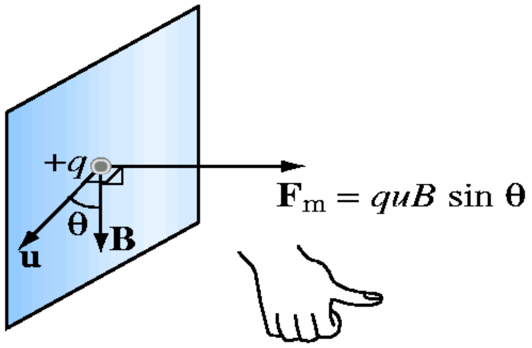
Chapter 6 Static Magnetic Fields

A magnetic flux in Webers is $\Phi = \int_S \vec{B} \cdot d\vec{s}$, where B is the *magnetic flux density* in Tesla or Weber/m^2.

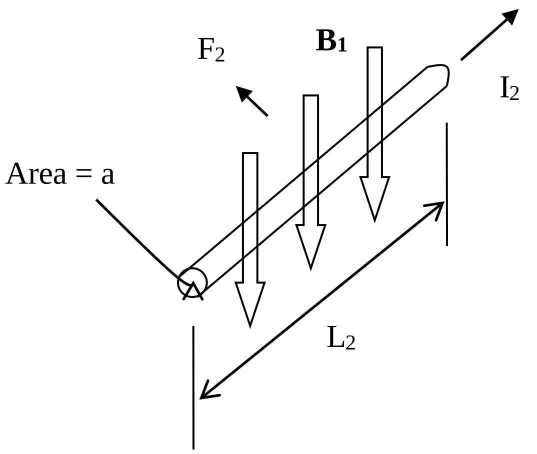
Recall the Lorentz's Force Equation

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}), \qquad (6\text{-}1)$$

In the E-field free region, a moving charge only feels the magnetic force $\vec{F} = q\vec{u} \times \vec{B}$. The sense of direction follows the right-hand rule.



Eg. Magnetic force on a DC electric wire in a uniform magnetic field.



Assume a conducting wire of length L_2 and cross section area of a

carrying a current I_2 in a region of a magnetic field $\vec{B}_1$. For N_{total} charged particles uniformly distributed over the wire, the total magnetic force on all charges is

$$\vec{F}_2 = N_{total} q\vec{u}_2 \times \vec{B}_1 = NaL_2 q\vec{u}_2 \times \vec{B}_1,$$

Where N is the number of charges per unit volume, and $\vec{u}_2$ is the average velocity of the charges.

But the current density and current are governed by

$Nq\vec{u}_2 a = \vec{J}_2 a = \vec{I}_2$. The magnetic force on the wire of length L_2 is therefore

$$\Rightarrow \vec{F}_2 = Nq\vec{u}_2 \times \vec{B}_1 = L_2 \vec{I}_2 \times \vec{B}_1$$

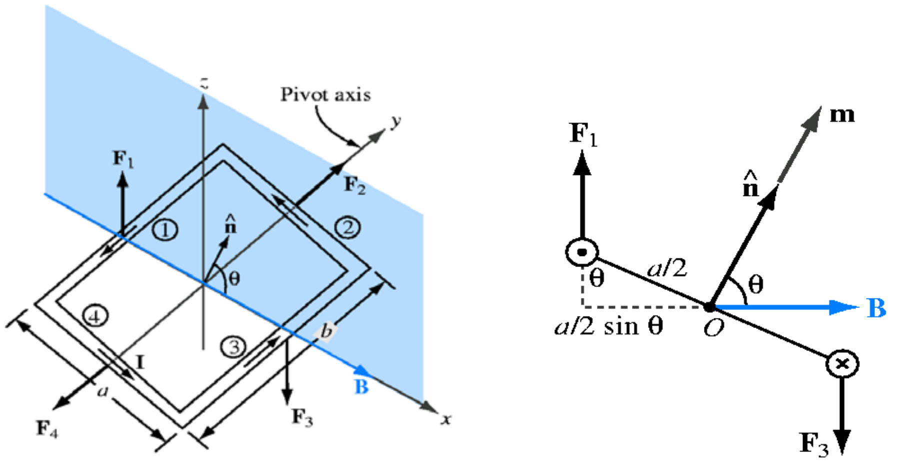
In terms of force per unit length, the expression is $\dfrac{\vec{F}_2}{L_2} = \vec{I}_2 \times \vec{B}_1$.

In our convention, I is not a vector but length is. Therefore we usually write $\vec{F}_2 = I_2 \vec{L}_2 \times \vec{B}_1$, where L_2 points to the direction of the current I_2.

Usually a current flows in a circuit loop. A force on a loop pivoted along an axis generates a torque. If a pivoted current loop is placed in a static magnetic field, as shown below, the current loop rotates due to the magnetic torque.

Refer to the following figure. Since B is along x and the pivoted axis is y, $F_{2,4}$ produce no torque to the current loop. Apply $\vec{F} = I\vec{L} \times \vec{B}$ to obtain $\vec{F}_1 = IbB\hat{a}_z$ and $\vec{F}_3 = IbB(-\hat{a}_z)$. A torque is defined through the expression

$\vec{T} = \vec{r} \times \vec{F}$,



where $\vec{r}$ is the radial vector from the pivoting axis to the point of the applied force $\vec{F}$. Therefore the total torque on the current loop is given by

$$\vec{T} = 2 \times \frac{a}{2} IbB\sin\theta\hat{a}_y = ISB\sin\theta\hat{a}_y ,$$

where $ab = S$ is the area of the current loop. Define the magnetic moment as

$$\vec{m} = I_{total}\vec{S} ,$$

where, if there are N current loops, I_{total} can be NI with each current loop carrying a current I. The direction of the area follows the right hand rule with the four fingers of your right hand point toward the direction of the current. The expression of the torque reduces to

$$\vec{T} = \vec{m} \times \vec{B} .$$

This expression, although obtained from a rectangular current loop, is applicable to current loops of any shape.

Fundamental Postulates for Magnetostatics

In vacuum

$$\nabla \cdot \vec{B} = 0 \tag{6-2}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{6-3}$$

$\mu_0 = 4\pi \times 10^{-7}$ Henry/m: vacuum permeability.

In $\nabla \times \vec{B} = \mu_0 \vec{J}$ the direction obeys the right-hand rule, according to the definition of a curl operator.

Apply the *divergence theorem* to Eq. (6-2) to obtain

$$\int_V \nabla \cdot \vec{B} dv = \oint_S \vec{B} \cdot d\vec{s} = 0 \tag{6-4}$$

$\Rightarrow$ *"no magnetic flow sources; magnetic flux lines always close upon themselves."* (D.K. Cheng)

$\Rightarrow$ *law of conservation of magnetic flux*

As a comparison, the following left figure shows the net outward flux of an electric field is proportional to the charge enclosed; whereas, the net magnetic flux of a magnet pole is always zero, as shown by the right figure below.

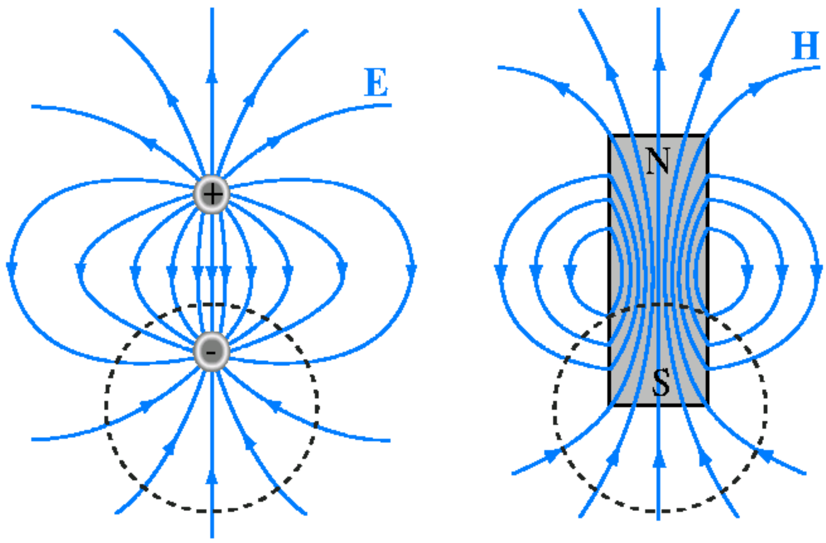
Apply Stoke's theorem to Eq. (6-3) to obtain

$$\int_s \nabla \times \vec{B} \cdot d\vec{s} = \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_s \vec{J} \cdot d\vec{s} = \mu_0 I$$

This is the well known Ampere's circuital law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \,, \tag{6-5}$$

which can be interpreted as “the circulation of magnetic fields is proportional to the current bounded by the circular path”. Like Gauss’s law, Ampere’s circuital law is particularly useful for solving a problem in which the magnetic field along a path has a constant magnitude. Summary of (6-2~5) gives rise to Maxwell’s Equations for Static Magnetic Field.



