last update Oct. 4th, 2020

電磁學（一）
Electromagnetics (I)

3. 電磁學的數學工具（二）向量運算
Mathematic Tools (II) - vector algebra

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Both electric and magnetic fields are vectors. This lecture is to introduce basic concepts of vector and vector algebra.

- 3.1 Scalar and vector 純量與向量
- 3.2 Multiplication of Vectors 向量乘積
- 3.3 Cartesian coordinate system 矩形座標系
- 3.4 Cylindrical coordinate system 圓柱座標系
- 3.5 Spherical coordinate system 圓球座標系

電磁學的數學工具 (二)- 向量運算
Mathematic Tools (II) – vector algebra

3.1 純量與向量 Scalar and vector

Scalar

- A scalar is the value or a symbol of a number.

  E.g. 1, 2, 3, –5, –12, x, y, z etc.

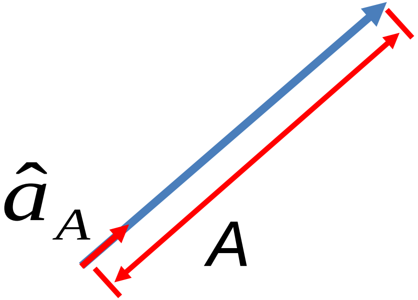
- A scalar describes the amplitude or magnitude of a physical quantity, such as

  charge q, charge density ρ, current I, flux Φ etc.

Vector

- A vector has a magnitude and direction, written as

$$\vec{A} = A\hat{a}_A$$



E.g. The unit vector of

$$\vec{A} = -3\hat{a}_x$$

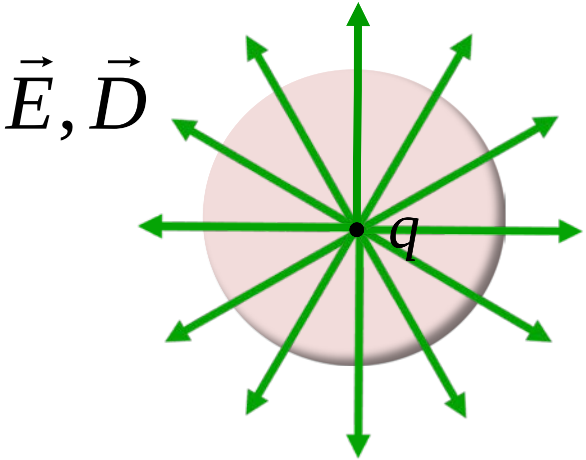
is $\hat{a}_A = -\hat{a}_x$ with $A = 3$.

$\hat{a}_A = \vec{A} / A$ is a unit vector, denoting the direction with a unit magnitude $|\hat{a}_A| = 1$

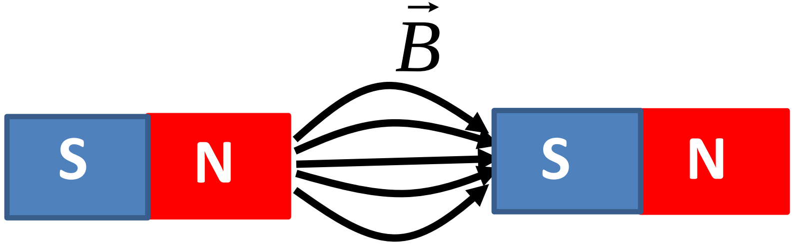
$|\vec{A}| \equiv A > 0$ is the magnitude or the length of the line

- A vector describes the magnitude and direction of a physics quantity.

E.g. Electric field intensity and flux density $\vec{E}, \vec{D}$



E.g. Magnetic flux intensity and field intensity $\vec{B}, \vec{H}$

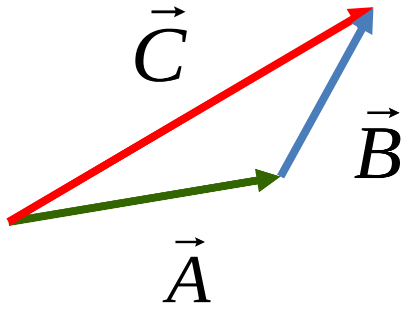


Vector Sum

Use the so-called head-and-tail construction

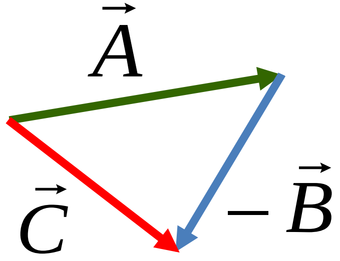
Vector Addition

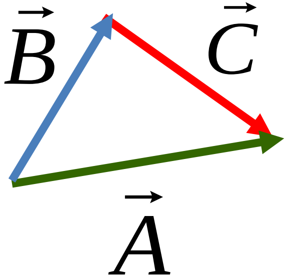
$$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$



Vector Subtraction

$$\vec{C} = \vec{A} - \vec{B}$$





3.1 純量與向量
Scalar and vector

- A scalar only has a magnitude or amplitude but a vector has a magnitude and a direction.
- The electric and magnetic fields are vectors.
- Vector sum can be accomplished geometrically by the so-called head-to-tail construction.

