EE2030Linear Algebra

Homework#4

Due: 04/26/2023 10:10(Wed)

1. The floor **V** and the wall **W** are not orthogonal subspaces,because they share a nonzero vector (along the line where they meet).No planes **V** and **W** in $\mathbf{R}^3$ can be orthogonal!Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector $A\mathbf{x}$ and also $B\,\widehat{\mathbf{x}}$. Think 3 by 4 with the matrix [A B].

2. If **S** is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\mathbf{S}^\perp$? If **S** is spanned by (1,1,1), what is $\mathbf{S}^\perp$? If **S** is spanned by (1,1,1) and (1,1,-1), what is a basis for $\mathbf{S}^\perp$?

$[\quad][\quad]=0$

3. Why is each of these statements false:
(a) (1,1,1) is perpendicular to (1,1,-2) so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal spaces.
(b) The subspace spanned by (1,1,0,0,0) and (0,0,1,1,1) is the orthogonal complement of the subspace spanned by (1,-1,0,0,0) and (2,-2,3,4,-4).
(c) Two subspaces that meet only in the zero vector are orthogonal.

$P\underline{x_1}\cdot(I-P)\underline{x_2}=\underbrace{(PI-P)}_{0}\,\underline{x_1}\,\underline{x_2}$

4. (Quick and Recommend) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b}$ = (1,2,3,4) onto the column space of A.What shape is the projection matrix P and what is P?

5. If $P^2 = P$ show that $(I\text{-}P)^2 = I\text{-}P$. When P projects onto the column space of A,I-P projects onto the ____ .

6. Use $P^T = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry P_{22}.This number is $\frac{2}{6} = \frac{4}{36}+\frac{4}{36}+\frac{4}{36}$ for

$$P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

7. The average of the four times is $\widehat{t} = \frac{1}{4}(0+1+3+4) = 2$. The average of the four b's is $\widehat{b} = \frac{1}{4}(0+8+8+20) = 9$.
(a) Verify that the best line goes through the center point $(\widehat{t},\widehat{b})$=(2,9).
(b) Explain why $C+D\widehat{t} = \widehat{b}$ comes from the first equation in $A^T A\widehat{\mathbf{x}} = A^T\mathbf{b}$.

1. Let $A\underline{x} = B\underline{\hat{x}}$ $\begin{bmatrix}1&2\\1&3\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}5&4\\6&3\\5&1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \underline{0}$

$\begin{bmatrix}1&2&-5&-4\\1&3&-6&-3\\1&2&-5&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\hat{x}_1\\\hat{x}_2\end{bmatrix} = \underline{0}$, $\begin{bmatrix}1&2&-5&-4\\1&3&-6&-3\\1&2&-5&-1\end{bmatrix} \Rightarrow \begin{bmatrix}1&2&-5&-4\\0&1&-1&1\\0&0&0&3\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&-3&0\\0&1&-1&0\\0&0&0&3\end{bmatrix} \Rightarrow \begin{cases}x_1 = 3\hat{x}_1\\x_2 = \hat{x}_1\\\hat{x}_2 = 0\end{cases}$

$\begin{bmatrix}x_1\\x_2\\\hat{x}_1\\\hat{x}_2\end{bmatrix} = \hat{x}_1\begin{bmatrix}3\\1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1&2\\1&3\\1&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5&4\\6&3\\5&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\6\\5\end{bmatrix}$ #

$(1,-1,0)\ (1,0,-1)$

$\begin{vmatrix}-1&0\\0&-1\end{vmatrix}, \begin{vmatrix}0&1\\-1&1\end{vmatrix}, \begin{vmatrix}1&-1\\1&0\end{vmatrix}$

$=(1,1,1)$

2. (a) $S^\perp = \{\text{all vectors in } \mathbb{R}^3\}$ (b) $S^\perp = \left\{a\begin{bmatrix}1\\-1\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\-1\end{bmatrix} : a,b\in\mathbb{R}\right\}$

(c) $(1,1,1)\times(1,1,-1) = (-2,2,0)$, Basis of $S^\perp$: $(1,-1,0)$

3. (a) $(x+y+z=0)\cap(x+y-2z=0)$ contains $(1,-1,0)$ $\Rightarrow$ They are not orthogonal.

(b) $(0,0,1,1,1)\cdot(2,-2,3,4,-4) = 3$ $\Rightarrow$ They are not orthogonal.

