last update: Jan. 3rd , 2021
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電磁學（一）
Electromagnetics (I)

17. 馬克士威爾方程式
Maxwell's Equations

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In this lecture, we will start to see how time-varying electric and magnetic fields are coupled together. The Maxwell’s equations describe such coupling.

- **17.1 Faraday’s Law of Electromagnetic Induction 法拉第電磁感應定理**
- **17.2 Electric Generator 發電機**
- **17.3 Maxwell’s Equations 馬克士威爾方程式**
- **17.4 Boundary Conditions for Time-varying Fields 時變場的邊界條件**
- **17.5 Review 單元回顧**

馬克斯威爾方程式
Maxwell’s Equations

17.1 法拉第電磁感應定理
Faraday’s Law of Electromagnetic Induction

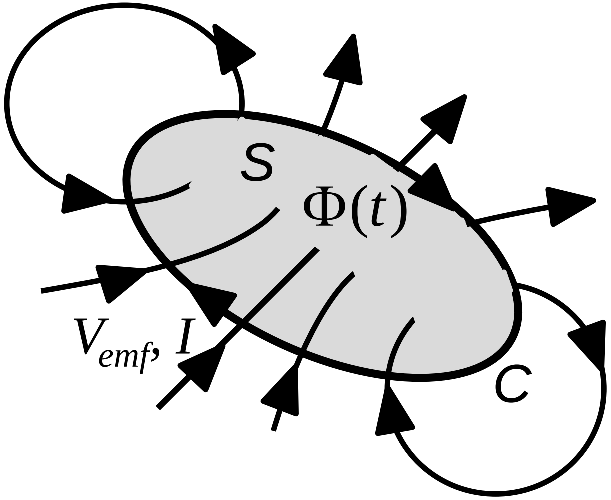
Faraday's Law of Electromagnetic Induction

For a single current loop in a time-varying magnetic flux, an emf is generated

Lenz's Law

$$V_{emf} = \boxed{-}\frac{d\Phi}{dt}$$

$$\Rightarrow V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$



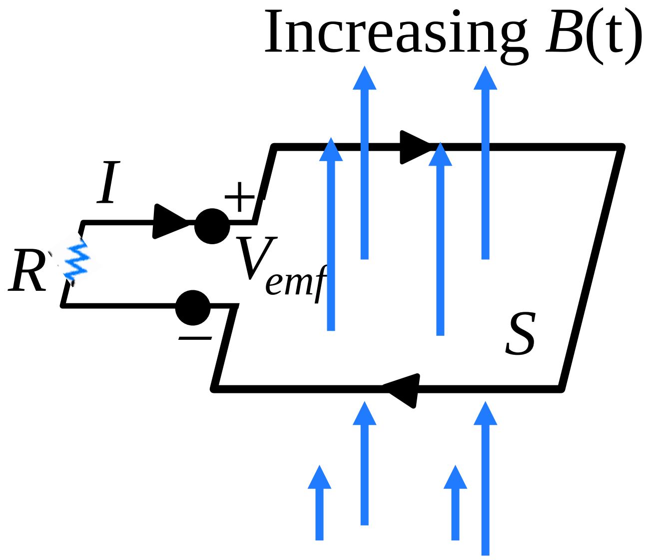
V_{emf} is the electromotive force (emf) induced along a conducting loop *C*, generating a current in such a way that the current opposes the change of the magnetic flux.

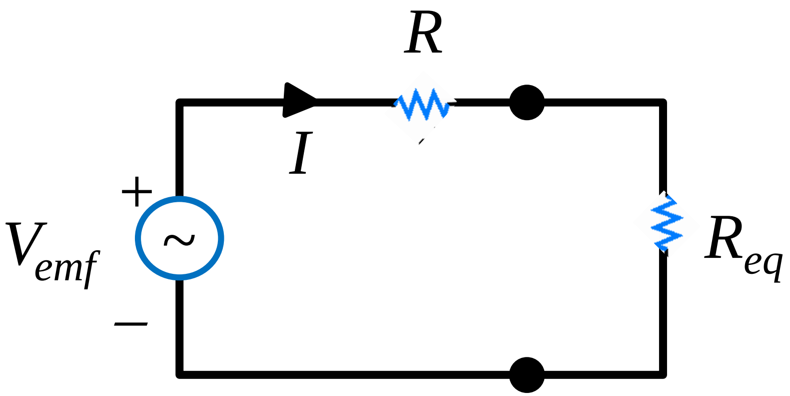
For N loops, $V_{emf} = -\dfrac{d\Lambda}{dt}$ $\Lambda = N\Phi$ is the magnetic linkage.

