Chapter 4 Boundary Value Problems in Electrostatics

Recall, the electrostatic Maxwell's equations

1. $\nabla \times \vec{E} = 0 \Rightarrow \qquad \vec{E} = -\nabla V$
2. $\nabla \cdot \vec{D} = \rho$

Combine 1 & 2 and assume a homogeneous medium (ε is independent of coordinates)

$\Rightarrow$ Poisson's Equation $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$,

In a source-free region

$\Rightarrow$ Laplace's Equation $\nabla^2 V = 0$

where the Laplacian operator is given by $\nabla^2 = \nabla \cdot \nabla$: *the divergent of the gradient of ..*

In Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

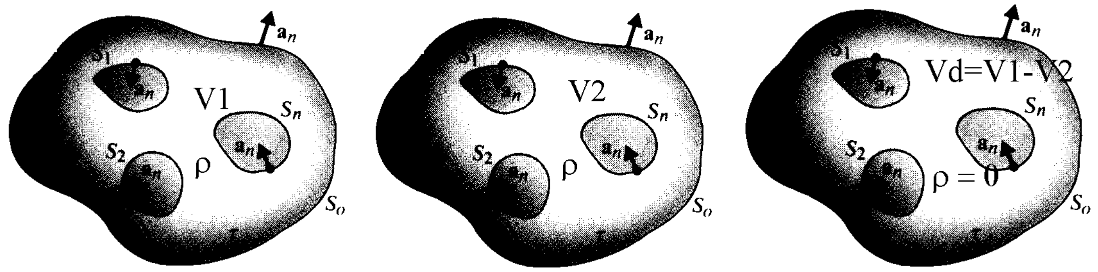
In cylindrical coordinates

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical coordinates

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Uniqueness Theorem: A solution satisfying the boundary conditions of a problem is the only solution.



1 2 3

Charges are enclosed in the surfaces S_1 , S_2 ….. S_n . Electric potentials *V* are all specified on the surface.

<u>Graphs (1,2)</u>

If $V_1 \& V_2$ are two different solutions to $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$, except $V_1 = V_2$ on the surfaces, $V_d \equiv V_1 - V_2$ is a solution of $\nabla^2 V_d = 0$.

<u>Creating Graph 3 by subtracting Graph 2 from 1</u>

One obtains $V_d = 0$ on the surfaces due to the same boundary conditions in Graphs 1 and 2. Also, $\nabla^2 V_d = 0$ and $\rho_d = 0$ everywhere. Therefore the surface charge $\rho_s = 0$ vanishes on the surfaces of S_1 , S_2 ….. S_n .

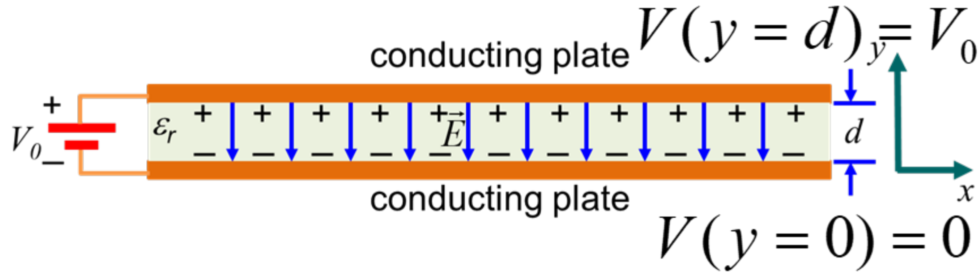
If $V_d \neq 0$ at any point between surfaces, there exits a gradient of potential in space $\vec{E} = -\nabla V_d \neq 0$. For a surface enclosing an object, the outward flux $\oint_S \vec{D} \cdot d\vec{s} = Q \neq 0$ won't be zero and the surface must enclose some charges, which contradicts $\rho_d = 0$ in Graph 3.

$\Rightarrow$ $V_1 = V_2$ everywhere if same boundary conditions are specified.

General Methodology for Solving an Electrostatic Problem

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \quad \Rightarrow V \Rightarrow \quad \vec{E} = -\nabla V \quad \Rightarrow \quad \rho_s = \varepsilon E_n$$

Eg. Given V_0 and d, find V everywhere and ρ_s



y = 0, V = 0

1. Electric potential in the capacitor

Between the two plates, there are no isolated charges. Use Laplace's equation $\nabla^2 V = 0$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2 V}{\partial y^2} = 0 \quad \Rightarrow \quad V = C_1 y + C_2$$

Boundary conditions: $V(y=0) = 0$, $V(y=d) = V_0$

$$\Rightarrow \; V = \frac{V_0}{d} y$$

2. Electric field in the capacitor $\vec{E} = -\nabla V = -\frac{V_0}{d}\hat{a}_y$

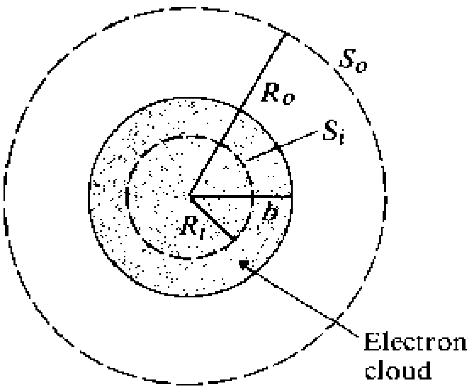
3. Surface charges on the conducting plates

$$\hat{a}_{n2} \cdot \vec{D} = \rho_s \ .$$

At the upper plate, $\hat{a}_{n2} = -\hat{a}_y$ and $\rho_{su} = \varepsilon \dfrac{V_0}{d}$

At the lower plate, $\hat{a}_{n2} = \hat{a}_y$ and $\rho_{sd} = -\varepsilon \dfrac{V_0}{d}$

Eg. Given a ball of electron cloud with a uniform volume charge density $\rho = -\rho_0$, find $\vec{E}$ everywhere.



Boundary Conditions

i. At $R = 0$, $E = 0$

ii. At $R \to \infty$, $E \,\&\, V \to 0$

iii. At $R = b$, D & V are continuous, because there's no surface charge density at the boundary and V (work) is a continuous function.

