電磁學（一）
Electromagnetics (I)

6. 物質靜電學
Electrostatics in Material

授課老師：國立清華大學 電機工程學系 黃衍介 教授
Yen-Chieh Huang, National Tsing Hua University, Taiwan

In this lecture, we will learn about how the charges in a material modify the electric field and electric potential.

- 6.1 Conductor 導體
- 6.2 Dielectric 介電物質
- 6.3 Polarization density vector 極化密度向量
- 6.4 Electrostatic boundary conditions 靜電邊界條件
- 6.5 Review 單元回顧

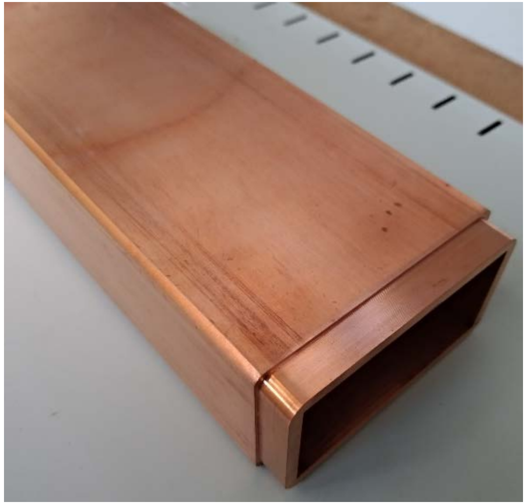
物質靜電學
Electrostatics in Material

6.1 導體
Conductor

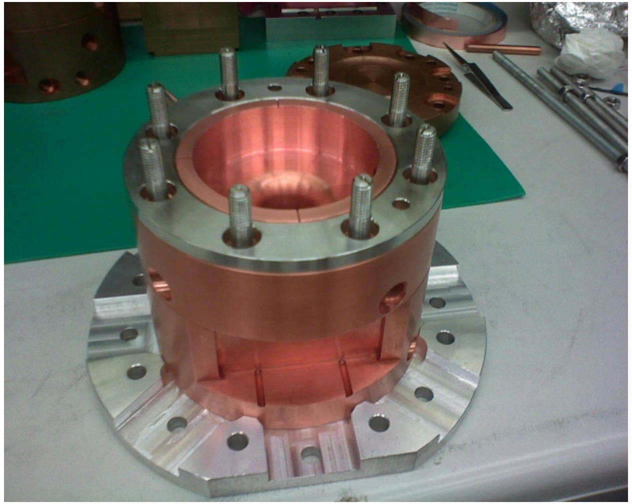
Ideal Conductor

Ideal Conductor: an ideal conductor is neutral and has an infinite amount of freely moving electrons

high-voltage inductor



copper waveguide



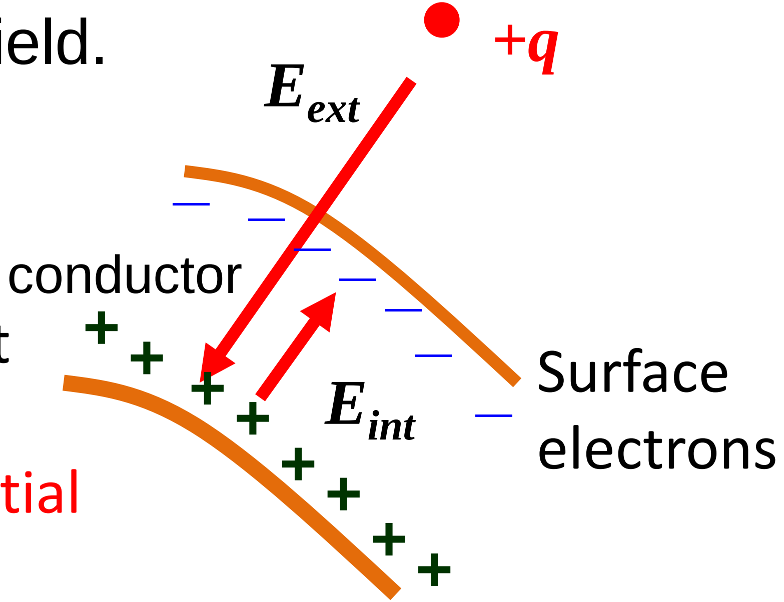
cavity of a microwave electron gun

Zero Electric field in a Conductor

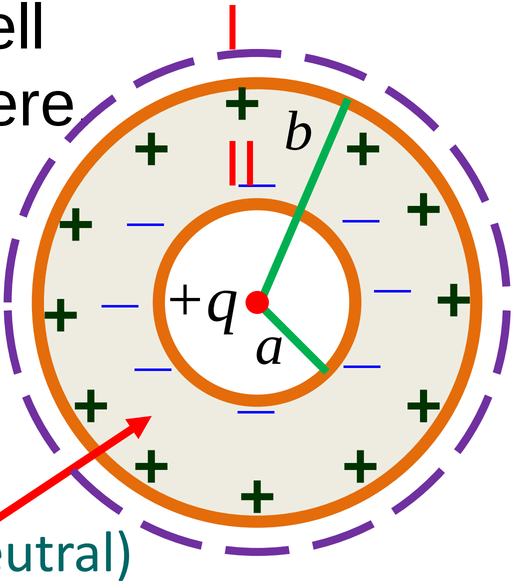
If there exists an electric field internal to a conductor, the electric field will induce motion of free electrons to balance the internal electric field.

$$E_{net} = E_{ext} - E_{int} = 0$$

As a result, (1) the net electric field in an ideal conductor must be zero,
(2) a conductor is an equipotential object.



E.g. A charge q is inside a conducting shell between $a \le R \le b$. Find V & E everywhere.



