EE 2030 Linear Algebra

Homework #6

Due to 06/07/2023

1.

Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = [A] \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

$$G_{k+1} = G_{k+1}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

2.

The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad and\ B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}S^{-1}$. Why is there no real matrix square root of B?

3.

A door is opened between rooms that hold $v(0) = 30$ people and $\omega(0) = 10$ people. The movement between rooms is proportional to the difference $v - \omega$:

$$\frac{dv}{dt} = \omega - v \quad and \quad \frac{d\omega}{dt} = v - \omega.$$

Show that the total $v + \omega$ is constant (40 people). Find the matrix in $\frac{du}{dt} = Au$
And its eigenvalues and eigenvectors. What are v and ω at $t = 1$ and $t = \infty$?

☆

Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check e^{At} and the derivative of e^{At} when $t = 0$.

5.

$A^T = -A$

(Recommended) This matrix M is skew-symmetric and also _____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($||Mx|| = ||x||$ for every x so $||\lambda x|| = ||x||$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6}$$

$$I + A + \frac{A^2 e^{At}}{2!}$$

1. (a) $A\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$, $G_{k+2} = \dfrac{G_k + G_{k+1}}{2}$ $\Rightarrow A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

$\det(A-\lambda I) = 0 \Rightarrow \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$, $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -\frac{1}{2} \end{cases}$

$\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \underline{x_1} = \underline{0}$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \underline{x_2} = \underline{0}$, $\underline{x_2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix} \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$ as $n \to \infty$ $A^n \to \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$

(c) $\frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

2. For A: $\begin{cases} \lambda_1 = 1, & \underline{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = 9, & \underline{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$, $R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{9} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = S\Lambda^{\frac{1}{2}}S^{-1} = A^{\frac{1}{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

For B: $\begin{cases} \lambda_1 = -1, & \underline{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = 9, & \underline{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$, $B^{\frac{1}{2}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{9} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3+i & 3-i \\ 3-i & 3+i \end{bmatrix} \cdot \frac{1}{2}$

$\because$ Not all the eigenvalues of $B \geq 0$ $\therefore \nexists$ real square root of B.

3. $\begin{cases} \dfrac{dv}{dt} = w - v \\ \dfrac{dw}{dt} = v - w \end{cases}$ $\begin{cases} v(0) = 30 \\ w(0) = 10 \end{cases}$, $\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix}$, $v - \dfrac{dv}{dt} + w - \dfrac{dw}{dt} = v + w = 40 \because \dfrac{dv}{dt} = -\dfrac{dw}{dt}$

$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\det(A-\lambda I) = 0$, $\lambda^2 + 2\lambda = 0$, $\begin{cases} \lambda_1 = -2, & \underline{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = 0, & \underline{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$

Let $\underline{u} = \begin{bmatrix} v \\ w \end{bmatrix}$, $A\underline{u} = \dfrac{du}{dt}$, $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

let $\underline{u_0} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} = 10\,\underline{x_1} + 20\,\underline{x_2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} = S\,\underline{c}$ $\quad \underline{u} = c_1 e^{\lambda_1 t}\,\underline{x_1} + c_2 e^{\lambda_2 t}\,\underline{x_2}$

$$\begin{cases} v = -e^{-2t} & \dfrac{dv}{dt} = 2e^{-2t} = w - v \\ w = e^{-2t} & \dfrac{dw}{dt} = -2e^{-2t} = v - w \end{cases}, \quad \begin{cases} v(1) = -e^{-2} \\ w(1) = e^{-2} \end{cases}, \quad \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \to \cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \text{ as } t \to \infty$$

$$\begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

$\Downarrow$

$$\begin{cases} v = -10e^{-2t} + 20 & \dfrac{dv}{dt} = 20e^{-2t} \\ w = 10e^{-2t} + 20 & \dfrac{dw}{dt} = -20e^{-2t} \end{cases}$$