1. $0 \le R \le b$

$$\nabla^2 V_i = -\frac{\rho}{\varepsilon}$$

$$\Rightarrow \quad \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_i}{\partial R}\right)=\frac{\rho_0}{\varepsilon_0}$$

$$\Rightarrow \quad \frac{dV_i}{dR}=\frac{\rho_0}{3\varepsilon_0}R+\frac{C_1}{R^2}=\frac{\rho_0}{3\varepsilon_0}R \tag{4-1}$$

take $C_1=0$ to avoid $E_R \propto \frac{dV_i}{dR}\to\infty$ as $R\to 0$

$$\vec{E}_i=-\frac{\rho_0}{3\varepsilon_0}R\hat{a}_R \tag{4-2}$$

2. $R\geq b$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_0}{\partial R}\right)=0$$

$$\frac{dV_o}{dR}=\frac{C_2}{R^2} \quad \Rightarrow \quad \vec{E}_o=\frac{-C_2}{R^2}\hat{a}_R \tag{4-3}$$

Recall the boundary condition, $D_{n1}-D_{n2}=\rho_s$, $\varepsilon_i=\varepsilon_o$, $\rho_s=0$.

Equate (4-2) and (4-3) at $R = b$ to obtain $C_2=\frac{\rho_0 b^3}{3\varepsilon_0}$

$$\Rightarrow \quad \vec{E}_o=\frac{-\rho_0 b^3}{3\varepsilon_0 R^2}\hat{a}_R \quad \text{for, } R\geq b \tag{4-4}$$

but $Q=-\frac{4}{3}\pi b^3\rho_0 \quad \Rightarrow \quad \vec{E}_o=\frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$

3. (4-1) $\Rightarrow$ $V_i = \dfrac{\rho_0}{6\varepsilon_0}R^2 + C_3$ (4-5)

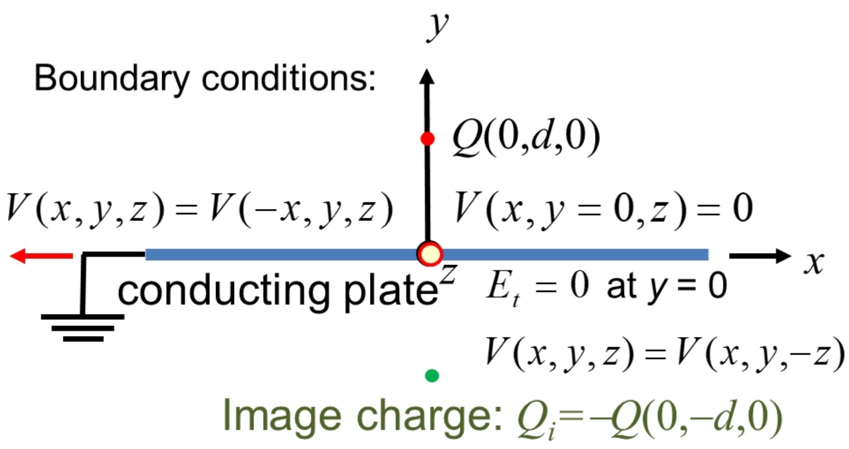
(5-4) $\Rightarrow$ $V_o = \dfrac{-\rho_0 b^3}{3\varepsilon_0 R} + C_4 = \dfrac{-\rho_0 b^3}{3\varepsilon_0 R}$ (4-6)

setting $C_4 = 0$ to obtain $V_o(R \to \infty) = 0$.

Equate (4-5) and (4-6) at $R = b$, $C_3 = -\dfrac{\rho_0 b^2}{2\varepsilon_0}$,

thus $V_i = \dfrac{\rho_0}{6\varepsilon_0}R^2 - \dfrac{\rho_0 b^2}{2\varepsilon_0}$

Method of Images: matching boundary conditions by creating image charges



Boundary conditions

1. $V(x, y = 0, z) = 0$
2. $V(x, y, z) = V(-x, y, z)$ and

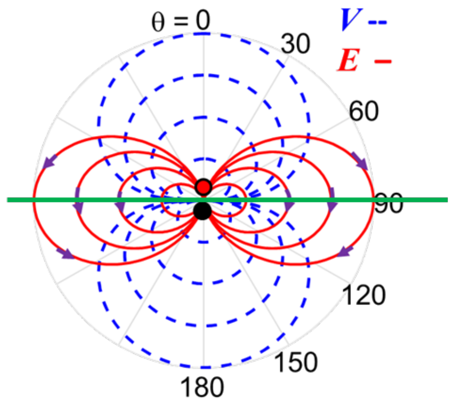
$V(x, y, z) = V(x, y, -z)$

3. $E_t = 0$ at $y = 0$

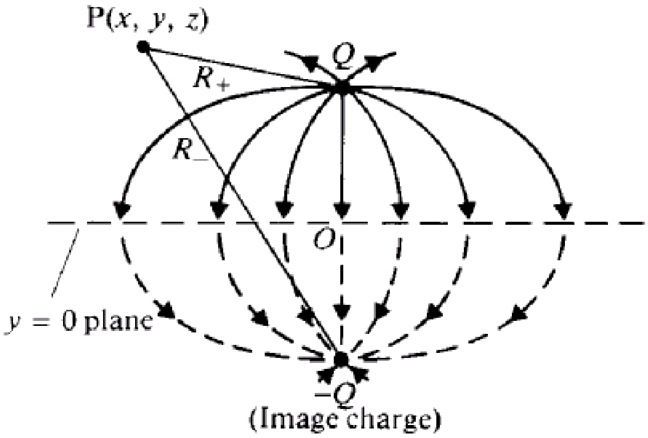
Recall the dipole field $\vec{E} = \dfrac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$

and dipole potential $V(R) = \dfrac{\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}$ at $\theta = 90^\circ$

Replace the problem with the electric dipole for the field above the plate



Matching the B.C.s by creating an *imaging charge* $-Q$ at $y = -d$



(b) Image charge and field lines.

In the region $y \geq 0$,

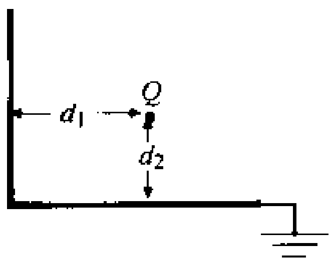
$$V(x,y,z)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)$$

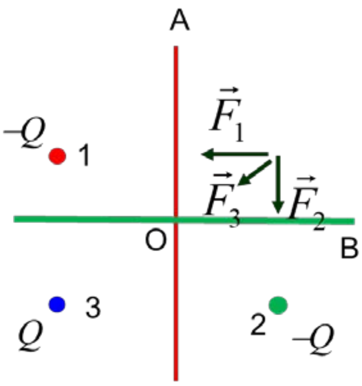
where $R_+=\left[x^2+(y-d)^2+z^2\right]^{1/2}$ and

$R_-=\left[x^2+(y+d)^2+z^2\right]^{1/2}$. The electric field above the $y=0$ plane can be obtained from $\vec{E}=-\nabla V$.

