Probability Part 2: Random Variables

Random Variables

A number associated to the outcome of a random experiment.

- The sum of two dice rolls
- The number of heads in 5 flips of a coin
- The average height of a sample of 10 people

Discrete: Outcomes can be listed

Continuous: Can take on any value in an interval

The Binomial Distribution

Binary outcome (yes/no).

A fixed number (*n*) of repeated **independent** trials with a fixed probability of “success” (*p*)

Eg. flipping a coin 3 times and recording the number of times it lands on heads.

Here, probability of “success” is 0.5 (if we define “success” as the coin landing on heads)

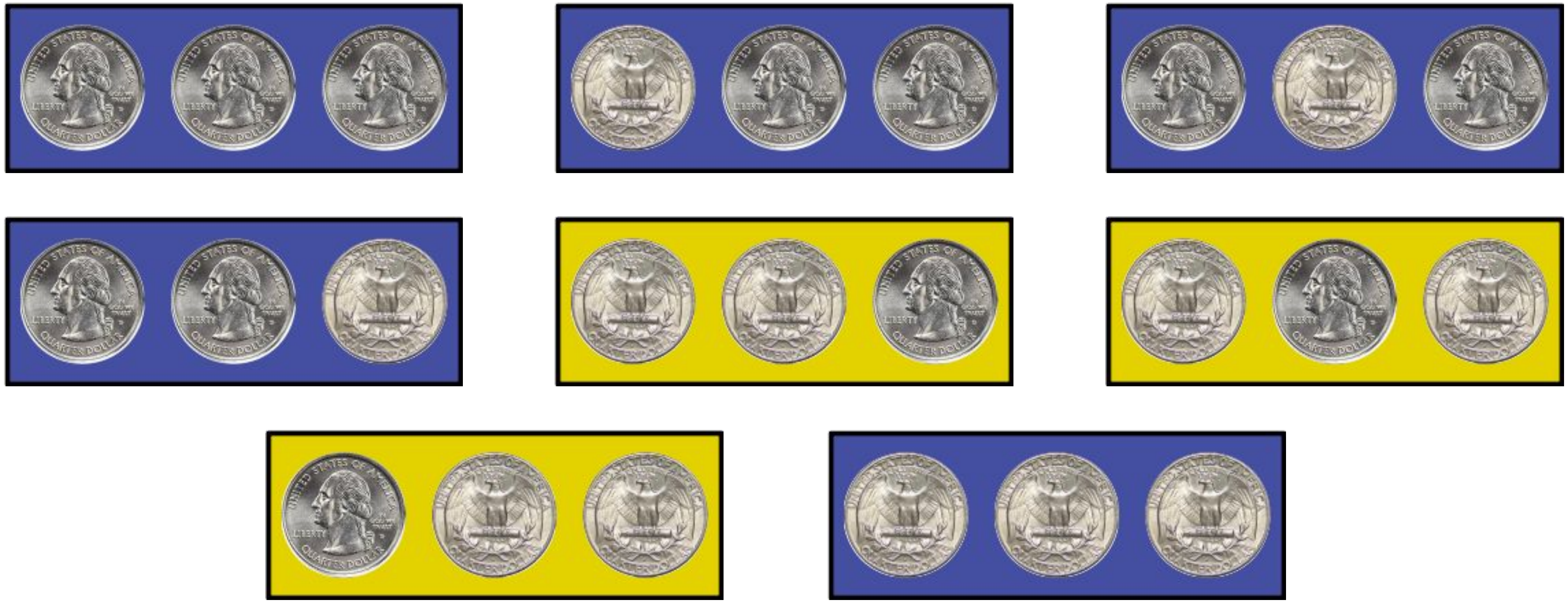
The Binomial Distribution

The binomial distribution lets us answer questions like “If I flip a coin three times, what is the probability of it landing on heads exactly one time?”

Before getting a general formula, let’s see how we could figure this probability “by hand”.

Here are all of the possible outcomes for a sequence of 3 coins flips:

And the ones that we care about:



How likely is each of these?

(0.5)*(0.5)*(0.5) = 0.125

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How likely is each of these?

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Since we have three cases where there is exactly one coin on heads, each with probability 0.125, the total probability of one heads is

3*(0.125) = 0.375

The Binomial Distribution

Let’s modify the setup slightly and say that we have a slightly bent coin which has probability of landing on heads equal to 0.7.

This means that the probability of landing on tails is 0.3.

If we flip our bent coin three times, what is the probability of it landing on heads exactly once?

We still have the same three relevant outcomes. How likely is each?

$$(0.3)*(0.3)*(0.7) = 0.063$$

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We still have the same three relevant outcomes. How likely is each?

$(0.3)*(0.3)*(0.7) = 0.063$

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This time, the total probability is

$3*(0.063) =$

0.189

Let's break down what went into this calculation:

$(0.3)*(0.3)*(0.7)$ $(0.3)*(0.7)*(0.3)$ $(0.7)*(0.3)*(0.3)$

All of these equal

$$0.063 = (0.7)^1 * (0.3)^2 =$$

$$(\text{probability of heads})^1 * (\text{probability of tails})^2$$

Let’s break down what went into this calculation:

$$3*(0.063) = 0.189$$

The three out front was equal to the number of ways to “choose” one coin out of three to land on heads.

In general, we can calculate the number of ways to choose x successes out of n trials using the **binomial coefficients:**

$$\binom{n}{x} := \frac{n!}{x! \cdot (n-x)!}$$

For example,

$$\binom{3}{1} = \frac{3!}{1! \cdot (3-1)!} = \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1} = 3$$

The Binomial Distribution

Now that we have some more tools, let's return to the case where we have a fair coin.

(That is, probability of heads = 0.5 = probability of tails)

If we flip this coin 10 times, what is the probability of it landing on heads exactly 6 times?

The Binomial Distribution

If we flip a coin 10 times, what is the likelihood of the coin landing on heads exactly 6 times?

Ans:

$$\underbrace{\binom{10}{6}}_{\text{Number of ways to choose 6 successes out of 10 options}} \cdot \underbrace{(0.5)^6}_{\text{Probability of heads}} \cdot \underbrace{(0.5)^4}_{\text{Probability of tails}} = 0.205$$

The Binomial Distribution

If we flip a coin 10 times, what is the likelihood of the coin landing on heads exactly 1 time?