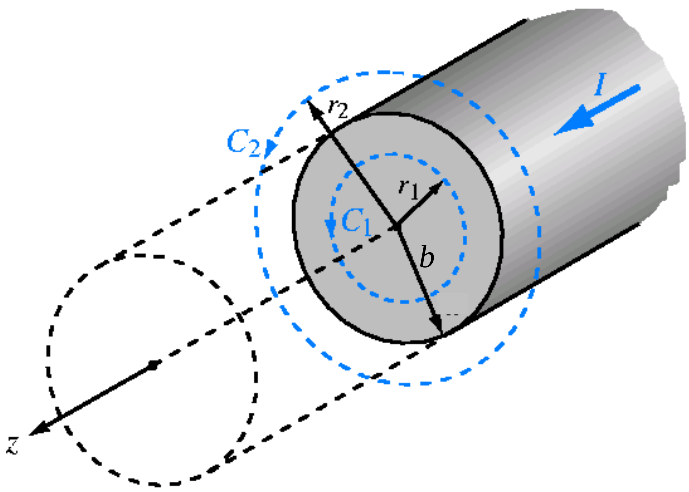
(a) Electric dipole (b) Bar magnet

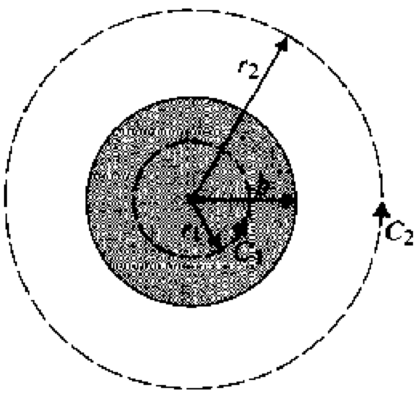
Maxwell’s Equations for Static Magnetic Field

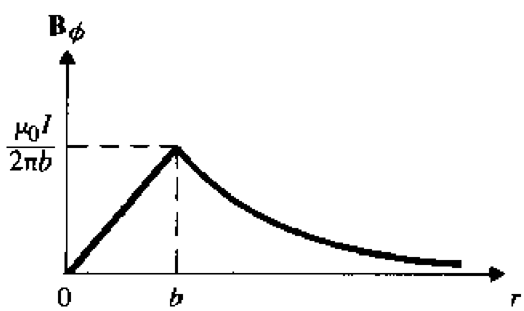
Differential form	Integral form
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$
$\nabla \times \vec{B} = \mu_0 \vec{J}$	$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$

Note that the expression $\nabla \times \vec{B} = \mu_0 \vec{J}$ has defined the relative directions between $\vec{B}$ and $\vec{J}$. Recall the definition of an curl operator $\nabla \times \vec{B} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{B} \cdot d\vec{l}}{\Delta s}$, where the direction of $\hat{a}_n$ or that of $\vec{J}$ follows the *right-hand* rule.

Eg. Magnetic flux density in a long wire of radius *b*, assuming a uniform current density $J = \frac{I}{\pi b^2}$.







This problem has a circular symmetry and only ϕ component magnetic field exists. Apply Ampere's circuital law (6-5), $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$, where C is a circular path of a constant radius.

i. In the region $r \le b$, the current enclosed by C is

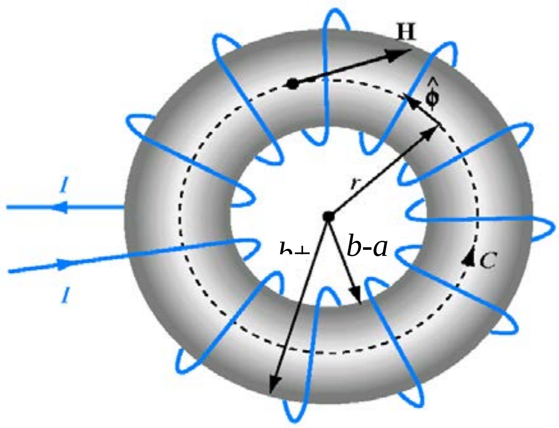
$$I(r) = J\pi r^2 = I\frac{r^2}{b^2}$$

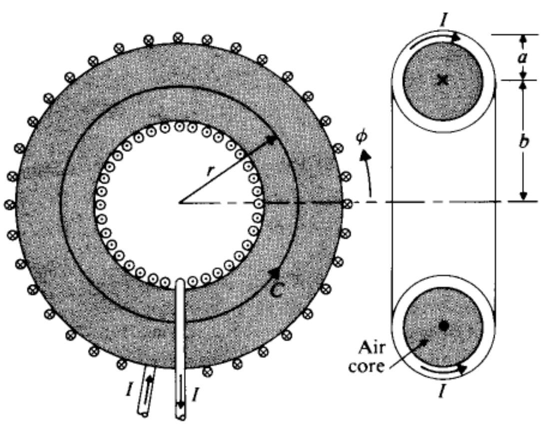
Thus $B_\phi 2\pi r = \mu_0 I(r) = \mu_0 I\frac{r^2}{b^2} \Rightarrow B_\phi = \mu_0 I\frac{r}{2\pi b^2}$

ii. In the region $r \ge b$,

$$B_\phi 2\pi r = \mu_0 I \quad \Rightarrow B_\phi = \mu_0 I\frac{1}{2\pi r}$$

Eg. Magnetic flux density in a toroid





This problem also has circular symmetry. The magnetic field at a constant *r* has a constant value along ϕ. From Ampere's law, $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 IN$, where *I* is the current in individual wires and *N* is the total number of wires around the toroid.

i. In the region $(b-a) \le r \le (b+a)$

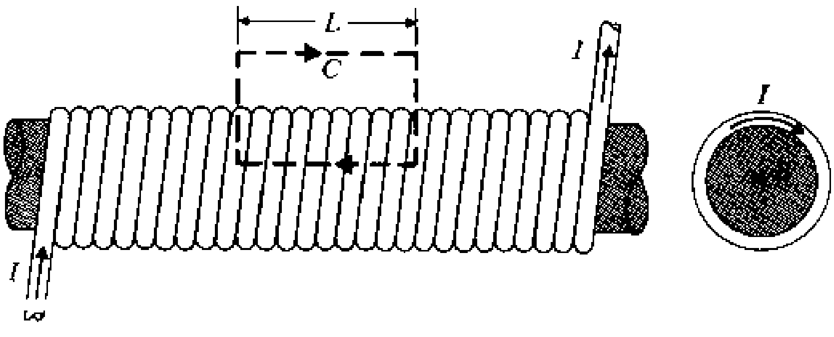
From symmetry, only B_ϕ exists and from Ampere's law $\Rightarrow$

$$B_\phi 2\pi r = \mu_0 NI \;\Rightarrow\; B_\phi = \frac{\mu_0 NI}{2\pi r}$$

ii. In the region $r < b-a$ and $r > a+b$,

$$\oint_C \vec{B}\cdot d\vec{l} = 0 \;\Rightarrow\; \vec{B} = 0$$

Eg. Magnetic flux density in a long solenoid of length *L*



The symmetry of the problem indicates a magnetic field in the longitudinal direction. If one constructs a rectangular path surrounding the outside of the solenoid, it is clear that there's no net current enclosed in the loop and no field outside the solenoid. However for the path shown above Ampere's law gives

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 IN$$

$\Rightarrow B_z L = \mu_0 nLI$, where n is the # of wires per unit length on the solenoid.

$\Rightarrow \ B_z = \mu_0 nI$

Note that this result ignores fringe fields at both ends of the solenoid. Therefore the calculated field is only valid near the center of a long solenoid. The same answer can be obtained by taking the limit $\lim_{r\to\infty} \frac{N}{2\pi r} = n$ in the last example, because when $r \to \infty$ a segment of a circle approaches a straight line.

Vector Magnetic Potential

From the vector identity

$\nabla \cdot \nabla \times \vec{A} = 0$, and (6-2) $\nabla \cdot \vec{B} = 0$. Define the *magnetic vector potential* $\vec{A}$ that satisfies

$$\vec{B} = \nabla \times \vec{A} \tag{6-6}$$

Substituting (6-6) into (6-3), one has

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} \tag{6-7}$$

or $\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$. (6-8)

Since $\vec{A}$ is a defined quantity, we still have the freedom to choose the divergence of it. Remember that Helmholtz's theorem requires $\nabla \times \vec{A}$, $\nabla \cdot \vec{A}$, and the boundary condition of A to be simultaneously specified to obtain a unique solution for A. Set

$$\nabla \cdot \vec{A} = 0 \tag{6-9}$$

(7-8) becomes $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ (*vector Poisson's equation*)

In the Cartesian coordinate system, the vector Laplacian operator results in

$$\nabla^2 A_x = -\mu_0 J_x, \ \nabla^2 A_y = -\mu_0 J_y, \ \nabla^2 A_z = -\mu_0 J_z \tag{6-10}$$

Recall $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ and the solution of it is $V = \dfrac{1}{4\pi\varepsilon_0}\int_{V'} \dfrac{\rho}{R} dv'$ in electrostatics. Similarly, the solution of $\vec{A}$ is

$$A_{x,y,z} = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_{x,y,z}}{R} dv' \qquad \text{(Weber/m)}$$

or in vector form

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv' \tag{6-11}$$

The direction of a vector potential is along the same direction of current. Therefore the definition of the magnetic vector potential provides a convenient formula for calculating a magnetic field from a known current distribution.

The magnetic flux is $\Phi = \int_S \vec{B} \cdot d\vec{s}$. But from $\vec{B} = \nabla \times \vec{A}$, one can express a magnetic flux in terms of a vector potential

$$\Phi = \int_S \nabla \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l} \tag{6-12}$$

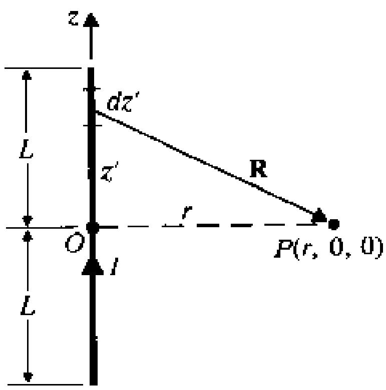
This equation means "*the circulation of magnetic vector potential equals* *the magnetic flux passing through the surface in the circulation path.”* This interpretation can be considered the physical meaning of a vector potential.

For a thin wire carrying a current $I = \vec{J} \cdot \vec{S}$, recast (6-11) into

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv' = \frac{\mu_0}{4\pi}\oint_{C'} \frac{\vec{J} \cdot \vec{S}}{R} d\vec{l}\,' = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R} \quad \text{(6-13).}$$

This expression can be convenient when solving a magnetic field from known wire geometry.

