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電磁學（一）
Electromagnetics (I)

10. Laplace Equation 的解
Solutions to Laplace Equation

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In this lecture, we will learn to solve electrostatic problems by using the Laplace equation subject to boundary conditions.

- **10.1 Laplace equations, Laplace 方程式**
- **10.2 Solutions in xyz coordinate system xyz座標系中的解**
- 10.3 Examples in xyz coordinate system xyz座標系中的實例解說
- 10.4 Solutions in cylindrical coordinate system 圓柱座標系中的解
- 10.5 Review 單元回顧

Laplace 方程式的解
Solutions to Laplace Equation

10.1 Laplace 方程式
Laplace Equations

Poisson's Equation & Laplace Equation

Recall, the two postulates for electrostatics

$$\left.\begin{aligned} \nabla \times \vec{E} = 0 &\Rightarrow \vec{E} = -\nabla V \\ \nabla \cdot \vec{D} = \rho &\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \end{aligned}\right\}$$

Poisson's Equation

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

In a charge-free region, $\nabla^2 V = -\cancelto{0}{\frac{\rho}{\varepsilon}} \Rightarrow \nabla^2 V = 0$

Laplace Equation

Laplacian Operator

In Cartesian coordinates, $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

In cylindrical coordinates, $$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical coordinates,

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Given $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$ ***and*** $\nabla^2 V = 0$***,***

problems in electrostatics can be solved from known boundary conditions.

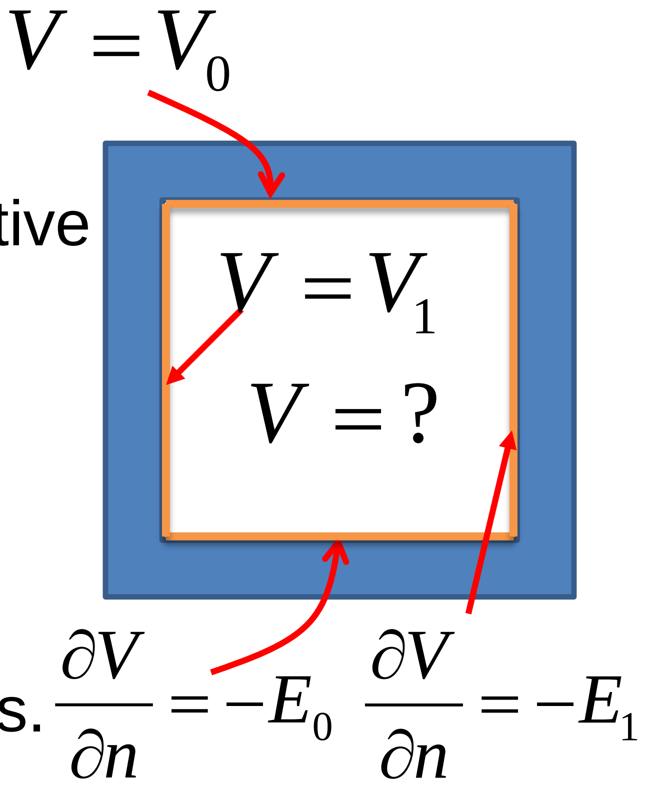
Types of Boundary Conditions for V

Need boundary conditions to solve a differential equation $\nabla^2 V = 0$

a. *Dirichlet Problems*: potential V is specified everywhere on the boundaries.

b. *Neumann Problems*: the normal derivative of the potential, $\partial V / \partial n$, is specified everywhere on the boundaries.

(normal component of E on an equipotential surface)



c. *Mixed Problems*: both *Dirichlet* and *Neumann* are specified on the boundaries.

10.1 Laplace 方程式

Laplace Equation

- In a charge-free region, the electric potential is governed by the Laplace equation $$\nabla^2 V = 0$$
- The electric potential in a region of space is solved from the Laplace equation subject to boundary conditions.
- Either the electric potential or the normal component of the electric field, or both are specified as the boundary conditions to solve the Laplace equation.

Laplace 方程式的解
Solutions to Laplace Equation

10.2 xyz 座標系中的解
Solutions in xyz Coordinate System

Laplace Equation in xyz coordinate system

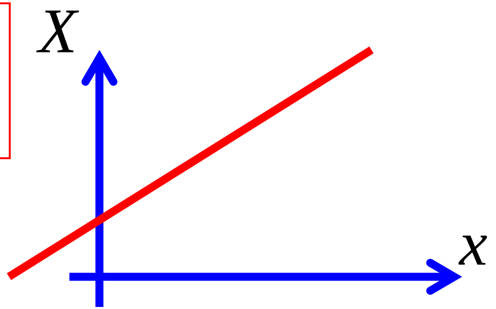
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Use separation of variables $V(x,y,z) = X(x)Y(y)Z(z)$

To obtain $YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$

Let $\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2,\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2,\ \frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2,$ with $k_x^2 + k_y^2 + k_z^2 = 0$

