EE306002: Probability **Spring 2022**
Department of Electrical Engineering
National Tsing Hua University

Homework #5 – Solutions
Coverage: Chapters 8 and 9
Due date: May 26, 2022

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Problem 8.1.3 (10 points) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} k(x^2+y^2) & \text{if } (x,y) = (1,1),\ (1,3),\ (2,3), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of the constant k.

(b) Determine the marginal probability mass functions of X and Y.

(c) Find $E(X)$ and $E(Y)$.

Solution:

(a) $k\big[(1+1)+(1+9)+(4+9)\big] = 1$ implies that $k = 1/25$.

(b)
$$p_X(1) = p(1,1) + p(1,3) = 12/25,\ p_X(2) = p(2,3) = 13/25.$$
$$p_Y(1) = p(1,1) = 2/25,\ p_Y(3) = p(1,3) + p(2,3) = 23/25.$$

Therefore,

$$p_X(x) = \begin{cases} 12/25 & \text{if } x = 1, \\ 13/25 & \text{if } x = 2. \end{cases}$$

$$p_Y(y) = \begin{cases} 2/25 & \text{if } y = 1, \\ 23/25 & \text{if } y = 3. \end{cases}$$

(c)
$$E(X) = 1 \cdot \frac{12}{25} + 2 \cdot \frac{13}{25} = \frac{38}{25},\ E(Y) = 1 \cdot \frac{2}{25} + 3 \cdot \frac{23}{25} = \frac{71}{25}.$$

Problem 8.1.14 (10 points) Let X be the proportion of customers of an insurance company who bundle their auto and home insurance policies. Let Y be the proportion of customers who insure at least their car with the insurance company. An actuary has discovered that for, $0 \le x \le y \le 1$, the joint distribution function of X and Y is $F(x, y) = x(y^2 + xy - x^2)$. Find the expected value of the proportion of the customers of the insurance company who bundle their auto and home insurance policies.

Solution:

$$F_X(x) = P(X \le x) = P(X \le x, Y \le 1) = F(x, 1) = x(1 + x - x^2),\ 0 \le x \le 1.$$

Therefore,

$$f_X(x) = F'_X(x) = 1 + 2x - 3x^2,\ 0 \le x \le 1.$$

So

$$E(X) = \int_0^1 x(1 + 2x - 3x^2)dx = \frac{5}{12}.$$

Problem 8.2.16 (10 points) Let X and Y be independent exponential random variables both with mean 1. Find $E[\max(X, Y)]$.

Solution:
Let F and f be the distribution and probability density functions of $\max(X, Y)$, respectively. Then

$$F(t) = P[\max(X, Y) \le t] = P(X \le t,\ Y \le t) = (1 - e^{-t})^2,\ t \ge 0.$$

Thus

$$f(t) = F'(t) = 2e^{-t}(1 - e^{-t}).$$

Hence

$$E[\max(X, Y)] = \int_0^\infty t \cdot 2e^{-t}(1 - e^{-t})\, dt = \frac{3}{2}.$$

Problem 8.3.10 (10 points) The random variable Y is selected at random from the interval (0,1); the random variable X is then selected at random from the interval $(Y, 1)$. Find the probability density function of X.

Solution:
Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\, dy.$$

Now,

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1-y} & 0 < y < 1,\ y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $0 < x < 1$,

$$f_X(x) = \int_0^x \frac{1}{1-y}\, dy = -\ln(1-x),$$

and hence

$$f_X(x) = \begin{cases} -\ln(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 8.3.13 (10 points) The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} ce^{-x} & \text{if } x \geq 0,\ |y| < x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(c) Calculate $E(Y|X=x)$ and $Var(Y|X=x)$.

