Homework 5 tutorial

Part 1

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HW5 Q3

- Let A = {∅, {∅}}. Determine whether each of the following statements is true or false.

Hint

$$e \in \{e\}$$
$$\{e\} \subseteq \{a, b, c, d, e\}$$

HW5 Q3

- Let A = {∅, {∅}}. Determine whether each of the following statements is true or false.

- (a). $\phi \in 2^A$ (T)
- (b). $\phi \subseteq 2^A$ (T)
- (c). $\{\phi\} \subseteq 2^A$ (T)
- (d). $\{\phi\} \subseteq A$ (T)

Hint

$$2^A = \{\{\}, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

ϕ = {} empty set

Hint

$$e \in \{e\}$$

$$\{e\} \subseteq \{a, b, c, d, e\}$$

HW5 Q3

- Let A = {Ø, {Ø}}. Determine whether each of the following statements is true or false.

Hint

2^A

$= \{\{\}, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

ϕ = {} empty set

- (e). $\{\phi\} \in 2^A$ (T)
- (f). $\{\phi\} \in A$ (T)
- (g). $\{\{\phi\}\} \subseteq 2^A$ (T)
- (h). $\{\{\phi\}\} \subseteq A$ (T)

Hint

$e \in \{e\}$

$\{e\} \subseteq \{a, b, c, d, e\}$

HWS Q3

- Let A = {∅, {∅}}. Determine whether each of the following statements is true or false.

- (i). $\{\{\phi\}\} \in 2^A$ (T)
- (j). $\{\{\phi\}\} \in A$ **(F)**

Hint

2^A

$= \{\{\}, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

ϕ = {} empty set

Hint

$e \in \{e\}$

$\{e\} \subseteq \{a, b, c, d, e\}$

HW5 Q5

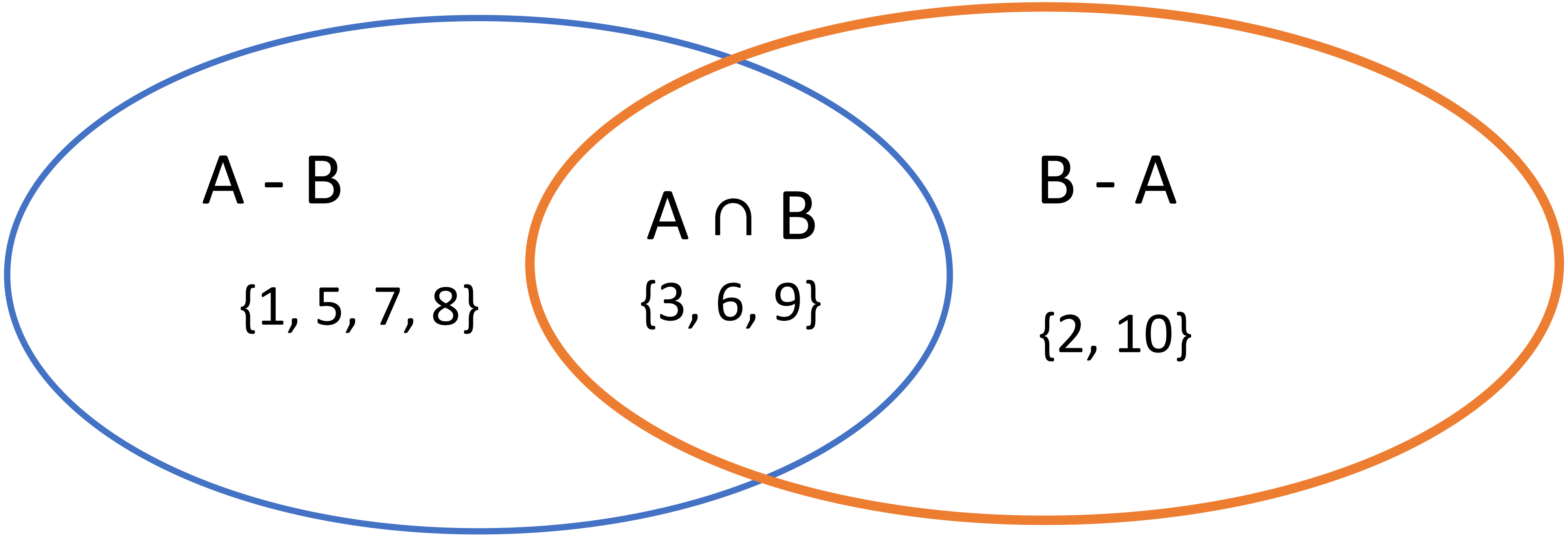
Hint :

A **–** B : A **without** B

P ∩ Q : P **A**nd Q

P ∪ Q : P **U**nion Q

- For two sets A and B, we use A – B to denote A ∩ B. Find the sets A and B if A – B = {1, 5, 7, 8}, B – A = {2, 10}, and A ∩ B = {3, 6, 9}.



