Signals and Systems

Homework 5 — Due : Mar. 29 2024

Problem 1 (5 pts)**.** Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t),$$

the system also satisfies the condition of initial rest. Given $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?

Problem 2 (25 pts)**.** Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.

(b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.

(c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.

(d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.

(e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

Problem 3 (30 pts)**.** Given the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t-\tau)u_1(\tau)d\tau = x'(t) \qquad \text{(eq. 1)}$$

for any signal $x(t)$, and from it we derived the relationship

$$\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau = -g'(0). \qquad \text{(eq. 2)}$$

(a) Show that (eq. 2) is an equivalent characterization of $u_1(t)$ by showing that (eq. 2) implies (eq. 1). [Hint: Fix t, and define the signal $g(\tau) = x(t-\tau)$.]

(b) Let $f(t)$ be a given signal. Show that

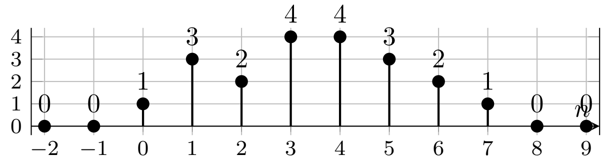
$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both functions have the same operational definitions.

Problem 4 (25 pts)**.** Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2x[n-2].$$

Plot the response of this system to the input depicted in the figure by solving the difference equation recursively.



Problem 5 (15 pts)**.** Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)dt$ (b) $\int_0^5 \sin(2\pi t)\delta(t+3)dt$ (c) $\int_{-5}^5 u_1(1-\tau)\cos(2\pi\tau)d\tau$

Problem 1 (5 pts). Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t),$$

the system also satisfies the condition of initial rest. Given $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?

$y_h' + 4y_h = 0$, $I(t) = e^{\int 4dt} = e^{4t}$, $e^{4t} \cdot y_h = \int 0\, dt = \text{const}$, $y_h = \text{const} \cdot e^{-4t}$

Let $y_p = Ae^{(-1+3j)t}$, $(3+3j)Ae^{(-1+3j)t} = e^{(-1+3j)t}$ for $t>0$, $A = \frac{1-j}{6}$

$y = [\text{const} \cdot e^{-4t} + \frac{1-j}{6} e^{(-1+3j)t}]u(t)$, initial rest $y(0)=0$

$y = [\frac{j-1}{6} \cdot e^{-4t} + \frac{1-j}{6} e^{(-1+3j)t}]\, u(t)$

Problem 2 (25 pts). Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$y_h = \text{Const.}\, e^{-2t}$

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.

(b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.

(c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.

(d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.

(e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

(a) $y = y_h + y_p = \text{const} \cdot e^{-2t} + y_p$

Let $x_1(t) = 1$, $y_1 = \frac{1}{2} e^2 e^{-2t} + \frac{1}{2}$

Let $x_2(t) = 0$, $y_2 = e^2 \cdot e^{-2t}$

Let $x_3(t) = x_1(t) + x_2(t) = x_1(t)$ implies that $y_3(t) = y_1(t) \neq y_1(t) + y_2(t)$

$\Rightarrow$ not linear

(b) Let $x_1(t) = e^{2t}$, $y_1 = \left(e^2 - \frac{e^4}{4}\right) e^{-2t} + \frac{1}{4} e^{2t}$

Let $x_2(t) = x_1(t-s)$, $y_2 = \left(e^2 - \frac{e^4}{4}\right) e^{-2(t-s)} + \frac{1}{4} e^{2(t-s)} \neq y_1(t-s)$

$\Rightarrow$ not time invariant

(c) Let $\frac{d}{dt} y_1(t) + 2y_1(t) = x_1(t)$, $y_1(1) = 0$

Let $\frac{d}{dt} y_2(t) + 2y_2(t) = x_2(t)$, $y_2(1) = 0$

Let α, β be constants

$$\frac{d}{dt}\left(\alpha y_1(t) + \beta y_2(t)\right) + 2\left(\alpha y_1(t) + \beta y_2(t)\right) = \alpha x_1(t) + \beta x_2(t)$$

$x_3(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow y_3(t) = y_1(t) + y_2(t)$ and $y_3(1) = y_1(1) + y_2(1) = 0$

$\Rightarrow$ linear for the auxiliary condition: $y(1)=0$

$\Rightarrow$ for $y(1)=1$ ，可視為 $y(1)=0$ 的系統加上一個固定的偏移量

$\Rightarrow$ incrementally linear

(d) From (c) we can see that for the condition $y(1)=0$, then the system is linear.

