CS2336 Discrete Mathematics

Exam 1

October 30, 2017 (2 hours)

Answer all questions. Total marks = 100. For all the proofs, if it is incomplete, large portion of marks may be deducted.

1. (15%) Consider the following compound proposition:

   $$[\neg q \oplus (p \wedge q)] \vee (p \rightarrow q).$$

   In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.

2. (15%) Use logical equivalences and rules of inferences to show that the following arguments are valid. Refer to the last page for some common equivalences and rules.

   - Premises: $\forall x(\neg(P(x) \vee Q(x)) \rightarrow R(x))$, $\exists x(\neg P(x) \wedge \neg Q(x))$
   - Conclusion: $\exists x R(x)$

3. (15%) Let x be an integer. Prove that if x is a multiple of 4, then x cannot be the sum of four consecutive integers.

4. (30%) Peter is a superstitious mathematician. He thinks that the number 13 is unlucky. So, by Peter's definition, if there exists a way to write a rational number x as p/q, where p and q are integers, and both are not divisible by 13, then x is called a *lucky* number. Otherwise, x is an *unlucky* number.

   (a) (10%) Show that 13 is an unlucky number.

   (b) (10%) Prove or disprove: The sum of two lucky numbers is always lucky.

   (c) (10%) Prove or disprove: The product of two lucky numbers is always lucky.

5. (15%) Fermat's little theorem states that for any prime number p and any integer n, the integer $n^p - n$ is always divisible by p.

   Show that Fermat's little theorem holds when $p = 3$.

6. (10%) [Adapted from R. Smullyan's book, *The Lady or The Tiger?*]

   There are three boxes A, B, and C. One box contains a diamond ring, and the other two each contains a roll of toilet paper.

   Box A is attached with a label, writing:

   > "This box contains a roll of toilet paper."

   Box B is also attached with a label, writing:

   > "This box contains a diamond ring."

   Box C is also attached with a label, writing:

   > "Box B contains a roll of toilet paper."

   It is known that at most one of the three labels is true. Which box contains the diamond ring? Justify your answer.

1.	Identity Laws:	$p \wedge T_0 \equiv p$	$p \vee F_0 \equiv p$
2.	Domination Laws:	$p \wedge F_0 \equiv F_0$	$p \vee T_0 \equiv T_0$
3.	Idempotent Laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
4.	Double Negation Law:	$\neg(\neg p) \equiv p$	
5.	Commutative Laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
6.	Associative Laws:	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
7.	Distributive Laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8.	De Morgan's Laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9.	Absorption Laws:	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
10.	Negation Laws:	$p \wedge \neg p \equiv F_0$	$p \vee \neg p \equiv T_0$
11.	De Morgan's Laws with Quantifiers:	$\neg \forall x P(x) \equiv \exists x \neg P(x)$	$\neg \exists x P(x) \equiv \forall x \neg P(x)$
12.	Conditional Statement Equivalences:	$p \rightarrow q \equiv \neg p \vee q$	$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Figure 1: Some useful logical equivalences

1. Modus Ponens:
   Premises: $p, \quad p \rightarrow q$ Conclusion: q
2. Modus Tollens:
   Premises: $\neg q, \quad p \rightarrow q$ Conclusion: $\neg p$
3. Hypothetical Syllogism:
   Premises: $p \rightarrow q, \quad q \rightarrow r$ Conclusion: $p \rightarrow r$
4. Disjunctive Syllogism:
   Premises: $\neg p, \quad p \vee q$ Conclusion: q
5. Addition:
   Premise: p Conclusion: $p \vee q$
6. Simplification:
   Premise: $p \wedge q$ Conclusion: p
7. Conjunction:
   Premises: $p, \quad q$ Conclusion: $p \wedge q$
8. Resolution:
   Premises: $p \vee q, \quad \neg p \vee r$ Conclusion: $q \vee r$
9. Universal Instantiation:
   Premise: $\forall x P(x)$ Conclusion: $P(c)$, for any c
10. Universal Generalization:
    Premise: $P(c)$, for any c Conclusion: $\forall x P(x)$
11. Existential Instantiation:
    Premise: $\exists x P(x)$ Conclusion: $P(c)$, for some c
12. Existential Generalization:
    Premise: $P(c)$, for some c Conclusion: $\exists x P(x)$

Figure 2: Some useful rules of inference