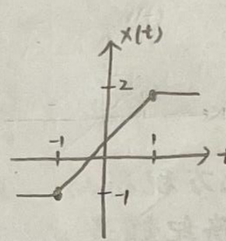
Signals and System HW4.

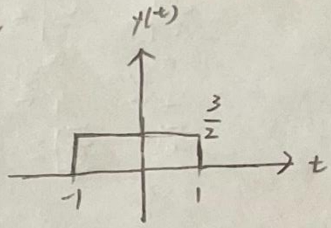
1.

$\boxed{4.8}$

$$x(t) = \begin{cases} -1, & t < -1 \\ \frac{3}{2}t + \frac{1}{2}, & -1 \le t \le 1 \\ 2, & t > 1 \end{cases}$$



$\because x(t) = \int_{-\infty}^{t} y(t)\,dt$



(a) $\because x(t) = \int_{-\infty}^{t} y(t)\,dt$

from the integration property,

$$X(j\omega) = \frac{1}{j\omega} Y(j\omega) + \pi Y(0)\,\delta(\omega)$$

$\because Y(j\omega) = \int_{-1}^{1} \frac{3}{2} \cdot e^{-j\omega t}\,dt$

$= 2 \cdot \int_{0}^{1} \frac{3}{2} \cos(\omega t)\,dt$

$= 3 \cdot \frac{1}{\omega} \sin(\omega t) \Big|_{0}^{1}$

$= \frac{3\sin(\omega)}{\omega}$

$\because Y(0) = \frac{0}{0} = \frac{3\cos(\omega)}{1} = 3$ ($\because$ L'Hopital's Rule)

$\therefore X(j\omega) = \frac{1}{j\omega} \times \frac{3\sin(\omega)}{\omega} + 3\pi\,\delta(\omega)$

$= \frac{3\sin(\omega)}{j\omega^2} + 3\pi\,\delta(\omega)$ #.

(b) $g(t) = x(t) - \frac{1}{2} \times 1$

$\downarrow \because 1 \xrightarrow{F.T.} 2\pi\,\delta(\omega)$

$\therefore G(j\omega) = X(j\omega) - \frac{1}{2} \times 2\pi \times \delta(\omega)$

$= \frac{3\sin(\omega)}{j\omega^2} + 3\pi\,\delta(\omega) - \pi\,\delta(\omega) = \frac{3\sin(\omega)}{j\omega^2} + 2\pi\,\delta(\omega)$ #

(c)

$x_e(t) \xrightarrow{F.T.} \mathrm{Re}\,(X(j\omega)) = 3\pi\,\delta(\omega)$ #.

$x_o(t) \xrightarrow{F.T.} j\,\mathrm{Im}\,(X(j\omega)) = j \cdot \left(-\frac{3\sin\omega}{\omega^2}\right) = \frac{-3\sin\omega}{\omega^2} j$ #.

2,

$\boxed{4,21}$

(e) $x(t)=[te^{-2t}\sin 4t]u(t)$

$=[te^{-2t}(\frac{e^{j4t}-e^{-j4t}}{2j})]u(t)$

$=\frac{1}{2j}te^{-2t}e^{j4t}u(t)-\frac{1}{2j}te^{-2t}e^{-j4t}u(t)$

$\because te^{-at}u(t)\xrightarrow{F.T.}\frac{1}{(a+jw)^2}$

$\therefore X(jw)=\frac{1}{2j}\left[\frac{1}{(2-4j+jw)^2}-\frac{1}{(2+4j+jw)^2}\right]$ #

(f)

$x(t)=\left[\frac{\sin\pi t}{\pi t}\right]\left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$

① let $x_1(t)=\frac{\sin\pi t}{\pi t}$

$\therefore X_1(jw)=\begin{cases}1, & |w|<\pi\\ 0, & \text{otherwise}\end{cases}$

