last update: Dec. 28th , 2020
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電磁學（一）
Electromagnetics (I)

15. 電感
Inductance

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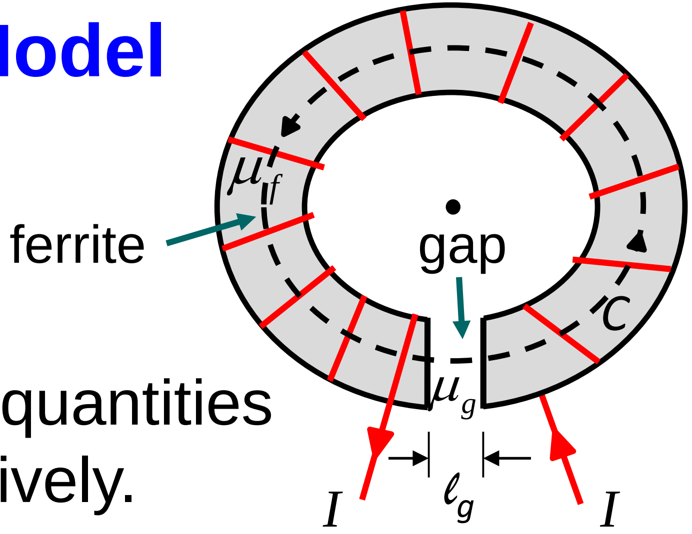
In this lecture, we will learn about magnetic circuits, including a magnetic-flux storage device, called an inductor.

- **15.1 Magnetic circuit 磁電路**
- **15.2 Self-inductance 自電感（自感）**
- **15.3 Mutual Inductance 互電感（互感）**
- **15.4 Inductor Circuit 電感電路**
- **15.5 Review 單元回顧**

電感
Inductance

15.1 磁電路
Magnetic Circuit

Magnetic Circuit Model



From Ampere's law, for small l_g

$$\oint_C \vec{H}\cdot d\vec{l} = NI \Rightarrow H_g l_g + H_f l_f = NI$$

where the subscripts, g and f, denote quantities in the gap and ferrite regions, respectively.

From the boundary condition, $B_{1n} = B_{2n}$, one has $\mu_g H_g = \mu_f H_f$

$$B_g = B_f = \frac{NI}{l_f/\mu_f + l_g/\mu_g} \Rightarrow \Phi = BS = \frac{NI}{l_f/\mu_f S + l_g/\mu_g S} \Rightarrow \Phi = \frac{V_m}{R_f + R_g}$$

where S is the cross sectional area, R is the magnetic reluctance, and $V_{\mathrm{m}} = NI$ is the the magnetomotive force (mmf).

*Recall resistance $R = l/\sigma S$

Analogy between Magnetic and Electric Circuits

Magnetic Circuit	Electric Circuit
$\nabla \times \vec{H} = \vec{J}$	$\nabla \times \vec{E} = \vec{f}$
mmf $V_m = NI$	*emf* V_{em}
magnetic flux Φ	*electric current* I
Magnetic reluctance $R = l/\mu S$	Electric resistance $R = l/\sigma S$

$$\frac{1}{R} = \mu S/l = \frac{\Phi}{NI}:$$ generated flux per unit current.

$\Phi \propto S$ & μ, because of $\Phi = \int_S \vec{B} \cdot d\vec{s} = \mu \int_S \vec{H} \cdot d\vec{s}$

$\Phi \propto 1/l$, because, from Ampere's law, the longer l, the weaker H & Φ for a given NI.

Kirchhoff's Voltage Law for a Magnetostatic Loop

$$\sum_j V_{m,j} = \sum_j N_j I_j = \sum_k R_k \Phi_k \quad \text{(resulting from } \nabla \times \vec{H} = \vec{J}\text{)}$$

