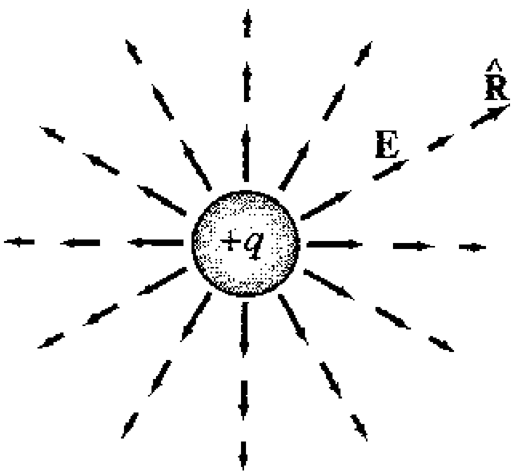
Chapter 3 Static Electric Fields

Electric Field Intensity*: The force, including magnitude and direction, experienced by a unit positive charge in space*

$$\vec{E} \equiv \lim_{q \to 0} \frac{\vec{F}}{q} \qquad \text{(V/m)} \qquad (3\text{-}1)$$

Electric-field lines are lines of force "felt" by a unit positive charge



Define *electric flux density* in vacuum as $\vec{D} = \varepsilon_0 \vec{E}$,

where v*acuum permittivity* is given by

$\varepsilon_0 \cong \dfrac{1}{36\pi} \times 10^{-9} \cong 8.854 \times 10^{-12}$ (Farad/m). This definition will come out handy when dealing with electric fields nearby a material.

Postulates of Electrostatics in vacuum

I. $\nabla \times \vec{E} = 0$ (3-2)

II. $\nabla \cdot \vec{D} = \rho$ (Gauss Law) (3-3)

or $\nabla \cdot \vec{E} = \rho / \varepsilon_0$ in vacuum.

where ρ is the volume charge density (C/m^3) of *free charges*.

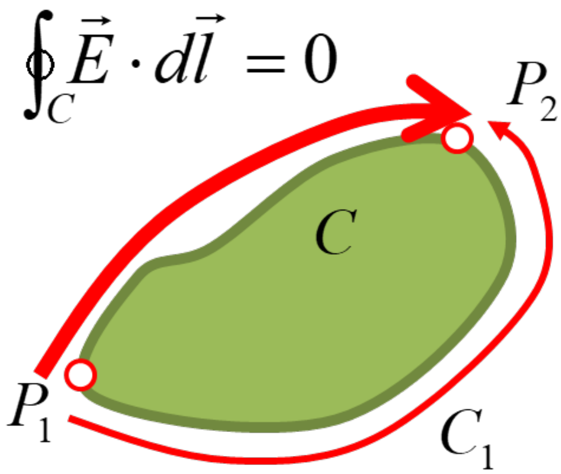
From postulate I and the physical meaning of curl, E is irrotational. Apply Stoke's theorem to Eq. (3-2) or $\int \nabla \times \vec{E} \cdot d\vec{s} = \oint_C \vec{E} \cdot d\vec{l} \Rightarrow$

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \qquad \text{(3-2.a)}$$

Refer to the following figure and recall that $\vec{E}$ is the electric force experienced by a unit positive charge. Since $\oint_C \vec{E} \cdot d\vec{l} = 0$, the work done by an electric field on a test particle with unit positive charge around a closed path is zero, or

$$\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} + \int_{P_2}^{P_1} \vec{E} \cdot d\vec{l} = 0 \Rightarrow \int_{P_1}^{P_2} \vec{E}_{C_1} \cdot d\vec{l} = \int_{P_1}^{P_2} \vec{E}_{C_2} \cdot d\vec{l} .$$

Therefore the work done by the electric field by moving a test charge between two arbitrary points in space is independent of the path. This situation is similar to moving a mass in a gravitation field and is the concept of a *conservative force.*



Conservative Force: Under a conservative force, free-space energy gain (loss) of an object is independent of its integration path. A conservative force is irrotational.

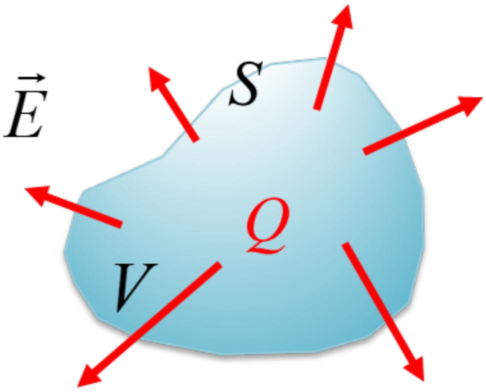
In vacuum, the often used expression for Eq. (3-3) is

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3-3.a}$$

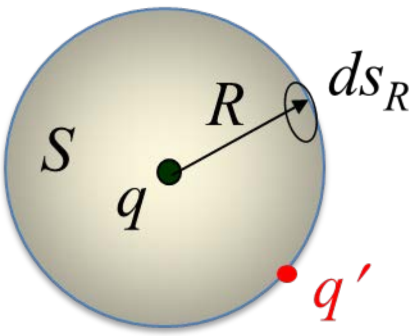
The electric field intensity from an aggregate of charges, according to the divergence theorem, is

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q \tag{3-3.b}$$

where Q is the free charges enclosed in the surface S. Clearly, Eq. (3-3.b) means that the total outward electric flux over a closed surface equals the charge enclosed by the surface.



Let's first use Eq. (3-3.b) to calculate the electric field surrounding a point charge.



Imagine a hypothetical spherical surface centered at a point charge q in space, shown above. On this surface the electric field must be a constant due to the symmetry of the problem. In Gauss's law, $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$, the differential surface is $d\vec{s} = d\vec{s}_R = \hat{a}_R R^2 \sin\theta d\theta d\phi$. In the dot product, $\vec{E} \cdot d\vec{s}$, only the $\vec{E}_R$ is preserved. Therefore, at a constant *R*, $\vec{E} = E_R \hat{a}_R$ is a constant and the integration $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$ gives $\vec{E} = \dfrac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R$.

If a charge of q_2 exists in an electric field generated by q_1, the force experienced by charge q_2 is given by

$$\vec{F}_{21} = \frac{q_2 q_1}{4\pi\varepsilon_0 R^2}\hat{a}_{R_{12}}$$

This is the famous *Coulomb's Law*: force exerted by q_1 on q_2.

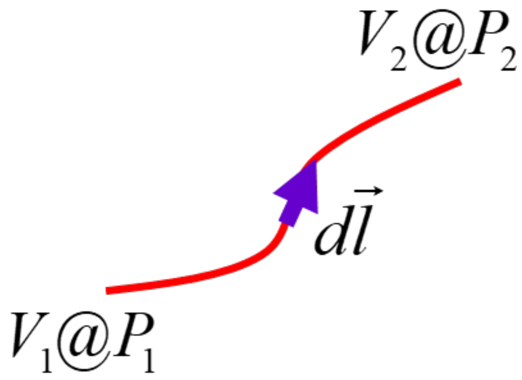
Electric Potential *V*

From the null identity of vector, $\nabla \times (\nabla V) = 0$ and postulate I, $\nabla \times \vec{E} = 0$, the electric field intensity can be written as

$$\vec{E} = \pm\nabla V \qquad \text{(3-2.b)}$$

Where *V* is a scalar. Note that both ± signs in the above expression are mathematically correct in $\nabla \times (\nabla V) = 0$. One can define the sign according the physical meaning that one would like to impose on *V*. Let's first write the following math

$$V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} \nabla V \cdot (\vec{a}_l dl) = \pm\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$



We recognize that the right hand side has the physical meaning of work done by the electric field on a unit charge. Therefore, one can define *V* as the

Work done externally by moving a unit positive (negative) charge from infinity to a positive (negative) charge is positive.

According to this definition, calculating V_{21} for moving a positive (negative) unit charge from infinity to a positive (negative) charge *q* must give a positive value. Let's try this calculation by assuming $q > 0$.

$$V(R_0)-V(\infty)=\pm\int_{\infty}^{R_0}\vec{E}\cdot d\vec{l}=\pm\int_{\infty}^{R_0}\frac{q}{4\pi R^2}\hat{a}_R\cdot\hat{a}_R dR=\mp\frac{q}{4\pi\varepsilon_0 R_0}$$

where $V(\infty)=0$, and R_0 is the final position the unit positive charge settles. In order to have $V(R_0)>0$ for this particular case, apparently we have to choose

$$\vec{E}=-\nabla V$$

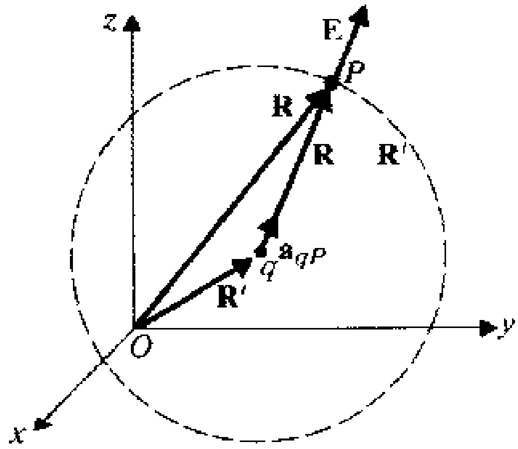
from the beginning. * Note that $d\vec{l}$ is always consistent with the definition in Chapter 3.

