HW1 Q4, 6, 8, 11

陳咨蓉

4. (*) Consider the expression $(p \wedge q) \vee \neg(p \to q)$.
In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.

solution :

$(p \wedge q) \vee \neg(p \to q)$

$\equiv (p \wedge q) \vee \neg(\neg p \vee q)$ by De Morgan's Laws

$\equiv (p \wedge q) \vee (p \wedge \neg q)$ by Distributive Laws

$\equiv p \wedge (q \vee \neg q)$ by Negation Laws

$\equiv p \wedge T$ by Identity Laws

$\equiv p$

6. (*) The following exercises involve the logical operator $\uparrow$ (read as NAND). The proposition $p \uparrow q$ is true when either p, or q, or both, are false.

(a) Show that $p \uparrow q \equiv \neg(p \land q)$.

(b) Show that $p \uparrow p \equiv \neg p$.

(c) Express $p \land q$ by using only $\uparrow$ operators.

(d) Express $p \lor q$ by using only $\uparrow$ operators.

6. (a) Show that $p \uparrow q \equiv \neg(p \wedge q)$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \uparrow q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

6. (b) Show that $p \uparrow p \equiv \neg p$

As (a), we get $p \uparrow p \equiv \neg(p \land p)$.

(a) Show that $p \uparrow q \equiv \neg(p \land q)$.

$p \uparrow p$

$\equiv \neg(p \land p)$ by Idempotent Laws

$\equiv \neg p$

6. (c) Express $p \land q$ by using only $\uparrow$ operators.

(a) Show that $p \uparrow q \equiv \neg(p \land q)$.

(b) Show that $p \uparrow p \equiv \neg p$.

By (a), we can get

$(p \land q)$

$\equiv \neg(p \uparrow q)$

By (b), we can get

$\neg(p \uparrow q)$

$\equiv (p \uparrow q) \uparrow (p \uparrow q)$

6. (d) Express p $\vee$ q by using only $\uparrow$ operators.

$p \vee q$

$\equiv \neg(\neg p \wedge \neg q)$ by De Morgan's Law

By (a), we can get

$\neg(\neg p \wedge \neg q)$

$\equiv (\neg p) \uparrow (\neg q)$

By (b), we can get

$(\neg p) \uparrow (\neg q)$

$\equiv (p \uparrow p) \uparrow (q \uparrow q)$

(a) Show that $p \uparrow q \equiv \neg(p \wedge q)$.

(b) Show that $p \uparrow p \equiv \neg p$.

8. (*) What is wrong with this argument?

Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists sS(s, Max)$ it follows that $S(Max, Max)$.

Then by existential generalization it follows that $\exists xS(x, x)$, so that someone is shorter than himself.

We know that there exists a person y such that S(y, Max) is true. However, we don't know if the person y is Max, thus we can't conclude S(Max, Max).

Moreover, S(Max, Max) can't be true, because Max can't be shorter than himself. This applies to everybody.

Thus, the statement $\exists xS(x, x)$ can't be true.

11. (*) Determine whether these are valid arguments.

(a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

(b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$.

Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

11. (a)

(a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

(a) Counterexample : -1

11. (b)

(b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$.

Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

(b)$x^2 \neq 0$, where x is a real number.

which means

$(x^2 \neq 0) \land (x \text{ is a real number}) \equiv (x \text{ is a real number}) \land (x^2 \neq 0)$

→Correct! (By Commutative Laws)

Assignment 1
14, 18, 20, 21

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14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(a) $\forall n \exists m (n^2 < m)$

(b) $\exists n \forall m (n < m^2)$

(c) $\forall n \exists m (n + m = 0)$

(d) $\exists n \forall m (nm = m)$

(e) $\exists n \exists m (n^2 + m^2 = 5)$

(f) $\exists n \exists m (n^2 + m^2 = 6)$

(g) $\exists n \exists m (n + m = 4 \wedge n - m = 1)$

(h) $\exists n \exists m (n + m = 4 \wedge n - m = 2)$

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(a) $\forall n \exists m (n^2 < m)$

