Introduction to Logistic Regression

Introduction

Recall that **linear regression** can be used to describe the relationship between two or more variables where the target variable is numeric.

For categorical targets, we can use **logistic regression**.

Example - ICU Admission

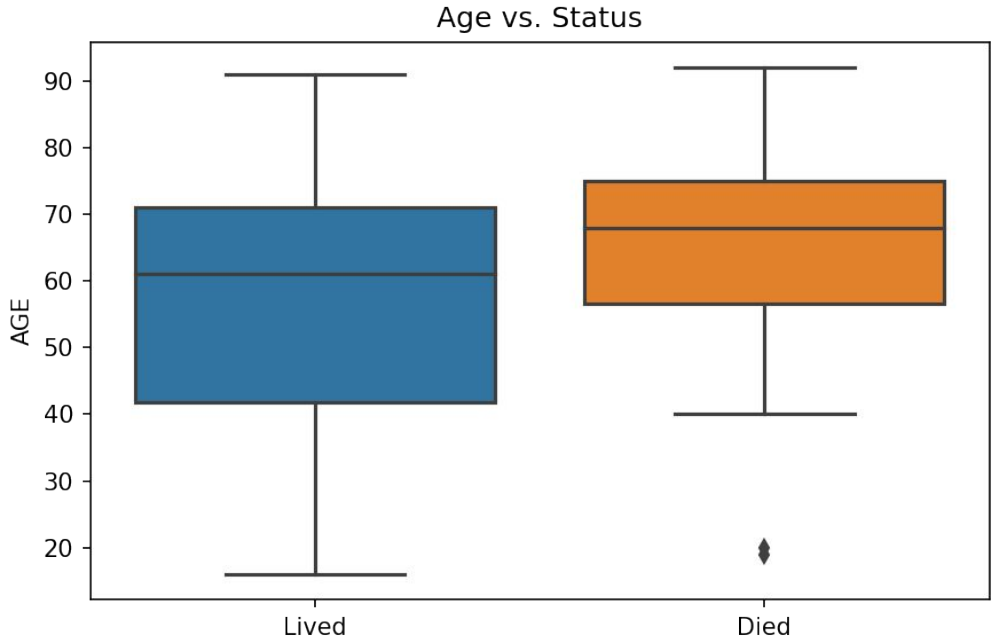
Let’s say we want to study survival rates of patients admitted to an intensive care unit. We gather several variables:

- Status: lived or died
- Age
- Sex
- Systolic Blood Pressure
- Type of Admission (Elective or Emergency)
- etc.

Example - ICU Admission

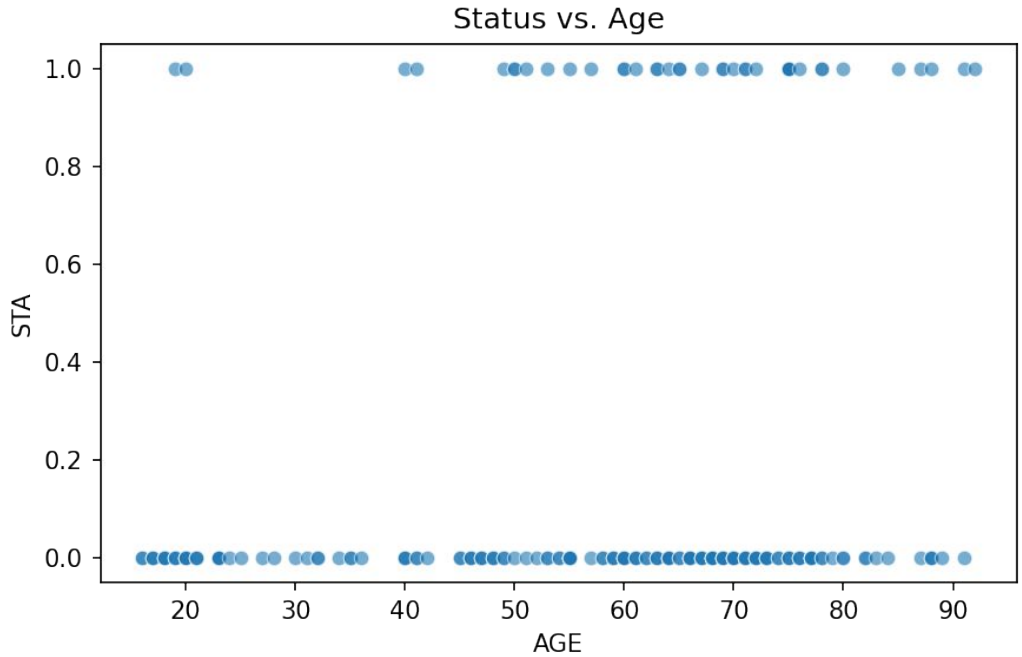
We might start by examining age vs. status.

