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Step 4: Apply B.C. & I.C. to delete unsatisfying terms

From B.C. ① & ② ($X(0)=0$, $X(L)=0$), we know $X(x)$ should be sinusoidal type:

$$X(x) = c_1\cos kx + c_2 \sin kx \xRightarrow{\text{B.C. ①\&②}} X(x) = A_n \sin\frac{n\pi}{L}x, \ (k=\frac{n\pi}{L}), n=1,2,3,\dots$$

$$T(t) = c_3\cos akt + c_4 \sin akt \underset{k=\frac{n\pi}{L}}{\Longrightarrow} T(t) = B_n\cos\frac{n\pi a}{L}t + C_n \sin\frac{n\pi a}{L}t, \quad n=1,2,3,\dots$$

$$u(x,t) = X(x)T(t) = \left(A_n \sin\frac{n\pi}{L}x\right)\left(B_n\cos\frac{n\pi a}{L}t + C_n\sin\frac{n\pi a}{L}t\right) \quad n=1,2,3,\dots$$

So from B.C. ① & ②, we obtain an infinite number of solutions (one for each n) as $u(x,t) = \left(D_n\cos\frac{n\pi a}{L}t + E_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$

(the coefficients are combined as D_n, E_n for simplicity)

Step 5: Use superposition and proper conditions to get the solution as a Fourier series

Although none of $u(x,t)$ obtained in step 4 satisfies the I.C. ③ & ④, it is possible to combine them as an infinite series that does.

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left(D_n\cos\frac{n\pi a}{L}t + E_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

From I.C. ③ ($u(x,0)=f(x)$)

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x = f(x)$$ (Note: $f(x)$ is expanded by a Fourier sine series)

"D_n" can be found as how we find "b_n" coefficient

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

From I.C. ④ ($\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$)

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-D_n\frac{n\pi a}{L}\sin\frac{n\pi a}{L}t + E_n\frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

$$\xrightarrow{\text{At } t=0} \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\left(E_n\frac{n\pi a}{L}\right)\sin\frac{n\pi}{L}x = g(x)$$ (Note: $g(x)$ is expanded by a Fourier sine series)

$$E_n\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \Rightarrow E_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

Final solution is $u(x,t) = \sum_{n=1}^{\infty}\left(D_n\cos\frac{n\pi a}{L}t + E_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$, where

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad E_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

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Remarks:

① When using SOV to solve PDEs, the trick of assuming solution as the

② In the examples, we repeatedly encountered the BVP

Such BVP actually belongs to a more general category of BVPs, called

Sturm-Liouville boundary-value problem

General form:

This general form can also be rewritten in a "self-adjoint form"

subject to B.C.

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Sturm-Liouville equation subject to the B.C. is referred to the "Sturm-Liouville boundary-value problem"

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☆ Properties of Sturm-Liouville BVP:

— There exists an infinite number of

— For each eigenvalue, there is one

— Eigenfunctions are $\left\{ \right.$

Remarks:

① Any homogeneous 2nd order linear DE can be rewritten

② Because the eigenfunctions form an orthogonal set, a given function $f(x)$ can be expanded by

Many DEs in engineering / physics are
and can be rewritten as a Sturm-Liouville equation. When solving these DEs with B.C., they all have

Example I: Legendre's equation $(1-x^2)y''-2xy'+n(n+1)y=0$, $-1<x<1$

1) has the
2) eigenvalues
3) eigenfunctions

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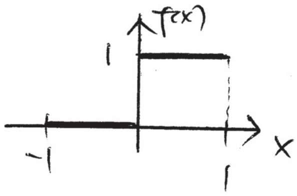
4) A function $f(x)$ can be expanded in terms of Legendre polynomials $P_n(x)$ as: $f(x) =$

Practice: Use Legendre polynomials to expand a function $f(x)$



Example II: Hermite's equation $y'' - 2xy' + 2ny = 0$

1) has the
2) eigenvalues
3) eigenfunctions

4) A function can be expanded in terms of Hermite polynomials $H_n(x)$

Example IV : BVP we encountered when solving Laplace's eq & wave eq

1) has the
2) eigenvalues
3) eigenfunctions

4) A function $f(x)$ can be expanded in terms of

More examples,

We already learned (in ch6) how to find the solution (eigenfunction) for some important special DEs. In the following, we will do more practices on solving the most basic yet important S-L BVP :

Ex1 : Find the eigenvalues and eigenfunctions for the BVP

$y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$

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Ex2: Same with Ex1 but with B.C. $y'(0)=0,\ y(1)+y'(1)=0$