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電磁學（一）
Electromagnetics (I)

2. 電磁學的數學工具（一）- 複變分析
Mathematic Tools (I) - complex analysis

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This lecture is to introduce complex analysis. Complex numbers are useful for studying phase-sensitive wave signals.

- 2.1 Complex number 複數
- 2.2 Polar form of complex number 極座標形式的複數
- 2.3 Complex algebra 複數運算
- 2.4 Complex signal 複數訊號
- 2.5 Phasor 相量
- 2.6 Review 單元回顧

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2.1 複數
Complex number

Imaginary Unit (虛數單位)

Define the imaginary unit $j=\sqrt{-1}$, with $j\times j=-1, -j\times j=+1$

Define a complex number $z = x + jy$, with x and y both real numbers

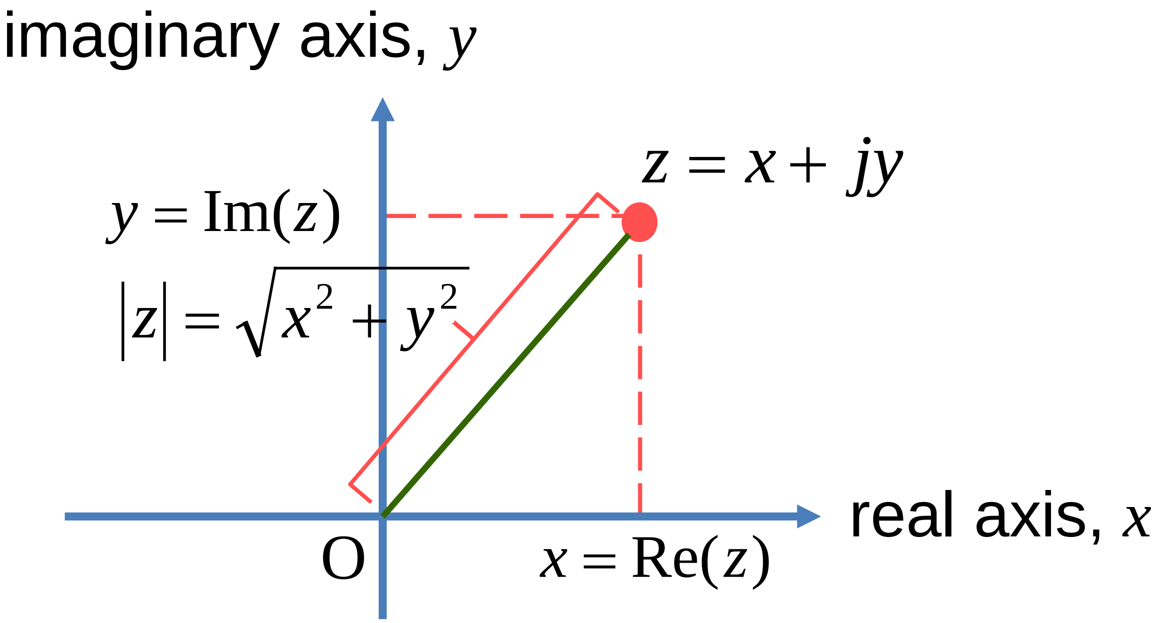
$x \equiv \mathrm{Re}(z)$ is the real part of the complex number z

$y \equiv \mathrm{Im}(z)$ is the imaginary part of the complex number z

The absolute value, magnitude, or modulus of z is defined as $|z| = \sqrt{x^2 + y^2}$

Graphic Representation – complex plane (複數平面)

A complex number can be shown on a complex plane.



Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta,\ e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\Downarrow$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\Downarrow$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

An establishment of trigonometric functions for complex analysis

2.1 複數

Complex number

- A complex number has a real part and an imaginary part.
- The absolute value or the modulus of a complex number is the square root of the square sum of the real and imaginary parts of the complex number.
- A complex number can be marked on an x-y plane, called the complex plane.

