Signals and Systems

Homework 4 — Due : Mar. 22 2024

Problem 1 (30 pts)**.** Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{3}\right)^{n-2} u[n-2] \quad \text{and} \quad h[n] = u[n+2].$$

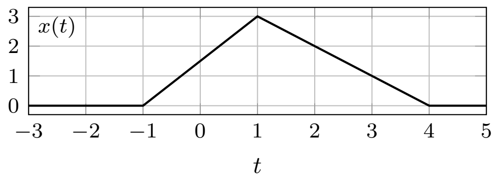
Determine and plot the output $y[n] = x[n] * h[n]$.

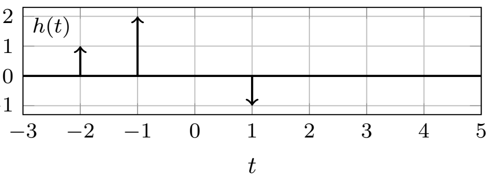
Problem 2 (20 pts)**.** Show that the causality for a continuous time linear system is equivalent to the following statement:

> For any time t_0 and any input $x(t)$ such that $x(t) = 0$ for $t < t_0$, the corresponding output $y(t)$ must also be zero for $t < t_0$.

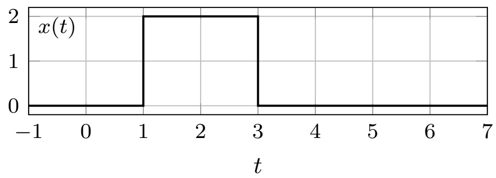
Problem 3 (30 pts)**.** Determine and sketch the convolution of the following signals:

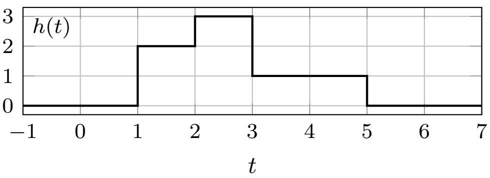
(a)





(b)





Problem 4 (20 pts)**.** Check if the following impulse responses correspond(s) to stable and/or causal LTI systems?

(a) $h_1(t) = e^{-6|t|}$

(b) $h_2(t) = e^{-4t}u(t-2)$

(c) $h_3[n] = 5^n u[3-n]$

(d) $h_4[n] = 3^{-n} n u[n-1]$

Problem 1 (30 pts). Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{3}\right)^{n-2} u[n-2] \quad \text{and} \quad h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{k-2} u[k-2]\, u[n-k+2]$$

$u[k-2]$: $0,\ k \le 1$ $\qquad$ $u[n-k+2]$: $\to 0,\ k \ge n+3$

$$= \sum_{k=2}^{n+2} \left(\frac{1}{3}\right)^{k-2}$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{n} \left(\frac{1}{3}\right)^{k} = \frac{1\left(1-\left(\frac{1}{3}\right)^{n+1}\right)}{1-\frac{1}{3}} = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n}, & \text{for } n \ge 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$\frac{3}{2} - \frac{1}{6} \quad \frac{9}{6} - \frac{1}{6} = \frac{4}{3}$

$\frac{3}{2} - \frac{1}{18} = \frac{26}{18} = \frac{13}{9}$

$\frac{3}{2} - \frac{1}{54} = \frac{81-1}{54}$

$= \frac{40}{27}$

Problem 2 (20 pts). Show that the causality for a continuous time linear system is equivalent to the following statement:

> For any time t_0 and any input $x(t)$ such that $x(t) = 0$ for $t < t_0$, the corresponding output $y(t)$ must also be zero for $t < t_0$.

causal $\rightarrow$ 只跟現在與過去相關

Denote t_0+s as t_s , $s\in\mathbb{R}$, and let k_s be the corresponding coefficient of t_s

For linear system, $\int_{-\infty}^{0} k_s x(t_s)\,ds \longrightarrow \int_{-\infty}^{0} k_s y(t_s)\,ds$

$\int_{-\infty}^{0} k_s x(t_s)\,ds = 0 \quad \forall s \in (-\infty, 0)$ and k_s are arbitary,

(linear)

which implies that $\int_{-\infty}^{0} k_s y(t_s)\,ds = 0 \quad \forall s \in (-\infty, 0)$ and k_s are arbitary

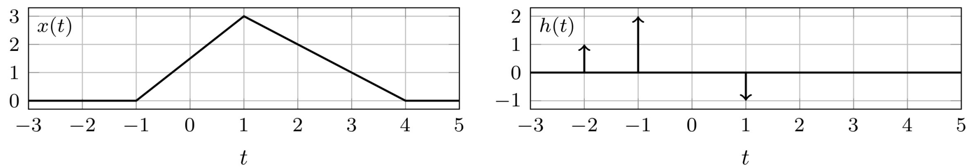
$\Rightarrow y(t_s) = 0 \quad \forall s < 0$ i.e. $\forall t < t_0$

因為 linear 的關係，若 $t<t_0$, $x(t)=0 \longrightarrow t<t_0$, $y(t)=0$

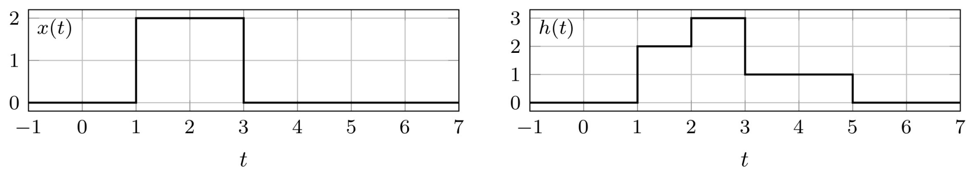
則此 linear system is causal.

