Signal and System HW 6

1、 $\boxed{7.43}$

① ∵ $\frac{d^2 y_c(t)}{dt^2} + 4\frac{dy_c(t)}{dt} + 3y_c(t) = x_c(t)$

∴ $(jw)^2 Y_c(jw) + 4(jw)Y_c(jw) + 3Y_c(jw) = X_c(jw)$

∴ $H(jw) = \frac{Y_c(jw)}{X_c(jw)} = \frac{1}{-w^2+4wj+3}$

$= \frac{1}{(3+jw)(1+jw)}$

$= \frac{\frac{1}{2}}{(1+jw)} - \frac{\frac{1}{2}}{3+jw}$

∵ $e^{-at}u(t) \to \frac{1}{a+jw}$

∴ $h(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-3t}u(t)$

② ∵ $x_p(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$ ∴ $X_p(jw) = X(e^{jwT})$

Also, $X_c(jw) = T \cdot X_p(jw) =$

$= T \cdot X(e^{jwT})$ for $-\frac{\pi}{T} \le w \le \frac{\pi}{T}$

∴ $Y_c(jw) = H(jw) \cdot X_c(jw)$

$= H(jw) \cdot T\, X(e^{jwT})$, $-\frac{\pi}{T} \le w \le \frac{\pi}{T}$

∴ $Y_p(jw)$: one period

$\Rightarrow Y_p(jw) = \frac{1}{T} Y_c(jw)$

$= H(jw) X(e^{jwT})$, $-\frac{\pi}{T} \le w \le \frac{\pi}{T}$

∴ $Y(e^{jwT}) = H(jw) \times X(e^{jwT})$

$\Omega = wT$ ↓

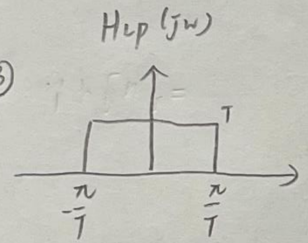
$Y(e^{j\Omega}) = H(j\frac{\Omega}{T}) \cdot X(e^{j\Omega})$, $-\pi \le \Omega \le \pi$

$x[n] \longrightarrow \boxed{H(e^{jw})} \rightarrow y[n]$

$(X(e^{j\Omega}))$ $(Y(e^{j\Omega}))$

∴ $H(jw) = H(j\frac{\Omega}{T})$, $-\pi \le \Omega \le \pi$ #.

③



∴ $h_{lp}(t) = \frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}$

∴ $h[n] = [\,h(t) * h_{lp}(t)\,]_{t=nT}$

$= \left[\left(\frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-3t}u(t)\right) * \left(\frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}\right) \right]_{t=nT}$

$= \left[\frac{T}{2}\int_0^{\infty} (e^{-\tau} - e^{-3\tau}) \left(\frac{\sin(\frac{\pi(t-\tau)}{T})}{\frac{\pi(t-\tau)}{T}} \right) d\tau \right]_{t=nT}$ #.

2,

(a) $x(t) = e^{-2t}u(t) + e^{-3t}(\sin 3t)u(t)$

$\because e^{-2t}u(t) \xrightarrow{L} \frac{1}{s+2}$, $Re\{s\} > -2$

$e^{-3t}\sin(3t)u(t) = \frac{1}{2j}[e^{-(3-3j)t} - e^{-(3+3j)t}]u(t)$

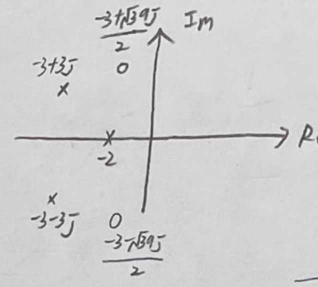
$\therefore e^{-3t}\sin(3t)u(t) \xrightarrow{L} \frac{3}{(s+3)^2+9}$, $Re\{s\} > -3$

