Introduction to Logistic Regression

Introduction

Recall that **linear regression** can be used to describe the relationship between two or more variables where the target variable is numeric.

For categorical targets, we can use **logistic regression**.

Example - ICU Admission

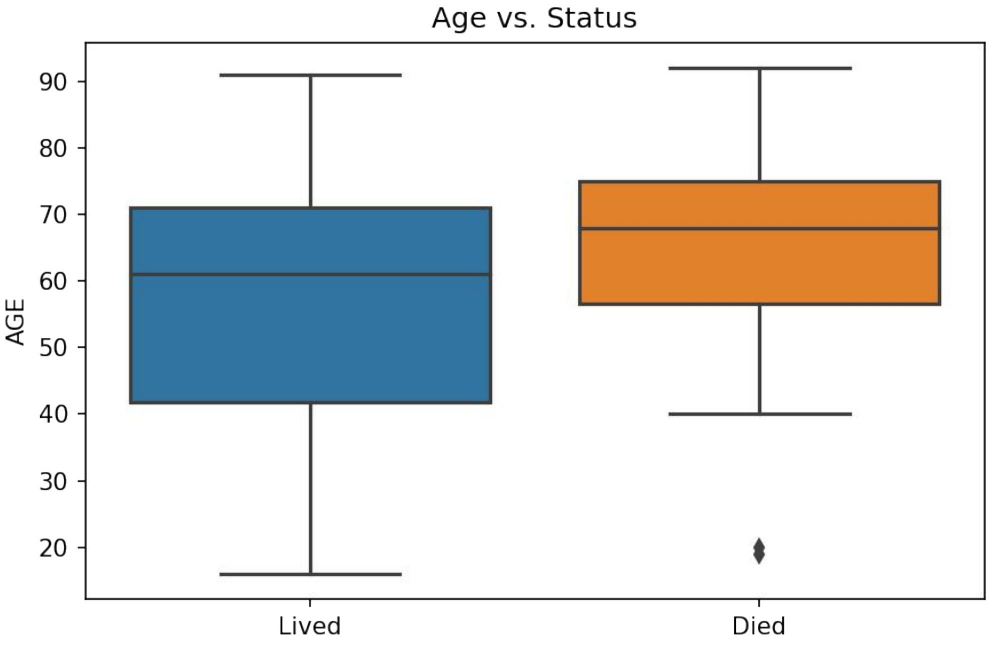
Let’s say we want to study survival rates of patients admitted to an intensive care unit. We gather several variables:

- Status: lived or died
- Age
- Sex
- Systolic Blood Pressure
- Type of Admission (Elective or Emergency)
- etc.

Example - ICU Admission

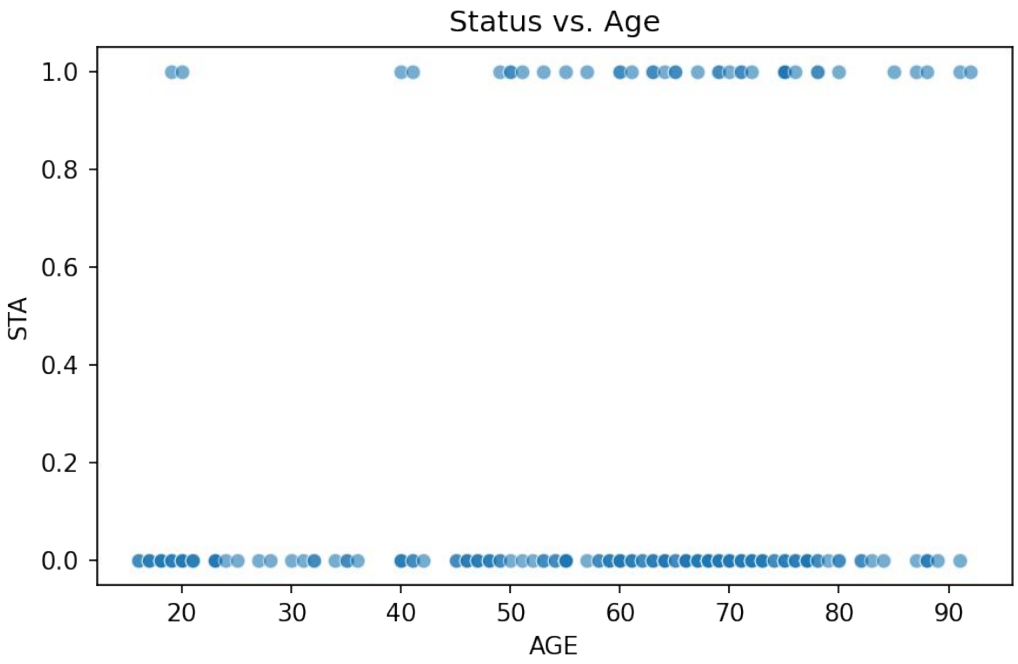
We might start by examining age vs. status.

