Chapter 9 - Complete Response of Circuits with Two Energy Storage Elements

Exercises

Exercise 9.2-1 Find the second-order differential equation for the circuit shown in Figure E 9.2-1 in terms of *i* using the direct method.

Answer: $\dfrac{d^2 i}{dt^2}+\dfrac{1}{2}\dfrac{di}{dt}+i=\dfrac{1}{2}\dfrac{di_s}{dt}$

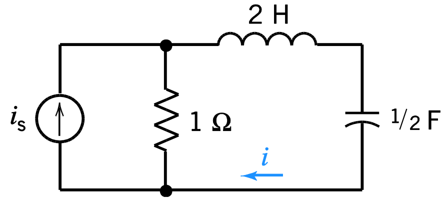
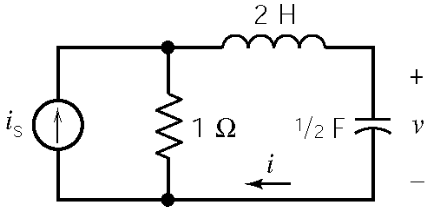


Figure E 9.2-1

Solution:



KVL a: $2\,{di}/{dt}\;+\;v\;+\;1(i-i_s)\;=\;0$

$\Rightarrow\; v\;=\;-2\,{di}/{dt}\;-i\;+\;i_s$

$$i\;=\;{1}/{2}\;{dv}/{dt}\;=\;{1}/{2}\;{d}/{dt}\left(-2\,{di}/{dt}-i+i_s\right)$$

$$=\;{1}/{2}\;{di_s}/{dt}-{1}/{2}\;{di}/{dt}-\frac{d^2 i}{dt^2}$$

$$\therefore\;\underline{\frac{d^2 i}{dt^2}\;+\;\frac{1}{2}\frac{di}{dt}\;+\;i\;=\;\frac{1}{2}\frac{di_s}{dt}}$$

Exercise 9.2-2 Find the second-order differential equation for the circuit shown in Figure E 9.2-2 in terms of *v* using the operator method.

Answer: $\frac{d^2v}{dt^2}+2\frac{di}{dt}+2v=2\frac{di_s}{dt}$

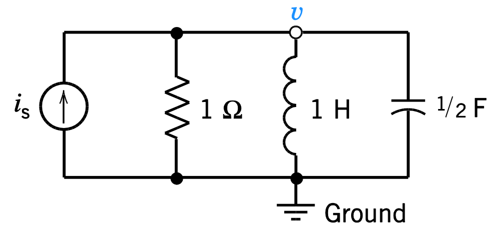
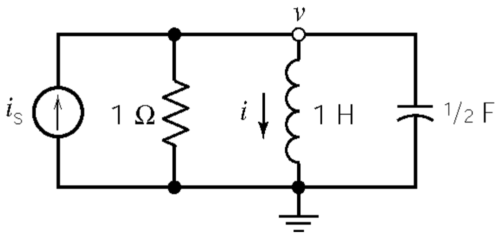


Figure E 9.2-2

Solution:



KCL at v : using $s = d/dt$

$$\underline{\frac{v}{1} + i + \tfrac{1}{2}sv = i_s} \qquad (1)$$

also $\underline{v = si}$ (2) Solving for i in (1) & plugging into (2)

yields $s^2v + 2sv + 2v = 2si_s$ *or* $\underline{\frac{d^2v}{dt} + 2\frac{dv}{dt} + 2v = 2\frac{di_s}{dt}}$

Exercise 9.3-1 Find the characteristic equation and the natural frequencies for the circuit shown in Figure E 9.3-1.

Answer: $s^2 + 7s + 10 = 0$; $s_1 = -2$, $s_2 = -5$

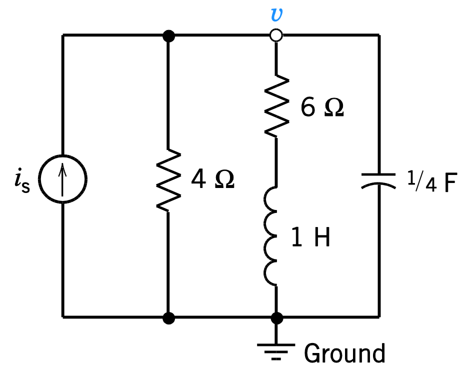
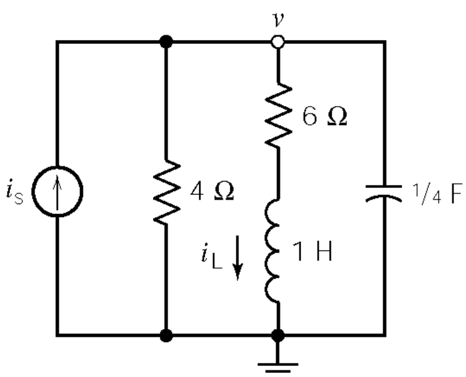


Figure E 9.3-1

Solution:



KCL at the top node : $\frac{v}{4} + i_L + \frac{1}{4}\frac{dv}{dt} = i_s$ (1)

KVL for the right mesh : $v = 6i_L + \frac{di_L}{dt}$ (2)

Plugging (2) into (1) yields

$$\frac{d^2 i_L}{dt^2} + 7\frac{di_L}{dt} + 10 i_L = 4 i_s$$

$\therefore$ characteristic equation $\Rightarrow$ $\underline{s^2 + 7s + 10 = 0}$ & natural frequencies $\Rightarrow$ $\underline{s = -2, -5}$

Exercise 9.4-1 Find the natural response of the *RLC* circuit of Figure 9.4-1 when $R = 6\ \Omega$, $L = 7$ H, and $C = 1/42$ F. The initial conditions are $v(0) = 0$ and $i(0) = 10$ A.

Answer: $v_n(t) = -84(e^{-t} - e^{-6t})$ V

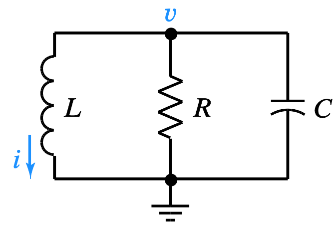


Figure 9.4-1

Solution:

This is a parallel *RLC* circuit with

$$\alpha = \frac{1}{2RC} = \frac{1}{2(6)(1/42)} = 7/2 \quad \text{and} \quad \omega_o^2 = \frac{1}{LC} = \frac{1}{(7)(1/42)} = 6$$

The roots of the characteristic equation are

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -7/2 \pm \sqrt{\left(7/2\right)^2 - 6} = -1,\ -6$$

so the natural response is

$$v_n(t) = A_1 e^{-t} + A_2 e^{-6t}$$

Need $v_n(0)$ and $\left.\frac{dv_n}{dt}\right|_{t=0}$ to evaluate A_1 & A_2. We are given $v_n(0) = 0$.

Apply KCL at the top node to get

$$i(t) + \frac{v(t)}{6} + \frac{1}{42}\frac{d}{dt}v(t) = 0 \quad \Rightarrow \quad \frac{d}{dt}v(t) = -\left(7\,v(t) + 42\,i(t)\right)$$

At time t=0,

$$\left.\frac{d}{dt}v(t)\right|_{t=0} = -\left(7\,v(0) + 42\,i(0)\right) = -\left(7\times 0 + 42\times 10\right) = -420\ \frac{\text{V}}{\text{s}}$$

So

$$\left.\begin{array}{l} v_n(0) = 0 = A_1 + A_2 \\ \left.\dfrac{dv_n}{dt}\right|_{t=0} = -420 = -A_1 - 6A_2 \end{array}\right\} A_1 = -84\,,\ A_2 = 84$$

and

$$\underline{v_n(t) = -84e^{-t} + 84e^{-6t}\ \text{V}}$$

Exercise 9.5-1 A parallel *RLC* circuit has $R = 10\ \Omega$, $C = 1$ mF, $L = 0.4$ H, $v(0) = 8$ V, and $i(0) = 0$. Find the natural response $v_n(t)$ for $t < 0$.

Answer: $v_n(t) = e^{-50t}(8 - 400t)$ V

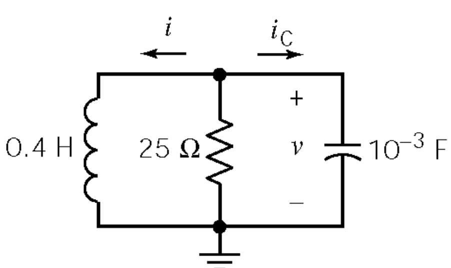
Solution:

For parallel RLC

$$\alpha = \frac{1}{2RC} = \frac{1}{2(10)(10^{-3})} = 50,\quad \omega_o^2 = \frac{1}{LC} = \frac{1}{(0.4)(10^{-3})} = 2500$$

$$\therefore s = -50 \pm \sqrt{(50)^2 - 2500} = -50,\ -50$$

$$\therefore v_n(t) = A_1 e^{-50t} + A_2 t e^{-50t}$$



with $i(0^+) = 0$ & $v(0^+) = 8$ V

$$i_C(0^+) = \frac{-v(0^+)}{10\Omega} = -0.8\ \text{V}$$

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = -800\ \frac{\text{V}}{\text{s}}$$

At $\underline{t = 0^+}$

$$v_n(0) = 8 = A_1 \Rightarrow v_n(t) = 8e^{-50t} + A_2 te^{-50t}$$

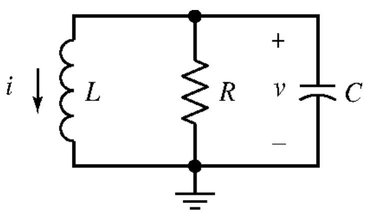
$$\frac{dv(0)}{dt} = -800 = -400 + A_2 \Rightarrow A_2 = -400$$

$$\therefore \underline{v_n(t) = 8e^{-50t} - 400t\, e^{-50t} \text{ V}}$$

Exercise 9.6-1 A parallel *RLC* circuit has $R = 62.5\ \Omega$, $L = 10$ mH, $C = 1\ \mu$F, $v(0) = 10$ V, and $i(0) = 80$ mA. Find the natural response $v_n(t)$ for $t > 0$.

Answer: $v_n(t) = e^{-8000t}\,[10\cos 6000t - 26.7\sin 6000t]$ V

Solution:

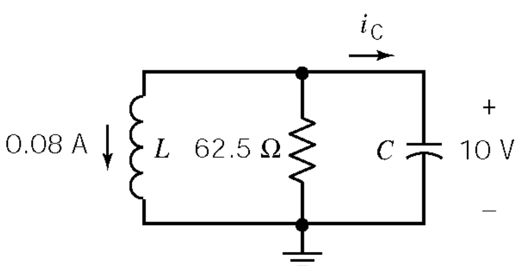


$$\alpha = \frac{1}{2RC} = \frac{1}{2(62.5)(10^{-6})} = 8000$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{(.01)(10^{-6})} = 10^8$$

$$\therefore\ s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8000 \pm \sqrt{(8000)^2 - 10^8} = -8000 \pm j\,6000$$

$$\therefore\ v_n(t) = e^{-8000t}[A_1 \cos 6000\,t + A_2 \sin 6000t\,]$$



$$\text{KCL at top}: 0.08 + \frac{10}{62.5} + i_C = 0$$

$$\Rightarrow i_C(0^+) = -0.24 \text{ A}$$

$$\therefore\ \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -2.4\times 10^{+5}\ \text{ V/s}$$

$$v_n(0) = 10 = A_1$$

$$\frac{dv_n(0)}{dt} = -2.4\times 10^5 = 6000A_2 - 8000(10) \Rightarrow A_2 = -26.7$$

$$\therefore\ \underline{v_n(t) = e^{-8000t}\ [10\ \cos\ 6000\,t - 26.7\ \sin\ 6000t\ ]\ \text{V}}$$

Exercise 9.7-1 A circuit is described for $t > 0$ by the equation

$$\frac{d^2 i}{dt^2} + 9\frac{di}{di} + 20i = 6i_s$$

where $i_s = 6 + 2t$ A. Find the forced response i_f for $t > 0$.

Answer: $i_f = 1.53 + 0.6t$ A

Solution:

$$i'' + 9i' + 20i = 36 + 12t$$

Try $i_f = A + Bt$ & plug into above

$$0 + 9B + 20(A + Bt) = 36 + 12t$$

Equating the constant coefficients and the coefficients of *t* gives

$$20Bt = 12t \Rightarrow B = 0.6 \text{ and } 9B + 20A = 36 \Rightarrow A = 1.53$$

$$\therefore \underline{i_f = 1.53 + 0.6t} \text{ A}$$

Exercise 9.9-1 Find $v_2(t)$ for $t > 0$ for the circuit of Figure E 9.9-1. Assume there is no initial stored energy.

Answer: $v_2(t) = -15e^{-2t} + 6e^{-4t} - e^{-6t} + 10$ V

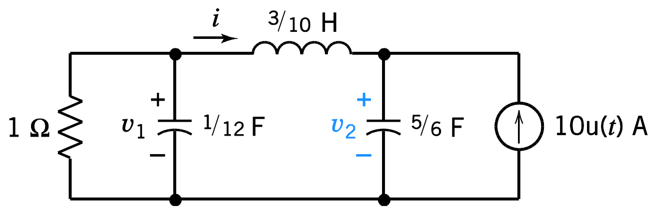
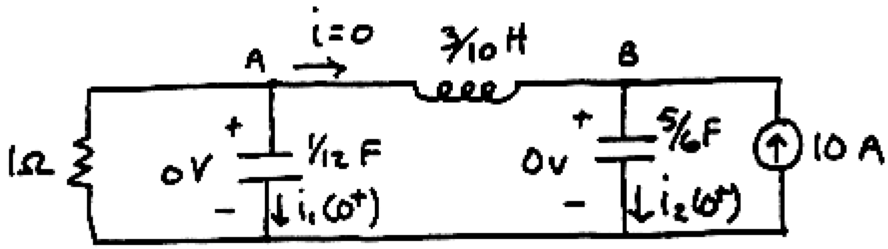


Figure E 9.9-1

Solution:

no initial stored energy $\Rightarrow v_1(0^+) = v_2(0^+) = i(0^+) = 0$

$\underline{t = 0^+}$

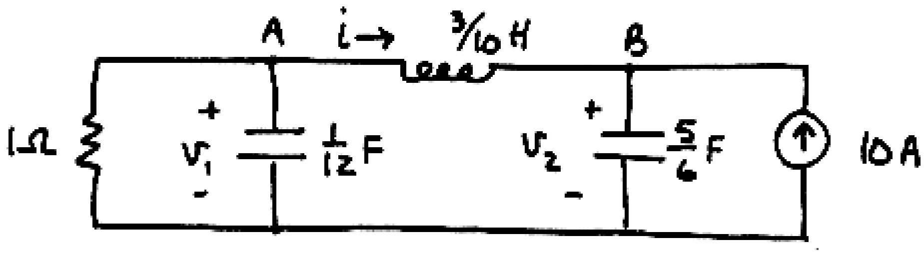


$$\text{KVL} : -0 + \tfrac{3}{10}\frac{di(0^+)}{dt} + 0 = 0 \quad \Rightarrow \quad \frac{di(0^+)}{dt} = 0$$

$$\text{KCL at A} : \frac{0V}{1\Omega} + i_1(0^+) + 0 = 0 \quad \Rightarrow \quad \frac{dv_1(0^+)}{dt} = 0$$

$$\text{KCL at B} : -0 + i_2(0^+) - 10 = 0 \Rightarrow i_2(0^+) = 5/6\,\frac{dv_2(0^+)}{dt} = 10 \quad \Rightarrow \quad \frac{dv_2(0^+)}{dt} = 12\,\text{V/s}$$

$\underline{t > 0}$



$$\text{KCL at A}: \frac{v_1}{1} + \frac{1}{12}v_1' + i = 0 \qquad (1)$$

$$\text{KCL at B}: -i + (5/6)v'_2 = 10 \qquad (2)$$

$$\text{KVL}: -v_1 + (3/10)i' + v_2 = 0 \qquad (3)$$

Eliminating i from (1) & (3) yields

$$v_1 + \frac{1}{12}v'_1 + (5/6)v_2' - 10 = 0 \qquad (4)$$

$$-v_1 + \frac{3}{10}\left(\frac{5}{6}v_2^{''}\right) + v_2 = 0 \qquad (5)$$

From (5)

$$v_1 = v_2 + \frac{1}{4}v_2^{''} \quad \Rightarrow \quad v_1^{'} = v_2^{''} + (1/4)v_2^{''}$$

Now substituting into (4) yields

$$v_2^{'} + \frac{1}{4}v_2^{''} + \frac{1}{12}\left(v_2^{'} + \frac{1}{4}v_2^{'''}\right) + \frac{5}{6}v_2' = 10$$

$$\underline{v_2^{'''} + 12v_2^{''} + 44v_2^{'} + 48v_2 = 480}$$

Natural Response: $v_{2n}: s^3 + 12s^2 + 44s + 48 = 0 \quad \Rightarrow s = -2, -4, -6$

$$\therefore \quad v_{2n} = A_1e^{-2t} + A_2e^{-4t} + A_3e^{-6t}$$

Forced Response: v_{2f}: try $v_{2f} = B$ and plug into Diff. Eq. $\Rightarrow B = 10$

Complete Response: $v_2(t) = A_1e^{-2t} + A_2e^{-4t} + A_3e^{-6t} + 10$

Recall $v_2(0^+) = 0$, $\dfrac{dv_2(0^+)}{dt} = 12$ V/s, then from (5) $\dfrac{d^2v_2(0^+)}{dt^2} = 4[v_1(0^+) - v_2(0^+)] = 0$.

$$v_2(0^+) = 0 = A_1 + A_2 + A_3 + 10 \qquad (6)$$

$$\frac{dv_2(0^+)}{dt} = 12 = -2A_1 - 4A_2 - 6A_3 \qquad (7)$$

$$\frac{d^2v_2(0^+)}{dt^2} = 0 = 4A_1 + 16A_2 + 36A_3 \qquad (8)$$

Solving (6)-(80 simultaneously gives $A_1 = -15,\ A_2 = 6,\ A_3 = -1$

Then

$$\underline{v_2(t) = -15e^{-2t} + 6e^{-4t} - e^{-6t} + 10 \text{ V}}$$

Exercise 9.10-1 A parallel *RLC* circuit has $L = 0.1$ H and $C = 100$ mF. Determine the roots of the characteristic equation and plot them on the *s*-plane when (a) $R = 0.4\ \Omega$ and (b) $R = 1.0\ \Omega$.

Answer: (a) $s = -5, -20$ (Figure E 9.10-1)

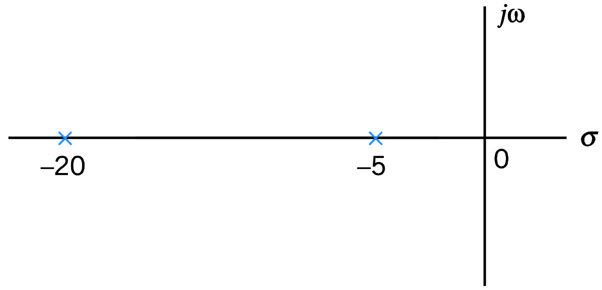


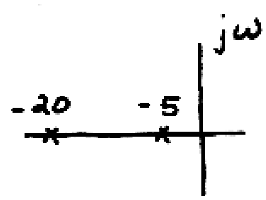
Figure E 9.10-1

Solution:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad \text{and} \quad L = 0.1,\ C = 0.1 \quad \Rightarrow \quad s^2 + \frac{10}{R}s + 100 = 0$$

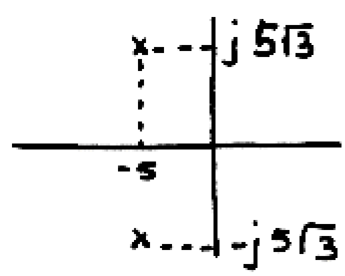
a)

$$R = 0.4\ \Omega \quad \Rightarrow \quad s^2 + 25s + 100 = 0$$
$$s = -5,\ -20$$



b)

$$R = 1\Omega \quad \Rightarrow s^2 + 10s + 100 = 0$$
$$s = -5 \pm j5\sqrt{3}$$



Section 9-2: Differential Equations for Circuits with Two Energy Storage Elements

P 9.2-1

Solution:

KCL: $i_L = \dfrac{v}{R_2} + C\dfrac{dv}{dt}$

KVL: $V_s = R_1 i_L + L\dfrac{di_L}{dt} + v$

$$v_s = R_1\left[\frac{v}{R_2} + C\frac{dv}{dt}\right] + \frac{L}{R_2}\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v$$

$$v_s = \left[\frac{R_1}{R_2} + 1\right]v + \left[R_1C + \frac{L}{R_2}\right]\frac{dv}{dt} + [LC]\frac{d^2v}{dt^2}$$

$$R_1 = 3\Omega,\ R_2 = 150\Omega,\ L = 1.5\text{mH}, C = 15\mu\text{F}$$

$$v_s = 1.02v + 5.5\times10^{-5}\frac{dv}{dt} + 2.25\times10^{-8}\frac{d^2v}{dt^2}$$

$$0.4\times10^8 v_s = 0.41\times10^8 v + 2200\frac{dv}{dt} + 0.9\frac{d^2v}{dt^2}$$

P 9.2-2
Solution:

KCL: $i_s = \dfrac{v}{R_1} + i_L + Csv$

KVL: $v = R_2 i_L + Lsi_L$

Solving Cramer's rule for i_L :

$$i_L = \frac{i_s}{\dfrac{R_2}{R_1} + \dfrac{Ls}{R_1} + R_2Cs + LCs^2 + 1}$$

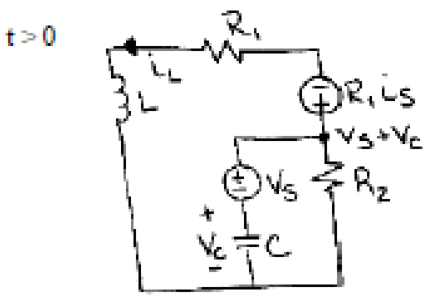
$$\left[1 + \frac{R_2}{R_1}\right] i_L + \left[\frac{L}{R_1} + R_2C\right] si_L + [LC]s^2 i_L = i_s$$

$$R_1 = 150\Omega,\ R_2 = 15\Omega,\ L = 1.5\,\text{mH},\ C = 15\mu\text{F}$$

$$1.1 i_L + 23.5\times10^{-5} si_L + 2.25\times10^{-8} s^2 i_L = i_s$$

$$\underline{0.48\times10^8 i_L + 10.4\times10^3 si_L + s^2 i_L = 0.4\times10^8 i_s}$$