HW5 Q6

- For each of the following lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

- Please ***give YOUR IDEA*** on exam.
- (a). 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, **123, 146, 171** (+3+5+7…)
- (b). 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011**, 1100, 1101, 1110** (binary)
- (c). 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, **59048, 177146, 531440** (3^n-1)
- (d). 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, **0, 0, 0** (1x1,2x0,3x1…)

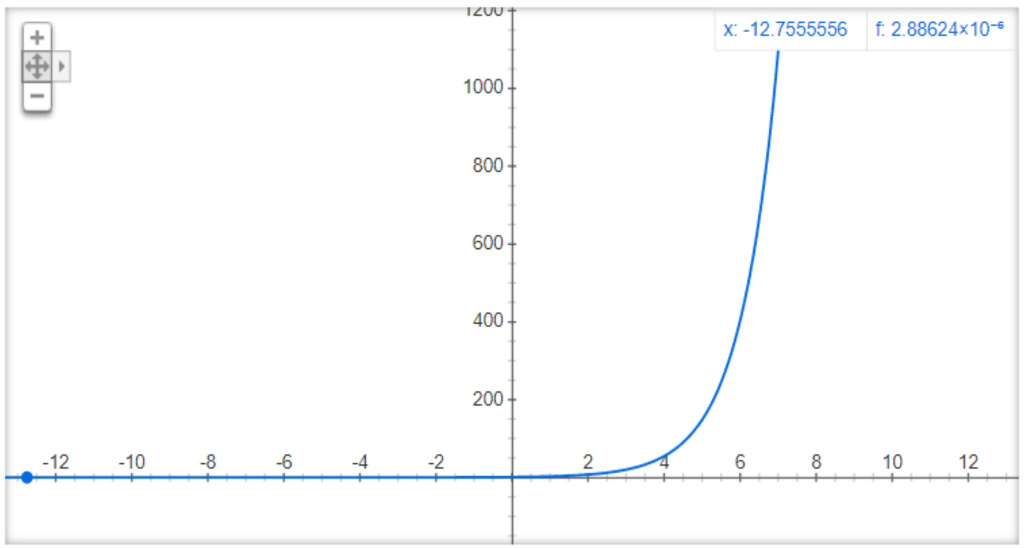
HW5 Q7

- Show that the set ℝ of real numbers and the set ℝ+ of positive real numbers have the same cardinality. (That is, give a one-to-one correspondence between the items in the two sets.)

- 解答不唯一，定義好對應關係即可

x in ℝ map to ℝ+ by $f(x) = e^x$

e^x 的圖表



HW5 Q8

- Peter show that is uncountable by diagonalization technique

$$X = \{ x \mid x \in (0, 1) \text{ and } x \text{ has } k \text{ decimal places and } k \in N \}$$

However, each number in X is a rational number; for instance, 0.33215 = 33215/100000. Thus, $X \subseteq \mathbb{Q}$ (where $\mathbb{Q}$ is countable), which implies X must be countable. So, what's wrong with Peter's proof?

HW5 Q8

X = { x | x ∈ (0, 1) and x has k decimal places and k ∈ N }

We prove this by contradiction. Assume to the contrary that there is a one-to-one correspondence between items in X and items in $\mathbb{N}$. Then, we can list the items in X one by one, say $x_1, x_2, x_3, \ldots$. Now, consider the number x such that its digit in the first decimal place is different from x_1, its digit in the second decimal place is different from x_2, and in general, its digit in the jth decimal palce is different from x_j for all j. Then, x is not listed by the correspondence, and a contradiction occurs as desired.

- x found by Peter is not in set X

HW5 Q8

X = { x | x ∈ (0, 1) and x has k decimal places and k ∈ N }

x found by Peter is not in set X

- If x in X, let x is k-digit number.
- Consider x_{k+1}, and let k+1 digit in x_{k+1} is 0
  - k+1 digit in x is the inverse of x_{k+1}.
  - But x is k-digit number, k+1 digit in x must be 0.
- So such x not exists.

HW5 Q9

- Give an example of three sets W, X, Y such that W ∈ X and X ∈ Y but W ∉ Y .

- W ∈ X
  - Let W = {a}, x = {{a}}
- X ∈ Y
  - Y = {{{a}}}
- So W ∉ Y

HW5 Q10

- Consider all the five-element subsets of {1, 2, 3, . . . , n}. It is known that one quarter of these subsets contain the element 7. What is the value of n?

- C(n-1,4) /C(n,5) = 0.25
- $\frac{(n-1)!}{4!(n-5)!} \times \frac{5!(n-5)!}{n!} = \frac{5}{n} = 0.25$
- $n = 20$

HW5 Q11

- Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

- (b). f(x) = 3x^2+4 No. because *f(x) = 0 not exists*.
- (c). f(x) = (x+1)/(x+2) No. because *f(x) = 1 not exists*.

HW5 Q11

Hint

If f is bijection $\Leftrightarrow f^{-1}$ exist and $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$

- Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

- (a). f(x) = -3x+4 Yes
  - $f^{-1}(x) = (4 - x)/3$
  - $f^{-1}\big(f(x)\big) = \frac{4-(-3x+4)}{3} = x$
  - $f(f^{-1}(x)) = -3\left(\frac{4-x}{3}\right) + 4 = x$

HW5 Q12

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function where $f(x) = x^2$. For any subset A of $\mathbb{R}$, we use $f(A)$ to denote the set $\{f(x) \mid x \in A\}$.
- Determine $f(A)$ for the following subsets A taken from the domain $\mathbb{R}$.

(a). $A = \{2, -3\}$, $f(A) = \{4, 9\}$
(b). $A = (-3,3)$, $f(A) = \{x|0 \leq x < 9\}$
(c). $A = (-3,2]$, $f(A) = \{x|0 \leq x < 9\}$

HW5 Q12

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function where $f(x) = x^2$. For any subset A of $\mathbb{R}$, we use $f(A)$ to denote the set $\{f(x) \mid x \in A\}$.
- Determine $f(A)$ for the following subsets A taken from the domain $\mathbb{R}$.