Ans: $$\binom{10}{1} \cdot (0.5)^1 \cdot (0.5)^9 = 0.0098$$

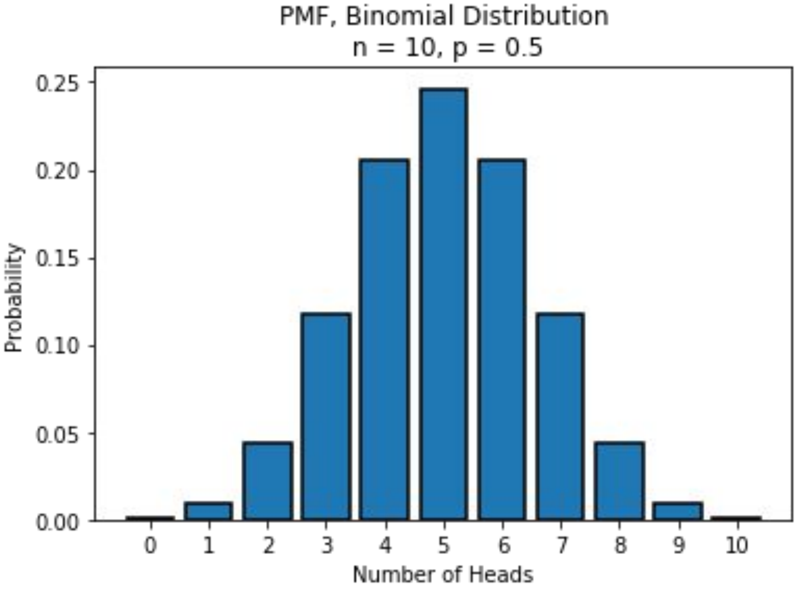
Number of ways to choose 1 success out of 10 options

Probability of heads

Probability of tails

The Binomial Distribution - Probability Mass Function

By looking at the probability associated with each possible outcome, we obtain the **probability mass function**.



The Binomial Distribution

What if we have a bent coin which has a 0.7 chance of landing on heads? What is the probability of it landing on heads on 4 out of 10 flips?

Ans: $$\binom{10}{4} \cdot (0.7)^4 \cdot (0.3)^6 = 0.0368$$

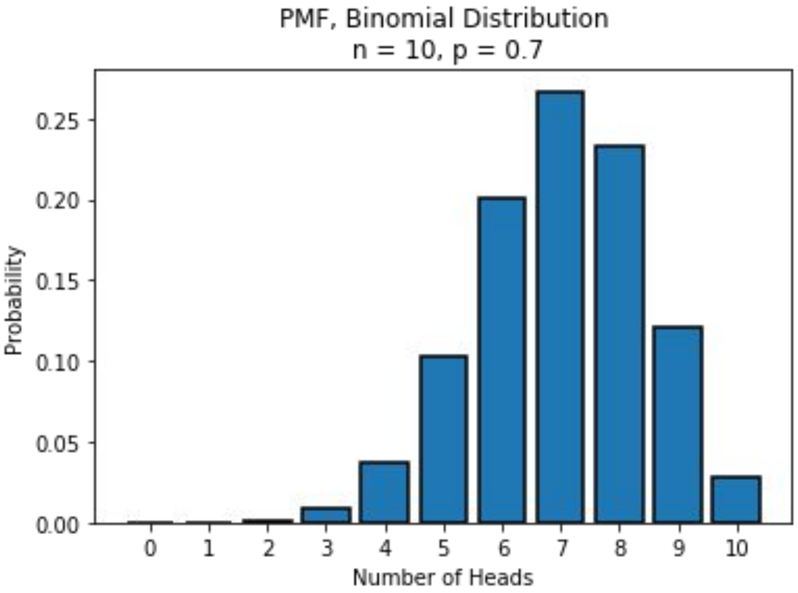
Number of ways to choose 4 successes out of 10 options

Probability of heads

Probability of tails

The Binomial Distribution - Probability Mass Function

Let's say we have a bent coin, where the probability of landing on heads is 0.7. Let's see how the PMF changes:



The Binomial Distribution

In general, a binomial distribution with *n* trials and probability of success *p* was a pmf of:

$$f(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$

Number of ways to choose *x* success out of *n* trials

Probability of success

Probability of failure

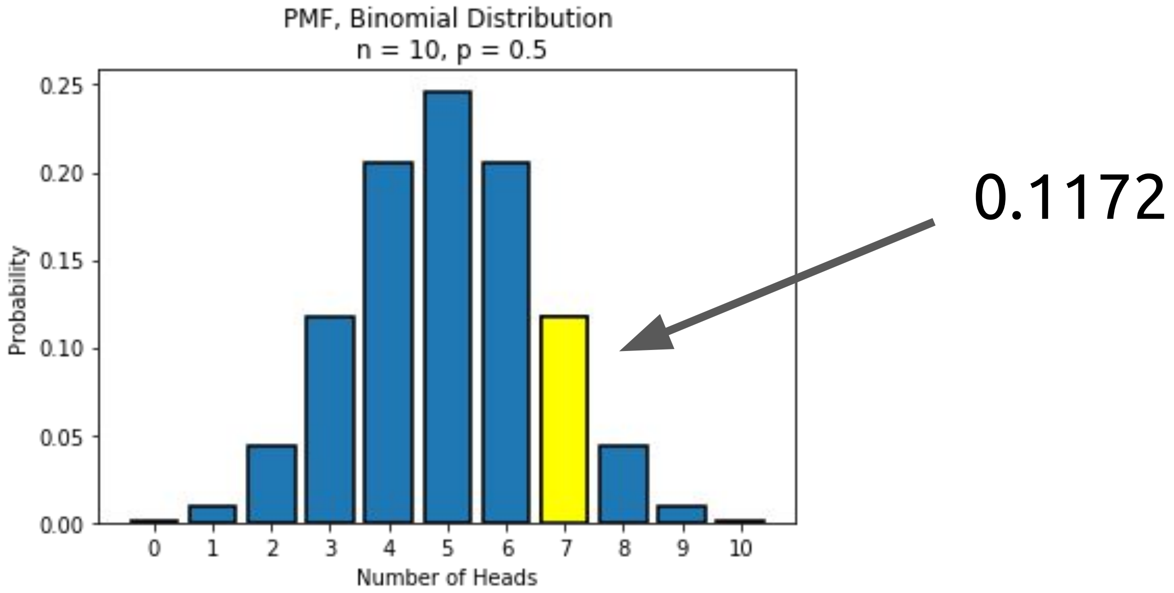
The Binomial Distribution

To carry out these types of calculations, we will usually use the *scipy stats* library rather than calculating the probabilities explicitly.

See the corresponding notebook to see what this looks like in practice.