P 9.2-3
Solution:



$$\text{KCL: } i_L + C\frac{dv_c}{dt} + \frac{v_s + v_c}{R_2} = 0$$

$$\text{KVL: } R_1 i_s + R_1 i_L + L\frac{di_L}{dt} - v_c - v_s = 0$$

Solving for i_L :

$$\underline{\frac{d^2 i_L}{dt^2} + \left[\frac{R_1}{L} + \frac{1}{R_2 C}\right]\frac{di_L}{dt} + \left[\frac{R_1}{LR_2C} + \frac{1}{LC}\right] i_L = \frac{-R_1}{LCR_2} i_s - \frac{R_1}{L}\frac{di_s}{dt} + \frac{1}{L}\frac{dv_s}{dt}}$$

P 9.2-4
Solution:
After the switch opens, apply KCL and KVL to get

$$R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t) = V_s$$

Apply KVL to get

$$v(t) = L\frac{d}{dt}i(t) + R_2\, i(t)$$

Substituting $v(t)$ into the first equation gives

$$R_1\left(i(t) + C\frac{d}{dt}\left(L\frac{d}{dt}i(t) + R_2\, i(t)\right)\right) + L\frac{d}{dt}i(t) + R_2\, i(t) = V_s$$

then

$$R_1 C L\frac{d^2}{dt^2}i(t) + \left(R_1 C R_2 + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t) = V_s$$

Dividing by $R_1 C L$:

$$\frac{d^2}{dt^2}i(t) + \left(\frac{R_1 C R_2 + L}{R_1 C L}\right)\frac{d}{dt}i(t) + \left(\frac{R_1 + R_2}{R_1 C L}\right)i(t) = \frac{V_s}{R_1 C L}$$

P 9.2-6
Solution:
After the switch closes, KVL and KCL give

$$v_1(t)+R_3\left(C_1\frac{d}{dt}v_1(t)+C_2\frac{d}{dt}v_2(t)\right)=v_s$$

KVL gives

$$v_1(t)=R_2C_2\frac{d}{dt}v_2(t)+v_2(t)$$

Using the operator method

$$v_1+R_3\left(C_1sv_1+C_2sv_2\right)=v_s$$
$$v_1=R_2C_2sv_2+v_2$$

so

$$v_1=\left(1+R_2C_2s\right)v_2$$
$$\left(1+R_2C_2s\right)v_2+R_3C_1s\left(1+R_2C_2s\right)v_2+R_3C_2sv_2=v_s$$

Then

$$R_2R_3C_1C_2s^2v_2+\left(R_2C_2+R_3C_1+R_3C_2\right)sv_2+v_2=v_s$$

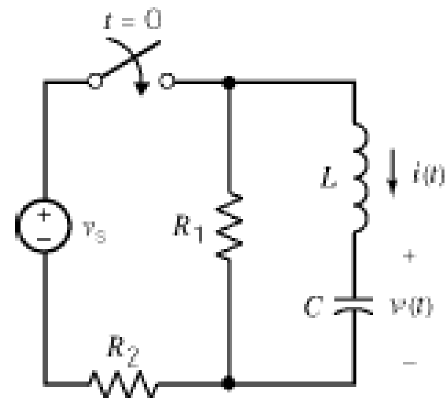
$$s^2v_2+\frac{R_2C_2+R_3C_1+R_3C_2}{R_2R_3C_1C_2}sv_2+\frac{1}{R_3R_2C_1C_2}v_2=\frac{v_s}{R_2R_3C_1C_2}$$

$$s^2v_2+\left(\frac{1}{R_3C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)sv_2+\frac{1}{R_2R_3C_1C_2}v_2=\frac{v_s}{R_2R_3C_1C_2}$$

so

$$\frac{v_s}{R_2R_3C_1C_2}=\frac{d^2}{dt^2}v_2(t)+\left(\frac{1}{R_3C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)\frac{d}{dt}v(t)+\frac{1}{R_2R_3C_1C_2}v(t)$$

P 9.2-7
Solution:
After the switch closes



$$i(t)=C\frac{d}{dt}v(t)$$

KCL and KVL give

$$v_s=R_2\left(i(t)+\frac{1}{R_1}\left(L\frac{d}{dt}i(t)+v(t)\right)\right)+L\frac{d}{dt}i(t)+v(t)$$

Substituting gives

$$v_s=\left(1+\frac{R_2}{R_1}\right)LC\frac{d^2}{dt^2}v(t)+R_2C\frac{d}{dt}v(t)+\left(1+\frac{R_2}{R_1}\right)v(t)$$
$$=\left(1+\frac{R_2}{R_1}\right)LC\frac{d^2}{dt^2}v(t)+R_2C\frac{d}{dt}v(t)+\left(1+\frac{R_2}{R_1}\right)v(t)$$

Finally

$$\frac{R_1v_s}{LC\left(R_1+R_2\right)}=\frac{d^2}{dt}v(t)+\frac{R_1R_2}{L\left(R_1+R_2\right)}\frac{d}{dt}v(t)+\frac{1}{LC}v(t)$$

P 9.2-8
Solution:

KVL gives
$$v_s(t) = R_1 C_1 \frac{d}{dt} v_1(t) + v_1(t)$$

KCL gives
$$C_1 \frac{d}{dt} v_1(t) + C_2 \frac{d}{dt} v_2(t) + \frac{v_2(t)}{R_2} = 0$$

KVL gives
$$v_o(t) = v_2(t)$$

Using the operator method
$$v_s = R_1 C_1 s v_1 + v_1$$
$$C_1 s v_1 + C_2 s v_2 + \frac{v_2}{R_2} = 0$$

Solving
$$v_1 = -\left(\frac{C_2}{C_1} v_2 + \frac{1}{R_2 C_1 s} v_2 \right)$$
$$s v_s = \left(s R_1 C_1 + 1 \right) \left(\frac{C_2}{C_1} s + \frac{1}{R_2 C_1} \right) v_o$$
$$\frac{1}{R_1 C_2} s v_s = s^2 v_o + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} \right) s v_o + \frac{1}{R_1 R_2 C_1 C_2} v_o$$

The corresponding differential equation is
$$\frac{1}{R_1 C_2} \frac{d}{dt} v_s(t) = \frac{d^2}{dt^2} v_o(t) + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} \right) \frac{d}{dt} v_o(t) + \frac{1}{R_1 R_2 C_1 C_2} v_o(t)$$

P 9.2-9

Solution:

After the switch opens, KCL gives

$$\frac{v_s(t)}{R_1} + C\frac{d}{dt}v(t) = 0$$

KVL gives

$$v(t) - v_o(t) = L\frac{d}{dt}i(t)$$

and Ohm's law gives

$$v_o(t) = R_2\, i(t)$$

so

$$\frac{d}{dt}v(t) = -\frac{1}{R_1 C}v_s(t)$$

and

$$\frac{d}{dt}v(t) - \frac{d}{dt}v_o(t) = L\frac{d^2}{dt^2}i(t)$$

Then

$$-\frac{1}{R_1C}v_s(t) = \frac{d}{dt}v(t) = L\frac{d^2}{dt^2}i(t) + R_2\frac{d}{dt}i(t)$$

or

$$-\frac{1}{R_1CL}v_s(t) = \frac{d^2}{dt^2}i(t) + \frac{R_2}{L}\frac{d}{dt}i(t)$$

P 9.2-10
Solution:
KCL gives

$$\frac{v_s(t)}{R_1} = \frac{v_1(t)}{R_2} + C_1 \frac{d}{dt} v_1(t)$$

and

$$\frac{v_2(t) + v_1(t)}{R_3} + C_2 \frac{d}{dt} v_2(t) = 0$$

so

$$v_1(t) + R_2 C_1 \frac{d}{dt} v_1(t) = \frac{R_2}{R_1} v_s(t)$$

and

$$v_1(t) = -\left(v_2(t) + R_3 C_2 \frac{d}{dt} v_2(t) \right)$$

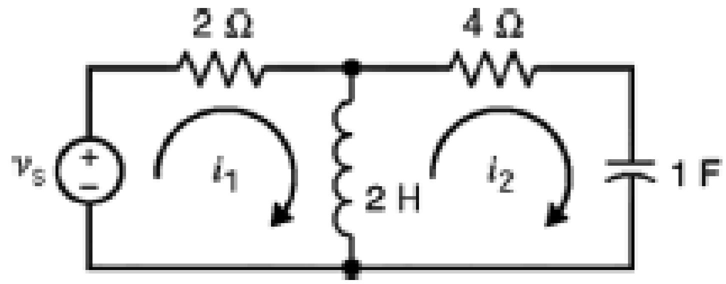
Substituting gives

$$\left[v_2(t) + R_3 C_2 \frac{d}{dt} v_2(t) + R_2 C_1 \frac{d}{dt} \left[v_2(t) + R_3 C_2 \frac{d}{dt} v_2(t) \right] \right] = -\frac{R_2}{R_1} v_s(t)$$

or

$$\frac{d^2}{dt^2} v_2(t) + \left(\frac{1}{R_2 C_1} + \frac{1}{R_3 C_2} \right) \frac{d}{dt} v_2(t) + \frac{1}{R_2 R_3 C_1 C_2} v_2(t) = -\frac{1}{R_1 R_3 C_1 C_2} v_s(t)$$

P 9.2-11
Solution:



Apply KVL to the left mesh : $\underline{2i_1 + 2s(i_1 - i_2) = v_s}$ (1)

where $s = d/dt$

Apply KVL to the right mesh : $4i_2 + \left(\frac{1}{s}\right)i_2 + 2s(i_2 - i_1) = 0$

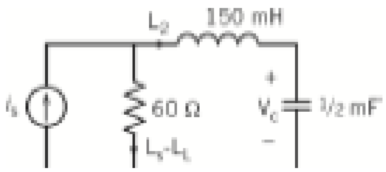
$$\Rightarrow \underline{i_1 = 2\left(\frac{1}{s}\right)i_2 + \frac{1}{2}\left(\frac{1}{s^2}\right)i_2 + i_2} \qquad (2)$$

Plugging (2) into (1) yields

$$6s^2 i_2 + 5si_2 + i_2 = 2s^2 v_s \quad \text{or} \quad \underline{6\frac{d^2 i_2}{dt^2} + 5\frac{di_2}{dt} + i_2 = 2\frac{d^2 v_s}{dt^2}}$$

Section 9-3: Solution of the Second Order Differential Equation - The Natural Response

P 9.3-1
Solution:



$$\text{KVL: } 60(i_s - i_L) = 150\text{m}\frac{di_L}{dt} + v_c \qquad \text{where } \text{m} = 10^{-3}$$

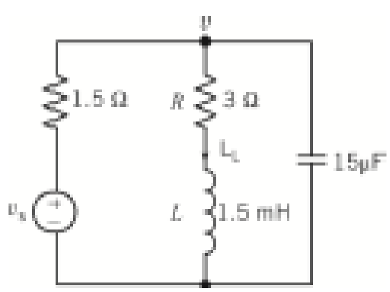
$$i_L = i_c = \frac{1}{2}\text{m}\frac{dv_c}{dt}$$

$$i_L = -75\text{m}^2\frac{d^2 i_L}{dt^2} + 30\text{m}\frac{di_s}{dt} - 30\text{m}\frac{di_L}{dt}$$

$$\frac{d^2 i_L}{dt^2} + 400\frac{di_L}{dt} + 13.3\times10^3 i_L = 400\frac{di_s}{dt}$$

$$\underline{s^2 + 400s + 13.3\times10^3 = 0} \Rightarrow \underline{s_1 = -36.60, \quad s_2 = -363.40}$$

P 9.3-2
Solution:



$$\text{KCL: } \frac{v - v_s}{1.5} + i_L + 15\mu\frac{dv}{dt} = 0 \qquad \text{where } \mu = 10^{-6}$$

KVL:

$$v = 3i_L + 1.5\text{m}\frac{di_L}{dt} \qquad \text{where m} = 10^{-3}$$

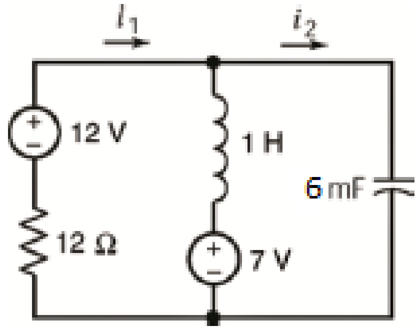
0 3i 1.5m v 1.5i 22.5

$$v_s = 4.5i_L + 0.00157\frac{di_L}{dt} + 3.4\times10^{-5}\frac{d^2 i_L}{dt^2}$$

$$\frac{d^2 i_L}{dt^2} + 46.2\frac{di_L}{dt} + 1.3\times10^5 i_L = 0.3\times10^5\, v_s$$

$$\underline{3.4s^2 + 157s + 4.5\times10^5 = 0}, \quad \therefore \underline{s = -23.09 \pm 363.07}$$

P 9.3-3
Solution:
Assume zero initial conditions



$$\text{loop 1}: 12i_1 + \frac{di_1}{dt} - \frac{di_2}{dt} = 12 - 7$$

$$\text{loop 2}: -\frac{di_1}{dt} + \frac{di_2}{dt} + 167\int i_2 dt = 7$$

$$\text{determinant}: \begin{bmatrix} (12+1s) & -1s \\ -1s & \left(1s + \frac{167}{s}\right) \end{bmatrix}$$

$$s^2 + 13.9s + 167 = 0, \quad \therefore\ s = -6.95 \pm j\,10.89$$

Section 9.4: Natural Response of the Unforced Parallel RLC Circuit

P 9.4-1
Solution:

$$v(0) = 9,\ \frac{dv(0)}{dt} = -4500$$

Using operators, the node equation is: $Csv + \frac{v}{R} + \frac{(v - v_s)}{sL} = 0$ or $\left(LCs^2 + \frac{L}{R}s + 1\right)v = v_s$

So the characteristic equation is: $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

$$\Rightarrow\ s_{1,2} = -222 \pm 150.39 = -71.61,\ -372.39$$

So $v(t) = Ae^{-71.61t} + Be^{-372.39t}$

$$v(0) = 9 = A + B$$

$$\left.\frac{dv(0)}{dt} = -4500 = -71.6A - 372.39B\right\} \quad \begin{matrix} A = -3.82 \\ B = 12.82 \end{matrix}$$

$$\therefore\ \underline{v(t) = -3.82e^{-71.6t} + 12.82e^{-372.39t}} \quad t > 0$$

P 9.4-2

Solution:

$v(0)= 2, \; i(0)= 0$

Characteristic equation $s^2+\frac{1}{RC}\,s+\frac{1}{LC} \;=\; 0 \;\Rightarrow\; s^2+4s+3 \;=\; 0 \;\Rightarrow s \;=\; -1,\; -3$

$v(t) \;=\; Ae^{-t}+Be^{-3t}$

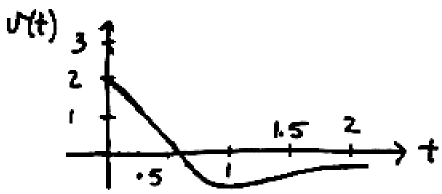
Use eq. $9.5-12 \;\Rightarrow\; s_1A+\; s_2B \;=\; -\frac{v(0)}{RC}-\frac{i(0)}{C}$

$$-1A-3B \;=\; -\frac{2}{1/4}-0 \;=\; -\,8 \qquad (1)$$

also have $v(0) \;=\; 2 \;=\; A+B \qquad (2)$

From (1) & (2) get $A=-1,\; B=3$

$\therefore\; \underline{v(t)=-e^{-t}+3e^{-3t}\;\text{V}}$



P 9.4-3
Solution:

$$\text{KVL}: 2i_1 + 10\frac{di_1}{dt} - 6\frac{di_2}{dt} = 0 \quad (1)$$

$$\text{KVL}: -6\frac{di_1}{dt} + 6\frac{di_2}{dt} + 4i_2 = 0 \quad (2)$$

in operator form

$$\left.\begin{array}{l}(2+10s)i_1 + (-6s)i_2 = 0 \\ (-6s)\,i_1 + (6s+4)\,i_2 = 0\end{array}\right\} \text{ thus } \Delta = (2+10s)\,(6s+4) - 36s^2 = 24s^2 + 52s + 8 = 0 \Rightarrow s = -\tfrac{1}{6}, -2$$

$$\text{Thus } i_1(t) = Ae^{-t/6} + Be^{-2t}$$

$$i_2(t) = Ce^{-t/6} + De^{-2t}$$

$$\text{Now } i_1(0) = 11 = A + B; \quad i_2(0) = 11 = C + D$$

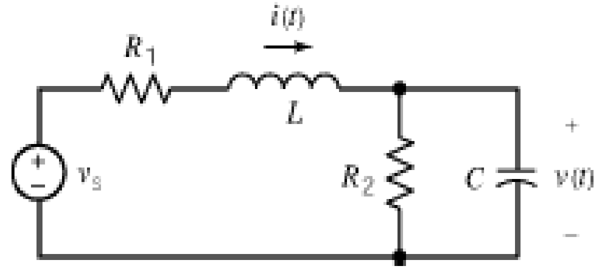
from (1) & (2) get

$$\frac{di_1(0)}{dt} = -\frac{33}{2} = -\frac{A}{6} - 2B\,; \quad \frac{di_2(0)}{dt} = -\frac{143}{6} = -\frac{C}{6} - 20$$

which yields $A = 3,\ B = 8,\ C = -1,\ 0 = 12$

$\underline{i_1(t) = 3e^{-t/6} + 8e^{-2t}\ A}$ & $\underline{i_2(t) = -e^{-t/6} + 12e^{-2t}\ A}$

P 9.4-4
Solution:
Represent this circuit by a differential equation.
(R_1 = 100 Ω when the switch is open and R_1 = 20 Ω when the switch is closed.)



Use KCL to get

$$i(t)=\frac{v(t)}{R_2}+C\frac{d}{dt}v(t)$$

Use KVL to get

$$v_s=R_1\, i(t)+L\frac{d}{dt}i(t)+v(t)$$

Substitute to get

$$v_s=\frac{R_1}{R_2}v(t)+R_1C\frac{d}{dt}v(t)+\frac{L}{R_2}\frac{d}{dt}v(t)+CL\frac{d^2}{dt^2}v(t)+v(t)$$

$$=CL\frac{d^2}{dt^2}v(t)+\left(R_1C+\frac{L}{R_2}\right)\frac{d}{dt}v(t)+\frac{R_1+R_2}{R_2}v(t)$$

Finally,

$$\frac{v_s}{CL}=\frac{d^2}{dt^2}v(t)+\left(\frac{R_1}{L}+\frac{1}{R_2C}\right)\frac{d}{dt}v(t)+\frac{R_1+R_2}{R_2CL}v(t)$$

Compare to

$$\frac{d^2}{dt^2}v(t)+2\alpha\frac{d}{dt}v(t)+\omega_0{}^2\, v(t)=f(t)$$

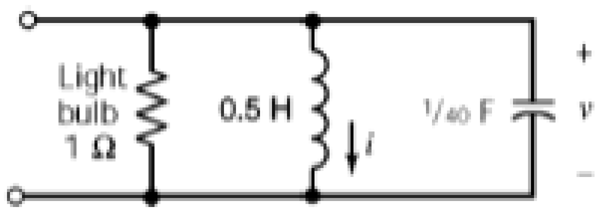
to get

$$2\alpha=\frac{R_1}{L}+\frac{1}{R_2C}\ \text{ and }\ \omega_0{}^2=\frac{R_1+R_2}{R_2CL}$$

(a) When the switch is open $\alpha = 13$, ω_0 = 7.07 rad/s and $\omega_d = 10.91j$ (the circuit is overdamped).

(b) When the switch is closed $\alpha = 3$, ω_0 = 5.48 rad/s and ω_d = 4.59 (the circuit is underdamped).

P 9.4-5
Solution:



$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$
$$s^2 + 40s + 80 = 0$$
$$s = -2.1\,,\ \ -37.9$$

The initial conditions are $v(0) = 0$, $i(0)\ 2$ A.

