Master Problems

1 Problem 1: Inconsistent System Analysis

Consider the following linear system:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

Part 1: Check if the solution exists or not. Show your work step by step, and explain how you can determine from the reduced row echelon form whether the vector $\mathbf{b}$ lies in the column space of matrix $\mathbf{A}$.

Part 2: If the system is inconsistent (has no solution), modify exactly one entry in the vector $\mathbf{b}$ to create a new consistent system.

Part 3: For the modified consistent system from Part 2:

- Find the null space $N(\mathbf{A})$ of the coefficient matrix
- Determine the complete solution set (general solution)
- Describe the geometry of the solution set (point, line, plane, etc.)
- If the solution set is not a single point, identify a particular solution and the homogeneous solution

2 Problem 2: System with Specified Nullity

Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 1 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}$$

Part 1: Find a basis for the null space $N(\mathbf{A})$ and hence find its nullity.

Part 2: Determine whether the system $\mathbf{Ax} = \mathbf{b}$ is consistent. If consistent, find the complete solution set.

Part 3: Describe the geometry of the solution set:

- What is the dimension of the solution set?
- Is it a point, line, or plane in $\mathbb{R}^3$?

- If the solution set is not a single point, express it in the form $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$ where $\mathbf{x_p}$ is a particular solution and $\mathbf{x_h}$ represents the homogeneous solution.

Part 4: Geometric interpretation:

- Explain the relationship between the dimension of the solution set and the nullity of $\mathbf{A}$
- Describe what this solution set looks like geometrically in $\mathbb{R}^3$

3 Problem 3: Multiple Choice Question

If a homogeneous system has nullity = 0, then which of the following is correct?

A. Trivial solution and $\det(A) \neq 0$

B. Non-trivial solution and $\det(A) = 0$

C. Trivial solution and $\det(A) = 0$

D. Non-trivial solution and $\det(A) \neq 0$

E. The system is inconsistent