Frequency domain operations on images

# Sl. No.	4
Topics	Fourier transform

Fourier transform of $f(x)$ in continuous domain

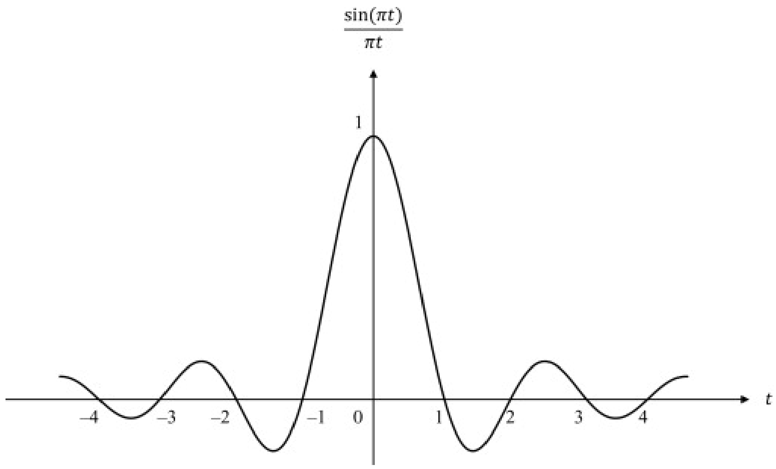
$$FT[f(x)] = F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-j2\pi ux} dx$$

Lets find Fourier transform of,

$$\begin{aligned} f(x) &= A \text{ if } -\omega/2 \leq x \leq \omega/2 \\ &= 0 \text{ otherwise} \end{aligned}$$

$$\begin{aligned} FT[f(x)] &= \int_{-\omega/2}^{\omega/2} f(x) \cdot e^{-j2\pi ux} dx \\ &= \int_{-\omega/2}^{\omega/2} A \cdot e^{-j2\pi ux} dx \\ &= -\frac{A}{j2\pi u}[e^{-j\pi u\omega} - e^{j\pi u\omega}] \\ &= -\frac{A}{2j\pi u}[-2j\sin(\pi uw)] \\ &= A\omega \cdot \frac{\sin(\pi u\omega)}{\pi u\omega} \end{aligned}$$

Which of form $\frac{\sin\theta}{\theta}$, which is also known as Sinc function,



- **IFT (Inverse Fourier transform)** in continuous domain,

$$IFT[F(u)] = f(x) = \int_{-\infty}^{\infty} F(u) \cdot e^{j2\pi ux} du$$

> ✍🏻 **Note:** If $f(x)$ is non-periodic it has infinite number of frequencies.

Show that Fourier transform is linear

$$FT(a\cdot f(x) + b\cdot g(x)) = a\cdot FT(f(x)) + b\cdot FT(g(x))$$

$$\begin{aligned}
\text{RHS} &= FT(a \cdot f(x) + b \cdot g(x)) \\
&= \int_{-\infty}^{\infty} (a \cdot f(x) + b \cdot g(x)) \cdot e^{-j2\pi ux} dx \\
&= a \cdot \int_{-\infty}^{\infty} f(x) \cdot e^{-j2\pi ux} dx + b \cdot \int_{-\infty}^{\infty} g(x) \cdot e^{-j2\pi ux} dx \\
&= a\cdot FT(f(x)) + b\cdot FT(g(x)) \\
&= \text{ LHS}
\end{aligned}$$

In Discrete Domain

$$DFT[f(x)] = F(u) = \sum_{x=0}^{M-1} f(x) \cdot e^{-\frac{j2\pi ux}{M}}$$

$$IDFT[F(u)] = f(x) = \sum_{x=0}^{M-1} F(u) \cdot e^{\frac{j2\pi ux}{M}}$$

Where, $F(u)$ is Fourier coefficient.

> ✍🏻 **Note:** If non-zero $F(u)$ Fourier coefficients are discrete then its number of frequencies is finite.