$v_n = A_1 e^{-2.1t} + A_2 e^{-37.9t}$, $v(0) = 0 = A_1 + A_2$ (1)

KCL at $t = 0^+$ yields : $\dfrac{v(0^+)}{1} + i(0^+) + \dfrac{1}{40}\dfrac{dv(0^+)}{dt} = 0$

$\therefore \dfrac{dv(0^+)}{dt} = -40v(0^+) - 40i(0^+) = -40(2) = -2.1A_1 - 37.9A_2$ (2)

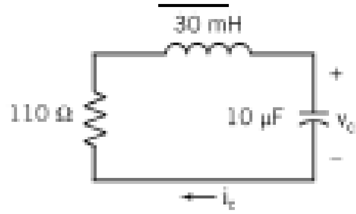
from (1) and (2) $\Rightarrow$ $A_1 = -2.23$, $A_2 = 2.23$

So $v(t) = v_n(t) = -2.23e^{-2.1t} + 2.23e^{-37.9t}$ V

Section 9.5: Natural Response of the Critically Damped Unforced Parallel RLC Circuit

P 9.5-1
Solution:

<u>$t > 0$</u>

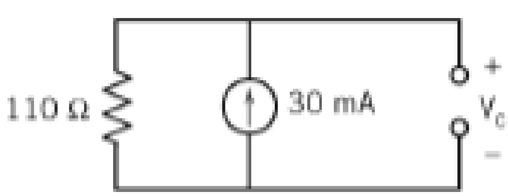


$$\text{KVL a: } 110 i_c + 0.030 \frac{di_c}{dt} + v_c = 0, \; i_c = 10^{-5} \frac{dv_c}{dt}$$

$$\therefore \; 0.03 \frac{d^2 v_c}{dt^2} + 110 \frac{dv_c}{dt} + 10^5 v_c = 0$$

$$0.03 s^2 + 110 s + 10^5 = 0 \Rightarrow s = -1667, \; -2000 \quad \therefore \; v_c(t) = A_1 e^{-1667t} + A_2 t e^{-2000t}$$

<u>$t = 0^-$</u> (Steady – State)



$$i_L = i_c(0^-) = 0 = i_c(0^+) \Rightarrow \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(0^-) = 3.3 \text{ V} = v_c(0^+)$$

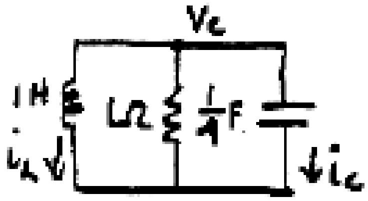
so $v_c(0^+) = 3.3 = A_1$

$$\frac{dv_c(0^+)}{dt} = 0 = -1667 A_1 + A_2 \Rightarrow A_2 = 5498$$

$$\therefore \underline{v_c(t) = 3.3 e^{-1667t} + 5498 t e^{-2000t} \text{ V}}$$

P 9.5-2
Solution:

$\underline{t>0}$

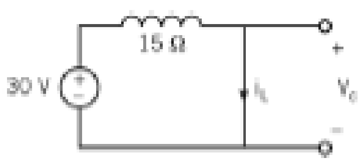


KCL at v_c : $\int_{-\infty}^{t} v_c\, dt + v_c + \left(\frac{1}{4}\right)\frac{dv_c}{dt} = 0$

$$\Rightarrow \frac{d^2 v_c}{dt} + 4\frac{dv_c}{dt} + 4v_c = 0$$

$$s^2 + 4s + 4 = 0,\ s = -2,\ -2 \ \Rightarrow\ v_c(t) = A_1 e^{-2t} + A_2\, t\, e^{-2t}$$

$\underline{t=0^-}$ (Steady-State)



$$v_c(0^-) = 0 = v_c(0^+) \ \&\ i_L(0^-) = \frac{30\text{ V}}{15\Omega} = 2\text{ A} = i_L(0^+)$$

Since $v_c(0^+) = 0$ then $i_c(0^+) = -i_L(0^+) = -2\text{ A}$

$$\therefore \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{\frac{1}{4}} = -8\text{ V/s}$$

So $v_c(0^+) = 0 = A_1$

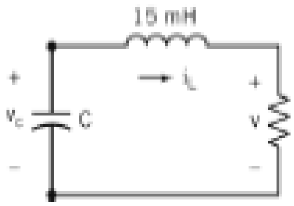
$$\frac{dv_c(0^+)}{dt} = -8 = A_2$$

$$\underline{\therefore v_c(t) = -8te^{-2t}\ V}$$

P 9.5-3
Solution:

Assume steady−state at $t = 0^-$ $\therefore v_c(0^-) = 15x10^3\text{V}$ & $i_L(0^-) = 0$

<u>$t > 0$</u>



$$\text{KVL a}: \; -v_c + .015\frac{di_L}{dt} + 15x10^5 i_L = 0 \qquad (1)$$

$$\text{Also} \; : i_L = -C\, {dv_c}/{dt} = -C\left[.015\frac{d^2 i_L}{dt^2} + 15x10^5\frac{di_L}{dt}\right] \qquad (2)$$

$$\therefore \; .015C\frac{d^2 i_L}{dt^2} + 15\times10^5 C\frac{di_L}{dt} + i_L = 0$$

Characteristic eq. $\Rightarrow$ $.015C\,s^2 + 15\times10^5 Cs + 1 = 0$

$$\Rightarrow s = \frac{-15\times10^5 C \pm \sqrt{\left(15\times10^5 C\right)^2 - 4(.015C)}}{2(.015C)}$$

for critically damped: $2.25\times10^{12}C^2 - .06C = 0$

$$\Rightarrow C = 0.03\text{ pF} \;\; \therefore s = -3.3\times10^7, -6.6\times10^7$$

So $i_L(t) = A_1 e^{-3.3\times10^7 t} + A_2 t e^{-6.6\times10^7 t}$

Now from (1) $\Rightarrow$ $\frac{di_L}{dt}(0^+) = \frac{1000}{15}\left[v_c(0^+) - 15\times10^5 i_L(0^+)\right] = 10^6\,{A}/{s}$

So $i_L(0) = 0 = A_1$ and $\frac{di_{L(0)}}{dt} = 10^6 = A_2$ $\therefore i_L(t) = 10^6 t e^{-6.6\times10^7 t}$ A

Now $\underline{v(t) = 15\times10^5 i_L(t) = 1.5\times10^{12} t e^{-6.6\times10^7 t}\text{ V}}$

P 9.5-4
Solution:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0 \quad \text{with } \frac{1}{RC}=500 \text{ and } \frac{1}{LC}=62.5\times10^3 \quad \text{yields } s=-250,-250$$

$$\begin{aligned} v(t) &= Ae^{-250t}+Bte^{-250t} \\ v(0) &= 9 = A \\ \frac{dv(0)}{dt} &= -4500 = -250A+B \Rightarrow B = -2250 \\ \therefore v(t) &= \underline{9e^{-250t}-2250\,te^{-250t}} \end{aligned}$$

P 9.5-5
Solution:

$$\text{KVL}: \frac{di}{dt}+Ri+\underbrace{3+2\int_0^t i\,dt}_{v(t)} = 9 \qquad (1)$$

taking the derivative with respect to t: $\quad \dfrac{d^2i}{dt^2}+R\dfrac{di}{dt}+2i = 0$

Characteristic equataion: $s^2+Rs+2 = 0$

Let $R=3$ for critical damping $\Rightarrow \quad (s+1)(s+2)=0$

So $i(t)=Ate^{-t}+Be^{-2t}$

$i(0)=0 \Rightarrow B=0$

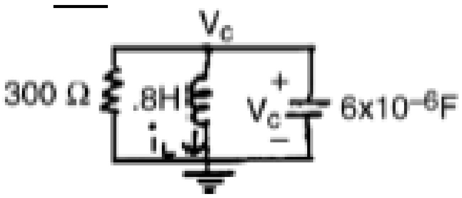
from (1) $\quad \dfrac{di(0)}{dt}=6-R(i(0))= 6-R(0)= 6 = A$

$\therefore \underline{i(t) = 6te^{-t}\text{A}}$

Section 9-6: Natural Response of an Underdamped Unforced Parallel RLC Circuit

P 9.6-1
Solution:

t > 0



KCL at v_c : $v_c/300 + i_L + 6\times10^{-6}\dfrac{dv_c}{dt} = 0 \quad (1)$

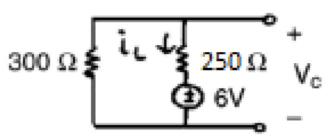
also : $v_c = .8\dfrac{di_L}{dt} \quad (2)$

Solving for i_L in (1) & plugging into (2)

$$\frac{d^2v_c}{dt^2}+556\frac{dv_c}{dt}+2.08\times10^5 v_c = 0 \;\Rightarrow s^2+556s+208{,}000=0,\; s=-278\pm j\,362$$

$$\therefore\; v_c(t) = e^{-278t}\left[A_1\cos 362t + A_2\sin 362t\right]$$

t = 0⁻ (Steady – State)



$i_L(0^-) = \dfrac{-6V}{550} = -6/ \quad A = i_L(0^+)$

$v \;\; 0 \qquad 360 \;\; {}^{6}{}_{550} \quad 6 \qquad 2.7V = v \;\; 0$

Now from (1): $\dfrac{dv_c(0^+)}{dt} = -\,1.6\times10^5 i_L(0^+) - 556 v_c(0^+) = 0$

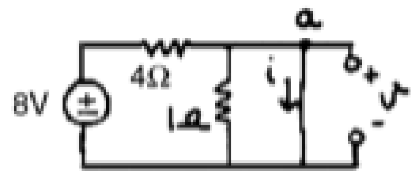
So $v_c(0^+) = 2.7 = A_1$

$$\frac{dv_c(0^+)}{dt} = 0 = -278A_1 + 362A_2 \;\Rightarrow\; A_2 = 2.07$$

$$\therefore\; v_c(t) = e^{-278t}\left[2.7\cos 362t + 2.07\sin 362t\right]\text{ V}$$

P 9.6-2
Solution:

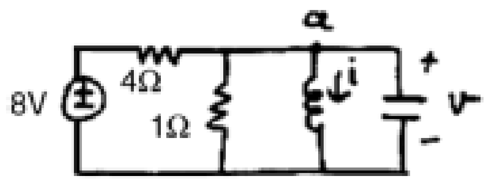
$\underline{t=0^-}$



$i(0) = 2\text{ A}$

$v(0) = 0$

$\underline{t=0^-}$



KCL at node a:

$$\frac{v}{1} + C\frac{dv}{dt} + \frac{1}{L}\int_0^t v\,dt + i(0) = 0 \quad (1)$$

in operator form have $v + Csv + \frac{1}{Ls}v + i(0) = 0$ *or* $\left(s^2 + \frac{1}{C}s + \frac{1}{LC}\right)v = 0$

with $s^2 + 4s + 16 = 0 \quad \Rightarrow \; s = -2 \pm j3.5$

$$v(t) = e^{-2t}\left[B_1 \cos 3.5t + B_2 \sin 3.5t\right]$$

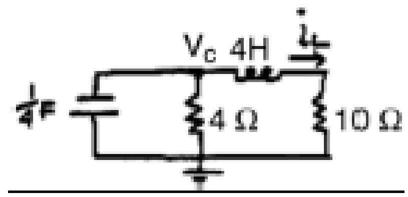
$$v(0) = 0 = B_1$$

From (1), $\frac{dv(0)}{dt} = \frac{1}{C}\left[-i(0) - v(0)\right] = -4[2] = -8 = 2B_2$ or $B_2 = -4$

So $\underline{v(t) = -4e^{-2t}\sin 3.5t \text{ V}}$

P 9.6-3
Solution:

t > 0



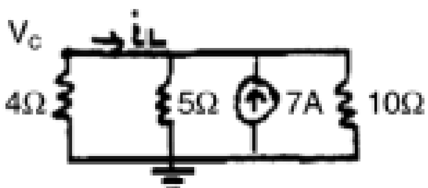
KCL at v_c : $\frac{1}{4}\frac{dv_c}{dt}+\frac{v_c}{4}+i_L=0$ (1)

KVL : $v_c=\frac{4di_L}{dt}+10i_L$ (2)

(2) into (1) yields $\frac{d^2i_L}{dt^2}+3.5\frac{di_L}{dt}+3.5i_L=0 \Rightarrow s^2+3.5s+3.5=0 \Rightarrow s=-1.75\pm i0.7$

$\therefore i_L(t)=e^{-1.75t}\left[A_1\cos 0.7t+A_2\sin 0.7t\right]$

$t=0^-$ (Steady−State)



$$\frac{v_c(0^-)}{4}=7\left(\frac{5\|10}{5\|10+4}\right)$$

$$\Rightarrow v_c(0^-)=12.7\text{ V}=v_c(0^+)$$

$$i_L(0^-)=\frac{-12.7\text{ V}}{4\Omega}=-3.2\text{ A}=i_L(0^+)$$

$$\therefore \text{ from (2) } \frac{di_L(0^+)}{dt}=\frac{v_c(0^+)}{4}-2.5i_L(0^+)=\frac{12.7\text{ V}}{4}-2.5(-3.2)=11.2\ \frac{\text{A}}{\text{s}}$$

So $i_L(0^+)=-3.2=A_1$

$$\frac{di_L(0^+)}{dt}=11.2=-1.75A_1+0.7A_2 \Rightarrow A_2=8$$

$$\therefore \underline{i_L(t)=e^{-1.75t}\left[-3.2\cos 0.7t+8\sin 0.7t\right]\text{ A}}$$

P 9.6-4

Solution:

The response is underdamped so

$$\therefore\ v(t) = e^{-\alpha t}\left[k_1 \cos\ \omega t + k_2 \sin \omega t\ \right] + k_3$$

$$v(\infty) = 0\ \Rightarrow\ k_3 = 0,\ \ v(0) = 0\ \Rightarrow\ k_1 = \ 0$$

$$\therefore\ v(t) = \ k_2 e^{-\alpha t} \sin \omega t$$

From Fig. P 9.6-4

$$t\ \approx 5\text{ms}\ \ \leftrightarrow\ \ v \approx 260\text{mV}\ \ \ (\text{max})$$

$$t\ \approx 7.5\text{ms} \leftrightarrow\ v \approx -200\text{ mV (min)}$$

$\therefore$ distance between adjacent maxima is $\approx\ \omega\ =\ \dfrac{2\pi}{T}\ =\ 1257\ \text{rad/s}$

so

$$0.26\ =\ k_2 e^{-\alpha\ (.005)} \sin\ \left(1257\ (.005)\right) \qquad (1)$$

$$-0.2\ =\ k_2 e^{-\alpha\ (.0075)} \sin\left(\ 1257\ (.0075)\right) \quad (2)$$

Dividing (1) by (2) gives

$$-1.3\ =\ e^{\alpha\ (0.0025)}\left(\frac{\sin\ (6.29\ rad)}{\sin\ (9.43\ rad)}\right)\ \ \Rightarrow\ \ \ e^{0.0025\ \alpha}\ \ = 1.95\ \Rightarrow\ \alpha = 267$$

From (1) $k_2 = 544$ so

$$\underline{v(t)\ \ =\ 544 e^{-267t} \sin 1257t} \qquad (\text{approx. answer})$$

P 9.6-5
Solution:

$$v(0) = 4\text{V}$$
$$i(0) = \tfrac{1}{20}\ \text{A}$$

Char. eq. $\Rightarrow\ s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$ or $s^2 + 2s + 5 = 0$ thus the roots are $s = -1 \pm j2$

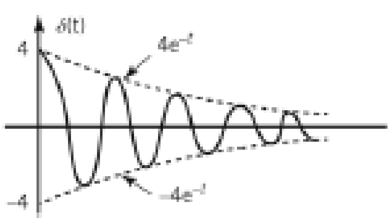
So have $v(t) = e^{-t}\left[B_1 \cos 2t + B_2 \sin 2t\right]$

now $v(0^+) = \underline{4 = B_1}$

Need $\frac{dv(0^+)}{dt} = \frac{1}{C} i_c(0^+)$ KCL yields $i_c(0^+) = -\frac{v(0^+)}{10} - i(0^+) = -\frac{9}{20}\ \frac{\text{V}}{\text{s}}$

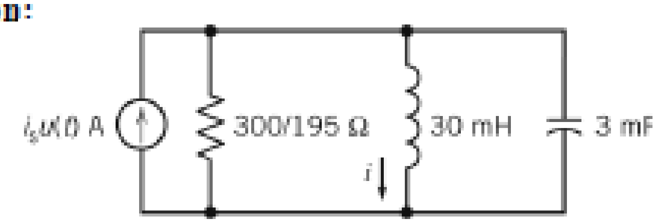
So $\frac{dv(0^+)}{dt} = 20\left(-\frac{9}{20}\right) = -B_1 + 2B_2 \Rightarrow \underline{B_2 = -\tfrac{5}{2}}$

Finally, have $\underline{v(t) = 4e^{-t}\cos 2t - \frac{5}{2}e^{-t}\sin 2t}$ V $\quad t > 0$



Section 9-7: Forced Response of an RLC Circuit

P 9.7-1
Solution:



$$\text{KCL}: i_s = \frac{v}{R} + i_L + C\frac{dv}{dt}$$

$$\text{KVL}: v = L\frac{di_L}{dt}$$

$$i_s = \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt^2}$$

(a) $i_s = 3\,u(t)\;\therefore$ assume $i_f = A$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = i_s$$

$$0 + 0 + A\frac{1}{(.03)\left(3\times10^{-3}\right)} = 1$$

$$\underline{A = 9\times10^{-5} = i_f}$$

(b)

$i_s = 1.5t\,u(t)\;\therefore$ assume $i_f = At + B$

$$0 + A\frac{195}{(300)\ (.003)} + (At+B)\frac{1}{(.03)(.003)} = 1.5t$$

$$\frac{650}{3}A + \frac{100000}{9}B = 0$$

$$\frac{100000}{9}At = 1.5t$$

$A = 1.35 \times 10^{-4}$

$B = -2.63 \times 10^{-6}$

$i_f = 1.35 \times 10^{-4} t - 2.63 \times 10^{-6} A$

(c) $i_s = 6e^{-750t}$ $\therefore$ assume $i_f = Ae^{-750t}$

This does not work $\therefore$ $i_f = Bte^{-750t}$

$$\frac{Be^{-750t}}{RC} + \frac{-750Bte^{-750t}}{RC} + \frac{Bte^{-750t}}{LC} = 6e^{-750t}$$

$$\frac{650}{3}B = 6$$

$B = 0.277$

$i_f = .0277te^{-750t} A$

P 9.7-2
Solution:

Represent the circuit by the differential equation: $\frac{d^2v}{dt} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = v_s$

(a) $v_s = 2$ $\therefore$ assume $v_f = A$

$0 + 0 + 6000A = 2$

$\underline{A = 1/3000 = v_f}$

(b) $v_s = .2t$ $\therefore$ assume $v_f = At + B$

$35A + 6000At + 6000B = 0.2t$

$35A + 6000B = 0$

$6000At = 0.2t$

$A = \frac{1}{30000}$, $B = \frac{35A}{6000}$, $B = 175$

$\therefore$ $\underline{v_f = \frac{t}{30000} + 175 \text{ V}}$

(c) $v_s = e^{-30t}$ $\therefore$ assume Ae^{-30t}

$900Ae^{-30t} - 1050Ae^{-30t} + 6000Ae^{-30t} = e^{-30t}$

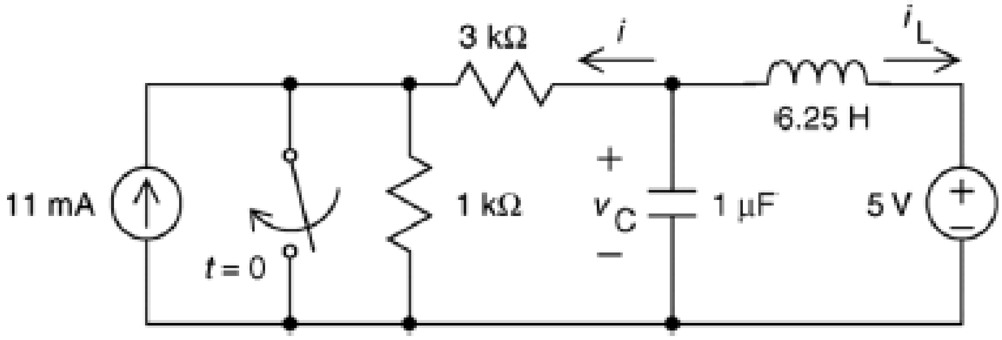
$5850Ae^{-30t} = e^{-30t}$

$A = \frac{1}{5850}$

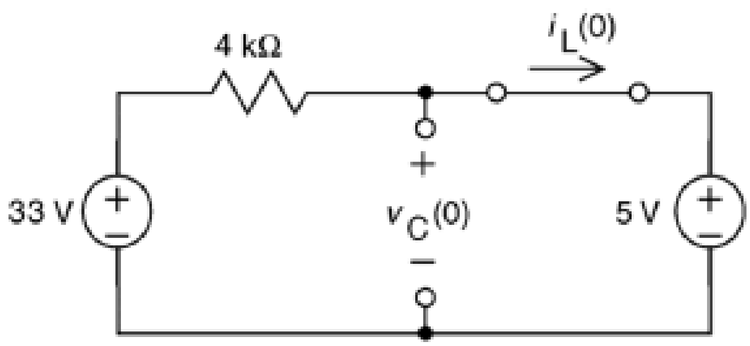
$\underline{v_f = \frac{e^{-30t}}{5850} \text{V}}$

Section 9-8: Complete Response of an RLC Circuit

P 9.8-1
Solution:



First, find the steady state response for t < 0, when the switch is open. Both inputs are constant so the capacitor will act like an open circuit at steady state, and the inductor will act like a short circuit. After a source transformation at the left of the circuit:



and

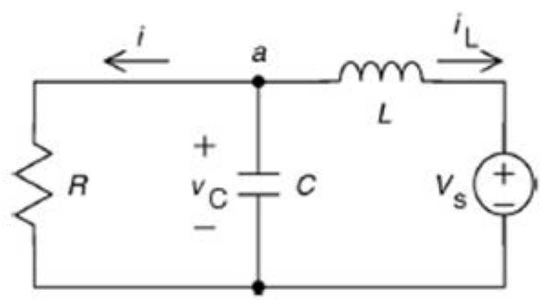
$$i_L(0) = \frac{33-5}{4000} = 7 \text{ mA}$$

$$v_C(0) = 5 \text{ V}$$

After the switch closes

Apply KCL at node a:

$$\frac{v_C}{R} + C\frac{d}{dt}v_C + i_L = 0$$



Apply KVL to the right mesh:

$$L\frac{d}{dt}i_L + V_s - v_C = 0 \quad \Rightarrow \quad v_C = L\frac{d}{dt}i_L + V_s$$

After some algebra:

$$\frac{d^2}{dt^2}i_L + \frac{1}{RC}\frac{d}{dt}i_L + \frac{1}{LC}i_L = -\frac{V_s}{RLC} \quad \Rightarrow \quad \frac{d^2}{dt^2}i_L + \left(10^3\right)\frac{d}{dt}i_L + \left(\frac{4}{25}\times 10^6\right)i_L = -\frac{20}{25}\times 10^3$$

The characteristic equation is

$$s^2+\left(10^3\right)s+\left(\frac{4}{25}\times 10^6\right)=0 \quad \Rightarrow \quad s_{1,2}=-200,-800 \text{ rad/s}$$

After the switch closes the steady-state inductor current is $i_L(\infty)$ = -5 mA so

$$i_L(t)=-0.005+A_1 e^{-200t}+A_2 e^{-800t}$$

$$v_C(t)=\left(\frac{4}{25}\right)\frac{d}{dt}i_L(t)+5=\frac{4}{25}\left[(-200)A_1 e^{-200t}+(-800)A_2 e^{-800t}\right]+5$$
$$=(-32)A_1 e^{-200t}+(-128)A_2 e^{-800t}+5$$

Let t = 0 and use the initial conditions:

$$0.007=-0.005+A_1+A_2 \quad \Rightarrow \quad 0.012=A_1+A_2$$

$$5=(-32)A_1+(-128)A_2+5 \quad \Rightarrow \quad A_1=(-4)A_2$$

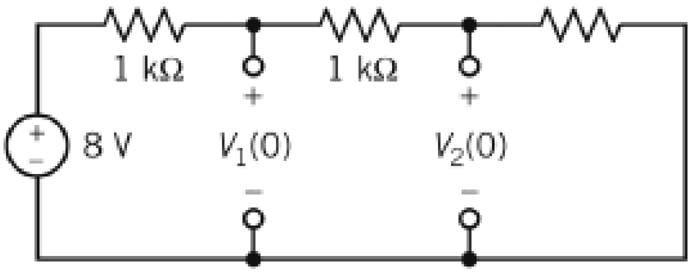
So $A_1 = 0.016$ and $A_2 = -0.004$ and

$$i_L(t) = -0.005 + (-0.004)e^{-200t} - 0.016e^{-800t} \text{ A}$$

$$v_C(t)=(-0.282)e^{-200t}+(0.282)e^{-800t}+5 \text{ V}$$

$$i(t)=\frac{v_C(t)}{1000}=\left[(-0.282)e^{-200t}+(0.282)e^{-800t}+5\right]\times 10^{-3} \text{ A}$$

