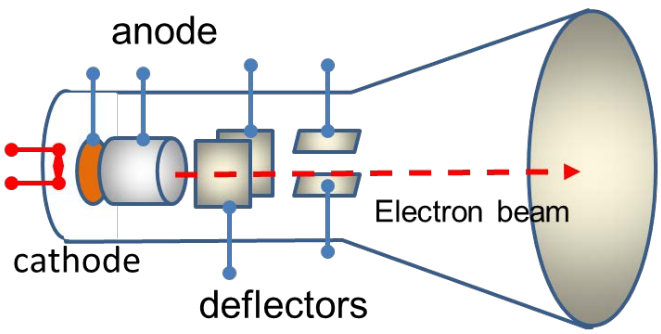
Chapter 5 Electric Circuits

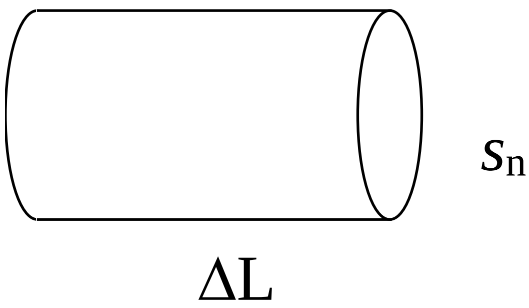
Currents:

1. Conduction current: resulting from motion of electrons and/or holes in a neutral material
2. Electrolytic current: resulting from migration of positive or negative ions in an aqueous environment.
3. Convection current: resulting from motion of charged particles in vacuum. The charge density can modify the potential that drives the particles.



cathode Ray Tube (CRT)

Volume Current Density *J* (A/m^2)



Total charge $\Delta Q = Nq\Delta V = Nq\Delta L s_n$

N : # of moving charges per unit volume (m^{-3})

q: charge unit (coulomb)

s_n : surface along the direction of the moving charges.

ΔV : a differential volume

but $\Delta L s_n = u_n \Delta t s_n = \vec{u} \cdot \vec{s} \Delta t$,

where $\vec{u}$ is the velocity of charges and the subscript *n* designates the **n**ormal direction of the cross sectional surface.

$\Rightarrow \Delta Q = Nq \Delta L s_n = Nq \vec{u} \cdot \vec{s} \Delta t$

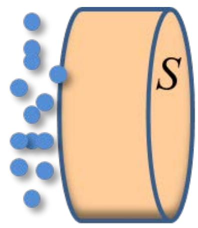
The definition of a *current* is the time rate change of charge

$\Rightarrow I = \dfrac{\Delta Q}{\Delta t} = Nq\vec{u} \cdot \Delta\vec{s} = \vec{J} \cdot \vec{s}$,

where $\vec{J} \equiv Nq\vec{u} = \rho\vec{u}$ is the *volume current density* in units of A/m^2.

Again ρ is the volume charge density.

$\Rightarrow$ total current $I = \int_S \vec{J} \cdot d\vec{s}$ (A)

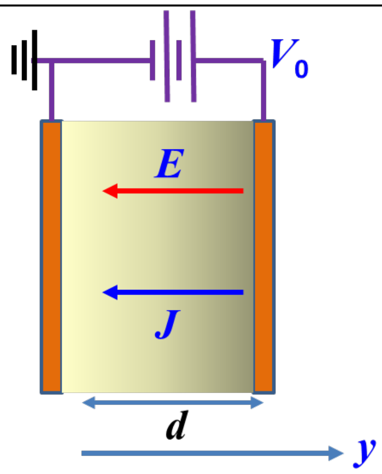


$\vec{u}$: velocity of charges

$$I = \frac{dq}{dt} = \int_s \vec{J} \cdot d\vec{s}$$

Eg. ***Convection current* across a parallel-plate accelerator**

Find *J* as a function of V_0



From energy conservation, electron kinetic energy = electric energy

Use Newton's mechanics for non-relativistic motion $\frac{1}{2}mu^2 = eV$

$$\Rightarrow \quad u = \sqrt{\frac{2eV}{m}}$$

Recall $\vec{J} \equiv Nq\vec{u} = \rho(y)\vec{u}(y) = \text{const.}$ at steady state.

