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電磁學（一）
Electromagnetics (I)

5. 靜電場與電位
Static Electric Field and Electric Potential

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In this lecture, we are to learn the concept and theory of static electric field and electric potential subject to isolated charges.

- 5.1 Postulates of electrostatics 靜電學基本假設
- 5.2 Electric potential 電位
- 5.3 Electric dipole 電耦極
- 5.4 Gaussian surface 高斯面
- 5.5 Integration of charges 電荷積分
- 5.6 Review 單元回顧

靜電場與電位
Static Electric Field and Electric Potential

5.1 靜電學基本假設
Postulates of electrostatics

Postulates for Electrostatics

At a point in space, the electric field $\vec{E}$ and the electric flux density $\vec{D}$ are governed by the two laws:

$$\nabla \times \vec{E} = 0, \quad \text{(Faraday's law of electrostatics)}$$

$$\nabla \cdot \vec{D} = \rho, \quad \text{(Gauss law)}$$

where ρ is the volume charge density.

In vacuum $\vec{D} = \varepsilon_0 \vec{E}$, where ε_0 is the vacuum permittivity

$$\varepsilon_0 \cong \frac{1}{36\pi} \times 10^{-9} \cong 8.854 \times 10^{-12} \quad \text{F/m}$$

Faraday's Law of Electrostatics

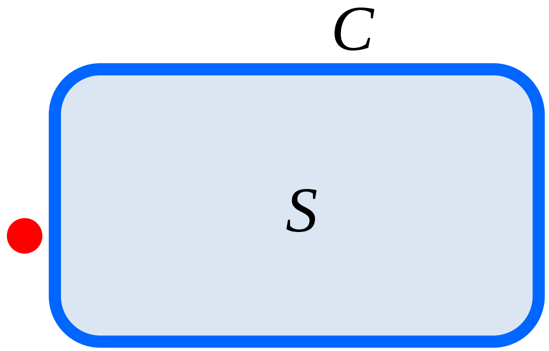
$$\nabla \times \vec{E} = 0 \quad \Rightarrow$$

A static electric field is irrotational

Apply Stokes' theorem to $\nabla \times \vec{E} = 0$

$$\int \nabla \times \vec{E} \cdot d\vec{s} = \oint_C \vec{E} \cdot d\vec{l} = \oint_C \underbrace{\frac{\vec{F}}{q} \cdot d\vec{l}}_{dW\,:\,\text{work}} = 0$$

force acting on a
unit positive charge



Physical meaning: the work done by an electric field on a unit positive charge around a closed path is zero

Conservative Force

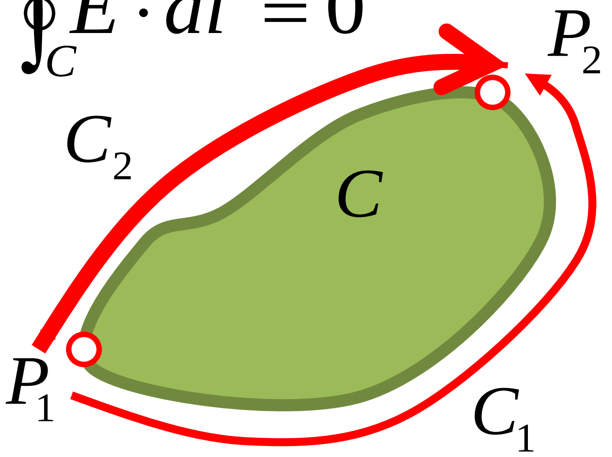
Consider the close-loop integration $\oint_C \vec{E}\cdot d\vec{l} = 0$

Divide the path into two sections

$$\underline{\int_{P_1}^{P_2} \vec{E}_{C_1} \cdot d\vec{l}} + \underline{\int_{P_2}^{P_1} \vec{E}_{C_2} \cdot d\vec{l}} = 0$$

$$\Rightarrow \int_{P_1}^{P_2} \vec{E}_{C_1} \cdot d\vec{l} - \int_{P_1}^{P_2} \vec{E}_{C_2} \cdot d\vec{l} = 0$$

$$\Rightarrow \int_{P_1}^{P_2} \vec{E}_{C_1} \cdot d\vec{l} = \int_{P_1}^{P_2} \vec{E}_{C_2} \cdot d\vec{l}$$



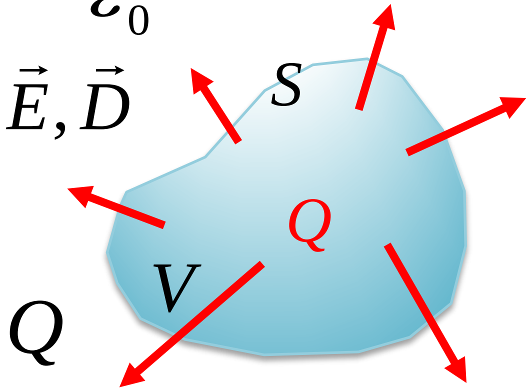
The work done by moving a test charge between two arbitrary points in space is independent of the path.