Once the electric field is found, one can use $\hat{a}_{n2}\cdot\vec{D}=\rho_s$ to calculate the surface charge on the conducting plate.

Eg. Find the force acting on Q





Physical arrangement image charge arrangement

Charge 1 results in $V=0$ along OA, but introduces $V_{1_{OB}}\neq 0$ along OB.

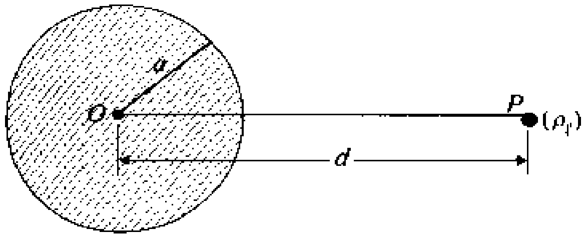
Charge 2 cancels $V_{1_{OB}}$ but gives rise to $V_{2_{OA}}$

Charge 3 cancels $V_{2_{OA}}$, but gives rise to $V_{3_{OB}}$, which is fortunately canceled by Q.

Force on charge Q is equal to that induced by the three image charges.

Use Coulomb's law to calculate the force.

Eg. A cylindrical equipotential surface and a line charge.



(a) Line charge and parallel conducting cylinder.

From Gauss's law $\oint \vec{D}\cdot d\vec{s} = Q$ to the line charge

$\Rightarrow$ $2\pi\varepsilon_0 r E_r L = \rho_l L$ $\Rightarrow$ $E_r = \dfrac{\rho_l}{2\pi\varepsilon_0 r}$

The potential at *r* from a line charge is

$V(r) = -\int_{r_0}^{r} E_r dr = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{r_0}{r}$, where r_0 is the location at which a constant potential is defined.

Assume an image charge of opposite sign and same density, $\rho_i = -\rho_l$ at P_i. The potential on the conducting surface is

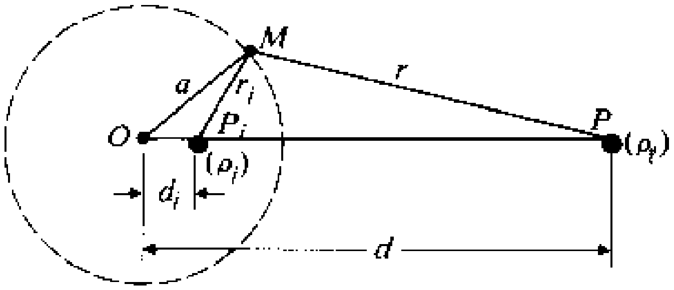
$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$

Apply the constant-potential boundary condition to the conducting cylinder $V_M = \text{const.}$ $\Rightarrow$ $\dfrac{r_i}{r} =$ const.

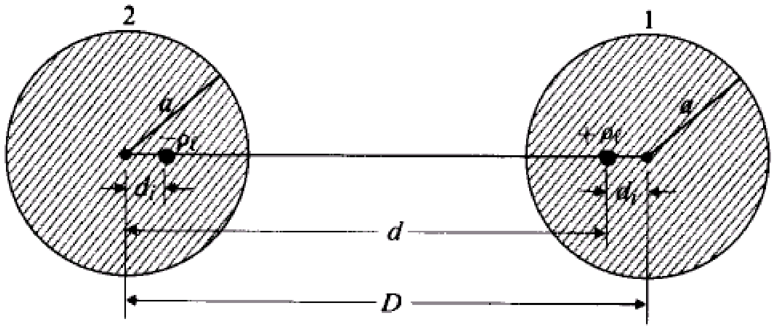
Choose d_i such that

ΔOPM ~ ΔOMP$_i$ $\Rightarrow$ $\dfrac{r_i}{r} = \dfrac{d_i}{a} = \dfrac{a}{d} = \text{const.}$

$\Rightarrow$ $d_i = \dfrac{a^2}{d}$, P_i is the *inverse point* of P that satisfies the boundary condition of this problem.



Eg. Two wire transmission line. The known parameters are *D* and *a*. Each surface of the conducting cylinder is at a constant potential.



The equipotential surfaces of the two conducting cylinders can be considered to be generated from two line charges in the longitudinal direction, yielding $V_1 = \dfrac{-\rho_l}{2\pi\varepsilon_0}\ln\dfrac{a}{d}$ and $V_2 = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{a}{d}$. The voltage difference between the two conducting cylinders can be calculated accordingly

$$V_{12} = V_1 - V_2 = \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{d}{a}$$

The capacitance per unit length is therefore

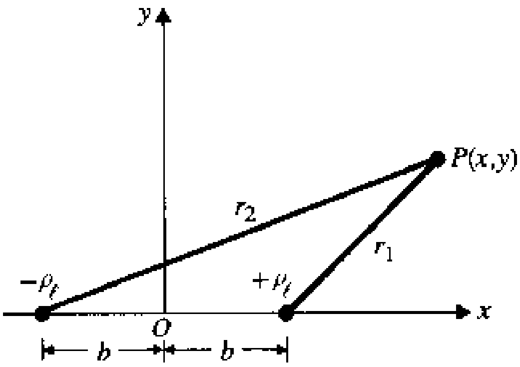
$$C_l = \rho_l / (V_1 - V_2) = \frac{\pi\varepsilon_0}{\ln(d/a)}$$

But from the last example $d = D - d_i = D - \dfrac{a^2}{d}$, the location of the image charge can be obtained, given by

$$d = \frac{1}{2}(D \pm \sqrt{D^2 - 4a^2})$$

The solution for the negative sign is discarded due to the usually arrangement of $d, D >> a$.

Question: Where are those equipotential circles with a pair of line charges separated by a distance 2*b*?



