# The Exam on Linear Algebra Jan 10th, 2021

1. (20%) Solve the initial value problem $Y' = AY, Y(0) = Y_0$ by computing $Y = e^{tA}\ Y_0$, where

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 1 \\ -2 & 2 & -2 \end{bmatrix}, \ Y_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

2. (15%) Use the $XDX^{-1}$ factorization to compute $A^6$, where A = $\begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$.

3. (15%)Find an orthogonal or unitary diagonalizing matrix for

$$A = \begin{bmatrix} 1 & -1-i \\ -1+i & 2 \end{bmatrix}.$$

4. (30%) Consider the inner product space C[0,1] with the inner product defined by

$$< f, g > = \int_0^1 f(x)g(x)dx$$

Let S be the subspace spanned by the vectors 1 and $2x - 1$

(a) Show that 1 and $2x - 1$ are orthogonal.

(b) Determine that ||1| and ||2x -1||.

(c) Find the best least square approximation to $\sqrt{x}$ by a function from the subspace of S.

**Hint: Gram-Schmidt Process**

H. (i) Let $\{\bar{x}_1 \ldots \bar{x}_n\}$ be a basis for an inner product space $V$ .

(ii) $\bar{u}_1 = \frac{1}{\|\bar{x}_1\|}\bar{x}_1,$

$$\bar{u}_{K+1} = \frac{1}{\|\bar{x}_{K+1} - \bar{p}_K\|}(\bar{x}_{K+1} - \bar{p}_K), K = 1, \ldots, n-1$$

$$where \quad \bar{p}_K = \sum_{j=1}^{K} \langle \bar{x}_{K+1}, \bar{u}_j \rangle \bar{u}_j$$

C. $\{\bar{u}_1 \ldots \bar{u}_n\}$ is an orthonormal basis.

5. (10%) For the vectors $\boldsymbol{z} = \begin{bmatrix} 3+4i \\ 12i \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2-i \\ 2+4i \end{bmatrix}$,

Compute

(a) ||**z**||

(b) **<z,w>**

6. (10%) Solve the initial value problem $Y' = AY, Y(0) = Y_0$ by computing $Y = e^{tA}\ Y_0$, where A = $\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$.,  $Y_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$