Discrete Fourier transform (DFT)

In mathematics, the discrete Fourier transform (DFT) converts a finite list of equally-spaced samples of a function into the list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies, that has those same sample values. It can be said to convert the sampled function from its original domain (often time or position along a line) to the frequency domain. The DFT differs from the discrete-time Fourier transform (DTFT) in that its input and output sequences are both finite; it is therefore said to be the Fourier analysis of finite-domain (or periodic) discrete-time functions.

Definition

The discrete Fourier transform can be regarded as a special case of the discrete-time Fourier transform or DTFT. The DTFT of a causal signal x(k) is defined as follows,

$$X(f)=\sum_{k=0}^{\infty} x(k)\exp(-jk2\pi fT). \quad -\frac{f_s}{2} < f \le \frac{f_s}{2}.$$

It has **two drawbacks** in practical. One drawback is that a direct evaluation of X(f) using above equation requires an infinite number of floating- point operations or FLOPs. This is compounded by the second computational drawback; namely, that the transform itself must be evaluated at an infinite number of frequencies, f**.** The first limitations can be removed by restricting our considerations to signals of finite duration.

$$X(f)\approx\sum_{k=0}^{N-1} x(k)\exp(-jk2\pi fT).$$

To address the second limitations, we evaluate X(f) at N discrete values of f. In particular, consider the following discrete frequencies equally spaced over one period of X(f):

$$f_i = \frac{if_s}{N}, \qquad 0 \le i < N;$$

It is of interest to view the complex points $z_i = exp(j2\pi f_i T)$ corresponding to the discrete frequencies.Then,

$$z_i = exp(ji2\pi/\mathrm{N})$$

Notice that $|z_i| = 1$.Thus the N evaluation points are equally spaced around the unit circle .Observe that, the unit circle is traversed in the counterclockwise direction with $z_0 = 1$,

$$Z_{\frac{N}{4}} = j, Z_{\frac{N}{2}} = -1, Z_{\frac{3N}{4}} = -j.$$

Roots of Unity

The formulating of the discrete Fourier transform can be simplified if we introduce of the following factor which corresponds to z_i for i = -1.

$$w_N \triangleq \exp\left(-\frac{j2\pi}{N}\right)$$

By using Euler's identity, we find that
$w_N^N = 1.$ Consequently, the factor w_N can be thought of as the Nth root of unity.

More generally, we can express w_N^k as follows using Euler's identity.

$w_N^k = \cos(2\pi k/N) - j\sin(2\pi k/N)$.

Table: Basic properties of w_N^k.

Property	**Description**
1	$w_N^{\frac{N}{4}} = -j$
2	$w_N^{\frac{N}{2}} = -1$
3	$w_N^{\frac{3N}{4}} = j$
4	$w_N^N = 1$
5	$w_N^{k+N} = w_N^N$
6	$w_N^{k+N/2} = -w_N^N$
7	$w_N^{2k} = w_{N/2}^N$
8	$w_N^* = w_N^{-1}$

DFT and its Inverse

DFT: It is a transformation that maps an N-point Discrete-time (DT) signal $x[n]$ into a function of the N complex discrete harmonics. That is, given $x[n]; \quad n = 0,1,2,\cdots,N-1$, an N-point Discrete-time signal $x[n]$ then DFT is given by **(analysis equation):**

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} \quad for \quad k = 0,1,2,\cdots,N-1$$

and the inverse DFT (**IDFT**) is given by **(synthesis equation):**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{+j\frac{2\pi}{N}nk} \quad for \quad n = 0,1,2,\cdots,N-1$$

Example : Compute the DFT of the following two sequences:

$h[n] = \{1,3,-1,-2\}$ and $x[n] = \{1,2,0,-1\}$

where $N = 4 \Rightarrow e^{j\frac{2\pi}{N}} = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$

Let us use this information in (6.1) to compute DFT values:

$$H(k) = \sum_{n=0}^{3} h[n] e^{-j\frac{\pi}{2}nk} \quad for \quad k = 0,1,2,3$$

$$H(0) = h[0] + h[1] + h[2] + h[3] = 1$$
$$H(1) = h[0] + h[1]e^{-j\pi/2} + h[2].e^{-j\pi} + h[3].e^{-j3\pi/2} = 2 - j5$$
$$H(2) = h[0] + h[1]e^{-j\pi} + h[2].e^{-j2\pi} + h[3].e^{-j3\pi} = -1$$
$$H(3) = h[0] + h[1]e^{-j3\pi/2} + h[2].e^{-j3\pi} + h[3].e^{-j9\pi/2} = 2 + j5$$