Apply Gauss law $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$

$$I.\, R \ge b \;\Rightarrow\; E_{R,I} = \frac{q}{4\pi\varepsilon_0 R^2}$$

$$V_I(R) = -\int_\infty^R E_{R,I}\, dR = \frac{q}{4\pi\varepsilon_0 R}$$

$II.\, a \le R \le b$ (inside conductor, equipotential), $\Rightarrow E_{R,II} = 0$

$$a \le R \le b \;\Rightarrow V_{II}(R) = -\int_\infty^b E_{R,I}\, dR - \cancelto{0}{\int_b^R E_{R,II}\, dR} = \frac{q}{4\pi\varepsilon_0 b}$$

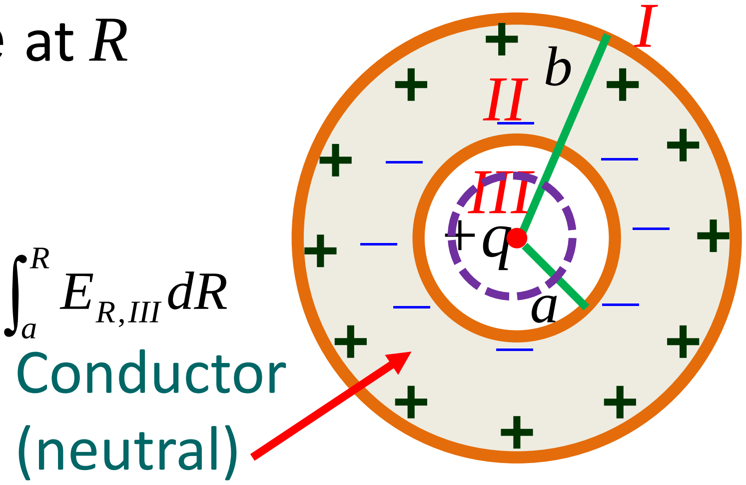
$III.\, R \le a$ (vacuum region)

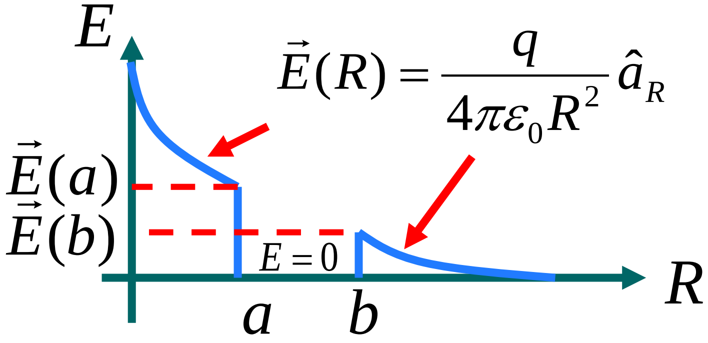
Find the Gaussian surface at R

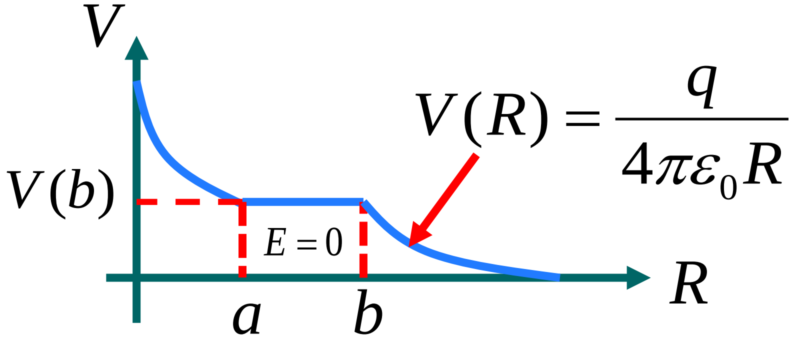
to obtain $E_{R,III} = \dfrac{q}{4\pi\varepsilon_0 R^2}$

$$V_{III}(R) = -\int_{\infty}^{b} E_{R,I}\, dR - \int_{b}^{a} \cancelto{0}{E_{R,II}}\, dR - \int_{a}^{R} E_{R,III}\, dR$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b}\right) + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right)$$







6.1 導體

Conductor

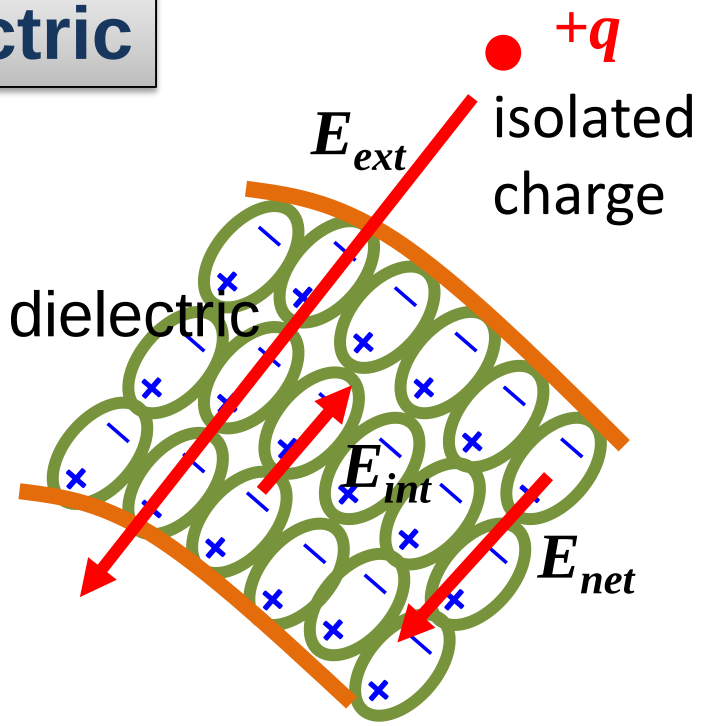
- A perfect or an ideal conductor has an infinite amount of free electrons.
- An external electric field moves the conducting electrons to the surface.
- The motion of the electrons stops when the internal field, built up from separation of charges, offsets the external electric field exactly.
- Eventually, the net electric field in an ideal conductor is 0.

物質靜電學
Electrostatics in Material

6.2 介電物質
Dielectric

Dielectric

- A dielectric only has bound charges or electric dipoles, which cannot move freely.
- An external electric field orients the dipole charges in a dielectric.
- The polarized dipole field partially cancel the external field, resulting a reduced electric field in a dielectric.



$$E_{net} = E_{ext} - E_{int} < E_{ext}$$

Electric Flux Density

In a dielectric, retain the Gauss law

$$\nabla \cdot \vec{D} = \rho \quad \text{(Gauss law)}$$

where ρ is the volume charge density for isolated charges.

