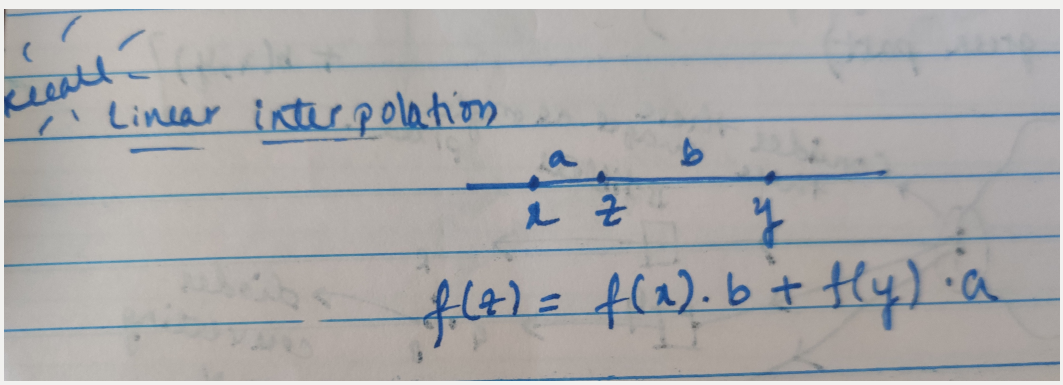
Basic Operations on images

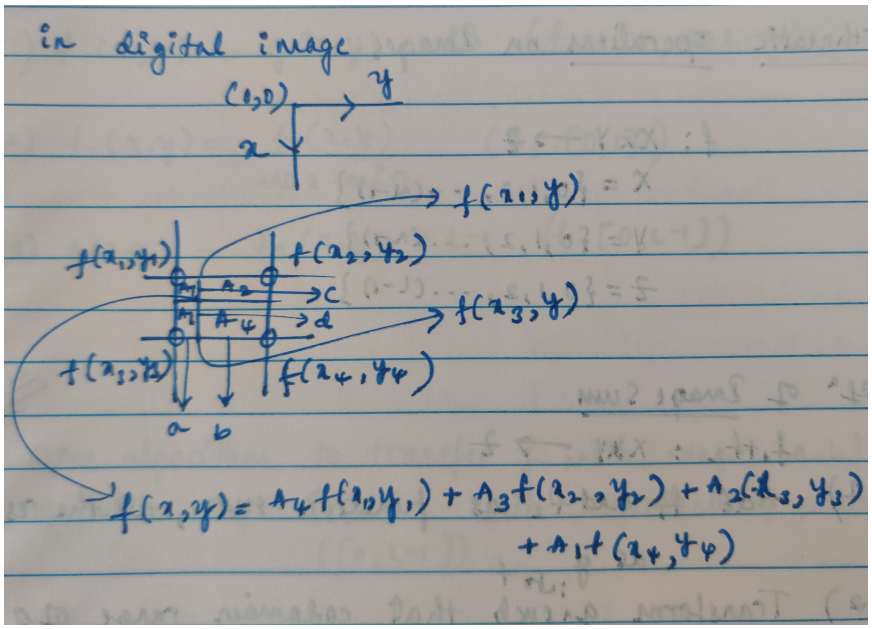
# Sl. No.	2
≣ Topics	Bilinear interpolation, arithmetic and unary operations, set theoretic operations and geometric transformations

Bilinear Interpolation

Linear Interpolation



$$f(z) = f(x) \cdot b + f(y) \cdot a$$



Derivation:

By linear interpolation,

$$f(x_1, y) = b\cdot f(x_1, y_1) + a\cdot f(x_2, y_2) \quad \rightarrow (1)$$

$$f(x_3, y) = b\cdot f(x_3, y_3) + a\cdot f(x_4, y_4) \quad \rightarrow (2)$$

$$f(x, y) = d\cdot f(x_1, y) + a\cdot f(x_3, y) \quad \rightarrow (3)$$

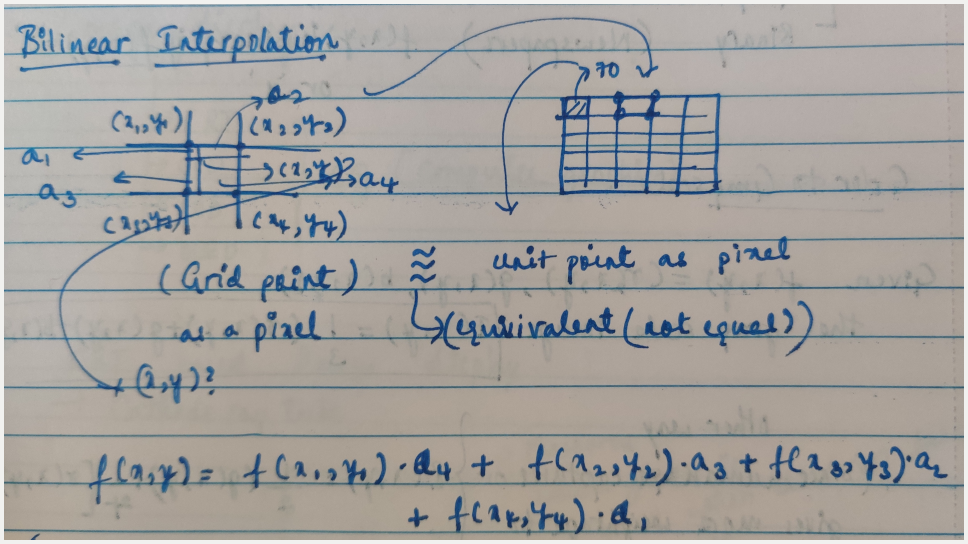
By substituting 1 and 2 in 3,

$$f(x, y) = d[b\cdot f(x_1, y_1) + a\cdot f(x_2, y_2)] + a[b\cdot f(x_3, y_3) + a\cdot f(x_4, y_4)]$$

$$f(x, y) = bd\cdot f(x_1, y_1) + ad\cdot f(x_2, y_2) + ab\cdot f(x_3, y_3) + ac\cdot f(x_4, y_4)$$

Which is the required equation for bilinear interpolation

✍🏻 **Grid Points**



HWP: Let $f(x, y) = ax + by + cxy + d$, given (x_i, y_i) and $f(x_i, y_i)$ for $1 \leq i \leq 4$, find the values of a, b, c and d.

Arithmetic operations on Images

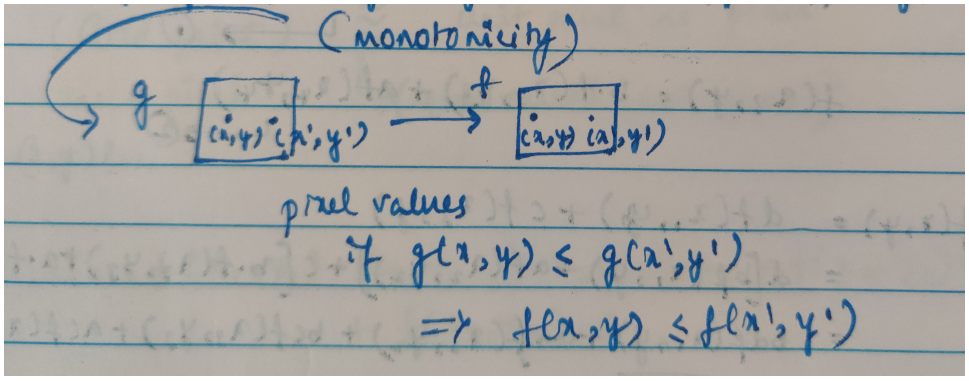
$$f : X \times Y \to Z$$
$$\text{where,}$$
$$X = \{0, 1, 2, ..., (M-1)\}$$
$$Y = \{0, 1, 2, ..., (N-1)\}$$
$$Z = \{0, 1, 2, ..., (L-1)\}$$

Image sum

1. Add f_1 and f_2 as function sum, call the result as g.

$$g(x, y) = f_1(x, y) + f_2(x, y)$$

2. Transform g intro f such that codomain range of f is $[0, L-1]$ with the property that if $g(x,y) \leq g(x',y')$ then $f(x,y) \leq f(x',y')$. This is also known as `monotonicity`.