It appears that those that died tended to be older.



Example - ICU Admission

We could also plot this as a scatterplot, where we encode status numerically, with lived = 0 and died = 1.



Example - ICU Admission

How can we build a model to describe the relationship between age and status?

Example - ICU Admission

Can we use a linear regression model?

Linear regression: The distribution of *Y*, given *X* is normal with mean

$$\mu = \beta_0 + \beta_1 X$$

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Idea: Instead of using 0/1 as our target,
let's make our target a *probability*.

Recall: Bernoulli Distribution

Setup: An experiment with exactly two outcomes, labeled "success" (denoted by 1) and "failure" (denoted by 0).

Probability of success = p
Probability of failure = $1 - p$

Example: A marketing company knows that historically, search ads have a click-through rate of 1.5%.
We can view each interaction as a Bernoulli trial with $p = 0.015$

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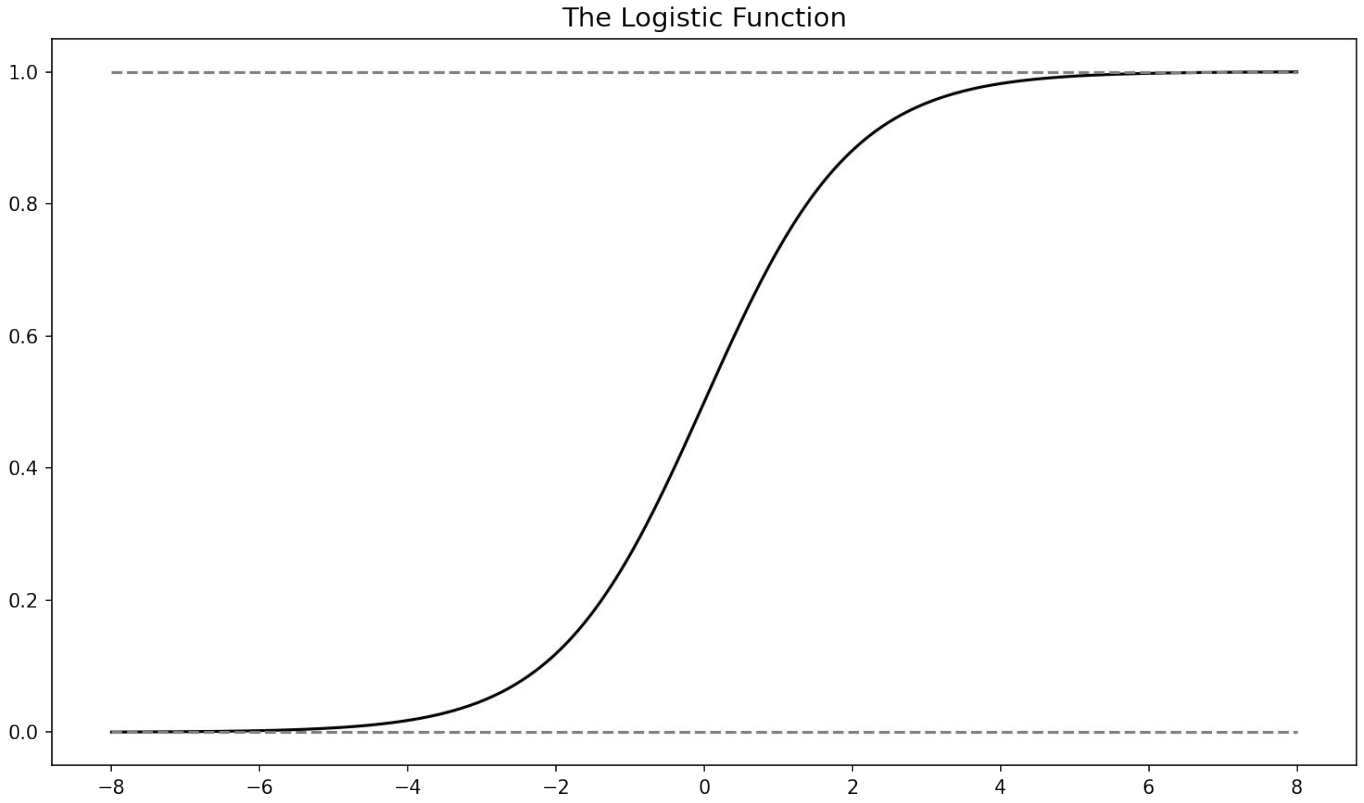
But wait, a probability must be between 0 and 1, and there is no guarantee that this expression will be.