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2.2 極座標形式的複數
Polar form of complex number

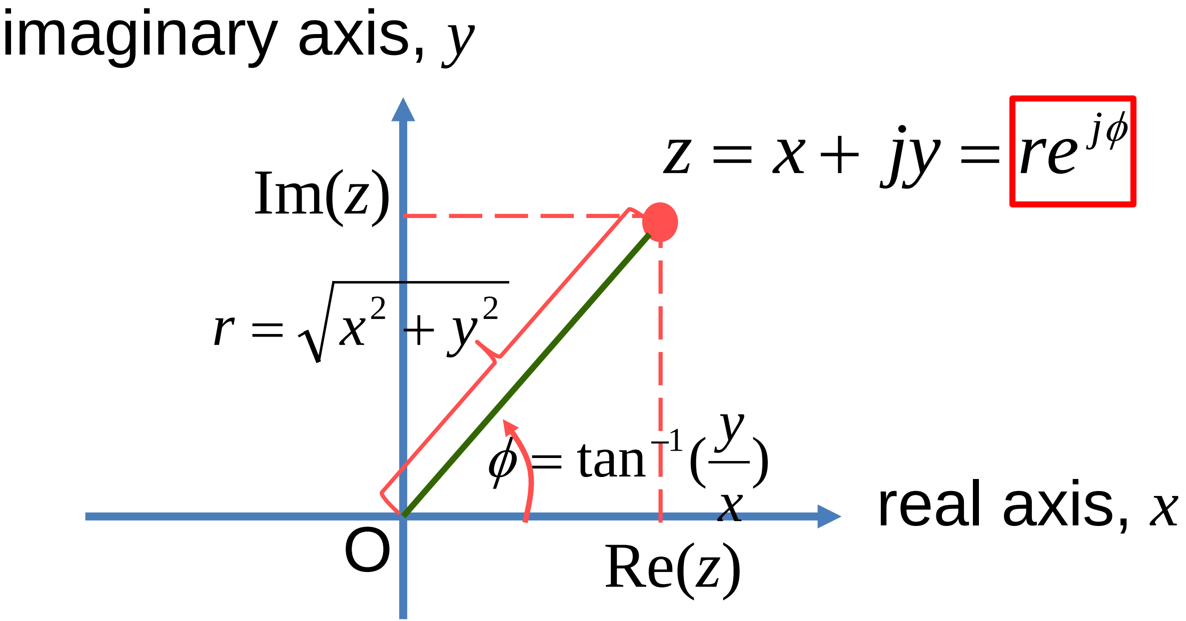
Apparently, $\left|e^{j\theta}\right| = \left|\cos\theta + j\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

Polar form of a complex number

$$z = x + jy = \underbrace{\sqrt{x^2+y^2}}_{r = |z|} \times \left(\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\phi} + j \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\sin\phi} \right) \boxed{= re^{j\phi}}$$

$\phi = \tan^{-1}(\frac{y}{x})$ is called the phase or phase angle of z

Graphic Representation – polar coordinate (極座標)



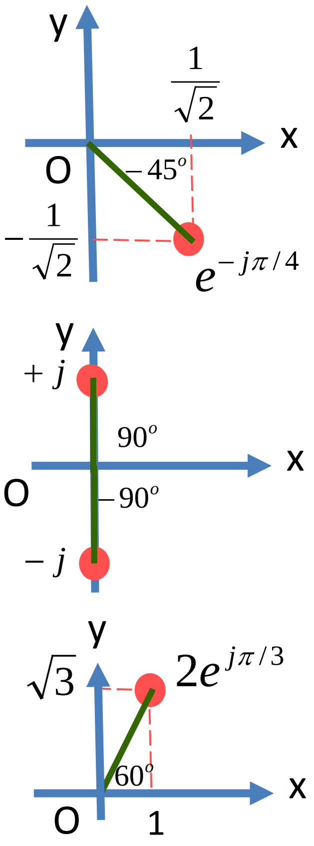
Examples

E.g. $e^{-j\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

E.g. $e^{\pm j\pi/2} = \pm j$

E.g. $1+\sqrt{3}j = 2(\frac{1}{2} + j\frac{\sqrt{3}}{2}) = 2e^{j\pi/3}$



2.2 極座標形式的複數
Polar form of complex number

- The polar form of a complex number comprises a radius and a phase angle.
- A polar-form complex number can be marked with ease in a polar-coordinate system.

