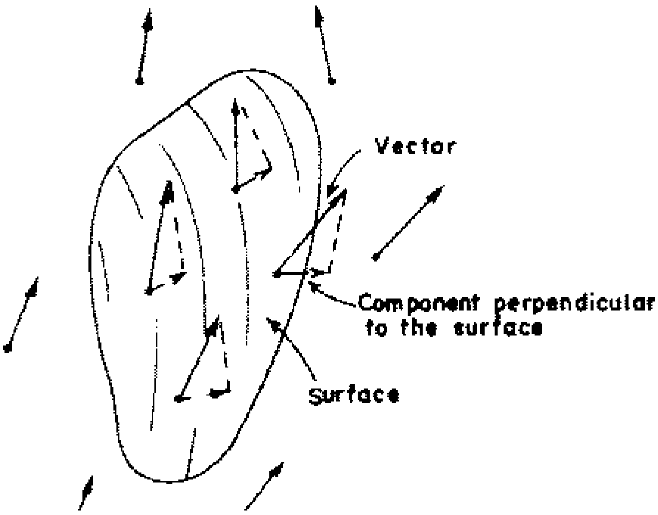
Chapter 2 Review on Vector Algebra and Vector Calculus

A *scalar* is specified by the magnitude of a numerical value, whereas a *vector* contains the information about the magnitude and the direction of a quantity. Vector analysis has to be invoked to properly describe the physical quantities in electromagnetics. For example, two important concepts, ***flux*** and ***circulation***, frequently occur in Electromagnetics.

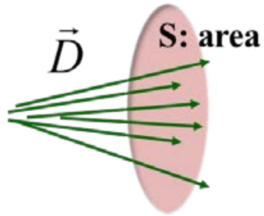
Flux: (average surface normal component)·(surface area)

In plain text, a flux is the total "amount" of a vector quantity going out of a surface area. In math, flux is the surface integral of a vector normal to a surface.



E.g. Water flux ∝ strength of water source

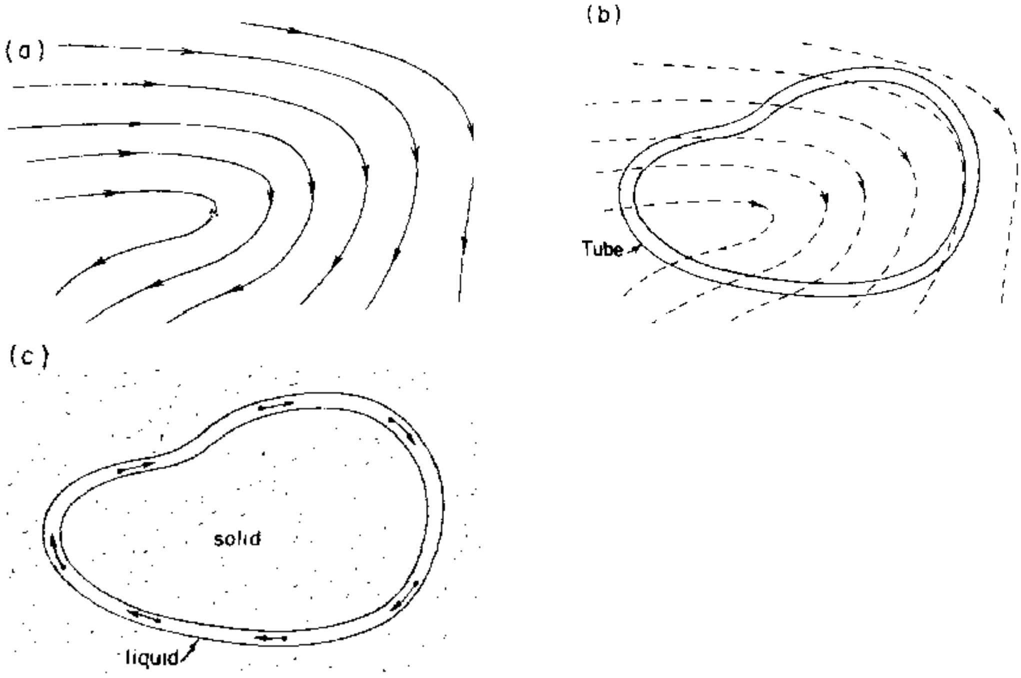
Electric Flux Φ_e

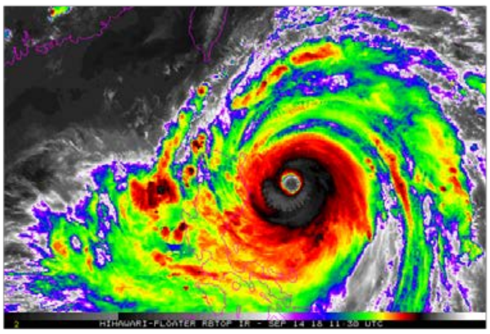


Circulation: (average tangential component)·(closed path)

In plain text, circulation is the strength of a vector quantity going around a closed path. In math, circulation is the line integral of a vector along a closes path.

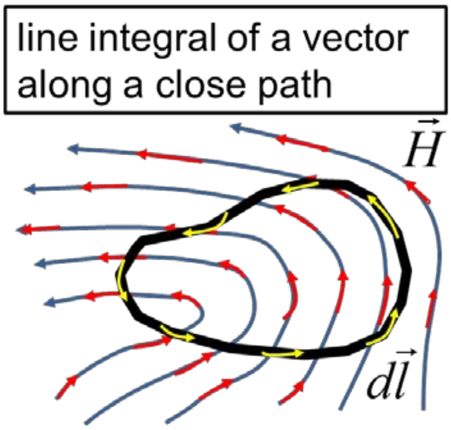
In the following we review the basic concept of vector algebra and calculus pertaining to the Cartesian, cylindrical, and spherical coordinates.





