CS5319 ADVANCED DISCRETE STRUCTURE

Exam 1 – November 03, 2020 (2 hours)

Answer all six questions. Total marks = 100. Maximum score = 100.

1. (20%) We are given a red box, a blue box, and a green box. We are also given 10 red balls, 10 blue balls, and 10 green balls. Balls of the same color are considered identical. We define the following two constraints.

   - Constraint 1: No box contains a ball that has the same color as the box.
   - Constraint 2: No box is empty.

   Determine the number of ways in which we can put all the 30 balls into boxes so that

   (a) (5%) No constraint has to be satisfied; that is, every combination is permitted.
   (b) (5%) Constraint 1 is satisfied.
   (c) (5%) Constraint 2 is satisfied.
   (d) (5%) Constraints 1 and 2 are both satisfied.

   Hint: Balls of different colors may be considered independently.

2. (20%) How many possible ways are there to select seven integers from $\{1, 2, 3, \ldots, 18\}$ such that no two of them are adjacent to each other? For example, $\{1, 3, 5, 8, 12, 16, 18\}$ is a legal selection while $\{1, 4, 6, 8, 9, 15, 17\}$ is not.

   Give your answers in the simplest form as possible. Explain how you derive your answer.

3. (20%) Show that the following equality is correct.

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j} = 3^n, \quad \text{where } \binom{0}{0} \text{ is considered as 1.}$$

   Hint: Use combinatorial argument.

4. (15%) Find the coefficient of x^n in the following generating function:

$$\frac{1}{(x-3)(1+2x)}.$$

5. (15%)

   (a) (10%) Find the exponential generating function

$$A(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots$$

   where a_n is the number of n-digit ternary strings such that there are odd number of `0`s, even number of `1`s, and at least one `2`.

(b) (5%) Find the exponential generating function

$$B(x) = b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + b_3\frac{x^3}{3!} + \cdots$$

where b_n is the number of $\underline{(n+1)\text{-digit}}$ ternary strings such that there are odd number of `0`s, even number of `1`s, and at least one `2`.

Hint: Discover the relationship between $A(x)$ and $B(x)$, and then compute $B(x)$.

6. Let n be a positive integer. Let a_n denote the number of ways to partition n into even number of integers. Let b_n denote the number of ways to partition n, whose largest part is an even integer.

(10%) Show that $a_n = b_n$.

Example: Consider $n = 6$. The following are the ways to partition n into even number of integers:

$$\{1,5\},\quad \{2,4\},\quad \{3,3\},\quad \{1,1,2,2\},\quad \{1,1,1,3\},\quad \{1,1,1,1,1,1\}$$

so that $a_6 = 6$. In contrast, we can partition n so that the largest part is an even number as follows:

$$\{1,1,1,1,2\},\quad \{1,1,2,2\},\quad \{2,2,2\},\quad \{1,1,4\},\quad \{2,4\},\quad \{6\}$$

so that $b_6 = 6$.