EE 203001 SES # 4

The idea of Elimination

A systematic way to solve lin. eqns

Recall our Ex:

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \Rightarrow \boxed{\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}} \begin{aligned} &(\text{eqn } 1 \times 3) \\ &(\text{substract to eliminate } 3x) \end{aligned}$$

(upper triangular)

($8y = 8 \Rightarrow y = 1$, plug in eqn 1

$x = 2y + 1 = 2 \cdot 1 + 1 = 3$)

(back substitution)

To eliminate x : substract a multiple of eqn 1 from eqn 2

(so that the system becomes triangular)

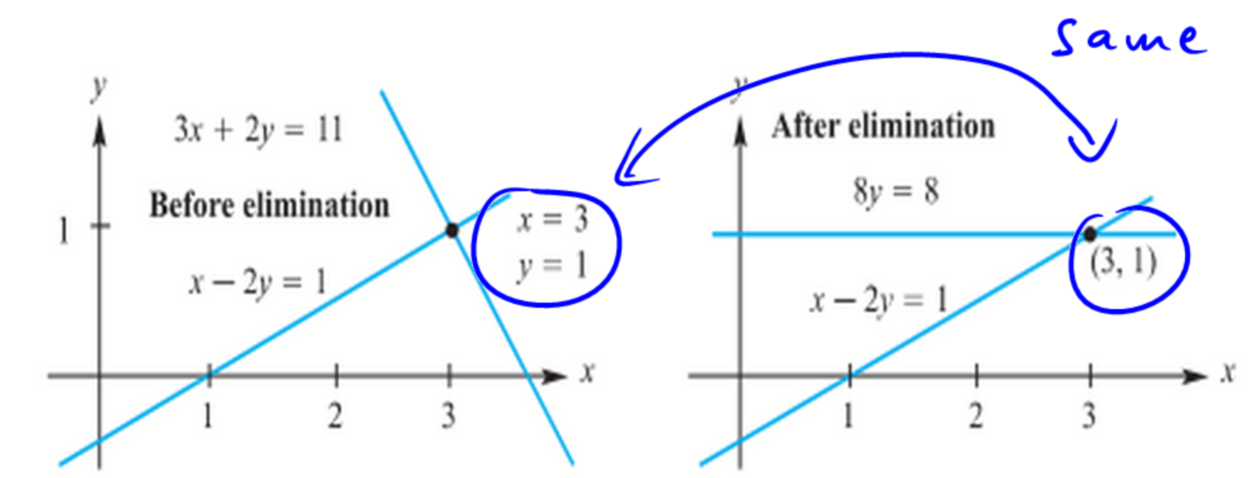

Figure 15: Eliminating x makes the second line horizontal. Then $8y = 8$ gives $y = 1$.

Q: How do you find the multiplier $l = 3$?

(first pivot)

$$\begin{aligned} 1x - 2y &= 1 \\ \underset{\sim}{3}x + 2y &= 11 \end{aligned} \Rightarrow \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

to eliminate $3x \Rightarrow l = 3/1 = 3$

$$\begin{aligned} 4x - 8y &= 4 \\ \underset{\sim}{3}x + 2y &= 11 \end{aligned} \Rightarrow \begin{aligned} 4x - 8y &= 4 \\ 8y &= 8 \end{aligned}$$

to eliminate $3x \Rightarrow l = 3/4$

Break down of Elimination

Break down when zero in pivot

$\therefore$ no sol. / too many sol.s

Ex 1:

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned} \Rightarrow \begin{aligned} x - 2y &= 1 \\ \underset{\sim}{0}y &= 8 \end{aligned}$$

2^{nd} pivot is zero $\Rightarrow$ fail!

(no sol. $\because 0y = 0 \neq 8$)

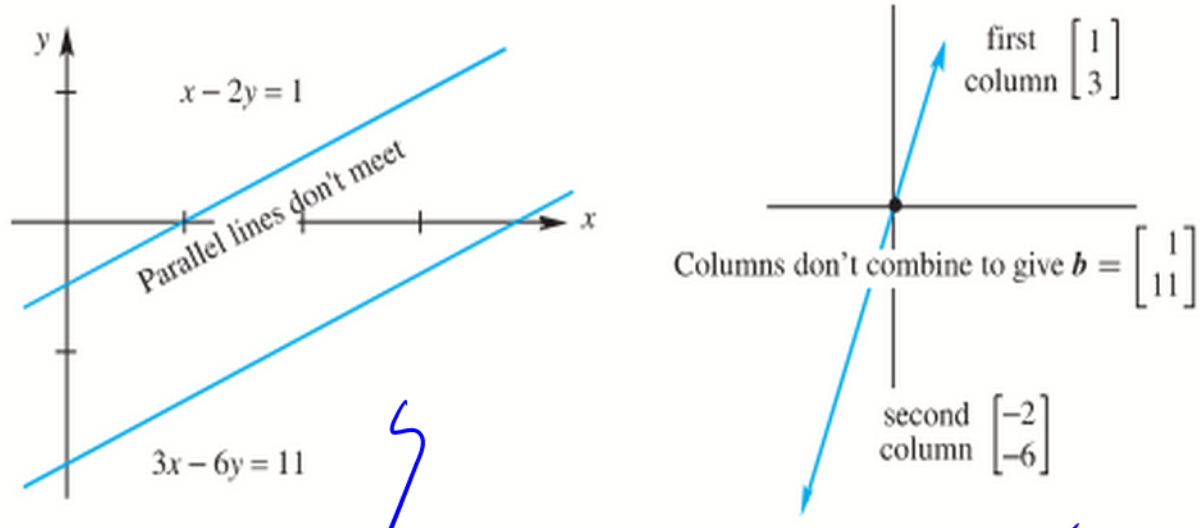

Figure 16: Row picture and column picture for Example 1: ***no solution***.

