Signals and Systems

Homework 6 — Due : Apr. 5 2024

Problem 1 (25 pts)**.** A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $N = 13$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1}^* = j, \quad a_2 = a_{-2} = -1, \quad \text{and} \quad a_5 = a_{-5}^* = e^{5j}.$$

Express $x(t)$ in the form $x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$.

Problem 2 (30 pts)**.** For the continuous-time periodic signal

$$x(t) = 2 + \sin\left(\frac{5}{3}\pi t\right) + 2\cos\left(\frac{8}{3}\pi t\right),$$

determine the *fundamental frequency* ω_0 and the *Fourier series coefficients* a_k such that $x(t) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.

Problem 3 (45 pts)**.** Determine the Fourier series representations for the following signals:

(a) $x(t)$ periodic with period 2 and $x(t) = \begin{cases} 3, & 0 \le t < 1 \\ -3, & 1 \le t < 2 \end{cases}$.

(b) $x(t)$ periodic with period 4 and $x(t) = t$ for $-2 \le t < 2$.

(c) $x(t)$ periodic with period 2 and $x(t) = \delta(t) - 3\delta(t-1)$ for $0 \le t < 2$.

Problem 1 (25 pts). A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $N = 13$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1}^* = j, \quad a_2 = a_{-2} = -1, \quad \text{and} \quad a_5 = a_{-5}^* = e^{5j}.$$

Express $x(t)$ in the form $x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_1 \cdot e^{j\frac{2\pi}{13}t} + a_{-1} e^{-j\frac{2\pi}{13}t} + a_2 e^{j\frac{4\pi}{13}t} + a_{-2} e^{-j\frac{4\pi}{13}t} + a_5 e^{j\frac{10}{13}\pi t} + a_{-5} e^{-j\frac{10}{13}\pi t}$$

$$= 2j^2 \sin\left(\frac{2\pi}{13}t\right) - 2\cos\left(\frac{4\pi}{13}t\right) + 2\cos\left(\frac{10}{13}\pi t + 5\right)$$

$$= -2\cos\left(\frac{2\pi}{13}t - \frac{\pi}{2}\right) - 2\cos\left(\frac{4\pi}{13}t\right) + 2\cos\left(\frac{10}{13}\pi t + 5\right)$$

Problem 2 (30 pts). For the continuous-time periodic signal

$$x(t) = 2 + \sin\left(\frac{5}{3}\pi t\right) + 2\cos\left(\frac{8}{3}\pi t\right),$$

determine the *fundamental frequency* ω_0 and the *Fourier series coefficients* a_k such that $x(t) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.

$$x(t) = 2 + \frac{e^{j\frac{5}{3}\pi t} - e^{-j\frac{5}{3}\pi t}}{2j} + e^{j\frac{8}{3}\pi t} + e^{-j\frac{8}{3}\pi t}$$

$$= 2 + \frac{1}{2j}e^{j5\cdot\frac{2\pi}{6}t} - \frac{1}{2j}e^{-j5\cdot\frac{2\pi}{6}t} + e^{j8\cdot\frac{2\pi}{6}t} + e^{-j8\frac{2\pi}{6}t}$$

Fundamental period is 6 $\Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$ $[\frac{6}{5}, \frac{3}{4}] \Rightarrow 6$

$$a_k = \begin{cases} 2, & k=0 \\ \frac{1}{2j}, & k=5 \\ -\frac{1}{2j}, & k=-5 \\ 1, & k=8 \text{ or } -8 \\ 0, & \text{else} \end{cases}$$

Problem 3 (45 pts). Determine the Fourier series representations for the following signals:

(a) $x(t)$ periodic with period 2 and $x(t) = \begin{cases} 3, & 0 \le t < 1 \\ -3, & 1 \le t < 2 \end{cases}$.

(b) $x(t)$ periodic with period 4 and $x(t) = t$ for $-2 \le t < 2$.

(c) $x(t)$ periodic with period 2 and $x(t) = \delta(t) - 3\delta(t-1)$ for $0 \le t < 2$.

(a) $$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\pi t}$$

$$a_k = \frac{1}{2}\int_0^2 x(t)\cdot e^{-jk\pi t}\,dt$$

$$= \frac{3}{2}\left(\int_0^1 e^{-jk\pi t}\,dt - \int_1^2 e^{-jk\pi t}\,dt\right)$$

$$= \frac{3}{-2jk\pi}\left(e^{-jk\pi} - 1 - e^{-2jk\pi} + e^{-jk\pi}\right)$$

$$= \frac{3}{2jk\pi}\left(e^{-jk\pi} - 1\right)^2 \quad \text{for } k \neq 0$$

$$a_0 = \frac{1}{2}\int_0^2 x(t)\,dt = 0$$

(b) $$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\frac{\pi}{2}t}$$

$$a_k = \frac{1}{4}\int_4 x(t)\cdot e^{-jk\frac{\pi}{2}t}\,dt$$

$$= \frac{1}{4}\int_{-2}^{2} t\, e^{-jk\frac{\pi}{2}t}\,dt$$

$$= \frac{1}{4}\left[\left(\frac{1}{-jk\frac{\pi}{2}}\, t\, e^{-jk\frac{\pi}{2}t}\right)\Big|_{-2}^{2} - \frac{1}{-jk\frac{\pi}{2}}\int_{-2}^{2} e^{-jk\frac{\pi}{2}t}\,dt\right]$$

$$= \frac{1}{4}\left[\left(\frac{2}{-jk\frac{\pi}{2}} e^{-jk\pi} - \frac{2}{jk\frac{\pi}{2}} e^{jk\pi}\right) - \frac{1}{(jk\frac{\pi}{2})^2}\left(e^{-jk\pi} - e^{jk\pi}\right)\right]$$

$$= \left(\frac{1}{-jk\pi} - \frac{1}{4(jk\frac{\pi}{2})^2}\right)e^{-jk\pi} + \left(\frac{1}{-jk\pi} + \frac{1}{4(jk\frac{\pi}{2})^2}\right)e^{jk\pi}$$

$$= \frac{-jk\pi - 1}{(jk\pi)^2}e^{-jk\pi} + \frac{-jk\pi + 1}{(jk\pi)^2}e^{jk\pi} \quad \text{for } k \neq 0$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} x(t)\,dt = \frac{1}{4}\int_{-2}^{2} t\,dt = \frac{1}{8}\left(t^2\right)\Big|_{-2}^{2} = 0$$

(c) $x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\pi t}$

$$a_k = \frac{1}{2}\int_{2} x(t) e^{-jk\pi t} dt$$

$$= \frac{1}{2}\int_{2} (\delta(t) - 3\delta(t-1)) e^{-jk\pi t} dt$$

$$= \frac{1}{2}\left(\int_0^2 \delta(t) dt - 3e^{-jk\pi}\int_0^2 \delta(t-1) dt\right)$$

$$= \frac{1}{2} - \frac{3}{2} e^{-jk\pi} \quad \text{for } k \neq 0$$

$$a_0 = \frac{1}{2}\int_0^2 x(t) dt$$

$$= \frac{1}{2}\left(\int_0^2 \delta(t) dt - \int_0^2 3\delta(t-1) dt\right)$$

$$= -1$$