Sampling Distributions

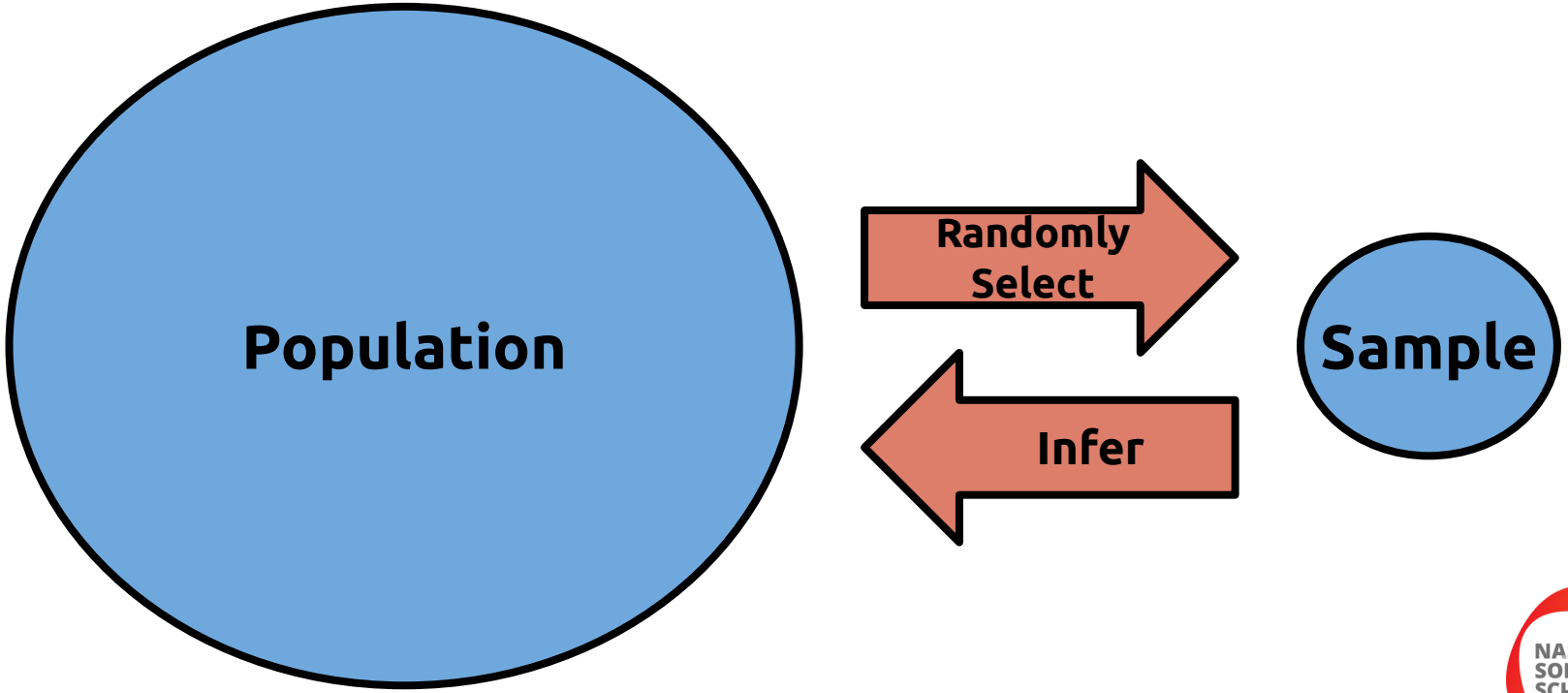
Estimation

When doing statistics, the goal is often to infer something about a population *parameter* using only a sample from that population.

Examples:

- Estimating the average household income in Putnam County.
- Predicting the percentage of votes a particular candidate will receive in an upcoming election, based on a poll.

Estimation



Confidence Intervals

Let’s say we take a sample of households in Putnam County and find that the average household income in our sample is $48,100.

How likely is it that our sample mean is the same as the population mean? Almost impossible!

But how far off are we likely to be from the actual population mean? How much margin of error do we need to allow to be confident that we are have included the true population mean?

Sampling Distributions

To answer this question, we need to take a step back and think about the bigger picture.

We will work to understand the distribution from which our sample came - not the population distribution but the *distribution of all possible samples*.

Sampling Distributions

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches.

$$\mu = 70 \qquad \sigma = 3$$

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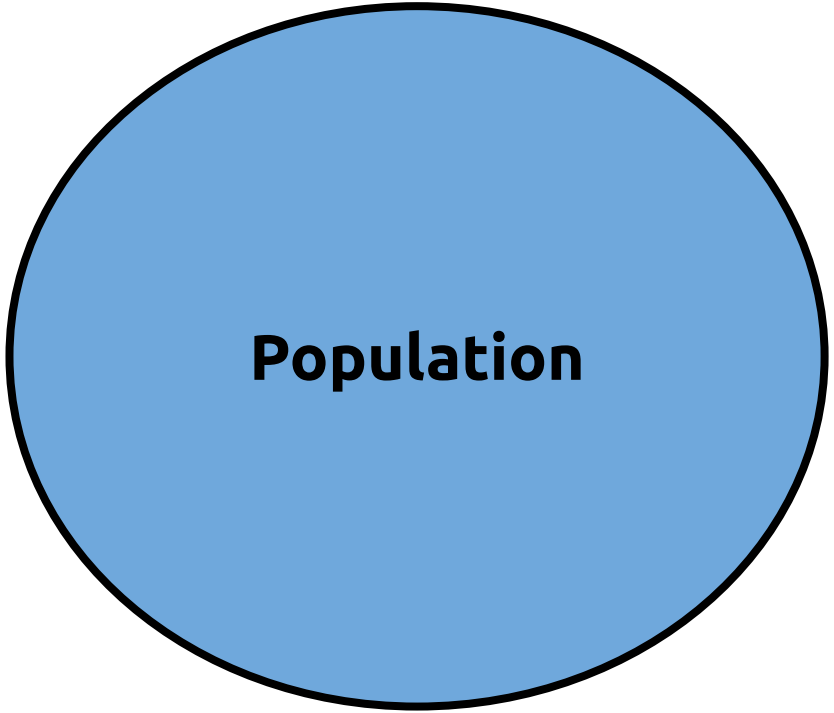
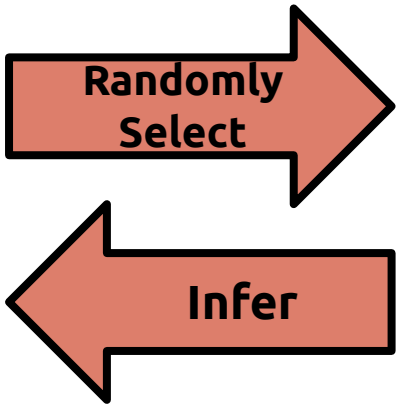
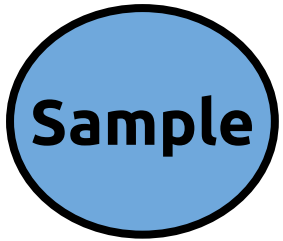
If we randomly choose 10 men, how close can we expect the **sample mean** to be to the population mean, 70?



$$\mu = 70$$

$$\sigma = 3$$





$$\bar{x} = ?$$

$$n = 10$$

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Sampling Distributions

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches.

$$\mu = 70 \qquad \sigma = 3$$

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To explore this, we can use the Sampling_Distribution.ipynb notebook.

Population Mean

$\mu = 70$

Population Standard Deviation

$\sigma = 3$

Sample Size	Mean of Sample Means	Standard Deviation of Sample Means
1		
10		
25		
100		
1000		

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Cool Fact:

If *X* is a normally-distributed variable with mean ***μ*** and standard deviation ***σ***, the distribution of sample means of size *n* is also normal with mean ***μ*** and standard deviation $\frac{\sigma}{\sqrt{n}}$

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The quantity $\frac{\sigma}{\sqrt{n}}$ is called the **standard error of the mean**.

Observe: As sample size increases, the standard error shrinks.

Really Cool Fact

Really Cool Fact

Central Limit Theorem: Let *X* be a random variable with mean μ and standard deviation σ.

For large enough sample size *n*, (as long as X is reasonably well-behaved), the distribution of sample means is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$

