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電磁學（一）
Electromagnetics (I)

12. 電阻與電容電路
Resistor and Capacitor Circuit

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In this lecture, we will learn about the connection between resistance and capacitance, and how they affect the temporal response of a circuit.

- **12.1 Charge Relaxation Time 電荷平衡時間**
- **12.2 Current Boundary Conditions 電流邊界條件**
- **12.3 Connection between Resistance and Capacitance 電阻與電容的關連**
- **12.4 Calculation for Resistance 計算電阻值**
- **12.5 Review 單元回顧**

電阻與電容電路
Resistor and Capacitor Circuit

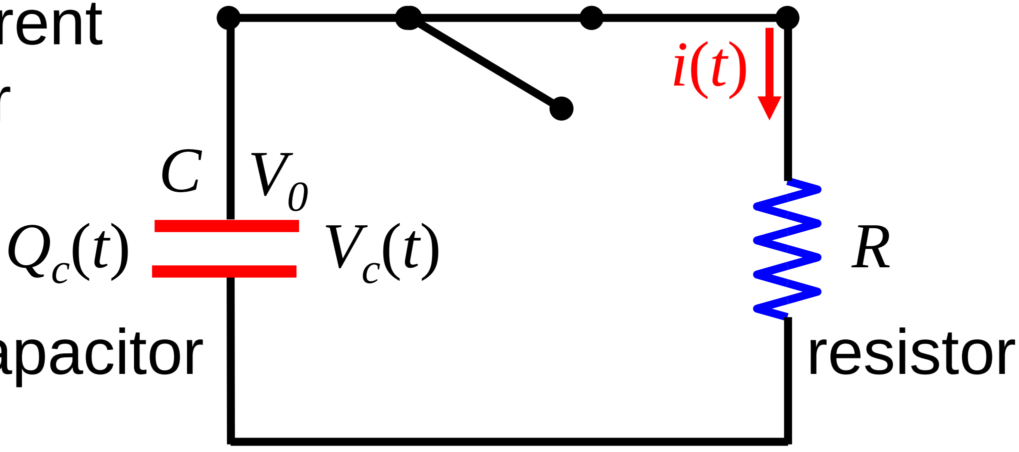
12.1 電荷平衡時間
Charge Relaxation Time

RC Discharging Circuit (Lecture 7)

capacitor discharging current
= current entering resistor

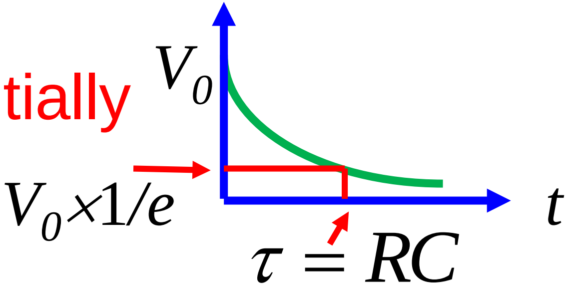
$$\frac{dQ_c}{dt} + \frac{V_c(t)}{R} = 0$$

Recall $C = \dfrac{Q_c}{V_c}$

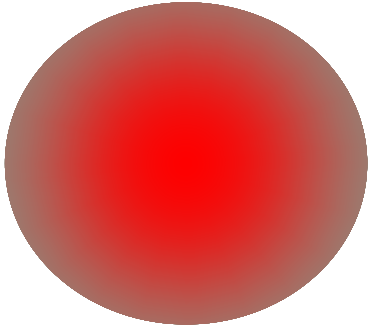


$$\Rightarrow \quad C\frac{dV_c}{dt} + \frac{V_c(t)}{R} = 0 \quad \Rightarrow \quad V_c(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

Capacitor voltage drops exponentially
with a time constant $\tau = RC$



Charge Relaxation



Charge relaxation time: Upon a perturbation, an excess charge density appears in a good conductor. How fast does the excess charge settle to zero?