4. $A=\begin{bmatrix}1&1\\0&3\end{bmatrix}$, $\det(A-\lambda I)=0$, $\lambda^2-4\lambda+3=0$, $\begin{cases}\lambda_1=1\\ \lambda_2=3\end{cases}$, $\underline{x_1}=\begin{bmatrix}1\\0\end{bmatrix}$, $\underline{x_2}=\begin{bmatrix}1\\2\end{bmatrix}$

$$A=\begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\frac{1}{2}\begin{bmatrix}2&-1\\0&1\end{bmatrix}$$

$$e^{At}=Se^{\Lambda t}S^{-1}=\begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}e^t&0\\0&e^{3t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}2&-1\\0&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^t&e^{3t}\\0&2e^{3t}\end{bmatrix}\begin{bmatrix}2&-1\\0&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2e^t&e^{3t}-e^t\\0&2e^{3t}\end{bmatrix}=\begin{bmatrix}e^t&\frac{e^{3t}-e^t}{2}\\0&e^{3t}\end{bmatrix}$$

$$e^{A\cdot 0}=I=\begin{bmatrix}e^0&\frac{e^{3\cdot 0}-e^0}{2}\\0&e^{3\cdot 0}\end{bmatrix},\quad Ae^{A\cdot 0}=\begin{bmatrix}1&1\\0&3\end{bmatrix}=\begin{bmatrix}e^0&\frac{3e^{3\cdot 0}-e^0}{2}\\0&3e^{3\cdot 0}\end{bmatrix}$$

$$\frac{1}{\det A}\begin{bmatrix}0&-\\ &0\end{bmatrix}$$

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

$$\frac{1}{1}\begin{bmatrix}1&0\\0&1\end{bmatrix}\quad \begin{bmatrix}d_1\\d_2\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

x x'

t t'

B^{-1} $\cancel{B^{-1}}$

$A\underline{c}=B\underline{d}$ $B^{-1}A\underline{c}=\underline{d}$

$\boxed{T}=\underline{(\ \ )}\,\underline{A}\,\underline{(\ \ )}$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}^{-1}$ $\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}T\begin{bmatrix}1&1\\0&1\end{bmatrix}^{-1}$

5. orthonormal. $\mathrm{trace}(M) = 0 \Rightarrow \lambda = i, i, -i, -i$

6. A is invertible, orthogonal, permutation, diagonalizable, Markov 轉移

B is projection, Markov

A is possible for : $\overset{PLU}{\cancel{LU}}$, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$

B is possible for : LU, $\cancel{QR}$ $S\Lambda S^{-1}$, $Q\Lambda Q^T$

7. $$[x_1\ x_2\ x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}\cdot x_i \cdot x_j$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For A: $\left(\det[2] = 2,\ \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3,\ \det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 4\right) > 0 \Rightarrow$ positive definite

For B: $\det[2] = 2,\ \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3,\ \det\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = 0) \geq 0 \Rightarrow$ semipositive definite

6.
(Recommended) Which of these classes of matrices do A and B belong to:
Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: $LU, QR, S\Lambda S^{-1}, Q\Lambda Q^T$?

7.
Which 3 by 3 symmetric matrices A and B produce these quadratics?

$x^TAx = 2({x_1}^2 + {x_2}^2 + {x_3}^2 - x_1x_2 - x_2x_3)$. Why is A positive definite?

$x^TBx = 2({x_1}^2 + {x_2}^2 + {x_3}^2 - x_1x_2 - x_1x_3 - x_2x_3)$. Why is B positive semidefinite?

8.
For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad and \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

$s(s^2-16)+4(-4s-16)-4(16+4s)$

$s^3-16s-16s-64-64-16s$

9.
There are sixteen 2 by 2 matrices whose entries are 0's and 1's. Similar matrices go into the same family. How many families? How many matrices (total 16) in each family?

$s^3-48s-128=0$

10.
These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors(one from each block). But the block size don't match and they are not *similar.*

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad and \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: $J\ is\ not\ similar\ to\ K$.

If $JM=MK$

$J=MKM^{-1}$

11.
Find the eigenvalues and unit eigenvectors v_1, v_2 of A^TA. Then find $u_1 = Av_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad and \quad A^TA = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \quad and \quad AA^T = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that u_1 is a unit eigenvector of AA^T. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = [u_1\ u_2] \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} [v_1\ v_2]^T.$$

12.
Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangulaur matrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ has the same shape as A.

8. For A: $\det[s]>0 \Rightarrow s>0$, $\det\begin{bmatrix} s & -4 \\ -4 & s \end{bmatrix} > 0 \Rightarrow s^2>16$

$$\det A = s\begin{vmatrix} -4 & s \\ -4 & -4 \end{vmatrix} + 4\begin{vmatrix} -4 & -4 \\ -4 & s \end{vmatrix} - 4\begin{vmatrix} -4 & s \\ -4 & -4 \end{vmatrix} = (16+4s)(s-4)+4(-4s-16) = (s-8)(16+4s)>0$$

$\Rightarrow s<-4$ or $s>8$

$\Rightarrow$ If $s>8$, then A is positive definite.

