Homework 3

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(1a) $\underbrace{\_\ \_\ \_\ \_\ \cdots\ n}_{n\text{个}}$ n不在第n位 而在第k位

1° k在n位，剩下n-2個排 ∴有D_{n-2}種

2° k不在n位，剩下n-1個排（視k=n，k不出現在n位）∴有D_{n-1}種

3° $1 \le k \le n-1$ ∴ k有$(n-1)$个

$$\therefore D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \#$$

(1b) Let $D_n = n!M_n$, $M_1 = 0$, $M_2 = \frac{1}{2}$

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \quad (\text{by 1a})$$

$$\Rightarrow n!M_n = (n-1)\left((n-1)!M_{n-1} + (n-2)!M_{n-2}\right)$$

$$\Rightarrow n!M_n = n!M_{n-1} - (n-1)!M_{n-1} + (n-1)!M_{n-2}$$

$$\Rightarrow nM_n = nM_{n-1} - M_{n-1} + M_{n-2}$$

$$\Rightarrow M_n - M_{n-1} = \frac{1}{n}(-M_{n-1} + M_{n-2}) = -\frac{1}{n}(M_{n-1} - M_{n-2})$$

$$= \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right)(M_{n-2} - M_{n-3})$$

$$\vdots$$

$$= \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right)\cdots\left(-\frac{1}{3}\right)\overset{\frac{1}{2}}{\cancel{(M_2 - M_1)}} = \frac{1}{n!} \times (-1)^{n-2}$$

$$= \frac{1}{n!} \times (-1)^n$$

$$\Rightarrow n!M_n = n!M_{n-1} + (-1)^n$$

$$\Rightarrow \underline{D_n = nD_{n-1} + (-1)^n} \quad \#$$

(2a) $a_r - 6a_{r-1} + 8a_{r-2} = 2^r,\ a_0 = 2,\ a_1 = 10.$

<sol> 1° $x^2 - 6x + 8 = 0 \Rightarrow x = 4$ or 2

$\Rightarrow a_r = a4^r + b2^r$ (homo)

2° $a_r = cr2^r$ (particular)

$\Rightarrow c \cdot r2^r - 6c(r-1)2^{r-1} + 8c(r-2)2^{r-2} = 2^r$

$\Rightarrow c(\cancel{4}r - 1\cancel{2}r + 12 + \cancel{8}r - 16) = 4$

$\Rightarrow c = -1 \Rightarrow a_r = -r2^r$

3° $a_r = -r2^r + a4^r + b2^r$

$\Rightarrow \begin{cases} a_0 = 0 + a + b = 2 \\ a_1 = -2 + 4a + 2b = 10 \end{cases} \Rightarrow \begin{cases} a = 4 \\ b = -2 \end{cases}$

$\therefore \underline{a_r = -r2^r - 2\cdot 2^r + 4\cdot 4^r}$ #

(2b) $a_r - 5a_{r-1} + 3a_{r-2} + 9a_{r-3} = 0,\ a_0 = 7,\ a_1 = 16,\ a_2 = 65.$

<sol> 1° $x^3 - 5x^2 + 3x + 9 = (x+1)(x^2 - 6x + 9) = 0$ (homo)

$\Rightarrow x = -1$ or 3 or 3

$\therefore a_r = a(-1)^r + b3^r + cr3^r$

2° $a_r = k$ (particular)

$\Rightarrow k - 5k + 3k + 9k = 0 \Rightarrow k = 0$

3° $a_r = a(-1)^r + b3^r + cr3^r$

$\Rightarrow \begin{cases} a_0 : 7 = a + b \\ a_1 = 16 = -a + 3b + 3c \\ a_2 = 65 = a + 9b + 18c \end{cases}$

$\Rightarrow a = 2,\ b = 5,\ c = 1$

$\therefore \underline{a_r = 2(-1)^r + 5\cdot 3^r + r\cdot 3^r}$ #

(3a) $a_r^2 - 2a_{r-1}^2 = 1, \quad a_0 = 2$

<sol> 1° $b_r = a_r^2 \Rightarrow b_r - 2b_{r-1} = 1, \quad b_0 = 4$

2° $x - 2 = 0 \Rightarrow x = 2 \Rightarrow b_r = a2^r$ (homo)

$b_r = b$ (particular)

$\Rightarrow b - 2b = 1 \Rightarrow b = -1$

3° $b_r = a2^r - 1$

$\Rightarrow b_0 = a - 1 = 4 \Rightarrow a = 5 \Rightarrow b_r = 5 \cdot 2^r - 1$