(c) Let $A=\{(1,1),(0,0)\}$, $B=\{(2,1),(0,0)\}$, $(1,1)\cdot(2,1) = 3 \neq 0 \Rightarrow$ counter case

4. $A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ $A^T\underline{b} = A^TA\cdot\underline{\hat{x}}$, $A^T\underline{b} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $P\underline{b} = \underline{p} \Rightarrow P = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$

$A^T\cdot A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}\hat{x}_1\\\hat{x}_2\\\hat{x}_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\underline{p} = \begin{bmatrix}1\\2\\3\\0\end{bmatrix}$,

5. $(I-P)^2 = I^2 - IP - PI + P^2 = I - P - P + P = I - P_{\#}$

$P\underline{x}\,(I-P)\underline{y} = (PI-P)\,\underline{x}\cdot\underline{y} = 0\cdot\underline{x}\cdot\underline{y} = 0 \Rightarrow$ orthogonal $\Rightarrow (I-P)$ projects onto $N(A^T)$

6. $P^T = P \Rightarrow P^TP = P^2 = P$

$P_{22} = \sum_{i=1}^{3} P_{2i}\cdot P_{i2} = \sum_{i=1}^{3} P_{i2}^2 \Rightarrow$ the length squared of column 2.

7. (a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} \neq \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$

$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$, $\begin{cases} C = 1 \\ D = 4 \end{cases}$, $1 + 4t = b$, $1 + 4\hat{t} = 1 + 8 = 9 = \hat{b}$

(b) $A^T \cdot A\hat{\underline{x}} = A^T\underline{b} \Rightarrow A^T \cdot A\begin{bmatrix} C \\ D \end{bmatrix} = A^T\underline{b} \Rightarrow \begin{bmatrix} 4 & \sum_{i=1}^{4} t_i \\ \sum_{i=1}^{4} t_i & \sum_{i=0}^{4} t_i^2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} b_i \\ \sum_{i=1}^{4} t_i b_i \end{bmatrix}$

$$\begin{cases} 4C + \sum_{i=1}^{4} t_i D = \sum_{i=1}^{4} b_i \\ \sum_{i=1}^{4} t_i C + \sum_{i=1}^{4} (t_i^2) D = \sum_{i=1}^{4} c_i t_i \end{cases} \Rightarrow C + \hat{t}D = \hat{b}$$

8. This problem projects $\mathbf{b} = (b_1, \ldots, b_m)$ onto the line through $\mathbf{a} = (1,\ldots,1)$. We solve m equations $\mathbf{a}x=\mathbf{b}$ in 1 unknown (by least squares).
(a) Solve $a^T a\widehat{x} = a^T\mathbf{b}$ to show that $\widehat{x}$ is the mean (the average) of the b's.
(b) Find $\mathbf{e} = \mathbf{b} - \mathbf{a}\widehat{x}$ and the variance $\|\mathbf{e}\|^2$ and the standard deviation $\|\mathbf{e}\|$.
(c) The horizontal line $\widehat{\mathbf{b}} = 3$ is closest to $\mathbf{b} = (1,2,6)$. Check that $\mathbf{p} = (3,3,3)$ is perpendicular to $\mathbf{e}$ and find the 3 by 3 projection matrix P.

9. Find the best line $C+Dt$ to fit b = 4,2,-1,0,0 at times t = -2,-1,0,1,2.

10. Find orthogonal vectors **A**,**B**,**C** by Gram-Schmidt from **a**, **b**,**c**:

$$\mathbf{a} = (1,-1,0,0)\ \mathbf{b} = (0,1,-1,0)\ \mathbf{c} = (0,0,1,-1)$$

A,**B**,**C** and **a**,**b**,**c** are bases for the vectors perpendicular to $\mathbf{d} = (1,1,1,1)$.

11. (a) Find a basis for the subspaces $\mathbf{S}$ in $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0$$

(b) Find a basis for the orthogonal complement $\mathbf{S}^\perp$.
(c) Find $\mathbf{b}_1$ in $\mathbf{S}$ and $\mathbf{b}_2$ in $\mathbf{S}^\perp$ so that $\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b} = (1,1,1,1)$

12. (a) Choose c so that Q is an orthogonal matrix:

$$\frac{a \cdot a^T}{a^T \cdot a}\underline{b} \qquad Q = c\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Projects $\mathbf{b}$ = (1,1,1,1) onto the first column. Then project $\mathbf{b}$ onto the plane of the first two columns.

$$\begin{matrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{matrix}$$

$$1\ -1\ -1\ -1 \quad \begin{matrix} 1 \\ -1 \\ -1 \\ -1 \end{matrix} \quad \frac{\quad}{4}$$

$$\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} \qquad a^T \cdot a = \frac{1}{4}$$

$$\begin{matrix} 1 \\ -1 \\ -1 \\ -1 \end{matrix} \quad 1\ -1\ -1\ -1 \qquad a \cdot a^T = \frac{1}{16}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

8. (a) $a^T \cdot a = m$, $m\hat{x} = \sum_{i=1}^{m} b_i$, $\hat{x} = \dfrac{\sum_{i=1}^{m} b_i}{m}$ $\Rightarrow$ $\hat{x}$ is the mean of b's.