Gauss Law

In vacuum, $\vec{D} = \varepsilon_0 \vec{E}$, where ε_0 is a constant

$$\nabla \cdot \vec{D} = \rho \quad \underset{\text{in vacuum}}{\Longrightarrow} \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Apply the divergence theorem



$$\int_V \nabla \cdot \vec{D} dv = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s}$$

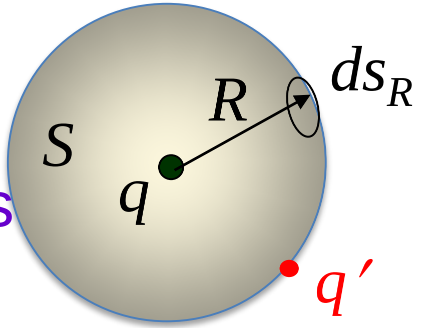
$$\& \quad \int_V \rho dv = Q \Longrightarrow \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q$$

The total outward electric flux of a volume equals the total charge in the volume, Q.

Coulomb's Law

Eg. Apply the integral form of the Gauss law to a point charge in a spherical space

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$$



Due to spherical symmetry, only E_R exists and is a constant at a constant R. The differential surface is $d\vec{s} = d\vec{s}_R = \hat{a}_R R^2 \sin\theta d\theta d\phi$

$$q = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 \oint_S E_R ds_R = \varepsilon_0 E_R \oint_S ds_R = \varepsilon_0 E_R 4\pi R^2$$

$$\Rightarrow \vec{E} = E_R \hat{a}_R = \frac{q}{4\pi\varepsilon_0 R^2} \hat{a}_R \Rightarrow \vec{F}_{q'} = q' E_R \hat{a}_R = \frac{q' q}{4\pi\varepsilon_0 R^2} \hat{a}_R$$

5.1 靜電學基本假設

Postulates of electrostatics

- Two postulates for electrostatics – Faraday’s law of electrostatics and Gauss’ law.
- Faraday’s law of electrostatics – work done on a charge by an electric field is independent of the moving path of the charge.
- Gauss’ law – total outward electric flux of a volume is equal to the amount of the charges in the volume.

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5.2 電位
Electric potential

Electric Potential V

Given the first postulate of electrostatics, $\nabla \times \vec{E} = 0$

For a scalar V in space, the null identity applies

$$\nabla \times (\nabla V) = 0$$

By comparison, one has $\vec{E} = \pm \nabla V$

The $\pm$ signs before the del operator, ∇, have different physical meanings. We will choose the sign from a correct physical definition.

What is V?

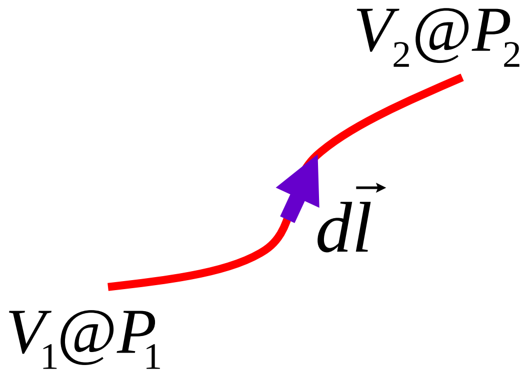
Recall the useful expression from **4.2**, $dV = (\nabla V)\cdot d\vec{l}$

Consider an integration path between P_1 and P_2

$$V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} \nabla V \cdot (\hat{a}_l dl)$$

But $\vec{E} = \pm \nabla V$

$$\Rightarrow \quad \underbrace{V_{21} = \pm \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} = \pm \int_{P_1}^{P_2} \frac{\vec{F}}{q} \cdot d\vec{l}}$$



work done (+) on or lost by (–) a unit positive charge.

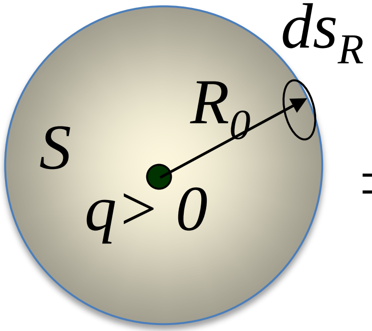
+ or –

● $q' = +1$

Define V as "Work done externally by moving a unit positive (negative) charge q' from infinity to a positive (negative) charge q is positive. (common sense!)"

$$V(R_0) - V(\infty) = \int_{\infty}^{R_0} \nabla V \cdot d\vec{l} = \pm \int_{\infty}^{R_0} \vec{E} \cdot d\vec{l}$$

$d\vec{l}$ was defined in the **Lecture 3** and has nothing to do with the moving direction of the test charge.