In vacuum $\vec{E} = \dfrac{\vec{D}}{\varepsilon_0}$, where ε_0 is the vacuum permittivity

In dielectric, introduce the relative permittivity or dielectric constant $\varepsilon_r > 1$, to account for the field reduction, so that

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r \varepsilon_0} < \frac{\vec{D}}{\varepsilon_0}$$

E.g. A charge q is inside a dielectric shell between $a \le R \le b$. Find V & E everywhere.

$I.\ R > b,\ III.\ R < a$ (vacuum region)

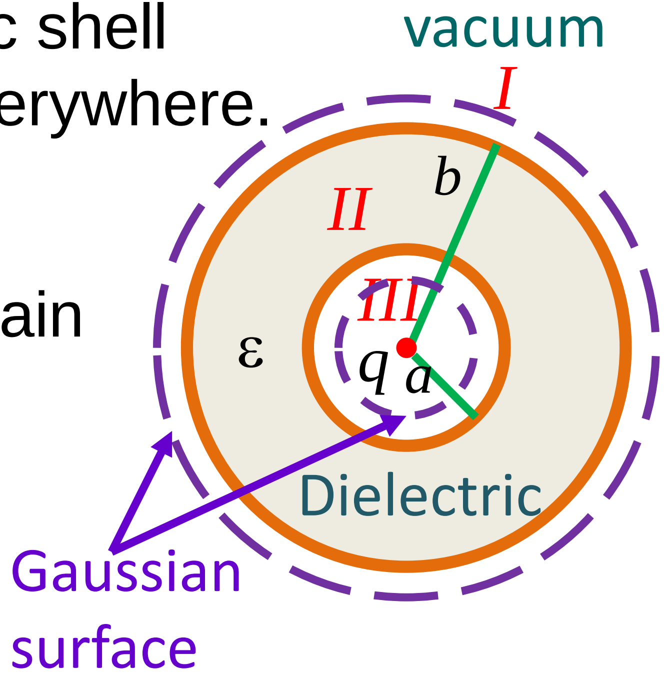
Use Gauss law $\oint_S \vec{D}\cdot d\vec{s} = q$ to obtain

$$\vec{D}_{R,I/III} = \frac{q}{4\pi R^2}\hat{a}_R$$

Use $\vec{D} = \varepsilon_0 \vec{E}$ in vacuum to obtain

$$\vec{E}_{R,I/III} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

The electric potential in I is $V_I(R) = -\int_{\infty}^{R} \vec{E}_{R,I}\cdot d\vec{R} = \dfrac{q}{4\pi\varepsilon_0 R}$



$II.\ a < R < b$ (dielectric region)

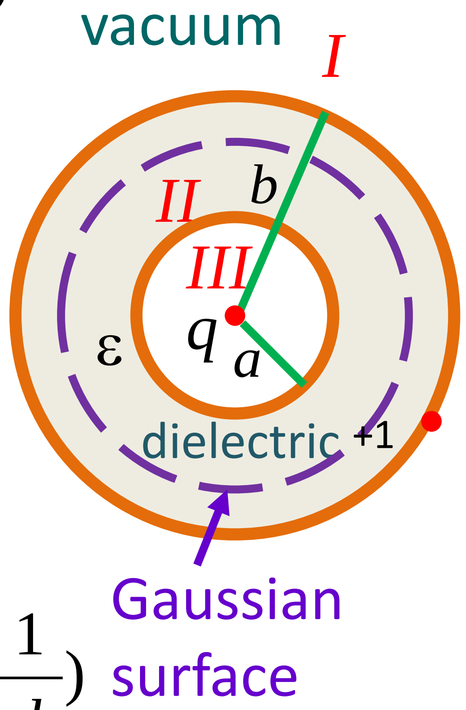
Again, use Gauss law $\oint_S \vec{D}\cdot d\vec{s} = q$

to obtain $\vec{D}_{R,II} = \dfrac{q}{4\pi R^2}\hat{a}_R$

Use $\vec{D} = \varepsilon\vec{E}$ to obtain $\vec{E}_{R,II} = \dfrac{q}{4\pi\varepsilon R^2}\hat{a}_R$

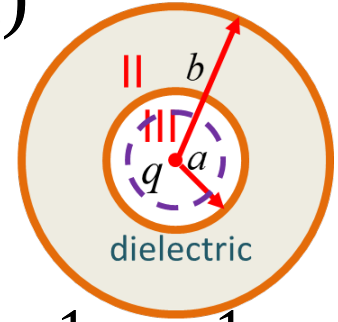
The electric potential in *II* is

$$V_{II}(R) = V_I(b) - \int_b^R E_{R,II}\,dR = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon R} - \frac{1}{\varepsilon b}\right)$$



+1

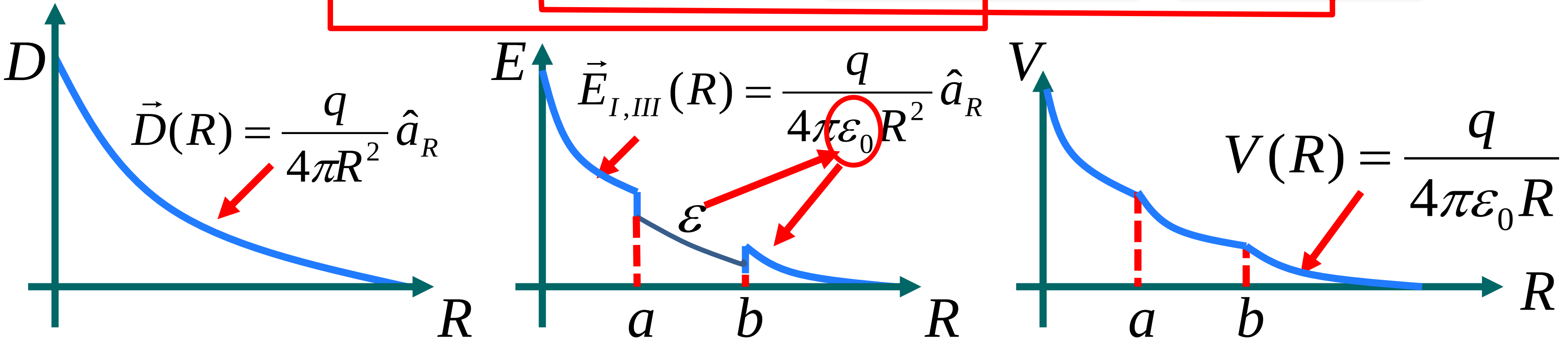
$III.\ R < a$ (vacuum region)



Recall $\vec{E}_{R,III} = \dfrac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R$

The electric potential is

$$V_{III}(R) = V_{II}(a) - \int_a^R E_{R,III}\,dR = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b} + \frac{1}{\varepsilon_0 R} - \frac{1}{\varepsilon_0 a}\right)$$



6.2 介電物質

Dielectric material

- A dielectric contains ordered or disordered dipoles.
- An external electric field can align the dipoles in a dielectric to partially cancel out the external electric field.
- For a dielectric, $\boxed{\nabla \cdot \vec{D} = \rho}$ and $\boxed{\vec{E} = \dfrac{\vec{D}}{\varepsilon_r \varepsilon_0}}$,

where $\boxed{\varepsilon_r > 1}$ is the field reduction factor in a dielectric.

