2.3.2 Resolution of a Discrete-Time Signal into Impulses

Suppose we have an arbitrary signal $x(n)$ that we wish to resolve into a sum of unit sample sequences. To utilize the notation established in the preceding section, we select the elementary signals $x_k(n)$ to be

$$x_k(n) = \delta(n-k) \tag{2.3.7}$$

where k represents the delay of the unit sample sequence. To handle an arbitrary signal $x(n)$ that may have nonzero values over an infinite duration, the set of unit impulses must also be infinite, to encompass the infinite number of delays.

Now suppose that we multiply the two sequences $x(n)$ and $\delta(n-k)$. Since $\delta(n-k)$ is zero everywhere except at $n = k$, where its value is unity, the result of this multiplication is another sequence that is zero everywhere except at $n = k$, where its value is $x(k)$, as illustrated in Fig. 2.22. Thus

$$x(n)\delta(n-k) = x(k)\delta(n-k) \tag{2.3.8}$$

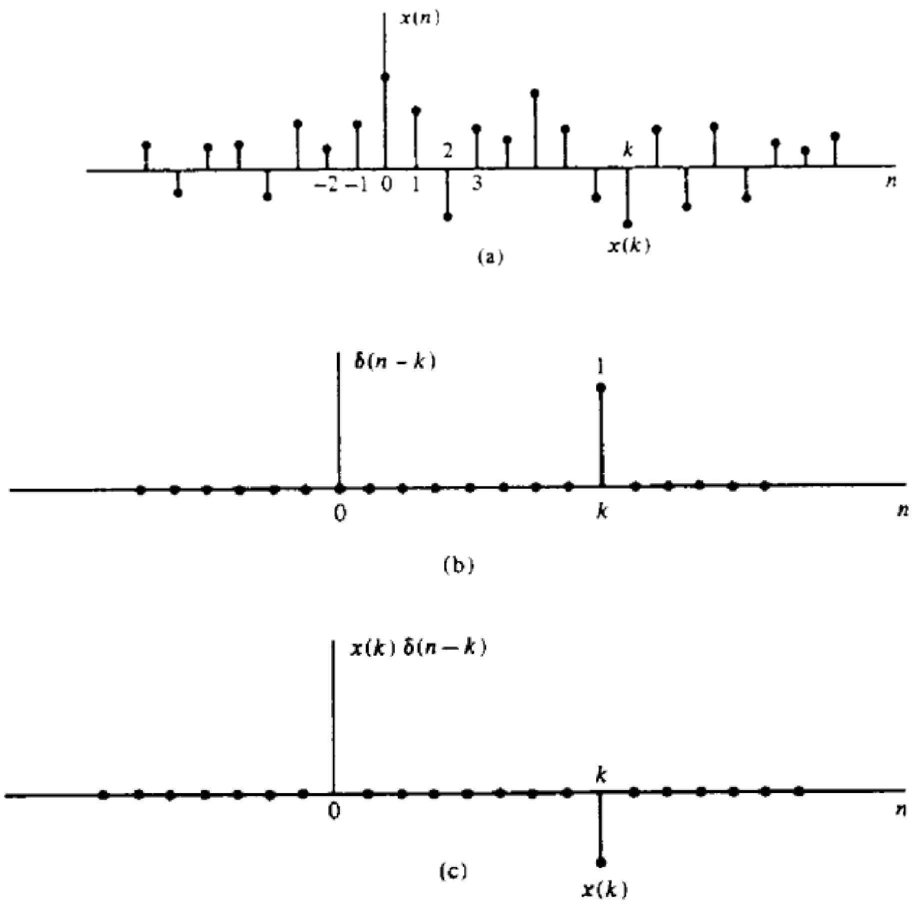

Figure 2.22 Multiplication of a signal $x(n)$ with a shifted unit sample sequence.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Consider the special case of a finite-duration sequence given as

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences.

Solution Since the sequence $x(n)$ is nonzero for the time instants $n = -1, 0, 2$, we need three impulses at delays $k = -1, 0, 2$. Following (2.3.10) we find that

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Finally, if the input is the arbitrary signal $x(n)$ that is expressed as a sum of weighted impulses, that is,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{2.3.13}$$

then the response of the system to $x(n)$ is the corresponding sum of weighted outputs, that is,

$$\begin{aligned} y(n) &= \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \qquad (2.3.14) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n,k) \end{aligned}$$

We note that (2.3.14) is an expression for the response of a linear system to any arbitrary input sequence $x(n)$. This expression is a function of both $x(n)$ and the responses $h(n, k)$ of the system to the unit impulses $\delta(n-k)$ for $-\infty < k < \infty$. In deriving (2.3.14) we used the linearity property of the system but not its time-invariance property. Thus the expression in (2.3.14) applies to any relaxed linear (time-variant) system.