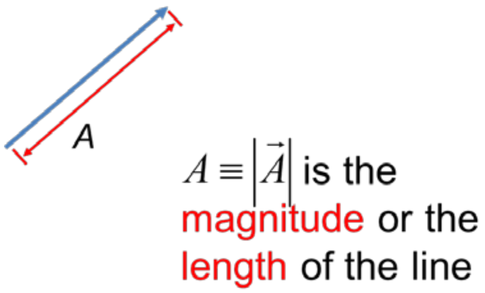
資料來源

zh.wikipedia.org/wiki/颱風山竹_(2018年)

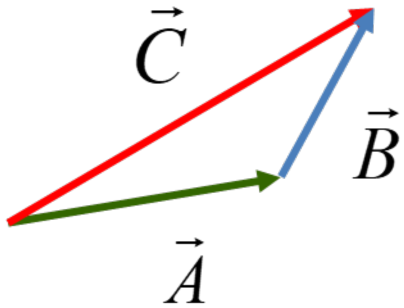


Basic Vector Algebra

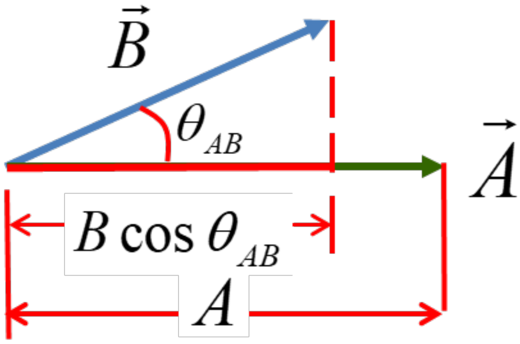
A Vector $\vec{A} = A\hat{a}_A$ has a magnitude of *A* pointing to the direction of $\hat{a}_A$ where $\hat{a}_A$ is a unit vector. A unit vector has a magnitude of 1.



Vector Sum $\vec{C} = \vec{A} + \vec{B}$, can be obtained by a head-to-tail rule shown below

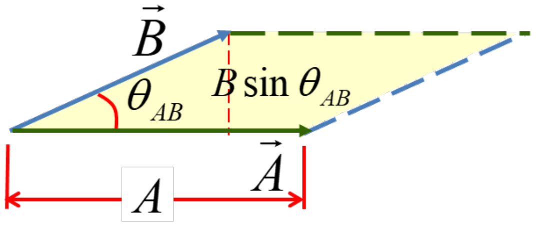


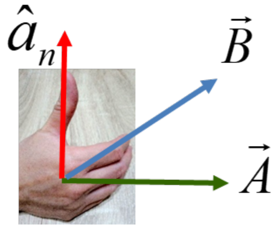
Scalar or Dot Product $\vec{A} \cdot \vec{B} = AB\cos\theta_{AB}$: multiplication of the projection of $\vec{B}$ along $\hat{a}_A$ and *A*.



Thus $A = \sqrt{\vec{A} \cdot \vec{A}}$ and $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (communicative)

Vector or Cross Product $\vec{A}\times\vec{B} = AB\sin\theta_{AB}\hat{a}_n$





Note:

1. $\left|\vec{A}\times\vec{B}\right| = AB\sin\theta_{AB}$ is equal to the area of the parallelogram shown above.

2. The direction of the unit vector $\hat{a}_n$ follows the so-called right-hand rule, with which you curl your right four fingers from the direction $\vec{A}$ toward $\vec{B}$ to find $\hat{a}_n$ along the thumb direction. Therefore, the direction of the surface area formed by $\vec{A}\times\vec{B}$ is normal to the surface or perpendicular to $\vec{A}$ and $\vec{B}$.

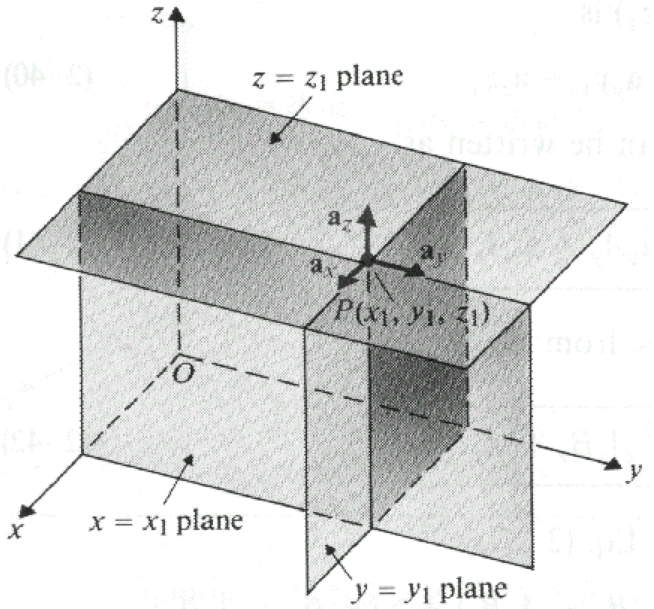
3. $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (anti-commutative, surface areas are the same, but sense of direction is reversed)

4. $(\vec{A}\times\vec{B})\times\vec{C} \neq \vec{A}\times(\vec{B}\times\vec{C})$

5. $(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}$ is the volume of the parallelepiped formed by the three vectors $\vec{A},\vec{B},\vec{C}$.

Orthogonal Coordinate Systems

Cartesian Coordinates (x, y, z): suitable for problems with rectangular symmetry. The three coordinates are within the range of $-\infty \le x \le \infty$, $-\infty \le y \le \infty$, $-\infty \le z \le \infty$.



A general expression of a vector:

$$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

The three unit vectors, $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are arranged such that $\hat{a}_x \times \hat{a}_y = \hat{a}_z$, $\hat{a}_y \times \hat{a}_z = \hat{a}_x$, $\hat{a}_z \times \hat{a}_x = \hat{a}_y$, $\hat{a}_x \cdot \hat{a}_y = 0$, $\hat{a}_y \cdot \hat{a}_z = 0$, $\hat{a}_z \cdot \hat{a}_x = 0$

A differential length:

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz = d\vec{l}_x + d\vec{l}_y + d\vec{l}_z$$

A differential surface:

$$\begin{aligned} d\vec{s} &= \hat{a}_x ds_x + \hat{a}_y ds_y + \hat{a}_z ds_z \\ &= d\vec{l}_y \times d\vec{l}_z + d\vec{l}_z \times d\vec{l}_x + d\vec{l}_x \times d\vec{l}_y , \end{aligned}$$

where $ds_x = \left| d\vec{l}_y \times d\vec{l}_z \right| = dydz$

$$ds_y = \left| d\vec{l}_z \times d\vec{l}_x \right| = dzdx$$

$$ds_z = \left| d\vec{l}_x \times d\vec{l}_y \right| = dxdy$$

*Note that a surface has a direction normal to the surface.

