EE2030Linear Algebra

Practice

June 12, 2023

1. The "cycle" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(v))$? What is $T^3(v)$? What is $T^{100}(v)$? Apply T a hundred times to v.

2. Suppose a linear T transforms (1, 1) to (2, 2) and (2, 0) to (0, 0). Find $T(v)$:
$$(a) v = (2, 2) \quad (b) v = (3, 1) \quad (c) v = (-1, 1) \quad (d) v = (a, b).$$

3. Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Show that the identity matrix I is not in the range of T. Find a nonzero matrix M such that $T(M) = AM$ is zero.

$\begin{matrix} a & b \\ c & d \end{matrix}$

4. Suppose $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [M] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find a matrix with $T(M) \neq 0$. Describe all matrices with $T(M) = 0$ (the kernel) and all output matrices $T(M)$ (the range).

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

5. M The transformation S takes the **second derivative.** Keep $1, x, x^2, x^3$ as the basis v_1, v_2, v_3, v_4 and also as w_1, w_2, w_3, w_4. Write Sv_1, Sv_2, Sv_3, Sv_4 in terms of the w's. Find the 4 by 4 matrix B for S.

$\begin{bmatrix} x & 0 \\ x & x \end{bmatrix}$

6. (a) The product TS of first and second derivatives produces the *third* derivative. Add zeros to make 4 by 4 matrices, then compute AB.
(b) The matrix B^2 corresponds to $S^2 =$ *fourth* derivative. Why is this zero?

7. Which bases v_1, v_2, v_3 and w_1, w_2, w_3, suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. T is a linear transformation. Find the matrix A and multiply by the vector (1, 1, 1). What is the output from T when the input is $v_1 + v_2 + v_3$?

8. Suppose $T(v_1) = w_1 + w_2 + w_3$ and $T(v_2) = w_2 + w_3$ and $T(v_3) = w_3$. Find the matrix A for T using these basis vectors. What input vector v gives $T(v) = w_1$?

9. (a) What matrix transforms (1, 0) into (2, 5) and transforms (0, 1) to (1, 3)?
(b) What matrix transforms (2, 5) to (1, 0) and (1, 3) to (0, 1)?
(c) Why does no matrix transform (2, 6) to (1, 0) and (1, 3) to (0, 1)?

10. The matrix that rotates the axis vectors (1, 0) and (0, 1) through an angle θ is Q. What are the coordinates (a, b) of the original (1, 0) using the new (rotated) axes? This *inverse* can be tricky. Draw a figure or solve for a and b:
$$Q = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} + b \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$$
.

1. $T(\underline{v}) = A\underline{v} = \underline{v}'$ $\quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$T(T(\underline{v})) = \begin{bmatrix} v_3 \\ v_1 \\ v_2 \end{bmatrix}$ $\quad T^3(\underline{v}) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad T^{100}(\underline{v}) = \begin{bmatrix} v_2 \\ v_3 \\ v_1 \end{bmatrix}$

2.

$T(\underline{v_1}) = \underline{w_1}$ $\quad \underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{w_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$T(\underline{v_2}) = \underline{w_2}$ $\quad \underline{v_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ \underline{w_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2T(\underline{v_1}) + 0\,T(\underline{v_2}) = 2T(\underline{v_1}) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = T(\underline{v_1}) + T(\underline{v_2}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = T(\underline{v_1}) - T(\underline{v_2}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow b\,T(\underline{v_1}) + \frac{a-b}{2} T(\underline{v_2}) = b \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$x + 2y = a \quad x = b \quad y = \frac{a-b}{2}$

$$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\2\end{bmatrix}$$

$$\begin{bmatrix}2\\0\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\} \rightarrow \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$$\left\{\overset{\underline{v_1}}{\begin{bmatrix}2\\0\end{bmatrix}}, \overset{\underline{v_2}}{\begin{bmatrix}1\\1\end{bmatrix}}\right\} \rightarrow \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$$T(\underline{v_1}) = 0\,\underline{w} = 0\,\underline{v_1} + 0\,\underline{v_2}$$

$$T(\underline{v_2}) = 2\,\underline{w} = 0\,\underline{v_1} + 2\,\underline{v_2}$$

$$A = \begin{bmatrix}0 & 2\end{bmatrix}$$

$\underline{v_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\underline{w_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\underline{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{w_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\underline{v_1} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$2T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

3. $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$, $\det A = 0$

$AM = I$ but A is not invertible.

$I \notin$ range of A

$M=\begin{bmatrix}-2&0\\1&0\end{bmatrix}$

4.

