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電磁學（一）
Electromagnetics (I)

13. 磁場與磁向量勢
Magnetic Field and Vector Potential

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In this lecture, we will introduce the concept of static magnetic field and how we will model it.

- 13.1 Postulates of Static Magnetism 靜磁學的假設
- 13.2 Ampere’s Law 安培定律
- 13.3 Magnetic Vector Potential 磁向量勢
- 13.4 Biot-Savart Law 磁場直接求解公式
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磁場與磁向量勢
Magnetic Field and Vector Potential

13.1 靜磁學的假設
Postulates of Static Magnetism

Postulates of Magnetostatics (vacuum)

$\vec{B}$: Magnetic Flux Density in Tesla

Postulate 1 $\quad \nabla \cdot \vec{B} = 0$

Postulate 2 $\quad \nabla \times \vec{B} = \mu_0 \vec{J}$ (Ampere's law)

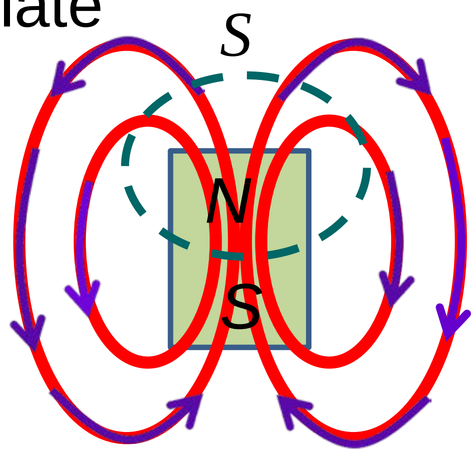
where $\mu_0 = 4\pi \times 10^{-7}$ Henry/m is the vacuum permeability

Apply the *divergence theorem* to the first postulate

$$\int_V \nabla \cdot \vec{B} dv = \oint_S \vec{B} \cdot d\vec{s} = 0$$

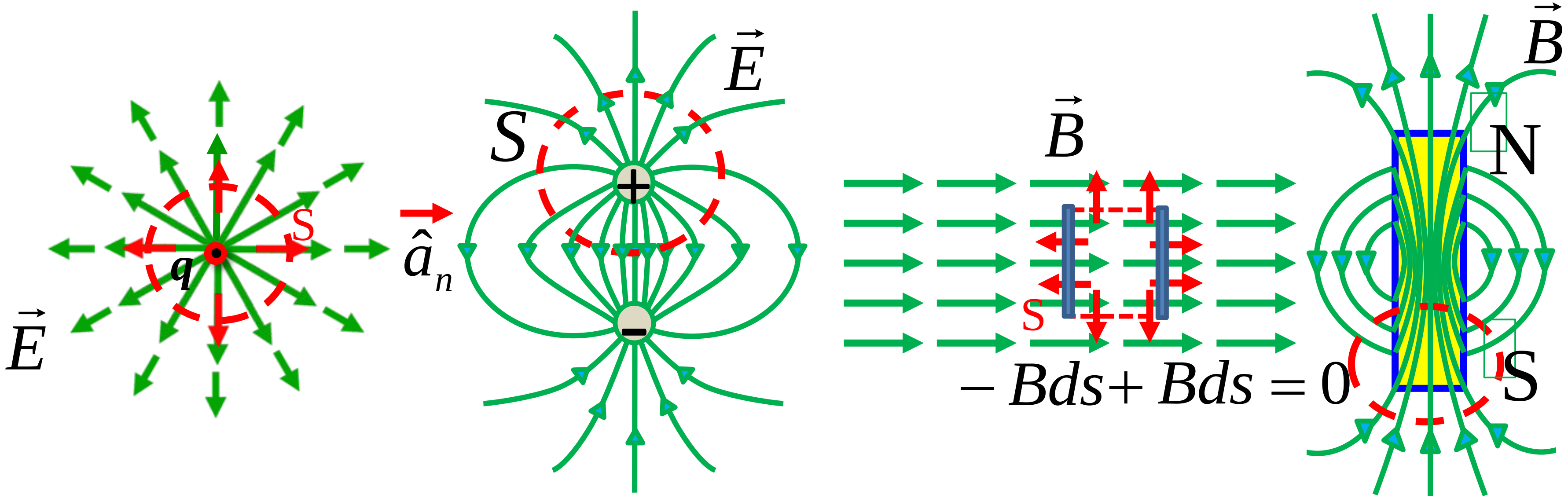
$\Rightarrow$ *1. magnetic field lines always return*