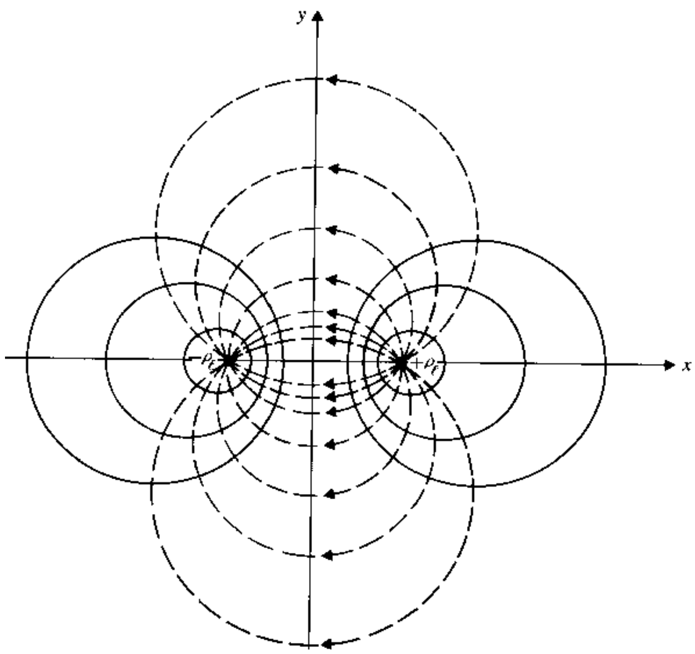
The potential at point *P* is $V_P = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{r_2}{r_1}$. Set $V_P = \text{const.}$ to obtain $\dfrac{r_2}{r_1} = k$.

$$\Rightarrow \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k$$

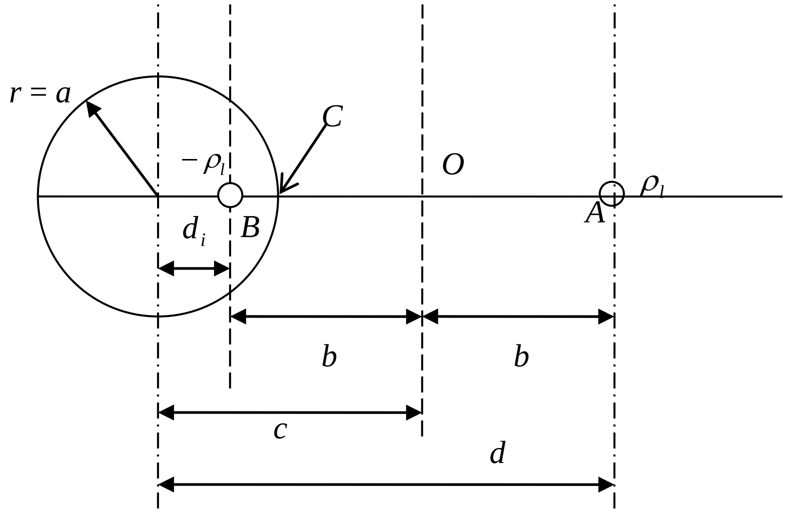
$$\Rightarrow \left(x - \frac{k^2+1}{k^2-1}b\right)^2 + y^2 = \left(\frac{2k}{k^2-1}b\right)^2$$

This expression indicates a set of equipotential circles with radii of $a \equiv \left|\frac{2kb}{k^2-1}\right|$ and centers at $c \equiv x_0 = \frac{k^2+1}{k^2-1}b$,

where $c^2 = a^2 + b^2$



Now let's come back to the first problem, in which the center of a conducting cylinder is placed a distance from a line charge of a charge density = ρ_l



Use the relation $c^2 = a^2 + b^2$ or $(c-b)(c+b) = a^2$. But $c - b = d_i$ and $c + b = d$. One obtains $d_i = a^2/d$ immediately.

The potential on the conducting cylinder is given by $V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1}$.

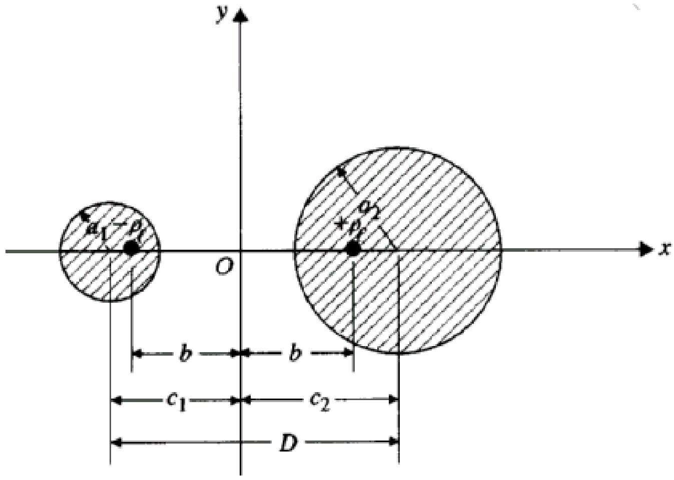
Because any point on the conductor has the same potential, we choose point C for calculation. Therefore $r_1 = \overline{AC} = d - a$ and $r_2 = \overline{BC} = a - d_i = (d-a)a/d$, which gives $V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$ on the conducting cylinder.

Eg. Two parallel conducting wires with different radii are shown below. Find its capacitance per unit length.

Step 1. Replace the two wires with two line charges of opposite sign at suitable locations.

Step 2. Calculate the voltage difference at the surface of the two wires.

Step 3. Calculate it capacitance per unit length.



In general a_1, a_2, D are known. Find b, c_1, c_2.

$$\Rightarrow \ V_1 = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{d_1} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{b+c_1}$$, and

$$V_2 = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{d_2} = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{b+c_2}$$

Solutions:

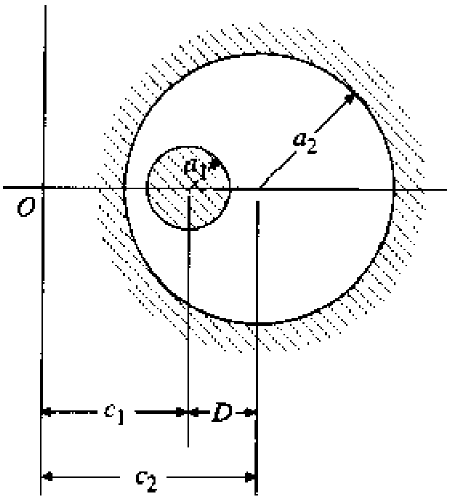
$$b^2 = c_1^2 - a_1^2, \ b^2 = c_2^2 - a_2^2, \ c_1 + c_2 = D$$

$$\Rightarrow \ c_1 = \frac{1}{2D}\left(D^2 + a_1^2 - a_2^2\right)$$ and

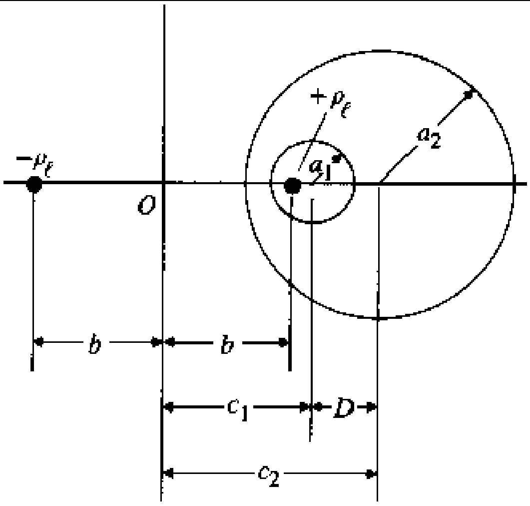
$$c_2 = \frac{1}{2D}\left(D^2 + a_2^2 - a_1^2\right)$$

With known c_1 and c_2, b can be obtained from $b^2 = c_1^2 - a_1^2$ or $b^2 = c_2^2 - a_2^2$. Substitute c_1, c_2, b into the expression of electric potentials to find $C_l = \rho_l / \left|V_1 - V_2\right|$.

