H.W. 1

#1. (a) When total is p red balls & n white balls, then it # of permutation is $\frac{(p+n)!}{p!\,n!}$. By rules of sum we can know the # of permutation is:

$$\frac{p!}{p!} + \frac{(p+1)!}{p!\,1!} + \frac{(p+2)!}{p!\,2!} + \cdots + \frac{(p+q)!}{p!\,q!}$$

(0白 or 1白 or 2白 or ..... q白) #

(b) To choose m items from $n+1$ items, we can list them:

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ ...... $\boxed{n+1}$

So the ways can be like:

1. must choose $\boxed{1}$ and choose $m-1$ from n
2. must choose $\boxed{2}$ and choose $m-1$ from $n-1$

(但不選 $\boxed{1}$ 因為 1. 已經把所有 $\boxed{1}$ 可能出現的次數算完)

⋮ (以此類推)

Then we know $C^{n+1}_{m} = \underset{\downarrow \atop 1.}{C^{n}_{m-1}} + \underset{\downarrow \atop 2.}{C^{n-1}_{m-1}} + C^{n-2}_{m-1} + \cdots + C^{m-1}_{m-1}$

......

If we see m as $p+1$ & n as $p+q$, the we get

$$C^{p+q+1}_{p+1} = C^{p+q}_{p} + C^{p+q-1}_{p} + \cdots + C^{p}_{p}$$

$$\Rightarrow \frac{(p+q+1)!}{(p+1)!\,q!} = \frac{(p+q)!}{p!\,q!} + \frac{(p+q-1)!}{p!(q-1)!} + \cdots + \frac{(p+1)!}{p!\,1!} + \frac{p!}{p!}$$

#

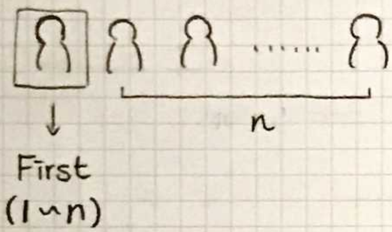
(c) By (b), we know $\frac{(p+q+1)!}{(p+1)!\,q!}$ stands for the # of permutation of p red balls and 0 or 1 or... or q white balls.

So since this time we are counting it's # of permutation when p is 0 or 1 or .... or p, we use rules of sum again and get:

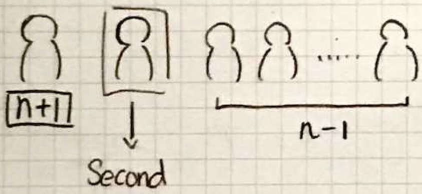
$$\frac{(q+1)!}{1!\,q!} + \frac{(1+q+1)!}{2!\,q!} + \frac{(2+q+1)!}{3!\,q!} + \cdots + \frac{(p+q+1)!}{(p+1)!\,q!}$$

same as (b), we can write the above formula into $\frac{(q+p+2)!}{(q+1)!\,(p+1)!}$

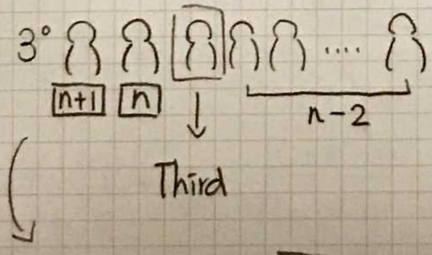
#2. (a) There are $n+1$ people need to be arranged, and the way is like this:

1° 

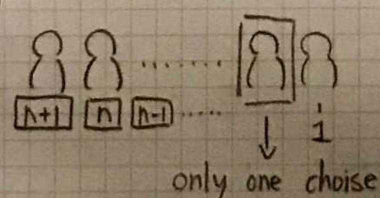
We list the people from $\boxed{1}$ to $\boxed{n+1}$. In this step, we choose $\boxed{1}$ ~ $\boxed{n}$ be the first, and arrange the left as $n!$. So the # is $n \times n!$.

2° 

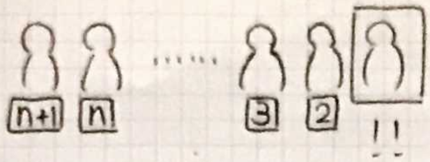
Because $\boxed{n+1}$ didn't be the first in 1°, we let $\boxed{n+1}$ be first and arrange the left. In this time, we also choose $\boxed{1}$ ~ $\boxed{n-1}$ to be first, and arrange the left as $n-1!$. So the # is $(n-1) \times (n-1)!$.