*Compare it with Kirchhoff's voltage law for electrostatics

$$\sum_j V_{em,j} = \sum_k R_k I_k$$

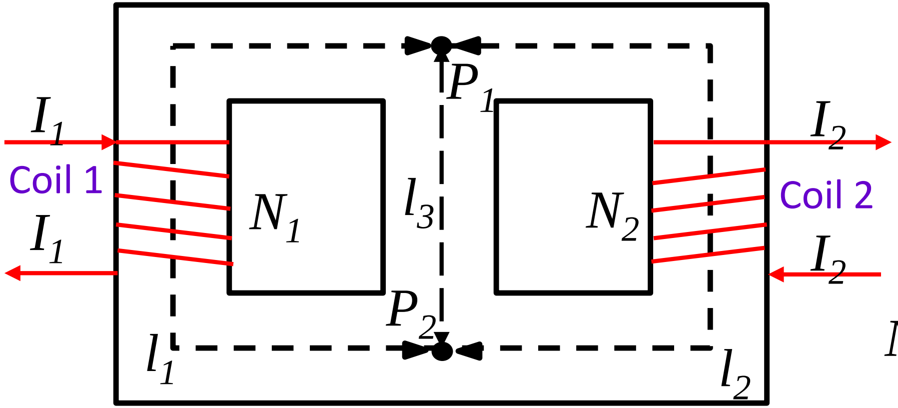
Kirchhoff's Current Law for a Magnetostatic Node

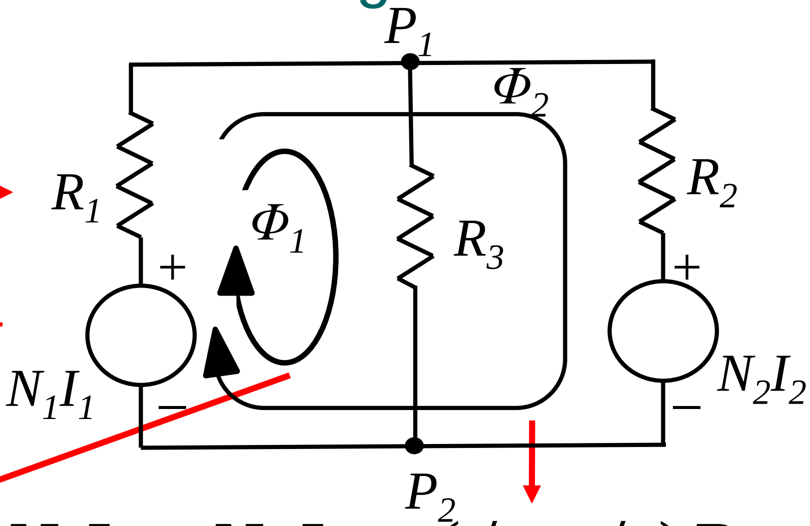
$$\sum_k \Phi_k = 0 \quad \text{(resulting from } \nabla \cdot \vec{B} = 0\text{)}$$

*Compare it with Kirchhoff's current law for electrostatics at a current node

$$\sum_k I_k = 0 \quad \text{(resulting from } \nabla \cdot \vec{J} = 0\text{)}$$

E.g. Determine the magnetic flux in the center leg in the following magnetic circuit with a ferromagnetic core.





$$N_1 I_1 = (\phi_1 + \phi_2) R_1 + \phi_1 R_3 \qquad N_1 I_1 - N_2 I_2 = (\phi_1 + \phi_2) R_1 + \phi_2 R_2$$

One can solve ϕ_1 (flux in the central leg) and ϕ_2 from the two equations.

* Recall the magnetic reluctance $R_i = \dfrac{l_i}{\mu S}$

15.1 磁電路

Magnetic Circuit

Magnetic Circuit	*Electric Circuit*
mmf $\boxed{V_m = NI}$	emf $\boxed{V_{em}}$
magnetic flux Φ	electric current I
Magnetic reluctance $\boxed{R = l/\mu S}$	Electric resistance $\boxed{R = l/\sigma S}$

Kirchhoff's Voltage Law for a Magnetostatic Loop $\boxed{\sum_j V_{m,j} = \sum_j N_j I_j = \sum_k R_k \Phi_k}$

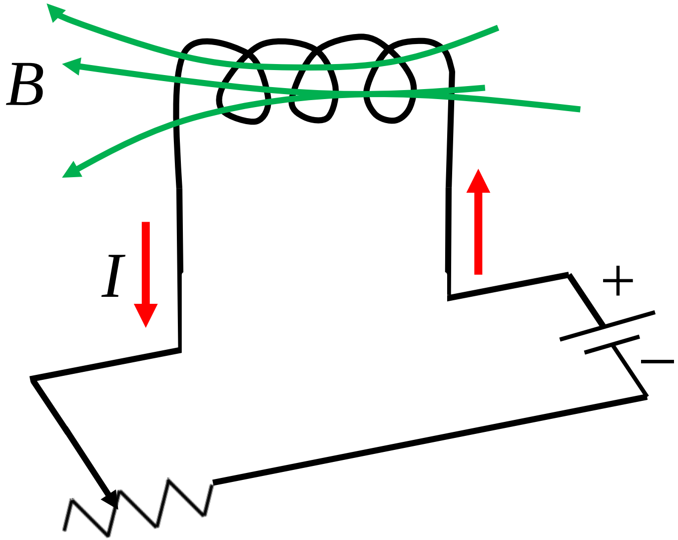
Kirchhoff's Current Law for a Magnetostatic Node $\boxed{\sum_k \Phi_k = 0}$

電感
Inductance

15.2 自電感（自感）
Self-inductance

Inductor: a device that stores magnetic flux and provides a certain amount of inductance.

