EECS 205003 Session 21

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Outline

Ch5 Determinants

- Ch 5.1 The Properties of Determinants
- Ch 5.2 Permutations and Cofactors
- Ch 5.3 Cramer's Rule, Inverses, and Volumes

Determinant formulas & cofactors

We learned properties of det. Now, we are ready to obtain formulas for det:

1. Products of pivots
2. The "big formula"
3. Cofactors

Products of pivots (use Elimination)

Recall from SES-20,

$$\begin{aligned} PA = LU &\Rightarrow (detP)(detA) = (detL)(detU) \\ &\Rightarrow \pm(detA) = 1 \cdot d_1 d_2 \cdots d_n \\ &\Rightarrow detA = \pm d_1 \cdots d_n \\ &\quad \text{(for invertible } A\text{)} \end{aligned}$$

For singular A, $detA = 0 \because detU = 0$
(zero rows in U)

The big formula

2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

break $[a\ b]$ into simple rows, $[a\ b] = [a\ 0] + [0\ b]$
break $[c\ d]$ into simple rows, $[c\ d] = [c\ 0] + [0\ d]$

Now apply linearity in rows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$ (row 1 with row 2 fixed)

$\Downarrow 3(b)$ $\Uparrow 3(b)$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$ (row 2 with row 1 fixed)

$3(a)$ =0 =0

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

1 & 2

$= ad - bc$ ($\#$ of terms $= 2^2 = 4$, $\#$ of nonzero terms $= 2! = 2$)

3×3:

break each row to simpe rows

e.g., $[a_{11}\ a_{12}\ a_{13}] = [a_{11}\ 0\ 0] + [0\ a_{12}\ 0] + [0\ 0\ a_{13}]$ (3 choices)

Same for row 2 & row 3

(3 choices) (3 choices)

$\Rightarrow$ a total of 3^3 simple det !

If a column choice is repeated, then the simple $det = 0$

e.g., $[a_{11}\ 0\ 0]\ [a_{21}\ 0\ 0]$

$$\Rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ \times & \times & \times \end{vmatrix} = 0$$

$\Rightarrow$ nonzero terms only comes from different columns

(3! ways to order columns)

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \underset{(1,\,2,\,3)}{\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix}} + \underset{(1,\,3,\,2)}{\begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}} + \underset{(2,\,1,\,3)}{\begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}}$$

$$+ \underset{(2,\,3,\,1)}{\begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}} + \underset{(3,\,1,\,2)}{\begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}} + \underset{(3,\,2,\,1)}{\begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}}$$

$$= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}$$

$$+ a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Ch 5.2 Permutations and Cofactors

An easy way to remember:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

But this only work for 2×2 & 3×3

Not for higher n

(e.g., for 4×4 this only produces 8 products but we actually have 4! = 24 products)

In general $(n \times n)$

There are n! column ordering
Let $(\alpha, \beta, \cdots, \omega)$ be one possible ordering

$\Rightarrow$ this simple $det = \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$
(± 1: determined by $P = (\alpha, \beta, \cdots, \omega)$)

(e.g., $P = (1, 2, 3) = \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}$

$P = (3, 1, 2) = \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$

$P = (2, 1, 3) = \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} \cdots)$

then

$$detA = \sum_{n!\ terms} (detP) a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$ (the "big formula")

where (α, β, $\cdots$, ω) is some permutations of (1, 2, $\cdots$, n)

Ex: $A = U$

The only nonzero term comes from the diagonal

$\Rightarrow detU = +u_{11}u_{22}\cdots u_{nn}$

(All other column orderings pick at least one entry below the diagonal
Since all entries of U below the diagonal is zero, $det = 0$)

$\Rightarrow detI = +(1)(1)\cdots(1) = 1$

(This formula satisfies property 1
You can check property 2, 3 are also true)

Ex: Z is the identity matrix except column 3

$$detZ = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = +(1)(1)(\mathrm{c})(1)$$ (Only nonzero term)

($\because$ If you pick a, b, or d, we used up column 3.
For row 3, we can only pick 0
$\Rightarrow$ row 3 $=$ zero row $\Rightarrow det = 0$)

Determinant by Cofactors

Recall: For 3×3 matrix A

$detA =$

$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

$= a_{11}\ (a_{22}a_{33} - a_{23}a_{32}) : C_{11}$

$+\ a_{12}\ (a_{23}a_{31} - a_{21}a_{33}) : C_{12}$

$+\ a_{13}\ (a_{21}a_{32} - a_{22}a_{31}) : C_{13}$

(cofactors: 2×2 det comes from matrices in row 2 & 3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

(Still choose one entry from each column and row when we split the det)

Let M_{1j} be a submatrix of size n-1 by crossing out 1st row & jth column of A

$\Rightarrow detA = a_{11}detM_{11} - a_{12}detM_{12} + a_{13}detM_{13}$

Note: $\begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & & \\ & a_{21} & a_{23} \\ & a_{31} & a_{33} \end{vmatrix} = -a_{12}detM_{12}$

(we need to watch signs) (one row change)

$$\begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix} = -\begin{vmatrix} & a_{13} & \\ a_{21} & & a_{22} \\ a_{31} & & a_{32} \end{vmatrix} = (-1)^2\begin{vmatrix} a_{13} & & \\ & a_{21} & a_{22} \\ & a_{31} & a_{32} \end{vmatrix}$$

(two row changes)

In general,

$C_{1j} = (-1)^{1+j} detM_{1j}$

Cofactor expansion:

$detA = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
(Just another form of the "big formula")

Note: we can do the expansion for any row

<u>**The most general form**</u> (cofactor formula)

$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$
where $C_{ij} = (-1)^{i+j} detM_{ij}$

Q : **Can we do cofactor expansion down a column ?**

Yes ! $\because det A^{\mathsf{T}} = det A$

$$det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Important Note:

We can find det of order n recursively via the cofactor formula

Application: tridiagonal matrices

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$|A_1| = |1| = 1$$

$$|A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1|A_3| - 1|A_2| = -1$$

In fact,

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$

we have a sequence which repeats every 6 terms:

$|A_1| = 1,\ |A_2| = 0,\ |A_3| = -1,\ |A_4| = -1$

$|A_5| = 0,\ |A_6| = 1,\ |A_7| = \quad 1,\ |A_8| = \quad 0$