(d). $A = (-4, -3] \cup [5,6], f(A) = \{x | 9 \leq x < 16 \cup 25 \leq x \leq 36\}$

HW5 Q13

- Define $g : \mathbb{Z} \rightarrow \mathbb{Z}$, where $g(n) = \lfloor n/2 \rfloor$. Is g a surjection? Is g a bijection?

- Bijection : No, Bexause $g(2) = g(3) = 1$
- Surjection : Yes, for any $p \in \mathbb{Z}$, exist $q = 2p \in \mathbb{Z}$ and $g(q) = p$

HW5 Q15~29
HW6 Q3~6

(part 2) 薛旻欣

HW5 - Q15

15. (*) What is wrong with the following argument that attempts to show that if R is a relation on a set S that is both symmetric and transitive, then R is also reflexive?

 "Since xRy implies yRx by the symmetric property,
 xRy and yRx imply xRx by the transitive property,
 thus, xRx is true for each x in S, and so R is reflexive."

- Transitive property: what if $\nexists y\ x\mathcal{R}y$ for some x?

HW5 - Q15

15. (*) What is wrong with the following argument that attempts to show that if R is a relation on a set S that is both symmetric and transitive, then R is also reflexive?

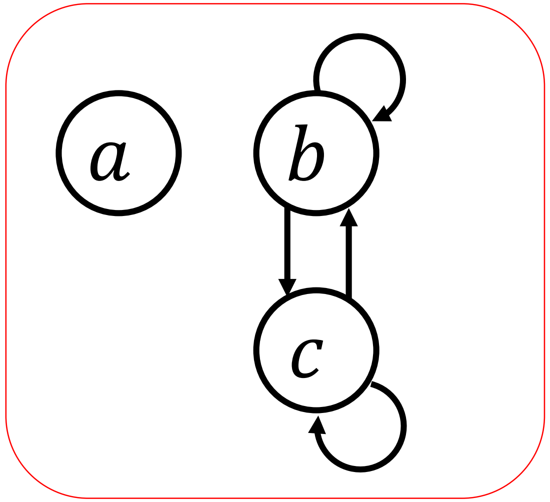
> "Since xRy implies yRx by the symmetric property,
> xRy and yRx imply xRx by the transitive property,
> thus, xRx is true for each x in S, and so R is reflexive."

- Consider some relation on a set $S = \{a, b, c\}$:
- It's symmetric

  $b\mathcal{R}c \rightarrow c\mathcal{R}b,\ c\mathcal{R}b \rightarrow b\mathcal{R}c$
- It's transitive

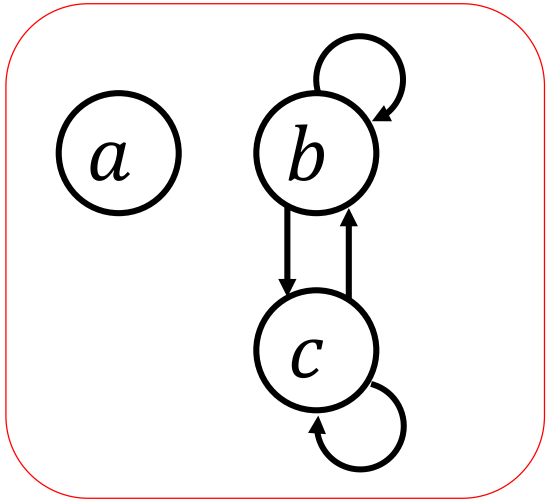
  $(b\mathcal{R}c \land c\mathcal{R}b) \rightarrow b\mathcal{R}b$

  $(c\mathcal{R}b \land b\mathcal{R}c) \rightarrow c\mathcal{R}c$



HW5 - Q15

- However, it's not reflexive.
- Because $\nexists y\ a\mathcal{R}y$ for the item a, it doesn't contradicts to the transitive property if $a\mathcal{R}a$ doesn't exist.

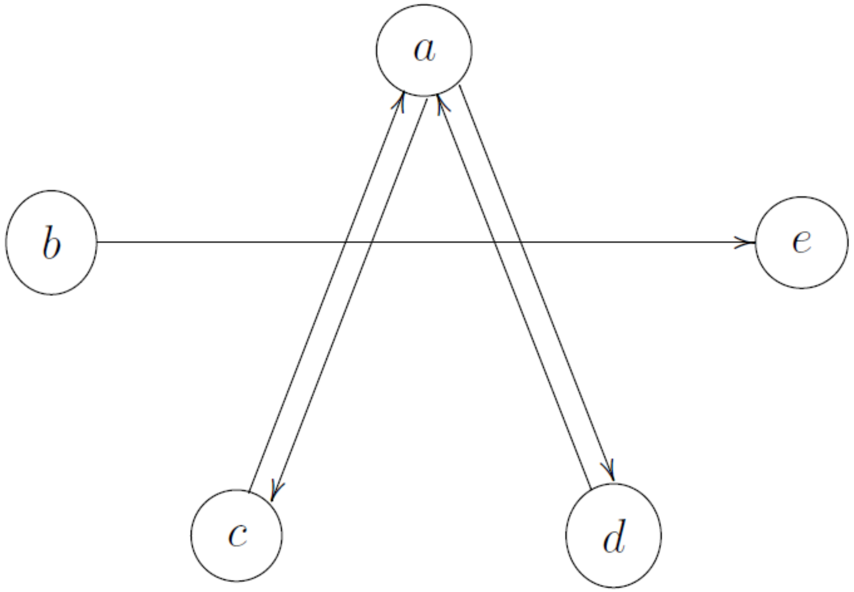


HW5 – Q16

16. (*)