A differential volume:

$$dv = dxdydz = (d\vec{l}_x \times d\vec{l}_y) \cdot d\vec{l}_z = (d\vec{l}_y \times d\vec{l}_z) \cdot d\vec{l}_x = (d\vec{l}_z \times d\vec{l}_x) \cdot d\vec{l}_y$$

Given $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$ and $\vec{B} = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z$

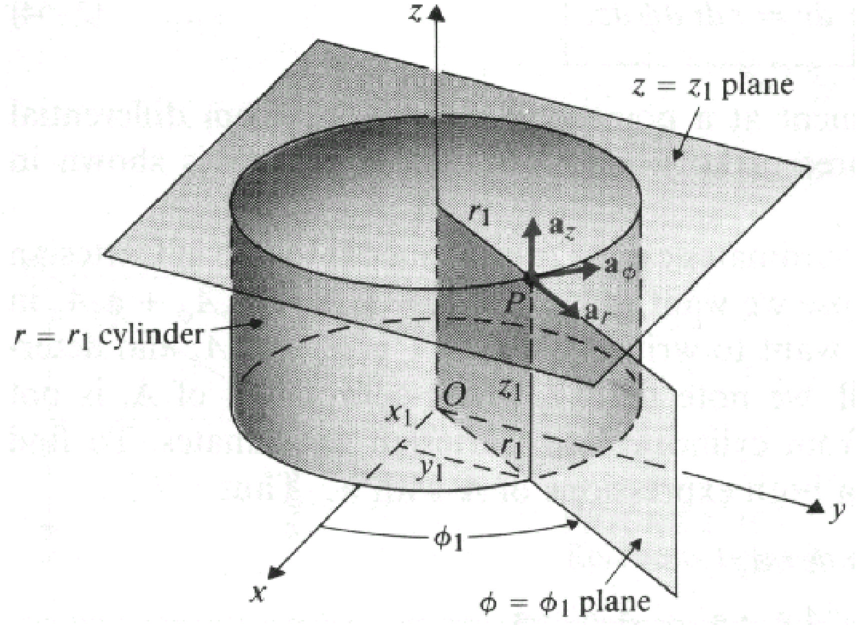
Vector scalar product:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Vector cross product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cylindrical Coordinates (r, ϕ, z): suitable for problems with cylindrical symmetry.



Radius distance: $0 \le r \le \infty$, azimuthal angle: $0 \le \phi < 2\pi$, longitudinal coordinate: $-\infty \le z \le \infty$. A general expression of a vector:

$$\vec{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z$$

The three unit vectors, $\hat{a}_r, \hat{a}_\phi, \hat{a}_z$ are arranged such that $\hat{a}_r \times \hat{a}_\phi = \hat{a}_z$, $\hat{a}_\phi \times \hat{a}_z = \hat{a}_r$, $\hat{a}_z \times \hat{a}_r = \hat{a}_\phi$, $\hat{a}_r \cdot \hat{a}_\phi = 0$, $\hat{a}_\phi \cdot \hat{a}_z = 0$, $\hat{a}_z \cdot \hat{a}_r = 0$

A differential length:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz = d\vec{l}_r + d\vec{l}_\phi + d\vec{l}_z$$

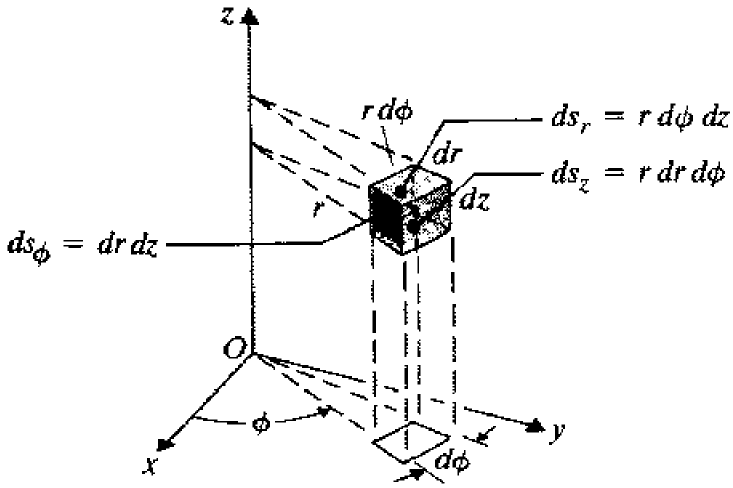
A differential surface:

$$d\vec{s} = \hat{a}_r ds_r + \hat{a}_\phi ds_\phi + \hat{a}_z ds_z$$
$$= d\vec{l}_\phi \times d\vec{l}_z + d\vec{l}_z \times d\vec{l}_r + d\vec{l}_r \times d\vec{l}_\phi ,$$

where $ds_r = rd\phi dz, ds_\phi = drdz,\ ds_z = rdrd\phi$.

A differential volume:

$$dv = rdrd\phi dz = (d\vec{l}_r \times d\vec{l}_\phi) \cdot d\vec{l}_z = (d\vec{l}_\phi \times d\vec{l}_z) \cdot d\vec{l}_r = (d\vec{l}_z \times d\vec{l}_r) \cdot d\vec{l}_\phi$$



Given $\vec{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ and $\vec{B} = \hat{a}_r B_r + \hat{a}_\phi B_\phi + \hat{a}_z B_z$