$\because$ (1) $n^2 < n^2 + 1, \forall n \in \mathbb{Z}$

(2) we can set $m = n^2 + 1 \in \mathbb{Z}$

$\therefore$ truth value is T

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(b) $\exists n \forall m (n < m^2)$

$\because$ (1) $0 \leq m^2, \forall m \in \mathbb{Z}$

(2) we can set $n = -1 \in \mathbb{Z}$

$\therefore$ truth value is T

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(c) $\forall n \exists m(n + m = 0)$

$\because$ (1) $n+(-n) = 0, \forall n \in \mathbb{Z}$

(2) we can set $m = -n \in \mathbb{Z}$ (additive inverse)

$\therefore$ truth value is T

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

$$\text{(d)}\ \exists n \forall m (nm = m)$$

$\because$ (1) $1 \cdot m = m, \forall m \in \mathbb{Z}$

(2) we can set $n = 1 \in \mathbb{Z}$ (multiplicative identity)

$\therefore$ truth value is T

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(e) $\exists n \exists m (n^2 + m^2 = 5)$

$\because$ (1) $1^2 + 2^2 = 5$

(2) we can set $n = 1 \in \mathbb{Z}$ and $m = 2 \in \mathbb{Z}$

$\therefore$ truth value is T

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(f) $\exists n \exists m (n^2 + m^2 = 6)$

$\because n^2 + m^2 = 6$ has no integral solution

$\therefore$ truth value is F

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

$$\text{(g)}\ \exists n \exists m(n+m=4\ \wedge n-m=1)$$

$$\because \begin{cases} n+m=4 \\ n-m=1 \end{cases} \Rightarrow n=\frac{5}{2}\notin\mathbb{Z} \text{ and } m=\frac{3}{2}\notin\mathbb{Z}$$

$\therefore$ truth value is F

14.(*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

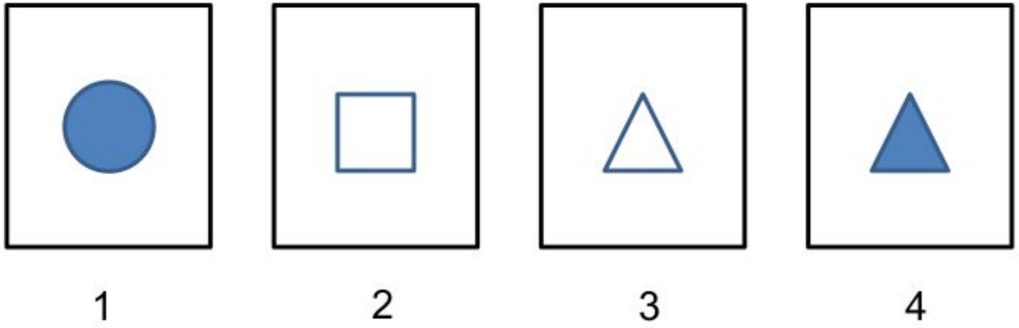
(h) $\exists n \exists m(n + m = 4 \wedge n - m = 2)$

$$\because \begin{cases} n + m = 4 \\ n - m = 2 \end{cases} \Rightarrow n = 3 \in \mathbb{Z} \text{ and } m = 1 \in \mathbb{Z}$$

$\therefore$ truth value is T

18.(*)

Four cards are displayed on the table as shown below.
It is known that for each card,
both faces are drawn with geometric shapes,
such that one is solid while the other is empty.



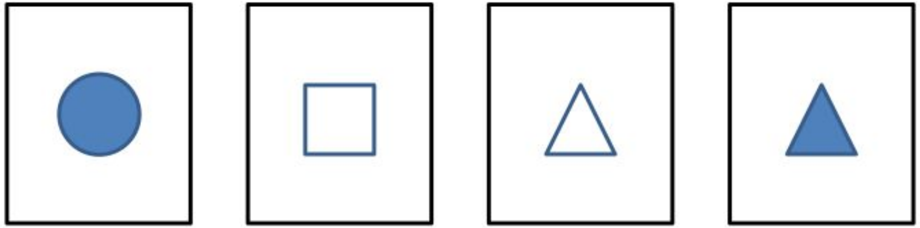
18.(*)

Peter took a look at the other face of each card and said,
"if one face is drawn with a solid circle,
then the other face must be drawn with an empty triangle".