Similarly,

$$X(0) = x[0] + x[1] + x[2] + x[3] = 2$$
$$X(1) = x[0] + x[1]e^{-j\pi/2} + x[2].e^{-j\pi} + x[3].e^{-j3\pi/2} = 1 - j3$$
$$X(2) = x[0] + x[1]e^{-j\pi} + x[2].e^{-j2\pi} + x[3].e^{-j3\pi} = 0$$
$$X(3) = x[0] + x[1]e^{-j3\pi/2} + x[2].e^{-j3\pi} + x[3]e^{-j9\pi/2} = 1 + j3$$

Matrix Representation of DFT

Let us write the variables involved in matrix form: $x = [x[0], x[1], \cdots, x[N-1]]^T$ and $W_N = e^{-j2\pi/N}$:

$$X(k) = \sum_{n=0}^{N-1} x[n].W_N^{kn} \quad for \quad k = 0,1,2,\cdots,N-1$$

Let x(n) be an N-point signal, and w_N be the Nth root of unity. Then the discrete Fourier transform of x(n) is denoted X(k)=DFT{x(n)} and defined

$X(k) \triangleq \sum_{n=0}^{N-1} x(n) w_N^{kn}$, for $k = 0, 1, 2, 3 \ldots \ldots N-1$

Inverse Discrete Fourier Transform (IDFT)

The continuous -time- Fourier transform has an inverse whose is always identical to the original signal .The inverse of the DFT, which is denoted x(n)=IDFT{X(k)},is computed as follows

$x(n) = 1/N \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$, for $n = 0, 1, 2, 3 \ldots \ldots N-1$

Here,W_N has been replaced by in the complex conjugate $W_N^* = W_N^{-1}$.

The unitary DFT

Another way of looking at the DFT is to note that in the above discussion, the DFT can be expressed as a Vandermonde matrix:

$$W=\begin{bmatrix} w_N^{0.0} & w_N^{0.1} & \ldots\ldots & w_N^{0.(N-1)} \\ w_4^{1.0} & w_4^{1.1} & \ldots\ldots & w_N^{1.(N-1)} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ w_N^{(N-1).0} & w_N^{(N-1).1} & \ldots\ldots & w_N^{(N-1).(N-1)} \end{bmatrix}$$

Where $W_N = e^{\frac{-2\pi i}{N}}$

DFT can be expressed in matrix form as X=W x

The matrix form of IDFT is $x = W^{*}X/N$.

Example:
As an example of computing a DFT using the matrix form, suppose the input samples are as follows. x=$[3,-1,0,2]^T$

Thus, N=4 and then

$$W_4 = \cos\left(\frac{2\pi}{4}\right) - j\sin\left(\frac{2\pi}{4}\right)$$
$$= -j$$

We know that, the DFT of x is

X= W_x

$$=\begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} x$$

$$=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$=\begin{bmatrix} 4 \\ 3+3j \\ 2 \\ 3-3j \end{bmatrix}$$

Note that even though the signal x(k) is real, its DFT X(i) is complex.

Example: As a numerical check, suppose we compute the IDFT of the results from before example. X=$[4, 3+\text{j}3, 2, 3-\text{j}3]^T$

Using N=4, the matrix W from before example, we have,

x = W*X/N

$$=1/4\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix} * \begin{bmatrix}4\\ 3+3j\\ 2\\ 3-3j\end{bmatrix}$$

$$=1/4\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}4\\ 3+3j\\ 2\\ 3-3j\end{bmatrix}$$

$$=1/4\begin{bmatrix}12\\ -4\\ 0\\ 8\end{bmatrix}$$

$$=\begin{bmatrix}3\\ -1\\ 0\\ 2\end{bmatrix}.$$

6.55. (*a*) Find the DFT $X[k]$ of $x[n] = \{0, 1, 2, 3\}$.

(*b*) Find the IDFT $x[n]$ from $X[k]$ obtained in part (*a*).

