Quantum Physics I Final Exam

2023/12/29 8:00~9:50

$\frac{1}{2} I\omega^2 = \frac{L^2}{2I}$

$m_1 \cdot \left(\frac{m_2}{m_1+m_2} l\right)^2 + m_2 \cdot \left(\frac{m_1}{m_1+m_2} l\right)^2 = \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2} l^2$

$L = I\omega = rp$

1. Consider the molecule CO, which can be described by two particles (mass m_1 and m_2) attached by a massless rigid rod of length l. The system can freely rotate about the center of mass. Classically, the energy of such a rigid rotator is given by $\frac{1}{2} I\omega^2$, where the moment of inertia $I = m_1 m_2 l^2/(m_1 + m_2)$ and ω is the angular velocity. $r_2 = \frac{m_1 l}{m_1+m_2}$ $L_2 = r_2 p_2$

25

   (a) Write down the Hamiltonian (in quantum mechanics) that describes this system. (5 pts.)
   (b) What are the allowed energies? (5 pts.)
   (c) Let θ and ϕ define the orientation of the rigid rotator. What are the normalized eigenfunctions? What is the degeneracy of the nth energy level? (10 pts.)
   (d) What will be the emission spectrum (namely, the formula for the frequencies of the spectral lines) of this system? (5 pts.)

2. Consider two noninteracting identical spin-1/2 particles in the infinite square well, of which the one-particle "position" wave functions and energies are

35

$$\psi_{n_{1,2}}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_{1,2}\pi}{a} x\right), \qquad E_{n_{1,2}} = \frac{\pi^2 \hbar^2 n_{1,2}^2}{2ma^2}, \qquad n_{1,2} = 1,2, \ldots$$

where a and m are the well's width and particle's mass, respectively.

   (a) What total spins can you get? (5 pts.)
   (b) Show that the constants $A_\pm$ associated with the raising and lowering operators $S_\pm = S_x \pm iS_y$ are $A_\pm = \hbar\sqrt{s(s+1) - m(m \pm 1)}$. (10 pts.)

$$S_\pm |s\ m\rangle = A_\pm |s\ (m \pm 1)\rangle$$

$S_\pm S_\mp = (S_x \pm i S_y)(S_x \mp i S_y) = S_x^2 + S_y^2 \mp i[S_x, S_y]$

   (c) Use the lowering operator of the total spins to construct the combined (coupled) spin states $|s\ m\rangle$ in terms of the composite (uncoupled) states $|s_1\ s_2\ m_1\ m_2\rangle$. (10 pts.)
   (d) Construct the ground and first-excited states $\psi_{n_1 n_2} |s\ m\rangle$, where $\psi_{n_1 n_2} = \psi_{n_1}\psi_{n_2}$. Explain your answers. (10 pts.)

3. System of particles with integer (half-integer) spins must have symmetric (antisymmetric) states. Explain why the (a) periodic table, (b) solidity of the solids, and (c) differences between the conductors, insulators, and semiconductors are the consequences of such an axiom or postulate. (15 pts.)

15

25

4. Consider a spin-1/2 particle with a magnetic moment $\mu = \gamma S$ prepared in the state $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ at $t = 0$ in a uniform magnetic field $\mathbf{B} = B_0\hat{k}$.

(a) What is the Hamiltonian of this system? (4 pts.)

(b) What are the eigenstates? (5 pts.)

(c) What is the particle's state at time t? (8 pts.)

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) Find the probability of measuring the state in $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ at time t. (8 pts.)

$\frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$

12

5. [加分題] The coherent states,

$-\frac{1}{\sqrt{2}} \quad + \frac{1}{\sqrt{2}} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(-1-\lambda)(1-\lambda) = 0 \quad \lambda = \pm 1$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

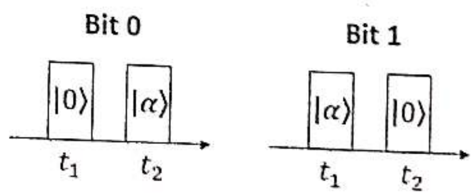
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are the eigenstates of the annihilation operator $\hat{a}$ with the eigenvalues α, where $|n\rangle$ are the number states.

(a) Show that $|\alpha|^2 = \bar{n}$, where $\bar{n}$ is the mean photon number or the expectation value of the number operator $\hat{a}^\dagger\hat{a}$. (4 pts.)

(b) Calculate the probability $P(n)$ that there are n photons in the coherent states. (4 pts.)

(c) To implement the quantum key distribution, Alice tries to encode the logical bits in time as shown in the figure. She encodes the logical bits 0 using the time slots at t_1 and t_2 containing vacuum state (0 photon) and coherent state $|\alpha\rangle$ with $\bar{n} = 0.01$, respectively, and the logical bits 1 using the time slots at t_1 and t_2 containing the coherent state $|\alpha\rangle$ ($\bar{n} = 0.01$) and vacuum state, respectively. Will such an encoding scheme be "protected" by the no-cloning theorem? Explain your answer. (4 pts.)



$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \cdot \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^\dagger\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\cdot\sqrt{n}\,|n\rangle$$

$$e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{|\alpha^n|^2}{n!} \cdot n$$