Really Cool Fact

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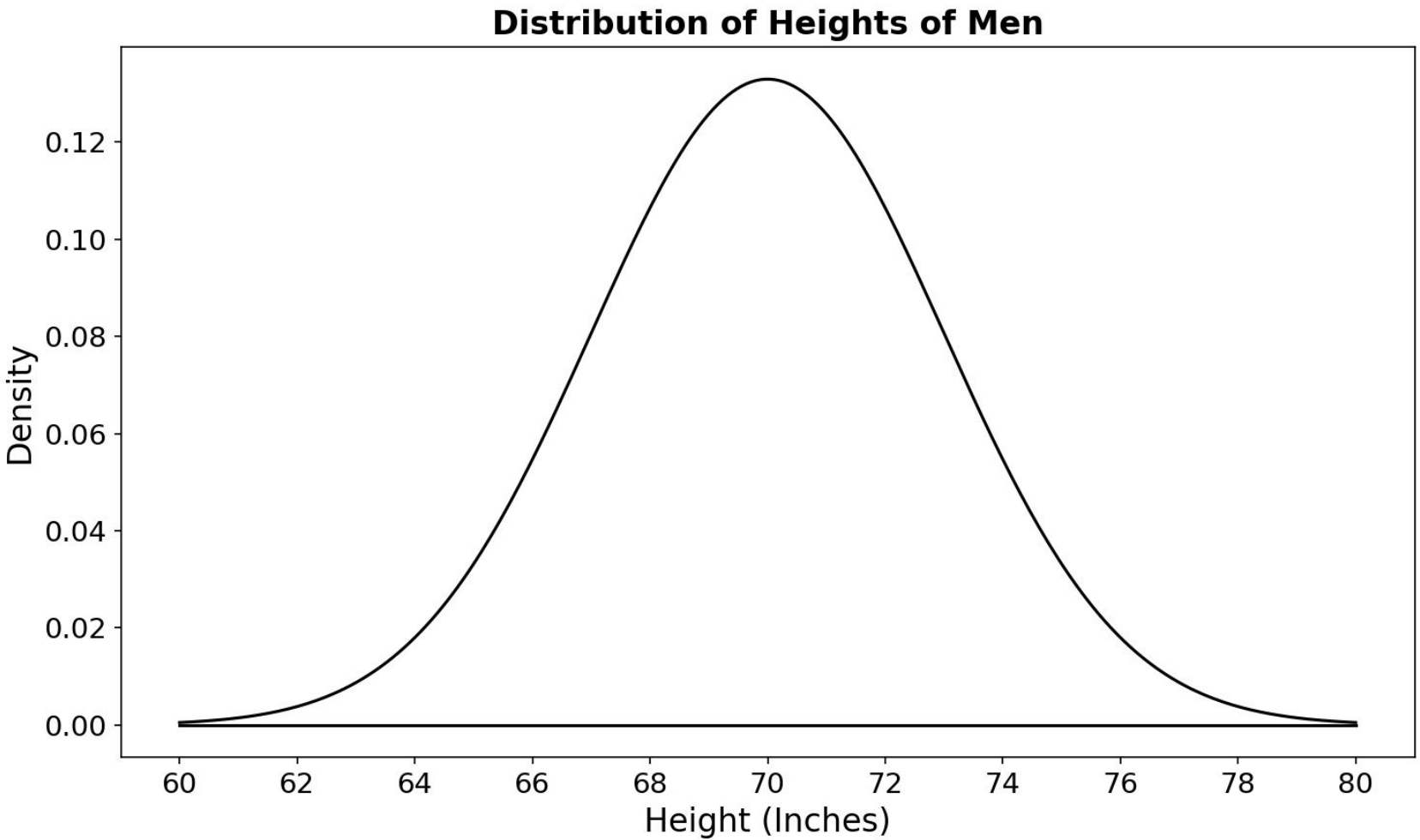
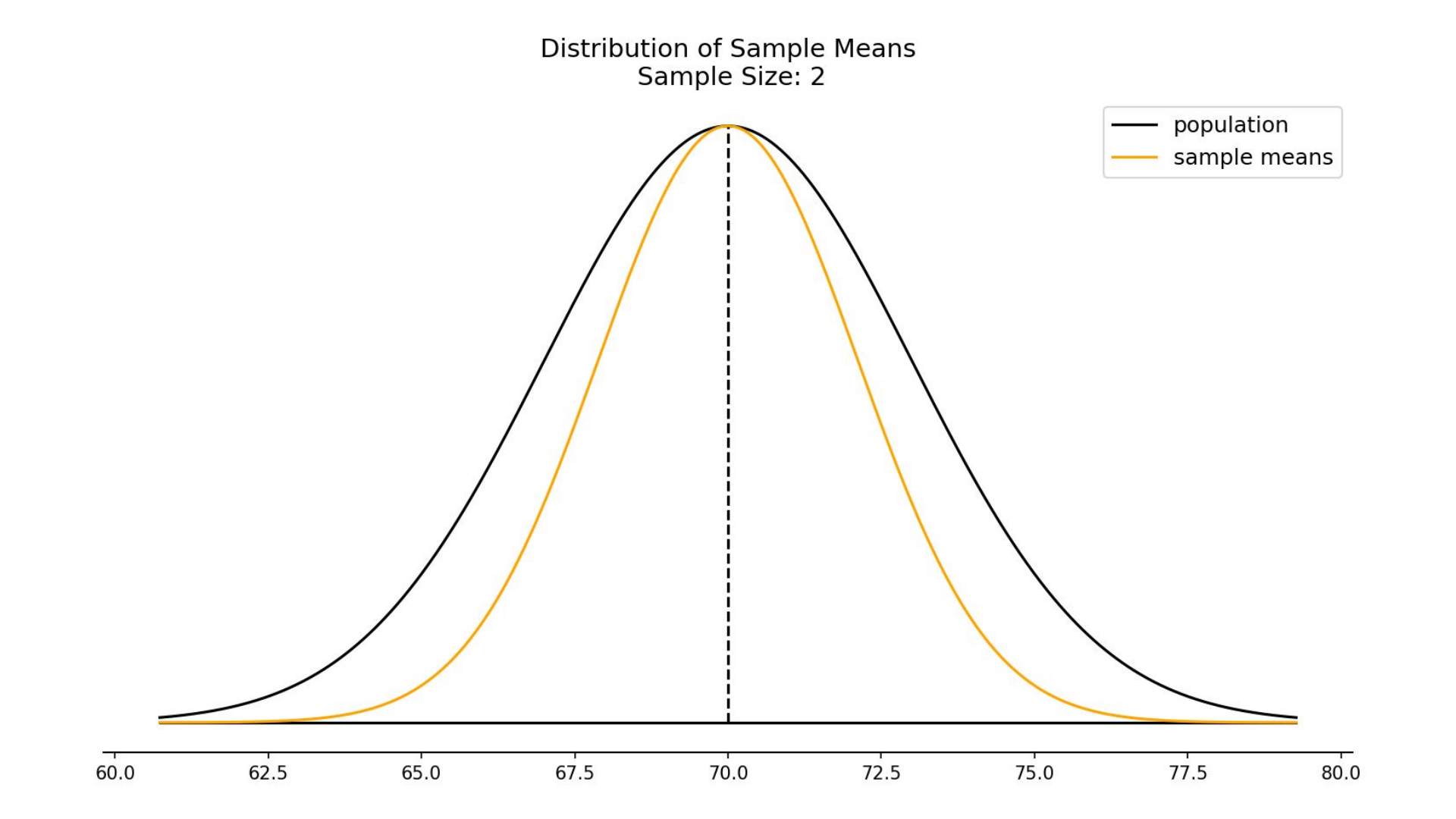
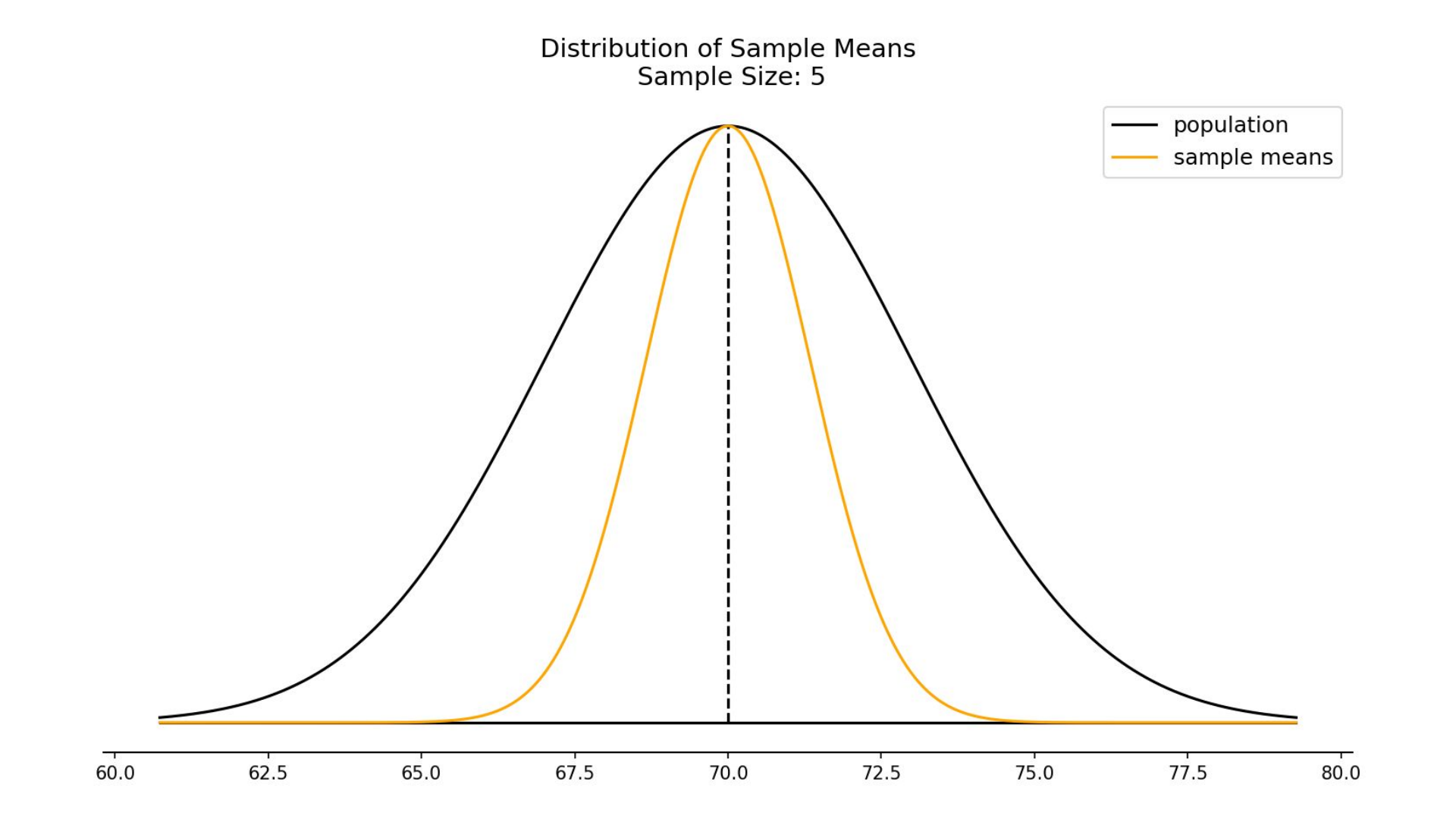
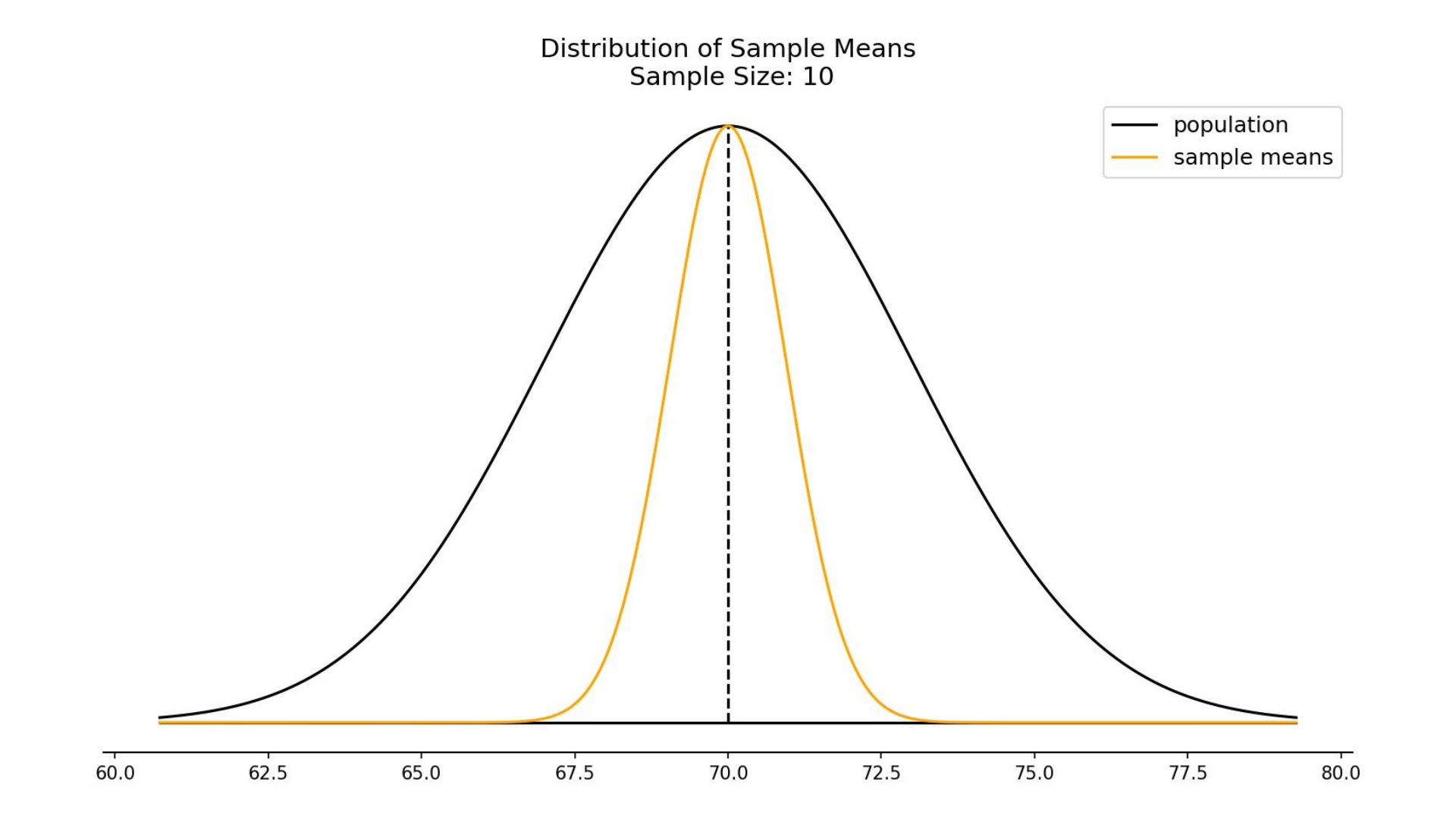
For large enough sample size n, (as long as X is reasonably well-behaved), the distribution of sample means is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$

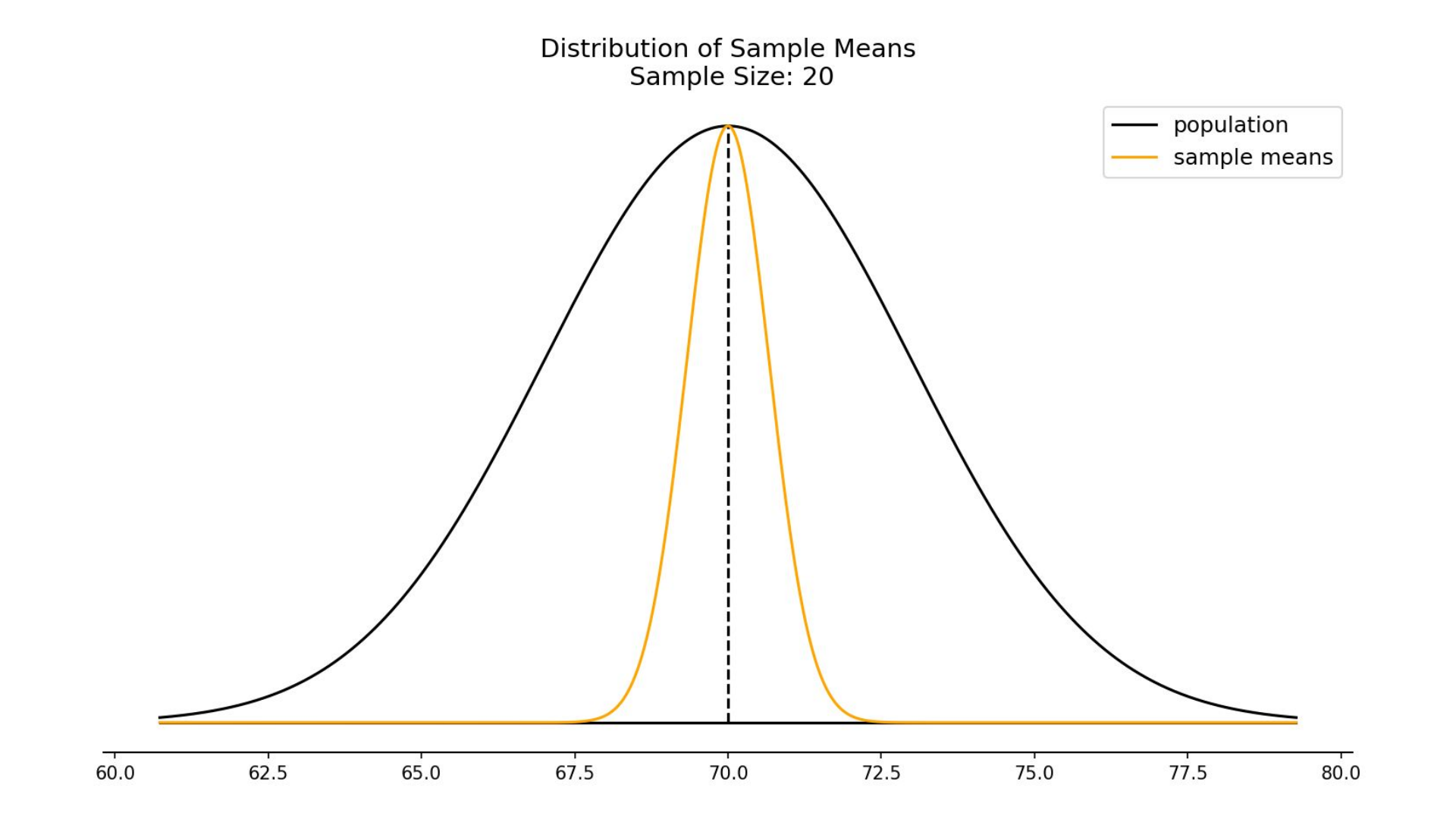
(In a lot of cases, sample size $n = 30$ is good enough for the central limit theorem to kick in).