(*a*) Using Eqs. (*6.206*) and (*6.212*), the DFT $X[k]$ of $x[n]$ is given by

$$\begin{bmatrix}X[0]\\ X[1]\\ X[2]\\ X[3]\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}0\\ 1\\ 2\\ 3\end{bmatrix} = \begin{bmatrix}6\\ -2+j2\\ -2\\ -2-j2\end{bmatrix}$$

(*b*) Using Eqs. (*6.209*) and (*6.212*), the IDFT $x[n]$ of $X[k]$ is given by

$$\begin{bmatrix}x[0]\\ x[1]\\ x[2]\\ x[3]\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}6\\ -2+j2\\ -2\\ -2-j2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}0\\ 4\\ 8\\ 12\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 2\\ 3\end{bmatrix}$$

6.54. The DFT definition in Eq. (*6.92*) can be expressed in a matrix operation form as

$$\mathbf{X} = \mathbf{W}_N \mathbf{x} \tag{6.206}$$

where

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{6.207}$$

The $N \times N$ matrix $\mathbf{W}_N$ is known as the DFT matrix. Note that $\mathbf{W}_N$ is symmetric; that is, $\mathbf{W}_N^T = \mathbf{W}_N$, where $\mathbf{W}_N^T$ is the transpose of $\mathbf{W}_N$.

(*a*) Show that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* \tag{6.208}$$

where $\mathbf{W}_N^{-1}$ is the inverse of $\mathbf{W}_N$ and $\mathbf{W}_N^*$ is the complex conjugate of $\mathbf{W}_N$.

(*b*) Find $\mathbf{W}_4$ and $\mathbf{W}_4^{-1}$ explicitly.

(*a*) If we assume that the inverse of $\mathbf{W}_N$ exists, then multiplying both sides of Eq. (*6.206*) by $\mathbf{W}_N^{-1}$, we obtain

$$\mathbf{x} = \mathbf{W}_N^{-1}\mathbf{X} \tag{6.209}$$

which is just an expression for the IDFT. The IDFT as given by Eq. (*6.94*) can be expressed in matrix form as

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^*\mathbf{X} \tag{6.210}$$

Comparing Eq. (*6.210*) with Eq. (*6.209*), we conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

(*b*) Let $W_{n+1,k+1}$ denote the entry in the $(n+1)$st row and $(k+1)$st column of the $\mathbf{W}_4$ matrix. Then, from Eq. (*6.207*)

$$W_{n+1,k+1} = W_4^{nk} = e^{-j(2\pi/4)nk} = e^{-j(\pi/2)nk} = (-j)^{nk} \tag{6.211}$$