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2.3 複數運算
Complex algebra

Given $$z_1 \equiv x_1 + jy_1 = r_1 e^{j\phi_1},$$
$$z_2 = x_2 + jy_2 = r_2 e^{j\phi_2}$$

If $z_1 = z_2$, $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

Specifically, if $z_1 = z_2 \Rightarrow x_1 = x_2, y_1 = y_2$
$$r_1 = r_2, \phi_1 = \phi_2$$

Summation $$z_1 + z_2 = z_2 + z_1 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction $$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Given $z_1 \equiv x_1 + jy_1 = r_1 e^{j\phi_1}$,

$z_2 = x_2 + jy_2 = r_2 e^{j\phi_2}$

Multiplication

$$z_1 \times z_2 = (x_1 + jy_1) \times (x_2 + jy_2)$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

$$= r_1 r_2 e^{j(\phi_1 + \phi_2)} = z_2 \times z_1$$

Division

$$z_1 \div z_2 = z_1 / z_2 = \frac{r_1 e^{j\phi_1}}{r_2 e^{j\phi_2}} = \frac{r_1}{r_2} e^{j(\phi_1 - \phi_2)}$$

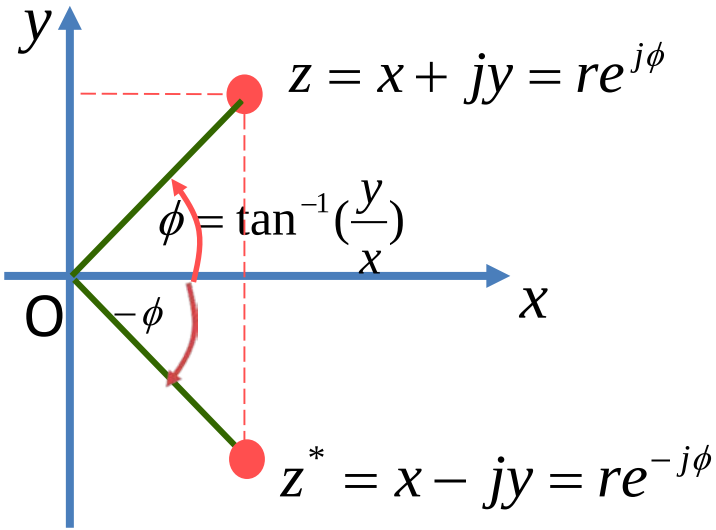
Evidently, the polar form is easier for complex multiplications and divisions.

Complex Conjugate 共軛複數

Define the complex conjugate of $z = x + jy = re^{j\phi}$ as

$$z^* = x - jy = re^{-j\phi}$$

In practice, just do the replacement $j \to -j$ or vice versa in a complex function to obtain its complex conjugate.



Given $z = x + jy = re^{j\phi}$,

the radius *r* can be calculated from

$r = \sqrt{x^2 + y^2} = \sqrt{z \times z^*}$ (a positive real number)

E.g. In a complex ratio, one quick way to separate the real and imaginary parts is to multiply its denominator with a complex conjugate.

$$z_3 = \frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{\left|z_2\right|^2} = \mathrm{Re}(z_3) + \mathrm{Im}(z_3)$$

$$= \frac{(x_1 + jy_1) \times (x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

2.3 複數運算

Complex algebra

- The polar form of a complex number makes multiplication and division of complex numbers much easier.
- Replacing j with $-j$ and vice versa converts a complex number into its complex conjugate.
- A complex number times its complex conjugate gives the square of its magnitude.