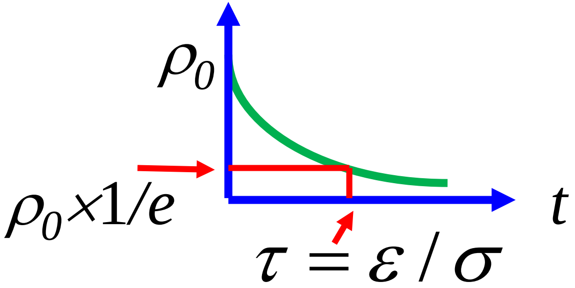
Use the equation of continuity

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

But $\vec{J} = \sigma \vec{E} \Rightarrow \sigma \nabla \cdot \vec{E} + \frac{\partial \rho}{\partial t} = 0$

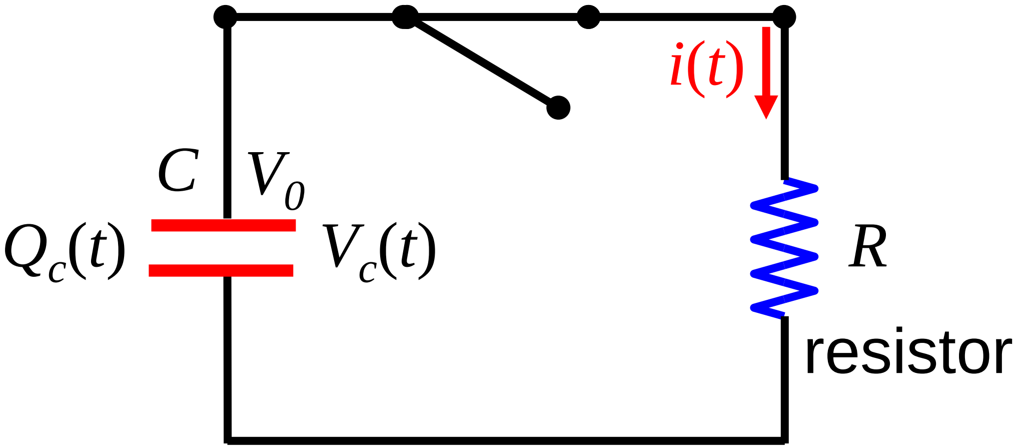
But $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon}\rho = 0$

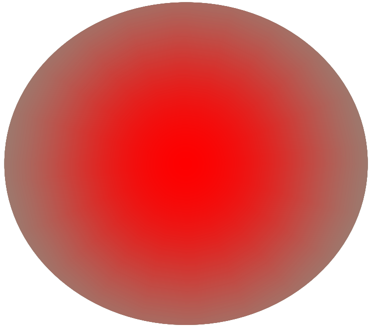
$$\Rightarrow \quad \rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$$



For Cu with $\sigma = 5.8 \times 10^7$ S/m, $\tau = 10^{-19}$ sec!

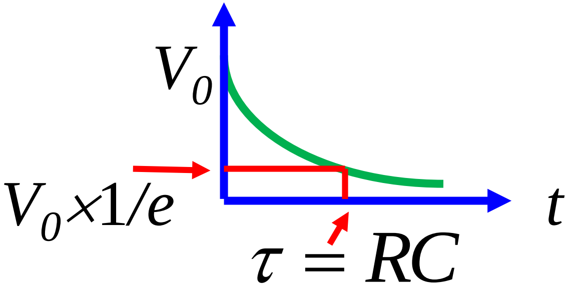
Analogy



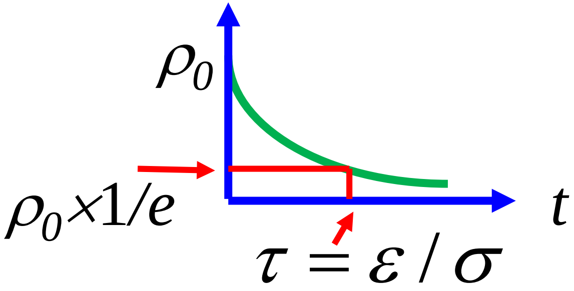


$$V_c(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

$$\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t} = \rho_0 e^{-t/\tau}$$



$$\boxed{\frac{\varepsilon}{\sigma} = RC\,?}$$



12.1 電荷平衡時間

Charge Relaxation Time

- Macroscopically, the characteristic time constant for a capacitor to discharge through a resistor is $\tau_d = RC.$
- Microscopically, the characteristic time constant for an excess charge to disappear in a conductor is $\tau_r = \varepsilon / \sigma$
- There must be some connection between τ_d and τ_r.