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Mathematic Tools (II) – vector algebra

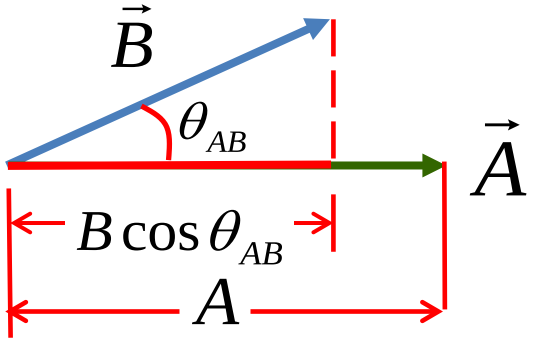
3.2 向量乘積 Multiplication of vectors

Scalar or Dot Product

$\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$ (dot product gives a scalar)

STEP 1 - project vector B along A to obtain $B\cos\theta_{AB}$

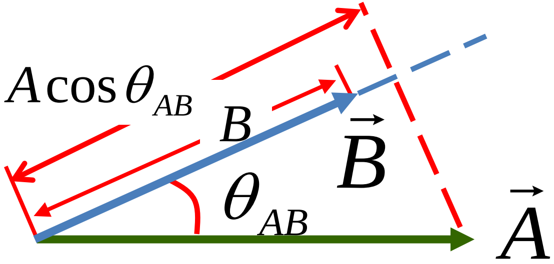
STEP 2 – multiply the projected magnitude $B\cos\theta_{AB}$ with the magnitude A to obtain $\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$



Alternatively, $\vec{A}\cdot\vec{B} = A\cos\theta_{AB} \times B,$

which is the projection of $\vec{A}$ along $\vec{B}$, $A\cos\theta_{AB}$,

multiplying $\left|\vec{B}\right|$



- Apparently, the scalar or dot product is commutative

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

E.g. $A = \sqrt{\vec{A}\cdot\vec{A}}$

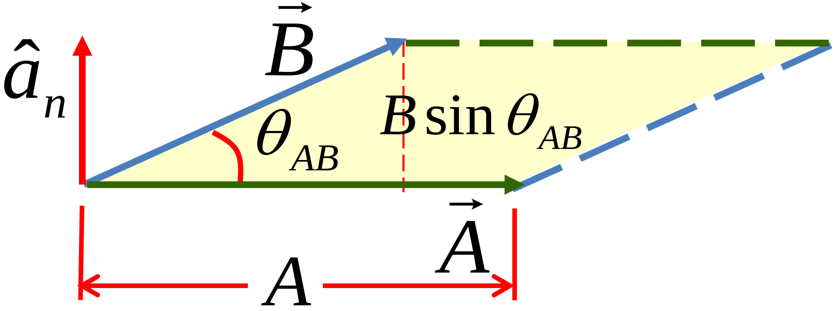
$$= \sqrt{A^2\cos 0^o} = \left|\vec{A}\right|$$

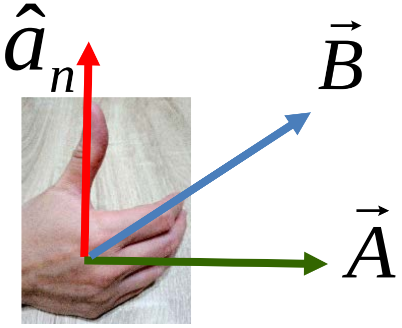
Cross Product

$$\vec{A}\times\vec{B} = \hat{a}_n \left| AB\sin\theta_{AB} \right|$$ (cross product gives a vector)

STEP 1 – The area of the parallelogram expanded by $\vec{A}$ & $\vec{B}$ is the magnitude of $\vec{A}\times\vec{B}$ or $AB\sin\theta_{AB}$.

STEP 2 – The direction of $\vec{A}\times\vec{B}$ is along the surface normal of the parallelogram determined by the right-hand rule*.





* Rotate your 4 fingers of your right hand from A to B, and find the direction of A x B along your thumb.

Properties of Cross Product

- Anti-commutative $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ *the sense of direction is reversed
- Order of products matters

$$(\vec{A}\times\vec{B})\times\vec{C} \neq \vec{A}\times(\vec{B}\times\vec{C})$$

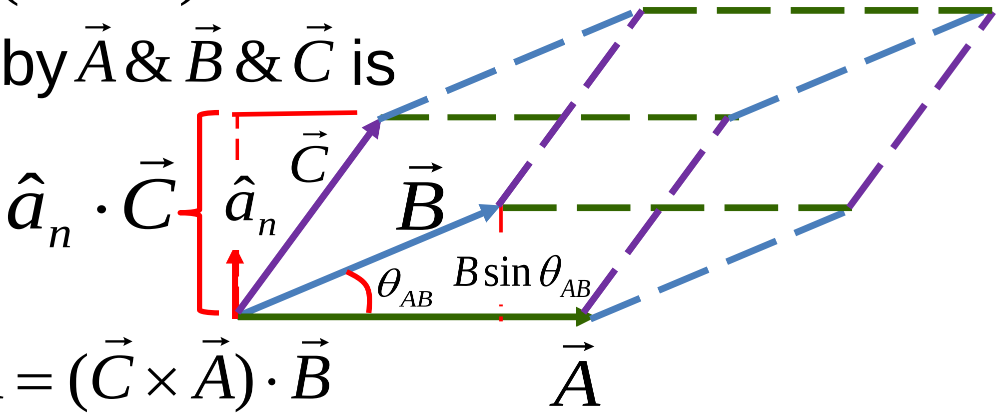
- The volume expanded by $\vec{A}$ & $\vec{B}$ & $\vec{C}$ is

$$\text{base area}\times\underline{\text{height}}$$

$$=[AB\sin\theta_{AB}]\times\underline{[\hat{a}_n\cdot\vec{C}]}$$

$$=(\vec{A}\times\vec{B})\cdot\vec{C}=(\vec{B}\times\vec{C})\cdot\vec{A}=(\vec{C}\times\vec{A})\cdot\vec{B}$$

a parallelepiped



3.2 向量乘積

Multiplication of vectors

- The vector dot or scalar product results in a scalar, which is the multiplication of the projection of a vector onto the other.
- The vector cross product results in a vector, which has a magnitude equal to the area of the parallelogram expanded by the two vector and a direction defined by the right-hand rule.