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2.4 複數訊號 Complex signal

Time-harmonic Signal

A time-harmonic signal assumes the sinusoidal form

$$\tilde{A}(t) = A_0 \cos(\omega t + \psi),$$

where t is the time variable, A_0 is the amplitude of A, ω is the angular frequency and ψ is the initial phase.

*In electromagnetics, ψ can be a function of position, R.

By using Euler's formula, A is expressed as

$$\tilde{A}(t) = A_0 \cos(\omega t + \psi) = \frac{A_0}{2}[e^{j(\omega t+\psi)} + e^{-j(\omega t+\psi)}] = \mathrm{Re}(A_0 e^{j\omega t + j\psi})$$

where $A_c = A_0 e^{j\omega t + j\psi}$ is called the complex signal of A.

Observations

Real-world calculation

$$\left.\begin{array}{ll} \mathrm{Re}(A_c) \pm \mathrm{Re}(B_c) & = \mathrm{Re}(A_c \pm B_c) \\ a\,\mathrm{Re}(A_c), a \text{ is real} & = \mathrm{Re}(aA_c), \\ \dfrac{\partial}{\partial x}\mathrm{Re}(A_c) & = \mathrm{Re}(\dfrac{\partial A_c}{\partial x}) \\ \int \mathrm{Re}(A_c)dx & = \mathrm{Re}(\int A_c dx) \\ \mathrm{Re}(A_c) = \mathrm{Re}(B_c) & \Leftarrow A_c = B_c \end{array}\right\}$$

Re(complex calculations)

The above is valid for linear operators, such as addition, subtraction, scaling, differentiation, integration etc.

Linear System

Assume a real-world linear system is characterized by a linear operator ℓ

Real-world scenario $\mathrm{Re}(A_c) \Rightarrow \boxed{\ell} \Rightarrow \ell[\mathrm{Re}(A_c)]$

Complex analysis

$$A_c \Rightarrow \boxed{\ell} \Rightarrow \ell[A_c] \Rightarrow \boxed{\mathrm{Re}} \Rightarrow \mathrm{Re}(\ell[A_c])$$

Since $\ell[\mathrm{Re}(A_c)] = \mathrm{Re}(\ell[A_c])$, a real-world solution can be obtained from complex calculations.

2.4 複數訊號 Complex signal

- A time-harmonic signal can be expressed as the real part of a time-harmonic complex signal.
- In a linear system, deriving a solution from a time-harmonic input signal is the same as taking the real part of that derived from a time-harmonic complex input signal.

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2.5 相量 Phasor

Phasor

For a time-harmonic signal $\tilde{A} = \mathrm{Re}(\overbrace{A_0 e^{j\psi + j\omega t}}^{A_c})$,

define the **phasor** of A as $\hat{A} = A_0 e^{j\psi}$, so that $\tilde{A} = \mathrm{Re}(\boxed{\hat{A}} e^{j\omega t})$

Temporal differentiation $\dfrac{\partial \tilde{A}}{\partial t} = \mathrm{Re}(\boxed{j\omega \hat{A}} e^{j\omega t})$

The operator $\boxed{\dfrac{\partial}{\partial t}}$ on A translates the phasor $\hat{A}$ into $\boxed{j\omega}\hat{A}$

Temporal integration $\int \tilde{A} dt = \mathrm{Re}(\boxed{\dfrac{\hat{A}}{j\omega}} e^{j\omega t})$

The operator $\boxed{\int dt}$ on A translates the phasor $\hat{A}$ into $\boxed{\dfrac{1}{j\omega}}\hat{A}$

Phasor Notations

$$A_c = \hat{A}\cancel{\times e^{j\omega t}} \Rightarrow \boxed{\ell} \Rightarrow \ell[A_c] = f_\ell(\hat{A}\times[1, j\omega, 1/j\omega])\cancel{\times e^{j\omega t}}$$