電阻與電容電路
Resistor and Capacitor Circuit

12.2 電流邊界條件
Current Boundary Conditions

Boundary Conditions for J (in a neutral material)

Differential Forms

$$\nabla \cdot \vec{J} = 0 \quad \text{(in steady state)}$$

$$\nabla \times \vec{E} = 0 \Rightarrow \nabla \times (\vec{J}/\sigma) = 0$$

Integral Forms

$$\oint_S \vec{J} \cdot d\vec{s} = 0$$

$$\oint_c \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = 0$$

Recall

$$\nabla \cdot \vec{D} = 0$$

$$\Downarrow$$

$$D_{1n} = D_{2n}$$

$$\nabla \times \vec{E} = 0$$

$$\Downarrow$$

$$E_{1t} = E_{2t}$$

I. Normal components of current density are continuous across boundaries

$$\oint_S \vec{J} \cdot d\vec{s} = 0 \Rightarrow \boxed{J_{1n} = J_{2n}}$$

II. Tangential components of J

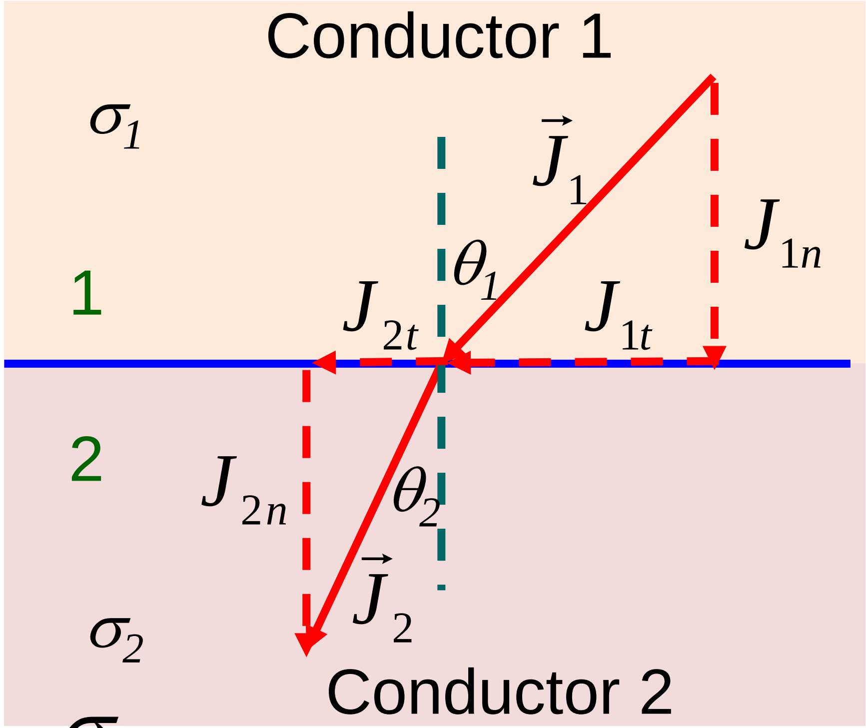
$$\oint_c \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = 0 \Rightarrow \boxed{J_{1t} / \sigma_1 = J_{2t} / \sigma_2}$$

Current Flowing through a Conducting Interface

$$J_{2n} = J_{1n}$$

$$J_{2t} = J_{1t} \times \frac{\sigma_2}{\sigma_1}$$

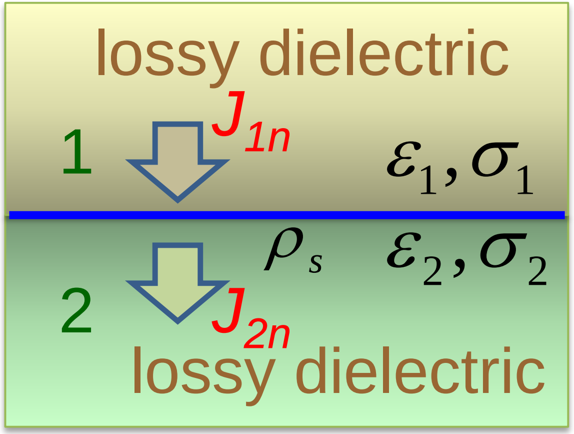
$$\tan\theta_2 = \frac{J_{2t}}{J_{2n}} = \frac{J_{1t} \times \frac{\sigma_2}{\sigma_1}}{J_{1n}} = \frac{\sigma_2}{\sigma_1}\tan\theta_1$$