Transformer *emf*

For a stationary current loop C,

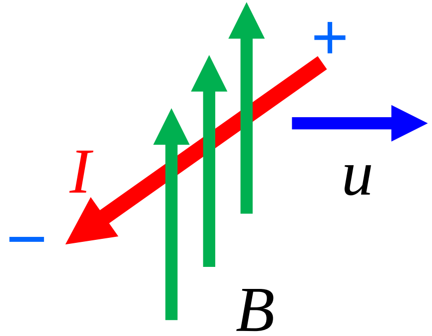
$$V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}(t) \cdot d\vec{s} \quad \Rightarrow V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$





Equivalent Circuit

Flux Cutting (motional) *emf*



An electric field is induced in a moving metal wire (contains charges) cutting a magnetic flux.

The charge in a wire moving with velocity u in a magnetic field B experiences a magnetic force, or an equivalent electric field given by

$$\vec{E} = \frac{\vec{F}}{q} = \vec{u} \times \vec{B}$$

*Recall the Lorentz force

$$\vec{F} = q\vec{u} \times \vec{B}$$

where $\vec{u}$ is the velocity vector of the moving wire.

The voltage induced on a wire of loop C Is therefore

$$\boxed{V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = \oint_C \vec{u} \times \vec{B} \cdot d\vec{l}}$$

17.1 法拉第電磁感應定理

Faraday's Law of Electromagnetic Induction

- A electromotive force (emf) is generated from a time-varying magnetic flux, according to $V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$
- For a stationary loop, the so-called transformer emf is given by

$$V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

- For a moving wire, the so-called flux-cutting emf is given by

$$V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = \oint_C \vec{u}\times\vec{B}\cdot d\vec{l}$$

馬克斯威爾方程式
Maxwell’s Equations

17.2 發電機
Electric Generator

Total emf

Recall the Faraday's Law of Electromagnetic Induction $V = \frac{-d\phi}{dt} \Rightarrow \oint_C \vec{E} \cdot d\vec{l} = -\underline{\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}}$

Transformer emf $V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$

Flux-cutting emf $V_{emf} = \oint_C \vec{u} \times \vec{B} \cdot d\vec{l}$

Indeed, mathematically, it can be shown that

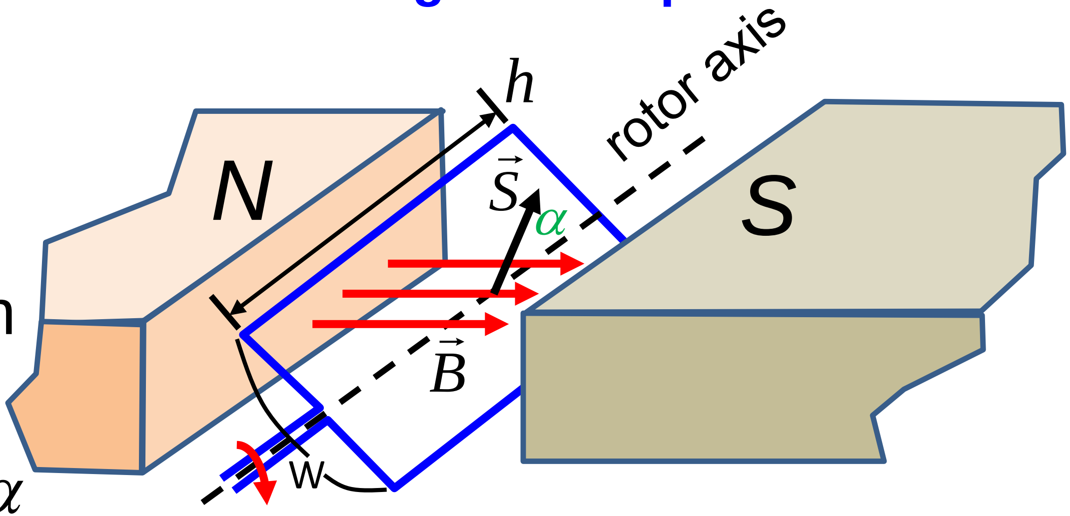
$$\underline{-\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}} = -\int_S \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{s}$$

Therefore, the total emf is the sum of the transformer emf and flux-cutting emf

$$\boxed{V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}}$$

Electric Generator – rotating wire loop in B

i. First Approach



Use $V = -\dfrac{d\Phi}{dt}$.