Problem 3 (30 pts). Determine and sketch the convolution of the following signals:

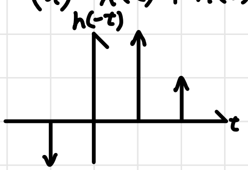
(a)

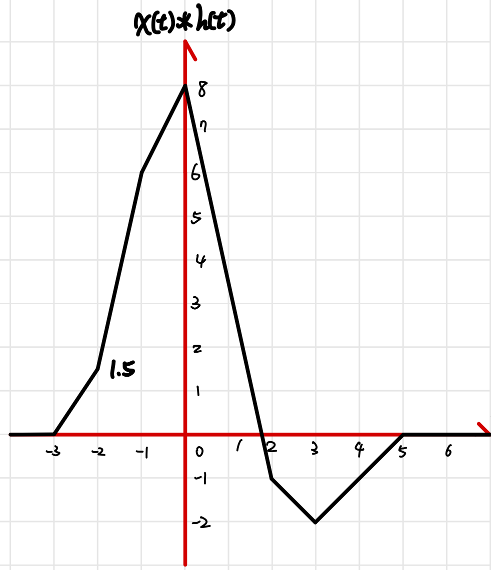


(b)



(a) $x(t) * h(t) = \begin{cases} \frac{3}{2}(t+3), & -3 \le t \le -2 \\ \frac{3}{2} + \frac{9}{2}(t+2), & -2 \le t \le -1 \\ 6 + 2(t+1), & -1 \le t \le 0 \\ 8 - \frac{9}{2}t, & 0 \le t \le 2 \\ -1 - (t-2), & 2 \le t \le 3 \\ -2 + (t-3), & 3 \le t \le 5 \\ 0, & t \le -3 \text{ or } t \ge 5 \end{cases}$





(b) $x(t) * h(t) = \begin{cases} 4(t-2), & 2 \le t \le 3 \\ 4 + 6(t-3), & 3 \le t \le 4 \\ 10 - 2(t-4), & 4 \le t \le 5 \\ 8 - 4(t-5), & 5 \le t \le 6 \\ 4 - 2(t-6), & 6 \le t \le 8 \\ 0, & t \le 2 \text{ or } t \ge 8 \end{cases}$

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(a) $h_1(t) = e^{-6|t|}$

(b) $h_2(t) = e^{-4t}u(t-2)$

(c) $h_3[n] = 5^n u[3-n]$

(d) $h_4[n] = 3^{-n} n u[n-1]$

(a) $$\int_{-\infty}^{\infty} \left|e^{-6|\tau|}\right| d\tau = 2\int_0^{\infty} \left|e^{-6\tau}\right| d\tau$$
$$= 2\int_0^{\infty} e^{-6\tau} d\tau$$
$$= 2 \cdot \left.\frac{e^{-6\tau}}{-6}\right|_0^{\infty}$$
$$= 2 \cdot \frac{0-1}{-6} = \frac{1}{3} < \infty$$

$\Rightarrow$ stable

for $t<0$, $h_1(t) \neq 0 \Rightarrow$ not causal

(b) $$\int_{-\infty}^{\infty} |h_2(\tau)| d\tau = \int_2^{\infty} e^{-4\tau} d\tau$$
$$= \left.\frac{e^{-4\tau}}{-4}\right|_2^{\infty}$$
$$= \frac{1}{2} < \infty$$

$\Rightarrow$ stable

for $t<0$, $h_2(t) = 0 \Rightarrow$ causal

(c) $$\sum_{k=-\infty}^{\infty} |h_3[k]| = \sum_{k=-\infty}^{3} 5^k = \sum_{k=-3}^{\infty} 5^{-k}$$
$$= \frac{125\left(1-\left(\frac{1}{5}\right)^{\infty}\right)}{1-\frac{1}{5}}$$
$$= 125 \times \frac{5}{4} = \frac{625}{4} < \infty$$

$\Rightarrow$ stable

for $n<0$, $x[n] \neq 0 \Rightarrow$ not causal

(d) $$\sum_{k=-\infty}^{\infty} |h_4[k]| = \sum_{k=1}^{\infty} 3^{-k} k = \sum_{k=1}^{\infty} \frac{k}{3^k}$$

$$\boxed{\begin{aligned} &\lim_{x\to\infty} \frac{x}{3^x} \\ &= \lim_{x\to\infty} \frac{1}{3^x \ln 3} \\ &= 0 \end{aligned}} \rightarrow \sum_{k=1}^{\infty} 3^{-k} k < \infty$$

$\Rightarrow$ stable

for $n<0$, $x[n] = 0 \Rightarrow$ causal