Surface Charges between Two Lossy Dielectrics

From the boundary condition for the normal components of *J*

$$J_{1n} = J_{2n} \Rightarrow \sigma_1 E_{1n} - \sigma_2 E_{2n} = 0$$



From the boundary condition for the normal components of *D*

$$D_{1n} - D_{2n} = \rho_s \Rightarrow \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

Combining the two results $\rho_s = (\varepsilon_1 \dfrac{\sigma_2}{\sigma_1} - \varepsilon_2) E_{2n} = (\varepsilon_1 - \varepsilon_2 \dfrac{\sigma_1}{\sigma_2}) E_{1n}$

Surface charges must exist unless $\dfrac{\sigma_1}{\sigma_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$

Physical meaning: charges relax at an equal rate $\tau_r = \dfrac{\varepsilon}{\sigma}$

12.2 電流邊界條件

Current Boundary Conditions

- The normal components of the current density at an interface are continuous or $J_{1n} = J_{2n}$
- The tangential components of the current density at an interface satisfy the ratio $J_{1t} / \sigma_1 = J_{2t} / \sigma_2$
- There won't be charge accumulation across two lossy dielectrics 1 & 2, if $\dfrac{\sigma_1}{\sigma_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$.

電阻與電容電路
Resistor and Capacitor in a Circuit

12.3 電阻與電容的關連
Connection between Resistance and Capacitance

Relationship between R and C

capacitance

$$C = \frac{Q}{\cancel{V}} = \frac{\oint_S \vec{D}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}} = \frac{\varepsilon\oint_S \vec{E}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}}$$

resistance

$$R = \frac{\cancel{V}}{I} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\int_{S_R} \vec{J}\cdot d\vec{s}} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\int_{S_R} \sigma\vec{E}\cdot d\vec{s}}$$

Multiply the two equations to obtain

$$RC = \frac{Q}{I} = \frac{\varepsilon}{\sigma}\frac{\oint_S \vec{E}\cdot d\vec{s}}{\int_{S_R} \vec{E}\cdot d\vec{s}}$$

If R and C are associated with the same volume enclosed by the same $S = S_R + S'_R$ and S'_R contributes no value to the surface integration of the electric field

$$\oint_S \vec{E}\cdot d\vec{s} = \int_{S_R} \vec{E}\cdot d\vec{s} + \underbrace{\int_{S'_R} \vec{E}\cdot d\vec{s}}_{0},$$

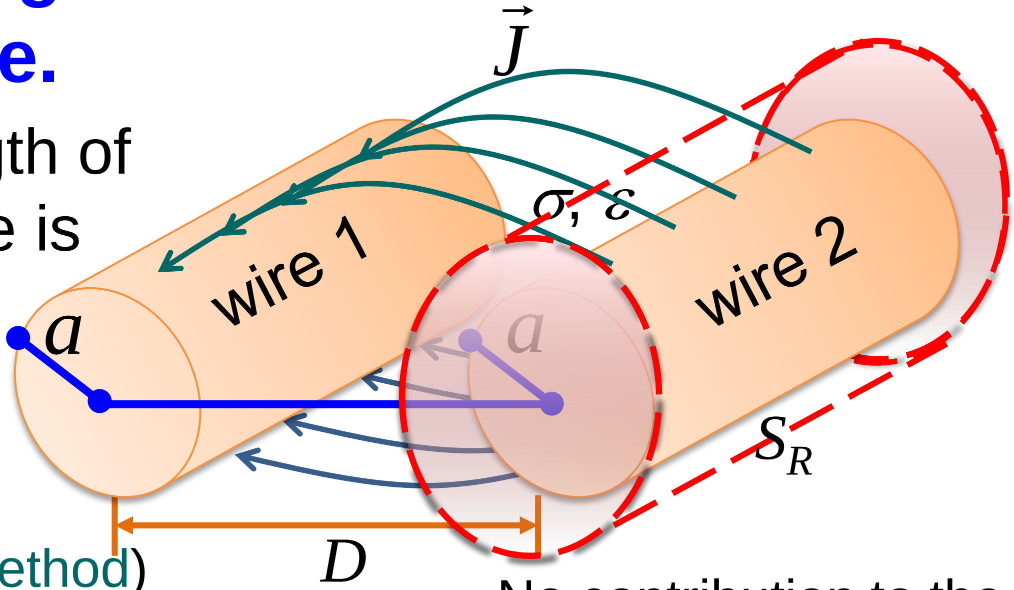
one obtains the relationship $\boxed{RC = \frac{\varepsilon}{\sigma}.}$

R can be obtained by knowing C or vice versa.