$\Rightarrow$ *2. no magnetic monopole*



Comparison between E and B Field Lines

Net outward flux surrounding q | No net flux over a volume



Electric monopole

$$\boxed{\nabla \cdot \vec{E} \neq 0}$$

Electric dipole

$$\oint_S \vec{E} \cdot d\vec{s} \neq 0$$

Solenoidal field

$$\boxed{\nabla \cdot \vec{B} = 0}$$

Magnet (dipole)

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \int_s \nabla \times \vec{B} \cdot d\vec{s} = \mu_0 \int_s \vec{J} \cdot d\vec{s} = \mu_0 I$$

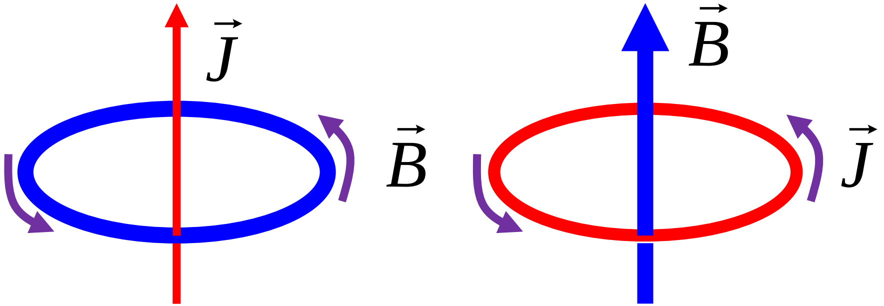
Apply the Stokes theorem and write

$$\int_s \nabla \times \vec{B} \cdot d\vec{s} = \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_s \vec{J} \cdot d\vec{s} = \mu_0 I$$

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I}$$ Ampere's circuital law for magnetostatics

The circulation of magnetic fields is proportional to the current bounded by the circular path.

* The directional relationship between B and I is understood from the curl operator or the right-hand rule.



13.1 靜磁學的假設

Postulates of Static Magnetism

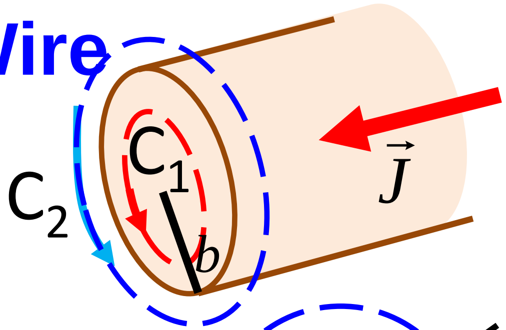
Maxwell's Equations for Static Magnetic Field (in vacuum)

Differential form	Integral form	
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$	Magnetic Gauss Law
$\nabla \times \vec{B} = \mu_0 \vec{J}$	$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$	Ampere's Law

磁場與磁向量勢
Magnetic Field and Vector Potential

13.2 安培定律
Ampere’s Law

Magnetic Flux Density of a Long Wire

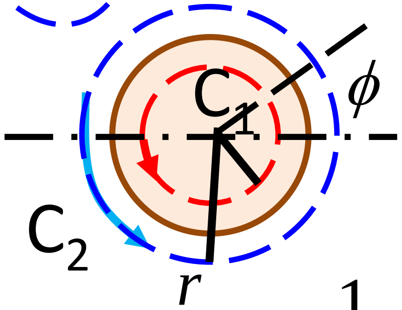


Assume a uniform current $\vec{J} = \dfrac{I}{\pi b^2}\hat{a}_z$

Circular symmetry $\Rightarrow$ only B_ϕ exists.

