EE205003 Session 11

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Solving $A\mathbf{x} = \mathbf{b}$: row reduced form R

Again,

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

(row3 = row1 + row2 $\Rightarrow b_3 = b_1 + b_2$
Otherwise, no sol. $A\mathbf{x} = \mathbf{b}$)

Q: How to find sol.?

Also use Elimination!

$A\mathbf{x} = \mathbf{b} \to U\mathbf{x} = \mathbf{c} \to R\mathbf{x} = \mathbf{d}$

Elimination with augmented matrix

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \rightarrow \begin{bmatrix} U & \mathbf{c} \end{bmatrix}$$

(need 0 = 0 for last row $\Rightarrow b_3 - b_2 - b_1 = 0$)

Ex: $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ $\Updownarrow$ Equivalant

Recall: $A\mathbf{x} = \mathbf{b}$ is solvable iff $\mathbf{b} \in \mathbf{C}(A)$

Complete solution

Step 1 : Check eqn. is solvable

Step 2 : Find a particular solution $\mathbf{x}_p$

Step 3 : Complete sol. $=$ particular sol. $+$ all vectors in $\mathbf{N}(A)$ $(\mathbf{x}_n)$

A Particular sol.

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \to \begin{bmatrix} U & \mathbf{c} \end{bmatrix} \to \begin{bmatrix} R & \mathbf{d} \end{bmatrix}$$

set all free var. $= 0$

$$\begin{bmatrix} U & \mathbf{c} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} & \uparrow & & \uparrow & \end{array}$$

free col.s $\Rightarrow$ set $x_2 = x_4 = 0$

$$\Rightarrow x_1 + 2x_3 = 1$$

$$2x_3 = 3 \Rightarrow x_3 = \frac{3}{2} \Rightarrow x_1 = -2$$

$$\Rightarrow \mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

Using $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$

$$\begin{bmatrix} U & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} R & \mathbf{d} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R\mathbf{x}_p = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left. \begin{aligned} x_1 = d_1 = -2 \\ x_3 = d_2 = \tfrac{3}{2} \end{aligned} \right) \Rightarrow \quad \mathbf{x}_{pivot} \text{ comes from } \mathbf{d}$$

Combine with nullspace

$\mathbf{x}_{complete} = \mathbf{x}_p + \mathbf{x}_n$

$\mathbf{x}_p \rightarrow$ one particular sol.s ($A\mathbf{x}_p = \mathbf{b}$)

$\mathbf{x}_n \rightarrow$ many sol. (a generic vector in $\mathbf{N}(A)$)

($A\mathbf{x}_n = \mathbf{0}$: comb. of $n - r$ special sol.)

Recall: special sol.s to $A\mathbf{x}_n = \mathbf{0}$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \text{complete sol. to } A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{x}_{complete} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

($\mathbf{N}(A)$ is a 2D subspace of R^4

$\Rightarrow$ Complete sol. forms a plane parallel to $\mathbf{N}(A)$ and passes through $\mathbf{x}_p = (-2,\ 0,\ \frac{3}{2},\ 0)$)

Q: If A is square, invertible, what are $\mathbf{x}_p$ & $\mathbf{x}_n$? $(m = n = r)$

$\mathbf{x}_p = A^{-1}\mathbf{b}$ (the only sol.)

$\#$ of free var.s $= n - r = 0$

$\Rightarrow$ no special sol.

$\Rightarrow R = I$ has no zero rows

$\Rightarrow$ $\mathbf{N}(A)$ contains only $\mathbf{0}$

$\Rightarrow \mathbf{x}_{complete} = A^{-1}\mathbf{b} + \mathbf{0} = A^{-1}\mathbf{b}$

(Situation in ch.2, $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \rightarrow \begin{bmatrix} I & A^{-1}\mathbf{b} \end{bmatrix}$)

(in general $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$)

Rank

rank = # of nonzero pivots

If $A_{m\times n}$ is of rank r $\Rightarrow r \leq m,\ r \leq n$

Full col. rank $(r = n)$

1. All col.s of A are pivot col.s
2. $\#$ of free var.s $= n - r = 0$ (no free var.s)
3. $\mathbf{N}(A) = 0$
4. $A\mathbf{x} = \mathbf{b}$: $\mathbf{x} = \mathbf{x}_p$ unique sol. if it exists (0 or 1 sol.)

Ex:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ (0 or 1 sol.), has sol. if $\mathbf{b} \in \mathbf{C}(A)$

Let $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ only unique sol.

(sum of 2 col.s)

In general

$\because r \leq m \;\&\; r = n \Rightarrow n \leq m$ ($n < m$: overdetermined)

A is tall & thin & $R = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}$

For any $\mathbf{b} \in R^m$ not a comb. of col.s of $A \Rightarrow$ no sol.

Full row rank ($r = m$)

can solve $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$

(no zero rows $\Rightarrow$ no constr. on $\mathbf{b}$)

$\#$ of free var.s $= n - r = n - m$

$\Rightarrow n - m$ special sol.s to $A\mathbf{x} = \mathbf{0}$

($m \leq n$, if $m < n$ underdetermined)

Ex:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \end{aligned} \qquad (r = m = 2)$$

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \boxed{\begin{matrix} 3 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} \end{bmatrix} = \begin{bmatrix} R & \mathbf{d} \end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_{complete} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

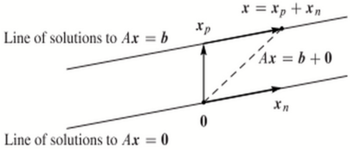

Figure 20: Complete solution = *one* particular solution + *all* nullspace solutions.

In general, if A is of full row rank

1. All rows have pivots, R has no zero rows
2. $A\mathbf{x} = \mathbf{b}$ has a sol. for every $\mathbf{b}$
3. $\mathbf{C}(A)$ is the entire R^m
4. There are $n - r = n - m$ special sols to $\mathbf{N}(A)$

Full row & col. rank ($r = m = n$)

1. A is invertible & square
2. $R = I$
3. $\mathbf{N}(A) = \{\mathbf{0}\}$
4. $A\mathbf{x} = \mathbf{b}$ has a unique sol. for every $\mathbf{b}$

$$\begin{pmatrix} \text{Full col. rank} \Rightarrow \text{uniqueness} \\ \text{Full row rank} \Rightarrow \text{existance} \end{pmatrix} \Rightarrow \text{both}$$

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \to R = I$$

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Summary

$r = m = n$	$R = I$	one sol. to $\mathbf{A}\mathbf{x} = \mathbf{b}$	square & invertible
$r = n < m$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	(0 or 1 sol.)	tall & thin
$r = n > m$	$R = \begin{bmatrix} I & 0 \end{bmatrix}$	infinitely many	short & wide
$r < m,\ r < n$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	0 or infinitely many	Not full rank