Calculate the total flux in the current loop:

$$\Phi = \int \vec{B} \cdot d\vec{s} = (wh) B \cos\alpha$$

Do the differentiation to Φ: $V = -\dfrac{d\Phi}{dt} = (wh) B \sin\alpha \dfrac{d\alpha}{dt}$

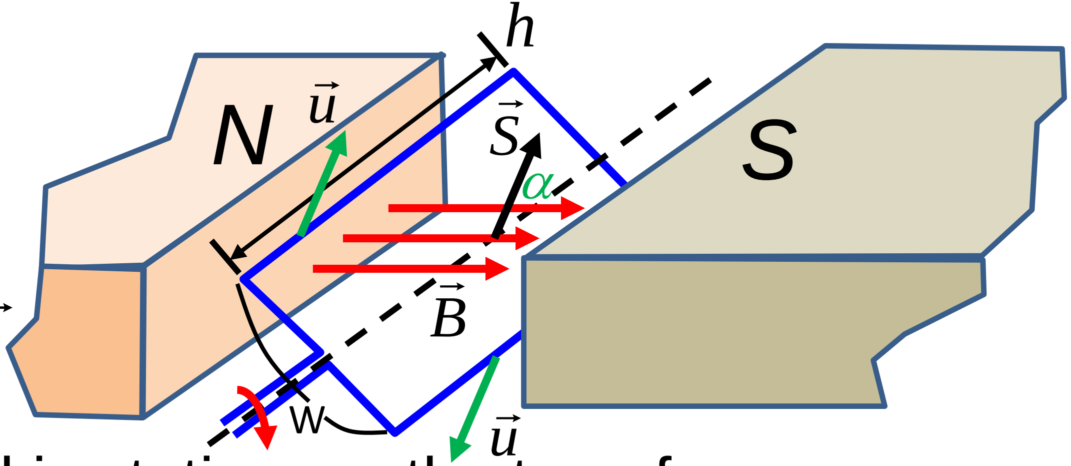
But $\alpha = \omega t$ with ω being the angular frequency of the rotor.

Finally, $V = -\dfrac{d\Phi}{dt} = (wh) B\omega \sin \omega t$ *Note that $-\int_S \dfrac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = 0$

ii. Second Approach

Start from

$$V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$



Because the magnetic field is stationary, the transformer emf is zero or $V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = 0$

Use $V = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$ for the flux-cutting emf along the 2 h-lengths (why not w?), where the velocity vector is $\vec{u} = \frac{w}{2}\omega \hat{a}_\phi$.

Therefore, $V = \frac{w}{2}\omega B \sin(\alpha) \times 2h = (wh)B\omega \sin \omega t$, as expected.

17.2 發電機

Electric Generator

- A rotating wire loop in a magnetic field generates an emf and thus a current around the wire loop.
- The emf can be calculated directly from the Faraday's law of magnetic induction, given by

$$V = -\frac{d\Phi}{dt}$$

- Since the problem does not involve a time-varying magnetic field, the emf can also be calculated from the flux-cutting emf.

馬克士威爾方程式
Maxwell's Equations

17.3 馬克士威爾方程式
Maxwell's Equations

In electrostatics and magentostatics, we have the postulates

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} = \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

However, for a stationary loop in a time-varying magnetic field, we have

$$V = \oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \qquad (1)$$

Apply the Stokes theorem to $V = \oint_C \vec{E} \cdot d\vec{l}$

and write $V = \oint_C \vec{E} \cdot d\vec{l} = \int_S \nabla \times \vec{E} \cdot d\vec{s}$ (2)

Comparing (1) and (2), we have at a point in space.

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

(only the transformer emf needs to be considered at a “point”).