E.g. Find R for a given C of a two-wire transmission line.

The capacitance per unit length of this two-wire transmission line is

$$C_l = \frac{\pi\varepsilon}{\cosh^{-1}\left(\dfrac{D}{2a}\right)}$$



(refer to Lecture 9 - image charge method)

No contribution to the surface integration

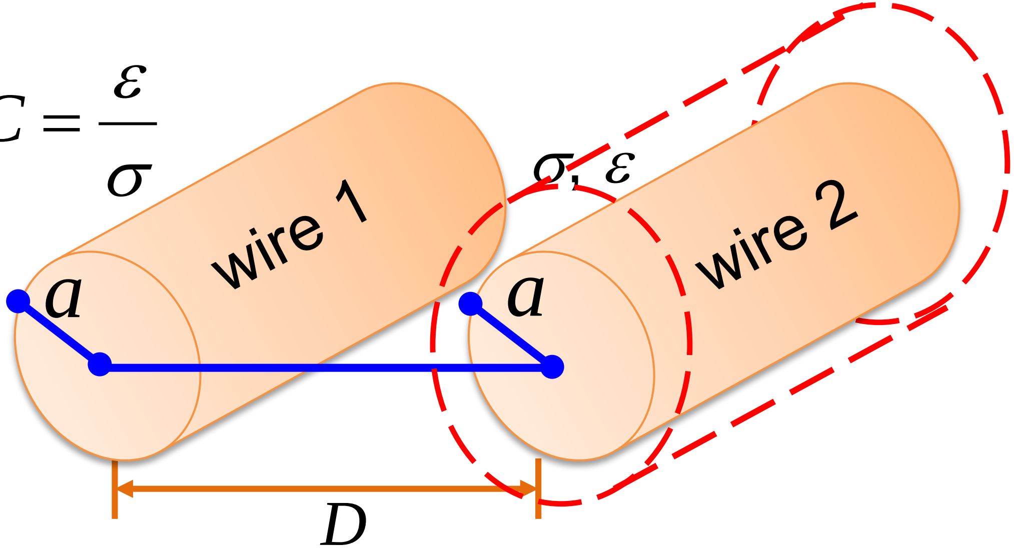
$$RC = \frac{Q}{I} = \frac{\varepsilon}{\sigma}\frac{\oint_S \vec{E}\cdot d\vec{s}}{\int_{S_R} \vec{E}\cdot d\vec{s}}$$

, where $S = S_R + S'_R = S_R$

$\Rightarrow$ $RC = \dfrac{\varepsilon}{\sigma}$ is valid.

Use the RC relationship $RC = \frac{\varepsilon}{\sigma}$ to obtain the total resistance of length L

$$R = \frac{1}{\pi\sigma L}\cosh^{-1}\left(\frac{D}{2a}\right)$$



Note that for a transmission line of length *L* in this case, the parallel resistance is inversely proportional to L .

It is easier to think in terms of conductance $G = \pi\sigma L / \cosh^{-1}\left(\frac{D}{2a}\right)$

The conductance per unit length is $G_l = \pi\sigma / \cosh^{-1}\left(\frac{D}{2a}\right)$.

12.3 電阻與電容的關連

Connection between Resistance and Capacitance

- When a capacitor and a resistor share the same volume of a device, the resistance and capacitance “often” have the relationship

$$RC = \frac{\varepsilon}{\sigma}$$

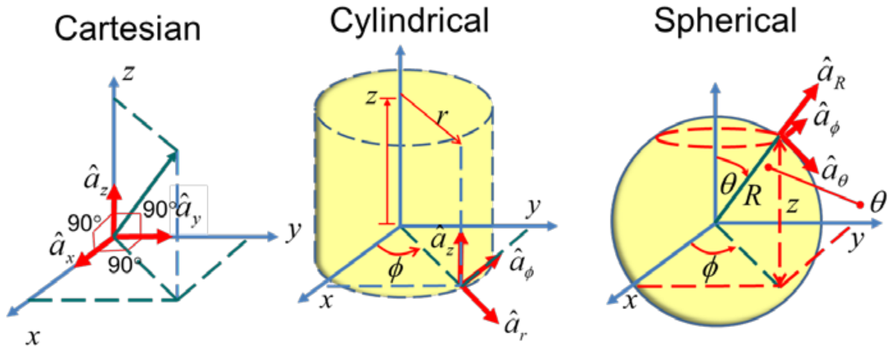
- By knowing the capacitance of the device, one can calculate its resistance or vice versa.

