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電磁學（一）
Electromagnetics (I)

18. 電磁波動現象
Wave Dynamics of Time-varying Fields

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In this lecture, we will learn about the propagation of time-varying electromagnetic fields as a wave.

- **18.1 Time-varying Potential Fields 時變位能場**
- **18.2 Propagation of Potential Fields 位能場的傳播**
- **18.3 Wave Equations for Electromagnetic Fields 電磁場波動方程式**
- **18.4 Time-harmonic Electromagnetic Wave 電磁弦波**
- **18.5 Review 單元回顧**

電磁波動現象
Wave Dynamics of Time-varying Fields

18.1 時變電位場
Time-varying Potential Fields

Time-varying Potential Functions

From $\nabla \cdot \vec{B} = 0$, it is straightforward to have $\boxed{\vec{B} = \nabla \times \vec{A}}$

From Faraday's law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, one can write

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \quad \text{or} \quad \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

From the null identity $\nabla \times (\nabla V) = 0$, the following relationship is obtained

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \quad \text{or} \quad \boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V}$$

If A and V are known, E and B can be derived.

Vector-Potential Wave Equation

Substitute $\vec{B} = \nabla \times \vec{A}$ into Ampere's law $\nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{J}$ (*)

Adopt the definition of vector Laplacian on the left hand side of (*)

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Use $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$ on the right hand side of (*)

Equate both sides to have $\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla(\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t})$

Adopt the Lorentz gauge $\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t} \equiv 0$ to define $\nabla \cdot \vec{A}$

We finally have the inhomogeneous wave equation for A

$$\boxed{\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}}$$

Scalar-Potential Wave Equation

The Gauss's law $\nabla \cdot \vec{D} = \rho$ can be written as $\nabla \cdot \vec{E} = \rho/\varepsilon$

Use $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$ to obtain $-\nabla \cdot (\nabla V + \partial \vec{A}/\partial t) = \rho/\varepsilon$

Apply Lorentz gauge $\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t} \equiv 0$ (self-consistency) to write

$$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}) = \rho/\varepsilon$$

or finally the inhomogeneous wave equation for electric scalar potential

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon}$$

18.1 時變電位場

Time-varying Potential Fields

- The time-varying electric and magnetic fields can be derived from known electric potential and magnetic vector potential via

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

- The electric potential satisfies the inhomogeneous wave equation:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon$$

- The magnetic vector potential satisfies the inhomogeneous equation:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}$$

電磁波動現象
Wave Dynamics of Time-varying Fields

18.2 位能場的傳播
Propagation of Potential Fields

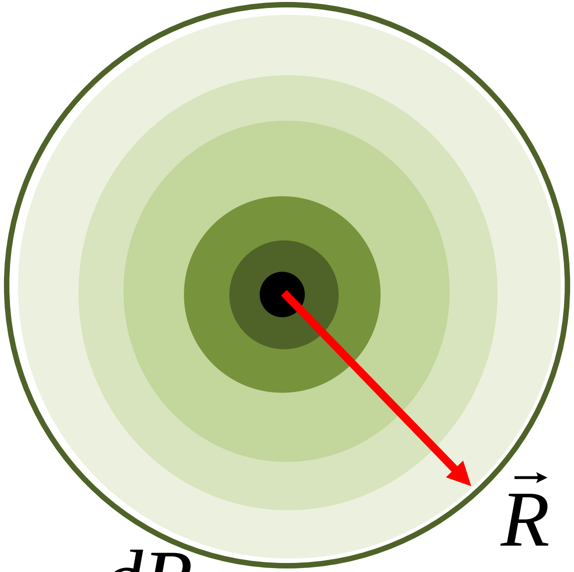
Wave Equation $\nabla^2\Psi - \dfrac{1}{u^2}\dfrac{d^2\Psi}{dt^2} = 0$ (!) Ψ can be A, V, ...

The general solutions to (!) is a wave function, having the form

$$\Psi(t, R) \propto f(t \pm R/u)$$

Wavefront is defined through

$$t \pm R/u = \text{constant}$$



Speed of the wavefront (phase velocity) is $\dfrac{dR}{dt} = \mp u$

$\Psi(t, R) \propto f(t \pm R/u)$ describes a propagating wave propagating along $\mp\hat{a}_R$.