In summary, we have the electric potential at *R* generated by a charge *q* at origin

$$V(R) = \frac{q}{4\pi\varepsilon_0 R}$$

Alternatively, an electric potential is the work done externally on a unit positive charge when brought from infinity.

If the point charge is not at origin, the electric field and potential associated with it are rewritten as



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R}'\right|^2}\hat{a}_{\vec{R}-\vec{R}'} \quad \text{and} \quad V(R) = \frac{q}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R}'\right|}$$

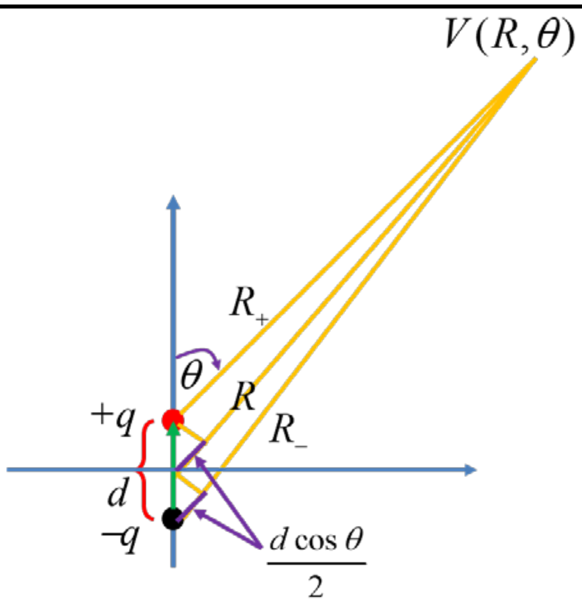
From here on, source coordinates are indicated with a ***prime*** $'$.

For many point charges, linear superposition applies as Gauss law is a linear equation.

$$\vec{E}_{total}(R) = \sum_i \frac{q_i}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R}_i'\right|^2}\hat{a}_{\vec{R}-\vec{R}_i'}$$

$$V_{total}(R) = \sum_i \frac{q_i}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R}_i'\right|}$$

Eg. Electric Dipole



Dipole potential:

Suppose we are only interested in the far-field quantities or assume R $>> d \Rightarrow$ $R_{\pm} = R \mp \dfrac{d\cos\theta}{2}$. In the far field

$$V(R) = \frac{q}{4\pi\varepsilon_0 R_+} - \frac{q}{4\pi\varepsilon_0 R_-}$$

$$\cong \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R-(d\cos\theta)/2} - \frac{1}{R+(d\cos\theta)/2}\right)$$

$$\cong \frac{q}{4\pi\varepsilon_0 R}\{[1+(d\cos\theta)/(2R)]-[1-(d\cos\theta)/(2R)]\}$$

$$= \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}$$

$\vec{p} \equiv q\vec{d}$ is called the *electric dipole moment*. The direction of the dipole moment points from a negative charge to a positive charge, which is in the opposite sense of an electric field.

The dipole field can be derived from $\vec{E} \equiv -\nabla V$, given by

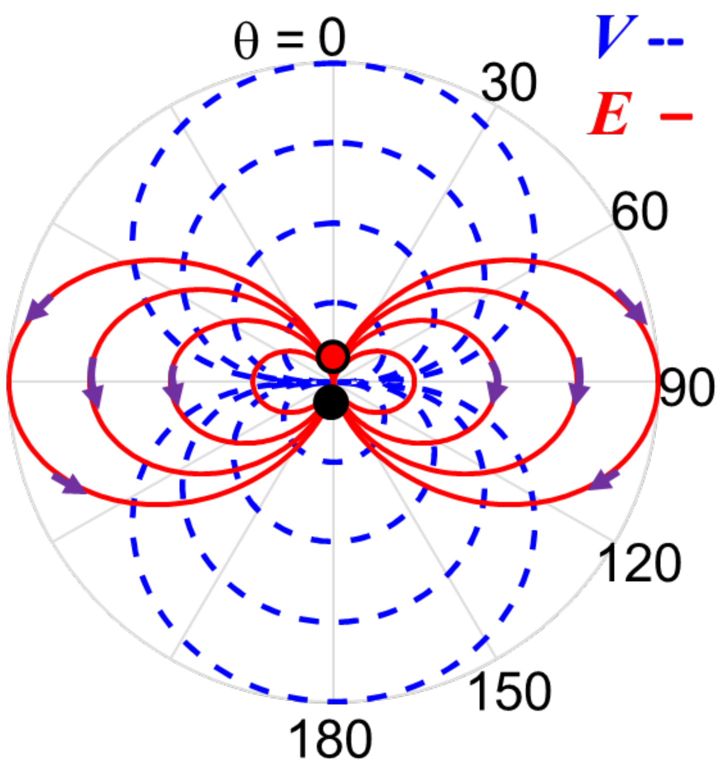
$$\vec{E} = -\nabla V = -\hat{a}_R \frac{\partial V}{\partial R} - \hat{a}_\theta \frac{\partial V}{R\partial\theta}$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

The following shows the far-field equipotential lines and the electric-field lines of an electric dipole. Since $\vec{E} \equiv -\nabla V$, these two families of lines are perpendicular to each other. One can rigorously drawing these lines:

Equipotential Surface of an Electric Dipole

To draw the equipotential surface, set the electric potential to a constant value $V(R) = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \text{const.}$ As a result, we obtain $R = \text{const.}\cdot\sqrt{\cos\theta}$



Electric-field Lines of an Electric Dipole

An electric field line has a direction along the electric field and a length proportional to the magnitude of the electric field $d\vec{l} = k\vec{E} \Rightarrow$

$$\hat{a}_{u_1} h_1 du_1 + \hat{a}_{u_2} h_2 du_2 + \hat{a}_{u_3} h_3 du_3$$
$$= \hat{a}_{u_1} kE_{u_1} + \hat{a}_{u_2} kE_{u_2} + \hat{a}_{u_3} kE_{u_3}$$

in spherical coordinate

$$\hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R \sin\theta d\phi$$
$$= \hat{a}_R kE_R + \hat{a}_\theta kE_\theta + \hat{a}_\phi kE_\phi$$

$\Rightarrow \dfrac{dR}{E_R} = \dfrac{Rd\theta}{E_\theta}$, $\quad E_\phi$ does not exist.

recall $\vec{E} = \dfrac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$

$$\frac{dR}{2\cos\theta} = \frac{Rd\theta}{\sin\theta} \quad \Rightarrow \quad \frac{dR}{R} = \frac{2d(\sin\theta)}{\sin\theta}$$

$$\Rightarrow \quad R = \text{const.} \cdot \sin^2\theta$$

Methods for Calculating an Electric Field

Gauss's Law:

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$$

is used whenever all the field vectors of equal magnitude are along the surface normal of a volume (this surface is called a Gaussian surface) that is well defined in one of the three coordinate systems.

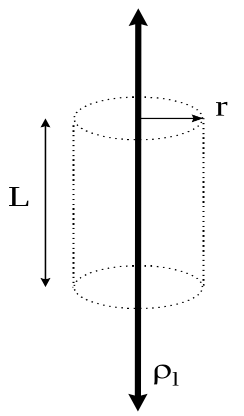
Distributive Integration:

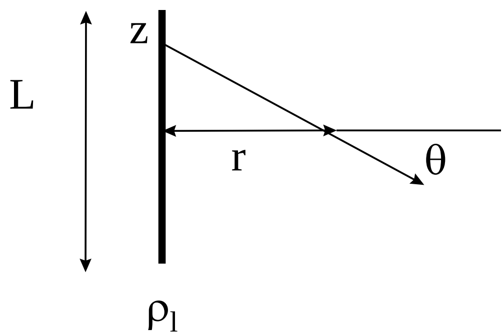
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{V',S',L'} \hat{a}_{\vec{R}-\vec{R}'}\frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|^2} dv', s', l' ,$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V',S',L'} \frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|} dv', s', l'$$

is otherwise used if a detailed charge distribution is given over a line, a surface, or a volume.

Eg. A Charged Wire





(a) infinite length (b) finite length

(a) From the geometry, we adopt the cylindrical coordinate system. Due to symmetry, only the radial components of *E* exist and are constant at a constant *r*.