The inductor bank of a high-voltage modulator at the NTHU HOPE Laboratory



In a linear medium, inductance is a function of spatial dimensions and permeability.

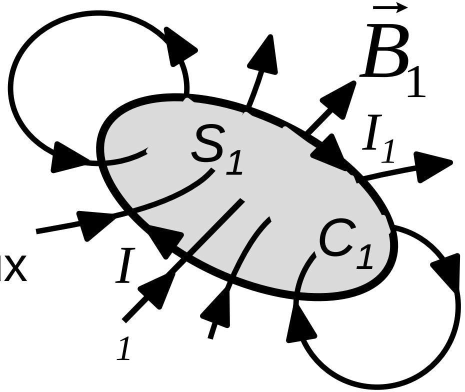
Inductance: magnetic linkage per unit current

Self-Inductance

$$L_{11} = \frac{N_1 \Phi_{11}}{I_1} = \frac{\Lambda_{11}}{I_1}$$

refer to flux loop

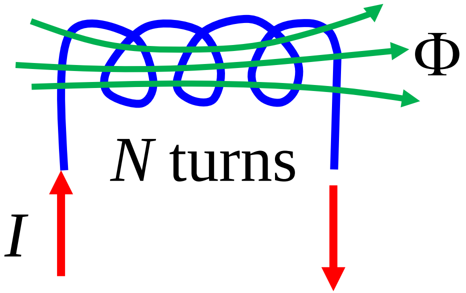
refer to current



where $\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}_1 = \oint_{C_1} \vec{A}_1 \cdot d\vec{l}_1$

is the magnetic flux through C_1 due to I_1,

and $$\Lambda_{11} = N_1 \Phi_{11}$$

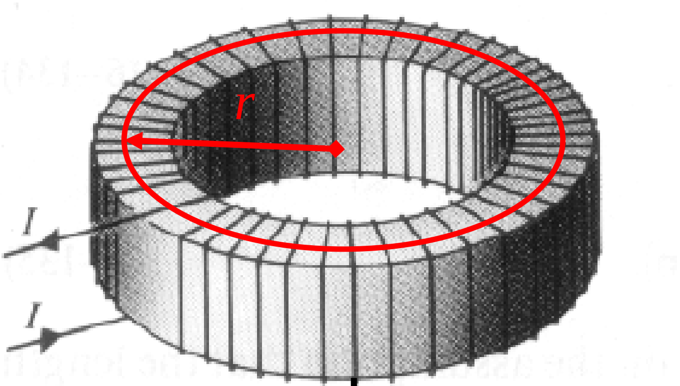


is the magnetic linkage associated with C_1 due to I_1.

Self-inductance L_{11} is the *magnetic linkage per unit current in the loop itself.*

E.g. Find the inductance of an *N*-turn toroid with μ_0 inside

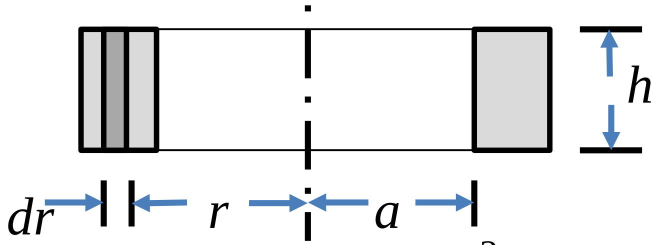
Apply Ampere's law $\oint_C \vec{H} \cdot d\vec{l} = NI$



$$H_\phi \cdot 2\pi r = NI \;\Rightarrow\; H_\phi = \frac{NI}{2\pi r}$$

$$\Rightarrow \quad B_\phi = \mu_0 H_\phi = \frac{\mu_0 NI}{2\pi r}$$

*adapted from Fig. 6-23 [1]



The magnetic flux is calculated from

$$\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot \underline{d\vec{s}_1} = \int_a^b B_\phi \underline{h dr} = \frac{\mu_0 NIh}{2\pi} \ln\frac{b}{a}$$

