Exam 1 - tutorial

10810EECS206001

Q1

林恩德

- Consider the proposition $(p \wedge \neg q) \leftrightarrow (p \vee q)$.

(15%) Find an equivalent proposition that is as short as possible. Show your steps.

Solution:

- Truth table
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Truth table

p	q	p ∧ ¬ q	p ∨ q	(p ∧ ¬ q) ↔ (p ∨ q)
T	T	F	T	F
T	F	T	T	T
F	T	F	T	F
F	F	F	F	T

Through the truth table we can get the answer: (p ∧ ¬ q) ↔ (p ∨ q) is equivalent to ¬ q

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

$$(p \land \neg q) \leftrightarrow (p \lor q)$$

$$\equiv ((p \land \neg q) \rightarrow (p \lor q)) \land ((p \lor q) \rightarrow (p \land \neg q))$$

$$\equiv (\neg p \lor q \lor p \lor q) \land ((\neg p \land \neg q) \lor (p \land \neg q))$$

$$\equiv T \land ((\neg p \lor p) \land \neg q)$$

$$\equiv T \land (T \land \neg q)$$

$$\equiv \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$(p \land \neg q) \leftrightarrow (p \lor q)$

$\equiv ((p \land \neg q) \land (p \lor q)) \lor (\neg(p \land \neg q) \land \neg (p \lor q))$

$\equiv ((p \land \neg q \land p) \lor (p \land \neg q \land q)) \lor ((\neg p \lor q) \land (\neg p \land \neg q))$

$\equiv ((p \land \neg q) \lor F) \lor ((\neg p \land \neg q \land \neg p) \lor (\neg p \land \neg q \land q))$

$\equiv (p \land \neg q) \lor ((\neg p \land \neg q) \lor F)$

$\equiv (p \land \neg q) \lor (\neg p \land \neg q)$

$\equiv (p \lor \neg p) \land \neg q$

$\equiv T \land \neg q$

$\equiv \neg q$

Grading policy

- The solution and the answer are clear -> get full credits
- The solution is clear but no answer -> get 10 pts
- The solution has some logic problem but get the right answer -> get 5 pts
- The solution has some writing problem and has the right answer -> - 1 pts
- The solution has problem and get the wrong answer -> 0 pts

Q2

林毓淇

(a)

Premises:

(i) p

(ii) q

(iii) $p \leftrightarrow q$

(iv) $p \rightarrow q$

<u>Background Knowledge</u>

If $A \rightarrow B,$ then

- If B is F, then A must be F.
- If B is T, then A don’t care.

(a)

Truth table:

p	q	$p \leftrightarrow q$	$p \rightarrow q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	T	T

Zero premise is impossible because we know nothing without premises.

From the truth table,
if one premise is the minimum,
then we should have one premise imply all three other promises.
That is, at least one of the following four cases should hold.
case 1 (p): (p → q) is F
case 2 (q): (q → p) is F
case 3 ($p \leftrightarrow q$): (($p \leftrightarrow q$) → p) is F
case 4 ($p \rightarrow q$): (($p \rightarrow q$) →p) is F
So, we know keeping one premise is not enough.

(a)

Given a collection of premises, say $p, q, r, s, \ldots$, we say a premise p is *redundant* if it is the case that when all premises except p are true would imply p is true. So, p is redundant if

$$\boxed{(q \wedge r \wedge s \wedge \cdots) \to p}$$

is always true.

Truth table

p	q	$p \leftrightarrow q$	$p \to q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	T	T

What if minimum is 2?

From the truth table,

1. $(p \wedge q \wedge (p \leftrightarrow q)) \to (p \to q)$

   $\Rightarrow$ $(p \to q)$ is redundant.

2. $(p \wedge q) \to (p \leftrightarrow q)$

   $\Rightarrow$ $(p \leftrightarrow q)$ is redundant.

$\Rightarrow$ We know that $(p \wedge q)$ implies $(p \to q)$ and $(p \leftrightarrow q)$.

$\Rightarrow$ minimum is 2. (Keep (i) and (ii))

If you can prove the other 2 premises that imply the other premises, your answer is also correct.