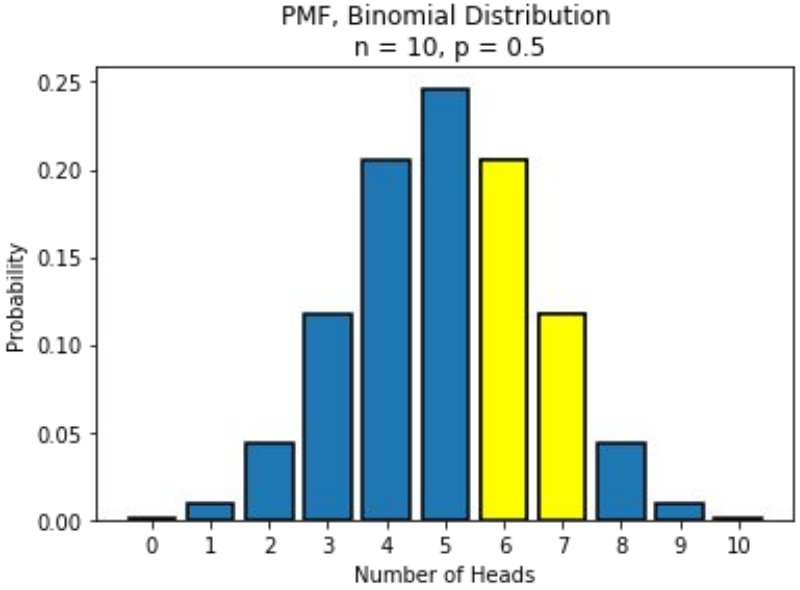
Let’s return to the case of the fair coin (p = 0.5).

The PMF lets us visualize the probabilities. If we want to know how likely it is that the coin will land on heads 7 times, it looks like this:



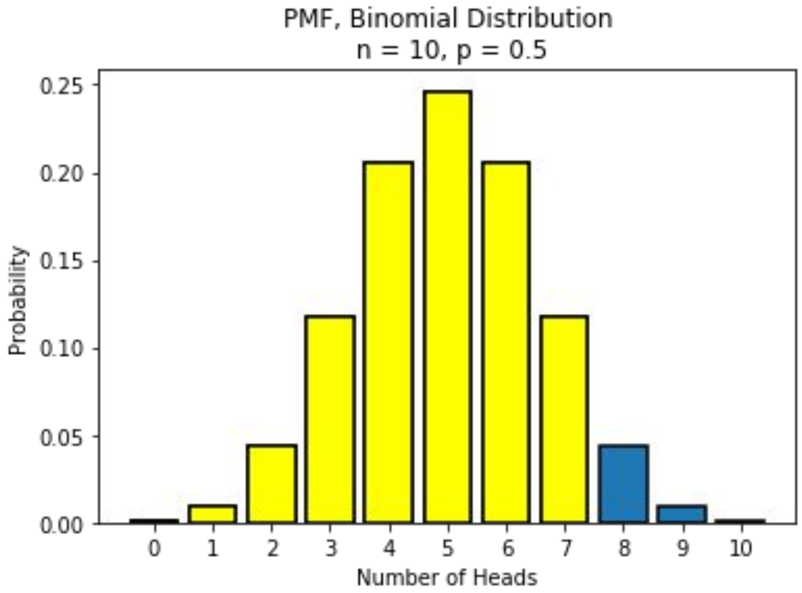
If we want to know the probability of either 6 or 7 heads, we can see that as well. To find the probability, we just add the probability of 6 and the probability of 7:

$$0.2051 + 0.1172 = 0.3223$$



What if we want the probability of 7 or fewer?

I could find the probability of 0, 1, 2, 3, 4, 5, 6, and 7 and then add these all together, but this would be cumbersome.

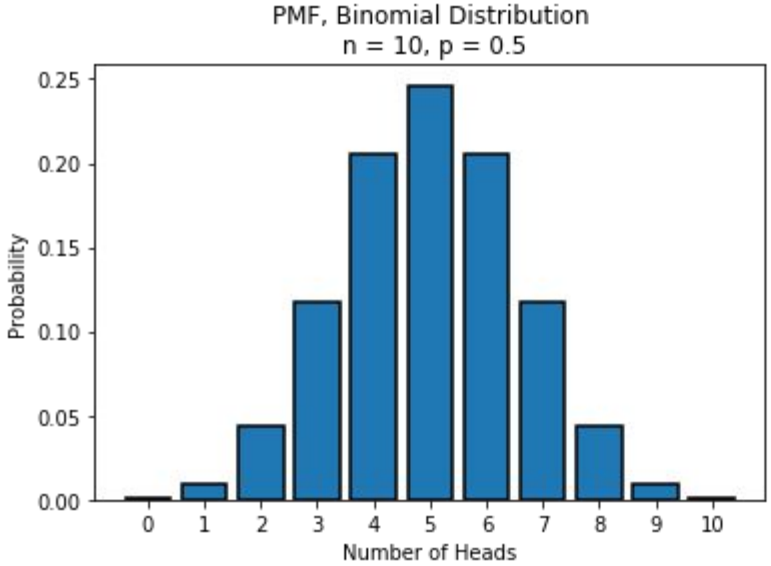


Cumulative Distribution Functions

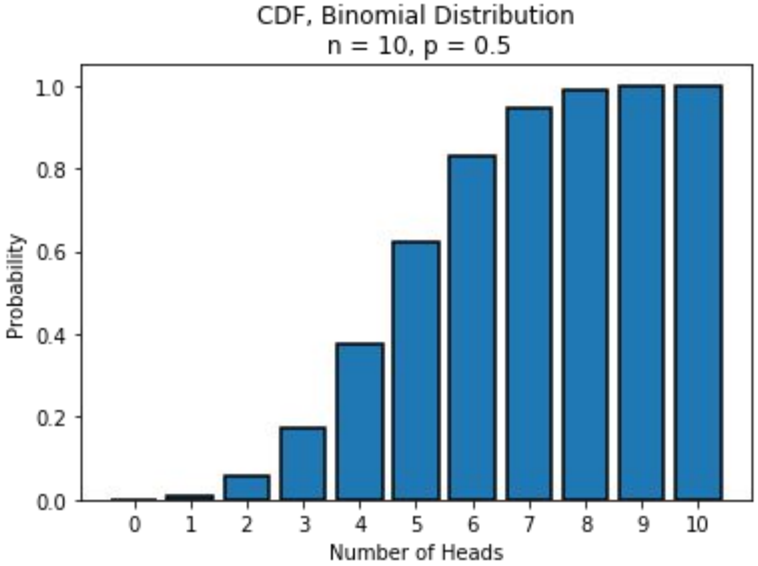
Random variables have cumulative distribution functions (and, in fact, are completely determined by them)

$F(x) = P(X \leq x)$ = The probability that the variable is less than or equal to x