$p :=$ one face is drawn with solid circle

$q :=$ the other face is drawn with empty triangle

Peter claims that $p \to q$ is true



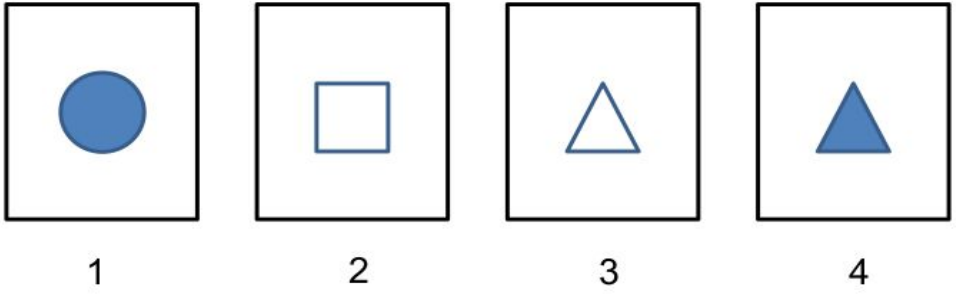
1 2 3 4

18.(*)

(a) Is it possible to check only some (but not all) of these cards, such that we can definitely ensure that Peter's claim is correct?

(b) What is the minimal number of cards we should check?

$p :=$ one face is drawn with solid circle
$q :=$ the other face is drawn with empty triangle
Peter claims that $p \to q$ is true



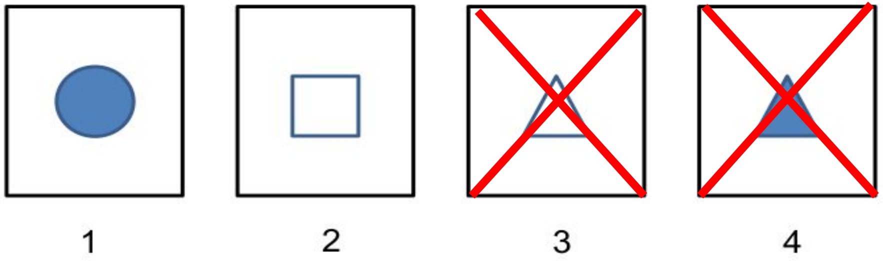
18.(*)

(a) Is it possible to check only some (but not all) of these cards, such that we can definitely ensure that Peter's claim is correct? yes

(b) What is the minimal number of cards we should check? 2

$p :=$ one face is drawn with solid circle
$q :=$ the other face is drawn with empty triangle
Peter claims that $p \to q$ is true

$p \to q$ is false $\iff$ p is true and q is false



20.(*)

We encounter three people named A, B, and C,
and each of them is either honest or dishonest but not both.
One of them said "A and B are liars",
and another one of them said "A and C are liars."
How many liars among A, B, and C?

$$p := \text{A and B are liars}$$

$$q := \text{A and C are liars}$$

20.(*)

$$p := \text{A and B are liars}$$

$$q := \text{A and C are liars}$$

0 liar $\Rightarrow$ both p and q are truths $\Rightarrow$ all of them are liars $(\rightarrow\leftarrow)$
1 liar $\Rightarrow$ at least one of p and q is truth $\Rightarrow$ at least 2 liars $(\rightarrow\leftarrow)$
3 liars $\Rightarrow$ both p and q are lies $\Rightarrow$ someone is honest $(\rightarrow\leftarrow)$

2 liars is possible:
A is liar, B is liar, but C is a honest guy,
where p is said by C, and q is said by either A or B

21. (extremely challenging)

There are two integers, x and y, such that

$$\begin{cases} 1 < x < y \\ x + y \leq 65 \end{cases}$$

There are two people, Peter and Sam, such that

$$\begin{cases} \text{Peter knows only } x \cdot y \\ \text{Sam knows only } x + y \end{cases}$$

21. (extremely challenging)

Peter and Sam subsequently chat in the following order:

(1) Peter: I don't know both x and y

(2) Sam: I knew that you don't know both x and y

(3) Peter: I know both x and y now

(4) Sam: I know both x and y now

Do you know both x and y now?