$\therefore x(t) \xrightarrow{L} \frac{1}{s+2} - \frac{3}{(s+3)^2+9}$

$= \frac{s^2+3s+12}{s^3+8s^2+30s+36}$, $Re\{s\} > -3$ #

$\therefore$ pole $\rightarrow (s+2)[(s+3)^2+9] = 0$, $s = -2, -3 \pm 3j$

zero $\rightarrow s^2+3s+12 = 0$, $s = \frac{-3 \pm \sqrt{39}j}{2}$



#

(b) $x(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 3-t, & 1 \le t \le 2 \end{cases}$

$x_1(t) = 2t[u(t) - u(t-1)]$

$x_2(t) = (3-t)[u(t-1) - u(t-2)]$

$\therefore x(t) = x_1(t) + x_2(t) = 2t[u(t)-u(t-1)] + (3-t)[u(t-1)-u(t-2)]$

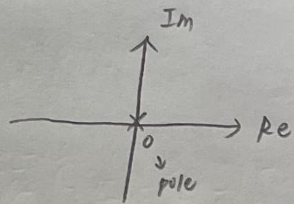
$= 2tu(t) - 2tu(t-1) + 3u(t-1) - 3u(t-2) - tu(t-1) + tu(t-2)$

$\therefore X(s) = \mathcal{L}\{2tu(t)\} - \mathcal{L}\{2(t+1)u(t)\}e^{-s} + \mathcal{L}\{3u(t)\}e^{-s} - \mathcal{L}\{3u(t)\}e^{-2s} - \mathcal{L}\{(t+1)u(t)\}e^{-s} + \mathcal{L}\{(t+2)u(t)\}e^{-2s}$

$= \frac{2}{s^2} - (\frac{2}{s^2} + \frac{2}{s})e^{-s} + \frac{3}{s}e^{-s} - \frac{3}{s}e^{-2s} - (\frac{1}{s^2} + \frac{1}{s})e^{-s} + (\frac{1}{s^2} + \frac{2}{s})e^{-2s}$

$= \frac{2}{s^2} - \frac{3}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-2s}$

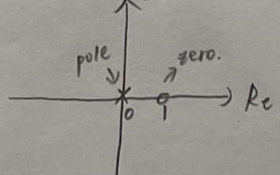
$= \frac{2 - 3e^{-s} + e^{-2s} - se^{-2s}}{s^2}$ # $Re\{s\} > 0$



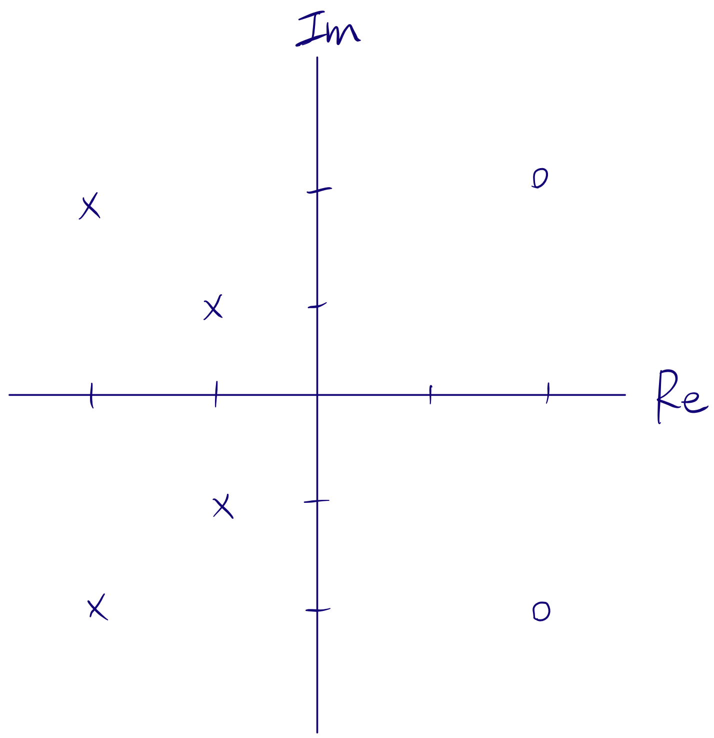
(c) $x(t) = \delta(2t) + u(-3t)$

$= \delta(t) + u(-t)$

$\therefore X(s) = 1 + (-\frac{1}{s}) = 1 - \frac{1}{s} = \frac{s-1}{s}$, $Re\{s\} > 0$ #



3.