Eg. Find $\vec{B}$ at *P* for a current wire of length 2*L*



The differential length along current flow is $d\vec{l}\,' = \hat{a}_z dz'$

From geometry, $R = \sqrt{r^2 + z'^2}$. Substitute them into Eq. (6-13) to calculate the vector potential

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R} = \hat{a}_z \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{dz'}{\sqrt{r^2 + z'^2}}$$

$$= \hat{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{r^2 + L^2} + L}{\sqrt{r^2 + L^2} - L}$$

With a known vector potential, calculate the magnetic field from

$\vec{B} = \nabla \times \vec{A}$, yielding

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\hat{a}_z A_z\right) = \hat{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \hat{a}_\phi \frac{\partial A_z}{\partial r}$$

$$= -\hat{a}_\phi \frac{\partial A_z}{\partial r} = \frac{\mu_0 IL}{2\pi r\sqrt{L^2 + r^2}}\hat{a}_\phi$$

The Biot-Savart Law

From (6-13) and $\vec{B} = \nabla \times \vec{A}$,

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \nabla \times \left(\frac{d\vec{l}\,'}{R}\right) \tag{6-14}$$

From the vector identity

$$\nabla \times (f\vec{G}) = f\nabla \times \vec{G} + (\nabla f) \times \vec{G} \quad \text{and} \quad \nabla \times d\vec{l}\,' = 0$$

Eq. (14) $\Rightarrow$ $\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \nabla \times \left(\frac{d\vec{l}\,'}{R}\right) = \frac{\mu_0 I}{4\pi}\oint_{C'} \left(\nabla \frac{1}{R}\right) \times d\vec{l}\,'$

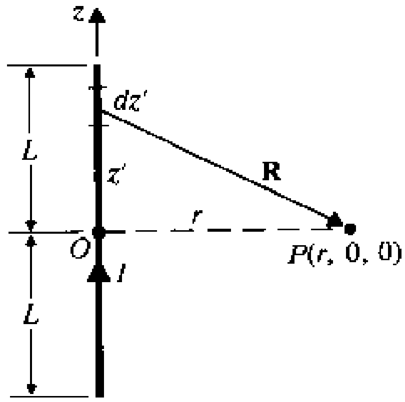
With $\nabla\left(\frac{1}{R}\right) = -\hat{a}_R \frac{1}{R^2}$ $\Rightarrow$ $\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,' \times \hat{a}_R}{R^2}$

The expression

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3} \tag{6-15}$$

is called the *Biot-Savart Law*. Previously we have to calculate the vector potential for a problem with a known current distribution, and use $\vec{B} = \nabla \times \vec{A}$ to obtain a magnetic field. Now, one can derive a magnetic field directly from a current distribution by using the *Biot-Savart Law*, although this approach involves a more complicated procedure to figure out the position vectors. .

Eg. Find $\vec{B}$ at *P* for the following current distribution using the Biot-Savart law, $\vec{B}=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\vec{l}\,'\times\vec{R}}{R^3}$



Again, the differential length along current is $d\vec{l}\,'=\hat{a}_z dz'$. The *R* vector is given by $\vec{R}=\hat{a}_r r-\hat{a}_z z'$.

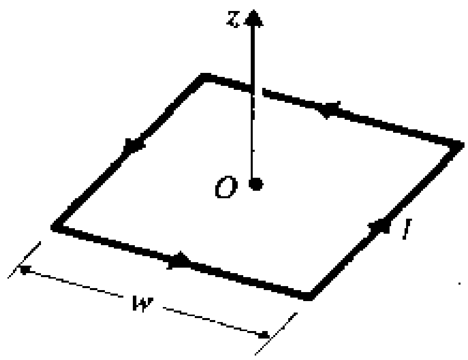
$\Rightarrow$ $d\vec{l}\,'\times\vec{R}=\hat{a}_\phi r dz'$. From geometry, $R=\sqrt{r^2+z'^2}$.

Substituting the above into the Biot-Savart law to obtain

$$\vec{B}=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\vec{l}\,'\times\vec{R}}{R^3}=\hat{a}_\phi\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{r dz'}{(z'^2+r^2)^{3/2}}$$

$$=\hat{a}_\phi\frac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}$$

Eg. Find $\vec{B}$ at *O* for the following current loop.

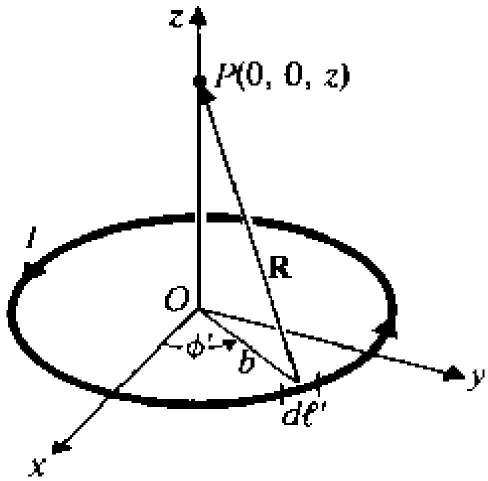


Use the result from the last example and the principle of superposition.

$$\vec{B} = \hat{a}_z 4 \times \left. \frac{\mu_0 IL}{2\pi r\sqrt{L^2 + r^2}} \right|_{L=\frac{w}{2}, r=\frac{w}{2}}$$

Eg. Find $\vec{B}$ at P for the current loop by using Biot-Savart law.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3}.$$



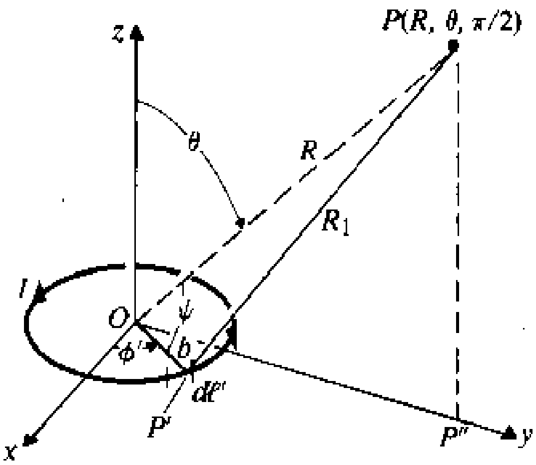
The differential length of the current is $d\vec{l}\,' = \hat{a}_\phi b d\phi'$. The R vector with a magnitude of $R = \sqrt{b^2 + z^2}$ in the cylindrical coordinate system is $\vec{R} = \hat{a}_z z - \hat{a}_r b$.

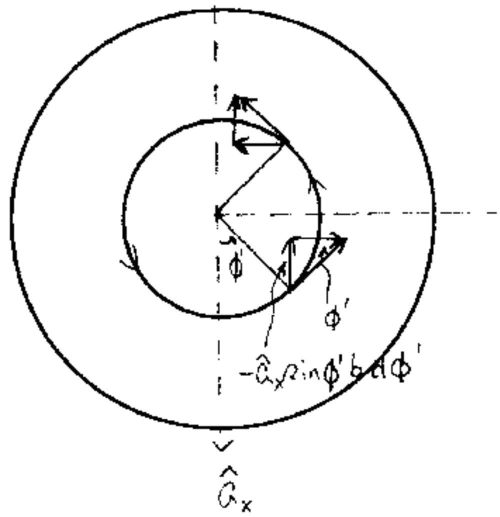
$\Rightarrow$ $d\vec{l}\,' \times \vec{R} = \hat{a}_r bzd\phi' + \hat{a}_z b^2 d\phi'$, in which the first term is ineffective in the integration from symmetry. Substituting the second term into the Biot-Savart law to obtain

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,' \times \vec{R}}{R^3} = \hat{a}_z \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{b^2}{(z^2+b^2)^{3/2}} d\phi' = \hat{a}_z \frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$$

A Magnetic Dipole: a small current loop.

(A magnetic dipole is the basic magnetic element in a magnetic material.)





Due to symmetry, the vector potential only has a component in the φ direction, expressed by $\vec{A} = A_\varphi \hat{a}_\varphi$.

Recall $\vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\vec{l}\,'}{R_1}$ (6-16)

The differential vector potential is given by

$$dA_\varphi = \frac{\mu_0 I}{4\pi} \times \frac{-b \sin\varphi' d\varphi'}{R_1}, \tag{6-17}$$

where $R_1^2 = R^2 + b^2 - 2bR\cos\psi = R^2 + b^2 - 2bR\sin\theta\sin\varphi'$

$$\frac{1}{R_1} \approx \frac{1}{R}\left(1 + \frac{b}{R}\sin\theta\sin\varphi'\right) \tag{6-18}$$

for $R >> b$. Substitute (6-18) into (6-16, 17) and integrate it over the whole current loop (φ over 2π) to obtain

$$A_\varphi = \frac{\mu_0 I \pi b^2}{4\pi R^2}\sin\theta \tag{6-19}$$

Define the *magnetic dipole moment* $\vec{m} \equiv I\pi b^2 \hat{a}_z$, where the directions of the current and the magnetic moment follow the right hand rule. Rewrite (6-19) as

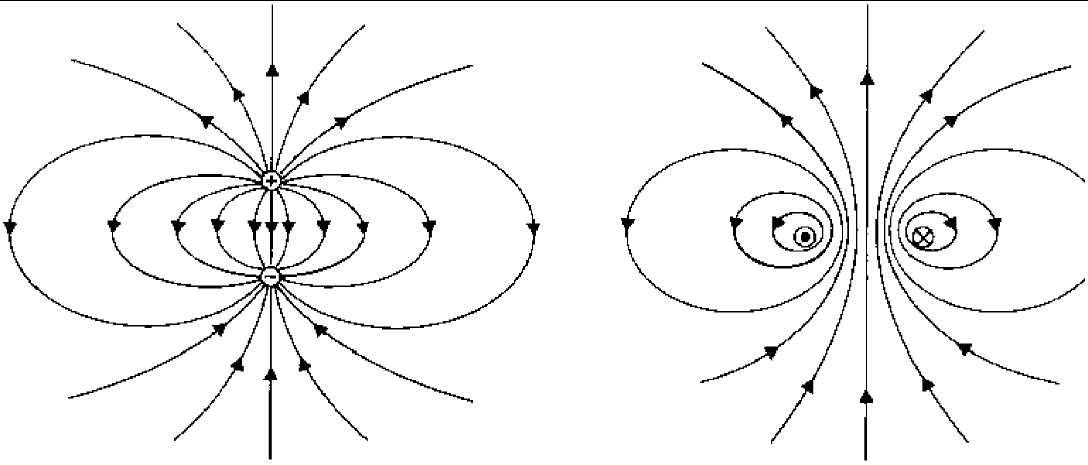
$$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{a}_R}{4\pi R^2}. \tag{6-20}$$

From $\vec{B} = \nabla \times \vec{A}$, the magnetic field can specifically calculated to be in the far-field region $R >> b$

$$\vec{B} = \frac{\mu_0 m}{4\pi R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta) \tag{6-21}$$

Notice that the electric dipole field in the far-field region has the same form

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta) \tag{6-22}$$

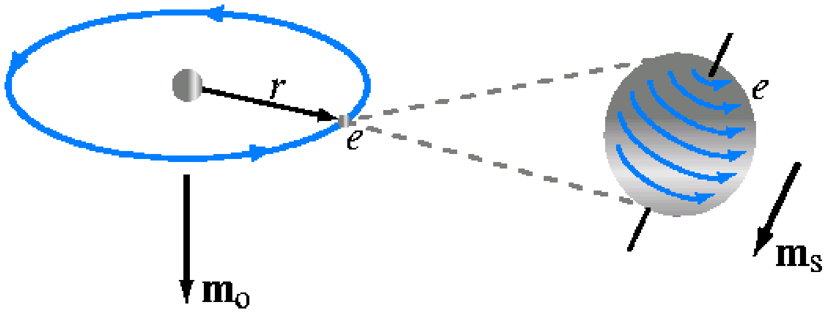


(a) Electric dipole. (b) Magnetic dipole.