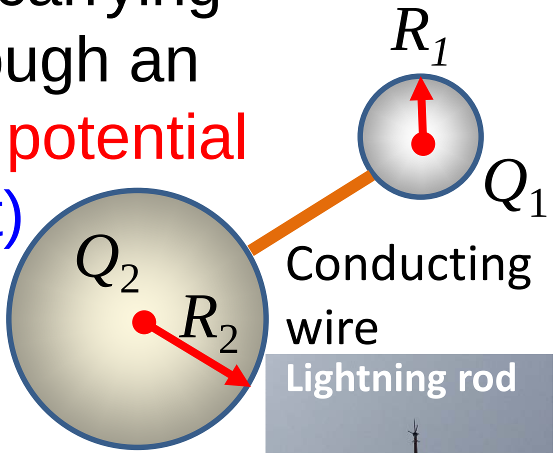
$$\Rightarrow \pm \int_{\infty}^{R_0} \vec{E} \cdot d\vec{l} = \pm \int_{\infty}^{R_0} \frac{q}{4\pi\varepsilon_0 R^2} dR = \mp \frac{q}{4\pi\varepsilon_0 R_0} > 0$$

for $q > 0$

Apparently, $V(R_0) = \dfrac{q}{4\pi\varepsilon_0 R_0}$ and $\boxed{\vec{E} \equiv -\nabla V}$

Sharp-tip Discharge

Two conducting balls of different radii carrying charges Q_1 and Q_2 are connected through an electric wire, having the same electric potential (a conductor is an equipotential object)



$$V_1 = V_2 \Rightarrow \frac{Q_1}{4\pi\varepsilon_0 R_1} = \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

The electric fields on the ball surfaces are

$$E_{1,2} = \frac{Q_{1,2}}{4\pi\varepsilon_0 R_{1,2}^2} \quad \Rightarrow \quad \frac{E_1}{E_2} = \frac{R_2}{R_1} \Rightarrow E \propto \frac{1}{R}$$

$\Rightarrow$ The smaller the ball, the higher the electric field

5.2 電位

Electric potential

- Mathematically, the electric field is the negative gradient of the electrical potential.
- Physically, the work done by moving a unit positive charge between two places is the difference of the electric potentials at the two places.
- A sharp metal tip with charges tends to have a strong electric field.

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5.3 電耦極
Electric dipole

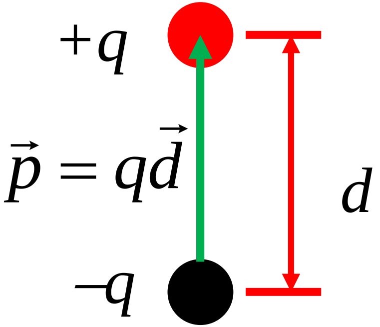
Electric Dipole Moment

An electric dipole consists of two opposite charges separated by a distance *d*, which is commonly found in a neutral dielectric.

The electric dipole moment is defined as

$$\vec{p} = q\vec{d}.$$

The direction of $\vec{p} = q\vec{d}$ is from the negative charge to the positive charge.



Electric-dipole Potential

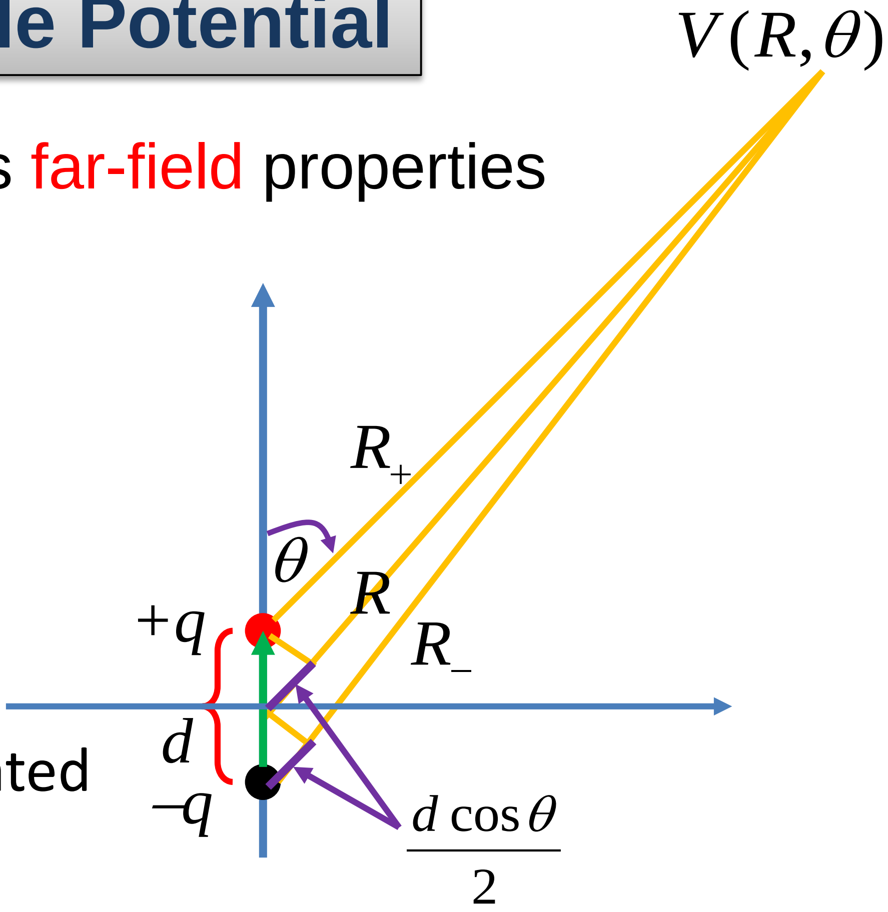
We are only interested in its far-field properties

$$R >> d \quad \Rightarrow \quad R_{\pm} \approx R \mp \frac{d\cos\theta}{2}$$

$$V(R,\theta) = \frac{q}{4\pi\varepsilon_0 R_+} - \frac{q}{4\pi\varepsilon_0 R_-}$$

$$= \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2} \propto \frac{1}{R^2}$$

*Note that $V \propto \dfrac{1}{R}$ for an isolated charge .