Retarded Potentials

For the static case $\nabla^2 V = -\rho/\varepsilon$ and $\nabla^2 \vec{A} = -\mu \vec{J}$

Recall the solutions in the static case, based on point-source integration, V and A are

$$V(R) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(R)}{R} dv' \quad \text{and} \quad \vec{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(R)}{R} dv'$$

However, we also expect a solution of the form $f(t - R\sqrt{\mu\varepsilon})$

for $\nabla^2 \vec{A} - \mu\varepsilon \dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}$, and $\nabla^2 V - \mu\varepsilon \dfrac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon$

Write the solution as the so-called retarded potentials

$$\vec{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(t - \sqrt{\mu\varepsilon}R)}{R} dv', \quad V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t - \sqrt{\mu\varepsilon}R)}{R} dv'$$

Time Retardation from Propagation

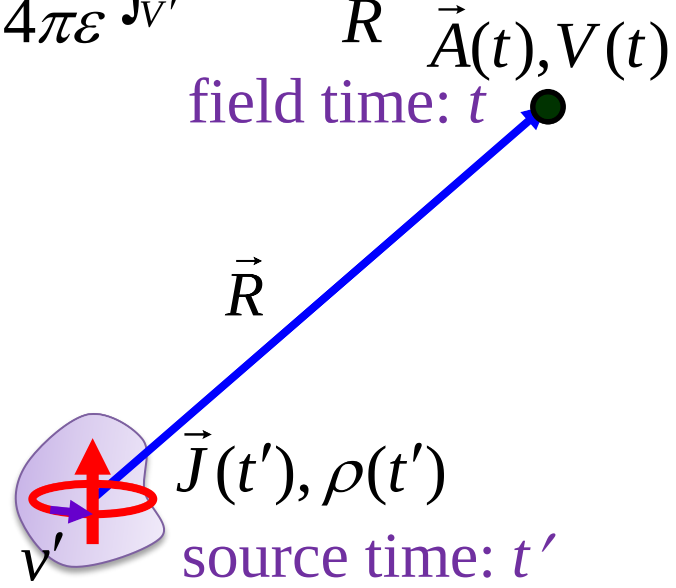
$$\vec{A}(R,t)=\frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(t-\sqrt{\mu\varepsilon}R)}{R}dv',\quad V(R,t)=\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(t-\sqrt{\mu\varepsilon}R)}{R}dv'$$

The potentials are retarded in time with respect to the source by

$$t'=t-R\sqrt{\mu\varepsilon}$$

E.g. an excitation of $J(t')$ or $\rho(t')$ at $t'=0$ allows a field to be detected at $\left.t'\right|_{t'=0}=t-R\sqrt{\mu\varepsilon}$

$$\Rightarrow t=R\sqrt{\mu\varepsilon}=\frac{R}{u}$$



The speed of propagation is

$$u=1/\sqrt{\mu\varepsilon}$$

18.2 位能場的傳播

Propagation of Potential Fields

- Upon excitation by a source, a potential field is generated and propagates at the speed

$$u = 1/\sqrt{\mu\varepsilon}$$

in space, which has a value of 3×10^8 m/s in vacuum.

- Specifically, the electric and magnetic vector potentials are

$$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t-\sqrt{\mu\varepsilon}R)}{R}dv', \quad \vec{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(t-\sqrt{\mu\varepsilon}R)}{R}dv',$$

電磁波動現象
Wave Dynamics of Time-varying Fields

18.3 電磁場的波動方程式
Wave Equations of Electromagnetic Fields

Source-free Wave Equation for Electric Field

In a simple medium, the source-free Maxwell's equations are

$$\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t},\quad \nabla\cdot\vec{E}=0,\quad \nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t},\quad \nabla\cdot\vec{H}=0$$

Use the 1st expression to write $\nabla\times\nabla\times\vec{E}=-\mu\dfrac{\partial\nabla\times\vec{H}}{\partial t}$ (*)

Left hand side becomes $\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\nabla^2\vec{E}$ for $\nabla\cdot\vec{E}=0$.