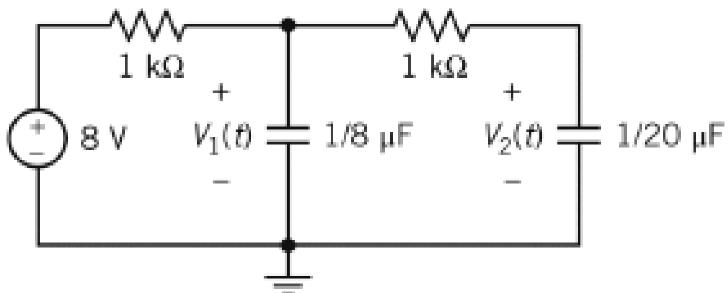
P 9.8-2
Solution:
First, find the steady state response for t < 0. The input is constant so the capacitors will act like an open circuit at steady state.



$$v_1(0)=\frac{1000}{1000+1000}(8)=4 \text{ V}$$
and
$$v_2(0)=0 \text{ V}$$

For $t>0$,

Node equations:



$$\frac{v_1-8}{1000}+\left(\frac{1}{8}\times10^{-6}\right)\frac{d}{dt}v_1+\frac{v_1-v_2}{1000}=0$$
$$\Rightarrow 2\,v_1+\left(\frac{1}{8}\times10^{-3}\right)\frac{d}{dt}v_1-8=v_2$$

$$\frac{v_1-v_2}{1000}=\left(\frac{1}{20}\times10^{-6}\right)\frac{d}{dt}v_2$$
$$\Rightarrow\ v_1-v_2=\left(\frac{1}{20}\times10^{-3}\right)\frac{d}{dt}v_2$$

After some algebra:
$$\frac{d^2v_1}{dt^2}+\left(3.6\times10^4\right)\frac{d}{dt}v_1+\left(16\times10^7\right)v_1=12.8\times10^8$$

The forced response will be a constant, $v_f = B$ so
$$\frac{d^2}{dt^2}B+\left(3.6\times10^4\right)\frac{d}{dt}B+\left(16\times10^7\right)B=12.8\times10^8\ \Rightarrow\ B=8\text{ V}.$$
To find the natural response, consider the characteristic equation:
$$s^2+\left(3.6\times10^4\right)s+\left(16\times10^7\right)=0\ \Rightarrow\ s_{1,2}=-5.2\times10^3,-3.1\times10^4$$
The natural response is
$$v_n=A_1\,e^{-5.2\times10^3 t}+A_2\,e^{-3.1\times10^4 t}$$
so

$$v_1(t) = A_1\, e^{-5.2\times10^3 t} + A_2\, e^{-3.1\times10^4 t} + 8$$

At $t = 0$

$$4 = v_1(0) = A_1\, e^{-5.2\times10^3 (0)} + A_2\, e^{-3.1\times10^4 (0)} + 8 = A_1 + A_2 + 8 \tag{1}$$

Next

$$2\, v_1 + \left(\frac{1}{8}\times10^{-3}\right)\frac{d}{dt}v_1 - 8 = v_2 \quad \Rightarrow \quad \frac{d}{dt}v_1 = 8000 v_2 - 16000\, v_1 + 6.4\times10^4$$

At $t = 0$

$$\begin{aligned}\frac{d}{dt}v_1(0) &= 8000 v_2(0) - 16000\, v_1(0) + 6.4\times10^4 \\ &= 8000(0) - 16000(4) + 6.4\times10^4 \\ &= 0\end{aligned}$$

so

$$\frac{d}{dt}v_1(t) = A_1\left(-5.2\times10^3\right)e^{-5.2\times10^3 t} + A_2\left(-3.1\times10^4\right)e^{-3.1\times10^4 t}$$

At $t = 0+$

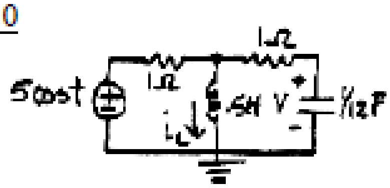
$$\begin{aligned}0 = \frac{d}{dt}v_1(0) &= A_1\left(-5.2\times10^3\right)e^{-5.2\times10^3 (0)} + A_2\left(-3.1\times10^4\right)e^{-3.1\times10^4 (0)} \\ &= A_1\left(-5.2\times10^3\right) + A_2\left(-3.1\times10^4\right)\end{aligned}$$

so $A_1 = -4.8$ and $A_2 = 0.8$. Finally

$$v_1(t) = 8 + 0.8e^{-3.1\times10^4 t} - 4.8e^{-5.2\times10^3 t} \ \text{V for } t > 0$$

P 9.8-3
Solution:

<u>t > 0</u>



KCL at top node : $\left(0.5\frac{di_L}{dt}-5\cos t\right)+i_L+\frac{1}{12}\frac{dv}{dt} = 0$ (1)

KVL at right loop : $0.5\frac{di_L}{dt} = \frac{1}{12}\frac{dv}{dt}+v$ (2)

$\frac{d}{dt}$ of (1) $\Rightarrow 0.5\frac{d^2i_L}{dt^2}+\frac{di_L}{dt}+\frac{1}{12}\frac{d^2v}{dt^2}=-5\sin t$ (3)

$\frac{d}{dt}$ of (2) $\Rightarrow 0.5\frac{d^2i_L}{dt^2} = \frac{1}{12}\frac{d^2v}{dt^2}+\frac{dv}{dt}$ (4)

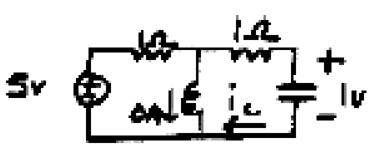
Solving for $\frac{d^2i_L}{dt^2}$ in (4) and $\frac{di_L}{dt}$ in (2) & plugging into (3)

$\underline{\frac{d^2v}{dt^2}+7\frac{dv}{dt}+12v = -30\sin t} \Rightarrow s^2+7s+12 = 0 \Rightarrow s = -3, -4$

so $v(t) = A_1e^{-3t}+A_2e^{-4t}+v_f$

Try $v_f = B_1\cos t+B_2\sin t$ & plug into D.E., equating like terms yields $B_1 = \frac{21}{17}$, $B_2 = -\frac{33}{17}$

<u>$t = 0^+$</u>



$i_c(0^+)=\frac{5-1}{1\ 1}=2A \quad \therefore \quad \frac{dv(0^+)}{dt} = \frac{2}{1/12} = 24\ \text{V/s}$

So $v(0^+) = 1 = A_1+A_2+\frac{21}{17}$

$\frac{dv(0^+)}{dt} = 24 = -3A_1-4A_2-\frac{33}{17}$

$A_1 = 25$

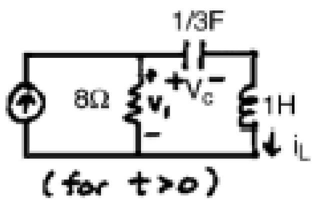
$A_2 = -\frac{429}{17}$

$\underline{\therefore\ v(t)=25e^{-3t}-\frac{1}{17}\left(429e^{-4t}-21\cos t+33\sin t\right)\ \text{V}}$

P 9.8-4

Solution:

Use superposition – first consider 2u(t) source



KVL at right mesh : $v_c + si_L + 8(i_L - 2) = 0$ (1)

also : $i_L = (1/3)\,sv_c \Rightarrow v_c = (3/s)\; i_L$ (2)

Plugging (2) into (1) yields $(s^2+8s+3)\,i_L = 0$, roots : $s = -0.4\,, -7.6$

So $i_L\;(t) = A_1 e^{-0.4t} + A_2 e^{-7.6t}$

$\underline{t=0^-} \Rightarrow$ circuit is dead $\therefore\; v_c(0) = i_L(0) = 0$

Now from (1) $\dfrac{di_L(0^+)}{dt} = 16 - 8i_L(0^+) - v_C(0^+) = 16$ A/s

$$\left.\begin{aligned} &\text{So}\;\; i_L(0) = 0 = A_1 + A_2 \\ &\frac{di_L(0)}{dt} = 16 = -0.4A_1 + (-7.6)A_2 \end{aligned}\right\} A_1 = 2.2\,,\; A_2 = -2.2$$

$\therefore\; i_L(t) = 2.2e^{-0.4t} - 2.2e^{-7.6t}$

$\therefore\; \underline{v_1(t) = 16 - 8\,i_L(t) = 16 - 17.6e^{-0.4t} + 17.6e^{-7.6t}}$ V

Now for $2u(t-2)$ source, just take above expression and replace $t \to t-2$ and flip signs

$\therefore\; v_2(t) = -16 + 17.6e^{-0.4(t-2)} - 17.6\, e^{-7.6(t-2)}$ V

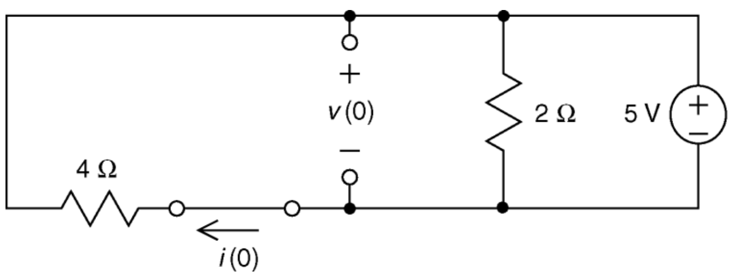
$\therefore\; v(t) = v_1(t) + v_2(t)$

$$\underline{v(t) = \left[16 - 17.6e^{-0.4t} + 17.6e^{-7.6t}\right] u(t) + \left[-16 + 17.6e^{-0.4(t-2)} - 17.6\, e^{-7.6(t-2)}\right] u\,(t-2) \text{ V}}$$

P 9.8-5

Solution:

First, find the steady state response for t < 0, when the switch is closed. The input is constant so the capacitor will act like an open circuit at steady state, and the inductor will act like a short circuit.

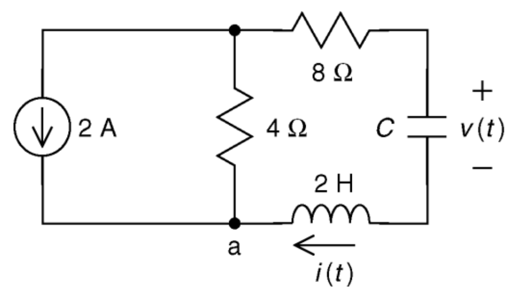


$$i(0) = -\frac{5}{4} = -1.25 \ \text{mA}$$

and

$$v(0) = 5 \ \text{V}$$

After the switch closes



Apply KCL at node a:

$$\frac{v}{2} + 0.125 \frac{d}{dt} v = i$$

Apply KVL to the right mesh:

$$-10 \cos t + v + 4 \frac{d}{dt} i + 4 i = 0$$

After some algebra: $$\frac{d^2}{dt^2} v + 5 \frac{d}{dt} v + 6 v = 20 \cos t$$

The characteristic equation is

$$s^2 + 5 s + 6 = 0 \quad \Rightarrow \quad s_{1,2} = -2, -3 \ \text{rad/s}$$

Try $$v_f = A \cos t + B \sin t$$

$$\frac{d^2}{dt^2}\left(A \cos t + B \sin t\right) + 5 \frac{d}{dt}\left(A \cos t + B \sin t\right) + 6\left(A \cos t + B \sin t\right) = 20 \cos t$$

$$\left(-A \cos t - B \sin t\right) + 5\left(-A \sin t + B \cos t\right) + 6\left(A \cos t + B \sin t\right) = 20 \cos t$$

$$\left(-A + 5 B + 6 A\right) \cos t + \left(-B - 5 A + 6 B\right) \sin t = 20 \cos t$$

So A =2 and B =2. Then $$v_f = 2 \cos t + 2 \sin t$$

$$v(t) = 2 \cos t + 2 \sin t + A_1 e^{-2t} + A_2 e^{-3t}$$

Next $$\frac{v(t)}{2} + 0.125 \frac{d}{dt} v(t) = i(t) \quad \Rightarrow \quad \frac{d}{dt} v(t) = 8 i(t) - 4 v(t)$$

$$\frac{d}{dt}v(0)=8\,i(0)-4\,v(0)=8\left(-\frac{5}{4}\right)-4\,(5)=-30\quad\frac{\text{V}}{s}$$

Let t = 0 and use the initial conditions:

$$5=v(0)=2\cos 0+2\sin 0+A_1\,e^{-0}+A_2\,e^{-0}=2+A_1+A_2$$

$$\frac{d}{dt}v(t)=-2\sin t+2\cos t-2\,A_1\,e^{-2t}-3\,A_2\,e^{-3t}$$

$$-30=\frac{d}{dt}v(0)=-2\sin 0+2\cos 0-2\,A_1\,e^{-0}-3\,A_2\,e^{-0}=2-2\,A_1-3\,A_2$$

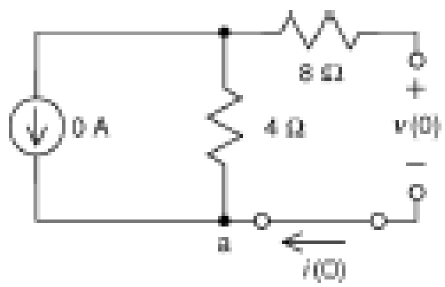
So A_1 = -23 and A_2 = 26 and

$$v(t)=2\cos t+2\sin t-23e^{-2t}+26e^{-3t}$$

P 9.8-6

Solution:

First, find the steady state response for t < 0, when the switch is closed. The input is constant so the capacitor will act like an open circuit at steady state, and the inductor will act like a short circuit.

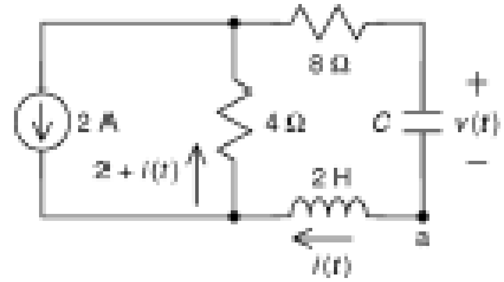


and

$$i(0)=0\ \text{A}$$

$$v(0)=0\ \text{V}$$

After the switch closes



Apply KCL at node a: $C\frac{d}{dt}v=i$

Apply KVL to the right mesh:

$$8\,i+v+2\frac{d}{dt}i+4(2+i)=0$$

$$12\,i+v+2\frac{d}{dt}i=-8$$

After some algebra: $$\frac{d^2}{dt^2}v+(6)\frac{d}{dt}v+\left(\frac{1}{2\,C}\right)v=-\frac{4}{C}$$

The forced response will be a constant, $v_f = B$ so

$$\frac{d^2}{dt^2}B+(6)\frac{d}{dt}B+\left(\frac{1}{2\,C}\right)B=-\frac{4}{C}\quad\Rightarrow\quad B=-8\text{ V}$$

(a) When C = 1/18 F the differential equation is $\frac{d^2}{dt^2}v+(6)\frac{d}{dt}v+(9)v=-72$

The characteristic equation is $s^2+6\,s+9=0\quad\Rightarrow\quad s_{1,2}=-3,-3$

Then $v(t)=\left(A_1+A_2\,t\right)e^{-3t}-8$.

Using the initial conditions:

$$0=v(0)=\left(A_1+A_2\,(0)\right)e^0-8\quad\Rightarrow\quad A_1=8$$

$$0=\frac{d}{dt}v(0)=-3\left(A_1+A_2\,(0)\right)e^0+A_2\,e^0\quad\Rightarrow\quad A_2=24$$

So

$$v(t)=(8+24\,t)\,e^{-3t}-8\text{ V}\quad\text{for } t>0$$

(b) When C = 1/10 F the differential equation is $\frac{d^2}{dt^2}v+(6)\frac{d}{dt}v+(5)v=-40$

The characteristic equation is $s^2+6\,s+5=0 \;\Rightarrow\; s_{1,2}=-1,-5$

Then $v(t)=A_1\,e^{-t}+A_2\,e^{-5t}-8$.

Using the initial conditions:

$$\left.\begin{array}{l} 0=v(0)=A_1\,e^0+A_2\,e^0-8 \;\Rightarrow\; A_1+A_2=8 \\ 0=\frac{d}{dt}v(0)=-A_1\,e^0-5A_2\,e^0 \;\Rightarrow\; -A_1-5A_2=0 \end{array}\right\} \Rightarrow A_1=10 \text{ and } A_2=-2$$

So

$$v(t)=10e^{-t}-2e^{-5t}-8\ \text{V} \quad \text{for } t>0$$

(c) When $C=1/20$ F the differential equation is $\frac{d^2}{dt^2}v+(6)\frac{d}{dt}v+(10)v=-80$

The characteristic equation is $s^2+6\,s+10=0 \;\Rightarrow\; s_{1,2}=-3\pm j$

Then $v(t)=e^{-3t}\left(A_1\cos t+A_2\sin t\right)-8$.

Using the initial conditions:

$$0=v(0)=e^0\left(A_1\cos 0+A_2\sin 0\right)-8 \;\Rightarrow\; A_1=8$$

$$0=\frac{d}{dt}v(0)=-3e^0\left(A_1\cos 0+A_2\sin 0\right)+e^0\left(-A_1\sin 0+A_2\cos 0\right) \;\Rightarrow\; A_2=24$$

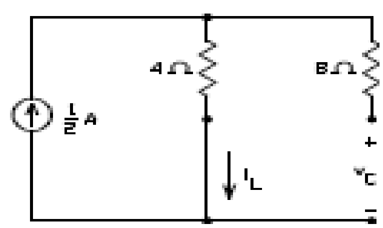
So

$$v(t)=e^{-3t}\left(8\cos t+24\sin t\right)-8\ \text{V} \quad \text{for } t>0$$

P 9.8-7
Solution:

The circuit will be at steady state for t<0:

so $i_L(0+)=i_L(0-)=0.5$ A and $v_C(0+)=v_C(0-)=2$ V.



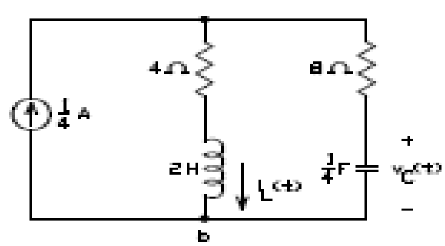
For t>0:

Apply KCL at node b to get:

$$\frac{1}{4}=i_L(t)+\frac{1}{4}\frac{d}{dt}v_C(t) \;\Rightarrow\; i_L(t)=\frac{1}{4}-\frac{1}{4}\frac{d}{dt}v_C(t)$$

Apply KVL to the right-most mesh to get:

$$4\,i_L(t)+2\frac{d}{dt}i_L(t)=8\left(\frac{1}{4}\frac{d}{dt}v_C(t)\right)+v_C(t)$$



Use the substitution method to get

$$4\left(\frac{1}{4}-\frac{1}{4}\frac{d}{dt}v_C(t)\right)+2\frac{d}{dt}\left(\frac{1}{4}-\frac{1}{4}\frac{d}{dt}v_C(t)\right)=8\left(\frac{1}{4}\frac{d}{dt}v_C(t)\right)+v_C(t)$$

or

$$2=\frac{d^2}{dt^2}v_C(t)+6\frac{d}{dt}v_C(t)+2v_C(t)$$

The forced response will be a constant, v_C= B so $2 = \frac{d^2}{dt^2}B + 6\frac{d}{dt}B + 2B \quad \Rightarrow \quad B = 1\text{ V}$.

To find the natural response, consider the characteristic equation:

$$0 = s^2 + 6s + 2 = (s+5.65)(s+0.35)$$

The natural response is

$$v_n = A_1 e^{-5.65t} + A_2 e^{-0.35t}$$

so

$$v_C(t) = A_1 e^{-5.65t} + A_2 e^{-0.35t} + 1$$

Then

$$i_L(t) = \frac{1}{4} + \frac{1}{4}\frac{d}{dt}v_C(t) = \frac{1}{4} + 1.41A_1 e^{-5.65t} + 0.0875A_2 e^{-0.35t}$$

At t=0+

$$2 = v_C(0+) = A_1 + A_2 + 1$$

$$\frac{1}{2} = i_L(0+) = \frac{1}{4} + 1.41A_1 + 0.0875A_2$$

so A_1 = 0.123 and A_2 = 0.877. Finally

$$v_C(t) = 0.123\,e^{-5.65t} + 0.877\,e^{-0.35t} + 1\text{ V}$$

P 9.8-8
Solution:

$$v = L\,{}^{di}\!/_{dt} \qquad (1)$$

$$KCL: C\frac{dv}{dt} + i + \frac{v}{3} = i_s \qquad (2)$$

Substituting (1) into (2) yields

$s^2 + 2.7s + 29 = 0$

roots : $s = -1.4 \pm j5.2$

$\therefore\ i_n = e^{-1.4t}\ [A\cos 5.2t + B\sin 5.2t]$

$i_f = {}^{174}\!/_{29} = 6$

So $i\ (t) = 6 + e^{-2t}\ [A\cos 5.2t\ + B\sin 5.2t]$

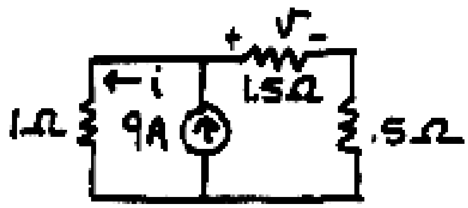
Now $i\ (0) = 0 = A + 6 \quad \Rightarrow A\ = -6$

from (1) $\dfrac{di\ (0)}{dt} = 0 = -2A + 5.2B \quad \Rightarrow B = -2.3$

$\therefore\ \underline{i\ (t) = 6 + e^{-2t}\ [-6\cos 5.2t - 2.3\sin 5.2t]}$

P 9.8-9
Solution:

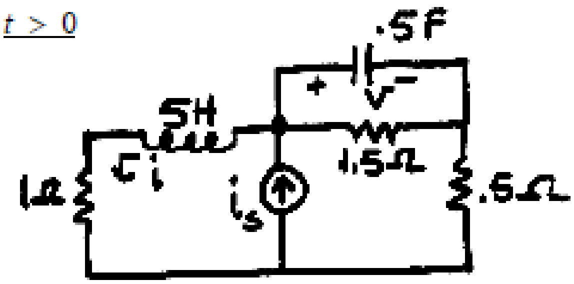
$\underline{t = 0^-}$



$$i(0^-) = \frac{2}{2+1} \times 9 = 6 \text{ A} = i\,(0^+)$$

$$\& \quad v\,(0^-) = \frac{1}{2+1} \times 9 \times 1.5 = 4.5 \text{ V} = v(0^+)$$

$\underline{t > 0}$



KCL at middle node: $\quad i + 0.5\,\dfrac{dv}{dt} + \dfrac{v}{1.5} = i_s \qquad (1)$

KVL : $\quad v + (0.5\,\dfrac{dv}{dt} + \dfrac{v}{1.5})\,(0.5) = \dfrac{5di}{dt} + i \qquad (2)$

Solving for i in (1) and plugging into (2) yields

$$\frac{d^2v}{dt^2} + \left(\frac{49}{30}\right)\frac{dv}{dt} + \left(\frac{4}{5}\right)v = \left(\frac{2}{5}\right)i_s + 2\frac{d_{is}}{dt} \quad \text{where } i_s = 9 + 3e^{-2t}\text{A}$$

So the characteristic equation is $s^2 + \dfrac{49}{30}s + \dfrac{4}{5} = 0$ and its roots are $s = -0.817 \pm j0.365$

$v_n(t) = e^{-.817t}\left[A_1 \cos(0.365t) + A_2 \sin(0.365t)\right]$

Try $v_f(t) = B_0 + B_1 e^{-2t}$ and substitute $v_f(t)$ into the differential equation and equate like terms

to get $B_0 = 4.5,\; B_1 = -7.04$

Then $v(t) = e^{-.817t}\,\left[A_1 \cos(0.365t) + A_2 \sin(0.365t)\right] + 4.5 - 7.04e^{-2t}$

Now using the initial conditions gives $v(0) = 4.5 = A_1 + 4.5 - 7.04 \Rightarrow A_1 = 7.04$

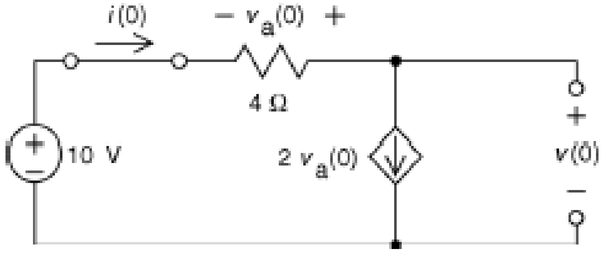
and $\dfrac{dv(0)}{dt} = 2i_s(0) - 2i(0) - \dfrac{4}{3}v(0) = 2(9+3) - 2(6) - \dfrac{4}{3}(4.5) = 6$

$\therefore\; 6 = -0.817A_1 + 0.365\,A_2 + 14.08 \Rightarrow A_2 = -22.82$

so $i(t) = i_s\,(t) - \dfrac{v(t)}{1.5} - 0.5\dfrac{dv(t)}{dt}$

$i(t) = \underline{e^{-0.817t}\left[2.37\cos(0.365t) + 7.14\sin(0.365t)\right] + 6 + 0.65e^{-2t} \text{ A}}$

P 9.8-10
Solution:
First, find the steady state response for t < 0, when the switch is closed. The input is constant so the capacitor will act like an open circuit at steady state, and the inductor will act like a short circuit.