The problem reduces to solving $\boxed{\frac{d^2X}{dx^2} + k_x^2 X = 0}$



CASE I $k_x = 0 \quad \Rightarrow X = A_0 x + B_0$

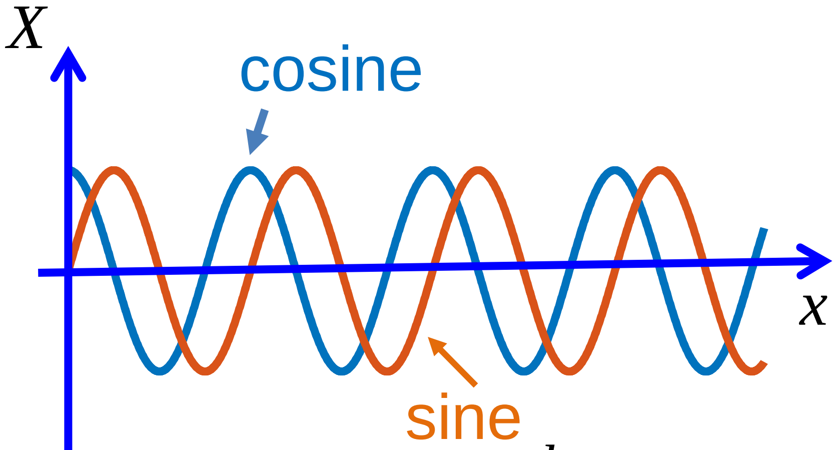
$$\boxed{\frac{d^2X}{dx^2} + k_x^2 X = 0}$$

CASE II $k_x^2 > 0$ let $k_x = k$

$$X = A_1 \sin kx + B_1 \cos kx$$

The solution has a periodic variation along x.

A_1, B_1 are decided by B.C.

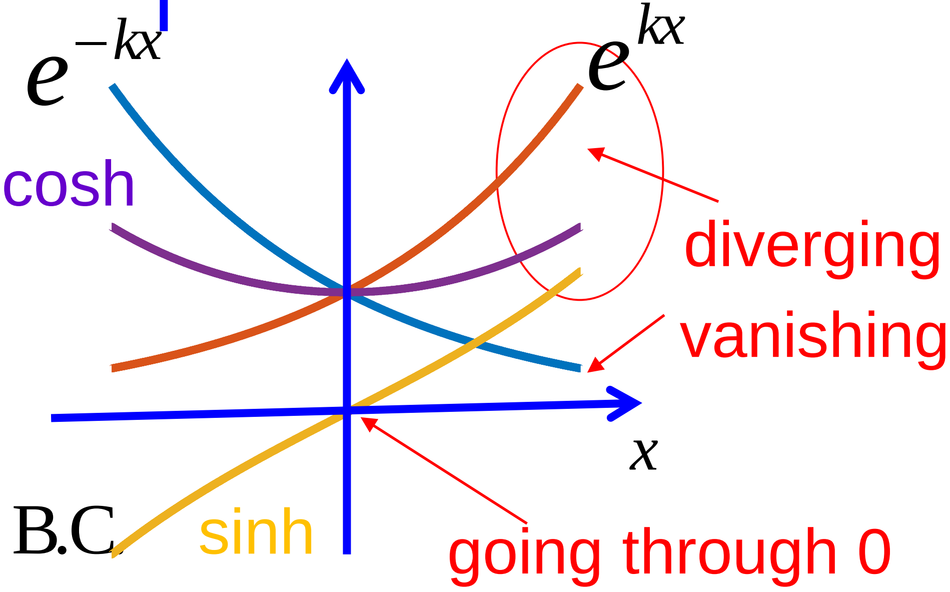


CASE III $k_x^2 < 0$ let $k_x = jk$

$$X = A_2 \sinh kx + B_2 \cosh kx$$

or $X = C_2 e^{kx} + D_2 e^{-kx}$

A_2, B_2, C_2, D_2 are decided by B.C.



2-D Problems
(no variation along z)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Again, use *separation of variables* $V(x,y) = X(x)Y(y)$

The resulting equations to be solved are

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2 \quad \text{and} \quad \frac{1}{Y}\frac{d^2 Y}{dy^2} = -k_y^2 \quad \text{subject to} \quad k_x^2 + k_y^2 = 0$$

CASE I $k_x = k_y = 0$

The solution is $V(x,y) = (Ax + B)(Cy + D)$

A, B, C, D are determined by B.C.

CASE II $k_x^2 = -k_y^2 = k^2 > 0$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$$ $\Rightarrow$ the solution is periodic in x and monotonic in y.

Specifically, $V(x,y) = (A\cos kx + B\sin kx)\times(C\cosh ky + D\sinh ky)$

or $V(x,y) = (A\cos kx + B\sin kx)\times(Ce^{-ky} + De^{ky})$

CASE III $k_x^2 = -k_y^2 = -k^2 < 0$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$ $\Rightarrow$ the solution is periodic in y and monotonic in x.

