1. $y'' + \lambda y = 0$, $y(0) = 0$, $y(\ell) = 0$

let $y_h = e^{\lambda x}$

$(D^2 + \lambda) y = 0$

$D^2 = -\lambda$

① $\lambda = -k^2$.

$D = \pm k$.

$y = d_1 \cdot e^{+kx} + d_2 e^{-kx}$

$= C_1 \cosh(kx) + C_2 \sinh(kx)$

$y(0) = 0 \Rightarrow C_1 = 0$.

$y(\ell) = 0 \Rightarrow C_2 = 0$

② $\lambda = 0$

$y = d_1 \cdot e^0 + d_2 \cdot e^0 \cdot x$

$= d_1 + x d_2$

$y(0) = 0 \Rightarrow d_1 = 0$

$y(\ell) = 0 \Rightarrow d_2 = 0$

③ $\lambda = k^2$.

$D = \pm ki$

$y = d_1 \cdot e^{+kix} + d_2 e^{-kix}$

$y = C_1 \cos(kx) + C_2 \sin(kx)$

$y(0) = 0 \Rightarrow C_1 = 0$.

$y(\ell) = 0 \Rightarrow C_2 \sin(k\ell) = 0$.

$\cancel{\sin}\ k\ell = n\pi$

$k = \frac{n\pi}{\ell}$

$\lambda = k^2$

$= \left(\frac{n\pi}{\ell}\right)^2$

$\phi_n = \sin\left(\frac{n\pi x}{\ell}\right)$, $n = 1, 2, 3, \ldots$

2. $y'' + 2y' + \lambda y = 0$, $y(0) = 0$, $y(\ell) = 0$

let $y_h = e^{\lambda x}$.

$(D^2 + 2D + \lambda) e^{\lambda x} = 0$

$D^2 + 2D + \lambda = 0$

let ① $\lambda = 1 - k^2$.

$D^2 + 2D + 1 = k^2$

$(D+1)^2 = k^2$

$D = -1 \pm k$

$y = d_1 e^{(-1+k)x} + d_2 e^{(-1-k)x}$

$= e^{-x}\left(d_1 e^{kx} + d_2 e^{-kx}\right)$

$= e^{-x}\left[C_1 \cosh(kx) + C_2 \sinh(kx)\right]$.

$y(0) = 0$

$\Rightarrow C_1 = 0$

$y(\ell) = 0 \Rightarrow C_2 = 0$.

② $\lambda = 1$

$(D+1)^2 = 0$.

$D = -1$.

$y = d_1 \cdot e^{-x} + d_2 \cdot x \cdot e^{-x}$.

$y(0) = 0 \Rightarrow d_1 = 0$

$y(\ell) = 0 \Rightarrow d_2 = 0$.

③ $\lambda = 1 + k^2$.

$D = -1 \pm ki$

$y = d_1 e^{(-1+ki)x} + d_2 e^{(-1-ki)x}$

$= e^{-x}\left(d_1 \cdot e^{kix} + d_2 \cdot e^{-kix}\right)$.

$= e^{-x}\left(C_1 \cos(kx) + C_2 \sin(kx)\right)$.

$y(0) = 0 \Rightarrow C_1 = 0$.

$y(\ell) = 0 \Rightarrow \sin k\ell = 0$

$k\ell = n\pi$

$k = \frac{n\pi}{\ell}$

$\lambda = 1 + k^2$.

$= 1 + \left(\frac{n\pi}{\ell}\right)^2$, $n = 1, 2, 3, \ldots$

$\phi_n = \sin\left(\frac{n\pi x}{\ell}\right) \cdot e^{-x}$

3. $y'' + \lambda y = 0$

$y(0) = 0$

$y'(1) - 2y(1) = 0$

Ans.