Cumulative Distribution Functions

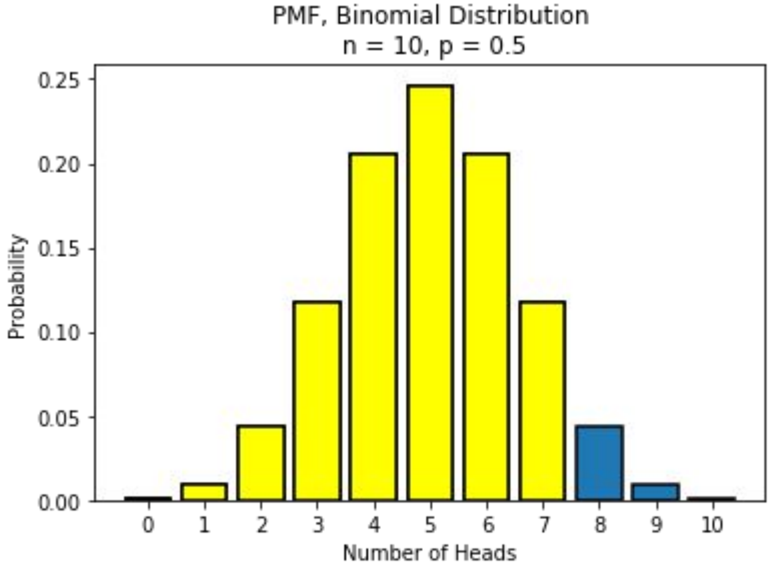


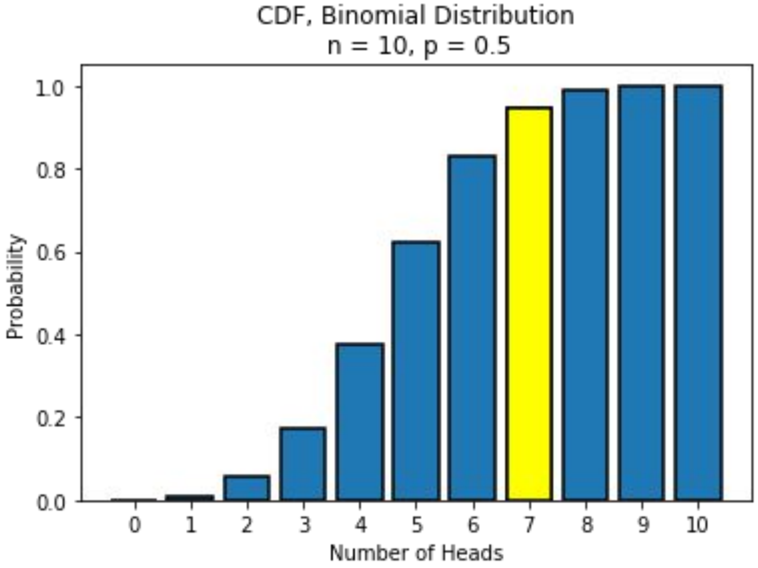
The probability of *exactly* this number of heads.



The probability of this number of heads *or fewer*.

Cumulative Distribution Functions



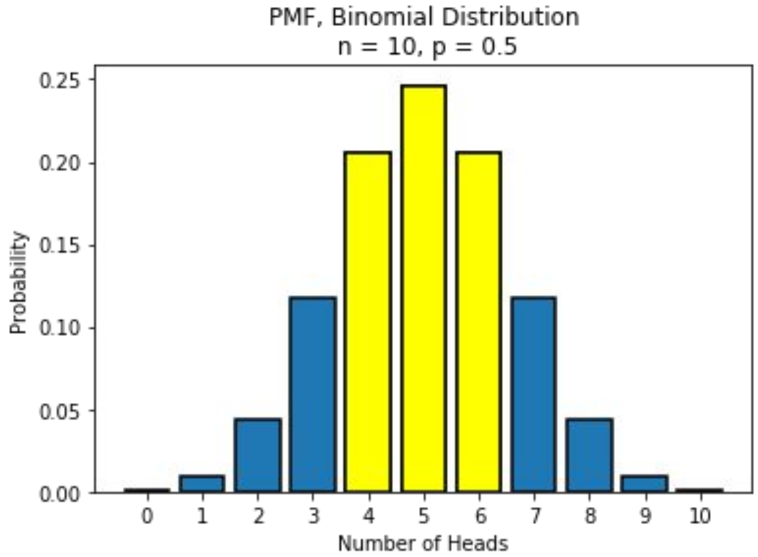


Using the cumulative distribution function, we see that the probability of 7 or fewer heads is 0.9453

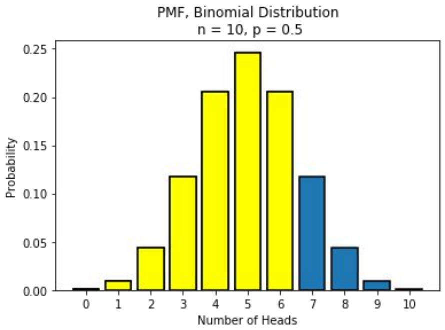
Cumulative Distribution Functions

CDFs can also be used to find other types of probabilities, besides just the probability of *x* or fewer.

For example, let’s say we want to know the probability of between 4 and 6 heads, inclusive.



Cumulative Distribution Functions



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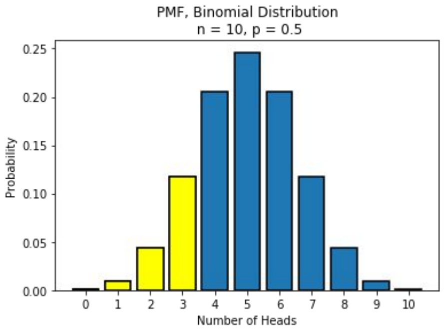

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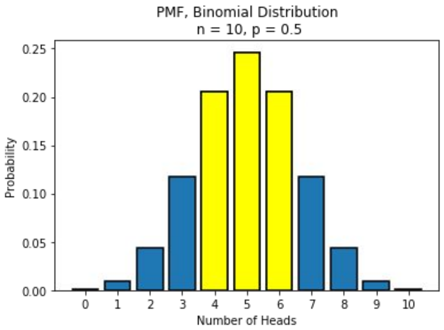

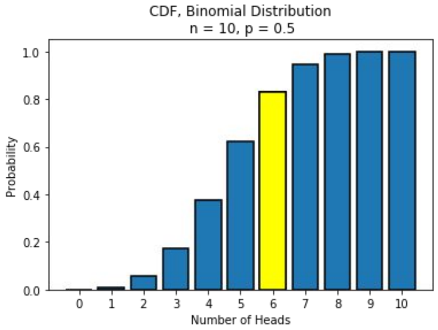

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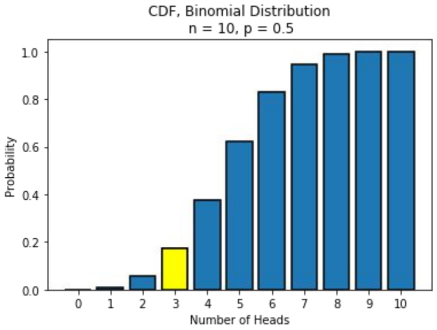

