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(changes are in purple)

電磁學（一）
Electromagnetics (I)

8. 靜電能與靜電力
Electrostatic Energy and Force

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In this lecture, we will study the energy and force of an electrostatic system.

- 8.1 Energy stored in discrete charges 點電荷系統能量
- 8.2 Energy stored in continuous charges 連續電荷系統能量
- 8.3 Energy stored in field 靜電場的能量
- 8.4 Electrostatic force 靜電力
- 8.5 Review 單元回顧

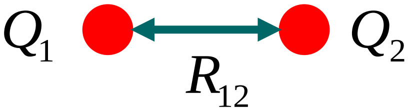
靜電能與靜電力
Electrostatic Energy and Force

8.1 點電荷系統能量
Energy Stored in Discrete Charges

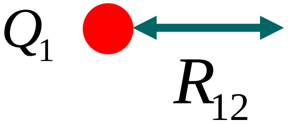
stored electrostatic energy in a charge system

= work done to assemble the charges in the system

Two-charge System



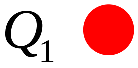
(1) moving Q_2 to a fixed Q_1



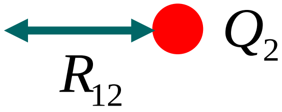
$$W_2 = Q_2 \times \frac{Q_1}{4\pi\varepsilon_0 R_{12}} = Q_2 \times V_2, \text{ where } V_2 = \frac{Q_1}{4\pi\varepsilon_0 R_{12}}$$

(V_2 is the electric potential in the absence of Q_2)

(2) moving Q_1 to a fixed Q_2



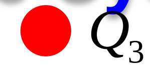
$$W_2 = Q_1 \times \frac{Q_2}{4\pi\varepsilon_0 R_{12}}, \text{where } V_1 = \frac{Q_2}{4\pi\varepsilon_0 R_{12}}$$



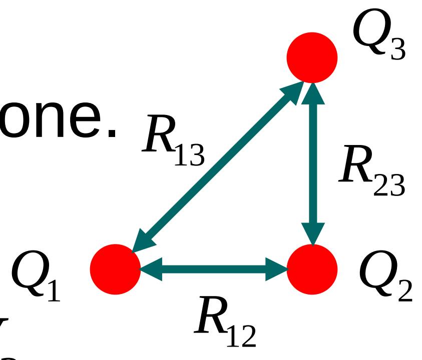
(V_1 is the electric potential at the absence of Q_1)

Thus, $W_2 = Q_2 V_2 = Q_1 V_1 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2)$

Three-charge System



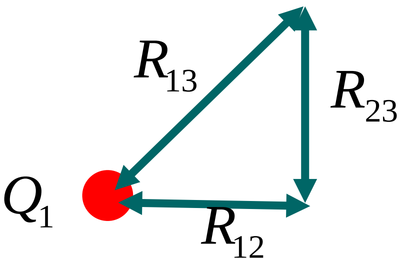
Keep Q_1 stationary and bring in Q_2, Q_3 one by one.



First assemble two charges Q_1 and Q_2…

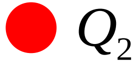
$$W_3 = W_2 + Q_3\left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}}\right) = W_2 + Q_3 V_3$$

Define $V_3 = \dfrac{Q_1}{4\pi\varepsilon_0 R_{13}} + \dfrac{Q_2}{4\pi\varepsilon_0 R_{23}}$ (electric potential without Q_3)



$$V_1 \equiv \frac{Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}}, \quad V_2 \equiv \frac{Q_3}{4\pi\varepsilon_0 R_{23}} + \frac{Q_1}{4\pi\varepsilon_0 R_{12}}$$

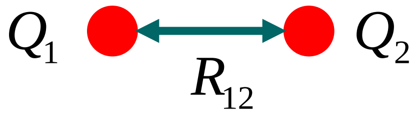
$$\Rightarrow \quad W_3 = \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2 + Q_3 V_3\right)$$



Many-charge System (1)

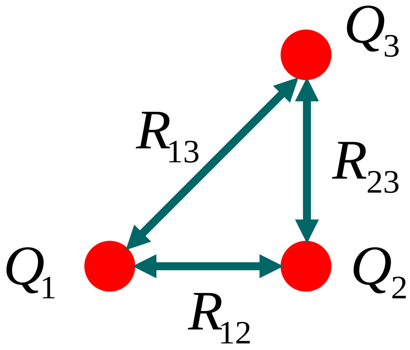
Two charges

$$W_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}}, \text{ for}$$



Three charges

$$W_3 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_1 Q_3}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\varepsilon_0 R_{23}}, \text{ for}$$



