# EE2030Linear Algebra

## Homework#3

## Due: 03/29/2023 10:10(Wed)

1. What conditions on $b_1$, $b_2$, $b_3$, $b_4$ make each system solvable? Find $\mathbf{x}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

2. Find by elimination the rank of $A$ and also the rank of $A^T$:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \text{(rank depends on q)}$$

3. Reduce to U$\mathbf{x} = \mathbf{c}$ (Gaussian elimination) and then R$\mathbf{x} = \mathbf{d}$ (Gauss-Jordan):

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \mathbf{b}.$$

Find a particular solution $\mathbf{x}_p$ and all homogeneous solutions $\mathbf{x}_n$.

4. $U$ comes from $A$ by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

5. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

4!

6. Choose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like $(x_2, x_1, x_3, x_4)$ and $(x_4, x_3, x_1, x_2)$. Those 24 vectors, including $\mathbf{x}$ itself, span a subspace $\mathbf{S}$. Find specific vectors $\mathbf{x}$ so that the dimension of $\mathbf{S}$ is: (a) zero, (b) one, (c) three, (d) four.

$b_4 - 3b_1 - 3b_3 + 6b_1$

$b_1 - 2b_3 + 4b_1$

1. $$\left[\begin{array}{cc:c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \Rightarrow \left[\begin{array}{cc:c} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{array}\right] \Rightarrow \left[\begin{array}{cc:c} 1 & 0 & 5b_1-2b_3 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_4-3b_3+3b_1 \end{array}\right]$$

If $b_2-2b_1=0$ and $b_4-3b_3+3b_1=0$, then it is solvable. $\underline{x}=\begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{bmatrix}$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 3 & 3 & b_4-3b_1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 1 & 5b_1-2b_3 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_4+3b_1-3b_3 \end{array}\right]$$

If $b_2-2b_1=0$ and $b_4+3b_1-3b_3=0$, then it is solvable. $\underline{x} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \\ 0 \end{bmatrix}$

2. $$A=\begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$ $A$ has rank 2.

$$A^T=\begin{bmatrix} 1 & 2 & -1 \\ 4 & 11 & 2 \\ 0 & 5 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$ $A^T$ has rank 2.

$$A=\begin{bmatrix}1&0&1\\1&1&2\\1&1&q\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&1\\0&1&1\\0&1&q-1\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&1\\0&1&1\\0&0&q-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&1\\0&1&0\\0&0&q-2\end{bmatrix}$$

If $q=2$, then $A$ has rank 2.

Otherwise, $A$ has rank 3.

$$A^T=\begin{bmatrix}1&1&1\\0&1&1\\1&2&q\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&1&1\\0&1&q-1\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&1&1\\0&0&q-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&0\\0&1&0\\0&0&q-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&q-2\end{bmatrix}$$

If $q=2$, then $A^T$ has rank 2.

Otherwise, $A^T$ has rank 3.

3. $$\left[\begin{array}{cccc:c}1&0&2&3&2\\1&3&2&0&5\\2&0&4&9&10\end{array}\right]\Rightarrow\left[\begin{array}{cccc:c}1&0&2&3&2\\0&3&0&-3&3\\0&0&0&3&6\end{array}\right]\Rightarrow\left[\begin{array}{cccc:c}1&0&2&0&-4\\0&3&0&0&9\\0&0&0&3&6\end{array}\right]\Rightarrow\left[\begin{array}{cccc:c}1&0&2&0&-4\\0&1&0&0&3\\0&0&0&1&2\end{array}\right]$$

$$\begin{cases}x_1=-2x_3-4\\x_2=3\\x_4=2\end{cases}\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2x_3-4\\3\\x_3\\2\end{bmatrix}=x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+\begin{bmatrix}-4\\3\\0\\2\end{bmatrix}$$

$$x_p=\begin{bmatrix}-4\\3\\0\\2\end{bmatrix},\quad x_n\Rightarrow\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}$$

4. $A=\begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix}$, $U=\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}$

$A\underline{x}=\underline{0}$

$\Downarrow$

$U\underline{x}=\underline{0}$

$\Downarrow$

$R\underline{x}=0$

$\begin{matrix}1&0&-1\\0&1&1\\0&0&0\end{matrix}$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}3\\1\\3\end{bmatrix}$ are a basis of $C(A)$