The logistic function: $f(x) = \frac{1}{1 + e^{-x}}$



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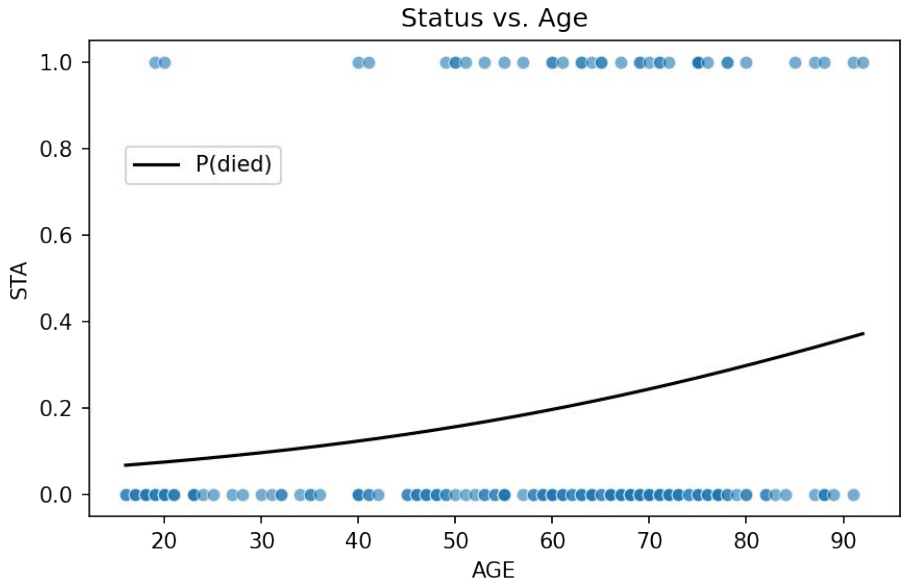
$$p = \text{logistic}(\beta_0 + \beta_1 X)$$

Example

If we fit a logistic regression model to the ICU data, using age, we get

P(died) = logistic(-3.0585 + 0.0275(age))



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How do we determine the coefficients?

On the basis of the **likelihood** of the resulting model.

For a single observation with outcome y_i and predicted probability π_i, the **likelihood** of this outcome is given by

$$(\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

This reduces to the predicted probability of the correct outcome.

Example

$$(\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

Age	Status	Predicted P(died)	Likelihood
49	lived	0.153	$(0.153)^0(0.847)^1 = 0.847$
80	died	0.298	$(0.298)^1(0.702)^0 = 0.298$
91	lived	0.365	$(0.365)^0(0.635)^1 = 0.635$

Example

The coefficients are the ones that maximize the likelihood across the dataset:

$$\prod_i (\pi_i)^{y_i} \cdot (1 - \pi_i)^{1-y_i}$$

Usually, it is actually the **log-likelihood** that is maximized:

$$\sum_i [y_i \log(\pi_i) + (1 - y_i) \log(1 - \pi_i)]$$

Inference for Logistic Regression Models

Types of questions that we can ask:

- How precise is our estimate of the coefficient associated with age?
- Is the coefficient associated with age statistically significant?
- If I add additional predictor variables, are their coefficients statistically significant, after controlling for age?

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The first two questions can be answered using the fact that coefficient estimates are approximately normally distributed.

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The last question can be answered using the **likelihood ratio test**, which compares the likelihood of the full model to a reduced model.

Likelihood Ratio Test

Procedure:

1. Defining a larger **full model** (which contains all predictors involved in the test)
2. Defining a smaller **reduced model** (which satisfies the assumptions of the null hypothesis)
3. Calculate the test statistic (which involves the ratio of likelihoods) and compare to a chi-square distribution.

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p-value: 0.00263

Conclusion: Reject the null hypothesis.

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Inference vs. Prediction

When building a statistical model, there are a number of possible objectives:

- **inference:** identifying key explanatory variables and understanding the relationship between these variables and the target
- **prediction:** predicting the outcome on new observations

Predictive analytics typically focuses on model-building for prediction rather than inference, and the techniques you can use in each differ.

Predictions on a New Observation

Once we have build a logistic regression model, we can use it to generate predictions on new observations.

To do this, we much translate our predicted probabilities π_i's into predictions.

A simple rule that can be used is to predict 1 if $\pi_i > 0.5$ and 0 otherwise.

Predictions on a New Observation

There are a number of metrics that can be used to evaluate predictions of a model on the basis of True Positives, False Positives, True Negatives, and False Negatives.

When evaluating the performance of a predictive model, it should be done by separating out a test set of data which the model is not fit on.

Logistic Regression

Let’s see all of this in action in a Jupyter notebook.