# Sampling Distributions

# Estimation

When doing statistics, the goal is often to infer something about a population *parameter* using only a sample from that population.

Examples:

- Estimating the average household income in Putnam County.
- Predicting the percentage of votes a particular candidate will receive in an upcoming election, based on a poll.

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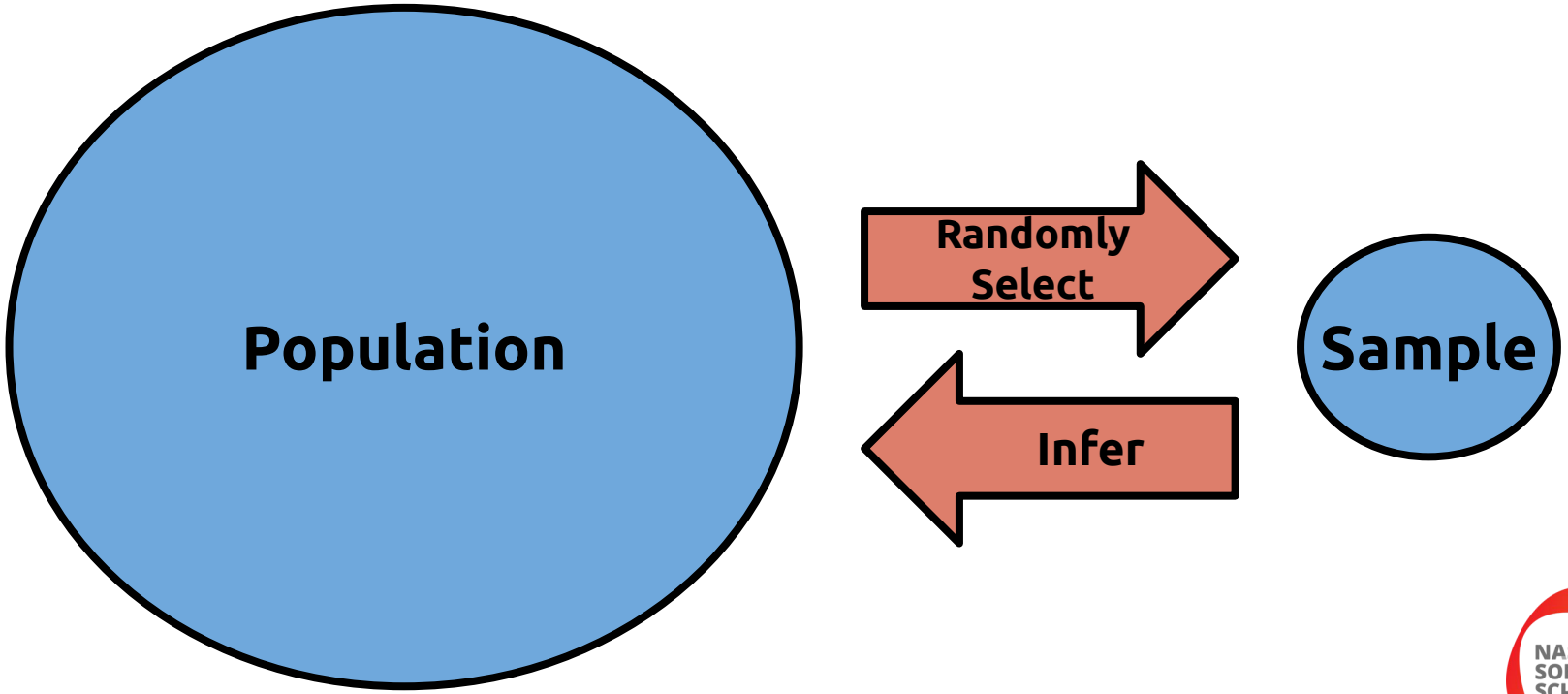
# Estimation

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# Sampling Distributions

When doing inference, we will have only a single sample to base our estimates off of.

However, we can work to understand the distribution from which our sample came - not the population distribution but the *distribution of all possible samples*.

For this, we can use the Sampling_Distribution.ipynb notebook.

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**Population Mean**

$$\mu = 100$$

**Population Standard Deviation**

$$\sigma = 10$$

| Sample Size | Mean of Sample Means | Standard Deviation of Sample Means |
|---|---|---|
| 1 | | |
| 10 | | |
| 25 | | |
| 100 | | |
| 1000 | | |

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# Cool Fact:

If *X* is a normally-distributed variable with mean ***μ*** and standard deviation ***σ***, the distribution of sample means of size *n* is also normal with mean ***μ*** and standard deviation $\frac{\sigma}{\sqrt{n}}$

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The quantity $\frac{\sigma}{\sqrt{n}}$ is called the **standard error of the mean**.

# Really Cool Fact

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**Central Limit Theorem:** Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$.

For large enough sample size $n$, (as long as X is reasonably well-behaved), the distribution of sample means is approximately normal with mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$

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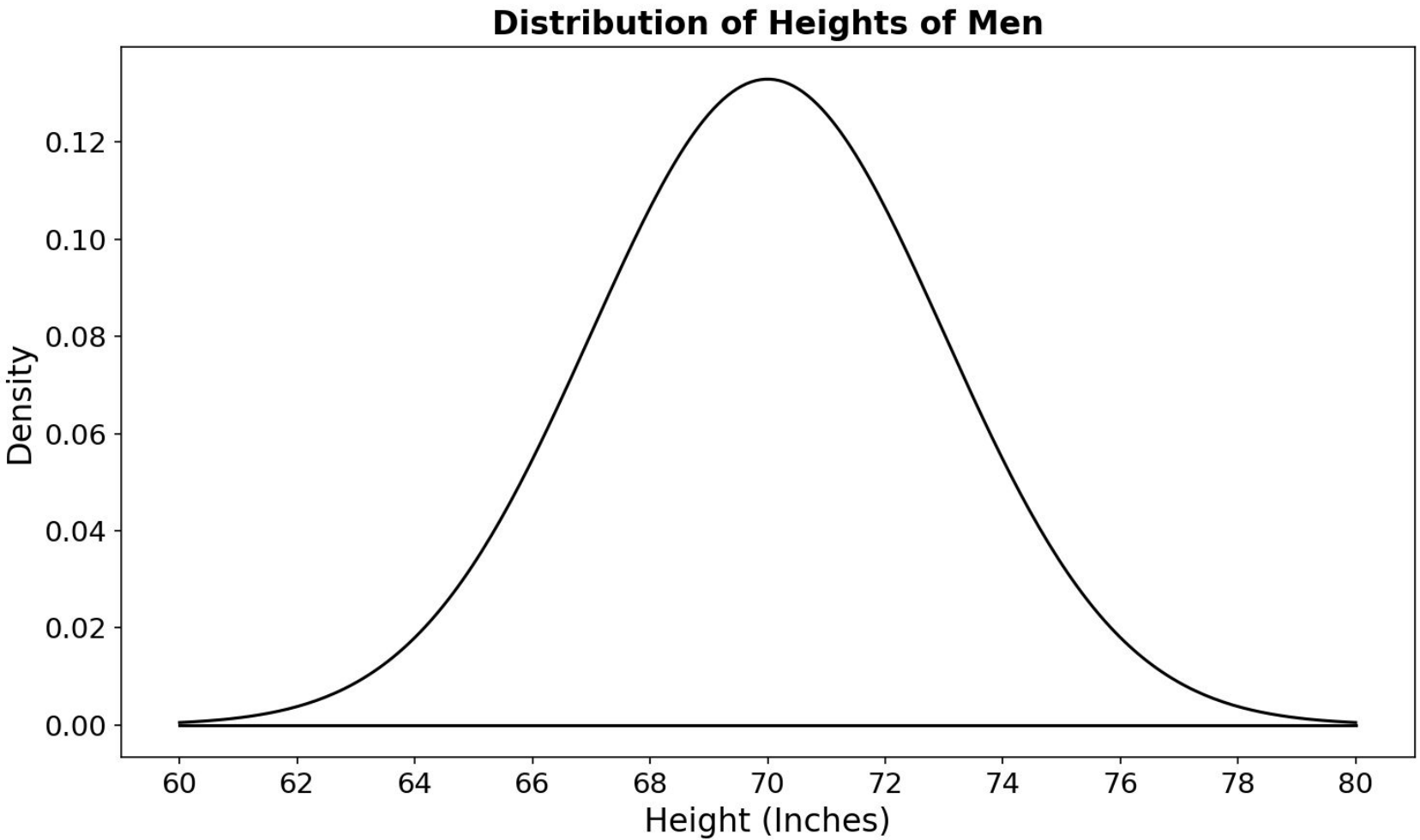
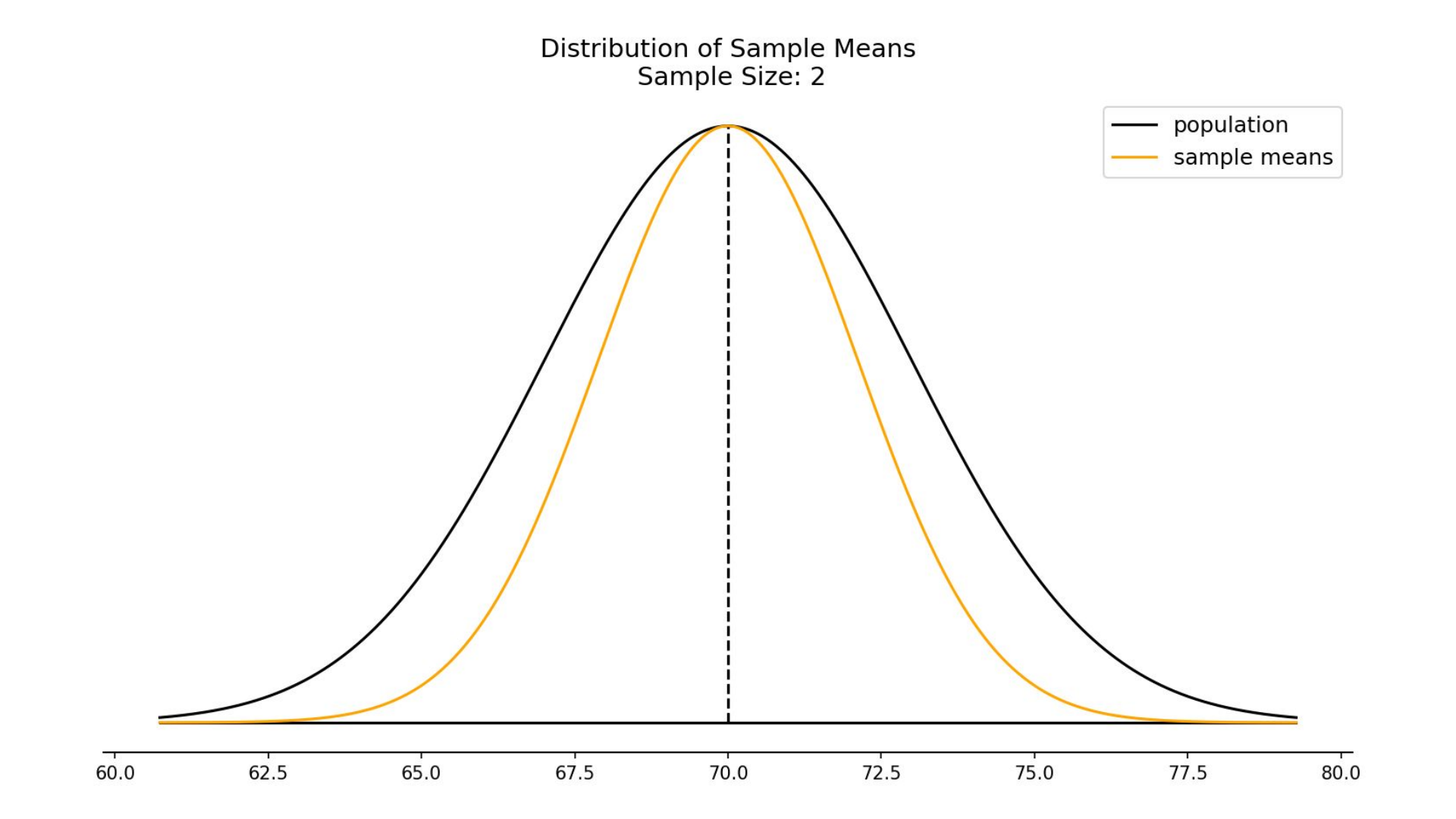
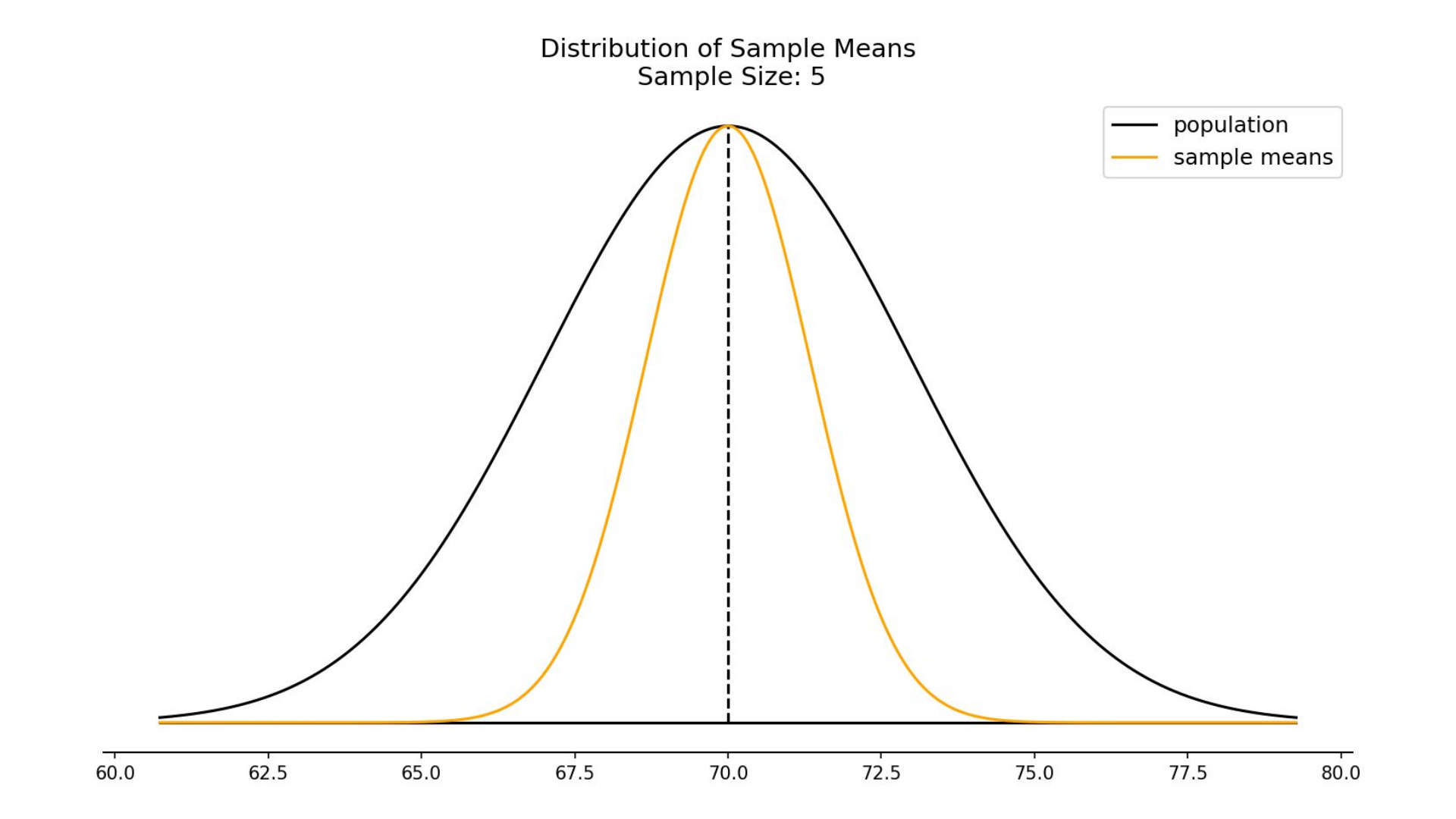
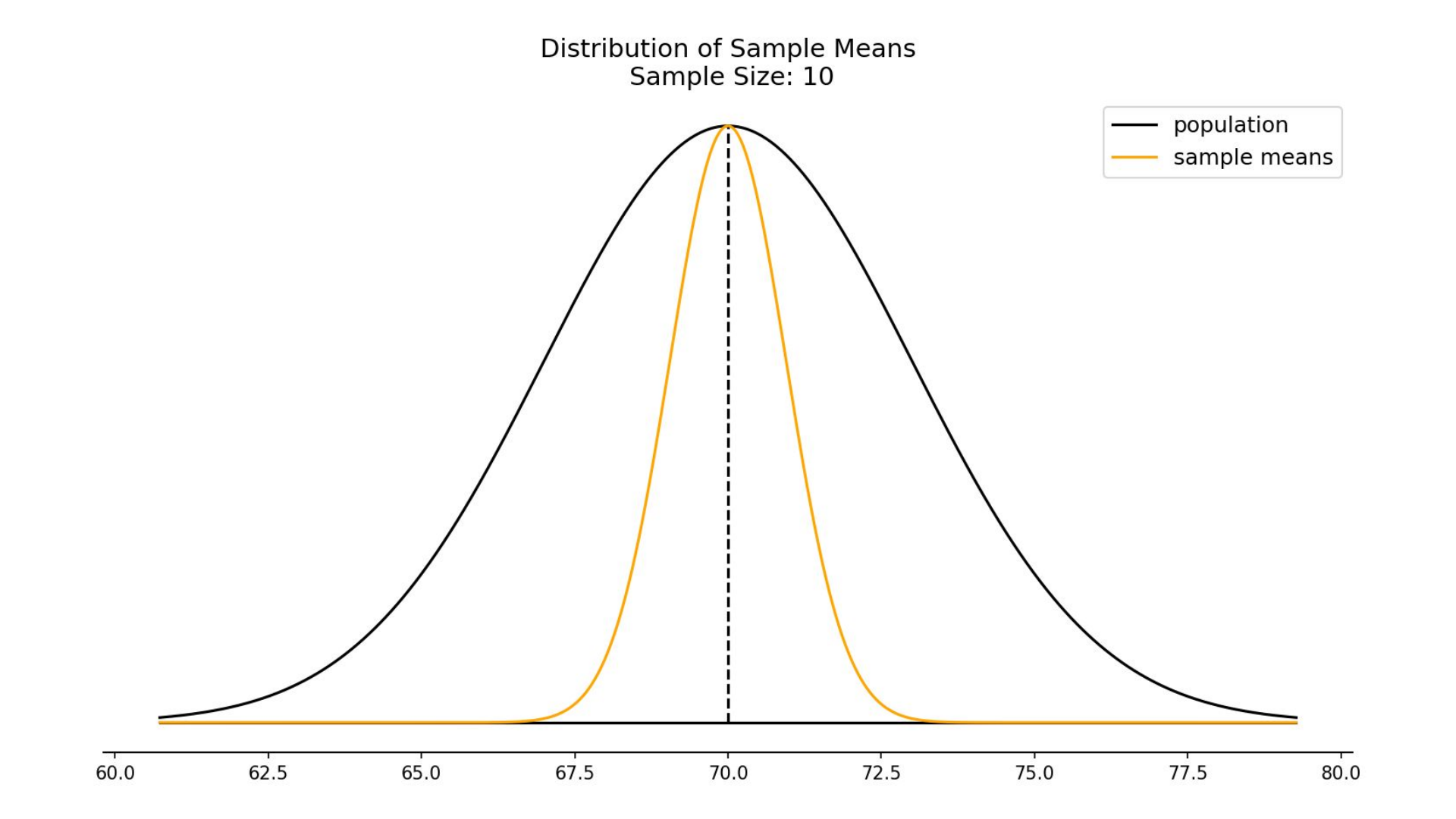
For large enough sample size $n$, (as long as X is reasonably well-behaved), the distribution of sample means is approximately normal with mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$