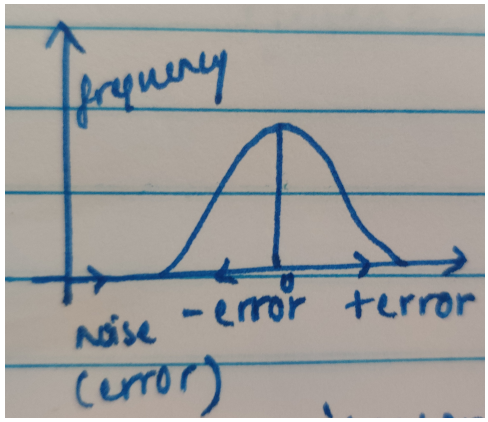
Monotonicity

> ✍🏻 **In general monotonicity** can be achieved via `min-max normalisation` and then multiplying with max value.
>
> Consider,
>
> input: g with range $[min, max]$
>
> output: f with range $[0, L-1]$
>
> 1. $f_1(x,y) = g(x,y) - min$
> 2. $f_2(x,y) = \frac{f_1(x,y)}{max - min}$
> 3. $f(x,y) = f_2(x,y) \cdot (L-1)$
>
> **HWP:** Give algorithm to transfer f with range $[a,b]$ to g with range $[c,d]$ with monotonicity property.

As there will be sensor noise (as no sensor is accurate), for images in digital cameras image taken will be the average of `n` clicks and average is close to the actual value, because the error positive and error negative gets compensated.

$$
\begin{aligned}
f(x,y) &= \frac{1}{n}\sum f_i(x,y) \\
&= \frac{1}{n}\sum [f(x,y) + \eta_i(x,y)]
\end{aligned}
$$

Where, η is noise and f is actual pixel.



Sensor noise can be modelled by Gaussian distribution

Image difference

1. $g(x,y) = f_1(x,y) - f_2(x,y)$
2. Transform g into f using monotonicity transform

Applications:

a. Motion detection

   if $f(x,y) > T$ where f is the difference of f_1 and f_2 and T is some threshold

b. Digital angiography (X-ray hard and soft tissues)

   As X-ray can't be used to find location of cholesterol, the image difference is used. Let f_1 be the image of blood vessel and f_2 be the image of blood vessel after injecting contrast material. Then $f = f_1 - f_2, f$ will give image of cholesterol content.

Image multiplication

1. $g(x,y) = f_1(x,y) \cdot f_2(x,y)$

2. Transform g to f using monotonicity transform

Image divison

Let f_1 and f_2 be two images with range $[0, L-1]$

1. $g(x,y) = \frac{f_1(x,y)}{f_2(x,y)}$ if $f_2(x,y) \neq 0$ other wise $g(x,y) = f_1(x,y)$ or else we can take, $g(x,y) = \frac{f_1(x,y)}{\epsilon}$
2. Transform g to f using monotonicity transform.

✍🏻

Applications:

1. Projection

   Let f be the image to be projected onto the screen and g be the projected image

   - Instead of scaling we use different weight matrix to different pixels (`not scaling!`)

     $$g(x,y) = w(x,y) \cdot f(x,y)$$

     Where w is weight matrix and f is our image.

   - Given $g(x,y) = w(x,y) \cdot f(x,y)$ i.e., projector and and $f(x,y)$ find a way such that when f' is projected $f'(x,y) \cdot w(x,y)$, the quality of display is good.

     i. Consider $f(x,y) = 1$ or some constant $f(x,y) = c$ and project it to get w, which is weight matrix.

     ii. Project $\frac{f(x,y)}{w(x,y)}$

Unary operations on images

Image negation or complementation

Given $f(x, y)$, the image negation is given by,

$$g(x, y) = (L - 1) - f(x, y)$$

e.g.:

- Digital x-ray process that converts negative to positive
- Swapping the background a2nd foreground in a binary image

Set theoretic operations on images

Representation of image as set

7	3
4	5

$$\implies \{((0,0),7),((0,1),3),((1,0),4),((1,1),5)\}$$

Gray scale image

1	1
0	1

$$\implies \{(0,0),(0,1),(1,1)\}$$

Binary image

Not required to represent all the values since other are just its complement (observe even pixel value isn't represented since they are binary and it implies 1 is intuitive.

Let S_1 and S_2 be set representation of binary images I_1 and I_2. Then,

1. **Union**

$$\text{Union}(I_1, I_2) = S_1 \cup S_2$$

2. **Intersection**

$$\text{Intersection}(I_1, I_2) = S_1 \cap S_2$$

Application: Consider road map of world and map of a particular city, if we need the road map of that particular city, we can apply intersection of both the set representation of those images and get the road map of corresponding city.