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3.3 矩形座標系 Cartesian coordinate system

Cartesian (x, y, z) Coordinate System

Three unit vectors, $\hat{a}_x, \hat{a}_y, \hat{a}_z$

A general expression of a vector:

$$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

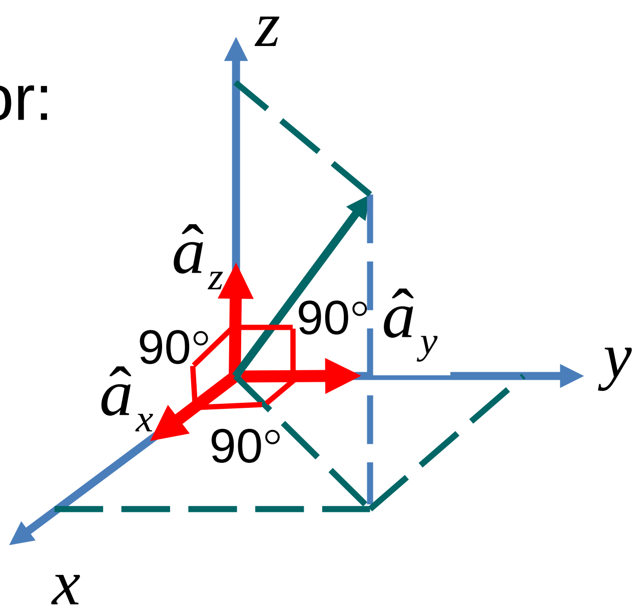
Orthogonality

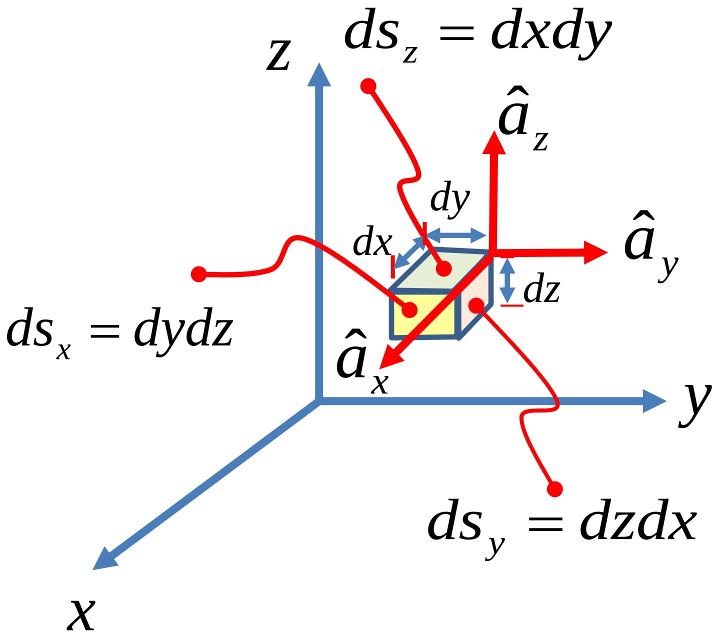
$$\hat{a}_x \cdot \hat{a}_y = |\hat{a}_x||\hat{a}_y|\cos 90^o = 0$$

$$\hat{a}_y \cdot \hat{a}_z = 0, \hat{a}_z \cdot \hat{a}_x = 0$$

$$\hat{a}_x \times \hat{a}_y = |\hat{a}_x||\hat{a}_y|\sin 90^o \times \hat{a}_z = \hat{a}_z$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x, \hat{a}_z \times \hat{a}_x = \hat{a}_y$$





A differential length:

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz = d\vec{l}_x + d\vec{l}_y + d\vec{l}_z$$

A differential surface:

$$d\vec{s} = \hat{a}_x ds_x + \hat{a}_y ds_y + \hat{a}_z ds_z$$

$$= d\vec{l}_y \times d\vec{l}_z + d\vec{l}_z \times d\vec{l}_x + d\vec{l}_x \times d\vec{l}_y$$

A differential volume:

$$dv = dxdydz = (d\vec{l}_x \times d\vec{l}_y) \cdot d\vec{l}_z$$

$$= (d\vec{l}_y \times d\vec{l}_z) \cdot d\vec{l}_x = (d\vec{l}_z \times d\vec{l}_x) \cdot d\vec{l}_y$$

Vector scalar product: $\hat{a}_i \cdot \hat{a}_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}$, where $i, j = x, y, z$

$$\vec{A} \cdot \vec{B} = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z) = A_x B_x + A_y B_y + A_z B_z$$

Vector cross product $\hat{a}_i \times \hat{a}_j = \begin{cases} 0 & \text{for} \quad i = j \\ \pm \hat{a}_k & \text{for} \quad i \neq j \end{cases}$, where $i, j, k = x, y, z$

Sign follows the right-hand rule

$$\vec{A} \times \vec{B} = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \times (B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z)$$

$$= (A_y B_z - A_z B_y)\hat{a}_x + (A_z B_x - A_x B_z)\hat{a}_y + (A_x B_y - A_y B_x)\hat{a}_z = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

