CS2336 Discrete Mathematics

Exam 2
November 19, 2018 (10:10–12:30)

Answer all questions. Total marks = 110. Maximum Score = 100. For all the proofs, if it is incomplete, a large portion of marks may be deducted.

1. Peter is studying a recurrence as shown below:

   $$a_n = a_{n-1} + 6a_{n-2}.$$

   He tries to plug in $a_0 = 4$ and $a_1 = -3$, and finds that using the recurrence, he will obtain $a_2 = 21$, $a_3 = 3$, $a_4 = 129$, and so on.

   Peter conjectures that for any integer $n \geq 0$, a_n would obey the following formula:

   $$a_n = 3 \times (-2)^n + 3^n.$$

   Indeed, the recurrence for a_n is called a *linear recurrence*, and there is a standard method to find a formula for a_n (without referring to other terms like a_{n-1} or a_{n-2}); however, Peter does not know this. Fortunately, Peter has learnt induction technique a few weeks ago.

   (15%) Use induction to show that Peter's formula is correct.

2. A graph is a structure that consists of vertices and edges. A graph is *simple* if (i) each edge connects two distinct vertices, and (ii) for every two vertices, there is at most one edge connecting them. Consider a simple graph G with 100 vertices. It is known that the degree of each vertex (i.e., number of edges connecting to each vertex) is a multiple of 3.

   (15%) Show that there are 4 vertices in G whose degrees are exactly the same.

3. Chef Nicholas is a very talented cook. Give him a frying pan, and a stack of pancakes and waffles, he can flip freely any pieces of pancakes and waffles at the top of the stack.

   For instance, suppose that the frying pan has a stack of 6 pieces of pancakes and waffles as follows (`P` for pancake, `W` for waffle):

   `P W P P W P` (reading from bottom to top)

   If Nichoas flips the top 5 pieces, the stack would become:

   `P P W P P W` (reading from bottom to top)

   Furthermore, if Nicholas continues to flip the top 3 pieces, the stack would become:

   `P P W W P P` (reading from bottom to top)

   And, with one more flip of top 4 pieces, the stack would become:

   `P P P P W W` (reading from bottom to top)

   (15%) Show that Nicholas can use at most $n - 1$ flips to flip any stack of n pieces of pancakes and waffles, so that all the pancakes are arranged together, while all the waffles are also arranged together.

4. Melissa conjectures that for any positive integer n, the product of any n consecutive positive integers is always divisible by $n!$. Her reasoning is as follows:

> *Any n consecutive positive integer must contain a multiple of n, so their product is divisible by n. Similarly, any n consecutive positive integer must contain a multiple of $n-1$, so their product is divisible by $n-1$. In general, we see that any n consecutive positive integer must contain a multiple of k, for any $k = 1, 2, 3, \ldots, n$. Thus, it follows that the product of any n consecutive positive integers must be divisible by k, for any $k = 1, 2, 3, \ldots, n$, so that it must be divisible by $n!$.*

   (5%) Explain why Melissa's reasoning is wrong.

   (10%) Give a correct proof showing that for any positive integer n, the product of any n consecutive integers is divisible by $n!$.

5. We want to arrange 6 boys and 4 girls in a circle. We consider two arrangements to be the same if one can be obtained from the other by rotation.

   (20%) Suppose that no two girls are adjacent to each other. How many different arrangemnets are there?

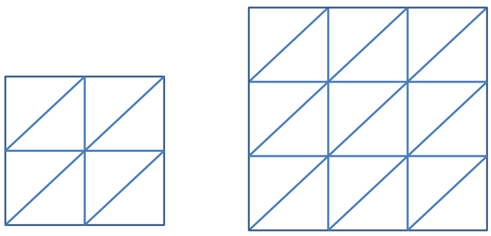
6. Consider a road network of an $m \times m$ grid, with roads running NS and EW. There are extra shortcuts, each links a grid point to its adjacent grid point in the NE direction.

   For instance, Figure 1(a) shows a road network of 2×2 grid with shortcuts, and Figure 1(b) shows a road network of 3×3 grid with shortcuts.

   If each move can go only E, N, or NE, how many ways can we travel from the SW corner to the NE corner in the road network:

   (a) (5%) when $m = 3$? (No explanation is needed.)
   (b) (5%) when $m = 4$? (No explanation is needed.)
   (c) (10%) for general m? (Express your solution in terms of m.)

   Check your understanding: There are 3 ways when $m = 1$, and 13 ways when $m = 2$.



(a) (b)

Figure 1: Figure for Question 6

7. (10%, Super Challenging) Consider two arrays $A[1..n]$ and $B[1..n]$, each with n entries. Each entry in these arrays contains an integer between 1 and n. Show that we can always find a contiguous subarray $A[i..j]$ and a contiguous subarray $B[i'..j']$, with $1 \leq i \leq j \leq n$ and $1 \leq i' \leq j \leq n$, such that the sum of all entries in these two subarrays are the same.

   For instance, set $n = 4$. Suppose $A[1..4] = \langle 2, 3, 2, 3 \rangle$ and $B[1..4] = \langle 4, 4, 4, 4 \rangle$. Then, entries in subarray $A[2..4]$ and entries in subarray $B[2..3]$ have the same sum.