If, in addition, the system is time invariant, the formula in (2.3.14) simplifies considerably. In fact, if the response of the LTI system to the unit sample sequence $\delta(n)$ is denoted as $h(n)$, that is,

$$h(n) \equiv \mathcal{T}[\delta(n)] \tag{2.3.15}$$

then by the time-invariance property, the response of the system to the delayed unit sample sequence $\delta(n-k)$ is

$$h(n-k) = \mathcal{T}[\delta(n-k)] \tag{2.3.16}$$

Consequently, the formula in (2.3.14) reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{2.3.17}$$

The formula in (2.3.17) that gives the response $y(n)$ of the LTI system as a function of the input signal $x(n)$ and the unit sample (impulse) response $h(n)$ is called a *convolution sum*. We say that the input $x(n)$ is convolved with the impulse response $h(n)$ to yield the output $y(n)$. We shall now explain the procedure for computing the response $y(n)$, both mathematically and graphically, given the input $x(n)$ and the impulse response $h(n)$ of the system.

Suppose that we wish to compute the output of the system at some time instant, say $n = n_0$. According to (2.3.17), the response at $n = n_0$ is given as

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0-k) \tag{2.3.18}$$

folded sequence is then shifted by n_0 to yield $h(n_0-k)$. To summarize, the process of computing the convolution between $x(k)$ and $h(k)$ involves the following four steps.

1. *Folding*. Fold $h(k)$ about $k=0$ to obtain $h(-k)$.
2. *Shifting*. Shift $h(-k)$ by n_0 to the right (left) if n_0 is positive (negative), to obtain $h(n_0-k)$.
3. *Multiplication*. Multiply $x(k)$ by $h(n_0-k)$ to obtain the product sequence $v_{n_0}(k) \equiv x(k)h(n_0-k)$.
4. *Summation*. Sum all the values of the product sequence $v_{n_0}(k)$ to obtain the value of the output at time $n=n_0$.

Example 2.3.2

The impulse response of a linear time-invariant system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\} \tag{2.3.19}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\} \tag{2.3.20}$$

The first step in the computation of the convolution sum is to fold $h(k)$. The folded sequence $h(-k)$ is illustrated in Fig. 2.23b. Now we can compute the output at $n=0$, according to (2.3.17), which is

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) \tag{2.3.21}$$

Since the shift $n=0$, we use $h(-k)$ directly without shifting it. The product sequence

$$v_0(k) \equiv x(k)h(-k) \tag{2.3.22}$$

is also shown in Fig. 2.23b. Finally, the sum of all the terms in the product sequence yields

$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

We continue the computation by evaluating the response of the system at $n = 1$. According to (2.3.17),

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) \tag{2.3.23}$$

The sequence $h(1-k)$ is simply the folded sequence $h(-k)$ shifted to the right by one unit in time. This sequence is illustrated in Fig. 2.23c. The product sequence

$$v_1(k) = x(k)h(1-k) \tag{2.3.24}$$

is also illustrated in Fig. 2.23c. Finally, the sum of all the values in the product sequence yields