3.3 矩形座標系

Cartesian coordinate system

- The 3 coordinates are x, y, z.
- The differential length is

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$$

- The differential surface is

$$d\vec{s} = \hat{a}_x dydz + \hat{a}_y dzdx + \hat{a}_z dxdy$$

- The differential volume is $dv = dxdydz$

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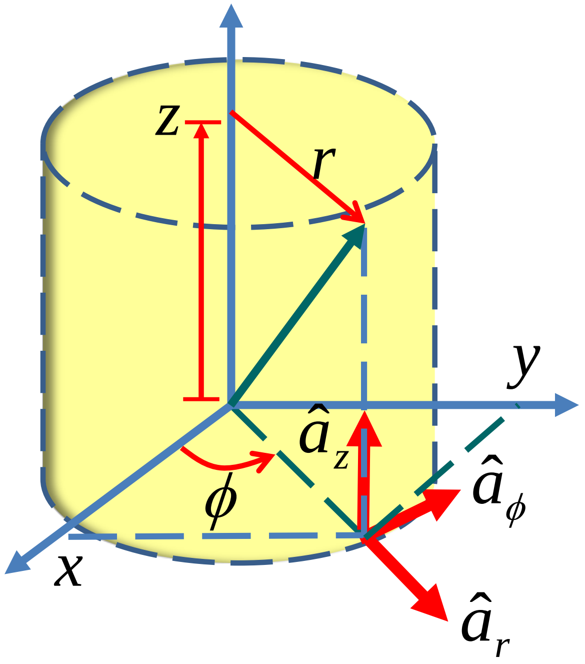
3.4 圓柱座標系 Cylindrical coordinate system

Cylindrical (r, ϕ, z) Coordinate System

Three unit vectors, $\hat{a}_r, \hat{a}_\phi, \hat{a}_z$

A general expression of a vector:

$$\vec{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z$$



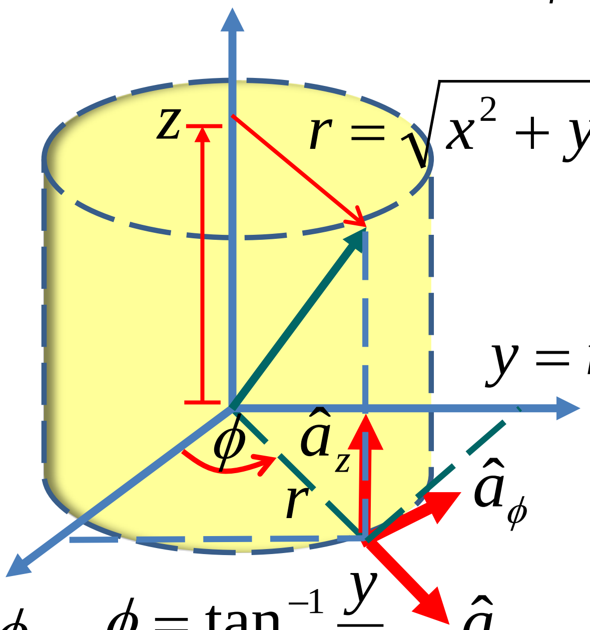
Orthogonality

$$\hat{a}_r \cdot \hat{a}_\phi = 0, \hat{a}_\phi \cdot \hat{a}_z = 0, \hat{a}_z \cdot \hat{a}_r = 0$$

$$\hat{a}_r \times \hat{a}_\phi = \hat{a}_z, \hat{a}_\phi \times \hat{a}_z = \hat{a}_r, \hat{a}_z \times \hat{a}_r = \hat{a}_\phi$$

Coordinate Transformation

$$\vec{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$



$$r = \sqrt{x^2 + y^2}$$

$$y = r \sin\phi$$

$$x = r\cos\phi$$

$$\phi = \tan^{-1}\frac{y}{x}$$

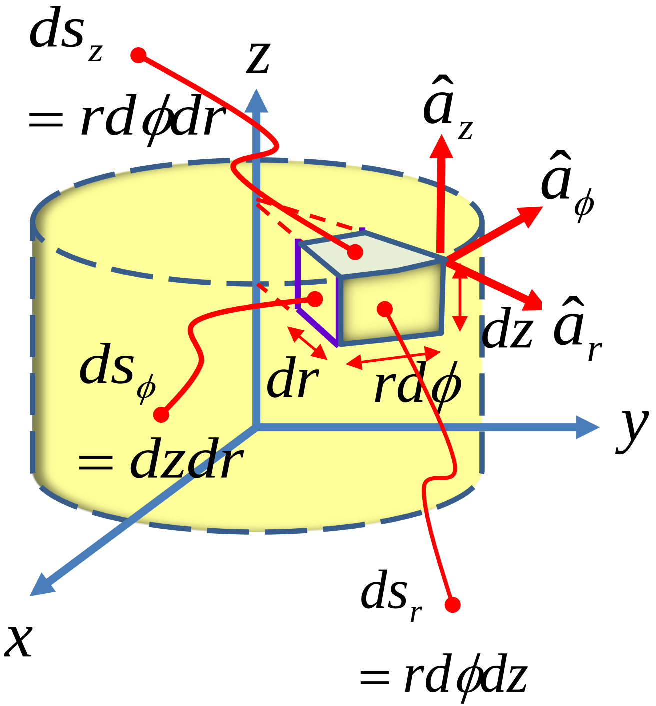
$$A_x = \hat{a}_x \cdot \vec{A} = \hat{a}_x \cdot \hat{a}_r A_r + \hat{a}_x \cdot \hat{a}_\phi A_\phi$$

$$= A_r \cos\phi - A_\phi \sin\phi$$

Similarly, $A_y = \hat{a}_y \cdot \vec{A}$

$$= A_r \sin\phi + A_\phi \cos\phi$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$