Note the same field distributions in the far field region, but different in the near field region.

Magnetization

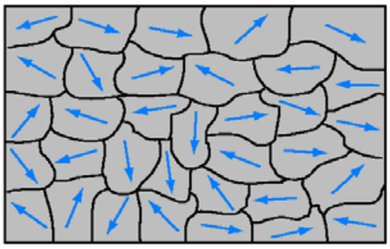
An atom has a nucleus surrounded by orbiting electrons. The orbiting electrons can be viewed as a current loop from a distance, having an orbital magnetic moment m_0. The electron itself also has an intrinsic spin magnetic moment m_s. Therefore each atom in a material constitutes a magnetic dipole.



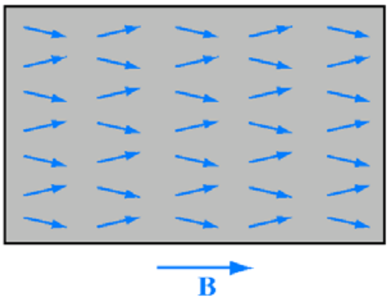
(a) Orbiting electron (b) Spinning electron

In most materials, all those tiny magnetic dipoles are randomly oriented and the ensemble average of the magnetic dipole moment is zero.

However the magnetic dipoles in a magnet or in a magnetic material under a magnetic field could align quite well in space.



(a) Unmagnetized domains



(b) Magnetized domains

For a small current loop, the *magnetic dipole moment* is defined to be $\vec{m} \equiv I\pi b^2 \hat{a}_z$. In a magnetic material, define the magnetization vector as the averaged sum of magnetic dipole moments per unit volume in a point volume.

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{m}_k}{\Delta v} \quad \text{(A/m)} \tag{6-23}$$

(This is just the same as the way one defines the electric polarization vector in a dielectric)

From Eq. (6-20) for a magnetic dipole

$$d\vec{A} = \frac{\mu_0 \vec{M} \times \hat{a}_r}{4\pi R^2} dv' = \frac{\mu_0 \vec{M} \times \nabla' \left(\frac{1}{R} \right)}{4\pi} dv' \tag{6-24}$$

Note that $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ and

$$\nabla'\left(\frac{1}{R}\right)=\frac{\partial(1/R)}{\partial x'}\hat{a}_x+\frac{\partial(1/R)}{\partial y'}\hat{a}_y+\frac{\partial(1/R)}{\partial z'}\hat{a}_z$$

$$=\frac{-1}{R^2}[\frac{\partial R}{\partial x'}\hat{a}_x+\frac{\partial R}{\partial y'}\hat{a}_y+\frac{\partial R}{\partial z'}\hat{a}_z]$$

$$=\frac{-1}{R^2}[\frac{-(x-x')}{R}\hat{a}_x+\frac{-(y-y')}{R}\hat{a}_y+\frac{-(z-z')}{R}\hat{a}_z]=\frac{\hat{a}_R}{R^2}$$

Use the vector identities

$\nabla\times(f\vec{G})=f\nabla\times\vec{G}+(\nabla f)\times\vec{G}$ and $\int_V\nabla\times\vec{F}dv=-\oint_S\vec{F}\times d\vec{s}$

(try $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot\nabla\times\vec{A}-\vec{A}\cdot\nabla\times\vec{B}$ and $\vec{A}\cdot\vec{B}\times\vec{C}=\vec{B}\cdot\vec{C}\times\vec{A}=\vec{C}\cdot\vec{A}\times\vec{B}$ to prove it)

(7-24) $\Rightarrow$ $\vec{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}_m}{R}dv'+\frac{\mu_0}{4\pi}\oint_{S'}\frac{\vec{J}_{ms}}{R}ds'$, (6-25)

where $\vec{J}_m\equiv\nabla'\times\vec{M}$ (A/m^2), (6-26)

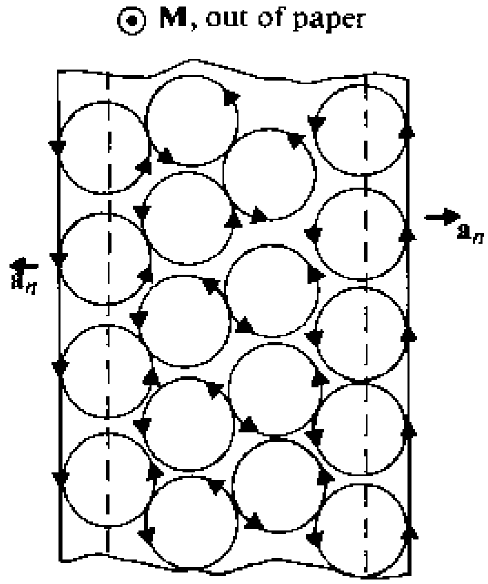
is a volume current density, and

$\vec{J}_{ms}\equiv\vec{M}\times\hat{a}_n$ (A/m) (6-27)

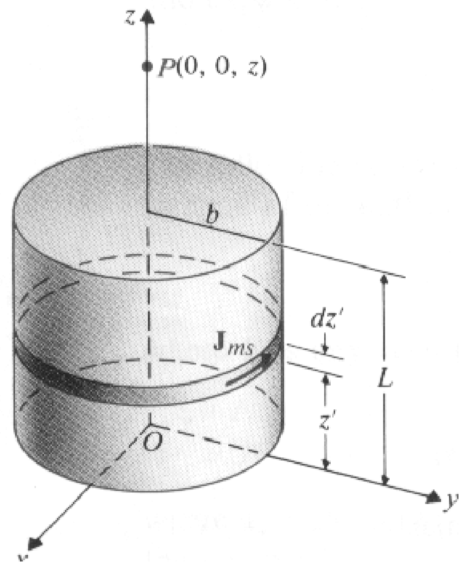
is a surface current density and $\hat{a}_n$ is the unit vector of the surface normal. $\vec{J}_m\equiv\nabla'\times\vec{M}$ and $\vec{J}_{ms}\equiv\vec{M}\times\hat{a}_n$ are called the magnetization current density. With Eq. (6-25), we have turned a microscopic problem dealing with individual magnetic dipoles into a macroscopic problem dealing with equivalent currents associated with a magnetic material.

In a homogeneously magnetized material, there is no spatial variation of magnetization and $\vec{J}_m \equiv \nabla' \times \vec{M} = 0$, which results in a simplified equation

$$\vec{A} = \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{J}_{ms}}{R} ds'$$

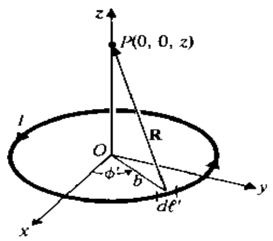


Eg. Find $\vec{B}$ at point $P(0, 0, z)$ with $\vec{M} = M_0 \hat{a}_z$ in a cylindrical magnet shown below.



Apparently the volume current density is $\vec{J}_m \equiv \nabla' \times \vec{M} = 0$ for this homogeneous magnet. But the surface current density is

$$\vec{J}_{ms} \equiv \vec{M} \times \hat{a}_n = M_0 \hat{a}_z \times \hat{a}_r = M_0 \hat{a}_\phi$$

Recall $\vec{B} = \hat{a}_z \dfrac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$ at $P(0, 0, z)$ for a single current loop carrying a current I .

The total magnetic field from summing all differential magnetization surface current is

$$\vec{B} = \hat{a}_z \int_0^L \frac{\mu_0 b^2 J_{ms} dz'}{2((z - z')^2 + b^2)^{3/2}} = \hat{a}_z \int_0^L \frac{\mu_0 b^2 M_0 dz'}{2((z - z')^2 + b^2)^{3/2}}$$

Modification of Maxwell's Equations in a Magnetic Material

In a material-free space,

$\nabla \times \vec{B} = \mu_0 \vec{J}$, where $\vec{J}$ is the free-current density

In a magnetic material, magnetization current has to be considered as well. Therefore

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_m = \mu_0 \vec{J} + \mu_0 \nabla \times \vec{M}$$

$$\Rightarrow \quad \nabla \times \vec{H} \equiv \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J} \qquad (6\text{-}28)$$

$\vec{H} \equiv \dfrac{\vec{B}}{\mu_0} - \vec{M}$, the magnetic field intensity, is defined to deal with the free-current $\vec{J}$ alone in a calculation. The Ampere's circuital law is modified to be

$$\Rightarrow \quad \int_{S'} \nabla \times \vec{H} \cdot d\vec{s} = \oint_{C'} \vec{H} \cdot d\vec{l} = \int_{S'} \vec{J} \cdot d\vec{s} = I$$

Ampere's Circuital Law*: Magnetic circulation over a loop equals the free-current going through the loop.*

Magnetostatic Maxwell's Equations

Differential form	Integral Form	
$\nabla \times \vec{H} = \vec{J}$	$\oint_C \vec{H} \cdot d\vec{l} = I$	(6-29.a,b)
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$	

In a linear, isotropic, and nondispersive magnetic medium

$$\vec{M} = \chi_m \vec{H}\ , \tag{6-30}$$

The proportional factor χ_m is called *magnetic susceptibility*.