② let $x_2(t)=\frac{\sin 2\pi(t-1)}{\pi(t-1)}$

$\therefore X_2(jw)=\begin{cases}e^{-jw}, & |w|<2\pi\\ 0, & \text{otherwise}\end{cases}$

③ $\because x(t)=x_1(t)x_2(t)$

$\therefore X(jw)=\frac{1}{2\pi}[X_1(jw)*X_2(jw)]$

④ (1) $-3\pi<w<-\pi$:

$X(jw)=\frac{1}{2\pi}\int_{-2\pi}^{\pi+w}1\cdot e^{-jw}dw$

$=\frac{1}{2\pi}\left[-\frac{1}{j}e^{-jw}\Big|_{-2\pi}^{\pi+w}\right]$

$=\frac{-1}{2\pi j}\left[e^{-j(\pi+w)}-e^{j2\pi}\right]$

$=\frac{-1}{2\pi j}\left[-e^{-jw}-1\right]$

$=\frac{1}{2\pi j}\left[1+e^{-jw}\right]$

(2) $-\pi<w<\pi$:

$X(jw)=\frac{1}{2\pi}\int_{-\pi+w}^{\pi+w}1\cdot e^{-jw}dw$

$=\frac{1}{2\pi}\left[-\frac{1}{j}e^{-jw}\Big|_{-\pi+w}^{\pi+w}\right]$

$=\frac{-1}{2\pi j}\left[e^{-j(\pi+w)}-e^{-j(-\pi+w)}\right]$

$=\frac{-1}{2\pi j}\left[-e^{-jw}-(-e^{-jw})\right]$

$=0$

(3) $\pi<w<3\pi$:

$X(jw)=\frac{1}{2\pi}\int_{-\pi+w}^{2\pi}1\cdot e^{-jw}dw$

$=\frac{1}{2\pi}\left[-\frac{1}{j}e^{-jw}\Big|_{-\pi+w}^{2\pi}\right]$

$=\frac{-1}{2\pi j}\left[e^{-j(2\pi)}-e^{-j(-\pi+w)}\right]$

$=\frac{-1}{2\pi j}\left[1-(-e^{-jw})\right]$

$=\frac{-1}{2\pi j}\left[1+e^{-jw}\right]$

⑤ $\therefore X(jw)=$

$\begin{cases}\frac{1}{2\pi j}[1+e^{-jw}], & -3\pi<w<-\pi\\ 0, & -\pi<w<\pi\\ \frac{-1}{2\pi j}[1+e^{-jw}], & \pi<w<3\pi\\ 0, & \text{otherwise}\end{cases}$ #

(g)

① $X_1(jw)=\int_{-2}^{-1}(-1)e^{-jwt}dt$

$=\frac{1}{jw}e^{-jwt}\Big|_{-2}^{-1}$

$=\frac{1}{jw}(e^{jw}-e^{j2w})$

② $X_2(jw)=\int_{-1}^{1}te^{-jwt}dt$

$=e^{-jwt}\left(\frac{-t}{jw}+\frac{1}{w^2}\right)\Big|_{-1}^{1}$

$=e^{-jw}\left(\frac{1}{w^2}-\frac{1}{jw}\right)-e^{jw}\left(\frac{1}{w^2}+\frac{1}{jw}\right)$

③ $X_3(jw)=\int_{1}^{2}1\cdot e^{-jwt}dt$

$=-\frac{1}{jw}e^{-jwt}\Big|_{1}^{2}$

$=-\frac{1}{jw}(e^{-2jw}-e^{-jw})$

④ $\therefore X(jw)=X_1(jw)+X_2(jw)+X_3(jw)$

$=\frac{1}{jw}(e^{jw}+e^{-jw})-\frac{1}{jw}(e^{j2w}+e^{-j2w})$

$-\frac{1}{w^2}(e^{jw}-e^{-jw})-\frac{1}{jw}(e^{jw}+e^{-jw})=-\frac{1}{jw}(2\cos(2w))-\frac{1}{w^2}(2j\sin w)=\frac{2j}{w}\left[\cos(2w)-\frac{\sin w}{w}\right]$ #

