

# Fourier analysis

Mainly two type signals are present in the world such as analog or continuous signal and digital or discrete signal. Both signals have two forms such as periodic and aperiodic. Fourier analysis is an excellent method to analysis every type of signal either continuous time or discrete time signal.

Fourier analysis for following case such as:

1. Continuous Time Periodic Signal
2. Discrete Time Periodic Signal
3. Continuous Time Aperiodic Signal
4. Discrete Time Aperiodic Signal

## 1. FOURIER SERIES FOR CONTINUOUS TIME PERIODIC

### 1.1 Fourier series for Continuous Time Periodic Signal in Trigonometric Form

Fourier given the concept all real periodic signal consists of dc signal which frequency is zero and one to infinity number of sinusoidal signal. Each sinusoidal wave has different amplitude and different frequency. In real world, sinusoidal signal means sine wave signal and cosine wave signal. This statement can be represented by mathematical form which is called Fourier series as given below.

$$f(t) = a_0 + (a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \dots \dots \dots \infty) + (b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \dots \dots \dots \infty)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \dots \dots \dots (1)$$

$$\text{Or } f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \dots \dots \dots (2)$$

*Calculation to find out the value of  $a_0$ :*

We know that from equation  $-(2)$ ,

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

$$\text{Or } \int_0^{2\pi} f(\theta) d\theta = \int_0^{2\pi} a_0 d\theta + \sum_{n=1}^{+\infty} (a_n \int_0^{2\pi} \cos n\theta d\theta + b_n \int_0^{2\pi} \sin n\theta d\theta)$$

$$\text{Or } \int_0^{2\pi} f(\theta) d\theta = \int_0^{2\pi} a_0 d\theta + 0 + 0$$

$$\text{Or } \int_0^{2\pi} f(\theta) d\theta = a_0 2\pi$$

$$\text{Or, } a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \text{ ----- (3)}$$

*Calculation to find out the value of  $a_n$ :*

We know that from initial (first) equation

$$f(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \text{-----} + a_n \cos n\theta \\ + b_1 \sin \theta + b_2 \sin 2\theta + \text{-----} + b_n \sin n\theta$$

$$\text{Or } \int_0^{2\pi} f(\theta) \cos n\theta d\theta = \int_0^{2\pi} a_0 \cos n\theta d\theta + a_1 \int_0^{2\pi} \cos \theta \cos n\theta d\theta \\ + a_2 \int_0^{2\pi} \cos 2\theta \cos n\theta d\theta + \text{-----} + \\ a_n \int_0^{2\pi} \cos n\theta \cos n\theta d\theta \\ + b_1 \int_0^{2\pi} \cos n\theta \sin \theta d\theta + b_2 \int_0^{2\pi} \cos n\theta \sin 2\theta d\theta + \text{-----} + \\ b_n \int_0^{2\pi} \cos n\theta \sin n\theta d\theta \text{ ----- (4)}$$

(We know that from geometrical function in **Appendix –A**,  $\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = 0$  [if  $m \neq n$ ] and  $\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = \pi$  [if  $m = n$ ]  $\int_0^{2\pi} \sin n\theta \cos m\theta d\theta = 0$  [either  $m \neq n$  nor  $m = n$ ]) Now apply the above this geometrical function in equation-(4) then we get

$$\text{Or } \int_0^{2\pi} f(\theta) \cos n\theta d\theta = 0 + 0 + 0 + \text{-----} - a_n \int_0^{2\pi} \cos n\theta \cos n\theta d\theta$$

$$+ 0 + 0 + 0 + \dots +$$

$$\text{Or } \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n \int_0^{2\pi} \cos n\theta \cos n\theta \, d\theta$$

$$\text{Or } \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n \pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \quad \text{-----} (5)$$

*Calculation to find out the value of  $b_n$ :*

We know that from initial (first) equation

$$f(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots + a_n \cos n\theta \\ + b_1 \sin \theta + b_2 \sin 2\theta + \dots + b_n \sin n\theta$$

Now we are multiply both side of above equation by  $\sin n\theta$  and apply integration on both side. So we get following equation

$$\text{Or } \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = \int_0^{2\pi} a_0 \sin n\theta \, d\theta + a_1 \int_0^{2\pi} \cos \theta \sin n\theta \, d\theta \\ + a_2 \int_0^{2\pi} \cos 2\theta \sin n\theta \, d\theta + \dots + a_n \int_0^{2\pi} \cos n\theta \sin n\theta \, d\theta \\ + b_1 \int_0^{2\pi} \cos \theta \sin n\theta \, d\theta + b_2 \int_0^{2\pi} \sin 2\theta \sin n\theta \, d\theta + \dots + \\ b_n \int_0^{2\pi} \sin n\theta \sin n\theta \, d\theta + \dots \quad (6)$$

(We know that from geometrical function  $\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0$  [if  $m \neq n$ ] and  $\int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \pi$  [if  $m = n$ ]  $\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0$  [either  $m \neq n$  nor  $m = n$ ]). Now apply the above this geometrical function in equation-(6) then we get

$$\text{Or } \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = 0 + 0 + 0 + 0 + 0 + 0 + \dots + b_n \int_0^{2\pi} \sin^2 n\theta \, d\theta$$

$$\text{Or } \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = b_n \pi$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \quad \text{-----} (7)$$

### Summary 1.1(A): Fourier series for Continuous Time Periodic Signal

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

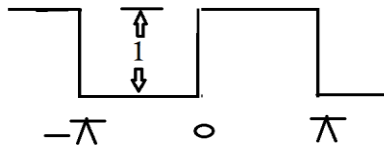
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta$$

Example-01: Find the mathematical equation for given signal using Fourier series as trigonometric form.