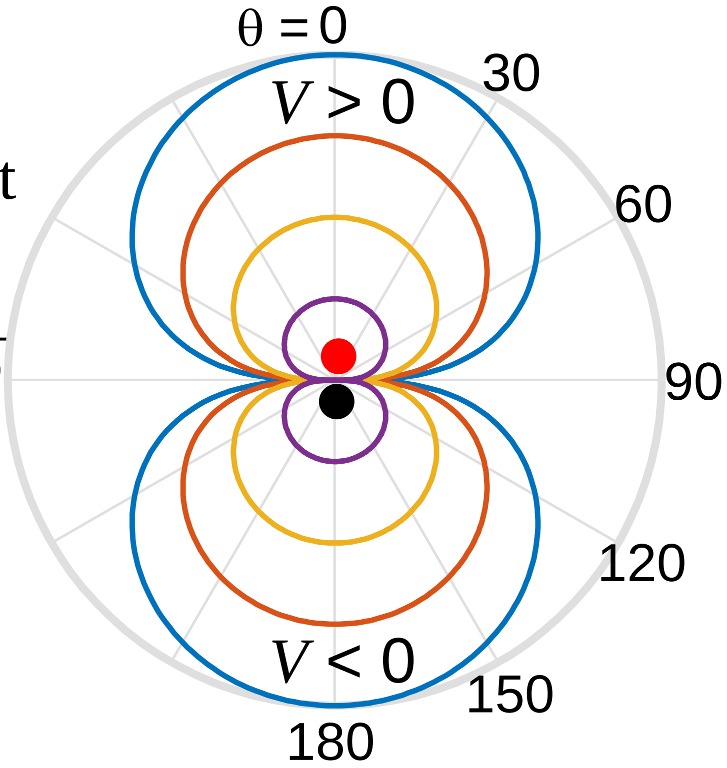


Equipotential Lines/Surfaces

set $V(R) = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \text{constant}$

to obtain $R = \text{constant} \times \sqrt{\cos\theta}$

for equipotential lines/surfaces



Electric dipole field

Use $\vec{E} \equiv -\nabla V$ to obtain the far-zone dipole field

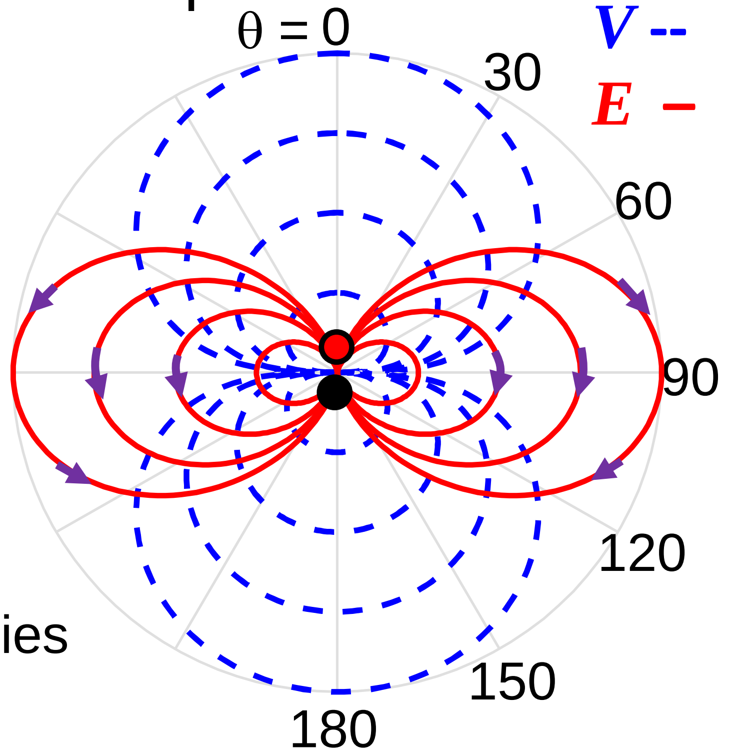
$$\vec{E} = -\nabla V = -\hat{a}_R \frac{\partial V}{\partial R} - \hat{a}_\theta \frac{\partial V}{R\partial \theta}$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

Set $d\vec{l} = k\vec{E}$ (recall the definition of the E field line)

to obtain $R = \text{constant} \times \sin^2\theta$

*note the orthogonality of the two families of curves due to $\vec{E} = -\nabla V$



5.3 電耦極

Electric dipole

- An electric dipole contains two opposite charges separated by a distance.
- Electric dipoles are the basic building blocks of a dielectic.
- In the far zone, the electric field and electric potential of an electric dipole are inversely proportional to R^3 and R^2, respectively.