A differential length:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz$$

$$= d\vec{l}_r + d\vec{l}_\phi + d\vec{l}_z$$

A differential surface:

$$d\vec{s} = \hat{a}_r ds_r + \hat{a}_\phi ds_\phi + \hat{a}_z ds_z$$

$$= d\vec{l}_\phi \times d\vec{l}_z + d\vec{l}_z \times d\vec{l}_r + d\vec{l}_r \times d\vec{l}_\phi$$

A differential volume:

$$dv = rdrd\phi dz = (d\vec{l}_r \times d\vec{l}_\phi) \cdot d\vec{l}_z$$

$$= (d\vec{l}_\phi \times d\vec{l}_z) \cdot d\vec{l}_r = (d\vec{l}_z \times d\vec{l}_r) \cdot d\vec{l}_\phi$$

Vector scalar product: $\hat{a}_i \cdot \hat{a}_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}$, where $i, j = r, \phi, z$

$$\vec{A} \cdot \vec{B} = (A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z) \cdot (B_r \hat{a}_r + B_\phi \hat{a}_\phi + B_z \hat{a}_z) = A_r B_r + A_\phi B_\phi + A_z B_z$$

Vector cross product $\hat{a}_i \times \hat{a}_j = \begin{cases} 0 & \text{for} \quad i = j \\ \pm \hat{a}_k & \text{for} \quad i \neq j \end{cases}$, where $i, j, k = r, \phi, z$

Sign follows the right-hand rule

$$\vec{A} \times \vec{B} = (A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z) \times (B_r \hat{a}_r + B_\phi \hat{a}_\phi + B_z \hat{a}_z)$$

$$= (A_\phi B_z - A_z B_\phi)\hat{a}_r + (A_z B_r - A_r B_z)\hat{a}_\phi + (A_r B_\phi - A_\phi B_r)\hat{a}_z = \begin{vmatrix} \hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

3.4 圓柱座標系

Cylindrical coordinate system

- The 3 coordinates are r, ϕ, z.
- The differential length is

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz$$

- The differential surface is

$$d\vec{s} = \hat{a}_r r d\phi dz + \hat{a}_\phi dz dr + \hat{a}_z r dr d\phi$$

- The differential volume is $dv = r dr d\phi dz$

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3.5 圓球座標系 Spherical coordinate system

Spherical (R, θ, ϕ) Coordinate System

Three unit vectors, $\hat{a}_R, \hat{a}_\theta, \hat{a}_\phi$

A general expression of a vector:

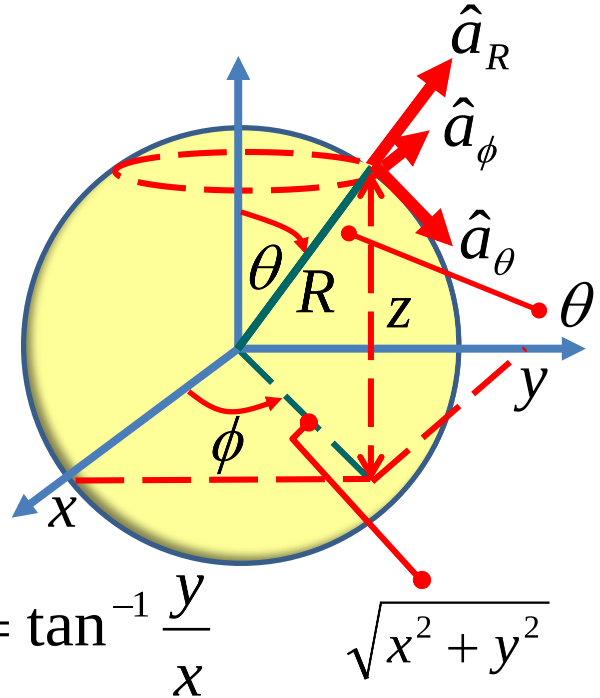
$$\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

Orthogonality

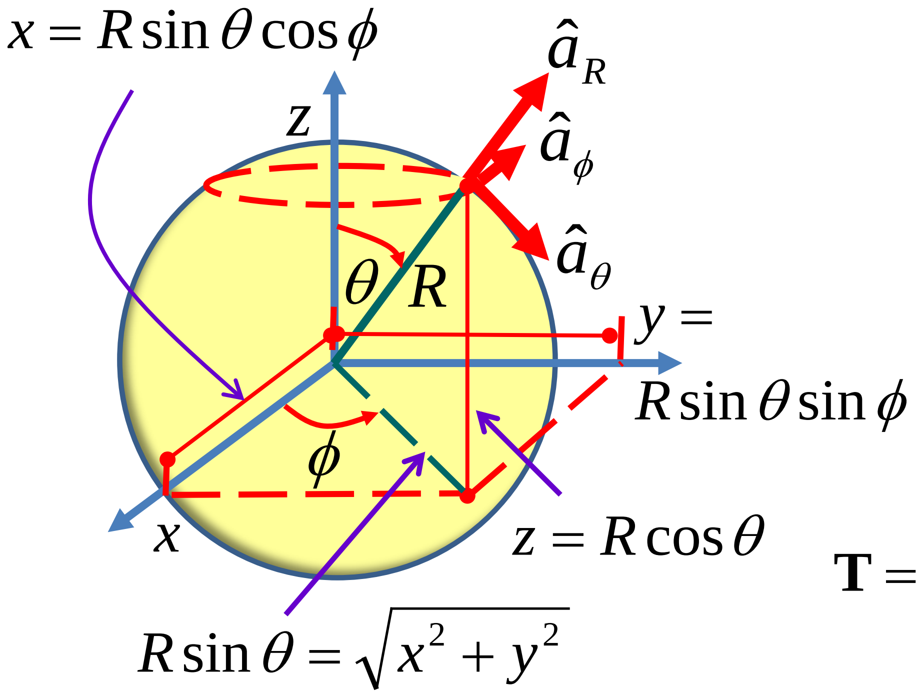
$$\hat{a}_R \cdot \hat{a}_\theta = 0, \hat{a}_\theta \cdot \hat{a}_\phi = 0, \hat{a}_\phi \cdot \hat{a}_R = 0$$