$f_\ell(\hat{A}\times[1, j\omega, 1/j\omega])$ is a linear function of $\hat{A}, j\omega\hat{A}, \hat{A}/j\omega$

$$\underbrace{\hat{A} \Rightarrow \boxed{\ell}}_{\text{leave out } e^{j\omega t}} \Rightarrow \underbrace{\boxed{\times e^{j\omega t}}}_{\text{multiply it back}} \Rightarrow \underbrace{\boxed{\mathrm{Re}} \Rightarrow \begin{array}{l} \mathrm{Re}(\ell[\hat{A}]\times e^{j\omega t}) \\ = \mathrm{Re}(f_\ell \times e^{j\omega t}) \end{array}}_{\text{return to the real world}}$$

$$A_c = \hat{A}\boxed{\times e^{j\omega t}} \Rightarrow \boxed{\ell} \Rightarrow \ell[A_c] = f_\ell(\hat{A}\times[1, j\omega, 1/j\omega])\boxed{\times e^{j\omega t}}$$

Example: wave equation with amplitude $A(R, t)$ and wave velocity c.

$$\nabla^2 A - \frac{1}{c^2}\cdot\frac{\partial^2 A}{\partial t^2} = 0$$

Assume a harmonic wave $A = \mathrm{Re}(\hat{A}\times e^{j\omega t})$

$$\ell \equiv \nabla^2 - \frac{1}{c^2}\cdot\boxed{\frac{\partial^2}{\partial t^2}}$$

$$f_\ell(\hat{A}\times[1, \boxed{j\omega}, 1/j\omega]) = \hat{A}[\nabla^2 - \frac{(j\omega)^2}{c^2}]$$

$$[\nabla^2\hat{A} - \frac{(j\omega)^2}{c^2}\hat{A}]\times \cancel{e^{j\omega t}} = [\nabla^2\hat{A} + \frac{\omega^2}{c^2}\hat{A}]\times \cancel{e^{j\omega t}} = 0 \Rightarrow \nabla^2\hat{A} + \frac{\omega^2}{c^2}\hat{A} = 0$$

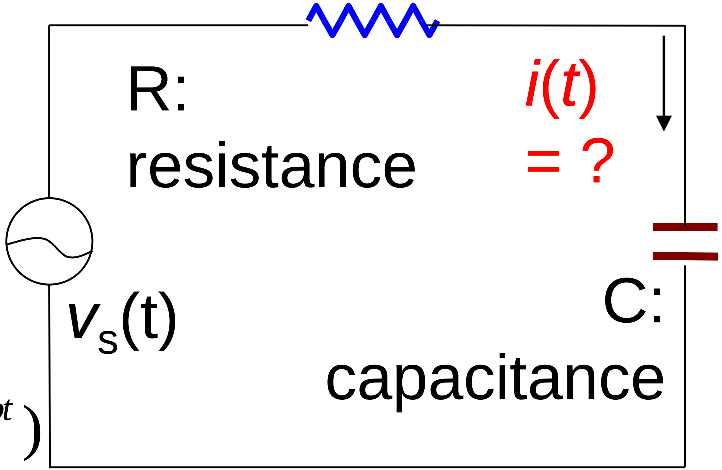
E.g. RC Circuit

Assume an AC voltage source

$$\tilde{v}_s(t) = v_0\cos(\omega t + \phi) = \mathrm{Re}(V_c) = \mathrm{Re}(\hat{V}_s e^{j\omega t})$$

with $V_c = v_0 e^{j\phi} \times e^{j\omega}, \;\; \hat{V}_s = v_0 e^{j\phi}$

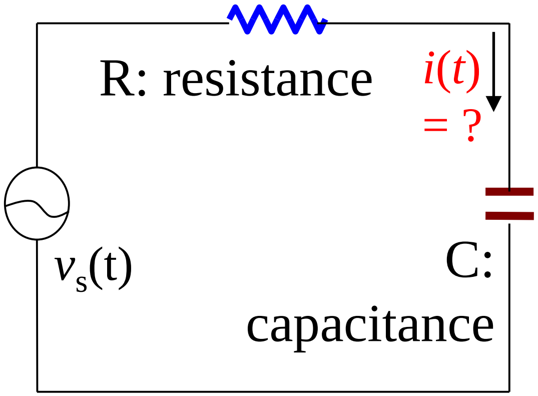
The current is $\tilde{i}\,(t) = \mathrm{Re}(I_c) = \mathrm{Re}(\hat{I} e^{j\omega t})$