3. **Complementation**

   Let S be set representation of binary image I. Then,

$$\text{Compliment}(I) = S^c$$

   Application: To swap foreground and background for binary images

Geometric transformations

Scaling

Scaling by factor (a, b):

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x' \\ y' \end{pmatrix}, \text{ where } x' = ax \text{ and } y' = by$$

- Only worries about pixel location where pixel value is determined via interpolation
- In matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  Therefore, scaling is a `linear transformation`.

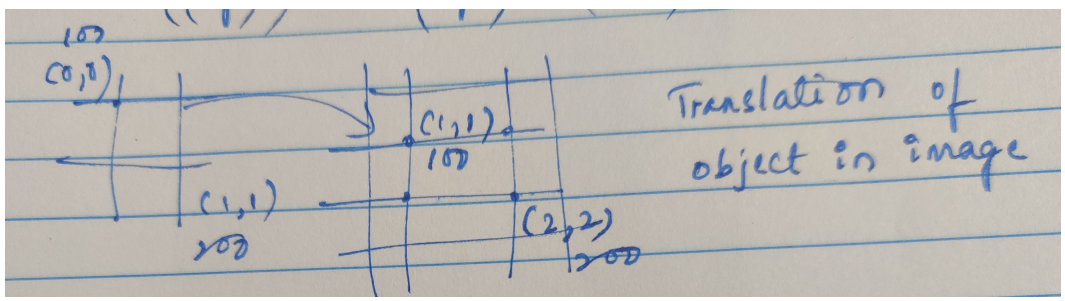
- In homogeneous coordinate system, (since translation can't be represented in matrix form in 2d we use homogeneous 3d form to represent all other geometric transformations for unity purpose while performing combined operations to avoid unnecessary additional calculations)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Translation

Translation by units (a, b):

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

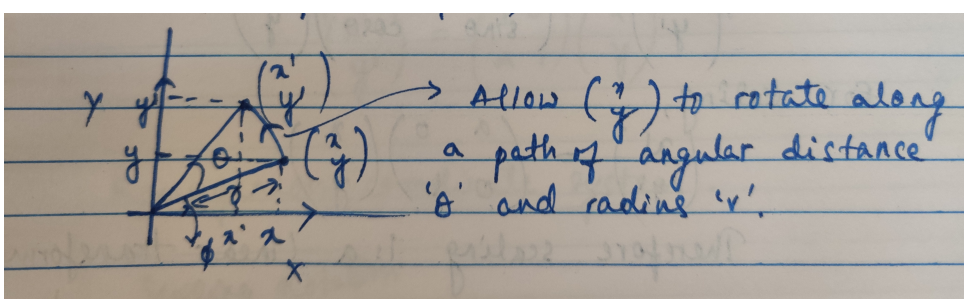


- Since translation can't be represented in 2d matrix form, it is `not a linear transformation`.
- In homogeneous coordinate system,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation

Rotation by θ, in 2D with respect to plane



From the above diagram,

$$\sin(\phi+\theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$$
$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

and also,

$$\sin(\phi+\theta) = \frac{y'}{r}, \quad \cos(\phi+\theta) = \frac{x'}{r}$$

combining both,

$$y' = r(\sin\phi\cos\theta + \cos\phi\sin\theta)$$
$$x' = r(\cos\phi\cos\theta - \sin\phi\sin\theta)$$

and again by the diagram,

$$\sin\phi = \frac{y}{r}, \quad \cos\phi = \frac{x}{r} \implies y = r\sin\phi, \quad x = r\cos\phi$$

by substituting,

$$y' = y\cos\theta + x\sin\theta$$
$$x' = x\cos\theta - y\sin\theta$$

This is `rotation transformation`.

- Rotation in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  Hence it is a linear transformation.

- In homogeneous coordinate system,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Shearing

1. **Shearing in x direction,**

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + ay \\ y \end{pmatrix}$$

ay, as y increases shearing increases

- In matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- In homogeneous coordinate system,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2. **Shearing in y direction,**

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y + ax \end{pmatrix}$$

ax, as x increases shearing increases

- In matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- In homogeneous coordinate system,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

✍🏻 **Pseudo inverse**

Pseudo inverse of $f(x)$ is $\frac{1}{f(x)}$ if $f(x) > \epsilon$ otherwise $\frac{1}{\epsilon}$