The charge density is $\rho(y) = -J\sqrt{\frac{m}{2eV(y)}}$

From Poisson's equation $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, one writes

$$\frac{d^2V}{dy^2} = \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2}$$

across two infinitely large electrode plates, where J is a constant.

Math trick: $\frac{d}{dy}\left(\frac{dV}{dy}\right)^2 = 2\frac{dV}{dy}\left(\frac{d^2V}{dy^2}\right)$

Thus $2\frac{dV}{dy}\left(\frac{d^2V}{dy^2}\right) = 2\frac{dV}{dy}\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2} = \frac{d}{dy}\left(\frac{dV}{dy}\right)^2$

$$\Rightarrow \quad d\left(\frac{dV}{dy}\right)^2 = 2\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2}dV$$

Boundary conditions

i. at $y = 0$, $V = 0$ and $\frac{dV}{dy} = 0$;

ii. at $y = d$, $V = V_0$

The current density can be solved to be $J = \frac{4\varepsilon_0}{9d^2}\sqrt{\frac{2e}{m}}V_0^{3/2}$ or the charge density is proportion to $V_0^{3/2}$ (note: Ohm's law has a linear dependence). This formula is called the Child-Langmuir law. Note that in Poisson's equation the charge density is associated with free charges and the charge density modifies local electric potential.

A Microscopic View of Current: Mobility $\mu_e\ (\mathrm{m^2/V\cdot s})$

For more than one kind of charged carriers, the total current density is a superposition of individual ones

$$\vec{J} = \sum_i N_i q_i \vec{u}_i$$

For most conducting materials, the average drifting velocity of charges ~ $10^{-3} \sim 10^{-4}\ m/s$ due to collisions, and can be expressed by

$$\vec{u} = -\mu_e \vec{E} \quad (\mathrm{m/s}),$$

where $\mu_e\ (\mathrm{m^2/V\cdot s})$ is the *mobility* of charged carries.

For example, copper (Cu), aluminum (Al), and silver (Ag) have mobility of $\mu_e = 3.2\times10^{-3}$, $\mu_e = 1.4\times10^{-4}$, and $\mu_e = 5.2\times10^{-3}\, m^2/V\cdot s$, respectively. Thus the expression of a current density can be written as

$$\vec{J} = \rho\vec{u} = -\rho\mu_e\vec{E} = \sigma\vec{E},$$

where σ in Siemens/m is the conductivity . The reciprocal of conductivity is *resistivity*.

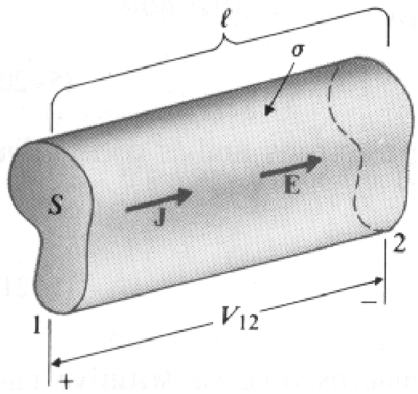
In semiconductor, there are two charge carriers, electrons and holes. The conductivity of a semiconductor is the sum of those of both electrons and holes, given by

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h$$

Material	Conductivity[a]	Material	Conductivity[a]
Silver	6.2×10^{7}	H_2O	2×10^{-4}
Copper	5.8×10^{7}	Marble	10^{-5}
Pure iron	1.0×10^{7}	Wood	10^{-9}
Steel	0.2×10^{7}	Glass	10^{-11}
Mercury	10^{6}	Oil	10^{-14}
Carbon	10^{4}	Polyethylene	10^{-15}
Silicon	10^{-2}	Fused quartz	10^{-17}
Alcohol	3×10^{-4}	True vacuum	?

For additional materials, consult Table B-2, Appendix B of Ulaby's textbook or Appendix B-4 of D. K. Cheng's textbook.

Ohm's Law



$V_{12} = El$, $I = JS = \sigma ES$

$$\Rightarrow \frac{V_{12}}{I} = \frac{l}{\sigma S} = R .$$

thus $R \equiv \frac{l}{\sigma S}$ is the resistance in Ω (ohms) for a conducting wire of length *l* and cross section *S*.