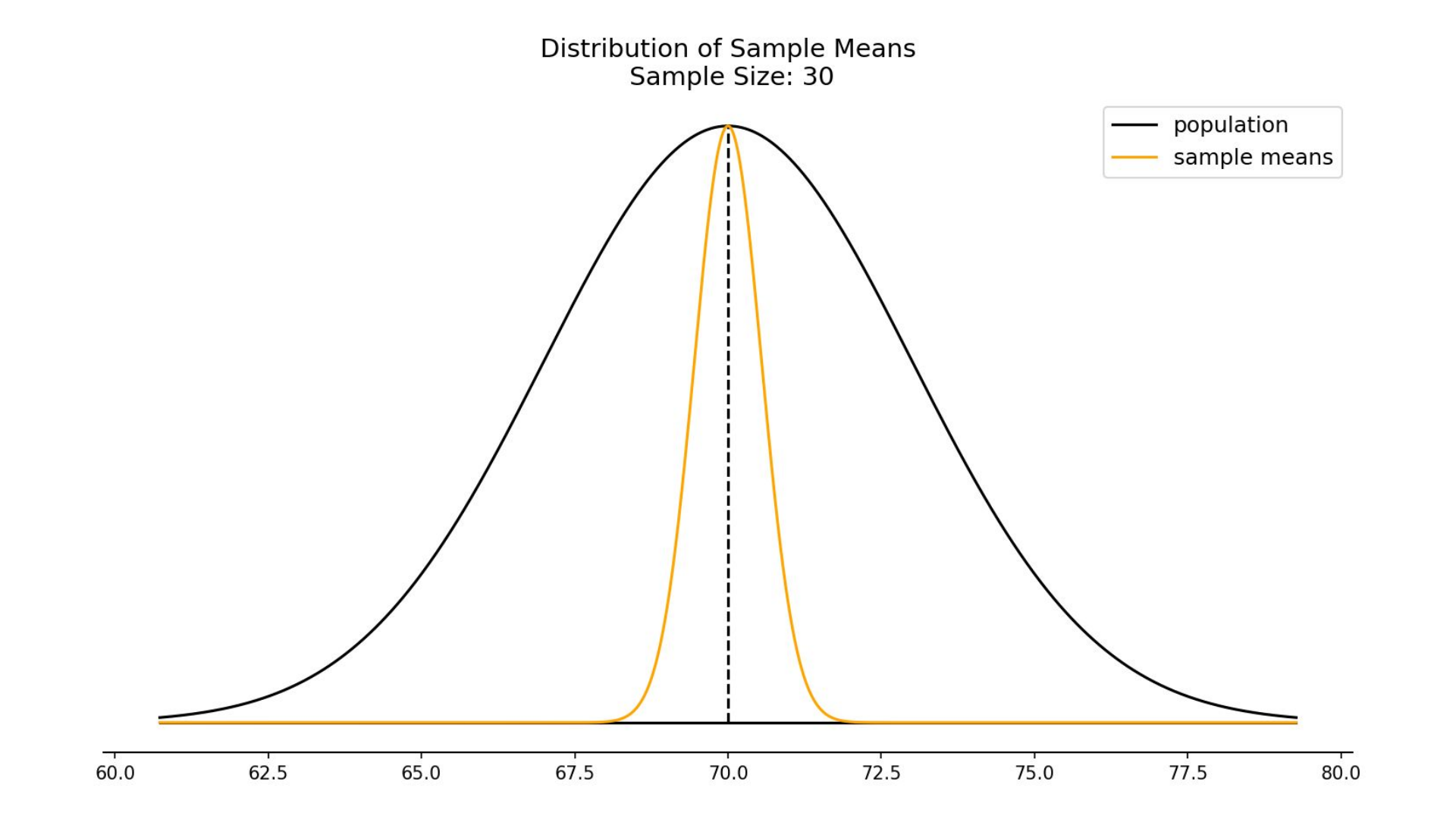

Distribution of Heights of Men
Density
0.12
0.10
0.08
0.06
0.04
0.02
0.00
60
62
64
66
68
70
72
74
76
78
80
Height (Inches)

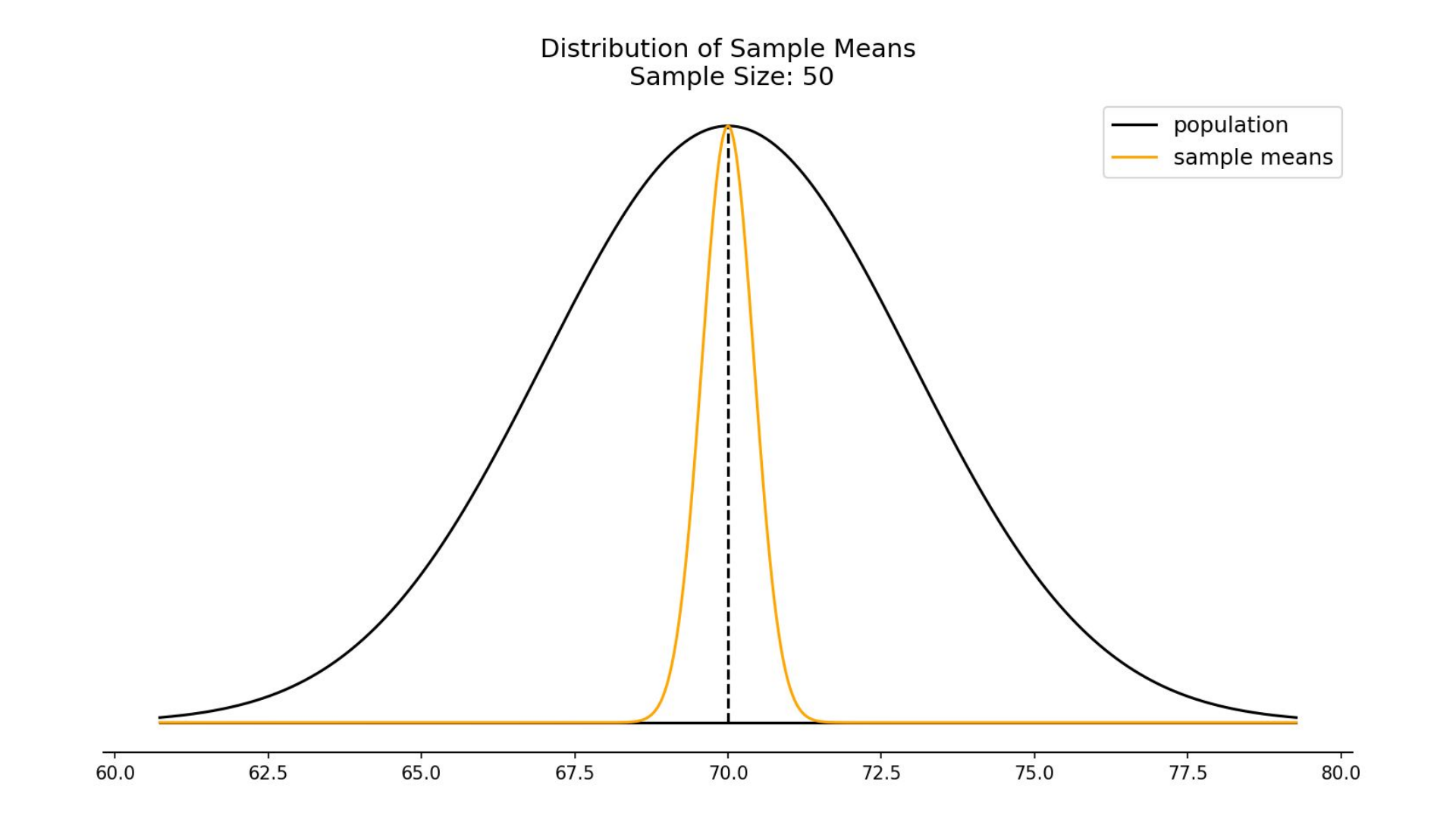

Distribution of Sample Means
Sample Size: 2
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

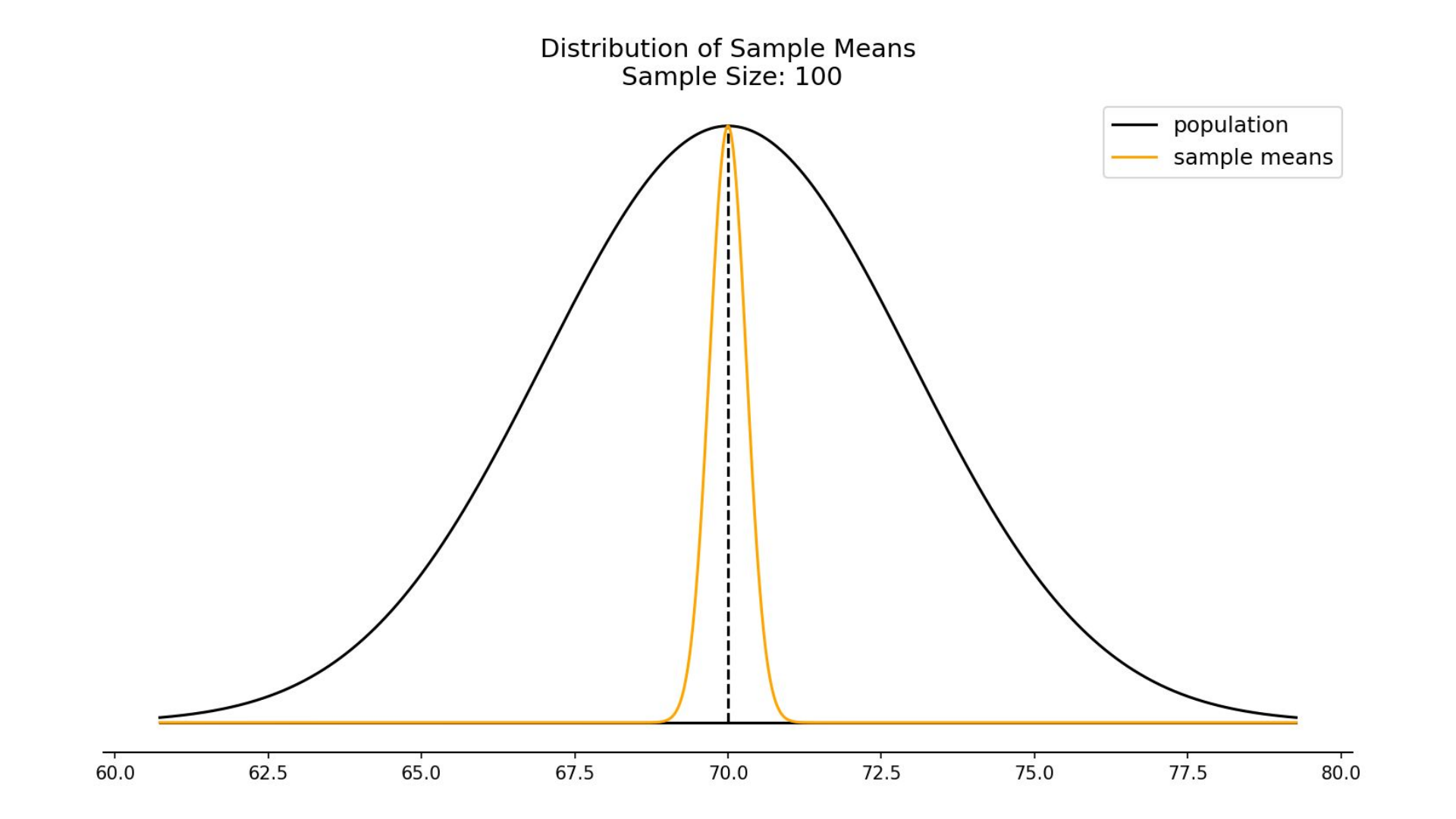

Distribution of Sample Means
Sample Size: 5
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 10
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 20
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