ROC of above plot: $Re\{s\} < -2$ or $Re\{s\} > -1$

① $e^{-3t} x(t) \xleftrightarrow{\mathcal{L}} X(s+3)$

ROC R_1 of ① is shifted by 3 to the left.

If $x(t)e^{-3t}$ is absolutely integrable, then R_1 must include the $j\omega$ axis.

R is $Re\{s\} > -1$

② $$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad Re\{s\}>-1$$

$$x(t)*(e^{-t}u(t)) \xleftrightarrow{\mathcal{L}} \frac{X(s)}{s+1}, \quad R_2 = R \cap \{Re\{s\}>-1\}$$

If $x(t)*(e^{-t}u(t))$ is absolutely integrable, then R_2 must include the $j\omega$ axis.

R is $Re\{s\}>-1$

③ If $x(t)=0,\ t>1$, then the signal is left-sided signal or a finite-duration signal.

R is $Re\{s\}<-2$

④ If $x(t)=0,\ t<-1$, then the signal is right-sided signal or a finite-duration signal.

R is $Re\{s\}>-1$

4. Assume $x_1(t) = u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s}$, $Re\{s\} > 0$

$\Rightarrow X_1(s)$ has a pole at $s=0$.

Laplace transform of the output $y_1(t)$

$$Y_1(s) = H(s) X_1(s)$$

Since ② $Y_1(s)$ is absolutely integrable, $H(s)$ must have a zero at $s=0$ which can cancels out the pole of $X_1(s)$ at $s=0$.

Assume $x_2(t) = tu(t) \xleftrightarrow{\mathcal{L}} X_2(s) = \frac{1}{s^2}$, $Re\{s\} > 0$

$\Rightarrow X_2(s)$ has two poles at $s=0$.

Laplace transform of the output $y_2(t)$

$$Y_2(s) = H(s) X_2(s)$$

Since ③ $Y_2(s)$ isn't absolutely integrable, $H(s)$ doesn't have two zeros at $s=0$.

$H(s)$ has one zero at $s=0$. (from ② & ③)

Assume $f(t) = \frac{d^2h(t)}{dt} + 2\frac{dh(t)}{dt} + 2h(t)$ (from ④)

Taking the Laplace transform of both sides

$$\Rightarrow F(s) = s^2 H(s) + 2s H(s) + 2H(s)$$

$$\Rightarrow \quad H(s) = \frac{F(s)}{s^2+2s+2}$$

Since ④ said that $f(t)$ is of finite duration,

$F(s)$ has no poles in the finite s-plane.

(from property 3 of ROC)

$$\Rightarrow \quad H(s) = \frac{A\prod_{i=1}^{N}(s-z_i)}{s^2+2s+2}$$

where z_i, $i=1,2,\ldots,N$ represent the zeros of $F(s)$,

and A is a constant.

Since ⑤ the degree of the denominator polynomial must be greater than the degree of the numerator polynomial.

$$\Rightarrow \quad H(s) = \frac{A(s-z_1)}{s^2+2s+2}$$

From above conclusion, we know that $H(s)$ has a zero at $s=0$,

$$\Rightarrow \quad H(s) = \frac{As}{s^2+2s+2}$$

Since ① $H(1) = 0.2 \quad \Rightarrow \quad H(1) = \frac{A}{1+2+2} = 0.2 \quad A=1$

$$\Rightarrow \quad H(s) = \frac{s}{s^2+2s+2}$$

the poles at $-1 \pm j$, the zero at 0.

since causal & stable $\Rightarrow$ $Re\{s\} > -1$

5. a)

$$X(s) = \frac{s+2}{s-2} = 1 + \frac{-2}{s-2}$$, We are given that $x(t)=0$ when $t>0$.