Adopt $\nabla\times\vec{H}=\varepsilon\dfrac{\partial\vec{E}}{\partial t}$ to the right of (*) to write $-\varepsilon\mu\dfrac{\partial^2\vec{E}}{\partial t^2}=-\dfrac{1}{u^2}\dfrac{\partial^2\vec{E}}{\partial t^2}$.

Equate both sides of (*) to obtain the wave equation for E

$$\boxed{\nabla^2\vec{E}-\frac{1}{u^2}\frac{\partial^2\vec{E}}{\partial t^2}=0,}\quad \text{where}\quad \boxed{u^2=\frac{1}{\mu\varepsilon}}$$

Source-free Wave Equation for Magnetic Field

In a simple medium, the source-free Maxwell's equations are

$$\underline{\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}}, \quad \nabla \cdot \vec{E} = 0, \quad \underline{\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}}, \quad \underline{\nabla \cdot \vec{H} = 0}$$

Use the 3rd expression to write $\underline{\nabla \times \nabla \times \vec{H}} = \varepsilon \underline{\frac{\partial \nabla \times \vec{E}}{\partial t}}$ (*)

Left hand side becomes $\underline{\nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}} = -\nabla^2 \vec{H}$ for $\underline{\nabla \cdot \vec{H} = 0}$.

Adopt $\underline{\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}}$ to the right of (*) to write $-\varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2}$.

Equate both sides of (*) to obtain the wave equation for H

$$\boxed{\nabla^2 \vec{H} - \frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0,} \quad \text{where} \quad \boxed{u^2 = \frac{1}{\mu\varepsilon}}$$

Propagating Electromagnetic Fields

The wave equation for the electric field intensity

$$\boxed{\nabla^2 \vec{E} - \frac{1}{u^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

The wave equation for the magnetic field intensity

$$\boxed{\nabla^2 \vec{H} - \frac{1}{u^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0}$$

The general solutions are

$$\vec{E}(t,R) = \vec{E}_0 \times f(t \pm R/u), \quad \text{and } \vec{H}(t,R) = \vec{H}_0 \times f(t \pm R/u)$$

Both are propagating waves with a speed of $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$

18.3 電磁場的波動方程式

Wave Equations of Electromagnetic Fields

- A time-varying electric field propagates in space according to

$$\nabla^2 \vec{E} - \frac{1}{u^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0.$$

- A time-varying magnetic field propagates in space according to

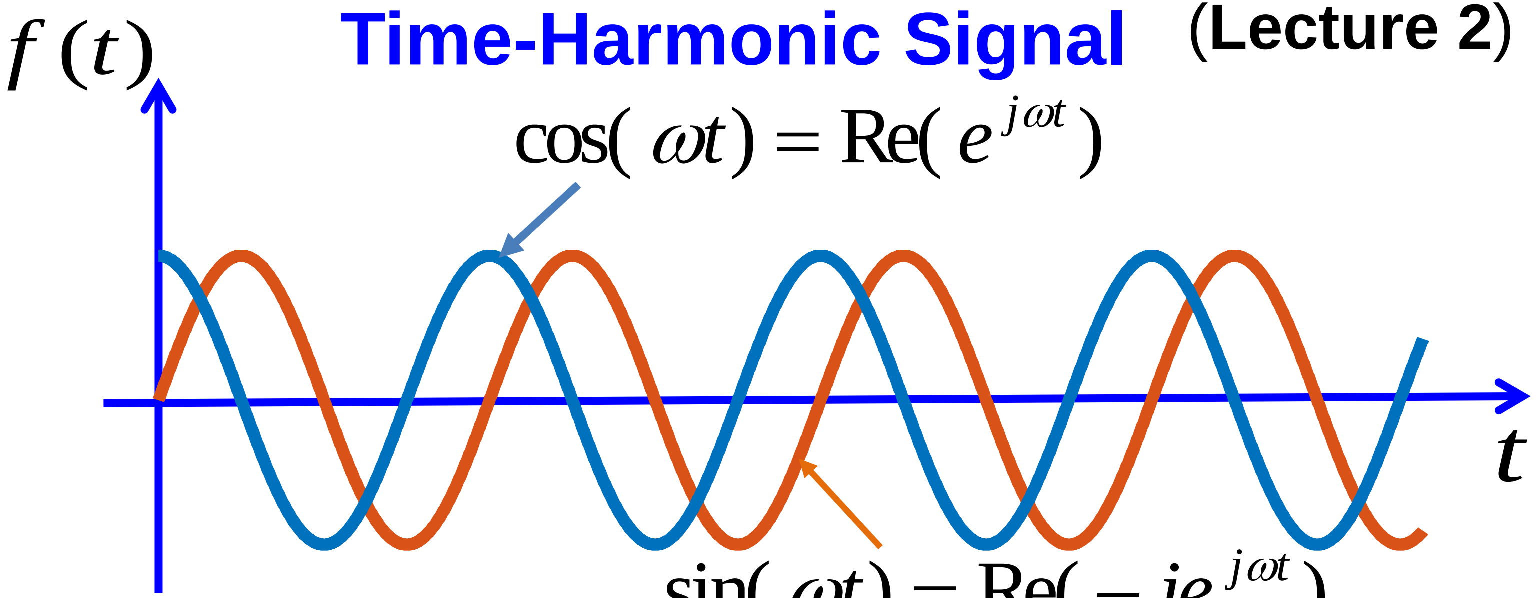
$$\nabla^2 \vec{H} - \frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0.$$