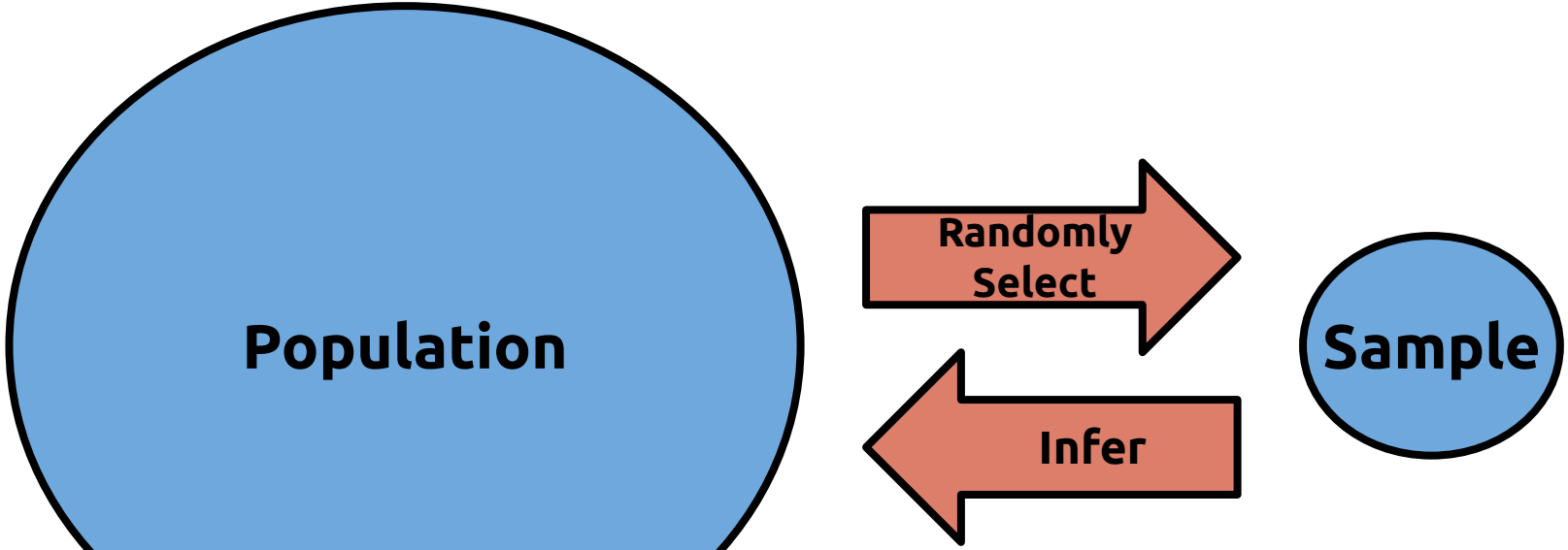

Distribution of Sample Means
Sample Size: 30
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 50
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0


Distribution of Sample Means
Sample Size: 100
population
sample means
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80.0

For the next series of questions, use the Calculations portion of Sampling_Distribution.ipynb to answer.

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches



$$\mu = 70$$

$$\sigma = 3$$

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The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

Question: If we randomly select a man, what is the probability that he is between 69 and 71 inches tall?

Answer:

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

Question: If we randomly select a group of 10 men, what is the probability that their average height is between 69 and 71 inches?

Answer:

The heights of men are approximately normally distributed with mean 70 inches and standard deviation of 3 inches

Question: If we randomly select a group of 100 men, what is the probability that their average height is between 69 and 71 inches?

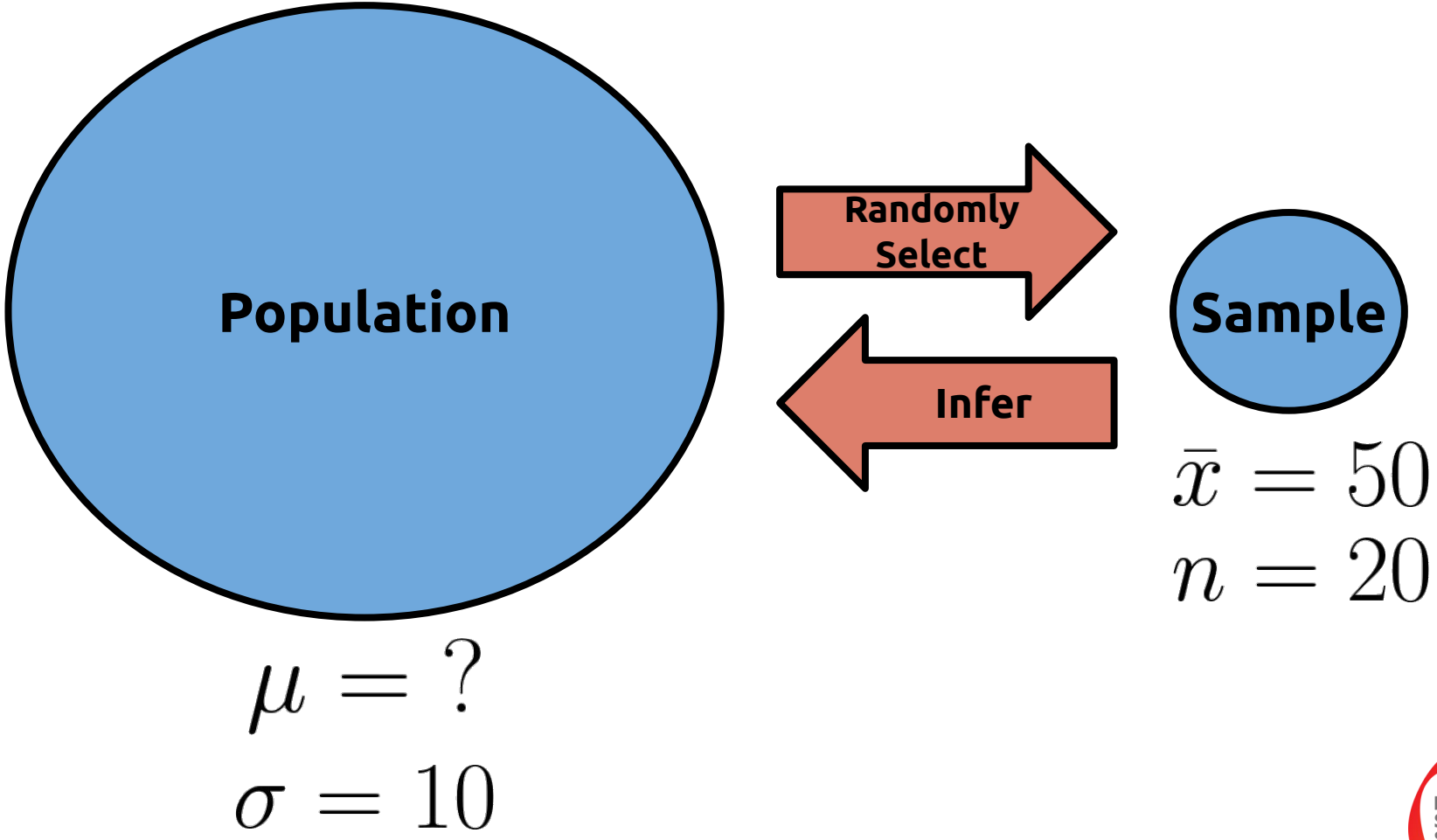
Answer:

We have now seen how we can use our understanding about the distribution of sample means to determine the probability that a sample mean will be in a certain range, if we know the population mean and standard deviation.

We have now seen how we can use our understanding about the distribution of sample means to determine the probability that a sample mean will be in a certain range, if we know the population mean and standard deviation.

But in practice, we will be starting with a single sample and trying to infer something about the population mean.

Difficult Question: Say we have a sample of size 20 from a normal distribution with **standard deviation** of 10, and the sample has a sample mean of 50.



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Is it possible that the *population* mean is actually 52?

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Does it seem plausible that the population mean is actually 52?

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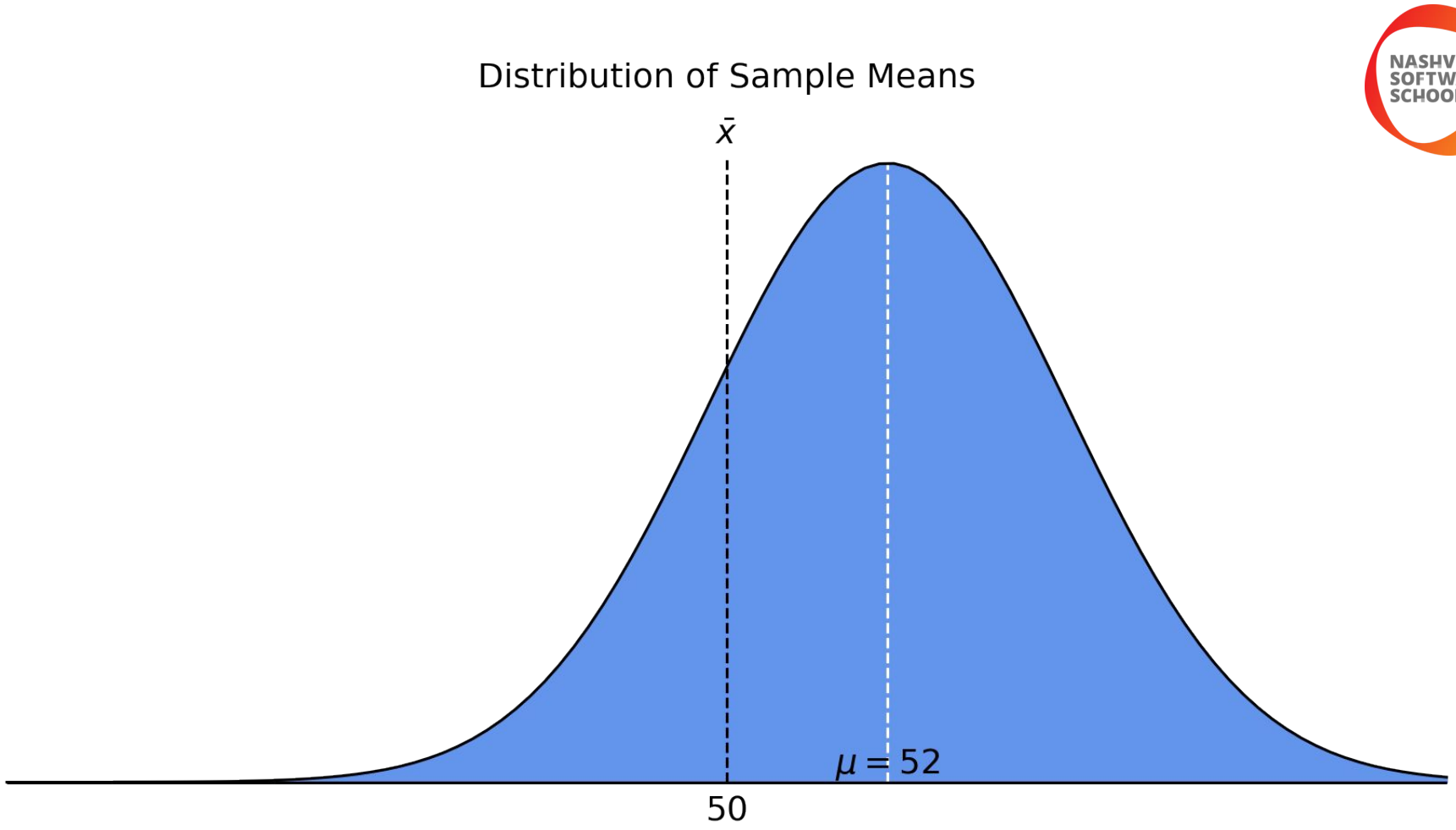
Does it seem plausible that the population mean is actually 52?

If the population mean is 52, what do we know about the distribution of sample means?


Distribution of Sample Means
$\bar{x}$
$\mu = 52$
50

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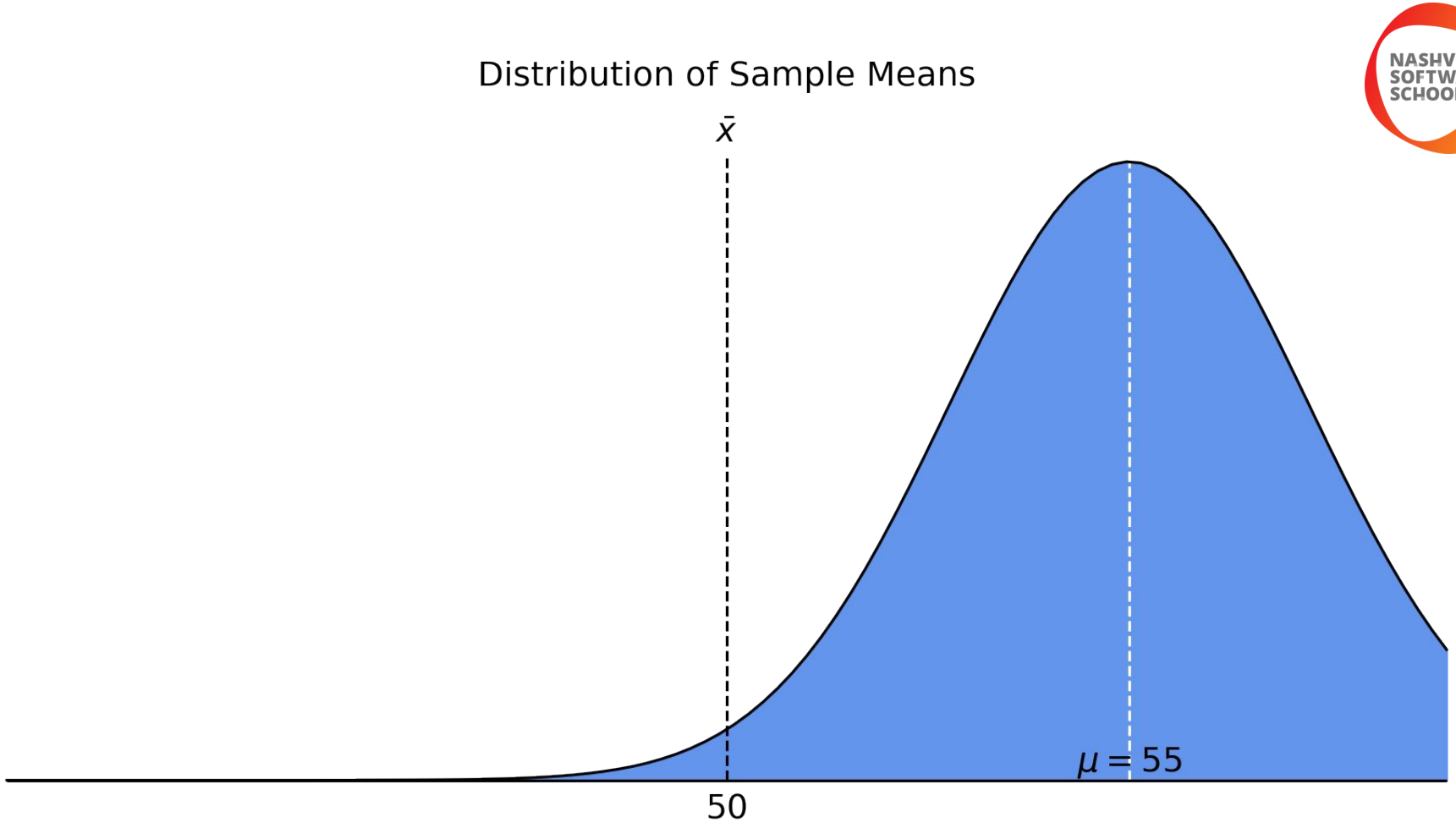
Distribution of Sample Means
$\bar{x}$
$\mu = 55$
50