In case of the following arrangement



or



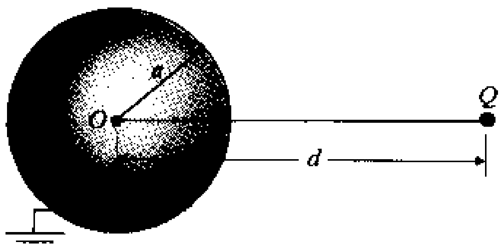
$c_2 - c_1 = D$ and one can derive

$$c_1 = \frac{1}{2D}\left(a_2^2 - a_1^2 - D^2\right) \text{ and}$$

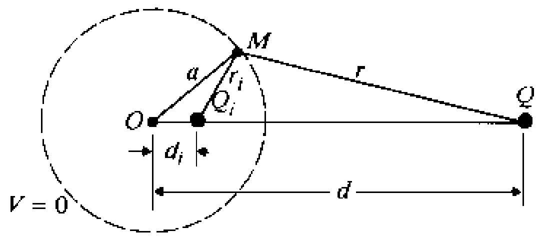
$$c_2 = \frac{1}{2D}\left(a_2^2 - a_1^2 + D^2\right)$$

Please find its capacitance per unit length as an exercise.

Point charge at *R* = *d* from a grounded conducting sphere of radius *a*



(a) Point charge and grounded conducting sphere.



(b) Point charge and its image.

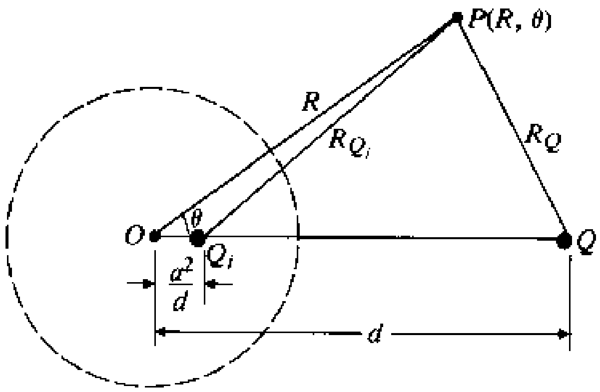
Because the conducting sphere is grounded $\Rightarrow$ The electric potential on the sphere is $V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0 \;\Rightarrow\; \frac{r_i}{r} = -\frac{Q_i}{Q} = \text{const.}$

Choose the location of the image charge d_i such that

ΔOQM ~ ΔOMQ$_i$ $\Rightarrow$ $\dfrac{r_i}{r} = -\dfrac{Q_i}{Q} = \dfrac{d_i}{a} = \dfrac{a}{d} = \text{const.}$

$\Rightarrow$ $Q_i = \dfrac{-a}{d}Q$ and therefore $d_i = \dfrac{a^2}{d}$.

Once the location of the image charge is determined, one can calculate the electric field everywhere outside the conducting sphere. The induced surface charge on the conducting sphere can also be found from $\hat{a}_{n2} \cdot \vec{D} = \rho_s$



Recall $\rho_s = \varepsilon_0 E_R(a,\theta)$ and $E_R = -\dfrac{\partial V(R,\theta)}{\partial R}$

Find $V(R,\theta) = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{R_Q} - \dfrac{a}{dR_{Q_i}}\right)$,

where $R_Q = \left[R^2 + d^2 - 2Rd\cos\theta\right]^{1/2}$ and

$$R_{Q_i} = \left[R^2 + \left(\frac{a^2}{d}\right)^2 - 2R\frac{a^2}{d}\cos\theta\right]^{1/2}$$

Question: What if the conducting sphere is not grounded and at some arbitrary potential V_0?

Hint: Place a second image charge at the origin to induce V_0 on the conducting sphere.

Solutions to Laplace's Equation

Boundary Conditions

a. *Dirichlet Problems:* potential V is specified everywhere on the boundaries.

b. *Neumann Problems*: the normal derivative of the potential, $\frac{\partial V}{\partial n}$, is specified everywhere on the boundaries.

c. *Mixed Problems*: either *Dirichlet* or *Neumann* is specified on each boundary.

Solutions to $\nabla^2 V = 0$ are those so-called harmonic functions, as will be shown below.

Cartesian Coordinates

The Laplace equation is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{4-7}$$

Use *separation of variables*

$$V(x, y, z) = X(x)Y(y)Z(z) \tag{4-8}$$

Substitute (4-8) into (4-7)

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

To satisfy all x, y, z

let $\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$, $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$, $\frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2$ (4-9)

with $k_x^2 + k_y^2 + k_z^2 = 0$. (4-10)

Now solving (4-7) is a matter of solving an ordinary differential equation of the form $\frac{d^2X}{dx^2} + k_x^2 X = 0$.

Possible Solutions

i. $k_x = 0$

$$X = A_0 x + B_0 \tag{4-11}$$

ii. $k_x^2 > 0 \quad \Rightarrow$ let $k_x = k$

$$X = A_1 \sin kx + B_1 \cos kx \quad \text{or} \tag{4-12}$$

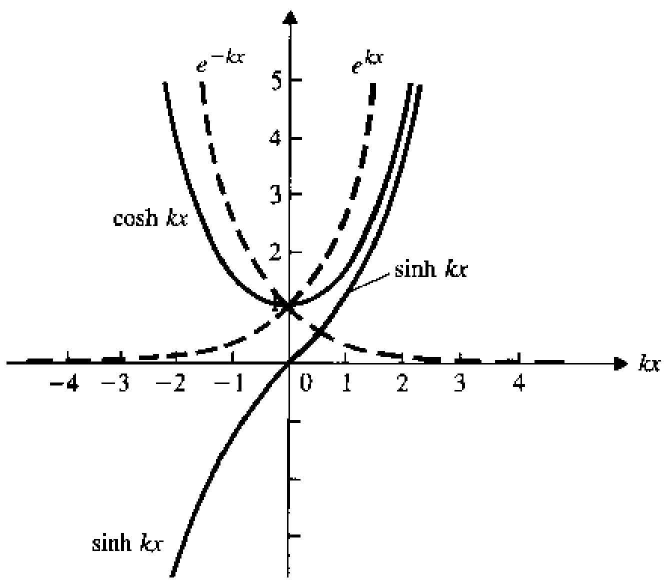
$$X = C_1 e^{jkx} + D_1 e^{-jkx} \tag{4-13}$$