Vector scalar product:

$$\vec{A} \cdot \vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

Vector cross product:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

Coordinate Transformation between the Cartesian and Cylindrical coordinates

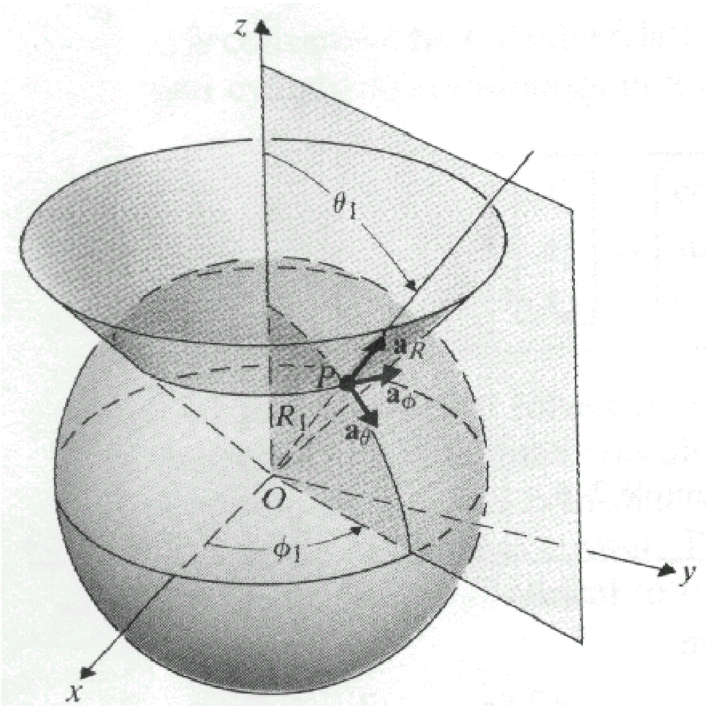
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

and

$$x = r\cos\phi,\ y = r\sin\phi,\ z = z$$

$$r = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x}, z = z$$

Spherical Coordinates (R, θ, ϕ): suitable for problems with spherical symmetry. Radius distance: $0 \le R \le \infty$, , colatitude angle: $0 \le \theta \le \pi$, azimuthal angle: $0 \le \phi < 2\pi$.



The three unit vectors, $\hat{a}_R, \hat{a}_\theta, \hat{a}_\phi$ are arranged such that $\hat{a}_R \times \hat{a}_\theta = \hat{a}_\phi$, $\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_R$, $\hat{a}_\phi \times \hat{a}_R = \hat{a}_\theta$, $\hat{a}_R \cdot \hat{a}_\theta = 0$,

$\hat{a}_\theta \cdot \hat{a}_\phi = 0$, $\hat{a}_\phi \cdot \hat{a}_R = 0$

A general expression of a vector

$$\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

A differential length:

$$d\vec{l} = \hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R \sin\theta d\phi = d\vec{l}_R + d\vec{l}_\theta + d\vec{l}_\phi$$ A

differential surface:

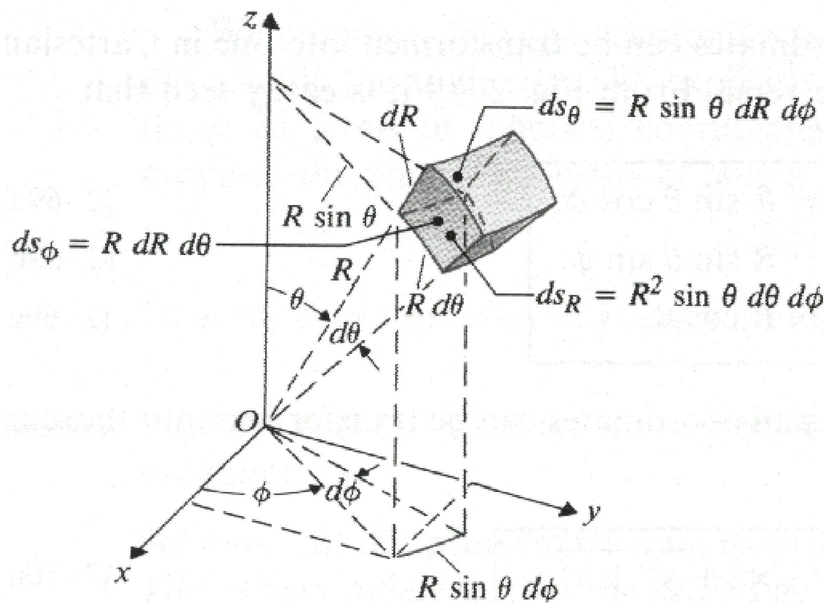
$$d\vec{s} = d\vec{s}_R + d\vec{s}_\theta + d\vec{s}_\phi ,$$

where

$$ds_R = R^2 \sin\theta d\theta d\phi, ds_\theta = R \sin\theta dR d\phi, \; ds_\phi = R dR d\theta$$

A differential volume:

$$dv = R^2 \sin\theta dR d\theta d\phi$$



Given $\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$ and $\vec{B} = \hat{a}_R B_R + \hat{a}_\theta B_\theta + \hat{a}_\phi B_\phi$

Vector dot product:

$$\vec{A}\cdot\vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

Vector cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_R & \hat{a}_\theta & \hat{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\varphi \end{vmatrix}$$

Coordinate Transformation between the Cartesian and spherical coordinates

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$x = R\sin\theta\cos\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\theta$$

$$R = \sqrt{x^2 + y^2 + z^2},\ \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

General Form

In an orthogonal coordinate system, the three unit vectors can be expressed by $\hat{a}_{u_1}, \hat{a}_{u_2}, \hat{a}_{u_3}$ are arranged such that $\hat{a}_{u_1}\times\hat{a}_{u_2} = \hat{a}_{u_3}$, $\hat{a}_{u_2}\times\hat{a}_{u_3} = \hat{a}_{u_1}$, $\hat{a}_{u_3}\times\hat{a}_{u_1} = \hat{a}_{u_2}$, $\hat{a}_{u_i}\cdot\hat{a}_{u_j} = 0$ for $i, j = 1,2,3$

A general expression of a vector

$$\vec{A} = \hat{a}_{u_1} A_{u_1} + \hat{a}_{u_2} A_{u_2} + \hat{a}_{u_3} A_{u_3}$$

A differential length:

$$d\vec{l} = \hat{a}_{u_1} h_1 du_1 + \hat{a}_{u_2} h_2 du_2 + \hat{a}_{u_3} h_3 du_3 = d\vec{l}_{u_1} + d\vec{l}_{u_2} + d\vec{l}_{u_3}$$

where h_1, h_2, h_3 are called *metric coefficients*.