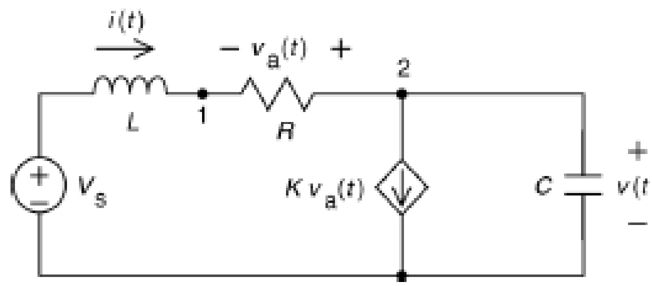


$$v_a(0) = -4\,i(0)$$

$$i(0) = 2(-4i(0)) \Rightarrow i(0) = 0 \text{ A}$$

and $\quad v(0) = 10 \text{ V}$

For $t > 0$
Apply KCL at node 2:



$$\frac{v_a}{R} + K v_a + C\frac{d}{dt}v = 0$$

KCL at node 1 and Ohm's Law:

$$v_a = -Ri$$

so

$$\frac{d}{dt}v = \frac{1+KR}{CR}i$$

Apply KVL to the outside loop: $\quad L\frac{d}{dt}i + R\,i + v - V_s = 0$

After some algebra:

$$\frac{d^2}{dt^2}v + \frac{R}{L}\frac{d}{dt}v + \frac{1+KR}{LC}v = \frac{1+KR}{LC}V_s \quad \Rightarrow \quad \frac{d^2}{dt^2}v + 40\frac{d}{dt}v + 144\,v = 2304$$

The forced response will be a constant, $v_f = B$ so

$$\frac{d^2}{dt^2}B + (40)\frac{d}{dt}B + (144)B = 2304 \quad \Rightarrow \quad B = 16 \text{ V}$$

The characteristic equation is $s^2 + 40s + 144 = 0 \quad \Rightarrow \quad s_{1,2} = -4, -36$.

Then $\qquad v(t) = A_1 e^{-4t} + A_2 e^{-36t} + 16$.

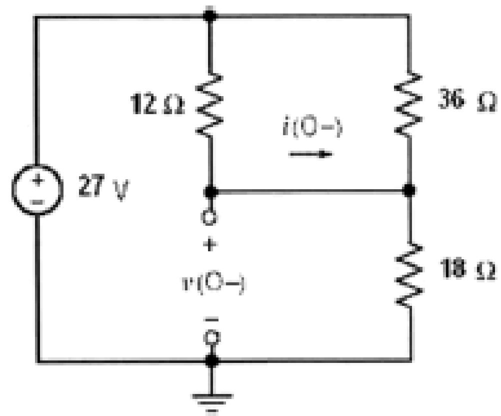
Using the initial conditions:

$$\left.\begin{array}{r} 10 = v(0) = A_1 e^0 + A_2 e^0 + 16 \quad \Rightarrow \quad A_1 + A_2 = -6 \\ 0 = \frac{d}{dt}v(0) = -4A_1 e^0 - 36A_2 e^0 \quad \Rightarrow \quad -4A_1 - 36A_2 = 0 \end{array}\right\} \Rightarrow \quad A_1 = 0.75 \text{ and } A_2 = -6.75$$

So

$$v(t) = 0.75e^{-4t} - 6.75e^{-36t} + 16 \text{ V} \quad \text{for } t > 0$$

P 9.8-11
Solution:
First, we find the initial conditions;



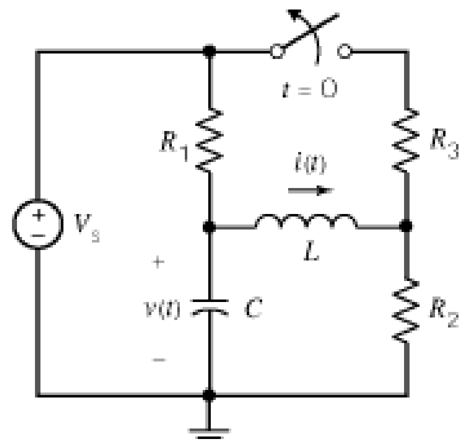
For $t < 0$, the switch is closed and the circuit is at steady state. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.

$$v(0-) = \frac{18}{(12 \| 36)+18} \times 27 = 18 \text{ V}$$

and

$$i(0-) = \frac{36}{12+36} \times \frac{27}{(12 \| 36)+18} = 0.75 \text{ A}$$

Next, represent the circuit by a differential equation.



After the switch opens, apply KCL and KVL to get

$$R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t) = V_s$$

Apply KVL to get

$$v(t) = L\frac{d}{dt}i(t) + R_2\, i(t)$$

Substituting $v(t)$ into the first equation gives

$$R_1\left(i(t) + C\frac{d}{dt}\left(L\frac{d}{dt}i(t) + R_2\, i(t)\right)\right) + L\frac{d}{dt}i(t) + R_2\, i(t) = V_s$$

then

$$R_1\, C\, L\frac{d^2}{dt^2}i(t) + \left(R_1\, C\, R_2 + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t) = V_s$$

Dividing by $R_1\, C\, L$:

$$\frac{d^2}{dt^2}i(t) + \left(\frac{R_1\, C\, R_2 + L}{R_1\, C\, L}\right)\frac{d}{dt}i(t) + \left(\frac{R_1 + R_2}{R_1\, C\, L}\right)i(t) = \frac{V_s}{R_1\, C\, L}$$

Compare to

$$\frac{d^2}{dt^2}i(t) + 2\alpha\frac{d}{dt}i(t) + \omega_0^2\, i(t) = f(t)$$

to get

$$2\alpha = \frac{R_1 C R_2 + L}{R_1 C L}, \quad \omega_0^{\,2} = \frac{R_1 + R_2}{R_1 C L} \quad \text{and} \quad f(t) = \frac{V_s}{R_1 C L}$$

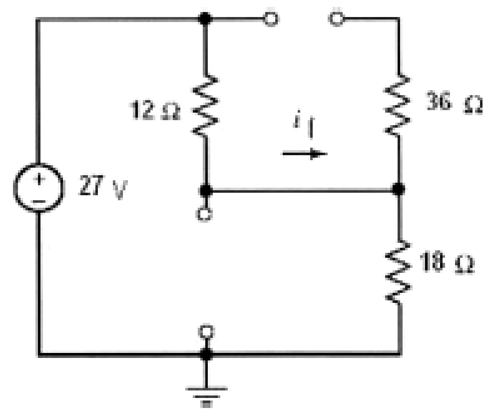
With the given element values, we have $\alpha = 16.11$ and $\omega_0^{\,2} = 111.11$. Consequently, the roots of the characteristic equation are $s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\,2}} = -28.29$ and $s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\,2}} = -3.93$

The natural response is

$$i_n(t) = A_1 e^{-3.93t} + A_2 e^{-28.29t}$$

Next, determine the forced response.

The steady state response after the switch opens will be used as the forced response. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.



$$i_f = \frac{27}{12+18} = 0.9 \text{ A}$$

So

$$i(t) = i_n(t) + i_f(t) = A_1 e^{-3.93t} + A_2 e^{-28.29t} + 0.9$$

It remains to evaluate A_1 and A_2 using the initial conditions. At $t = 0$ we have

$$0.75 = i(0) = A_1 + A_2 + 0.9$$

The other initial condition comes from

$$\frac{d}{dt} i(t) = \frac{v(t)}{L} - \frac{R_2}{L} i(t) \quad \Rightarrow \quad \frac{d}{dt} i(0) = \frac{18}{0.6} - \frac{18}{0.6} \times 0.75 = 7.5$$

then

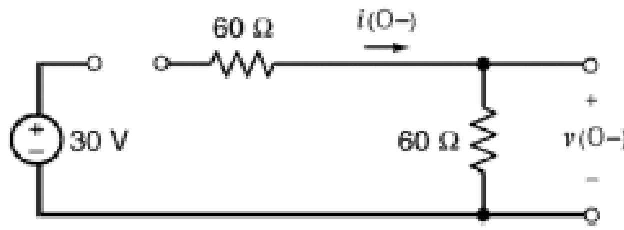
$$7.5 = \frac{d}{dt} i(0) = -3.93\, A_1 - 28.29\, A_2$$

Solving these equations gives $A_1 = 0.13$ and $A_2 = -0.28$ so

$$i(t) = 0.13 e^{-3.93t} - 0.28 e^{-28.29t} + 0.9 \text{ A} \quad \text{for } t > 0$$

P 9.8-12
Solution:
First, we find the initial conditions;

For $t < 0$, the switch is open and the circuit is at steady state. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.



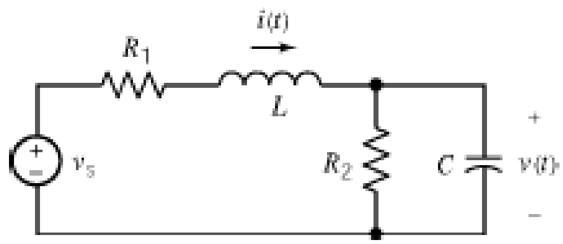
$$v(0-) = 0 \text{ V and } i(0-) = 0 \text{ A}$$

also

$$\frac{d}{dt}v(0) = \frac{i(0)}{0.005} - \frac{v(0)}{60 \times 0.005} = 0$$

Next, represent the circuit after the switch closes by a differential equation.

After the switch closes, use KCL to get



$$i(t) = \frac{v(t)}{R_2} + C\frac{d}{dt}v(t)$$

Use KVL to get

$$v_s = R_1\, i(t) + L\frac{d}{dt}i(t) + v(t)$$

Substitute to get

$$v_s = \frac{R_1}{R_2}v(t) + R_1 C\frac{d}{dt}v(t) + \frac{L}{R_2}\frac{d}{dt}v(t) + CL\frac{d^2}{dt^2}v(t) + v(t)$$

$$= CL\frac{d^2}{dt^2}v(t) + \left(R_1 C + \frac{L}{R_2}\right)\frac{d}{dt}v(t) + \frac{R_1 + R_2}{R_2}v(t)$$

Finally,

$$\frac{v_s}{CL} = \frac{d^2}{dt^2}v(t) + \left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)\frac{d}{dt}v(t) + \frac{R_1 + R_2}{R_2 CL}v(t)$$

Compare to

$$\frac{d^2}{dt^2}i(t) + 2\alpha\frac{d}{dt}i(t) + \omega_0^2\, i(t) = f(t)$$

to get

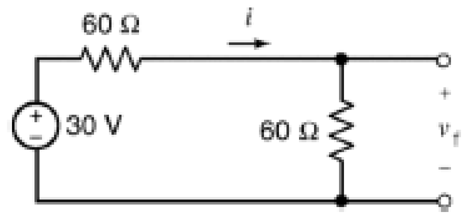
$$2\alpha = \frac{R_1}{L} + \frac{1}{R_2 C}, \quad \omega_0^2 = \frac{R_1 + R_2}{R_2 CL} \quad \text{and} \quad f(t) = \frac{v_s}{CL}$$

With the given element values, we have $\alpha = 16.65$ and $\omega_0^2 = 200$. Consequently, the roots of the characteristic equation are $s_1 = -7.9$ and $s_2 = -25.4$ so the circuit is overdamped. The natural response is

$$v_n(t) = A_1 e^{-7.9t} + A_2 e^{-25.4t}$$

Next, determine the forced response.

The steady state response after the switch opens will be used as the forced response. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.



$$v_f = \frac{1}{2} v_s = 15 \text{ V}$$

So

$$v_n(t) = 15 + A_1 e^{-7.9t} + A_2 e^{-25.4t}$$

It remains to evaluate A_1 and A_2 using the initial conditions. At $t = 0$ we have

$$0 = v(0) = 15 + A_1 + A_2$$

and

$$0 = \frac{d}{dt} v(0) = -7.9\, A_1 - 25.4\, A_2$$

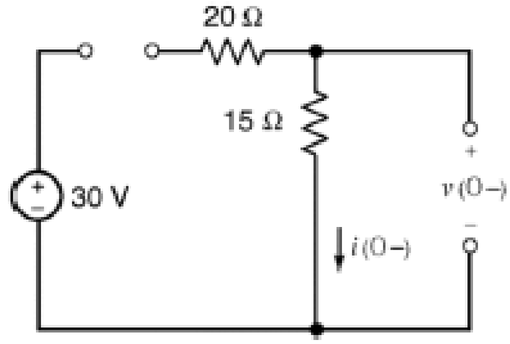
Solving these equations gives

$$A_1 = -21.8 \text{ and } A_2 = 6.8$$

Finally,

$$v(t) = 15 - 21.8 e^{-7.9t} + 6.8 e^{-25.4t}$$

P 9.8-13
Solution:
First, we find the initial conditions;



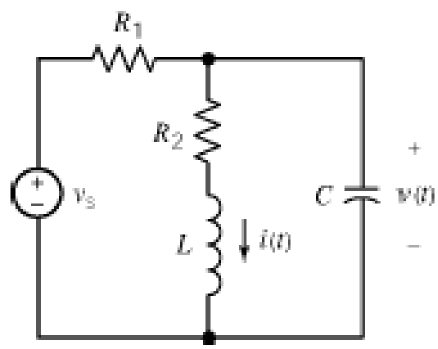
For $t < 0$, the switch is closed and the circuit is at steady state. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.

$$v(0-) = 0 \text{ V and } i(0-) = 0 \text{ A}$$

Also

$$15\,i(0) + 0.4\frac{d}{dt}i(0) = v(0) \quad \Rightarrow \quad \frac{d}{dt}i(0) = 0$$

Next, represent the circuit by a differential equation.



After the switch closes use KVL to get

$$R_2\, i(t) + L\frac{d}{dt}i(t) = v(t)$$

Use KCL and KVL to get

$$v_s = R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t)$$

Substitute to get

$$v_s = R_1 i(t) + R_1 C R_2 \frac{d}{dt}i(t) + R_1 C\, L\frac{d^2}{dt^2}i(t) + R_2 i(t) + L\frac{d}{dt}i(t)$$

$$= R_1 C L\frac{d^2}{dt^2}i(t) + \left(R_1 R_2 C + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t)$$

then

$$\frac{v_s}{R_1 C L} = \frac{d^2}{dt^2}i(t) + \left(\frac{R_2}{L} + \frac{1}{R_1 C}\right)\frac{d}{dt}i(t) + \frac{R_1 + R_2}{R_1 C L}i(t)$$

Compare to

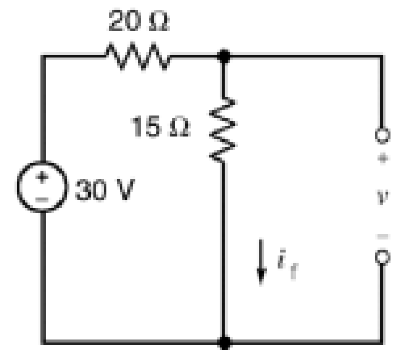
$$\frac{d^2}{dt^2}i(t) + 2\alpha\frac{d}{dt}i(t) + \omega_0^2\, i(t) = f(t)$$

to get

$$2\alpha = \frac{R_2}{L} + \frac{1}{R_1 C}, \quad \omega_0^2 = \frac{R_1 + R_2}{R_1\, C\, L} \quad \text{and} \quad f(t) = \frac{V_s}{R_1\, C\, L}$$

With the given element values, we have $\alpha = 19.75$ and $\omega_0{}^2 = 17.5$. Consequently, the roots of the characteristic equation are $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2} = -5.05,\ -34.45$ so the circuit is critically damped. The natural response is

$$A_1 e^{-5.05t} + A_2 t\, e^{34.45t}$$



Next, determine the forced response.

The steady state response after the switch opens will be used as the forced response. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.

$$i_f = \frac{30}{20+15} = 0.9 \text{ A}$$

So

$$i(t) = i_n(t) + i_f(t) = A_1 e^{-5.05t} + A_2 t\, e^{-34.45t} + 0.9$$

It remains to evaluate A_1 and A_2 using the initial conditions. At $t = 0$ we have

$$0 = i(0) = A_1 + 0.9 \quad \Rightarrow \quad A_1 = -0.9$$

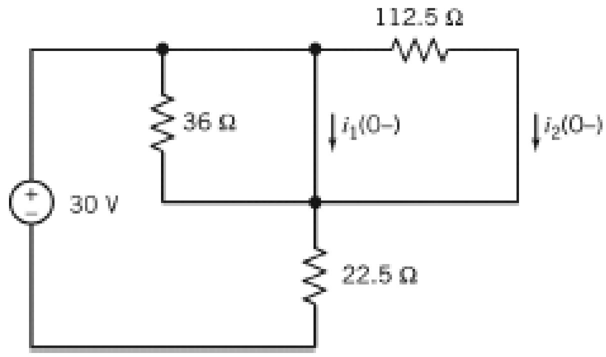
And

$$0 = \frac{d}{dt} i(0) = -5.05\, A_1 + A_2 \quad \Rightarrow \quad A_2 = -4.5$$

Thus

$$i(t) = \left(-0.9 e^{-5.05t} - 4.5 t\, e^{-34.45t}\right) + 0.9 \quad \text{for } t > 0$$

P 9.8-14
Solution:
First, we find the initial conditions;



For $t < 0$, the switch is closed and the circuit is at steady state. At steady state, the inductors act like short circuits.

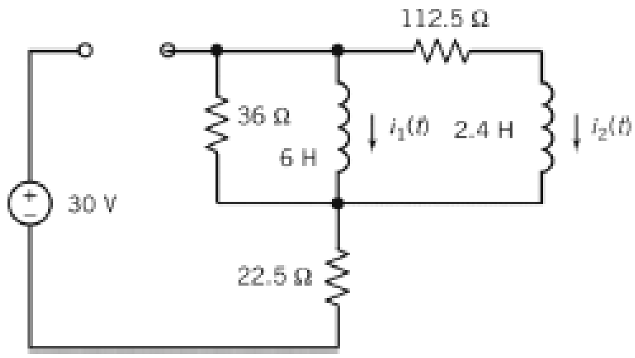
$$i_1(0-) = \frac{30}{22.5} = 1.333 \text{ A}$$

and

$$i_2(0-) = 0 \text{ A}$$

Next, represent the circuit by a differential equation.

After the switch opens, KVL gives



$$L_1 \frac{d}{dt} i_1(t) = R_2\, i_2(t) + L_2 \frac{d}{dt} i_2(t)$$

KVL and KCL give

$$L_1 \frac{d}{dt} i_1(t) + R_1\left(i_1(t) + i_2(t)\right) = 0$$

Use the operator method to get

$$L_1 s\, i_1 = R_2\, i_2 + L_2 s\, i_2$$

$$L_1 s\, i_1 + R_1\left(i_1 + i_2\right) = 0$$

$$L_1 s^2 i_1 + R_1 s\, i_1 + R_1 s\, i_2 = 0$$

$$s\left(R_2 i_2 + L_2 s\, i_2\right) + \frac{R_1}{L_1}\left(R_2 i_2 + L_2 s\, i_2\right) + R_1 s\, i_2 = 0$$

$$L_2 s^2\, i_2 + \left(R_2 + R_1 \frac{L_2}{L_1} + R_1\right) s\, i_2 + \frac{R_1 R_2}{L_1} i_2 = 0$$

$$s^2 i_2 + \left(\frac{R_2}{L_2} + \frac{R_1}{L_2} + \frac{R_1}{L_1}\right) s\, i_2 + \frac{R_1 R_2}{L_1 L_2} i_2 = 0$$

so

$$\frac{d^2}{dt^2} i_2(t) + \left(\frac{R_2}{L_2} + \frac{R_1}{L_2} + \frac{R_1}{L_1}\right) \frac{d}{dt} i_2(t) + \frac{R_1 R_2}{L_1 L_2} i_2(t) = 0$$

Compare to

$$\frac{d^2}{dt^2}i(t)+2\alpha\frac{d}{dt}i(t)+\omega_0^{\,2}\,i(t)=f(t)$$

to get

$$2\alpha=\frac{R_2}{L_2}+\frac{R_1}{L_2}+\frac{R_1}{L_1},\quad \omega_0^{\,2}=\frac{R_1R_2}{L_1L_2}\quad \text{and}\quad f(t)=0$$

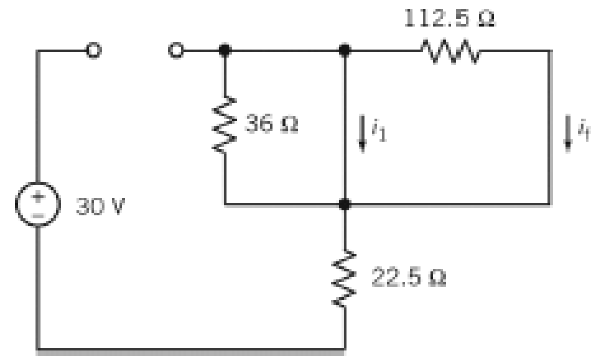
With the given element values, we have $\alpha=33.9$ and $\omega_0^{\,2}=281.25$. Consequently, the roots of the characteristic equation are $s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^{\,2}}=-4.4,\ -63.4$ so the circuit is overdamped. The natural response is

$$i_{\mathrm{n}}(t)=A_1\,e^{-4.4t}+A_2\,e^{-63.4t}$$

Next, determine the forced response.