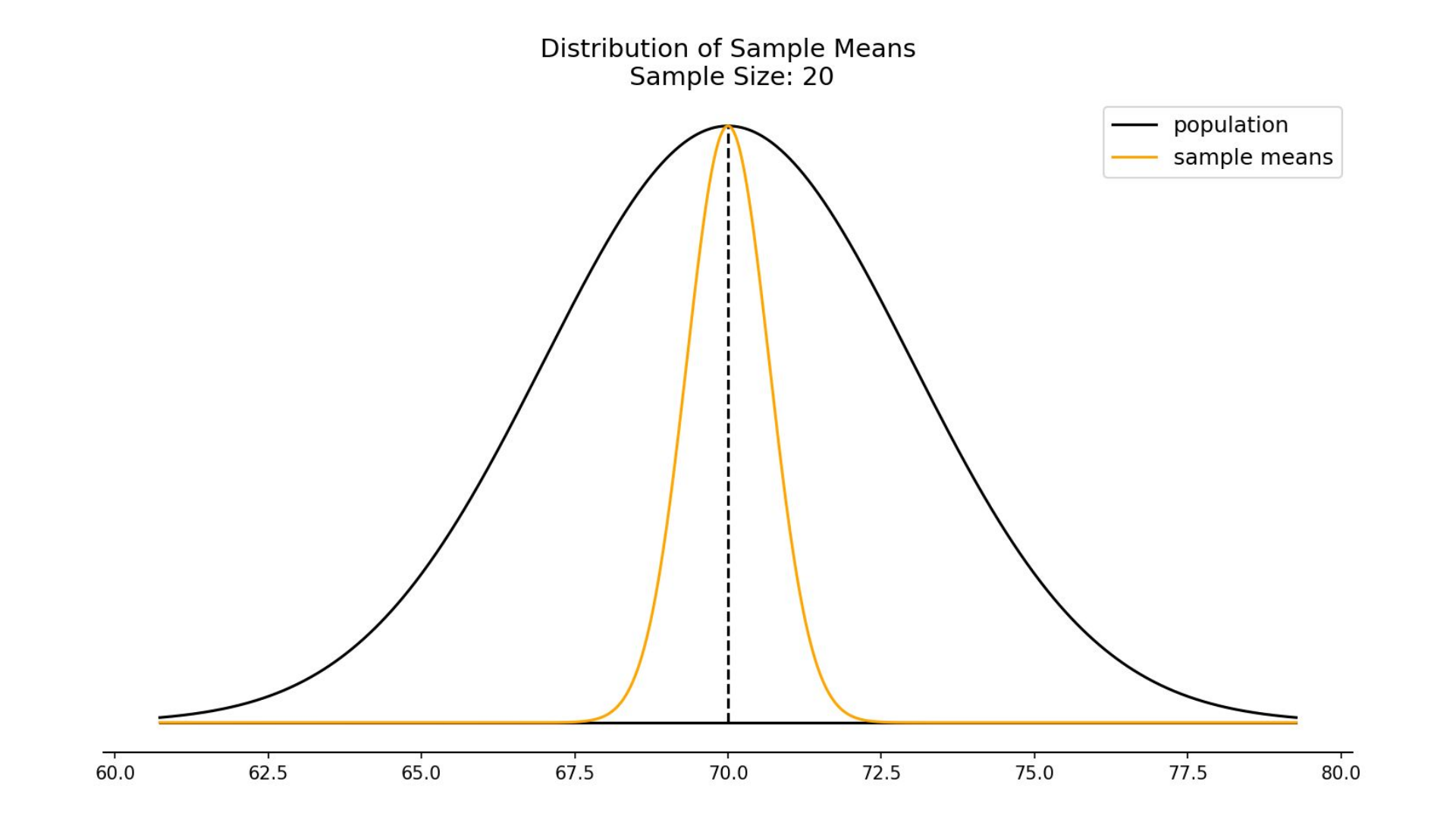
(In a lot of cases, sample size $n = 30$ is good enough for the central limit theorem to kick in).