物質靜電學
Electrostatics in Material

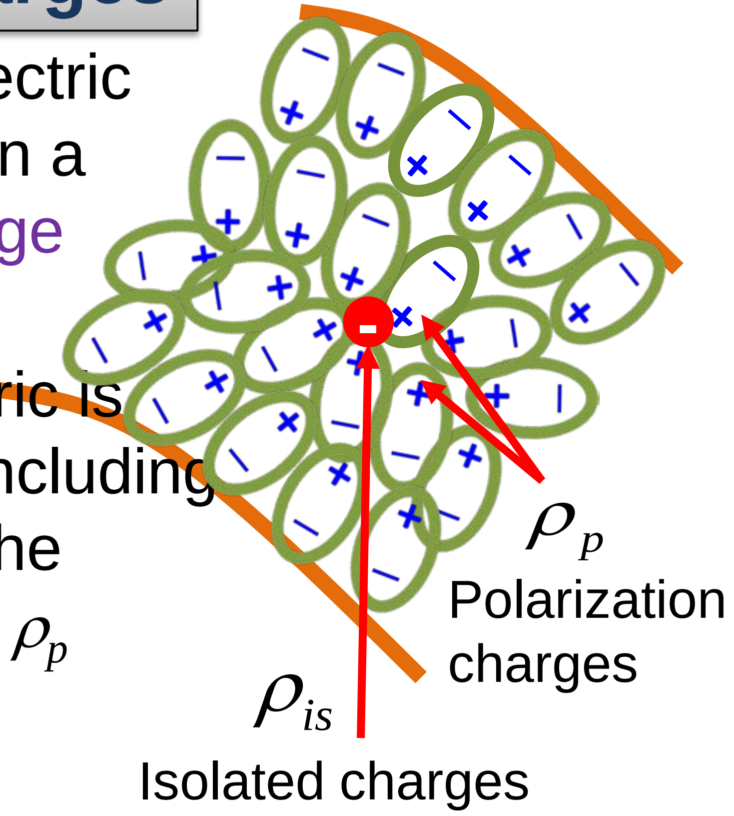
6.3 極化密度向量
Polarization Density Vector

Polarization Charges

An isolated charge generates an electric field that aligns the electric dipoles in a dielectric, creating polarization charge density ρ_p.

The net electric field $\vec{E}$ in a dielectric is related to the total charge density, including the isolated charge density ρ_{is} and the induced polarization charge density ρ_p

$$\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_{is} + \rho_p$$



Polarization (Density) Vector

Define the electric flux density ***D*** as a quantity dealing with isolated charges through

$$\nabla \cdot \vec{D} = \rho_{is}$$

Define the polarization vector $\vec{P}$ as $\nabla \cdot \vec{P} = -\rho_p$

$$\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_{is} + \rho_p = \nabla \cdot \vec{D} - \nabla \cdot \vec{P}$$

$$\Rightarrow \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

In a simple (linear, isotropic, non-dispersive) dielectric, P and E has the linear relationship,

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where χ_e is called electric susceptibility.

Recall $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$

$$= \varepsilon_0 \varepsilon_r \vec{E}$$

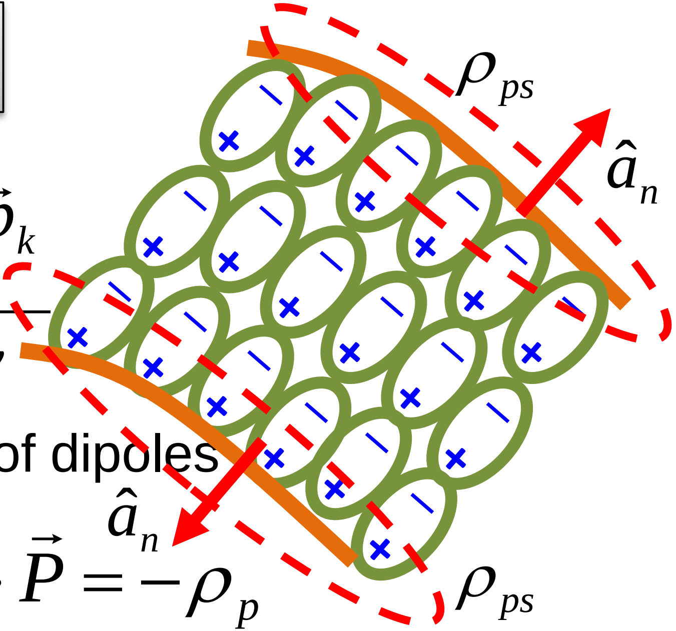
where the relative permittivity or dielectric constant $\varepsilon_r \equiv 1 + \chi_e$ is a material parameter.

What is P ?

$\vec{P}$ is in fact *the average vector sum over all the* *electric dipole moments per unit volume at a point in a dielectric.*

$$\vec{P} \equiv \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{p}_k}{\Delta v}$$

n: number density of dipoles



Apply the divergence theorem to $\nabla \cdot \vec{P} = -\rho_p$

$$\Rightarrow \int_V \nabla \cdot \vec{P} dv = \oint_S \vec{P} \cdot d\vec{s} = -\int_V \rho_p dv \Rightarrow \int_V \rho_p dv + \oint_S \vec{P} \cdot d\vec{s} = 0$$

For a neutral material $\int_V \rho_p dv + \oint_S \rho_{ps} ds = 0 \Rightarrow \vec{P} \cdot \hat{a}_n = \rho_{ps}$

ρ_{ps} : polarization surface charge density

Dielectric Strength

Dielectric breakdown - an electric field is so strong that it pulls apart and accelerate the dipole charges to cause disintegration of a dielectric.