Specifically $V(x,y) = (A\cosh kx + B\sinh kx)\times(C\cos ky + D\sin ky)$

or $V(x,y) = (Ae^{-kx} + Be^{kx})\times(C\cos ky + D\sin ky)$

Orthogonality properties of harmonic functions

$$\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = 0 \quad \text{for } n \neq m$$

$$\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = \frac{a}{2} \quad \text{for } n = m \neq 0$$

$$\int_0^a \cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{a}x)dx = 0 \quad \text{for } n \neq m$$

$$\int_0^a \cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{a}x)dx = \frac{a}{2} \quad \text{for } n = m \neq 0$$

E.g. Consider the series $f(x)=\sum_{n=1}^{\infty} A_n \sin(n\pi x/a)$

Suppose $f(x)$ is known. What is the coefficient A_n?

Multiply both sides of $f(x)=\sum_{n=1}^{\infty} A_n \sin(n\pi x/a)$

by $\sin(\frac{m\pi}{a}x)$ and integrate it over $x=[0,a]$

$$\cancel{\sum_{n=1}^{\infty}} A_n \underbrace{\int_0^a \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{a}x)dx}_{\boxed{n=m}} = \int_0^a f(x)\sin(\frac{m\pi}{a}x)dx$$

$\boxed{n \neq m}$

$$\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = \frac{a}{2}$$

Apply the orthogonality property to obtain the coefficient A_m

$$A_m = \frac{2}{a}\int_0^a f(x)\sin(\frac{m\pi}{a}x)dx \text{ , } m = 1, 2, 3\ldots$$

10.2 xyz 座標系中的解

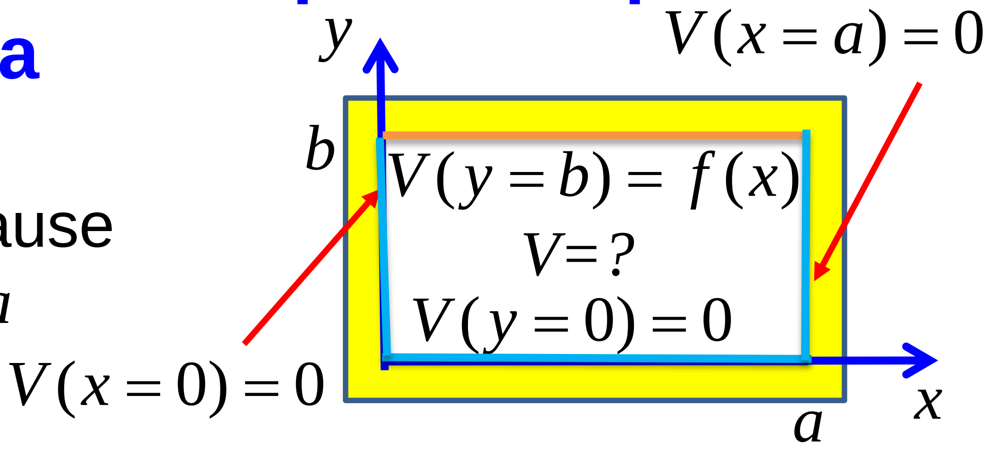
Solutions in xyz Coordinate System

- Model the problem with the separation of variables or $V(x,y,z) = X(x)Y(y)Z(z)$
- The general form of solutions for XYZ is superposition of sine, cosine, sinh, cosh, and exponential functions.
- For a 2-D problem, a periodic solution in one direction means a monotonic (sinh/cosh/exponential) solution in the other.

Laplace 方程式的解
Solutions to Laplace Equation

10.3 xyz 座標系中的實例解說
Examples in xyz Coordinate System

E.g. Solve the 2-D static potential problem in the boxed area



Observation: The solution must be periodic in *x*, because $V(x)$ returns to zero at $x = a$

Adopt the solution

$$V(x,y)=(A\cos kx+B\sin kx)\times(C\cosh ky+D\sinh ky)$$

Apply the boundary condition $V(x=0,y)=0$ ⇨ $A = 0$

$$\Rightarrow V(x,y)=\sin kx\times(C\cosh ky+D\sinh ky)$$

Apply the boundary condition $V(x=a,y)=0 \Rightarrow \sin ka=0$

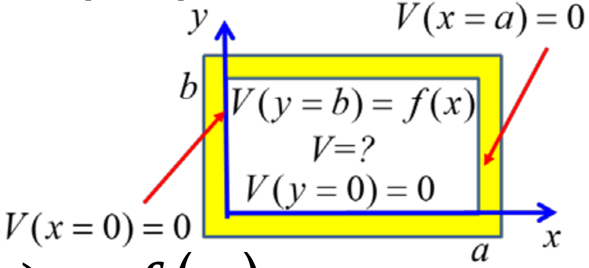
$$\Rightarrow ka=n\pi \quad \Rightarrow V_n(x,y)=\sin\frac{n\pi}{a}x\times(C\cosh\frac{n\pi}{a}y+D\sinh\frac{n\pi}{a}y)$$

