Linear Algebra, EE 10810/EECS 205004

1st Exam

(Dated: Fall, 2021)

Total scores: 120%

-6

1. ($\pm$30%) [**True or False**] Note that: a Correct answer gaining +3%; but a Wrong answer loosing -3% (答錯倒扣).

   (1) : In any vector space, $a\,\vec{x} = b\,\vec{x}$ implies that $a = b$. F: if $\vec{x}=0$

   (2) The zero vector space has no basis. F

   (3) Any set containing the zero vector is linearly dependent. T

   (4) If $\hat{T}$ is linear, then $\hat{T}(\vec{0}_V) = \vec{0}_W$. T

   (5) $\overline{\overline{A}}^2 = \overline{\overline{I}}$ implies that $\overline{\overline{A}} = \overline{\overline{I}}$ or $\overline{\overline{A}} = -\overline{\overline{I}}$. T

   (6) If $\overline{\overline{A}}$ is invertible, then $(\overline{\overline{A}}^{-1})^{-1} = \overline{\overline{A}}$. T

   (7) Every change of coordinate matrix is invertible. T

   (8) A linear functional defined on a field may be represented as a 1×1 matrix. T

   (9) The transpose of an elementary matrix is an elementary matrix. T

   (10) If $\overline{\overline{AB}} = \overline{\overline{0}}$, then either $\overline{\overline{A}} = \overline{\overline{0}}$ or $\overline{\overline{B}} = \overline{\overline{0}}$, where $\overline{\overline{0}}$ is the zero matrix. F

2. (10%) [**Linearly dependent**]

   Let $\mathcal{V}$ be a vector space, and let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{V}$. If $\mathcal{S}_1$ is linear dependent. Prove that $\mathcal{S}_2$ is linearly dependent.

3. (10%) [**Matrix representation**]

   $-\frac{7}{3} + \frac{4}{3} = -\frac{3}{3} = -1 \quad -\frac{2}{3}\ 12 + \frac{4}{3}$

   Let $\hat{T} : \mathcal{R}^2 \to \mathcal{R}^3$ be defined by

   $$\hat{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2). \tag{1}$$

   Let $\alpha = \{(1, 2), (2, 3)\}$ be the basis for $\mathcal{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $\left[\hat{T}\right]_\alpha^\gamma$.

4. (15%) [**Basis and Dimension**]

   Let S be the set of all positive real numbers. Now, we want to make S as a vector space in $\mathcal{V}$ by asking the following definitions for vectors, vector addition and scalar multiplication:

   - Each element of S will be considered as a "vector" in $\mathcal{V}$.
   - For $A, B \in S$, a "vector sum" is defined as

   $$A + B \equiv A\,B, \tag{2}$$

     where the product on the right is the usual product of two real numbers.
   - For $c \in \mathcal{R}$(real), and $A \in S$, a "scalar multiplication" is defined as

   $$c \cdot A \equiv A^c, \tag{3}$$

     that is the real number A raised to the c power.

   Now the questions are

   (a) (5%) What is the zero vector in $\mathcal{V}$?

   (b) (5%) Give an example of a set of basis vectors for $\mathcal{V}$.

   (c) (5%) What is the dimension of $\mathcal{V}$?

5. (15%) [**System of Linear Equations**]

Find the solution(s) of the following system of linear equations

$$\begin{array}{rcrcrcrcrcl} 2x_1 & - & 2x_2 & - & x_3 & + & 6x_4 & - & 2x_5 & = & 1 \\ x_1 & - & x_2 & + & x_3 & + & 2x_4 & - & x_5 & = & 2 \\ 4x_1 & - & 4x_2 & + & 5x_3 & + & 7x_4 & - & x_5 & = & 6 \end{array} \tag{4}$$

63135 - 92

98-92=6

6. (20%) [**Linear Transformation**]

Let $\mathcal{V}$ be the subspace of 2×2 real matrices defined by Find the solution(s) of the following system of linear equations

$$\mathcal{V} = \left\{ \overline{\overline{\mathrm{A}}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{where} \quad a - 2b - 3c - 4d = 0 \right\} \tag{5}$$

and let Find the solution(s) of the following system of linear equations

$$\mathcal{W} = \{ f(x) = l\, e^{x}. + m\, e^{2x} + n\, e^{3x}, \quad \text{where} \quad l, m, n \in \mathcal{R} \quad \text{and} \quad 0 \le x \le 1 \}. \tag{6}$$

Also, define a linear transformation $\hat{T} : V \to W$ by Find the solution(s) of the following system of linear equations

$$\hat{T}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a - 2b)\, e^{x} + (a - 3c)\, e^{2x} + (a - 4d)\, e^{3x}. \tag{7}$$

x

$a-2b=\lambda$ $a-3c=m$

(a) (5%) Find the range of $\hat{T}$.

(b) (5%) Find the null space of $\hat{T}$. $a-2b=\alpha$

(c) (10%) Is $\hat{T}$ invertible? Justify your answer and if yes, Find $\hat{T}^{-1}$. 2λ λ $3m$ m

$a-d=2b$

7. (20%) [**Parabola transformation with a Rotation**]

As shown in FIG. 1(a), let $\hat{T}_0$ be the transformation sending every horizontal line $y = c$ into the parabola $y = x^2 + c$

(a) (5%) Find $\hat{T}_0$, i.e., $\begin{pmatrix} x' \\ y' \end{pmatrix} = \hat{T}_0 \begin{pmatrix} x \\ y \end{pmatrix}$. Try to have the matrix representation of $\left[\hat{T}_0\right]_{\beta}^{\beta'}$ in the standard ordered basis. If not, explain why.

(b) (5%) Now, if we rotation the $x - y$ coordinate by an angle Θ, find the corresponding matrix for the change of coordinate.

(c) (10%) Let $\hat{T}_1$ be the parabola transformation about the new coordinate, i.e., about the line in Red-color shown in FIG. 1(b). Find the corresponding transformation r $\hat{T}_1$, i.e., $\begin{pmatrix} x' \\ y' \end{pmatrix} = \hat{T}_1 \begin{pmatrix} x \\ y \end{pmatrix}$

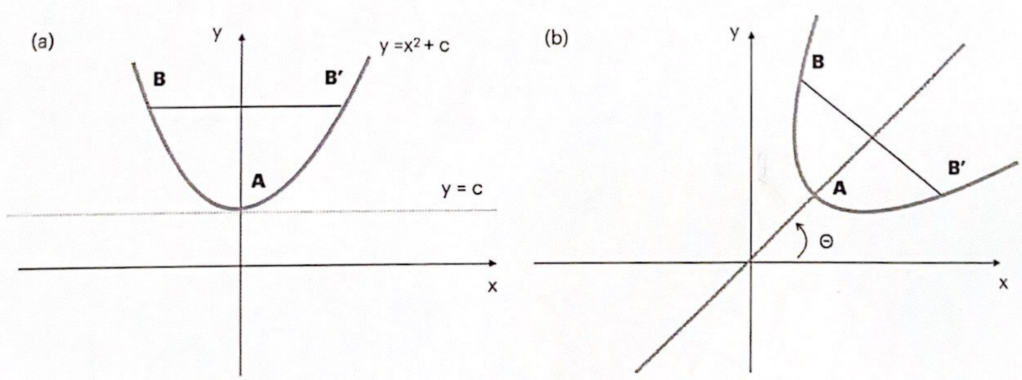

FIG. 1: Problem 7.