21. (extremely challenging)

possible $(x+y)$: corresponding possible $(x \cdot y)$ to fixed $(x+y)$

$$\begin{cases} 5: 2\cdot 3 \\ 6: 2\cdot 4 \\ 7: 2\cdot 5, 3\cdot 4 \\ 8: 2\cdot 6, 3\cdot 5 \\ \vdots \\ 63: 2\cdot 61, 3\cdot 60, \cdots, 30\cdot 33, 31\cdot 32 \\ 64: 2\cdot 62, 3\cdot 61, \cdots, 30\cdot 34, 31\cdot 33 \\ 65: 2\cdot 63, 3\cdot 62, \cdots, 31\cdot 34, 32\cdot 33 \end{cases}$$

$$\boxed{\begin{cases} 1 < x < y \\ x+y \leq 65 \end{cases}}$$

21. (extremely challenging)

(1) Peter: I don't know both x and y

$$\equiv x \cdot y \text{ can be re-written as } x' \cdot y',\ \text{where } x \neq x', y \neq y',\ \text{and } \begin{cases} 1 < x' < y' \\ x' + y' = 65 \end{cases}$$

$$x \cdot y = 8 \Rightarrow \text{Peter knows } \begin{cases} x = 2 \\ y = 4 \end{cases}$$

$$x \cdot y = 12 \Rightarrow \text{Peter doesn't know whether } \begin{cases} x = 2 \\ y = 6 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 4 \end{cases}$$

$$\begin{cases} 5 : 2 \cdot 3 \\ 6 : 2 \cdot 4 \\ 7 : 2 \cdot 5, 3 \cdot 4 \\ 8 : 2 \cdot 6, 3 \cdot 5 \\ \vdots \\ 63 : 2 \cdot 61, 3 \cdot 60, \cdots, 30 \cdot 33, 31 \cdot 32 \\ 64 : 2 \cdot 62, 3 \cdot 61, \cdots, 30 \cdot 34, 31 \cdot 33 \\ 65 : 2 \cdot 63, 3 \cdot 62, \cdots, 31 \cdot 34, 32 \cdot 33 \end{cases} \Rightarrow \begin{cases} 5 : \cancel{2 \cdot 3} \\ 6 : \boxed{\cancel{2 \cdot 4}} \\ 7 : \cancel{2 \cdot 5}, \boxed{3 \cdot 4} \\ 8 : \boxed{2 \cdot 6}, \cancel{3 \cdot 5} \\ \vdots \\ 63 : \cancel{2 \cdot 61}, 3 \cdot 60, \cdots, \cancel{30 \cdot 33}, \cancel{31 \cdot 32} \\ 64 : 2 \cdot 62, \cancel{3 \cdot 61}, \cdots, \cancel{30 \cdot 34}, \cancel{31 \cdot 33} \\ 65 : 2 \cdot 63, 3 \cdot 62, \cdots, \cancel{31 \cdot 34}, \cancel{32 \cdot 33} \end{cases}$$

21. (extremely challenging)

(2) Sam: I knew that you don't know both x and y

$\equiv$ every possible $x \cdot y$ corresponding to the $x + y$ cannot be re-written as $x' \cdot y'$,

$$\text{where } x \neq x', y \neq y', \text{ and } \begin{cases} 1 < x' < y' \\ x' + y' \leq 65 \end{cases}$$

$$x{+}y = 8 \Rightarrow \text{Sam can't say that since Peter can figure out} \begin{cases} x = 3 \\ y = 5 \end{cases} \text{if } x{\cdot}y = 15$$