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5.4 高斯面
Gaussian surface

Gaussian Surface

Consider the Gauss's Law $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = q$

Suppose

- all the field vectors of equal magnitude are along the surface normal of a volume, and
- this volume has symmetry in one of the three coordinate systems.

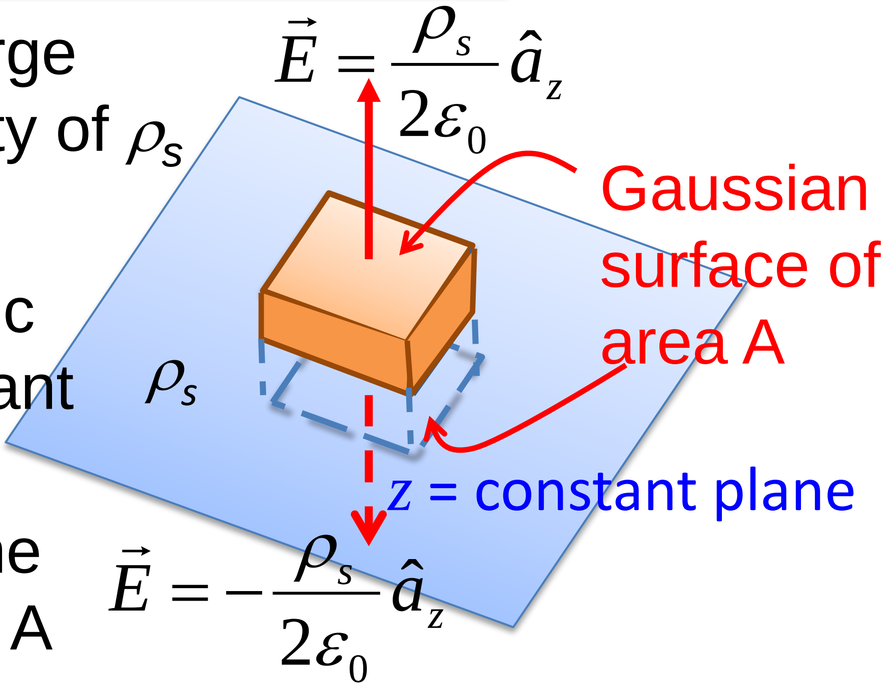
$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 E_i \underbrace{\oint_S ds_i}_{\text{Gaussian surface}} = q \Rightarrow E_i = \frac{q}{\varepsilon_0 \oint_S ds_i}, \vec{E} = E_i \hat{a}_i$$

constant

Infinite Planar Charge

Given an infinite planar charge with a surface charge density of ρ_s

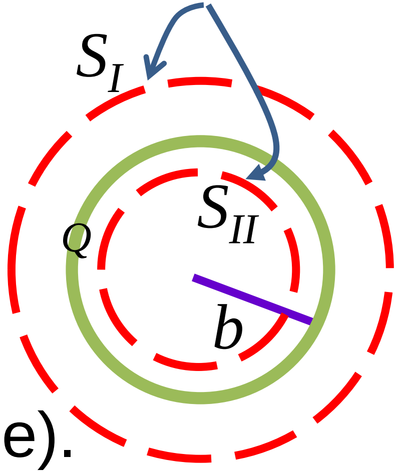
- Due to symmetry, only normal-component electric fields exist and are constant at a constant z.
- Apply the Gauss law to the Gaussian surface of area A



$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 E_z A + \varepsilon_0 (-E_z)(-A) = A\rho_s \Rightarrow E_z = \frac{\rho_s}{2\varepsilon_0}$$

Spherical Charges

Given a spherical shell of uniform charges Q at radius b. Find E and V everywhere.



- Due to spherical symmetry, only R-component electric fields exist and are constant at a constant R (Gaussian surface).

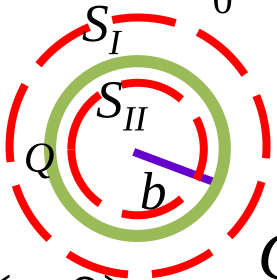
- Apply the Gauss law to the Gaussian surfaces S_I and S_{II}

  * Charges enclosed in S_I is Q and that in S_{II} is 0.

Region I: $R > b$

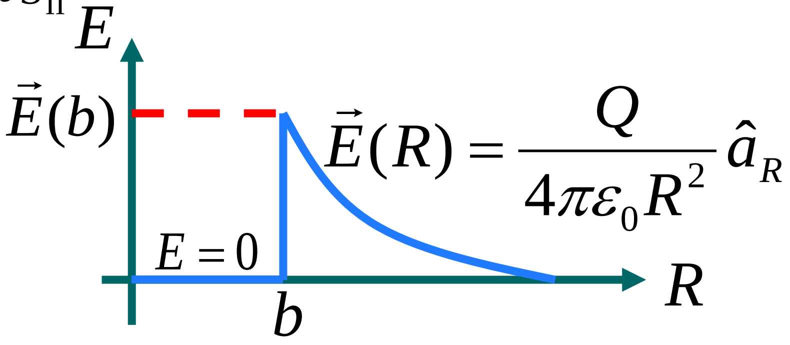
$$\varepsilon_0 \oint_{S_I} \vec{E} \cdot d\vec{s} = Q \Rightarrow \varepsilon_0 4\pi R^2 E_R = Q \quad \Rightarrow \vec{E} = E_R \hat{a}_R = \frac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R$$