Let $x_1(t)=e^{2t}$, $y_1(t)=-\frac{e^4}{4}e^{-2t}+\frac{1}{4}e^{2t}$

Let $x_2(t)=x_1(t-s)=\frac{e^{2t}}{e^{2s}}$, $y_2(t)=-\frac{e^4}{4}e^{-2(t+s)}+\frac{1}{4}e^{2(t-s)}\neq y_1(t-s)$

$\Rightarrow$ not time invariant

(e) Let $\frac{d}{dt}y_1(t)+2y_1(t)=x_1(t)$, $y_1(0)+y_1(4)=0$

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Let α , β be constants

$$\frac{d}{dt}\left(\alpha y_1(t)+\beta y_2(t)\right)+2\left(\alpha y_1(t)+\beta y_2(t)\right)=\alpha x_1(t)+\beta x_2(t)$$

$$y_3(0)+y_3(4)=y_1(0)+y_1(4)+y_2(0)+y_2(4)$$

$\Rightarrow x_3(t)=\alpha x_1(t)+\beta x_2(t)\longrightarrow y_3(t)=\alpha y_1(t)+\beta y_2(t)$

$\Rightarrow$ linear

Let $x_1(t)=e^{2t}$, $y_1(t)=\frac{1}{4}e^{2t}+\text{const}\cdot e^{-2t}$

$$y(0) = \frac{1}{4} + const \quad , y(4) = \frac{e^8}{4} + const \cdot e^{-8}$$

$$\frac{1+e^8}{4} + (1+e^{-8})\, const = 0 \quad , \quad const = -\frac{1+e^8}{4(1+e^{-8})}$$

$$y_1(t) = \frac{-1-e^8}{4(1+e^{-8})} e^{-2t} + \frac{1}{4} e^{2t}$$

Let $x_2(t) = x_1(t-s)$, $y_2(t) = const \cdot e^{-2t} + \frac{1}{4} e^{2t-2s}$

$$const + \frac{1}{4e^{2s}} + \frac{const}{e^8} + \frac{e^8}{4e^{2s}} = 0$$

$$const\,(1+e^{-8}) = \frac{-1-e^8}{4e^{2s}} \quad , \quad const = \frac{-1-e^8}{4(1+e^{-8})} \cdot e^{-2s}$$

$$y_2(t) = \frac{-1-e^8}{4(1+e^{-8})} e^{-2(t+s)} + \frac{1}{4} e^{2(t-s)} \neq y_1(t-s)$$

$\Rightarrow$ not time invariant

Problem 3 (30 pts)**.** Given the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t-\tau)u_1(\tau)d\tau = x'(t) \qquad \text{(eq. 1)}$$

for any signal $x(t)$, and from it we derived the relationship

$$g(-t) * u(t) \quad \int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau = -g'(0). \qquad \text{(eq. 2)}$$

(a) Show that (eq. 2) is an equivalent characterization of $u_1(t)$ by showing that (eq. 2) implies (eq. 1). [Hint: Fix t, and define the signal $g(\tau) = x(t-\tau)$.]

(b) Let $f(t)$ be a given signal. Show that

$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both functions have the same operational definitions.