Eq. (6-28) is recast into a more compact form, given by

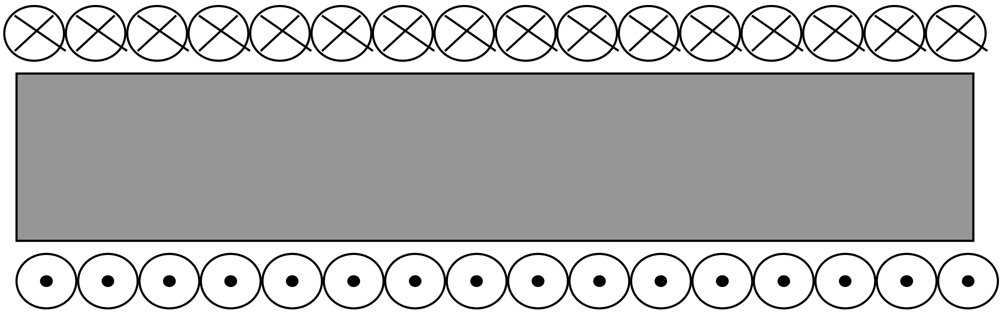
$$\vec{B} = \mu \vec{H} \equiv \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}\ , \tag{6-31}$$

where the relative permeability is defined as $\mu_r \equiv 1 + \chi_m = \dfrac{\mu}{\mu_0}$. μ_r can be 10^6 for some ferromagnetic materials (Fe, Ni, Co)!!

Analogy in electrostatics and magnetostatics: Systematic substitutions of the following table of symbols transform an equation in one system into another. In other words, the equations in electrostatics and magnetostatics are *dual equations.*

Electrostatics	*Magnetostatics*
E	***B***
D	***H***
$\boldsymbol{\varepsilon}$	$\boldsymbol{1/\mu}$
P	***-M***
$\boldsymbol{\rho}$	***J***
V	***A***
$\bullet$	$\times$
$\times$	$\bullet$

Eg. A long solenoid with and without a ferromagnetic core.



Find $\vec{H}$, $\vec{B}$ with and without the ferromagnetic material?

i. Without the ferromagnetic material

From Ampere's law: $\oint_C \vec{H} \cdot d\vec{l} = NI$

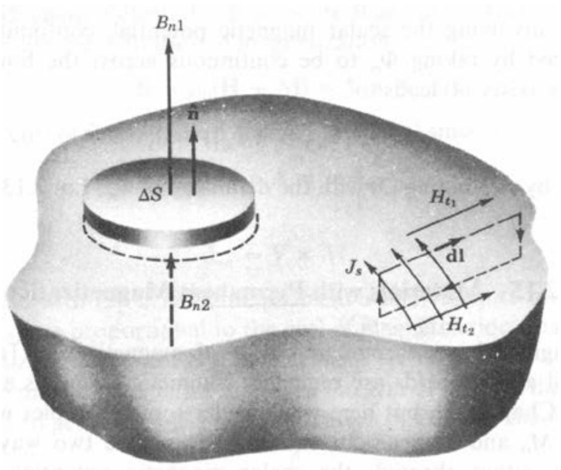
$$\Rightarrow \vec{H} = H_z \hat{a}_z = \hat{a}_z nI \quad \Rightarrow \quad \vec{B} = \mu_0 \vec{H} = \hat{a}_z \mu_0 nI$$

ii. With the ferromagnetic material

$$\vec{H} = H_z \hat{a}_z = \hat{a}_z nI$$

$\Rightarrow \ \vec{B} = \mu_r \mu_0 \vec{H} = \hat{a}_z \mu_r \mu_0 nI$

Boundary Conditions for Static Magnetic fields



i. Boundary condition of normal components of magnetic fields

From $\oint_S \vec{B} \cdot d\vec{s} = 0 \quad \Rightarrow \quad B_{n1}\Delta S = B_{n2}\Delta S$

$\Rightarrow \ B_{n1} = B_{n2}$ or $\mu_1 H_{n1} = \mu_2 H_{n2}$ (6-32.a,b)

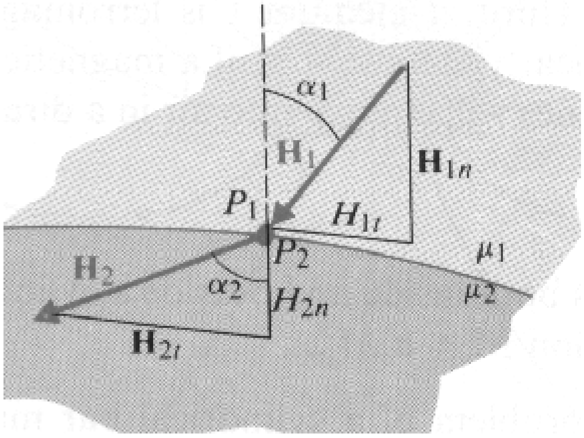
ii. Boundary condition of tangential components of magnetic fields

From $\oint_C \vec{H} \cdot d\vec{l} = I \ \Rightarrow \ \oint_C \vec{H} \cdot d\vec{l} = H_{1t}\Delta l - H_{2t}\Delta l = J_s \Delta l$

$\Rightarrow \ H_{1t} - H_{2t} = J_s$ the relative direction between *H* and *J* follows the right hand rule or

$$\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \qquad (6\text{-}33)$$

Eg. At the interface of two magnetic materials 1 & 2, the magnetic field intensity H_1 in the first material forms an angle of α_1 at the interface. Assume no free current at the interface. Find $\vec{H}_2$ or H_{2t}, H_{2n} between the boundaries of two magnetic materials



Recall the boundary conditions $H_{1t} = H_{2t}$ and $B_{1n} = B_{2n}$.

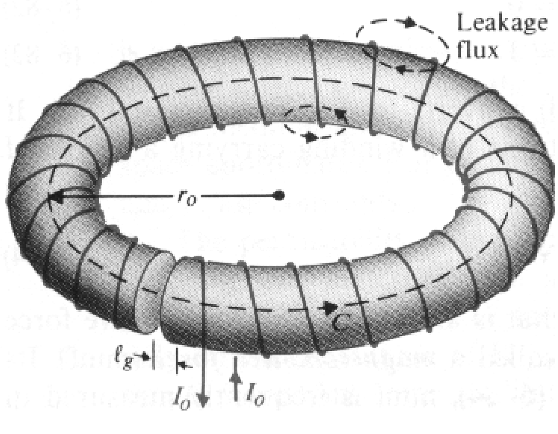
$$H_{1t} = H_1 \sin\alpha_1 \;\Rightarrow\; H_{2t} = H_1 \sin\alpha_1$$

$$H_{1n} = H_1 \cos\alpha_1 \;\Rightarrow\; B_{2n} = B_{1n} = \mu_1 H_1 \cos\alpha_1$$

Thus $H_{2n} = \dfrac{B_{2n}}{\mu_2} = \dfrac{\mu_1}{\mu_2} H_1 \cos\alpha_1$

and $\tan\alpha_2 = \dfrac{H_{2t}}{H_{2n}} = \dfrac{\mu_2}{\mu_1}\tan\alpha_1$

Magnetic Circuits



Refer to the above toroid with a ferrite core of length l_f and an air gap of length l_g. From Ampere's law, $\oint_C \vec{H} \cdot d\vec{l} = NI$, one writes the expression $H_g l_g + H_f l_f = NI$, where the subscripts, *g* and *f*, designate quantities in the gap and ferrite regions, respectively. From the boundary condition, $B_{1n} = B_{2n}$, one also has $\mu_g H_g = \mu_f H_f$. With above, H_g, H_f, B_g, B_f can be found. For example, the magnetic flux density is found to be

$$B_g = B_f = \frac{NI}{l_f/\mu_f + l_g/\mu_g},$$

resulting in the magnetic flux

$$\Phi = \frac{NI}{l_f/\mu_f S + l_g/\mu_g S} \quad \text{or} \quad \Phi = \frac{V_m}{R_f + R_g},$$

where *S* is the cross sectional area of the ferrite core, the *magnetomotive force* (*mmf*) is defined to be $V_m \equiv NI$, and the *magnetic reluctance* is defined to be

$$R_f = l_f/\mu_f S \text{ and } R_g = l_g/\mu_g S.$$

** Recall the expression for *resistance* $R = l/\sigma S \Rightarrow$

Analogy

Magnetic Circuit	*Electric Circuit*
$\nabla \times \vec{H} = \vec{J}$	$\nabla \times \vec{E} = \vec{f}$
mmf $V_m = NI$	*emf* V_{em}
magnetic flux Φ	electric current I
magnetic reluctance $R = l/\mu S$	electric resistance $R = l/\sigma S$

Kirchhoff's Voltage Law for a Magnetostatic Loop

$$\sum_j V_{m,j} = \sum_j N_j I_j = \sum_k R_k \Phi_k \, ,$$

compared with Kirchhoff's voltage law for electrostatics

$$\sum_j V_{em,j} = \sum_k R_k I_k$$

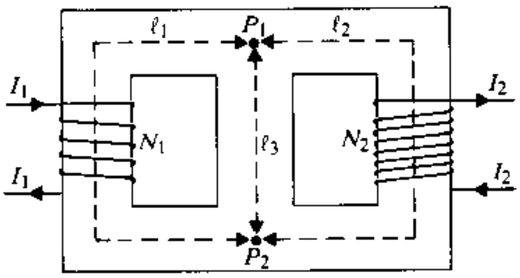
Kirchhoff's Current Law for a Magnetostatic Node

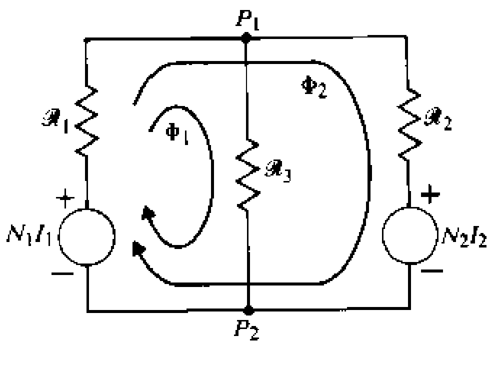
$$\sum_k \Phi_k = 0$$ (resulted from $\nabla \cdot \vec{B} = 0$),

compare with Kirchhoff's current law for electrostatics at a current node

$$\sum_k I_k = 0$$

Exercise: Determine the magnetic flux in the center leg in the following magnetic circuit with a ferromagnetic core.





