Updating Formula for Centerline Integration

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We want to integrate the following PDE:

$$\mathbf{M}\ddot{\mathbf{x}} = -\frac{\partial \mathbf{E}(\mathbf{x})}{\partial \mathbf{x}},$$

we derive the variational formula for various integrators for our ribbon solver, since we use L-BFGS to solve the timestepping equation.

1 Implicit Euler Integrator

The updating formula is:

$$\begin{aligned}\mathbf{M}\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t} &= -\frac{\partial \mathbf{E}(\mathbf{x}^{n+1})}{\partial \mathbf{x}^{n+1}} \\ \mathbf{x}^{\mathbf{n+1}} &= \mathbf{x}^n + \Delta t \mathbf{v}^{n+1}.\end{aligned}$$

The variational formula is:

$$\begin{aligned}\mathbf{M}\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t} + \frac{\partial \mathbf{E}(\mathbf{x}^{n+1})}{\partial \mathbf{x}} &= \\ \frac{\partial}{\partial \mathbf{x}^{n+1}}\left(\frac{1}{2}(\mathbf{v}^{n+1}-\mathbf{v}^n)^T\mathbf{M}(\mathbf{v}^{n+1}-\mathbf{v}^n) + \mathbf{E}(\mathbf{x}^{n+1})\right) &= 0.\end{aligned}$$

2 Second Order BDF2 Integrator

The updating formula is:

$$\begin{aligned}\mathbf{x}^{n+1} &= \frac{4}{3}\mathbf{x}^n - \frac{1}{3}\mathbf{x}^{n-1} + \frac{2}{3}\Delta t \mathbf{v}^{n+1} \\ \mathbf{v}^{n+1} &= \frac{4}{3}\mathbf{v}^n - \frac{1}{3}\mathbf{v}^{n-1} - \frac{2}{3}\Delta t \mathbf{M}^{-1}\frac{\partial \mathbf{E}(\mathbf{x}^{n+1})}{\partial \mathbf{x}^{n+1}}.\end{aligned}$$

By rearranging, we have:

$$\begin{aligned}\mathbf{M}\frac{\frac{3}{2}\mathbf{v}^{n+1} - 2\mathbf{v}^n + \frac{1}{2}\mathbf{v}^{n-1}}{\Delta t} + \frac{\partial \mathbf{E}(\mathbf{x}^{n+1})}{\partial \mathbf{x}} &= \\ \frac{\partial}{\partial \mathbf{x}^{n+1}}\left(\frac{1}{2}\mathbf{A}^T\mathbf{M}\mathbf{A} + \mathbf{E}(\mathbf{x}^{n+1})\right) &= 0 \\ \mathbf{v}^{n+1} - \frac{4}{3}\mathbf{v}^n + \frac{1}{3}\mathbf{v}^{n-1} &= \mathbf{A} \\ \frac{3}{2\Delta t}\mathbf{x}^{n+1} - \frac{2}{\Delta t}\mathbf{x}^n + \frac{1}{2\Delta t}\mathbf{x}^{n-1} - \frac{4}{3}\mathbf{v}^n + \frac{1}{3}\mathbf{v}^{n-1} &= \mathbf{A}.\end{aligned}$$

3 Implicit Newmark Scheme

Our updation formula is:

$$\mathbf{M}\ddot{\mathbf{S}} + \mathbf{C}\dot{\mathbf{S}} = \mathbf{f}_I(\mathbf{S}) + \frac{\partial \mathbf{X}}{\partial \mathbf{S}}^T \mathbf{f}_E(\mathbf{X})$$

The updating rule for Newmark Scheme is:

$$\begin{aligned}\dot{\mathbf{S}}_{n+1} &= \dot{\mathbf{S}}_n + (1-\gamma)\Delta t \ddot{\mathbf{S}}_n + \gamma \Delta t \ddot{\mathbf{S}}_{n+1} \\ \mathbf{S}_{n+1} &= \mathbf{S}_n + \Delta t \dot{\mathbf{S}}_n + \frac{(1-2\beta_2)\Delta t^2}{2}\ddot{\mathbf{S}}_n + \beta \Delta t^2 \ddot{\mathbf{S}}_{n+1}\end{aligned}$$

Multiply both side by $\mathbf{M}$ and write:

$$\begin{aligned}\mathbf{M}\dot{\mathbf{S}}_{n+1} &= \Phi_n + \gamma \Delta t \mathbf{M}\ddot{\mathbf{S}}_{n+1} \\ \mathbf{M}\mathbf{S}_{n+1} &= \Psi_n + \beta \Delta t^2 \mathbf{M}\ddot{\mathbf{S}}_{n+1} \\ \Phi_n &= \mathbf{M}\dot{\mathbf{S}}_n + (1-\gamma)\Delta t \mathbf{M}\ddot{\mathbf{S}}_n \\ \Psi_n &= \mathbf{M}\mathbf{S}_n + \Delta t \mathbf{M}\dot{\mathbf{S}}_n + \frac{(1-2\beta_2)\Delta t^2}{2}\mathbf{M}\ddot{\mathbf{S}}_n \\ \mathbf{M}\mathbf{S}_{n+1} &= \Psi_n + \frac{\beta}{\gamma}\Delta t(\mathbf{M}\dot{\mathbf{S}}_{n+1} - \Phi_n) \\ \Delta \mathbf{S} &= \frac{\beta}{\gamma}\Delta t \Delta \dot{\mathbf{S}}\end{aligned}$$

