Chapter 7 Time-varying Fields

In a single current loop, Faraday's Law of electromagnetic induction states that the electromotive force is

$$V_{emf} = -\frac{d\Phi}{dt} \tag{7-1}$$

or

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} \tag{7-2}$$

V_{emf} is the electromotive force (*emf*) induced along a conducting loop *C*, generating a current in such a way that the current opposes the change of the magnetic flux (Lenz's law). Note that the electromotive force and the electric voltage we previously learned about differ by a negative sign due to Kirchohhoff's voltage law (resulting from $\nabla \times \vec{E} = \vec{f}$).

For loops of *N*,

$$\text{(7-1)} \quad \Rightarrow \quad V_{emf} = -\frac{d\Lambda}{dt}, \tag{7-3}$$

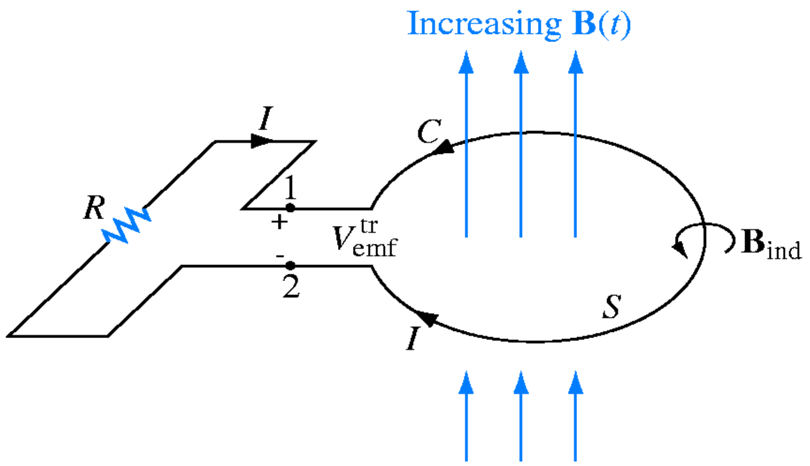
where $\Lambda = N\Phi$ is the magnetic linkage.

Transformer emf: For a stationary current loop *C*,

$$\text{(8-2)} \ \Rightarrow \quad V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \tag{7-4}$$

From the Stokes theorem, one obtains

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7-5}$$



(a) Loop in a changing **B** field

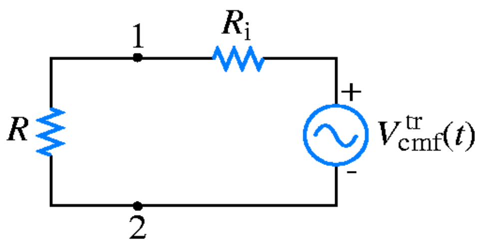


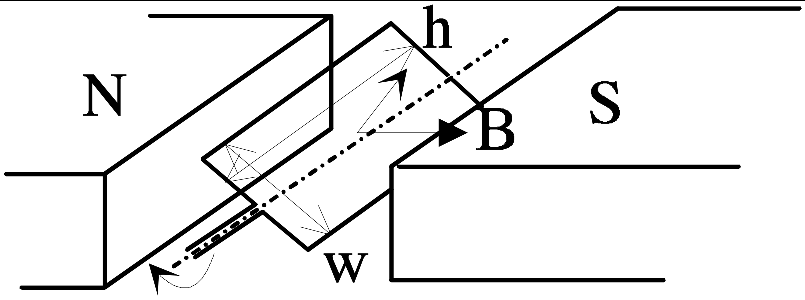
Figure 6-2 (b) Equivalent circuit

Flux Cutting (motional) emf : The electric field induced in a moving wire in a static magnetic field.

The electrons in a moving wire experience a magnetic force, or an equivalent electric field given by $\vec{E} = \dfrac{\vec{F}}{q} = \vec{u} \times \vec{B}$, where $\vec{u}$ is the velocity vector of the moving wire. The voltage induced in the moving wire is therefore

$$V = \oint_C \vec{E} \cdot d\vec{l} = \oint_C \vec{u} \times \vec{B} \cdot d\vec{l} \qquad (7\text{-}6)$$

Ex. AC generator: a rotating loop in a stationary magnetic field.



i. Solve from (7-1) $V = -\dfrac{d\Phi}{dt}$

Calculation for the magnetic flux $\Phi = \int \vec{B}\cdot d\vec{s} = whB_0 \cos\alpha$

Apply Eq. (7-1) $V = -\dfrac{d\Phi}{dt} = whB_0 \sin\alpha \dfrac{d\alpha}{dt}$, but $\alpha = \omega t$,

where ω is the angular frequency of the rotating loop.

