EE 205003 Session 14

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New vector space

All 3×3 matrices form a vector space $\mathbf{M}$

(we can add matrices, multiply by scalar & there is a new matrix)

($A + B$, cA) (not AB for now)

(All of rules are satisfied)

Subspaces

- All upper triangular matrices ($\mathcal{U}$)
- All symmetric matrices ($\mathbf{S}$)
- All diagonal matrices ($\mathbf{D}$)

Note : $\mathbf{D} = \mathcal{U} \cap \mathbf{S}$

Q: What is the dimension of $\mathbf{D}$?

$\dim(\mathbf{D}) = 3$

basis : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q: What is the dimension of $\mathbf{M}$?

$\dim(\mathbf{M}) = 9$

basis : (standard)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Very similar to $\mathbb{R}^9$ just arrange in a matrix form

Q: What is the dimension of the subspace $\mathbf{S}$?

$\dim(\mathbf{S}) = 6$ (pick 3 diagonal elements + 3 in upper right)
(lower left determined by upper right)

basis : (Also basis for $\mathbf{M}$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(Not basis for $\mathbf{M}$)

Q: What is the dim. of the subspace $\mathcal{U}$?

dim$(\mathcal{U}) = 6$ (3 diagonal + 3 upper right)

basis : (different from $\mathbf{S}$)

$$\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$ (happens to be a subset of basis of $\mathbf{M}$)

Other subspaces

$$\mathbf{S} \cap \mathcal{U} = \text{symmetric \& upper triangular}$$
$$= \mathbf{D}$$
$$\dim(\mathbf{S} \cap \mathcal{U}) = 3$$

$$\mathbf{S} \cup \mathcal{U} = \text{symmetric or upper triangular}$$

(Not a subspace since a symmetric matrix + a upper triangular matrix is NOT in $\mathbf{S} \cup \mathcal{U}$ in general)

(Analogy : two lines in $\mathbb{R}^2$ is NOT a subspace. Need to fill in between them)

Other subspaces (cont.)

Instead,

$\mathbf{S} + \mathcal{U}$ = any element of $\mathbf{S}$ + any element of $\mathcal{U}$

(sum subspace)

= All $3 \times 3 = \mathbf{M}$

$$\left(\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ & 0 & 2 \\ & & 0 \end{bmatrix} \right)$$

$\dim(\mathbf{S} + \mathcal{U}) = 9$

In general,

$$\dim(\mathbf{S}) + \dim(\mathcal{U}) = \dim(\mathbf{S} + \mathcal{U}) + \dim(\mathbf{S} \cap \mathcal{U})$$

$$6 \quad + \quad 6 \qquad\qquad 9 \quad + \quad 3$$

Differential equations as a vector space

$\frac{d^2y}{dx^2} + y = 0$, solution to this equation is a element of the nullspace
possible solutions :

$$y = \underbrace{\cos x, \sin x}_{\text{basis}}, e^{ix} \text{ (special solutions)}$$

complete solutions : $y = c_1 \cos x + c_2 \sin x$

dim(solution space) $= 2$ (since this is a 2^{nd}-order equation)

(Don't look like vectors, but we can build a vector space from it since we can add & multiply by a scalar)

Rank one matrices

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

Q: What is the rank of A ?

rank$(A) = 1$ (row2 $= 2 \cdot$ row1)

$$\dim(\mathbf{C}(A)) = \underset{\substack{\shortparallel \\ 1}}{\text{rank}} = \dim(\mathbf{C}(A^\intercal))$$

or

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}_{1\times 3}$$

(each col. is a multiple of col.1 or
each row is a multiple of row1)

Q: What is the rank of A ? (cont.)

In general, for every rank-1 matrix

$$A = \mathbf{u}\mathbf{v}^\intercal = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} & & \end{bmatrix}$$

(building blocks for more complicated matrix,
e.g., a 5×17 rank-4 matrix can be written as
comb. of 4 rank-1 matrices)

(To be discussed later)

Q: Is subset of rank-1 matrices a subspace ?

No, since sum of two rank-1 matrices may NOT be rank-1

Another example

In $\mathbb{R}^4$, the set of all vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ for which $v_1 + v_2 + v_3 + v_4 = 0$

Q: Is this a subspace ?

Yes ! it contains $\mathbf{0}$ & closed under ADD & scalar MUL

$$\left(\mathbf{w} + \mathbf{v} = \begin{bmatrix} w_1 + v_1 \\ w_2 + v_2 \\ w_3 + v_3 \\ w_4 + v_4 \end{bmatrix} \text{ sum of all components} = 0 \right)$$

Q: What is the dimension ?

This is a null space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

rank$(A) = 1$

$\Rightarrow$ dim$(\mathbf{N}(A)) = n - r = 4 - 1 = 3$

Basis :

Find special solutions

$$\left.\begin{array}{l} col_2 = 1 \cdot col_1 \\ col_3 = \cdots \\ col_4 = \cdots \end{array}\right) \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Q: What is $\mathbf{C}(A)$?

1 pivot & $m = 1 \Rightarrow \mathbf{C}(A) = \mathbb{R}^1$

Q: What is $\mathbf{N}(A^\intercal)$?

$$\mathbf{y}^\intercal \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y = 0 \Rightarrow \mathbf{N}(A^\intercal) = \{0\}$$

(basis is empty set)

Q: What is $\mathbf{C}(A^\intercal)$?

$$\dim(\mathbf{C}(A^\intercal)) = r = 1$$

$$\text{basis} : \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Check dimension

$$\dim(\mathbf{C}(A^\intercal)) + \dim(\mathbf{N}(A)) = 1 + 3 = 4 = n$$

$$\dim(\mathbf{C}(A)) + \dim(\mathbf{N}(A^\intercal)) = 1 + 0 = 1 = m$$