- The propagation speed of the fields is $u = 1/\sqrt{\mu\varepsilon},$ which has value of 3×10^8 m/s in vacuum.

電磁波動現象
Wave Dynamics of Time-varying Fields

18.4 電磁弦波
Time-harmonic Electromagnetic Wave

Time-Harmonic Signal (Lecture 2)



$\omega = 2\pi f$ is the angular frequency with f being the frequency in Hertz.

For a general time-harmonic signal

$$\tilde{A} = \mathrm{Re}(A_0 e^{j\psi + j\omega t}) = A_0 \cos(\omega t + \psi),$$

The phasor of A is $\hat{A} = A_0 e^{j\psi}$, so that $\tilde{A} = \mathrm{Re}(\boxed{\hat{A}} e^{j\omega t})$

Potentials with Harmonic Source

Assume harmonic sources and adopt the phasor notations

$$\rho(t') = \mathrm{Re}(\hat{\rho} e^{j\omega t'}) \quad \vec{J}(t') = \mathrm{Re}(\hat{\vec{J}} e^{j\omega t'})$$

Accordingly, all the field variables are varying in time harmonics.

$$V(R,t) = \mathrm{Re}(\hat{V}(R) e^{j\omega t}) \quad \vec{A}(R,t) = \mathrm{Re}(\hat{\vec{A}}(R) e^{j\omega t})$$

Substitute $\rho(t')$ and $J(t')$ into $\vec{A}(R,t), V(R,t)$ with $t' = t - R\sqrt{\mu\varepsilon}$,

one obtains the phasors of the potential fields

$$\hat{\vec{A}}(R) = \frac{\mu}{4\pi} \int_{V'} \frac{\hat{\vec{J}} e^{-jkR}}{R} dv' \qquad \hat{V}(R) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\hat{\rho} e^{-jkR}}{R} dv'$$

where $k = \omega\sqrt{\mu\varepsilon} = \omega / u = 2\pi/\lambda$ is called the wave number with λ being the wavelength of the *radiation* wave.

Helmholtz's Equations

Assume time-harmonic fields $\vec{E},\vec{H}(R,t)=\mathrm{Re}(\vec{\hat{E}},\vec{\hat{H}}(R)e^{j\omega t})$,
and apply the phasor notation to the wave equations:

Electric field **(Lecture 2)**

$$\nabla^2\vec{E}-\frac{1}{u^2}\frac{\partial^2\vec{E}}{\partial t^2}=0\Rightarrow\nabla^2\vec{\hat{E}}-\frac{1}{u^2}(j\omega)^2\vec{\hat{E}}=0 \Rightarrow \boxed{\nabla^2\vec{\hat{E}}+k^2\vec{\hat{E}}=0,}$$

where $k=\omega\sqrt{\mu\varepsilon}=\omega/u=2\pi/\lambda$ is the wave number.