The induced voltage is therefore

$$V = -\frac{d\Phi}{dt} = whB_0 \omega \sin \omega t \qquad (7\text{-}7)$$

This solution can't be obtained from $V = -\int_S \dfrac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$, because $\dfrac{\partial \vec{B}}{\partial t} = 0$.

ii. Solve from (7-6) $V = \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$ where the velocity vector $\vec{u}$ is expressed by $\vec{u} = \dfrac{w}{2}\omega \hat{a}_\phi$.

$$\Rightarrow\ V = \frac{w}{2}\omega B_0 \sin(\alpha)\times 2h = whB_0\omega \sin \omega t \qquad (7\text{-}8)$$

$\Rightarrow$ (7-7) = (7-8), as expected.

The last problem can't be solved from $V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$, because $\partial \vec{B} / \partial t = 0$. There must be another term (in fact, $V = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$) in $V = -\frac{d\phi}{dt}$ other than $-\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$.

Indeed, for a moving current loop *C* in a time-varying $\vec{B}$ field, the time derivative in (7-1) predicts both the transformer emf and the flux cutting emf. Mathematically it can be found that

$$V = \frac{-d\phi}{dt} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$$

$$\Leftrightarrow \qquad \oint_C \vec{E} \cdot d\vec{l} = -\int_S \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{s}$$

$$\Leftrightarrow \qquad V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \tag{7-9}$$

Therefore the electromagnetic induction consists of the transformer emf and the motional emf.

Maxwell Equations

From what we have learned so far, we write the following for electric- and magnetic-field quantities

$$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} \tag{7-10}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

According to one of the two null identities of vectors, the left hand side of Eq. (7-10) gives $\nabla \cdot (\nabla \times \vec{H}) = 0$.

However taking divergence to the right hand side of (7-10) gives

$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ according to the equation of continuity . This contradiction prompts the need for modifying Eq. (7-10). To be consistent, let's pretend adding a term "?" to the left hand side of Eq. (7-10) to achieve

$$\Rightarrow \nabla \cdot (\nabla \times \vec{H} + ?) = -\frac{\partial \rho}{\partial t}$$

The term "?" must satisfy $\nabla \cdot ? = -\frac{\partial \rho}{\partial t}$.

Recall the Gauss law $\nabla \cdot \vec{D} = \rho$

One immediately finds the assumed term "?" equal to

$$? = -\frac{\partial \vec{D}}{\partial t} = -\vec{J}_d$$

This reasoning concludes an equivalent current density from the time derivative of the *D* vector, called the *displacement current density*. Equation (7-10) is modified as

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

With the modified Eq. (7-10), we have the following set of equations governing electromagnetics in both the static and time-varying regimes. This set of equations is called Maxwell's equations.

A Complete Set of Maxwell's Equations

Differential Form Integral Form

1. Faraday's Law (7-11.a, b)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\frac{d\Phi}{dt}$$

2. Ampere's circuital law (7-12.a, b)

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \qquad \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$

3. Gauss's law (7-13.a, b)

$$\nabla \cdot \vec{D} = \rho \qquad \oint_S \vec{D} \cdot d\vec{s} = Q$$

4. no magnetic charges (7-14.a, b)

$$\nabla \cdot \vec{B} = 0 \qquad \oint_S \vec{B} \cdot d\vec{s} = 0$$

Maxwell's equations, together with the Lorentz force equation

$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$ and the equation of continuity $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$,

describe **all** the known phenomena in electromagentics.