For B: $\det[t]>0 \Rightarrow t>0$, $\det\begin{bmatrix} t & 3 \\ 3 & t \end{bmatrix} >0 \Rightarrow t^2>9$,

$$\det\begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix} = t\begin{vmatrix} t & 4 \\ 4 & t \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 4 & t \end{vmatrix} = t^3-25t = t(t+5)(t-5)>0 \Rightarrow t>5 \ , \ -5<t<0$$

$\Rightarrow$ If $t>5$, then B is positive definite.

9. three families : $\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$, $\left(\begin{bmatrix} 0 & x \\ x & 1 \end{bmatrix} \text{or} \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix}\right)$, $\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$ (x is don't care)

$\Downarrow$ 4 matrices $\Downarrow$ 8 matrices $\Downarrow$ 4 matrices

$\begin{bmatrix}0&0\\0&0\end{bmatrix}$ $\lambda^2=0$ ✓ $\begin{cases}\lambda_1=0\\\lambda_2=0\end{cases}$ $J=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ 1

$\begin{bmatrix}0&0\\0&1\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $\begin{cases}\lambda_1=1\\\lambda_2=0\end{cases}$ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $\lambda^2=0$ ✓ $\begin{bmatrix}0\\1\end{bmatrix}$ $J=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ 3

$\begin{bmatrix}0&0\\1&1\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $\lambda^2=0$ ✓ $J=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ 3

$\begin{bmatrix}0&1\\0&1\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $\begin{bmatrix}1\\0\end{bmatrix}$ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\lambda^2-1=0$ ✓ $J=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ 6

$\begin{bmatrix}0&1\\1&1\end{bmatrix}$ $\lambda^2-\lambda-1=0$ ✓ $J=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$ 4

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\lambda^2-2\lambda+1=0$ ✓ $J=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ 7

$\begin{bmatrix}1&0\\1&0\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}1&0\\1&1\end{bmatrix}$ $\lambda^2-2\lambda+1=0$ ✓ $J=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ 5

$\begin{bmatrix}1&1\\0&0\end{bmatrix}$ $\lambda^2-\lambda=0$ ✓ $J=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 2

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $\lambda^2-2\lambda+1=0$ ✓ $J=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ 5

$\begin{bmatrix}1&1\\1&0\end{bmatrix}$ $\lambda^2-\lambda-1=0$ ✓ $J=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$ 4

$\begin{bmatrix}1&1\\1&1\end{bmatrix}$ $\lambda^2-2\lambda=0$ ✓ $J=\begin{bmatrix}2&0\\0&0\end{bmatrix}$ 8

$$\frac{1\pm\sqrt{5}}{2}$$

8 families

10. If $JM = MK$, and M is invt. $\Rightarrow$ $M^{-1}JM = K$ $\Rightarrow$ J and K are similar (contradictio)

$\Rightarrow M$ is not invt.

11. $A^T \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$, $\lambda^2 - 50\lambda = 0$, $\begin{cases} \lambda_1 = 50 \\ \lambda_2 = 0 \end{cases}$, $\underline{v_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\underline{v_2} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$A\underline{v_1} = \sigma_1 \underline{u_1} \Rightarrow \underline{u_1} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{\sqrt{50}} = \frac{1}{5\sqrt{10}}\begin{bmatrix} 5 \\ 15 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$AA^T = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$, $\begin{cases} \lambda_1 = 50 \\ \lambda_2 = 0 \end{cases}$, $\begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underline{0}$, $\underline{u_1} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\underline{u_2} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^T$$

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda^2 - 4\lambda + 3 = 0$, $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$, $\underline{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$, $\underline{v_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v_2} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$AV = U\Sigma \Rightarrow \quad AV = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{12}}\begin{bmatrix} \sqrt{6} & \sqrt{2} \\ 0 & 2\sqrt{2} \\ -\sqrt{6} & \sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} \sqrt{6} & 3\sqrt{2} \\ -\sqrt{6} & 3\sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{bmatrix}$$

$$U\Sigma = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{bmatrix}$$

$r^2 - r\cos\theta = 0$

$r = 0$ or $\cos\theta$

$$\int_0^{\cos\theta} \int_0 1 - r^2$$

$\cos\theta = 0 \quad \theta = \pm \frac{\pi}{2}$