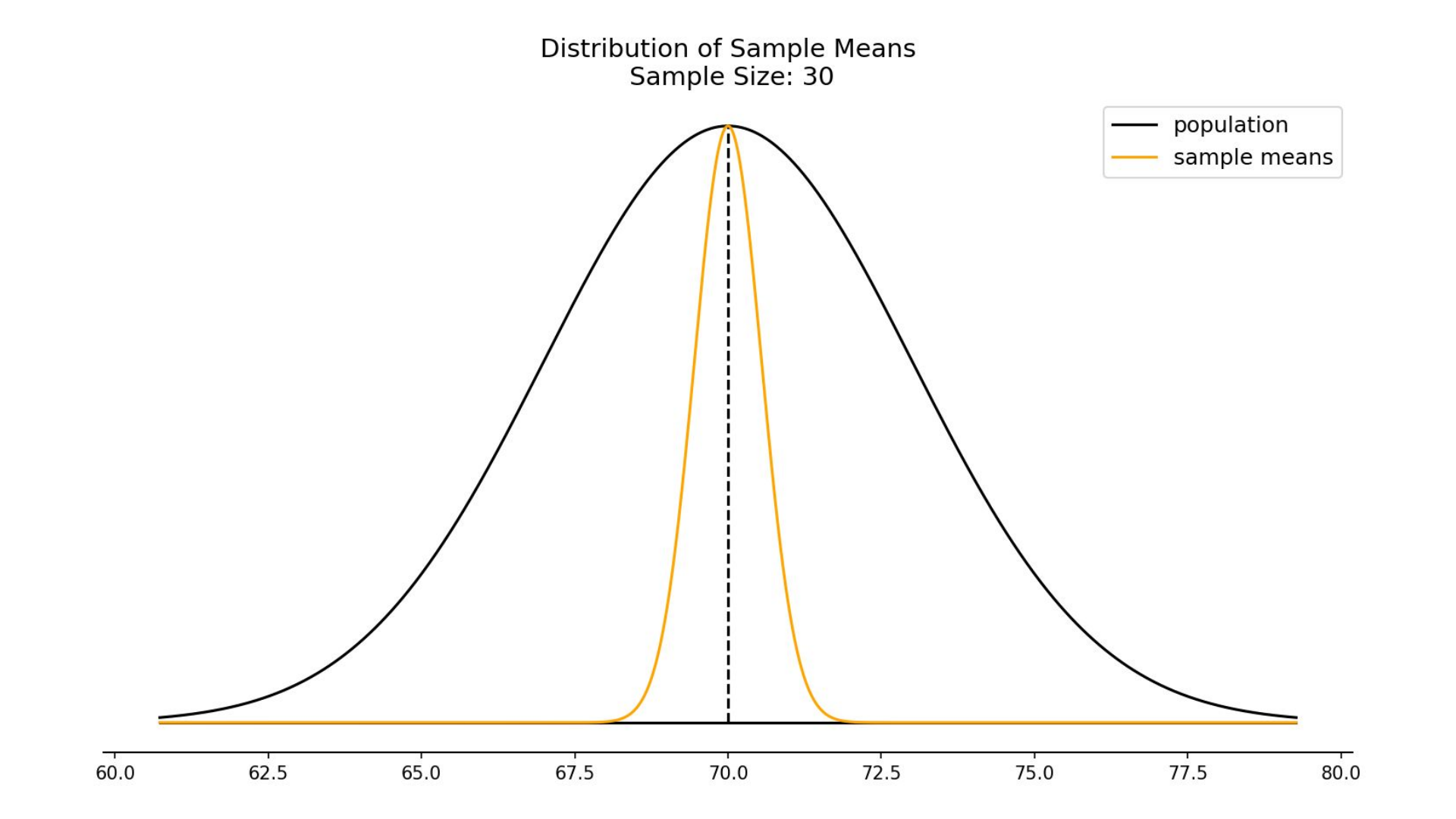

Distribution of Heights of Men
Density
0.12
0.10
0.08
0.06
0.04
0.02
0.00
60
62
64
66
68
70
72
74
76
78
80
Height (Inches)

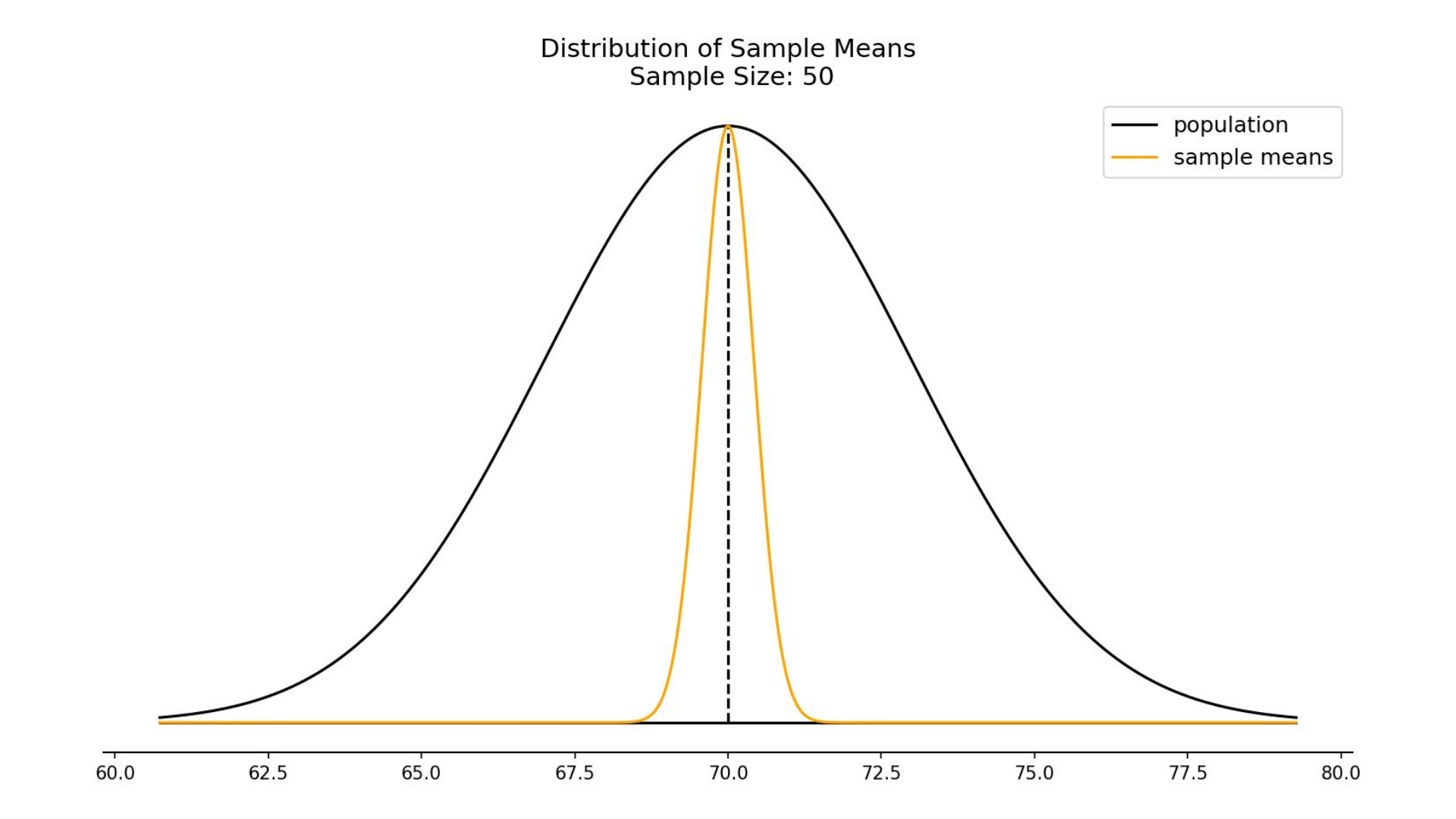

Distribution of Sample Means
Sample Size: 2
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

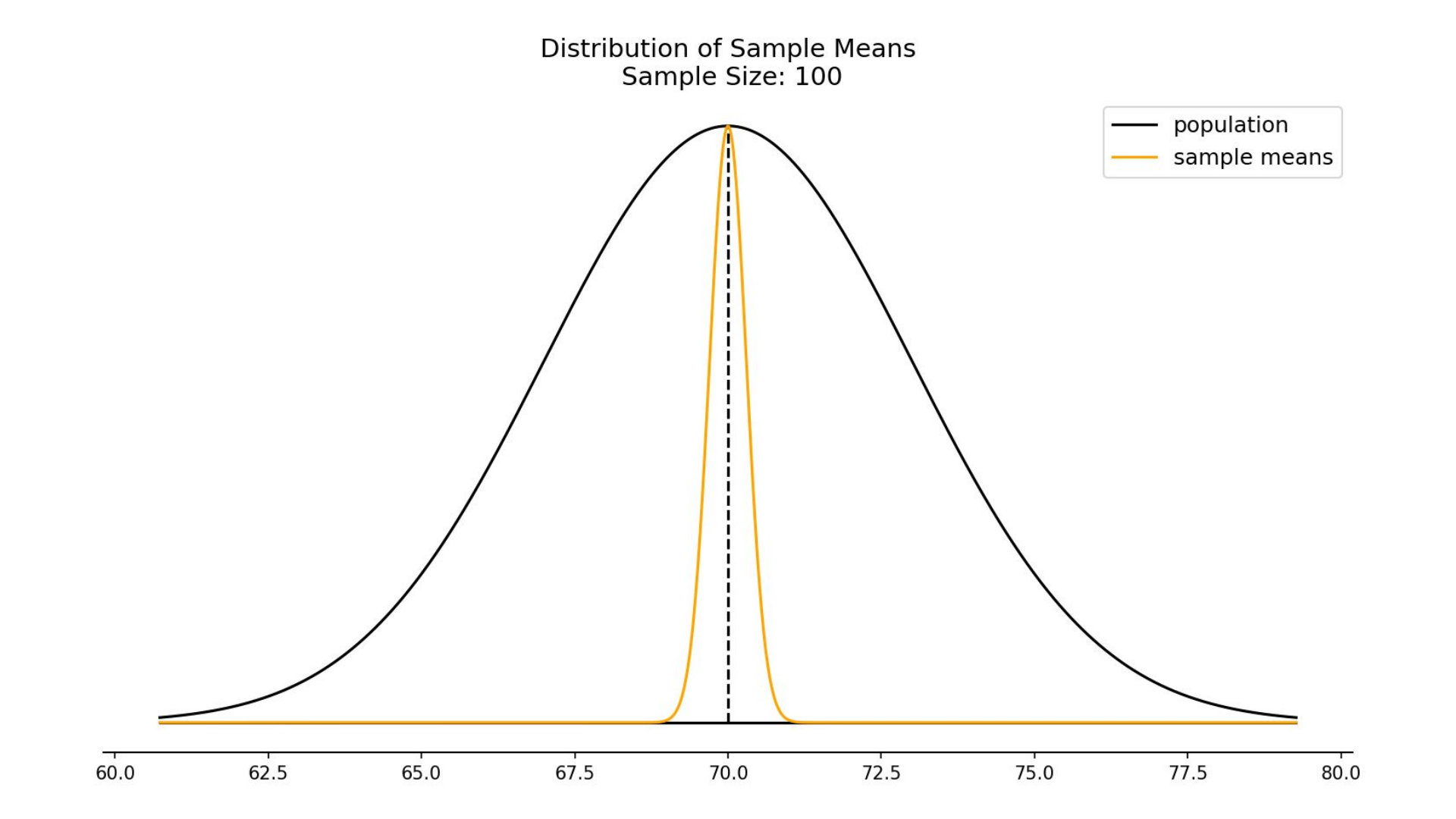

Distribution of Sample Means
Sample Size: 5
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 10
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

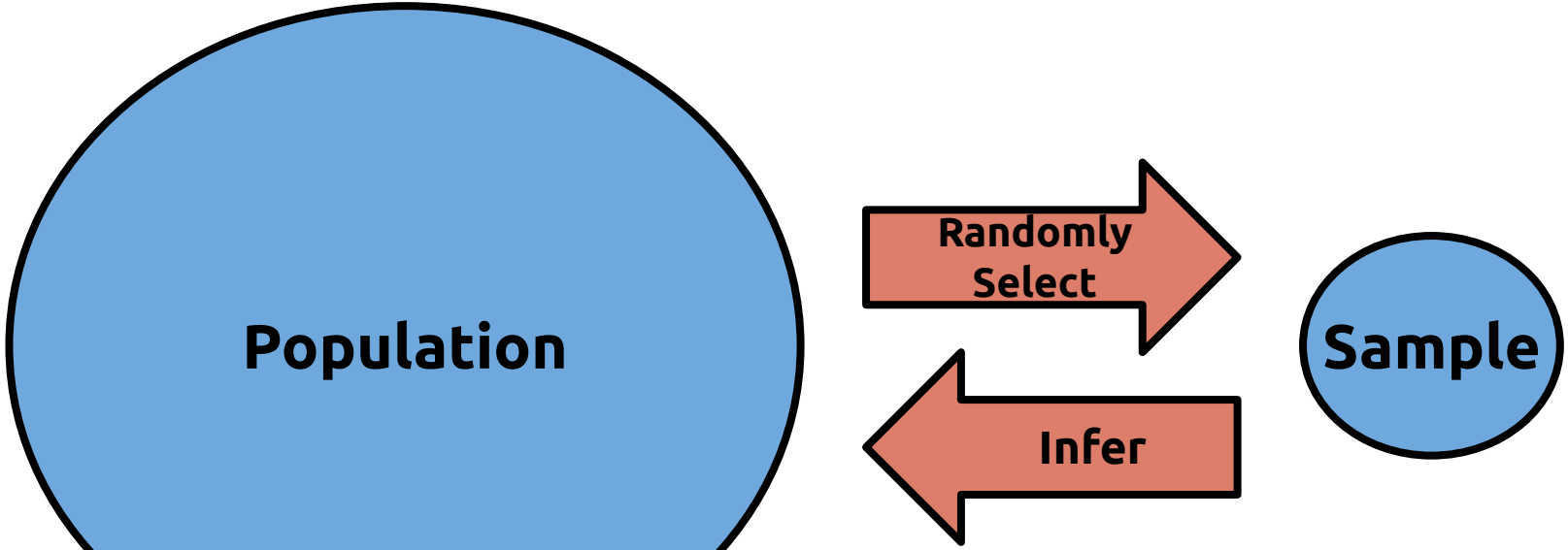

Distribution of Sample Means
Sample Size: 20
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 30
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 50
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 100
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches




$$\mu = 70$$

$$\sigma = 3$$

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The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

