Solutions of Exam #2

Signals and Systems

11020 EECS 202001

1

$$x(t) = e^{-\left(\frac{t}{t_0}\right)^2} \tag{1}$$

Fourier Transform is:

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-\left(\frac{t}{t_0}\right)^2} e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{t_0^2}\left(t^2+j\omega t_0^2 t+(\frac{1}{2}j\omega t_0^2)^2-(\frac{1}{2}j\omega t_0^2)^2\right)}dt \\
&= e^{(\frac{1}{2}j\omega t_0)^2}\int_{-\infty}^{\infty} e^{-\frac{1}{t_0^2}\left(t^2+j\omega t_0^2 t\right)}dt \\
X(j\omega) &= t_0\sqrt{\pi}e^{-\frac{1}{4}\omega^2 t_0^2}
\end{aligned} \tag{2}$$

where integral of Gaussian is $t_0\pi$. **Note:** Integration of a Gaussian Function: Let

$$I = \int_{-\infty}^{\infty} e^{-ax^2}dx \qquad \text{and} \qquad I^2 = \int_{-\infty}^{\infty} dxe^{-ax^2}\int_{-\infty}^{\infty} dye^{-ay^2} \tag{3}$$

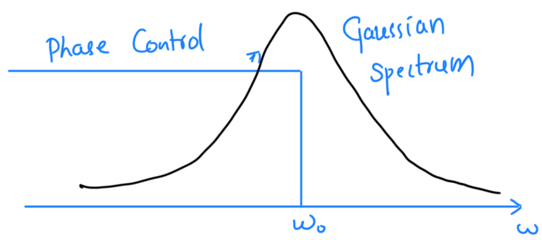
To find out the I^2:

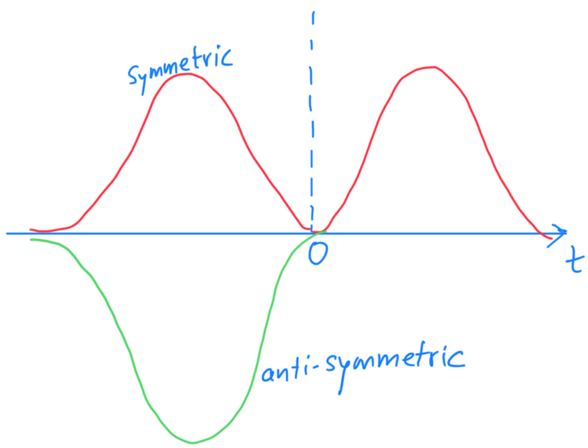
$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdye^{-a(x^2+y^2)} \\
\text{Changing into radial coordinates} \Longrightarrow I^2 &= 2\pi\int_{0}^{+\infty} rdre^{-ar^2} \\
&= \pi\int_{0}^{+\infty} d(r^2)e^{-ar^2}
\end{aligned} \tag{4}$$

$$I^2 = \frac{\pi}{a} \qquad \Longrightarrow \qquad I = \frac{\sqrt{\pi}}{a} \tag{5}$$

2

Gaussian spectrum with spectral phase control applied in the picture.





The time domain signal can be obtained by inverse s transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega t} d\omega \tag{6}$$

At $t = 0$:

$$\begin{aligned} x(0) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega . 0} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) d\omega \\ &= 0 \end{aligned} \tag{7}$$

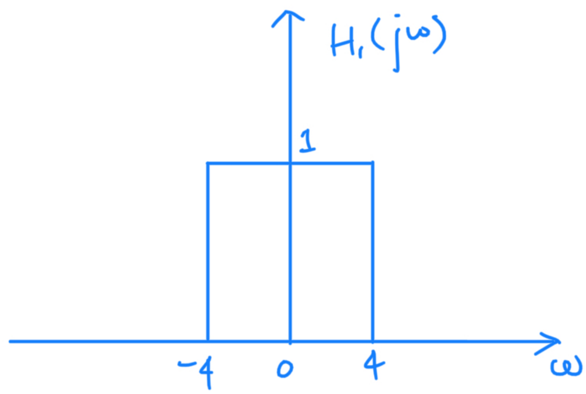
It is because the area of Gaussian over the entire interval is zero as it is symmetric about zero. The time envelop which are considered should satisfy this condition. While sketching if the absolute value is sketched then, it would have a symmetric double pulse otherwise it is antisymmetric with $x(0) = 0$.