Det $\overline{b}$ = b's mean

(b) $\underline{e} = \underline{b} - \underline{a}\hat{x} = (b_1 - \overline{b}, b_2 - \overline{b}, \cdots, b_m - \overline{b})$

$\|\underline{e}\|^2 = \sum_{i=1}^{m} (b_i - \overline{b})^2$, $\|e\| = \sqrt{\sum_{i=1}^{m} (b_i - \overline{b})^2}$

(c) $\hat{x} = \dfrac{1+2+6}{3} = 3$, $(3,3,3)\cdot(\underline{b} - \underline{p}) = (3,3,3)(-2,-1,3) = 0 = \underline{p}\cdot\underline{e}$

$$P = \frac{a \cdot a^T}{a^T \cdot a} = \frac{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}{3} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \neq \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} , \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} , \begin{cases} C = 1 \\ D = \frac{3}{5} \end{cases}$$

best line : $b = 1 + \frac{3}{5}t$

10.

$A = a = (1, -1, 0, 0)$

$B = b - \frac{1}{2}(A^T b) A = \left(\frac{1}{2}, \frac{1}{2}, -1, 0\right)$

$C = c - \frac{2}{3}(B^T c) B - \frac{1}{2}(A^T c) \cdot A = C + \frac{2}{3} B = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1\right)$

11. $x_1 + x_2 + x_3 - x_4 = 0$

(a)

$$x_1 = -x_2 - x_3 + x_4 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis : $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b)

$$y \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [y_1 \ y_2 \ y_3 \ y_4] \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \ 0 \ 0] \quad \begin{cases} y_1 = y_2 \\ y_1 = y_3 \\ y_1 = -y_4 \end{cases}$$

$\Rightarrow S^\perp$ basis : $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

(c) $A\underline{b} = A\underline{br} + A\underline{bn} = A\underline{br}$

$$A\underline{b} = [1\ 1\ 1\ -1]\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 2$$

$$A\underline{br} = [1\ 1\ 1\ -1]\ \underline{br} = 2\ ,\ \underline{br} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = \underline{b_2}\ ,\quad \underline{bn} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\end{bmatrix} = \underline{b_1}$$

12. $$Q^T \cdot Q = I = c^2 \cdot \begin{bmatrix}1 & -1 & -1 & -1\\-1 & 1 & -1 & -1\\-1 & -1 & 1 & -1\\-1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & -1\\-1 & 1 & -1 & -1\\-1 & -1 & 1 & -1\\-1 & -1 & -1 & 1\end{bmatrix} = c^2\begin{bmatrix}4 & 0 & 0 & 0\\0 & 4 & 0 & 0\\0 & 0 & 4 & 0\\0 & 0 & 0 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \Rightarrow c = \frac{1}{2}$$

Project $\underline{b} = (1,1,1,1)$ onto the first column:

$$\underline{p} = \frac{\underline{a} \cdot \underline{a}^T}{\underline{a}^T \cdot \underline{a}}\,\underline{b} = \frac{1}{4}\begin{bmatrix}1 & -1 & -1 & -1\\-1 & 1 & 1 & 1\\-1 & 1 & 1 & 1\\-1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-2\\2\\2\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

Project $\underline{b}$ onto the plane of the first two column:

$$\text{let } A = \frac{1}{4}\begin{bmatrix}1 & -1\\-1 & 1\\-1 & -1\\-1 & -1\end{bmatrix},\ A^T A = \frac{1}{4}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ A\underline{b} = \frac{1}{4}\begin{bmatrix}-2\\-2\end{bmatrix},\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\underline{\hat{x}} = \begin{bmatrix}-2\\-2\end{bmatrix} \Rightarrow \underline{\hat{x}} = \begin{bmatrix}-2\\-2\end{bmatrix}$$

$$\underline{p} = A\underline{\hat{x}} = \frac{1}{4}\begin{bmatrix}0\\0\\4\\4\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$