>> Prove the following as an exercise

1. for two serially connected resistors R_1 and R_2 the total resistance is $R_{total} = R_1 + R_2$
2. for two parallel connected resistors R_1 and R_2 the total resistance is $\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$ or $G_{total} = G_1 + G_2$, where $G_i = \frac{1}{R_i}$ is termed conductance.

Electromotive Force (from a non-conservative field)

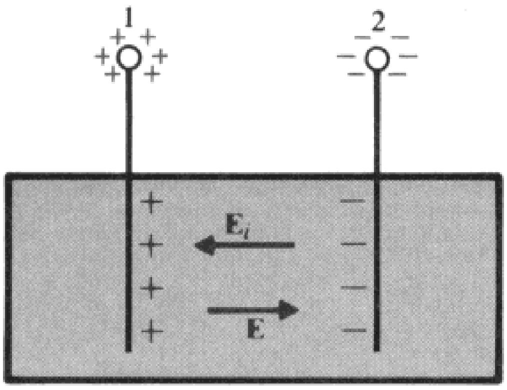
conservative field $\nabla \times \vec{E} = 0 \quad \Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$ or $\oint \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = 0$. This means no unidirectional current can be sustained in a closed circuit loop by an electrostatic field, because the energy of the moving charges in an ohmic material (with a finite σ) has to come from somewhere.

For a non-conservative field, $\nabla\times\vec{E}=0$ is written as

$\nabla\times\vec{E}=f$,

where *f* can be related to, say, time-varying magnetic field in a power generator or chemical potential in a battery.

$\nabla\times\vec{E}=f \quad \Rightarrow \quad \oint\frac{1}{\sigma}\vec{J}\cdot d\vec{l}=V_{em}$, where V_{em} is called the *electromotive force*.



Electric battery

$\vec{E}_i$: non-conservative field from, say, chemical energy, separating positive and negative charges.

$\vec{E}=-\vec{E}_i$ at steady state (no current flow)

Electromotive force $V_{em}=\int_2^1\vec{E}_i\cdot d\vec{l}=-\int_2^1\vec{E}\cdot d\vec{l}\Big|_{inside}$

Conservative field

$$\oint_C\vec{E}\cdot d\vec{l}=\int_1^2\vec{E}\cdot d\vec{l}\Big|_{outside}+\int_2^1\vec{E}\cdot d\vec{l}\Big|_{inside}=0$$

thus $V_{em}=\int_2^1\vec{E}_i\cdot d\vec{l}=\int_1^2\vec{E}\cdot d\vec{l}\Big|_{outside}$

Closed loop integration on total electric field $\vec{E} = \dfrac{\vec{J}}{\sigma}$

$$\oint_C \vec{E} \cdot d\vec{l} = \int_2^1 \vec{E}_i \cdot d\vec{l}\Big|_{inside} + \int_2^1 \vec{E} \cdot d\vec{l}\Big|_{inside} + \int_1^2 \vec{E} \cdot d\vec{l}\Big|_{outside}$$

$\Rightarrow\ V_{em} = \oint_C \dfrac{\vec{J}}{\sigma} \cdot d\vec{l} = RI$ in a circuit.

Kirchhoff's Voltage Law

around a closed-loop circuit, voltage rises = voltage drops

$$\sum_j V_{em_j} = \sum_k R_k I_k$$

Note that this result is consistent with energy conservation, because a voltage by its definition is the work done externally when moving a unit positive charge from one location to another.

Equation of Continuity

In a closed volume,

Positive (negative) time-rate change of charges = current flowing outward (*inward*).

$$\oint_S \vec{J} \cdot d\vec{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho dv$$

From the *divergence theorem,*

$$\int_V \nabla \cdot \vec{J} dv = -\int_V \frac{\partial \rho}{\partial t} dv \ ,$$

one can obtain the so-called *equation of continuity*

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \ .$$

Kirchhoff's current law

In equilibrium, no charge is generated or annihilated in a neutral conductor and therefore $\oint_S \vec{J} \cdot d\vec{s} = 0 \;\Rightarrow\; \sum_j I_j = 0$

It is equivalent to say that the *algebraic sum of all current flowing out of a circuit node is zero*

Relaxation Time in a Conductor

Upon a perturbation of a charge density, how fast does an *E* field (induced by the perturbation of charges) settle to zero inside a good conductor?