Thus, $x(t)$ is left sided ⇒ According to Property 5 (p.666), the ROC of $X(s)$ is $Re\{s\} < 2$

Let $y_1(t) = -\frac{2}{3}e^{2t}u(-t)$, $y_2(t) = \frac{1}{3}e^{-t}u(t)$

$$\begin{cases} Y_1(s) = \int_{-\infty}^{\infty} -\frac{2}{3}e^{2t}u(-t)e^{-st}dt = -\frac{2}{3}\int_{-\infty}^{0} e^{t(2-s)}dt = \frac{2}{3}\frac{1}{s-2}, \quad Re\{s\} < 2 \\ Y_2(s) = \int_{-\infty}^{\infty} \frac{1}{3}e^{-t}u(t)e^{-st}dt = \frac{1}{3}\int_{0}^{\infty} e^{t(-1-s)}dt = \frac{1}{3}\frac{1}{1+s}, \quad Re\{s\} > -1 \end{cases}$$

$$Y(s) = Y_1(s) + Y_2(s) = \frac{2}{3}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s+1} = \frac{s}{(s+1)(s-2)}, \quad -1 < Re\{s\} < 2.$$

$$\boxed{H(s) = \frac{Y(s)}{H(s)} = \frac{s}{(s+1)(s+2)}}$$

By convolution property (p.687), ROC of $Y(s)$ contains ROC(H(s)) ∩ ROC(X(s)).

$\Rightarrow$ $\boxed{\text{ROC of } H(s) \text{ is } \{s \mid s \in \mathbb{C},\ Re\{s\} > -1\}}$ #

b)

$$H(s) = \frac{s}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$, $Re\{s\} > -1 \Rightarrow h(t)$ is right-sided.

$$\boxed{h(t) = 2e^{-t}u(t) - e^{-2t}u(t)}$$ #

(c) $x(t) = e^{3t}$, $t \in (-\infty, \infty)$

$$x(t) = e^{3t} \rightarrow \boxed{h(t)} \rightarrow y(t)$$

eigenvalue eigenfunction

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{3(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\, e^{st}\Big|_{s=3} = H(s)\,e^{st}\Big|_{s=3}$$

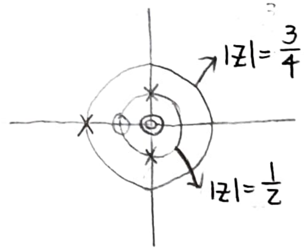
$$\Rightarrow \boxed{y(t) = H(3)e^{3t} = \frac{3}{20}e^{3t}}$$ #

6.

$$X(z) = \frac{1-\frac{1}{4}z^{-2}}{(1+\frac{1}{4}z^{-2})(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})} = \frac{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})}{(1+j\frac{1}{2}z^{-1})(1-j\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} = \frac{1+\frac{1}{2}z^{-1}}{(1+j\frac{1}{2}z^{-1})(1-j\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$$

zeros : $z = -\frac{1}{2}$, 0 (Repeated root)

poles : $z = -\frac{3}{4}$, $i\frac{1}{2}$, $-i\frac{1}{2}$.



$$X(z) = \frac{A}{1+j\frac{1}{2}z^{-1}} + \frac{B}{1-j\frac{1}{2}z^{-1}} + \frac{C}{1+\frac{3}{4}z^{-1}}$$

① $|z| > \frac{3}{4}$

$$x[n] = A(-j\tfrac{1}{2})^n u[n] + B(j\tfrac{1}{2})^n u[n] + C(-\tfrac{3}{4})^n u[n]$$

② $\frac{1}{2} < |z| < \frac{3}{4}$

$$x[n] = A(-j\tfrac{1}{2})^n u[n] + B(j\tfrac{1}{2})^n u[n] + C(-\tfrac{3}{4})^{-n-1} u[-n-1]$$

③ $|z| < \frac{1}{2}$

$$x[n] = A\cdot(-j\tfrac{1}{2})^{-n-1} u[-n-1] + B(j\tfrac{1}{2})^{-n-1} u[-n-1] + C(-\tfrac{3}{4})^{-n-1} u[-n-1]$$

(where $A = \frac{1+j}{2+3j}$, $B = \frac{1-j}{2-3j}$, $C = \frac{3}{13}$) #

7.

$$x_1[n]=\left(\tfrac{1}{2}\right)^n u[n] \overset{ZT}{\rightleftharpoons} X_1(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|>\tfrac{1}{2}$$

$$x_2[n]=\left(\tfrac{1}{3}\right)^n u[n] \overset{ZT}{\rightleftharpoons} X_2(z)=\frac{1}{1-\frac{1}{3}z^{-1}},\quad |z|>\tfrac{1}{3}$$