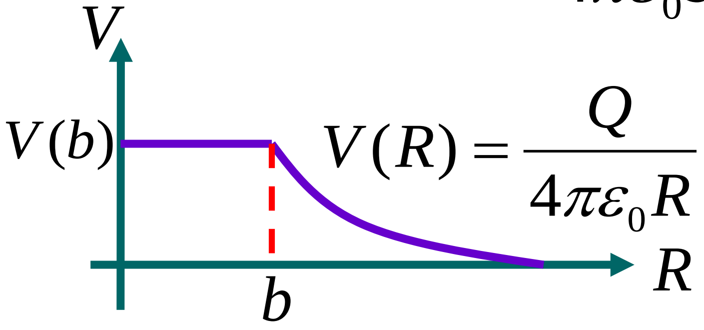
$$V(R) = V_{R\infty} = -\int_{\infty}^{R} \vec{E} \cdot d\vec{R} = \frac{Q}{4\pi\varepsilon_0 R}$$



Region II: $0 \le R < b$

$$\varepsilon_0 \oint_{S_{\mathrm{II}}} \vec{E} \cdot d\vec{s} = 0 \Rightarrow \vec{E} = 0 \qquad V(R) = V(b) + V_{Rb}(=0) = \frac{Q}{4\pi\varepsilon_0 b}$$





5.4 高斯面

Gaussian Surface

- On a Gaussian surface, the electric field along the surface normal is a constant.
- A Gaussian surface is convenient for calculating the electric field by using the integral form of the Gauss Law:

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 E_i \oint_S ds_i = q$$

$$\Rightarrow E_i = \frac{q}{\varepsilon_0 \oint_S ds_i}, \vec{E} = E_i \hat{a}_i$$

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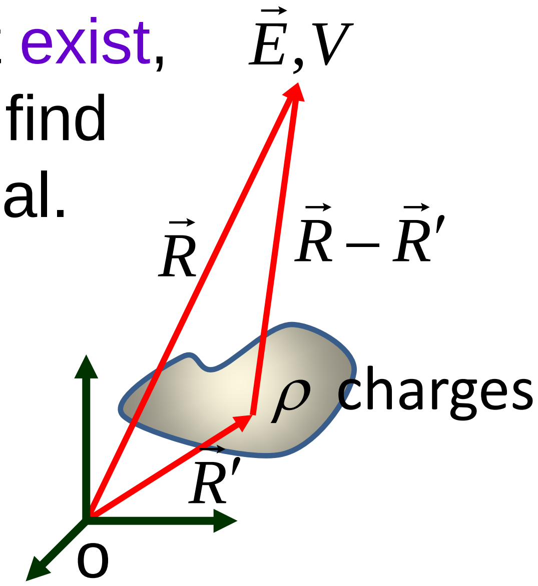
5.5 電荷積分
Integration of Charges

Distributive Integration of Charges

When a Gaussian surface does not exist, integrate the distributed charges to find the electric field and electric potential.



$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{V',S',L'}\hat{a}_{\vec{R}-\vec{R}'}\frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|^2}dv',s',l'$$

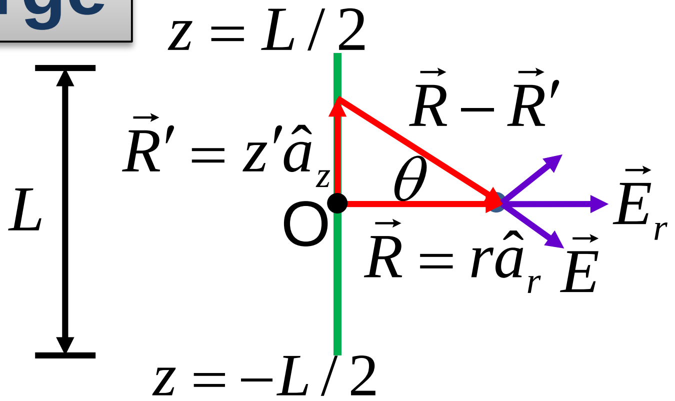
One can calculate V first and then E

$$V=\frac{1}{4\pi\varepsilon_0}\int_{V',S',L'}\frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|}dv',s',l' \Rightarrow \vec{E}=-\nabla V$$

* note: a prime ′ is used to denote the source coordinates.

Line Charge

Given a line charge with a finite length L and charge density ρ_l, find E at r and $z = 0$.