With this equation, we can solve the first one for $\dot{\mathbf{S}}_{n+1}$ by:

$$\mathbf{M}\dot{\mathbf{S}}_{n+1} = \Phi_n + \gamma \Delta t(\mathbf{f}_I(\mathbf{S}_{n+1}) + \frac{\partial \mathbf{X}}{\partial \mathbf{S}}^T \mathbf{f}_E(\mathbf{X}_{n+1}) - \mathbf{C}\dot{\mathbf{S}}_{n+1})$$

We apply first order taylor series on the above equation to get:

$$\begin{aligned}\mathbf{M}(\dot{\mathbf{S}}_{n+1} + \Delta \dot{\mathbf{S}}) &= \Phi_n \\ &+ \gamma \Delta t(\mathbf{f}_I(\mathbf{S}_{n+1}) - \mathbf{K}_I \Delta \mathbf{S} - \mathbf{C}(\dot{\mathbf{S}}_{n+1} + \Delta \dot{\mathbf{S}})) \\ &+ \gamma \Delta t \frac{\partial \mathbf{X}}{\partial \mathbf{S}}^T (\mathbf{f}_E(\mathbf{X}_{n+1}) - \mathbf{K}_E \frac{\partial \mathbf{X}}{\partial \mathbf{S}}\Delta \mathbf{S})\end{aligned}$$

After arrangement, we get:

$$\begin{aligned}\mathbf{LHS} &= \mathbf{M} + \beta \Delta t^2 (\mathbf{K}_I + \frac{\partial \mathbf{X}}{\partial \mathbf{S}}^T \mathbf{K}_E \frac{\partial \mathbf{X}}{\partial \mathbf{S}}) + \gamma \Delta t \mathbf{C} \\ \mathbf{RHS} &= \Phi_n - \mathbf{M}\dot{\mathbf{S}}_{n+1} + \gamma \Delta t(\mathbf{f}_I(\mathbf{S}_{n+1}) + \frac{\partial \mathbf{X}}{\partial \mathbf{S}}^T \mathbf{f}_E(\mathbf{X}_{n+1}) - \mathbf{C}\dot{\mathbf{S}}_{n+1})\end{aligned}$$

3.1 Implicit Euler for S

In this section, we introduce the approach to convert the Implicit Newmark Scheme mentioned above to the Implicit Euler Scheme, and use S as the unknows in the resulting linear equation.

When $\gamma = \beta = 1$ and $\beta_2 = 0.5$, we can obtain an Implicit Euler Integrator for $\Delta \dot{S}$. And we need to solve the following linear equation for $\Delta \dot{S}$,

$$\mathbf{LHS}\Delta \dot{\mathbf{S}} = \mathbf{RHS}$$

Because

$$\mathbf{S}_{n+1} = \mathbf{S}_n + \Delta \mathbf{S} = \mathbf{S}_n + \frac{\beta}{\gamma}\Delta t \Delta \dot{\mathbf{S}}$$

we have

$$\Delta \dot{\mathbf{S}} = \frac{\gamma}{\beta \Delta t}(\mathbf{S}_{n+1} - \mathbf{S}_n)$$

Thus at each time step, we need to solve

$$(\mathbf{LHS})\mathbf{S}_{n+1} = \frac{\beta \Delta t}{\gamma}\mathbf{RHS} + (\mathbf{LHS})\mathbf{S}_n$$

4 Local Basis for Subspace STVK

In this case, the energy term is: $\mathbf{E} = \mathbf{E}_N(\mathbf{U}\mathbf{z} + \mathbf{U}_l\mathbf{z}_l)$. We cannot find the forth order coefficients for $\left(\mathbf{U}\ \mathbf{U}_l\right)$. So that we have to use taylor approximation at $\mathbf{U}\mathbf{z}$.

5 A Simple Friction Model

Let's consider one point collides with a surface with normal $\mathbf{n}$. Suppose the velocity of this point is $\mathbf{v}$, and the external force is $\mathbf{f}$, the static and kinetic friction coefficients are μ_s and μ_k respectively, and the mignitude for the normal pressure force is λ. Then, the friction force of this point is

$$\mathbf{f}_r = \begin{cases} -\min(\|\mathbf{f}_t\|, \lambda\mu_k)\frac{\mathbf{v}_t}{\|\mathbf{v}_t\|}, & \mathbf{v}_t \neq 0 \\ \\ -\min(\|\mathbf{f}_t\|, \lambda\mu_s)\frac{\mathbf{f}_t}{\|\mathbf{f}_t\|}, & \mathbf{v}_t = 0 \end{cases}$$

where $\mathbf{f}_t$ and $\mathbf{v}_t$ is calculated by using

$$\begin{aligned} \mathbf{f}_t &= \mathbf{f} - (\mathbf{n} \cdot \mathbf{f})\mathbf{n} \\ \mathbf{v}_t &= \mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{n} \end{aligned}$$

respectively. Finally, we need to update the force $\mathbf{f}$ using $\mathbf{f}_r$, i.e

$$\mathbf{f} = \mathbf{f} + \mathbf{f}_r$$

And when $\|\mathbf{f}_t\| < \lambda\mu$, we also need to set $\mathbf{v} = \mathbf{v} - \mathbf{v}_t$.

References