Apply Ampere's law $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I,$

where C is a circular path of a constant radius r.



i. In the region $r \le b,$

the current enclosed by C is

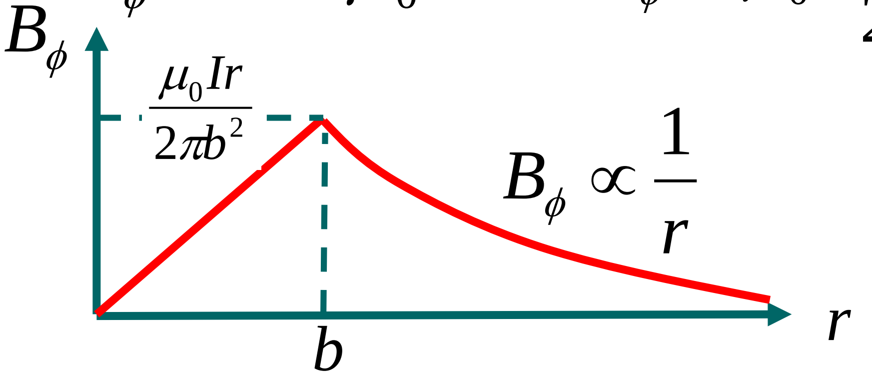
$$I(r) = J\pi r^2 = I\frac{r^2}{b^2}$$

Thus, $B_\phi 2\pi r = \mu_0 I(r) = \mu_0 I\dfrac{r^2}{b^2}$

$$\Rightarrow \quad B_\phi = \mu_0 I\frac{r}{2\pi b^2}$$

ii. In the region $r \ge b,$

$$\Rightarrow \quad B_\phi 2\pi r = \mu_0 I \quad \Rightarrow \quad B_\phi = \mu_0 I\frac{1}{2\pi r}$$



Magnetic Flux Density of a Toroid

The magnetic field at a constant r has a constant value along ϕ.

Again, apply Ampere's law, $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 IN$

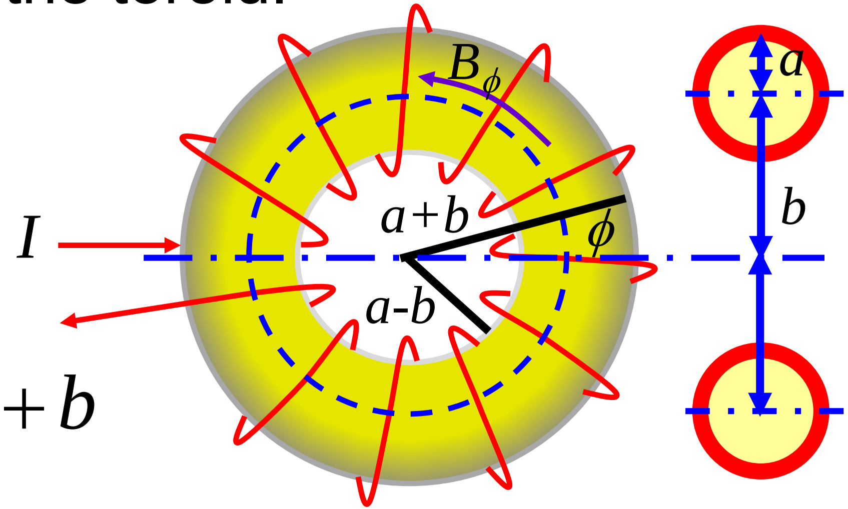
where I is the current in individual wires and *N* is the total number of wires around the toroid.