The steady state response after the switch opens will be used as the forced response. At steady state the inductors act like short circuits.

$$i_{\mathrm{f}}=0\ \ \mathrm{A}$$



So

$$i_2(t)=i_{\mathrm{n}}(t)+i_{\mathrm{f}}(t)=A_1\,e^{-4.4t}+A_2\,e^{-63.4t}$$

It remains to evaluate A_1 and A_2 using the initial conditions. At $t = 0$ we have

$$0=i_2(0)=A_1+A_2$$

$$L_2\frac{d}{dt}i_2(0)+R_2\,i_2(0)+R_1\,i_1(0)+R_1\,i_2(0)\ \Rightarrow\ \frac{d}{dt}i_2(0)=-30$$

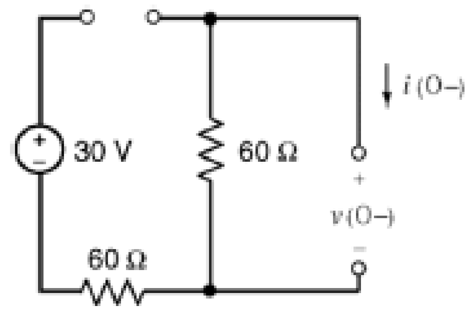
and

$$-30=\frac{d}{dt}i(0)=-4.4\,A_1-63.4\,A_2$$

Solving these equations gives $A_1 = -0.508$ and $A_2 = 0.508$ so

$$i_2(t)=-0.508e^{-4.4t}+0.508e^{-63.4t}\quad \text{for } t\geq 0$$

P 9.8-15
Solution:
First, we find the initial conditions;



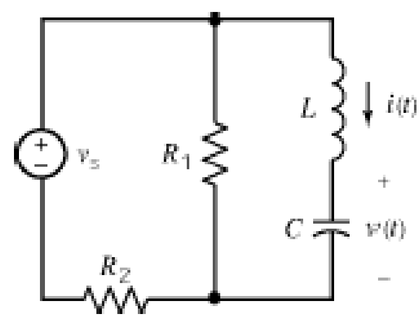
For $t < 0$, the switch is open and the circuit is at steady state. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.

$$v(0-)=0 \text{ V and } i(0-)=0 \text{ A}$$

also

$$\frac{d}{dt}v(0)=\frac{i(0)}{0.005}=0$$

Next, represent the circuit after the switch closes by a differential equation.



After the switch closes

$$i(t)=C\frac{d}{dt}v(t)$$

KCL and KVL give

$$v_s=R_2\left(i(t)+\frac{1}{R_1}\left(L\frac{d}{dt}i(t)+v(t)\right)\right)+L\frac{d}{dt}i(t)+v(t)$$

Substituting gives

$$v_s=\frac{R_2}{R_1}LC\frac{d^2}{dt^2}v(t)+R_2C\frac{d}{dt}v(t)+\left(1+\frac{R_2}{R_1}\right)v(t)=\left(1+\frac{R_2}{R_1}\right)LC\frac{d^2}{dt^2}v(t)+R_2C\frac{d}{dt}v(t)+\left(1+\frac{R_2}{R_1}\right)v(t)$$

So the differential equation is

$$\frac{R_1v_s}{LC(R_1+R_2)}=\frac{d^2}{dt}v(t)+\frac{R_1R_2}{L(R_1+R_2)}\frac{d}{dt}v(t)+\frac{1}{LC}v(t)$$

Compare to

$$\frac{d^2}{dt^2}i(t)+2\alpha\frac{d}{dt}i(t)+\omega_0{}^2\,i(t)=f(t)$$

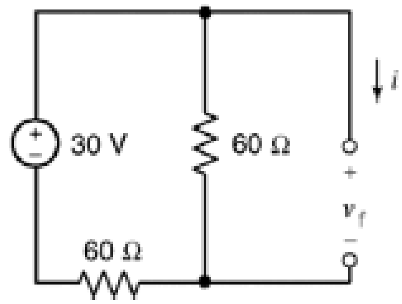
to get

$$2\alpha=\frac{R_1R_2}{L(R_1+R_2)},\quad \omega_0{}^2=\frac{1}{CL}\quad \text{and}\quad f(t)=\frac{R_1v_s}{LC(R_1+R_2)}$$

With the given element values, we have $\alpha = 7.5$ and $\omega_0^{\ 2} = 100$. Consequently, the roots of the characteristic equation are $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\ 2}} = -7.5 \pm j\,6.614$ and the circuit is underdamped. The damped resonant frequency is $\omega_d = \sqrt{\omega_0^{\ 2} - \alpha^2} = 6.614$ rad/s. The natural response is

$$v_n(t) = e^{-7.5t}\left(A_1 \cos 6.614t + A_2 \sin 6.614t\right)$$

Next, determine the forced response.



The steady state response after the switch opens will be used as the forced response. At steady state, the capacitor acts like an open circuit and the inductor acts like a short circuit.

$$v_f = \frac{60}{60+60} \times 30 = 15 \ \text{V}$$

So

$$v(t) = 15 + e^{-7.5t}\left(A_1 \cos 6.614t + A_2 \sin 6.614t\right)$$

It remains to evaluate A_1 and A_2 using the initial conditions. At $t = 0$ we have

$$0 = v(0) = 15 + A_1 \quad \Rightarrow \quad A_1 = -15$$

and

$$0 = \frac{d}{dt}v(0) = -7.5\,A_1 + 6.614\,A_2 \quad \Rightarrow \quad A_2 = -\frac{7.5 \times 15}{6.614} = -17.009$$

Finally,

$$\begin{aligned} v(t) &= 15 + e^{-7.5t}\left(-15\cos 6.614t - 17.009\sin 6.614t\right) \\ &= 15 + 22.68\,e^{-7.5t}\left(\ 6.614t + 131.5°\right) \ \ \text{V} \quad \text{for } t \geq 0 \end{aligned}$$

P 9.8-16

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives: $$\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t) = i_L(t)$$

KVL around the outside loop gives: $$v_s(t) = L\frac{d}{dt}i_L(t) + R_1\, i_L(t) + v_C(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= L\frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + v_C(t) \\ &= LC\frac{d^2}{dt^2}v_C(t) + \left(\frac{L}{R_2} + R_1 C\right)\frac{d}{dt}v_C(t) + \left(1 + \frac{R_1}{R_2}\right)v_C(t) \end{aligned}$$

(a) $C = 1$ F, $L = 0.25$ H, $R_1 = R_2 = 1.309\ \Omega$

Use the steady state response as the forced response:

$$v_f = v_C(\infty) = \frac{R_2}{R_1 + R_2} 1 = \frac{1}{2}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 6s + 8 = (s+2)(s+4)$$

so the natural response is

$$v_n = A_1\, e^{-2t} + A_2\, e^{-4t}\ \text{V}$$

The complete response is

$$v_c(t) = \frac{1}{2} + A_1\, e^{-2t} + A_2\, e^{-4t}\ \text{V}$$

$$i_L(t) = \frac{v_C(t)}{1.309} + \frac{d}{dt}v_C(t) = -1.236\, A_1 e^{-2t} - 3.236\, A_2 e^{-4t} + 0.3819$$

At t = 0+

$$0 = v_c(0+) = A_1 + A_2 + 0.5$$
$$0 = i_L(0+) = -1.236\,A_1 - 3.236\,A_2 + 0.3819$$

Solving these equations gives A_1 = -1 and A_2 = 0.5, so

$$v_c(t) = \frac{1}{2} - e^{-2t} + \frac{1}{2}e^{-4t} \text{ V}$$

(b) C = 1 F, L = 1 H, R_1 = 3 Ω, R_2 = 1 Ω

Use the steady state response as the forced response:

$$v_f = v_C(\infty) = \frac{R_2}{R_1 + R_2}1 = \frac{1}{4}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 4s + 4 = (s+2)^2$$

so the natural response is

$$v_f = (A_1 + A_2 t)\,e^{-2t} \text{ V}$$

The complete response is

$$v_c(t) = \frac{1}{4} + (A_1 + A_2 t)\,e^{-2t} \text{ V}$$

$$i_L(t) = v_C(t) + \frac{d}{dt}v_C(t) = \frac{1}{4} + \left((A_2 - A_1) - A_2 t\right)e^{-2t}$$

At t = 0+

$$0 = v_c(0+) = A_1 + \frac{1}{4}$$
$$0 = i_L(0+) = \frac{1}{4} + A_2 - A_1$$

Solving these equations gives A_1 = -0.25 and A_2 = -0.5, so

$$v_c(t) = \frac{1}{4} - \left(\frac{1}{4} + \frac{1}{2}t\right)e^{-2t} \text{ V}$$

(c) C = 0.125 F, L = 0.5 H, R_1 = 1 Ω, R_2 = 4 Ω

Use the steady state response as the forced response:

$$v_f = v_C(\infty) = \frac{R_2}{R_1 + R_2}1 = \frac{4}{5}$$

The characteristic equation is

$$s^2+\left(\frac{1}{R_2 C}+\frac{R_1}{L}\right)s+\left(\frac{1+\frac{R_1}{R_2}}{LC}\right)=s^2+4s+20=(s+2-j4)(s+2+j4)$$

so the natural response is

$$v_f=e^{-2t}\left(A_1\cos 4t+A_2\sin 4t\right)\text{ V}$$

The complete response is

$$v_c(t)=0.8+e^{-2t}\left(A_1\cos 4t+A_2\sin 4t\right)\text{ V}$$

$$i_L(t)=\frac{v_C(t)}{4}+\frac{1}{8}\frac{d}{dt}v_C(t)=0.2+\frac{A_2}{2}e^{-2t}\cos 4t-\frac{A_1}{2}e^{-2t}\sin 4t$$

At t = 0+

$$0=v_c(0+)=0.8+A_1$$

$$0=i_L(0+)=0.2+\frac{A_2}{2}$$

Solving these equations gives A_1 = -0.8 and A_2 = -0.4, so

$$v_c(t)=0.8-e^{-2t}(0.8\cos 4t+0.4\sin 4t)\text{ V}$$

Section 9-9: State Variable Approach to Circuit Analysis

P 9.9-1
Solution:

$\underline{t=0^-}$ circuit is source free $\therefore\ i_L(0)=0\ \&\ v(0)\ =\ 0$

$\underline{t>0}$

KCL at top node: $i_L + \left(\frac{1}{5}\right) dv/dt\ =\ 4$ (1)

KVL at right loop: $(v-1)\ di_L/dt - 6i_L\ =\ 0$

Solving for i_1 in (1) & plugging into (2) $\Rightarrow\ \frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 5v = 120$

The characteristic equation is: $s^2+6s+5=0$,

The roots of the characteristic equation are $s=-1,-5$

$\therefore$ Tthe natural response is: $v_n\ (t) = A_1\ e^{-t} + A_2\ e^{-5t}$

Try $v_f = B$ & plug into D.E. $\Rightarrow\ B = 24 = v_f$

From (1) $\frac{dv(0)}{dt} = 20 - 5i_L(0) = 20\ \text{V}/\text{s}$

So $v(0) = 0 = A_1 + A_2 + 24$

$\frac{dv(0)}{dt} = 20\ = -A_1 - 5A_2$

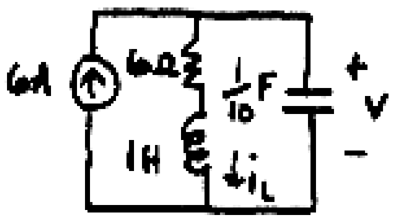
$A_1 = -25,\quad A_2 =\ 1$

$\underline{\therefore\ v(t) = -25e^{-t} + e^{-5t} +\ 24\ \text{V}}$

P 9.9-2
Solution:

At $\underline{t=0^-}$ the circuit is source free $\therefore\ i_L(0)=0,\ \&\ v(0)=0$

At $\underline{t>0}$



KCL at top node: $i_L = 4-\left(\frac{1}{10}\right)\frac{dv}{dt}$ (1)

KVL at right node: $v-\frac{di_L}{dt}-6i_L=0$ (2)

(1) into (2) yields $\frac{d^2v}{dt^2}+6\frac{dv}{dt}+10v=240$

$\Rightarrow s^2+6s+10=0,\ s=-3\pm j \quad \therefore\ v_n(t)=e^{-3t}\left[A_1\cos t+A_2\sin t\right]$

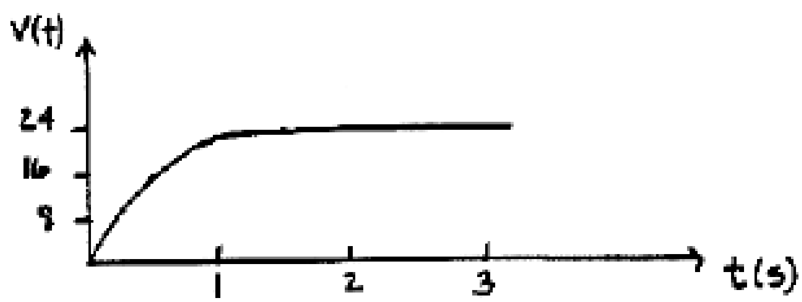
Try $v_f=B$ & plug into D.E. $\Rightarrow v_f=B=24$

From (1) $\frac{dv(0)}{dt}=40-10\,i_L(0)=40\ \text{V/s}$

So $v(0)=0=A_1+24 \Rightarrow A_1=-24$ & $\frac{dv(0)}{dt}=40=-3A_1+A_2$

$\Rightarrow A_2=-32$

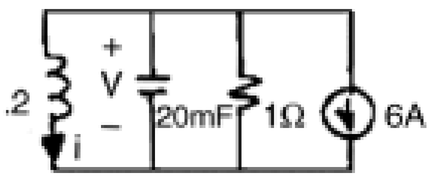
$\underline{\therefore v(t)=e^{-3t}\left[-24\cos t-32\sin t\right]+24\ \text{V}}$



P 9.9-3
Solution:

$i(0) = -3,\ v(0) = 0$

$t > 0$



$$\text{KCL: } i + C\frac{dv}{dt} + \frac{v}{R} + 6 = 0$$

$$\text{KVL: } v = L\frac{di}{dt}$$

$$\frac{d^2i}{dt^2} + 50\frac{di}{dt} + 250i = -1500$$

$$s = -5.6,\ -44.3$$

$$i_f(t) = \frac{-1500}{250} = -6$$

$$i(t) = A_1 e^{-5.6t} + A_2 e^{-44.3t} - 6$$

$$\left.\begin{array}{l} i(0) = A_1 + A_2 - 6 = -3 \\ \dfrac{di(0)}{dt} = 0 = -5.6A_1 - 44.3A_2 \end{array}\right\} \begin{array}{l} A_1 = 3.434 \\ A_2 = -0.434 \end{array}$$

$$\underline{i(t) = 3.434\, e^{-5.6t} - 0.434 e^{-44.3t} - 6 \text{ A}}$$

$$\underline{v(t) = 0.2\frac{di}{dt} = -3.84 e^{-5.6t} + 3.84 e^{-44.3t} \text{ V}}$$

P 9.9-4
Solution:
Apply KCL to the supernode corresponding to the VCVS to get

$$i_x+\frac{v_x}{2}=2i_x+C\frac{dv}{dt} \quad\Rightarrow\quad -i_x+\frac{v_x}{2}=0.01\frac{dv}{dt} \tag{1}$$

Apply KCL to the right node of the CCCS to get $i+\frac{v_x}{2}=2i_x$. (2)

Apply KVL to the mesh consisting of the 2-Ω resistor, inductor and capacitor to get

$$v_x+v-L\frac{di}{dt}=0 \quad\Rightarrow\quad v_x+v=0.1\frac{di}{dt} \tag{3}$$

Apply KVL to the outside loop to get $v+i_x+2v_x=0$ (4)

Combine equations (2) and (4) to get

$$i_x=\frac{4}{9}i-\frac{1}{9}v \quad\text{and}\quad v_x=-\frac{2}{9}i-\frac{4}{9}v \tag{5}$$

Use (5) to eliminate i_x and v_x from (1) and (3) to get

$$0.01\frac{dv}{dt}=-\frac{5}{9}i-\frac{1}{9}v \quad\text{and}\quad 0.1\frac{di}{dt}=-\frac{2}{9}i+\frac{5}{9}v \tag{6}$$

Use operators to write

$$s\,v=-\frac{500}{9}i-\frac{100}{9}v \quad\text{and}\quad s\,i=-\frac{20}{9}i+\frac{50}{9}v \tag{7}$$

The characteristic equation is : $s^2+13.33s+333.33=0 \quad\Rightarrow\quad s_1,\ s_2=-6.67\pm j\,17$

The natural response is $v(t)=e^{-6.67t}\left[A\cos(17t)+B\sin(17t)\right]$ and there is no forced because there is no forcing function. The constants A and B are evaluated using the initial conditions:

$$v(0)=10=A \quad\text{and}\quad \frac{dv(0)}{dt}=-111=-6.67A+17\,B \quad\Rightarrow\quad B=-2.6$$

Then
$$v(t)=e^{-6.67t}\left[10\cos(17t)-2.6\sin(17t)\right]\ \text{V}$$

Similarly, the natural response is $i(t)=e^{-6.67t}\left[A\cos(17t)+B\sin(17t)\right]$ and again there is no forced because there is no forcing function. The constants A and B are evaluated using the initial conditions:

$$i(0)=0=A \quad\text{and}\quad \frac{di(0)}{dt}=55.6=-6.67A+17\,B \quad\Rightarrow\quad B=-3.27$$

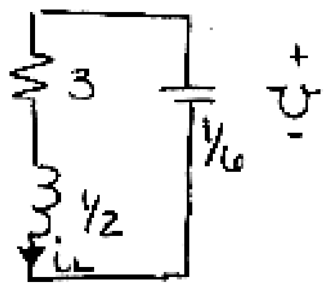
Then
$$i(t)=e^{-6.67t}\left[3.27\sin(17t)\right]\ \text{V}$$

P 9.9-5

Solution

First consider $t < 0$: $v(0) = 10$ V, $i_L(0) = \dfrac{10}{3}$ A

Next consider $0 < t < 0.5$s



$$\alpha = \frac{R}{L} = \frac{3}{2} \quad \text{and} \quad \omega_0^2 = LC = \frac{1}{12}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow s_1 = -.028 \text{ and } s_2 = -2.97$$

The natural response is $v(t) = Ae^{-0.028t} + Be^{-2.97t}$ and the forced response is $v_f = 0$.