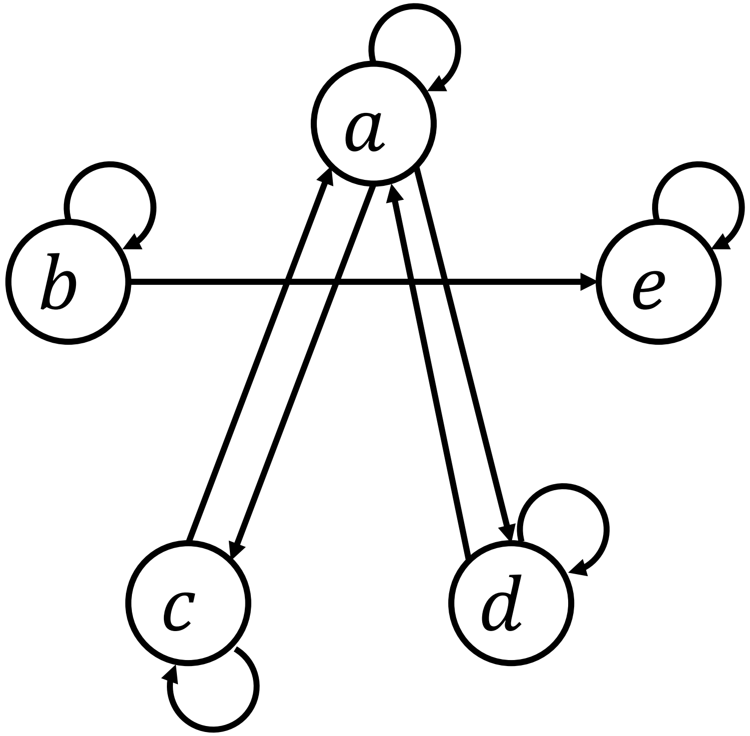
(a) Find the reflexive closure of R as represented by the following directed graph.

(b) Find the transitive closure of R as represented by the following directed graph.



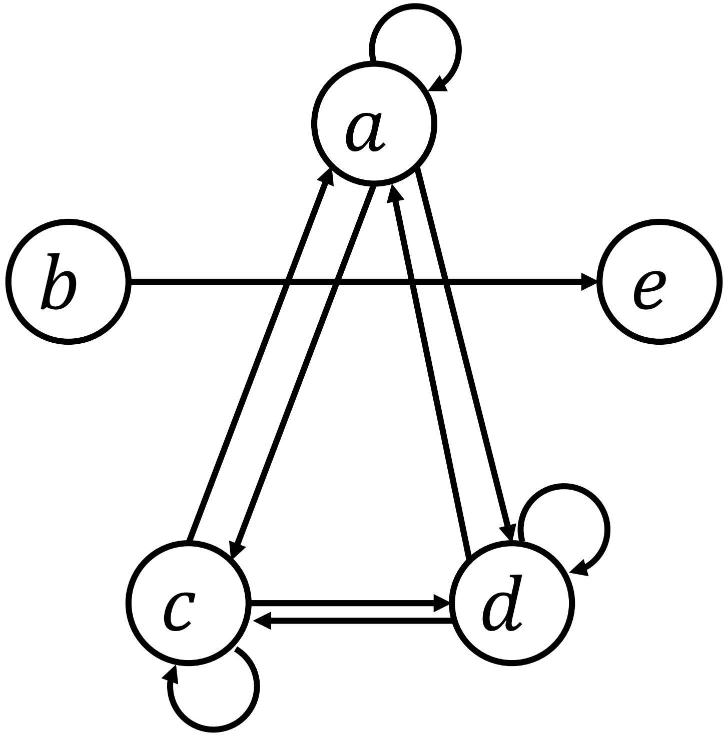
HW5 – Q16 (a)

- Find the reflexive closure – **note** that it also contains the original graph.



HW5 – Q16 (a)

- Find the transitive closure – **note** that it also contains the original graph.



HW5 – Q20 (a)

- Let $A = \{0,1,2,3,4\}$ and $B = \{1,3,5,7,9\}$.
- If the relation $P: aPb \Leftrightarrow a \equiv b(mod\ 4)$

- Then, $P = \{(1,1), (1,5), (1,9), (3,3), (3,7)\}$
- Note that for a relation aPb, $a \in A\ and\ b \in B$.

HW5 – Q20 (b)

- Let $A = \{0,1,2,3,4\}$ and $B = \{1,3,5,7,9\}$.
- If the relation $P: aPb \Leftrightarrow 2\ divides\ (a + b)$
- Then, $P = \{(1,1), (1,3), (1,5), (1,7), (1,9), (3,1), (3,3), (3,5), (3,7), (3,9)\}$

HW5 – Q20 (c)

- Let $A = \{0,1,2,3,4\}$ and $B = \{1,3,5,7,9\}$.
- If the relation $P{:}\, aPb \Leftrightarrow a = b \; or \; a - 1 = b$

- Then, $P = \{(1,1), (2,1), (3,3), (4,3)\}$

HW5 – Q26:

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Equivalence relation: reflexive, symmetric, and transitive.

