

Homework 3

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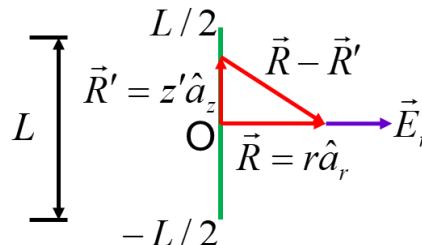
Problem 1 (6%)

Refer to the following figure. In Lecture 5, we derived the electric potential a distance r from the mid-point of a line with length L carrying a line charge of ρ_l , given by

$$V(z=0, r) = \frac{\rho_l}{4\pi\epsilon_0} \ln(z' + \sqrt{r^2 + z'^2}) \Big|_{-L/2}^{L/2}. \quad (1) \text{ Show that in the limit } L \gg r, \text{ the electric}$$

potential approximates $V(z=0, r) \sim \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{L}{r}$. (2%) (2) Explain why you are getting

a potential that is unphysical when $L \rightarrow \infty$. (2%) (3) Use the expression in (1) to calculate the electric field intensity at $(z=0, r)$ when $L \gg r$. (2%)



Ans: (1) $V(z=0, r) = \frac{\rho_l}{4\pi\epsilon_0} \ln \frac{L/2 + L/2\sqrt{1+(2r/L)^2}}{-L/2 + L/2\sqrt{1+(2r/L)^2}}$. In the limit of $L \gg r$,

$$\sqrt{1+(2r/L)^2} \sim 1 + \frac{1}{2}(2r/L)^2. \quad V(z=0, r) \sim \frac{\rho_l}{4\pi\epsilon_0} \ln \frac{1+(r/L)^2}{(r/L)^2} \sim \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{L}{r}$$

(2) Apparently, the voltage diverges for $L \rightarrow \infty$ $V(z=0, r) = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{L}{r} \xrightarrow{L \rightarrow \infty} \infty$.

When $L \rightarrow \infty$, the total charge is also $\rightarrow \infty$, which adds up to an infinite potential at $(z=0, r)$ and is unphysical.

(3) $\vec{E} = -\nabla V \Rightarrow \vec{E} = \frac{-\rho_l}{2\pi\epsilon_0} \frac{\partial}{\partial r} \ln\left(\frac{L}{r}\right) = \frac{\rho_l}{2\pi\epsilon_0 r} \hat{a}_r$. So, for a long enough line charge L

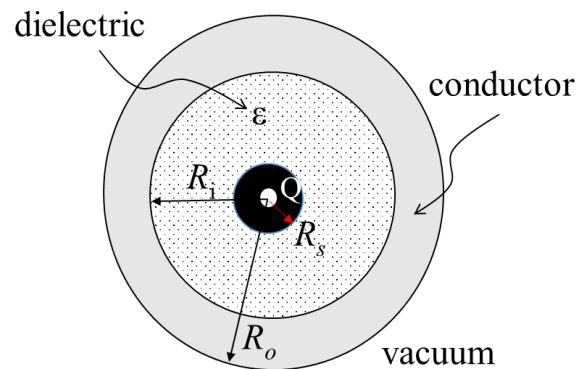
$\gg r$, $\vec{E} = \frac{\rho_l}{2\pi\epsilon_0 r} \hat{a}_r$ obtained from $L \rightarrow \infty$ is a good approximation for the electrical

field intensity close to the mid-point of the line.

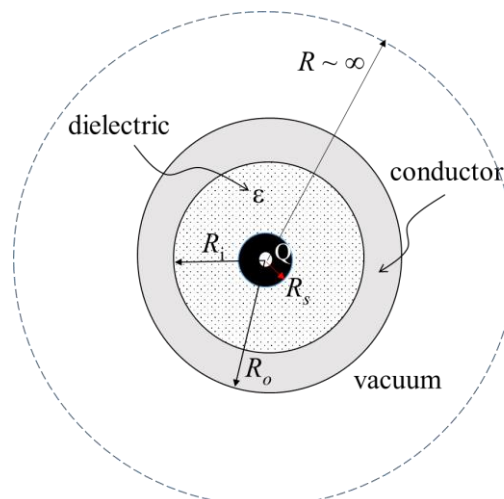
Problem 2 (16%)

A charged ball electrode of radius R_s is embedded into the center of a dielectric ball with an outer radius of R_i . The permittivity of the dielectric between $R_s < R < R_i$ is ϵ . A total charge Q is uniformly distributed on the surface of the ball electrode. There is a

spherical shell of conductor with $R_i < R < R_o$ covering the dielectric, as shown below.