Recall

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{V',S',L'} \hat{a}_{\vec{R}-\vec{R}'} \frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|^2} dv', s', l'$$

where $\left|\vec{R}-\vec{R}'\right|^2 = r^2 + z'^2$

Due to symmetry, only E_r exits. $\vec{E}_r = \cos\theta \times \vec{E}$

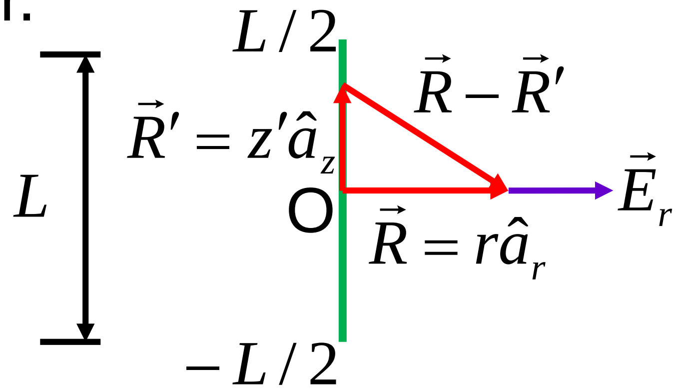
$(\cos\theta = r\Big/\sqrt{r^2 + z'^2})$

$$E_r(z=0,r) = \frac{\boxed{2}}{4\pi\varepsilon_0} \int_0^{L/2} \frac{\rho_l}{r^2 + z'^2} \underline{\cos\theta} dz' = \frac{\rho_l}{2\pi\varepsilon_0 r} \frac{L/2}{\sqrt{(L/2)^2 + r^2}}$$

Scalar integration is often easier.

Alternatively, use

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V',S',L'} \frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|} dv', s', l'$$



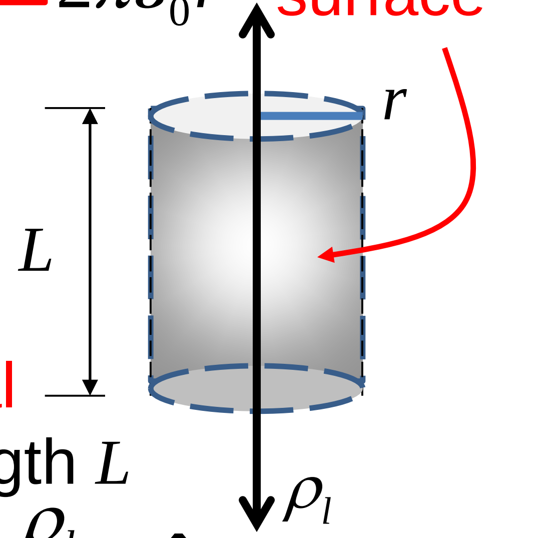
to obtain

$$V(z=0,r) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2} \frac{\rho_l}{\sqrt{r^2+z'^2}} dz' = \tfrac{\rho_l}{4\pi\varepsilon_0}\ln(z'+\sqrt{r^2+z'^2})\Big|_{-L/2}^{L/2}$$

Then, use $\vec{E} = -\nabla V$ to obtain $\vec{E}$

Infinite Line Charge

$$E_r(z=0,r)=\frac{\rho_l}{2\pi\varepsilon_0 r}\frac{L/2}{\sqrt{(L/2)^2+r^2}}\xrightarrow[\boxed{L\to\infty}]{}\frac{\rho_l}{2\pi\varepsilon_0 r}$$



- Due to symmetry, only the radial components of E exist and are constant at a constant r.
- Apply the Gauss law to the cylindrical Gaussian surface of radius r and length L

$$\varepsilon_0\oint_S \vec{E}\cdot d\vec{s}=\varepsilon_0 E_r 2\pi r L = L\rho_l \Rightarrow \vec{E}=\frac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$$

5.5 電荷積分

Integration of charges

- When a Gaussian surface cannot be found, use distributive integration to calculate the electric field from a detailed charge distribution.
- It is often easier to first calculate the scalar integration for V and then take the gradient of V to get E.

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5.6 單元回顧 Review

單元回顧

1. Two postulates for electrostatics

Faraday's law of electrostatics

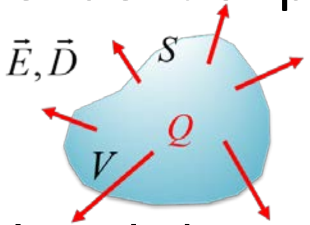
$$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \oint_L \vec{E} \cdot d\vec{l} = 0$$

An electrostatic field is irrotational

Work done on a charge is independent of path

Gauss law

$$\nabla \cdot \vec{D} = \rho \quad \Rightarrow \quad \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q$$



The total outward electric flux of V equals total enclosed charge in V.

單元回顧

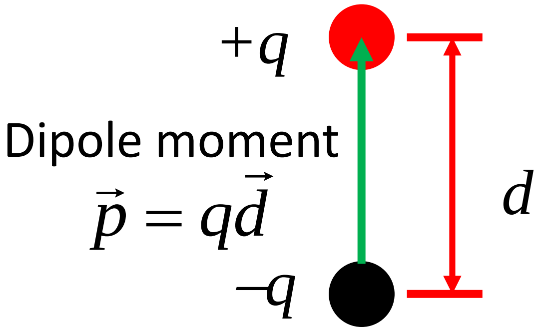
2. The electric potential V is defined in $\vec{E} = -\nabla V$