$$\begin{cases} 5 : \cancel{2 \cdot 3} \\ 6 : \cancel{2 \cdot 4} \\ 7 : \cancel{2 \cdot 5}, 3 \cdot 4 \\ \boxed{8 : 2 \cdot 6, \cancel{3 \cdot 5}} \\ \vdots \\ 63 : \cancel{2 \cdot 61}, 3 \cdot 60, \cdots, \cancel{30 \cdot 33}, \cancel{31 \cdot 32} \\ 64 : 2 \cdot 62, \cancel{3 \cdot 61}, \cdots, \cancel{30 \cdot 34}, \cancel{31 \cdot 33} \\ 65 : 2 \cdot 63, 3 \cdot 62, \cdots, \cancel{31 \cdot 34}, \cancel{32 \cdot 33} \end{cases} \Rightarrow \begin{cases} 11 : 2 \cdot 9, 3 \cdot 8, 4 \cdot 7, 5 \cdot 6 \\ 17 : 2 \cdot 15, 3 \cdot 14, 4 \cdot 13, 5 \cdot 12, 6 \cdot 11, 7 \cdot 10, 8 \cdot 9 \\ 23 : 2 \cdot 21, 3 \cdot 20, \cdots, 10 \cdot 13, 11 \cdot 12 \\ 27 : 2 \cdot 25, 3 \cdot 24, \cdots, 12 \cdot 15, 13 \cdot 14 \\ 29 : 2 \cdot 27, 3 \cdot 26, \cdots, 13 \cdot 16, 14 \cdot 15 \\ 35 : 2 \cdot 33, 3 \cdot 32, \cdots, 16 \cdot 19, 17 \cdot 18 \\ 37 : 2 \cdot 35, 3 \cdot 34, \cdots, 17 \cdot 20, 18 \cdot 19 \end{cases}$$

21. (extremely challenging)

(3) Peter: I know both x and y now

$\equiv$ Peter can determine the only corresponding $x+y$ of the $x \cdot y$

$$x \cdot y = \boxed{30} \Rightarrow \text{ Peter can't determine } x+y = \boxed{11} \text{ or } \boxed{17}$$

$$\begin{cases} 11: 2\cdot 9, 3\cdot 8, 4\cdot 7, 5\cdot 6 \\ 17: 2\cdot 15, 3\cdot 14, 4\cdot 13, 5\cdot 12, 6\cdot 11, 7\cdot 10, 8\cdot 9 \\ 23: 2\cdot 21, 3\cdot 20, \cdots, 10\cdot 13, 11\cdot 12 \\ 27: 2\cdot 25, 3\cdot 24, \cdots, 12\cdot 15, 13\cdot 14 \\ 29: 2\cdot 27, 3\cdot 26, \cdots, 13\cdot 16, 14\cdot 15 \\ 35: 2\cdot 33, 3\cdot 32, \cdots, 16\cdot 19, 17\cdot 18 \\ 37: 2\cdot 35, 3\cdot 34, \cdots, 17\cdot 20, 18\cdot 19 \end{cases} \Rightarrow \begin{cases} \boxed{11}: 2\cdot 9, 3\cdot 8, 4\cdot 7, \boxed{\cancel{5\cdot 6}} \\ \boxed{17}: \boxed{\cancel{2\cdot 15}}, \cancel{3\cdot 14}, 4\cdot 13, \cancel{5\cdot 12}, \cancel{6\cdot 11}, \cancel{7\cdot 10}, \cancel{8\cdot 9} \\ 23: \cancel{2\cdot 21}, \cancel{3\cdot 20}, \cdots, 10\cdot 13, \cancel{11\cdot 12} \\ 27: 2\cdot 25, \cancel{3\cdot 24}, \cdots, \cancel{12\cdot 15}, 13\cdot 14 \\ 29: 2\cdot 27, 3\cdot 26, \cdots, 13\cdot 16, \cancel{14\cdot 15} \\ 35: \cancel{2\cdot 33}, 3\cdot 32, \cdots, 16\cdot 19, 17\cdot 18 \\ 37: \cancel{2\cdot 35}, \cancel{3\cdot 34}, \cdots, 17\cdot 20, 18\cdot 19 \end{cases}$$

21. (extremely challenging)

