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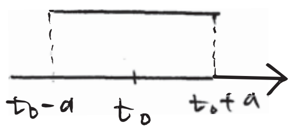
when the impulse is getting shorter:

unit impulse



$\xrightarrow{a\to 0}$

as $a\to 0$, the unit impulse is called

LT of impulse function & delta function

ex: $\mathcal{L}\{2\delta_4\} =$

III. Periodic function

ex:

Def: periodic function with period T

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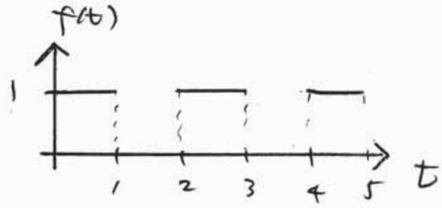
LT of periodic function $f(t)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^T f(t)\, e^{-st}\, dt +$$

$$\Rightarrow \quad \mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-sT}}$$

ex: LT of a periodic square wave



So far, we already learn some important functions and the LT. We can now solve DEs with these functions as the forcing term.

How to use LT to solve DEs with $\begin{cases}\text{discontinuous} \\ \text{periodic}\end{cases}$ forcing

We'll use the same DE with different forcing terms for examples.

$$y'' + 4y = f(t), \text{ where } f(t) =$$

with initial conditions

$$y(0) = y'(0) = 0$$

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Example 1 : With impulse forcing $f(t)=\delta_0(t)$

D.E. $y''+4y=$

step 1 : Take $\mathcal{L}$ and find $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{\delta_0(t)\}$$

step 2 : Take $\mathcal{L}^{-1}$ and find y

$$y=$$

☆ is the response of the system when input forcing is an " ". So $\frac{1}{2}\sin 2t$ is called

Example 2 : With step-like forcing $f(t)=\begin{cases} 1, & 0\le t<1 \\ -1, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$

D.E. $y''+4y=f(t)$

step 1 : Take $\mathcal{L}$ and find $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{$$

$$(s^2+4)\mathcal{L}\{y\} =$$

$$\Rightarrow \mathcal{L}\{y\} =$$

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Step 2 : Take $\mathcal{L}^{-1}$ and find y

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)}\left(\frac{1}{s}-\frac{2e^{-s}}{s}+\frac{e^{-2s}}{s}\right)\right\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)(1-2e^{-s}+e^{-2s})\right\}$$

Example 3 : With a ramp forcing $f(t)=\begin{cases}0 & ,0\le t<5\\ \frac{1}{5}(t-5) & ,5\le t<10\\ 1 & ,t\ge 10\end{cases}$

D.E. $y''+4y=f(t)$

This ramp function can be expressed by step function as

Step 1 : Take $\mathcal{L}$ and find $\mathcal{L}\{y\}$

$\mathcal{L}\{y''+4y\}=$

$(s^2+4)\,\mathcal{L}\{y\}=$

$\Rightarrow \mathcal{L}\{y\}=$

Step 2 : Take $\mathcal{L}^{-1}$ and find y

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Example 4: with a periodic square wave forcing $f(t)$

D.E. $y'' + 4y = f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$

and outside the interval by $f(t+2) = f(t)$

Step 1: Take $\mathcal{L}$ and find $\mathcal{L}\{y\}$

Step 2: Take $\mathcal{L}^{-1}$ and find y

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\left(\frac{1}{1+e^{-s}}\right)\right\} = \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right)\right.$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s}\left(\qquad \right) - \frac{1}{4}\frac{s}{s^2+4}\left(\qquad \right)\right\}$$

$$= \frac{1}{4}\left(\qquad \right) - \frac{1}{4}\left(\qquad \right)$$

Remark: In these examples, $\mathcal{L}\{y\}$ has the form of

In a general form, for a system described by $y'' + py' + qy = f(t)$,
if the impulse response of this system is given as
Then for any forcing function $f(t)$,