Around the circuit loop, voltage drop = voltage rise

$$R\tilde{i}\,(t) + \frac{\tilde{q}(t)}{C} = \tilde{v}_s(t) \;\Rightarrow R\tilde{i}\,(t) + \frac{1}{C}\int \tilde{i}\,(t)dt = \tilde{v}_s(t)$$

Phasor Calculation



1. Based on the governing equation

$$R\tilde{i}(t) + \frac{1}{C}\int \tilde{i}(t)dt = \tilde{v}_s(t)$$

2. Use phasor notations for the equation

$$\hat{I}(R + \frac{1}{j\omega C}) = \hat{V}_s$$

3. Obtain the phasor solution

$$\hat{I} = \frac{\hat{V}_s}{R + \frac{1}{j\omega C}}$$

4. Convert it to a real-world solution

$$\tilde{i}(t) = \mathrm{Re}[\frac{\hat{V}_s}{R + \frac{1}{j\omega C}} e^{j\omega t}]$$

2.5 相量 Phasor

- A phasor is a time-harmonic complex signal leaving out the temporal part.
- Phasor notation simplifies the analysis of a time-harmonic linear system without involving the temporal part of the variables.
- The real part of a phasor solution multiplied by a time-harmonic complex exponential gives the real-signal solution.

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單元回顧

單元回顧

1. A complex number has a real part x and an imaginary part y, expressed as

$$z = x + jy, \text{ where } j = \sqrt{-1}$$

2. The Euler's formula connect complex analysis to trigonometric functions:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta,$$

單元回顧

3. A complex number can be expressed by the polar form, given by

$$z = x + jy = re^{j\phi}$$

where $r = |z| = \sqrt{x^2 + y^2}$ is the radius and $\phi = \tan^{-1}(\frac{y}{x})$ is the phase of z.

4. Complex analysis is an excellent mathematic tool to study a physical parameter with a phase.

單元回顧

5. A time-harmonic signal is described by the sinusoidal function

$$\tilde{A}(t) = A_0 \cos(\omega t + \psi) = \mathrm{Re}(A_0 e^{j\psi} \times e^{j\omega t}) = \mathrm{Re}(A_c)$$

where $A_c = A_0 e^{j\psi} \times e^{j\omega t}$ is called a complex signal of $A(t)$.

6. A time-harmonic complex variable can be written as

$$A_c = A_0 e^{j\psi} \times e^{j\omega t} = \hat{A} \times e^{j\omega t},$$

where $\boxed{\hat{A} = A_0 e^{j\psi}}$ is called the phasor of the time-harmonic signal $\tilde{A}(t) = A_0 \cos(\omega t + \psi) = \mathrm{Re}(\hat{A} \times e^{j\omega t})$.

單元回顧

7. The phasor notation simplifies calculations of a time-harmonic linear system without involving the temporal variable *t*.

E.g. an RC circuit $R\tilde{i}(t)+\frac{1}{C}\int\tilde{i}(t)dt=\tilde{v}_s(t)=\boxed{\mathrm{Re}(\hat{V}_s e^{j\omega t})}$

$$\Rightarrow \quad \hat{I}\left(R+\frac{1}{j\omega C}\right)=\hat{V}_s \quad \Rightarrow \quad \hat{I}=\frac{\hat{V}_s}{R+\frac{1}{j\omega C}}$$

$$\Rightarrow \quad \tilde{i}(t)=\boxed{\mathrm{Re}}\left[\frac{\hat{V}_s}{R+1/j\omega C}\boxed{e^{j\omega t}}\right]$$

THANK YOU FOR YOUR ATTENTION