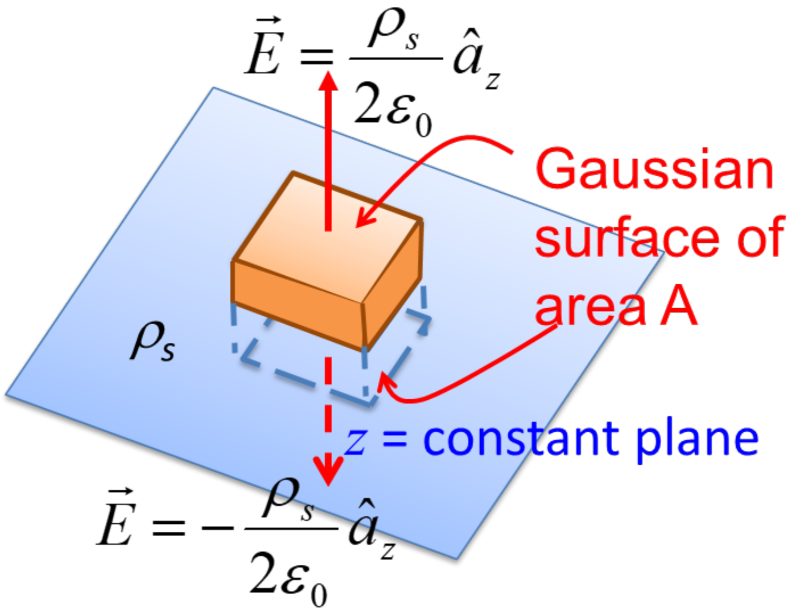
$$\varepsilon_0 \oint_S \vec{E}\cdot d\vec{s} = \varepsilon_0 E_r 2\pi r L = L\rho_l$$

(b) Due to symmetry, *z* component electric fields are all canceled.

$$E_r(z=0,r) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{\rho_l}{r^2+z'^2}\cos\theta dz' ,$$ where

$\cos\theta = r\big/\sqrt{r^2+z'^2}$ and ρ_l is the line charge density.

Eg. Infinite Planar Charge



From the geometry, we adopt the Cartesian coordinate system. Due to symmetry, only z-component electric fields exist and are constant at a constant z. $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = 2\varepsilon_0 E_z A = A\rho_s$, where ρ_s is the surface charge density.

Eg. A charged, concentric spherical shells in vacuum contain charges of q at $R = a$ and $-q$ at $R = b$. Find E and V everywhere.

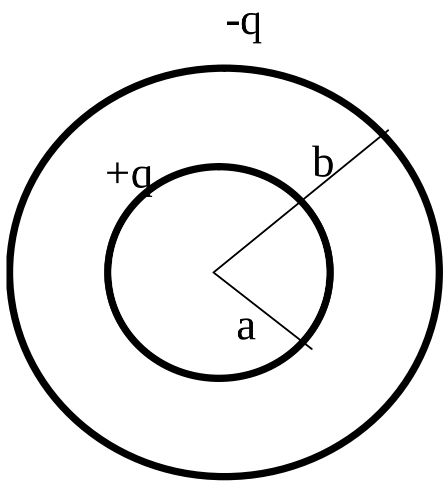
Due to symmetry, only R-component electric fields exist and are constant at a constant R. Gauss's law: $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q$

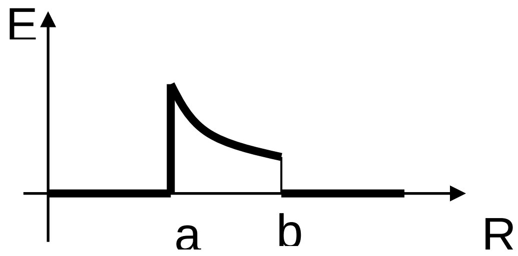
$$b \le R \Rightarrow \vec{E} = \frac{+q-q}{4\pi\varepsilon_0 R^2}\hat{a}_R = 0, \quad V(R) = 0$$

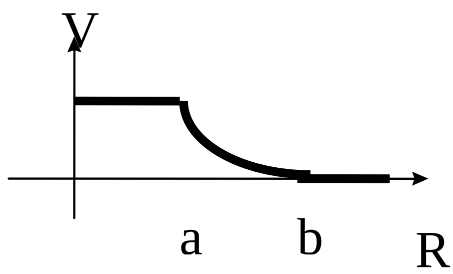
$a \le R \le b \;\Rightarrow\; \vec{E} = \dfrac{+q}{4\pi\varepsilon_0 R^2}\hat{a}_R$,

$$V_R(R) = V_b + V_{Rb} = 0 + (-\int_b^R \vec{E}\cdot d\vec{R}) = \frac{+q}{4\pi\varepsilon_0}(\frac{1}{R} - \frac{1}{b})$$

$0 \le R \le a \;\Rightarrow\; E = 0$, $V(R) = \dfrac{+q}{4\pi\varepsilon_0}(\dfrac{1}{a} - \dfrac{1}{b})$







Vacuum-Conductor Boundary Conditions

Vacuum: nothing (This statement is within the scope of classical electromagnetics. However, it is now widely believed that vacuum has a structure and is the last frontier of modern physics.)

Conductor: materials in which copious electrons can move freely but remain neutral at all time. Thus inside a conductor, there is no net electric field $E = 0$ and no net *bulk* charge $\rho = 0$. If $E \neq 0$ inside a conductor, electrons will flow to compensate the electric field. If $\rho \neq 0$ inside a conductor, an electric field will be built to cause the charge to flow. Therefore a conductor at a steady state is an equi-potential volume.

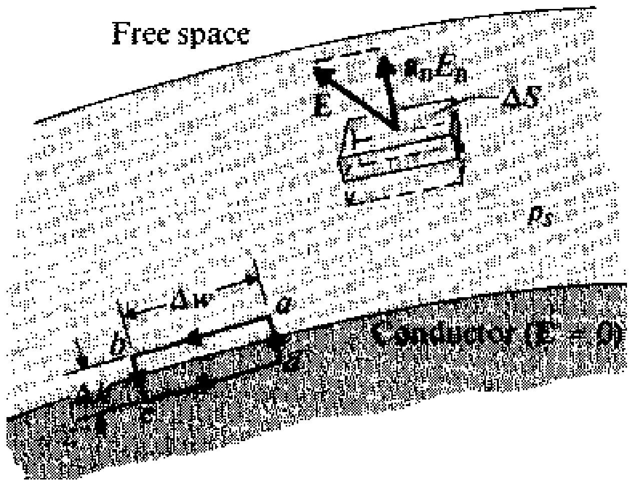
On the vacuum side of a vacuum-conductor interface,

$$\oint_{abcda} \vec{E} \cdot d\vec{l} = E_t \Delta w = 0 \qquad \Rightarrow \quad E_t = 0$$

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 \vec{E}_{vacuum} \cdot \hat{a}_n \Delta s + \varepsilon_0 \vec{E}_{conductor} \cdot (-\hat{a}_n) \Delta s = \varepsilon_0 \hat{a}_n \cdot \vec{E} \Delta s = \rho_s \Delta S$$

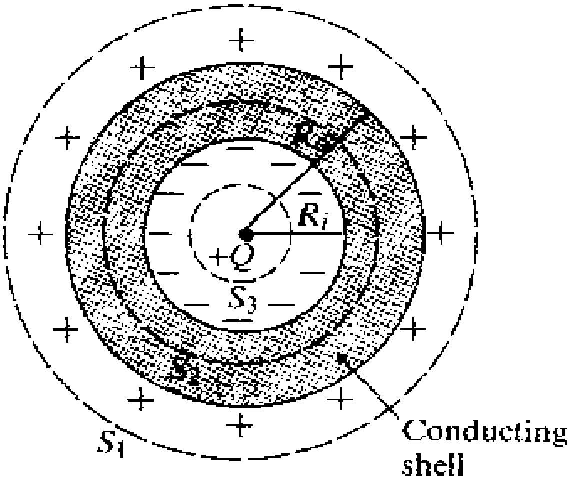
$$\Rightarrow \quad \hat{a}_n \cdot \vec{E}_{vacuum} = \frac{\rho_s}{\varepsilon_0} \Rightarrow \varepsilon_0 |E_n| = |\rho_s|$$

where ρ_s is the surface charge and E_n is the electric field normal to the conductor surface (in the outward direction if ρ_s is positive). Note that the surface unit vector a_n always points outward a material volume and surface charges can exist on the surface of a conductor.



Eg. A Charged Hollow Conducting Sphere: A point charge of Q is placed inside a hollow spherical conductor between the radial range

$R_i \le R \le R_0$. Owing to the inner charge *Q*, surface charges are induced at $R = R_i, R = R_0$ with opposite signs. Find *E* and *V* everywhere.



