# CS2336 Discrete Mathematics

Exam 2
December 11, 2017 (2 hours)

**Answer all questions. Total marks = 100. A large portion of marks may be deducted from incomplete proofs or wrong arguments.**

1. Fermat once conjectured that for $n \geq 0$, all numbers $F_n = 2^{2^n} + 1$ are primes. Indeed, the numbers
$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$
are all primes. However, in 1732, Euler showed that $F_5 = 4294967297 = 641 \times 6700417$, thus disproving Fermat's claim.

   (20%) Here, you are asked to show an interesting property of $F_n$:

   For all integer $n \geq 1$, $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$.

2. A standard chessboard contains $8 \times 8$ squares. A king controls the squares immediately adjacent to the square that it is placed, in all eight directions. See Figure 1 for an example. A king can attack a piece if it is placed on the squares it controls.

   (a) (5%) If 17 pieces of kings are placed on a chessboard, show that there must be two kings attacking each other.

   (b) (5%) If only 16 pieces of kings are placed on the board, show that it is possible that no kings are attacking any other.

   (c) (5%) If 17 pieces of kings are placed on a chessboard, show that we can find five kings such that they are not attacking any other.

   (d) (5%) If only 16 pieces of kings are placed on a chessboard, show that it is possible that we cannot find five kings such that they are not attacking any other.

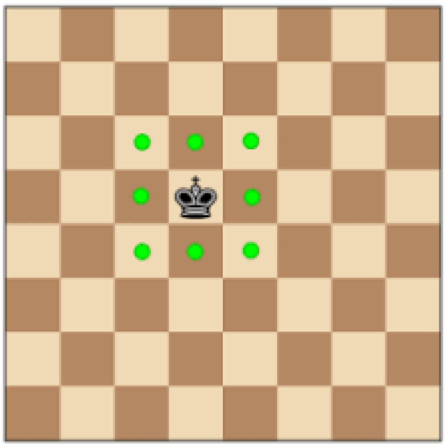

Figure 1: A $8 \times 8$ chessboard and a king, with squares controlled by the king marked

3. A contiguous sequence of characters in a string $X$ is called a *substring* of $X$. For instance, `ana` is a substring of `banana`, but `aa` is not a substring of `banana`.

   (10%) Consider all the 5-bit binary strings. How many of them contains `11` but not `101` as its substring?

   For example, `11011` contains both `11` and `101` as its substring, `10001` does not contain `11` and `101` as its substring, while `10011` contains `11` but not `101` as its substrng.

   *Hint:* Use a tree diagram.

4. (10%) Given that $x \geq 2, y \geq 1,$ and $z \geq 0$, how many integral solutions are there for the equation

$$x + y + z = 11?$$

5. Consider the diagram in Figure 2, where each vertex represents a city, and each edge represents a one-way road.

   (a) (5%) How many ways are there to travel from $A$ to $B$? $\boxed{47}$

   (b) (5%) How many ways are there to travel from $A$ to $B$ that must pass through $X$?

   (c) (5%) How many ways to travel from $A$ to $B$ that must pass through both $X$ and $Y$?

   Note: For each part of this question, no explanation is needed.

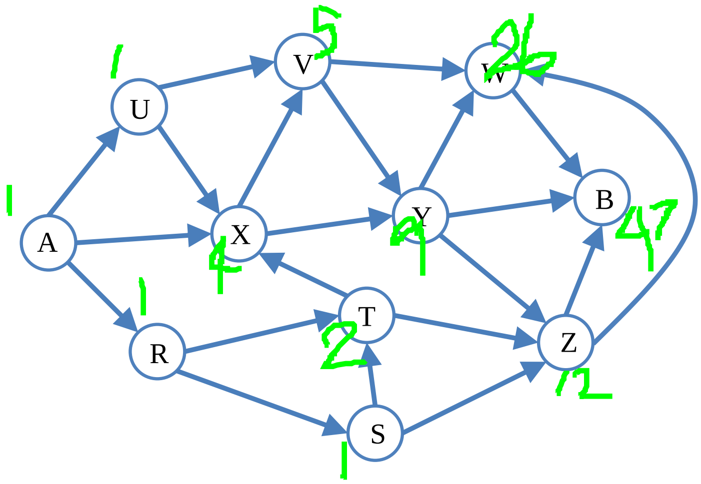


Figure 2: Diagram for Question 5

6. (15%) How many ways we can select 4 distinct integers from $\{1, 2, 3, \ldots, 100\}$, so that their sum is divisible by 3?

7. (10%) Give a combinatorial argument to show that

$$\binom{2n}{3} = 2 \times \binom{n}{3} + 2n \times \binom{n}{2}.$$

   Note: No marks will be given if you are not using a combinatorial argument.