(h)

let $p(t)=\sum_{k=-\infty}^{\infty}\delta(t-2k)\xleftrightarrow{F.T.}P(jw)=\frac{2\pi}{2}\sum_{k=-\infty}^{\infty}\delta(w-\frac{2\pi k}{2})$

$\therefore x(t)=2p(t)+p(t-1)$

$\Rightarrow X(jw)=P(jw)[2+e^{-jw}]$

$=\pi\sum_{k=-\infty}^{\infty}\delta(w-\pi k)\left[2+(-1)^k\right]$ #

3. (6%+6%) Problem 4.26 of the textbook.

4.26. (a) Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

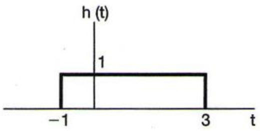


Figure P4.26

(a)

Some useful FT properties：

$$tx(t) \xrightarrow{\mathfrak{F}} j\frac{d}{d\omega}X(j\omega) \qquad x(at) \xrightarrow{\mathfrak{F}} \frac{1}{|a|}X(j\frac{\omega}{a}) \qquad x(t-t_0) \xrightarrow{\mathfrak{F}} e^{-j\omega t_0}X(j\omega)$$

(i)

$$te^{-2t}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{(2+j\omega)^2} \qquad e^{-4t}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{4+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1/2}{(2+j\omega)^2} + \frac{-1/4}{2+j\omega} + \frac{1/4}{4+j\omega}$$

$$Y(j\omega) \xrightarrow{\mathfrak{F}^{-1}} \left[\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}e^{-4t}\right]u(t) \qquad \blacksquare$$

(ii)

$$te^{-2t}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{(2+j\omega)^2} \qquad te^{-4t}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{(4+j\omega)^2}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1/4}{(4+j\omega)^2} + \frac{-1/4}{4+j\omega} + \frac{1/4}{(2+j\omega)^2} + \frac{1/4}{2+j\omega}$$

$$Y(j\omega) \xrightarrow{\mathfrak{F}^{-1}} \left[\frac{1}{4}te^{-4t} - \frac{1}{4}e^{-4t} + \frac{1}{4}te^{-2t} + \frac{1}{4}e^{-2t}\right]u(t) \qquad \blacksquare$$

(iii)

$$e^{-t}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{1+j\omega} \qquad e^{t}u(-t) \xrightarrow{\mathfrak{F}} \frac{1}{1-j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega}$$

$$Y(j\omega) \xrightarrow{\mathfrak{F}^{-1}} \frac{1}{2}e^{t}u(-t) + \frac{1}{2}e^{-t}u(t) = \frac{1}{2}e^{-|t|} \qquad \blacksquare$$

(b)

$$e^{t-2}u(t-2) \xrightarrow{\mathfrak{F}} \frac{e^{-j2\omega}}{1+j\omega}$$

$$h(t) = u(t+1) - u(t-3) \xrightarrow{\mathfrak{F}} \frac{2e^{-j\omega}\sin(2\omega)}{\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{e^{-j2\omega}}{1+j\omega}\frac{2e^{j\omega}\sin(2\omega)}{\omega} \quad -(1)$$

Obtain y(t) via direct convolution,

$$y(t) = x(t) * h(t) = \begin{cases} 0 & ,t \in (-\infty, 1] \\ 1 - e^{-(t-1)} & ,t \in (1, 5] \\ e^{-(t-5)} - e^{-(t-1)} & ,t \in (5, +\infty) \end{cases}$$

and then take the FT of y(t).

$$Y(j\omega) = \frac{2e^{-j3\omega}}{\omega(1+j\omega)} = \frac{e^{-j2\omega}}{1+j\omega}\frac{2e^{-j\omega}\sin(2\omega)}{\omega} \quad -(2)$$

(2) should be same with (1). ■

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4. (6%+6%+6%) Problem 4.33 of the textbook.