Fast Fourier Transformation

> ✍🏻 Prove that: **Discrete Fourier Transformation is Periodic or $F(u+M) = F(u)$**

$$\begin{aligned}
\text{LHS} = F(u+M) &= \sum_{x=0}^{M-1} f(x) \cdot e^{\frac{-j2\pi}{M}(u+M)x} \\
&= \sum_{x=0}^{M-1} f(x) \cdot e^{-j\frac{2\pi ux}{M} - 2\pi jx} \\
&= \sum_{x=0}^{M-1} f(x) \cdot \left(e^{-j\frac{2\pi ux}{M}} \cdot e^{-2\pi jx}\right) \text{ but } e^{-2\pi j} = 1 \\
&= \sum_{x=0}^{M-1} f(x) \cdot e^{-j\frac{2\pi ux}{M}} \\
&= F(u) \\
&= \text{RHS}
\end{aligned}$$

We know that,

$$F(u) = \sum_{x=0}^{M-1} f(x) \cdot e^{-j\frac{2\pi ux}{M}}; 0 \leq u \leq M-1$$

Now lets split it into sum of even and odd terms,

$$F(u) = \sum_{l=0}^{M/2-1} f(2l) \cdot e^{-j\frac{2\pi u(2l)}{M}} + \sum_{l=0}^{M/2-1} f(2l+1) \cdot e^{-j\frac{2\pi u(2l+1)}{M}}$$

Consider,-

$$f(2l) = g(l) \text{ and } f(2l+1) = h(l)$$

Now,

$$F(u) = \sum_{l=0}^{M/2-1} g(l) \cdot e^{-j\frac{2\pi u(2l)}{M}} + \sum_{l=0}^{M/2-1} h(l) \cdot e^{-j\frac{2\pi u(2l)}{M}} \cdot e^{-j\frac{2\pi u}{M}}$$

$$F(u) = G(u) + e^{-j\frac{2\pi u}{M}} H(u); 0 \leq u \leq M/2-1$$

Since $F(u)$ is periodic with period M, consider $u = v + M/2$;$\forall$ $0 \leq v \leq M/2 - 1$

$$F(v+M/2) = G(v+M/2) + e^{-j\frac{2\pi(v+M/2)}{M}} H(v+M/2)$$

As, $G(v+M/2)$ and $H(v+M/2)$ are periodic with period $M/2$

$$\begin{aligned} F(v+M/2) &= G(v) + e^{-j\frac{2\pi(v+M/2)}{M}} H(v) \\ &= G(v) + e^{-j\frac{2\pi M}{2M}} \cdot e^{-j\frac{2\pi v}{M}} \cdot H(v) \\ &= G(v) - e^{-j\frac{2\pi v}{M}} \cdot H(v) \; ; 0 \leq v \leq M/2-1 \end{aligned}$$

Algorithm:

✍🏻 $FFT(f(0), f(1), ...f(M-1))$

If $(M == 1)$:

return $DFT(f(0))$

else:

$$G = FFT(f(0), f(2), ...f(M-2))$$
$$H = FFT(f(1), f(3), ...f(M-1))$$

$$F1(u) = G(u) + e^{-j\frac{2\pi v}{M}} \cdot H(u) \rightarrow \text{ for odd terms}$$
$$F2(u) = G(u) - e^{-j\frac{2\pi v}{M}} \cdot H(u) \rightarrow \text{ for even terms}$$

return $(F1 \cdot F2)$ [Concatenation of $F1$ and $F2$)

Time complexity:

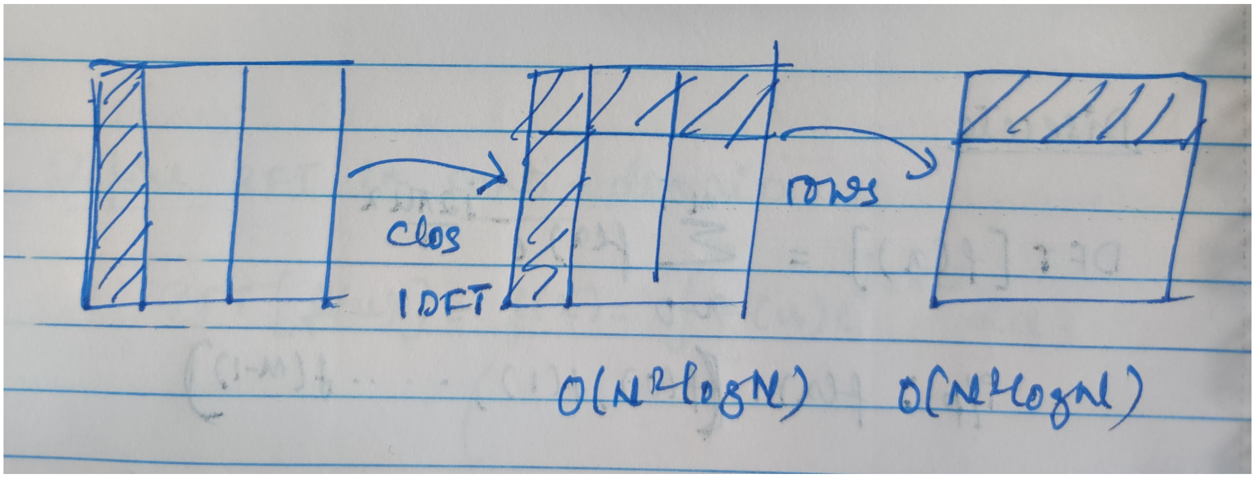
$$T(M) = 2T(M/2) + O(M)$$

Hence Time complexity is $\Theta(n \log n)$

Fourier Transform in 2D

$$F(u,v) = DFT[f(x,y)] = \sum_{y=0}^{N-1}\sum_{x=0}^{M-1} f(x,y) \cdot e^{-j\left(\frac{2\pi ux}{M} + \frac{2\pi vy}{N}\right)}$$
$$IDFT[F(u,v)] = \frac{1}{MN}\sum_{y=0}^{N-1}\sum_{x=0}^{M-1} f(x,y) \cdot e^{j\left(\frac{2\pi ux}{M} + \frac{2\pi vy}{N}\right)}$$

$F(0,0)$ takes $O(n^2)$ → hence for all u, v it will take $O(n^4)$



✍🏻 **Prove that:**

$$DFT[f(x,y)] = DFT_y[DFT_x[f(x,y)]]$$

Where,

$$DFT_x[f(x,y)] = \sum_{x=0}^{M-1} f(x,y) \cdot e^{-j\frac{2\pi}{M}ux}$$

and

$$DFT_y[DFT_x[f(x,y)]] = \sum_{y=0}^{N-1} DFT_x[f(x,y)] \cdot e^{-j\frac{2\pi}{N}uy}$$

$$
\begin{aligned}
\text{RHS} &= DFT_y[DFT_x[f(x,y)]] \\
&= DFT_y[\sum_{x=0}^{M-1} f(x,y) \cdot e^{-j\frac{2\pi}{M}ux}] \\
&= \sum_{y=0}^{N-1} \left[\sum_{x=0}^{M-1} f(x,y) \cdot e^{-j\frac{2\pi}{M}ux} \right] \cdot e^{-j\frac{2\pi}{N}uy} \\
&= \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f(x,y) \cdot e^{-j\left(\frac{2\pi ux}{M} + \frac{2\pi vy}{N}\right)} \\
&= DFT[f(x,y)] \\
&= \text{LHS}
\end{aligned}
$$

Convolution Theorem

$$FT[f(x,y) * g(x,y)] = FT[f(x,y)] \cdot FT[g(x,y)]$$

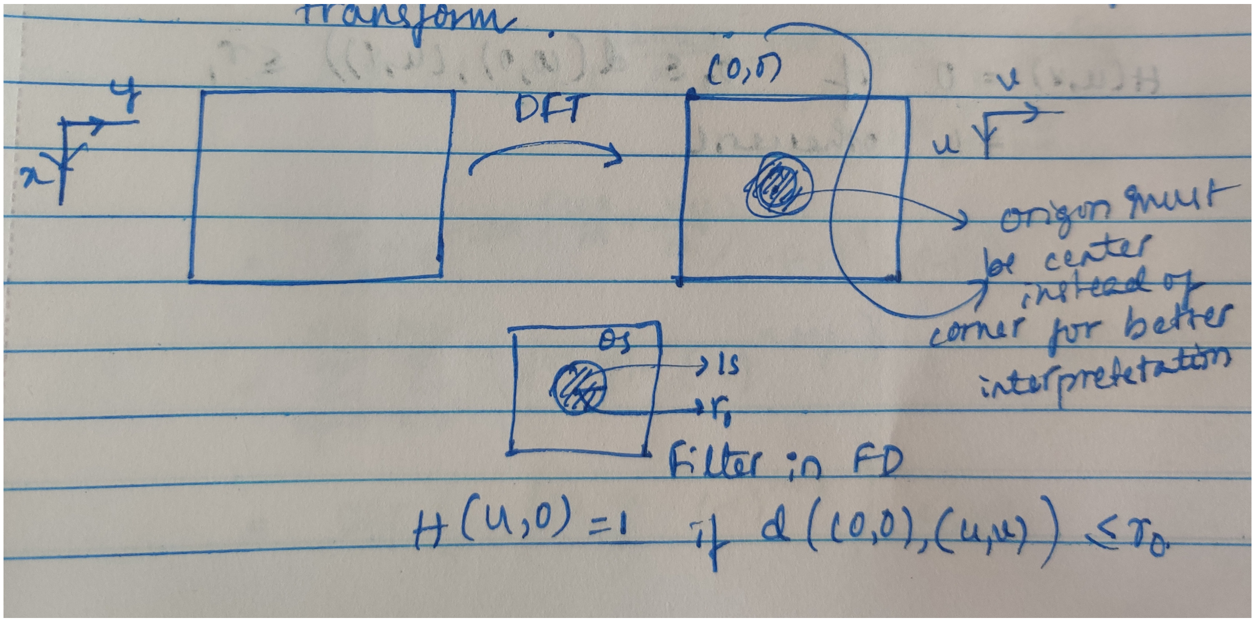
✍🏻 **Note:** In the above equation `*` indicates convolution where as `.` indicates element wise multiplication.