(i) p, q

(ii) p, $(p \leftrightarrow q)$

(iii) p, $(p \to q)$

(iv) q, $(p \leftrightarrow q)$

(a)

Score:

1. Why 1 premise is not enough (5 points)
2. Exist 1 case for 2 premises (5 points)

Common mistakes:

1. Not showing why 1 is not enough (simply write "1 is not enough")(-5 points)
2. Write truth table or answer without explanation
3. Wrong derivation

(b)

Premises:

(i) $p \rightarrow q$

(ii) $r \rightarrow (p \rightarrow q)$

(iii) $(p \rightarrow r) \vee ((\sim r) \rightarrow q)$

Zero premise is impossible.

If (i) is the premise, (i) is true.
Since anything imply true is always true, (ii) is true.

(i) is T
≡ ((p is F) or (q is T))
≡ $((p \rightarrow r)$ or$((\sim r) \rightarrow q))$
≡ (iii) is T.
So keep (i) is enough. (minimum = 1)

Q3

洪嘉陽

Prove or disprove

1. Ans: Prove
2. a , b are integer

   If a+b is a multiple of 3 $\Leftrightarrow 3 \mid (a+b)$

   $\Leftrightarrow \exists\, k \in \mathbb{Z}\ s.t.\ a+b=3z$
3. Want to show: $3|(a^3+b^3)$
4. i.e., $\exists l \in \mathbb{Z}\ s.t\ a^3+b^3=3l$

Proof

1. Let $a + b = 3k \ where \ k \in \mathbb{Z}$
2. And we have $a^3 + b^3$
$$= (a + b)(a^2 - ab + b^2) \ \forall a, b \in \mathbb{Z}$$
3. Then $a^3 + b^3 = 3k(a^2 - ab + b^2)$
$$= 3(ka^2 - kab - kb^2)$$
4. Therefore, $3 \mid (a^3 + b^3)$

Common mistake

1. Suppose $a + b = 3k \; where \; k \in \mathbb{N}$ (-1)
2. Mistake in calculation (-1)
3. In Proof by case, Suppose
   $a = 3k + 1 \,, b = 3k + 2 \; for \; k \in \mathbb{Z}$ (-10)

   You can suppose
   $$a = 3k_1 + 1 \,, b = 3k_2 + 2 \;\; for \; k_1, k_2 \in \mathbb{Z}$$

Q4

陳弘欣

Solution

Find three different pairs of integers n and m such that $2^n = 3 + 5^m$.

Sol. $(n, m) = (2, 0), (3, 1), (7, 3)$

Grading: each correct answer for 5 points.

Q5

李沛倫

5. Consider the equation $z^{13} - z^2 - 15015 = 0$.

(a) (10%) Show that the equation does not have any integral root.

Hint: Show that for any z that satisfies the above equation, (i) z cannot be an odd number, and (ii) z cannot be an even number.

(b) (10%) Show that the equation does not have any rational root.

(a)

(i) Assume z is an odd number. (5pts)

Then z^{13} and z^2 are odd numbers, since the product of odd numbers is odd.

Then $z^{13} - z^2$ is even number, since odd number minus odd numbers is even.

Since 15015 is an odd number, $z^{13} – z^2 – 15015$ will never be zero.

Contradiction occurs, so z is not an odd number.

(ii) Assume z is an even number. (5pts)

Then z^{13} and z^2 are even numbers, since the product of even numbers is even.

Then $z^{13} - z^2$ is even number, since even number minus even numbers is even.

Since 15015 is an odd number, $z^{13} – z^2 – 15015$ will never be zero.

Contradiction occurs, so z is not an even number.

Since z is not odd or even number, z is not an integer.

The equation does not have any integral root.

(b)

Method 1:

(2pts)

Let the rational root $z = \frac{p}{q}$, where p, q $\in \mathbb{Z}$ and gcd(p, q) = 1.

$z^{13} - z^2 - 15015 = 0$

$\Rightarrow (\frac{p}{q})^{13} - (\frac{p}{q})^2 - 15015 = 0$

$\Rightarrow p^2 \times (p^{11} - q^{11}) = 15015 \times q^{13}$

Consider 4 cases: (each 2pts)

(p, q) = (odd, even), (even, odd), (odd, odd), (even, even)

(1) p is odd, q is even

Then p^2 and p^{11} are odd numbers. q^{11} and q^{13} are even numbers.