(4) Sam: I know both x and y now

$\equiv$ Sam can determine the only corresponding $x \cdot y$ of the $x + y$

$$x + y = \boxed{11} \Rightarrow \text{Peter can't determine } x \cdot y = \boxed{18 \text{ or } 24 \text{ or } 28}$$

$$\begin{cases} 11 : 2 \cdot 9, 3 \cdot 8, 4 \cdot 7, \cancel{5 \cdot 6} \\ 17 : \cancel{2 \cdot 15}, \cancel{3 \cdot 14}, 4 \cdot 13, \cancel{5 \cdot 12}, \cancel{6 \cdot 11}, \cancel{7 \cdot 10}, \cancel{8 \cdot 9} \\ 23 : \cancel{2 \cdot 21}, \cancel{3 \cdot 20}, \cdots, 10 \cdot 13, \cancel{11 \cdot 12} \\ 27 : 2 \cdot 25, \cancel{3 \cdot 24}, \cdots, \cancel{12 \cdot 15}, 13 \cdot 14 \\ 29 : 2 \cdot 27, 3 \cdot 26, \cdots, 13 \cdot 16, \cancel{14 \cdot 15} \\ 35 : \cancel{2 \cdot 33}, 3 \cdot 32, \cdots, 16 \cdot 19, 17 \cdot 18 \\ 37 : \cancel{2 \cdot 35}, \cancel{3 \cdot 34}, \cdots, 17 \cdot 20, 18 \cdot 19 \end{cases} \Rightarrow \begin{cases} \boxed{11} : \boxed{2 \cdot 9, 3 \cdot 8, 4 \cdot 7} \\ 17 : 4 \cdot 13 \\ 23 : 4 \cdot 19, 5 \cdot 18, 7 \cdot 16, 10 \cdot 13 \\ 27 : 2 \cdot 25, 4 \cdot 23, 5 \cdot 22, 7 \cdot 20, 8 \cdot 19, 9 \cdot 18, \\ \quad 10 \cdot 17, 11 \cdot 16, 13 \cdot 14 \\ 29 : 2 \cdot 27, 3 \cdot 26, 4 \cdot 25, 6 \cdot 23, 7 \cdot 22, 8 \cdot 21, \\ \quad 10 \cdot 19, 11 \cdot 18, 12 \cdot 17, 13 \cdot 16 \\ 35 : 3 \cdot 32, 4 \cdot 31, 5 \cdot 30, 6 \cdot 29, 7 \cdot 28, 8 \cdot 27, \\ \quad 9 \cdot 26, 10 \cdot 25, 11 \cdot 24, 12 \cdot 23, 14 \cdot 21, 16 \cdot 19, 17 \cdot 18 \\ 37 : 5 \cdot 32, 6 \cdot 31, 8 \cdot 29, 9 \cdot 28, 13 \cdot 24, 14 \cdot 23, \\ \quad 15 \cdot 22, 16 \cdot 21, 17 \cdot 20, 18 \cdot 19 \end{cases}$$

21. (extremely challenging)

$$\begin{cases} 11 : 2\cdot 9, 3\cdot 8, 4\cdot 7 \\ \boxed{17} : \boxed{4\cdot 13} \\ 23 : 4\cdot 19, 5\cdot 18, 7\cdot 16, 10\cdot 13 \\ 27 : 2\cdot 25, 4\cdot 23, 5\cdot 22, 7\cdot 20, 8\cdot 19, 9\cdot 18, \\ \quad 10\cdot 17, 11\cdot 16, 13\cdot 14 \\ 29 : 2\cdot 27, 3\cdot 26, 4\cdot 25, 6\cdot 23, 7\cdot 22, 8\cdot 21, \\ \quad 10\cdot 19, 11\cdot 18, 12\cdot 17, 13\cdot 16 \\ 35 : 3\cdot 32, 4\cdot 31, 5\cdot 30, 6\cdot 29, 7\cdot 28, 8\cdot 27, \\ \quad 9\cdot 26, 10\cdot 25, 11\cdot 24, 12\cdot 23, 14\cdot 21, 16\cdot 19, 17\cdot 18 \\ 37 : 5\cdot 32, 6\cdot 31, 8\cdot 29, 9\cdot 28, 13\cdot 24, 14\cdot 23, \\ \quad 15\cdot 22, 16\cdot 21, 17\cdot 20, 18\cdot 19 \end{cases}$$

$$\begin{cases} \text{Peter knows } \boxed{4\cdot 13} = 52 \\ \text{Sam knows } 4 + 13 = \boxed{17} \end{cases}$$

$$\Rightarrow \begin{cases} x = 4 \\ y = 13 \end{cases}$$

HW2 Q1.2.7.9

白崇佑

Q1. Show that: If n is perfect square, then n + 2 is not a perfect square.