Apply the boundary condition $V(x, y=0) = 0$

$$\Rightarrow V_n(x,y) = \sin\frac{n\pi}{a}x \times (\underbrace{C}_{0}\cosh\frac{n\pi}{a}y + A_n\sinh\frac{n\pi}{a}y)$$

Include all possible solutions and write their superposition

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\cdot\sinh(n\pi y/a)}$$



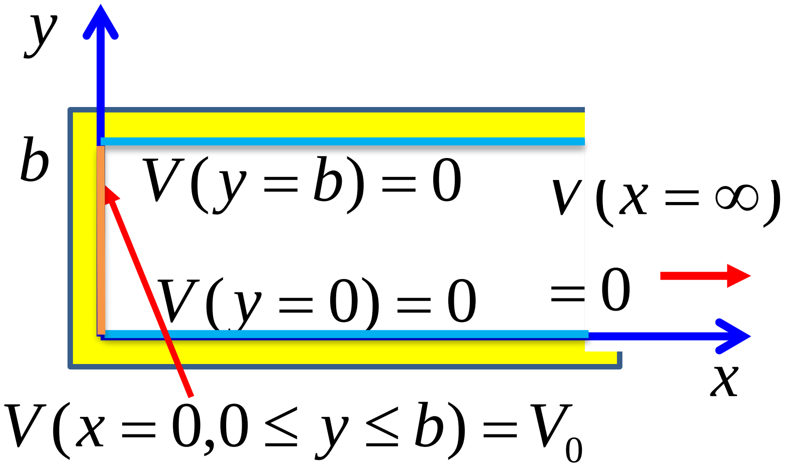
Apply the last boundary condition $V(x, y=b) = f(x)$

$$\Rightarrow V(x, y=b) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\cdot\sinh(n\pi b/a) = f(x) \text{ for } 0 \le x \le a$$

Apply the orthogonality property of sine to solve for

$$A_n = \frac{2}{a\times\sinh(n\pi b/a)}\int_0^a f(x)\sin(\frac{n\pi}{a}x)dx \quad , n = 1, 2, 3\ldots$$

E.g. Solve the 2-D static potential problem in the following figure



The solution must be periodic in y, because $V(y)$ returns to zero at $y = b$. Also, $V(y = 0) = 0$ $\Rightarrow$ choose $\sin(ky)$ solution in y with .

The chosen form of the solution is

$$\Rightarrow V(x,y) = (Ae^{-kx} + \cancel{Be^{kx}}) \times \sin ky \;\Rightarrow\; k_n = \frac{n\pi}{b}$$

$\boxed{V(x \to \infty) = 0}$ $\boxed{V(y=b)=0}$

$$\Rightarrow V(x,y) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y,$$

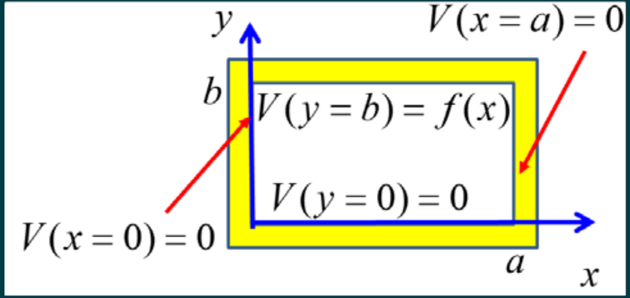
find A_n via orthogonality from the B.C. $V(x=0, 0\le y\le b)=V_0$

10.3 xyz 座標系中的實例解說

Examples in xyz Coordinate System

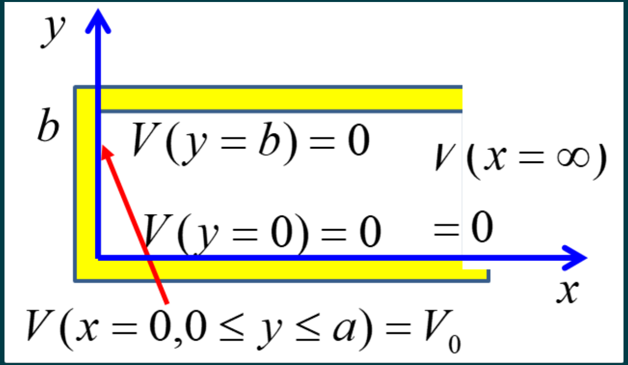
- The "guessing" solution for the problem on the right can be

$$V(x,y)=\sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\cdot\sinh(n\pi y/a)$$



- The "guessing" solution for the problem on the right can be.