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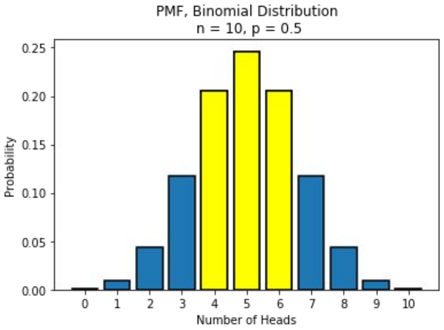
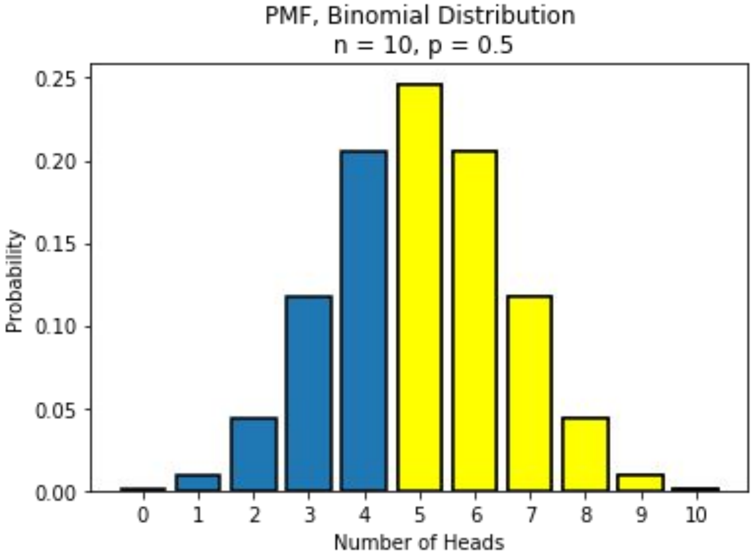


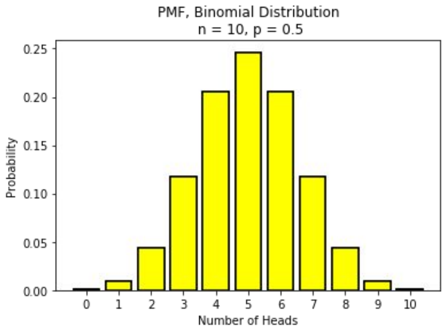
0.8281 - 0.1719 = 0.6563

Cumulative Distribution Functions

What if we want to know the probability of 5 or more heads?



Cumulative Distribution Functions



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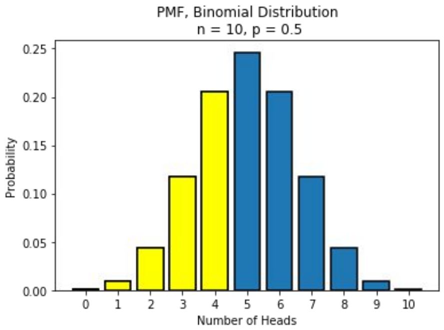

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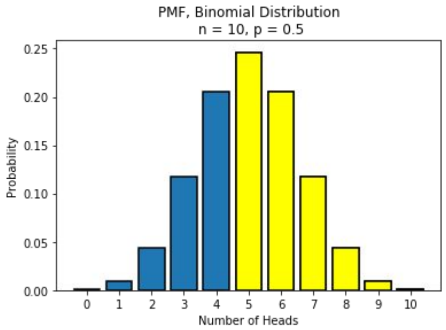

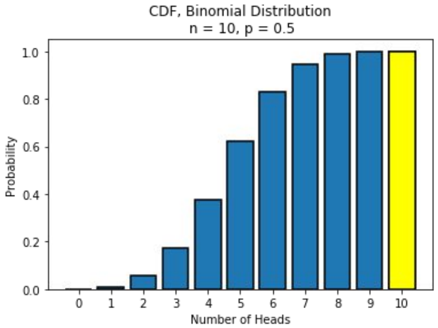

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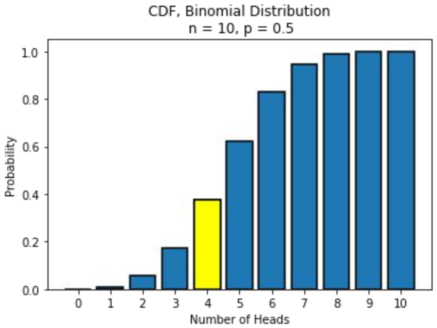

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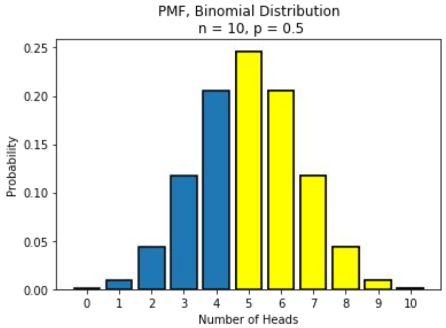


1 - 0.3770 = 0.6230

Continuous Random Variables

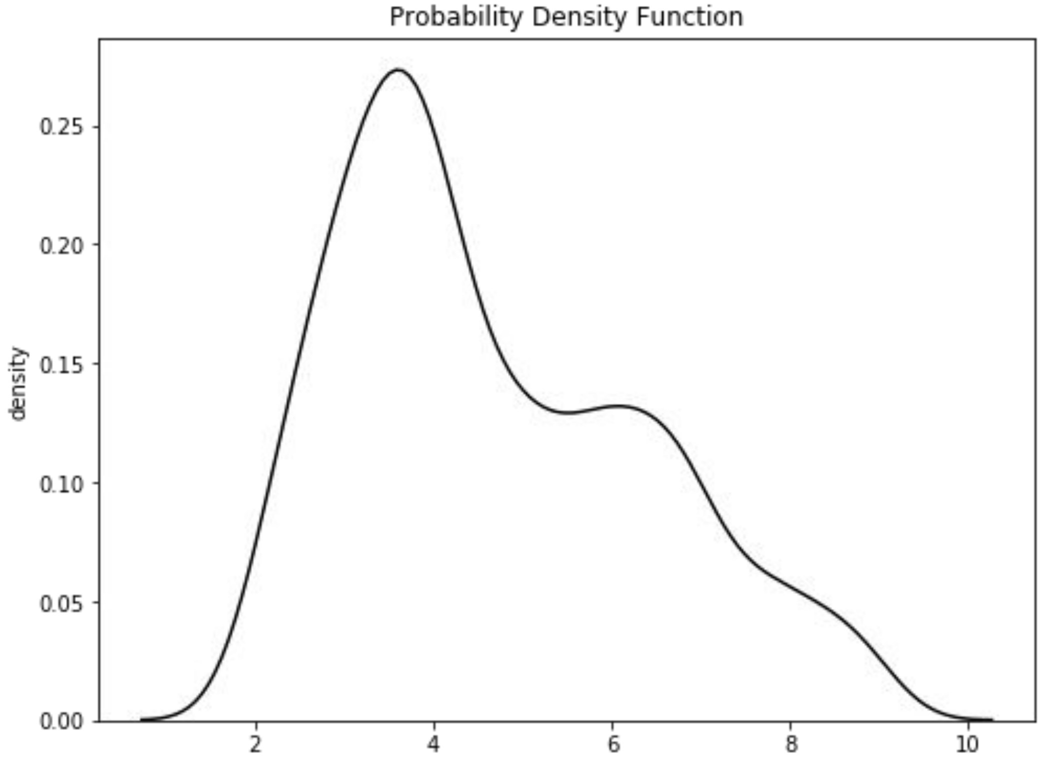
What about random variables which can take on any value in a range?