Row picture:

two lines are parallel

$\Rightarrow$ can never meet!

Col. picture: $\begin{bmatrix}1\\3\end{bmatrix}$ $\begin{bmatrix}-2\\-6\end{bmatrix}$ on the same line $\Rightarrow$ all comb. form a line

But $\begin{bmatrix}1\\11\end{bmatrix}$ in diff. direction $\Rightarrow$ no comb. can produce $\begin{bmatrix}1\\11\end{bmatrix}$

Ex2: change $\underline{b} = (1, 11)$ to $(1, 3)$

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x - 2y &= 1 \\ 0y &= 0 \end{aligned}$$

zero in pivot $\Rightarrow$ fail!

unknown y is free

infinitely many sol.s

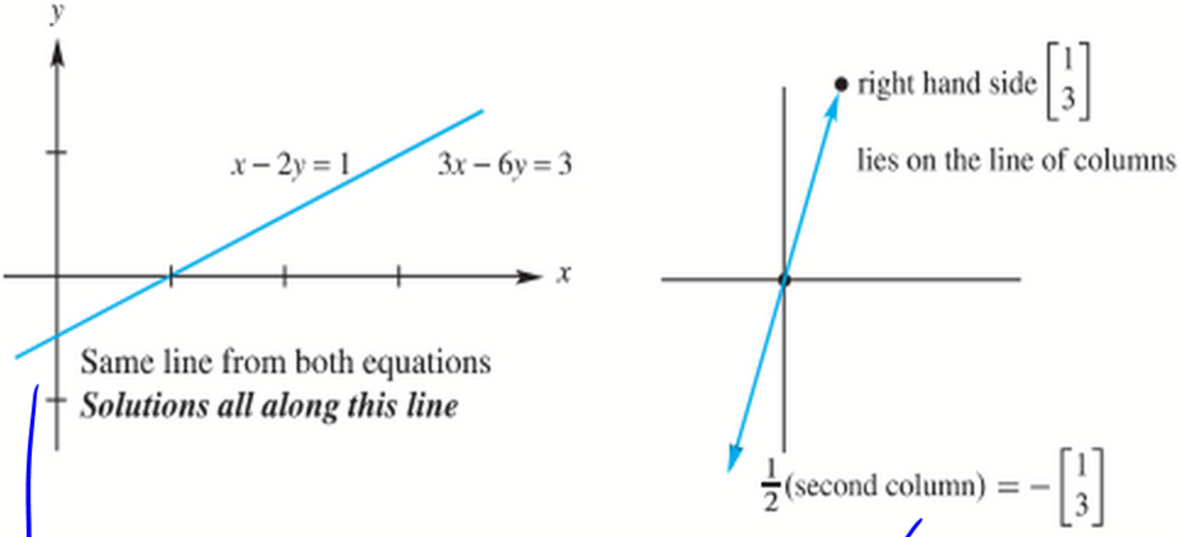

Figure 17: Row and column pictures for Example 2: *infinitely many solutions.*

Row picture:
a whole line of sol.s

Col. picture: $\underline{b} = (1,3)$ same as col. 1
so we can choose $x=1,\ y=0$, or $x=0,\ y=-\frac{1}{2}$

In general

For n eqns, we don't get n pivots

$\Rightarrow$ Failure

$\Rightarrow$ Elimination leads to $\begin{cases} \underline{0} \neq \underline{0} & \text{(no sol.)} \\ \underline{0} = 0 & \text{(many sol.)} \end{cases}$

Success comes with n pivots but we may have to exchange the n eqns.

Ex 3: temporary failure but a row exchange fixes it

0 in first pivot

$$0x + 2y = 4$$
$$3x - 2y = 5$$
$$\Rightarrow \quad 3x - 2y = 5$$
$$2y = 4$$

(exchange two equs)

(both row & col. picture are normal but a row exchange is required)

Three equs in three unknowns

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8 \qquad A\underline{x} = \underline{b}$$
$$-2x - 3y + 7z = 10$$

$$\Downarrow$$

$$2x + 4y - 2z = 2$$
$$1y + 1z = 4 \qquad U\underline{x} = \underline{c}$$
$$4z = 8$$

(upper triangular)

(hidden in the original system)

By back substitution,

$$4z = 8 \Rightarrow z = 2, \quad y + z = 4 \Rightarrow y = 2$$

$$2x + 4y - 2z = 2 \Rightarrow x = -1$$

In general

- Use 1^{st} eqn to create zeros below 1^{st} pivot
- Use 2^{nd} " " " " 2^{nd}
- keep going to find all n pivots and the triangular matrix U

(multiplier $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$)