The solution has a periodic variation along *x*

iii. $k_x^2 < 0 \quad \Rightarrow$ let $k_x = jk$

$$X = A_2 \sinh kx + B_2 \cosh kx \quad \text{or} \tag{4-14}$$

$$X = C_2 e^{kx} + D_2 e^{-kx} \tag{4-15}$$



Without variation in *z*

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Again use *separation of variables*

$$V(x, y) = X(x)Y(y) \qquad (*)$$

The resulting equations are

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2, \quad \frac{1}{Y}\frac{d^2 Y}{dy^2} = -k_y^2$$

with $k_x^2 + k_y^2 = 0$.

i. $k_x = k_y = 0$

$$(*) \quad \Rightarrow \quad V(x, y) = (Ax + B)(Cy + D) \qquad (4\text{-}16)$$

ii. $k_x^2 = -k_y^2 = k^2 > 0$

$$V(x, y) = (A\cos kx + B\sin kx) \times (C\cosh ky + D\sinh ky) \qquad (4\text{-}17.a)$$

or

$$V(x, y) = (A\cos kx + B\sin kx) \times (Ce^{-ky} + De^{ky}) \qquad (4\text{-}17.b)$$

The solution is periodic in *x*.

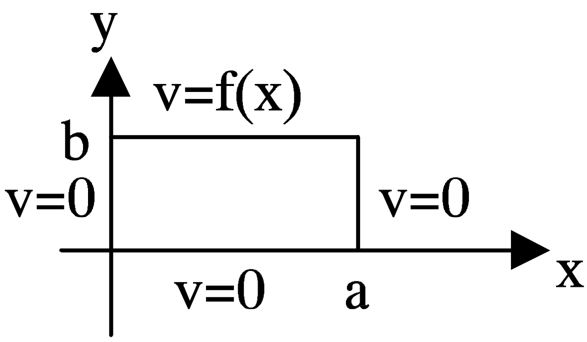
iii. $k_x^2 = -k_y^2 = -k^2 < 0$

$$V(x, y) = (A\cosh kx + B\sinh kx) \times (C\cos ky + D\sin ky) \qquad (4\text{-}18.a)$$

$$V(x, y) = (Ae^{-kx} + Be^{kx}) \times (C\cos ky + D\sin ky) \qquad (4\text{-}18.b)$$

The solution is periodic in *y*.

Eg. two dimensional static potential problem



From boundary conditions, solution (4-16, 5-18) are not suitable (why?)

Since $V(x=0,y)=0$, use (4-17),

$\Rightarrow$ $V(x,y)=\sin kx \times (C\cosh ky + D\sinh ky)$

Because $V(x=a,y)=0$, *k* can be determined and

$\Rightarrow$ $V_n(x,y)=\sin\dfrac{n\pi}{a}x \times (C\cosh\dfrac{n\pi}{a}y + D\sinh\dfrac{n\pi}{a}y)$

In addition, $V(x,y=0)=0$

$\Rightarrow$ $V_n(x,y)=A_n\sin(n\pi x/a)\cdot\sinh(n\pi y/a)$

Therefore the general solution is given by the superposition for all possible *n*

$$V(x,y)=\sum_{n=1}^{\infty}A_n\sin(n\pi x/a)\cdot\sinh(n\pi y/a)$$

The fourth boundary condition requires

$V(x,y=b)=f(x)$ or

$$V(x,y=b)=\sum_{n=1}^{\infty}A_n\sin(n\pi x/a)\cdot\sinh(n\pi b/a)=f(x)$$

for $0 \le x \le a$ (4-19)

Note the orthogonality properties

1. $\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = 0$ for $n \ne m$

2. $\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = \frac{a}{2}$ for $n = m$

3. $\int_0^a \cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{a}x)dx = 0$ for $n \ne m$

4. $\int_0^a \cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{a}x)dx = \frac{a}{2}$ for $n = m$

(verify the above)

Multiply both sides of (4-19) with $\sin(\frac{m\pi}{a}x)$ and integrate over the range in *x* $[0,a]$,

$$\sum_{n=1}^{\infty} A_n \sinh(n\pi b/a)\int_0^a \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{a}x)dx$$

$$= \int_0^a f(x)\sin(\frac{m\pi}{a}x)dx$$

The coefficient A_m can be determined from

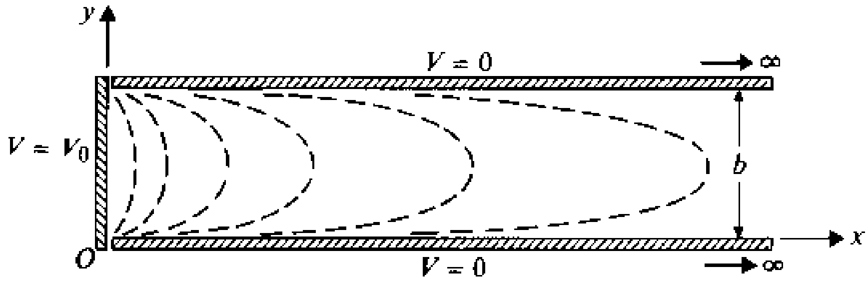
$$A_m = \frac{2}{a \times \sinh(m\pi b/a)}\int_0^a f(x)\sin(\frac{m\pi}{a}x)dx \text{ , } m = 1, 2, 3\ldots$$

Thus the solution of potential subject to the boundary conditions in Fig (*) is established:

$$V(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\cdot\sinh(n\pi y/a) \text{ , where}$$

$$A_n = \frac{2}{a\times\sinh(n\pi b/a)}\int_0^a f(x)\sin(\frac{n\pi}{a}x)dx$$

Eg. Find the electric potential within the rectangular boundaries.



From the boundary conditions, the suitable form of solution is Eq.

(4-18.b) $V(x,y) = (Ae^{-kx} + Be^{kx})\times(C\cos ky + D\sin ky)$

Because $V(x, y=0) = 0 \Rightarrow$ *C = 0*. Also, because the term $e^{kx}\to\infty$ when $x\to\infty$, *B* = 0.

$$V(x,y) = \sum_{n=1}^{\infty} A_n e^{-k_n x}\sin k_n y\ ,$$

where $k_n = \frac{n\pi}{b}$ due to $V(x, y=b) = 0$.

