EE2030Linear Algebra

Homework#2

Due: 03/22/2023 10:10(Wed)

1. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?
   (a) The plane of vectors (b_1,b_2,b_3) with $b_1 = b_2$.
   (b) The plane of vectors with $b_2 = 1$.
   (c) The vectors with $b_1b_2b_3$=0.
   (d) All linear combination of $\boldsymbol{\nu}$ = (1,4,0) and $\boldsymbol{\omega}$ = (2,2,2).
   (e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
   (f) All vectors with $b_1 \leq b_2 \leq b_3$.

2. If we add an extra column $\mathbf{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{x}$=$\boldsymbol{b}$ solvable exactly when the column space doesn't get larger — it is the same for A and $[A\ \mathbf{b}]$?

3. Construct a 3 by 3 matrix whose column space contains (1,1,0) and (1,0,1) but not (1,1,1). Construct a 3 by 3 matrix whose column space is only a line.

4. The equation $x - 3y - z = 0$ determines plane in $\mathbf{R}^3$.What is the matrix A in this equation? Which are the free variables? The special solutions are (3,1,0) and _____.

5. The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in former problem. One particular point on this plane is (12,0,0). All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}$$

6. Suppose column 1 +column 3 +column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)?What is the special solution? What is the nullspace?

7. Construct a matrix whose column space contains (1,1,0) and (0,1,1) and whose nullspace contains (1,0,1) and(0,0,1).

8. Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} and B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} and M = \begin{bmatrix} a & b \\ c & \end{bmatrix}$$

1.

(a) Set $S = \{(b_1, b_2, b_3) : (b_1 = b_2 = 0, b_3 \in \mathbb{R}) \text{ or } (b_1 = b_2 \neq 0 \in \mathbb{R}, b_3 = 0)\}$

If $b_1 = b_2 = 0$, then for each $n \in \mathbb{R}$

$n(b_1, b_2, b_3) + n(b_1, b_2, b_3) = (0, 0, 2nb_3) = 2n(b_1, b_2, b_3) \in S$

If $b_1 = b_2 \neq 0$, then for each $k \in \mathbb{R}$

$k(b_1, b_2, b_3) + k(b_1, b_2, b_3) = (2kb_1 + 2kb_2 + 0) = 2k(b_1, b_2, b_3) \in S$

$\therefore$ S is a subspace

(b) Set $S = \{(b_1, b_2, b_3) : b_2 = 1, (b_1 \in \mathbb{R}, b_3 \in \mathbb{R}, b_1 \neq b_3) \text{ or } (b_1 = b_3 = 0)\}$

If $b_1 \neq b_3$, then for each $n \in \mathbb{R}$

$n(b_1, b_2, b_3) + n(b_1, b_2, b_3) = (2nb_1 + 2n + 2nb_3) \notin S \quad \because 2n \neq 1 \text{ for } n \neq \frac{1}{2}$

$\therefore$ S is not a subspace

(c) Set $S = \{(b_1, b_2, b_3) : \text{at least one of } b_1, b_2, b_3 \text{ is } 0\}$

$(1, 0, 1) + (0, 1, 0) = (1, 1, 1) \notin S$

$\therefore S$ is not a subspace

(d) Set $S = \{ a(1,4,0) + b(2,2,2) : a \in \mathbb{R}, b \in \mathbb{R} \}$

let $n, k \in \mathbb{R}$, $[k(1,4,0) + n(2,2,2)] + [n(1,4,0) + k(2,2,2)]$

$= (n+k)(1,4,0) + (n+k)(2,2,2) \in S$ $\therefore S$ is a subspace

(e) Set $S = \{(b_1, b_2, b_3) : b_1 + b_2 + b_3 = 0\}$

let $n \in \mathbb{R}$, $n(b_1, b_2, b_3) + n(b_1, b_2, b_3) = 2n(b_1, b_2, b_3) = (2nb_1, 2nb_2, 2nb_3)$

$2n(b_1 + b_2 + b_3) = 0$, $\therefore S$ is a subspace.

(f) Set $S = \{(b_1, b_2, b_3) : b_1 \le b_2 \le b_3\}$

$-1(b_1, b_2, b_3) = (-b_1, -b_2, -b_3) \notin S$

$\therefore S$ is not a subspace

2. (1) b is not a pivot column, ie. b is a free column

(2) getting larger: $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, column space: $x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$

no getting larger: $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix}$, column space: $x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+\cancel{x_1\begin{bmatrix}1\\0\\0\end{bmatrix}}+\cancel{2x_2\begin{bmatrix}0\\1\\0\end{bmatrix}}$

(3) $[A|b]$ is soluable iff b is not pivot column iff $[A|b]$ doesn't get larger from $[A]$

3. (1) set $A\underline{x} = x_1\begin{bmatrix}1\\1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\0\\1\end{bmatrix}+x_3\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&0\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

(2) set $A\underline{x} = x_1\begin{bmatrix}a\\b\\c\end{bmatrix}+x_2\begin{bmatrix}0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}a&0&0\\b&0&0\\c&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

4. $A=\begin{bmatrix}1&-3&-1\end{bmatrix}$, y and z are free variables

$A\underline{v}=\underline{0}$, $\underline{v}=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $\Rightarrow x-3y-z=0$, $x=3y+z$, $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3y+z\\y\\z\end{bmatrix}=y\begin{bmatrix}3\\1\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix}$

(1, 0, 1) is the other special solution.

5. $\left[\begin{array}{ccc:c}1&-3&-1&12\\0&0&0&0\\0&0&0&0\end{array}\right] \Rightarrow x=3y+z+12 \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3y+z+12\\y\\z\end{bmatrix}=y\begin{bmatrix}3\\1\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}12\\0\\0\end{bmatrix}$

6. (1) Column 5

(2) Special solution is $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(3) Nullspace $= \left\{ n \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} . \ n \in \mathbb{R} \right\} \Rightarrow$ a line in R^5

7. 2 pivot column and 2 free column is needed. But there are only 3 columns.

8. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 9 & -\frac{9}{2} \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$

9. $\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

10. A: $\begin{bmatrix} I & I \end{bmatrix} N = \underline{0}$, $N = \begin{bmatrix} -I \\ I \end{bmatrix}$

B: $\begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} N = \underline{0}$, $N = \begin{bmatrix} -I \\ I \end{bmatrix}$

C: $\begin{bmatrix} I & I & I \end{bmatrix} N = \underline{0}$, $N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$

9. Choose vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $A = \boldsymbol{u}\boldsymbol{v}^T$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} and A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}$$

$A = \boldsymbol{u}\boldsymbol{v}^T$ is the natural form for every matrix that has rank r=1.

10. What is the nullspace matrix N (containing the special solutions) for A,B,C ?

$$A = \begin{bmatrix} I & I \end{bmatrix} and B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} and C = \begin{bmatrix} I & I & I \end{bmatrix}$$