Revising the Ampere's Law

Consider the equation of continuity $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

However, if we apply the divergence to the Ampere's law in magnetostatics $\nabla \cdot (\nabla \times \vec{H} = \vec{J})$ $\Rightarrow$ $\underline{\nabla \cdot \nabla \times \vec{H}} = \underline{\nabla \cdot \vec{J}}$

$$\nabla \cdot \nabla \times \vec{H} = 0 \qquad \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$$

We have to modify the Ampere's law by writing

$$\nabla \cdot (\nabla \times \vec{H} + ?) = \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \nabla \cdot (?) = -\frac{\partial \rho}{\partial t}$$

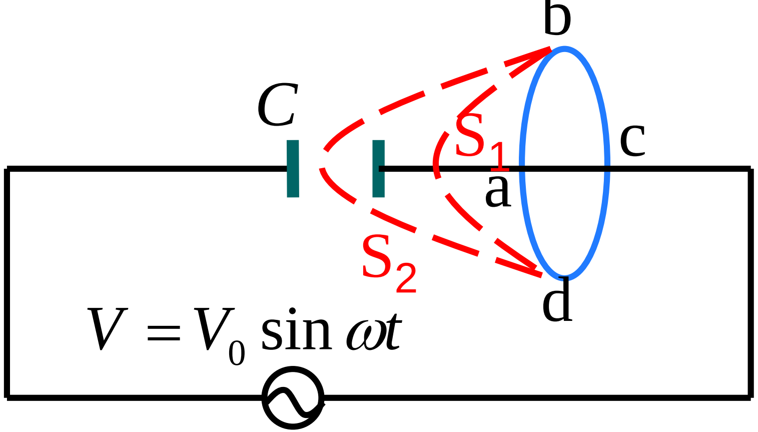
Recall $\nabla \cdot \vec{D} = \rho$ $\Rightarrow$ $? = -\frac{\partial \vec{D}}{\partial t}$

Therefore, the revised Ampere's Law is $\boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}}$

Displacement Current

Note that $\frac{\partial \vec{D}}{\partial t} = \vec{J}_d$ can be considered as a kind of current, called displacement current.

Consider the capacitive circuit -



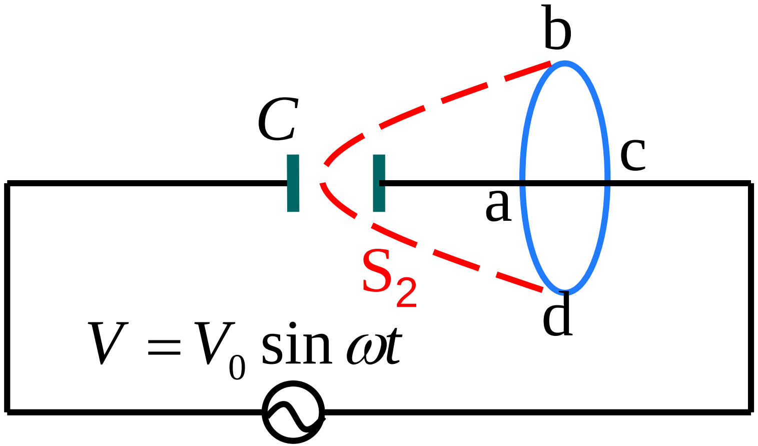
According to Ampere's law

$$\oint_{a,b,c,a} \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$$

The *abcd* loop can be defined by (1) the surface S_1 intercepting a physical current density J or (2) the surface S_2 intercepting a displacement current J_{d} through the capacitor C.

(1) For S_1 $\int_{S_1} \vec{J} \cdot d\vec{s} = I_c = C\frac{dV}{dt} = C\frac{d(V_0 \sin \omega t)}{dt} = CV_0 \omega \cos \omega t$

In $\oint_{a,b,c,a} \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$



The *abcd* loop can also be defined by the surface S_2 intercepting a displacement current J_d through the capacitor C.

For S_2, use $C = \varepsilon S / d$ and apply surface integration to the displacement current density

$$\int_{S_2} \vec{J}_d \cdot d\vec{s} = \int_{S_2} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} = I_d = \frac{\partial(\varepsilon V/d)}{\partial t} \cdot S = \frac{\varepsilon S}{d} \frac{\partial(V_0 \sin \omega t)}{\partial t}$$

$$= C V_0 \omega \cos \omega t = \int_{S_1} \vec{J} \cdot d\vec{s}$$

Therefore, $\int_{S_1} \vec{J} \cdot d\vec{s} = \int_{S_2} \vec{J}_d \cdot d\vec{s}$ J_d is indeed a kind of current!