Boundary condition $V(0,0<y<b) = \sum_{n=1}^{\infty} A_n \sin k_n y = V_0$ can be used to determine A_n by using the orthogonality property of harmonic functions.

Cylindrical Coordinates

The Laplace equation is given by

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (4\text{-}20)$$

<u>Axial Symmetry with Longitudinal Invariance</u>: $\frac{\partial^2 V}{\partial \phi^2} = 0$ and

$\frac{\partial^2 V}{\partial z^2} = 0$

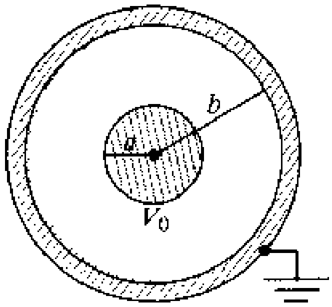
Thus only radial dependence has to be considered

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \quad r\frac{\partial V}{\partial r} = C_1 = \text{const.}$$

$$\Rightarrow \quad V = C_1 \ln r + C_2 \qquad (4\text{-}21)$$

Eg. The potential in a coaxial cable has axial symmetry with longitudinal invariance, as shown below.



Boundary conditions: $V(b) = 0$ and $V(a) = V_0$.

recall the general solution of Eq. (4-21)

$$V = C_1 \ln r + C_2 .$$

Substitute the conditions into (4-21) to obtain

$$C_1 \ln b + C_2 = 0 ,$$

$$C_1 \ln a + C_2 = V_0$$

The potential in the coaxial cable is therefore $V(r) = \frac{V_0}{\ln(b/a)} \ln\left(\frac{b}{r}\right)$

Longitudinal Invariance: $\frac{\partial^2 V}{\partial z^2} = 0$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \qquad \text{(4-22)}$$

Use *separation of variables*

$$V(r,\phi) = R(r)\Phi(\phi) \qquad \text{(4-23)}$$

substitute (4-23) into (4-22) to obtain

$$\Phi(\phi)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{R(r)}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

$$\Rightarrow \quad \frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

To satisfy all r and ϕ, assume

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = n^2 = \text{constant} \quad , \qquad \text{(4-24)}$$

and

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -n^2 \qquad \text{(4-25)}$$

$\Phi(\phi)$ must be a periodic function of ϕ. Recall Eq. (4-12, 13), the solution to (4-25) is of the form

$$\Phi(\phi) = A_\phi \cos n\phi + B_\phi \sin n\phi \text{ , } n \text{ is an integers} \qquad \text{(4-26)}$$

The solution to (4-24), by direct substitution, is given by

$$R(r) = A_r r^n + B_r r^{-n} \qquad \text{(4-27)}$$

Thus z-independent solution to $\nabla^2 V = 0$ is therefore

$$V(r,\phi) = r^n (A_n \cos n\phi + B_n \sin n\phi) + r^{-n}(A_{-n} \cos n\phi + B_{-n} \sin n\phi) \tag{4-28}$$

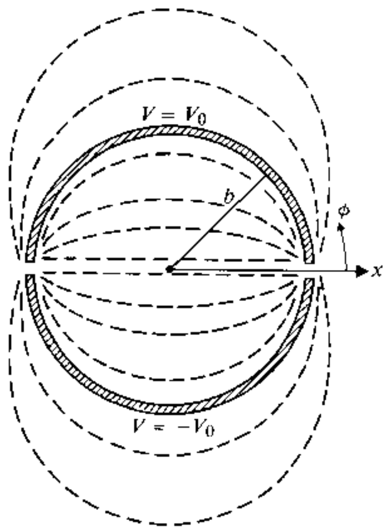
1. In the region including $r = 0$, the term including r^{-n} has to vanish. The solution is of the form

$$V(r,\phi) = r^n (A_n \cos n\phi + B_n \sin n\phi) \tag{4-29}$$

2. In the region including $r = \infty$, the term including r^n has to vanish. The solution is of the form

$$V(r,\phi) = r^{-n} (A_{-n} \cos n\phi + B_{-n} \sin n\phi) \tag{4-30}$$

Eg. The following is an example of a potential with longitudinal invariance



There are two boundary conditions

$$V(b,\phi) = V_0 \qquad \text{for} \quad 0 < \phi < \pi$$

$$V(b,\phi) = -V_0 \qquad \text{for} \quad -\pi < \phi < 0$$

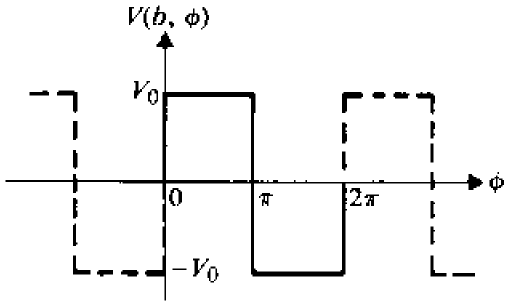
$\Rightarrow$ $V(r,-\phi) = -V_0(r,\phi)$ must be an odd function

a) In the region $r < b$

Recall (4-23), the form of solution is

$$V(r,\phi) = \sum_{n=1}^{\infty} B_n r^n \sin n\phi \,, \tag{4-31}$$

with the boundary Condition



Recall the orthogonality of trigonometry functions

$$\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = 0 \quad \text{for} \quad n \neq m$$

$$\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx = \frac{a}{2} \quad \text{for} \quad n = m$$

For this example, $a = \pi$. Apply the boundary conditions to (4-31), one can calculate the coefficient B_m in (4-31).

$$\sum_{n=1}^{\infty} B_n b^n \int_0^{\pi} \sin n\phi \times \sin m\phi \cdot d\phi = \int_0^{\pi} \sin m\phi \cdot V_0 d\phi$$

$B_m = \dfrac{4V_0}{m\pi b^m}$ if m is odd, and $B_m = 0$ if m is even.

Thus the potential is solved to be $V(r,\phi) = \sum_{n=odd}^{\infty} \dfrac{4V_0}{n\pi b^n} r^n \sin n\phi$ for $r < b$

b) In the region $r > b$

recall (4-30) for solution of the form $V(r,\phi)=\sum_{n=1}^{\infty} r^{-n} B_n' \sin n\phi$

Follow the same derivation in (a) and solve the coefficient B_n' as an exercise by yourselves.

Axial Symmetry

The Laplace equation is given by

$$\nabla^2 V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial z^2}=0$$

Use separation of variables of the form

$$V(r,z)=R(r)Z(z)$$

to obtain

$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+T^2R=0$ and $\frac{1}{Z}\frac{d^2Z}{dz^2}=T^2$, where T^2 is a constant.

