EE205003 session 10

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Null space of A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

The nullspace of A ($\mathbf{N}(A)$) is the collection of all sol.s $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the eqn. $A\mathbf{x} = \mathbf{0}$

Recall : $\mathbf{C}(A)$ is a subspace of R^4
But $\mathbf{N}(A)$ is a subspace of R^3

In general,

$\boxed{\text{Def}}$ $A_{m\times n}$, the nullspace of A, $\mathbf{N}(A)$, consists of all sol.s to $A\mathbf{x} = \mathbf{0}$
$\mathbf{x} \in R^n \Rightarrow \mathbf{N}(A)$ is a subspace of R^n

Q: Is $\mathbf{N}(A)$ a subspace ?

If $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathbf{N}(A)$, then $A\mathbf{x}_1 = \mathbf{0},\ \ A\mathbf{x}_2 = \mathbf{0}$

$\Rightarrow A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$

$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$

Back to Ex:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ col.1 + col.2 = col.3

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in$ N(A), in fact $\mathbf{N}(A)$ is a line through $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ & origin

Other value of $\mathbf{b}$

sol. to the eqn:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Q: Do the sol. form a subspace ?

No, $\because \mathbf{0}$ is NOT a sol. to this eqn.

In fact, the set of sol. forms a line in R^3 through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

but NOT $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Q: How do we compute $\mathbf{N}(A)$?

Use elimination (even for singular & rectangular matrices)!

(Elimination does NOT change sol. to $A\mathbf{x} = \mathbf{0}$ so nullspace unchanged, but col. space changed)

Solving $A\mathbf{x} = \mathbf{0}$: pivot variables, special sols

Recall : nullspace of A is made up of vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

(col.s of A are NOT lin. indep., col.2 $=$ 2·col.1)

(We don't need to use an augmented matrix $\because \mathbf{b} = \mathbf{0}$)

(any operation still $\mathbf{0}$)

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

(0 pivot & no row exchange saves it)

(this col. depends on previous col.s)

Q: What to do next ?

Move on to next col. in the same row

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form
(staircase) (2 pivots)

rank of A = # of pivots = 2 (= # of nonzero rows)
($\because$ each pivot for each nonzero row)

Pivot col.s & free col.s

$$U=\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

(col.1,3 are pivots columns, col.2,4 are free columns)

row 3 (eqn. 3) is a linear comb. of row 1 & row 2
$\Rightarrow$ We can assign any number to x_2 & x_4 (free variable)
(free col. is comb. of previous col.s)

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}2\\0\\0\end{bmatrix}+x_3\begin{bmatrix}2\\2\\0\end{bmatrix}+x_4\begin{bmatrix}2\\4\\0\end{bmatrix}$$

Special Sol.s ($U\mathbf{x} = \mathbf{0}$)

$(x_2, x_4) = (0, 1)$ or $(1, 0)$ set one of free variables to 0

If $(x_2, x_4) = (1, 0)$, then by back substitution,

$$2x_3 + 4x_4 = 0 \Rightarrow x_3 = 0$$
$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \Rightarrow x_1 = -2$$

$\Rightarrow$ one special sol. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Similiarly, $(x_2, x_4) = (0, 1) \Rightarrow \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Complete Sol.

(For $A\mathbf{x} = \mathbf{0}$)

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

special sol. $(1, 0)$ special sol. $(0, 1)$

(The nullspace contains comb. of all special sol.s)

(# special sol.s = # of free variables $= n - r = 4 - 2 = 2$)

# of pivots = rank of A

$\boxed{\text{Fact}}$ Suppose $A\mathbf{x} = \mathbf{0}$ has more unknowns than eqn.s
($n > m$, more col.s than rows)

$\Rightarrow \exists$ free col.s

(At least n-m free variables)
($\because$ # of pivots $\leq m$)

Note: When there is free variable it can be set to 1
$\Rightarrow \exists$ nonzero $\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{0}$
(infinite # of sol.s since $c\mathbf{x}$ also a sol.)
($\mathbf{N}(A)$ contains at least a line. If more free variables, $\mathbf{N}(A)$ is larger)