HW5 – Q26: $R_1 \cap R_2$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Let's denote $R_1 \cap R_2$ as R_3.
- Reflexive: For any item x in set S, the tuple (x, x) exists in R_1 and R_2(because they are both reflexive). Therefore, (x, x) also exists in R_3.

HW5 – Q26: $R_1 \cap R_2 = R_3$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Symmetric: For any tuple $(x, y) \in \mathrm{R}_3$, it must exist in R_1 and R_2, which gives (y, x) existing in R_1 and R_2. Therefore (y, x) also exists in R_3.

HW5 – Q26: $R_1 \cap R_2 = R_3$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Transitive: For any tuples $(x, y), (y, z) \in \mathrm{R}_3$, they must exist in R_1 and R_2, which gives (x, z) existing in R_1 and R_2. Therefore (x, z) also exists in R_3.

- So $R_3 = R_1 \cap R_2$ is an equivalence relation.

HW5 – Q26: $R_1 \cup R_2$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Let's denote $R_1 \cup R_2$ as R_3.
- Reflexive: For any item x in set S, the tuple (x, x) must exists in R_1 and R_2 (Note that the relations are on the same set S). Therefore, (x, x) also exists in R_3.

HW5 – Q26: $R_1 \cup R_2 = R_3$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Symmetric: For any tuple $(x, y) \in R_3$, it must exist in R_1 or R_2 (or both), giving us (y, x) existing in R_1 or R_2 (or both). Therefore (y, x) must exist in R_3.

HW5 – Q26: $R_1 \cup R_2 = R_3$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

- Not transitive (we give a counter-example) : For any tuples $(x, y), (y, z) \in R_3$, if (x, y) exists in $R_1 - R_2$ and (y, z) exists in $R_2 - R_1$, or vice versa. Then neither R_1 nor R_2 gives the existence of (x, z), so R_3 may not be transitive.

- Therefore, it is not necessarily true that $R_1 \cup R_2$ is an equivalence relation.

HW5 – Q28 (a)

28. (*) Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

    (a) Verify that R is an equivalence relation on A.

    (b) Determine the equivalence classes $[(1, 3)]$, $[(2, 4)]$, and $[(1, 1)]$.

    (c) Determine the partition induced by R.

- Equivalence relation: reflexive, symmetric, and transitive.

HW5 – Q28 (a) reflexive

28. (*) Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

    (a) Verify that R is an equivalence relation on A.

- For any tuple (x, y), $x + y = x + y$, therefore $(x, y\ )\mathcal{R}(x, y)$ always exists.

HW5 – Q28 (a) symmetric

28. (*) Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

(a) Verify that R is an equivalence relation on A.

- If exists some $(x_1, y_1)\mathcal{R}(x_2, y_2)$, then $x_1 + y_1 = x_2 + y_2 = x_1 + y_1$

  $\rightarrow (x_2, y_2)\mathcal{R}(x_1, y_1)$

HW5 – Q28 (a) transitive

28. (*) Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

    (a) Verify that R is an equivalence relation on A.

- If exists some relations $(x_1, y_1)\mathcal{R}(x_2, y_2)$ and $(x_2, y_2)\mathcal{R}(x_3, y_3)$, we have $x_1 + y_1 = x_2 + y_2 = x_3 + y_3$

  $\rightarrow (x_1, y_1)\mathcal{R}(x_3, y_3)$

- Now we've shown that $\mathcal{R}$ is an equivalence relation on A.

HW5 – Q28 (b)

28. (*) Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

   (b) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$, and $[(1,1)]$.

- $[(1,3)]_R = \{(1,3), (3,1), (2,2)\}$
- $[(2,4)]_R = \{(2,4), (4,2), (3,3), (5,1), (1,5)\}$
- $[(1,1)]_R = \{(1,1)\}$

HW5 – Q28 (c)

28. (*) Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define R on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

(c) Determine the partition induced by R.

- The partition consists of all equivalence classes.

HW5 – Q28 (c) partition

- I would like to call them $P_2, P_3, \ldots, P_{10}$ (regarding to the sum).
- $P_2 = \{(1,1)\}$
- $P_3 = \{(1,2), (2,1)\}$
- $P_4 = \{(1,3), (3,1), (2,2)\}$
- $P_5 = \{(1,4), (4,1), (2,3), (3,2)\}$
- $\mathrm{P}_6 = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$
- $\mathrm{P}_7 =$…
- …

HW5 – Q29

29. (*) If $|A| = 30$ and the equivalence relation R on A partitions A into equivalence classes A_1, A_2, and A_3, where $|A_1| = |A_2| = |A_3|$, what is $|R|$?

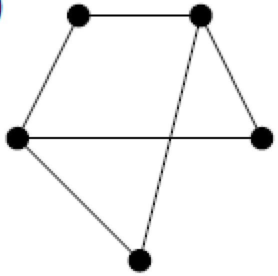
- $|A_1| = |A_2| = |A_3| = \frac{|A_1|}{3} = 10$
- $|R| = \underbrace{10 \times 10}_{\text{The tuples derived from a single equivalence class}} \times 3 = 300$

The tuples derived from
a single equivalence class

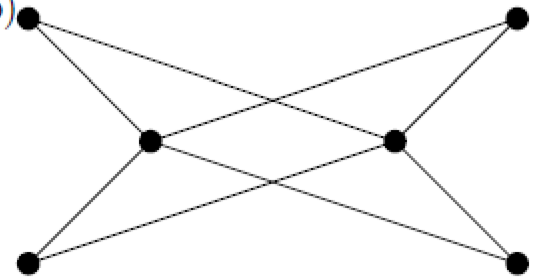
HW 6 – Q3

3. (*) For each of the following graphs, determine whether it is bipartite.