Maxwell's Equations

	Differential form	Integral form
Faraday's induction law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\frac{d\Phi}{dt}$
Gauss law	$\nabla \cdot \vec{D} = \rho$	$\oint_S \vec{D} \cdot d\vec{s} = Q$
Ampere's circuital law	$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$	$\oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$
Magnetic Gauss law	$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$

Two fundamental equations

Lorentz force equation

$$\boxed{\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}}$$

Equation of continuity

$$\boxed{\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

The Maxwell's equations, along with the two equations above, describe ALL known phenomena in electromagnetism.

*Note that,

in a Ohmic material, the relationship applies $\boxed{\vec{J} = \sigma \vec{E}}$

in a simple medium, the relationships apply

$$\boxed{\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}} \qquad \boxed{\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}}$$

17.3 馬克士威爾方程式

Maxwell's Equations

	Differential form	Integral form
Faraday's induction law	$\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\dfrac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\dfrac{d\Phi}{dt}$
Gauss law	$\nabla \cdot \vec{D} = \rho$	$\oint_S \vec{D} \cdot d\vec{s} = Q$
Ampere's circuital law	$\nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J}$	$\oint_C \vec{H} \cdot d\vec{l} = I + \int_S \dfrac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$
Magnetic Gauss law	$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$

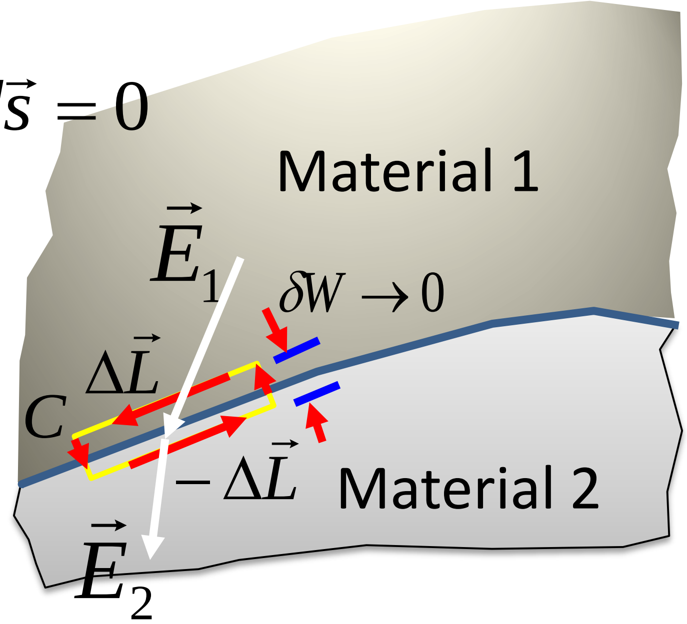
馬克士威爾方程式
Maxwell's Equations

17.4 Boundary Conditions for Time-varying Fields
時變場的邊界條件

Tangential Components of Electric Field

(1) Apply Faraday's law to a loop path/surface at the boundary

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = 0$$



The surface area under integration is zero, as $\delta W \rightarrow 0$.

$\oint_C \vec{E} \cdot d\vec{l} = 0$ gives $\boxed{E_{t1} = E_{t2}}$

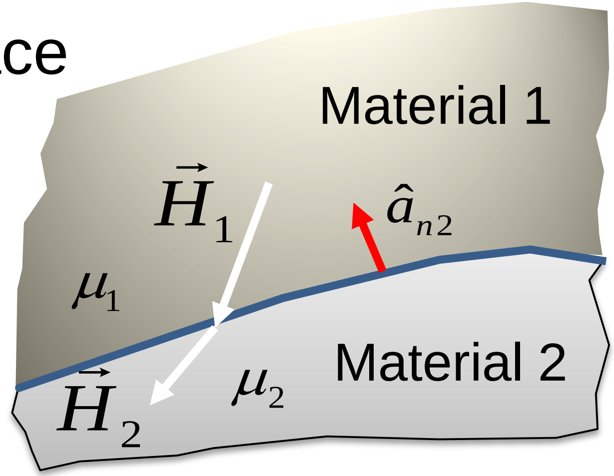
(Lecture 6)

Tangential components of electric field intensity are continuous across the boundary.