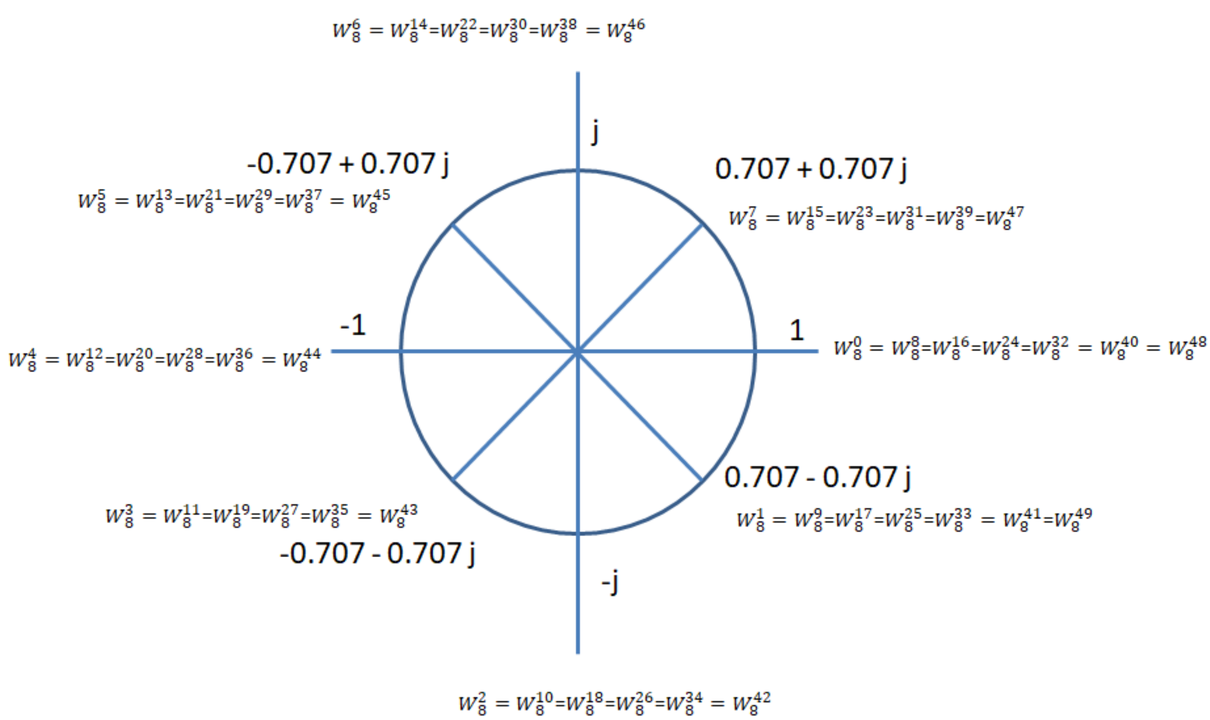
and we have

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \tag{6.212}$$

Example: Find 8 point DFT of $x(n) = \{1, 1, 2, 2, 3, 3, 4, 4\}$ using Matrix method

$X_N = W_N x_N$; N=8

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$



$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .707-.707j & -j & -.707-.707j & -1 & -.707+.707j & j & .707+.707j \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -.707+.707j & j & .707-.707j & -1 & .707+.707j & -j & -.707-.707j \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -.707-.707j & -j & .707+.707j & -1 & .707-.707j & j & -.707+.707j \\ 1 & j & -1 & -j & 1 & j & -1 & j \\ 1 & .707+.707j & j & -.707-.707j & W_8^{28} & -.707-.707j & W_8^{42} & -.707-.707j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

$$\boldsymbol{X(k) = \{20, -2+4.82j, -2+2j, -2+0.828j, 0, -2-0.828j, -2-2j, -2-4.82j\}}$$

Applications of the DFT

The Discrete Fourier Transform (DFT) is one of the most important tools in Digital Signal Processing. This chapter discusses three common ways it is used. First, the DFT can calculate a signal's frequency spectrum. This is a direct examination of information encoded in the frequency, phase, and amplitude of the component sinusoids. For example, human speech and hearing use signals with this type of encoding. Second, the DFT can find a system's frequency response from the system's impulse response, and vice versa. This allows systems to be analyzed in the frequency domain, just as convolution allows systems to be analyzed in the time domain. Third, the DFT can be used as an intermediate step in more elaborate signal processing techniques. The classic example of this is FFT convolution, an algorithm for convolving signals that is hundreds of times faster than conventional methods.

- Spectral Analysis of Signals
- Frequency Response of Systems
- Convolution via the Frequency Domain

Find 6-Point DFT of $x(n) = \{1,1,0,0,0,2\}$ using Matrix method [linear Transformation]

$X_N = W_N x_N \quad \therefore N = 6$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix}$$

$W_N = e^{-j\frac{2\pi}{N}}$; $W_N^0 = 1 \Rightarrow W_6^0 = 1$

$W_6^1 = e^{-j\frac{2\pi}{N}\cdot 1} = \cos\left(\frac{2\pi}{6}\right) - j\sin\left(\frac{2\pi}{6}\right)$
$= 0.5 - 0.866j$

$W_6^2 = -0.5 - 0.866j$

$W_6^3 = -1$ $\qquad W_6^9 = -1$

$W_6^4 = -0.5 + 0.886j$ $\qquad W_6^{10} = -0.5 + 0.886j$

$W_6^5 = 0.5 + 0.886j$ $\qquad W_6^{12} = 1$

$W_6^6 = 1$ $\qquad W_6^{15} = -1$

$W_6^8 = -0.5 - 0.866j$ $\qquad W_6^{16} = -0.5 + 0.886j$

$W_6^{20} = -0.5 - 0.886j$

$W_6^{25} = 0.5 - 0.886j$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5-0.866j & -0.5-0.866j & -1 & -0.5+0.866j & 0.5+0.866j \\ 1 & -0.5-0.866j & -0.5+0.866j & 1 & -0.5-0.866j & -0.5+0.866j \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.5+0.866j & -0.5-0.866j & 1 & -0.5+0.866j & -0.5-0.866j \\ 1 & 0.5+0.866j & -0.5+0.866j & -1 & -0.5-0.866j & 0.5-0.866j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2.5+0.866j \\ -0.5+0.866j \\ -2 \\ -0.5-0.866j \\ 0.5-j0.866j \end{bmatrix}$$

$X(K) = \{4,\ 2.5+0.866j,\ -0.5+0.866j,\ -2,\ -0.5-0.866j,\ 0.5-$[redacted]$.866j\}$