$$f(x,y) * g(x,y) = IFT[FT[f(x,y)] \cdot FT[g(x,y)]]$$

Where, $g(x,y)$ is filter in spatial domain and $FT[g(x,y)]$ is filter in frequency domain if $f(x,y)$ is considered as image and $g(x,y)$ as kernel or filter.

✍🏻 **Note:** We define filter in frequency domain directly instead of defining in spatial domain and finding its Fourier transform to avoid another Fourier transform cause of computation costs.

We shift the origin of input image to centre before converting to frequency domain (DFT), to get a better interpretation for the filter defined in frequency domain from the centre instead of edges, since `element wise multiplication` will be done.



✍🏻 **Prove:**

$$DFT[(-1)^{x+y} \cdot f(x,y)] = F(u - M/2, v - N/2)$$
$$\text{where, } F(u,v) = DFT(f(x,y))$$

$$\begin{aligned}
\text{RHS} &= F(u - M/2, v - N/2) \\
&= \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f(x,y) \cdot e^{-j2\pi\left(\frac{(u-M/2)x}{M} + \frac{(v-N/2)x}{N}\right)} \\
&= \sum\sum f(x,y) \cdot e^{-j\frac{2\pi ux}{M}} \cdot e^{-j2\pi(\frac{-xM}{2M})} \cdot e^{-j\frac{2\pi vy}{N}} \cdot e^{-j2\pi(\frac{-yN}{2N})} \\
&= \sum\sum f(x,y) \cdot e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \cdot (-1)^x \cdot (-1)^y \\
&= \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} (f(x,y) \cdot (-1)^{x+y}) \cdot e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\
&= DFT[f(x,y) \cdot (-1)^{x+y}] \\
&= \text{LHS}
\end{aligned}$$

✍🏻 **Algorithm for frequency domain filtering:**

$$i/p : f(x,y), H(u,v) \text{ [filter in frequency domain]}$$
$$o/p : g(x,y) \text{ [filtered image for } i/p\ f\text{]}$$

- Find $F(u,v) = DFT[(-1)^{x+y} \cdot f(x,y)]$
- Find $G(u,v) = F(u,v) \cdot H(u,v)$
- Find $\hat{f}(x,y) = IDFT(G(u,v))$
- Find $g(x,y) = \hat{f}(x,y) \cdot (-1)^{x+y}$

Ideal frequency domain filters

Ideal low pass filter

$$\begin{aligned} H(u,v) &= 1 \text{ if } d((0,0),(u,v)) \leq r_0 \\ &= 0 \text{ otherwise} \end{aligned}$$

where d indicates distance from origin of that particular point in the image and r_0 is the radius upto which we wanna pass.

Ideal high pass filter

$$\begin{aligned} H(u,v) &= 0 \text{ if } d((0,0),(u,v)) \leq r_0 \\ &= 1 \text{ otherwise} \end{aligned}$$

Ideal band pass filter

$$\begin{aligned} H(u,v) &= 1 \text{ if } r_0 \leq d((0,0),(u,v)) \leq r_1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Where r_0 and r_1 are the radii of circles from which we wanna make a band and only pass values which are between those radii.

Ideal band reject filter

$$\begin{aligned} H(u,v) &= 0 \text{ if } r_0 \leq d((0,0),(u,v)) \leq r_1 \\ &= 1 \text{ otherwise} \end{aligned}$$

✍🏻 **Note:**

- Low pass filter: for smoothing
- High pass filter: for edge detection
- Band reject filter: for removing periodic noise