Magnetic Materials

All materials more or less respond to an external magnetic field. The magnetic flux density *B* in a material can be increased or decreased due to a magnetic field intensity *H* or an external current *I*.

Diamagnetism $\mu_r \le 1$ or $\chi_m \le 0$ (all materials)

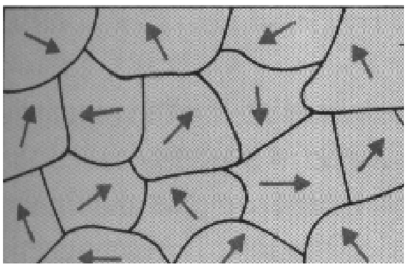
Lenz's law: The induced magnetic dipole moment from electron orbits always opposes the applied external field. This effect is very weak, $|\chi_m| \approx 10^{-8} \sim 10^{-5}$, and is often obscured in materials with intrinsic magnetism.

Paramagnetism $\mu_r \ge 1$ or $\chi_m \ge 0$

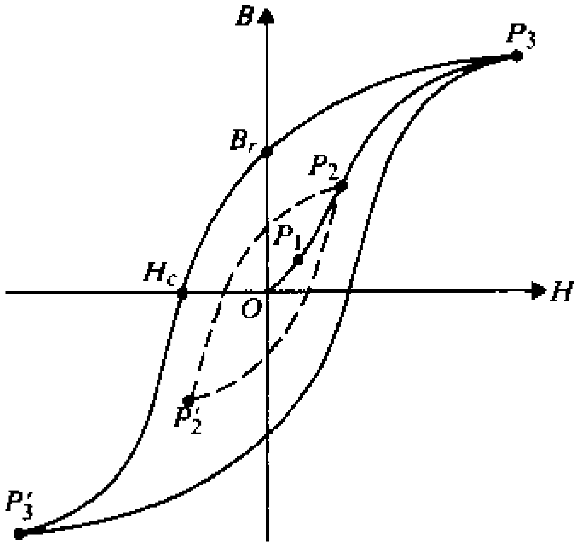
Dipole moments in electrons, atoms, or molecules tend to partially align with an applied external field.

Ferromagnetism $\mu_r >> 1$ or $\chi_m >> 0$

Magnetic domains exist in such materials. Permanent magnetization in discrete domains results from electron spinning. A small external magnetic field intensity (from, say, a current loop) aligns the domains, resulting in a large magnetic flux density *B*. Above the so-called *Curie temperature* the aligned magnetic domains are disorganized.



Hysteresis Loop



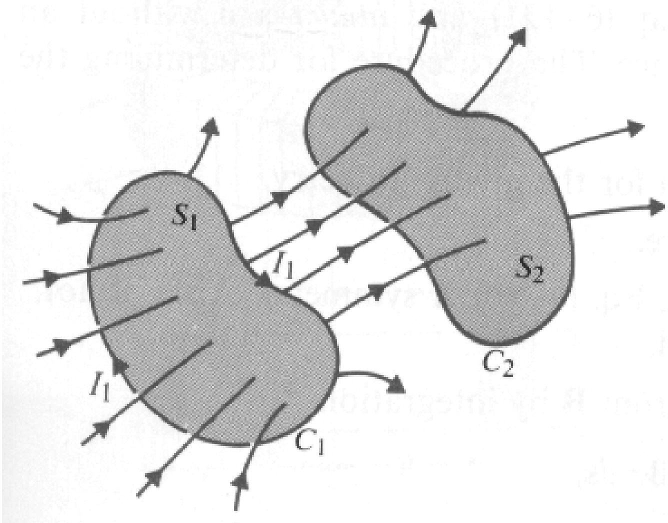
B_r : *residue or remanent flux density*

H_c : *coercive field intensity*

A hysteresis loop is the B-field response curve of a ferromagnetic material under a current (note that *H* is proportional to the external current *I*). Initially *B* field is zero without *I*, corresponding to the point at the origin. As *I* or *H* is increased, magnetization occurs and *B* gradually saturates. If one reduces the current back to zero, *B* does not return to the zero point but shows a nonzero residue value, because not all the ferromagnetic domains return to their original positions due to friction between domain walls. It takes some backward current to cancel out the residue *B*. Once we learn about the magnetic energy density near the end of this chapter, it is not hard to see that the energy loss per unit volume in the ferromagnetic material is the area of the hysteresis loop. This energy can cook food by an induction cooker when a ferromagnetic pot is used. .

Inductance

magnetic linkage per unit current



Self-Inductance

$$L_{11} = \frac{N_1\Phi_{11}}{I_1} = \frac{\Lambda_{11}}{I_1}, \tag{6-34}$$

where $\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}_1 = \oint_{C_1} \vec{A}_1 \cdot d\vec{l}_1$ is the magnetic flux through C_1 due to I_1, and $\Lambda_{11} = N_1\Phi_{11}$ is the magnetic linkage associated with C_1 due to I_1.

Self-inductance $\boldsymbol{L_{11}}$ is the *magnetic linkage per unit current in the loop itself.*

Mutual Inductance

$$L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}, \tag{6-35}$$

where $\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2$ is the magnetic flux through C_2 due to I_1, and $\Lambda_{12} = N_2\Phi_{12}$ is the magnetic linkage associated with C_2 due to I_1. Accordingly

$$L_{12}=\frac{N_2}{I_1}\oint_{C_2}\vec{A}_1\cdot d\vec{l}_2=\frac{N_2}{I_1}\oint_{C_2}\left(\vec{A}_1=\frac{\mu_0 N_1 I_1}{4\pi}\oint_{C_1}\frac{d\vec{l}_1}{R}\right)\cdot d\vec{l}_2$$

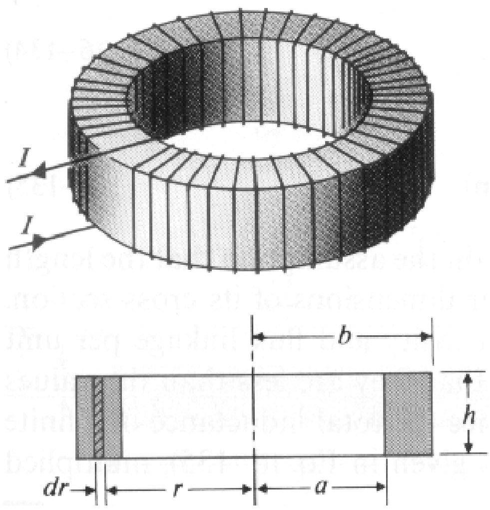
$$\Rightarrow\ L_{12}=\frac{\mu_0 N_1 N_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R}=L_{21} \tag{6-36}$$

Mutual inductance $\boldsymbol{L_{12}}$ is the *magnetic linkage through* C_2 *per unit current in loop* C_1.

In a linear medium, inductance is a function of space dimensions and permeability.

Inductor: a device that provides a certain amount of inductance.

Eg. Find the inductance of an *N*-turn toroid with μ_0 inside.



Use Ampere's law $\oint_C \vec{H}\cdot d\vec{l}=NI$ to calculate the magnetic field intensity and magnetic flux density

$$H_\phi \cdot 2\pi r = NI \quad \Rightarrow \quad H_\phi = \frac{NI}{2\pi r} \quad \Rightarrow \quad B_\phi = \mu_0 H_\phi = \frac{\mu_0 NI}{2\pi r}$$

The magnetic flux is calculated from

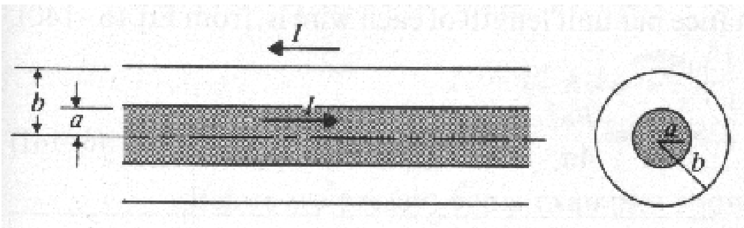
$$\Phi_{11} = \int_{S_1} \vec{B}_1 \cdot d\vec{s}_1 = \int_a^b B_\phi h dr = \frac{\mu_0 NIh}{2\pi} \ln \frac{b}{a}$$

The magnetic linkage is readily calculated from

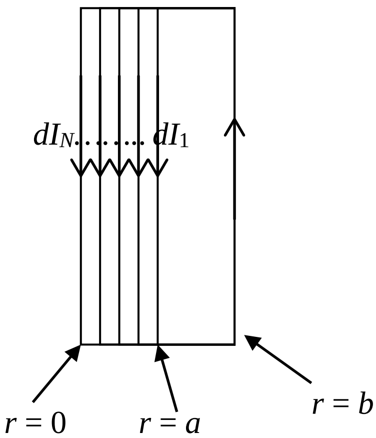
$$\Lambda_{11} = N_1 \Phi_{11} = \frac{\mu_0 N^2 Ih}{2\pi} \ln \frac{b}{a}$$

The inductance is therefore $L_{11} = \dfrac{\Lambda_{11}}{I_1} = \dfrac{\mu_0 N^2 h}{2\pi} \ln \dfrac{b}{a}$

Eg. Find the inductance of a coaxial cable with a uniform current flowing in the core and a returning current in a thin outer conductor.



This problem can be tackled by imagining the superposition of many rectangular coils, as shown schematically below.



