Linear Algebra, EE 10810/EECS 205004

2nd Exam

(Dated: Fall, 2021)

Total scores: 120%

+12

1. ($\pm$30%) **[True or False]** Note that: a Correct answer gaining +3%; but a Wrong answer loosing -3% (答錯倒扣).

X (1) Any determinant $\delta : \overline{\overline{M}}_{n\times n}(\mathbf{F}) \rightarrow \mathbf{F}$ is a linear transform.

X (2) If $\overline{\overline{E}}$ is an elementary matrix, then $\det(\overline{\overline{E}}) = \pm 1$.

X (3) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

O (4) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

O (5) Similar matrices always have the same eigenvalues.

O (6) If λ is an eigenvalue of a linear operator $\hat{T}$, then each element of E_λ is an eigenvector of $\hat{T}$.

X (7) If 2 is an eigenvalue of $\overline{\overline{A}} \in \overline{\overline{M}}_{n\times n}(\mathbf{C})$, then $\lim_{n\to\infty} \overline{\overline{A}}^m$ exists.

X (8) An inner product is linear in both components.

O (9) If $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{x}$ in an inner product space, then $\vec{y} = \vec{0}$.

X (10) The triangle inequality only holds in finite-dimensional inner product spaces.

+10

2. (20%) **[Determinant of Hessenberg Matrix]**

A *Hessenberg* matrix is a $n \times n$ matrix, $\overline{\overline{A}}_{n\times n}$, with the matrix element $a_{ij} = 0$ for $i > j+1$.

(a) (10%) Find the determinant of a 4×4 Hessenberg matrix, $\overline{\overline{A}}_{4\times 4}$ i.e.,

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \tag{1}$$

$a_{44} + a_{34} \det(\overline{\overline{A}}_{2\times 2})$
$- a_{24}\, a_{11} a_{32}$
$+ a_{14}\, a_{21} a_{32}$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{matrix}$$

(b) (10%) Based on (a), by filling the empty box, show that the determinant of a $n \times n$ Hessenberg matrix has the form:

$$\det\left[\overline{\overline{A}}_{n\times n}\right] = a_{nn} \det\left[\overline{\overline{A}}_{n-1\times n-1}\right] + \sum_{i=1}^{n-1} [\quad ?? \quad] \tag{2}$$

+20

3. (20%) **[Eigenvalues of Fibonacci Series]**
Fibonacci sequence $\{F_i\}$ starts

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. In terms of a 2-dimensional system of linear difference equations to describes the Fibonacci sequence, we have

$$\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}, \tag{3}$$

$\frac{1+\sqrt{5} - 1+\sqrt{5}}{2}$

or alternatively $\vec{F}_{k+1} = \overline{\overline{A}}\vec{F}_k$. With Eq. (3), show that $F_n = \frac{1}{\sqrt{5}}\left[\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\right](F_1 + 2F_0)$

(a) (10%) Find the n-th element in the Fibonacci series in a closed-form expression, i.e., $F_n = ?$ F_0

(b) (10%) Show that the ratio of consecutive Fibonacci numbers converges to the *Golden ratio*, i.e.,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} \tag{4}$$

$ab = \frac{1-5}{2} = -2$

$$\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

$$\begin{pmatrix} a^n & b^n \\ a^{n-1} & b^{n-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ -1 & a \end{pmatrix}$$

$a^n - b^n \quad 2a^{n-1}$

$-aba^{n-1}$
$-aba^{n-1} + abb^{n-1}$
$aba^{n-1} + abb^{n-1}$
$= ab(a^{n-1} - b^{n-1})$

+50

4. (15%) [**System of Differential Equations**]
Find the general solution to the system of differential equations:

$$\frac{dx_1}{dt} = x_1 + x_3 \tag{5}$$

$$\frac{dx_2}{dt} = x_2 + x_3 \tag{6}$$

$$\frac{dx_3}{dt} = 2x_3 \tag{7}$$

$$x = \begin{pmatrix} C_1 e^{t} + C_3 e^{2t} \\ C_2 e^{t} + C_3 e^{2t} \\ 2C_3 e^{2t} \end{pmatrix}$$

5. (20%) [**Cayley-Hamilton Theorem**]

Suppose that a 2×2 matrix $\overline{\overline{\mathbf{M}}}$ satisfies

$$\overline{\overline{\mathbf{M}}}^2 + 5\overline{\overline{\mathbf{M}}} + 6\bar{\bar{\mathbf{I}}} = \overline{\overline{\mathbf{O}}}, \tag{8}$$

where $\bar{\bar{\mathbf{I}}}$ is a 2×2 identity matrix and $\overline{\overline{\mathbf{O}}}$ is a 2×2 zero matrix.

(a) (5%) Determine the eigenvalues of $\overline{\overline{\mathbf{M}}}$. $-2, -3$

(b) (10%) Is $\overline{\overline{\mathbf{M}}}^{-1}$ diagonalizable? If yes, find $\overline{\overline{\mathbf{M}}}^{-1}$; If not, explain your answer. $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$

(c) (5%) Calculate $\overline{\overline{\mathbf{M}}}^{(-2)}$ with the help of Cayley-Hamilton theorem. $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$

6. (15%) [**Gram-Schmidt orthogonalization process**]
For the given subset S of the inner product space $\mathcal{V} = \mathcal{R}^3$,

$$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1,0,1), (0,1,1), (1,3,3)\} \tag{9}$$

(a) (5%) Show that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent to each other.

(b) (10%) Apply the Gram-Schmidt process to obtain an orthonormal basis.

$\frac{1}{\sqrt{2}}(1,0,1)$ $\frac{1}{\sqrt{6}}(-1,2,1)$ $\frac{1}{\sqrt{3}}(1,1,-1)$