**Question:** If we randomly select a man, what is the probability that he is between 69 and 71 inches tall?

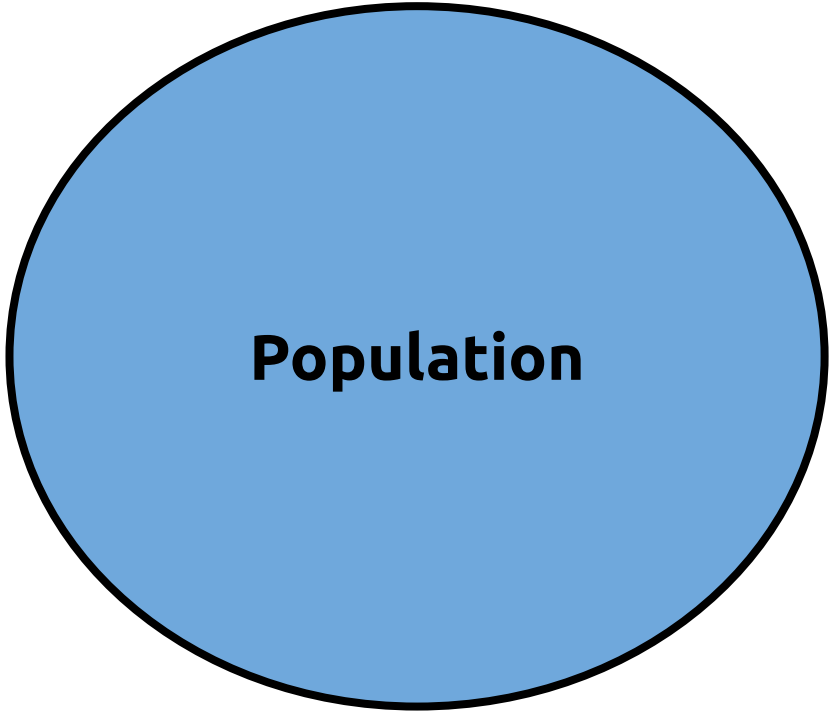
The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

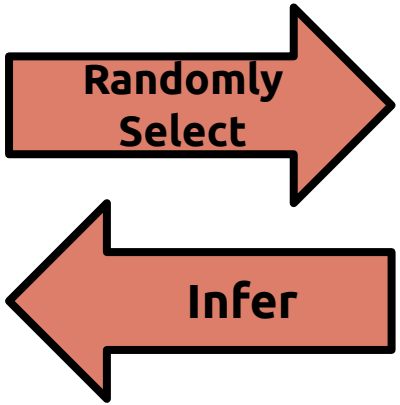
**Question:** If we randomly select a group of 10 men, what is the probability that their average height is between 69 and 71 inches?

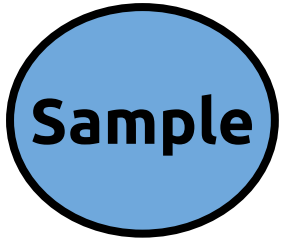
The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

**Question:** If we randomly select a group of 100 men, what is the probability that their average height is between 69 and 71 inches?

**Difficult Question:** Say we have a sample of size 20 from a normal distribution with **standard deviation** of 10, and the sample has a sample mean of 50.




$$\bar{x} = 50$$
$$n = 20$$

$$\mu = ?$$
$$\sigma = 10$$

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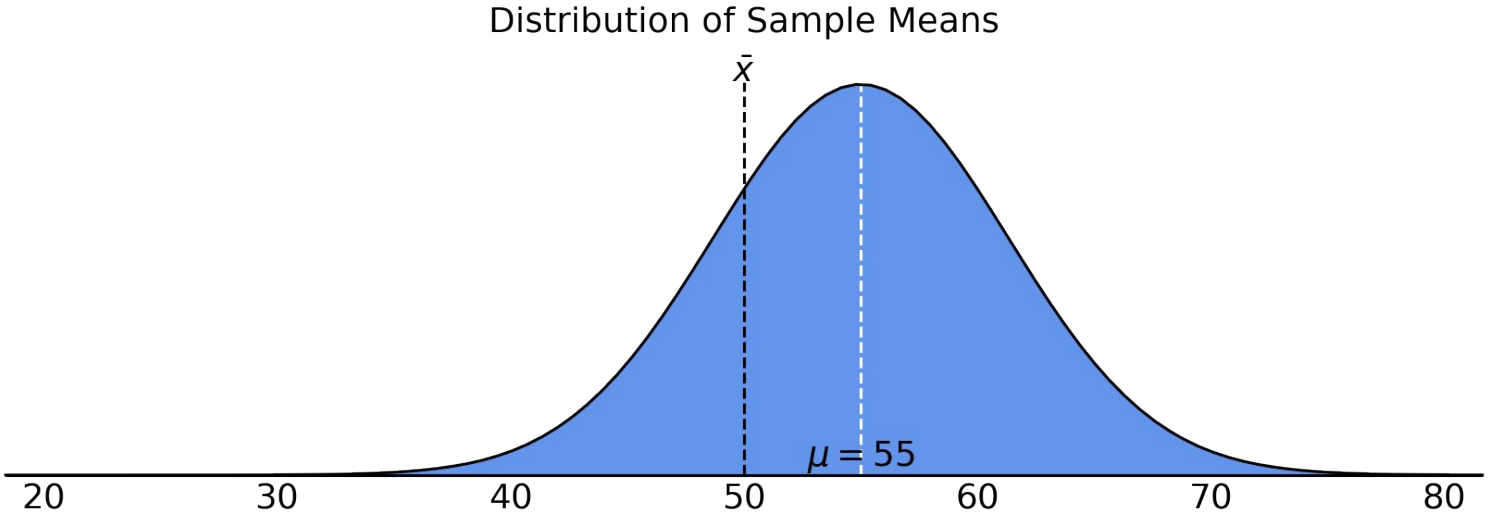
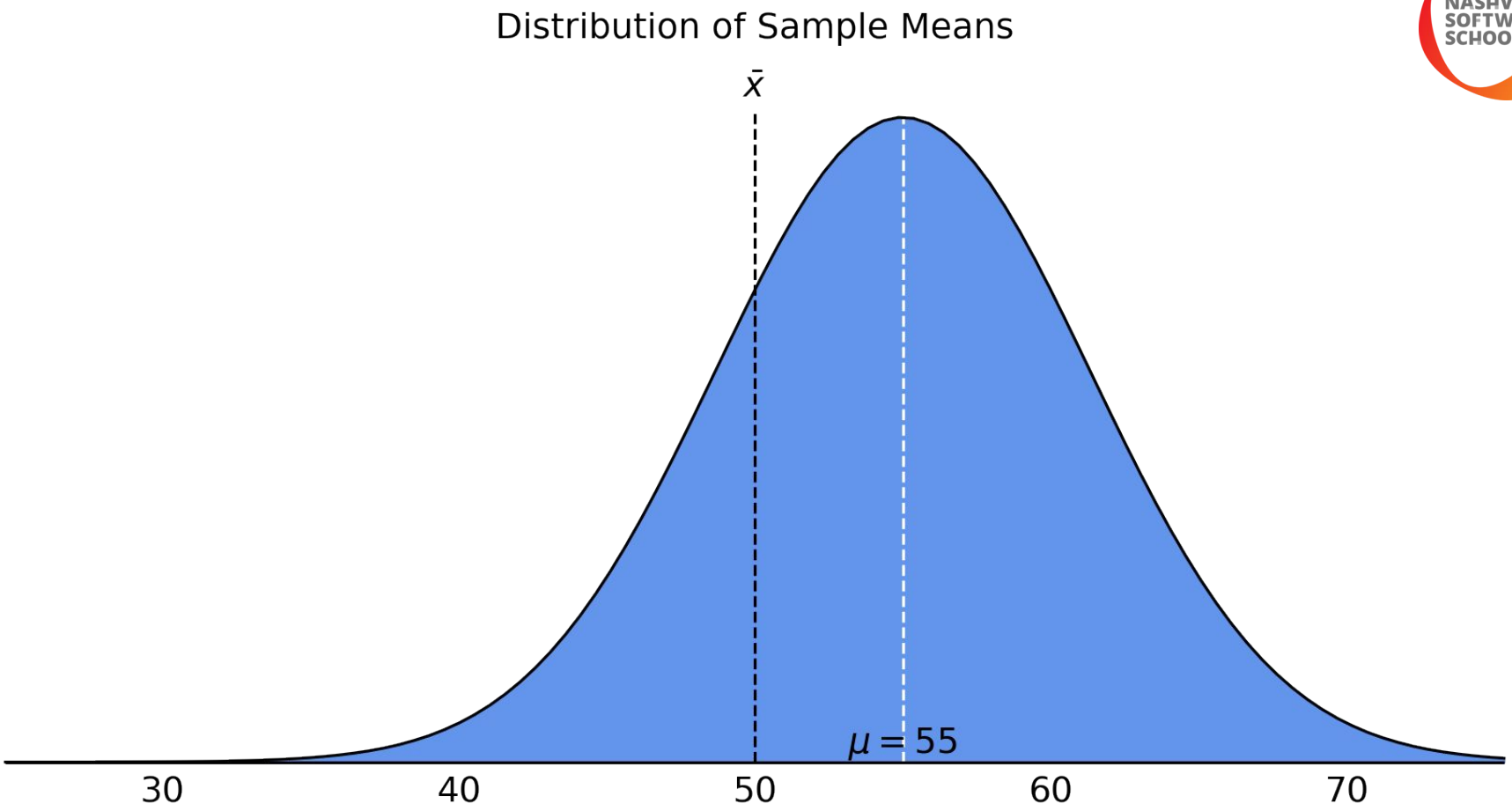
**Difficult Question:** Say we have a sample of size 20 from a normal distribution with **standard deviation** of 10, and the sample has a sample mean of 50.

Is it possible that the *population* mean is actually 55? Does it seem plausible that the population mean is actually 55?