We can no longer talk about the exact probability for any particular value, but instead can talk about the probability *density* at a particular point.

To find probabilities, we can only find probabilities for the value landing in a particular *range* of values.

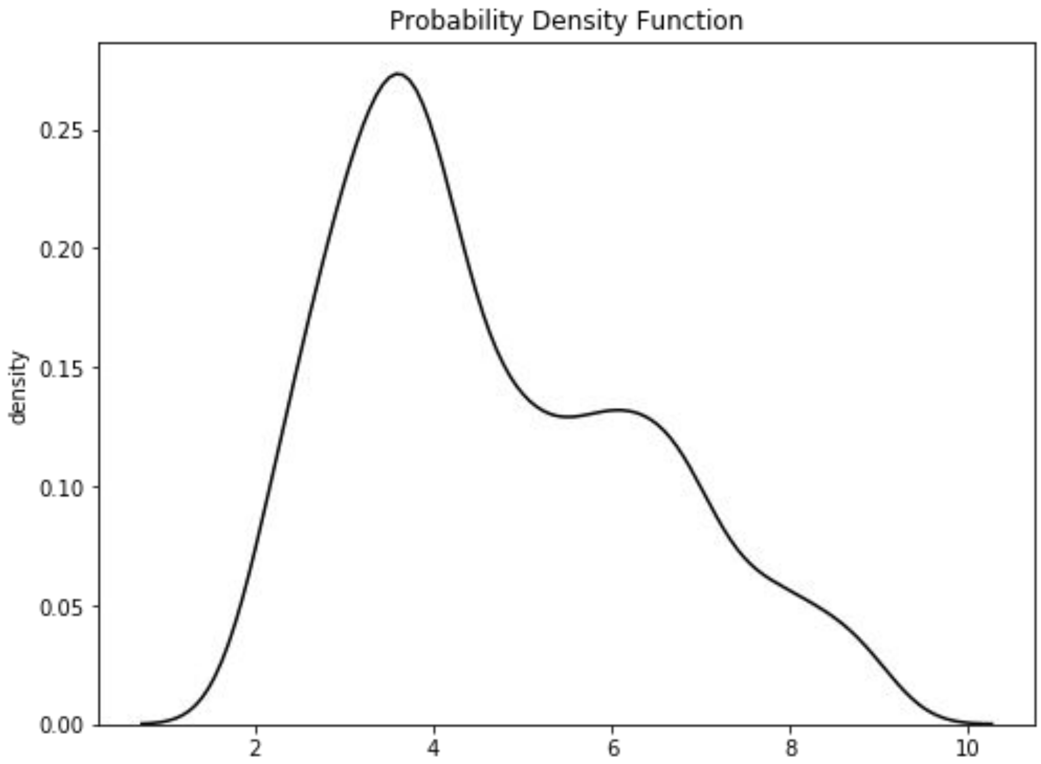
Probability Density Functions

The probability density at each possible value can be specified by a **probability density function**, or **PDF.**



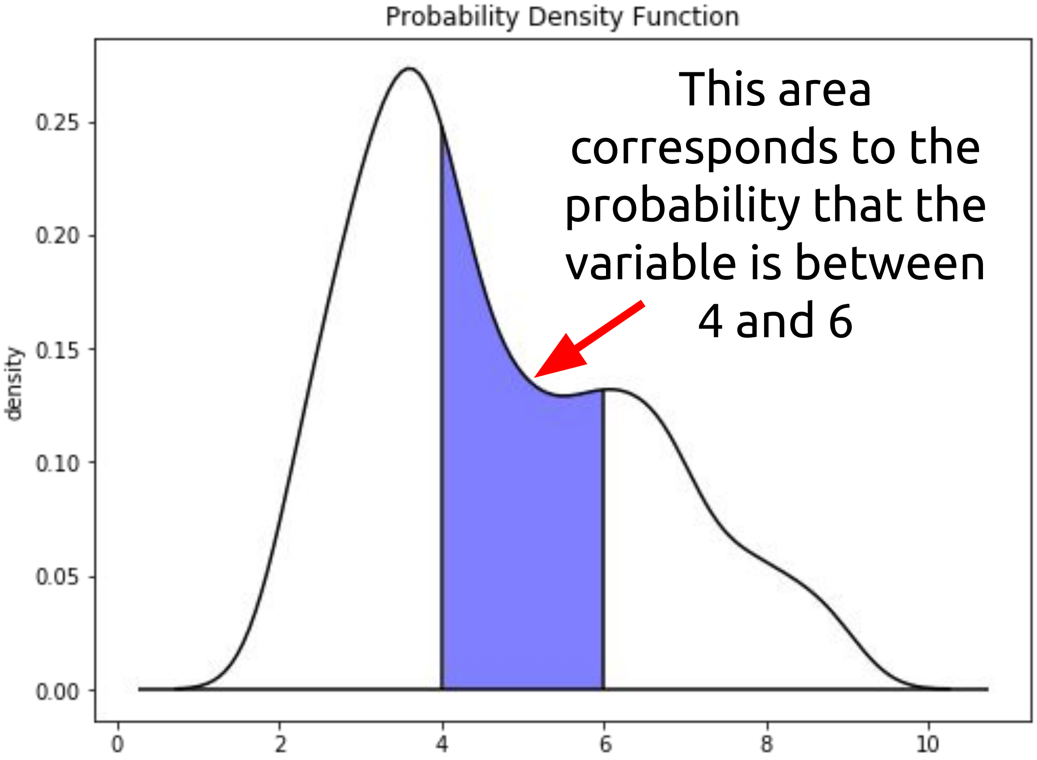
Probability Density Functions

We can no longer talk about the probability of a specific value, but instead talk about the probability of the variable being in a particular range.