A differential surface:

$$d\vec{s} = d\vec{s}_{u_1} + d\vec{s}_{u_2} + d\vec{s}_{u_3} \ ,$$

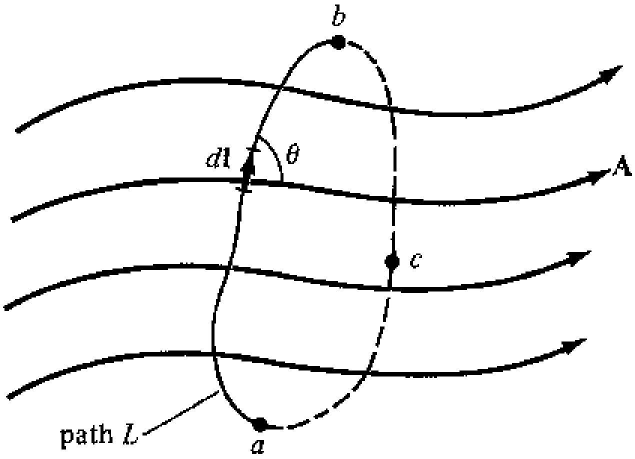
coordinate system relations	Cartesian Coor. (x, y, z)	Cylindrical Coor. (r,ϕ,z)	Spherical Coor. (R,θ,ϕ)
Base vectors			
$\hat{a}_{u_1}$	$\hat{a}_x$	$\hat{a}_r$	$\hat{a}_R$
$\hat{a}_{u_2}$	$\hat{a}_y$	$\hat{a}_\phi$	$\hat{a}_\theta$
$\hat{a}_{u_3}$	$\hat{a}_z$	$\hat{a}_z$	$\hat{a}_\phi$
Metric coefficient			
h_1	1	1	1
h_2	1	r	R
h_3	1	1	$R\sin\theta$

Vector Calculus

Line Integral

$\int_L \vec{F} \cdot d\vec{l} = \int_a^b F\cos\theta dl$: the integral of vector $\vec{F}$ along path *L*.

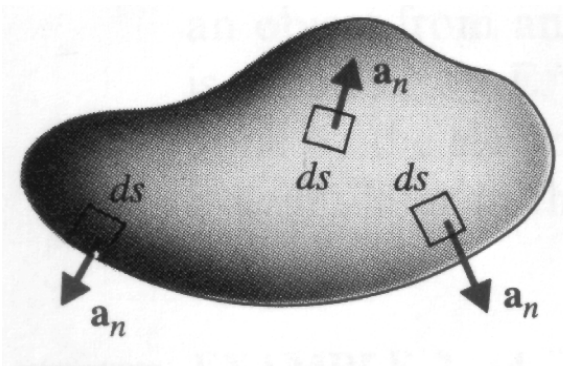
$\oint_L \vec{F} \cdot d\vec{l}$: integral of $\vec{F}$ along a closed path *L*. Note that a closed path defines a surface.



Surface Integral

$\int_s \vec{A}\cdot d\vec{s} = \int_S A\cos\theta ds = \int_S \vec{A}\cdot \hat{a}_n ds$: the integral of $\vec{A}$ across an open surface *S*, or the *flux* of $\vec{A}$ through *S*.

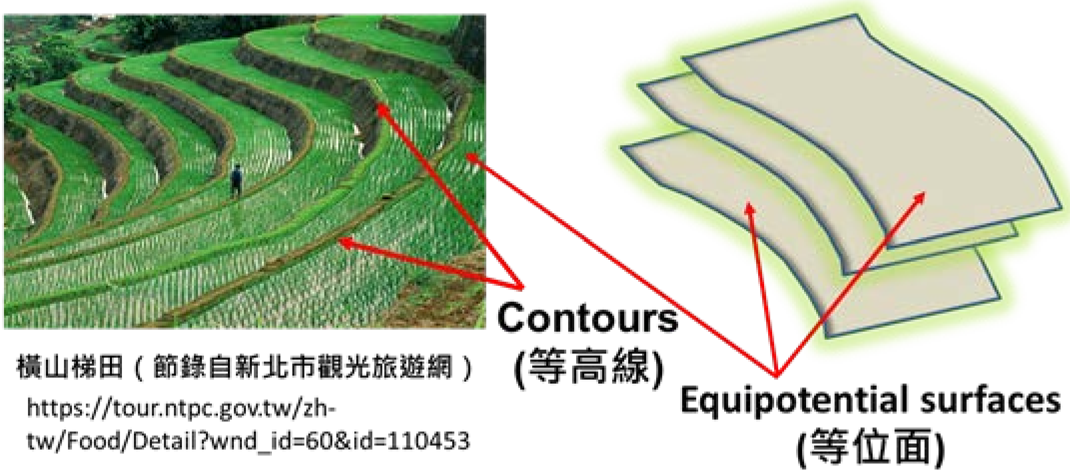
$\oint_S \vec{A}\cdot d\vec{s}$: the integral of $\vec{A}$ across a closed surface *S*. Note that a closed surface defines a volume. The direction of a closed surface is defined outward a volume.



Volume Integral

$\int_V \rho_v dv$: the integral of the function ρ_v over a volume *V*.

Gradient of a Scalar : a vector having a magnitude equal to the maximum rate of change of a scalar in space, and a direction along the maximum change.



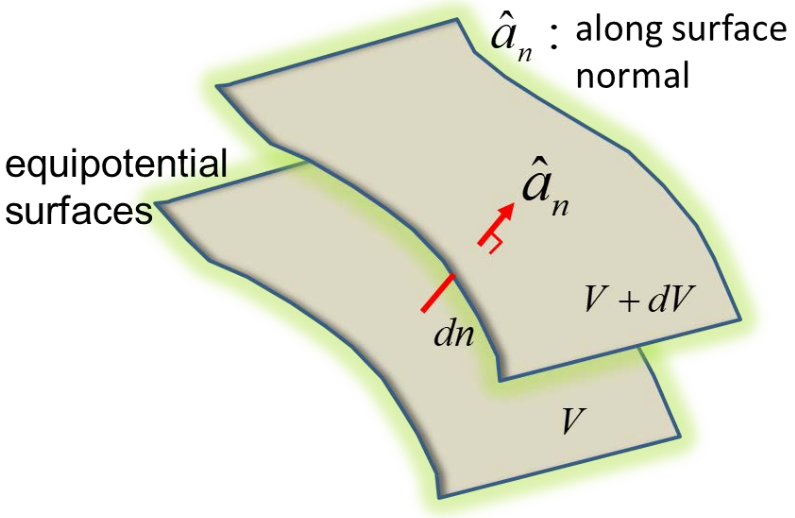
Consider the directional derivative of a scalar function V along an arbitrary direction $\hat{a}_l$