$$\hat{a}_R \times \hat{a}_\theta = \hat{a}_\phi, \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_R, \hat{a}_\phi \times \hat{a}_R = \hat{a}_\theta$$

$$R = \sqrt{x^2 + y^2 + z^2}, \ \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \ \phi = \tan^{-1}\frac{y}{x}$$



Coordinate Transformation

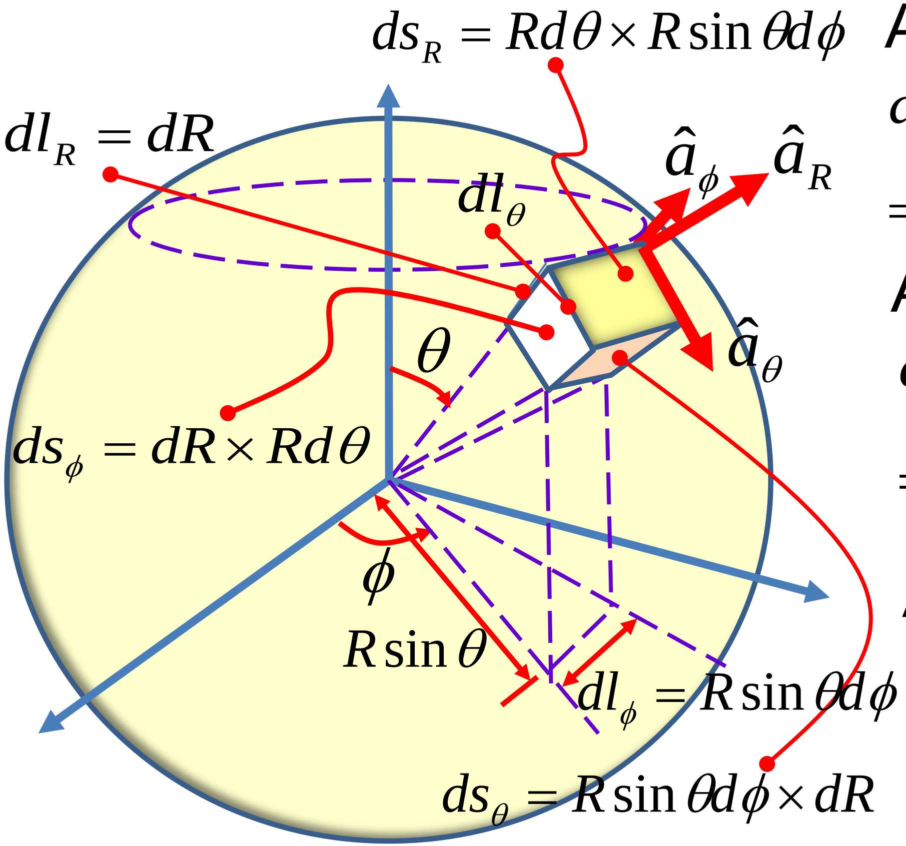


$$\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

$$= \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \mathbf{T} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$



A differential length:

$$d\vec{l} = \hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R\sin\theta d\phi$$

$$= d\vec{l}_R + d\vec{l}_\theta + d\vec{l}_\phi$$

A differential surface:

$$d\vec{s} = d\vec{s}_R + d\vec{s}_\theta + d\vec{s}_\phi$$

$$= d\vec{l}_\theta \times d\vec{l}_\phi + d\vec{l}_\phi \times d\vec{l}_R + d\vec{l}_R \times d\vec{l}_\theta$$

A differential volume:

$$dv = d\vec{s}_{R,\theta,\phi} \cdot d\vec{l}_{R,\theta,\phi}$$

$$= R^2 \sin\theta dR d\theta d\phi$$

Vector scalar product: $\hat{a}_i \cdot \hat{a}_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}$, where $i, j = R, \theta, \phi$

$$\vec{A} \cdot \vec{B} = (A_R \hat{a}_R + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi) \cdot (B_R \hat{a}_R + B_\theta \hat{a}_\theta + B_\phi \hat{a}_\phi) = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

Vector cross product $\hat{a}_i \times \hat{a}_j = \begin{cases} 0 & \text{for} \quad i = j \\ \pm \hat{a}_k & \text{for} \quad i \neq j \end{cases}$, where $i, j, k = R, \theta, \phi$

Sign follows the right-hand rule

$$\vec{A} \times \vec{B} = (A_R \hat{a}_R + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi) \times (B_R \hat{a}_R + B_\theta \hat{a}_\theta + B_\phi \hat{a}_\phi)$$

$$= (A_\theta B_\phi - A_\phi B_\theta)\hat{a}_R + (A_\phi B_R - A_R B_\phi)\hat{a}_\theta + (A_R B_\phi - A_\phi B_R)\hat{a}_\phi = \begin{vmatrix} \hat{a}_R & \hat{a}_\theta & \hat{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\varphi \end{vmatrix}$$