Electric Field: use Gauss's law $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q$

For $R_0 \le R \quad \Rightarrow \quad E_{R1} = \dfrac{Q}{4\pi\varepsilon_0 R^2}$

For $R_i \le R \le R_0$ (inside the conductor), $E_{R2} = 0$

For $R \le R_i$, $E_{R3} = \dfrac{Q}{4\pi\varepsilon_0 R^2}$

Electric Potential: use $V(R) = -\int_{\infty}^{R} \vec{E} \cdot d\vec{l}$

For $R_0 \le R$, $V_1(R) = -\int_{\infty}^{R} E_{R1} dR = \dfrac{Q}{4\pi\varepsilon_0 R}$

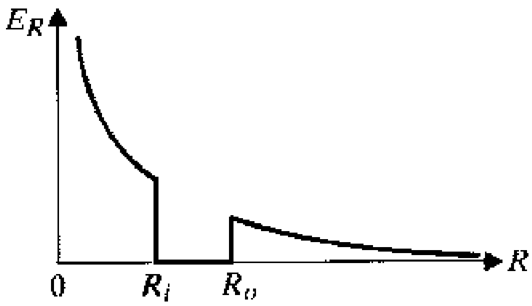
For $R_i \le R \le R_0$,

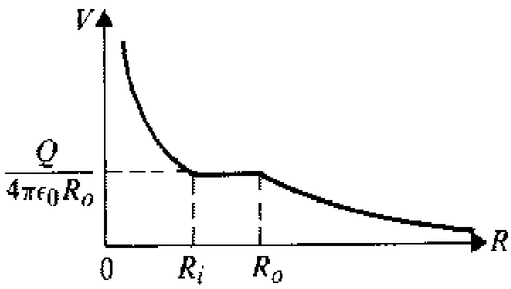
$$V_2(R) = -\int_{\infty}^{R_0} E_{R1} dR - \int_{R_0}^{R} E_{R2} dR = \frac{Q}{4\pi\varepsilon_0 R_0}$$

For $R \le R_i$,

$$V_3(R) = -\int_{\infty}^{R_0} E_{R1} dR - \int_{R_0}^{R_i} E_{R2} dR - \int_{R_i}^{R} E_{R3} dR$$

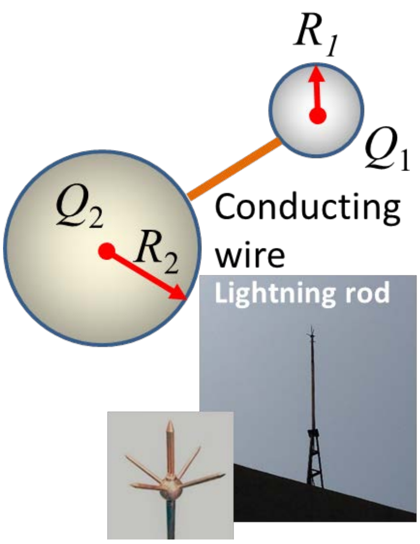
$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_0} + \frac{1}{R} - \frac{1}{R_i}\right)$$





Questions: What are the surface charge densities at the conductor surfaces $R = R_0$ and $R = R_i$?

Eg. Principle of Sharp-tip Discharge and Lightning Rod



At equilibrium, the two spheres are at an equal potential $V_1 = V_2$.

Assume the two spheres are far from each other ⇒

$$\frac{Q_1}{4\pi\varepsilon_0 b_1} = \frac{Q_2}{4\pi\varepsilon_0 b_2}$$

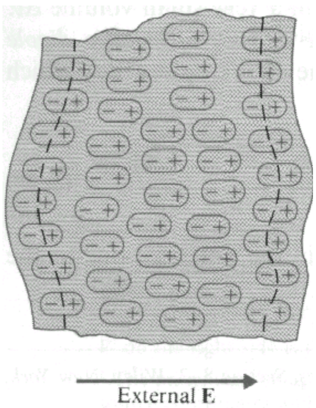
electrical fields can be calculated to be

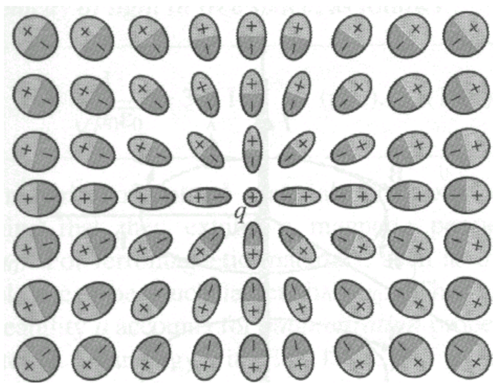
$$E_{1R} = \frac{Q_1}{4\pi\varepsilon_0 b_1^2}, \quad E_{2R} = \frac{Q_2}{4\pi\varepsilon_0 b_2^2}$$

There ratio is $\dfrac{E_{1R}}{E_{2R}} = \dfrac{b_2}{b_1}$ ⇒ small-radius conducting ball has a higher electric field.

Static Field in Dielectric

Dielectric: no free-moving charges, nonconducting





Locally, induced polarization charges modify the internal $\vec{E}$. The divergence of the electric field is modified accordingly

$$\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_{free} + \rho_p$$

ρ_{free} : volume density of *free charges* or *isolated charges*

ρ_p : volume density of polarization charges

Define the *electric flux density* $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ to keep the compact expression $\nabla \cdot \vec{D} = \rho_{free}$ for all electrostatic problems. This compact expression $\nabla \cdot \vec{D} = \rho_{free}$ has the advantage of only dealing with free/isolated charges in a problem. If so, $\nabla \cdot \vec{D} = \nabla \cdot (\varepsilon_0 \vec{E}) + \nabla \cdot \vec{P}$, but $\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_{free} + \rho_p$ and $\nabla \cdot \vec{D} = \rho_{free}$, one obtains the physical meaning of the polarization vector $\nabla \cdot \vec{P} = -\rho_p$. The polarization vector $\vec{P}$ can be further understood by calculating the total polarization charges for a neutral dielectric material according to

$$\int_V \rho_p dv + \oint_S \rho_{ps} ds = 0 \quad \Rightarrow \quad -\int_V \nabla \cdot \vec{P} dv + \oint_S \rho_{ps} ds = 0 ,$$

where ρ_{ps} is the surface density of polarization charges on a dielectric surface and the zero on the right hand side is due to charge conservation for a neutral material. Applying the divergence theorem to obtain

$$-\oint_S \vec{P} \cdot d\vec{s} + \oint_S \rho_{ps} ds = 0 \quad \Rightarrow \quad \vec{P} \cdot \hat{a}_n = \rho_{ps}.$$

Therefore, another physical meaning of the polarization vector is that its projection along the surface normal of a dielectric is equal to the surface charge density.

In D. K. Cheng's text, one can start with the microscopic view of a polarization vector and reach the same conclusion of $\nabla \cdot \vec{P} = -\rho_p$ and $\vec{P} \cdot \hat{a}_n = \rho_{ps}$. In this microscopic view, the polarization density vector is the average volume density of electric dipoles at a point volume

$\Delta v \to 0$ or

$$\vec{P} \equiv \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{p}_k}{\Delta v} \qquad (\mathrm{C \cdot m/m^3})$$

where n is the number of dipoles per unit volume in a dielectric. Since the polarization vector is the volume density of electric dipole moment, sometimes it is also called *polarization density vector*.

In a simple (linear, isotropic, nondispersive) medium,

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$

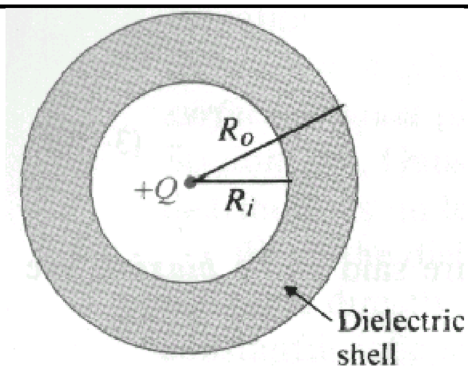
where ε_r is called *relative permittivity*, χ_e is called *electric susceptibility*.

The modified Gauss's law in materials is therefore

$$\nabla \cdot \vec{D} = \rho_{free} \quad \text{or} \quad \oint_S \vec{D} \cdot d\vec{s} = Q_{free}$$

where, again, ρ_{free} and Q_{free} are only related to free or isolated charges. With no concern of confusion, we drop the subscript "*free*" from the charge from here on. The creation of the $\vec{D}$ vector makes the calculation of electric fields particularly simple. Unlike $\vec{D}$, the electric field vector $\vec{E}$ was defined from the Lorentz force equation and has a specific physical meaning.

