To say that Σ is parametrized by $\bar{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$ for all u, v within the region R in the uv-plane means that if you take all possible u and v with in your region R then you get the entire surface with the resulting points $(x(u,v), y(u,v), z(u,v))$. In other words think of the vectors $\bar{r}(u,v)$ as just being points.

For example, consider the parametrization $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + 2\,\hat{k}$ with $0 \leq x \leq 2$ and $0 \leq y \leq 3$. As x varies and y varies within their allowable ranges we get all the points $(x, y, 2)$ with $0 \leq x \leq 2$ and $0 \leq y \leq 3$. This gives us a small rectangular piece of the plane $z = 2$.

This is a very simple example but is a good start. Here are a series of ideas you can consider when presented with a description of Σ. Following each are some problems which fit that criteria. Some have solutions, some have hints, some have notes.

1. Is Σ a part of the graph of a function $z = f(x,y)$ defined on some x, y which are themselves nicely parametrized by rectangular coordinates? If so then we can use

   $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + f(x,y)\,\hat{k}$ with R the region of allowable x and y.

   (a) **Example:** Σ is the part of the cone $z = \sqrt{x^2+y^2}$ above the rectangle in the xy-plane with opposite corners $(1,0)$ and $(2,5)$.

   Solution: $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + \sqrt{x^2+y^2}\,\hat{k}$ with $1 \leq x \leq 2$ and $0 \leq y \leq 5$.

   (b) **Example:** Σ is the part of the paraboloid $z = 9 - x^2 - y^2$ above the triangle in the xy-plane with corners $(0,0)$, $(4,0)$ and $(0,2)$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ above R, where R is the region in the xy-plane between $y = x^2$ and $y = 4$.

   Hint: You'll need to parametrize R as vertically simple.

2. Is Σ a part of the graph of a function $z = f(x,y)$ defined on some x, y which are themselves nicely parametrized by polar coordinates? If so then we can use

   $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + f(r\cos\theta, r\sin\theta)\,\hat{k}$, with R the region of allowable r and θ.

   Try not to think of r and θ as polar coordinates here though, just think of them as variables with a certain range and as they vary over that range the function $\bar{r}(r,\theta)$ gives all the points on the surface. For example if your parametrization for some problem turned out to be $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^3\,\hat{k}$ for $0 \leq \theta \leq \pi$ and $0 \leq r \leq \sin\theta$ then you could just as readily use any variables, for example $\bar{r}(t,q) = t\cos q\,\hat{\imath} + t\sin q\,\hat{\jmath} + t^3\,\hat{k}$ for $0 \leq q \leq \pi$ and $0 \leq t \leq \sin q$. No difference. You're just using what you know about polar coordinates to come up with the parametrization.

   (a) **Example:** Σ is the part of the cone $z = 2 + \sqrt{x^2+y^2}$ inside the cylinder $x^2 + y^2 = 4$.

   (b) **Example:** Σ is the part of the parabolic sheet $z = y^2$ inside the cylinder $r = \sin\theta$.

   Solution: $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^2\sin^2\theta\,\hat{k}$ for $0 \leq \theta \leq \pi$ and $0 \leq r \leq \sin\theta$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ in the first octant and inside $r = 2$.

3. In some cases the above two situations can also work with the variables switched around in the cases where Σ is part of a surface given by $x = f(y, z)$ or $y = f(x, z)$. This is rare but it's useful to work some out.

   (a) **Example:** Σ is the part of the paraboloid $y = x^2 + z^2$ to the right of the square in the xz-plane with corners $(0, 0)$, $(2, 0)$, $(0, 2)$ and $(2, 2)$.

   Hint: Your two variables will be x and z. The region R will be in the xz-plane and y will depend upon x and z.

   (b) **Example:** Σ is the part of the parabolic sheet $x = 16 - z^2$ inside the cylinder $y^2 + z^2 = 9$.

   Hint: Since x depends on z and since y and z always lie within a circle we should use what we know about polar coordinates but with the variables switched. Try using $y = r\cos\theta$ and $z = r\sin\theta$. What would x be? How would R be described and in what plane?

4. If none of these are the case then we need to custom-design a parametrization based upon the surface in question. It may also be the case that a problem can be done in one of the previous ways but it simply works out better this way.

   (a) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$.

   Hint: z is free to vary between 0 and 2 independent of x and y so it should be its own variable. Can x and y both be determined by some other variable, perhaps θ?