The magnetic linkage is calculated from $\Lambda_{11} = N_1 \Phi_{11} = \dfrac{\mu_0 N^2 Ih}{2\pi} \ln\dfrac{b}{a}$

The inductance is therefore $L_{11} = \dfrac{\Lambda_{11}}{I_1} = \dfrac{\mu_0 N^2 h}{2\pi} \ln\dfrac{b}{a}$

15.2 自電感（自感）

Self-inductance

- An inductor is a magnetic-flux storage device.
- Specifically, inductance is the magnetic linkage per unit current.
- Self-inductance is the magnetic linkage per unit current without considering coupling from other current loops.

電感
Inductance

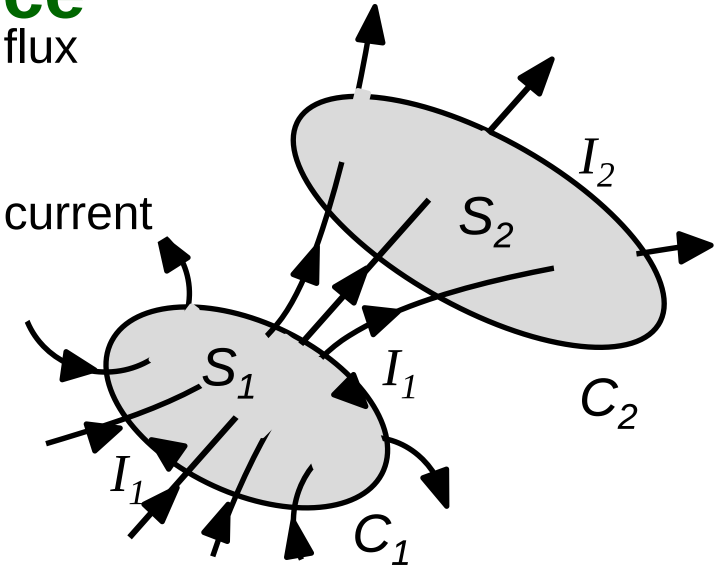
15.3 互電感（互感）
Mutual Inductance

Mutual Inductance

$$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}$$

refer to flux loop

refer to current



where $\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2$

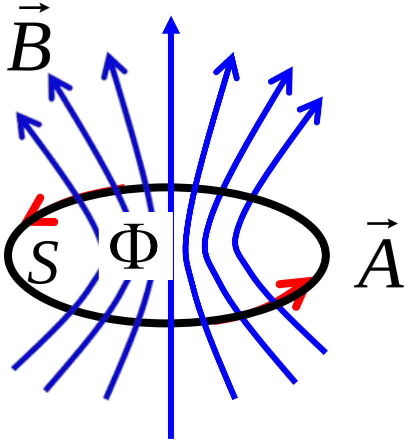
is the magnetic flux through C_2 due to I_1, and

$\Lambda_{12} = N_2 \Phi_{12}$ is the magnetic linkage associated with C_2 due to I_1.

Mutual inductance L_{12} is the *magnetic linkage through* C_2 *per unit current in loop* C_1.

Given the mutual inductance $L_{12} = \dfrac{N_2 \Phi_{12}}{I_1} = \dfrac{\Lambda_{12}}{I_1}$

Recall the total magnet flux $\Phi = \int_S \vec{B} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$



$$\Rightarrow L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2$$

Recall the vector potential $\vec{A} = \dfrac{\mu_0 I}{4\pi} \oint_{C'} \dfrac{d\vec{l}\,'}{R}$

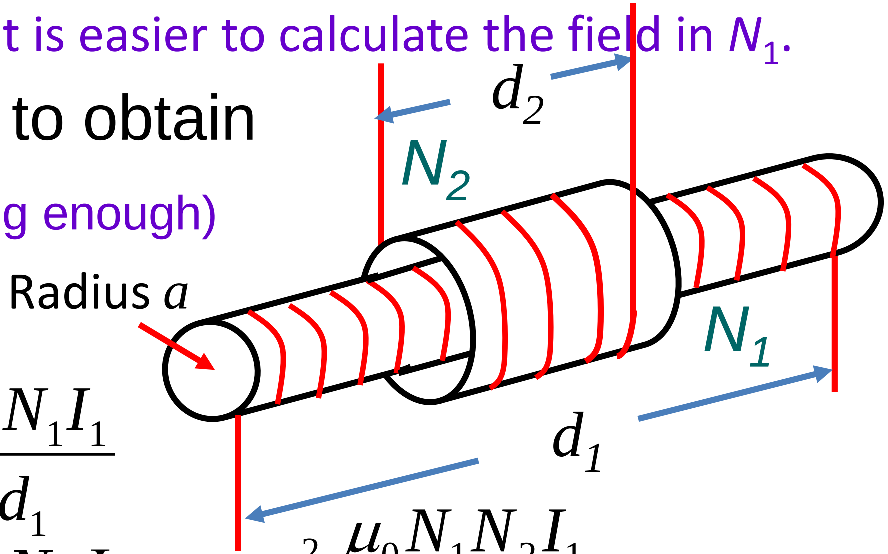
$$L_{12} = \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2 = \frac{N_2}{I_1} \oint_{C_2} \left(\vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{R} \right) \cdot d\vec{l}_2$$