Eg. Hollow Dielectric Sphere



For $R_0 < R$, use $\oint_S \vec{D} \cdot d\vec{s} = Q$ to calculate

$$\vec{D}_{R1} = \frac{Q}{4\pi R^2}\hat{a}_R, \quad \vec{E}_{R1} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R, \text{ and } P_{R1} = 0$$

The electric potential $V_1(R) = -\int_{\infty}^{R} \vec{E}_R \cdot d\vec{R} = \frac{Q}{4\pi\varepsilon_0 R}$

For $R_i < R < R_0$, again use $\oint_S \vec{D} \cdot d\vec{s} = Q$ to calculate

$$\vec{D}_{R2} = \frac{Q}{4\pi R^2}\hat{a}_R, \quad \vec{E}_{R2} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R, \text{ and}$$

$$\vec{P}_{R2} = \vec{D}_{R2} - \varepsilon_0\vec{E}_{R2} = (1 - \frac{1}{\varepsilon_r})\frac{Q}{4\pi R^2}\hat{a}_R$$

The electric potential is given by

$$V_2(R) = V_1(R_0) - \int_{R_0}^{R} E_{R2} dR$$

$$= \frac{Q}{4\pi}(\frac{1}{\varepsilon_0 R_0} + \frac{1}{\varepsilon R} - \frac{1}{\varepsilon R_0})$$

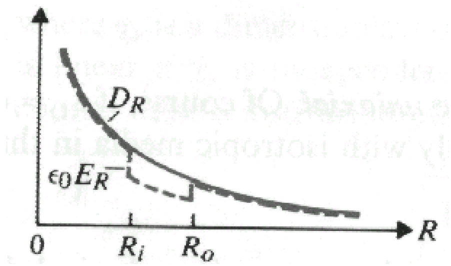
For $R < R_i$, use $\oint_S \vec{D} \cdot d\vec{s} = Q$ to calculate

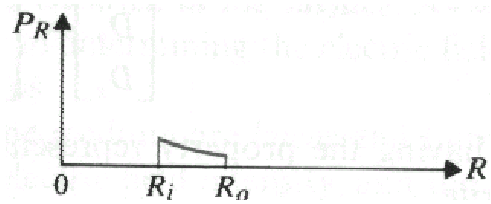
$$\vec{D}_{R3} = \frac{Q}{4\pi R^2}\hat{a}_R, \quad \vec{E}_{R3} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R, \text{ and } P_{R3} = 0.$$

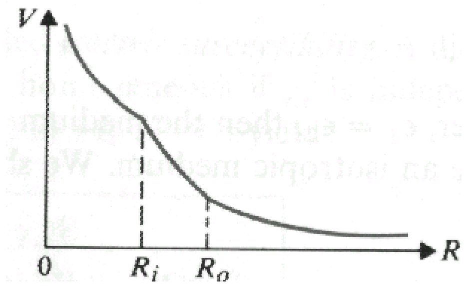
The electric potential is given by

$$V_3(R) = V_2(R_i) - \int_{R_i}^{R} E_{R3}dR$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 R_0} + \frac{1}{\varepsilon R_i} - \frac{1}{\varepsilon R_0} + \frac{1}{\varepsilon_0 R} - \frac{1}{\varepsilon_0 R_i}\right)$$





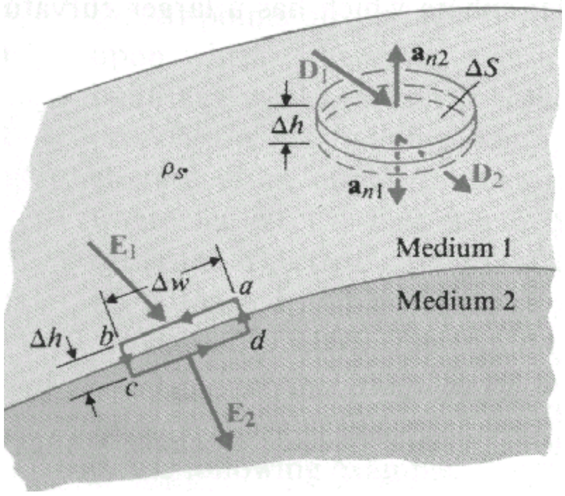


Questions: What are the densities of the surface polarization charges at the dielectric surfaces $R = R_0$ and $R = R_i$?

Dielectric Strength: dielectric breakdown field

materials	dielectric constant ε_r	dielectric strength (kV/mm)
air	~1.0	3
mineral oil	2.3	15
paper	2-4	15
polystyrene	2.6	20
rubber	2.3 ~ 4.0	25
glass	4-10	30
mica	6.0	200

General Boundary Conditions for Electrostatics



Apply Faraday's law for electrostatics

$$\oint_{abcda} \vec{E}\cdot d\vec{l} = \vec{E}_1 \cdot \Delta\vec{w} + \vec{E}_2 \cdot (-\Delta\vec{w})$$

$$= E_{1t}\Delta w - E_{2t}\Delta w = 0$$

$$\Rightarrow \; E_{1t} = E_{2t} \quad \text{or} \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

*Tangential components of the electric field intensity across a dielectric boundary have to be continuous.

Gauss's law

$$\oint_S \vec{D}\cdot d\vec{s} = (\vec{D}_1 \cdot \hat{a}_{n2} + \vec{D}_2 \cdot \hat{a}_{n1})\Delta S = \rho_s \Delta S$$

$\Rightarrow \; \hat{a}_{n2}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$ or $D_{1n} - D_{2n} = \rho_s$ with reference to the direction of the 2nd medium

* ρ_s is the free or isolated charges at the boundary

* Normal components of the electric flux density across a dielectric boundary are discontinuous, if surface charges exist.

Note that the general boundary conditions $E_{1t} = E_{2t}$ and

$\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$ are reduced to two subsets of boundary conditions at dielectric/dielectric and dielectric/conductor interfaces, specifically shown below.

i. Dielectric/dielectric boundary conditions

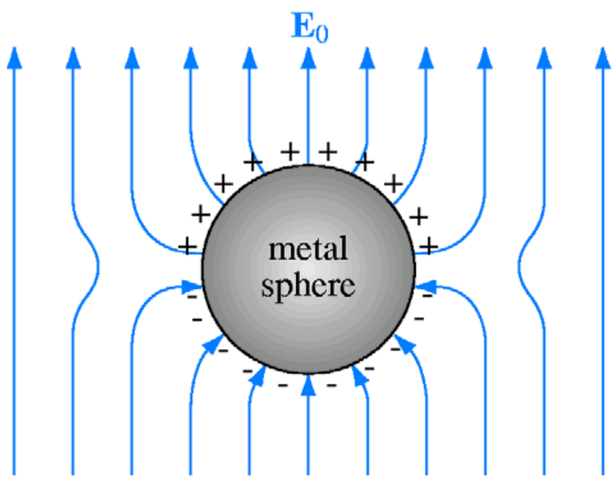
There's no free surface charge at a dielectric/dielectric boundary. The two conditions become

$$E_{1t} = E_{2t} \text{ and } D_{1n} = D_{2n} \ (\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n})$$

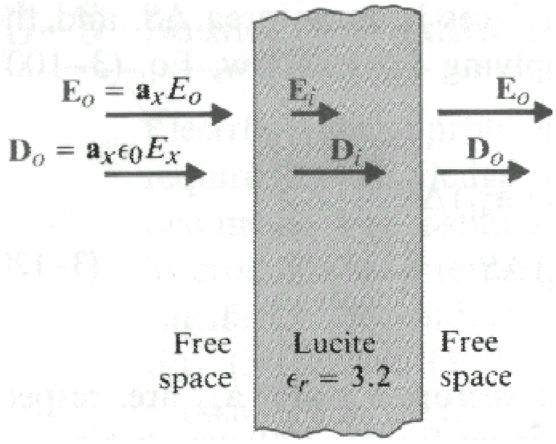
ii. Dielectric/conductor boundary conditions

Suppose the second medium is a good conductor and thus $E_{2t} = 0$ and $E_{2n} = 0$. However, at the conductor boundary, free surface charge ρ_s could exist. The two conditions become

$E_{1t} = E_{2t} = 0$ and $\hat{a}_{n2} \cdot \vec{D}_1 = \rho_s$, $D_{2n} = 0$. The conditions are also applicable to a vacuum/conductor interface by specifying $\varepsilon_1 = \varepsilon_0$. The dielectric/conductor boundary condition results in a consequence of requiring all the electric field lines normal to the surface of a conductor.



Eg. $\vec{E}_0 = E_0\hat{a}_x$ is known in vacuum, find $\vec{E}_L$, $\vec{D}_L$, $\vec{P}_L$ in Lucite.