It appears that those that died tended to be older.



Example - ICU Admission

We could also plot this as a scatterplot, where we encode status numerically, with lived = 0 and died = 1.



Example - ICU Admission

How can we build a model to describe the relationship between age and status?

Example - ICU Admission

Can we use a linear regression model?

Linear regression: The distribution of *Y*, given *X* is normal with mean

$$\mu = \beta_0 + \beta_1 X$$

Recall: Bernoulli Distribution

Setup: An experiment with exactly two outcomes, labeled "success" (denoted by 1) and "failure" (denoted by 0).

Probability of success = p
Probability of failure = $1 - p$

Example: A marketing company knows that historically, search ads have a click-through rate of 1.5%.
We can view each interaction as a Bernoulli trial with $p = 0.015$

Example - ICU Admission

Can we use a linear regression model?

Linear regression: The distribution of *Y*, given *X* is **normal** with mean

$$\mu = \beta_0 + \beta_1 X$$

Logistic regression: The distribution of *Y*, given *X* is **Bernoulli** with probability of success (mean)

$$p = \beta_0 + \beta_1 X$$

Example - ICU Admission

Can we use a linear regression model?

Linear regression: The distribution of *Y*, given *X* is **normal** with mean

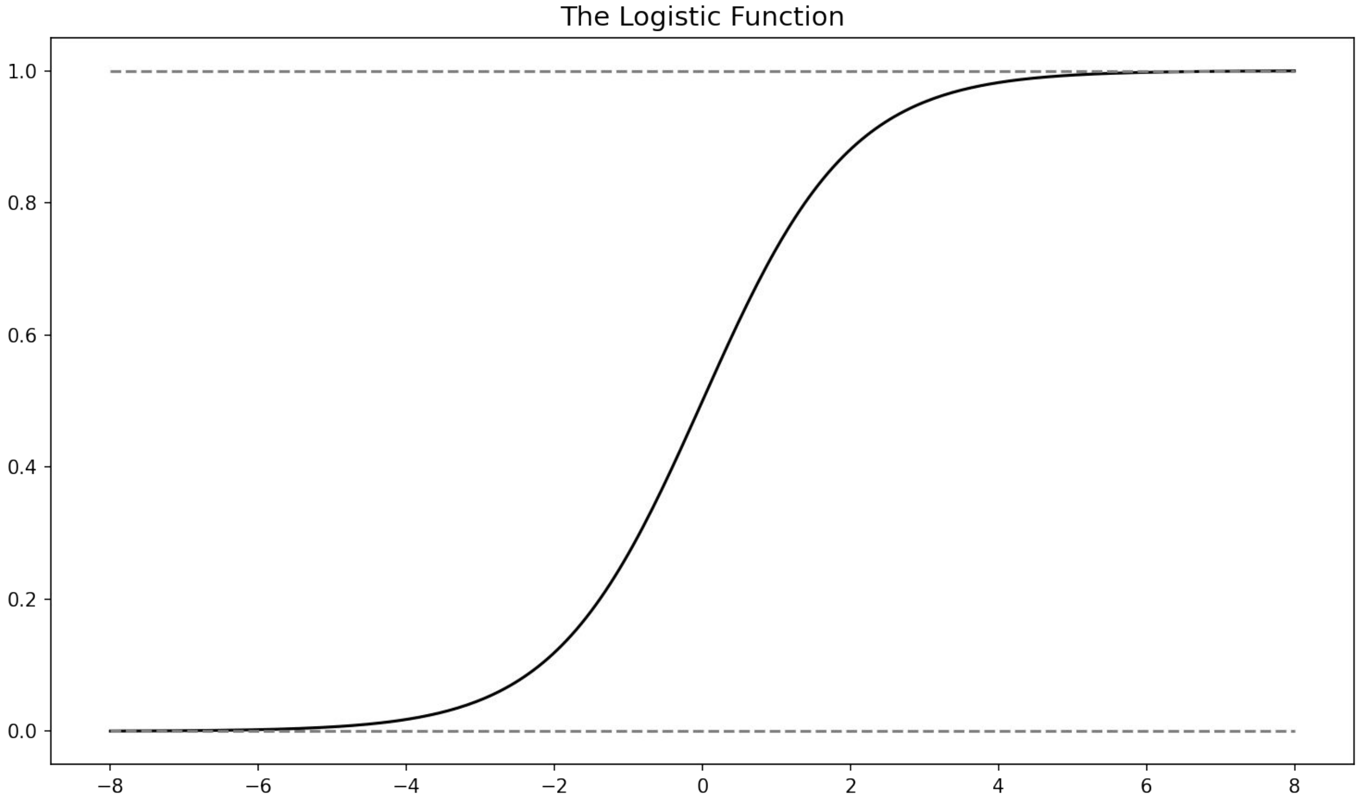
$$\mu = \beta_0 + \beta_1 X$$

Logistic regression: The distribution of *Y*, given *X* is **Bernoulli** with probability of success (mean)

$$p = \beta_0 + \beta_1 X$$

But wait, a probability must be between 0 and 1, and there is no guarantee that this expression will be.

The logistic function: $f(x) = \dfrac{1}{1 + e^{-x}}$



Example - ICU Admission

Can we use a linear regression model?

Linear regression: The distribution of *Y*, given *X* is **normal** with mean

$$\mu = \beta_0 + \beta_1 X$$

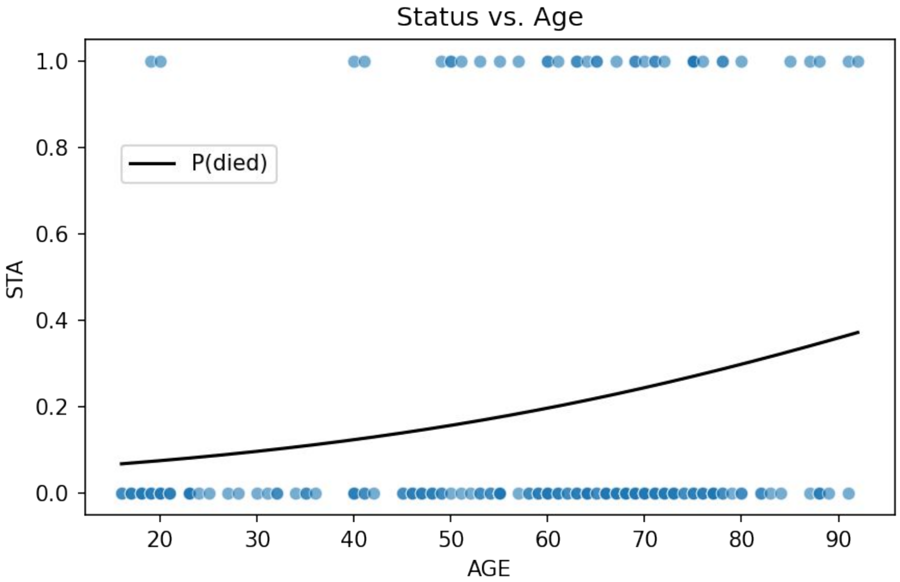
Logistic regression: The distribution of *Y*, given *X* is **Bernoulli** with probability of success (mean)

$$p = \text{logistic}(\beta_0 + \beta_1 X)$$

Example

If we fit a logistic regression model to the ICU data, using age, we get

$$P(\text{died}) = \text{logistic}(-3.0585 + 0.0275(\text{age}))$$



Example

$$P(\text{died}) = \text{logistic}(-3.0585 + 0.0275(\text{age}))$$

How do we determine the coefficients?

Example

P(died) = logistic(-3.0585 + 0.0275(age))

How do we determine the coefficients?