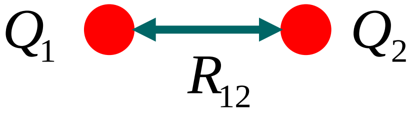
Extend it to a system with N charges

$$\boxed{W_N = \sum_{i=1}^{N-1} \sum_{j>i}^{N} \frac{Q_i Q_j}{4\pi\varepsilon_0 R_{ij}}}$$

Many-charge System (2)

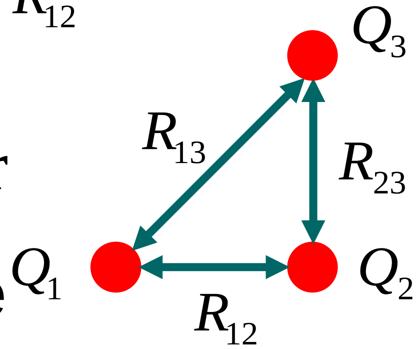
Two Charges

$$W_2 = \frac{1}{2}(Q_1V_1 + Q_2V_2), \text{ for}$$



Three Charges

$$W_3 = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3), \text{ for}$$



For a system with N charges, first define

$$V_k = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}$$

(the total electric potential without charge Q_k)

and write the energy stored in an N-charge system as

$$\boxed{W_N = \frac{1}{2}\sum_{k=1}^{N} Q_kV_k}$$

8.1 電荷系統能量

Energy Stored in Discrete Charges

- The energy stored in a charge system is equal to the work done to assemble the charges.
- For a system with N charges, the stored electrostatic energy is

$$W_N = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k,$$

where V_k is the total electric potential without charge Q_k.

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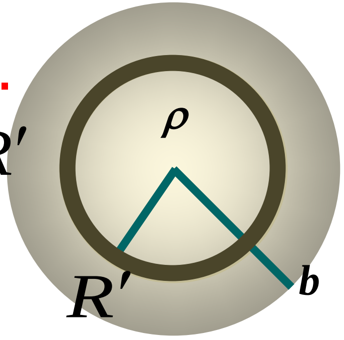
8.2 連續電荷系統能量
Energy Stored in Continuous Charges

Energy stored in a charge ball of radius b and volume charge density of ρ



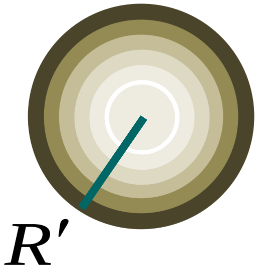
Solution 1: assemble the charge ball layer by layer.

The differential charge at R' is $dq = \rho dV' = \rho 4\pi R'^2 dR'$

The electric potential at R' for a ball radius of R' is

$$V = \frac{\overbrace{(4/3)\pi R'^3\rho}^{q(R')}}{4\pi\varepsilon_0 R'}$$

The differential work to move a sphere of charge from $R=\infty$ to a charge ball of R' is



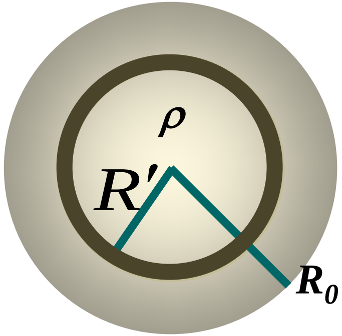
$$dW = Vdq = \frac{(4/3)\pi R'^3\rho}{4\pi\varepsilon_0 R'} \times \underbrace{\rho 4\pi R'^2 dR'}_{dq} = \frac{4\pi R'^4\rho^2}{3\varepsilon_0} dR'$$

The total work to assemble it is $W = \int_0^b \frac{4\pi\rho^2 R'^4}{3\varepsilon_0} dR' = \frac{4\pi\rho^2 b^5}{15\varepsilon_0}$

Solution 2: decompose the charge ball layer by layer.