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{a}_n \cdot \hat{a}_l \, ,$$

where $\hat{a}_n$ is a unit vector normal to the V = constant surface. Refer to the following plot, $\hat{a}_n \cdot \hat{a}_l = \cos\alpha$ has a value less than 1. Therefore $dV/dn \geq dV/dl$ $\Rightarrow$ the gradient of a scalar can be defined as

$$\nabla V \equiv \hat{a}_n \frac{dV}{dn} \, ,$$

and the directional derivative can be rewritten as $\dfrac{dV}{dl} = (\nabla V) \cdot \hat{a}_l$



It follows that $dV = (\nabla V)\cdot d\vec{l}$ or

$$dV = \frac{\partial V}{\partial l_{u_1}} dl_{u_1} + \frac{\partial V}{\partial l_{u_2}} dl_{u_2} + \frac{\partial V}{\partial l_{u_3}} dl_{u_3}$$

$$= (\frac{\partial V}{\partial l_{u_1}}\hat{a}_{u_1} + \frac{\partial V}{\partial l_{u_2}}\hat{a}_{u_2} + \frac{\partial V}{\partial l_{u_3}}\hat{a}_{u_3})\cdot(dl_{u_1}\hat{a}_{u_1} + dl_{u_2}\hat{a}_{u_2} + dl_{u_3}\hat{a}_{u_3})$$

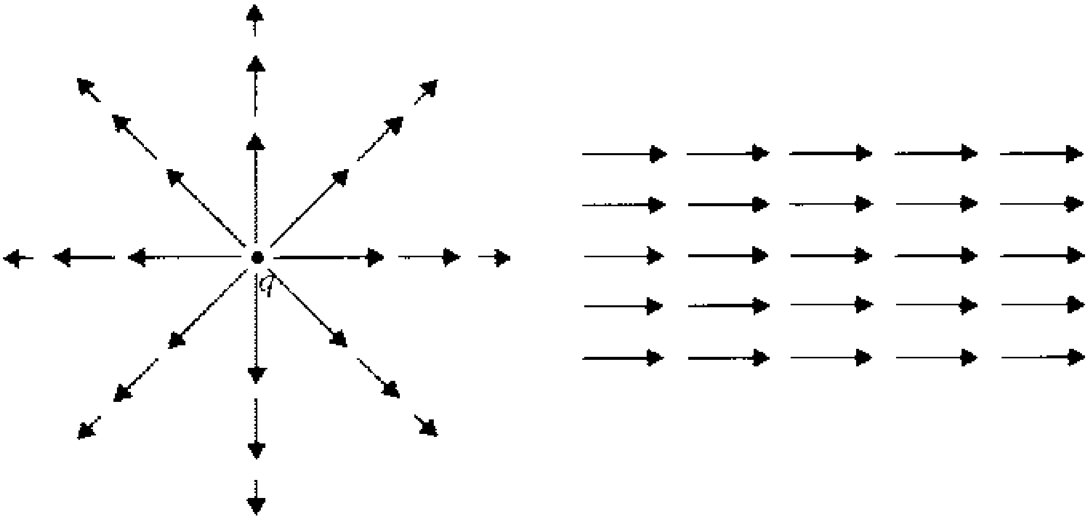
$$= (\frac{\partial V}{\partial l_{u_1}}\hat{a}_{u_1} + \frac{\partial V}{\partial l_{u_2}}\hat{a}_{u_2} + \frac{\partial V}{\partial l_{u_3}}\hat{a}_{u_3})\cdot d\vec{l}$$

$$\Rightarrow \quad \nabla V = (\frac{\partial V}{\partial l_{u_1}}\hat{a}_{u_1} + \frac{\partial V}{\partial l_{u_2}}\hat{a}_{u_2} + \frac{\partial V}{\partial l_{u_3}}\hat{a}_{u_3})$$

In the Cartesian coordinate system, the gradient of a scalar V can be found to be

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

Divergence of a Vector Field

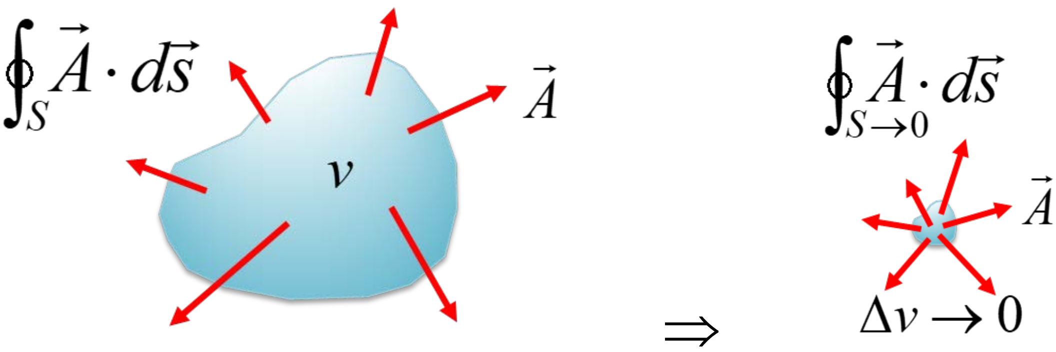


(a) (b)