The constants are evaluated using the initial conditions:

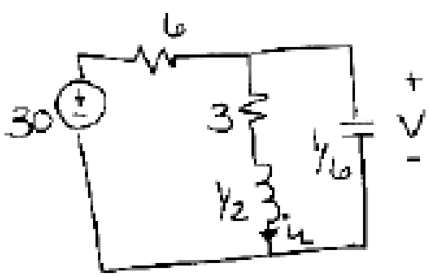
$$\left.\begin{aligned} v(0) = 10 &= A + B \\ \frac{dv(0)}{dt} = 20 &= -0.028A - 2.97B \end{aligned}\right\} \begin{aligned} A &= 16.89 \\ B &= -6.89 \end{aligned}$$

so $v(t) = 16.89\, e^{-0.028t} - 6.89e^{-2.97t}$

Similarly $i(t) = -.079e^{-0.028t} + 3.41e^{-2.97t}$

At $t = 0.5$ s, $v(0.5) = 15.1$ V and $i(0.5) = 0.7$ A

For $t > 0.55$ s:



KCL: $\dfrac{v - 30}{6} + i_L + \dfrac{1}{6}\dfrac{dv}{dt} = 0$

KVL: $v = 3i_L + \frac{1}{2}\dfrac{di_L}{dt}$

Characteristic equation: $0 = s^2 - 7s - 18 \Rightarrow s = -1, 9$

$v_f = 10 \text{ V}$

$v(t) = Ae^{9t} + Be^{-t} + 10$

$$\left.\begin{aligned} v(0.5) &= 15.1 = 90A + 0.61B + 10 \\ \frac{dv(0.5)}{dt} &= 10.7 = 810A - 0.61B \end{aligned}\right\} \quad \begin{aligned} A &= 17.6 \times 10^{-3} \\ B &= 5.77 \end{aligned}$$

t	$v(t)$
0	$16.89e^{-0.28\tau} - 6.89e^{-2.97\tau}$ V
$\rightarrow .5$	
$.5$	$17.6 \times 10^{-3}e^{9t} + 5.77e^{-t} + 10$ V
$\rightarrow 2$	

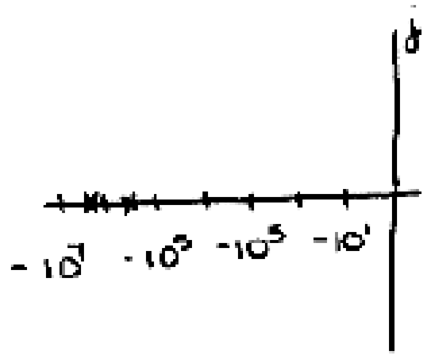
Section 9-10: Roots in the Complex Plane

P 9.10-1
Solution:

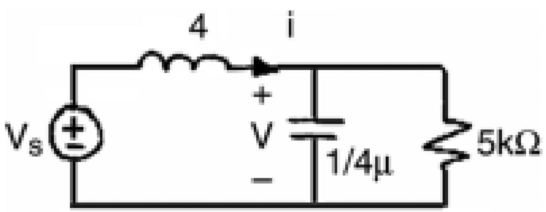
$$s^2 + 3.5\times10^6 s + 1.5\times10^{12} = 0$$

$$s_1 = -5\times10^5$$

$$s_2 = -3\times10^6$$



P 9.10-2
Solution:

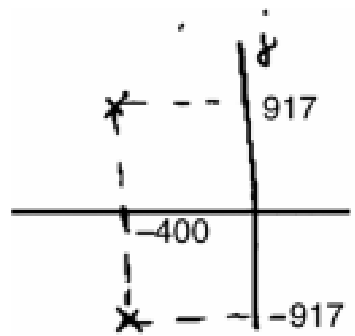


KCL: $i = \frac{1}{4}\times10^{-6}\frac{dv}{dt} + \frac{v}{5000}$

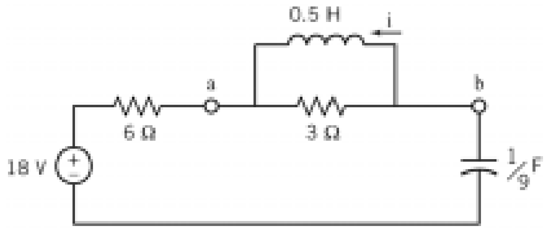
KVL: $v_S = 4\frac{di}{dt} + v$

Characteristic equation: $s^2 + 800s + 1\times10^6 = 0$

$$s = -400 \pm j\,917$$



P 9.10-3
Solution:



at $t = 0$

$v(0) = v_b\ (0) = 0$

$i(0) = 0$ and $C\dfrac{dv_b}{dt} + \dfrac{v_b - v_a}{3} + = 0$ (1)

$t = 0$

Node a: $\dfrac{v_a(0) - 18}{6} - i(0) + \dfrac{v_a(0) - v_b\ (0)}{3} =$

then $v_a(0) + 2v_a(0) = 18$ so $v_a(0) = 6\text{V}$

$t\ \geq\ 0$

Node a : $\dfrac{v_a\ \ - v_s}{12} + \dfrac{1}{L}\int (v_a\ - v_b)\, dt + \dfrac{v_a\ \ - v_b}{6} = 0$

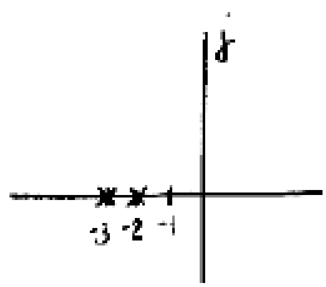
Node b : $C\dfrac{dv_b}{dt} + \dfrac{v_b - v_a}{6} + \dfrac{1}{L}\int (v_a - v_b)dt = 0$

Using operators $\left(\dfrac{1}{6} + \dfrac{1}{3} + \dfrac{1}{s}\right) v_a + \left(-\dfrac{1}{3}\ -\dfrac{1}{s}\right) v_b = \dfrac{v_s}{6}$

$(-\dfrac{1}{3} - \dfrac{1}{s})\ v_a\ + (\dfrac{1}{9}\,s + \dfrac{1}{3} + \dfrac{1}{s})\, v_b = 0$

Cramers rule

$$(s^2 + 5s\ +\ 6)v_b = (s + \ 6)1 v_s$$



Then $v_b = 18 + A_1\, e^{-2t} + A_2\, e^{-3t}$

$v_b(0) = 18 + A_1 + A_2$ (2)

need $\dfrac{dv_b}{dt}(0) = -2A_1 - 3A_2$

Use 1 above: $C\dfrac{dv_b(0)}{dt} = \dfrac{1}{9}(-2A_1 - 3A_2) = \dfrac{v_a(0) - v_b(0)}{3} - i(0) = \dfrac{6}{3} = 2$ (3)

Use (2) and (3) to get

$A_1 = -36$ $A_2 = 18$ so $v_b = 18 - 36e^{-2t} + 18e^{-3t},\ \ t\ \geq\ 0$

Section 9-11 How Can We Check…?

P 9.11-1

Solution:

This problem is similar to the verification example in this chapter. First, check the steady-state inductor current

$$i(t) = \frac{v_s}{100} = \frac{25}{100} = 250 \text{ mA}$$

This agrees with the value of 250.035 mA shown on the plot. Next, the plot shows an underdamped response. That requires

$$12 \cdot 10^{-3} = L < 4R^2C = 4(100)^2 \left(2 \cdot 10^{-6}\right) = 8 \cdot 10^{-2}$$

This inequality is satisfied, which also agrees with the plot. The damped resonant frequency is given by

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2} = \sqrt{\frac{1}{\left(2 \cdot 10^{-6}\right)\left(12 \cdot 10^{-3}\right)} - \left(\frac{1}{2(100)\left(2 \cdot 10^{-6}\right)}\right)^2} = 5.95 \cdot 10^3$$

The plot indicates a maxima at 550.6μs and a minima at 1078.7μs. The period of the damped oscillation is

$$T_d = 2\,(1078.7\,\mu s - 550.6\,\mu s) = 1056.2\,\mu s$$

Finally, check that $5.95 \cdot 10^3 = \omega_d = \dfrac{2\pi}{T_d} = \dfrac{2\pi}{1056.2 \cdot 10^{-6}} = 5.949 \cdot 10^3$

The value of ω_d determined from the plot agrees with the value obtained from the circuit.

The plot is correct

P 9.11-2

Solution:

This problem is similar to the verification example in this chapter. First, check the steady-state inductor current.

$$i(t) = \frac{v_s}{100} = \frac{15}{100} = 150 \text{ mA}$$

This agrees with the value of 149.952 mA shown on the plot. Next, the plot shows an under damped response. This requires

$$8 \cdot 10^{-3} = L < 4R^2C = 4\,(100)^2 \left(0.2 \cdot 10^{-6}\right) = 8 \cdot 10^{-3}$$

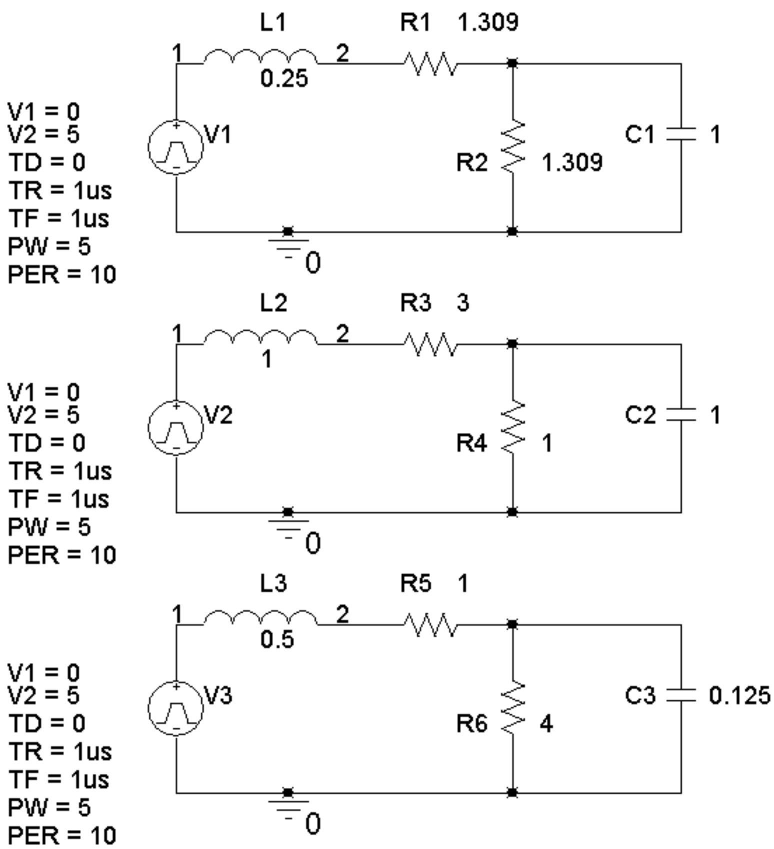
This inequality is not satisfied. The values in the circuit would produce a critically damped, not underdamped, response. This plot is not correct.

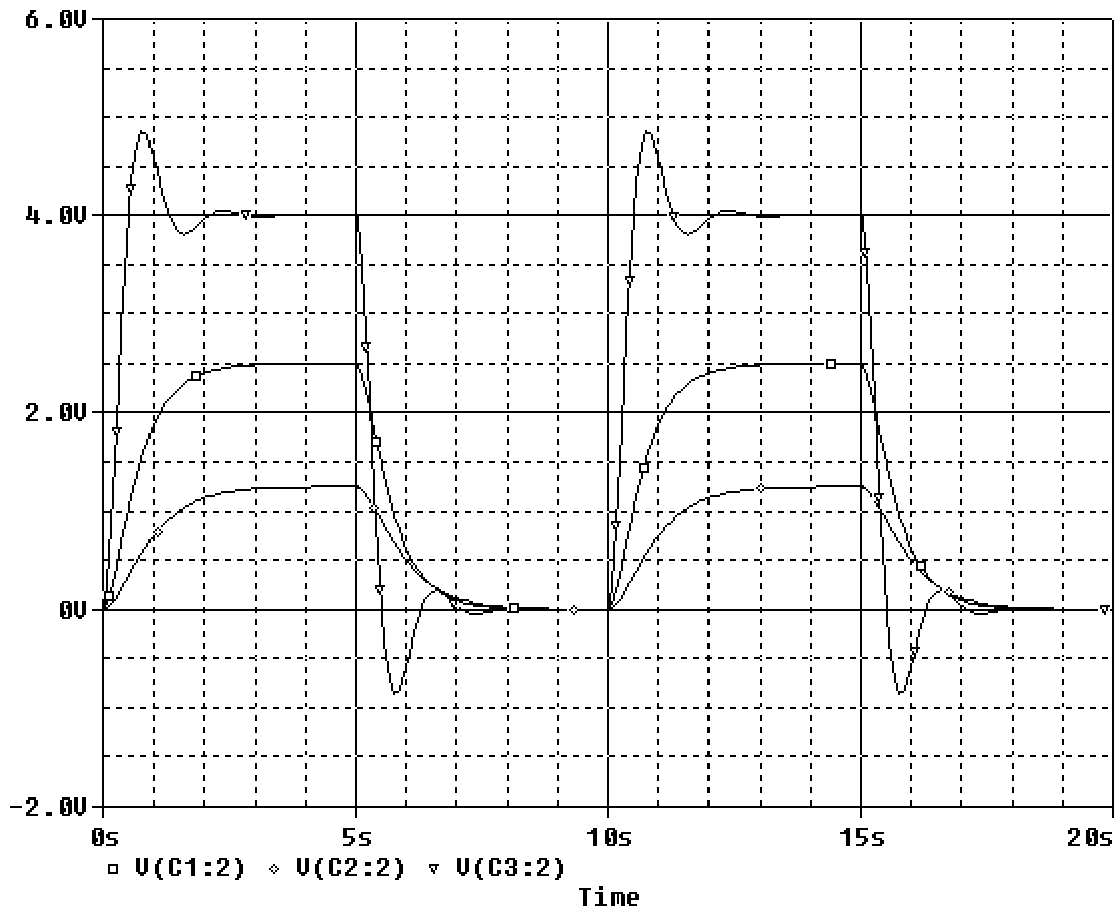
PSpice Problems

SP 9-1

Solution:

Make three copies of the circuit: one for each set of parameter values. (Cut and paste, but be sure to edit the labels of the parts so, for example, there is only one R1.)



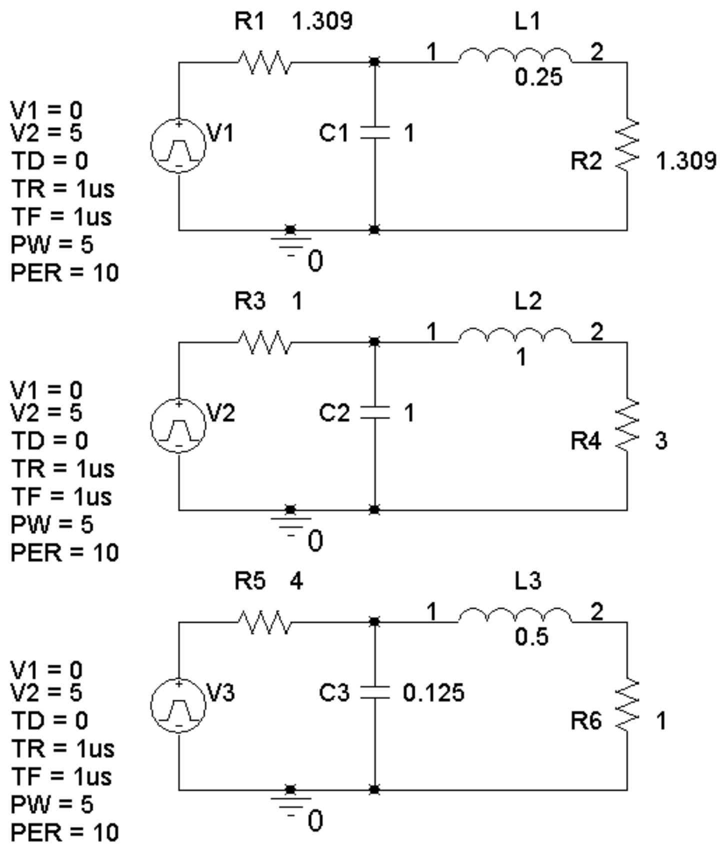


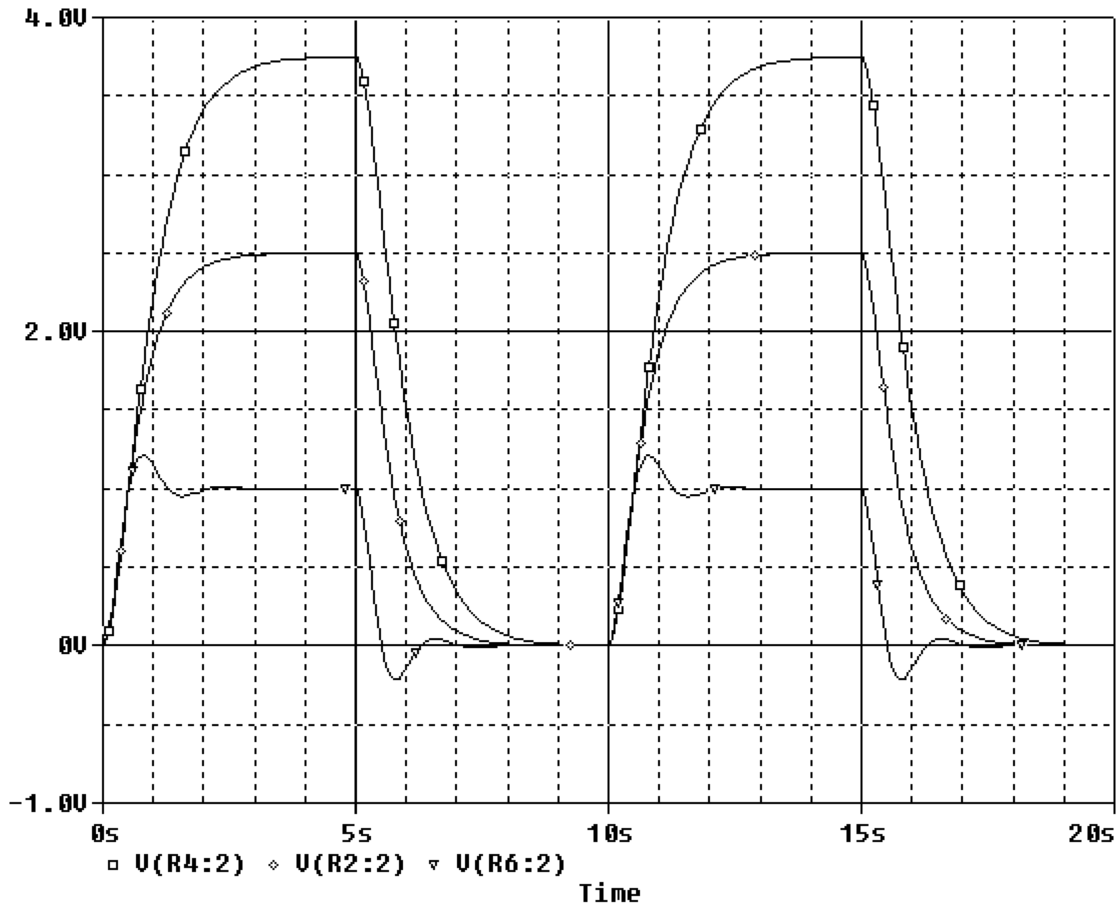
V(C1:2), V(C2:2) and V(C3:2) are the capacitor voltages, listed from top to bottom.

SP 9-2

Solution:

Make three copies of the circuit: one for each set of parameter values. (Cut and paste, but be sure to edit the labels of the parts so, for example, there is only one R1.)

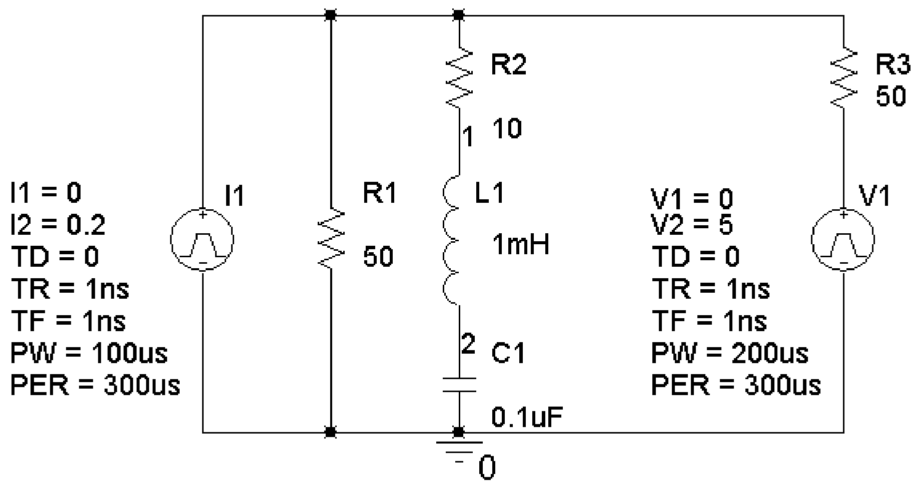


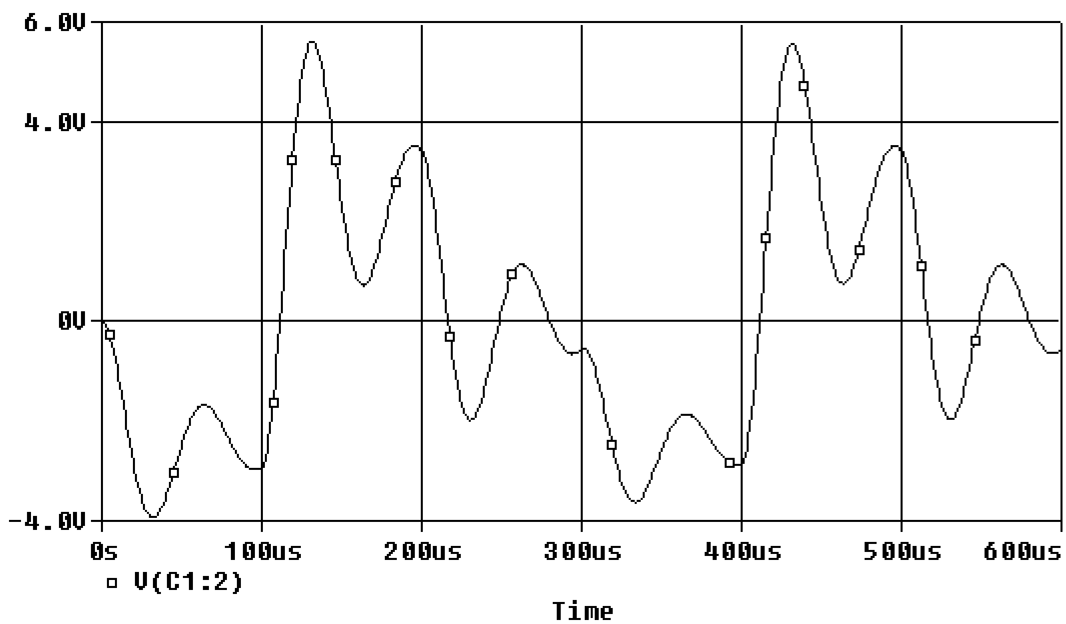


V(R2:2), V(R4:2) and V(R6:2) are the output voltages, listed from top to bottom.