Note: The students can choose their on time envelops. Gaussian pulse is considered as an example in the figure with magnitude would give symmetric double peaks or else anti-symmetric double pulse.

3

Because of the periodicity $h(t) = h_1(t-2)$, where

$$h_1(t) = \frac{\sin 4t}{\pi t} \tag{8}$$



$H_1(j\omega)$ is an impulse response of a low pass filter with the frequency band $|\omega_c| < 4$.
$h(t) \xrightarrow{\mathcal{F}} H(j\omega)$ is $H_1(j\omega)$ shifted by 2 to the right.

$$H(j\omega) = \begin{cases} e^{-2j\omega} & |\omega_c| < 4 \\ 0 & \text{otherwise} \end{cases}$$

when

$$x(t) = \frac{\sin 3(t+3)}{\pi(t+3)}$$

Fourier transform is given by

$$X(j\omega) = \begin{cases} e^{3j\omega} & |\omega_c| < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \begin{cases} e^{j\omega} & |\omega_c| < 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{9}$$

Taking the inverse transform,

$$y(t) = x(t+1) = \frac{\sin 3(t+1)}{\pi(t+1)} \tag{10}$$

4

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \tag{11}$$

(a)

$$\begin{aligned} H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ \implies (6 - \omega^2 + 5j\omega)Y(j\omega) &= (j\omega + 4)X(j\omega) \end{aligned} \tag{12}$$

Taking the inverse transform, we get the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t) \tag{13}$$

(b) Using partial fractions

$$H(j\omega) = \frac{2}{2 + j\omega} - \frac{1}{3 + j\omega} \tag{14}$$

Taking the inverse transform,

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t) \tag{15}$$

(c)

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t) \tag{16}$$

Fourier transform of the above equation:

$$X(j\omega) = \frac{1}{4 + j\omega} - \frac{1}{(4 + j\omega)^2} \tag{17}$$

The output spectrum

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{(j\omega + 2)(j\omega + 4)} \end{aligned} \tag{18}$$

The output can be found out by taking inverse transform. To make it easier, we rewrite **??** using partial fractions.

$$Y(j\omega) = \frac{A}{(j\omega + 2)} + \frac{B}{(j\omega + 4)}$$

A and B are found out to be $\frac{1}{2}$ and $\frac{-1}{2}$ respectively.

$$Y(j\omega) = \frac{\frac{1}{2}}{(j\omega + 2)} + \frac{\frac{-1}{2}}{(j\omega + 4)} \tag{19}$$

Taking the inverse transform, we get output as

$$y(t) = \frac{1}{2}\left[e^{-2t}u(t) - e^{-4t}u(t)\right] \tag{20}$$

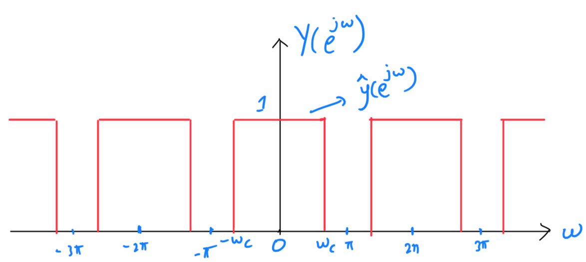
5

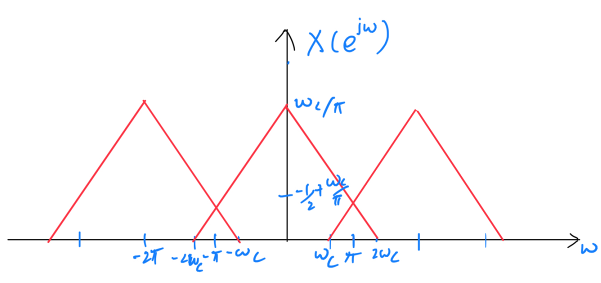
$$x[n] = \left(\frac{\sin \omega_c n}{\pi n}\right)^2 \xrightarrow{\mathcal{F}} X(e^{j\omega}) \qquad 0 < \omega_c < \pi$$

