

Signals and Systems

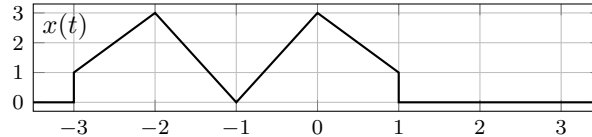
Homework 9 — Due : May 10, 2024

Problem 1 (20 pts). Given the relationships

$$y(t) = x(t) * h(t-t) \quad \text{and} \quad g(t) = 2x(-3t) * h(3t).$$

Given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B .

Problem 2 (30 pts). Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in the figure.



- $X(j\omega)$ can be expressed as $A(j\omega)e^{j\Theta(j\omega)}$, where $A(j\omega)$ and $\Theta(j\omega)$ are both real-values. Find $\Theta(j\omega)$.
- Find $X(j0)$.
- Find $\int_{-\infty}^{\infty} X(j\omega) d\omega$.
- Evaluate $\int_{-\infty}^{\infty} X(j\omega) \frac{2\sin(\omega)}{\omega} e^{-j\omega} d\omega$.
- Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.
- Sketch the inverse Fourier transform of $\text{Re}\{X(j\omega)\}$.

Note: You should perform all these calculations without explicitly evaluating $X(j\omega)$.

Problem 3 (20 pts). Consider an LTI system whose response to the input $x(t) = [e^{-t} + e^{-3t}]u(t)$ is $y(t) = [2e^{-t} - 2e^{-4t}]u(t)$.

- Find the frequency response and the impulse response of the system.
- Find the differential equation relating the input and the output of this system.

Problem 4 (20 pts). The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{d}{dt}y(t) + 10y(t) = x(t) + \int_{-\infty}^{\infty} z(\tau)x(t-\tau)d\tau.$$

where $z(t) = e^{-2t}u(t) + \delta(t)$. Find the frequency response and the impulse response of the system.

Problem 5 (10 pts). Find the Fourier transform of

$$x(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \\ -1, & \text{if } t < 0 \end{cases}$$

Problem 1 (20 pts). Given the relationships

$$y(t) = x(t) * h(t-t) \quad \text{and} \quad g(t) = 2x(-3t) * h(3t).$$

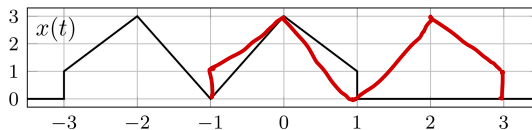
Given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B .

$$Y(j\omega) = X(j\omega) H(-j\omega)$$

$$G(j\omega) = 2 \cdot \frac{1}{3} X\left(-\frac{j\omega}{3}\right) \cdot \frac{1}{3} H\left(\frac{j\omega}{3}\right) = \frac{2}{3} \cdot \frac{1}{3} Y\left(-\frac{j\omega}{3}\right)$$

$$\Rightarrow g(t) = \frac{2}{3} \cdot y(-3t) \quad A = \frac{2}{3} \quad B = -3$$

Problem 2 (30 pts). Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in the figure.



(a) $X(j\omega)$ can be expressed as $A(j\omega)e^{j\Theta(j\omega)}$, where $A(j\omega)$ and $\Theta(j\omega)$ are both real-values. Find $\Theta(j\omega)$.

$$\text{Let } a(t) = x(t-1) \Rightarrow x(t) = a(t+1)$$

$$X(j\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{a(t+1)\} = A(j\omega) \cdot e^{j\omega}$$

$$\theta(j\omega) = \omega$$

(b) Find $X(j0)$.

$$X(j0) = \int_{-\infty}^{\infty} x(t) e^{j0t} dt = \int_{-\infty}^{\infty} x(t) dt = 7$$

(c) Find $\int_{-\infty}^{\infty} X(j\omega) d\omega$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\int_{-\infty}^{\infty} X(j\omega) d\omega = 2\pi x(0) = 6\pi$$

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega) \frac{2 \sin(\omega)}{\omega} e^{-j\omega} d\omega$.

$$\mathcal{F}^{-1} \left\{ \frac{2 \sin \omega}{\omega} \right\} = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} = h(t)$$

$$\chi(-1) * h(-1) = \int_{-\infty}^{\infty} \chi(\tau) h(-1-\tau) d\tau = \int_{-2}^0 \chi(\tau) d\tau = 3$$