Dielectric Strength = dielectric breakdown field

materials	dielectric strength (kV/mm)
dry air	~3
mineral oil	~15
glass	~30
mica	~200

6.3 極化密度向量

Polarization density vector

- The polarization $\vec{P}$ density vector is the average vector sum of the electric dipole moments per unit volume in a dielectric.
- In a “simple” material, $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ and $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

  where $\varepsilon_r \equiv 1 + \chi_e$.
- Dielectric strength is the breakdown electric field of a dielectric.

物質靜電學
Electrostatics in Material

6.4 靜電邊界條件
Electrostatic boundary conditions

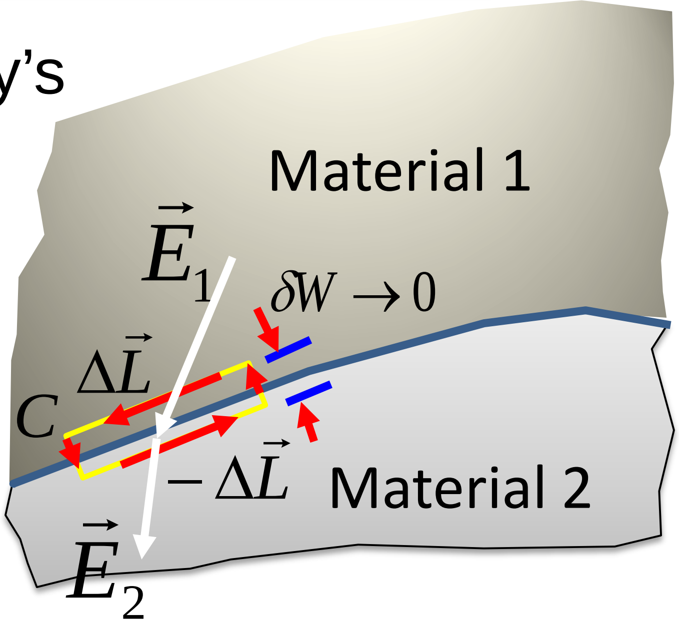
Tangential Component of Fields

Use the integral form of the Faraday's law $\oint \vec{E} \cdot d\vec{l} = 0$ at the interface.

$$\oint_C \vec{E} \cdot d\vec{l} = \vec{E}_1 \cdot \Delta\vec{L} + \vec{E}_2 \cdot (-\Delta\vec{L})$$

$$= E_{1t}\Delta L - E_{2t}\Delta L = 0$$

$$\Rightarrow \quad E_{1t} = E_{2t} \qquad \text{or} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$



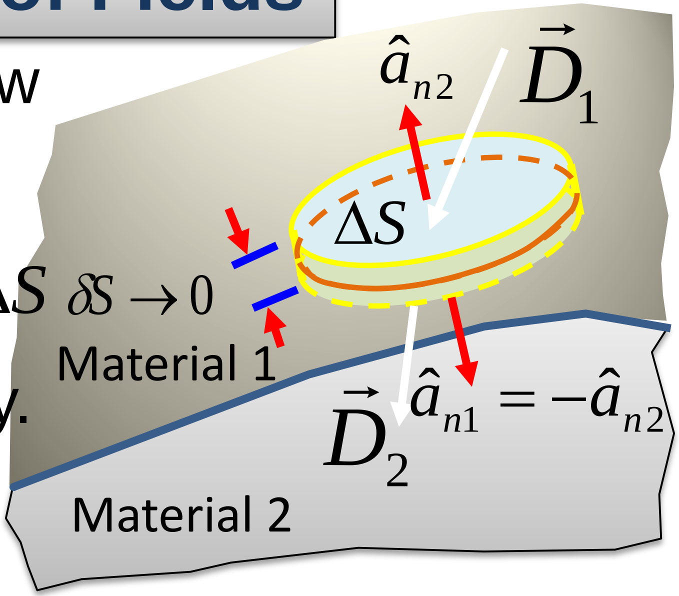
*Tangential components of the electric field intensity across a boundary are continuous

Normal Component of Fields

Use the integral form of the Gauss law

$\oint_S \vec{D}\cdot d\vec{s} = \int \rho_s ds$ at the interface.

$$\oint_S \vec{D}\cdot d\vec{s} = (\vec{D}_1\cdot\hat{a}_{n2} + \vec{D}_2\cdot\hat{a}_{n1})\Delta S = \rho_s \Delta S$$



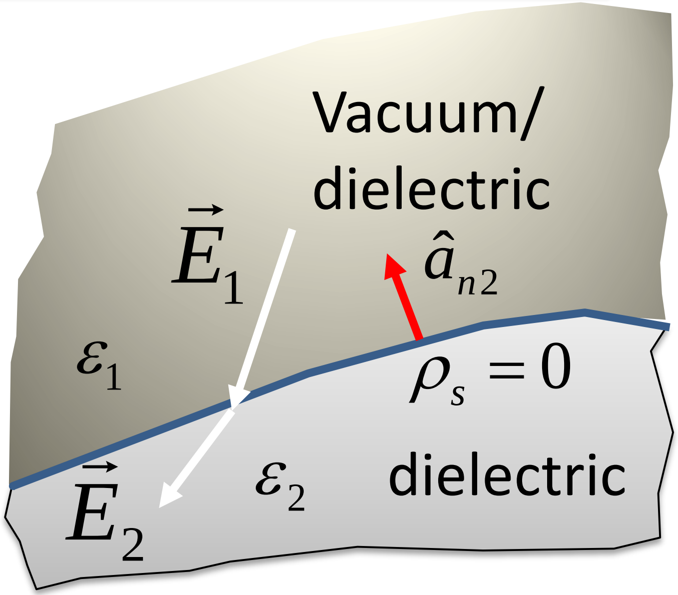
where ρ_s is the surface charge density.
With reference to $\hat{a}_{n2}$, the boundary condition is

$$\left|D_{1n} - D_{2n}\right| = \left|\rho_s\right| \quad \text{or} \quad \hat{a}_{n2}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$$

*Normal components of the electric flux density across a dielectric boundary are discontinuous, if there exist isolated surface charges.