i. In the region $(b-a)\leq r\leq(b+a)$

$$B_\phi 2\pi r = \mu_0 NI \;\Rightarrow\; B_\phi = \frac{\mu_0 NI}{2\pi r}$$

ii. In the regions $r < b-a \;\&\; r > a+b$

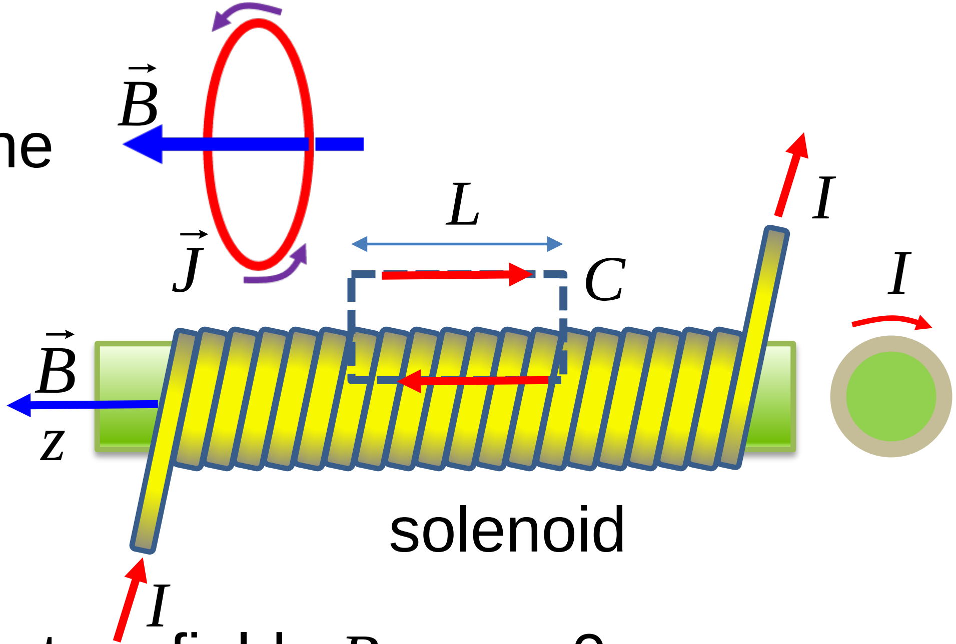
$$\oint_C \vec{B}\cdot d\vec{l} = 0 \Rightarrow \vec{B} = 0$$



Magnetic Flux Density of a Solenoid

1. The symmetry of the problem indicates a magnetic field in the longitudinal direction.
2. For the dashed-line path shown in the figure, Ampere's law gives

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 IN$$



3. For a very long solenoid, the return field $B_{outside} \sim 0$

*Assumption: solenoid length is long, fringe fields are ignored

$\Rightarrow B_z L = \mu_0 nLI$, where $n = N/L$ is the # of wire loops per unit length on the solenoid.

$\Rightarrow B_z = \mu_0 nI$

13.2 安培定律

Ampere's Law

- With enough symmetry, the Ampere's law is useful to calculate the magnetic field subject to a current.
- A long wire with a uniform current, the magnetic field outside the wire drops with $1/r$.
- In a toroid or a solenoid, the magnetic field is proportional to the density of wires carrying the current.

磁場與磁向量勢
Magnetic Field and Vector Potential

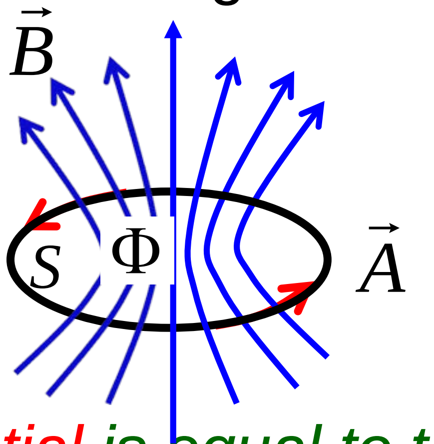
13.3 磁向量勢
Magnetic Vector Potential

Vector Potential

From the postulate $\nabla \cdot \vec{B} = 0$, one can use the vector identity $\nabla \cdot \nabla \times \vec{A} = 0$ to write

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

where A is the so-called vector potential of the magnetic field in units of Weber/m.



Physical meaning of $\vec{A}$?

Recall the total magnet flux

$$\Phi = \int_S \vec{B} \cdot d\vec{s} \Rightarrow \Phi = \int_S \nabla \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

The circulation of a magnetic vector potential is equal to the total magnetic flux going through the circulation path of A.

Solution to Vector Potential *A*

Insert $\vec{B} = \nabla \times \vec{A}$ into the Ampere's law $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\Rightarrow \nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} \quad \Rightarrow \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

(refer to the definition of vector Laplacian in Lecture 4)

Recall that, to be unique for a vector solution, the Helmoltz's theorem requires simultaneous definitions of the divergence and curl of a vector and boundary conditions.