- (1) Find E , D , P , V in the three regions, $R_o < R < \infty$, $R_i < R < R_o$, $R_s < R < R_i$. (4%)
- (2) Sketch E , D , P , V versus R in the 3 regions. (4%)
- (3) What are the surface charge densities on the conductor at R_i and R_o ? Specify the signs of the charges. (2%)
- (4) What is the surface polarization charge density on the dielectric at R_i ? Specify the sign of the charge in your answer. (2%)
- (5) Refer to the following plot. Imagine that another spherical electrode is placed at $R \sim \infty$. Calculate the capacitance of this “capacitor” by first deriving the voltage V between $R = R_s$ and $R = \infty$ and then use $C = Q/V$ to obtain C . (2%)
- (6) Calculate the capacitances in $R_o < R < \infty$ and $R_s < R < R_i$ to prove that the answer in “(5)” can be calculated from two serial capacitors connected through the conductor between $R_i < R < R_o$. (2%)



Ans:

- (1) In the vacuum region $R_o < R < \infty$, use the Gauss law to obtain

$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}$, $\vec{E} = \hat{a}_R \frac{Q}{4\pi\epsilon_0 R^2}$. In a vacuum region, $P = 0$. The work to bring a

unit positive charge from infinity to R is $V = \frac{Q}{4\pi\epsilon_0 R}$.

In the conductor shell region, $R_i < R < R_o$, $D = 0$, $E = 0$, and $\vec{P} = \vec{D} - \epsilon_0 \vec{E} = 0$.

A neutral conductor is an equipotential object. Therefore $V = \frac{Q}{4\pi\epsilon_0 R_o}$ remains a constant equal to that at the outer surface of the conductor.

In the dielectric region, $R_s < R < R_i$, again apply the Gauss law to obtain

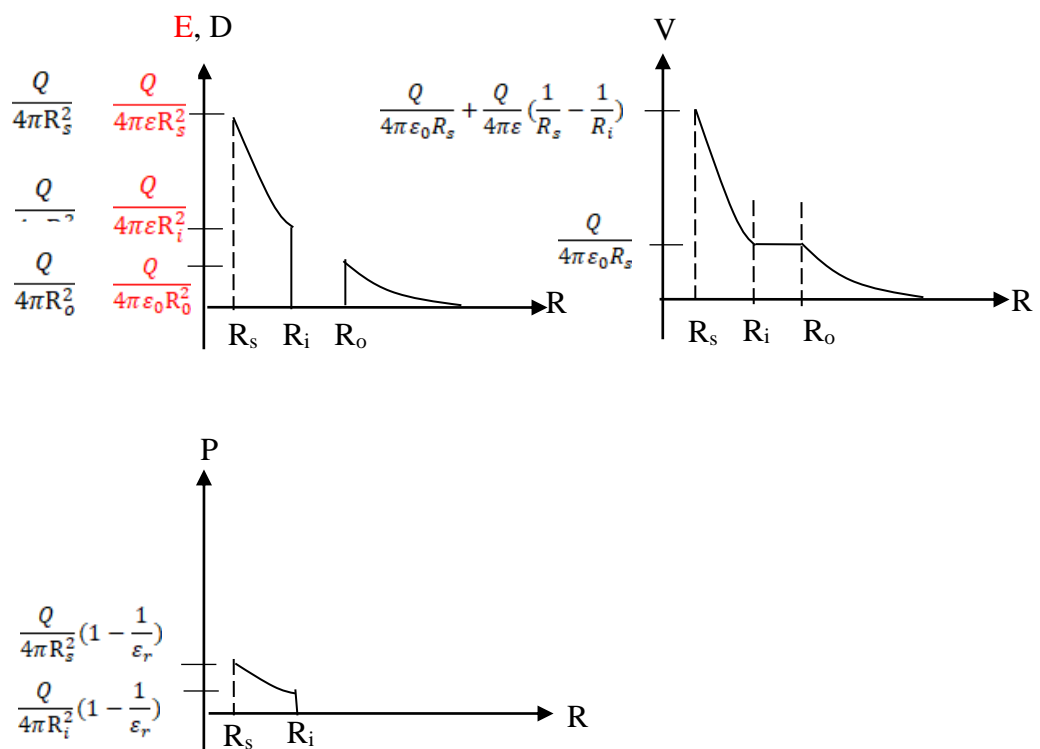
$$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}, \quad \vec{E} = \hat{a}_R \frac{Q}{4\pi\epsilon R^2}.$$

