Linear Algebra, EE 10810/EECS 205004

2nd Exam (Dated: Fall, 2021)

Total scores: 120%

+12

- 1. (±30%) [True or False] Note that: a Correct answer gaining +3%; but a Wrong answer loosing -3% (答錯倒扣).
- X (1) Any determinant $\delta : \overline{\overline{M}}_{n \times n}(\mathbf{F}) \to \mathbf{F}$ is a linear transform.
- (2) If \overline{E} is an elementary matrix, then $\det(\overline{E}) = \pm 1$.
- \emptyset (3) Every system of n linear equations in n unknowns can be solved by Cramer's rule.
- (4) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
- (3) Similar matrices always have the same eigenvalues.
- \mathcal{H} (6) If λ is an eigenvalue of a linear operator \hat{T} , then each element of E_{λ} is an eigenvector of \hat{T} .
- χ (7) If 2 is an eigenvalue of $\overline{\overline{A}} \in \overline{\overline{M}}_{n \times n}(\mathbf{C})$, then $\lim_{n \to \infty} \overline{\overline{A}}^m$ exists.
- (8) An inner product is linear in both components.
- $\mathcal{O}(9)$ If $\langle \vec{x}, \vec{y} \rangle = 0$ for all \vec{x} in an inner product space, then $\vec{y} = \vec{0}$.
- χ (10) The triangle inequality only holds in finite-dimensional inner product spaces.

(20%) [Determinant of Hessenberg Matrix]

A Hessenberg matrix is a $n \times n$ matrix, $\overline{\overline{A}}_{n \times n}$, with the matrix element $a_{ij} = 0$ for i > j + 1.

(b) (10%) Based on (a), by filling the empty box, show that the determinant of a $n \times n$ Hessenberg matrix has the

$$\det\left[\overline{\overline{A}}_{n\times n}\right] = a_{nn} \det\left[\overline{\overline{A}}_{n-1\times n-1}\right] + \sum_{i=1}^{n-1} [??]$$
 (2)

3. (20%) [Eigenvalues of Fibonacci Series]

Fibonacci sequence $\{F_i\}$ starts

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for n > 1. In terms of a 2-dimensional system of linear difference equations to describes the Fibonacci sequence, we have

 $\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix},$ or alternatively $\vec{F}_{k+1} = \overline{\overline{A}} \vec{F}_k$. With Eq. (3), show that $\begin{bmatrix} \gamma & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{\sqrt{5}} \\ \frac{1-\sqrt{5}}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{\sqrt{5}} \\ \frac$

- (b) (10%) Show that the ratio of consecutive Fibonacci numbers converges to the Golden ratio, i.e.,

$$\left(\begin{array}{c} \left(\begin{array}{c} 1+JS\\ \overline{2}\end{array}\right)^{n}, \quad \left(\begin{array}{c} 1-JS\\ \overline{2}\end{array}\right)^{n} \\ \left(\begin{array}{c} 1+JS\\ \overline{2}\end{array}\right)^{n-1}, \quad \left(\begin{array}{c} 1-JS\\ \overline{2}\end{array}\right)^{n-1} \\ \left(\begin{array}{c} 1+JS\\ \overline{2}\end{array}\right)^{n-1}, \quad \left(\begin{array}$$

4. (15%) [System of Differential Equations]

Find the general solution to the system of differential equations:

$$\frac{dx_1}{dt} = x_1 + x_3 \tag{5}$$

$$\frac{dx_2}{dt} = x_2 + x_3 \tag{6}$$

$$\frac{dx_3}{dt} = 2x_3 \tag{7}$$

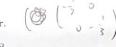
5. (20%) [Cayley-Hamilton Theorem]

Suppose that a 2×2 matrix $\overline{\overline{M}}$ satisfies

$$\overline{\overline{M}}^2 + 5 \overline{\overline{M}} + 6 \overline{\overline{I}} = \overline{\overline{O}},$$
 (8)

where $\overline{\overline{I}}$ is a 2 × 2 identity matrix and $\overline{\overline{O}}$ is a 2 × 2 zero matrix.

- (a) (5%) Determine the eigenvalues of $\overline{\overline{M}}$. $\overline{\overline{M}}^{-1}$ $\overline{\overline{M}}^{-1}$; If not, explain your answer.



(b) (10%) Is $\overline{\overline{M}}^{-1}$ diagonalizable? It yes, nmu , , , , , , , , (c) (5%) Calculate $\overline{\overline{M}}^{(-2)}$ with the help of Cayley-Hamilton theorem. $\begin{pmatrix} \zeta_{i}^{1} & \circ & \vdots \\ \circ & \zeta_{i}^{1} \end{pmatrix}$

6. (15%) [Gram-Schmidt orthogonalization process] For the given subset S of the inner product space $V = \mathbb{R}^3$,

$$S = {\vec{v}_1, \vec{v}_2, \vec{v}_3} = {(1,0,1), (0,1,1), (1,3,3)}$$
(9)

- (a) (5%) Show that \vec{v}_1 , \vec{v}_2 , and \vec{v}_3 are linearly independent to each other.
- (b) (10%) Apply the Gram-Schmidt process to obtain an orthonormal basis.