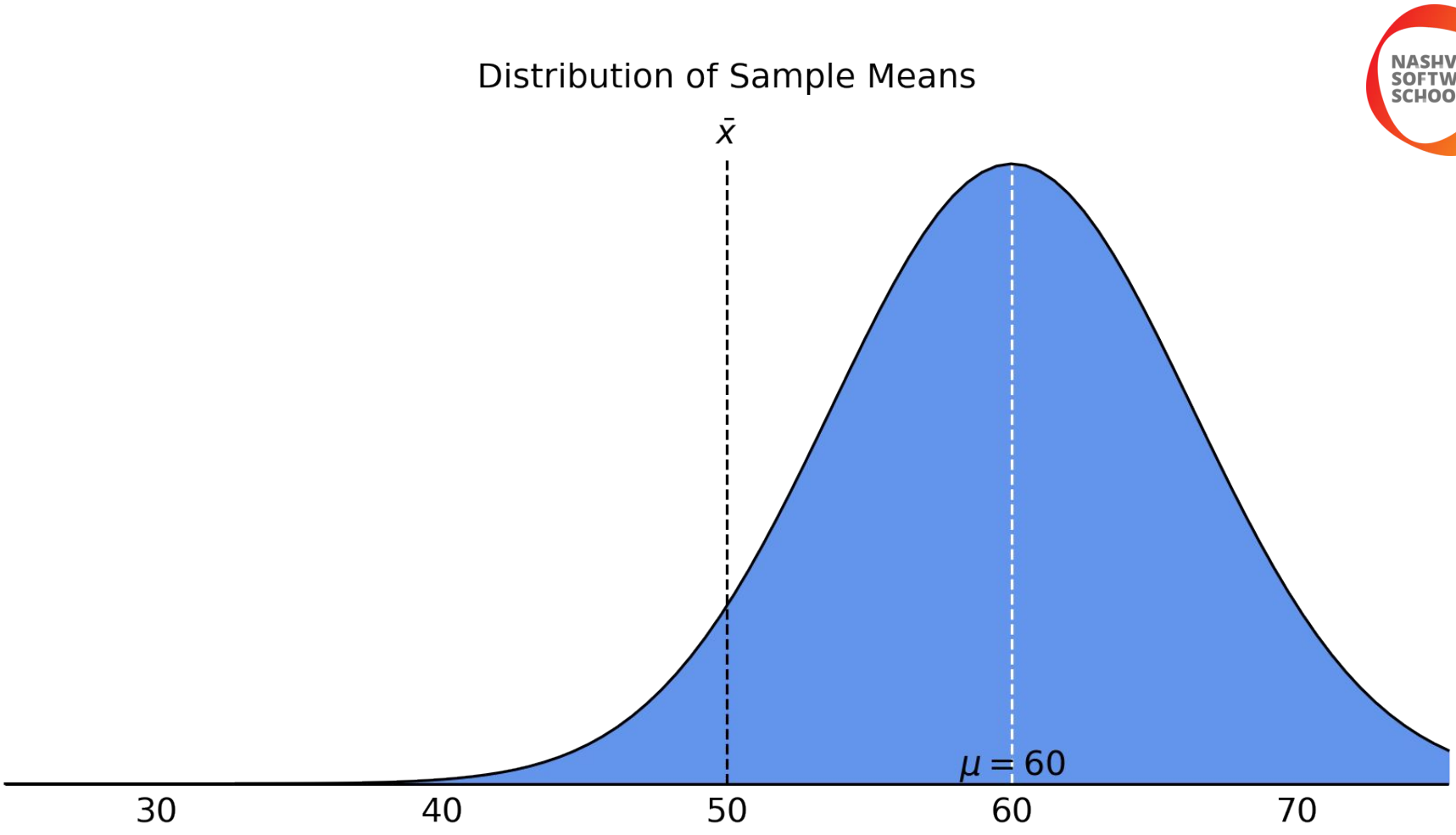



Distribution of Sample Means
$\bar{x}$
$\mu = 55$
30
40
50
60
70

**Difficult Question:** Say we have a sample of size 20 from a normal distribution with **standard deviation** of 10, and the sample has a sample mean of 50.

Is it possible that the *population* mean is actually 55? Does it seem plausible that the population mean is actually 55?

Is it possible that the *population* mean is actually 60? Does it seem plausible that the population mean is actually 60?

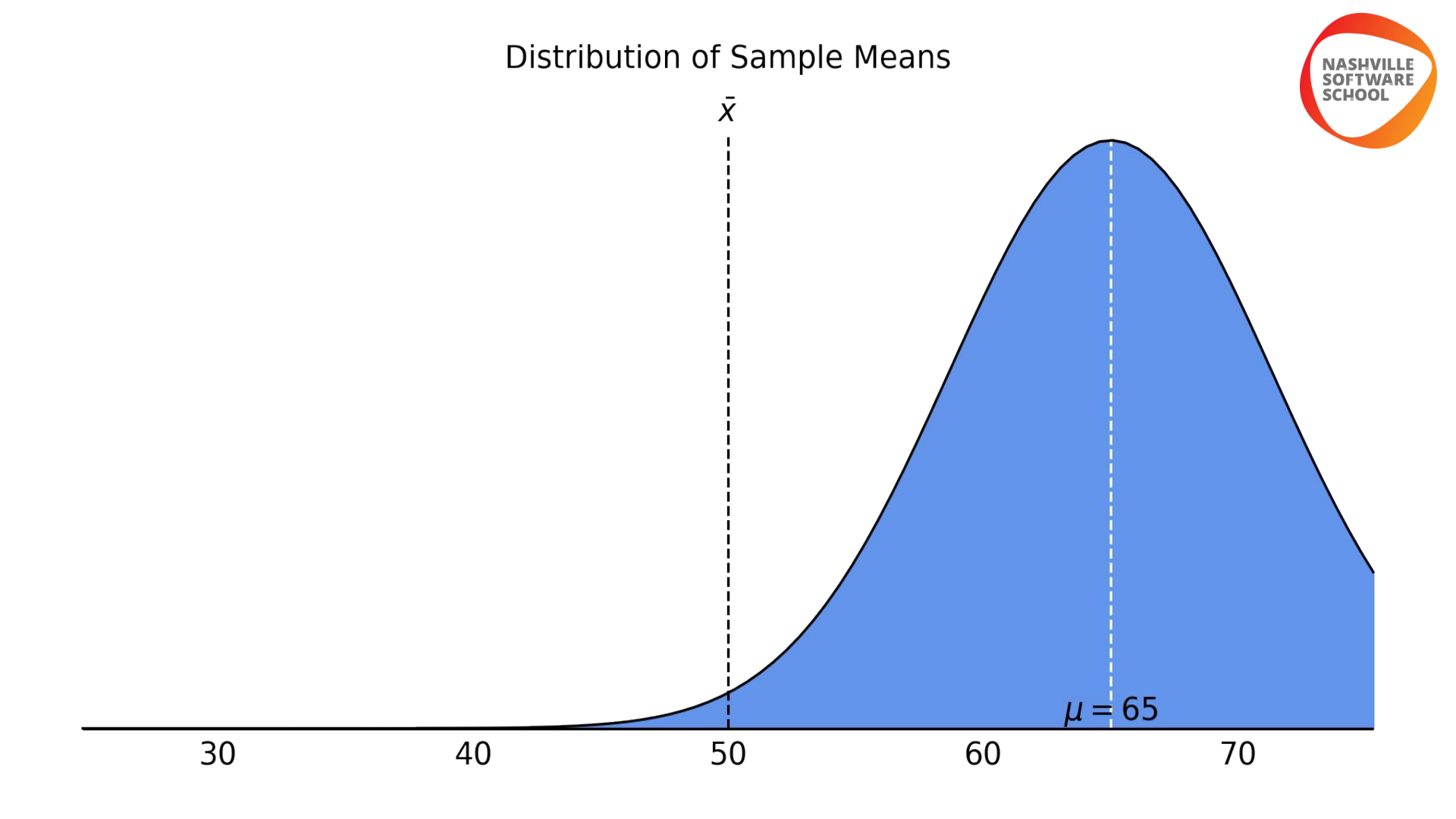

Distribution of Sample Means
$\bar{x}$
$\mu = 60$
30
40
50
60
70

**Difficult Question:** Say we have a sample of size 20 from a normal distribution with **standard deviation** of 10, and the sample has a sample mean of 50.

Is it possible that the *population* mean is actually 55? Does it seem plausible that the population mean is actually 55?

Is it possible that the *population* mean is actually 60? Does it seem plausible that the population mean is actually 60?

Is it possible that the *population* mean is actually 65? Does it seem plausible that the population mean is actually 65?


Distribution of Sample Means
$\bar{x}$
$\mu = 65$
30
40
50
60
70

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This means that $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ approximately follows a standard normal distribution (mean 0 and standard deviation 1).

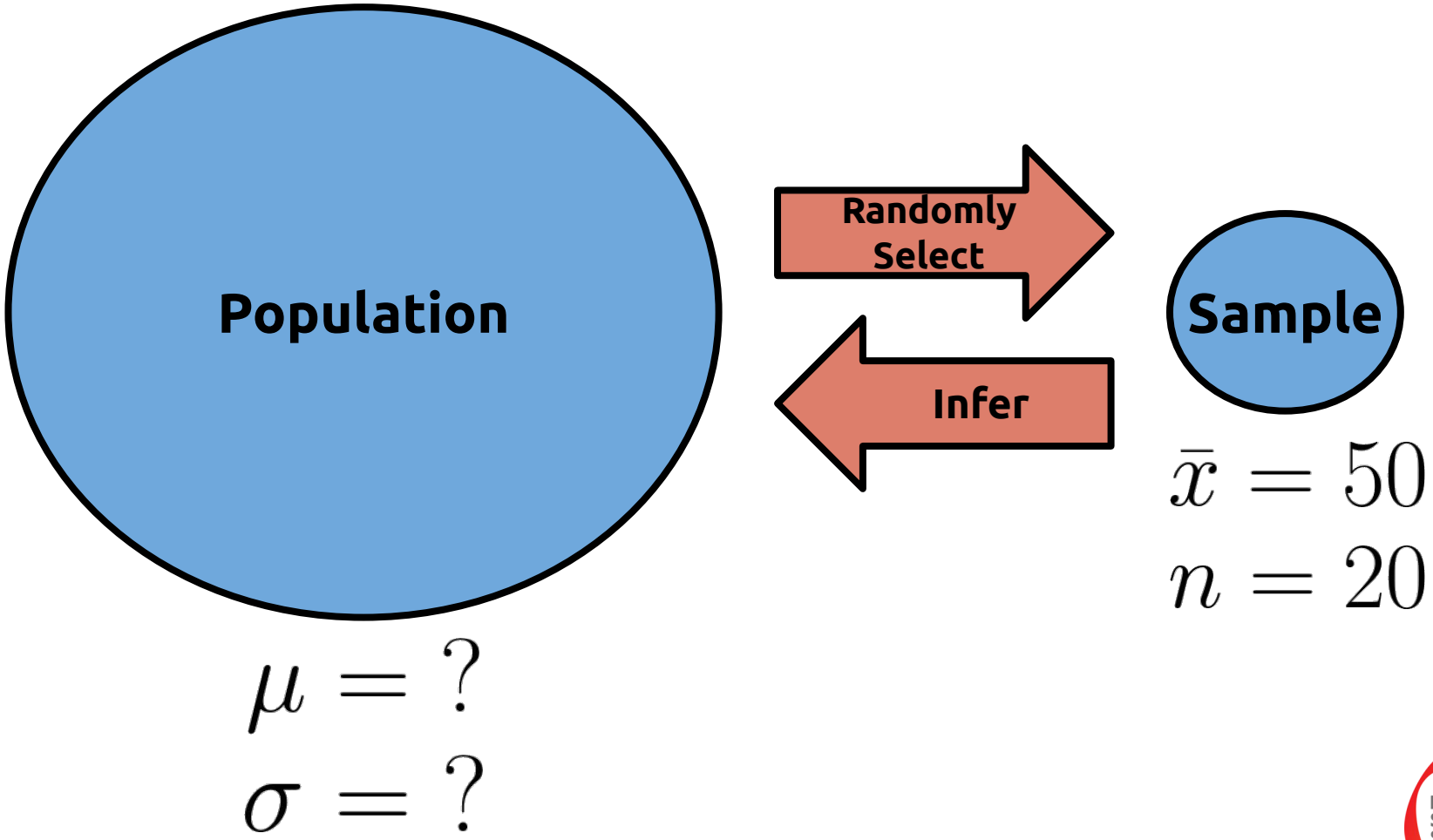
# Why do we care about all of this?

Knowing something about the distribution of sample means, we can determine what a “plausible” range of values for $\mu$.

More specifically, knowing about the standard deviation of the sample means and the shape of its distribution (normal), helps us determine a plausible range of values.

The key thing was knowing the **standard error** - the standard deviation of the distribution of sample means - is equal to $\frac{\sigma}{\sqrt{n}}$ , where $\sigma$ is the population standard deviation.

But wait - why would we know the population standard deviation but not the mean?

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If we don’t know the population standard deviation, how could we estimate it using just our single sample?