Useful Fact:

If X follows a normal distribution with mean μ and standard deviation σ, then (X - μ) / σ is a follows a standard normal distribution (mean 0 and standard deviation 1).

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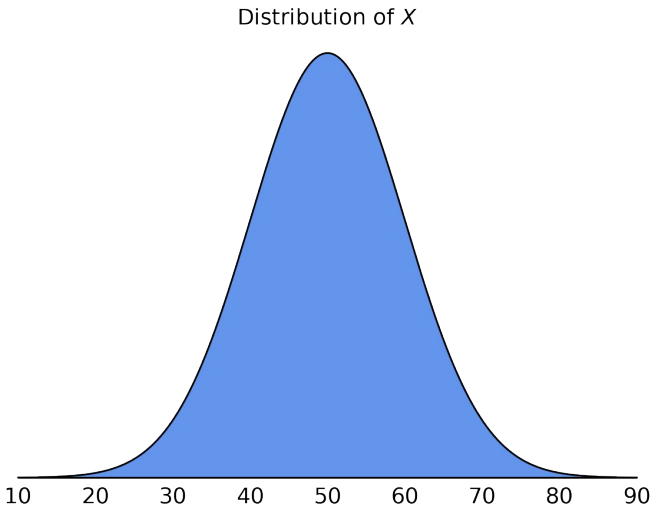
Example: *X* normally distributed with μ = 50 and σ = 10



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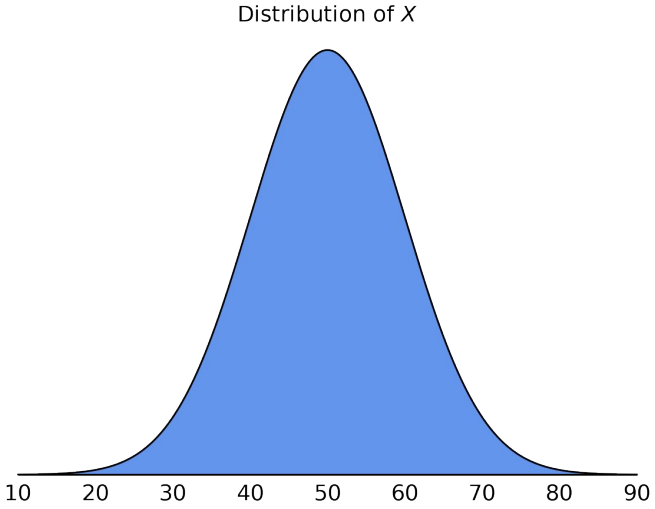

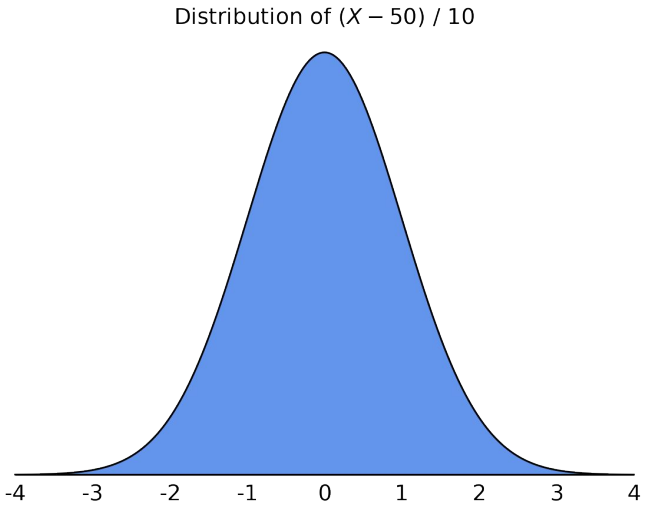

Recall - Cool Fact:

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If *X* is a normally-distributed variable with mean ***μ*** and standard deviation ***σ***, the distribution of sample means of size *n* is also normal with mean ***μ*** and standard deviation $\frac{\sigma}{\sqrt{n}}$

This means that $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ follows a standard normal distribution (mean 0 and standard deviation 1).

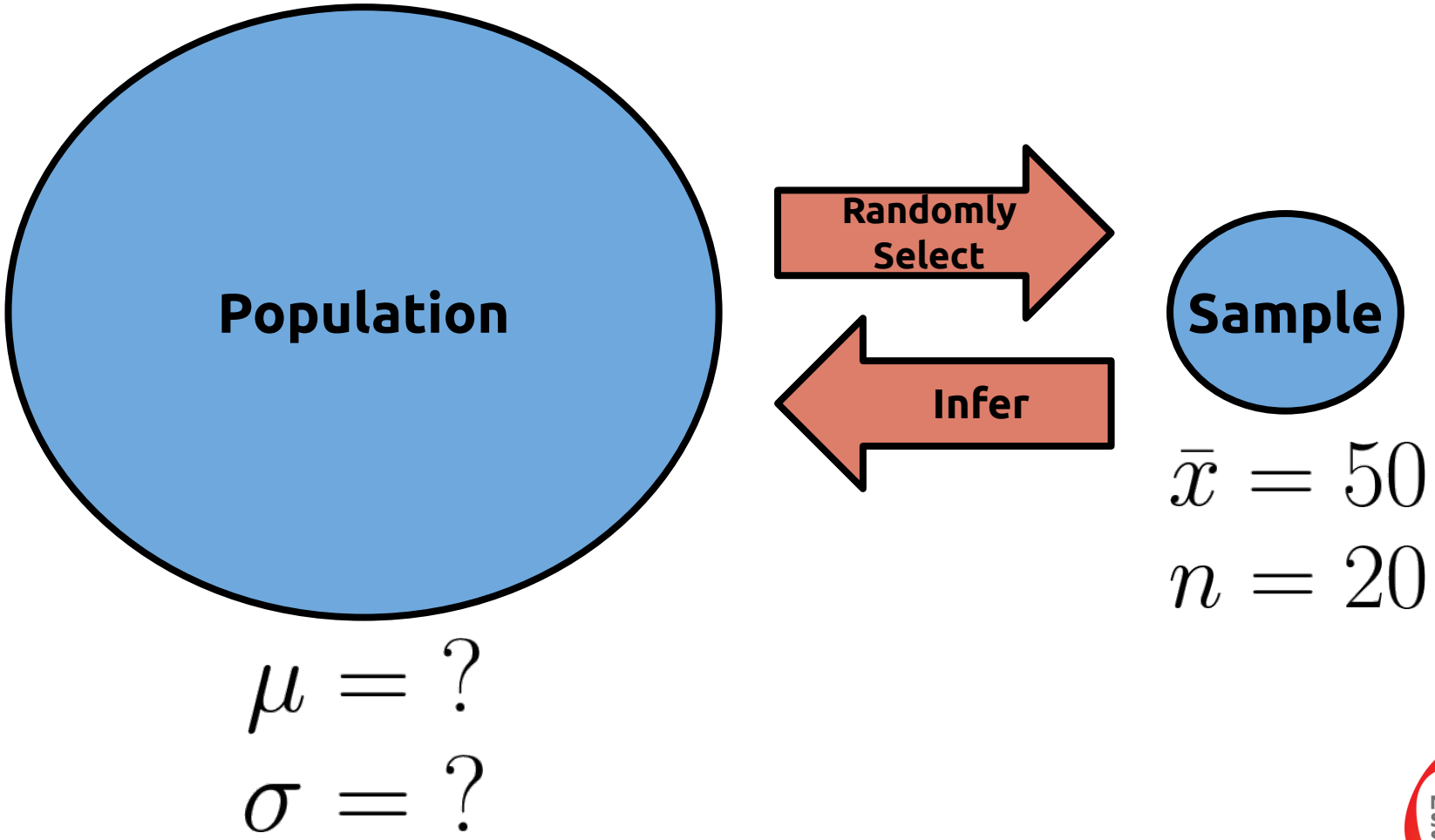
Why do we care about all of this?

Knowing something about the distribution of sample means, we can determine what a “plausible” range of values is for μ.

More specifically, knowing about the standard deviation of the sample means and the shape of its distribution (normal), helps us determine a plausible range of values.

The key thing was knowing the **standard error** - the standard deviation of the distribution of sample means - is equal to $\frac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation.

But wait - why would we know the population standard deviation but not the mean?



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If we don’t know the population standard deviation, how could we estimate it using just our single sample?

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$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$ follows a standard normal distribution

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By using the sample standard deviation, *s*.

$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

follows a standard normal distribution

$$\frac{\bar{x} - \mu}{s / \sqrt{n}}$$

follows a Student's *t*-distribution with n-1 degrees of freedom

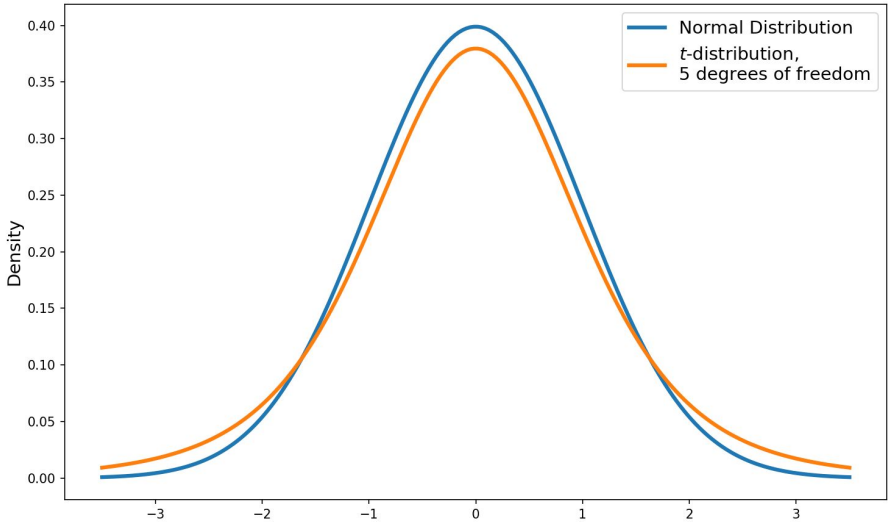
Estimation

The family of Student's *t*-distribution is named after statistician William Sealy Gosset, who published his research under the pseudonym "Student".

Gosset worked for the Guinness Brewery where he worked on determining the quality of raw materials. Gosset was interested in the problem of small samples, as he would sometimes have to draw inferences from samples with as few as 3 observations.

Estimation

The family of Student's *t*-distributions is a family of symmetric, bell-shaped distributions, which are similar to normal distributions, but have wider tails. That is, more extreme observations are more common. This family is parametrized by the number of degrees of freedom.