Be reminded that in a conductor the current density and electric field intensity is related by

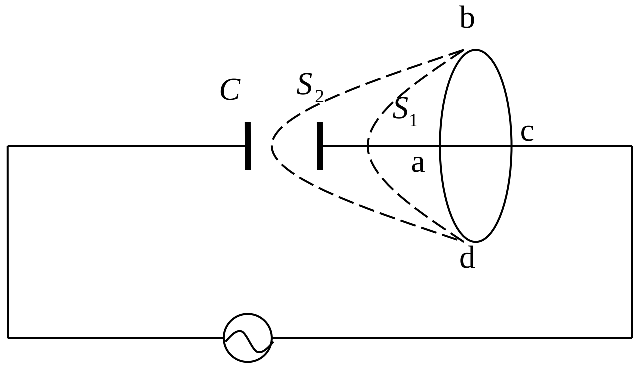
$$\vec{J} = \sigma\vec{E}\,, \qquad \text{(7-15.a, b, c)}$$

and in a simple medium (linear, isotropic), the polarization and magnetization vectors give rise to the relationships

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_0\vec{E} \quad \text{and} \quad \vec{B} = \mu\vec{H} = \mu_r\mu_0\vec{H}$$

Physical Picture of the Displacement Current

$$\frac{\partial\vec{D}}{\partial t} = \vec{J}_d$$



For a capacitor driven by an AC voltage source $V = V_0 \sin\omega t$, there's a magnetic field intensity generated along the *abcd* loop in space. The *abcd* loop can be defined by the surface S_1 intercepting a physical current density or the surface S_2 intercepting a displacement current through the capacitor *C*. According to Ampere's law

$$\oint_{a,b,c,a} \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$$

Question:

. Is $\int_{S_1} \vec{J} \cdot d\vec{s} = \int_{S_2} \vec{J}_d \cdot d\vec{s}$?

For the surface S_1

$$\int_{s_1} \vec{J} \cdot d\vec{s} = I_c = C\frac{dV}{dt} = C\frac{d(V_0 \sin \omega t)}{dt} = CV_0 \omega \cos \omega t \quad (7\text{-}16)$$

For the surface S_2

$$\int_{s_2} \vec{J}_d \cdot d\vec{s} = \int_{s_2} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} = I_d = \frac{\partial (\varepsilon V/d)}{\partial t} \cdot A = \frac{\varepsilon A}{d} \frac{\partial (V_0 \sin \omega t)}{\partial t} \quad (7\text{-}17)$$
$$= CV_0 \omega \cos \omega t$$

(the parallel-plate capacitor formula, $C = \varepsilon A / d$, has been used)

$\Rightarrow$ It turns out that (7-16) = (7-17)

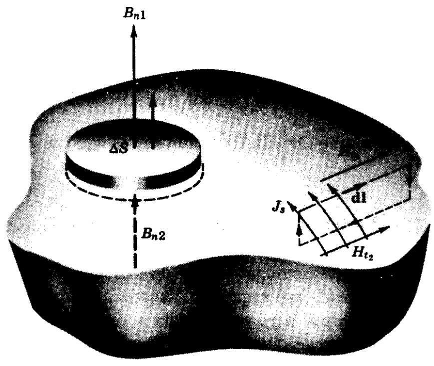
This concludes that a displacement current is equivalent to a physical current.

A straightforward way is to calculate the capacitor current under a driving voltage *V*.

$i_c = C\dfrac{dV}{dt}$, but $V = \dfrac{Dd}{\varepsilon}$ and $C = \dfrac{\varepsilon A}{d}$ at the capacitor. Therefore

$i_c = A\dfrac{dD}{dt} \Rightarrow J_c = \dfrac{dD}{dt} = J_d$.

Boundary Conditions for Time-varying Fields



(figure adopted from the reference book by Ramo, Whinnery, and van Duzer.)

1. Apply Faraday's law to the boundary $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$.

Since the surface at the boundary can arbitrarily small, the integration on the right hand side goes to zero. One obtains the boundary condition for the tangential electric field intensities

$$\Rightarrow \; E_{t1} = E_{t2} \tag{7-18}$$

2. Apply Ampere's law to the boundary $\oint_C \vec{H}\cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s}$.

Again the surface integral for the displacement current goes to zero. One obtains the boundary condition for the tangential magnetic field intensities

$$\Rightarrow \; H_{t1} - H_{t2} = J_s \quad \text{or} \quad \hat{a}_{n2}\times(\vec{H}_1 - \vec{H}_2) = \vec{J}_s \tag{7-19}$$