Dielectric/Vacuum-Dielectric Interface

On a dielectric, there is no isolated surface charge $\rho_s = 0$



Tangential components

$$E_{1t} = E_{2t}$$

Tangential components of $\vec{E}$ are continuous.

Normal components

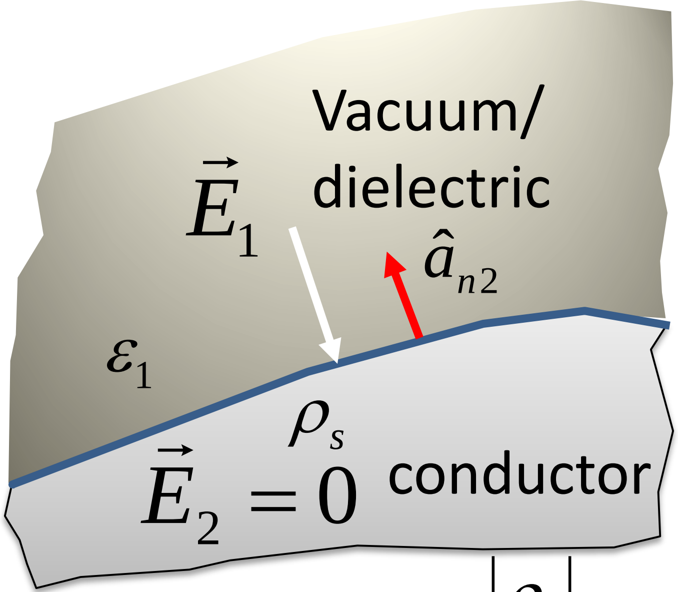
$$\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s = 0 \Rightarrow D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

Normal components of the electric flux density $\vec{D}$ are continuous.

Dielectric/Vacuum-Conductor Interface

In a conductor, $\vec{E}_2 = 0, \vec{D}_2 = 0$

On a conductor, there exists a surface charge ρ_s .



Tangential components

$$E_{1t} = E_{2t} = 0$$

Normal components

$$\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad \Rightarrow \quad \hat{a}_{n2} \cdot \vec{D}_1 = \rho_s \quad \text{or} \quad |E_{1n}| = \frac{|\rho_s|}{\varepsilon_1}$$

The electric field lines are along the normal direction of a conducting surface.

E.g. A charge q is inside a conducting shell between $a \le R \le b$. Find the surface charge density at $R = a$ and b.

$$@R = a$$

$$\hat{a}_{n2} = -\hat{a}_R, \ \vec{D} = \varepsilon_0 \vec{E} = \frac{q}{4\pi a^2}\hat{a}_R$$

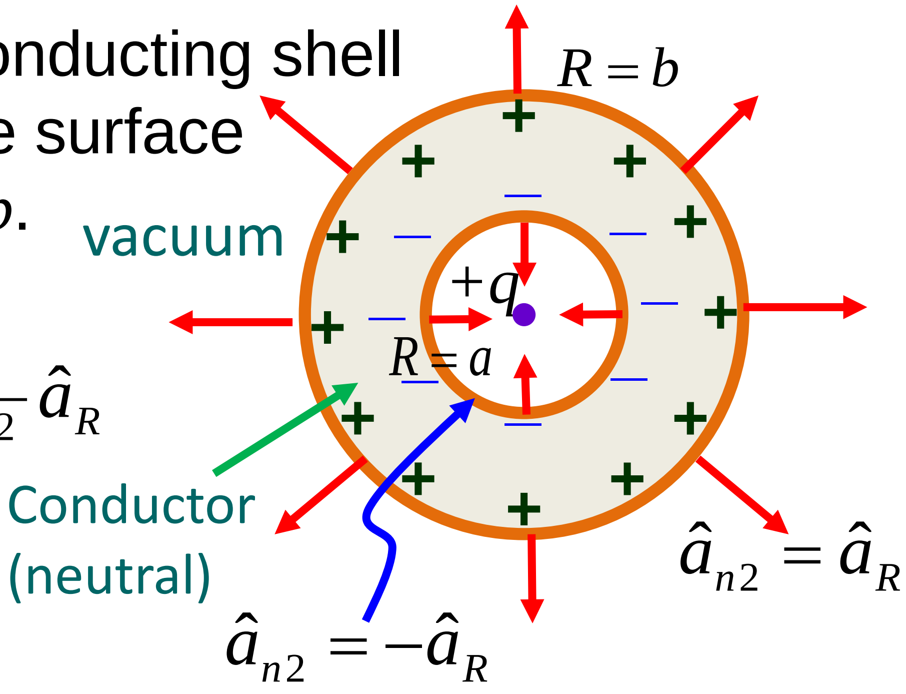
$$\rho_s = \hat{a}_{n2} \cdot \vec{D} = -\frac{q}{4\pi a^2}$$

*total surface charge = -*q*

$$@R = b$$

$$\hat{a}_{n2} = \hat{a}_R, \ \vec{D} = \varepsilon_0 \vec{E} = \frac{q}{4\pi b^2}\hat{a}_R \Rightarrow \rho_s = \hat{a}_{n2} \cdot \vec{D} = \frac{q}{4\pi b^2}$$

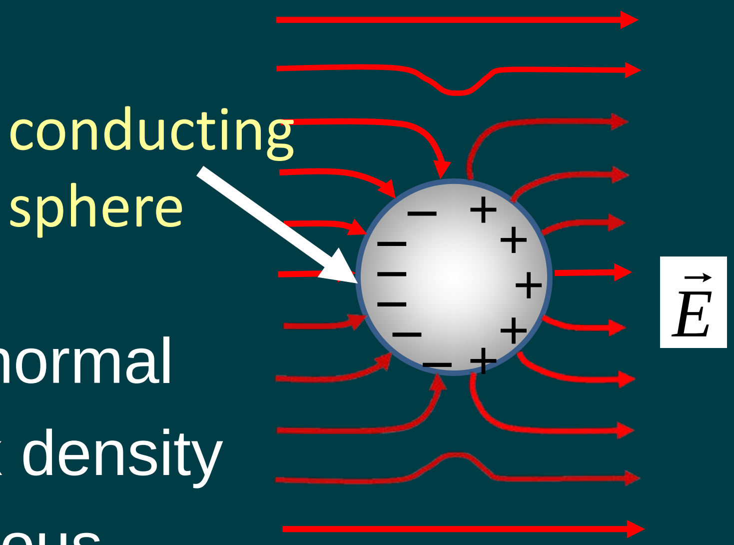
*total surface charge = +*q*



6.4 靜電邊界條件

Electrostatic boundary conditions

- At an interface, the tangential components of the electric field intensity across a boundary are continuous.
- On a conducting surface, the electric fields are normal to the surface, inducing surface charges on the surface.
- Without surface charges, the normal components of the electric flux density across a boundary are continuous.