電阻與電容電路
Resistor and Capacitor Circuit

12.4 計算電阻值
Calculation for Resistance

Approach I

1.From the symmetry of the problem, choose a coordinate system.



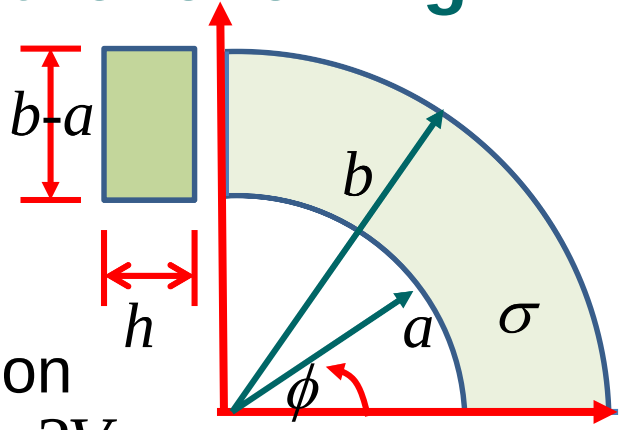
2. Find V from $\nabla^2 V = 0$ subject to $V = V_0$ at a suitable boundary.

3. Find the electric field E from $\vec{E} = -\nabla V$ and then $\vec{J} = \sigma \vec{E}$

4. Find current I from $I = \int_S \vec{J} \cdot d\vec{s} = \int_S \sigma \vec{E} \cdot d\vec{s}$

5. Calculate resistance R from $R = \dfrac{V_0}{I}$

E.g. Find the resistance of the following conductor.



Boundary conditions: $V = 0$ at $\phi = 0$

$V = V_0$ at $\phi = \pi/2$

1. Choose cylindrical coordinate system
2. No variation in z and r. The Laplace equation becomes

$$\frac{d^2V}{d\phi^2} = 0 \Rightarrow V = c_1\phi + c_2 \Rightarrow V = \frac{2V_0}{\pi}\phi$$

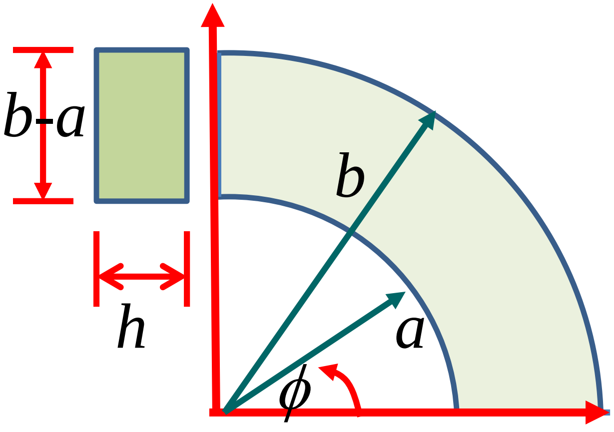
Apply boundary conditions

3. The electric field $\vec{E} = -\nabla V = -\hat{a}_\phi \dfrac{\partial V}{r\partial\phi} = -\dfrac{2V_0}{\pi r}\hat{a}_\phi$

4. Current density is $\vec{J} = \sigma\vec{E} = -\sigma\dfrac{2V_0}{\pi r}\hat{a}_\phi$

Total current is the integration

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_a^b \sigma \frac{2V_0}{\pi r} h dr = \frac{2\sigma h V_0}{\pi} \ln \frac{b}{a}$$