For continuous charges, the stored energy is written as

$$W_N = \frac{1}{2}\sum_{k=1}^{N} q_k V_k \Rightarrow W_e = \frac{1}{2}\int_{V'} \rho V dv'$$



where V = electric potential without the charge

$$dq = \rho dv' \sim 0 \Rightarrow V\big|_{Q-dq} \sim V\big|_{Q}$$

To move a sphere of charge $dq(R') = \rho 4\pi R'^2 dR'$ from $R = \infty$ to R'

inside a charge ball of R_0, the work to be done is $V(R')dq(R')$

where $V(R') = [V_{b\infty} = \dfrac{Q}{4\pi\varepsilon_0 b}] - \displaystyle\int_b^{R'} [E(R) = \dfrac{\rho \frac{4}{3}\pi R^3}{4\pi\varepsilon_0 R^2}] dR$

The stored energy is then $W_e = \dfrac{1}{2}\displaystyle\int_{V'} \rho V dv = \dfrac{1}{2}\rho \int_0^b V(R') 4\pi R'^2 dR'$

8.2 連續電荷系統能量

Energy Stored in Continuous Charges

There are two ways to calculate the electrostatic energy stored in continuous charges:

- Assembling charges bit by bit, where $V = V(q)$ and q = charge as is
  $$W = \int Vdq$$
- Decomposing all charges bit by bit, where $V = V(Q)$ and Q = total charge
  $$W_e = \frac{1}{2}\int Vdq$$

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8.3 靜電場的能量
Energy Stored in Electric Field

When all the charges are in place, an equivalence between the field energy and mechanic energy must exist, because a charge generates a field through $\nabla \cdot \vec{D} = \rho$

Use $\boxed{W_e = \frac{1}{2}\int_{V'} \rho V dv}$, which assumes all charges are already in place.

Recall $\nabla \cdot \vec{D} = \rho \;\Rightarrow\; W_e = \frac{1}{2}\int_{V'} \rho V dv = \frac{1}{2}\int_{V} (\nabla \cdot \vec{D}) V dv$

V' is the volume of the charges and V is that of ALL space.

Use $\nabla \cdot (V\vec{D}) = V\nabla \cdot \vec{D} + \vec{D} \cdot \nabla V$ to re-write

$$\Rightarrow \quad W_e = \frac{1}{2}\int_V [\nabla\cdot(V\vec{D}) - \vec{D}\cdot\nabla V]dv = \frac{1}{2}\oint_S \underline{V\vec{D}}\cdot\underline{d\vec{s}} + \frac{1}{2}\int_V \vec{D}\cdot\vec{E}dv$$

Apply the divergence theorem

$$V \propto \frac{1}{R}, D \propto \frac{1}{R^2} \Rightarrow V\times D \propto \frac{1}{R^3} \times \; ds \propto R^2 \; \boxed{\propto \frac{1}{R} \xrightarrow[R\to\infty]{} 0}$$

V is the all space volume and S is the surface at $R \to \infty$, as the field extends to infinity.

$$\Rightarrow \; W_e = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}dv = \frac{\varepsilon}{2}\int_V E^2 dv = \frac{1}{2\varepsilon}\int_V D^2 dv \quad \underline{(\text{recall } \vec{D} = \varepsilon\vec{E})}$$

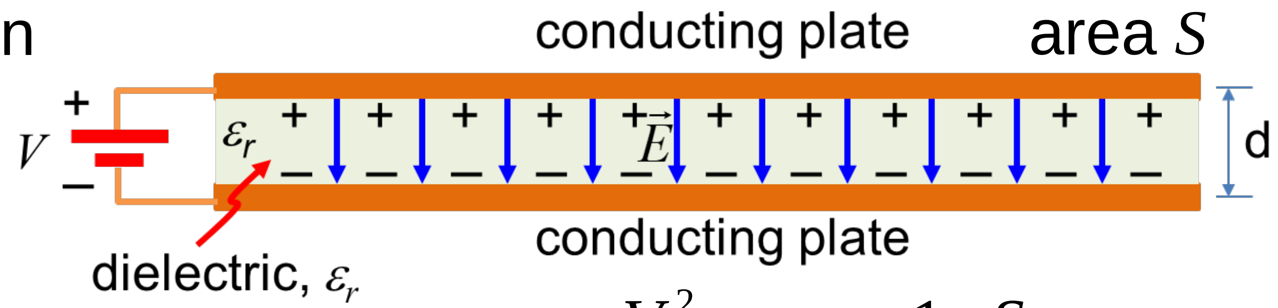
The electrostatic energy density (energy per unit volume) is therefore

$$w_e \equiv \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\varepsilon}{2}E^2 = \frac{D^2}{2\varepsilon}$$

Energy Stored in Capacitor

The electric field in the capacitor is

$$E \sim \frac{V}{d}$$



The stored energy is $W_e = \frac{\varepsilon}{2}\int_V E^2 dv = \frac{\varepsilon}{2}\frac{V^2}{d^2}(Sd) = \frac{1}{2}\frac{\varepsilon S}{d}V^2$

Recall the expressions, $C = \frac{\varepsilon S}{d}$ and $C = \frac{Q}{V}$

The stored energy in a capacitor is $W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}$

This formula is valid for any kind of capacitors and offers another path to derive the capacitance of a capacitor.

