Characteristic function

If A is a subset of the universal set U, the characteristic function denoted by f_A is defined as follows:

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

We may add and multiply characteristic functions, since their values are numbers, and these operations sometimes help us prove theorems about properties of subsets.

Theorem.

$$f_{A\cap B} = f_A f_B$$

$$f_{A\cup B} = f_A + f_B - f_A f_B$$

$$f_{A \oplus B} = f_A + f_B - 2 f_A f_B$$

Proof: (a) $f_A(x)\,f_B(x)$ equals 1 if and only if both $f_A(x)$ and $f_B(x)$ are equal to 1, and this happens if and only if x is in A and x is in B, that is, x is in $A \cap B$. Since $f_A f_B$ is 1 on $A \cap B$ and 0 otherwise, it must be $f_{A\cap B}$.

(b) If $x \in A$, then $f_A(x) = 1$, so $f_A(x) + f_B(x) - f_A(x)f_B(x) = 1 + f_B(x) - f_B(x) = 1$. Similarly, when $x \in B$, $f_A(x) + f_B(x) - f_A(x)f_B(x) = 1$. If x is not in A or B, then $f_A(x)$ and $f_B(x)$ are 0, so $f_A(x) + f_B(x) - f_A(x)f_B(x) = 0$. Thus $f_A + f_B - f_A f_B$ is 1 on $A \cup B$ and 0 otherwise, so it must be $f_{A\cup B}$.

(c) We leave the proof of (c) as an exercise. ◆

(c) If $x \in A \Rightarrow f_A(x) = 1 \Rightarrow f_A(x) + f_B(x) - 2 f_A f_B = 1 + f_B(x) - 2 f_B(x)$, and if x also $\in B \Rightarrow f_B(x) = 1 \Rightarrow 1 + f_B(x) - 2 f_B(x) = 1 + 1 - 2 = 0$, but in case $x \notin B \Rightarrow 1 + f_B(x) - 2 f_B(x) = 1 + 0 - 2 \cdot 0 = 1 - 0 = 1$ (1) If $x \in B \Rightarrow f_B(x) = 1 \Rightarrow f_A(x) + f_B(x) - 2 f_A f_B = 1 + f_A(x) - 2 f_A(x)$, and if $x \notin A \Rightarrow 1 + f_A(x) - 2 f_A(x) = 1 + 0 - 2\ 0 = 1$, but we shown the case in witch $x \in A \wedge x \in B$ If $x \notin A \wedge x \notin B \Rightarrow f_A(x) + f_B(x) - 2 f_A f_B = 0 + 0 - 2\ 0 = 0$, so $f_A + f_B - 2 f_A f_B = 1 \iff$

$\iff (x \in A \wedge x \notin B)$ or $(x \in B \wedge x \notin A)$

$\Rightarrow f_A + f_B - 2 f_A f_B$ is 1 on $A \oplus B$ and 0 otherwise,

so $f_{A \oplus B} = f_A + f_B - 2 f_A f_B$ $\blacksquare$