Each loop carrying a current I_i in the region from r to b is associated with a magnetic flux $\phi_i(r)$. The expressions hold

$LI = \Lambda$, $LI_i = \phi_i$, and

$$L\sum_i I_i = \sum_i \phi_i \Rightarrow L\int_0^a I\frac{2\pi r}{\pi a^2}dr = \int_0^a \phi(r)\frac{2\pi r}{\pi a^2}dr$$

In the inner conductor, the magnetic field is $\vec{B}_1 = \hat{a}_\phi \dfrac{\mu_0 rI}{2\pi a^2}$.

In the vacuum region, the magnetic field is $\vec{B}_2 = \hat{a}_\phi \dfrac{\mu_0 I}{2\pi r}$.

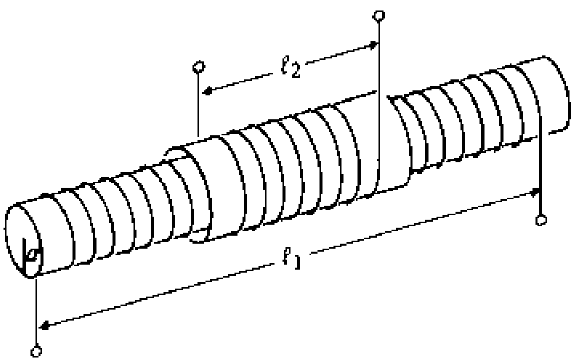
The magnetic flux from r to b per unit length is given by

$\phi(r) = \int_r^a B_1 dr + \int_a^b B_2 dr$. Substitute $\phi(r)$ into

$L\int_0^a I\dfrac{2\pi r}{\pi a^2}dr = \int_0^a \phi(r)\dfrac{2\pi r}{\pi a^2}dr$ to find L.

* see a better approach later.

Eg. A wire of N_1 turns is wound inside a wire of N_2 turns carrying currents I_1 and I_2 respectively. Find mutual inductance.



To find Φ_{12}, use Ampere's law to obtain $B_1 = \dfrac{\mu_0 N_1 I_1}{l_1}$

The magnetic flux is $\Phi_{12} = B_1 S_2 = B_1 S_1 = \pi a^2 \dfrac{\mu_0 N_1 I_1}{l_1}$

The magnetic linkage is $\Lambda_{12} = N_2 \Phi_{12} = \pi a^2 \dfrac{\mu_0 N_1 N_2 I_1}{l_1}$

The mutual inductance is $L_{12} = \dfrac{\Lambda_{12}}{I_1} = \pi a^2 \dfrac{\mu_0 N_1 N_2}{l_1}$

Magnetic Energy Stored in Inductors

Without considering the sign of the induced voltage, for a *single current loop*, $v_1 = \dfrac{d\phi_{11}}{dt}$ thus $v_1 = L_{11} \dfrac{di_1}{dt}$.

The energy is stored to the current loop when current increases from 0 to I_1, given by

$$w_1 = \int v_1 i_1 \cdot dt = \int_0^{I_1} L_{11} i_1 di_1 = \frac{1}{2} L_{11} I_1^2 = \frac{1}{2} \Phi_{11} I_1 \qquad (6\text{-}37)$$

For two current loops C_1 and C_2, the total stored energy is

$w_m = w_1 + w_{12} + w_2 = w_1 + w_{21} + w_2$,

where

1. $w_1 = \dfrac{1}{2} L_{11} I_1 I_1$ is the energy for pumping in current I_1 while keeping $I_2 = 0$

2. $w_2 = \frac{1}{2} L_{22} I_2 I_2$ is the energy for pumping in current I_2

3. $w_{21} = \int v_{21} I_1 dt = \int_0^{I_2} L_{21} \frac{di_2}{dt} I_1 dt = L_{21} I_1 I_2 = w_{12}$ is the energy necessary for maintaining I_1 in loop 1 when current i_2 is increased from 0 to I_2 in loop 2.

A short-hand expression for the energy stored in these two loops is

$$w_m = \frac{1}{2} \sum_{k=1}^{2} \sum_{j=1}^{2} L_{jk} I_j I_k \tag{6-38}$$

The above can be generalized to the energy stored in *N* inductor loops, given by

$$w_m = \frac{1}{2} \sum_{k=1}^{N} \sum_{j=1}^{N} L_{jk} I_j I_k = \frac{1}{2} \sum_{k=1}^{N} \Phi_k I_k \tag{6-39}$$

where $\Phi_k = \sum_{j=1}^{N} L_{jk} I_j$ is the total magnetic flux going through loop *k*.

Magnetic Energy in Space

In a distributed inductive system, the calculation of the stored energy becomes an integration

$$w_m = \frac{1}{2}\sum_{j=1}^{N}\Phi_j I_j = \frac{1}{2}\sum_{j=1}^{N}\vec{J}_j \cdot \Delta\vec{s}_j \oint_{C_j} \vec{A}\cdot d\vec{l}_j$$

$$w_m \to \frac{1}{2}\int_V \vec{A}\cdot\vec{J}dv$$

Starting from the vector identity

$$\nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H}) = \vec{H}\cdot\vec{B} - \vec{A}\cdot\vec{J}$$

and take volume integral on both sides.

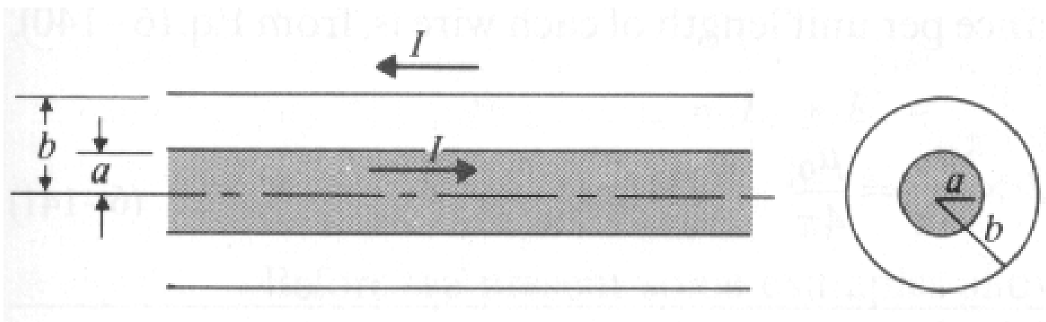
LHS goes to zero when $R \to \infty$, because $A \propto 1/R, H \propto 1/R^2$.

$$\int_V \nabla\cdot(\vec{A}\times\vec{H})dv = \oint_s \vec{A}\times\vec{H}\cdot d\vec{s} \underset{R\to\infty}{\longrightarrow} 0$$

RHS gives the stored magnetic energy as

$$w_m = \frac{1}{2}\int_V \vec{A}\cdot\vec{J}dv = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}dv = \frac{\mu}{2}\int_V H^2 dv = \frac{1}{2\mu}\int_V B^2 dv \quad (6\text{-}40)$$

Eg. Find the inductance of the following coaxial cable.



***Internal inductance L_i*:** arises from the magnetic linkage in the current-flow region

The magnetic field intensity in the core is given by $H_i 2\pi r = I\dfrac{r^2}{a^2}$

$$\Rightarrow \quad H_i = I\frac{r}{2\pi a^2}$$

along the φ direction. The stored energy associated with the internal inductance is given by

$$w_{m,i} = \frac{\mu}{2}\int_{V'} H_i^{\,2} dv' = \frac{\mu_0}{2}\int_{V'} H_i^{\,2} l 2\pi r dr = \frac{\mu_0 l I^2}{16\pi}$$

From the expression, $w_{m,i} = \dfrac{L_i I^2}{2} = \dfrac{\mu_0 l I^2}{16\pi}$, one can calculate the

internal inductance per unit length $\dfrac{L_i}{l} = \dfrac{\mu_0}{8\pi}$.

External inductance L_e: arises from the magnetic linkage in the zero-current region.

The magnetic field intensity between the two conductors, according to the Ampere's law is

$$H_e 2\pi r = I \quad \Rightarrow \quad H_i = \frac{I}{2\pi r}$$

also along the φ direction.

The stored energy associated with the external inductance is given by

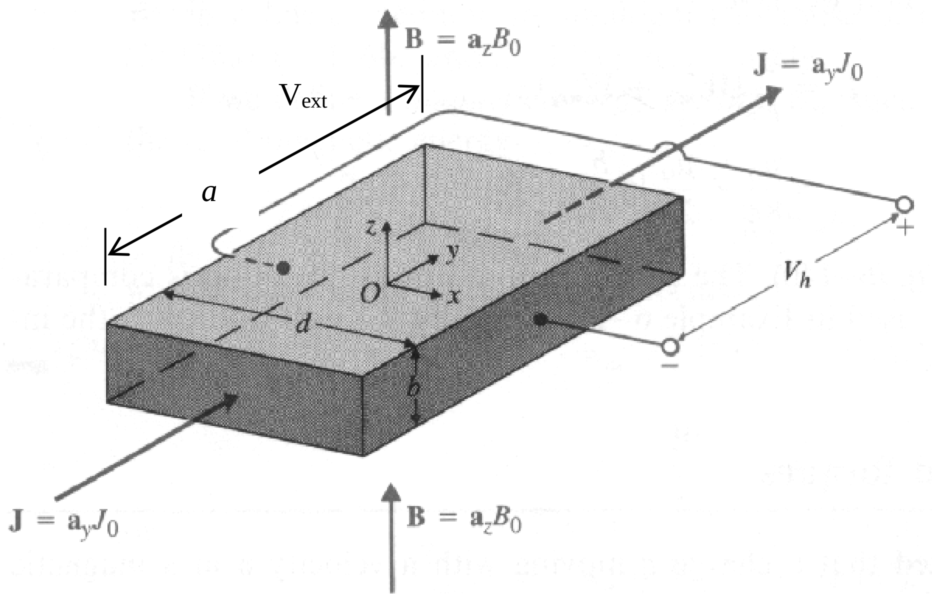
$$w_{m,e} = \frac{\mu}{2}\int_{V'} H_e^{\,2} dv' = \frac{\mu_0}{2}\int_{V'} H_e^{\,2} l 2\pi r dr = \frac{\mu_0 l I^2}{4\pi}\ln\frac{b}{a}$$

From the expression, $w_{m,e} = \dfrac{L_e I^2}{2} = \dfrac{\mu_0 l I^2}{4\pi}\ln\dfrac{b}{a}$, one can calculate the *external inductance per unit length* $\frac{L_e}{l} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$.