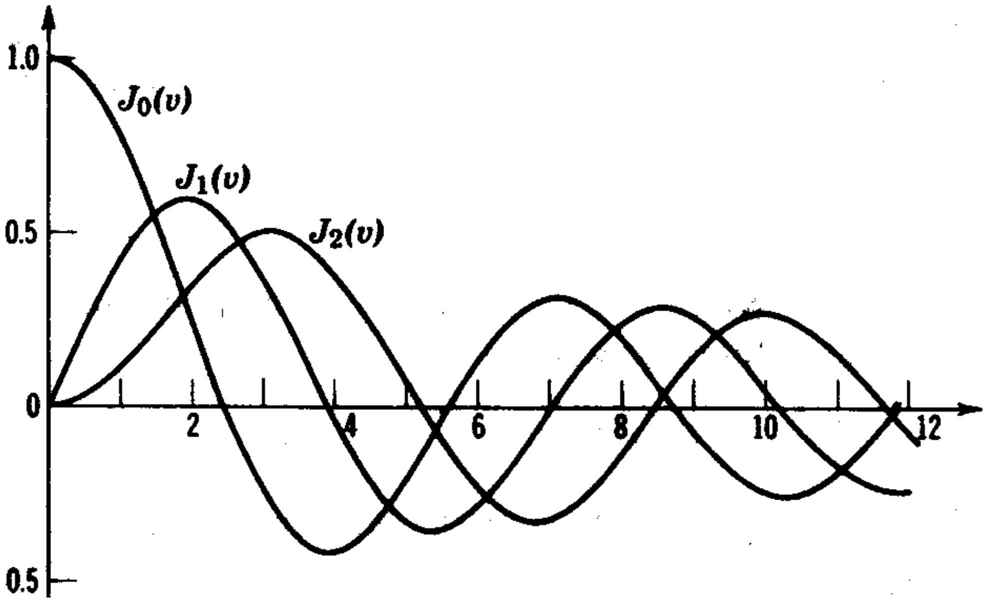
(1) $T^2>0$

$R(r)=C_1J_0(Tr)+C_2N_0(Tr)$,

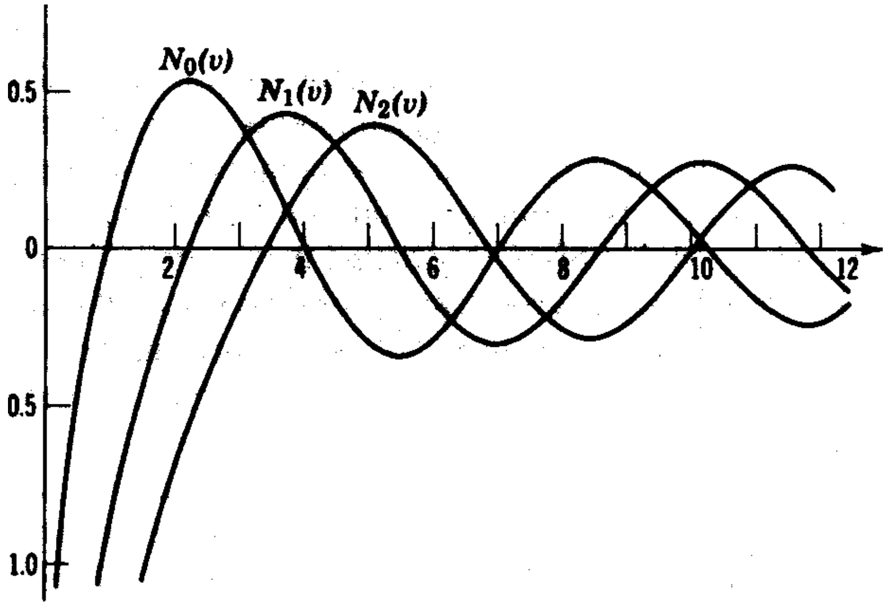
and $Z(z)=C_3\cosh(Tz)+C_4\sinh(Tz)$

where $J_0(..)$ is a Bessel function of the first kind and of zero order,

and $N_0(..)$ is a Bessel function of the second kind and of zero order.



For a large υ, J_n approaches a periodic function and can be approximated by sine or cosine functions.



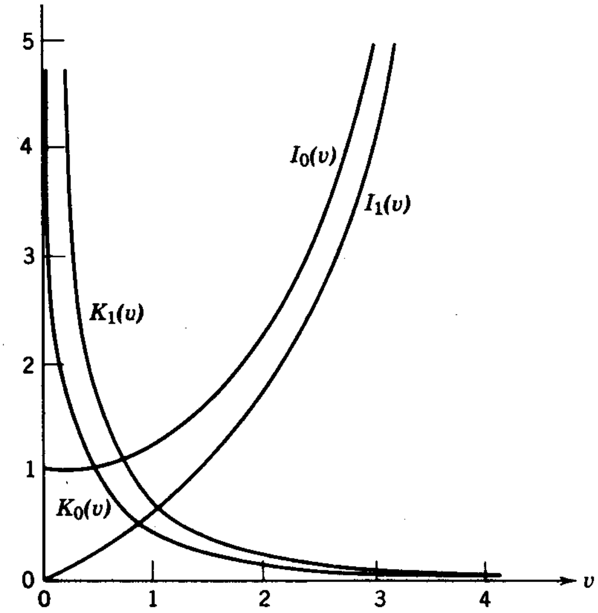
Note that $N_0(\upsilon)$ diverges at $\upsilon = 0$. For the region involving $r = 0$, the solution $N_0(Tr)$ should be avoided.

(1) $T^2 = -\tau^2 < 0$

$$R(r) = C_1 I_0(\tau r) + C_2 K_0(\tau r),$$

and $Z(z) = C_3 \cos(\tau z) + C_4 \sin(\tau z)$

where $I_0(..)$ and $K_0(..)$ are modified Bessel functions.



Note that $K_0(\upsilon)$ diverges at $\upsilon = 0$; for a solution involving $r = 0$, the solution $K_0(\tau r)$ should be avoided. On the other hand, $I_0(\upsilon)$ diverges at $\upsilon = \infty$; for a solution involving $r = \infty$, the solution $I_0(\tau r)$ should be avoided.

For the situation where all three variables r, φ, z vary, the solution to the Laplace's equation is too complicated and beyond the scope of this course.

Spherical Coordinates

Laplace's equation is given by

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

The techniques of solving Laplace's equation by using harmonic functions in the spherical coordinate system are the same as those used in the other two coordinate systems. Interested students may consult Sec. 4-7 of the textbook by D. K. Cheng.