We prove it by contradiction:

let $n = k^2$, where $k \in Z$

assume $n + 2 = m^2 \Rightarrow 2 = m^2 - n = m^2 - k^2, where\ m \in Z$

hence $2 = (m + k)(m - k)$

We aren't able to find integers m and k satisfy the equation

above, so n + 2 can't be perfect square when n is perfect square.

Q2. Show that any odd integer is the difference of two squares.

Let $m = 2k + 1 \quad \forall k \in Z$

$\because a^2 - b^2 = (a + b)(a - b)$

$\therefore m = 2k + 1 = 1 \times (2k + 1) = (k + 1 - k)(k + 1 + k)$

$m = (k + 1)^2 - k^2$

Q7. Show that $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$ has an integral root.

According to Newton's method, we take x = 1 into the equation

$$1^5 - 1^4 + 1^3 - 1^2 + 1^1 - 1 = 0$$

Hence 1 is the root of $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$

We then prove that the equation has an integral root.

Q9.

(*, Challenging) Let α be an angle such that $\alpha = \tan^{-1}(1/3) + \tan^{-1}(1/2)$ and $0 \leq \alpha < 2\pi$. Show that $\alpha = \pi/4$ without using a calculator.

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\times\tan(b)}$$

$$\Rightarrow a+b = \tan^{-1}\left(\frac{\tan(a)+\tan(b)}{1-\tan(a)\times\tan(b)}\right)$$

$$\text{let } a = \tan^{-1}(x) \quad b = \tan^{-1}(y)$$

$$\Rightarrow x = \tan(a) \quad y = \tan(b)$$

$$\Rightarrow \tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-x\times y}\right)$$

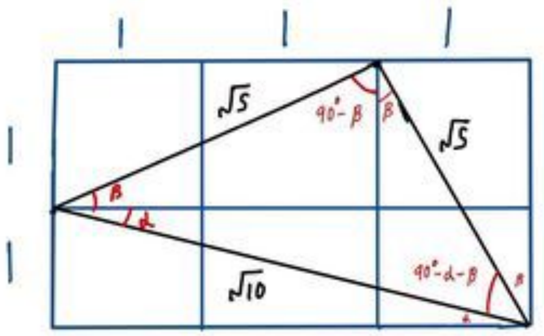
Q9.(continue)

$$\because \tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-x\times y}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right) = \tan^{-1}(1)$$

$$\tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}$$

Q9.(approach provided by Jamie Deng)



$$\tan^{-1}\left(\frac{1}{3}\right) = \alpha \quad \tan^{-1}\left(\frac{1}{2}\right) = \beta$$

$$90° - (\alpha + \beta) = \alpha + \beta \Rightarrow \alpha + \beta = \frac{\pi}{4}$$

HW2 Q10.11

李明哲

Q10: Prove or disprove the following:

If $p_1, p_2, \ldots, p_n$ are the n smallest primes, then $k = p_1 p_2 \cdots p_n + 1$ is prime.

Sol:

We disprove the statement.

It is sure that k cannot be divided by $p_1, p_2, \ldots, p_n$.

However, it is not necessarily true that k must be a prime.

There are some chances that k can be divided p_{n+1} or p_{n+2} ....

Consider a counterexample:

$n = 6,\ p_1 = 2,\ p_2 = 3, p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ p_6 = 13$

$k = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031$

Amazingly, 30031 $can\ be\ divided\ by$ 59! $(30031 = 59 \times 509)$

So k is not a prime.