Solution:
(a)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^{\infty}\int_{-x}^{x} ce^{-x}dydx = \int_0^{\infty} 2cxe^{-x}dx = 2c(\big[-xe^{-x}\big]_0^{\infty} - \big[e^{-x}\big]_0^{\infty}) = 2c,$$

then

$$c = \frac{1}{2}.$$

(b)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{2}e^{-x}}{\int_{|y|}^{\infty}\frac{1}{2}e^{-x}dx} = e^{-x+|y|},\ x > |y|.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{2}e^{-x}}{\int_{-x}^{x}\frac{1}{2}e^{-x}dy} = \frac{1}{2x},\quad -x < y < x.$$

(c) Given $X = x$, Y is a uniform random variable over $(-x, x)$ by (b). Therefore,

$$E(Y|X=x) = 0,\quad Var(Y|X=x) = \frac{[x-(-x)]^2}{12} = \frac{x^2}{3}.$$

Problem 9.1.14 (10 points) Let $X_1, X_2, ..., X_n$ be identically distributed, independent, exponential random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that

$$E[\min(X_1, ..., X_n)] < \min\{E(X_1), ..., E(X_n)\}.$$

Solution:
We know that

$$P(X_i \geq t) = e^{-\lambda_i t}, \text{ for } i = 1, ..., n, \; t \geq 0.$$

Therefore,

$$\begin{aligned} P[\min(X_1, X_2, ..., X_n) > t] &= P(X_1 > t, X_2 > t, ..., X_n > t) \\ &= P(X_1 > t)P(X_2 > t) \cdots P(X_n > t) \\ &= (e^{-\lambda_1 t})(e^{-\lambda_2 t}) \cdots (e^{-\lambda_n t}) \\ &= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}, \end{aligned}$$

which implies $\min(X_1, X_2, ..., X_n)$ is a exponential random variable with parameter $(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$. The inequality follows from the fact that for $i = 1, 2, ..., n$,

$$E[\min(X_1, ..., X_n)] = \frac{1}{\lambda_1 + \cdots + \lambda_n} < \frac{1}{\lambda_i} = E(X_i).$$

Problem 9.2.5 (10 points) Let $X_1, X_2, ..., X_n$ be a sequence of nonnegative, identically distributed, and independent random variables. Let F be the distribution function of X_i, $1 \leq i \leq n$. Prove that

$$E[X_{(n)}] = \int_0^\infty (1 - [F(x)]^n)dx.$$

Solution:
By Remark 6.4, we have

$$E[X_{(n)}] = \int_0^\infty P(X_{(n)} > x)dx.$$

Now

$$\begin{aligned} P(X_{(n)} > x) &= 1 - P(X_{(n)} \leq x) \\ &= 1 - P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x) = 1 - [F(x)]^n. \end{aligned}$$

Hence

$$E[X_{(n)}] = \int_0^\infty (1 - [F(x)]^n)dx.$$

Problem Ch9-Review 8 (10 points) A system consists of n components whose lifetimes form an independent sequence of random variables. Suppose that the system works as long as at least one of its components works. Let $F_1, F_2, ..., F_n$ be the cumulative distribution functions (CDF) of the lifetimes of the components of the system. In terms of $F_1, F_2, ..., F_n$, find the CDF of the lifetime of the system.

Solution:
For $1 \le i \le n$, let X_i be the lifetime of the ith component. The lifetime of the system is $\max(X_1, X_2, ..., X_n)$. Let $F(t)$ be the CDF of the lifetime of the system.

$$\begin{aligned} F(t) &= P[\max(X_1, X_2, ..., X_n) \le t] \\ &= P(X_1 \le t, X_2 \le t, ..., X_n \le t) \\ &= P(X_1 \le t)P(X_2 \le t) \cdots P(X_3 \le t) \\ &= F_1(t)F_2(t) \cdots F_n(t). \end{aligned}$$

Problem 9.2.9 (10 points) Let X_1 and X_2 be two independent random variables $N(0, \sigma^2)$, and $\{X_{(1)}, X_{(2)}\}$ be the ordered statistics of $\{X_1, X_2\}$. Let $f_{12}(x_1, x_2)$ be the joint probability density function of $X_{(1)}$ and $X_{(2)}$. Find $E[X_{(1)}] = \int\int x_1 f_{12}(x_1, x_2)dx_1dx_2$, where the integration is taken over an appropriate region.