$\Rightarrow (p^{11} - q^{11})$ is odd.

$\Rightarrow p^2 \times (p^{11} - q^{11})$ is odd .

However, $15015 \times q^{13}$ is even. Thus contradiction occurs.

(2) p is even, q is odd

Then p^2 and p^{11} are even numbers. q^{11} and q^{13} are odd numbers.

$\Rightarrow$ $(p^{11} - q^{11})$ is odd.

$\Rightarrow$ $p^2 \times (p^{11} - q^{11})$ is even.

However, $15015 \times q^{13}$ is odd. Thus contradiction occurs.

(3) p is odd, q is odd

Then p^2 and p^{11}, q^{11} and q^{13} are odd numbers.

$\Rightarrow$ $(p^{11} - q^{11})$ is even.

$\Rightarrow$ $p^2 \times (p^{11} - q^{11})$ is even.

However, $15015 \times q^{13}$ is odd. Thus contradiction occurs.

(4) p is even, q is even

It contradicts to the assumption that $\gcd(p, q) = 1$.

Thus, the equation does not have any rational root.

Method 2:

By rational root theorem.
Given an equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$, where $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$.
Then all the possible rational roots can be written as $z = \pm \dfrac{\text{factor of } a_0}{\text{factor of } a_n}$

Consider the equation $z^{13} - z^2 - 15015 = 0$, where $a_0 = 15015$ and $a_n = 1$.
The possible rational roots $z = \pm$ (factor of 15015),
which are all integers ($\pm 1, \pm 3, \pm 5, \pm 7, \pm 11, \pm 13$).

By part (a), z can not be an integer.
Thus, the equation does not have any rational roots.

Method 3:

Let the rational root $z = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. **(2pts)**

(5pts)

$(\frac{p}{q})^{13} - (\frac{p}{q})^2 - 15015 = 0$

$\Rightarrow p^2 \times (p^{11} - q^{11}) = 15015 \times q^{13}$

$\Rightarrow p^2 \times (p^{11} - q^{11}) = 1 \times 3 \times 5 \times 7 \times 11 \times 13 \times q^{13}$

Since in the right side of the formula,

we can't find two identical numbers such that p is an integer except ± 1.

(3pts)

Let $p = \pm 1$.

$\Rightarrow 1 \times (\pm 1 - q^{11}) = 15015 \times q^{13}$

$\Rightarrow 15015 \times q^{13} + q^{11} = \pm 1$

(i) Let q be an even number.

Then $15015 \times q^{13} + q^{11} = \text{odd} \times \text{even} + \text{even} = \text{even} + \text{even} = \text{even}$.

The contradiction occurs.

(ii) Let q be an odd number.

Then $15015 \times q^{13} + q^{11} = \text{odd} \times \text{odd} + \text{odd} = \text{odd} + \text{odd} = \text{even}$.

The contradiction occurs.

$\Rightarrow$ We can't find any integer q that satisfies this equation.

$\Rightarrow p \neq \pm 1$.

Thus, the equation does not have any rational root.

Q6

陳咨蓉

6. (Adapted from a logical puzzle in an online competition)

Raymond is visiting the famous country, *Pureland*, where citizens there are either honest (always tell the truth) or dishonest (always lie). Raymond sees three people, let us identify them as A, B, and C, and chats with them. Suddenly, one of them said "A and B are liars", and then another one of them said "A and C are liars."

(15%) How many liars are there among these three people? Justify your answer.

Discuss with :

1. How many people are liars?
2. The sentences they say are truth or false?
3. Who are liars?

Sol:
2. The sentences they say are true or false?

I suppose the sentence “A and B are liars.” is X,

And the sentence “A and C are liars.” is Y.

F means false, and T means true.

Then we discuss the four cases below.

(1)X, Y are both T.

(2)X is F, Y is T.

(3)X is T, Y is F.

(4)X, Y are both F.

(1) X, Y are both T. ->impossible

If X, Y are T, we can know that A, B and C are liars, and two of them say the two sentences are not liars.

-> makes a contradiction(impossible)

(2)X is F, Y is T.->2 liars

If X is F, we can know that at least one of A and B is not a liar.

If Y is T, we can know that both A and C are liars.