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$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

follows a standard normal distribution

$$\frac{\bar{x} - \mu}{s / \sqrt{n}}$$

follows a Student's *t*-distribution with n-1 degrees of freedom

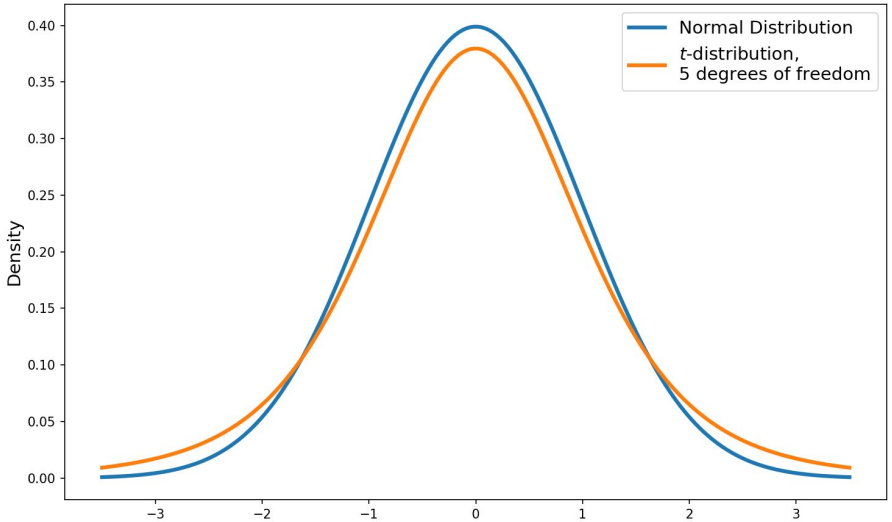
# Estimation

The family of Student's *t*-distribution is named after statistician William Sealy Gosset, who published his research under the pseudonym "Student".

Gosset worked for the Guinness Brewery where he worked on determining the quality of raw materials. Gosset was interested in the problem of small samples, as he would sometimes have to draw inferences from samples with as few as 3 observations.

# Estimation

The family of Student's *t*-distributions is a family of symmetric, bell-shaped distributions, which are similar to normal distributions, but have wider tails. That is, more extreme observations are more common. This family is parametrized by the number of degrees of freedom.



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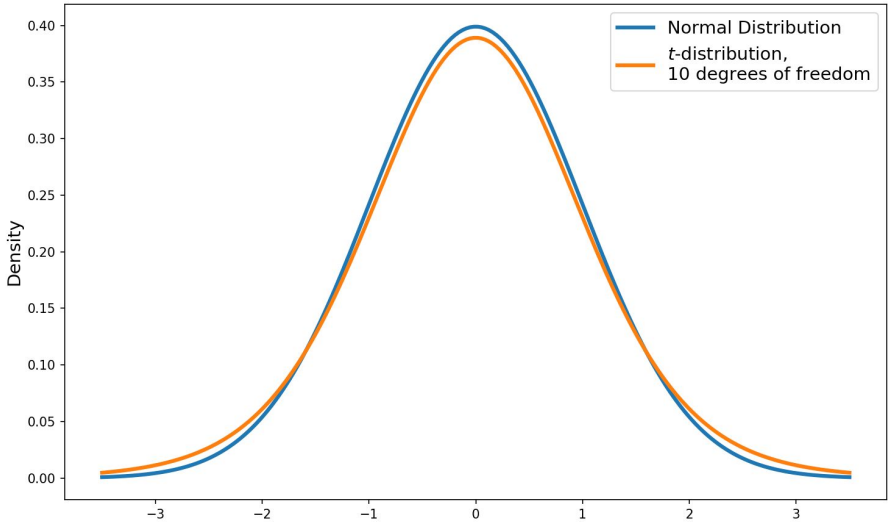

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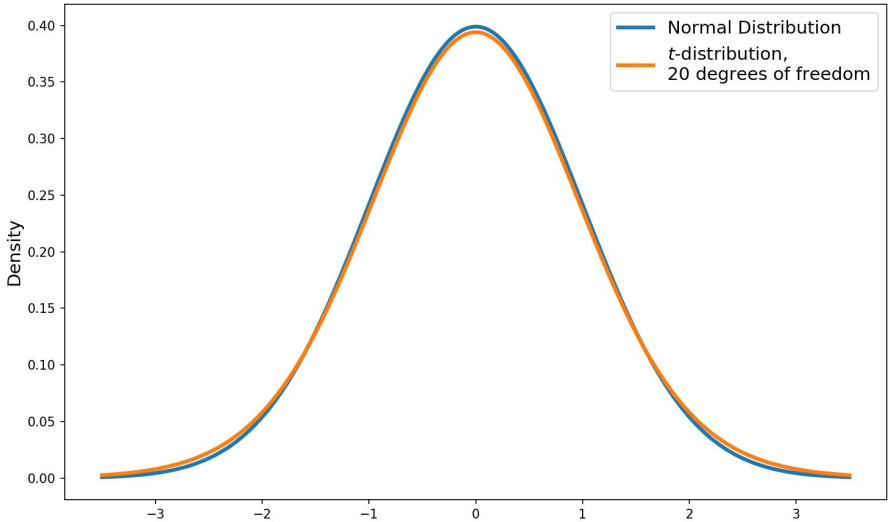

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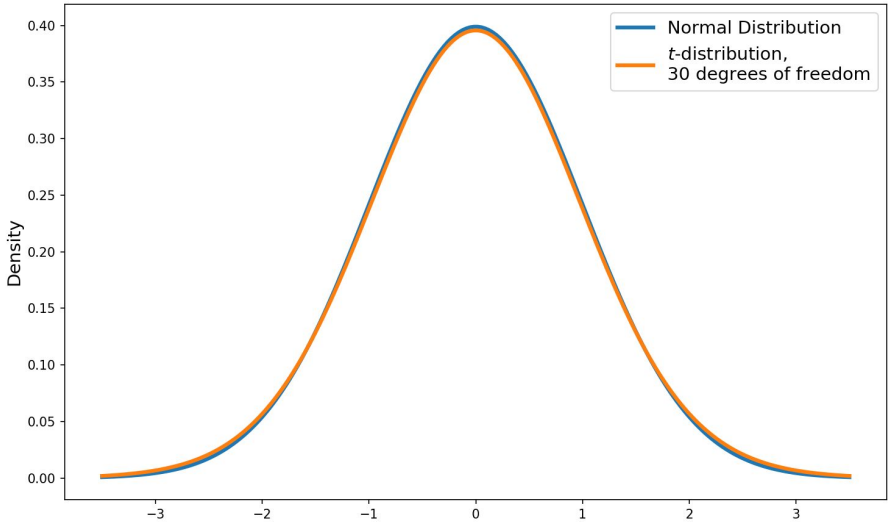

# Confidence Intervals

**Goal:** For a given sample, construct an interval around the sample mean just wide enough so that for 95% of samples, the interval constructed in this way will contain the true population mean $\mu$.

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**Goal:** For a given sample, construct an interval around the sample mean just wide enough so that for 95% of samples, the interval constructed in this way will contain the true population mean $\mu$.

To do this, forget about our single sample temporarily and think about *all* possible samples.

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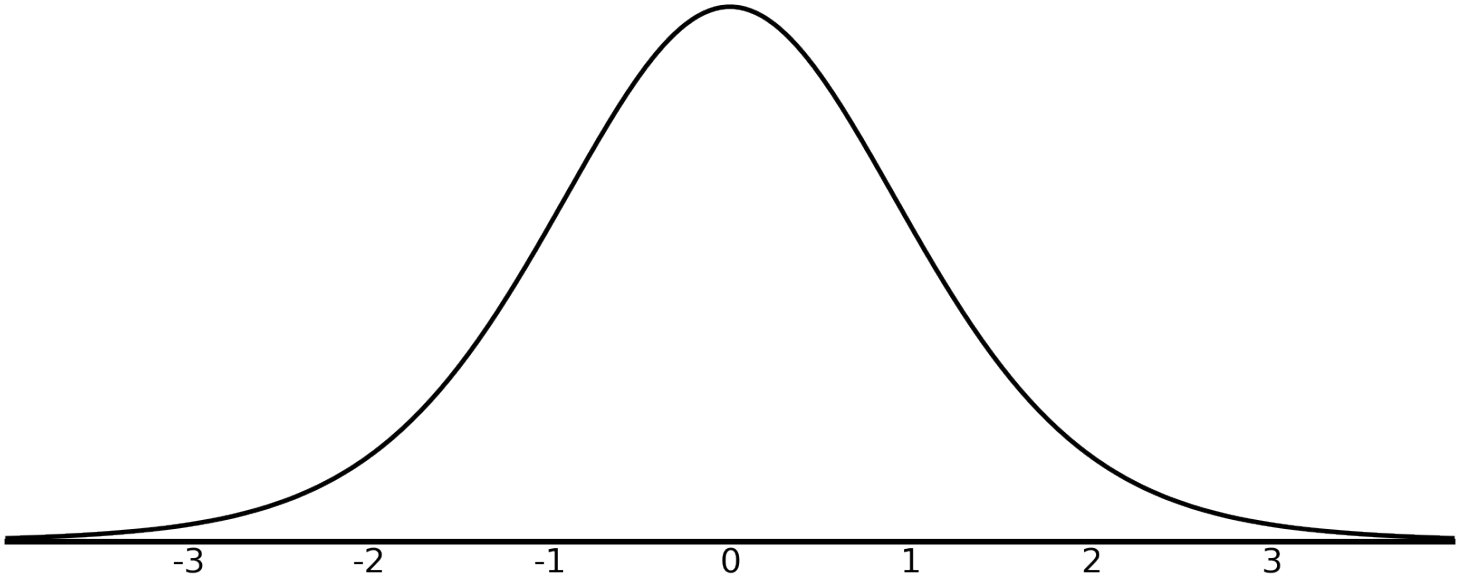
# Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$

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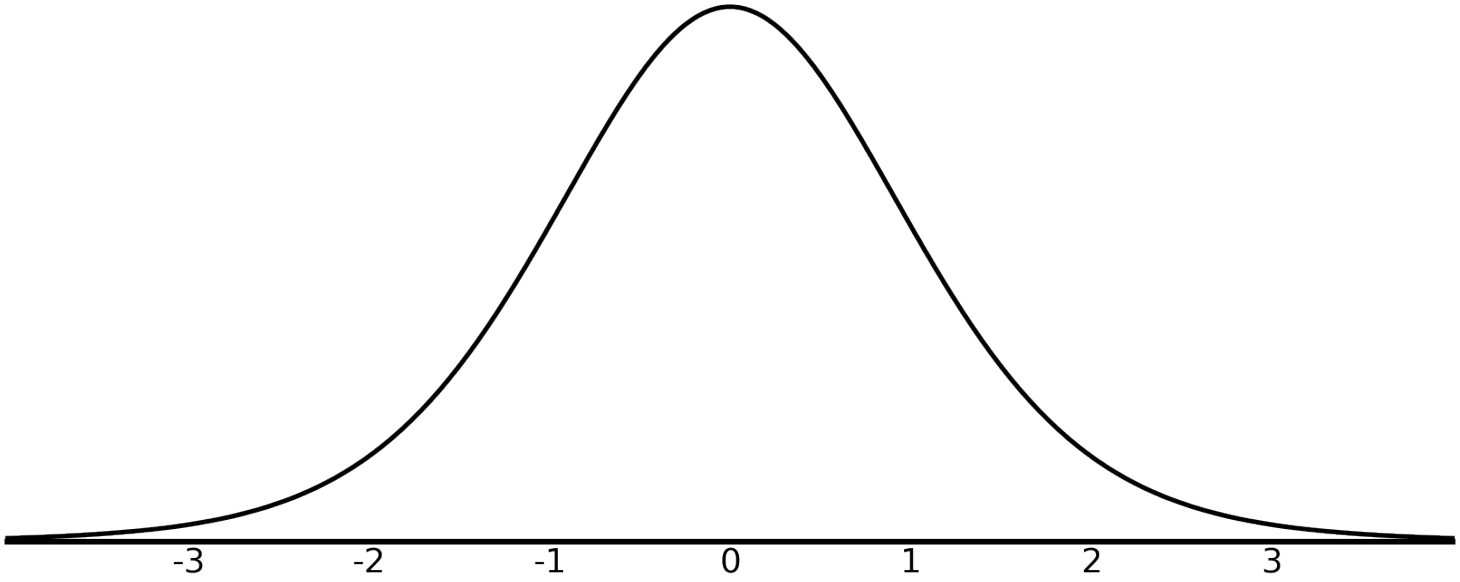
Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$