(a)



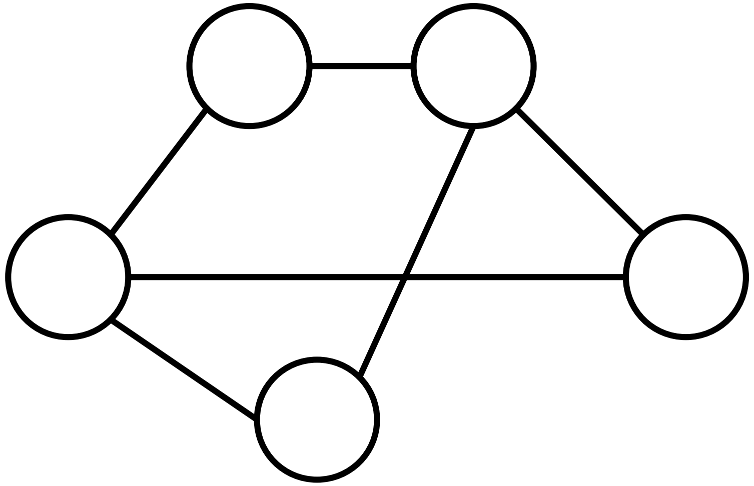
(b)



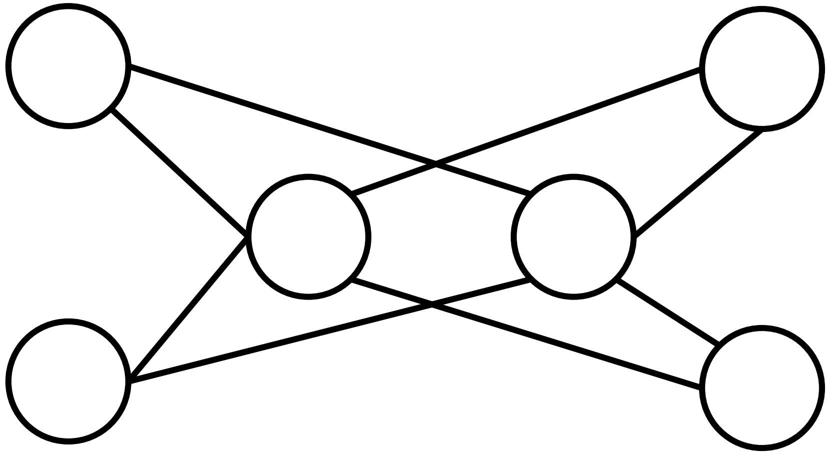
HW 6 – Q3

- We can draw the vertices Black/White.

(a)

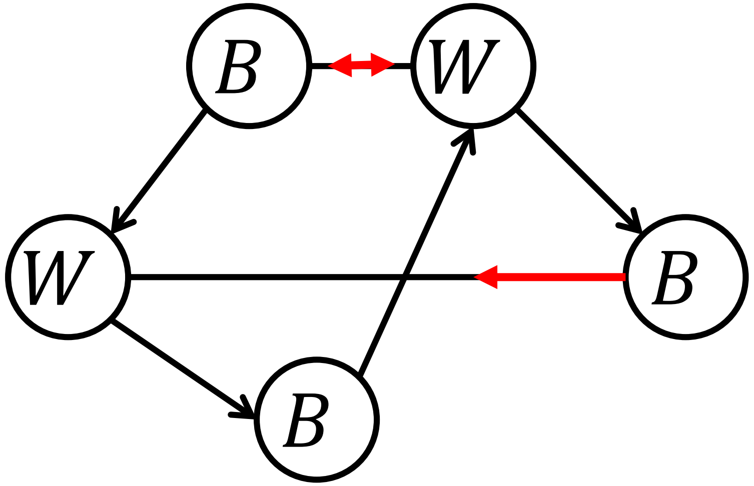


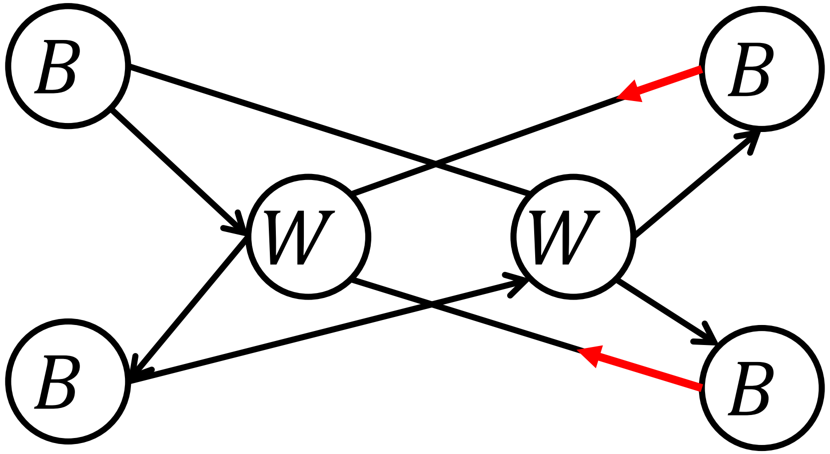
(b)