8.3 靜電場的能量

Energy Stored in Field

- The energy stored in charges is manifested by a field distribution in space.
- The energy density of an electric field (energy per unit volume) is described by

$$w_e \equiv \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\varepsilon}{2}E^2 = \frac{D^2}{2\varepsilon}$$

- The energy stored in a capacitor is given by

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}$$

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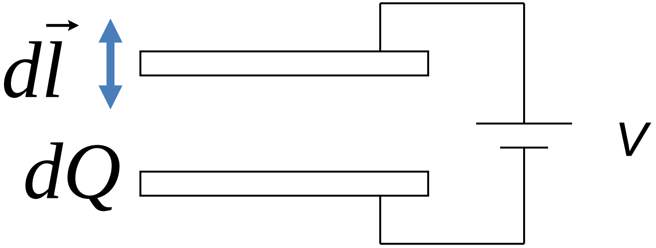
8.4 靜電力
Electrostatic Force

Force (F) and Work (W)

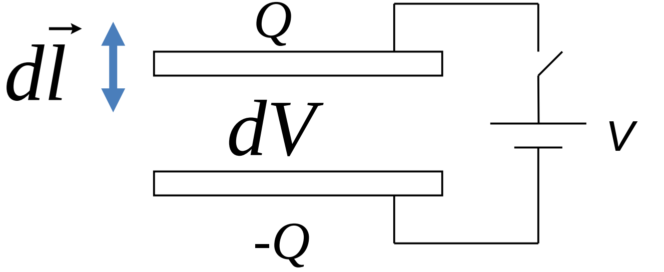
Differential work $dW = \vec{F} \cdot d\vec{l}$ ⇒ $dW = \nabla W \cdot d\vec{l}$ ⇒ $\vec{F} = \nabla W$

Recall $dV = (\nabla V) \cdot d\vec{l}$ from **Lec. 4**

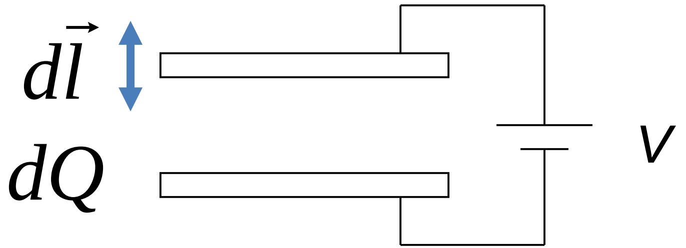
1. **Fixed-voltage system**: a system connected to batteries (forcing a displacement causes a flow of charges)



2. **Fixed-charge system**: an isolated system (forcing a displacement causes a change in voltages)



Thought Experiment I: System with fixed voltage (maintained by battery sources)



Energy conservation requires

$$dW_s = dW + dW_e\Big|_{V=const.}$$

energy supplied by the sources

mechanical work done to the system

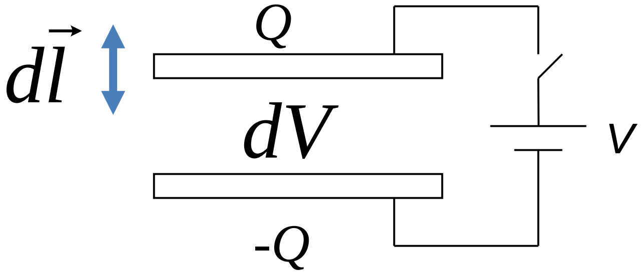
change in internal energy

$$dW_s = \sum_k V_k dQ_k \qquad dW = \vec{F}_{V=const} \cdot d\vec{l} \qquad dW_e\Big|_{V=const} = \frac{1}{2}\sum_k V_k dQ_k$$

x 1/2

$$\left.\begin{aligned} &\vec{F}_{V=const} \cdot d\vec{l} = dW_e\Big|_{V=const} \\ &\text{But } dW_e\Big|_{V=const} = (\nabla W_e\Big|_{V=const}) \cdot d\vec{l} \end{aligned}\right\} \Rightarrow \boxed{\vec{F}_{V=const} = \nabla W_e\Big|_{V=const}}$$

Thought Experiment II: System with fixed charges (isolated system)