Q11: Consider the equation $z^{13} - z^2 - 15015 = 0$.

(a) Show that the equation does not have any **integral root**.

Sol:

(i) Suppose z is an odd number

The product of odd numbers is odd,

so z^{13} and z^2 are odd.

$z^{13} - z^2 - 15015$ → (odd)-(odd)-(odd) = odd

But 0 is an even number. That's the contradiction !

Thus z cannot be an odd number. #

z	z^{13}	z^2	$z^{13} - z^2$	$z^{13} - z^2 - 15015$	0
odd	odd	odd	$even$	odd	$even$

Q11: Consider the equation $z^{13} - z^2 - 15015 = 0$.

(a) Show that the equation does not have any **integral root**.

Sol:

(ii) Suppose z is an even number

The product of even numbers is even,

so z^{13} and z^2 are even.

$z^{13} - z^2 - 15015$ → (even)-(even)-(odd) = odd

But 0 is an even number. That's the contradiction !

Thus z cannot be an even number. #

z	z^{13}	z^2	$z^{13} - z^2$	$z^{13} - z^2 - 15015$	0
$even$	$even$	$even$	$even$	odd	$even$

Q11: Consider the equation $z^{13} - z^2 - 15015 = 0$.

(a) Show that the equation does not have any **integral root**.

Sol:

(iii) Combine (i) 、(ii).

$$\left.\begin{array}{l} z \text{ is not an odd number.} \\ z \text{ is not an even number.} \end{array}\right\} z \text{ cannot be an integer.}$$

Thus, the equation does not have any integral root. #

Q11: Consider the equation $z^{13} - z^2 - 15015 = 0$.

(b) Show that the equation does not have **rational root**.

Sol:

Apply **Rational Root Theorem**.

For a polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$

with $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$,

its rational roots have the form of $\mathrm{x} = \frac{p}{q}$, where

p is the integer factor of a_0, q is the integer factor of a_n.

Q11: Consider the equation $z^{13} - z^2 - 15015 = 0$.

(b) Show that the equation does not have **rational root**.

(cont'd)

For $z^{13} - z^2 - 15015 = 0, a_0 = -15015, a_n = 1$.

Possible rational roots are:

$$z = \frac{p}{q} = \frac{integer\ factor\ of\ a_0}{integer\ factor\ of\ a_n} = \frac{integer\ factor\ of\ 15015}{\pm 1} \in \mathbb{Z}$$

But from the previous question,

the equation does not have any integral root.

Therefore, there is no rational root.

Q11 Method 2

Assume that the equation a has rational root $z = p/q$, where p,q has no common factor ohter than 1.

And we have $(p/q)^{13} - (p/q)^2 - 15015 = 0$

$\rightarrow p^{13} - (p^2)(q^{11}) - 15015(q^{13}) = 0$.

Case 1: p is odd, q is odd.

$\rightarrow p^{13}$: odd, p^2 : odd, q^{11} : odd, q^{13} : odd.

For the result of $p^{13} - (p^2)(q^{11}) - 15015(q^{13})$

$\rightarrow$ (odd) - (odd)*(odd) - 15015*(odd) = odd

But 0 is an even number! So this case is impossible.

Q11 Method 2

Case 2: p is odd, q is even.

→ p^13 : odd, p^2 : odd, q^11 : even, q^13 : even.

For the result of p^13 - (p^2)(q^11) - 15015(q^13)

→ (odd) - (odd)*(even) - 15015*(even) = odd

But 0 is an even number! So this case is impossible.

Case 3: p is even, q is odd.

→ p^13 : even, p^2 : even, q^11 : odd, q^13 : odd.

For the result of p^13 - (p^2)(q^11) - 15015(q^13)

→ (even) - (even)*(odd) - 15015*(odd) = odd

But 0 is an even number! So this case is impossible.

Q11 Method 2

Case 4: p is even, q is even.

p, q are both even numbers → p, q have common factor of 2.

This case violates our ogrinal assumption that p, q has no common factor ohter than 1.

That’s the contradiction! So this case is impossible!

Combine all of the 4 cases, there is no chance that the equation has the rational root z = p/q.