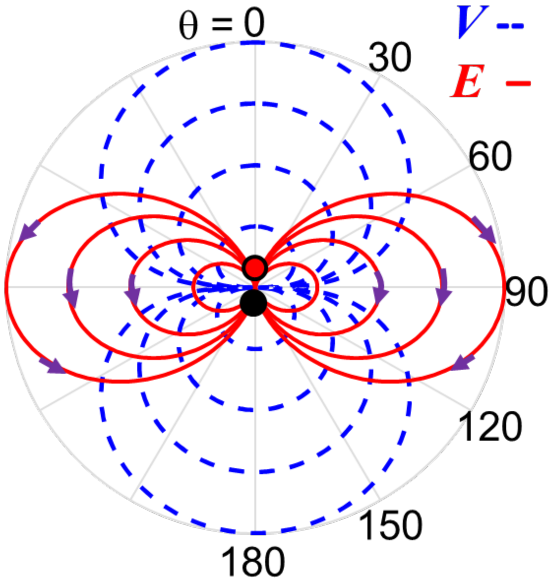
3. The total work done by moving a charge q from Point 1 to 2 in space is therefore $q(V_2 - V_1)$

4. A sharp metal tip tends to have a high electric field, which could induce electric discharge to air.

單元回顧

5. An electric dipole consists of two opposite charges separated by a distance *d*, which is the basic electromagnetic element in a neutral dielectric.

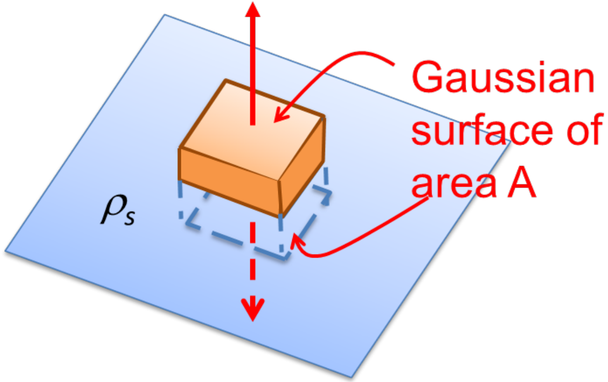




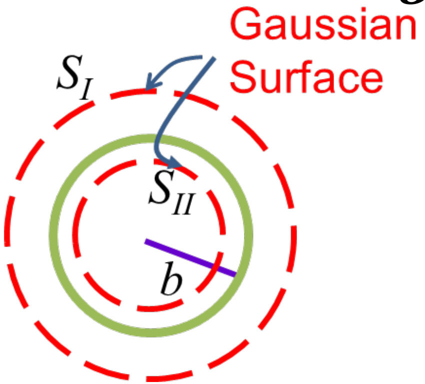
單元回顧

7. A Gaussian surface is convenient for calculating the electric field by using the integral form of the Gauss law.

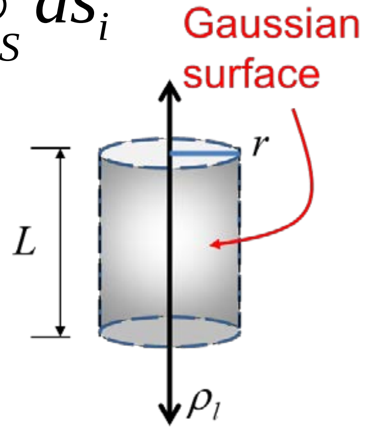
$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 E_i \oint_S ds_i = q \quad \Rightarrow E_i = \frac{q}{\varepsilon_0 \oint_S ds_i}, \vec{E} = E_i \hat{a}_i$$



Infinite sheet charges



Ball charges

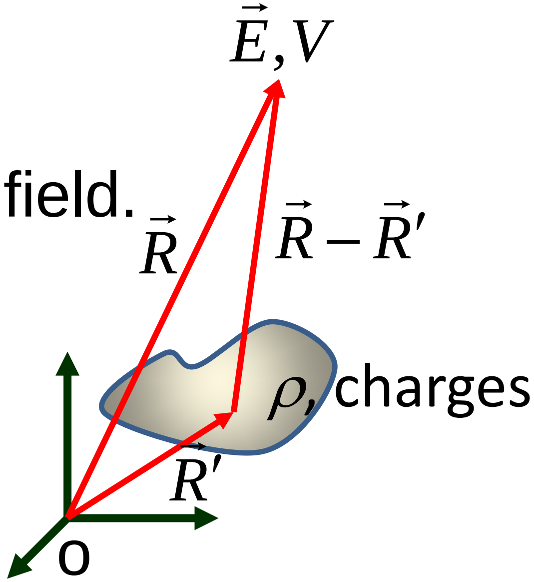


Infinite line charges

單元回顧

8. When a Gaussian surface does not exit, integrate the point-charge field over the distributed charges to find the total electric field.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{V',S',L'} \hat{a}_{\vec{R}-\vec{R}'} \frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|^2} dv', s', l'$$



It is often easier to calculate V first and then E

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V',S',L'} \frac{\rho_{v,s,l}}{\left|\vec{R}-\vec{R}'\right|} dv', s', l' \quad \Rightarrow \quad \vec{E} = -\nabla V$$

THANK YOU FOR YOUR ATTENTION