Choose $\nabla \cdot \vec{A} = 0$ to obtain the vector Poisson's equation

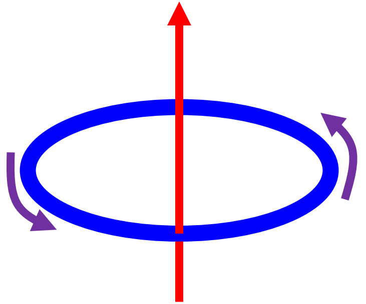
$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

In the Cartesian coordinate system, the vector Laplacian gives

$$\nabla^2 A_x = -\mu_0 J_x, \; \nabla^2 A_y = -\mu_0 J_y, \; \nabla^2 A_z = -\mu_0 J_z$$

Recall, in electrostatics, the Poisson's equation

$\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ with the solution $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{V'} \frac{\rho}{R} dv'$



Similarly, the solution of $\vec{A}$ is

$$A_{x,y,z} = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_{x,y,z}}{R} dv' \quad \text{or} \quad \vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv'$$

The direction of a vector potential is along the same direction of current $\Rightarrow$ easier to calculate A than B.

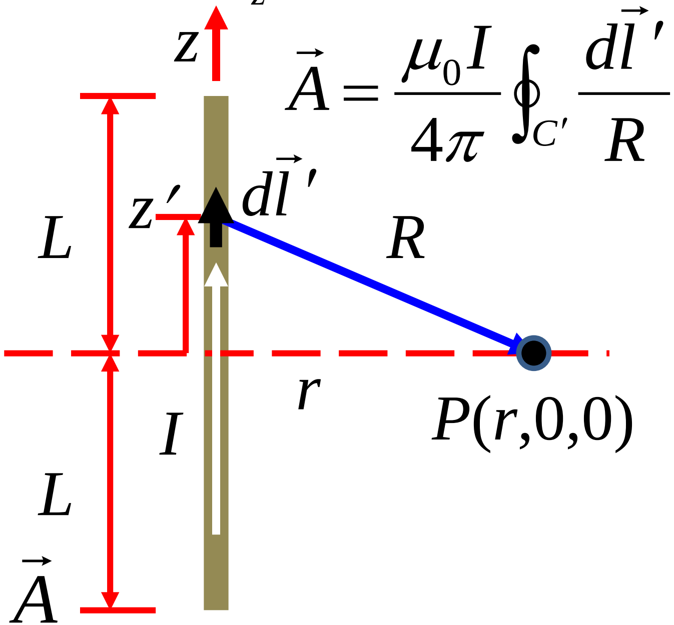
For a thin wire carrying a current $I = \vec{J} \cdot \vec{S}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv' = \frac{\mu_0}{4\pi}\oint_{C'} \frac{\vec{J}\cdot\vec{S}}{R} d\vec{l}\,' = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R}$$

E.g. Find $\vec{B}$ at P for a current element of length $2L$.

The differential length along current flow is $d\vec{l}\,' = \hat{a}_z dz'$

From the geometry, $R = \sqrt{r^2 + z'^2}$



$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R} = \hat{a}_z \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{dz'}{\sqrt{r^2 + z'^2}}$$

$$= \hat{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{r^2 + L^2} + L}{\sqrt{r^2 + L^2} - L}$$

Calculate the magnetic field from $\vec{B} = \nabla \times \vec{A}$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\hat{a}_z A_z\right) = \hat{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \hat{a}_\phi \frac{\partial A_z}{\partial r} = -\hat{a}_\phi \frac{\partial A_z}{\partial r} = \frac{\mu_0 I L}{2\pi r\sqrt{L^2 + r^2}}\hat{a}_\phi$$

13.3 磁向量勢

Vector Potential

- Because of $\nabla \cdot \vec{B} = 0$, we define the vector potential A from $\vec{B} = \nabla \times \vec{A}$.
- The vector potential is along the direction of a driving current,

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}}{R} dv' = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,'}{R},$$

which simplifies the calculation for A and then B.