3.5 圓球座標系

Spherical coordinate system

- The 3 coordinates are R, θ, ϕ
- The differential length is

$$d\vec{l} = \hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R \sin\theta d\phi$$

- The differential surface is

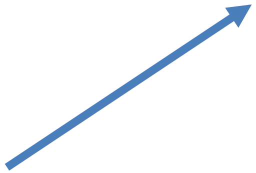
$$d\vec{s} = \hat{a}_R R^2 \sin\theta d\theta d\phi + \hat{a}_\theta R \sin\theta dR d\phi + \hat{a}_\phi R dR d\theta$$

- The differential volume is $dv = R^2 \sin\theta dR d\theta d\phi$

單元回顧

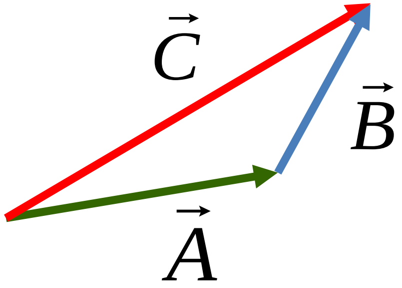
1. A vector consists of a magnitude and a direction.

$$\vec{A} = A\hat{a}_A$$



2. Vector addition can be completed by using the head-to-tail rule.

$$\vec{C} = \vec{A} + \vec{B}$$

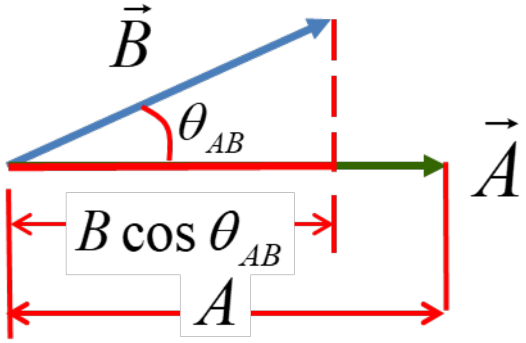


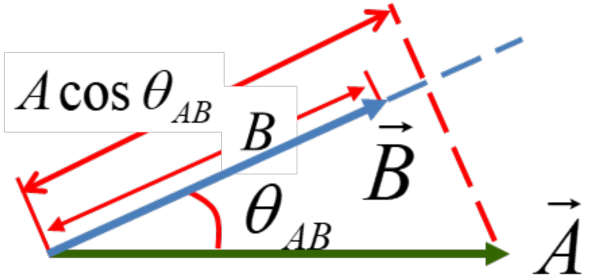
單元回顧

3. The scalar or dot product of two vectors is defined as

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}=AB\cos\theta_{AB}$$

where $A\cos\theta_{AB}$ is the projection of $\vec{A}$ along $\vec{B}$ or $B\cos\theta_{AB}$ is the projection of $\vec{B}$ along $\vec{A}$.



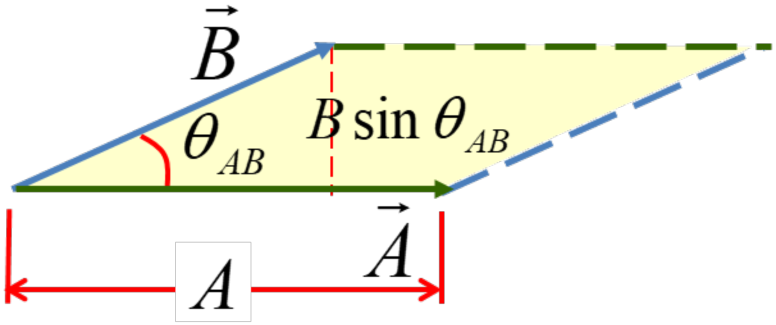


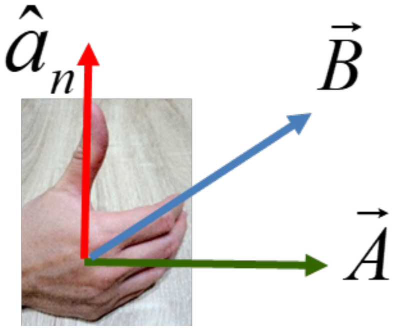
單元回顧

4. The cross product of two vectors is defined as

$$\vec{A}\times\vec{B}=\hat{a}_n\left|AB\sin\theta_{AB}\right|=-\vec{B}\times\vec{A},$$

where the magnitude $\left|AB\sin\theta_{AB}\right|$ is the area of the parallelogram expanded by $\vec{A}\,\&\,\vec{B}$ and the direction follows the right hand rule.