$$\Rightarrow \quad L_{12} = \underbrace{\frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R}}_{} = L_{21}$$

No difference when 1 and 2 are exchanged.

E.g. A wire of N_1 turns is wound inside a wire of N_2 turns carrying currents I_1 and I_2 respectively. Find the mutual inductance of such a device.

*To avoid dealing with the fringe fields, it is easier to calculate the field in N_1.



To find Φ_{12}, use Ampere's law to obtain

$$B_1 = \frac{\mu_0 N_1 I_1}{d_1}$$ (valid only when d_1 is long enough)

The magnetic flux in Coil 2 is

$$\Phi_{12} = B_1 S_2 = B_1 S_1 = \pi a^2 \frac{\mu_0 N_1 I_1}{d_1}$$

The magnetic linkage is $\Lambda_{12} = N_2 \Phi_{12} = \pi a^2 \frac{\mu_0 N_1 N_2 I_1}{d_1}$

The mutual inductance is $L_{12} = \frac{\Lambda_{12}}{I_1} = \pi a^2 \frac{\mu_0 N_1 N_2}{d_1}$

15.3 互電感（互感）
Mutual Inductance

- Mutual inductance L_{12} is the magnetic linkage through loop 2, per unit current in loop 1.
- In general, $L_{12} = L_{21}$
- Therefore, in a problem, you choose to calculate either L_{12} or L_{21}, whichever easier.

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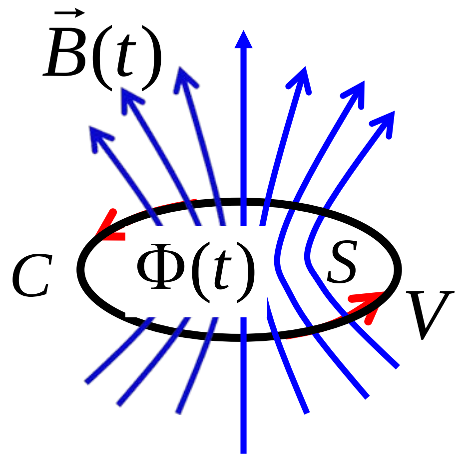
15.4 電感電路
Inductor Circuit

Faraday's Law of Magnetic Induction

Consider a single contour loop C in a space with a time-varying magnetic flux $\Phi(t)$ through it, where

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

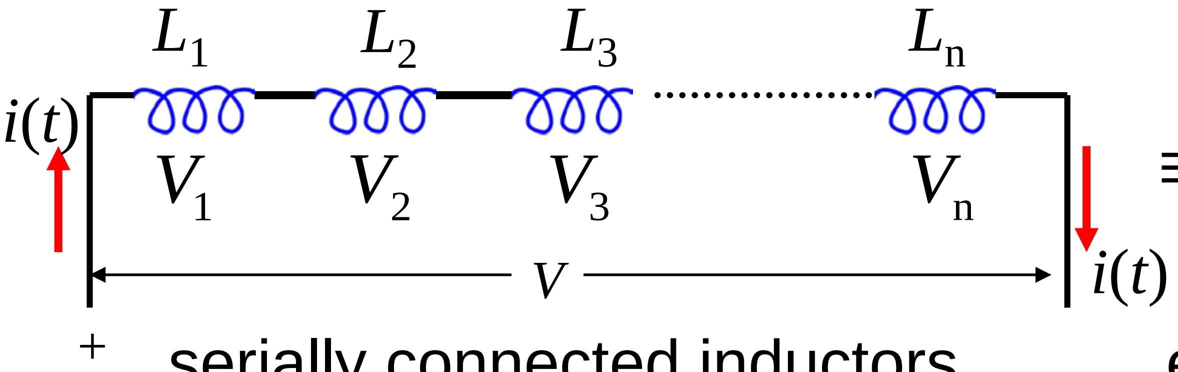


Based on experimental observation, a voltage $V(t)$ is induced around the loop C, given by

