1) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\Omega_x, \Omega \in \mathbb{R}$

$$H = \frac{\Omega_x}{2}\sigma_x + \frac{\Omega_y}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & \Omega_x - \Omega_y i \\ \Omega_x + \Omega_y i & 0 \end{pmatrix}$$

$$H = H^{\dagger} \Rightarrow H\underline{\Psi} = \lambda\underline{\Psi}, \ \lambda \in \mathbb{R}$$

$$\Rightarrow \det(H - \lambda \mathbb{1}) = \det\begin{vmatrix} -2\lambda & \Omega_x - \Omega_y i \\ \Omega_x + \Omega_y i & -2\lambda \end{vmatrix} = 0$$

$$\Rightarrow 4\lambda^2 = \Omega_x^2 + \Omega_y^2 \quad (\geq 0)$$

$$\Rightarrow \lambda = \pm\frac{\sqrt{\Omega_x^2 + \Omega_y^2}}{2} \quad (\in \mathbb{R})$$

$$\begin{bmatrix} 0 & \Omega_x - \Omega_y i \\ \Omega_x + \Omega_y i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \pm\sqrt{\Omega_x^2 + \Omega_y^2}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} (\Omega_x - \Omega_y i)\, b \\ (\Omega_x + \Omega_y i)\, a \end{bmatrix}$$

Defining $\quad e^{i\Theta_{xy}} = \dfrac{\Omega_x + \Omega_y i}{|\Omega_x + \Omega_y i|}$, $\quad |\Omega_x + \Omega_y i| = \sqrt{\Omega_x^2 + \Omega_y^2}$

$$\Theta_{xy} = \arctan(\Omega_y / \Omega_x)$$

$$\Rightarrow \quad \begin{bmatrix} e^{-i\Theta_{xy}}\, b \end{bmatrix} = \pm\begin{bmatrix} a \end{bmatrix}$$

$$\begin{bmatrix} \\ e^{i\theta_{xy}} a \end{bmatrix} \quad \begin{bmatrix} \\ b \end{bmatrix}$$

$$\therefore \lambda_+ = \frac{\sqrt{\Omega_x^2 + \Omega_y^2}}{2}, \; H\Psi_+ = \lambda_+ \Psi_+ \Rightarrow \Psi_+ = \begin{bmatrix} 1 \\ e^{i\theta_{xy}} \end{bmatrix}$$

as required

$$\lambda_- = -\frac{\sqrt{\Omega_x^2 + \Omega_y^2}}{2}, \; H\Psi_- = \lambda_- \Psi_- \Rightarrow \Psi_- = \begin{bmatrix} 1 \\ -e^{i\theta_{xy}} \end{bmatrix}$$

2) $H \underline{e}_n = \lambda_n \underline{e}_n$

$$U(t)\underline{e}_n = e^{-i\lambda_n t} \underline{e}_n$$

$$\Psi = \begin{pmatrix} a \\ b \end{pmatrix}, \; a, b \in \mathbb{C}, \; |a|^2 + |b|^2 = 1$$

$$U(t)\underline{e}_n = e^{-i\lambda_n t} \underline{e}_n \Rightarrow U(t) = e^{-iHt}$$

$$H = \frac{1}{2}\begin{pmatrix} 0 & \Omega_x - \Omega_y i \\ \Omega_x + \Omega_y i & 0 \end{pmatrix}, \; H^2 = \frac{\Omega_x^2 + \Omega_y^2}{4} \mathbb{1}, \; \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \lambda_+^2 \mathbb{1}$$

$$H^3 = \frac{\Omega_x^2 + \Omega_y^2}{4} H, \; H^4 = \left(\frac{\Omega_x^2 + \Omega_y^2}{4}\right)^2 \mathbb{1}, \ldots\ldots$$

$$= \lambda_+^2 H \qquad\qquad = \lambda_+^4 \mathbb{1}$$

$$e^{-iHt} = \mathbb{1} - iHt - H^2 t^2 + iH^3 t^3 + H^4 t^4 + \ldots$$

$$\qquad\quad 2 \qquad\qquad 3! \qquad\qquad 4!$$

$$= \mathbb{1} - iHt - \frac{\lambda_+^2 t^2}{2}\mathbb{1} + \frac{\lambda_+^2 t^3}{3!} iH + \frac{\lambda_+^4 t^4}{4!}\mathbb{1} + \dots$$

$$= \left(1 - \frac{\lambda_+^2 t^2}{2} + \frac{\lambda_+^4 t^4}{4!} + \dots\right)\mathbb{1} - \frac{i}{\lambda_+}\left(\lambda_+ t - \frac{\lambda_+^3 t^3}{3!} + \dots\right)H$$

$$= \cos(\lambda_+ t)\,\mathbb{1} - \frac{i}{\lambda_+}\sin(\lambda_+ t)\,H$$

$$\Rightarrow U(t)\underline{\Psi} = \begin{bmatrix}\cos(\lambda_+ t)a \\ \cos(\lambda_+ t)b\end{bmatrix} - \frac{i}{2\lambda_+}\sin(\lambda_+ t)\begin{bmatrix}(\Omega_x - \Omega_y i)b \\ (\Omega_x + \Omega_y i)a\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\lambda_+ t)a - i\sin(\lambda_+ t)(\Omega_x - \Omega_y i)b/2\lambda_+ \\ \cos(\lambda_+ t)a - i\sin(\lambda_+ t)(\Omega_x + \Omega_y i)a/2\lambda_+\end{bmatrix}$$

as required