The polarization density vector is $\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \hat{a}_R \frac{Q}{4\pi R^2} \left(1 - \frac{\epsilon_0}{\epsilon}\right)$.

The electric potential is obtained by adding the work needed to bring the charge into

the dielectric from R_i to R , given by $V = \frac{Q}{4\pi\epsilon_0 R_o} + \frac{Q}{4\pi\epsilon} \left(\frac{1}{R} - \frac{1}{R_i}\right)$.

(2)



(3) At R_i , $\rho_s = -\hat{a}_R \cdot \vec{D} = \frac{-Q}{4\pi R_i^2}$. At R_o , $\rho_s = \hat{a}_R \cdot \vec{D} = \frac{Q}{4\pi R_o^2}$.

(4) The surface polarization charge at R_i is $\rho_s = +\hat{a}_R \cdot \vec{P} = \frac{Q}{4\pi R_i^2} (1 - \frac{\epsilon_0}{\epsilon})$

(5) The voltage between the two electrodes is $V = \frac{Q}{4\pi\epsilon_0 R_o} + \frac{Q}{4\pi\epsilon} (\frac{1}{R_s} - \frac{1}{R_i})$. The

capacitance is $C = \left(\frac{V}{Q} \right)^{-1} = \left(\frac{1}{4\pi\epsilon_0 R_o} + \frac{1}{4\pi\epsilon} (\frac{1}{R_s} - \frac{1}{R_i}) \right)^{-1}$

(6) For the capacitor in the vacuum region $R_o < R < \infty$, $V = \frac{Q}{4\pi\epsilon_0 R_o}$ and thus

$$\frac{1}{C_{R_o-\infty}} = \frac{V}{Q} = \frac{1}{4\pi\epsilon_0 R_o}.$$

For the capacitor in the dielectric region, the voltage between the electrodes at $R = R_s$

and R_i is $V = \frac{Q}{4\pi\epsilon} (\frac{1}{R_s} - \frac{1}{R_i})$ and thus $\frac{1}{C_{R_s-R_i}} = \frac{V}{Q} = \frac{1}{4\pi\epsilon} (\frac{1}{R_s} - \frac{1}{R_i})$.

For the two serially connected capacitors, the total capacitance is

$$C = \left(\frac{1}{C_{R_s-R_i}} + \frac{1}{C_{R_o-\infty}} \right)^{-1} = \left[\frac{1}{4\pi\epsilon_0 R_o} + \frac{1}{4\pi\epsilon} (\frac{1}{R_s} - \frac{1}{R_i}) \right]^{-1}, \text{ which is the same as}$$

the answer in (5)

Problem 3 (7%) A large-area parallel-plate capacitor is charged up to store an amount of charge Q and then disconnected from the power supply, as shown in (a) below. The parallel-plate capacitor has an area A and an electrode separation d in vacuum. If one inserts a perfect dielectric of area A , thickness $d/2$, and relative permittivity $\epsilon_r = 2$ into the capacitor, as shown in (b) with an arbitrary y_0 ,