Tangential components of Magnetic Field

(2) Apply Ampere's law to a loop path/surface at the boundary

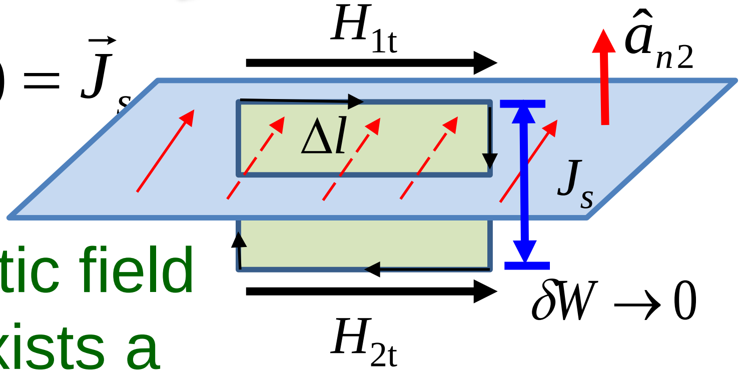
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \Rightarrow \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$



Again the surface area is taken to approach zero, as $\delta W \to 0$.

$$\oint_C \vec{H} \cdot d\vec{l} = \int \vec{J}_s \cdot d\vec{l} \quad \Rightarrow \hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

(Lecture 14)



The tangential components of magnetic field intensity are discontinuous, if there exists a surface current at the boundary.

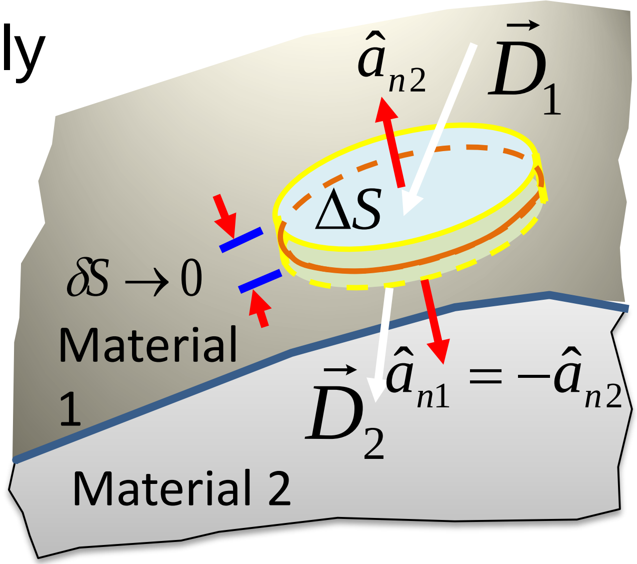
Normal Components of Electric Field

(3) Apply Gauss's law to a infinitesimally thin pillbox on the boundary

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

to obtain $$\boxed{\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s}$$

(Lecture 6)



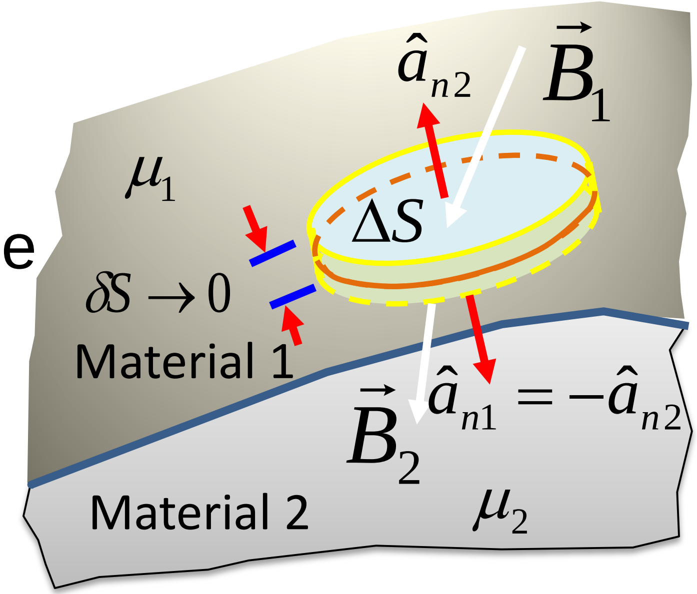
The normal components of electric flux density are discontinuous, if there exists a surface charge at the boundary.