Magnetic field

$$\nabla^2\vec{H}-\frac{1}{u^2}\frac{\partial^2\vec{H}}{\partial t^2}=0\Rightarrow\nabla^2\vec{\hat{H}}-\frac{1}{u^2}(j\omega)^2\vec{\hat{H}}=0 \Rightarrow \boxed{\nabla^2\vec{\hat{H}}+k^2\vec{\hat{H}}=0}$$

Equations of the form $\boxed{\nabla^2\hat{\Psi}+k^2\hat{\Psi}=0}$ are called the Helmholtz's equation.

Solutions to Helmholtz's Equations

One possible solution to

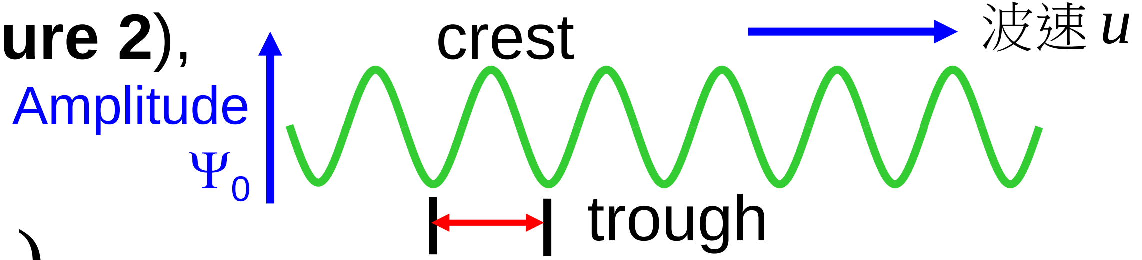
$$\nabla^2\Psi + k^2\Psi = 0$$ is

$$\hat{\Psi} = \hat{E}, \hat{H} = \Psi_0 e^{-jkz + j\phi_0}$$

In the real domain (**Lecture 2**),

$$\Psi(t, z) = \mathrm{Re}[\hat{\Psi} e^{j\omega t}]$$

$$= \Psi_0 \cos(\omega t - kz + \phi_0)$$



Spatial period (Wavelength) λ

Temporal period $T = 1/\upsilon$

* **A wavefront is a constant phase plane.** Set

$$\omega t - kz + \phi_0 = \text{constant}$$

to obtain the phase velocity

$$dz/dt = \frac{\omega}{k} = u = \frac{1}{\sqrt{\mu\varepsilon}}$$

18.4 電磁弦波

Time-harmonic Electromagnetic Wave

- A time-harmonic signal is a sinusoidal signal with an angular frequency ω, described by

$$\tilde{A} = \mathrm{Re}(A_0 e^{j\psi + j\omega t}) = \mathrm{Re}(\hat{A} e^{j\omega t}) = A_0 \cos(\omega t + \psi),$$

where $\hat{A} = A_0 e^{j\psi}$ is the phasor of the signal.

- In phasor notations, a time-varying harmonic field satisfies the Helmholtz's equation $\nabla^2 \hat{\Psi} + k^2 \hat{\Psi} = 0$, which is often the starting equation to solve a sinusoidal electromagnetic wave.

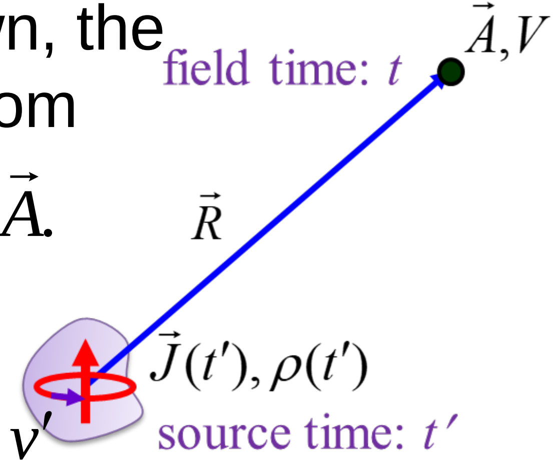
電磁波動現象
Wave Dynamics of Time-varying Fields

