EE 203001 Session 15

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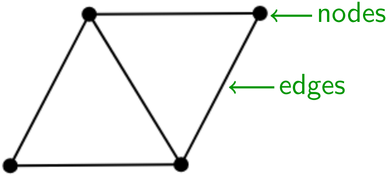
Small world graphs

$$\mathbf{G} = \{\text{nodes, edges}\}$$
$$= \text{collection of nodes joined by edges}$$



Social Network

Each node is a person, and two nodes are connected by edge if they are friends

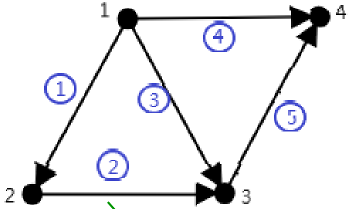
Q: What is the farthest distance between two people in the graph ?

Six degree of separation
$\Rightarrow$ It's a small world!

Other Example

www : nodes are websites
edges are links

Electrical Network



n $= 4$ nodes
m $= 5$ edges

(direction of currents) (Directed graph)

Incident Matrix

one col. for each node, and one row for each edge

If edge runs from node $1 \to$ node 2

$(-1$ in col $1)$ $(+1$ in col $2)$

$$\begin{array}{cc} A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} 1 \\ 2 \\ 3 \quad \text{edge} \\ 4 \\ 5 \end{array} \\ \text{Node} \quad 1 \qquad 2 \qquad 3 \qquad 4 & \end{array}$$

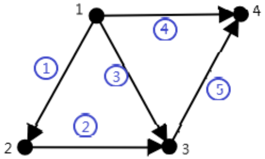
(Incident matrix A if large is sparse in general

$\Rightarrow$ most entries are zero)

(Each row only has two non-zero entries)

Ch 8.2 Graphs and Networks

Loops :



n $= 4$ nodes

m $= 5$ edges

$$A = \begin{array}{cc} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \\ \begin{array}{cccc} \text{Node } 1 & 2 & 3 & 4 \end{array} & \end{array}$$

linearly dependent

(row $3 =$ row $2 +$ row 1)

Null space of A

$\mathbf{x} = (x_1, x_2, x_3, x_4)$: potentials at nodes

$A\mathbf{x} = \mathbf{0}$

$$\Rightarrow A\mathbf{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ \underbrace{x_4 - x_3}_{} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(difference of potentials)

$\Rightarrow$ dim$(\mathbf{N}(A)) = 1$ with basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(Nothing will move if all potentials are the same)

(or potential difference $= 0$)

(But potentials can only be determined up to a constant)

(If we ground node 4, $x_4 = 0 \Rightarrow x_1 = x_2 = x_3 = 0$)

Q: What is rank(A) ?

$$\text{rank}(A) + \dim(\mathbf{N}(A)) = n = 4$$
$$\Rightarrow \text{rank}(A) = 4 - 1 = 3$$

(We can also see this via Elimination)

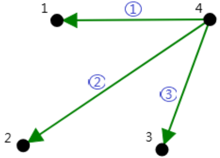
$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$ basis for $\mathbf{C}(A)$

$\uparrow \quad \uparrow \quad \uparrow$ pivot col.

(Top 3 rows of R are independent
$\Rightarrow$ the graph it forms has no loops
$\Rightarrow$ It's a tree !)

(This is a tree
with no loops)



Left nullspace $\mathbf{N}(A^\intercal)$

$$A^\intercal \mathbf{y} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim(\mathbf{N}(A^\intercal)) = m - r = 5 - 3 = 2$

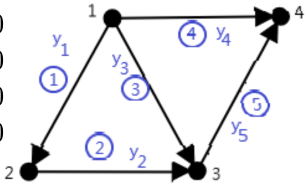
($\mathbf{y} = (y_1, y_2, y_3, y_4, y_5)$ are currents & $A^\intercal \mathbf{y} = \mathbf{0}$ is Kirchhoff's current law)

(will come back to this later)

Left nullspace $\mathbf{N}(A^{\mathsf{T}})$ (cont.)

Basis :

$$\begin{array}{lrrrrrl}
\text{node 1 :} & -y_1 & & -y_3 & -y_4 & & = 0 \\
\text{node 2 :} & y_1 & -y_2 & & & & = 0 \\
\text{node 3 :} & & y_2 & +y_3 & & -y_5 & = 0 \\
\text{node 4 :} & & & & y_4 & +y_5 & = 0
\end{array}$$

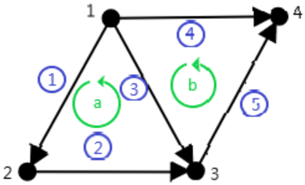


(current in $=$ current out)

($-y_1$: current out, $+y_1$: current in)

Left nullspace $\mathbf{N}(A^\intercal)$ (cont.)