$$V(x,y)=\sum_{n=1}^{\infty} A_n e^{-(n\pi x/b)} \sin(n\pi y/b),$$



Laplace 方程式的解
Solutions to Laplace Equation

10.4 圓柱座標系中的解
Solutions in Cylindrical Coordinate System

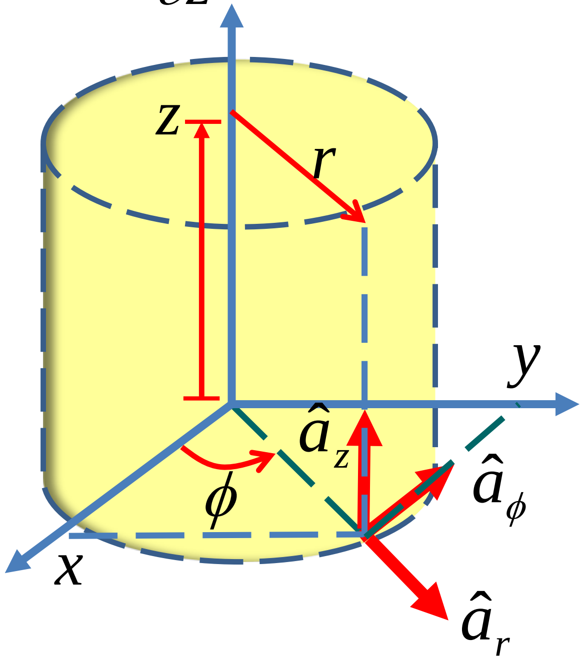
Longitudinal Invariance System $\frac{\partial^2 V}{\partial z^2} = 0$

The Laplace Equation reduces to

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Again, use separation of variables

$V(r,\phi) = R(r)\Phi(\phi)$ to write



$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

To satisfy all r and ϕ, set

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -n^2 \qquad \frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = n^2 = \text{constant}$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -n^2$$

$\Phi(\phi)$ must be a periodic function of $\phi = 2\pi$.

$$\boxed{\Rightarrow \Phi(\phi) = A_\phi \cos n\phi + B_\phi \sin n\phi}$$

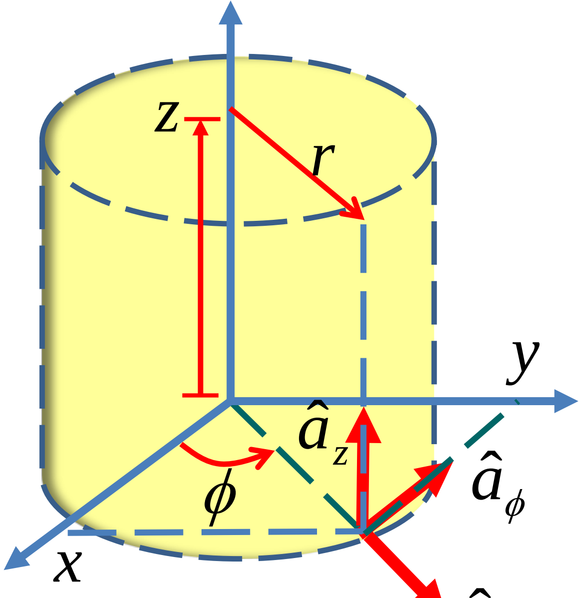
with n = 1, 2, 3…

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = n^2$$

$$\boxed{\Rightarrow R(r) = \cancel{A_r} r^n + \cancel{B_r} r^{-n}}$$

0 for region involving $r = 0$

0 for region involving $r \to \infty$



General Solutions

$$V(r,\phi) = \sum_{n=1}^{\infty} r^n (A_n \cos n\phi + B_n \sin n\phi) +$$
$$r^{-n}(A_{-n}\cos n\phi + B_{-n}\sin n\phi)$$

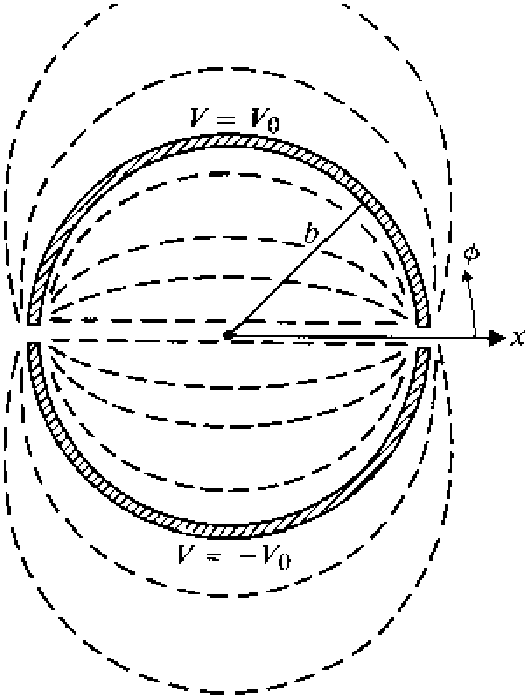
$$*\,use \int_0^\pi \begin{matrix}\cos(m\phi)\cos(n\phi)\\ \sin(m\phi)\sin(n\phi)\end{matrix}\, d\phi = \frac{\pi}{2}, m = n \neq 0$$