5. Calculate the ratio $R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln \frac{b}{a}}$

Approach II

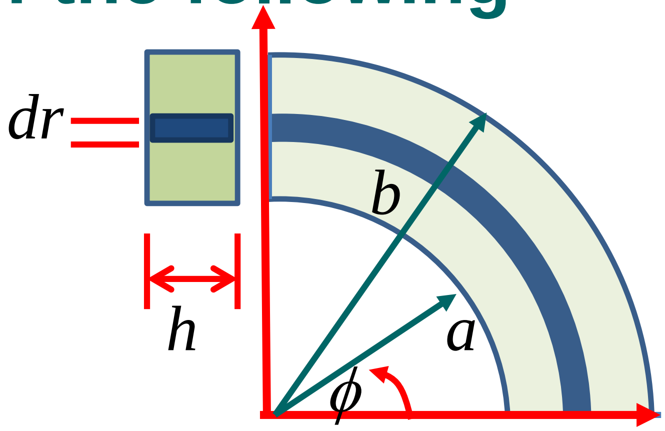
Partition the device into

serial resistors $\quad R_{total} = R_1 + R_2 ... + R_N \Rightarrow R = \int dR$

or

parallel resistors $\quad G_{total} = G_1 + G_2 ... G_N \Rightarrow G = \int dG$

E.g. Find the resistance of the following conductor.



Consider the device is formed by layers of parallel resistors

Given $R \equiv \dfrac{l}{\sigma S} \Rightarrow G \equiv \dfrac{\sigma S}{l}$, the differential conductance of each layer resistor is $dG = \dfrac{\sigma ds}{l}$,

where the differential surface is $ds = hdr$

& the length of the layer resistor at r is $l = \dfrac{\pi}{2} r$

Sum over the conductance of all the layer resistors

$$dG = \frac{\sigma hdr}{\pi r/2} \Rightarrow G = \int_a^b \frac{2\sigma h}{\pi r} dr = \frac{2\sigma h}{\pi} \ln\frac{b}{a} \Rightarrow R = \frac{1}{G} = \frac{\pi}{2\sigma h \ln(b/a)}$$

12.4 計算電阻值

Calculation of Resistance

- If a resistor has enough symmetry and boundary conditions in an orthogonal coordinate system, one can calculate its resistance by going down the steps, $\nabla^2 V = 0 \Rightarrow V \Rightarrow E \Rightarrow J \Rightarrow I \Rightarrow R = V/I$.
- Alternatively, one can model the problem as integration of serial resistance or parallel conductance.

$$\boxed{R_{total} = R_1 + R_2 \ldots + R_N \Rightarrow R = \int dR}$$, given $\boxed{R \equiv \frac{l}{\sigma S}.}$

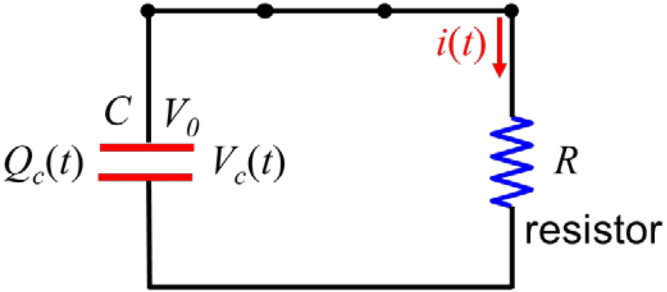
$$\boxed{G_{total} = G_1 + G_2 \ldots G_N \Rightarrow G = \int dG}$$, given $\boxed{G \equiv \frac{\sigma S}{l}.}$

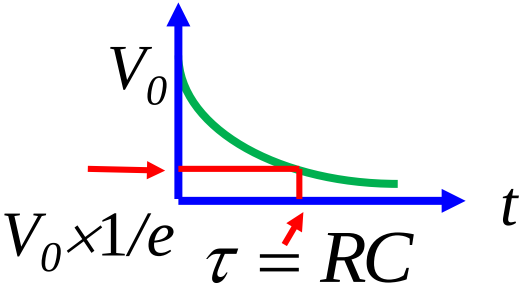
電阻與電容電路
Resistor and Capacitor Circuit

12.5 單元回顧 Review

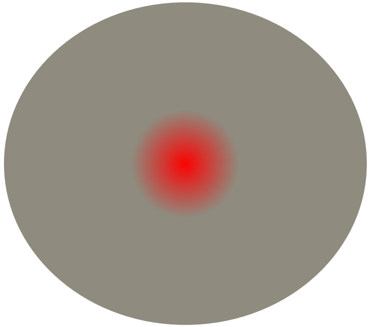
單元回顧

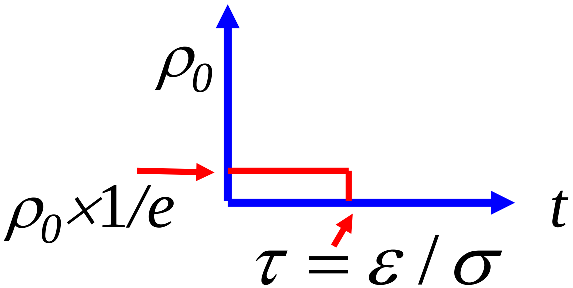
1. The characteristic discharging time of an RC circuit is $\tau_d = RC$