18.5 單元回顧 Review

單元回顧

1. For time-varying fields, if V and A are known, the electric and magnetic fields can be derived from

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}.$$



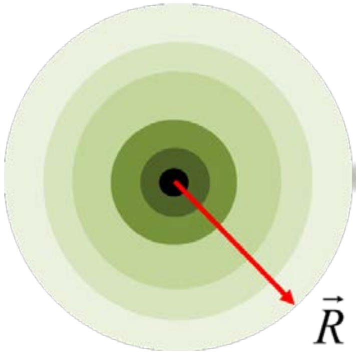
2. Upon excitation by a source, the potential functions at R and t are governed by

$$\vec{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(t-\sqrt{\mu\varepsilon}R)}{R}dv', \quad V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t-\sqrt{\mu\varepsilon}R)}{R}dv'$$

The potentials are retarded in time by an amount equal to the propagation time or $t = t' + R/u$, where $u = 1/\sqrt{\mu\varepsilon}$ is the speed of the propagation.

單元回顧

3. $\nabla^2\Psi - \frac{1}{u^2}\frac{d^2\Psi}{dt^2} = 0$ is called the wave equation, which has a general solution of the form $\Psi(t,R) \propto f(t \pm R/u)$ describing a wave propagating at a speed u.

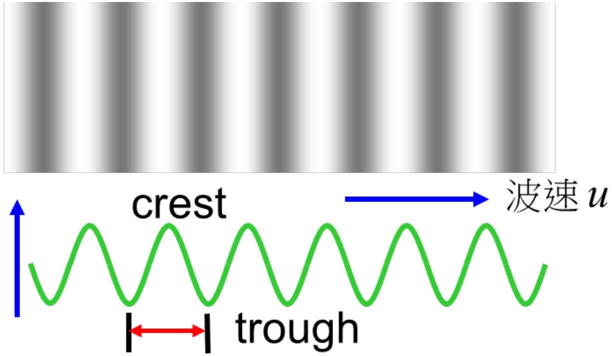


4. Both time-varying electric and magnetic fields satisfy the wave equation, propagating as a wave in space at a speed of $u = 1/\sqrt{\mu\varepsilon}$.

單元回顧

5. The phasors of the time-harmonic electromagnetic fields,

$\hat{\vec{E}}, \hat{\vec{H}}$ of $\vec{E}, \vec{H}(R,t) = \mathrm{Re}(\hat{\vec{E}}, \hat{\vec{H}}(R)e^{j\omega t}),$



satisfy the Helmholtz's equation

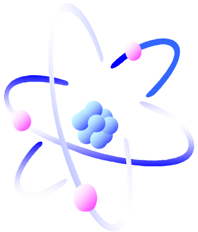
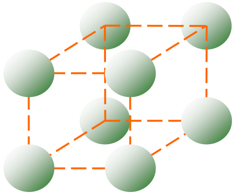
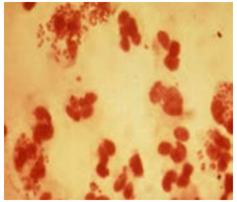
$$\nabla^2 \hat{\vec{E}} + k^2 \hat{\vec{E}} = 0, \text{ and } \nabla^2 \hat{\vec{H}} + k^2 \hat{\vec{H}} = 0$$

where $k = \omega\sqrt{\mu\varepsilon} = \omega / u = 2\pi/\lambda$ is called the wave number

with λ being the wavelength of the *electromagnetic* wave.

單元回顧

Electromagnetic Spectrum

原子	晶格	病毒	細菌	網球	房子
					
$<10^{-10}$ m	$\sim 10^{-9}$ m	$10^{-7}\sim 10^{-6}$ m	$10^{-6}\sim 10^{-3}$ m	$10^{-3}\sim 1$ m	> 1 m
γ-射線	x-光	可見光	紅外線	微波	無線電波

不同電磁波長的電磁波有不同的名稱

References

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[2] Fawwaz T. Ulaby, Fundamentals of Applied Electromagnetics 6th Ed., PEARSON Prentice Hall, 2007.

[3] Ramo, Whinnery, and van Duzer, Fields and Waves in Communication Electronics, 2nd Ed., John Wiley & Sons, 1984.

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