CS5319 Advanced Discrete Structure

Exam 1 – November 02, 2021 (13:20–15:10)

Answer all six questions. Total marks = 100. Maximum score = 100.

1. (20%) Suppose we have n pairs of measuring weights, where their weights are the same. We have also a pan balance, where one side can hold the object to be weighed, and the other side can hold the measuring weights.

   We can use the pan balance and the measuring weights for weighing items. For example, if we have two pairs of measuring weights $(1, 1)$ and $(2, 2)$, then we can measure items with integral weights ranging from 1 to 6.

   Suppose now we need to measure items with integral weights ranging from 1 to 100.

   (a) (10%) Show that at least 5 pairs of measuring weights are needed.

   (b) (10%) Find 5 pairs of measuring weights that serves our purpose. Justify your answer.

2. (20%) Consider arranging 3 identical red balls and B identical blue balls on a circle. Two arrangements are considered the same, if one can be transformed to the other by rotation.

   For instance, when $B = 0$ or $B = 1$, there is only 1 way. When $B = 2$, there are exactly two ways.

   (a) (15%) How many ways are there when $B = 10$?

   (b) (5%: Challenging) How many ways are there when $B = 100$?
   Remark. No need to compute the exact value. A correct formula, with clear explanation, will be sufficient.

3. (20%) Find the coefficient of x^n in the following generating function:

$$\frac{x+1}{(x-2)(1+5x)}.$$

4. (20%) Consider distributing r distinct balls one by one into three boxes, namely Box 1, Box 2, and Box 3, such that ordering of balls within Box 1 does not matter, while ordering of balls within Box 2 matters, and ordering of balls within Box 3 matters. (For the latter two cases, balls are ordered by the time they are distributed to the boxes.)

   Furthermore, we require that Box 1 and Box 2 cannot be empty.

   Let a_r denote the number of ways of distributing the balls, so we have $a_0 = a_1 = 0$ since Box 1 and Box 2 cannot be empty, while $a_2 = 2$ since we can throw the two distinct balls, one to each of Box 1 and Box 2.

   (a) (10%) Give the EGF for the sequence $(a_0, a_1, a_2, \ldots)$.

   (b) (10%) Hence, or otherwise, compute the exact value of a_6.

5. Let n be a positive integer. Let a_n denote the number of ways to partition n into at least three integers. Let b_n denote the number of ways to partition n, whose largest part is at least 3.

   (15%) Show that $a_n = b_n$.

   Example: Consider $n = 6$. The following are the ways to partition n into at least three integers:

$$\{1,1,4\},\quad \{1,1,1,3\},\quad \{1,1,1,1,2\},\quad \{1,1,1,1,1,1\},\quad \{1,1,2,2\},\quad \{1,2,3\},\quad \{2,2,2\}$$

   so that $a_6 = 7$. In contrast, we can partition n so that the largest part is at least 3:

$$\{6\},\quad \{1,5\},\quad \{1,1,4\},\quad \{2,4\},\quad \{1,1,1,3\},\quad \{1,2,3\},\quad \{3,3\}$$

   so that $b_6 = 7$.

6. (5%: Tricky) Let $F(x)$ denote the following function:

$$F(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \frac{4x^3}{5!} + \cdots$$

   Show that $F(1) = 1$.

   Hint: Discover the relationship between e^x and $F(x)$, and then simplify $F(x)$.