2. The charge relaxation time in a conductor is $\tau_r = \varepsilon / \sigma$





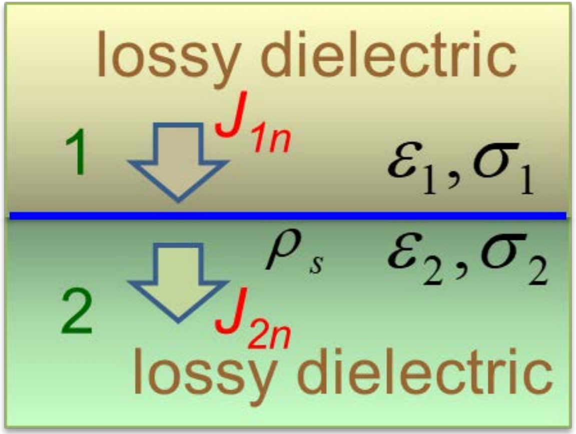
單元回顧

3. For a resistor and capacitor sharing the same volume, the relationship "often" holds:

$$RC = \frac{\varepsilon}{\sigma}.$$

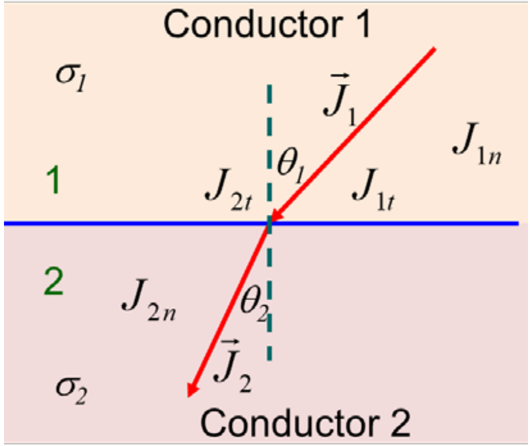
4. The condition for zero charge accumulation $\rho_s = 0$ at the interface of two lossy dielectrics is

$$\frac{\varepsilon_1}{\sigma_1} = \frac{\varepsilon_2}{\sigma_2} \quad \text{or} \quad \underbrace{\tau_1 = \tau_2}_{\text{equal charge relaxation time}}$$



單元回顧

5. The boundary conditions for currents:



Normal components are continuous at the interface $$J_{1n} = J_{2n}$$

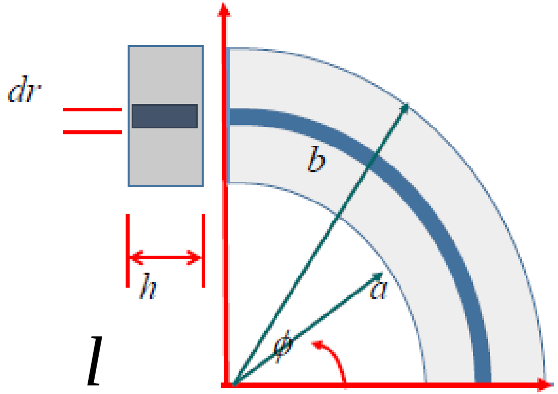
Tangential components satisfy the ratio

$$J_{1t} / \sigma_1 = J_{2t} / \sigma_2$$

6. If a resistor has enough symmetry and boundary conditions in the right-angle coordinate systems, one can calculate its resistance by going down the steps,

$$\nabla^2 V = 0 \Rightarrow V \Rightarrow E \Rightarrow J \Rightarrow I \Rightarrow R = V/I .$$

單元回顧



7. Alternatively, one can model the problem as integration of serial resistance

$$R_{total} = R_1 + R_2 ... + R_N \Rightarrow R = \int dR \text{ with } R \equiv \frac{l}{\sigma S}.$$

or integration of parallel conductance

$$G_{total} = G_1 + G_2 ... G_N \Rightarrow G = \int dG \text{ with } G \equiv \frac{\sigma S}{l}.$$

THANK YOU FOR YOUR ATTENTION