(a) Let $g(\tau) = x(t-\tau)$, $-g'(0) = \int_{-\infty}^{\infty} g(\tau)u_1(\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)u_1(\tau)d\tau = x(t)*u_1(t) = x'(t)$

(b) $x(t) * f(t)u_1(t)$

$= \int_{-\infty}^{\infty} x(t-\tau) f(\tau)u_1(\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(t-\tau) g(t-\tau) u_1(\tau) d\tau$

$= \int_{-\infty}^{\infty} p(t-\tau)u_1(\tau) d\tau$

$= p(t) * u_1(t) = \frac{d}{dt} p(t)$

$= \frac{d}{dt}\left(x(t)\, g(t)\right)$

$= g(t)x'(t) + g'(t)x(t)$

$= \underline{f(0)\, x'(t) - f'(0)\, x(t)}$

let $g(t-\tau) = f(\tau)$

let $p(t-\tau) = x(t-\tau)\, g(t-\tau)$

from (a), we can have $-g'(t) = f'(0)$

$x(t) * \left(f(0)u_1(t) - f'(0)\, \delta(t)\right)$

$= f(0) \int_{-\infty}^{\infty} x(t-\tau)u_1(\tau)d\tau - f'(0)\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau$

$= \underline{f(0)\, x'(t) - f'(0)\, x(t)}$

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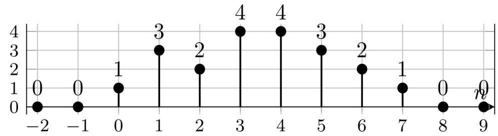
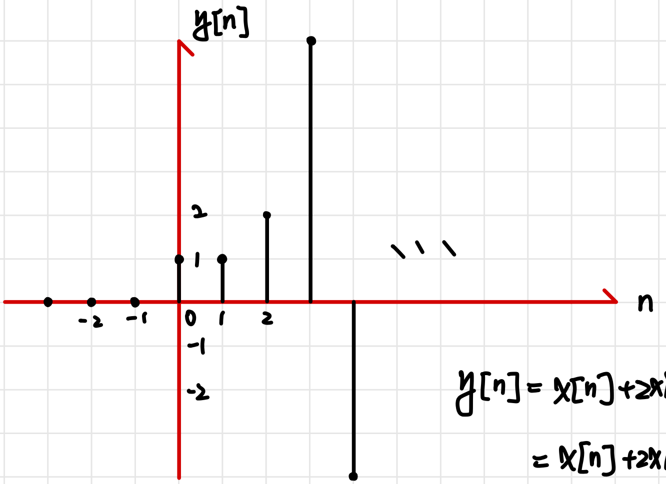
Plot the response of this system to the input depicted in the figure by solving the difference equation recursively.





$$y[n] = x[n] + 2x[n-2] - 2y[n-1]$$

$-8 \cdot 3 \quad +16(-?)$

$$= x[n] + 2x[n-2] - 2x[n-1] - 4x[n-3] + 4x[n-2] + 8x[n-4]$$

$$= x[n] - 2x[n-1] + 6x[n-2] - 12x[n-3] \cdots$$

$$= x[n] - 2x[n-1] + \sum_{k=2}^{\infty}\left(3 \cdot 2^{k-1}(-1)^k x[n-k]\right)$$

$y[n] = 0$ for $n \leq -1$

$y[0] = 1$, $y[1] = 1$

$y[0] = 1$

$y[1] = 3 - 2 = 1$

$y[2] = 4 - 2 = 2$

$y[3] = 10 - 4 = 6$

$y[4] = 8 - 12 = -4$

$y[5] = 11 + 8 = 19$

$y[6] = 10 - 38 = -28$

$y[7] = 7 + 56 = 63$

$y[8] = 4 - 126 = -122$

$y[9] = 2 + 244 = 246$

$\vdots$

Problem 5 (15 pts). Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)dt$ (b) $\int_0^5 \sin(2\pi t)\delta(t+3)dt$ (c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau$

(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt = 1$

(b) $\int_0^5 \sin(2\pi t)\delta(t+3)\,dt = 0$

(c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau$

$= -\delta(1-\tau)\cos(2\pi\tau)\Big|_{\tau=-5}^{5} - 2\pi\int_{-5}^{5}\delta(1-\tau)\sin(2\pi\tau)\,d\tau$

$= 0 - 2\pi\sin(2\pi)\int_{-5}^{5}\delta(1-\tau)\,d\tau = 0$