EECS 205003 Session 20

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Outline

Ch5 Determinants

- Ch 5.1 The Properties of Determinants
- Ch 5.2 Permutations and Cofactors
- Ch 5.3 Cramer's Rule, Inverses, and Volumes

Determinants

The determinant is an important number
associated with any square matrix

e.g., the matrix is invertible iff its determinant is nonzero

Notation:

$$det(A) \text{ or } |A|$$

Properties

start with properties $\rightarrow$ Big formula

(e.g., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$)

Basic rules (1-3)
Rule (4-10) follows from (1-3)

1. $det I = 1$ for any n×n identity matrix I
2. The determinant changes sign when two rows are exchanged (sign reversal)

e.g., $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Note: we can find $det\ P$ from rule 2

$\Rightarrow$ exchange rows of I to reach P
$\Rightarrow$ $det P = +1$ (even number of row change)
$det P = -1$ (odd number of row change)

3. The determinant is a linear function of each row separately (orther rows unchanged)

check 2×2:

(a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

(true for any row since by rule 2 we can put any row as row 1 then exchange it back and determinant won't change)

Note: $det\ 2I \neq 2detI$

$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4, \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2$

(Just like area & volume)

From Rule 1-3, we can deduce many others (Rule 4-10)

4. If two rows of A are equal, then $detA = 0$

check 2×2:

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0, \text{ (ab - ab} = 0)$$

Reason: By rule 2, we can exchange these two rows $\Rightarrow$ -D (if $detA = D$)

But A stays the same when we exchange two identical rows $\Rightarrow$ D

So we have -$D = D \Rightarrow D = 0$

5. Substracting a multiple of one row from another row leaves $det\ A$ unchanged

check 2×2:

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Reason: (for 2×2)

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \underset{3(b)}{\underset{\uparrow}{=}} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} \underset{3(a)}{\underset{\uparrow}{=}} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underset{=0}{\underset{\uparrow}{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}}$$

(Proof for higher dim is similar)

Conclusion: Determinant not changed by Elimination $det\ A = \pm\ detU$

(if row change)

6. A matrix with a row of zeros has $det\ A = 0$

check 2×2: $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0\,, \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$

Reason: 2×2

$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} = 0$ or $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$

$\nwarrow 5 \qquad \nwarrow 4 \qquad \nwarrow 3(a)$

7. If A is triangular then $det\ A = a_{11}a_{22}\cdots a_{nn} =$ product of diagonal entries

check 2×2:

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad\ , \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$$

Reason: Do Gauss-Jordan elimination to eliminate entries in upper triangular for U (lower triangular for L)

$\Rightarrow$ We reach D with entries of diagonal of U By Rule 5, det stays the same & $det\ D = a_{11}\cdots a_{nn}\ det\ I$ by rule 1

Note: If $a_{ii} = 0$ for some i, Elimination produces a zero row $\Rightarrow det\ A = 0$

$\nwarrow 6$

8. $det\ A = 0$ iff A is singular

Reason:

If A is singular , we can use elimination to get zero rows $\underset{6}{\Rightarrow}$ $det\ A = 0$

If A is not singular, elimination produces
a full set of pivots $d_1, \cdots, d_n$ on U

$$\Rightarrow det\ A \overset{5}{=} \pm\ det\ U \overset{7}{=} \pm(d_1 d_2 \cdots d_n)$$

(possible row exchange)

<u>Derive 2×2 formula</u>

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{8}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc$$

(In fact, we know how to derive determinant for any n×n invertible A
$detA = \pm\ detU = \pm(d_1 \cdots d_n)$.
This is how MATLAB compute det !)

9. $det(AB) = det(A)det(B)$

check 2×2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

Reason: When $|B| \neq 0$, Let $D(A) = \dfrac{|AB|}{|B|}$

check if D(A) satisfies Rule $1-3 \Rightarrow D(A) = |A|$

Rule 1: If $A = I$, $D(A) = \dfrac{|B|}{|B|} = 1$ ($\surd$)

Rule 2: When two rows of A are exchanged $\Rightarrow$ same two rows of AB are exchanged $|AB|$ changes sign $\Rightarrow D(A) = \dfrac{|AB|}{|B|}$ changes sign

Rule 3:

(a) when row 1 of A is multiplied by t so is row 1 of AB

$$detA'B = tdetAB \Rightarrow D(A') = tD(A) \ (\surd)$$

(b) Add row 1 of A to row 1 of A' to get row 1 of A''

$\Rightarrow$ row 1 of $A''B$ = row 1 of AB + row 1 of $A'B$

$\downarrow$3(b)

$$\Rightarrow |A''B| = |AB| + |A'B| \Rightarrow \frac{|A''B|}{|B|} = \frac{|AB|}{|B|} + \frac{|A'B|}{|B|}$$

$$\Rightarrow D(A'') = D(A) + D(A') \ (\surd)$$

(trivial for $|B| = 0$)

Note: $AA^{-1} = I \Rightarrow det(A)det(A^{-1}) = 1$

$$\Rightarrow det(A^{-1}) = \frac{1}{det(A)}$$

Note: $det(A^2) = (detA)^2$

10. $det(A^{\mathsf{T}}) = det(A)$

check 2×2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Reason:

If $|A| = 0 \Rightarrow A$ is singular $\Rightarrow A^{\mathsf{T}}$ is singular $\Rightarrow |A^{\mathsf{T}}| = 0$

For invertible A, $PA = LU$

$\Rightarrow (PA)^{\mathsf{T}} = (LU)^{\mathsf{T}}$

$\Rightarrow A^{\mathsf{T}} P^{\mathsf{T}} = U^{\mathsf{T}} L^{\mathsf{T}}$

Compare

$det\ P \cdot det\ A = det\ L \cdot det\ U$

& ↑9↓

$det\ A^{\mathsf{T}} \cdot det\ P^{\mathsf{T}} = det\ U^{\mathsf{T}} \cdot det\ L^{\mathsf{T}}$

— $det\ L = 1 = det\ L^{\mathsf{T}}$

(both have 1's on diagonal)

— $det\ U = d_1 \cdots d_n = detU^{\mathsf{T}}$

(both U, U^{T} are triangular & have same diagonal entries)

— $det\ P = det\ P^{\mathsf{T}} = \pm\ 1$

$(P^{\mathsf{T}} = P^{-1} \Rightarrow det\ P^{\mathsf{T}} = det\ P^{-1} = \frac{1}{detP})$

$\Rightarrow det\ A = det\ A^{\mathsf{T}}$

Note: By this property, every rules for rows can be applied to columns, e.g., exchange two columns $\Rightarrow$ determinant changes sign