EE203001 SES #5

Elimination using matrices

$$2x_1 + 4x_2 - 2x_3 = 2$$
$$4x_1 + 9x_2 - 3x_3 = 8$$
$$-2x_1 - 3x_2 + 7x_3 = 10$$

In matrix form

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\underline{b}}$$

$$A \quad \underline{x} = \underline{b}$$

(Sol. is $\underline{x} = (-1, 2, 2)$)

Col. form:

$$A\underline{x} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

In general

$$A\underline{x} = x_1(\text{col. } 1) + \cdots + x_n(\text{col. } n)$$

Row form

ith component of $A\underline{x}$ = dot product of i^{th} row of A $[a_{i1}\ a_{i2}\ \cdots\ a_{in}]$ with $\underline{x}$

$$= \sum_{j=1}^{n} a_{ij} x_j$$

The matrix form of one elimination step

Recall: the 1^{st} step of Elimination subtracts (eqn 2) − 2× (eqn 1)

Focus on the right side of $A\underline{x} = \underline{b}$

$$\underline{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \Rightarrow \underline{b}_{new} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Q: Can we represent this step using a matrix?

Yes! Elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

1^{st} & 3^{rd} row of identity matrix

⇒ Row 1 & 3 of $\underline{b}$ stay the same

Q: How to construct a Elimination matrix?

Use an identity matrix I. E_{ij} that substracts a multiple ℓ of row j from row i has the extra nonzero entry $-\ell$ in the i,j position

Ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix}$$

$$I\underline{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$E_{31}\underline{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9-4=5 \end{bmatrix}$$

Q: How about the left side of $A\underline{x} = \underline{b}$?

The purpose of E_{31} is to produce a zero in the (3,1) position of the matrix

Elimination using matrices

Apply E's to produce zeros below the pivot

Q: What is the first E?

$$E_{21} \to E_{31} \to E_{32}$$

Note: the vector $\underline{x}$ stays the same

coeff. matrix is changed

Start with:

$$A\underline{x} = \underline{b}$$

multiply by E:

$$EA\underline{x} = E\underline{b}$$

Q: How do we multiply two matrices?

We expect E acting on A: subtracts $2\times$ (row 1) from (row 2) of A

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

Note: $A\underline{x} = \underline{b}$

$$E(A\underline{x}) = E\underline{b}$$

Same as $(EA)\underline{x} = E\underline{b}$

For matrices,

Associative law is true, i.e.,

$$A(BC) = (AB)C$$

Commutative law is false, i.e.,

often $AB \neq BA$

Another requirement for matrix multiplication

If B has only one col. ($\underline{b}$)

then EB should agree with $E\underline{b}$

In fact, if $B = [\underline{b}_1\ \underline{b}_2\ \underline{b}_3]$

$$\Rightarrow EB = [E\underline{b}_1\ E\underline{b}_2\ E\underline{b}_3]$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

P_{ij} for a row exchange

(permutation matrix)

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ exchange component 2 & 3 for any vector

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

So it also exchanges row 2 & 3 for any matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

In general

P_{ij} is the identity matrix with row i & j exchanged

The augmented matrix

Elimination does same row operations to A & to b $\Rightarrow$ we can include $\underline{b}$ as an extra col.

$$[A\ \underline{b}] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$E_{21}[A\ \underline{b}] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$x_2 + x_3 = 4$

<u>By rows</u>

Each row of E acts on $[A\ b]$

to give a row of $[EA\ Eb]$

<u>By col.s</u>

E acts of each col. of $[A\ b]$

to give a col. of $[EA\ Eb]$

<u>Step by step</u>

$$A \to E_{21}A \to E_{31}E_{21}A \to E_{32}E_{31}E_{21}A$$

p. 61 : 2.3 A

More on Matrix multiplication

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{array}{c} 1 \cdot \text{col } 1 \\ + \\ 2 \cdot \text{col } 2 \\ + \\ 3 \cdot \text{col } 3 \end{array}$$

lin. comb. of col.s

$$\begin{bmatrix} 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{array}{c} 3 \cdot \text{row } 1 \\ + \\ 5 \cdot \text{row } 2 \\ + \\ 7 \cdot \text{row } 3 \end{array}$$

lin. comb. of rows

$$E_{21}\,[A \ \underline{b}] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$(\text{row}_2^{\text{new}} = \text{row } 2 - 2 \cdot \text{row } 1)$

$\Rightarrow$ Each row of E acts on $[A \ b]$ to give a row of $[EA \ Eb]$

$\Rightarrow$ E acts of each col. of $[A \ b]$ to give a col. of $[EA \ Eb]$