EE2030Linear Algebra

Homework#5

Due: 05/05/2023 10:10(Fri)

1. Use row operations to simplify and compute these determinants:

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \qquad \text{and} \qquad \det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}.$$

2. Elimination reduces A to U. Then $A = LU$.

$$A = \begin{bmatrix}3 & 3 & 4\\ 6 & 8 & 7\\ -3 & 5 & -9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 4 & 1\end{bmatrix}\begin{bmatrix}3 & 3 & 4\\ 0 & 2 & -1\\ 0 & 0 & -1\end{bmatrix} = LU.$$

Find the determinants of L, U, A, U^{-1}, L^{-1}, and $U^{-1}L^{-1}A$.

3. True or False(give a reason if true or a 2 by 2 example if false):
(a)If A is not invertible then AB is not invertible.
(b)The determinant of A is always the product of its pivots.
(c)The determinant of A-B equals detA - detB
(d)AB and BA have the same determinant.

4. The n by n determinant C_n has 1's above and below the main diagonal:

$C_2 = -|1|$

$$C_1 = \begin{vmatrix}0\end{vmatrix}, C_2 = \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}, C_3 = \begin{vmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{vmatrix}, C_4 = \begin{vmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{vmatrix}.$$

(a) What are these determinants C_1, C_2, C_3, C_4?
(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

5. Find G_2 and G_3 and then by row operations G_4.Can you predicet G_n?

$$G_2 = \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} \qquad G_3 = \begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix} \qquad G_4 = \begin{vmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{vmatrix}$$

1. $\begin{vmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{vmatrix} = \begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$ $\quad$ $\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} = \begin{vmatrix} 1-t^2 & 0 & 0 \\ t & 1 & t \\ 0 & 0 & 1-t^2 \end{vmatrix} = t^4-2t^2+1$

2.

$\det L = 1$, $\det U = \det A = 3\times 2\times(-1) = -6$

$\det(U^{-1}L^{-1}A) = \det I = 1$

$\det U^{-1} \cdot \det U = \det(U^{-1}U) = \det I = 1$

$\det U^{-1} = -\frac{1}{6}$, $\det L^{-1} = 1$

3. (a) True

A is not invertible iff A is signular iff $\det A = 0$

$\det(AB) = \det A \cdot \det B = 0 \Rightarrow AB$ is not invertible.

(b) False

If A is singular, then $\det A = 0 \neq$ the product of its pivots.

(c) False

$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = 1 \neq \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \det\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = -5$

(d) True

$$\det(AB) = \det A \cdot \det B = \det B \cdot \det A = \det(BA)$$

4. (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$

(b) $C_2 = -\begin{vmatrix} 1 \end{vmatrix},\ C_3 = 0,\ C_4 = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -C_2$

$\Rightarrow C_n = -C_{n-2} \Rightarrow C_{10} = -1$

5. $G_2 = -1,\ G_3 = 2,\ G_4 = -3 \Rightarrow G_n = (-1)^{(n-1)} \cdot (n-1)$ for $n \geq 2$

$$G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$G_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix} = -3$$

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{matrix} = -\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = -3\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

6. Find the determinant of this cyclic P by cofactors of row 1 and then the "big formula". How many exchanges reorder 4, 1, 2, 3 into 1, 2, 3, 4? Is $|P^2| = 1$ or -1?

$$(-1) \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

7. Find the cofactors of A and multiply AC^T to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ -3 & -1 & 1 \\ -6 & 2 & 1 \end{bmatrix} \quad \text{and} \quad AC^T = \underline{\quad\quad}.$$

$\det A = 3$

If you change that 4 to 100, why is $\det A$ unchanged?

平行四边形

8. The parellelogram with sides (2, 1) and (2, 3) has the same area as the parallelogram with sides (2, 2) and (1, 3). Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

9. The box with edges i and j and w=$2i$+$3i$+$4k$ has height $\underline{\ 4\ }$.What is the volume?What is the matrix with determinant?What is $i \times j$ and what is its dot product with w?

4 4 (0,0,1) 4

(1,0,0)

(0,1,0)

(2,3,4)

$$\left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) (2,3,4)$$

$$= (0,0,1)(2,3,4) = 4$$

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 4 - 0 = 4$$

6. (a) $P = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$ $\quad P = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)(1 \cdot 1 \cdot 1 \cdot 1) = -1$

(b) 3 exchanges $\quad$ (c) $\det P^2 = \det P \cdot \det P = 1$

7. (a) $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$, $AC^T = \det A \cdot I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \det A = 3$

(b) $AC^T = \begin{bmatrix} \Sigma a_{1i} C_{1i} & 0 & 0 \\ 0 & \Sigma a_{2i} C_{2i} & 0 \\ 0 & 0 & \Sigma a_{3i} C_{3i} \end{bmatrix} \Rightarrow a_{13} \cdot C_{13} = 4 \times 0 = a_{13}' \cdot C_{13} = 100 \times 0 = 0$

8. (a) $\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$, (b) because $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}^T$ and $\det A = \det A^T$

9. $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (2,3,4)(0,0,1) = 4$

height : 4, volume : 4, det : 4, $i \times j = (0,0,1)$, $(i \times j) w = 4$