Continuity of the tangential electric field intensity $E_{1t} = E_{2t}$ $\Rightarrow$ $E_{Lt} = 0$

no free charges in Lucite $\Rightarrow$ $D_{1n} - D_{2n} = \rho_s = 0$

$\Rightarrow$ $\vec{D}_0 = \varepsilon_0 E_0 \hat{a}_x = \vec{D}_L = \varepsilon_0 \varepsilon_r E_L \hat{a}_x$

Thus $\vec{E}_L = \dfrac{E_0}{\varepsilon_r}\hat{a}_x = \dfrac{\vec{E}_0}{\varepsilon_r}$

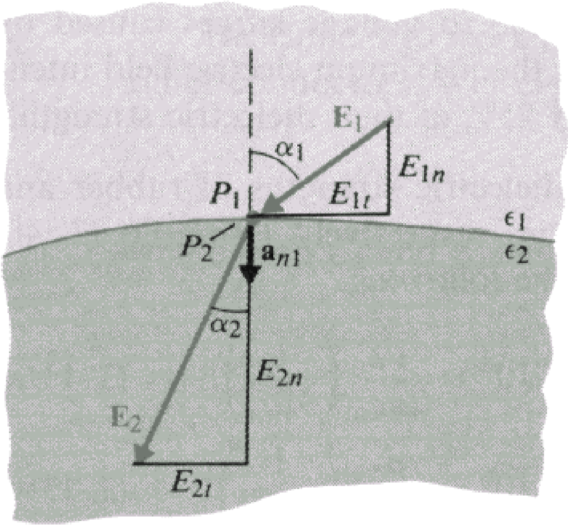
$$\vec{D}_L = \varepsilon_0 E_0 \hat{a}_x = \varepsilon_0 \vec{E}_0$$

Recall $\vec{D}_L = \varepsilon_0 \vec{E}_L + \vec{P}_L$

$$\Rightarrow \vec{P}_L = \vec{D}_L - \varepsilon_0 \vec{E}_L = \varepsilon_0 \vec{E}_0 - \frac{\varepsilon_0}{\varepsilon_r}\vec{E}_0 = \varepsilon_0 (1 - 1/\varepsilon_r)\vec{E}_0 .$$

Note that for $\varepsilon_r > 1$ in natural materials $\vec{P}_L // \vec{E}_0$. Could you explain this result?

Eg. The electric field $\vec{E}_1$, incident angle α_1, permittivity ε_1, in the 1st dielectric, and the permittivity in the 2nd dielectric ε_2 are known, find the electric field $\vec{E}_2$, and the deflect angle α_2 in the 2nd material.



tangential electric field intensity is continuous $E_{2t} = E_{1t} = E_1 \sin\alpha_1$

In the surface-charge free region, the normal components of *D* are continuous $E_{2n} = \dfrac{D_{2n}}{\varepsilon_2} = \dfrac{D_{1n}}{\varepsilon_2} = \dfrac{\varepsilon_1 E_{1n}}{\varepsilon_2} = \dfrac{\varepsilon_1 E_1 \cos\alpha_1}{\varepsilon_2}$

$$\tan\alpha_2 = \frac{E_{2t}}{E_{2n}} = \frac{\varepsilon_2}{\varepsilon_1}\tan\alpha_1 \quad \Rightarrow \quad \frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

$$E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = E_1\left[\sin^2\alpha_1 + \frac{\varepsilon_1^2}{\varepsilon_2^2}\cos^2\alpha_1\right]^{1/2}$$

Capacitance and Capacitors

<u>Observation:</u> $-\nabla\cdot\nabla V = \dfrac{\rho}{\varepsilon}$ or $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$

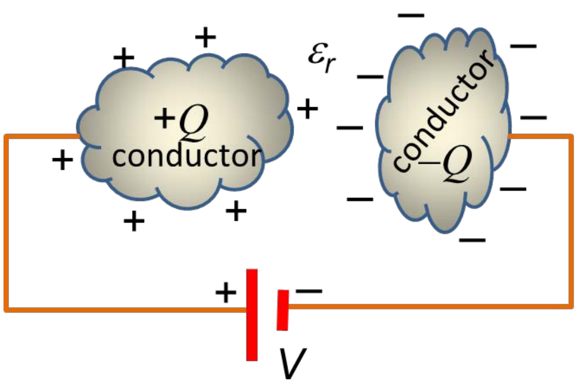
or from previous calculations $$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho}{R} dv',$$

one finds the electric potential or *the voltage* is proportional to charge $\Rightarrow$

$$Q = CV,$$

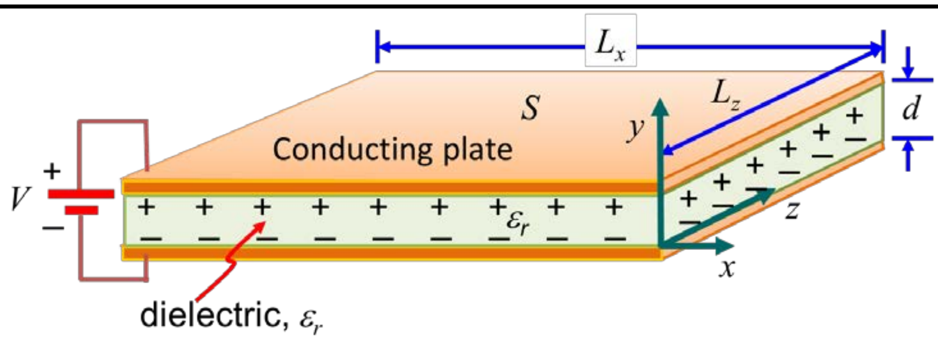
where C, the *capacitance*, is chosen to be always positive and a function of device geometry and material property (ε_r). Capacitance is the amount of stored charges per unit voltage in a device (called a capacitor).

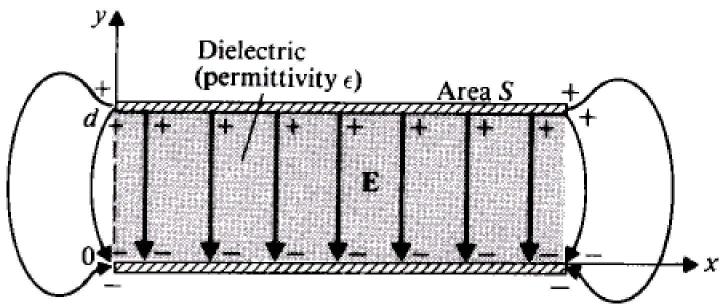
The following is a configuration of a capacitor



$$C = \left|\frac{Q}{V}\right|$$

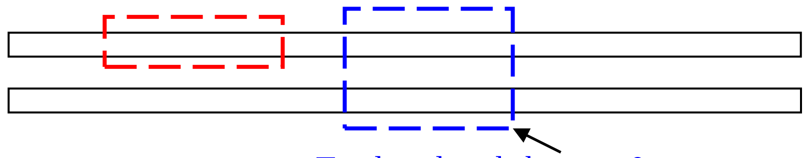
Eg. A parallel-plate capacitor: two large conducting plates enclose a dielectric. The two conducting plates are connected to a power supply. An equal amount of charges with opposite sign is induced in the two conducting plates.





Apply Gauss's Law: $\varepsilon \oint_S \vec{E} \cdot d\vec{s} = Q$ Assume the two plates are huge.

Choose a small pillbox surface (blue dashed line) enclosing the two metal plates. The net charge in the pillbox is zero, and thus no electric field outside the capacitor.



Now, choose a pillbox with surfaces (red dashed line) just enclosing the boundary of the top plate. In the surface an amount of charge Q is enclosed. Assume a big enough plate and ignore fringe fields. The transverse electric field is constant along y

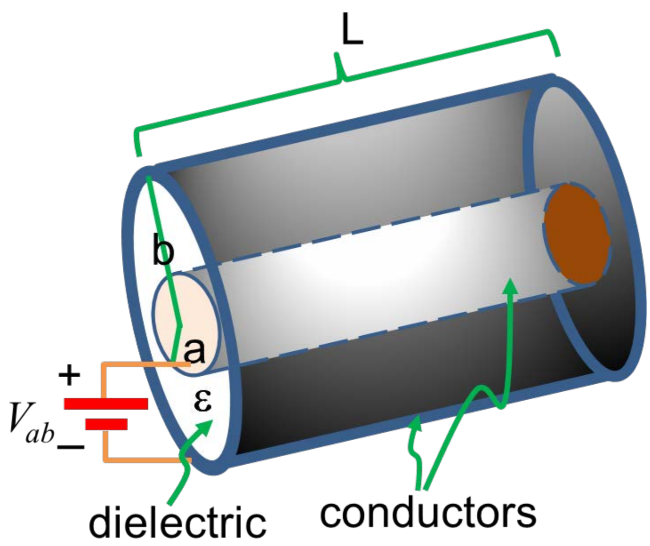
$$\varepsilon E_y(-\hat{a}_y)\cdot(-S\hat{a}_y) = Q \quad \Rightarrow \quad E_y = \frac{Q}{\varepsilon S}$$

The voltage across the two electrodes is $V = E_y d = \dfrac{dQ}{\varepsilon S}$ $\Rightarrow$

$$C \equiv \frac{Q}{V} = \frac{\varepsilon S}{d}$$

So, a large surface area, small gap between the two electrodes, and large permittivity (polarizability of a dielectric) favor charge storage.