4.33. The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of this system.
(b) What is the response of this system if $x(t) = te^{-2t}u(t)$?
(c) Repeat part (a) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

(a)

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t) \xrightarrow{\mathfrak{F}} (j\omega)^2Y(j\omega) + 6(j\omega)Y(j\omega) + 8Y(j\omega) = 2X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{2}{(j\omega)^2 + 6j\omega + 8} = -\frac{1}{j\omega+4} + \frac{1}{j\omega+2}$$

$$\Rightarrow h(t) = -e^{-4t}u(t) + e^{-2t}u(t)$$ ■

(b)

$$x(t) = te^{-2t}u(t) \xrightarrow{\mathfrak{F}} X(j\omega) = \frac{1}{(2+j\omega)^2}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(2+j\omega)^2}\left[\frac{-1}{j\omega+4} + \frac{1}{j\omega+2}\right]$$

$$= \frac{1}{(j\omega+2)^3} + \frac{-1/2}{(j\omega+2)^2} + \frac{1/4}{j\omega+2} + \frac{1/4}{j\omega+4} \quad -(3)$$

$$y(t) = [\frac{t^2}{2}e^{-2t} - \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} - \frac{1}{4}e^{-4t}]u(t) \qquad -(4)$$

$$(\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xrightarrow{\mathfrak{F}} \frac{1}{(a+j\omega)^n})$$

(3) is the response in frequency domain, and (4) is the response in time domain. ■

(c)

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

$$\xrightarrow{\mathfrak{F}} (j\omega)^2Y(j\omega) + \sqrt{2}(j\omega)Y(j\omega) + Y(j\omega) = 2(j\omega)^2X(j\omega) - 2X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{2(j\omega)^2 - 2}{(j\omega)^2 + \sqrt{2}(j\omega) + 1}$$

$$= 2 + \frac{\gamma}{j\omega - \lambda} + \frac{\gamma^*}{j\omega - \lambda^*}$$

$$\Rightarrow h(t) = 2\delta(t) + \gamma e^{\lambda t}u(t) + \gamma^* e^{\lambda^* t}u(t)\,,$$

where $\lambda = \dfrac{-\sqrt{2} + j\sqrt{2}}{2}, \quad \gamma = -\sqrt{2} - \sqrt{2}j$

and λ^*, γ^* are the conjugates of λ, γ respectively. ■

5.(a) Take the Fourier transform of both sides, we have

$$Y(j\omega)[10+j\omega] = X(j\omega)[Z(j\omega)-1]$$

where $Z(j\omega) = \frac{1}{1+j\omega} + 3$, then we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)} \quad \#$$

(b) Use the partial fraction expansion of $H(j\omega)$

$$H(j\omega) = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{10+j\omega}$$

$$A(10+j\omega) + B(1+j\omega) = 3+2j\omega$$

$$\begin{cases} 10A+B=3 \\ A+B=2 \end{cases} \quad 9A=1 \quad A=\frac{1}{9},\ B=\frac{17}{9} \qquad H(j\omega) = \frac{\frac{1}{9}}{1+j\omega} + \frac{\frac{17}{9}}{10+j\omega}$$

Take its inverse Fourier transform we obtain

$$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t) \quad \#$$

6.

(a)

$$x_1[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\omega})e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[2\pi\delta(\omega-2\pi k)+\pi\delta(\omega-\tfrac{\pi}{2}-2\pi k)+\pi\delta(\omega+\tfrac{\pi}{2}-2\pi k)\right]e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\times 2\pi e^{j(2\pi k)n}+\frac{1}{2\pi}\times\pi e^{j(\frac{\pi}{2}+2\pi k)n}+\frac{1}{2\pi}\times\pi e^{j(-\frac{\pi}{2}+2\pi k)n}$$

$$=1+\frac{1}{2}e^{j\frac{\pi n}{2}}+\frac{1}{2}e^{-j\frac{\pi}{2}n}$$

$$=1+\cos(\tfrac{\pi n}{2})$$

(b)