On the basis of the **likelihood** of the resulting model.

For a single observation with outcome y_i and predicted probability π_i, the **likelihood** of this outcome is given by

$$(\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

This reduces to the predicted probability of the correct outcome.

Example

$$(\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

Age	Status	Predicted P(died)	Likelihood
49	lived	0.153	$(0.153)^0(0.847)^1 = 0.847$
80	died	0.298	$(0.298)^1(0.702)^0 = 0.298$
91	lived	0.365	$(0.365)^0(0.635)^1 = 0.635$

Example

The coefficients are the ones that maximize the likelihood across the dataset:

$$\prod_i (\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

Usually, it is actually the **log-likelihood** that is maximized:

$$\sum_i [y_i \log(\pi_i) + (1 - y_i) \log(1 - \pi_i)]$$

Inference for Logistic Regression Models

Types of questions that we can ask:

- How precise is our estimate of the coefficient associated with age?
- Is the coefficient associated with age statistically significant?
- If I add additional predictor variables, are their coefficients statistically significant, after controlling for age?

Inference for Logistic Regression Models

Types of questions that we can ask:

- How precise is our estimate of the coefficient associated with age?
- Is the coefficient associated with age statistically significant?
- If I add additional predictor variables, are their coefficients statistically significant, after controlling for age?

The first two questions can be answered using the fact that coefficient estimates are approximately normally distributed.

Inference for Logistic Regression Models

Types of questions that we can ask:

- How precise is our estimate of the coefficient associated with age?
- Is the coefficient associated with age statistically significant?
- If I add additional predictor variables, are their coefficients statistically significant, after controlling for age?

The last question can be answered using the **likelihood ratio test**, which compares the likelihood of the full model to a reduced model.

Likelihood Ratio Test

Procedure:

1. Defining a larger **full model** (which contains all predictors involved in the test)
2. Defining a smaller **reduced model** (which satisfies the assumptions of the null hypothesis)
3. Calculate the test statistic (which involves the ratio of likelihoods) and compare to a chi-square distribution.

Example:

Question: Is the coefficient for systolic blood pressure statistically significant, after controlling for age?

Null Hypothesis: $\beta_{\text{sys}} = 0$

Alternative Hypothesis: $\beta_{\text{sys}} \neq 0$

Example:

Question: Is the coefficient for systolic blood pressure statistically significant, after controlling for age?

Null Hypothesis: $\beta_{sys} = 0$

Alternative Hypothesis: $\beta_{sys} \neq 0$

Log-likelihood of reduced model: -96.15

Log-likelihood of full model: -91.63

Example:

Question: Is the coefficient for systolic blood pressure statistically significant, after controlling for age?

Null Hypothesis: $\beta_{sys} = 0$

Alternative Hypothesis: $\beta_{sys} \neq 0$

Log-likelihood of reduced model: -96.15

Log-likelihood of full model: -91.63

There is an increase in the log-likelihood, but is it significant?

Example:

Question: Is the coefficient for systolic blood pressure statistically significant, after controlling for age?

Null Hypothesis: $\beta_{sys} = 0$

Alternative Hypothesis: $\beta_{sys} \neq 0$

Log-likelihood of reduced model: -96.15

Log-likelihood of full model: -91.63

There is an increase in the log-likelihood, but is it significant?

p-value: 0.00263

Conclusion: Reject the null hypothesis.

Inference vs. Prediction

When building a statistical model, there are a number of possible objectives:

- **inference:** identifying key explanatory variables and understanding the relationship between these variables and the target
- **prediction:** predicting the outcome on new observations

Predictive analytics typically focuses on model-building for prediction rather than inference, and the techniques you can use in each differ.

Predictions on a New Observation

Once we have build a logistic regression model, we can use it to generate predictions on new observations.

To do this, we much translate our predicted probabilities π_i's into predictions.

A simple rule that can be used is to predict 1 if π_i > 0.5 and 0 otherwise.

Predictions on a New Observation

There are a number of metrics that can be used to evaluate predictions of a model on the basis of True Positives, False Positives, True Negatives, and False Negatives.

When evaluating the performance of a predictive model, it should be done by separating out a test set of data which the model is not fit on.

Logistic Regression

Let’s see all of this in action in a Jupyter notebook.