- (1) what is the ratio $|V_b/V_a|$, where V_a , V_b are the voltages across the conducting electrodes before and after inserting the perfect dielectric, respectively. (2%)

Ans: It is well known that $C_a = \frac{\epsilon_0 A}{d}$. With the inserted dielectric slab, consider the capacitance as that equivalent to 3 serially connected capacitors with thickness y_0 , $d/2$, and $d/2 - y_0$, given by

$$C_b = \frac{1}{\frac{y_0}{\epsilon_0 A} + \frac{d/2 - y_0}{\epsilon_0 A} + \frac{d/2}{\epsilon_0 \epsilon_r A}} = \frac{4\epsilon_0 A}{3d} = \frac{4}{3} C_a, \text{ which is independent of } y_0.$$

Since $V = Q/C$ with a constant Q , the voltage ratio is $|V_b / V_a| = C_a / C_b = \frac{3}{4}$.

- (2) what is the ratio $|F_b / F_a|$, where F_a, F_b are the forces that are needed for a structure to hold the conducting electrodes before and after inserting the perfect dielectric, respectively. (2%)

Ans: $|F| = \frac{dW_e|_Q}{dy} = \frac{1}{2} Q^2 \frac{d(1/C)}{dy} \Rightarrow 1/C_a(y) = \frac{y}{\epsilon_0 A}, \quad |F_a| = \frac{1}{2} \frac{Q^2}{dC_a};$

$$1/C_b(y) = \frac{3y}{4\epsilon_0 A}, \quad |F_b| = \frac{1}{2} \frac{Q^2}{dC_b} \Rightarrow |F_b / F_a| = \frac{C_a}{C_b} = \frac{3}{4}.$$

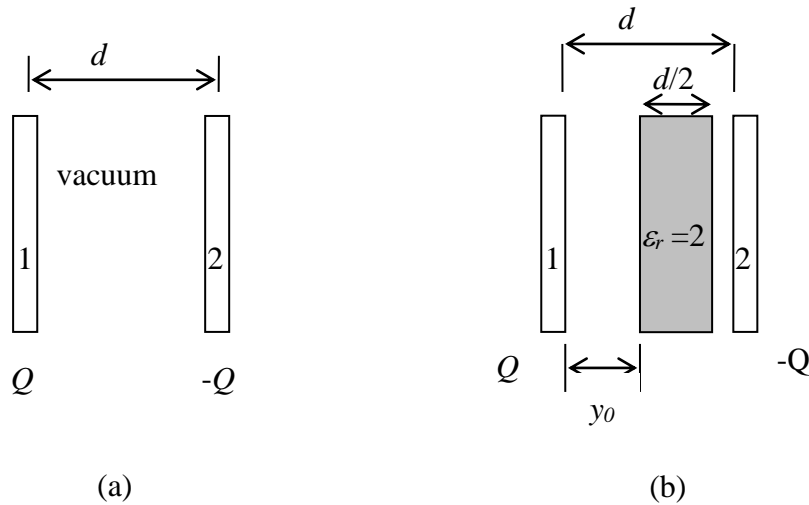
- (3) what is the polarization charge density induced on the surface of the dielectric facing electrode 1? (3%)

Ans: The electric flux density D in the capacitor is the normal component and is continuous across the boundary. The voltage across the two conducting plates for capacitor (b) is $V_b = \frac{Q}{C_b} = \frac{3Qd}{4\epsilon_0 A} = \frac{D}{2\epsilon_0} \frac{d}{2} + \frac{D}{\epsilon_0} \frac{d}{2} \Rightarrow D = \frac{Q}{A}$. According to

$$P = D - \epsilon_0 E = (1 - \frac{1}{\epsilon_r}) D = \frac{1}{2} D = \frac{Q}{2A}. \text{ Given the positive charge on electrode 1,}$$

the induced polarization charge facing it must be negative. The surface polarization charge density on the dielectric facing electrode 1 is therefore

$$\vec{P} \cdot \hat{a}_s = -\frac{Q}{2A}.$$



Problem 4 (9%)

Find the electrostatic energy stored in a sphere of charges with its charge density varying with $\rho = \rho_0 \frac{b}{R}$ between the radius distance $0 < R < R_0$, where ρ_0 and b are constants. The sphere of charges is in a vacuum. Present your calculation by using (1) the assembling technique, (3%) (2) the decomposing technique, (3%) (3) field-energy-density technique. (3%) Cross check your results.