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

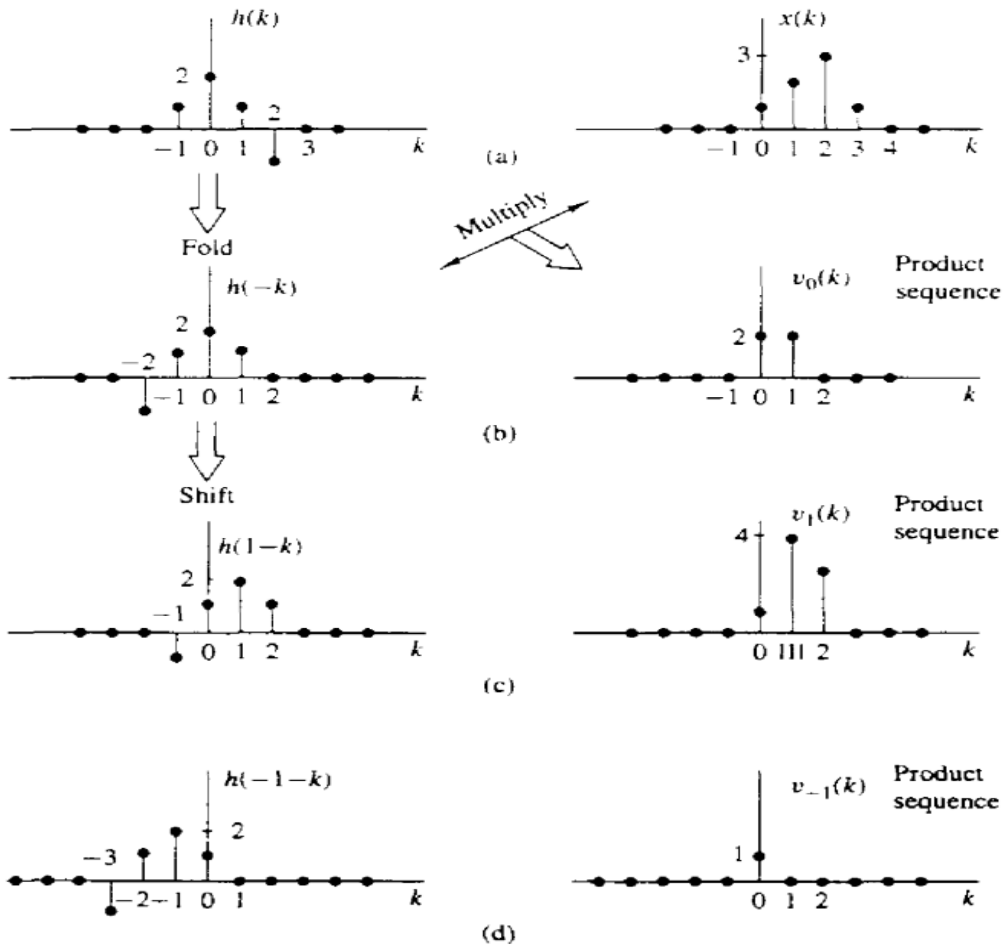

Figure 2.23 Graphical computation of convolution.

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

■ LINEAR CONVOLUTION USING CROSS-TABLE METHOD

Let us consider the convolution,

$$y(n) = x(n) * h(n)$$

where $x(n) = \{x_1(n), x_2(n), x_3(n), \ldots\}$ is the input signal and $h(n) = \{h_1(n), h_2(n), h_3(n), \ldots,\}$ is the impulse response.

The convolution of $x(n)$ and $h(n)$ can be performed as

	$x_1(n)$	$x_2(n)$	$x_3(n)$
$h_1(n)$	$x_1(n)\, h_1(n)$	$x_2(n)\, h_1(n)$	$x_3(n)\, h_1(n)$
$h_2(n)$	$x_1(n)\, h_2(n)$	$x_2(n)\, h_2(n)$	$x_3(n)\, h_2(n)$
$h_3(n)$	$x_1(n)\, h_3(n)$	$x_2(n)\, h_3(n)$	$x_3(n)\, h_3(n)$

Fig. 3.7 Convolution of x(n) and h(n)

Procedure

1. Multiply each row element with column element
2. Draw a diagonal line as shown in the Fig. 3.7.
3. Add the diagonal terms

   $y(n) = \{x_1(n)\, h_1(n), \{(x_1(n)\, h_2(n) + x_2(n)\ h_1(n)\}, \{(x_1(n)\, h_3(n) + x_2(n)\, h_2(n) + x_3(n)\, h_1(n)\}, \ldots\}$

Problem 3.8 Perform the convolution of $x(n)$ and $h(n)$, where $x(n) = \{1,2,3,4\}$ and $h(n) = \{1,1,1,1\}$.

Solution

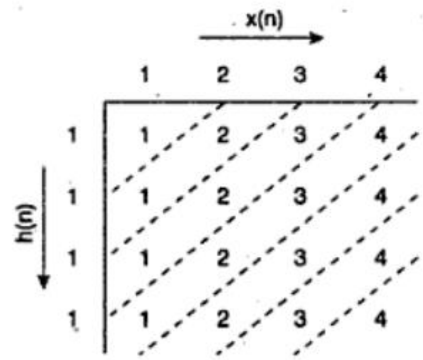

Fig. 3.8

$$y(n) = x(n) * h(n)$$

$$y(n) = \{1,(1+2),(1+2+3),(1+2+3+4),(2+3+4),(3+4),4\}$$

$$y(n) = \{1,\ 3,\ 6,\ 10,\ 9,\ 7,\ 4\}$$

Problem 3.9 Perform the convolution of $x(n)$ and $h(n)$. $x(n) = \{1,-2,\ 3,-4\}$, $h(n) = \{4,-3,\ 2,-1\}$

Solution

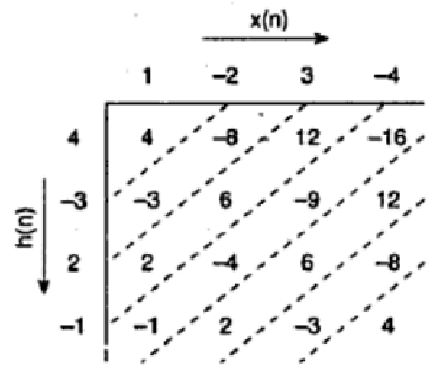

Fig. 3.9

$$y(n) = x(n) * h(n)$$

$$y(n) = \{4,-11,20,-30,20,-11,4\}$$

■ 3.6 LINEAR CONVOLUTION USING MATRIX METHOD

In this method, the data sequences are represented as a matrix. If the length of signal $x(n)$ is N_1 and the length of the impulse response $h(n)$ is N_2, then the matrix X can be obtained from $x(n)$, whose order will be $(N_1+N_2-1)\times N_1$ and the matrix H can be obtained from $h(n)$, whose order will be $N_2\times 1$ such that $Y = XH$.