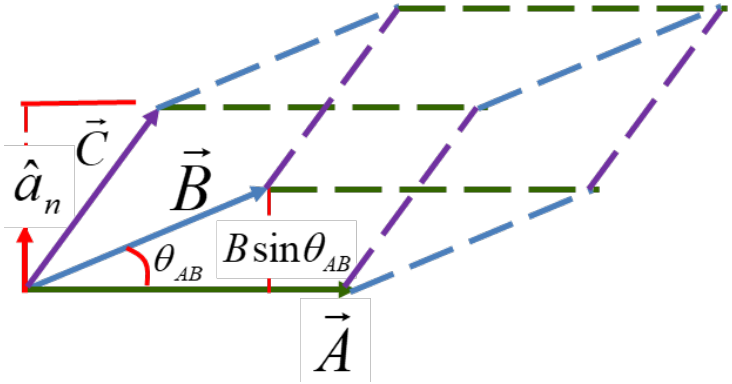




單元回顧

5. The volume of the parallelepiped expanded by $\vec{A} \& \vec{B} \& \vec{C}$ is given by

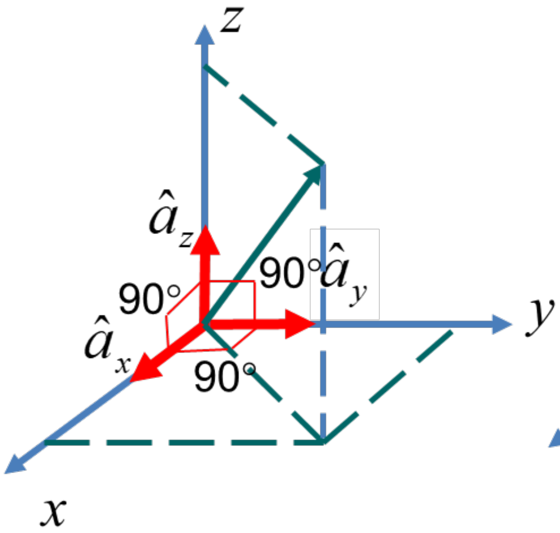
$$(\vec{A}\times\vec{B})\cdot\vec{C}=(\vec{B}\times\vec{C})\cdot\vec{A}=(\vec{C}\times\vec{A})\cdot\vec{B}$$



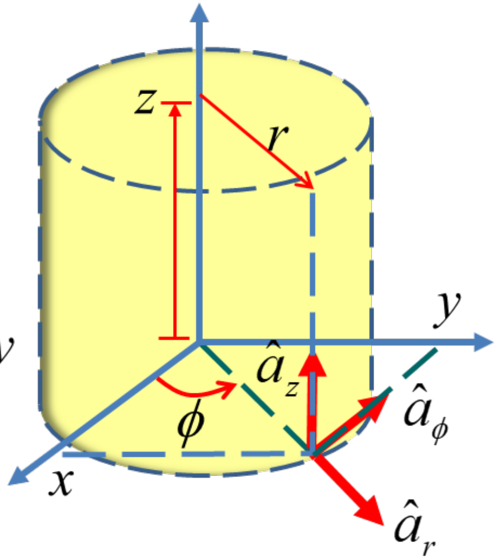
單元回顧

6. Coordinate systems

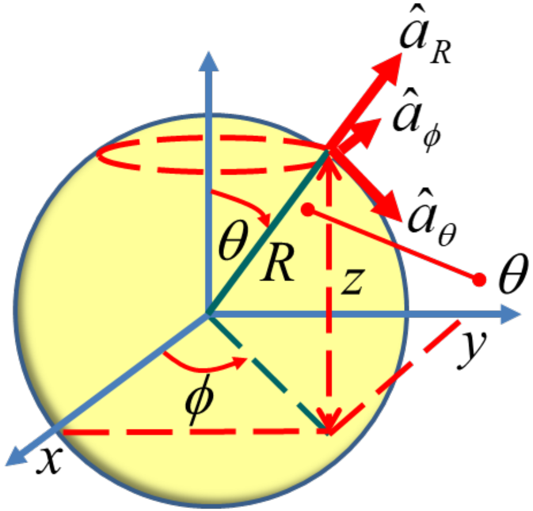
Cartesian



Cylindrical



Spherical



單元回顧

Simplified labels	Cartesian Coordinates (x, y, z)	Cylindrical Coordinates $(\mathrm{r}, \phi, \mathrm{z})$	Spherical Coordinates $(\mathrm{R}, \theta, \phi)$
$\hat{a}_{u_1}$ $\hat{a}_{u_2}$ $\hat{a}_{u_3}$	$\hat{a}_x$ $\hat{a}_y$ $\hat{a}_z$	$\hat{a}_r$ $\hat{a}_\phi$ $\hat{a}_z$	$\hat{a}_R$ $\hat{a}_\theta$ $\hat{a}_\phi$
h_1 h_2 h_3	1 1 1	1 r 1	1 R $R\sin\theta$

單元回顧

- A general expression of a vector: $\vec{A} = \hat{a}_{u_1} A_{u_1} + \hat{a}_{u_2} A_{u_2} + \hat{a}_{u_3} A_{u_3}$
- Orthogonality $\hat{a}_{u_i} \cdot \hat{a}_{u_j} = 0 \quad i, j = 1,2,3$

$$\hat{a}_{u_1} \times \hat{a}_{u_2} = \hat{a}_{u_3}, \quad \hat{a}_{u_2} \times \hat{a}_{u_3} = \hat{a}_{u_1}, \quad \hat{a}_{u_3} \times \hat{a}_{u_1} = \hat{a}_{u_2}$$

- A differential length:

$$d\vec{l} = \hat{a}_{u_1} h_1 du_1 + \hat{a}_{u_2} h_2 du_2 + \hat{a}_{u_3} h_3 du_3 = d\vec{l}_{u_1} + d\vec{l}_{u_2} + d\vec{l}_{u_3}$$

where h_1, h_2, h_3 are called metric coefficients.

- A differential surface: $d\vec{s} = d\vec{s}_{u_1} + d\vec{s}_{u_2} + d\vec{s}_{u_3}$

  where $d\vec{s}_{u_i} = d\vec{l}_{u_j} \times d\vec{l}_{u_k}$
- A differential volume: $dv = h_1 h_2 h_3 du_1 du_2 du_3$