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \text{ but } \quad \vec{J} = \sigma \vec{E}$$

$$\Rightarrow \quad \sigma \nabla \cdot \vec{E} + \frac{\partial \rho}{\partial t} = 0, \text{ but } \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon}\rho = 0 \qquad \Rightarrow \quad \rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$$

note that ρ is a localized excessive charges under disturbance of an electric field. ρ settles to almost zero with a time constant (called *relaxation time*) $\tau \approx \dfrac{\varepsilon}{\sigma}$. For Cu with $\sigma = 5.8 \times 10^7$ S/m, $\tau \approx 10^{-19}$ sec!

Ohmic Loss and Joule's Law

Charges move under an electric force, because an electric source does work on charges.

power $P = \vec{F} \cdot \vec{u} \quad \Rightarrow P = \int q\vec{E} \cdot \vec{u} N dv$, but $\vec{J} = Nq\vec{u} \quad \Rightarrow$

$P = \int_V \vec{E} \cdot \vec{J} dv$,

where $\vec{E}\cdot\vec{J}$ is a volume power density in Watt/m^3.

The formula $P=\int_V \vec{E}\cdot\vec{J}dv$ is called ***Joule's Law***

Note that $dv = dlds$ in a circuit

$$P=\int_V \vec{E}\cdot\vec{J}dv=\int_L \vec{E}\cdot d\vec{l}\int_S \vec{J}\cdot d\vec{s}=VI=I^2R=\frac{V^2}{R}$$

Or simply $P=\dfrac{d(QV)}{dt}=IV=I^2R=\dfrac{V^2}{R}$ This power dissipation in a conductor is called Ohmic loss.

Boundary Conditions for *J* at a Steady State $\partial\rho/\partial t=0$

Differential Forms	Integral Forms
$\nabla\cdot\vec{J}=0$	$\oint_S \vec{J}\cdot d\vec{s}=0$
$\nabla\times\vec{E}=0 \Rightarrow \nabla\times(\vec{J}/\sigma)=0$	$\oint_c \frac{1}{\sigma}\vec{J}\cdot d\vec{l}=0$

Boundary Conditions:

i. Normal components across a boundary

$$J_{1n}=J_{2n}$$

iii. Tangential components across a boundary

$$J_{1t}/\sigma_1=J_{2t}/\sigma_2$$

Recall $\varepsilon_1E_{1n}-\varepsilon_2E_{2n}=\rho_s$ and $E_{1t}=E_{2t}$

Surface charges between two Lossy Dielectrics

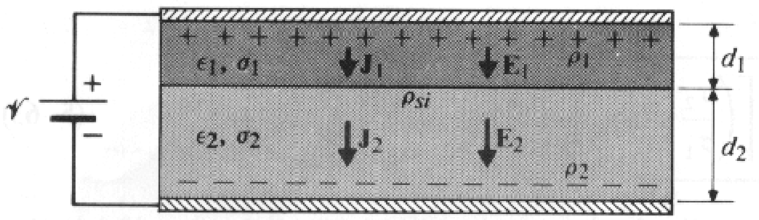
$J_{1n}=J_{2n} \quad\Rightarrow\quad \sigma_1E_{1n}-\sigma_2E_{2n}=0$

Recall $D_{1n}-D_{2n}=\rho_s \Rightarrow \varepsilon_1E_{1n}-\varepsilon_2E_{2n}=\rho_s$

$\Rightarrow \quad \rho_s = (\varepsilon_1 \frac{\sigma_2}{\sigma_1} - \varepsilon_2) E_{2n} = (\varepsilon_1 - \varepsilon_2 \frac{\sigma_1}{\sigma_2}) E_{1n}$

Surface charges must exist unless $\frac{\sigma_1}{\sigma_2} = \frac{\varepsilon_1}{\varepsilon_2}$. This result is not surprising, if one notices that the charge relaxation time $\tau = \frac{\varepsilon}{\sigma}$ was previously calculated. Later, one will also find $\frac{\varepsilon}{\sigma}$ is in fact the RC time constant of a material system. The condition $\frac{\sigma_1}{\sigma_2} = \frac{\varepsilon_1}{\varepsilon_2}$ indicates that the charges at the two sides of the interface dissipate at the same rate and the interface charge cannot be built up.