HW 6 – Q3

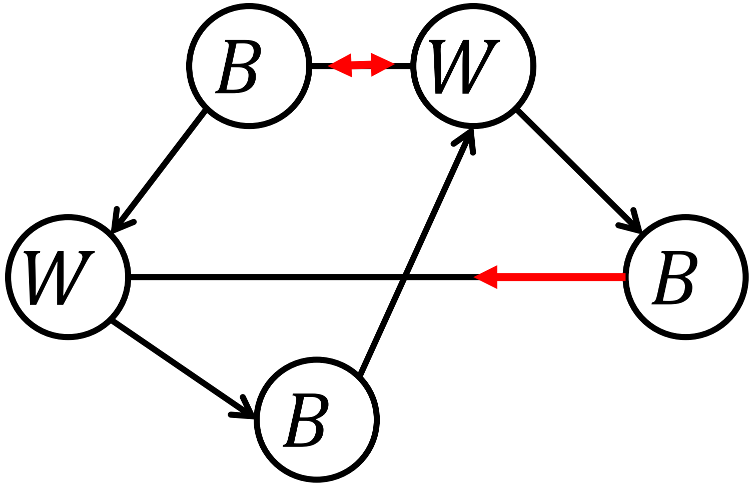
- We can draw the vertices with Black/White
- Suppose we always start at the up-left vertex and draw it Black. We can move on drawing the adjacent vertices in any order (the arrows denotes one).

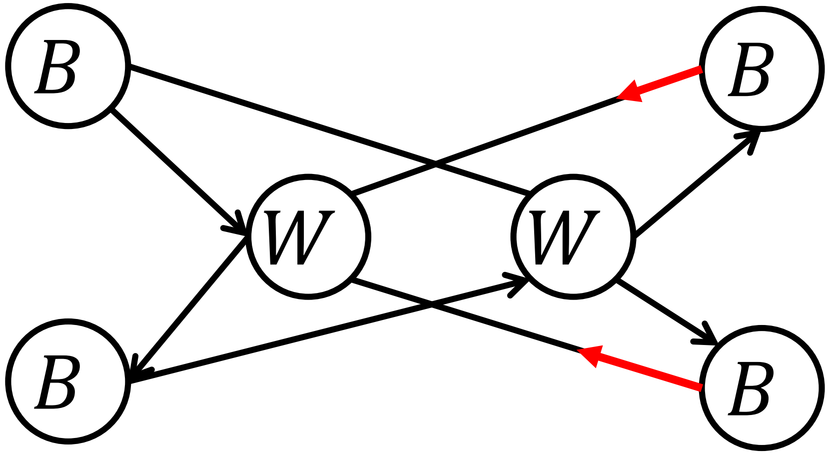




HW 6 – Q3

- (a) is bipartite.
- (b) is also bipartite.





HW 6 – Q4

4. (*) If G is a graph, the *complement* of G, denoted by $\overline{G}$, is a graph with the same vertex set, such that an edge e exists in $\overline{G}$ if and only if e does not exist in G.

   Suppose it is known that a simple graph G has 15 edges and its complement graph $\overline{G}$ has 13 edges. How many vertices does G have?

- Definition: In graph theory, the complement of a graph G is a graph $\bar{G}$ on the same vertices such that **two distinct** vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

  →we don’t consider self loops.

HW 6 – Q4

4. (*) If G is a graph, the *complement* of G, denoted by $\overline{G}$, is a graph with the same vertex set, such that an edge e exists in $\overline{G}$ if and only if e does not exist in G.

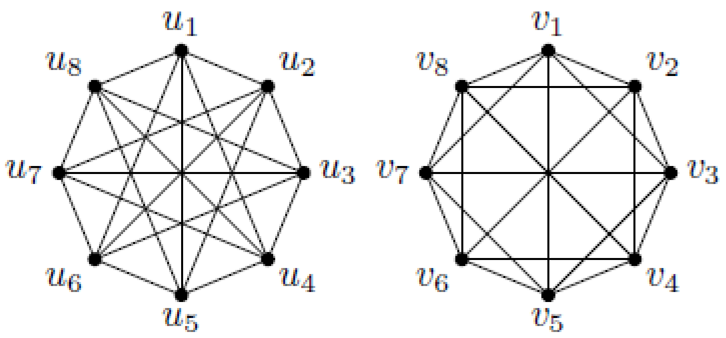
   Suppose it is known that a simple graph G has 15 edges and its complement graph $\overline{G}$ has 13 edges. How many vertices does G have?

- Denoting the number of vertices in G by n.
- We know that $G \cap \bar{G} = \phi$ and $G \cup \bar{G} = K_n$. (related to Q5)
- The number of edges in K_n is E $= \binom{n}{2}$.
- Solving $E = \binom{n}{2} = 15 + 13$ gives $n = 8$ as the answer.