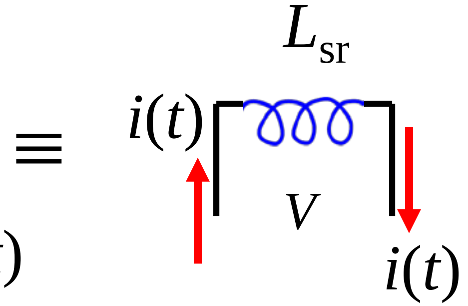
$$V = -\frac{d\Phi}{dt} = -L\frac{di}{dt}$$

(The "–" sign is from the so-called Lenz's law – the induced voltage is to oppose the change of the magnetic flux)

Serial Inductors



serially connected inductors



equivalence circuit

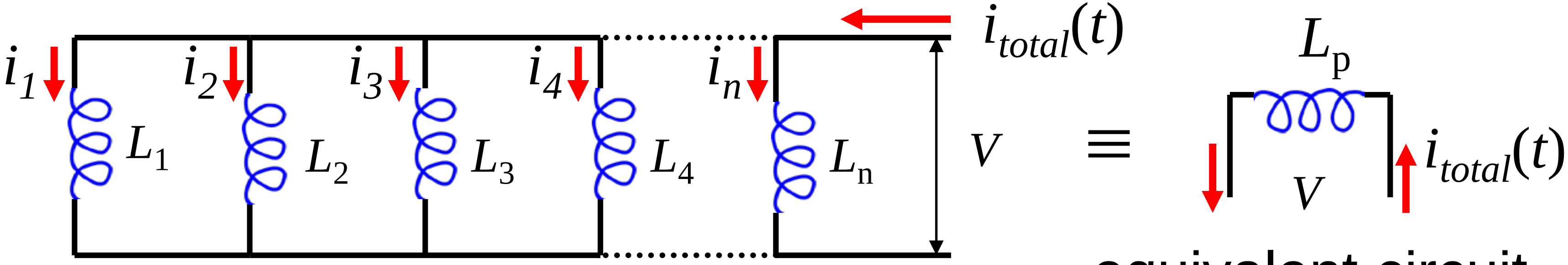
Circuit expression

$$V = L_{sr}\frac{di}{dt} = \sum_i V_i = L_1\frac{di}{dt} + L_2\frac{di}{dt} \ldots + L_N\frac{di}{dt}$$

Equivalent inductance = sum of individual inductances L_i

$$L_{sr} = L_1 + L_2 \ldots + L_N$$

Parallel Inductors



parallel connected inductors

equivalent circuit

Circuit expression

$$V = L_p \frac{di_{total}}{dt} = L_p\left(\frac{di_1}{dt} + \frac{di_2}{dt} + \frac{di_3}{dt} + \ldots + \frac{di_n}{dt}\right) = L_p\left(\frac{V}{L_1} + \frac{V}{L_2} + \frac{V}{L_3} \ldots + \frac{V}{L_n}\right)$$

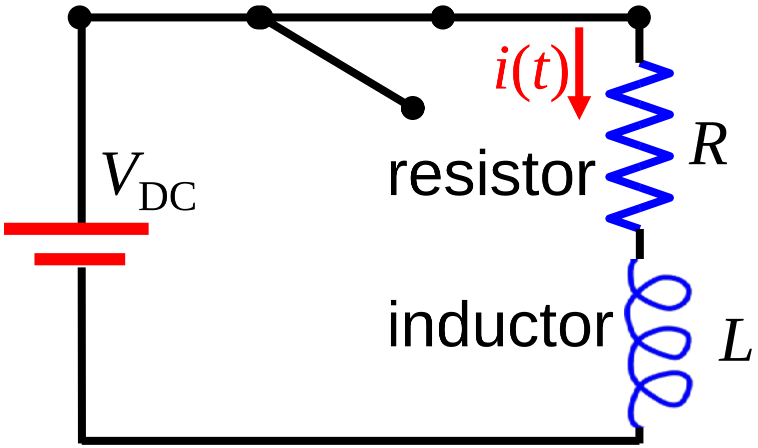
$\frac{V}{L_1}$ $\frac{V}{L_2}$ $\frac{V}{L_n}$ $\frac{V}{L_n}$