Eg. Given an emf V, find $\rho_{s1}, \rho_{s2}, \rho_{si}, E_1, E_2, J$



From the geometry, no tangential components of J or E.

Boundary conditions at the dielectric interface

$$\sigma_1 E_{1n} - \sigma_2 E_{2n} = 0$$

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_{si}$$

thus $E_{1n} = \frac{\sigma_2 \rho_{si}}{\varepsilon_1 \sigma_2 - \varepsilon_2 \sigma_1}$, and $E_{2n} = \frac{\sigma_1 \rho_{si}}{\varepsilon_1 \sigma_2 - \varepsilon_2 \sigma_1}$. (5-1)

But the voltage across the two plates is fixed at V or $E_{1n} d_1 + E_{2n} d_2 = V$,

$$\Rightarrow \quad \rho_{si} = \frac{\varepsilon_1\sigma_2 - \varepsilon_2\sigma_1}{d_1\sigma_2 + d_2\sigma_1} V \quad . \tag{5-2}$$

Substituting (5-2) into (5-1) to obtain E_{1n} and E_{2n}.

Calculate charge densities from $\rho_{s1} = \varepsilon_1 E_{1n}$ and $\rho_{s2} = -\varepsilon_2 E_{2n}$.

Obtain J from $J = \sigma_1 E_{1n} = \sigma_2 E_{2n}$.

Relationship between *R* and *C*

From the definition of capacitance, one has

$$C = \frac{Q}{V} = \frac{\oint_S \vec{D}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}} = \frac{\varepsilon\oint_S \vec{E}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}}$$

From the definition of resistance, one has

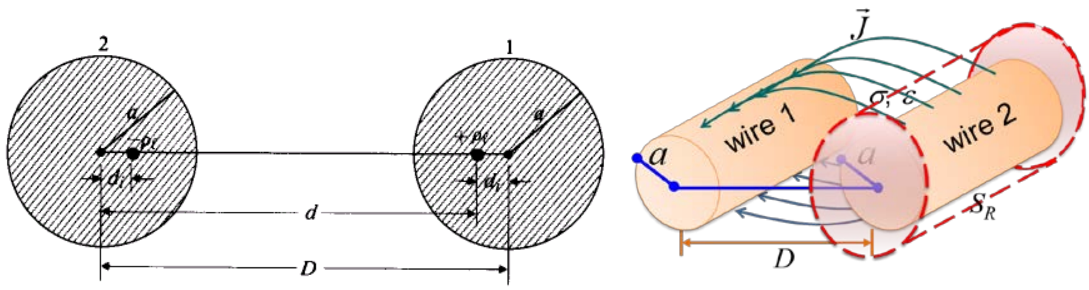
$$R = \frac{V}{I} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\oint_S \vec{J}\cdot d\vec{s}} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\oint_S \sigma\vec{E}\cdot d\vec{s}}$$

If R and C are associated with the same volume, the following relationship can be derived immediately

$$RC = \frac{\varepsilon}{\sigma},$$

from which R can be derived by knowing C or vice versa.

Eg. Find R for a given C of the parallel conducting cylinders



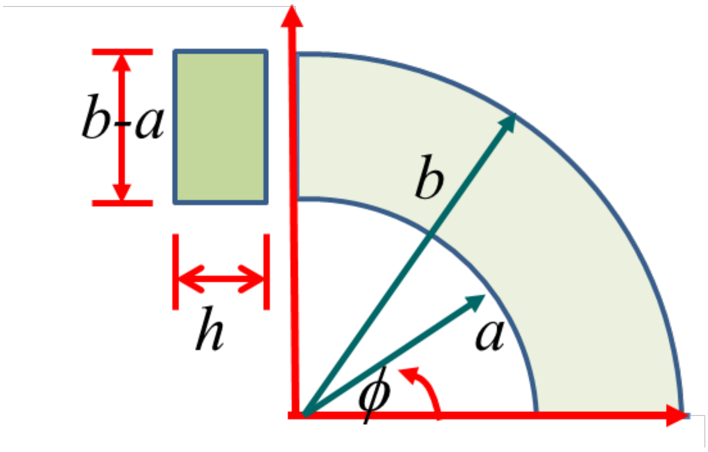
Recall the capacitance per unit length $C_l = \dfrac{\pi\varepsilon}{\cosh^{-1}\left(\dfrac{D}{2a}\right)}$ or $C = \dfrac{\pi\varepsilon L}{\cosh^{-1}\left(\dfrac{D}{2a}\right)}$ for a transmission line of length *L*. From the RC relationship $RC = \dfrac{\varepsilon}{\sigma}$, $\Rightarrow$ the resistance for a transmission line of length L is $R = \dfrac{1}{\pi\sigma L}\cosh^{-1}\left(\dfrac{D}{2a}\right)$.