E.g. Solve for the potential with longitudinal invariance

Boundary Conditions

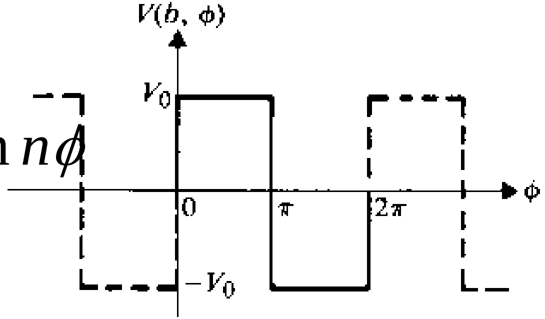
$$\left.\begin{aligned} V(b,\phi) &= V_0 \qquad & 0<\phi<\pi \\ V(b,\phi) &= -V_0 \qquad & -\pi<\phi<0 \end{aligned}\right\} \quad V(r,-\phi) = -V_0(r,\phi)$$

odd function of ϕ



In the region $r < b$

$$V(r,\phi) = \sum_{n=1}^{\infty} (\cancelto{0}{A_n} r^{-n} + B_n r^n)\sin n\phi$$



Take $a = \pi$ in the orthogonality to write

$$\sum_{n=1}^{\infty} B_n b^n \int_0^{\pi} \sin n\phi \times \sin m\phi \cdot d\phi = \int_0^{\pi} \sin m\phi \cdot V_0 d\phi$$

One obtains $B_m = \dfrac{4V_0}{m\pi b^m}$ for odd m and $B_m = 0$ for even m

The solution for $r < b$ is therefore $V(r,\phi) = \displaystyle\sum_{n=odd}^{\infty} \frac{4V_0}{n\pi b^n} r^n \sin n\phi.$

Ex. Solve V in region $r > b$

Axially Symmetric System $\dfrac{\partial^2 V}{\partial \phi^2} = 0$

The Laplace Equation $\Rightarrow \nabla^2 V = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial V}{\partial r}\right) + \dfrac{\partial^2 V}{\partial z^2} = 0$

Use separation of variables of the form $V(r,z) = R(r)Z(z)$ to obtain

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = T^2 \quad \text{and} \quad \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + T^2 R = 0$$

$$Z = \exp(..), \sinh, \cosh, \underbrace{\text{sine}, \text{cosine}}_{\text{periodic functions}}$$

$$R = \text{Bessel functions}$$

$$(\underbrace{J_0, N_0}_{\sim\text{periodic functions}}, \underbrace{I_0, K_0}_{\text{Modified Bessel functions}})$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = T^2$$

CASE I For $T^2 > 0$

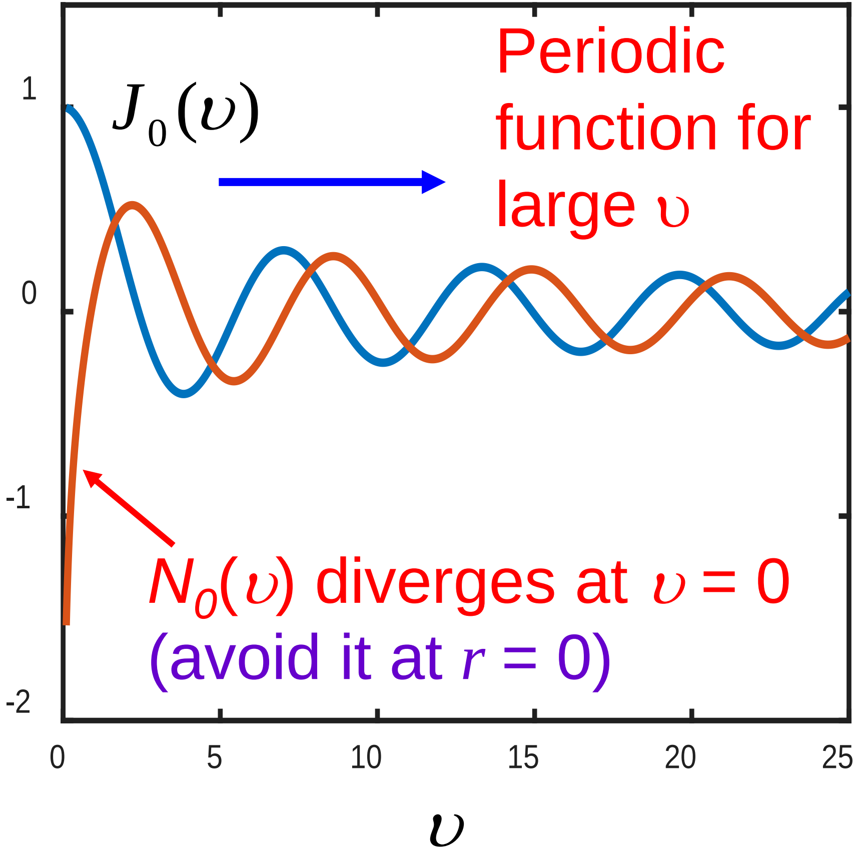
$$Z(z) = C_3 \cosh(Tz) + C_4 \sinh(Tz)$$