$\boxed{\text{Fact}}$ Dimension of N(A) = # of free variables

Reduced row echelon form

R = reduced row echelon form (rref)

zeros below & above (pivots = 1)

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(scale to 1)

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot cols

$$A\mathbf{x} = \mathbf{0} \rightarrow U\mathbf{x} = \mathbf{0} \rightarrow R\mathbf{x} = \mathbf{0} \quad (\mathbf{N}(A) = \mathbf{N}(U) = \mathbf{N}(R))$$

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With proper col. change

$$I \leftarrow \begin{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}2 & -2\\ 0 & 2\end{bmatrix} \\ \text{pivot col.s} & \text{free col.s} \\ 0 \quad 0 & 0 \quad 0 \end{bmatrix} \rightarrow F$$

Recall:

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} = -F$$

(Obtain special sol. too!)

Q: Why?

Let N contains col.s of special sol.
(nullspace matrix)

$$R = \begin{bmatrix} I_{r\times r} & F \\ 0 & 0 \end{bmatrix} \begin{array}{l} \text{) pivot rows} \\ \end{array}$$

pivot col.s

$n-r$ free col.s

Since N contains col.s of special sol.s
s.t. $R\mathbf{s} = \mathbf{0} \Rightarrow RN = \mathbf{0}_{m\times(n-r)}$

Let $N = \begin{bmatrix} -F \\ I_{(n-r)\times(n-r)} \end{bmatrix}$

$\Rightarrow RN = 0$ (so special sol. has the form $-F$)

More on pivot & free col.s

$\boxed{\text{Fact}}$ The # of the pivot col. of A is the same as R & pivot col.s of A is the first r col.s of E^{-1} where $EA = R$

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\end{matrix}} & \begin{matrix}3\\0\\3\end{matrix} & \boxed{\begin{matrix}0\\1\\1\end{matrix}} & \begin{matrix}2\\4\\6\end{matrix} & \begin{matrix}-1\\-3\\-4\end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 1 & 3 & 1 & 6 & -4 \\ 0 & 0 & -1 & 4 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}3\\0\\0\end{matrix} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}2\\4\\0\end{matrix} & \begin{matrix}-1\\-3\\0\end{matrix} \end{bmatrix} = R$$

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$$(P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix})$$

$$E = E_{32}E_{21}P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$E^{-1} = P_{23}^{-1}E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

pivot col.s of A

($EA = R \Rightarrow A = E^{-1}R$, I in R picks out the first two col.s of E^{-1} to form the corr. col.s of A)

$\boxed{\text{fact}}$ The pivots cols are NOT comb. of earlier col.s. But the free col.s are comb. of earlier col.s & the comb.s are given by special sol.s

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{ccccc} & & F & & \end{array} \\ \begin{bmatrix} 1 & \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \end{matrix} & \boxed{\begin{matrix} 2 & -1 \\ 4 & -3 \end{matrix}} \\ 0 & 0 & 0 & \begin{matrix} 0 & \;\;0 \end{matrix} \end{bmatrix} \end{array} = R$$

For R, col.2 $=$ 3 $\cdot$ col.1 $+$ 0 $\cdot$ col.3

col.4 $=$ 2 $\cdot$ col.1 $+$ 4 $\cdot$ col.3

col.5 $=$ -1 $\cdot$ col.1 - 3 $\cdot$ col.3

$\Rightarrow$ -3 $\cdot$ col.1 $+$ 0 $\cdot$ col.3 $+$ 1 $\cdot$ col.2 $=$ 0

$\Rightarrow$ -2 $\cdot$ col.1 $+$ -4 $\cdot$ col.3 $+$ 1 $\cdot$ col.4 $=$ 0

$\Rightarrow$ $+$1 $\cdot$ col.1 $+$ 3 $\cdot$ col.3 $+$ 1 $\cdot$ col.5 $=$ 0

$$\Rightarrow \mathbf{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Same for A since $A\mathbf{x} = \mathbf{0}$ exactly when $R\mathbf{x} = \mathbf{0}$