3° 

Same as 2°, we keep arrange the left in this way. So we get $(n-2)\times(n-2)! + (n-3)\times(n-3)! + \cdots$



On and on, we come to this condition. So we get $1 \times 1!$ at last.

4°


But there still have one condition with no left people to arrange. So we can add $1\times 0!$ or minus 1 from $(n-1)!$.

5° By 1° ~ 4° we know that

$$n\times n! + (n-1)\times(n-1)! + (n-2)\times(n-2)! + \cdots + 1\times 1! = (n-1)! - 1$$

#

(b) 1° When $m=0$, $0\times 1! + 0\times 2! + \cdots = 0$ ∴成立

When $m=1$, $1\times 1! + 0\times 2! + \cdots = 1$ ∴成立

2° 假設當 $m=2\sim k$, 且 $k=\sum^{p} i\cdot i!$ 時成立

(即 $m=2\sim k$ 時皆可以 $\sum a_i\cdot i!$ 形式表示)

當 $m=k+n$ $(n\le (p+1)(k+1))$

$$k+n = (k+1)+(n-1) = \left(\sum^{p} i\cdot i! + 1\right) + (n-1)$$

$$= (p+1)! + \underline{(n-1)}$$

∴ $n-1$ 可以整理成 $x(p+1)!+y$ 且 $x\le p$, $y\le k$

∴ $n-1$ 可以用 $\sum a_i\cdot i!$ 表示, 且當 $n=(p+1)(k+1)$ ~

又可以用 $\sum^{p+1} i\cdot i!$ 表示

∴成立

3° By 1°, 2° ∴ $m=\sum a_i\cdot i!$

4° 假設 $m=\sum a_i\cdot i! = \sum b_i\cdot i!$, 且 $\sum a_i \ne \sum b_i$

且 a_i 和 b_i 在 k 值以後皆相等 $(a_{k+1}=b_{k+1}, a_{k+2}=b_{k+2}\cdots)$

$a_k \ne b_k$ ∴至少差 $k!$

但因(a) $1\times 1! + 2\times 2! + \cdots + (k-1)\times(k-1)! = k! - 1 < k!$

所以不可能用前面補

∴ $\sum a_i = \sum b_i$, 必唯一

#3. Because of chousing m for first group and k for second group, we choose $m+k$ from n first. In these chosen numbers, we always can classify into two groups.

small $\longrightarrow$ big

$$\underbrace{\square\ \square\ \square}_{m}\ \overset{}{\underset{\text{cut}}{\Big|}}\ \underbrace{\square\ \square\ \square\ \square}_{k}$$

We can arange the numbers from small to big. Then cut them in any place to make them into 2 groups while $m, k \geq 1$.

So we only need to choose $m+k$ from n, and the answer is $\underline{C(n, m+k)}$ ※

#4. 1° $\square\square\square\square\square \rightarrow$ 5 letters sequences

$\downarrow$

each of them can put A ~ D

$\therefore$ the # of ways is 4^5

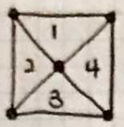
2° BAD$\square\square$
$\square\square$BAD
$\square$BAD$\square$
$\rightarrow$ All of them only have 2 places to choose and 4 letters can be put.

$\therefore$ the # of ways is $4^2 \times 3$

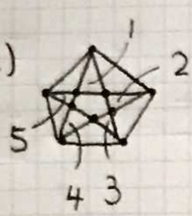
3° By 1°, 2°, we get the # of ways that exclude "BAD" is $\underline{4^5 - 3\times 4^2}$ ※

#5. (a) Since 3 vertices are all on the polygon and none of 3 vertices on polygon make a line, the only need to do is to choose 3 vertices out of n vertices.

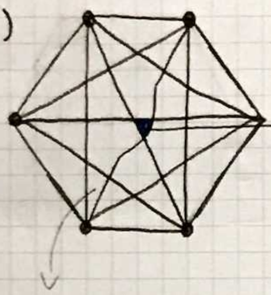
$\Rightarrow \underline{C(n, 3)}$ ※

(b) 

Since choosing 4 vertices from n vertices we can get 1 intersection and 4 triangles, the # of triangles is $\underline{4 \times C(n, 4)}$ #

(c) 

Since choosing 5 vertices from n vertices we can get 5 intersections and 5 triangles that meet the requirement, the # of triangles is $\underline{5 \times C(n, 5)}$ #

(d) 

→ The only 1 triangle.

※ Because none of 3 diagonals meet at same point.

Since choosing 6 vertices from n vertices can get only 1 triangle that only use intersections to be it's vertices, the # of triangles is $\underline{C(n, 6)}$ #