Eg. A Cylindrical Capacitor: the two electrodes form a coaxial configuration.



Apply Gauss's law $\varepsilon \oint_S \vec{E} \cdot d\vec{s} = Q$ at a constant r

$$\Rightarrow \quad \hat{a}_r E_r = \hat{a}_r \frac{Q}{2\pi\varepsilon L r}$$

Calculate the potential across the two conductors

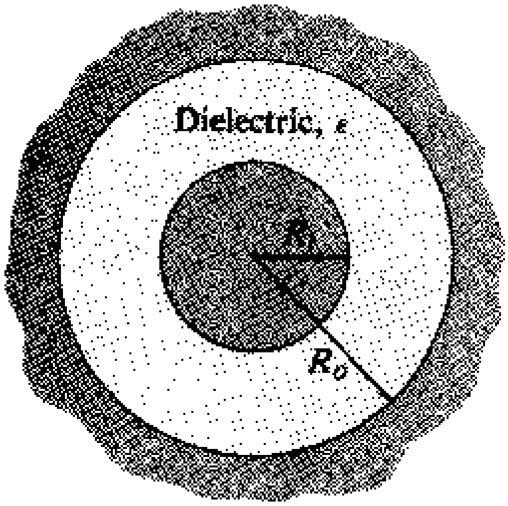
$$V_{ab} = -\int_{r=b}^{r=a} \vec{E}_r \cdot d\vec{r} = \frac{Q}{2\pi\varepsilon L} \ln\frac{b}{a}$$

Take the ratio of Q to V and obtain the capacitance $C \equiv \dfrac{Q}{V} = \dfrac{2\pi\varepsilon L}{\ln(b/a)}$

For a transmission line, what we care is the capacitance per unit length or

$$C_l \equiv \frac{\rho_l}{V} = \frac{2\pi\varepsilon}{\ln(b/a)}.$$

Ex. A Dielectric-filled Spherical Capacitor



Again, apply Gauss's law $\varepsilon\oint_S \vec{E}\cdot d\vec{s} = Q$ to obtain

$$4\pi R^2 E_R = \frac{Q}{\varepsilon} \quad \Rightarrow \quad E_R = \frac{Q}{4\pi\varepsilon R^2}$$

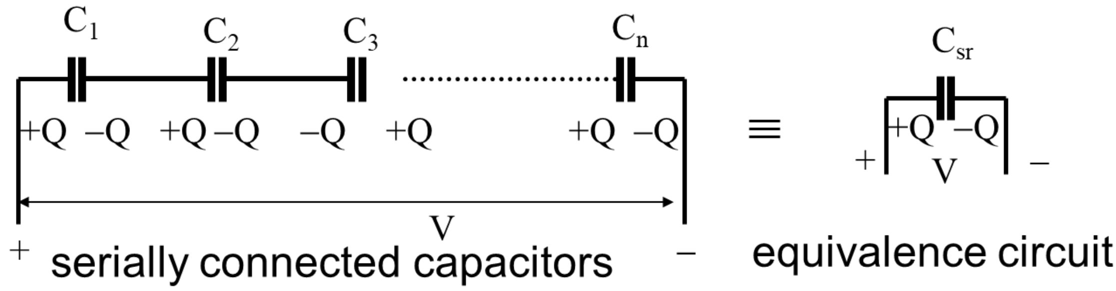
Calculate the electric potential in relation to the charge

$$V_{10} = -\int_{R=R_0}^{R=R_1} \vec{E}_R \cdot d\vec{R} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_0}\right)$$

Take the ratio of charge to voltage to obtain the capacitance

$$C \equiv \frac{Q}{V} = \frac{4\pi\varepsilon}{\left(1/R_1 - 1/R_0\right)}$$

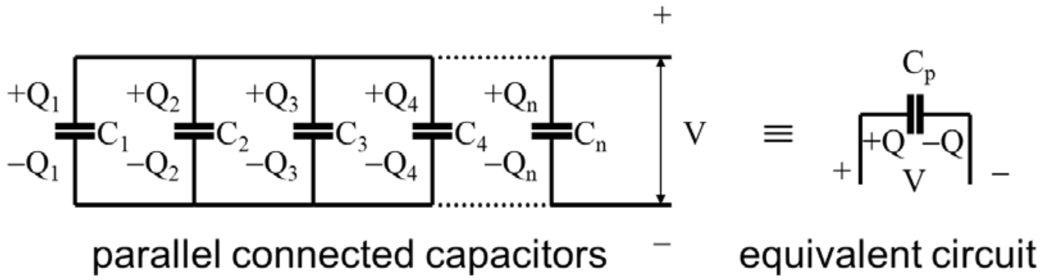
<u>Serial Capacitors</u>



$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \ldots + \frac{Q}{C_n}$$

$$\Rightarrow \quad \frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots + \frac{1}{C_n}$$

<u>Parallel Capacitors</u>

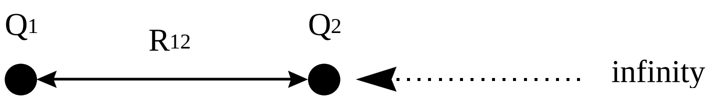


$$C_p V = Q = Q_1 + Q_2 + Q_3 + \ldots + Q_n$$
$$= C_1 V + C_2 V + C_3 V \ldots + C_n V$$

$$C_p = C_1 + C_2 + C_3 \ldots + C_n$$

Electrostatic Energy: The electrostatic energy stored in a charge system is equivalent to the work necessary for assembling the system by moving charges from infinity to their locations.

Two-charge System:



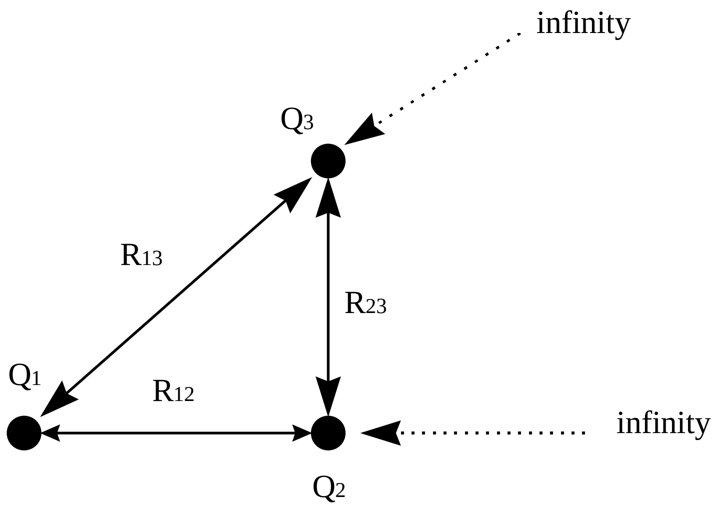
The work necessary for bringing Q_1 and Q_2 from infinity to a separation distance of R_{12} is

$$W_2 = Q_2 \frac{Q_1}{4\pi\varepsilon_0 R_{12}} = Q_2 V_2 \quad \text{or} \quad W_2 = Q_1 \frac{Q_2}{4\pi\varepsilon_0 R_{12}} = Q_1 V_1$$

Thus $W_2 = Q_2 V_2 = Q_1 V_1 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2)$

V_i : electric potential in the absence of Q_i

Three charge System:



Keep Q_1 stationary and bring in Q_2, Q_3 one by one.

$$W_3 = W_2 + \left(Q_3 \frac{Q_1}{4\pi\varepsilon_0 R_{13}} + Q_3 \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right) = W_2 + Q_3 V_3$$

Thus

$$W_3 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_1 Q_3}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\varepsilon_0 R_{23}}.$$

Therefore, if we assemble a system of *N* charges by bringing in charges one by one, we obtain

$$W_N = \sum_{i=1}^{N-1} \sum_{j>i}^{N} \frac{Q_i Q_j}{4\pi\varepsilon_0 R_{ij}}$$

Another way to calculate the stored energy for the three-charge system is to define

$$V_1 \equiv \frac{Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}}$$

$$V_2 \equiv \frac{Q_3}{4\pi\varepsilon_0 R_{23}} + \frac{Q_1}{4\pi\varepsilon_0 R_{12}}$$

$$V_3 \equiv \frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}}$$

or V_i is electric potential in the absence of Q_i. The corresponding expression of the stored electrostatic energy is

$$W_3 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

For N charges, the total stored potential energy can be expressed by

$$W_e = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k$$

This is the energy required for assembling $Q_1, Q_2 \ldots\ldots Q_N$. Again, the expression $V_k = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{j=1 \\ (J \neq k)}}^{N} \frac{Q_j}{R_{jk}}$ is the electric potential of the charge system excluding Q_k.