or $\quad = C_3 e^{-Tz} + C_4 e^{Tz}$

$$R(r) = C_1 J_0(Tr) + C_2 N_0(Tr)$$

where $J_0(..)$ is a Bessel function of the first kind and of zero order

and $N_0(..)$ a Bessel function of the second kind and of zero order.



$$\frac{1}{Z}\frac{d^2Z}{dz^2} = T^2$$

CASE II For $T^2 = -\tau^2 < 0$

$$Z(z) = C_3\cos(\tau z) + C_4\sin(\tau z)$$

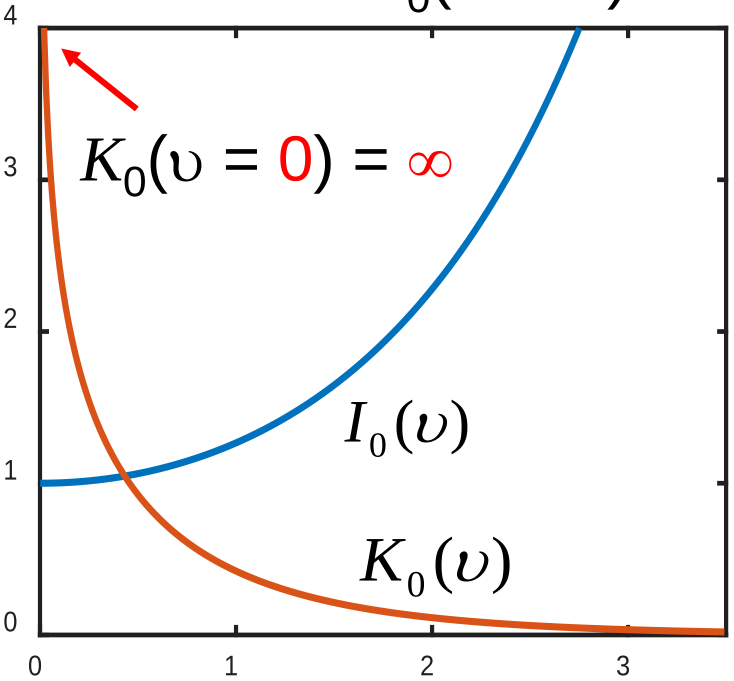
(periodic in z)

$$R(r) = C_1 I_0(\tau r) + C_2 K_0(\tau r)$$

where $I_0(..)$ and $K_0(..)$

are modified Bessel functions of the 1st and 2nd kinds, respectively.

$I_0(\upsilon = \infty) = \infty$



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1) $I_0(\upsilon = \infty) = \infty$

A region including $r = \infty$ should avoid this solution

2) $K_0(\upsilon = 0) = \infty$

A region including $r = 0$ should avoid this solution

10.4 圓柱座標系中的解

Solutions in Cylindrical Coordinate System

- The solution of the Laplace equation in a cylindrical or spherical system involves special functions.
- In a 2-D system, if the boundary condition requires a periodically varying solution along a direction, then the solution along the other is monotonic.
- In general, the methodology to solve the Laplace equation in the cylindrical and spherical systems is the same as that in an xyz coordinate system.

Laplace Equation 的解
Solutions to Laplace Equation

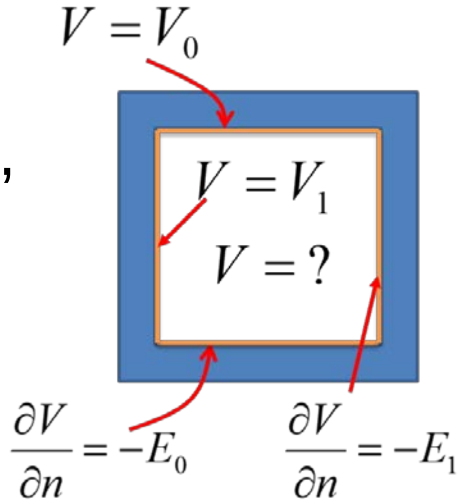
10.5 單元回顧 Review

單元回顧

1. In a charge-free region, a static electric potential is governed by the Laplace equation, given by

$$\nabla^2 V = 0$$

2. The static electric potential is solved from the Laplace equation, subject to boundary conditions (specified with V or E_n at the boundaries).