   (b) **Example:** Σ is the part of the cylinder $x^2 + z^2 = 9$ between $y = 0$ and $y = 2$.

   Hint: Tweak the previous example.

   (c) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ below the cone $z = \sqrt{x^2 + y^2}$.

   Hint: Your knowledge of spherical coordinates should give you a parametrization $\bar{r}(\phi, \theta)$.

   (d) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$ and in the first octant.

   Note: We could treat this part of the cylinder as $y = \sqrt{9 - x^2}$ then do $\bar{r}(x, z) = x\,\hat{\imath} + \sqrt{9 - x^2}\,\hat{\jmath} + z\,\hat{k}$ for $0 \leq x \leq 3$ and $0 \leq z \leq 2$ but this is not so pretty.

   Instead how about $\bar{r}(z, \theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + z\,\hat{k}$ for $0 \leq \theta \leq \pi/2$ and $0 \leq z \leq 2$.

   (e) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane.

   Note: This can be done solving for z and treating it as function of x and y and using polar but it's certainly much easier using spherical coordinates to get a parametrization.

Math 241 Parametrization of Surfaces - Solutions

1. (a) **Example:** Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ above the rectangle in the xy-plane with opposite corners $(1,0)$ and $(2,5)$.

   Solution: $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + \sqrt{x^2+y^2}\,\hat{k}$ with $1 \le x \le 2$ and $0 \le y \le 5$.

   (b) **Example:** Σ is the part of the paraboloid $z = 9 - x^2 - y^2$ above the triangle in the xy-plane with corners $(0,0)$, $(4,0)$ and $(0,2)$.

   Solution: $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + (9 - x^2 - y^2)\,\hat{k}$ with $0 \le x \le 4$ and $0 \le y \le 2 - \frac{1}{2}x$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ above R, where R is the region in the xy-plane between $y = x^2$ and $y = 4$.

   Solution: $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + (20 - x - 2y)\,\hat{k}$ with $-2 \le x \le 2$ and $x^2 \le y \le 4$.

2. (a) **Example:** Σ is the part of the cone $z = 2 + \sqrt{x^2+y^2}$ inside the cylinder $x^2 + y^2 = 4$.

   Solution: $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + (2+r)\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 2$.

   (b) **Example:** Σ is the part of the parabolic sheet $z = y^2$ inside the cylinder $r = \sin\theta$.

   Solution: $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^2\sin^2\theta\,\hat{k}$ for $0 \le \theta \le \pi$ and $0 \le r \le \sin\theta$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ in the first octant and inside $r = 2$.

   Solution: $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + (20 - r\cos\theta - 2r\sin\theta)\,\hat{k}$ with $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$.

3. (a) **Example:** Σ is the part of the paraboloid $y = x^2 + z^2$ to the right of the square in the xz-plane with corners $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$.

   Solution: $\bar{r}(x,z) = x\,\hat{\imath} + (x^2+z^2)\,\hat{\jmath} + z\,\hat{k}$ with $0 \le x \le 2$ and $0 \le z \le 2$.

   (b) **Example:** Σ is the part of the parabolic sheet $x = 16 - z^2$ inside the cylinder $y^2 + z^2 = 9$.

   Solution: $\bar{r}(r,\theta) = (16 - r^2\sin^2\theta)\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

4. (a) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$.

   Solution: $\bar{r}(z,\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + z\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le z \le 2$.

   (b) **Example:** Σ is the part of the cylinder $x^2 + z^2 = 9$ between $y = 0$ and $y = 2$.

   Solution: $\bar{r}(y,\theta) = 3\cos\theta\,\hat{\imath} + y\,\hat{\jmath} + 3\sin\theta\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le y \le 2$.

   (c) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ below the cone $z = \sqrt{x^2+y^2}$.

   Solution: $\bar{r}(\phi,\theta) = 3\sin\phi\cos\theta\,\hat{\imath} + 3\sin\phi\sin\theta\,\hat{\jmath} + 3\cos\phi\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $\pi/4 \le \phi \le \pi$.

   (d) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$ and in the first octant.

   Solution: $\bar{r}(z,\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + z\,\hat{k}$ for $0 \le \theta \le \pi/2$ and $0 \le z \le 2$.

   (e) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane.

   Solution: $\bar{r}(\phi,\theta) = 3\sin\phi\cos\theta\,\hat{\imath} + 3\sin\phi\sin\theta\,\hat{\jmath} + 3\cos\phi\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$.