Let

$$y[n] = \left(\frac{\sin \omega_c n}{\pi n}\right) \qquad \text{and} \qquad y[n] \xrightarrow{\mathcal{F}} Y(e^{j\omega}) \tag{21}$$

The Fourier transform of $y[n]$ is $Y(e^{j\omega})$ is given in the figure below:





Note that $x[n] = y[n]y[n]$. The Fourier transform of $X(e^{j\omega})$ is

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{<2\pi>} Y(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta \tag{22}$$

Converting periodic $Y(e^{j\omega})$ into a periodic one as

$$\hat{Y}(e^{j\omega}) = \begin{cases} Y(e^{j\omega}) & -\pi < \omega_c \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Rewriting ??

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Y}(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta \tag{23}$$

This is the aperiodic convolution of period square wave $Y(e^{j\omega})$ with rectangular $\hat{Y}(e^{j\omega})$ as shown in the figure. From figure, we need $-1 + \frac{2\omega_c}{\pi} = \frac{2}{5} \implies \omega_c = \frac{7\pi}{10}$

6

Decomposing given signal into two rectangular signals of the form

$$x[n] = y[n] + y'[n]$$

First signal: Take a rectangular signal in the interval $\pm N_1$ centered at zero as in fig. a.
The fourier transform of it is

$$y[n] \xrightarrow{\mathcal{F}} \frac{\sin\left(\omega(N_1 + \frac{1}{2})\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{24}$$

The above signal is shifted to the left by one as fig. b. and $N_1 = 2$

$$y[n-1] \xrightarrow{\mathcal{F}} e^{-j\omega} \frac{\sin\left(\omega(\frac{5}{2})\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{25}$$

Time expanded by inserting zeroes to give fig. c.

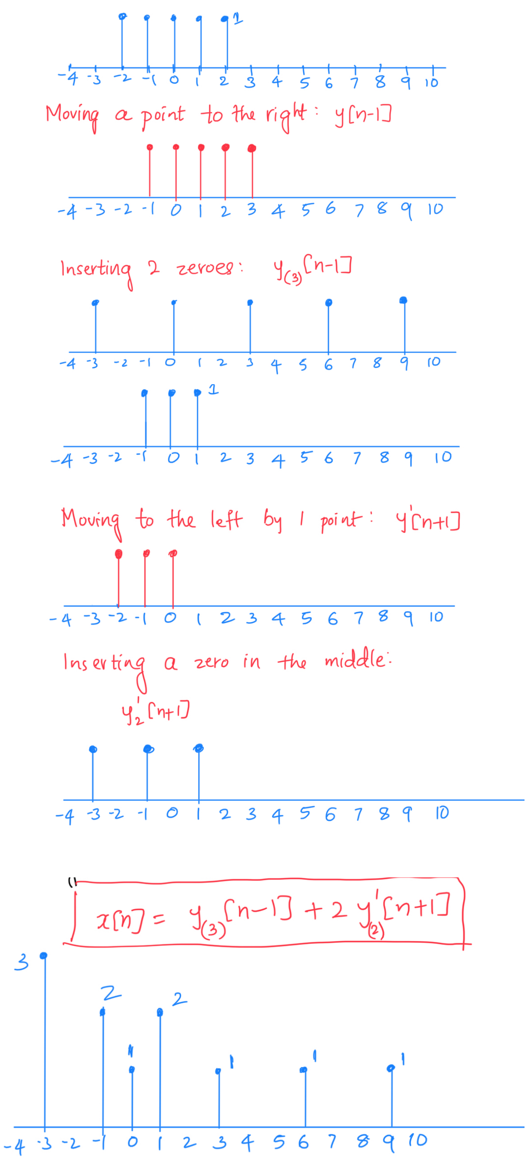
$$\begin{aligned} y_{(3)}[n-1] &\xrightarrow{\mathcal{F}} e^{-j3\omega} \epsilon^{-j\omega} \frac{\sin\left(\omega(\frac{5}{2})\right)}{\sin\left(\frac{\omega}{2}\right)} \\ &= e^{-j4\omega} \frac{\sin\left(\frac{15\omega}{2}\right)}{\sin\left(\frac{3\omega}{2}\right)} \end{aligned} \tag{26}$$