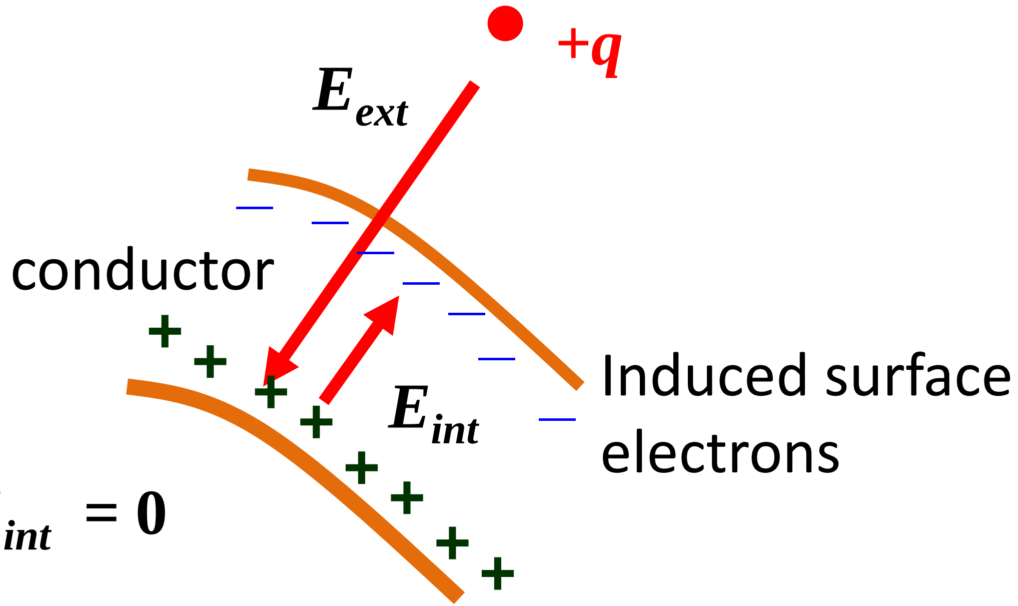


物質靜電學
Electrostatics in Material

6.5 單元回顧 Review

單元回顧

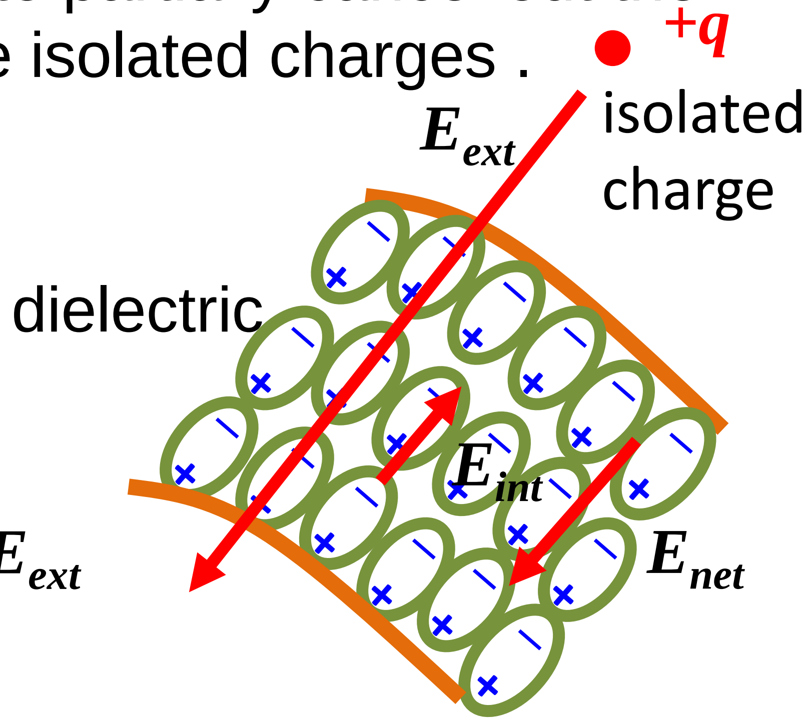
1. The net electric field in an ideal conductor is zero.



$$E_{net} = E_{ext} - E_{int} = 0$$

單元回顧

2. In a dielectric, polarization charges are induced by isolated charges to partially cancel out the electric field from the isolated charges .



$$E_{net} = E_{ext} - E_{int} < E_{ext}$$

單元回顧

3. In general, the Gauss law is given by

$$\nabla \cdot \vec{D} = \rho$$

where ρ is the volume charge density for isolated charges.

4. In a linear dielectric, $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$, where ε_0 is the vacuum permittivity, ε_r is the relative permittivity or dielectric constant of the dielectric.

單元回顧

5. The electric flux density vector can be expressed as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E}$$

with the polarization density vector $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

where χ_e is called the electric susceptibility and the relative permittivity $\varepsilon_r \equiv 1 + \chi_e$.

6. ε_r is usually large than 1 in a dielectric, which can be considered as the reduction factor of an electric field in a dielectric, manifested from $\vec{E} = \dfrac{\vec{D}}{\varepsilon_0} \times \dfrac{1}{\varepsilon_r}$.