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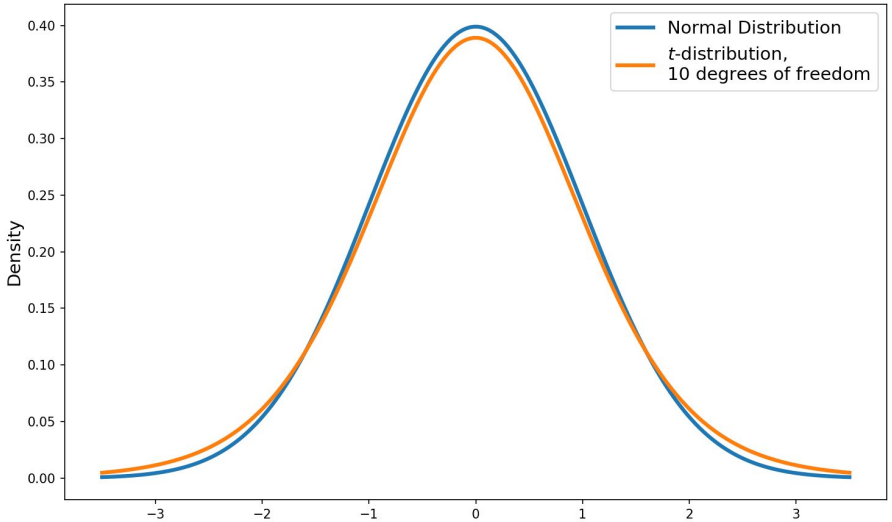

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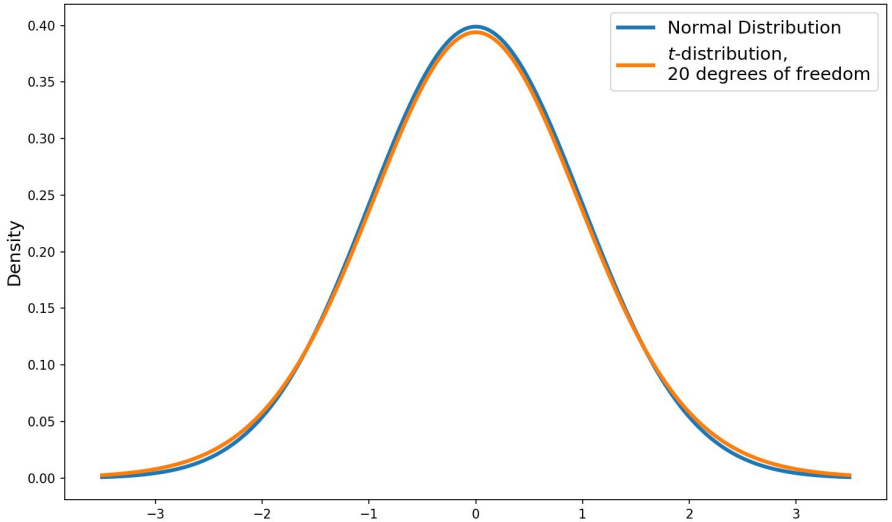

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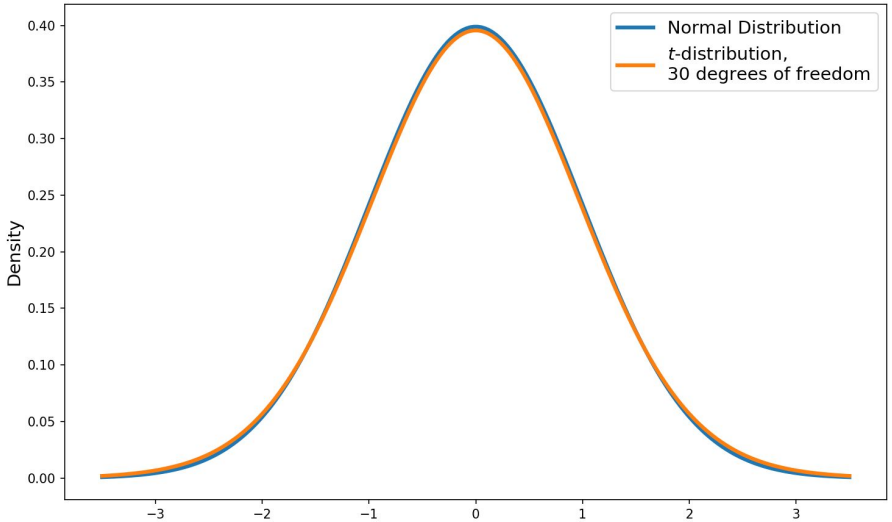

Confidence Intervals

Goal: For a given sample, construct an interval around the sample mean just wide enough so that for 95% of samples, the interval constructed in this way will contain the true population mean μ.

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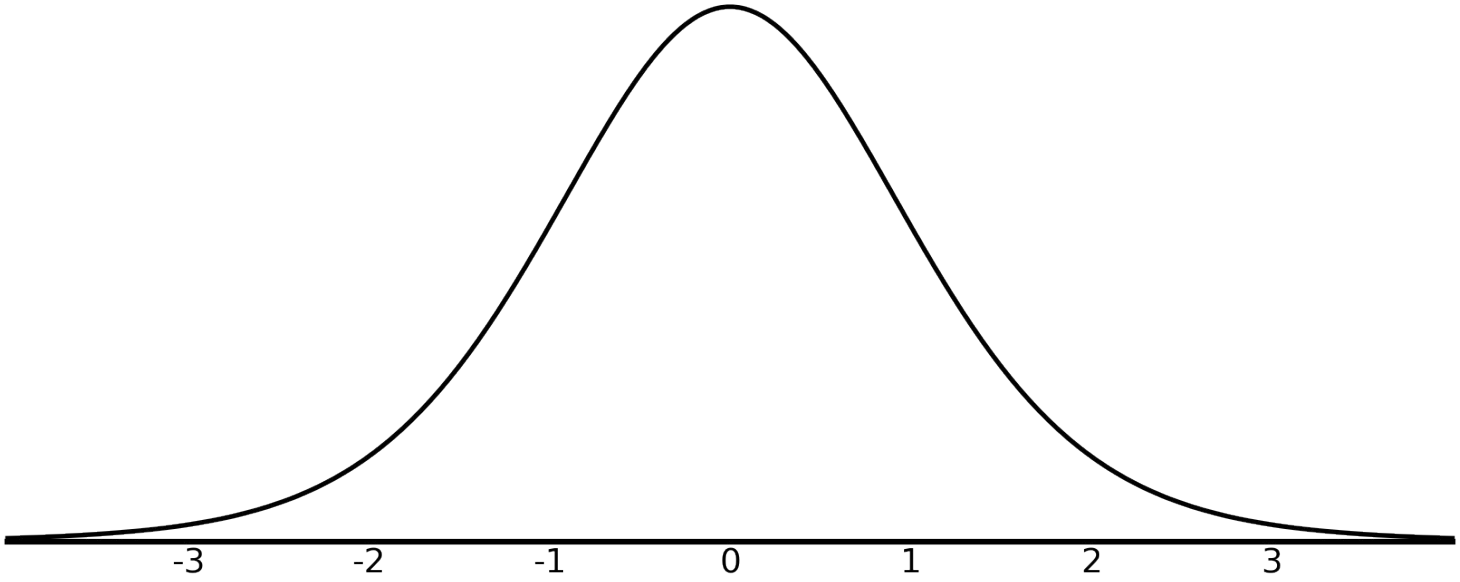
To do this, forget about our single sample temporarily and think about *all* possible samples.

Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$



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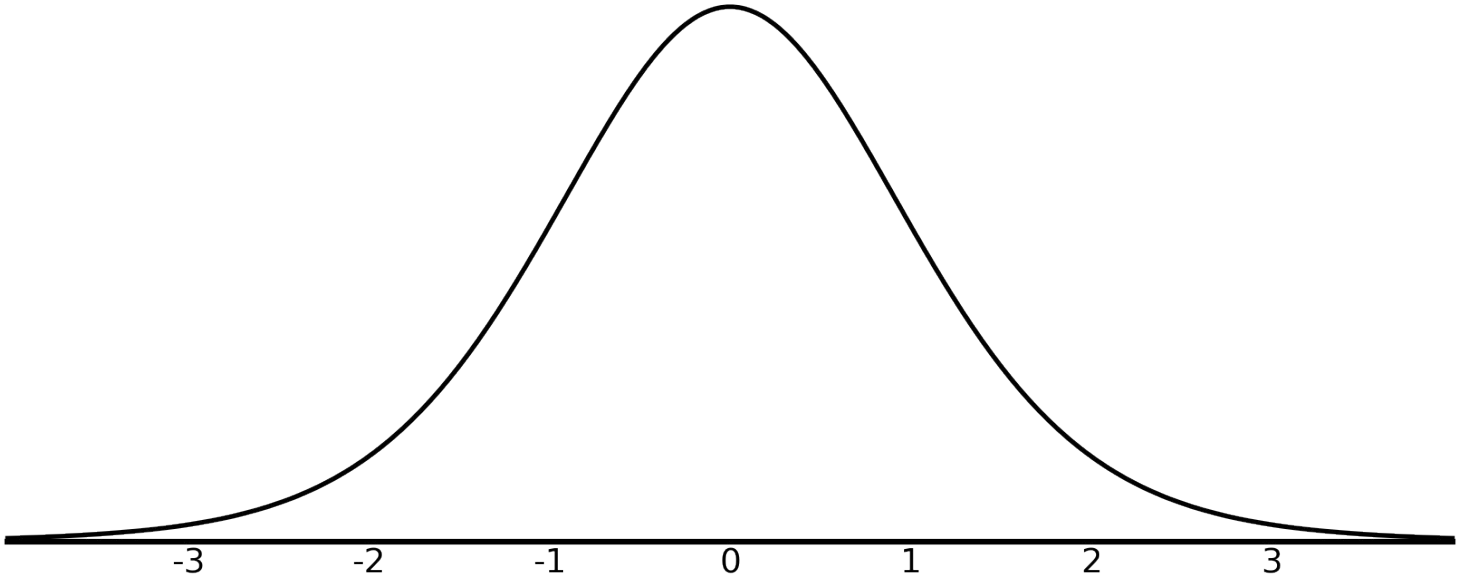
Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$



Student's *t*-distribution with n-1 degrees of freedom

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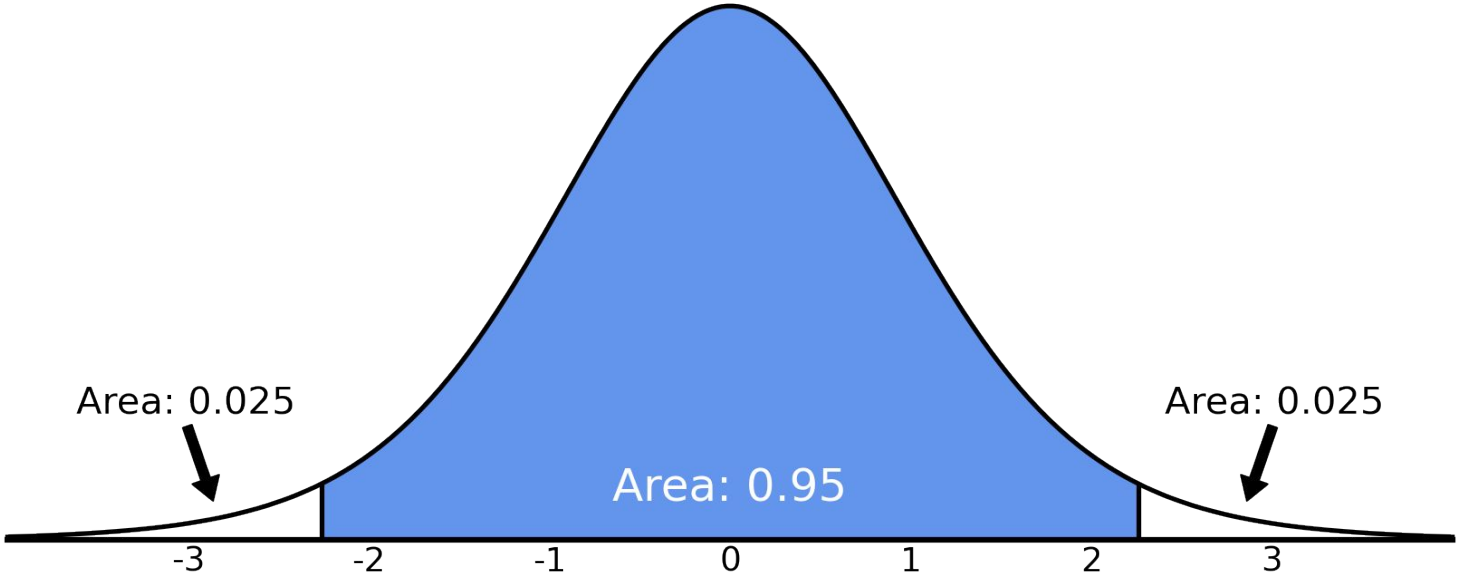
Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$