If $A^T = -A$, then $\underline{x}^T A \underline{x} = 0$

$(\underline{x}^T A \underline{x})^T = \underline{x}^T A^T \underline{x} = -\underline{x}^T A \underline{x}$

$\underline{x}^T A \underline{x} = \lambda \underline{x}^T \underline{x} = \lambda (\underline{x}^T \underline{x})^T = \lambda \underline{x}^T \underline{x} = (\underline{x}^T A \underline{x})^T$

$= \underline{x}^T A^T \underline{x} = -\underline{x}^T A \underline{x}$

$2\underline{x}^T A \underline{x} = 0$

$\underline{x}^T A \underline{x} = 0$

5. $S(\underline{v}) = \underline{w}$

$$\frac{d(1)}{dx} = 0 \quad \frac{dx}{dx} = 1 \quad \frac{dx^2}{dx} = 2x \quad \frac{dx^3}{dx} = 3x^2$$

$B\underline{v} = \underline{w}$

$$0 \quad 0 \quad 2 \quad 6x$$

$$B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{v_1} = A^{-1}\underline{w_2}$

$\underline{v_2} = A^{-1}\underline{w_1} + A^{-1}\underline{w_2} = \underline{v_3}$

$A\underline{v_1} = \underline{w_2}$

$A\underline{v_2} = \underline{w_1} + \underline{w_2}$ $\quad A\underline{v_3} = \underline{w_1} + \underline{w_3}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{v_1} = \underline{w_2}$$

7. Which bases v_1, v_2, v_3 and w_1, w_2, w_3, suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. T is a linear transformation. Find the matrix A and multiply by the vector (1, 1, 1). What is the output from T when the input is $\underline{v_1} + \underline{v_2} + \underline{v_3}$?

coordinate

$$\begin{bmatrix} T(\underline{v_1}) & T(\underline{v_2}) & T(\underline{v_3}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$$

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$T(\underline{v_1}+\underline{v_2}+\underline{v_3}) = T(\underline{v_1}) + T(\underline{v_2}) + T(\underline{v_3})$$
$$= 2\underline{w_1} + \underline{w_2} + 2\underline{w_3}$$

$$\begin{bmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \underline{w_1} & \underline{w_2} & \underline{w_3} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{bmatrix}^{-1} \begin{bmatrix} \underline{w_1} & \underline{w_2} & \underline{w_3} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

8. Suppose $T(v_1) = w_1 + w_2 + w_3$ and $T(v_2) = w_2 + w_3$ and $T(v_3) = w_3$. Find the matrix A for T using these basis vectors. What input vector v gives $T(v) = w_1$?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \underline{w_1} & \underline{w_2} & \underline{w_3} \end{bmatrix}^{-1} \begin{bmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\underline{v} = \underline{v_1} - \underline{v_2}$$

$$T(\underline{v_1} - \underline{v_2}) = T(\underline{v_1}) - T(\underline{v_2}) = \underline{w_1}$$

$$\Downarrow$$

$$\underline{v}$$

9. (a) What matrix transforms (1, 0) into (2, 5) and transforms (0, 1) to (1, 3)?
(b) What matrix transforms (2, 5) to (1, 0) and (1, 3) to (0, 1)?
(c) Why does no matrix transform (2, 6) to (1, 0) and (1, 3) to (0, 1)?

(a) let $\underline{v_1} = \begin{bmatrix}1\\0\end{bmatrix}$, $\underline{w_1} = \begin{bmatrix}2\\5\end{bmatrix}$

$\underline{v_2} = \begin{bmatrix}0\\1\end{bmatrix}$, $\underline{w_2} = \begin{bmatrix}1\\3\end{bmatrix}$

$$\left[\underset{\text{column}}{T(\underline{v_1})} \quad T(\underline{v_2})\right] = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$$

(b) let $\underline{v_1} = \begin{bmatrix}2\\5\end{bmatrix}$ $\underline{w_1} = \begin{bmatrix}1\\0\end{bmatrix}$

$\underline{v_2} = \begin{bmatrix}1\\3\end{bmatrix}$ $\underline{w_2} = \begin{bmatrix}0\\1\end{bmatrix}$

$$T(\underline{v_1}) = \underline{w_1} = 3\underline{v_1} - 5\underline{v_2}$$

$$T(\underline{v_2}) = \underline{w_2} = -\underline{v_1} + 2\underline{v_2}$$

$$A = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$$

(c) let $\underline{v_1} = \begin{bmatrix}2\\6\end{bmatrix}$ $\underline{w_1} = \begin{bmatrix}1\\0\end{bmatrix}$

$\underline{v_2} = \begin{bmatrix}1\\3\end{bmatrix}$ $\underline{w_2} = \begin{bmatrix}0\\1\end{bmatrix}$

$$T(\underline{v_1}) = \underline{w_1} = c_1\underline{v_1} + c_2\underline{v_2}$$

c_1, c_2 can't be soluted

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

10. The matrix that rotates the axis vectors (1, 0) and (0, 1) through an angle θ is Q. What are the coordinates $(a,\ b)$ of the original (1, 0) using the new (rotated) axes? This *inverse* can be tricky. Draw a figure or solve for a and b:

$$Q = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} + b \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$$

$a=1$, $b=0$