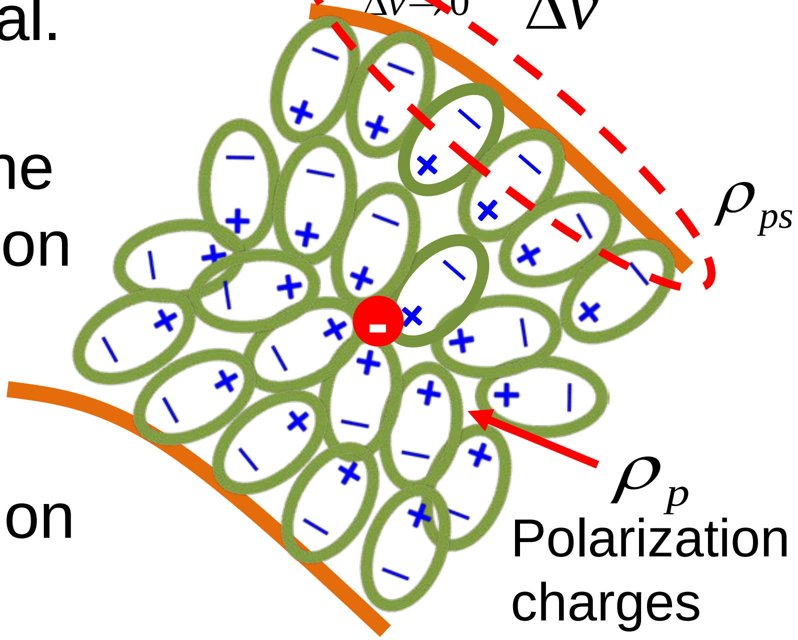
單元回顧

$$\vec{P} \equiv \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{p}_k}{\Delta v}$$

7. The polarization (density) vector is the volume average of electric dipole moment at a "point" in a material.

8. $\nabla \cdot \vec{P} = -\rho_p$, where ρ_p is the volume density of the polarization charges.

9. $\hat{a}_{ns} \cdot \vec{P} = \rho_{ps}$, where ρ_{ps} is the surface density of the polarization charges.



單元回顧

10. General boundary conditions for electrostatics are:

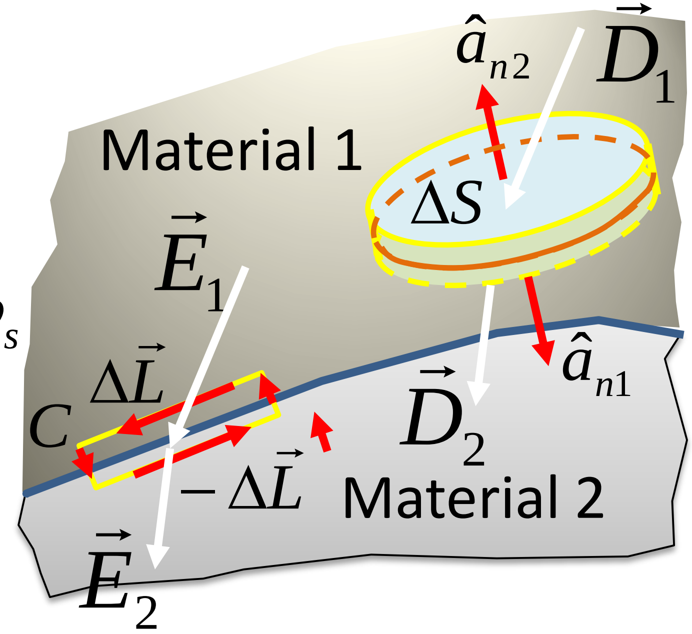
Tangential components $E_{1t} = E_{2t}$

Normal components $\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$

Surface charges ρ_s only exit on a conducting surface.

In an ideal conductor, $\vec{E} = 0, \vec{D} = 0$

An electric field line reaches a conducting surface along the surface normal direction.



THANK YOU FOR YOUR ATTENTION

Review Questions

1. Explain why the static electric field in an ideal conductor must be zero.

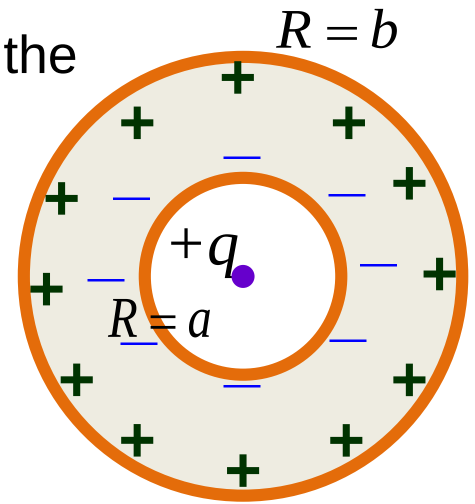
Ans: By definition, an ideal or perfect conductor has an infinite amount of free electrons in it. Under the excitation of an external electric field, if the electric field inside the perfect conductor is not zero, the free electrons will be moved by the field to build up a field in the opposite direction until the two fields cancel each other exactly to stop the moving of the free electrons. In equilibrium, the net electric field inside a perfect conductor will settle to zero.

2. Explain why the electric field lines entering a perfect conductor must be along the surface normal of the conductor.

Ans: For electrostatics, the tangential electric fields across a material boundary must be continuous. However, the electric field inside a perfect conductor is zero and thus the tangential component of the electric field on the outer surface of the conductor can only be zero. Therefore, there only exists a normal component of the electric field on the outer surface of the conductor.

3. In Sec. 6.4, we first derived the surface charge of the conducting sphere at $R = a$, given by

$$\rho_s = \hat{a}_{n2} \cdot \vec{D} = -\frac{q}{4\pi a^2}$$



Argue from charge conservation that the surface charge at $R = b$ must be $\rho_s = \dfrac{q}{4\pi b^2}$.

Ans: The conductor between $a \le R \le b$ is neutral. The total amount of surface charge at $R = a$ surface must be the same as that at $R = b$, except that the signs of the charges are opposite. Apparently, the total charge at $R = a$ is

$$-\frac{q}{4\pi a^2} \times 4\pi a^2 = -q$$

From charge conservation, there will be $+q$ distributed uniformly on the surface of $R = b$. Therefore, the surface charge at $R = b$ must be $\rho_s = \dfrac{q}{4\pi b^2}$.

4. Compare the electric field intensity at a distance R from a point charge q in vacuum and in a space filled with a dielectric with relative permittivity ε_r.

Ans: In vacuum, from Gauss law with ε_r =1, the electric field intensity at R is given by

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

In dielectric, from Gauss law with ε_r > 1, the electric field intensity at R is given by

$$\vec{E} = \frac{q}{4\pi\varepsilon_0\varepsilon_r R^2}\hat{a}_R.$$

For a typical ε_r > 1 in a dielectric, the electric field in a dielectric is reduced by a factor of ε_r.