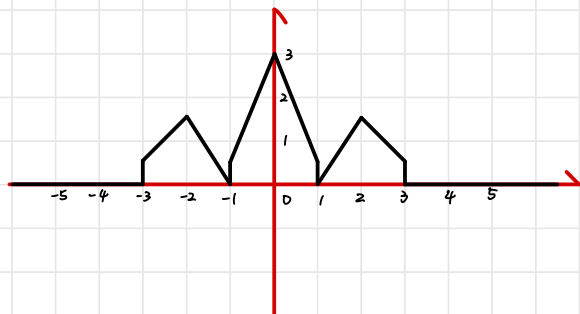
$$\int_{-\infty}^{\infty} \chi(j\omega) \cdot \frac{2 \sin(\omega)}{\omega} e^{-j\omega} d\omega = 2\pi \cdot 3 = 6\pi$$

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} |\chi(t)|^2 dt &= 4\pi \left(\int_0^1 9t^2 dt + \int_0^1 (4t^2 - 12t + 9) dt \right) \\ &= 4\pi \left(3 + \frac{13}{3} \right) = \frac{88}{3}\pi \end{aligned}$$

(f) Sketch the inverse Fourier transform of $\text{Re}\{X(j\omega)\}$.

$$\mathcal{F}^{-1} \{ \text{Re} \{ \chi(j\omega) \} \} = \frac{1}{2} [\chi(t) + \chi(-t)]$$



Problem 3 (20 pts). Consider an LTI system whose response to the input $x(t) = [e^{-t} + e^{-3t}]u(t)$ is $y(t) = [2e^{-t} - 2e^{-4t}]u(t)$.

(a) Find the frequency response and the impulse response of the system.

$$\begin{aligned} F\{x(t)\} &= \int_0^{\infty} e^{-t} \cdot e^{j\omega t} dt + \int_0^{\infty} e^{-3t} \cdot e^{j\omega t} dt \\ &= \int_0^{\infty} e^{-(j\omega+1)t} dt + \int_0^{\infty} e^{-(j\omega+3)t} dt \\ &= \frac{1}{j\omega+1} + \frac{1}{j\omega+3} = \frac{2j\omega+4}{(j\omega+1)(j\omega+3)} \end{aligned}$$

$$F\{y(t)\} = \frac{2}{j\omega+1} - \frac{2}{j\omega+4} = \frac{6}{(j\omega+1)(j\omega+4)}$$

Frequency Response

$$H(j\omega) = \frac{6 \cdot (j\omega+3)}{(j\omega+4) \cdot 2 \cdot (j\omega+2)} = \frac{3}{2} \cdot \frac{2j\omega+6}{(j\omega+4) \cdot (j\omega+2)} = \underline{\underline{\frac{3}{2} \left(\frac{1}{j\omega+4} + \frac{1}{j\omega+2} \right)}}$$

$$h(t) = F^{-1}\{H(j\omega)\} = \underline{\underline{\frac{3}{2} (e^{-2t} + e^{-4t}) u(t)}}$$

Impulse Response

(b) Find the differential equation relating the input and the output of this system.

$$H(j\omega) = \frac{3j\omega+9}{(j\omega)^2+6j\omega+8} = \frac{Y(j\omega)}{X(j\omega)}$$

$$\Rightarrow 3 \cdot \frac{d}{dt} x(t) + 9 x(t) = \frac{d^2}{dt^2} y(t) + 6 \cdot \frac{d}{dt} y(t) + 8 y(t)$$

Problem 4 (20 pts). The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{d}{dt}y(t) + 10y(t) = x(t) + \int_{-\infty}^{\infty} z(\tau)x(t-\tau)d\tau.$$

where $z(t) = e^{-2t}u(t) + \delta(t)$. Find the frequency response and the impulse response of the system.

$$\frac{d}{dt} h(t) + 10 h(t) = z(t) + \int_{-\infty}^{\infty} [e^{-2\tau} u(\tau) + z(\tau)] z(t-\tau) d\tau$$

$$j\omega H(j\omega) + 10 H(j\omega) = 1 + \frac{1}{j\omega+2} + 1$$

$$H(j\omega) = \frac{2j\omega+5}{(j\omega+2)(j\omega+10)} = \frac{1}{8} \cdot \frac{1}{j\omega+2} + \frac{15}{8} \cdot \frac{1}{j\omega+10}$$

$$h(t) = \frac{1}{8} e^{-2t} u(t) + \frac{15}{8} e^{-10t} u(t)$$

Problem 5 (10 pts). Find the Fourier transform of

$$x(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \\ -1, & \text{if } t < 0 \end{cases}$$

$$F\{u(t)\} = \frac{1}{j\omega}, \quad F\{u(-t)\} = -\frac{1}{j\omega}$$

$$\begin{aligned} F\{x(t)\} &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^0 (-1) e^{-j\omega t} dt + \int_0^{\infty} 1 \cdot e^{-j\omega t} dt \\ &= \frac{1}{j\omega} + \pi\delta(\omega) - \frac{1}{-j\omega} - \pi\delta(-\omega) = \frac{2}{j\omega} \end{aligned}$$