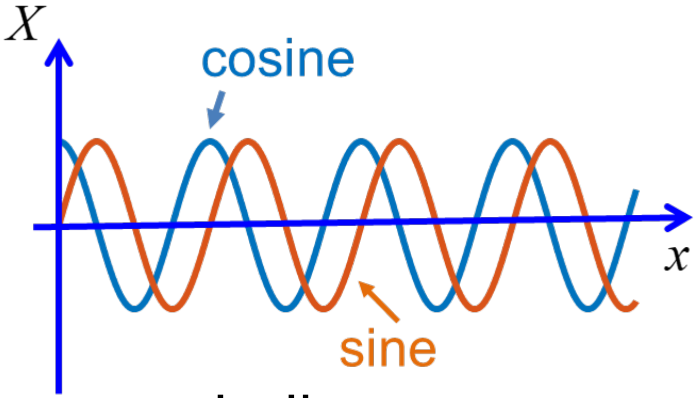


單元回顧

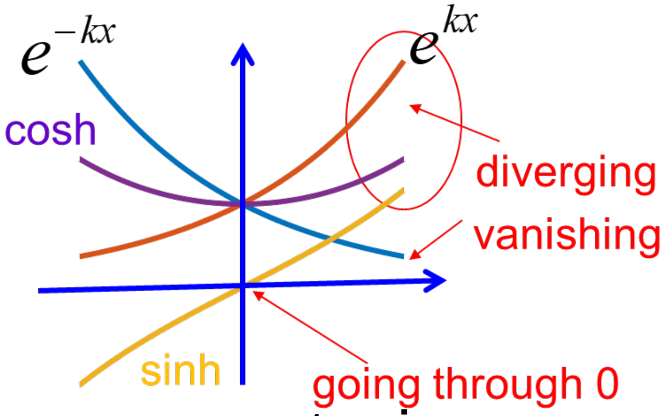
3. In the Cartesian coordinate system, the separation-of-variable technique, $V(x,y,z)=X(x)Y(y)Z(z)$, reduces the solving of

$$\nabla^2 V = 0$$

to the solving of the eigen equation $\dfrac{1}{X}\dfrac{d^2X}{dx^2}=-k_x^2$, which has two possible types of solutions:



periodic

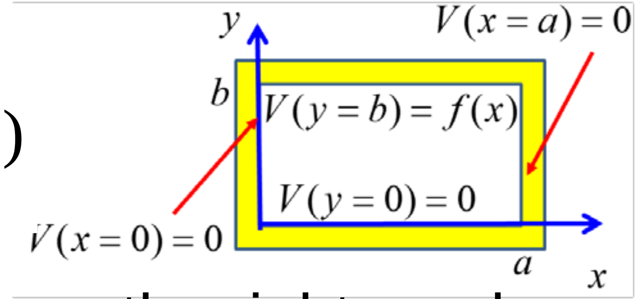


monotonic

單元回顧

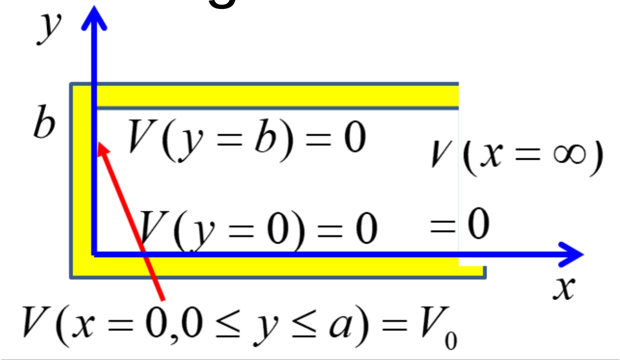
4. The “guessing” solution for the problem on the right can be

$$V(x,y)=\sum_{n=1}^{\infty}A_n\sin(n\pi x/a)\cdot\sinh(n\pi y/a)$$



5. The “guessing” solution for the problem on the right can be

$$V(x,y)=\sum_{n=1}^{\infty}A_n e^{-(n\pi x/b)}\sin(n\pi y/b),$$



6. The coefficient A_n is solved from the orthogonality property of harmonic functions subject to boundary conditions.

單元回顧

7. In cylindrical and spherical systems, the solutions to the Laplace equation involve special functions. The eigen solution along ϕ must be periodic.

8. In general, the solutions to the 3-D Laplace equation are mixed with multiplications of periodic functions (sinusoidal-like) and monotonic functions (exponential-like).

9. In a 2-D problem, if the eigen solution in one direction is periodic, the other must be monotonic. This is true for all the problems in the Cartesian, cylindrical, and spherical coordinate systems.

單元回顧

10. Choice of the "guessing solutions and the eigen values" depends on the boundary conditions.

E.g.1 In a cylindrical system involving $r = 0$, $r^{-n}, N_0(Tr), K_0(\tau r)$ are not suitable solutions, because they diverge at $r = 0$.

E.g.2 In a system involving $r = \infty$, $I_0(\tau r), e^{kx}$.. are not suitable solutions, because they diverge at $r = \infty$.

THANK YOU FOR YOUR ATTENTION