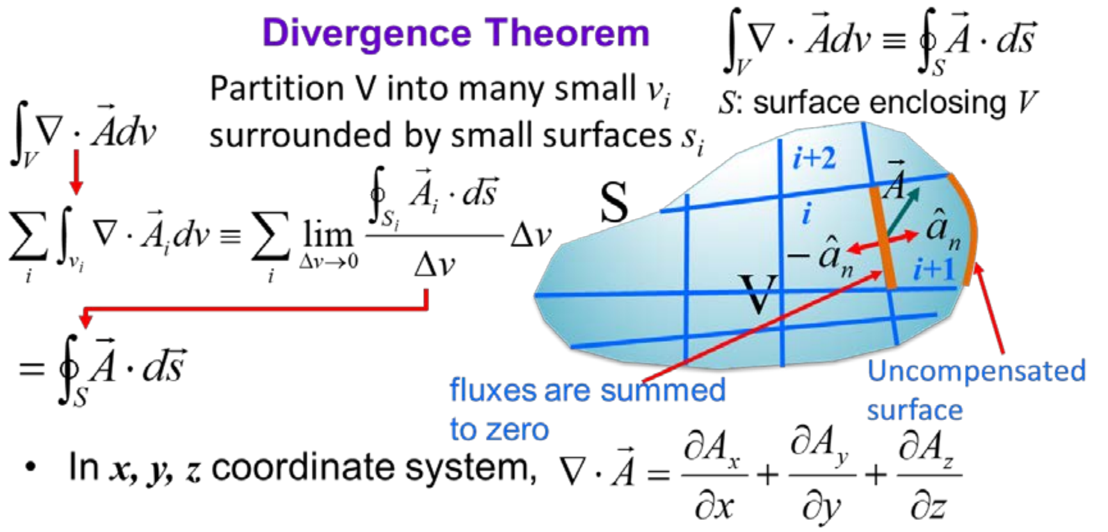
(a) A net outward flux surrounding q.

(b) An uniform flux of fields to the right.

Divergence of $\vec{A}$: $\nabla\cdot\vec{A}\equiv\lim_{\Delta v\to 0}\dfrac{\oint_S \vec{A}\cdot d\vec{s}}{\Delta v}$ is a scalar equal to the net outward flux of $\vec{A}$ per unit volume at a point in space. Therefore $\nabla\cdot\vec{A}$ is a point function that describes the aforementioned physical quantity at a point location.



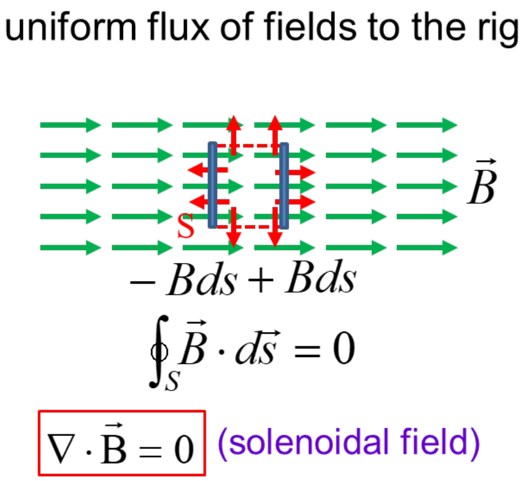
$\Rightarrow$ ***Divergence Theorem*** $\int_V \nabla\cdot\vec{A}dv \equiv \oint_S \vec{A}\cdot d\vec{s}$



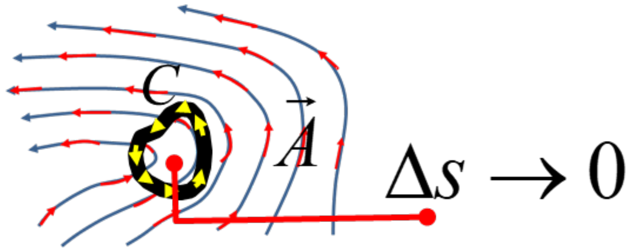
In the Cartesian coordinate system, the divergence of a vector $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$ is expressed by

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

For a vector satisfying $\nabla\cdot\vec{B}=0$, vector $\vec{B}$ is said to be solenoidal.

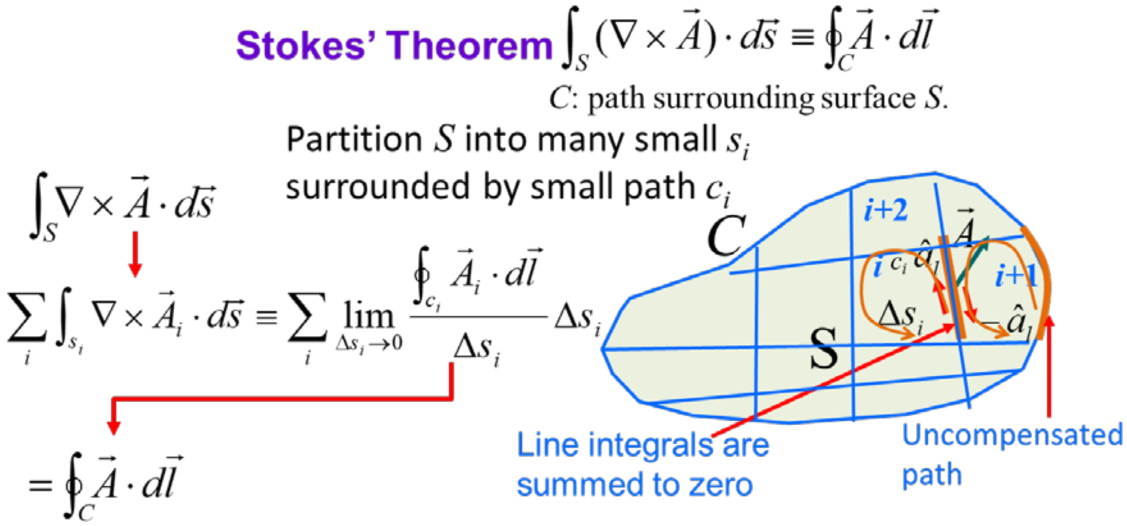


Curl of a Vector Field



$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{A} \cdot d\vec{l}}{\Delta s}$: *maximum* net circulation of $\vec{A}$ per unit area at a point in space. The direction of $\nabla \times \vec{A}$ is chosen to be the surface normal direction of the infinitesimal area Δs with which the net circulation is a maximum. Therefore $\nabla \times \vec{A}$ is also a point function that describes the aforementioned physical quantity at a point location.

$\Rightarrow$ ***Stoke's Theorem*** $\int_S (\nabla \times \vec{A}) \cdot d\vec{s} \equiv \oint_C \vec{A} \cdot d\vec{l}$, where C is the path surrounding the surface S.



For a vector satisfying $\nabla \times \vec{A} = 0$, vector $\vec{A}$ is said to be irrotational.