磁場與磁向量勢
Magnetic Field and Vector Potential

13.4 磁場直接求解公式
Biot-Savart Law

The Biot-Savart Law

The magnetic field can be calculated from

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left(\frac{d\vec{l}\,'}{R} \right) \quad \text{with} \quad \vec{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,'}{R}$$

Use the formula $\nabla \times (f\vec{G}) = f\nabla \times \vec{G} + (\nabla f) \times \vec{G}$ and $\nabla \times d\vec{l}\,' = 0^*$

(*differentiation of unprimed coordinates on primed ones)

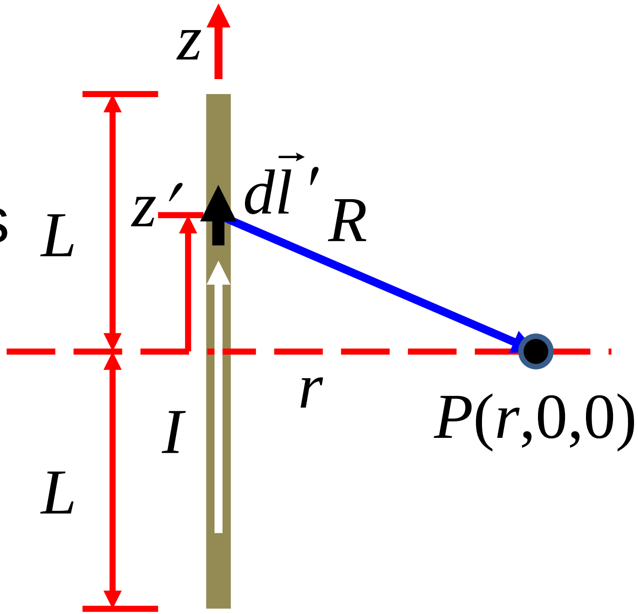
$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left(\frac{d\vec{l}\,'}{R} \right) = \frac{\mu_0 I}{4\pi} \oint_{C'} \left(\nabla \frac{1}{R} \right) \times d\vec{l}\,' \quad \text{But,} \quad \nabla \left(\frac{1}{R} \right) = -\hat{a}_R \frac{1}{R^2}$$

$$\Rightarrow \quad \boxed{\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \hat{a}_R}{R^2}} \quad \textbf{(Biot-Savart Law)}$$

- A direct way to calculate a magnetic field from a given current-carrying wire.

E.g. Find $\vec{B}$ at P for the following current element using the Biot-Savart law

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3}$$



The differential length along the current is

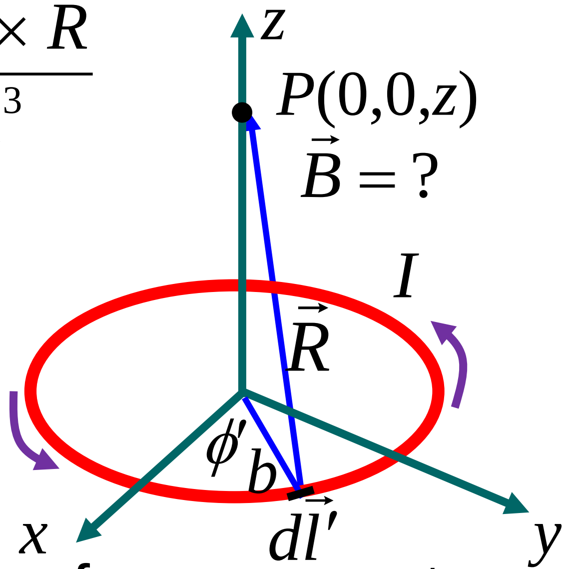
$$d\vec{l}\,' = \hat{a}_z dz'$$

The R vector is given by $\vec{R} = \hat{a}_r r - \hat{a}_z z'$

with $R = \sqrt{r^2 + z'^2} \Rightarrow d\vec{l}\,' \times \vec{R} = \hat{a}_\phi r dz'$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3} = \hat{a}_\phi \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{r dz'}{(z'^2 + r^2)^{3/2}} = \hat{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2 + r^2}}$$

E.g. Find $\vec{B}$ at P for the current loop of radius b by using Biot-Savart law - $\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3}$



The differential length of the current is

$$d\vec{l}\,' = \hat{a}_\phi b d\phi'$$

Also, $\vec{R} = \hat{a}_z z - \hat{a}_r b$ with $R = \sqrt{b^2 + z^2}$

$$\Rightarrow d\vec{l}\,' \times \vec{R} = \hat{a}_r bz d\phi' + \hat{a}_z b^2 d\phi'$$

The first term is ineffective in the integration from symmetry.