Then we gather the information above, we can get

->A and C are liars. B is not liars.(Because at least one of A and B is not a liar, and A is liar, so B must be not liar.)

Then we check the sentences with people.

-> Y is said by B and X is said by A or C.

-> No contradiction

(3)X is T, Y is F.->2 liars

If X is T, we can know that both A and B are liars.

If Y is F, we can know that at least one of A and C is not a liar.

Then we gather the information above, we can get

->A and B are liars. C is not liars.(Because at least one of A and C is not a liar, and A is liar, so C must be not liar.)

Then we check the sentences with people.

-> Y is said by C and X is said by A or B.

-> No contradiction

(4)X, Y are both F.->2 liars

X and Y are F which means

->At least one of A and B is not a liar.

->At least one of A and C is not a liar.

a. A is a liar.

->B and C are not liars.

b. B is a liar.

->A is not liar, and C is liar, because there are two liars say the two sentences.

c. C is a liar.

->A is not liar, and B is liar, because there are two liars say the two sentences.

Prove requirement

1. How many people are liars?
2. The sentences they say are truth or false?
3. Who are liars?

Pf requirement :
1. How many people are liars?

You should prove 0, 1, 3 liars are impossible cases.

Then, you should take an example(or more) for the case which is there are two liars.

However , if you say there is an “only” one case that A is a liar, B and C are not liars.->wrong

But if you say there is a “possible” case that A is a liar, B and C are not liars.->correct

Pf requirement :
2. The sentences they say are true or false?

You should prove every case I mention before.

Pf requirement :
3. Who are liars?

You should prove every case is possible or impossible.

If you only write truth table and don’t explain why, you will get 0 point.

If you only explain one case and say all cases are in the same reason, you will be subtract 9 points.

Deduction

1. Every “X” represents your analysis is wrong, and -10.

(If you didn’t think the opposite of “A and B are liars.” is both A,B are honest or “A and C are liars.” is both A,C are honest, then you get “X” : -3)

2. Every “why”, “?” represent I think your explain is not complete, and -3 ,at most -9(more than 3 “why” or “?”).

3. If there is losing case, every case I mention before will -7, the small case will -3.

4. If you are subtracted more than 15 above already, I will not correct your answer below.

Reminding

1. If you are uncertain the word in English, you should write the word in Chinese. Because if you spell in wrong way, I may not know what you mean.

2. Please don’t write the word together, because I can’t understand your words correctly.

Q7

薛旻欣

Q7 The Sum and Product Puzzle

- Given x and y, where $1 < x < y$ and $x + y \leq 65$.
- Sam knows the sum $x + y$.
- Peter knows the product $x \cdot y$.

Then, they make four statements in order:

① Peter: I don’t know the numbers x and y.

② Sam: I already knew that you didn’t know.

③ Peter: Now I know x and y.

④ Sam: And so do I.

① Peter: I don't know the value of x and y.

- The product P has 2 or more ordered factorizations that satisfy $1 < x < y$.
  - $12 = 2 \times 6 = 3 \times 4, 60 = 2 \times 30 = 3 \times 20 = \cdots$
- P can't be some values such that…
  - $6 = 2 \times 3,\ 55 = 5 \times 11, 589 = 19 \times 31$
  - $8 = 2 \times 4, 27 = 3 \times 9$
  - $188 = 4 \times 47 = 2 \times 94$
  - …

- We suggest you start with statement ②, because it implies statement ①.

② Sam: I have known that you don't know before you make statement ①.

- Why could Sam confidently state ②?

1. The sum S can't be factorized to be $S = a + b$ for some a, b are both prime numbers.
2. Some other conditions…

The possible S

5	6	7	8	9	10	11	12	13	14
15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34
35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63	64

Sum of any two prime numbers…

x\y	3	5	7	11	13	17	19	23	29	31	…
2	5	7	9	13	15	19	21	25	31	33	…
3		8	10	14	16	20	22	26	32	34	…
5			12	16	18	22	24	28	34	36	…
7				18	20	24	26	30	36	38	…
11					24	28	30	34	40	42	…
13						30	32	36	42	44	…
17							36	40	46	48	…
19								42	48	50	…
…											…

These are even numbers…
Can they cover all even numbers?