3. Apply Gauss's law to the boundary $\oint_S \vec{D}\cdot d\vec{s} = Q$ and obtain the boundary condition for the normal components of the electric flux density.

$$\Rightarrow \; D_{n1} - D_{n2} = \rho_s \quad \text{or} \quad \hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \qquad (7\text{-}20)$$

4. Apply $\oint_S \vec{B} \cdot d\vec{s} = 0$ to the boundary and obtain the boundary condition for the normal components of the magnetic flux density.

$$\Rightarrow \; B_{n1} - B_{n2} = 0 \quad \text{or} \quad \hat{a}_{n2,1} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \qquad (7\text{-}21)$$

For **time-varying** cases at a dielectric-conductor boundary

1. dielectric

2. perfect conductor

i. inside the conductor

$$\vec{E} = 0, \vec{D} = 0, \vec{H} = 0, \vec{B} = 0 \qquad (7\text{-}22)$$

From $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, if $\vec{E} = 0$ in a conductor, $\vec{B}$ can not be a time-varying field. Note that a static magnetic field in a conductor is not necessarily zero. A static current in a conductor can certainly sustain a static magnetic field in it.

ii. at a dielectric surface just above a perfect conductor

$$\hat{a}_{n2} \times \vec{E}_1 = 0, \; \hat{a}_{n2} \cdot \vec{B}_1 = 0,$$

$$\hat{a}_{n2} \cdot \vec{D}_1 = \rho_s, \;\; \hat{a}_{n2} \times \vec{H}_1 = \vec{J}_s. \qquad (7\text{-}23)$$

Time-varying Potential Functions

From $\nabla \cdot \vec{B} = 0$, it is straightforward to have $\nabla \times \vec{A} = \vec{B}$.

From Faraday's law $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, one can write

$\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{A})$ or $\nabla\times(\vec{E}+\frac{\partial\vec{A}}{\partial t}) = 0$. From the null identity of a scalar field $\nabla\times(\nabla V) = 0$, the above expression is reduced to

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\nabla V$$

Very often, one derives the electric field of an electromagnetic wave from known scalar and vector potentials according to

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla V . \qquad (7\text{-}24)$$

In the following, we find the governing equations for $\vec{A}$ and V.

Retarded Vector Potential

Recall Ampere's law

$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}$ or $\nabla\times\vec{B} = \mu\varepsilon\frac{\partial\vec{E}}{\partial t} + \mu\vec{J}$.

Use $\vec{B} = \nabla\times\vec{A}$ $\Rightarrow$ Left hand side (LHS) of Ampere's law becomes $\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

Use (7-24) $\Rightarrow$ Right hand side (RHS) of Ampere's law becomes

$$\mu\vec{J} + \mu\varepsilon\,\partial/\partial t\,(-\nabla V - \partial\vec{A}/\partial t) = \mu\vec{J} - \nabla(\mu\varepsilon\frac{\partial V}{\partial t}) - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

LHS = RHS gives

$\Rightarrow$ $$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \nabla(\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t})$$

Because the vector potential is a defined symbol through the expression $\vec{B} = \nabla\times\vec{A}$, one still has the freedom to define the divergence of $\vec{A}$ according to the Helmholtz theorem in the second chapter.

A vector field is well defined with its divergence and curl. Given $\vec{B} = \nabla\times\vec{A}$, we still have a freedom of defining the divergence of $\nabla\cdot\vec{A}$.

Set the so-called ***Lorentz gauge***: $\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t} \equiv 0$ (7-25)

Ampere's law is reduced to the so-called nonhomogeneous wave equation for the magnetic vector potential, given by

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} \qquad (7\text{-}26)$$

It will become clear later as why we call Eq. (7-26) a *wave equation*.