Solution:
By Theorem 9.6,

$$\begin{aligned} f_{12}(x_1, x_2) = 2! f_{X_1}(x_1) f_{X_2}(x_2) &= 2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-x_1^2/2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-x_2^2/2\sigma^2} \\ &= \frac{1}{\sigma^2\pi} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2}, \quad -\infty < x_1 < x_2 < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} E[X_{(1)}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1 \frac{1}{\sigma^2\pi} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2} dx_1 dx_2 \\ &= \frac{1}{\sigma^2\pi} \int_{-\infty}^{\infty} e^{-x_2^2/2\sigma^2} \Big(\int_{-\infty}^{x_2} x_1 e^{-x_1^2/2\sigma^2} dx_1 \Big) dx_2 \\ &= \frac{1}{\sigma^2\pi} \int_{-\infty}^{\infty} e^{-x_2^2/2\sigma^2} (-\sigma^2) e^{-x_2^2/2\sigma^2} dx_2 \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-{x_2}^2/\sigma^2} dx_2 \\ &= -\frac{1}{\pi} \cdot \sigma\sqrt{\pi} \cdot \frac{1}{\frac{\sigma}{\sqrt{2}}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\dfrac{x_2^2}{2(\sigma/\sqrt{2})^2}} dx_2 \\ &= -\frac{1}{\pi} \cdot \sigma\sqrt{\pi} \cdot 1 = -\frac{\sigma}{\sqrt{\pi}}. \end{aligned}$$

Problem Ch9-Review 9 (10 points) A bar of length ℓ is broken into three pieces at two random spots. What is the probability that the length of at least one piece is less than $\ell/20$?

Solution:
Let X and Y be two random variables. X and Y are selected independently and at random

from the interval $(0,\ \ell)$.

$$\begin{aligned}
&P\Big[\min(X, Y-X, \ell-Y) < \frac{\ell}{20}\,\Big|\, X<Y\Big]P(X<Y)\\
&+P\Big[\min(Y, X-Y, \ell-X) < \frac{\ell}{20}\,\Big|\, X>Y\Big]P(X>Y)\\
&\quad= 2P\Big[\min(X, Y-X, \ell-Y) < \frac{\ell}{20}\,\Big|\, X<Y\Big]P(X<Y)\\
&\quad= 2P\Big[\min(X, Y-X, \ell-Y) < \frac{\ell}{20}\,\Big|\, X<Y\Big]\cdot\frac{1}{2}\\
&\quad= 1-P\Big[\min(X, Y-X, \ell-Y) \geq \frac{\ell}{20}\,\Big|\, X<Y\Big]\\
&\quad= 1-P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20},\ \ell-Y\geq\frac{\ell}{20}\,\Big|\, X<Y\Big)\\
&\quad= 1-P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20},\ Y\leq\frac{19\ell}{20}\,\Big|\, X<Y\Big).
\end{aligned}$$

Now $P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20}, Y\leq\frac{19\ell}{20}\,\Big|\, X<Y\Big)$ is the area of the region

$$\Big\{(x,y)\in R^2\,|\,0<x<\ell,\ 0<y<\ell,\ x\geq\frac{\ell}{20},\ y-x\geq\frac{\ell}{20},\ y\leq\frac{19\ell}{20}\Big\},$$

divided by the area of the region

$$\Big\{(x.y)\in R^2\,|\,0<x<\ell,\ 0<y<\ell,\ y>x\Big\}.$$

Therefore,

$$1-P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20}, Y\leq\frac{19\ell}{20}\,\Big|\, X<Y\Big) = 1-\frac{\frac{1}{2}\cdot(\frac{17\ell}{20})^2}{\ell^2/2} = 1-\frac{289}{400}=\frac{111}{400}.$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)