Ans:

Method 1 – assembling technique

First calculate the charge in a radius of R , then the potential V at R for a charge ball of radius R , and finally the work to assemble the charges layer by layer.

$$Q_R = \int_0^{2\pi} \int_0^\pi \int_0^R \rho_0 \frac{b}{R} R^2 \sin \theta dR d\theta d\phi = 2\pi b R^2 \rho_0$$

$$V_R = \frac{Q_R}{4\pi\epsilon_0 R} = \frac{Rb\rho_0}{2\epsilon_0}$$

$$dW_e = V_R dQ_R$$

$$W_e = \int_0^{R_0} V_R dQ_R = \int_0^{R_0} \frac{2\pi\rho_0^2 b^2 R^2}{\epsilon_0} dR = \frac{2\pi\rho_0^2 b^2 R_0^3}{3\epsilon_0}$$

Method 2 – decomposing technique

Assume all the charges are fully in place

$$R \geq R_0 \quad E_R = \frac{Q_{R_0}}{4\pi\epsilon_0 R^2} = \frac{\rho_0 b R_0^2}{2\epsilon_0 R^2}$$

$$0 < R \leq R_0 \quad E_R = \frac{Q_R}{4\pi\epsilon_0 R^2} = \frac{\rho_0 b}{2\epsilon_0}$$

$$V = -\int_{\infty}^R E dR = -\int_{\infty}^{R_0} \frac{\rho_0 b R_0^2}{2\epsilon_0 R^2} dR - \int_{R_0}^R \frac{\rho_0 b}{2\epsilon_0} dR = \frac{\rho_0 b R_0}{\epsilon_0} - \frac{\rho_0 b R}{2\epsilon_0}$$

$$W_e = \frac{1}{2} Q \times V = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^{R_0} \rho_0 \frac{b}{R} \left(\frac{\rho_0 b R_0}{\epsilon_0} - \frac{\rho_0 b R}{2\epsilon_0} \right) \times R^2 \sin \theta \times dR d\theta d\phi = \frac{2\pi\rho_0^2 b^2}{\epsilon_0} \left(\frac{R_0^3}{2} - \frac{R_0^3}{6} \right) = \frac{2\pi\rho_0^2 b^2 R_0^3}{3\epsilon_0}$$

Method 3 – field-energy-density technique

$$R \geq R_0 \quad E_R = \frac{Q_{R_0}}{4\pi\epsilon_0 R^2} = \frac{\rho_0 b R_0^2}{2\epsilon_0 R^2}$$

$$0 < R \leq R_0 \quad E_R = \frac{Q_R}{4\pi\epsilon_0 R^2} = \frac{\rho_0 b}{2\epsilon_0}$$

$$W_e = \frac{1}{2} \oint_V \epsilon_0 E^2 dV = \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^{\pi} \left(\int_{R_0}^{\infty} \left(\frac{\rho_0 b R_0^2}{2\epsilon_0 R^2} \right)^2 R^2 dR + \int_0^{R_0} \left(\frac{\rho_0 b}{2\epsilon_0} \right)^2 R^2 dR \right) \sin \theta d\theta d\phi \right)$$

$$= \frac{\epsilon_0}{2} \left(\frac{\rho_0 b}{2\epsilon_0} \right)^2 4\pi \left(\int_{R_0}^{\infty} \frac{R_0^4}{R^2} dR + \int_0^{R_0} R^2 dR \right) = \frac{\pi\rho_0^2 b^2}{2\epsilon_0} \left(R_0^3 + \frac{R_0^3}{3} \right) = \frac{2\pi\rho_0^2 b^2 R_0^3}{3\epsilon_0}$$