Goldbach's conjecture

- Every even integer greater than 2 can be expressed as the sum of two primes.
- For example:

$$6 = 3 + 3$$

$$10 = 3 + 7 = 5 + 5$$

- We still need to check the double of prime numbers (to satisfy $1 < x < y, \mathrm{x} + \mathrm{y} \leq 65$): 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

$4, 6, 10 = 3 + 7, 14 = 3 + 11, 22 = 3 + 19, 26 = 3 + 23, 34 = 3 + 31, 38 = 7 + 31, \ 46 = 3 + 43, 58 = 5 + 53, 62 = 3 + 59, 74$

Eliminating sum of primes

- $2 + p$, where p is a prime number
- Even numbers.

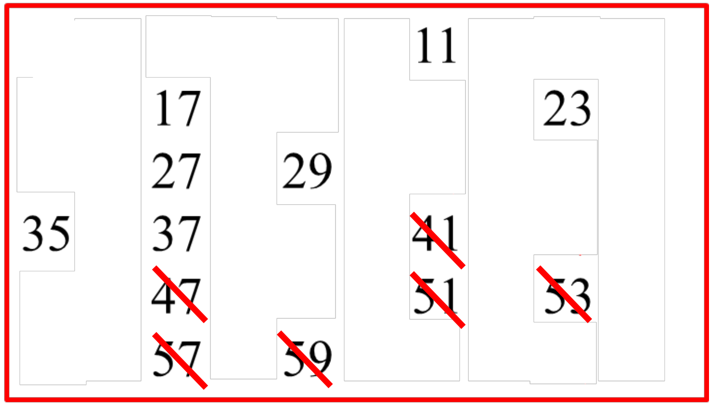
The possible S

~~5~~	~~6~~	~~7~~	~~8~~	~~9~~	~~10~~	11	~~12~~	~~13~~	~~14~~
~~15~~	~~16~~	17	~~18~~	~~19~~	~~20~~	~~21~~	~~22~~	23	~~24~~
~~25~~	~~26~~	27	~~28~~	29	~~30~~	~~31~~	~~32~~	~~33~~	~~34~~
35	~~36~~	37	~~38~~	~~39~~	~~40~~	41	~~42~~	~~43~~	~~44~~
~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~	51	~~52~~	53	~~54~~
~~55~~	~~56~~	57	~~58~~	59	~~60~~	~~61~~	~~62~~	~~63~~	~~64~~

Some other conditions…

- If $S = 59$, we can rewrite it as $59 = 4 + 55$, then Peter sees $4 \times 55 = 220$; however we can't factorize $220 = 2 * 110$, which is against $x + y \leq 65$, in this case, Peter could have known the answer.
  - $57 \rightarrow 4 \times 53, 53 \rightarrow 6 \times 47, 51 \rightarrow 4 \times 47,$
  - $47 \rightarrow 6 \times 41, 41 \rightarrow 4 \times 37,$
  - $37 \rightarrow 6 \times 31 = 3 \times 62$

The possible S



The possible Sum/Product

- Now (after statement ②), Peter knows that the possible S are 11,17,23,…
- The corresponding possible P is …

Sum	Product
11	$2 \times 9 = 18, 3 \times 8 = 24, 4 \times 7 = 28, 5 \times 6 = 30$
17	$2 \times 15 = 30, 3 \times 14 = 42, \ldots$
23	
27	
29	
35	
37	

The possible Sum/Product

- If you patiently list the numbers…

11: [18, 24, 28, 30]
17: [30, 42, 52, 60, 66, 70, 72]
23: [42, 60, 76, 90, 102, 112, 120, 126, 130, 132]
27: [50, 72, 92, 110, 126, 140, 152, 162, 170, 176, 180, 182]
29: [54, 78, 100, 120, 138, 154, 168, 180, 190, 198, 204, 208, 210]
35: [66, 96, 124, 150, 174, 196, 216, 234, 250, 264, 276, 286, 294, 300, 304, 306]
37: [70, 102, 132, 160, 186, 210, 232, 252, 270, 286, 300, 312, 322, 330, 336, 340, 342]

- If $P = 18$, Peter knows the numbers ($S = 11$).
- If $P = 30$, Peter doesn't know $S = 11$ *or* 17, then his information is not enough to make statement ③.