Retarded Scalar Potential

In a homogeneous medium, Gauss's law $\nabla\cdot\vec{D} = \rho$ can be written as $\nabla\cdot\vec{E} = \rho/\varepsilon$,

From (7-24), Gauss's law becomes

$$-\nabla\cdot(\nabla V + \partial\vec{A}/\partial t) = \rho/\varepsilon$$

Use Lorentz gauge $\Rightarrow$ $$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla\cdot\vec{A} = -\mu\varepsilon\frac{\partial V}{\partial t}) = \rho/\varepsilon$$

$$\Rightarrow \quad \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon \tag{7-27}$$

This is the so-called nonhomogeneous wave equation for the electric scalar potential. In the spherical coordinate system, the solution for the so-called *wave equation*,

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

has the form[1]

$$V(t,R) = \frac{f(t - R\sqrt{\mu\varepsilon})}{R}, \tag{7-28}$$

where $f(t - R\sqrt{\mu\varepsilon})$ is any arbitrary function (called *wave function*) of $t - R\sqrt{\mu\varepsilon}$. At a later time $t + \Delta t$ corresponding to a farther location $R + \Delta R$, the wave function becomes $f(t + \Delta t - (R + \Delta R)\sqrt{\mu\varepsilon})$. If an observer is looking at the same point at the wave function, the observe will see $f(t + \Delta t - (R + \Delta R)\sqrt{\mu\varepsilon}) = f(t - R\sqrt{\mu\varepsilon})$ subject to $\Delta t - \Delta R\sqrt{\mu\varepsilon} = 0$ or equivalently see a moving wave propagating at a velocity of $u = \Delta R / \Delta t = 1/\sqrt{\mu\varepsilon}$.

In the static electromagnetics, the solutions for $\nabla^2 \vec{A} = -\mu\vec{J}$

[1] Another form $f(t + \sqrt{\mu\varepsilon})/R$ means propagation in the opposite direction but follows the same

and $\nabla^2 V = -\rho/\varepsilon$ are

$$\vec{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(R)}{R} dv' \quad \text{and} \quad V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(R)}{R} dv',$$

respectively, where *R* is the distance between the location of interest and the source.

Based on the discussion on Eq. (7-28), the solution of $\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}$ is a ***retarded vector potential***, given by

$$\vec{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(t-\sqrt{\mu\varepsilon}R)}{R} dv'; \tag{7-29}$$

the solution of $\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon$ is a ***retarded scalar potential***, given by

$$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t-\sqrt{\mu\varepsilon}R)}{R} dv' \tag{7-30}$$

Both Eqs. (7-29, 30) indicate propagating potential waves at a speed of $u = 1/\sqrt{\mu\varepsilon}$. The potential wave is induced by a source and arrives at *R* with a time delay of $t = R/u$. Therefore the Maxwell's equation predicts a wave of time varying fields moving at a speed of $u = 1/\sqrt{\mu\varepsilon}$.

With known $\vec{A}$ and V, the electromagnetic fields of a wave can be calculated from

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physical argument.

$\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla \times \vec{A}$

In phasor notation (harmonic field only), the sources at the origin assume the forms

$\rho(t) = \mathrm{Re}(\hat{\rho}e^{j\omega t})$ and $\vec{J}(t) = \mathrm{Re}(\hat{\vec{J}}e^{j\omega t})$.

The wave equation with a source excitation

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

reduces to

$$\nabla^2\hat{\vec{A}} + \omega^2\mu\varepsilon\hat{\vec{A}} = -\mu\hat{\vec{J}} \quad \text{or} \quad \nabla^2\hat{\vec{A}} + k^2\hat{\vec{A}} = -\mu\hat{\vec{J}} \qquad (7\text{-}31)$$

where $k = \omega\sqrt{\varepsilon\mu} = \dfrac{2\pi}{\lambda}$ is called the *wave number* with λ being the wavelength of the time-harmonic wave. The solution of (7-31) is given by

$$\hat{\vec{A}}(R) = \frac{\mu}{4\pi}\int_{V'}\frac{\hat{\vec{J}}e^{-jkR}}{R}dv' , \qquad (7\text{-}32)$$

The real-time solution can be converted from the above complex field, given by

$$\vec{A}(R,t) = \frac{\mu}{4\pi}\int_{V'}\frac{\mathrm{Re}[\hat{\vec{J}}e^{-jkR}e^{j\omega t}]}{R}dv' .$$

Similarly, in phasor notation (harmonic field only), Eq. (7-30) gives

$$\hat{V}(R) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\hat{\rho}e^{-jkR}}{R}dv' \qquad (7\text{-}33)$$

Source-free Wave Equations

Assume there's no charge or current in space, the source-free Maxwell's equations are

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}, \qquad \nabla\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \text{(7-34.a,b)}$$

$$\nabla\cdot\vec{E} = 0 \qquad \nabla\cdot\vec{H} = 0 \qquad \text{(7-34.c,d)}$$

Use the first expression to write

$$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial\nabla\times\vec{H}}{\partial t}$$

$\Rightarrow$ LHS = $\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$, because $\nabla\cdot\vec{E} = 0$ in a source-free space.