Normal Components of Magnetic Field

(4) Apply $\oint_S \vec{B} \cdot d\vec{s} = 0$

to a infinitesimally thin pillbox on the boundary to obtain

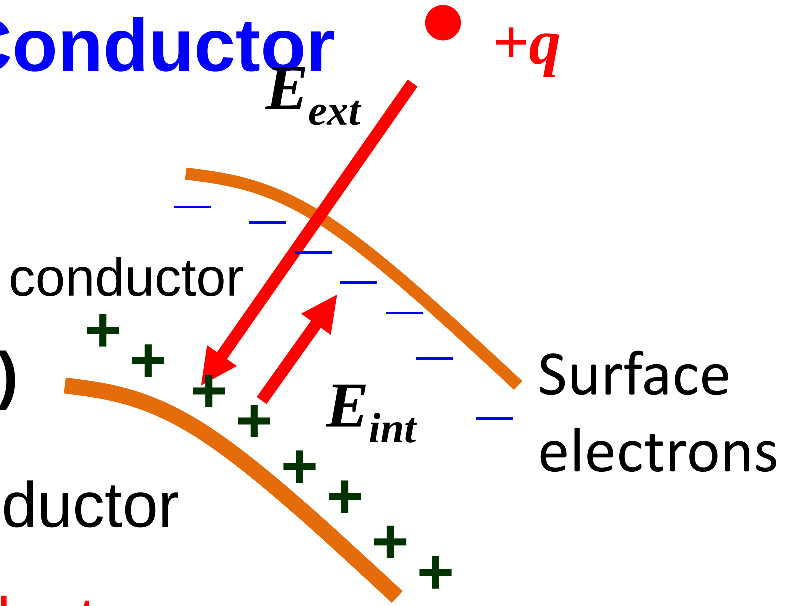
$$\boxed{B_{n1} - B_{n2} = 0}$$ **(Lecture 14)**



The normal components of magnetic flux density are continuous across a boundary.

Fields in a Perfect Conductor

A time-varying $E = 0$ in a perfect conductor for the same reason as that for a static $E = 0 \Rightarrow \; D = 0.$

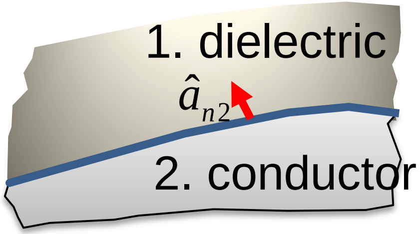


$$\boldsymbol{E_{net} = E_{ext} - E_{int} = 0} \text{ \textbf{(Lecture 6)}}$$

From $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, $E = 0$ in a conductor

$\Rightarrow$ time-varying B is zero in a conductor.

E.g. Dielectric-conductor interface



Inside a conductor $\vec{E} = 0, \vec{D} = 0, \vec{H} = 0, \vec{B} = 0$

$$E_{t1} = E_{t2} \Rightarrow E_{1t} = 0, \quad \hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \Rightarrow \hat{a}_{n2} \times \vec{H}_1 = \vec{J}_s$$

$$\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \Rightarrow \hat{a}_{n2} \cdot \vec{D}_1 = \rho_s, \quad B_{n1} - B_{n2} = 0 \Rightarrow B_{n1} = 0$$

17.4 時變場的邊界條件

Boundary Conditions for Time-varying Field

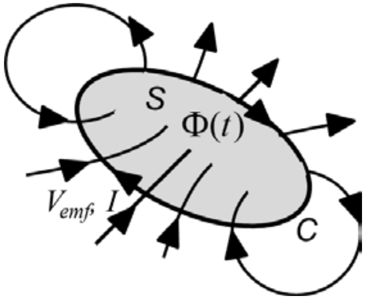
- Tangential components of the electric-field intensity are continuous.
- Normal components of the electric flux density differ by the surface charge density.
- Tangential components of the magnetic-field intensity differ by the surface current density.
- Normal components of the magnetic-flux density are continuous.