HW 6 – Q6

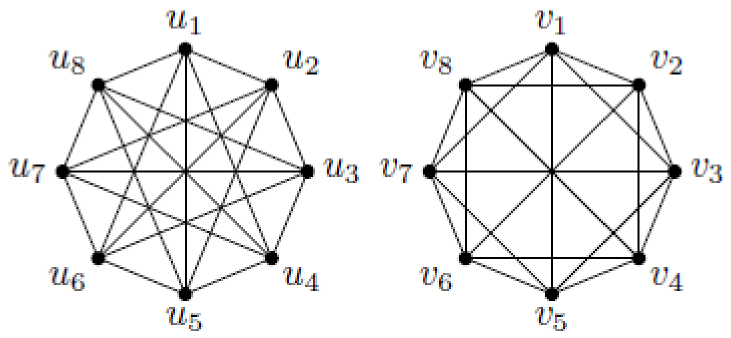
6. (*) Two graphs G and H are *isomorphic* if there is a one-to-one correspondence f between the vertices of G and the vertices of H, such that u, v are adjacent in G if and only if $f(u), f(v)$ are adjacent in H. (See Lecture Note 13, pages 26 and 27 for some examples.)

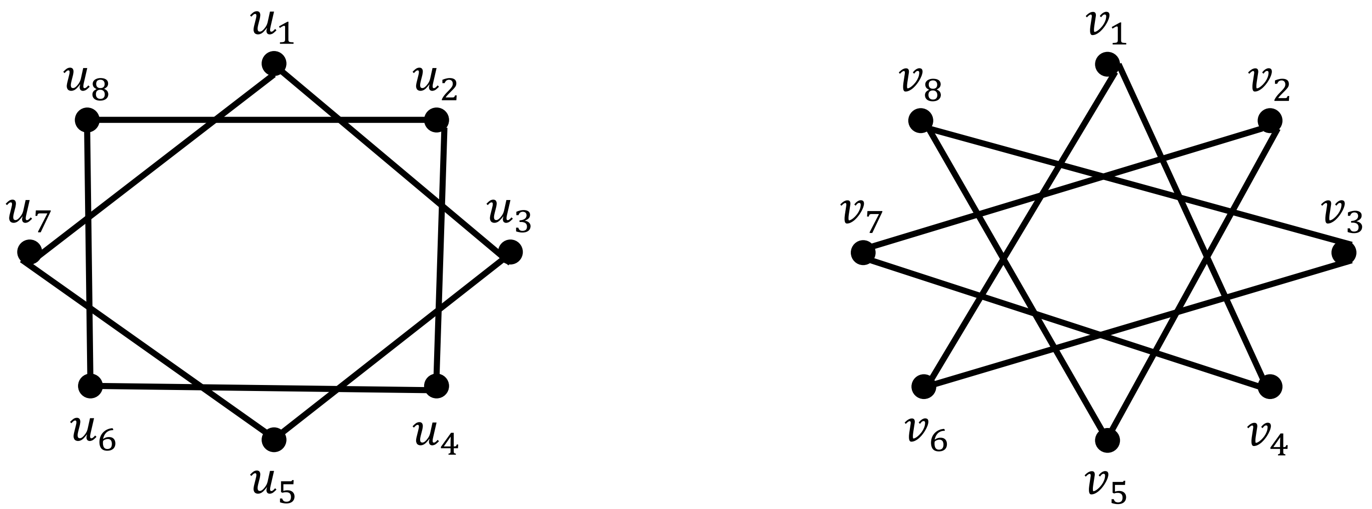
   Determine whether the following two graphs are isomorphic.



HW 6 – Q6

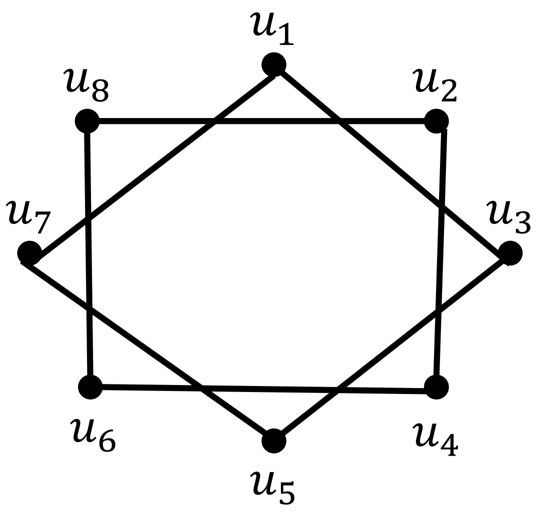
- The complement of the graphs:

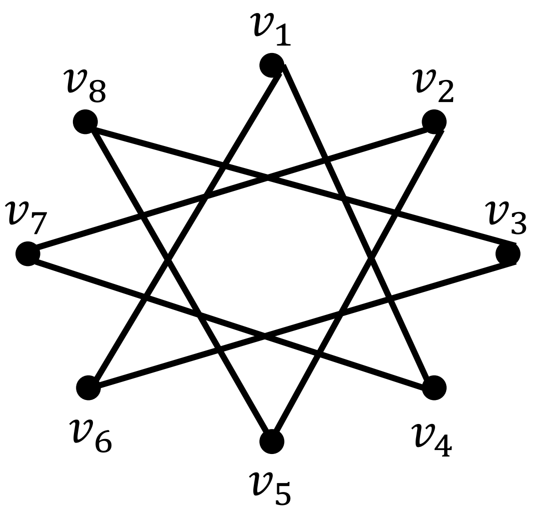




HW 6 – Q6

- The complement of the graphs:





- $\overline{G_u}$ is not connected, but $\overline{G_v}$ is connected. There doesn't exist a one-to-one correspondence between them.
- Therefore $\overline{G_u}$ and $\overline{G_v}$ are not isomorphic, and their complement graphs G_u and G_v are neither isomorphic. (related to Q7)