$\Rightarrow$ RHS = $-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{u^2}\frac{\partial^2\vec{E}}{\partial t^2}$

Equating LHS and RHS to obtain the ***Homogeneous Vector Wave Equations***

$$\Rightarrow \qquad \nabla^2\vec{E} - \frac{1}{u^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad \text{(7-35)}$$

For the magnetic field, similarly, on can derive

$$\nabla^2\vec{H} - \frac{1}{u^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad \text{(7-36)}$$

Helmholtz's Equations

In phasor notations, the source-free Maxwell's equations are

$$\nabla\times\hat{\vec{E}} = -j\omega\mu\hat{\vec{H}}, \qquad \nabla\times\hat{\vec{H}} = j\omega\varepsilon\hat{\vec{E}}$$

$\nabla \cdot \hat{\vec{E}} = 0$, $\qquad \nabla \cdot \hat{\vec{H}} = 0$

Follow the above derivation, one can obtain the ***Homogeneous Vector Helmholtz's Equations***

$\nabla^2 \hat{\vec{E}} + k^2 \hat{\vec{E}} = 0$, and $\nabla^2 \hat{\vec{H}} + k^2 \hat{\vec{H}} = 0$,

where $k = \omega\sqrt{\varepsilon\mu} = 2\pi / \lambda$ is the wave number. From Eq. (1-25), the solution of the fields $\Psi = E, H$ propagating along z has the form

$\Psi(t,z) = \Psi_0 \cos(\omega t - kz + \phi_0) = \mathrm{Re}[\hat{\Psi} e^{j\omega t}]$ with

$\hat{\Psi} = \hat{E}, \hat{H} = \Psi_0 e^{-jkz + j\phi_0}$.

For a harmonic field in a conducting material, there will be current induced by the time-varying field, according to

$\nabla \times \hat{\vec{H}} = j\omega\varepsilon\hat{\vec{E}} + \hat{\vec{J}} = j\omega\varepsilon\hat{\vec{E}} + \sigma\hat{\vec{E}} = j\omega\varepsilon_c\hat{\vec{E}}$,

where the permittivity becomes a complex number $\varepsilon_c \equiv \varepsilon' - j\varepsilon''$

with $\sigma = \omega\varepsilon''$ (S/m).

The wave number is also a complex number, given by

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} = k_r - jk_i, \qquad (7\text{-}37)$$

where k_r and $-k_i$ are the real and imaginary parts of k_c, respectively. Note that k has a role of determining the spatial variation of a wave during propagation, as can be seen from (7-32) and (7-33). Substitute k_c into $\hat{\Psi} = \hat{E}, \hat{H} = \Psi_0 e^{-jkz + j\phi_0}$ to obtain

$$\hat{\Psi} = \hat{E}, \hat{H} = \left(\Psi_0 e^{-jk_r z + j\phi_0}\right) \times e^{-k_i z}.$$

Therefore, if k turns out to be a complex number, the wave is attenuated in a conductor due to ohmic loss.

To see the degree of attenuation, one can define *loss tangent* as

$$\tan\delta_c \equiv \frac{\varepsilon''}{\varepsilon'} \approx \frac{\sigma}{\omega\varepsilon}, \tag{7-38}$$

where δ_c is called *loss angle***.**

The complex form of permittivity $\varepsilon_c \equiv \varepsilon' - j\varepsilon''$ not only applies to a conducting material, but also applies to any lossy dielectric material, because the motion of charges or dipoles in a lossy dielectric is also a form of a current. ε'' becomes significant when the frequency of the field is close to the resonant frequency of the dipole.

It can be concluded from this chapter that a time-varying field generates an electromagnetic wave in space moving with a speed of $u = \frac{1}{\sqrt{\varepsilon\mu}}$. For a time harmonic wave, there's a specific wavelength λ and frequency ν, satisfying $u = \lambda\nu$ or $u = \omega / k$ where $\omega = 2\pi\nu$ and $k = 2\pi / \lambda$.