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$



**Solution:**

We know that,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{2\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} 1 dx \right]$$

$$= \frac{1}{2}$$

$$\text{Again } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

[sin nπ=0 for n=1,2,.....] [cos nπ=-1 for n=1,3,5,..... and =1 for n=2,4,.....]

$$= \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \cos nx dx + \int_0^{\pi} 1 \cos nx dx \right]$$

$$= 0$$

$$\begin{aligned}
 \text{Aging } b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \sin nx \, dx + \int_0^{\pi} 1 \sin nx \, dx \right] \\
 &= \frac{1}{\pi} \left[ -\frac{\cos nx}{n} \right]_0^{\pi} \\
 &= \frac{1}{n\pi} [1 - \cos n\pi]
 \end{aligned}$$

$$= \begin{cases} 0 & \text{for even } n \\ \frac{2}{n\pi} & \text{for odd } n \end{cases}$$

Again we know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Putting the values of  $a_0$ ,  $a_n$  and  $b_n$  in above equation and we get below expression

$$\begin{aligned}
 f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} 0 \cos nx + \sum_{n=1(\text{odd})}^{\infty} \frac{2}{n\pi} \sin nx = \frac{1}{2} + 0 + \sum_{n=1(\text{odd})}^{\infty} \frac{2}{n\pi} \sin nx \\
 f(x) &= \frac{1}{2} + \frac{2}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots + \infty \right)
 \end{aligned}$$

The above this equation represent mathematical equation for example 01

## 1.2 Complex Form Fourier series for Continuous Time Periodic Signal

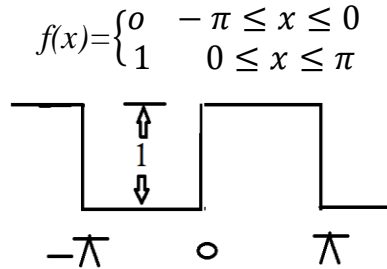
### Summary 1.1(B): Fourier series for Continuous Time Periodic Signal

$$f(\theta) = \sum_{n=-\infty}^{\infty} C_n e^{jn\theta} \quad \text{and} \quad C_n = 1/2\pi \int_0^{2\pi} f(\theta) e^{-jn\theta} d\theta$$

$$\text{Or } f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi F_0 t} \quad \text{and} \quad C_n = 1/T \int_0^T f(t) e^{-jn2\pi F_0 t} dt$$

$$\text{Or } f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk2\pi F_0 t} \quad \text{and} \quad C_k = 1/T \int_0^T f(t) e^{-jk2\pi F_0 t} dt$$

Example-02: Find the mathematical equation for given signal using Fourier series as complex form.



**Solution:**

We know that complex form Fourier series for continuous time periodic signal is given by following expression

$$f(x) = \sum_{n=-\infty}^{n=\infty} C_n e^{inx} \quad [x=\theta] \quad [\text{See summary 1.1(B), page-9}]$$

$$\text{Where } C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$\Rightarrow C_n = \frac{1}{2\pi} \left[ \int_{-\pi}^0 f(x) e^{-inx} dx + \int_0^{\pi} f(x) e^{-inx} dx \right]$$

$$\Rightarrow C_n = \frac{1}{2\pi} \left[ \int_{-\pi}^0 0 e^{-inx} dx + \int_0^{\pi} 1 e^{-inx} dx \right] \quad (*)$$

$$C_n = -\frac{1}{2in\pi} (e^{in\pi} - 1)$$

$$\text{If } n \text{ is odd then } C_n = \frac{1}{in\pi}$$

$$\text{If } n \text{ is even then } C_n = 0 \quad [n \neq 0]$$

$$(*) \text{ If } n=0 \text{ then } C_n = \frac{1}{2\pi} \int_0^{\pi} 1 dx = \frac{1}{2}$$

Putting the value of  $C_n$  in equation (12) and we get

$$\begin{aligned} \Rightarrow f(x) &= \frac{1}{2} + \frac{1}{i\pi} \left( \frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \frac{e^{i5x}}{5} + \dots \right) \\ &+ \frac{1}{i\pi} \left( \frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \frac{e^{-i5x}}{-5} + \dots \right) \end{aligned}$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{1}{1} \left( \frac{e^{ix} - e^{-ix}}{2i} \right) + \frac{1}{3} \left( \frac{e^{3ix} - e^{-3ix}}{2i} \right) + \right. \\ \left. \frac{1}{5} \left( \frac{e^{5ix} - e^{-5ix}}{2i} \right) + \dots \right]$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right)$$

The above this equation represent mathematical equation for example-02

NOTE: Example-01 and Example-02 are same problem. Example-01 solved by using of trigonometrical form Fourier series and Example-02 solved by using of complex form Fourier series but we get same result. So it is clear that the trigonometrically form Fourier series and complex form Fourier series is same case but only representation form is different.