EECS 205003 Session 22

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Ch5 Determinants

- Ch 5.1 The Properties of Determinants
- Ch 5.2 Permutations and Cofactors
- Ch 5.3 Cramer's Rule, Inverses, and Volumes

Many applications of determinant.
Let's see how it is used!

Formula for A^{-1}

For 2×2:

we know $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$det A$ involves cofactors of A

$(C_{11} = det(d) = d, C_{12} = -c, C_{21} = -b, C_{22} = a$

$\Rightarrow$ cofactor matrix $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix})$

Guess A^{-1} for general n$\times$n matrix:

$A^{-1} = \underbrace{\frac{1}{detA}}_{\text{(product of n entries)}} C^{\mathsf{T}} \rightarrow$ (product of n$-$1 entries)

(Now, it is possible that A^{-1} cancels with A)

(For 2$\times$2, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

(Much easier to see from this than elimination)

Proof of inverse formula:

same as proving $AC^{\mathsf{T}} = (detA)I$

$$D = AC^{\mathsf{T}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

$$d_{11} = \sum_{j=1}^{n} a_{1j} C_{1j} = detA$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$d_{nn} = \sum_{j=1}^{n} a_{nj} C_{nj} = detA$$

Next, we want to show that all off-diagonal terms are zero

Say, row 2 of A & row 1 of C

(column 1 of C^T)

$$d_{21} = a_{21}C_{11} + a_{22}C_{12} + \cdots + a_{2n}C_{1n}$$

This is cofactor rule of a new matrix A^{21}

$$A^{21} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Obviously, $det A^{21} = 0$

In general,

$d_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$

$det A^{ij}$ (replace jth row of A by ith row of A)

$\Rightarrow det A^{ij} = 0$ for all i $\neq$ j

$\Rightarrow AC^{\mathsf{T}} = (det A)I$

$\Rightarrow A^{-1} = \frac{1}{det A}C^{\mathsf{T}}$

(This formula helps answer how inverse changes when the matrix changes)

Cramer's rule for $\mathbf{x} = A^{-1}\mathbf{b}$

If A is nonsingular & $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

Applying inverse formula $A^{-1} = \frac{C^{\mathsf{T}}}{detA}$

$\Rightarrow \boldsymbol{x} = \frac{C^{\mathsf{T}}\boldsymbol{b}}{detA}$

$\Rightarrow x_j = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$

$= \frac{detB_j}{detA}$

Where we get B_j from A by replacing the jth column from $\boldsymbol{b}$

(Usually less efficient than elimination but more insights)

$|detA|$ = volume of box

Claim: $|detA|$ = volume of box whose edges are the row vectors of A
(or column vector since $detA = detA^{\mathsf{T}}$)

proof: Show that volume of box satisties property 1-3 of $|detA|$

property 1: If $A = I$, the box is a unit cube $\Rightarrow$ volume = 1 = $|detI|$

(If $A = Q$, the box is a unit cube with different orientation & volume = 1 = $|detQ|$)

($\because Q$ is an orthogonal matrix $\Rightarrow Q^{\mathsf{T}}Q = I$
$\Rightarrow (\text{det}Q)^2 = 1 \Rightarrow detQ = \pm 1$)

property 2: exchanging two rows of A does NOT change the volume & $|detA|$

property 3:
check 2×2 first:

$$\begin{vmatrix} tx_1 & ty_1 \\ x_2 & y_2 \end{vmatrix} = t \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \; ?$$

$$\begin{vmatrix} x_1 + x_1' & y_1 + y_1' \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1' & y_1' \\ x_2 & y_2 \end{vmatrix} \; ?$$

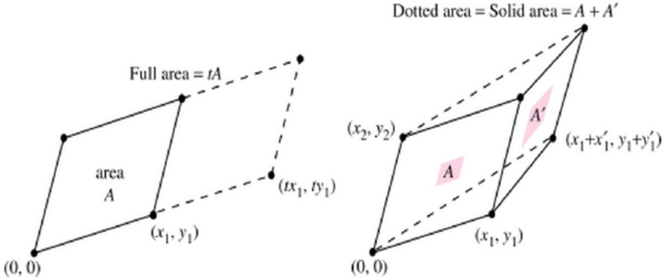


Figure 36: Areas obey the rule of linearity (keeping the side (x_2, y_2) constant).

Can be generalized to n dimension box, e.g., 3×3

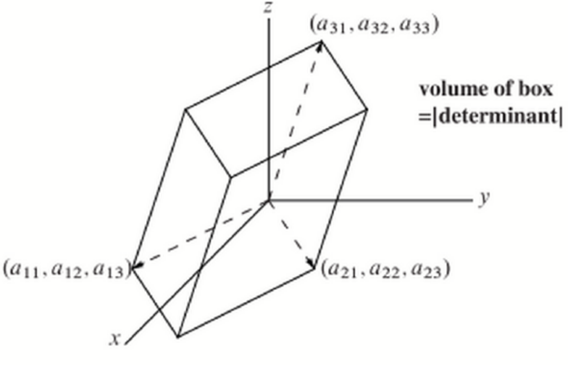


Figure 37: Three-dimensional box formed from the three rows of A.

Interesting to see: (not necessary for our proof)

If two edges of a box are equal, the box flatten out

$\Rightarrow$ volume $= 0$ (property 4)

Important note:
If you know the corners of a box, then computing volume is as easy as computing det
Ex:

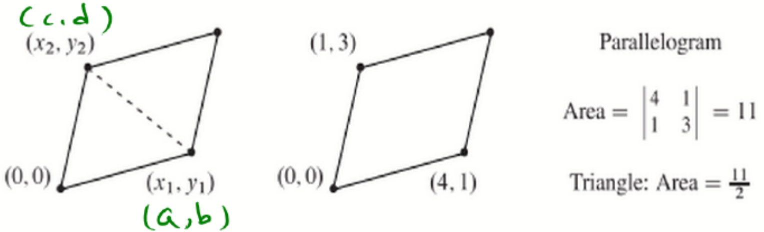


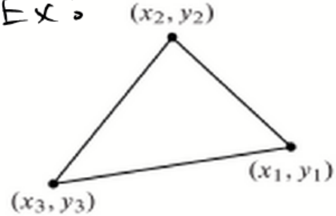
Figure 35: A triangle is half of a parallelogram. Area is half of a determinant.

$$\text{area of parallelogram} = \left| det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right| = |ad - bc|$$

$$\text{area of triangle} = \frac{1}{2} |ad - bc|$$

general triangle

Ex:



$$Area = \frac{1}{2}\left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

$$\because \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

(shift (x_1, y_1) to origin)