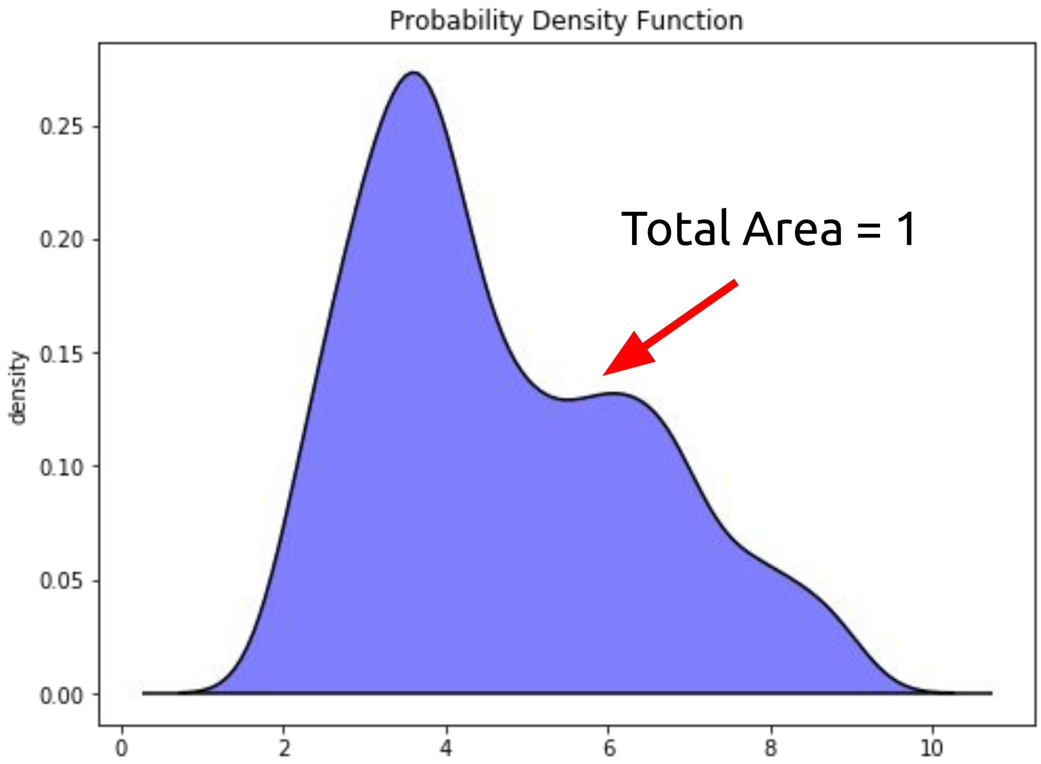
Probability Density Functions

The probability of the variable being in a particular range corresponds to the are under the PDF in that range.



Probability Density Functions

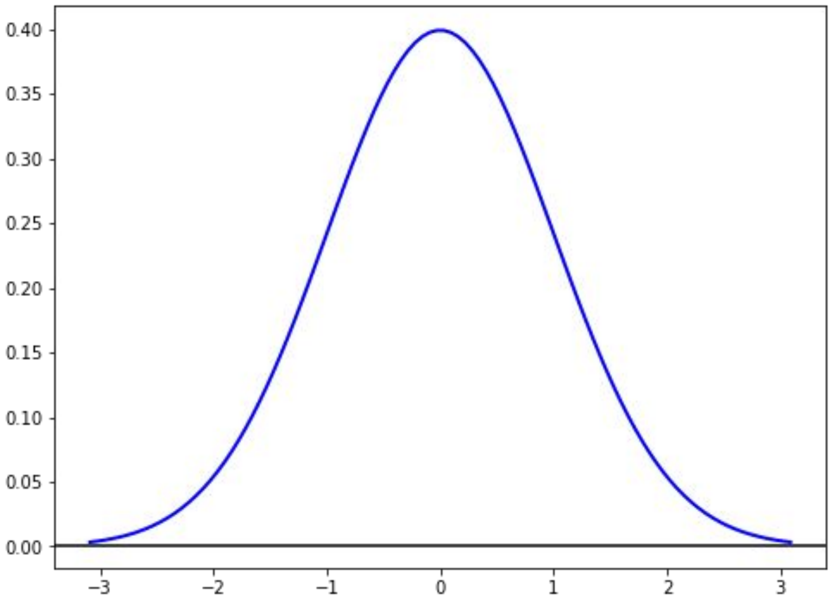
This means that the total area under the curve is 1.



The Normal Distribution

Perhaps the most well-known distribution is the Normal distribution, aka, the “Gaussian” distribution.

It is a symmetric, bell-shaped distribution.



The Standard Normal Distribution

The Normal Distribution

Bell-shaped distribution, described by two parameters: mean ***μ*** and standard deviation ***σ***

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The **standard normal** distribution has mean 0 and standard deviation 1.

See the notebook for a demo of how ***μ*** and ***σ*** affect the distribution.

The Normal Distribution

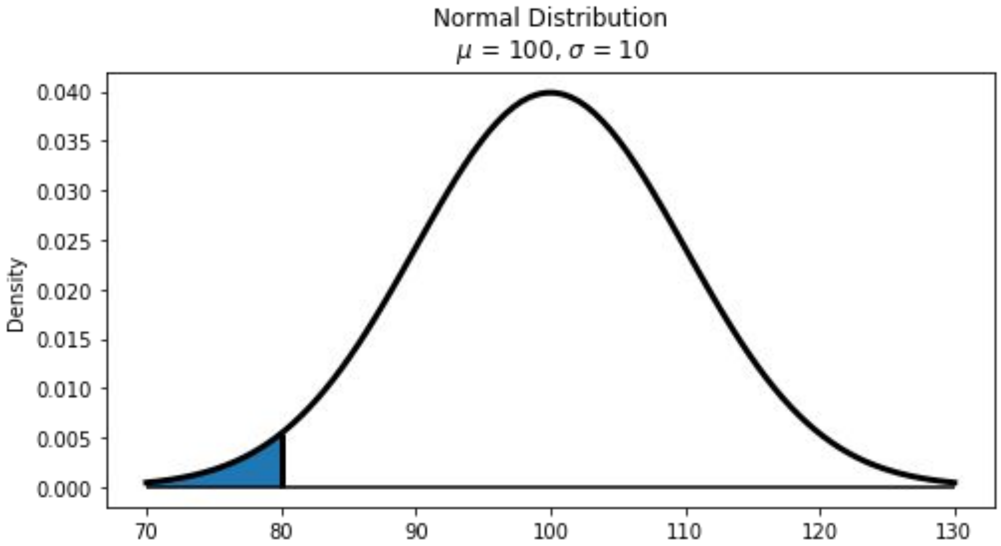
(It is thought that) many things can be described by a normal distribution.

Eg. IQs, test scores, heights, weights, random variations in industrial processes

However, these can only be approximately true (the normal distribution has nonzero density for all real numbers, including negatives)

The Normal Distribution

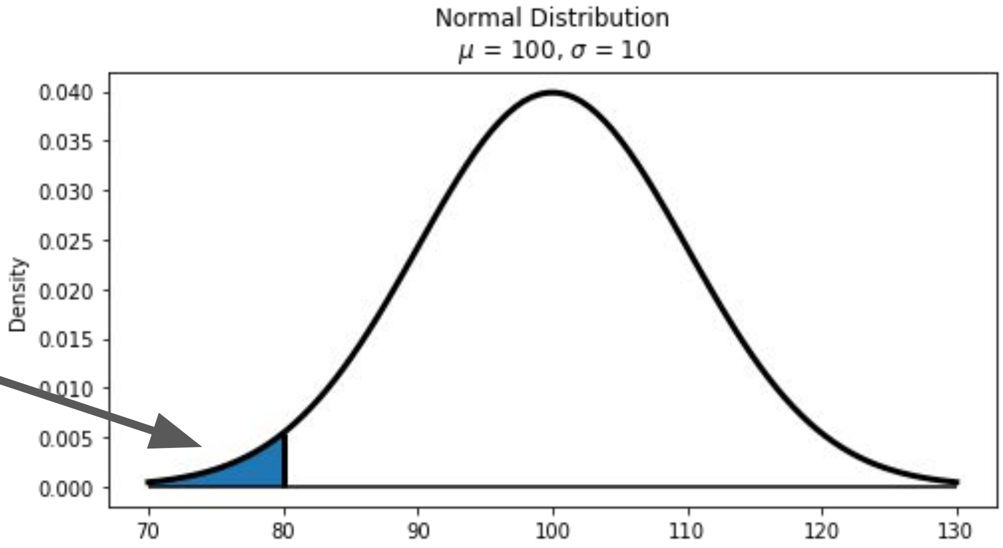
Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is less than 80?