In the Cartesian coordinate system, the curl of a vector

$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$ is expressed by

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\hat{a}_x + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\hat{a}_y + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\hat{a}_z$$

Laplacian operator of a scalar field $\nabla^2 V \equiv \nabla \cdot (\nabla V)$

In the *xy*z coordinate system, the expression is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian operator of a vector field

$$\nabla^2 \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$$

In the *xy*z coordinate system, the expression is given by

$$\nabla^2 \vec{A} = \hat{a}_x \nabla^2 A_x + \hat{a}_y \nabla^2 A_y + \hat{a}_z \nabla^2 A_z$$

Two Null Identities

$\nabla \times (\nabla V) = 0$, no net circulation around a vector normal to an equipotential surface.

$\nabla \cdot (\nabla \times \vec{A}) = 0$, no net outward flux around the maximum circulation of a vector.

A Quick Reference

$$\nabla V = \hat{a}_{u_1}\frac{\partial V}{h_1\partial u_1} + \hat{a}_{u_2}\frac{\partial V}{h_2\partial u_2} + \hat{a}_{u_3}\frac{\partial V}{h_3\partial u_3}$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{a}_{u_1}h_1 & \hat{a}_{u_2}h_2 & \hat{a}_{u_3}h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

***Helmholtz's Theorem*:**

A vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere.

A similar statement is "if two vector fields have the same values for their curl, divergence, and surface dot products at a boundary, the two vectors fields are the same within an additive constant."

In mathematics, the *Helmholtz's Theorem* is equivalent to: If

(i) $\nabla\cdot\vec{A} = \nabla\cdot\vec{B}$, (ii) $\nabla\times\vec{A} = \nabla\times\vec{B}$, and (iii) $\vec{A}\cdot d\vec{s} = \vec{B}\cdot d\vec{s}$

on the surface surrounding the volume in question, then $\vec{A} = \vec{B} +$ a constant vector.

Lemma: Green's first identity

The divergence theory, $\int_V \nabla\cdot\vec{G}dv = \oint_S \vec{G}\cdot d\vec{s}$, where $\vec{G}$ is any vector field in space. Define $\vec{G} = \phi\nabla\psi$ and substitute into $\int_V \nabla\cdot\vec{A}dv = \oint_S \vec{A}\cdot d\vec{s}$, with $\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$. One has

$\int_V (\nabla\phi\cdot\nabla\psi+\phi\nabla^2\psi)dv=\oint_S \phi\nabla\psi\cdot d\vec{s}$, which is called *Green's first identity*. For the special case $\phi=\psi$, Green's first identity becomes

$$\int_V \left|\nabla\phi\right|^2 dv+\int_V \phi\nabla^2\phi dv=\oint_S \phi\nabla\phi\cdot d\vec{s} \qquad (*)$$

Proof of Helmholtz's Theorem:

Define $\vec{C}=\vec{A}-\vec{B}$.

a. $\nabla\times\vec{C}=\nabla\times\vec{A}-\nabla\times\vec{B}=0 \Rightarrow \vec{C}=-\nabla\phi$ (the minus sign is immaterial)

b. $\nabla\cdot\vec{C}=\nabla\cdot\vec{A}-\nabla\cdot\vec{B}=0 \qquad \Rightarrow \nabla\cdot\vec{C}=-\nabla^2\phi=0$

c. $\vec{C}\cdot d\vec{s}=\vec{A}\cdot d\vec{s}-\vec{B}\cdot d\vec{s}=0 \Rightarrow \vec{C}\cdot d\vec{s}=-\nabla\phi\cdot d\vec{s}=0$

Substitute *a*-*c* into (*) $\Rightarrow \int_V \left|\nabla\phi\right|^2 dv=0$. But $\left|\nabla\phi\right|^2>0$ and therefore $\vec{C}=\vec{A}-\vec{B}=-\nabla\phi=0 \Rightarrow \vec{A}=\vec{B}$. Note that if one adds a constant to $\vec{C}=-\nabla\phi$, it does not change the final answer. We thus prove the Helmholtz theorem. A simple statement for the theorem is that there exits a unique solution for the vector field in a space if its divergence, curl, and boundary values are specified uniquely. Therefore in electromagnetics the divergence and curl of a field are specified by so-called Maxwell's equations and the boundary condition of a field is specified in a real-world problem.

The following two are extracted from S. Ramo, J. Whinnery, and T. van Duzer, Fields and Waves I Communication Electronics, John Wiley & Sons.

VECTOR DIFFERENTIAL OPERATIONS

Rectangular Coordinates (x, y, z)

$$\nabla\Phi = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla\times\mathbf{H} = \hat{\mathbf{x}}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z$$

Cylindrical Coordinates (r, φ, z)

$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\mathbf{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$\nabla\times\mathbf{H} = \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right] + \hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\phi}\right]$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{r}}\left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi} - \frac{A_r}{r^2}\right) + \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial\phi} - \frac{A_\phi}{r^2}\right) + \hat{\mathbf{z}}(\nabla^2 A_z)$$

Spherical Coordinates (r, θ, φ)

$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{H} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$\nabla^2\mathbf{A} = \hat{\mathbf{r}}\left[\nabla^2 A_r - \frac{2}{r^2}\left(A_r + \cot\theta A_\theta + \csc\theta\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_\theta}{\partial\theta}\right)\right] + \hat{\boldsymbol{\theta}}\left[\nabla^2 A_\theta - \frac{1}{r^2}\left(\csc^2\theta A_\theta - 2\frac{\partial A_r}{\partial\theta} + 2\cot\theta\csc\theta\frac{\partial A_\phi}{\partial\phi}\right)\right] + \hat{\boldsymbol{\phi}}\left[\nabla^2 A_\phi - \frac{1}{r^2}\left(\csc^2\theta A_\phi - 2\csc\theta\frac{\partial A_r}{\partial\phi} - 2\cot\theta\csc\theta\frac{\partial A_\theta}{\partial\phi}\right)\right]$$

VECTOR FORMULAS

$$\nabla(\Phi + \psi) = \nabla\Phi + \nabla\psi$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla(\Phi\psi) = \Phi\nabla\psi + \psi\nabla\Phi$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

$$\nabla \times (\Phi\mathbf{A}) = \nabla\Phi \times \mathbf{A} + \Phi\nabla \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla\Phi = 0$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$