Equivalent inductance = sum of individual inductances

$$\Rightarrow \frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \ldots + \frac{1}{L_n}$$

RL Circuit

Kirchhoff's voltage law over the whole loop:

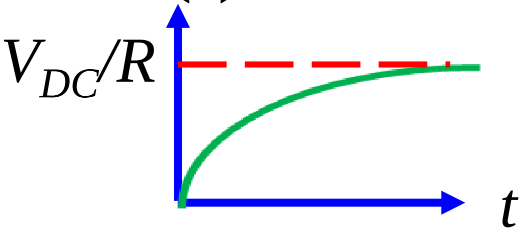


$$V_{DC} = V_R(t) + V_L(t)$$

Recall $V_L = L\frac{di}{dt}$

$$\Rightarrow \quad Ri + L\frac{di}{dt} = V_{DC} \quad \Rightarrow \quad i(t) = \frac{V_{DC}}{R}\left(1 - e^{-Rt/L}\right)$$

Inductor current increases to a steady-state value of V_{DC}/R with a time constant of $\tau = L/R$



15.4 電感電路

Inductor Circuit

- The equivalent inductance of serial inductors is the sum of all the individual inductances.

$$L_{sr} = L_1 + L_2 \ldots + L_N$$

- The inverse of the equivalent inductance of parallel inductors is the inverse sum of all the individual inductances.

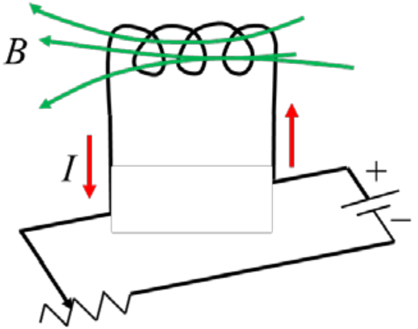
$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \ldots + \frac{1}{L_n}$$

- A DC-voltage powered RL circuit reaches a steady-state current with a characteristic time constant equal to $\tau = L/R$.

電感
Inductance
15.5 單元回顧 Review

單元回顧

1. An inductor, driven by a current, is a magnetic-flux storage device.



2. The analogy between magnetic and electric circuits can be summarized as

Magnetic Circuit	***Electric Circuit***
$\nabla \times \vec{H} = \vec{J}$	$\nabla \times \vec{E} = \vec{f}$
mmf $V_m = NI$	*emf* V_{em}
magnetic flux Φ	*electric current I*
Magnetic reluctance $R = l/\mu S$	Electric resistance $R = l/\sigma S$

單元回顧

3. The Kirchhoff's laws for a magnetostatic circuit are written as:

Kirchhoff’s Voltage Law for a Magnetostatic Loop

$$\sum_j V_{m,j} = \sum_j N_j I_j = \sum_k R_k \Phi_k$$

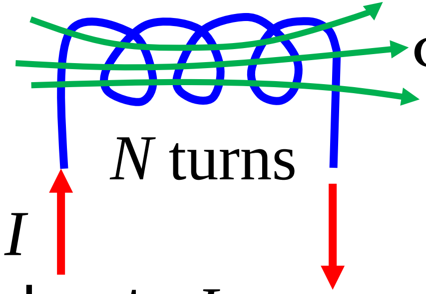
Kirchhoff’s Current Law for a Magnetostatic Node

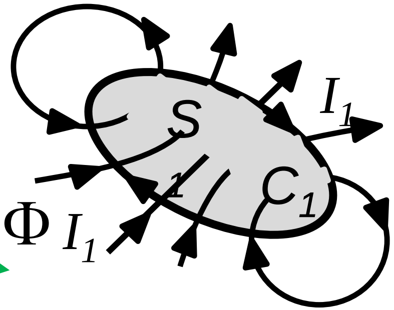
$$\sum_k \Phi_k = 0$$

單元回顧

4. **Self-inductance** is the magnetic linkage per unit current in the current loop itself.