The Normal Distribution

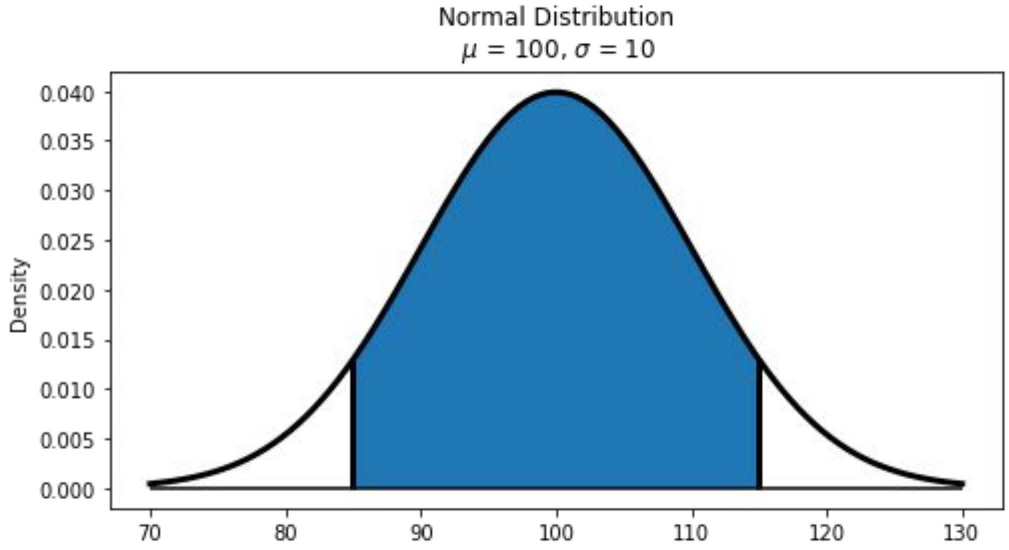
Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is less than 80?

Probability = Area = 0.0228



The Normal Distribution

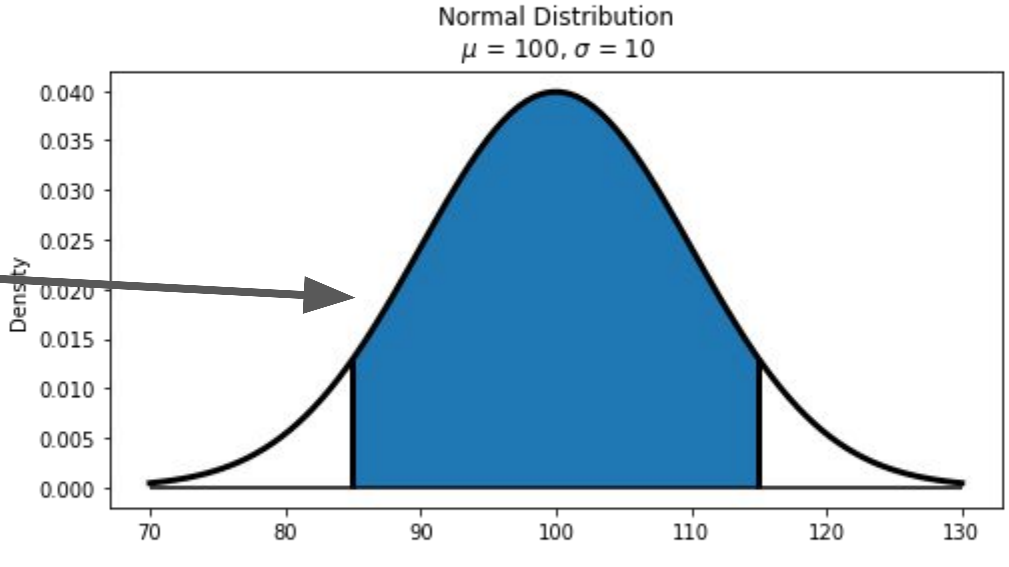
Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is more than 85 but less than 115?



The Normal Distribution

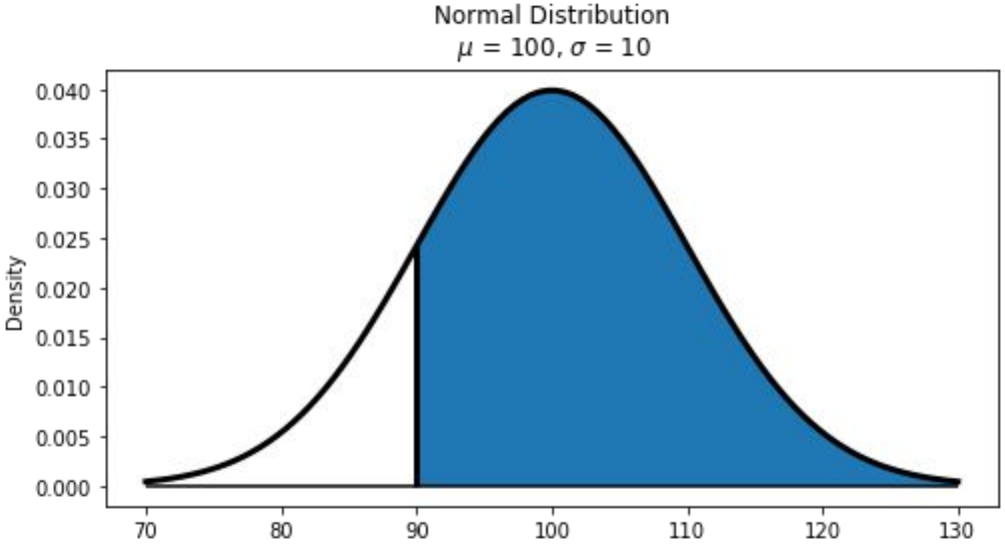
Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is more than 85 but less than 115?

Probability = Area = 0.08664



The Normal Distribution

Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is more than 90?



The Normal Distribution

Example: For a random variable which is normally distributed with a mean of 100 and standard deviation of 10, what is the probability that the variable is more than 90?

Probability = Area = 0.8413