Substitute the above into the Biot-Savart law

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3} = \hat{a}_z \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{b^2}{(z^2 + b^2)^{3/2}} d\phi' = \hat{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$$

13.4 磁場直接求解公式

Biot-Savart Law

- The magnetic flux density generated by a current element can be calculated directly from the following integration:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3}$$

where $d\vec{l}\,'$ is the differential length of the current element, $\vec{R}$ is the position vector between the current and the point of interest.

磁場與磁向量勢
Magnetic Field and Vector Potential

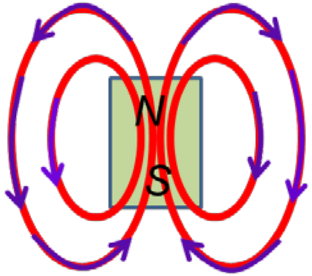
13.5 單元回顧 Review

單元回顧

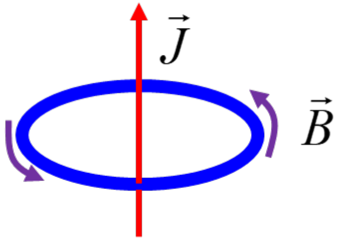
1. There are two postulates for static magnetism:

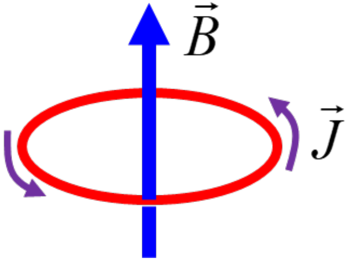
Differential form	Integral form
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$
$\nabla \times \vec{B} = \mu_0 \vec{J}$	$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$

$\Rightarrow$ Magnetic field lines always close upon themselves



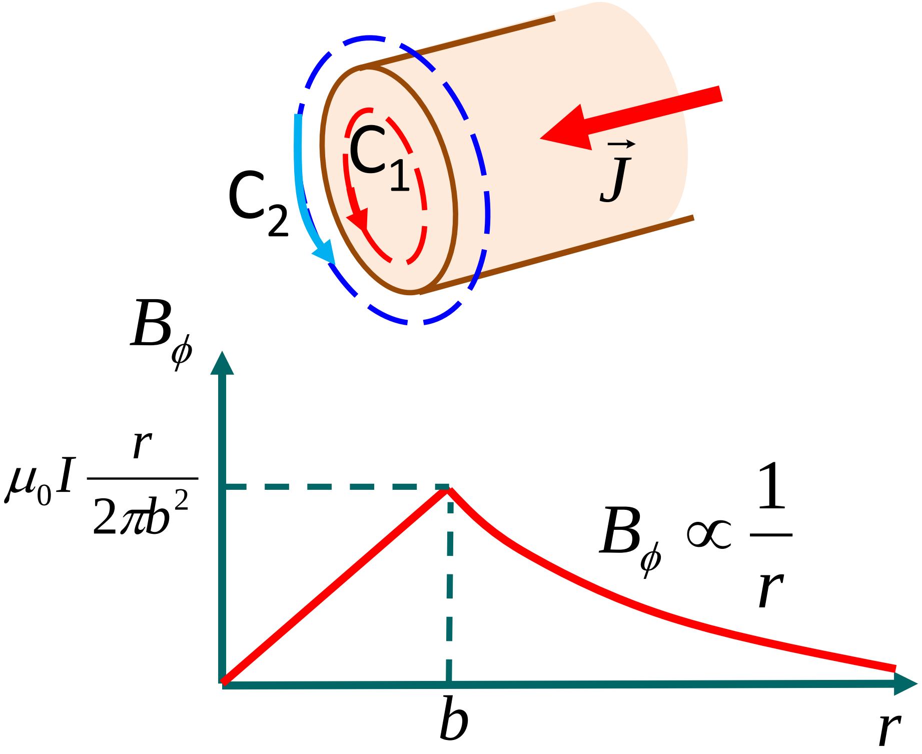
(Ampere's circuital law)





單元回顧

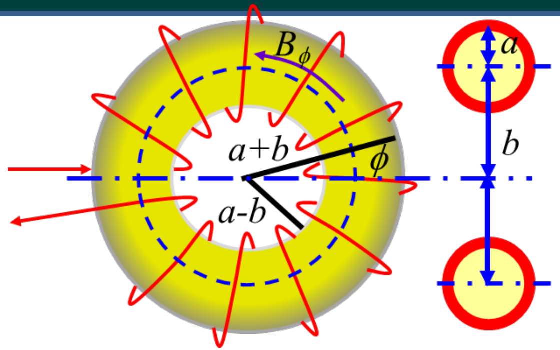
2. The magnetic flux density of a long wire carrying a uniform current increases over r inside the wire and decreases with $1/r$ outside the wire.