SP 9-3

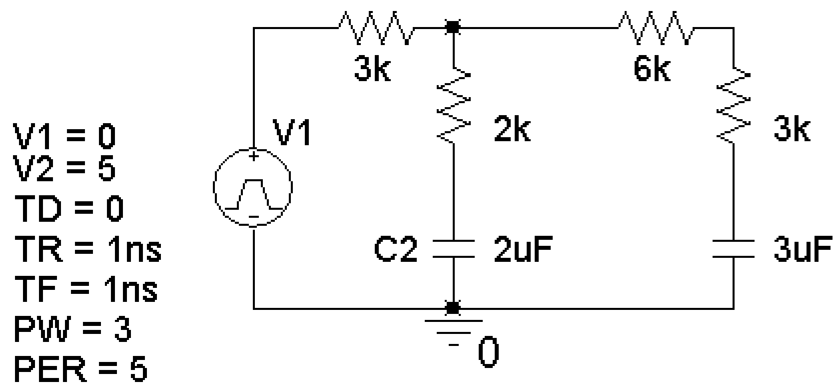
Solution:

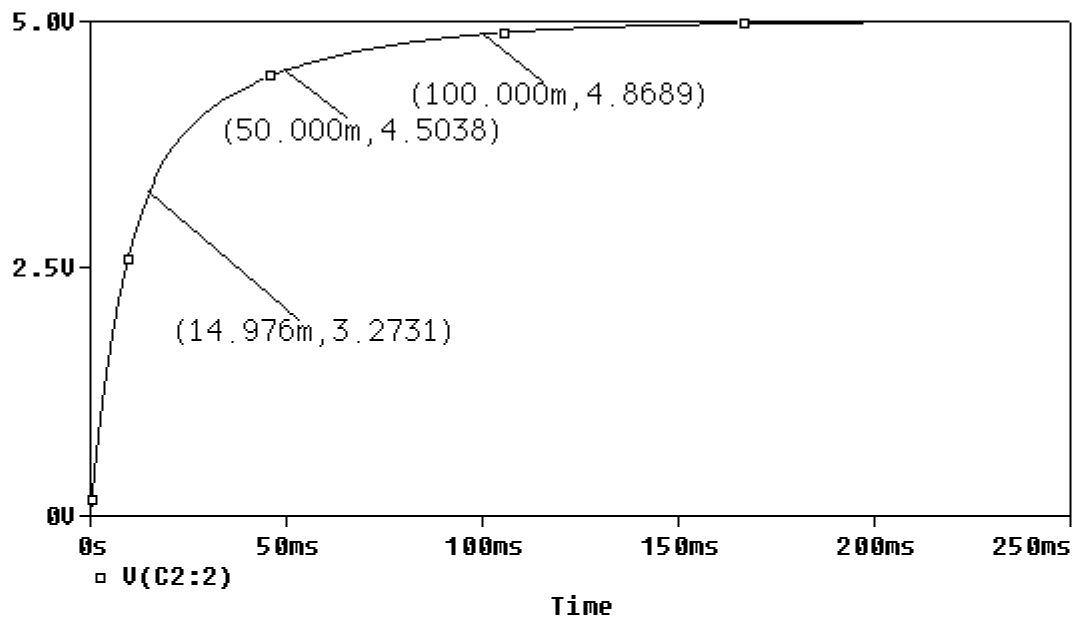




SP 9-4

Solution:





Design Problems

DP 9-1

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{1}{2}$ so

$$\frac{1}{2} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives:
$$\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t) = i_L(t)$$

KVL around the outside loop gives:
$$v_s(t) = L\frac{d}{dt}i_L(t) + R_1\, i_L(t) + v_C(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= L\frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + v_C(t) \\ &= LC\frac{d^2}{dt^2}v_C(t) + \left(\frac{L}{R_2} + R_1 C\right)\frac{d}{dt}v_C(t) + \left(1 + \frac{R_1}{R_2}\right)v_C(t) \end{aligned}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 6s + 8 = (s+2)(s+4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_2 C}+\frac{R_1}{L}=6 \text{ and } \frac{1+\frac{R_1}{R_2}}{LC}=8$$

Using $R_1 = R_2 = R$ gives

$$\frac{1}{R\,C}+\frac{R}{L}=6 \text{ and } \frac{1}{LC}=4$$

These equations do not have a unique solution. Try C = 1 F. Then $L=\frac{1}{4}\ \mathrm{H}$ and

$$\frac{1}{R}+4R=6 \quad\Rightarrow\quad R^2-\frac{3}{2}R+\frac{1}{4}=0 \quad\Rightarrow\quad R=1.309\,\Omega \text{ or } R=0.191\,\Omega$$

Pick R = 1.309 Ω. Then

$$v_c(t)=\frac{1}{2}+A_1\,e^{-2t}+A_2\,e^{-4t}\ \mathrm{V}$$

$$i_L(t)=\frac{v_C(t)}{1.309}+\frac{d}{dt}v_C(t)=-1.236\,A_1\,e^{-2t}-3.236\,A_2\,e^{-4t}+0.3819$$

At t = 0+

$$0=v_c(0+)=A_1+A_2+0.5$$

$$0=i_L(0+)=-1.236\,A_1-3.236\,A_2+0.3819$$

Solving these equations gives A_1 = -1 and A_2 = 0.5, so

$$v_c(t)=\frac{1}{2}-e^{-2t}+\frac{1}{2}e^{-4t}\ \mathrm{V}$$

DP 9-2

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{1}{4}$ so

$$\frac{1}{4} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad 3R_2 = R_1$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives: $$\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t) = i_L(t)$$

KVL around the outside loop gives: $$v_s(t) = L\frac{d}{dt}i_L(t) + R_1\, i_L(t) + v_C(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= L\frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + v_C(t) \\ &= LC\frac{d^2}{dt^2}v_C(t) + \left(\frac{L}{R_2} + R_1 C\right)\frac{d}{dt}v_C(t) + \left(1 + \frac{R_1}{R_2}\right)v_C(t) \end{aligned}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 4s + 4 = (s+2)^2$$

Equating coefficients of like powers of s:

$$\frac{1}{R_2 C}+\frac{R_1}{L}=4 \text{ and } \frac{1+\dfrac{R_1}{R_2}}{LC}=4$$

Using $R_2 = R$ and $R_1 = 3R$ gives

$$\frac{1}{R C}+\frac{3R}{L}=4 \text{ and } \frac{1}{LC}=1$$

These equations do not have a unique solution. Try C = 1 F. Then L = 1 H and

$$\frac{1}{R}+3R=4 \quad \Rightarrow \quad R^2-\frac{4}{3}R+\frac{1}{3}=0 \quad \Rightarrow \quad R=1\,\Omega \text{ or } R=\frac{1}{3}\,\Omega$$

Pick R = 1 Ω. Then $R_1 = 3\,\Omega$ and $R_2 = 1\,\Omega$.

$$v_c(t)=\frac{1}{4}+\left(A_1+A_2\,t\right)e^{-2t} \text{ V}$$

$$i_L(t)=v_C(t)+\frac{d}{dt}v_C(t)=\frac{1}{4}+\left(\left(A_2-A_1\right)-A_2\,t\right)e^{-2t}$$

At t = 0+

$$0=v_c(0+)=A_1+\frac{1}{4}$$

$$0=i_L(0+)=\frac{1}{4}+A_2-A_1$$

Solving these equations gives A_1 = -0.25 and A_2 = -0.5, so

$$v_c(t) = \frac{1}{4} - \left(\frac{1}{4} + \frac{1}{2}t\right)e^{-2t} \text{ V}$$

DP 9-3

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{4}{5}$ so

$$\frac{4}{5} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad 4R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives: $$\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t) = i_L(t)$$

KVL around the outside loop gives: $$v_s(t) = L\frac{d}{dt}i_L(t) + R_1 i_L(t) + v_C(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= L\frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + v_C(t) \\ &= LC\frac{d^2}{dt^2}v_C(t) + \left(\frac{L}{R_2} + R_1 C\right)\frac{d}{dt}v_C(t) + \left(1 + \frac{R_1}{R_2}\right)v_C(t) \end{aligned}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 4s + 20 = (s + 2 - j4)(s + 2 + j4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_2 C}+\frac{R_1}{L}=4 \text{ and } \frac{1+\frac{R_1}{R_2}}{LC}=20$$

Using $R_1 = R$ and $R_2 = 4R$ gives

$$\frac{1}{4RC}+\frac{R}{L}=4 \text{ and } \frac{1}{LC}=16$$

These equations do not have a unique solution. Try $C=\frac{1}{8}\text{ F}$. Then $L=\frac{1}{2}\text{ H}$ and

$$\frac{2}{R}+2R=4 \quad \Rightarrow \quad R^2-2R+2=0 \quad \Rightarrow \quad R=1\,\Omega$$

Then $R_1 = 1\,\Omega$ and $R_2 = 4\,\Omega$. Next

$$v_c(t)=0.8+e^{-2t}\left(A_1\cos 4t+A_2\sin 4t\right)\text{ V}$$

$$i_L(t)=\frac{v_C(t)}{4}+\frac{1}{8}\frac{d}{dt}v_C(t)=0.2+\frac{A_2}{2}e^{-2t}\cos 4t-\frac{A_1}{2}e^{-2t}\sin 4t$$

At t = 0+

$$0=v_c(0+)=0.8+A_1$$

$$0=i_L(0+)=0.2+\frac{A_2}{2}$$

Solving these equations gives A_1 = -0.8 and A_2 = -0.4, so

$$v_c(t)=0.8-e^{-2t}\left(0.8\cos 4t+0.4\sin 4t\right)\text{ V}$$

DP 9-4

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{1}{2}$ so

$$\frac{1}{2} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives: $$\frac{v_C(t)}{R_2} + C \frac{d}{dt} v_C(t) = i_L(t)$$

KVL around the outside loop gives: $$v_s(t) = L \frac{d}{dt} i_L(t) + R_1 i_L(t) + v_C(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= L \frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C \frac{d}{dt} v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C \frac{d}{dt} v_C(t)\right) + v_C(t) \\ &= LC \frac{d^2}{dt^2} v_C(t) + \left(\frac{L}{R_2} + R_1 C\right) \frac{d}{dt} v_C(t) + \left(1 + \frac{R_1}{R_2}\right) v_C(t) \end{aligned}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right) s + \left(\frac{1 + \frac{R_1}{R_2}}{LC}\right) = s^2 + 4s + 20 = (s + 2 - j4)(s + 2 + j4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_2 C}+\frac{R_1}{L}=4 \text{ and } \frac{1+\frac{R_1}{R_2}}{LC}=20$$

Using $R_1 = R_2 = R$ gives

$$\frac{1}{R C}+\frac{R}{L}=4 \text{ and } \frac{1}{LC}=10$$

Substituting $L=\dfrac{1}{10C}$ into the first equation gives

$$(RC)^2-\frac{4}{10}(RC)+\frac{1}{10}=0 \quad\Rightarrow\quad RC=\frac{0.4\pm\sqrt{0.4^2-4(0.1)}}{2}$$

Since *RC* cannot have a complex value, the specification cannot be satisfied.

DP 9-5

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty)=\frac{R_2}{R_1+R_2}1, \quad i_L(\infty)=\frac{1}{R_1+R_2} \quad \textit{and} \quad v_o(\infty)=\frac{R_2}{R_1+R_2}1$$

The specifications require that $v_o(\infty)=\dfrac{1}{2}$ so

$$\frac{1}{2}=\frac{R_2}{R_1+R_2} \quad\Rightarrow\quad R_1=R_2$$

Next, represent the circuit by a 2nd order differential equation:

KVL around the right-hand mesh gives: $v_C(t)=L\dfrac{d}{dt}i_L(t)+R_2\, i_L(t)$

KCL at the top node of the capacitor gives: $\dfrac{v_s(t)-v_C(t)}{R_1}-C\dfrac{d}{dt}v_C(t)=i_L(t)$

Use the substitution method to get

$$v_s(t) = R_1 C \frac{d}{dt}\left(L \frac{d}{dt} i_L(t) + R_2 i_L(t) \right) + \left(L \frac{d}{dt} i_L(t) + R_2 i_L(t) \right) + R_1 i_L(t)$$
$$= R_1 LC \frac{d^2}{dt^2} i_L(t) + \left(L + R_1 R_2 C \right) \frac{d}{dt} i_L(t) + \left(R_1 + R_2 \right) i_L(t)$$

Using $v_o(t) = \frac{i_L(t)}{R_2}$ gives

$$v_o(t) = \frac{R_1}{R_2} LC \frac{d^2}{dt^2} i_L(t) + \left(\frac{L}{R_2} + R_1 C \right) \frac{d}{dt} i_L(t) + \left(\frac{R_1 + R_2}{R_2} \right) i_L(t)$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_1 C} + \frac{R_2}{L} \right) s + \left(\frac{1 + \frac{R_2}{R_1}}{LC} \right) = s^2 + 6s + 8 = (s+2)(s+4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_1 C} + \frac{R_2}{L} = 6 \text{ and } \frac{1 + \frac{R_2}{R_1}}{LC} = 8$$

Using $R_1 = R_2 = R$ gives

$$\frac{1}{R\,C} + \frac{R}{L} = 6 \text{ and } \frac{1}{LC} = 4$$

These equations do not have a unique solution. Try C = 1 F. Then $L = \frac{1}{4}\ \mathrm{H}$ and

$$\frac{1}{R} + 4R = 6 \quad \Rightarrow \quad R^2 - \frac{3}{2} R + \frac{1}{4} = 0 \quad \Rightarrow \quad R = 1.309\,\Omega \text{ or } R = 0.191\,\Omega$$

Pick R = 1.309 Ω. Then

$$v_o(t) = \frac{1}{2} + A_1\, e^{-2t} + A_2\, e^{-4t} \text{ V}$$

$$i_L(t) = \frac{v_o(t)}{1.309} = \frac{1}{2.618} + \frac{A_1}{1.309}\, e^{-2t} + \frac{A_2}{1.309}\, e^{-4t} \text{ V}$$

$$v_C(t) = 1.309\, i_L(t) + \frac{1}{4}\frac{d}{dt} i_L(t) = \frac{1}{2} + 0.6167\, A_1\, e^{-2t} + 0.2361\, A_2\, e^{-4t}$$

At t = 0+

$$0 = i_L(0+) = \frac{1}{2.618} + \frac{A_1}{1.309} + \frac{A_2}{1.309}$$

$$0 = v_C(0+) = \frac{1}{2} + 0.6167\, A_1 + 0.2361\, A_2$$

Solving these equations gives A_1 = -1 and A_2 = 0.5, so

$$v_o(t) = \frac{1}{2} - e^{-2t} + \frac{1}{2} e^{-4t} \text{ V}$$

DP 9-6

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1, \quad i_L(\infty) = \frac{1}{R_1 + R_2} \quad and \quad v_o(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_o(\infty) = \frac{3}{4}$ so

$$\frac{3}{4} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad 3R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KVL around the right-hand mesh gives: $v_C(t) = L\frac{d}{dt}i_L(t) + R_2\, i_L(t)$

KCL at the top node of the capacitor gives: $\frac{v_s(t) - v_C(t)}{R_1} - C\frac{d}{dt}v_C(t) = i_L(t)$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= R_1\, C\frac{d}{dt}\left(L\frac{d}{dt}i_L(t) + R_2\, i_L(t)\right) + \left(L\frac{d}{dt}i_L(t) + R_2\, i_L(t)\right) + R_1\, i_L(t) \\ &= R_1 LC\frac{d^2}{dt^2}i_L(t) + \left(L + R_1\, R_2\, C\right)\frac{d}{dt}i_L(t) + \left(R_1 + R_2\right)i_L(t) \end{aligned}$$

Using $v_o(t) = \frac{i_L(t)}{R_2}$ gives

$$v_o(t) = \frac{R_1}{R_2}LC\frac{d^2}{dt^2}i_L(t) + \left(\frac{L}{R_2} + R_1\, C\right)\frac{d}{dt}i_L(t) + \left(\frac{R_1 + R_2}{R_2}\right)i_L(t)$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_1\, C} + \frac{R_2}{L}\right)s + \left(\frac{1 + \frac{R_2}{R_1}}{LC}\right) = s^2 + 4s + 4 = (s+2)^2$$

Equating coefficients of like powers of s:

$$\frac{1}{R_1\, C} + \frac{R_2}{L} = 4 \text{ and } \frac{1 + \frac{R_2}{R_1}}{LC} = 4$$

Using $R_1 = R$ and $R_2 = 3R$ gives

$$\frac{1}{R\, C} + \frac{3R}{L} = 4 \text{ and } \frac{1}{LC} = 1$$

These equations do not have a unique solution. Try *C* = 1 F. Then *L* = 1 H and

$$\frac{1}{R}+3R=4 \quad \Rightarrow \quad R^2-\frac{4}{3}R+\frac{1}{3}=0 \quad \Rightarrow \quad R=1\,\Omega \text{ or } R=\frac{1}{3}\,\Omega$$

Pick R = 1 Ω. Then $R_1=1\,\Omega$ and $R_2=3\,\Omega$.

$$v_o(t)=\frac{3}{4}+\left(A_1+A_2\,t\right)e^{-2t}\ \text{V}$$

$$i_L(t)=\frac{v_o(t)}{3}=\frac{1}{4}+\left(\frac{A_1}{3}+\frac{A_2}{3}t\right)e^{-2t}\ \text{V}$$

$$v_C(t)=3i_L(t)+\frac{d}{dt}i_L(t)=\frac{3}{4}+\left(\left(\frac{A_1}{3}+\frac{A_2}{3}\right)+\frac{A_2}{3}t\right)e^{-2t}$$

At t = 0+

$$0=i_L(0+)=\frac{A_1}{3}+\frac{1}{4}$$

$$0=v_C(0+)=\frac{3}{4}+\frac{A_1}{3}+\frac{A_2}{3}$$

Solving these equations gives A_1 = -0.75 and A_2 = -1.5, so

$$v_o(t)=\frac{3}{4}-\left(\frac{3}{4}+\frac{3}{2}t\right)e^{-2t}\ \text{V}$$

DP 9-7

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1+R_2}1, \quad i_L(\infty) = \frac{1}{R_1+R_2} \quad and \quad v_o(\infty) = \frac{R_2}{R_1+R_2}1$$

The specifications require that $v_o(\infty) = \frac{1}{5}$ so

$$\frac{1}{5} = \frac{R_2}{R_1+R_2} \quad \Rightarrow \quad R_1 = 4R_2$$

Next, represent the circuit by a 2nd order differential equation:

KVL around the right-hand mesh gives: $$v_C(t) = L\frac{d}{dt}i_L(t) + R_2\,i_L(t)$$

KCL at the top node of the capacitor gives: $$\frac{v_s(t)-v_C(t)}{R_1} - C\frac{d}{dt}v_C(t) = i_L(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= R_1C\frac{d}{dt}\left(L\frac{d}{dt}i_L(t) + R_2\,i_L(t)\right) + \left(L\frac{d}{dt}i_L(t) + R_2\,i_L(t)\right) + R_1\,i_L(t) \\ &= R_1LC\frac{d^2}{dt^2}i_L(t) + \left(L + R_1R_2C\right)\frac{d}{dt}i_L(t) + \left(R_1+R_2\right)i_L(t) \end{aligned}$$

Using $v_o(t) = \frac{i_L(t)}{R_2}$ gives

$$v_o(t) = \frac{R_1}{R_2}LC\frac{d^2}{dt^2}i_L(t) + \left(\frac{L}{R_2} + R_1C\right)\frac{d}{dt}i_L(t) + \left(\frac{R_1+R_2}{R_2}\right)i_L(t)$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_1C} + \frac{R_2}{L}\right)s + \left(\frac{1+\frac{R_2}{R_1}}{LC}\right) = s^2 + 4s + 20 = (s+2-j4)(s+2+j4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_1 C}+\frac{R_2}{L}=4 \text{ and } \frac{1+\dfrac{R_2}{R_1}}{LC}=20$$

Using $R_2 = R$ and $R_1 = 4R$ gives

$$\frac{1}{4RC}+\frac{R}{L}=4 \text{ and } \frac{1}{LC}=16$$

These equations do not have a unique solution. Try $C=\frac{1}{8}\text{ F}$. Then $L=\frac{1}{2}\text{ H}$ and

$$\frac{2}{R}+2R=4 \quad \Rightarrow \quad R^2-2R+2=0 \quad \Rightarrow \quad R=1\,\Omega$$

Then $R_1 = 4\,\Omega$ and $R_2 = 1\,\Omega$. Next

$$v_o(t)=0.2+e^{-2t}\left(A_1\cos 4t+A_2\sin 4t\right)\text{ V}$$

$$i_L(t)=\frac{v_o(t)}{1}=0.2+e^{-2t}\left(A_1\cos 4t+A_2\sin 4t\right)\text{ V}$$

$$v_C(t)=i_L(t)+\frac{1}{2}\frac{d}{dt}i_L(t)=0.2+2A_2e^{-2t}\cos 4t-2A_1e^{-2t}\sin 4t$$

At t = 0+

$$0=i_L(0+)=0.2+A_1$$

$$0=v_C(0+)=0.2+2A_2$$

Solving these equations gives A_1 = -0.8 and A_2 = -0.4, so

$$v_c(t)=0.2-e^{-2t}\left(0.2\cos 4t+0.1\sin 4t\right)\text{ V}$$

DP 9-8

Solution:

When the circuit reaches steady state after t = 0, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1, \quad i_L(\infty) = \frac{1}{R_1 + R_2} \quad and \quad v_o(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{1}{2}$ so

$$\frac{1}{2} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KVL around the right-hand mesh gives: $$v_C(t) = L\frac{d}{dt} i_L(t) + R_2\, i_L(t)$$

KCL at the top node of the capacitor gives: $$\frac{v_s(t) - v_C(t)}{R_1} - C\frac{d}{dt} v_C(t) = i_L(t)$$

Use the substitution method to get

$$\begin{aligned} v_s(t) &= R_1\, C\frac{d}{dt}\left(L\frac{d}{dt} i_L(t) + R_2\, i_L(t)\right) + \left(L\frac{d}{dt} i_L(t) + R_2\, i_L(t)\right) + R_1\, i_L(t) \\ &= R_1 LC\frac{d^2}{dt^2} i_L(t) + \left(L + R_1\, R_2\, C\right)\frac{d}{dt} i_L(t) + \left(R_1 + R_2\right) i_L(t) \end{aligned}$$

Using $v_o(t) = \frac{i_L(t)}{R_2}$ gives

$$v_o(t) = \frac{R_1}{R_2} LC\frac{d^2}{dt^2} i_L(t) + \left(\frac{L}{R_2} + R_1\, C\right)\frac{d}{dt} i_L(t) + \left(\frac{R_1 + R_2}{R_2}\right) i_L(t)$$

The characteristic equation is

$$s^2+\left(\frac{1}{R_1 C}+\frac{R_2}{L}\right)s+\left(\frac{1+\frac{R_2}{R_1}}{LC}\right)=s^2+4s+20=(s+2-j4)(s+2+j4)$$

Equating coefficients of like powers of s:

$$\frac{1}{R_1 C}+\frac{R_2}{L}=4 \text{ and } \frac{1+\frac{R_2}{R_1}}{LC}=20$$

Using $R_1=R_2=R$ gives

$$\frac{1}{R C}+\frac{R}{L}=4 \text{ and } \frac{1}{LC}=10$$

Substituting $L=\frac{1}{10C}$ into the first equation gives

$$(RC)^2-\frac{4}{10}(RC)+\frac{1}{10}=0 \quad \Rightarrow \quad RC=\frac{0.4\pm\sqrt{0.4^2-4(0.1)}}{2}$$

Since *RC* cannot have a complex value, the specification cannot be satisfied.

DP 9.9

Solution:

Characteristic equation: $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

Select L so fast response and i achieve maximum at $t = 0.5$ s

$$s^2 + \frac{4}{L}s + \frac{3}{L} = 0 \quad \text{Try } L = 1\text{ H} \quad \Rightarrow s^2 + 4s + 3 = 0 \text{ or } s = -3, -1$$

$$i(t) = A_1 e^{-t} + A_2 e^{-3t}$$

$$\left.\begin{array}{l} i(0) = 0 = A_1 + A_2 \\ \dfrac{di(0)}{dt} = \dfrac{v(0)}{L} = \dfrac{10}{1} = -A_1 - 3A_2 \end{array}\right\} A_1 = 5,\ A_2 = -5$$

$$i(t) = 5e^{-t} - 5e^{-3t} \qquad at\ t = 0.5\text{ s} \;\; i = 1.92$$