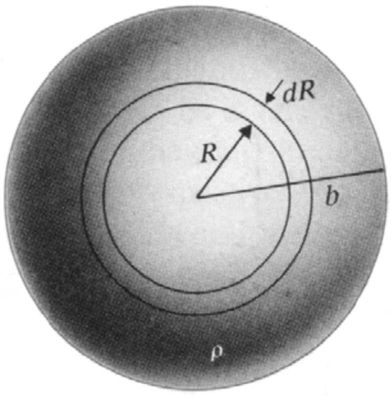
The above can be generalized to the case with a continuous charge distribution, given by

$$W_e = \frac{1}{2}\int_{V'} \rho V dv'$$

where V is the electric potential in the absence of $\rho dv'$

Eg. Calculate the energy stored by a charged ball of radius b and volume charge density of ρ.

$\boxed{\text{Solution 1}}$: assemble the charge ball by moving charges from infinity layer by layer



To move a sphere of charge $\rho 4\pi R'^2 dR'$ from $R = \infty$ to a charge ball of R', the work to be done is

$$dW = Vdq = \rho 4\pi R'^2 dR' \times \frac{(4/3)\pi R'^3 \rho}{4\pi\varepsilon_0 R'} = \frac{4\pi R'^4 \rho^2}{3\varepsilon_0} dR'$$

For a charge ball of radius b, the total work to assemble it is

$$W = \int_0^b \frac{4\pi\rho^2 R'^4}{3\varepsilon_0} dR' = \frac{4\pi\rho^2 b^5}{15\varepsilon_0}$$

$\boxed{\text{Solution 2}}$: use $W_e = \frac{1}{2}\int_{V'} \rho V dv$

To move a sphere of charge $\rho 4\pi R'^2 dR'$ from *R* = ∞ to a charge ball of *b*, the work to be done is

$$W_e = \frac{1}{2}\int_{V'} \rho V dv = \frac{1}{2}\rho \int_0^b V(R') 4\pi R'^2 dR'$$

$V(R')$ is the work of moving a positive unit charge from infinity to R' within the charge ball. Please complete the rest of calculation and compare the result with the one derived above.

Energy in terms of E and D

We choose $W_e = \frac{1}{2}\int_{V'} \rho V dv$ to calculate the equivalent energy stored in the field, because $W_e = \frac{1}{2}\int_{V'} \rho V dv$ was derived under the scenario that all the charges are in place already. Of course, we have to use the total field for this calculation. The calculation can be meaningful only when all the charges are in place.

Recall $\nabla \cdot \vec{D} = \rho \quad \Rightarrow \quad W_e = \frac{1}{2}\int_V (\nabla \cdot \vec{D}) V dv$

but $\nabla \cdot (V\vec{D}) = V\nabla \cdot \vec{D} + \vec{D} \cdot \nabla V$

$$W_e = \frac{1}{2}\int_V (\nabla \cdot (V\vec{D}) - \vec{D} \cdot \nabla V) dv = \frac{1}{2}\oint_S V\vec{D} \cdot d\vec{s} + \frac{1}{2}\int_V \vec{D} \cdot \vec{E} dv$$

but $\frac{1}{2}\oint_S V\vec{D} \cdot d\vec{s} \to 0$ as $R \to \infty$, because for large *R*, $V \propto 1/R$, $D \propto 1/R^2$, whereas $S \propto R^2$. On the other hand,

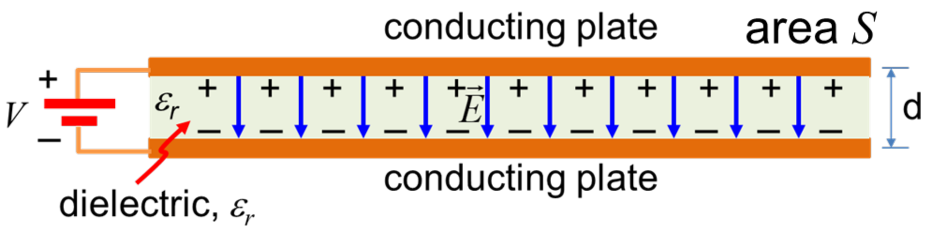
$\frac{1}{2}\int_V \vec{D}\cdot\vec{E}dv$ remains finite as the volume integration always starts from small R. After dropping the first term $\Rightarrow$

$$W_e = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}dv = \frac{\varepsilon}{2}\int_V E^2 dv = \frac{1}{2\varepsilon}\int_V D^2 dv$$

Define the *electrostatic energy density* (energy per unit volume)

$$w_e \equiv \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\varepsilon}{2}E^2 = \frac{D^2}{2\varepsilon} \qquad \text{(J/m}^3\text{)}$$

Eq. Energy Stored in a Capacitor



Ignore the fringe fields and use $E = \frac{V}{d}$ $\Rightarrow$

The stored electrostatic energy is

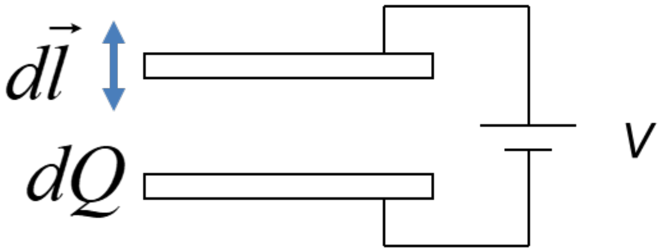
$$W_e = \frac{\varepsilon}{2}\int_V E^2 dv = \frac{\varepsilon}{2}\frac{V^2}{d^2}(Sd) = \frac{1}{2}\frac{\varepsilon S}{d}V^2$$

Recall the expressions, $C = \frac{\varepsilon S}{d}$ and $C = \frac{Q}{V}$. The store energy can take other forms

$$\Rightarrow \quad W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C} \qquad \text{(a general result)}$$

Electrostatic Force

System with fixed potential (maintained by a source)



Imagine that a charged system is connected to a fixed-voltage source and one tries to pull apart charges. The basis of this experiment of imagination is called the *principle of virtual displacement.*

Whenever there's a change in stored charge, the source can only sense the amount of change of charges at a constant voltage. The energy done by source is equal to

$$dW_s = \sum_k V_k dQ_k$$

The electrostatic energy change of the system excluding the source is

$$dW_e\Big|_{V=const} = \frac{1}{2}\sum_k V_k dQ_k = \frac{1}{2} dW_s$$

The mechanical work done by a small displacement =

$$dW = \vec{F}_{V=const} \cdot d\vec{l}$$

From energy conservation, the total energy change to the system has to be supplied by the source

$$dW_s = dW + dW_e\Big|_{V=const.}$$

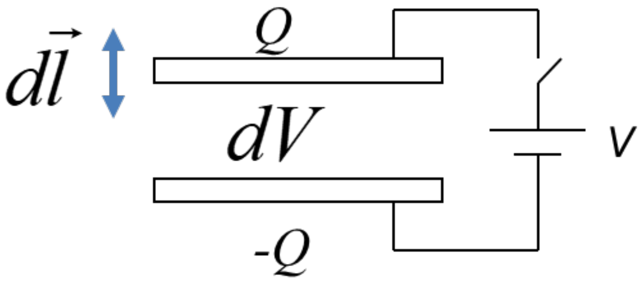
$$\Rightarrow \quad \vec{F}_{V=const} \cdot d\vec{l} = dW_e\Big|_{V=const} = (\nabla W_e\Big|_{V=const}) \cdot d\vec{l}$$

The electrostatic force holding the charges is $\Rightarrow$

$$\vec{F}_{V=const} = \nabla(W_e\Big|_{V=const})$$

Similarly, the electrostatic torque holding the charges long z for a virtual angular displacement in ϕ is $(\vec{T}_V)_z = \dfrac{\partial W_e|_{V=const}}{\partial \phi}$

System with fixed charges (an isolated system)



Imagine that a charged system is isolated from a source (the outside world) and one tries to pull apart charges.

Recall $dW_s = dW + dW_e|_{V=const}$ from the last example,

but no source exits now. Therefore $dW_s = 0 \Rightarrow dW_e|_{Q=const}$ and $dW + dW_e|_{Q=const} = 0$.

This means that work is done by the system at the expense of the stored $W_e|_{Q=const}$ or $-dW_e|_{Q=const} = \vec{F}_{Q=const} \cdot d\vec{l}$.

but $dW_e = (\nabla W_e) \cdot d\vec{l} \Rightarrow$ The electrostatic force holding the charges is $\vec{F}_{Q=const} = -\nabla W_e|_{Q=const}$

Similarly, the electrostatic torque holding the charges along z for a virtual angular displacement in ϕ is $(T_Q)_z = -\dfrac{\partial W_e|_{Q=const}}{\partial \phi}$