Let us understand the convolution using matrix method with the following examples.

Problem 3.10 Find $y(n) = x(n) * h(n)$ using the matrix method. $x(n) = \{1, 2, 3, 4\}$; $h(n) = \{1, 1, 1, 1\}$

Solution

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Y = XH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+1 \\ 3+2+1 \\ 4+3+2+1 \\ 4+3+2 \\ 4+3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}$$

Problem 3.11 Find the convolution of the following data sequences using the matrix method. $x(n) = \{1, -2, 3, -4\}$; $h(n) = \{4, -3, 2, -1\}$

Solution

$$Y = XH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -4 & 3 & -2 & 1 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 20 \\ -30 \\ 20 \\ -11 \\ 4 \end{bmatrix}$$

■ 3.8 DECONVOLUTION

Deconvolution is "undo" procedure of convolution. In order to understand the deconvolution, let us consider an ideal system whose impulse response is $h(n)$ and system output is $y(n)$. This relation is expressed as,

$$y(n) = x(n) * h(n) \tag{3.29}$$

where $x(n)$ is the input signal.

The basic problem of deconvolution is to find $x(n)$ by deconvolute $h(n)$ with $y(n)$. Deconvolution has many practical applications, such as input pressure measurement by considering the output of the systems and system response.

By the definition of convolution sum as,

$$y(n) = \sum_{m=0}^{n} x(m)h(n-m) \tag{3.30}$$

Let us expand the convolution sum for $n = 0, 1, \ldots \infty$

$$\begin{aligned}
y(0) &= x(0)\ h(0) \\
y(1) &= x(0)\ h(1) + x(1)\ h(0) \\
y(2) &= x(0)\ h(2) + x(1)\ h(1) + x(2)\ h(0) \\
y(3) &= x(0)\ h(3) + x(1)\ h(2) + x(2)\ h(1) + x(3)\ h(0) \\
&\ \vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

The matrix form of above equations,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & 0 & & 0 \\ h(2) & h(1) & h(0) & 0 & & 0 \\ h(3) & h(2) & h(1) & h(0) & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \end{bmatrix} \tag{3.31}$$

The input signal $x(n)$ can be directly computed by considering $h(n)$ and $y(n)$. The equations can be rewritten as

$$x(0) = \frac{y(0)}{h(0)}$$

$$x(1) = \frac{y(1) - x(0)h(1)}{h(0)}$$

$$x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)}$$

$$x(3) = \frac{y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1)}{h(0)}$$

$$\vdots$$

$$x(n) = \frac{y(n) - \sum_{m=0}^{n-1} x(m)h(n-m)}{h(0)} \tag{3.32}$$

Equation (4.32) is called the general deconvolution equation.

Problem 3.21 What is the input signal $x(n)$ that will generate the output sequence $y(n) = \{\underset{\uparrow}{1}, 5, 10, 11, 8, 4, 1\}$ for a system with impulse response $h(n) = \{\underset{\uparrow}{1}, 2, 1\}$

Solution

$$y(n) = \{\underset{\uparrow}{1}, 5, 10, 11, 8, 4, 1\}\text{ ; } h(n) = \{\underset{\uparrow}{1}, 2, 1\}$$

The total number of samples in the output response is $N_1 + N_2 - 1 = 7$.

The number of samples in the impulse response is $N_2 = 3$.

The number of samples in the input signal is $N_1 = 7 - N_2 + 1 = 5$.

Let us consider the general deconvolution equation

$$x(n) = \frac{y(n) - \sum_{m=0}^{n-1} x(m)h(n-m)}{h(0)}$$

For $n=0$, $$x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1$$

For $n=1$, $$x(1) = \frac{y(1) - x(0)h(1)}{h(0)} = \frac{5 - 1\times 2}{1} = 3$$

For $n=2$, $$x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)} = \frac{10 - 1\times 1 - 3\times 2}{1} = 3$$

For $n=3$, $$x(3) = \frac{y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1)}{h(0)} = \frac{11 - 1\times 0 - 3\times 1 - 3\times 2}{1} = 2$$

For $n=4$, $$x(4) = \frac{y(4) - x(0)h(4) - x(1)h(3) - x(2)h(2) - x(3)h(1)}{h(0)}$$
$$= \frac{8 - 1\times 0 - 3\times 0 - 3\times 1 - 2\times 2}{1} = 1$$

The input sequence is $x(n) = \{1, 3, 3, 2, 1\}$.