Student's *t*-distribution with n-1 degrees of freedom

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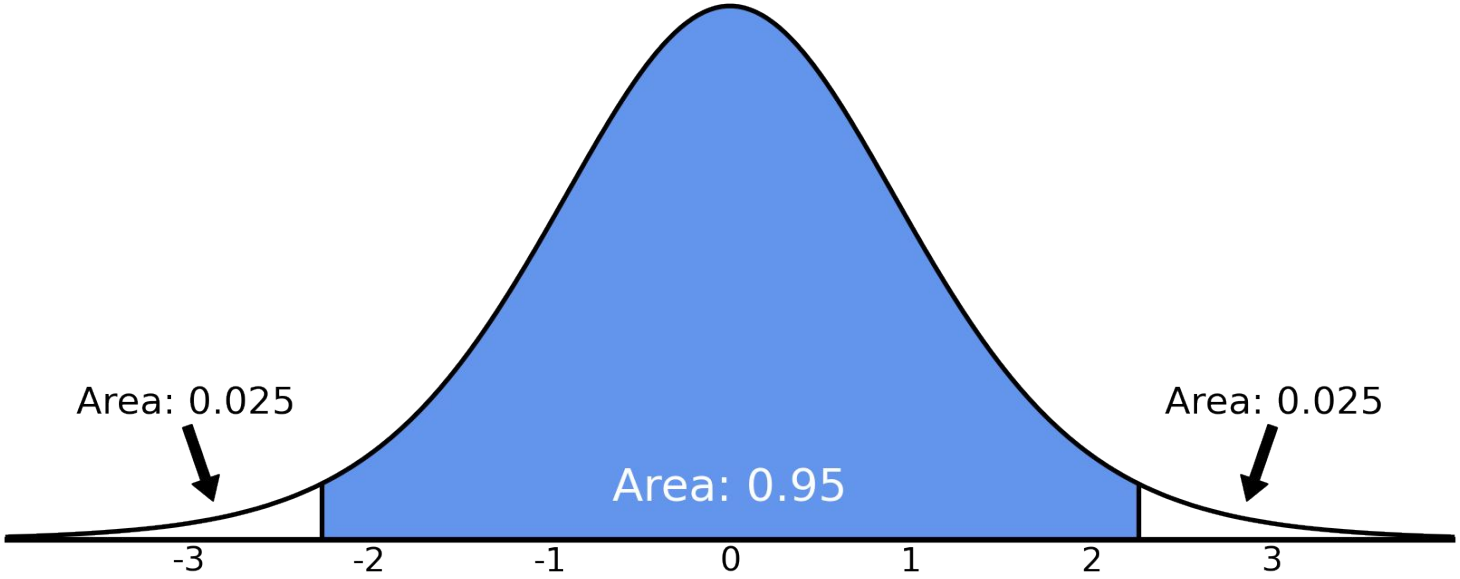
# Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$

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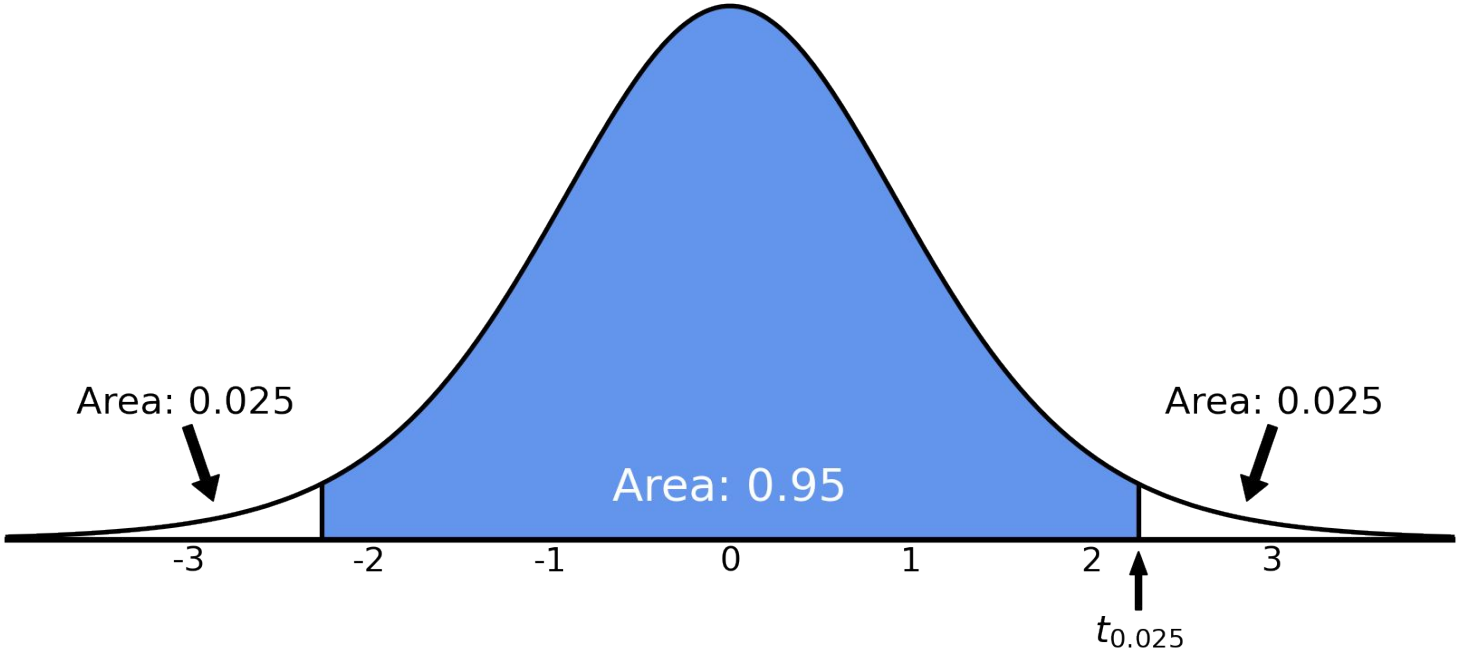
# Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$

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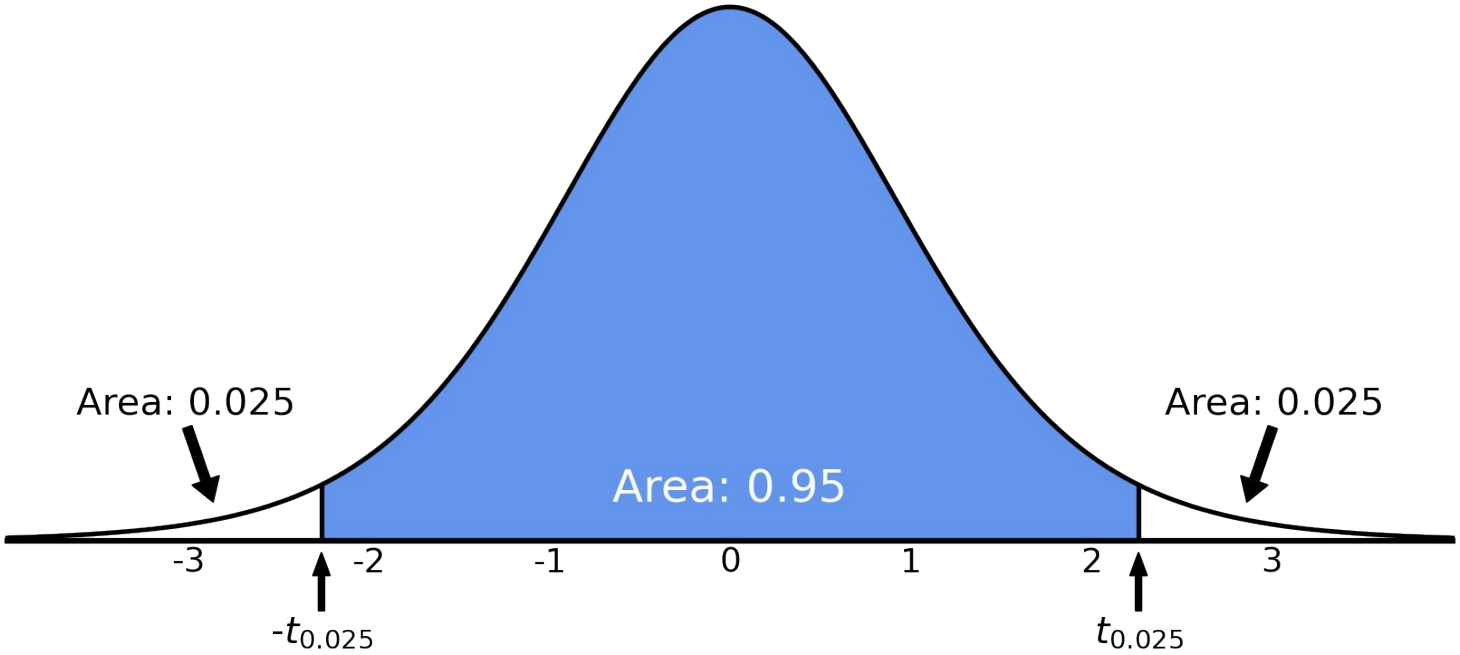
# Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$

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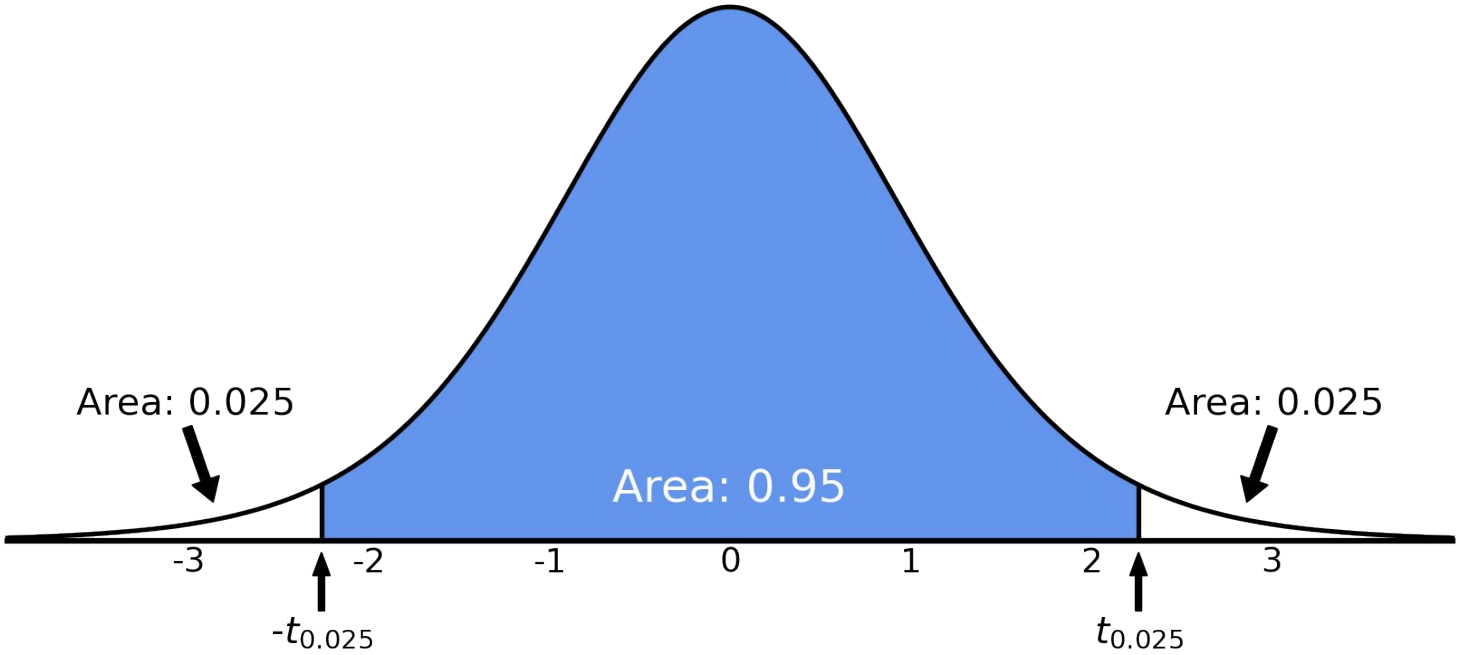
Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$




For 95% of samples, $-t_{0.025} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < t_{0.025}$

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For 95% of samples, $-t_{0.025} \cdot \frac{s}{\sqrt{n}} < \bar{x} - \mu < t_{0.025} \cdot \frac{s}{\sqrt{n}}$

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For 95% of samples, $-\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < -\mu < -\bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$

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Hence, we can use this recipe to build a 95% confidence interval:

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Point Estimate

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For 95% of samples,

$$\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Hence, we can use this recipe to build a 95% confidence interval:

$$\bar{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Point Estimate

Margin of Error

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# Estimation - Proportions

What does the sampling distribution for a proportion look like?

That is, if we take a sample from a population and calculate the proportion of observations in our sample that meet a particular condition, what does the distribution of the possible sample proportions look like?

Eg. When we conduct a poll, we might find the proportion of respondents planning to vote for a particular candidate.