After statement ③

- We should eliminate the duplicated numbers.

11: [18, 24, 28]
17: [52]
23: [76, 90, 112, 130]
27: [50, 92, 110, 140, 152, 162, 170, 176, 182]
29: [54, 78, 100, 138, 154, 168, 190, 198, 204, 208]
35: [96, 124, 150, 174, 196, 216, 234, 250, 264, 276, 294, 304, 306]
37: [160, 186, 232, 252, 270, 312, 322, 330, 336, 340, 342]

- If $P = 18$, Peter knows the numbers ($S = 11$).
- Can Sam make statement ④ ?

Grading Policy

- Correct answer ($x = 4, y = 13$) (5%)
- No partial credits.

Further Studying

- WIKIPEDIA
- If the limit of $x + y \leq 65$ is altered (is 65 a magic number?)

$$x + y < 65$$

- The possible Sum buckets are 11,17,23,27,29,35

```
11: [18, 24, 28]
17: [52, 70]
23: [76, 90, 102, 112, 130, 132]
27: [50, 92, 110, 140, 152, 162, 170, 176, 182]
29: [54, 78, 100, 138, 154, 168, 190, 198, 204, 208, 210]
35: [96, 124, 150, 174, 196, 216, 234, 250, 264, 276, 286, 294, 300, 304, 306]
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- Peter knows the answer, because he know the product P.
- But Sam will never know the answer (statement ④) without the information about P.

$$x + y < 80 \; or \; 100$$

- None of the larger Sum buckets will eliminate the number 52 in bucket_17.

```
11: [18, 24, 28]
17: [52]
23: [76, 90, 112, 130]
27: [50, 92, 110, 140, 152, 162, 170, 176, 182]
29: [54, 100, 138, 154, 168, 190, 198, 204, 208]
35: [96, 124, 150, 174, 196, 216, 234, 250, 276, 294, 304, 306]
37: [160, 186, 232, 252, 270, 312, 322, 336, 340, 342]
41: [114, 148, 238, 288, 310, 348, 364, 378, 390, 400, 408, 414, 418, 420]
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11: [18, 24, 28]
17: [52]
23: [76, 112, 130]
27: [50, 92, 110, 140, 152, 162, 170, 176, 182]
29: [54, 100, 138, 154, 168, 190, 198, 204, 208]
35: [96, 124, 150, 174, 196, 216, 234, 250, 276, 294, 304, 306]
37: [160, 186, 232, 252, 322, 336, 340]
41: [114, 148, 238, 288, 310, 348, 364, 390, 400, 408, 414, 418]
47: [172, 246, 280, 370, 396, 442, 462, 480, 496, 510, 522, 532, 546, 550, 552]
51: [98, 144, 188, 230, 308, 344, 410, 440, 468, 494, 518, 560, 578, 594, 608, 620, 630, 638, 644, 648, 650]
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$$x + y \leq \cancel{865}1685$$

- The second solution occurs as $(x, y) = (4{,}61)$
- Peter knows the answer.
  (with knowledge of P)
- And Sam knows, too.
  (with information of S)
- But we'll never know the
  exact values. multiple solutions: (4,13) and (4,61)

The upperbound of $x + y$

- The second solution occurs as $(x, y) = (4,61)$

- But it's hard to determine the upper bound of $x + y$.
- Please consider if $x + y < 1685$, and note that $4 \times 61 = 203 \times 2$; while $203 + 2 = 205$, we need bucket 205 to eliminate the product 406 in bucket 65. Otherwise the product 406 still exists in the bucket of sum=65.

$x + y \leq \cancel{865}\ 1685$

- We need bucket 205 to eliminate the product 406 in bucket 65. Otherwise $(x, y) = (4,61)$ can't be an valid solution.
- However, consider $205 = 41 + 164$, while the product is $41 \times 164 = 4 \times 1681 = 2 \times 3362$
- The corresponding sums are $205, 1685, 3364$
- Then, we can see that if $x + y < 1685$ and Peter receives $P = 6724$, he immediately knows that $P = 41 \times 164$ is the only valid factorization.
- Without the information of bucket 205, Sam will never know the answer.

Score Distribution

- Average: 61.38554