$\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are a basis of $C(U)$

$\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is a basis of $N(A)$

$\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is a basis of $N(U)$

$A^T=\begin{bmatrix}1&0&1\\3&1&3\\2&1&2\end{bmatrix}$

$U^T=\begin{bmatrix}1&0&0\\3&1&0\\2&1&0\end{bmatrix}$

$[1\ 3\ 2]$, $[0\ 1\ 1]$ are a basis of $C(A^T)$

$[1\ 3\ 2]$, $[0, 1, 1]$ are a basis of $C(U^T)$

Row spaces and Nullspaces stay fixed in Elimination

5.

$$A=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}, \quad B=\begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

If $c=0$, $d=2$, $A$ has rank 2.

If $c,d \in \mathbb{R}$, $c \neq \pm d$, $B$ has rank 2.

$\Rightarrow$ If $c=0$, $d=2$, then $A$, $B$ have rank 2.

6. (a) If $x_1=x_2=x_3=x_4=0$, then $\dim(S)=0$

(b) If $x_1=x_2=x_3=x_4\neq 0$, then $\dim(S)=1$

$\because \frac{4!}{2!2!}\times\frac{1}{2}=3$

(c) If 2 components of $\underline{x}$ are positive, the other two are negative, and $x_1^2=x_2^2=x_3^2=x_4^2$

(d) Except above situation, $\dim(s)=4$

7. Without computing $A$, find bases for its four fundamental subspaces:

1 2 3 4
6 13 20 27
9 26 44 62

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

1 2 0 -2
0 1 0 -1
0 0 1 2

1 0 0 0
0 1 0 -1
0 0 1 2

44 18+16+1

8. (Left nullspace) Add the extra column $\mathbf{b}$ and reduce $A$ to echelon form:

$$\begin{bmatrix} 0 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

1 4 7
2 5 8
3 6 9

$$[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

1 4 7
0 -3 -6
~~0 -6 -12~~

A combination of the rows of $A$ has produced the zero row. What combination is it? (Look at $b_3$ - $2b_2$ + $b_1$ on the right side.) Which vectors are in the nullspace of $A^T$ and which vectors are in the nullspace of A?

9. $\mathbf{M}$ is the space of 3 by 3 matrices. Multiply every matrix $X$ in $\mathbf{M}$ by

1 -2 1

1 4 7
0 -3 -6
0 0 0

$$x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1 2 3
-1 -2

1 0 -1
0 1 2
0 0 0

(a)which matrices $X$ leads to $AX$ = zero matrix?
(b)which matrices have the form $AX$ for some matrix $X$?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n - r) + r = 9$?

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

1 2 3
0 -3 -6
0 0 0
-1

$\Rightarrow$

1 0 $\frac{3}{2}$
0 3 6
0 0 0

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

0 1 0

7.

Basis of $C(A)$: $\begin{bmatrix}1\\6\\9\end{bmatrix}, \begin{bmatrix}2\\13\\26\end{bmatrix}, \begin{bmatrix}3\\20\\44\end{bmatrix}$

Basis of $N(A)$: $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$

Basis of $C(A^T)$: $\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}, \begin{bmatrix}6 & 13 & 20 & 27\end{bmatrix}, \begin{bmatrix}9 & 26 & 44 & 62\end{bmatrix}$

Basis of $N(A^T)$: $\begin{bmatrix}0 & 0 & 0\end{bmatrix}$

8. (1) $\begin{bmatrix}1 & 0 & 0\\-4 & 1 & 0\\1 & -2 & 1\end{bmatrix}$

(2) $\begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}$

$N(A) = \left\{ a \cdot \begin{bmatrix}1\\-2\\1\end{bmatrix}, a \in \mathbb{R} \right\}$

$N(A^T) = \left\{ b \cdot \begin{bmatrix}1 & -2 & 1\end{bmatrix}, b \in \mathbb{R} \right\}$

9\. (a) $x = a\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, a \in \mathbb{R}$

(b)

(a) $N(A) = \left\{ a\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, a \in \mathbb{R} \right\}$

(b) $C(A) =$