Energy conservation requires

$$\cancel{dW_s}^{\,0} = dW + dW_e\Big|_{Q=const.}$$

No sources

mechanical work done to the system

$$dW = \vec{F}_{Q=const} \cdot d\vec{l}$$

change in internal energy

$$dW_e\Big|_{Q=const} = \frac{1}{2}\sum_k Q_k dV_k$$

$$-dW_e\Big|_{Q=const} = \vec{F}_{Q=const} \cdot d\vec{l}$$

But $dW_e = (\nabla W_e) \cdot d\vec{l}$

$$\Rightarrow \vec{F}_{Q=const} = -\nabla W_e\Big|_{Q=const}$$

8.4 靜電力

Electrostatic Force

- The relationship between force and work is given by

$$dW = \vec{F} \cdot d\vec{l} \quad \text{or} \quad \vec{F} = \nabla W$$

- In a fixed voltage system, the electrostatic force is

$$\vec{F}_{V=const} = \nabla W_e \Big|_{V=const}$$

- In an isolated system, the electrostatic force is

$$\vec{F}_{Q=const} = -\nabla W_e \Big|_{Q=const}$$

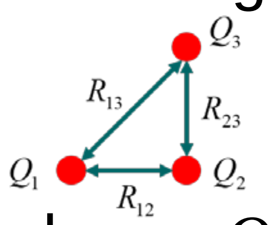
靜電能與靜電力
Electrostatic Energy and Force

8.5 單元回顧 Review

單元回顧

1. The energy stored in a system with N discrete charges is given by

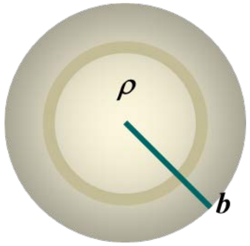
$$W_N = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k,$$



where V_k is the total electric potential without charge Q_k.

2. The energy stored in a continuous-charge system is given by

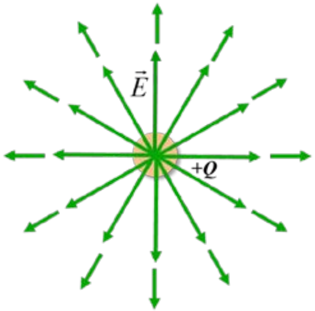
$$W_e = \frac{1}{2}\int V dq$$



$V = V(Q)$, where Q is the total charges.

單元回顧

3. The electrostatic energy density (energy per unit volume) is given by



$$w_e \equiv \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\varepsilon}{2}E^2 = \frac{D^2}{2\varepsilon}$$

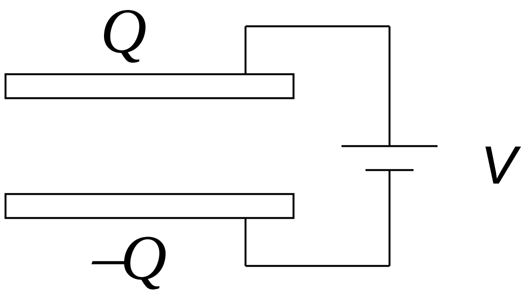
4. The electrostatic energy stored in a volume V is the integration of the energy density over volume:

$$W_e = \int_V w_e dv = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}dv = \frac{1}{2}\int_V \varepsilon E^2 dv = \frac{1}{2}\int_V \frac{D^2}{\varepsilon}dv$$

單元回顧

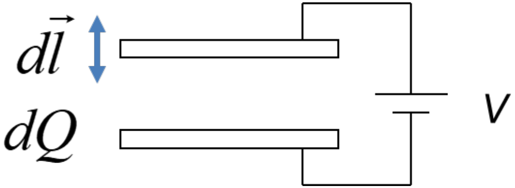
5. A capacitor with capacitance C, voltage V, and charge Q, stores the energy

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}$$



6. The electrostatic force holding a system is given by

$$\vec{F}_{V=const} = \nabla W_e \Big|_{V=const}$$

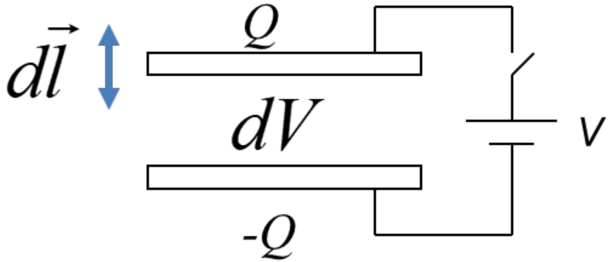


for a thought experiment with fixed voltage sources

單元回顧

7. The electrostatic force holding a system is given by

$$\vec{F}_{Q=const} = -\nabla W_e\Big|_{Q=const}$$



for a thought experiment with an isolated system.

THANK YOU FOR YOUR ATTENTION