Using time reversal and shift properties

$$x_1[-n+2] \overset{ZT}{\rightleftharpoons} z^{-2}X_1(z^{-1}),\quad |z|<2$$

$$\underset{-(n-2)}{} $$

$$x_2[n-1] \overset{ZT}{\rightleftharpoons} z^{-1}X_2(z),\quad |z|>\tfrac{1}{3}$$

$$\therefore\ y[n]=x_1[-n+2]*x_2[n-1] \overset{ZT}{\rightleftharpoons} z^{-3}X_1(z^{-1})X_2(z),\quad \tfrac{1}{3}<|z|<2$$

$$\Rightarrow Y(z)=\frac{z^{-3}}{(1-\frac{1}{2}z)(1-\frac{1}{3}z^{-1})}$$

8.

$$H(z)=\frac{1}{z-\frac{5}{2}+z^{-1}}=\frac{z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}=\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}=\frac{-\frac{2}{3}}{1-\frac{1}{2}z^{-1}}+\frac{\frac{2}{3}}{1-2z^{-1}}$$

poles: $z=\frac{1}{2}$, $z=2$

zeros: $z=0$

For ROC: $|z|<\frac{1}{2}$

$$h[n]=-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^n u[n]+\tfrac{2}{3}(2)^n u[n]$$

$\frac{1}{2}<|z|<2$

$$h[n]=-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^n u[n]-\tfrac{2}{3}(2)^n u[-n-1]$$

$|z|>2$

$$h[n]=\tfrac{2}{3}\left(\tfrac{1}{2}\right)^n u[-n-1]-\tfrac{2}{3}(2)^n u[-n-1]$$

By substituting $y[n]=h[n]$, we can verify that whether $x[n]=\delta[n]$.

Case $|z|<\frac{1}{2}$,

$$h[n-1]-\tfrac{5}{2}h[n]+h[n+1]$$

$$=-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n-1}u[n-1]+\tfrac{2}{3}(2)^{n-1}u[n-1]+\tfrac{5}{3}\left(\tfrac{1}{2}\right)^n u[n]-\tfrac{5}{3}(2)^n u[n]-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n+1}u[n+1]+\tfrac{2}{3}(2)^{n+1}u[n+1]$$

$$=\begin{cases} 0 & ,n<-1 \\ -\frac{2}{3}+\frac{2}{3}=0 & ,n=-1 \\ \frac{5}{3}-\frac{5}{3}-\frac{2}{3}(\frac{1}{2})+\frac{2}{3}(2)=1, & n=0 \\ -\frac{2}{3}+\frac{2}{3}+\frac{5}{3}(\frac{1}{2})-\frac{5}{3}(2)-\frac{2}{3}\times(\frac{1}{2})^2+\frac{2}{3}(2)^2=0, & n=1 \\ 0, & n>1 \end{cases}$$

$$-\tfrac{2}{3}\left(\tfrac{1}{2}\right)+\tfrac{2}{3}\times 2$$

Case $\frac{1}{2}<|z|<2$

$$h[n-1]-\tfrac{5}{2}h[n]+h[n+1]$$

$$=-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n-1}u[n-1]-\tfrac{2}{3}(2)^{n-1}u[-n]+\tfrac{5}{3}\left(\tfrac{1}{2}\right)^n u[n]+\tfrac{5}{3}(2)^n u[-n-1]-\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n+1}u[n+1]-\tfrac{2}{3}(2)^{n+1}u[-n-2]$$

$$=\delta[n]$$

Case $|z|>2$

$$h[n-1]-\tfrac{5}{2}h[n]+h[n+1]$$

$$=\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n-1}u[-n]-\tfrac{2}{3}(2)^{n-1}u[-n]-\tfrac{5}{3}\left(\tfrac{1}{2}\right)^n u[-n-1]-\tfrac{5}{3}(2)^n u[-n-1]+\tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n+1}u[-n-2]-\tfrac{2}{3}(2)^{n+1}u[-n-2]$$

$$=\delta[n]$$