let $y_h = e^{\lambda x}$

$(D^2 + \lambda) = 0$

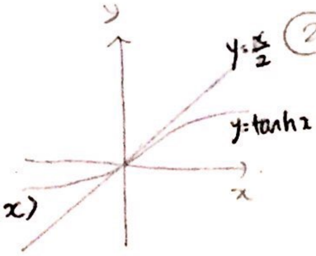
$D^2 = -\lambda$

① $\lambda = -k^2$

$D = \pm k$

$y = d_1 e^{kx} + d_2 e^{-kx}$

$= c_1 \cosh(kx) + c_2 \sinh(kx)$



$y(0) = 0$,

$\Rightarrow c_1 = 0$

$y = c_2 \sinh(kx)$

$y' = c_2 k \cosh(kx)$

$y'(1) - 2y(1) = c_2 k \cosh(k) - 2c_2 \sinh(k) = 0$

$\tanh(k) = \frac{k}{2}$

② $\lambda = 0$

$D = 0$

$y = d_1 e^0 + d_2 x e^0$

$= d_1 + d_2 x$

$y(0) = 0$

$\Rightarrow d_1 = 0$

$y = d_2 x$

$y' = d_2$

$y'(1) - 2y(1) = d_2 - 2d_2 = 0$

$\Rightarrow d_2 = 0$

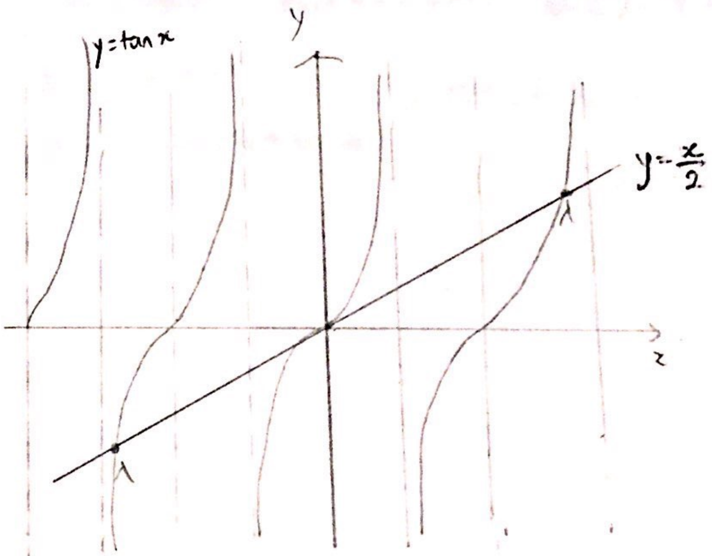
③ $\lambda = k^2$

$D^2 = -k^2$

$D = \pm ki$

$y = d_1 e^{kxi} + d_2 e^{-kxi}$

$= c_1 \cos(kx) + c_2 \sin(kx)$



$y(0) = 0$

$c_1 \cos(0) = 0$

$\Rightarrow c_1 = 0$

$y = c_2 \sin(kx)$

$y' = c_2 k \cos(kx)$

$y'(1) - 2y(1) = c_2 k \cos k - 2 c_2 \sin k = 0$

$\tan k = \frac{k}{2}$

$\lambda_k = k^2$; $n = 1, 2, 3, \ldots$

$\phi_n = \sin k_n x$

4. $x^2y'' + xy' + \lambda y = 0$

$y(1) = 0 \quad y(\ell) = 0$

Ans:

let $y = x^m$

$[m(m-1) + m + \lambda]\, x^m = 0$

$m^2 + \lambda = 0$

$m^2 = -\lambda$

① $\lambda = -k^2$

$m = \pm k$

$y = c_1 x^k + c_2 x^{-k}$

$y(1) = 0$ $\quad c_1 + c_2 = 0$ — ①

$y(\ell) = 0$ $\quad e^k c_1 + e^{-k} c_2 = 0$ — ②

$c_1 = -c_2$

from ②,

$e^k(-c_2) + e^{-k}(c_2) = 0.$

$c_2(e^{-k} - e^k) = 0$

$c_2 = 0$

$\Rightarrow c_1 = 0$

② $\lambda = 0$

$m = 0$

$y = c_1 x^0 + c_2 \ln x \cdot x^0$

$= c_1 + \ln x \cdot c_2$

$y(1) = 0$

$\Rightarrow c_1 = 0$

$y = c_2 \cdot \ln x$

$y(e) = 0$

$\Rightarrow c_2 = 0$

③ $\lambda = k^2$

$m = \pm jk$

$y = d_1 x^{+jk} + d_2 x^{-jk}$

$= d_1 \cdot e^{jk\ln x} + d_2 e^{jk\ln x}$

$y = c_1 \cos(k\ln x) + c_2 \sin(k\ln x)$

$y(1) = 0$

$\Rightarrow c_1 \cos(0) = 0$

$c_1 = 0$

$y = c_2 \sin(k\ln x)$

$y(e) = 0$

$\Rightarrow c_2 \sin k\ln \ell = 0$

$\sin k\ln \ell = 0$

$k = \frac{n\pi}{\ln \ell}$, $n = 1, 2, 3$

if $\ell = e$

$k = n\pi$

$\therefore \lambda_n = (\frac{n\pi}{\ln \ell})^2$, $\phi_n(x) = \sin(n\pi \frac{\ln x}{\ln \ell})$, if $\ell = e$

$\lambda_n = (n\pi)^2$ $\quad ; n = 1, 2, 3, \ldots$

$\phi_n(x) = \sin(n\pi \cdot \ln x)$

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5. $y''+\lambda y=0$. $y'(0)=0$, $y'(l)=0$.