Second signal: Take a rectangular signal in the interval $\pm N_1 = 1$ centered at zero as in fig. d.
Shifted to the right by one point as shown in fig. e.

$$y'[n+1] \xrightarrow{\mathcal{F}} e^{j\omega} \frac{\sin\left(\omega(\frac{3}{2})\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{27}$$

After time expansion we have:

$$\begin{aligned} y_{(2)}[n+1] &\xrightarrow{\mathcal{F}} e^{-j2\omega} \epsilon^{j\omega} \frac{\sin(3\omega)}{\sin(\omega)} \\ &= e^{-j\omega} \frac{\sin(3\omega)}{\sin(\omega)} \end{aligned} \tag{28}$$



The given signal can be written as

$$x[n] = y_{(3)}[n-1] + 2y_{(2)}[n+1] \tag{29}$$

Taking the Fourier transform we get:

$$X(e^{j\omega}) = e^{-j4\omega}\frac{\sin\left(\frac{15\omega)}{2}\right))}{\sin\left(\frac{3\omega}{2}\right)} + 2e^{-j\omega}\frac{\sin(3\omega)}{\sin(\omega)} \tag{30}$$

7

$$H(e^{j\omega}) = e^{-2j\omega}\frac{1-\frac{1}{2}e^{j\omega}}{1+\frac{1}{2}e^{-j\omega}} \tag{31}$$

(a)

$$\begin{aligned} |H(e^{j\omega})| &= \sqrt{e^{-j2\omega\frac{(1-\frac{1}{2}e^{j\omega})}{(1+\frac{1}{2}e^{j\omega})}} \cdot e^{j2\omega\frac{(1-\frac{1}{2}e^{j\omega})*}{(1+\frac{1}{2}e^{j\omega})*}}} \\ &= \sqrt{\frac{1+\frac{1}{4}-\cos\omega}{1+\frac{1}{4}-\cos\omega}} \\ \implies |H(e^{j\omega})| &= \sqrt{\frac{5-4\cos\omega}{5+4\cos\omega}} \end{aligned} \tag{32}$$

(b)

$$\begin{aligned} \sphericalangle H(e^{j\omega}) &= \sphericalangle e^{-2j\omega} + \sphericalangle(1-\frac{1}{2}e^{j\omega}) - \sphericalangle(1+\frac{1}{2}e^{-j\omega}) \\ &= \sphericalangle e^{-2j\omega} + \sphericalangle(1-\frac{1}{2}\cos\omega - \frac{1}{2}j\sin\omega) - \sphericalangle(1+\frac{1}{2}\cos\omega - \frac{1}{2}j\sin\omega) \\ &= -2\omega + \tan^{-1}\left[\frac{-\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega}\right] - \tan^{-1}\left[\frac{-\frac{1}{2}\sin\omega}{1+\frac{1}{2}\cos\omega}\right] \end{aligned} \tag{33}$$

This equation can be further simplified using $\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a-b}{1-ab}$

(c)

$$\tau(\omega) = -\frac{d}{d\omega}(\sphericalangle H(e^{j\omega})) \tag{34}$$

From **??**:

$$\begin{aligned} \tau(\omega) &= -\frac{d}{d\omega}\Big[-2\omega + \tan^{-1}\left[\frac{-\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega}\right] - \tan^{-1}\left[\frac{-\frac{1}{2}\sin\omega}{1+\frac{1}{2}\cos\omega}\right]\Big] \\ &= 2 + \frac{1-2\cos[\omega]}{5-4\cos[\omega]} - \frac{1+2\cos[x]}{5+4\cos[x]} \end{aligned} \tag{35}$$