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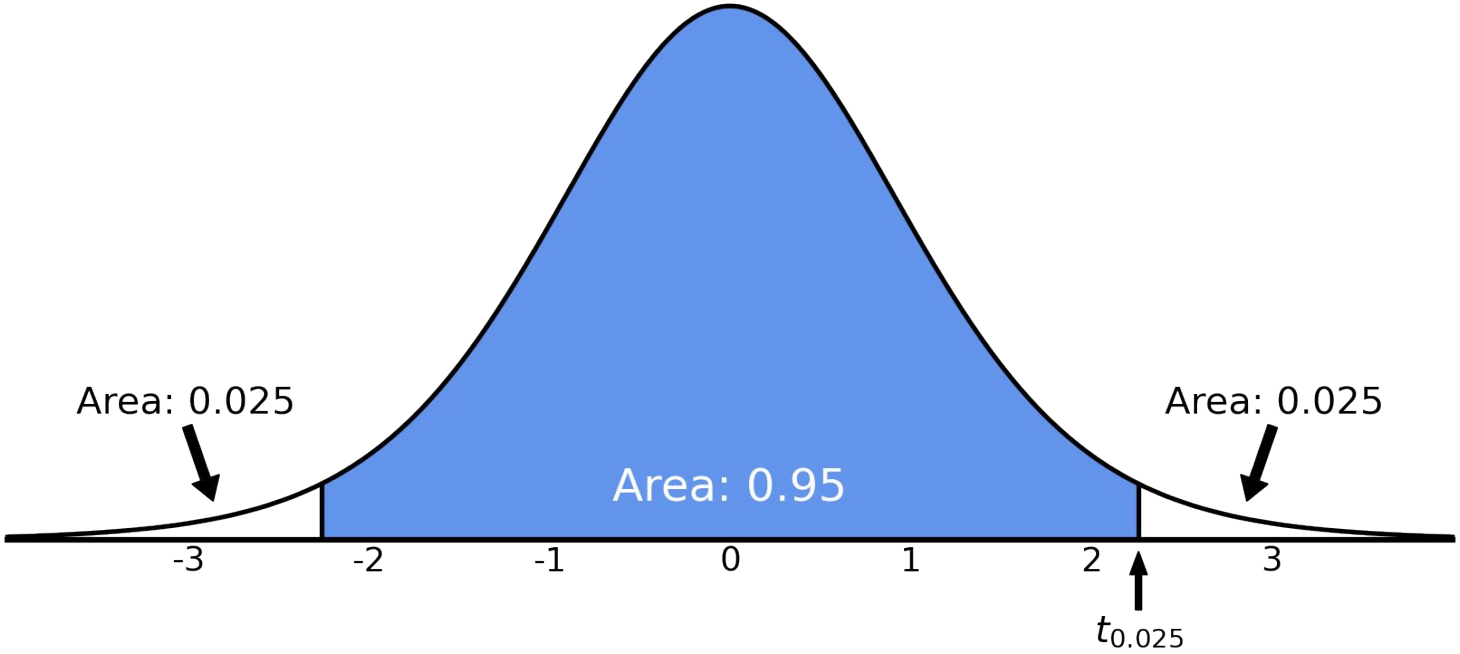
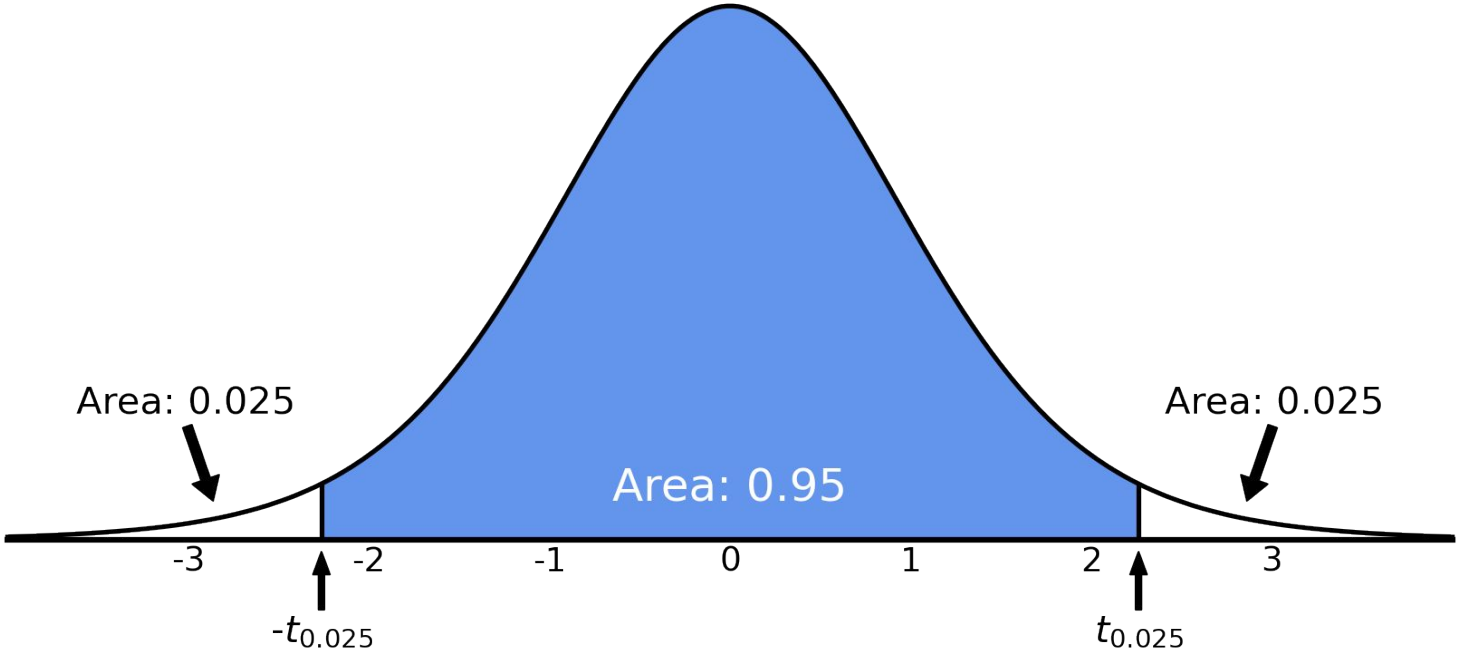


Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$



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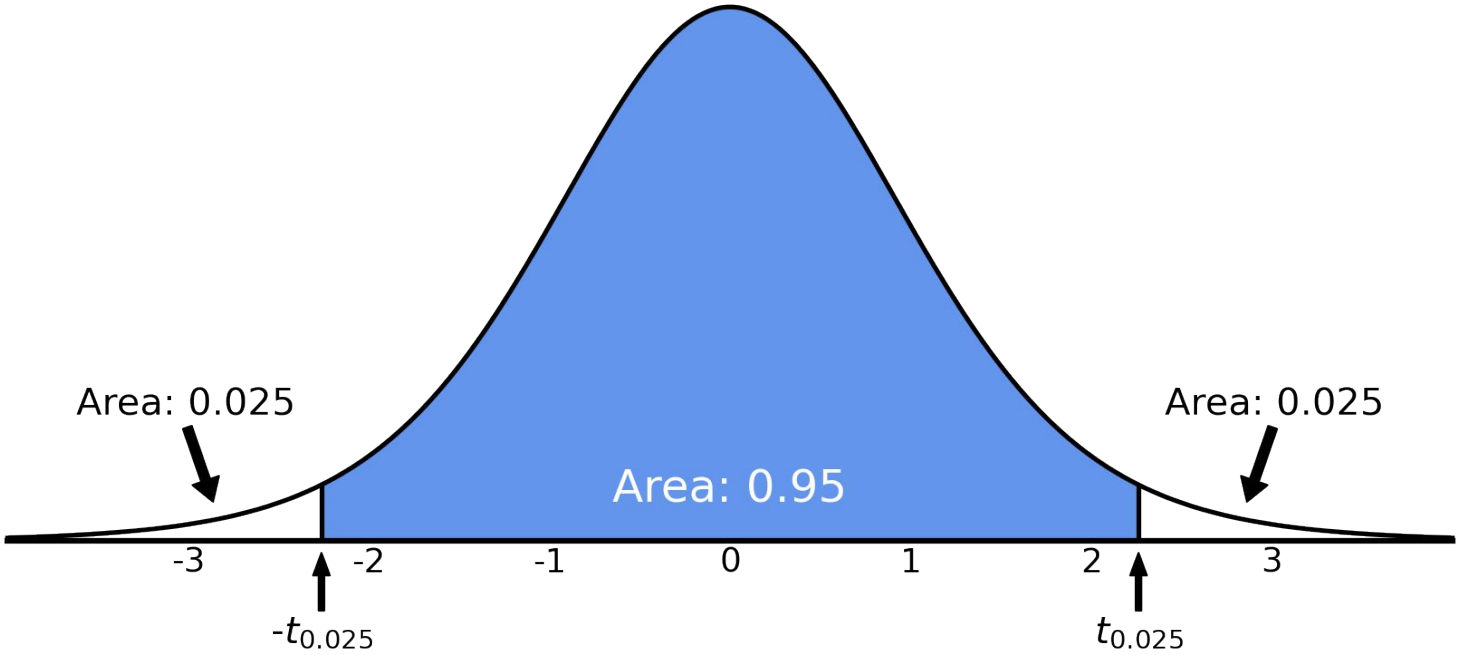
Distribution of $\frac{\bar{x} - \mu}{s/\sqrt{n}}$



For 95% of samples, $-t_{0.025} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < t_{0.025}$

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For 95% of samples, $-\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < -\mu < -\bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$

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For 95% of samples,

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For 95% of samples, $$\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Hence, we can use this recipe to build a 95% confidence interval:

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For 95% of samples,

$$\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Hence, we can use this recipe to build a 95% confidence interval:

$$\bar{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

For 95% of samples,

$$\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Hence, we can use this recipe to build a 95% confidence interval:

$$\bar{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Point Estimate

For 95% of samples,

$$\bar{x} - t_{0.025} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Hence, we can use this recipe to build a 95% confidence interval:

$$\bar{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Point Estimate

Margin of Error

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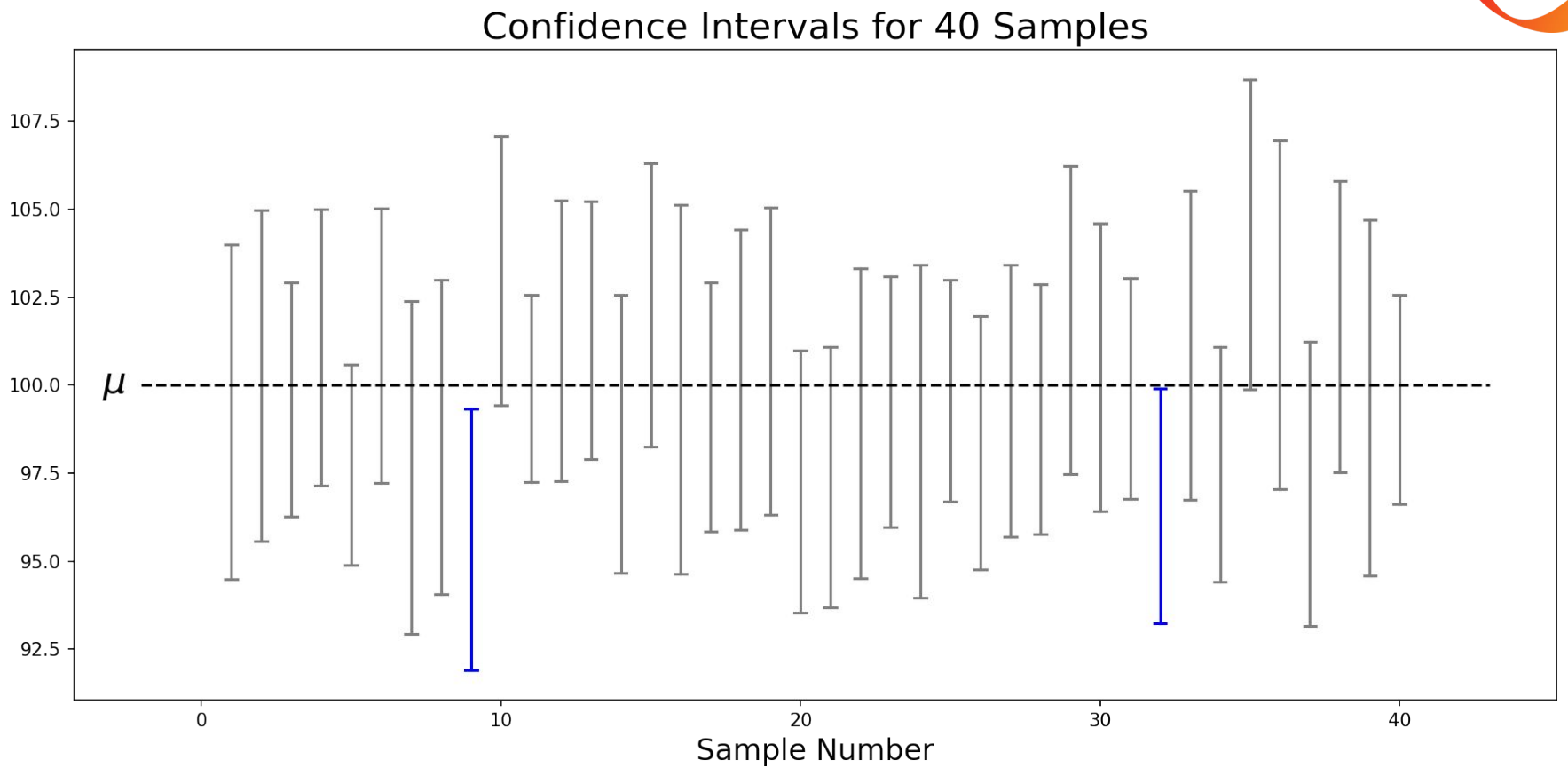
Confidence Intervals - Simulation



Estimation

The general recipe for a confidence interval is

$$\text{point estimate} \pm \text{margin of error}$$

The margin of error depends on the confidence level:

- Higher confidence = wider margin of error
- Lower confidence = smaller margin of error

Estimation

The general recipe for a confidence interval is

$$\text{point estimate} \pm \text{margin of error}$$

Let’s say we want to build a 95% confidence interval. The idea is that we’ll build it in such a way that the confidence intervals do not contain the true population parameter for only 5% of possible samples.