Ans.

let $y_h=e^{\lambda x}$

$(D^2+\lambda)e^{\lambda x}=0$

$D^2=-\lambda$.

① $\lambda=-k^2$.

$D=\pm k$

$y=d_1 e^{kx}+d_2 e^{-kx}$.

$=c_1\cosh(kx)+c_2\sinh(kx)$

$y'=c_1 k\sinh(kx)+c_2 k\cosh(kx)$

$y'(0)=0 \Rightarrow c_2=0$

$y'(l)=0 \Rightarrow c_1=0$

② $\lambda=0$

$y=d_1+d_2 x$

$y'=d_2$

$y'(0)=0 \Rightarrow d_2=0$.

$\lambda_0=0$

$\phi_0=d_1$

③. $\lambda=k^2$.

$D=\pm ki$

$y=d_1 e^{kix}+d_2 e^{-kix}$

$=c_1\cos(kx)+c_2\sin(kx)$

$y'=-kc_1\sin(kx)+kc_2\cos(kx)$

$y'(0)=0 \Rightarrow c_2=0$.

$y'(l)=-kc_1\sin(kl)=0$.

$\sin kl=0$

$kl=n\pi$

$k=\frac{n\pi}{l}$

$\Rightarrow \lambda=k^2$; $\lambda_0=0$

$=\left(\frac{n\pi}{l}\right)^2$; $\phi_0=d_1$

$\phi_n=\cos\left(\frac{n\pi x}{l}\right)$, $n=1,2,3,$

6. $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$= \frac{2}{\pi}\int_0^{\pi} \cancel{f(x)}\,(1).\sin\frac{n\pi x}{\pi}\,dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$

$= -\frac{2}{n\pi}\cos(nx)\Big]_0^{\pi}$

$= -\frac{2}{n\pi}[\cos(n\pi)-1]$

$= -\frac{2}{n\pi}[(-1)^n-1]$, $n=1,2,3,\ldots$

Ans: $f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}[(-1)^n-1]\sin(nx)$, $n=1,2,3,\ldots$

7. $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{2}{L}\int_0^L x\cdot\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

$\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{L}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big]_0^L - \int_0^L \sin\frac{n\pi x}{L}\,dx\right)$

$= \frac{L}{n\pi}\times\frac{L}{n\pi}\left(\cos\frac{n\pi x}{L}\right)\Big]_0^L$; $L=\pi$

$= \frac{1}{n^2}[\cos(n\pi)-\cos(0)]$

$= \frac{1}{n^2}[(-1)^n-1]$

$a_n = \frac{2}{\pi n^2}[(-1)^n-1]$

$a_0 = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$; $n=0$

$= \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left(\frac{1}{2}\right)\pi^2 = \frac{1}{2}\pi$

Ans:

$f(x) = \frac{1}{2}\pi + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}[(-1)^n-1]\cos nx$

, $n=1,2,3,\ldots$

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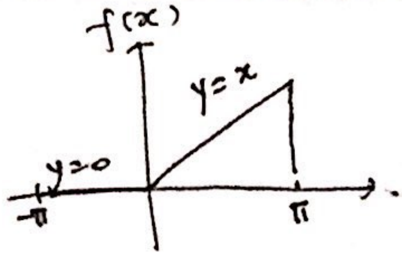
8.



$-\pi \le x \le \pi$

$$f(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

$$a_0 = \frac{1}{2l}\int_{-l}^{+l} f(x)\,dx$$

$$= \frac{1}{2l}\left(\int_{-l}^{+l} 0\,dx + \int_0^l x\,dx\right)$$

$$= \frac{1}{2l}\left(\frac{1}{2}x^2\right)\Big]_0^l \quad , l = \pi$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{n}\left(x\sin x\Big]_0^{\pi} - \int_0^{\pi}\sin nx\,dx\right)$$

$$= \frac{1}{n\pi}\left(\frac{1}{n}\cos nx\right)\Big]_0^{\pi}$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\right)\left(x\cos nx\Big]_0^{\pi} - \int_0^{\pi}\cos nx\,dx\right)$$

$$= -\frac{1}{\pi n}\left(x\cos nx\Big]_0^{\pi} - \frac{1}{n}\sin(nx)\Big]_0^{\pi}\right)$$

$$= -\frac{1}{\pi n}\pi\cos n\pi$$

$$= \frac{(-1)^{n+1}}{\cancel{\pi} n}$$

$$\text{Ans} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx)$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin(nx)$$

$n = 1, 2, 3, \ldots$