EE306002: Probability **Spring 2022**
College of Electrical Engineering
National Tsing Hua University

Homework #1 – Solution

Coverage: chapter 1–2

Due date: 17 March, 2022

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Problem 1.2.10. (10 points) Define a sample space for the experiment of drawing two coins from a purse that contains two quarters, three nickels, one dime, and four pennis. For the same experiment describe the following events:
(a) drawing 26 cents;
(b) drawing more than 9 but less than 25 cents;
(c) drawing 29 cents.

Solution:
Let $\{q, n, d, p\}$ be the outcome that each coin is drawn, and we define the sample space $S = \{qq, qn, qd, qp, nn, nd, np, dp, pp\}$ being the set of all outcomes that drawing two coins from a purse occurs. Then we can express the following events:
(a) $\{qp\}$
(b) $\{nd, dp, nn\}$
(c) $\varnothing$

Problem 1.4.6. (10 points) In a probability test, for two events E and F of a sample space, Tina's calculations resulted in $P(E) = 1/4$, $P(F) = 1/2$, and $P(EF) = 3/8$. Is it possible that Tina made a mistake in her calculations? Why or why not?

Solution:
Clearly, she made a mistake. Since $EF \subseteq E$, we must have $P(EF) \leq P(E)$. However, in Tina's calculations, $P(EF) = \frac{3}{8} > \frac{1}{4} = P(E)$.

Problem 1.7.12. (10 points) A point is selected at random from the interval $(0, 1)$. What is the probability that it is rational? What is the probability that it is irrational?

Solution:
Let $\mathbf{Q} = \{r_1, r_2, r_3, ...\}$ be the set of all rational numbers in the interval $S = (0, 1)$. Then $P(\{r_i\}) = 0$ for all r_i in $\mathbf{Q}$. Moreover, $\mathbf{Q} = \cup_i \{r_i\}$ is the union of countable mutually exclusive events $\{r_i\}$. By the Axiom 3 of the probability function, we have

$$P(\mathbf{Q}) = P(\{r_1, r_2, r_3, ...\}) = \sum_{i=1}^{\infty} P(\{r_i\}) = 0.$$

The probability of the event $\mathbf{Q}^c = S - \mathbf{Q}$ (event of all the irrational numbers in (0,1)) is

$$P(\mathbf{Q}^c) = 1 - P(\mathbf{Q}) = 1,$$

Problem Ch1-Review 30. (10 points) A number is selected at random from the set of natural numbers $\{1, 2, 3, \ldots, 1000\}$. What is the probability that it is not divisible by 4, 7, or 9?

Solution:
Let F, S, and N be the events that the number selected is divisible by 4, 7, and 9, respectively. We are interested in $P(F^c S^c N^c)$ which is equal to $1 - P(F \cup S \cup N)$ by DeMorgan's law. Now

$$\begin{aligned} P(F \cup S \cup N) &= P(F) + P(S) + P(N) - P(FS) - P(SN) - P(FN) + P(FSN) \\ &= \frac{250}{1000} + \frac{142}{1000} + \frac{111}{1000} - \frac{35}{1000} - \frac{15}{1000} - \frac{27}{1000} + \frac{3}{1000} = 0.429. \end{aligned}$$

So the desired probability is 0.571.

Problem 2.2.36 (10 points) What is the probability that a random r-digit number ($r \geq 3$) contains at least one 0, at least one 1, and at least one 2?

Solution:
Let A be the event that the number selected contains at least one 0. Let B be the event that it contains at least one 1 and C be the event that it contains at least one 2. By DeMorgan's law, the desired quantity is $P(ABC) = 1 - P(A^c \cup B^c \cup C^c)$, where
$P(A^c \cup B^c \cup C^c) = P(A^c) + P(B^c) + P(C^c) - P(A^c B^c) - P(A^c C^c) - P(B^c C^c) + P(A^c B^c C^c)$

$$= \frac{9^r}{9 \cdot 10^{r-1}} + \frac{8 \cdot 9^{r-1}}{9 \cdot 10^{r-1}} + \frac{8 \cdot 9^{r-1}}{9 \cdot 10^{r-1}} - \frac{8^r}{9 \cdot 10^{r-1}} - \frac{8^r}{9 \cdot 10^{r-1}} - \frac{7 \cdot 8^{r-1}}{9 \cdot 10^{r-1}} + \frac{7^r}{9 \cdot 10^{r-1}}$$

(Note: The first digit cannot be zero.)