Estimation

Big Idea: If we want a 95% confidence interval for the mean, we just need to find the distance $t_{0.025}$ from the center of the *t* distribution with *n-1* degrees of freedom to the point where the area to the right is 0.025 (that is, $t_{0.025}$ is the 97.5th percentile).

Then, multiply by the standard deviation of the sampling distribution, $\frac{s}{\sqrt{n}}$

$$\bar{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

Point Estimate

Margin of Error

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Estimation - Proportions

What does the sampling distribution for a proportion look like?

That is, if we take a sample from a population and calculate the proportion of observations in our sample that meet a particular condition, what does the distribution of the possible sample proportions look like?

Eg. When we conduct a poll, we might find the proportion of respondents planning to vote for a particular candidate.

Estimation - Proportions

Say we have a sample size of *n* and the the true population proportion is *p*.

Let $\hat{p}$ be the sample proportion.

We know that $n \cdot \hat{p}$ follows a binomial distribution with *n* trials and probability of success *p*.

This distribution has mean $n \cdot p$ and standard deviation $\sqrt{n \cdot p \cdot (1-p)}$

Fun Fact: As *n* increases, the normal distribution approximates the binomial distribution.

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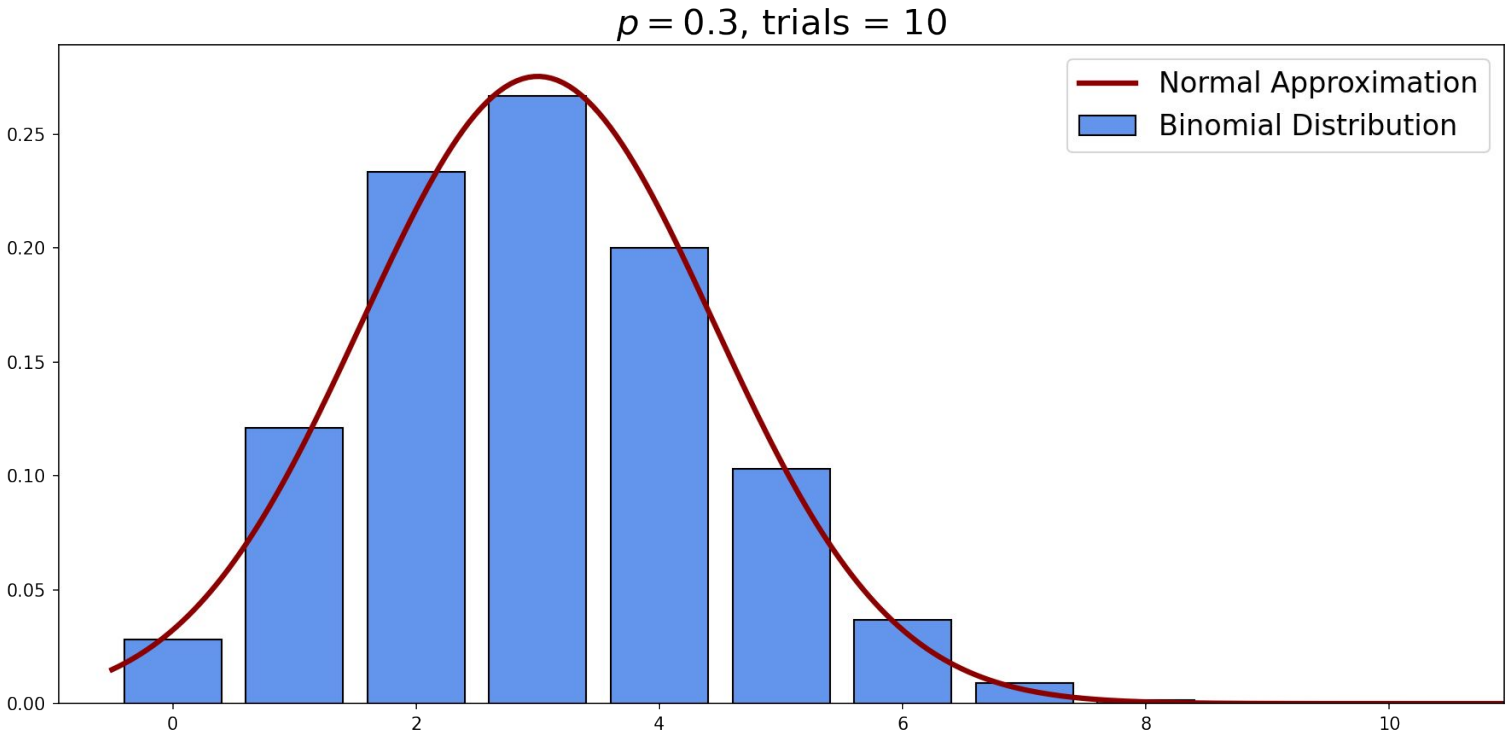

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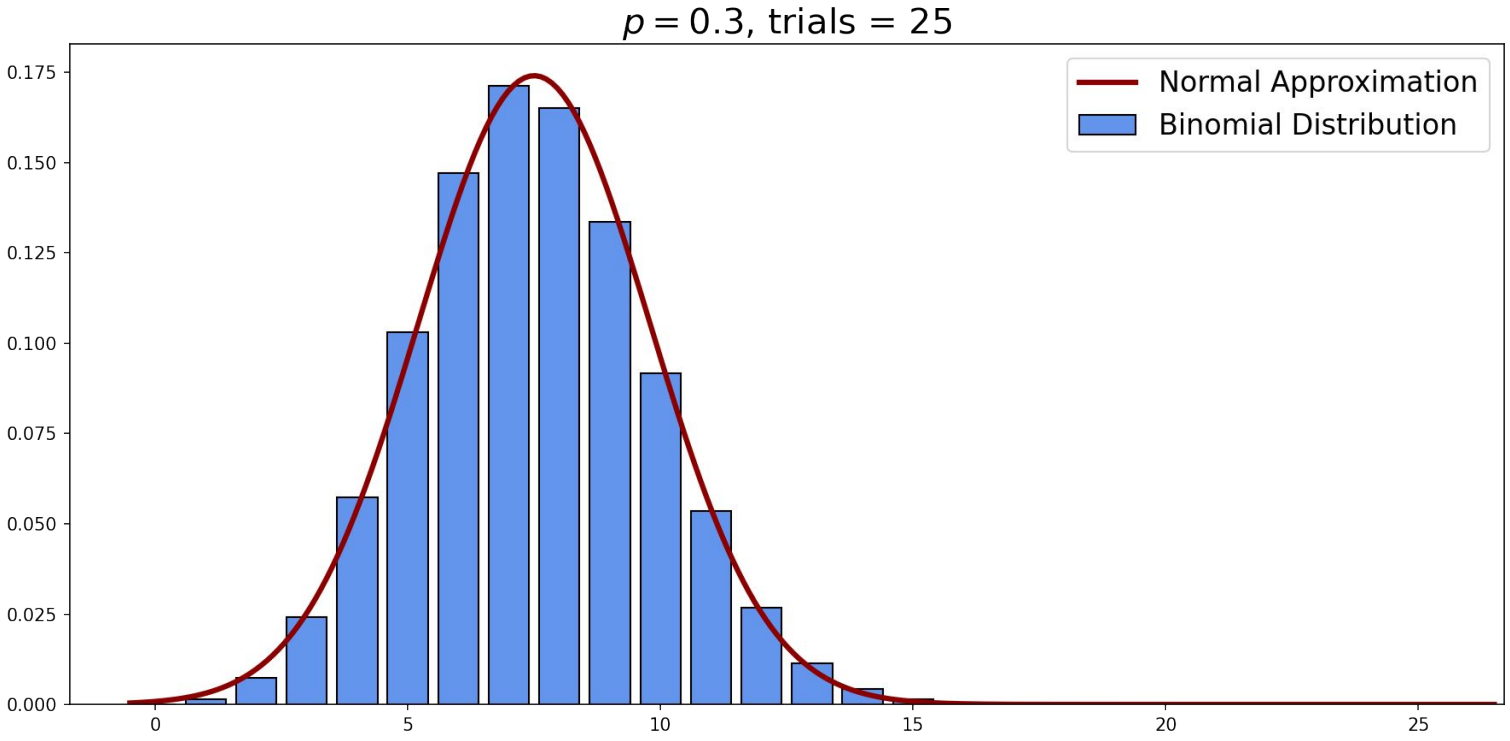
Fun Fact: As *n* increases, the normal distribution approximates the binomial distribution.



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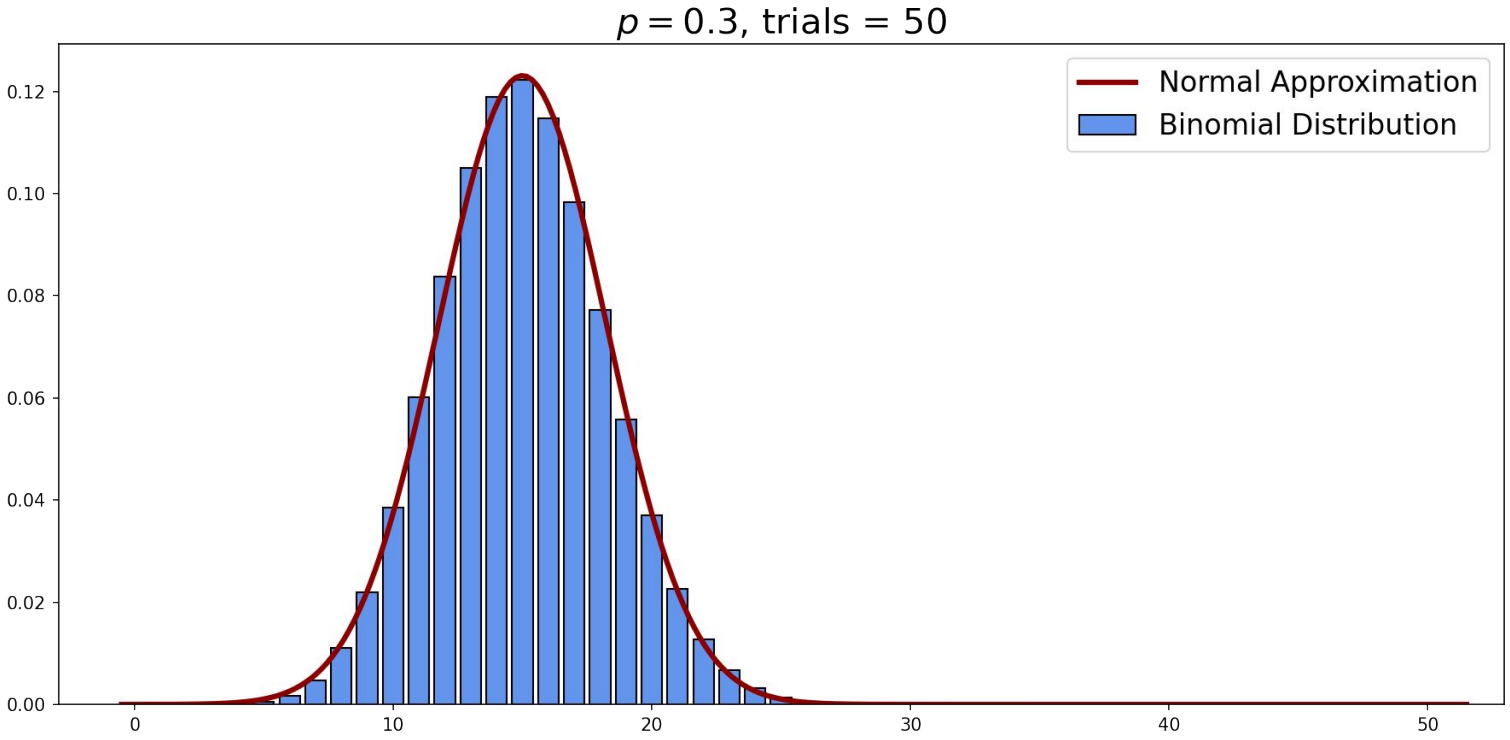
Fun Fact: As *n* increases, the normal distribution approximates the binomial distribution.



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Estimation - Proportions

If *np ≥ 10* and *n(1-p) ≥ 10*, the distribution of $n \cdot \hat{p}$ is approximately normal with mean *np* and standard deviation $\sqrt{n \cdot p \cdot (1-p)}$

Divide by *n* and we can see that the distribution of $\hat{p}$ is approximately normal with mean *p* and standard deviation $\sqrt{\frac{p(1-p)}{n}}$

Estimation - Proportions

Recall that we can build a confidence interval if we know about the variability of the distribution of test statistics.

And now we know that for $\hat{p}$, this has a standard deviation of $\sqrt{\frac{p(1-p)}{n}}$

But, there is a problem - it relies on using the population proportion - the quantity we are trying to approximate.

Estimation - Proportions

Just sub in the sample proportion!

$$\sqrt{\frac{p(1-p)}{n}} \quad \Rightarrow \quad \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Estimation - Proportions

Confidence Interval for the Proportion:

$$\hat{p} \pm z_\alpha \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Estimation - Proportions

Confidence Interval for the Proportion:

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Point Estimate

Estimation - Proportions

Confidence Interval for the Proportion:

$$\hat{p} \pm z_{\alpha} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Point Estimate

Critical Value (from the standard normal distribution)