Previously we have $R \propto l$, the length of a conducting wire. Note that for a transmission line of length *L* in this case, the resistance is inversely proportional to *L* (why?)

Calculating Resistance for a Conductor

1. Choose a coordinate system
2. Find *V* from $\nabla^2 V = 0$ subject to $V = V_0$ at a suitable boundary.
3. Find *E* from $\vec{E} = -\nabla V$.
4. Find *I* from $I = \int_S \vec{J} \cdot d\vec{s} = \int_S \sigma\vec{E} \cdot d\vec{s}$
5. Calculate *R* from $R = \dfrac{V_0}{I}$

Exercise: Find the resistance of the following conductor..



Boundary Conditions:

$V=0$ at $\phi=0$ and $V=V_0$ at $\phi=\pi/2$. No dependence on *r* and *z*, thus

Turn Laplace's equation $\nabla^2 V=0$ into $\dfrac{d^2V}{d\phi^2}=0$

General solution $V=c_1\phi+c_2$

Apply boundary conditions $V=\dfrac{2V_0}{\pi}\phi$

Electric field can be found from

$$\vec{E}=-\nabla V=-\hat{a}_\phi\frac{\partial V}{r\partial\phi}=-\frac{2V_0}{\pi r}$$

Current density is $\vec{J}=\sigma\vec{E}=-\sigma\dfrac{2V_0}{\pi r}\vec{a}_\phi$

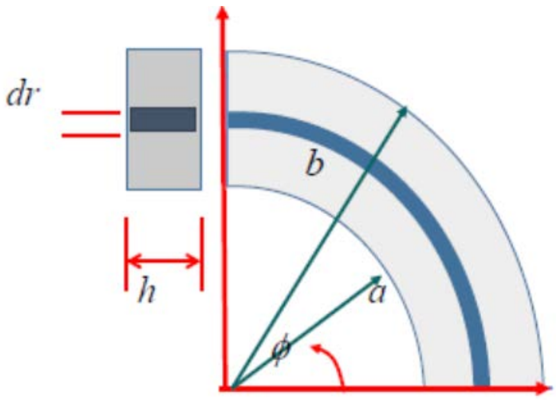
Total current is the integration

$$I=\int_S \vec{J}\cdot d\vec{s}=\int_a^b \sigma\frac{2V_0}{\pi r}hdr=\frac{2\sigma hV_0}{\pi}\ln\frac{b}{a}$$

Therefore the resistance can be obtained

$$R = \frac{V}{I} = \frac{\pi}{2\sigma h \ln\frac{b}{a}}$$

A Simple Trick



Given $R \equiv \dfrac{l}{\sigma S} \Rightarrow G \equiv \dfrac{\sigma S}{l}$. Thus the differential conductance is $dG = \dfrac{\sigma ds}{l}$, where the differential surface is $ds = hdr$, and the length of the conductor is $l = \dfrac{\pi}{2} r$. The differential conductance becomes

$dG = \dfrac{2\sigma h dr}{\pi r} \Rightarrow G = \displaystyle\int_a^b \frac{2\sigma h}{\pi r} dr = \frac{2\sigma h}{\pi} \ln\frac{b}{a}$. Take the inverse of the conductance to obtain resistance

$\Rightarrow R = \dfrac{1}{G} = \dfrac{\pi}{2\sigma h \ln(b/a)}$ same as above.