$$x_2[n]=\frac{1}{2\pi}\int_0^{\pi}2je^{j\omega n}d\omega+\frac{1}{2\pi}\int_{-\pi}^{0}(-2j)e^{j\omega n}d\omega$$

$$=\frac{j}{\pi}\left(\frac{1}{jn}\right)\left[e^{j\omega n}\right]_0^{\pi}-\frac{j}{\pi}\left(\frac{1}{jn}\right)\left[e^{j\omega n}\right]_{-\pi}^{0}$$

$$=\frac{1}{\pi n}\left[e^{j\pi n}-1\right]-\frac{1}{\pi n}\left[1-e^{-j\pi n}\right]$$

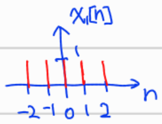
$$=\frac{1}{\pi n}\left[e^{j\pi n}-e^{-j\pi n}-2\right]$$

(c)

$$\because X_1(e^{j\omega})=\frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}}=\frac{e^{j\frac{5}{2}\omega}-e^{-j\frac{5}{2}\omega}}{e^{j\frac{1}{2}\omega}-e^{-j\frac{1}{2}\omega}}=\frac{e^{j\frac{5}{2}\omega}}{e^{j\frac{\omega}{2}}}\frac{1-e^{-j5\omega}}{1-e^{-j\omega}}$$

$$=e^{j2\omega}\sum_{m=0}^{4}e^{-j\omega m}$$

Take $m'=m-2$ $\quad=\sum_{m'=-2}^{2}e^{-j\omega m'}$ which is a rectangular function in time domain



And $u[n]\overset{FT}{\longleftrightarrow}\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega+2\pi k)$

$\because$ the multiplication in freq. domain implies convolution in time domain.

and $X_1(e^{j0})=\left.\frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}}\right|_{\omega=0}=\left.\frac{\frac{5}{2}\cos\frac{5}{2}\omega}{\frac{1}{2}\cos\frac{\omega}{2}}\right|_{\omega=0}=5$

Therefore, in $-\pi\le\omega\le\pi$

$$x_1[n]*u[n]\longleftrightarrow\frac{1}{1-e^{-j\omega}}X_1(e^{j\omega})+\pi X_1(e^{j0})\delta(\omega)$$

$$=\frac{1}{1-e^{-j\omega}}\frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}}+5\pi\delta(\omega)$$

To satisfy $X(e^{j\omega})$, we add $4\pi\delta(\omega)$ to the right hand side, which implies adding 2 to the left hand side.

$$\therefore X(e^{j\omega})\overset{FT}{\rightleftharpoons}x[n]$$

$$=x_1[n]*u[n]+2$$

$$=\sum_{k=-\infty}^{\infty}x_1[k]u[n-k]+2$$

$$=\sum_{k=-\infty}^{n}x_1[k]+2$$

$$=\begin{cases}2, & n<-2\\ n+2, & -2\le n\le 2\\ 7, & n>2\end{cases}$$

7. From given the fact, knowing that

$$\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\mathcal{F}} \frac{1-\frac{1}{4}}{1-\cos w+\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{5}{4}-\cos w} = \frac{3}{5-4\cos w}$$

Use the Fourier transform analysis equation to write

$$\frac{3}{5-4\cos w} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-jwn}$$

Assume $\omega=-2\pi t$ in this equation, and change the variable n by the variable k

$$\frac{3}{5-4\cos(2\pi t)} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} e^{j2\pi k t}$$

$$\Rightarrow \frac{1}{5-4\cos(2\pi t)} = \sum_{k=-\infty}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{|k|} e^{j2\pi k t}$$

By comparing this with the continuous-time Fourier series synthesis equation, we can find that $a_k = \frac{1}{3}\left(\frac{1}{2}\right)^{|k|}$ is the Fourier series coefficients of the signal $\frac{1}{5-4\cos(2\pi t)}$. #