$$L_{11} = \frac{N_1 \Phi_{11}}{I_1} = \frac{\Lambda_{11}}{I_1}$$



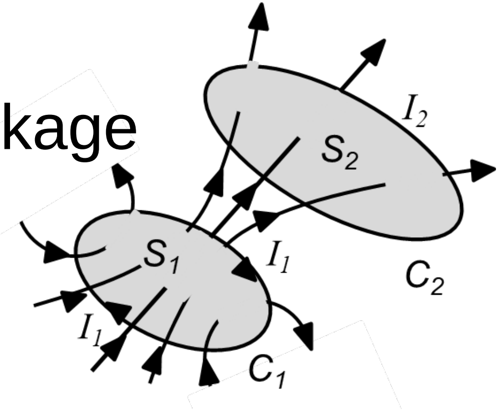


where $\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}_1 = \oint_{C_1} \vec{A}_1 \cdot d\vec{l}_1$

is the magnetic flux through C_1 due to I_1,

5. **Mutual inductance** L_{12} is the magnetic linkage through loop 2, per unit current in loop 1 or

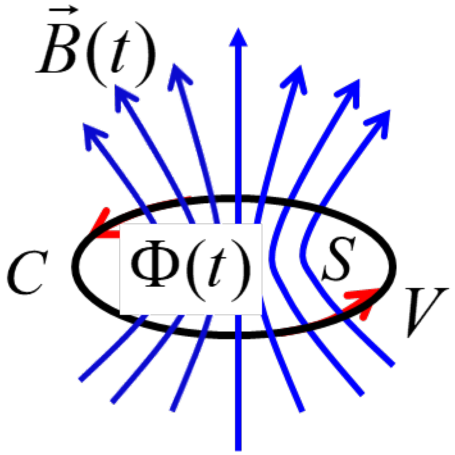
$$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}$$



單元回顧

6. In general, $L_{12} = L_{21}$.

7. Faraday's law of magnetic induction: A time varying magnetic flux can induce a voltage $V(t)$ around a contour loop C, according to

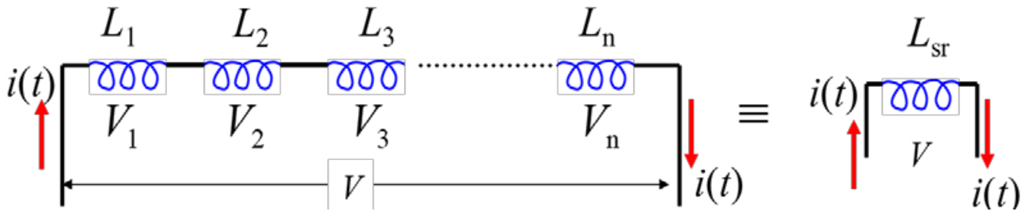


$$V = -\frac{d\Phi}{dt} - L\frac{di}{dt}$$

The induced current opposes the change of Φ.

8. The equivalent inductance of serial inductors is the sum of all the individual inductances.

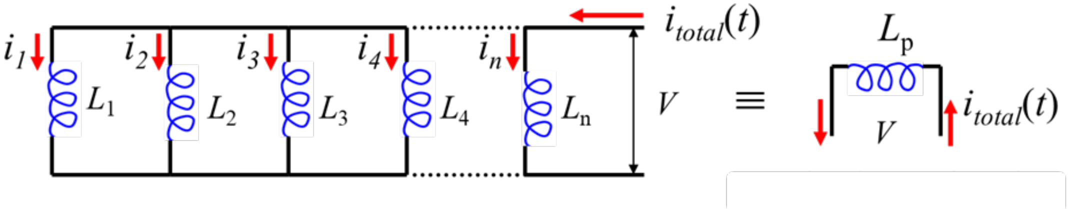
$$L_{sr} = L_1 + L_2 \ldots + L_N$$



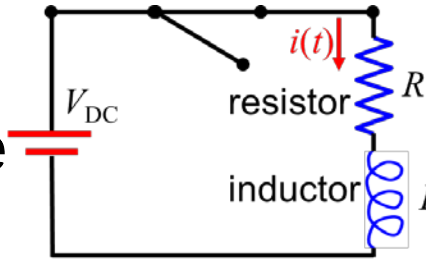
單元回顧

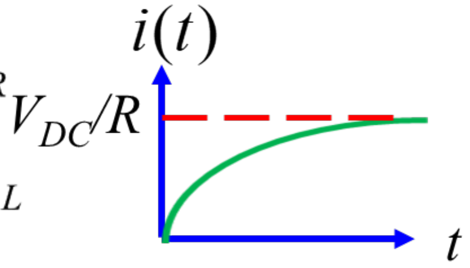
9. The inverse of the equivalent inductance of parallel inductors is the inverse sum of all the individual inductances.

$$\frac{1}{L_p}=\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}\dots+\frac{1}{L_n}$$



10. A DC-voltage powered RL circuit reaches a steady-state current with a characteristic time constant equal to $\tau = L/R$.





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