ODE Final Exam: Open Book Test June 14, 2022.

1. (30%)

   (a) (10%) Assume that f(t) has the Fourier series $f(t)=a_0/2+\sum_{n=1}^{\infty} a_n cos(\frac{n\pi t}{L}) + b_n sin(\frac{n\pi t}{L})$. Find $\frac{1}{L}\int_{\tau}^{\tau+2L} f^2(t)\,dt$ in terms of a_n and b_n.

   (b) (10%) Find the Fourier series of the following periodic function $f(t) = t^2$ over the interval $[-\pi, \pi]$.

   (c) (10%) Use (a) to prove that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

2. (30%) Let $F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt,\ s > 0$ be the Laplace Transform of f(t).

   (a) (8%) Show that $\frac{d}{ds}F(s) = L[-tf(t)](s)$

   (b) (12%) Let y=f(t) and Y(s)=L[f(t)] be its Laplace Transform. Show that $L(ty') = -\frac{d}{ds}(sY(s))$ and $L(ty'') = -\frac{d}{ds}[s^2Y(s)-sy(0)]$.

   (c) (10%) Calculate $L^{-1}[1/(s^2(s^2+1))]$.

3. (15%) Use the method of <u>exact</u> equations to solve the differential equation $0.5x \cot(y)\, y'=-1$.
4. (15%) Solve the differential equation $y''-y'-2y=4x^2$ using the method of <u>variation of parameters</u>.
5. (10%) Find the general solution of $y^{(4)} + 2y''+ y = 0$.