4° $\underline{a_r = \sqrt{5 \cdot 2^r - 1}}$ #

(3b) $a_r^2 - 2a_{r-1} = 0, \quad a_0 = 4$

<sol> 1° $b_r = \log_2 a_r$

$\because a_r^2 = 2a_{r-1}$

$\therefore 2b_r = 1 + b_{r-1}, \quad b_0 = 2$

2° $2x - 1 = 0 \Rightarrow x = \frac{1}{2} \Rightarrow b_r = c \cdot (\frac{1}{2})^r$ (homo)

$b_r = b$ (particular)

$\Rightarrow 2b = 1 + b \Rightarrow b = 1$

3° $b_r = c \cdot (\frac{1}{2})^r + 1$

$\Rightarrow b_0 = c \cdot 1 + 1 = 2 \Rightarrow c = 1 \Rightarrow b_r = (\frac{1}{2})^r + 1$

4° $\underline{a_r = 2^{(\frac{1}{2})^r + 1}}$ #

(4a) $a_r - 5a_{r-1} + 6a_{r-2} = 0, \quad a_0 = 6, \quad a_1 = 30$

<sol> $1°\ \sum_{r=2} (a_r - 5a_{r-1} + 6a_{r-2})\, x^r = 0$

$$\Rightarrow (A(x) - a_1 x - a_0) - 5x(A(x) - a_0) + 6x^2(A(x)) = 0$$

$$\Rightarrow A(x)\cdot(1-5x+6x^2) - a_1 x - a_0(1-5x) = 0$$

$$\Rightarrow A(x)(1-5x+6x^2) = a_1 x + a_0 - 5x a_0$$

$$= \cancel{30x} + 6 - \cancel{30x} = 6$$

$$\Rightarrow A(x) = \frac{6}{1-5x+6x^2} \quad \#$$

(4b) $a_r - 2a_{r-1} - 3a_{r-2} = 4^r + 6, \quad a_0 = 20, \quad a_1 = 60.$

<sol> $1°\ \sum_{r=2} (a_r - 2a_{r-1} - 3a_{r-2})\, x^r = \sum_{r=2} (4^r + 6)\, x^r$

$$\Rightarrow (A(x) - a_1 x - a_0) - 2x(A(x) - a_0) - 3x^2(A(x))$$

$$= \sum_{r=2} (4x)^r + 6\sum_{r=2} x^r = \frac{16x^2}{1-4x} + \frac{6x^2}{1-x}$$

$$\Rightarrow A(x)(1-2x-3x^2) = \frac{16x^2}{1-4x} + \frac{6x^2}{1-x} + \overset{60}{\cancel{a_1}}x + \overset{20}{\cancel{a_0}} - \overset{40}{\cancel{2x a_0}}$$

$$= \frac{16x^2}{1-4x} + \frac{6x^2}{1-x} + 20(x+1)$$

$$\Rightarrow A(x) = \frac{20x + 20 + \frac{16x^2}{1-4x} + \frac{6x^2}{1-x}}{1-2x-3x^2} \quad \#$$

(5a) $\underline{01001} \longrightarrow b_n + b_{n-3} = 2^{n-5}$

$$\Rightarrow \sum_{n=5}^{\infty} b_n x^n + b_{n-3} x^n = \sum_{n=5} 2^{n-5} x^n$$

$$\Rightarrow \left(B(x) - \cancel{b_4}^{\,0} x^4 - \cancel{b_3}^{\,0} x^3 - \cancel{b_2}^{\,0} x^2 - \cancel{b_1}^{\,0} x - \cancel{b_0}^{\,1}\right) + x^3\left(B(x) - \cancel{b_1}^{\,0} x - \cancel{b_0}^{\,1}\right)$$

$$= (B(x) - 1) + x^3(B(x) - 1) = (1 + x^3)(B(x) - 1)$$

$$= \sum_{n=5} 2^{n-5} x^n = x^5 + 2x^6 + \cdots = \frac{x^5}{1-2x}$$

$$\Rightarrow B(x) - 1 = \frac{x^5}{(1-2x)(1+x^3)}$$

$$\Rightarrow B(x) = \frac{x^5}{(1-2x)(1+x^3)} + 1$$

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(5b) first occurs at nth

$$\Rightarrow b_n = C_n b_0 + C_{n-1} b_1 + \cdots + C_1 b_{n-1} + C_0 b_n$$

$$\begin{cases} C_0 = C_1 = C_2 = C_3 = C_4 = 0 \\ b_0 = 1,\ b_1 = b_2 = b_3 = b_4 = 0 \end{cases}$$

$$\Rightarrow \sum_{n=5}^{\infty} b_n x^n = \sum_{n=5}^{\infty} (C_n b_0 + C_{n-1} b_1 + \cdots + C_0 b_n) x^n$$

$$\Rightarrow (B(x) - 1) = B(x) \cdot C(x)$$

$$\Rightarrow C(x) = 1 - \frac{1}{B(x)}$$

$$= 1 - \frac{(1-2x)(1+x^3)}{x^5 + (1-2x)(1+x^3)}$$

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