Magnetic Force and Torque

Recall the Lorentz force equation $\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$. A torque is the cross product of a length and a force, $\vec{T} = \vec{r} \times \vec{B}$. A magnetic force is associated with a moving charge.

Hall Effect



In a conductor, a current density $\vec{J} = \hat{a}_y J_0$ flows in the *y* direction and a magnetic field $\vec{B} = \hat{a}_z B_0$ is applied along the *z* direction. If the moving charges are holes, the charge velocity is expressed as $\vec{u} = \hat{a}_y u_h$; whereas, if the moving charges are electrons, the velocity is $\vec{u} = -\hat{a}_y u_e$

($u_{e,h}$ are both positive).At the steady state, the net force on a charge equals zero. That is, the magnetic force in the *x* direction is balanced by the electric force due to accumulation of charges.

$$q\vec{E}_H + q\vec{u}\times\vec{B} = 0 \Rightarrow \vec{E}_{H,h} = -(\hat{a}_y u_h)\times\hat{a}_z B_0 = -\hat{a}_x u_h B_0$$

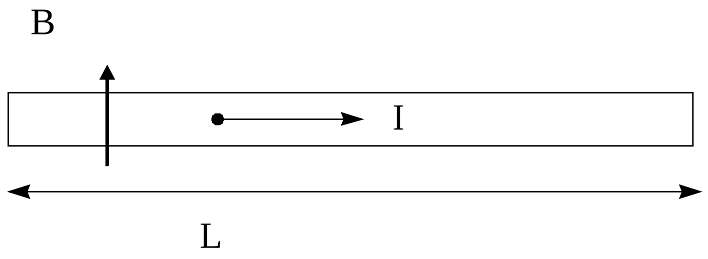
for holes and $\vec{E}_{H,e} = -(-\hat{a}_y u_e)\times\hat{a}_z B_0 = \hat{a}_x u_e B_0$ for electros.

By measuring the polarity of the Hall voltage $V_H = E_H d$, one can determine the type of the moving charges in the conducting material.

The *Hall coefficient* is defined to be $\dfrac{E_H}{JB_0} = \dfrac{V_H b}{IB_0} = \dfrac{1}{Nq} = \dfrac{1}{\rho}$,

from which the charge density ρ can be determined. The Hall voltage $V_H = E_H d = u_{e,h} B_0 d$ can be easily measured, and the charge velocity $u_{e,h}$ can be deduced. Given an external field $E_{ext} = V_{ext}/a$ and the expression $\vec{u} = -\mu_e \vec{E}_{ext}$ or $\vec{u} = +\mu_h \vec{E}_{ext}$, the carrier mobility can be known and the conductivity is obtained from the expression $\sigma = \rho\mu_e$.

Force Between Two Electric Wires



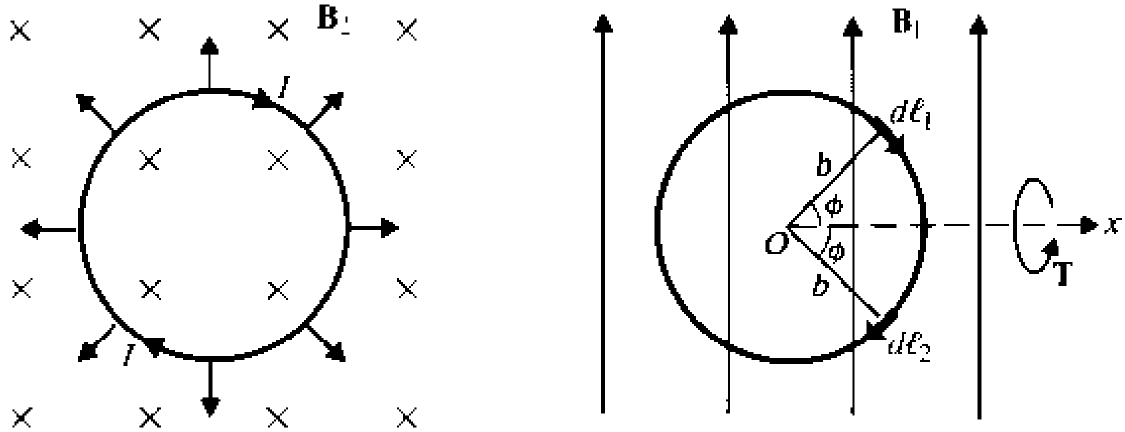
At the very beginning of this chapter, we derive the magnetic force experienced by a current-carrying wire in a uniform magnetic field, given by

$$\vec{F}_2 = Nq\vec{u}_2 \times \vec{B}_1 = L_2\vec{I}_2 \times \vec{B}_1$$

or the force per unit length $\frac{\vec{F}_2}{L_2} = \vec{I}_2 \times \vec{B}_1$

In a two-wire system, $\vec{B}_1$ is generated from a wire carrying a current of $\vec{I}_1$ at a distance *d* away. As an exercise, apply Ampere's law to obtain the force per unit length on two parallel wires carrying I_1 and I_2 in the same and opposite directions.

Magnetic Torques



B_1 B_2

No net force if B is parallel to $\vec{m}$

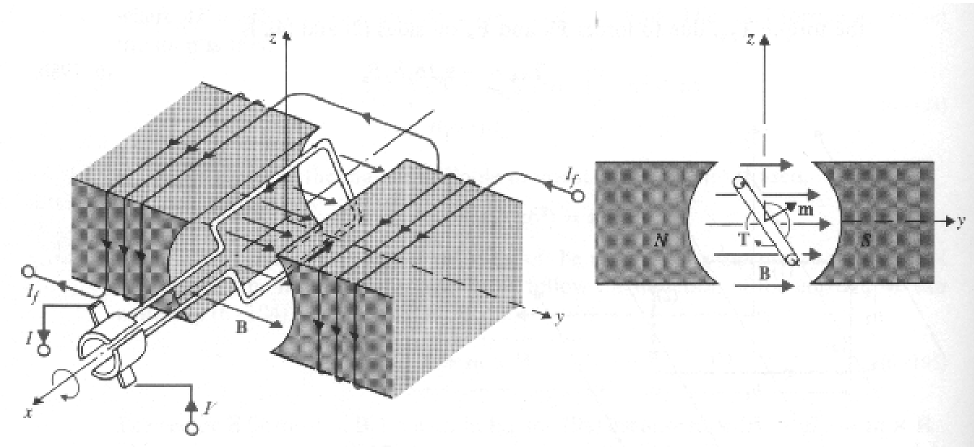
According to the definition of a *torque* $\vec{T} = \vec{r} \times \vec{F}$

The differential torque is $d\vec{T} = 2\hat{a}_x (dF) b \sin\phi$, given a differential magnetic force $dF = I \cdot dl \cdot B_2 \sin\phi$.

After integration, the total m*agnetic Torque* becomes

$$\vec{T} = \hat{a}_x I(\pi b^2)B_2 = \vec{m} \times \vec{B} \, .$$

This magnetic torque is being used to make a DC Motor



(a) Perspective view. (b) Schematic view from $+x$ direction.

Magnetostatic Force

Still, we use the *principle of virtue displacement* to solve a magnetostatic force problem. The energy conservation requires

$$dW_s = dW + dW_m\big|_{I,\Phi}$$

where dW_s is the work done by a source, dW is the work done by a mechanical force, $dW_m\big|_{I,\Phi}$ is the change on stored magnetic energy under a fixed current or a fixed magnetic flux.

System with fixed currents (maintained by a source)

Work done by a source is given by

$$dW_s = \sum_k I_k (v_k = \frac{d\Phi_k}{dt})dt$$

The source has to provide energy for keeping a constant *I* in response to a varying magnetic flux. From $W_m = \frac{1}{2}\sum_k I_k \Phi_k$, the stored magnetic energy is changed by

$$dW_m\big|_I = \frac{1}{2}\sum_k I_k d\Phi_k = \frac{1}{2} dW_s$$

Mechanical work done by a virtual displacement is $dW = \vec{F}_I \cdot d\vec{l}$.

From energy conservation $dW_s = dW + dW_m\big|_I$

$$\Rightarrow \vec{F}_I \cdot d\vec{l} = dW_m\big|_I = (\nabla W_m\big|_I) \cdot d\vec{l}$$

$$\Rightarrow \vec{F}_I = \nabla (W_m\big|_I)$$

In practice $\vec{F}_I = \nabla (W_m\big|_I = \frac{1}{2} L I^2) = \frac{I^2}{2}\nabla L$

Similarly, the magnetic torque can be evaluated from $(\vec{T}_I)_z = \frac{\partial W_m\big|_I}{\partial \phi}$.

System with fixed flux linkages (no voltage change = no source work)

Recall the energy conservation $dW_s = dW + dW_m\big|_\Phi$. But $dW_s = 0$ for $V = d\phi / dt = 0$ $\Rightarrow$ $dW + dW_m\big|_\Phi = 0$. Should there be a displacement, the work is done by the system at the expense of the stored $W_m\big|_\Phi$.

$$\Rightarrow -dW_m\big|_\Phi = \vec{F}_\Phi \cdot d\vec{l} \ ,$$

But $dW_m = (\nabla W_m) \cdot d\vec{l} \;\Rightarrow\; \vec{F}_\Phi = -\nabla W_m\big|_\Phi$

In practice, $\vec{F}_\Phi = -\nabla(W_m\big|_\Phi) = -\nabla(\frac{1}{2\mu}\int_V B^2 dv)$

Similarly, the magnetic torque $(\vec{T}_\Phi)_z = -\frac{\partial W_m\big|_\Phi}{\partial \phi}$ can be deduced.

Note that $\vec{F}_I = \vec{F}_\Phi$ and $(\vec{T}_\Phi)_z = (\vec{T}_I)_z$, as they are all derived from virtual displacements and rotation.