Problem 2.3.18. (10 points) The letters in the word SUPERCALIFRAGILISTICEXPIALIDOCIOUS are arranged randomly. (a) How many of the distinguishable arrangements begin with G and end with X? (b) What is the probability that the outcome begins with G and ends with X?

Solution:
The word consists of letter I (7 places), letters S, C, A, L (3 places each), letters U, P, E, R, O (2 places each), and letters F, T, D, G, X (1 place each). Therefore,
(a) The number of the arrangements under the condition that the first letter G and the last letter X is given by

$$N(G, X) = \frac{32!}{(2!)^5 \cdot (3!)^4 \cdot (7!)}$$

(b) The probability that the arrangements begins with G and end with X is $N(G, X)$ divided by the total number of the arrangements $|S|$ as follows:

$$\frac{N(G, X)}{|S|} = \frac{\frac{32!}{(2!)^5 \cdot (3!)^4 \cdot 7!}}{\frac{34!}{(2!)^5 \cdot (3!)^4 \cdot 7!}} = \frac{32!}{34!} = \frac{1}{1122}$$

Problem 2.3.26. (10 points) If we put five math, six biology, eight history, and three literature books on a bookshelf at random, what is the probability that all the math books are together?

Solution:
Consider the five math books as one entity. The answer is $\frac{5! \cdot 18!}{22!}$.

Problem 2.4.38. (10 points) A history professor who teaches three classes of the same course every semester decides to make several tests and use them for the next 10 years (20 semesters) as final exams. The professor has two policies: (1) not to give the same test to more than one class in a semester, and (2) not to repeat the same combination of three tests for any two semesters. Determine the minimum number of different tests that the professor should prepare.

Solution:
Let x be the total number of tests, and it is the solution of the equation $\binom{x}{3} = 20$. This equation is equivalent to $x(x-1)(x-2) = 120$ and its minimum solution is $x = 6$.

Problem 2.4.50. (10 points) In a closet there are 10 pairs of shoes. If six shoes are selected at random, what is the probability of (a) no complete pairs; (b) exactly one complete pair; (c) exactly two complete pairs; (d) exactly three complete pairs?

Solution:
(a) There are $\binom{10}{6}$ combinations of 6 different pairs, and each pair has 2 different shoes. So the probability is given by

$$\frac{\binom{10}{6} \times 2^6}{\binom{20}{6}} = 0.347$$

(b) Following the same logical reasoning as part (a), we have the probability of exactly one complete pair given by

$$\frac{\binom{10}{1} \times \binom{9}{4} \times 2^4}{\binom{20}{6}} = 0.520$$

(c) Following the same logical reasoning as part (a), we have the probability of exactly two complete pairs given by

$$\frac{\binom{10}{2} \times \binom{8}{2} \times 2^2}{\binom{20}{6}} = 0.130$$

(d) The probability of exactly three complete pairs is given by computing the arrangements of selecting 3 pairs from 10 pairs as follows:

$$\frac{\binom{10}{3}}{\binom{20}{6}} = 0.0031.$$

Problem Ch2-Review 36. (10 points) An ordinary deck of 52 cards is dealt, 13 each, at random among A, B, C, and D. What is the probability that (a) A and B together get two aces; (b) A gets all the face cards; (c) A gets five hearts and B gets the remaining eight hearts?

Solution:
(a) It is straightforward to see that the probability that A and B together get 2 aces is given by

$$\frac{\binom{4}{2} \times \binom{52-4}{24}}{\binom{52}{26}} = 0.390$$

(b) Since A has all the 12 face cards, choose any one card from the remains. The probability for this event is given by

$$\frac{\binom{52-12}{1}}{\binom{52}{13}} = 6.299 \times 10^{-11}$$

(c) Through the logical reasoning as in Parts (a) and (b), it can be inferred that the probability for Part (c) is

$$\frac{\binom{13}{5} \times \binom{52-13}{8} \times \binom{8}{8} \times \binom{52-13-8}{5}}{\binom{52}{13} \times \binom{39}{13}} = 0.00000261.$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)