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# Estimation - Proportions

Say we have a sample size of *n* and the the true population proportion is *p*.

Let $\hat{p}$ be the sample proportion.

We know that $n \cdot \hat{p}$ follows a binomial distribution with $n = n$ and $p = p$.

We also know that this distribution has mean $np$ and standard deviation $\sqrt{n \cdot p \cdot (1 - p)}$

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**Fun Fact:** As *n* increases, the normal distribution approximates the binomial distribution.

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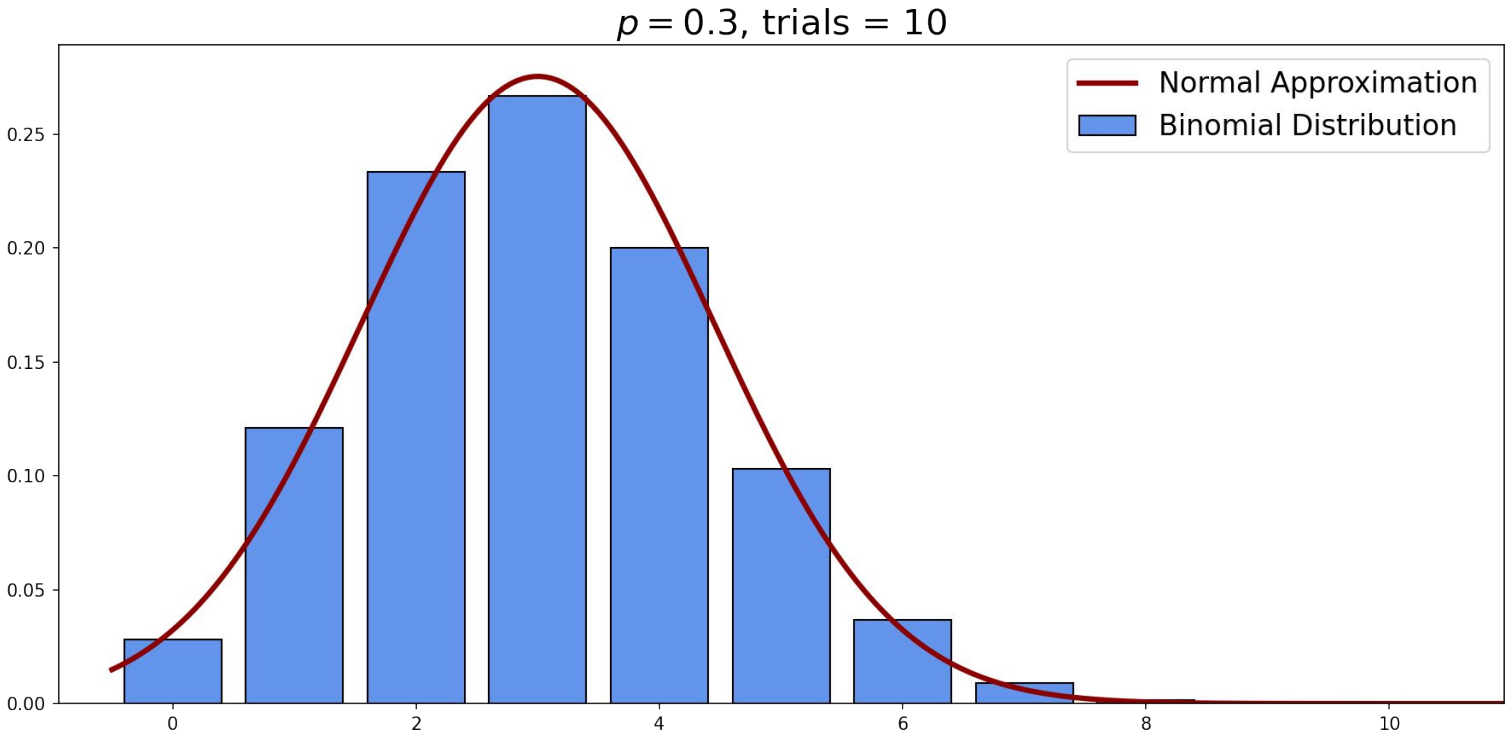


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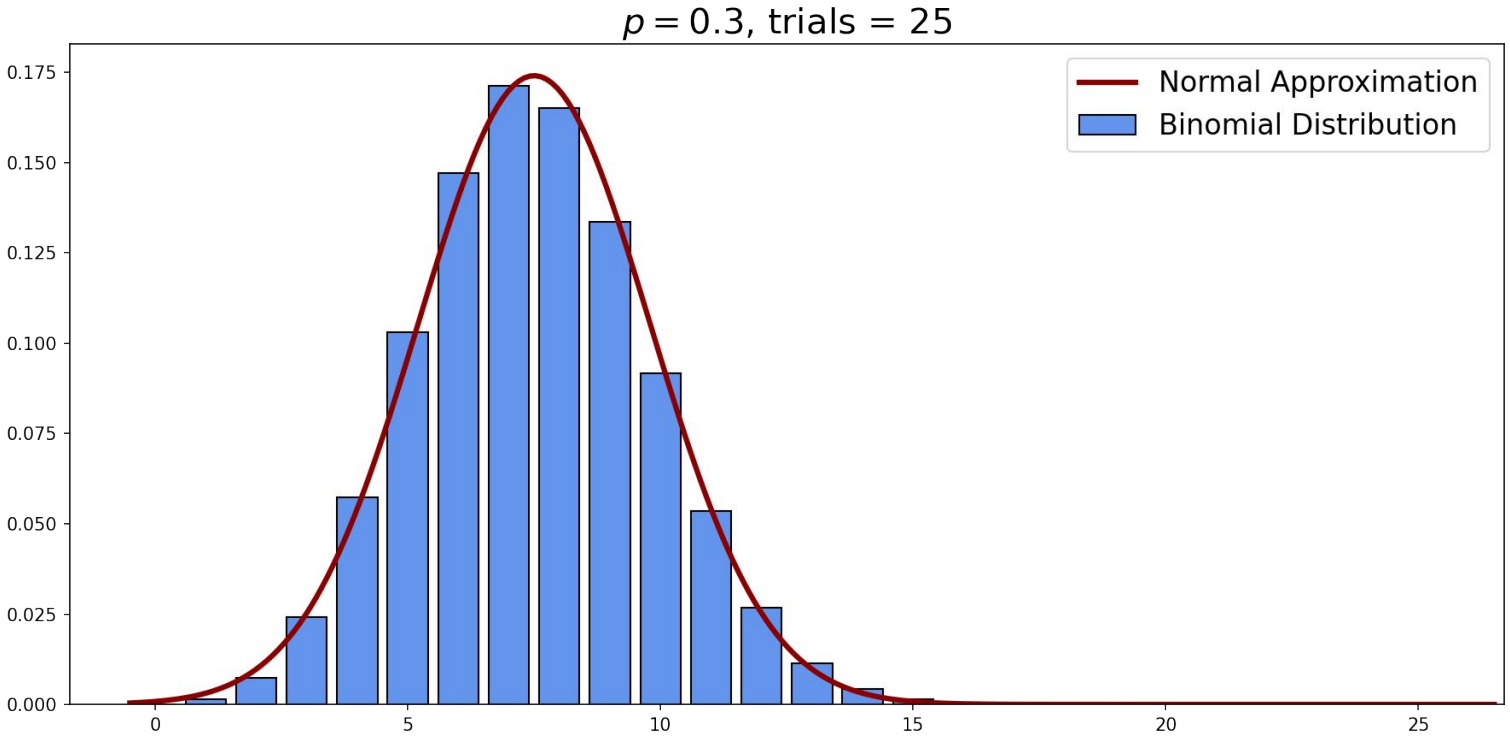


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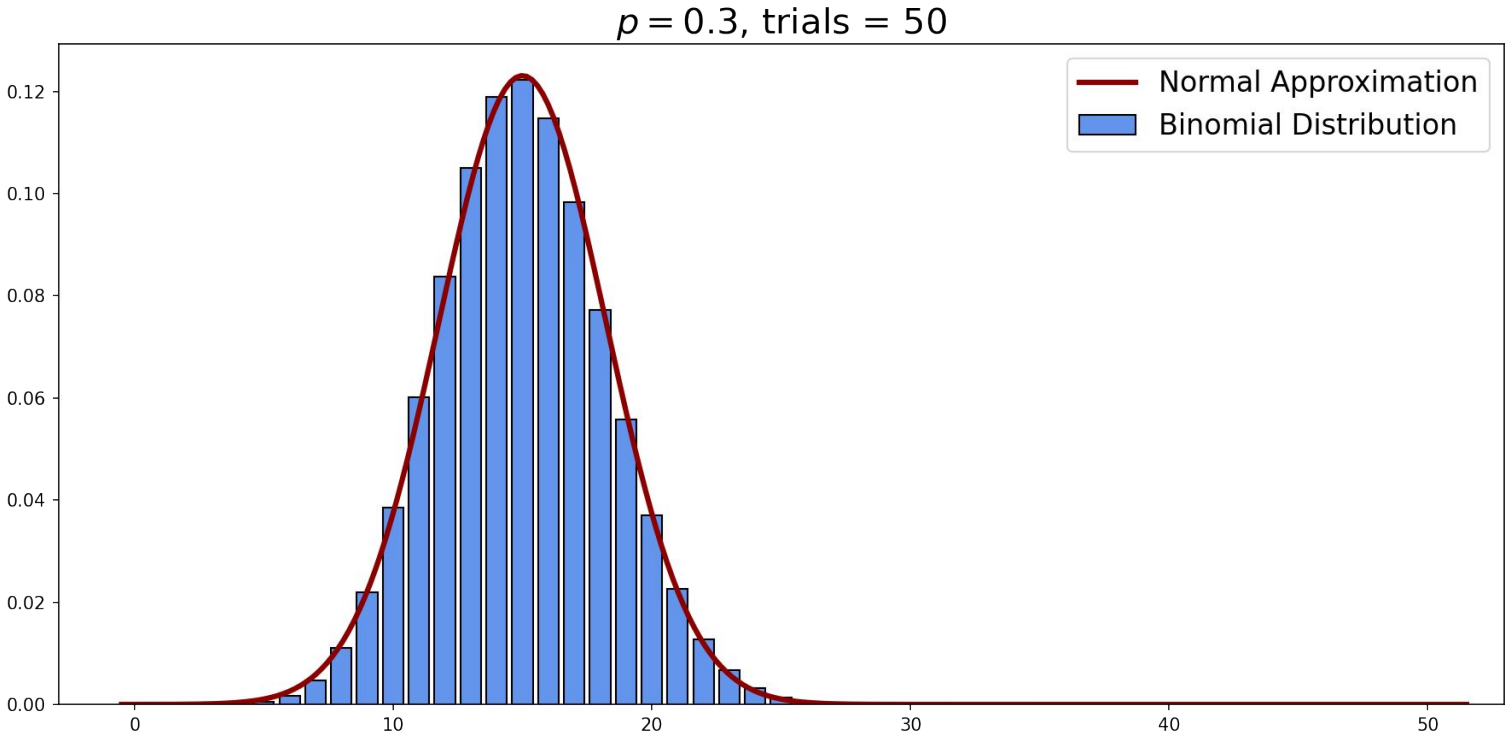


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# Estimation - Proportions

If *np ≥ 10* and *n(1-p) ≥ 10*, the distribution of $n \cdot \hat{p}$ is approximately normal with mean *np* and standard deviation $\sqrt{n \cdot p \cdot (1-p)}$

Divide by *n* and we can see that the distribution of $\hat{p}$ is approximately normal with mean *p* and standard deviation $\sqrt{\frac{p(1-p)}{n}}$

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# Estimation - Proportions

Recall that we can build a confidence interval if we know about the variability of the distribution of test statistics.

And now we know that for $\hat{p}$ , this has a standard deviation of $\sqrt{\frac{p(1-p)}{n}}$

But, there is a problem - it relies on using the population proportion - the quantity we are trying to approximate.

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# Estimation - Proportions

Just sub in the sample proportion!

$$\sqrt{\frac{p(1-p)}{n}} \quad \Rightarrow \quad \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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# Estimation - Proportions

**Confidence Interval for the Proportion:**

$$\hat{p} \pm z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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# Estimation - Proportions

**Confidence Interval for the Proportion:**

$$\hat{p} \pm z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Point Estimate

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# Estimation - Proportions

**Confidence Interval for the Proportion:**

$$\hat{p} \pm z_{\alpha} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Point Estimate

Critical Value (from the standard normal distribution)

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