單元回顧

3. The magnetic flux density in a toroid is given by

$$B_\phi = \frac{\mu_0 NI}{2\pi r}$$

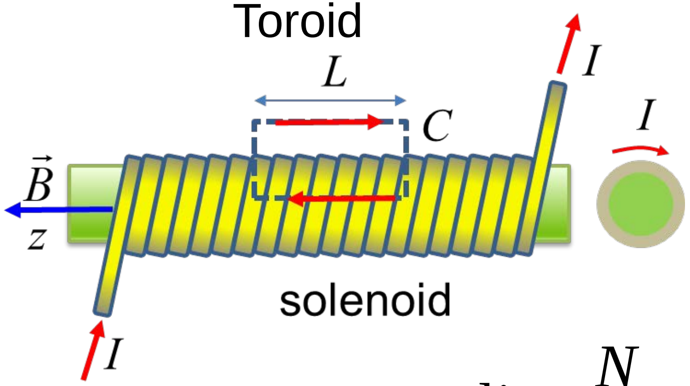


Toroid

4. The magnetic flux density in a solenoid is given by

$$B_z = \mu_0 nI$$

where n is the number density of the current loops.

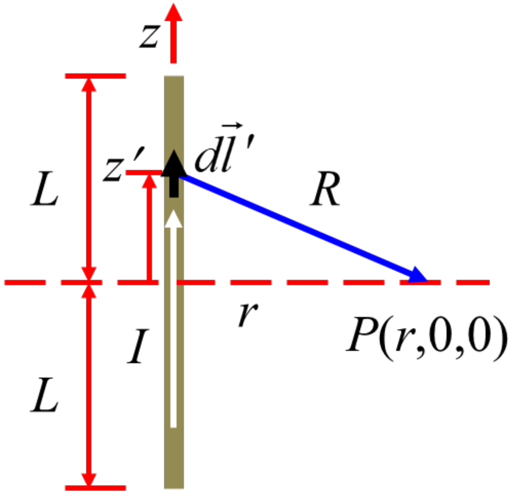


Remark: a toroid becomes a solenoid when $r \to$ infinity. Use $\lim_{r\to\infty} \frac{N}{2\pi r} = n$ for the solution of a toroid to obtain the same solution for a solenoid.

單元回顧

5. Because of $\nabla \cdot \vec{B} = 0$, we can define the vector potential A from $\vec{B} = \nabla \times \vec{A}$.

6. Given a current, the vector potential is along the direction of the current,



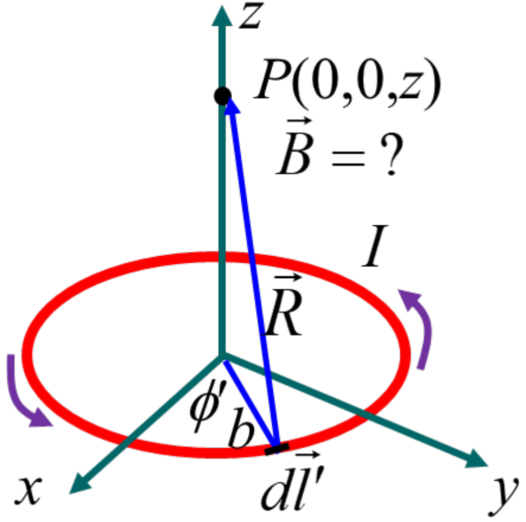
$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv' = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R},$$

which simplifies the calculation for A and then B.

單元回顧

7. It is possible to calculate the magnetic field directly for a given circuit element by using the so-called Biot-Savart law

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3}$$



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