Basic by inspection :



current in loops

$$\text{loop : } \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

(loop a) (loop b)

(Dim($\mathbf{N}(A^\intercal)$) $= 2$ so only need these two vectors for a basis)
(Outer loop also gives a special solution $(1, 1, 0, -1, 1)$)

Row space $\mathbf{C}(A^\intercal)$

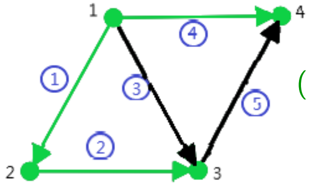
not a pivot col. since node 1, 2, 3 forms a loop

$$A^\intercal = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow \qquad \quad \uparrow$

pivot col.s

$\text{rank}(A) = 3 \Rightarrow \dim(\mathbf{C}(A^\intercal)) = 3$



(linear independent edges 1, 2, 4 form a tree)

Ch 8.2 Graphs and Networks

Complete picture

$$\begin{array}{ccc}
\begin{array}{c}\mathbf{x} = (x_1, x_2, x_3, x_4) \\ \text{potentials at nodes}\end{array} & & \begin{array}{c}A^\intercal \mathbf{y} = \mathbf{0} \\ \text{Kirchott's current law}\end{array} \\
\Big\downarrow \; A\mathbf{x} = \mathbf{e} & & \Big\uparrow \; A^\intercal \mathbf{y} \\
\begin{array}{c}x_2 - x_1, \text{ etc.} \\ \text{potential differences}\end{array} & \xrightarrow[\text{Ohm's law}]{\mathbf{y} = -C\mathbf{e}} & \begin{array}{c}y_1, y_2, y_3, y_4, y_5 \\ \text{currents on edges}\end{array}
\end{array}$$

(conductance matrix)

Euler's Formula

$\dim(\mathbf{N}(A^{\intercal})) = m - r$

\# loops $=$ \# edges - (\# nodes - 1)

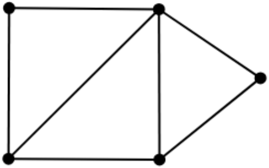
(rank $= n-1$) (dim$(\mathbf{N}(A^{\intercal}))$ always $= 1$)

$\Rightarrow$ \# nodes - \# edges $+$ \# loops $= 1$

(small loops)

(True for any connected graph)

Ex:



$5 - 7 + 3 = 1$

One more thing

Still need a outside source to drive circuit

Current source $\mathbf{f}$

$\mathbf{x} = (x_1, x_2, x_3, x_4)$
potentials at nodes

$\downarrow$ $A\mathbf{x} = \mathbf{e}$

$x_2 - x_1$, etc.
potential differences

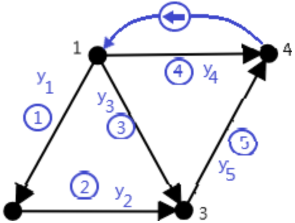
(conductance matrix)

$\xrightarrow[\text{Ohm's law}]{\mathbf{y} = -C\mathbf{e}}$

y_1, y_2, y_3, y_4, y_5
currents on edges

$\uparrow$ $A^\intercal \mathbf{y}$

$A^\intercal \mathbf{y} = \mathbf{f}$
Kirchhoff's current law



(outside current source $\mathbf{f} = \begin{bmatrix} -s \\ 0 \\ 0 \\ 0 \end{bmatrix}$)

Ch 8.2 Graphs and Networks

Combing all 3 equations

$$\underbrace{A^{\mathsf{T}} C A}_{\text{symmetric matrix}}\mathbf{x} = -\mathbf{f}$$

Ex1: in textbook (p. 427)
All conductances are $c = 1$ so $C = I$

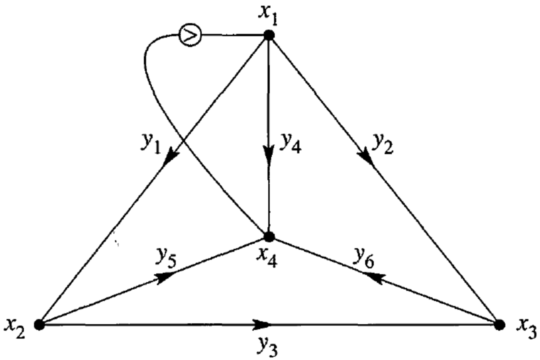


Figure 8.5: The currents in a network with a source S into node 1.

Ex1: (cont.)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^\intercal C A = A^\intercal A$$

$$= \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Ex1: (cont.)

Ground node 4 $\Rightarrow x_4 = 0$

$\Rightarrow$ remove col. 4 and row 4 from $A^{\mathsf{T}}CA$

Solve

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ \frac{s}{4} \\ \frac{s}{4} \end{bmatrix}$$

Ex1: (cont.)

By Ohm's law $\mathbf{y} = -CA\mathbf{x}$ $(x_4 = 0)$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{s}{2} \\ \frac{s}{4} \\ \frac{s}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s}{4} \\ \frac{s}{4} \\ 0 \\ \frac{s}{2} \\ \frac{s}{4} \\ \frac{s}{4} \end{bmatrix}$$