馬克士威爾方程式
Maxwell’s Equations

17.5 單元回顧 Review

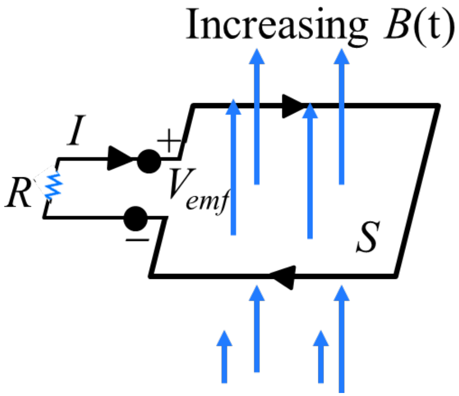
單元回顧

1. Faraday's law of electromagnetic induction: An electromotive force (emf) is generated from a time-varying magnetic flux, according to



$$V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$

2. For a stationary loop, a transformer emf is generated from a time-varying magnetic flux

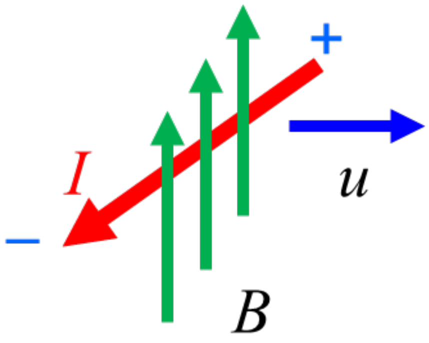


$$V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$

單元回顧

3. For a moving wire in a magnetic field, a flux-cutting emf is generated from the wire boundary cutting the magnetic flux, given by

$$V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = \oint_C \vec{u} \times \vec{B} \cdot d\vec{l}$$

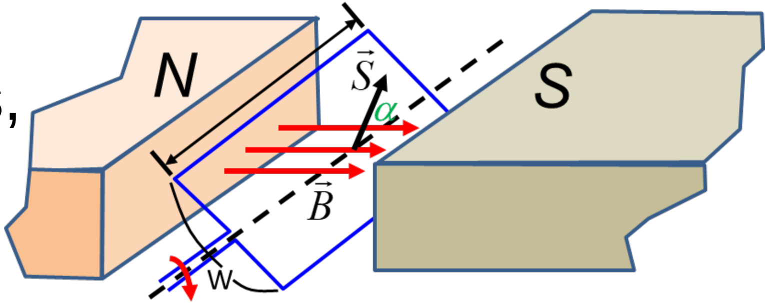


4. The total emf associated with a general current loop is therefore

$$V_{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} \Rightarrow V = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

單元回顧

5. An electric generator can be a rotating wire loop between magnets, generating flux-cutting emf across the loop.



6. The Ampere's law for time-varying field is given by

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J},$$

where $\vec{J}_d = \dfrac{\partial \vec{D}}{\partial t}$ is call the displacement current and $\vec{D}$ is sometimes called the displacement vector.

單元回顧

7. The Maxwell's equations describe all known electromagnetic phenomena

	Differential Form	Integral Form
Faraday's induction law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\frac{d\Phi}{dt}$
Gauss law	$\nabla \cdot \vec{D} = \rho$	$\oint_S \vec{D} \cdot d\vec{s} = Q$
Ampere's circuital law	$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$	$\oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$
Magnetic Gauss law	$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$

單元回顧

8. The boundary conditions for time-varying fields:

- Tangential components of the electric-field intensity are continuous:

$$E_{t1} = E_{t2}$$

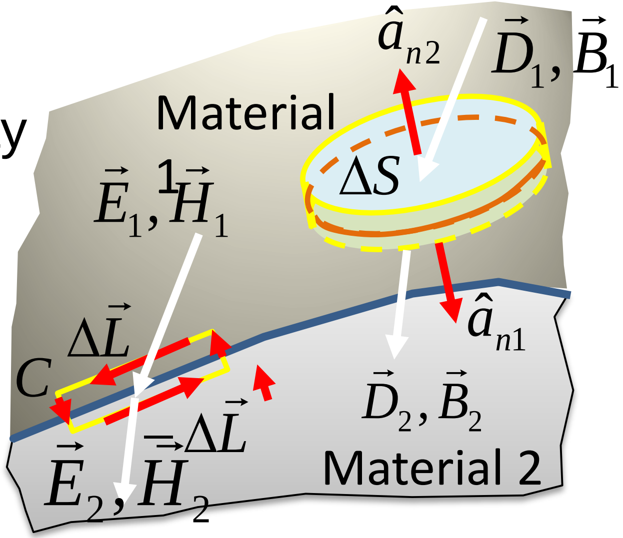
- Normal components of the electric flux density differ by the surface charge density:

$$\hat{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

- Tangential components of the magnetic-field intensity differ by the surface current density:

$$\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

- Tangential components of the magnetic-flux density are continuous: $$B_{n1} = B_{n2}$$

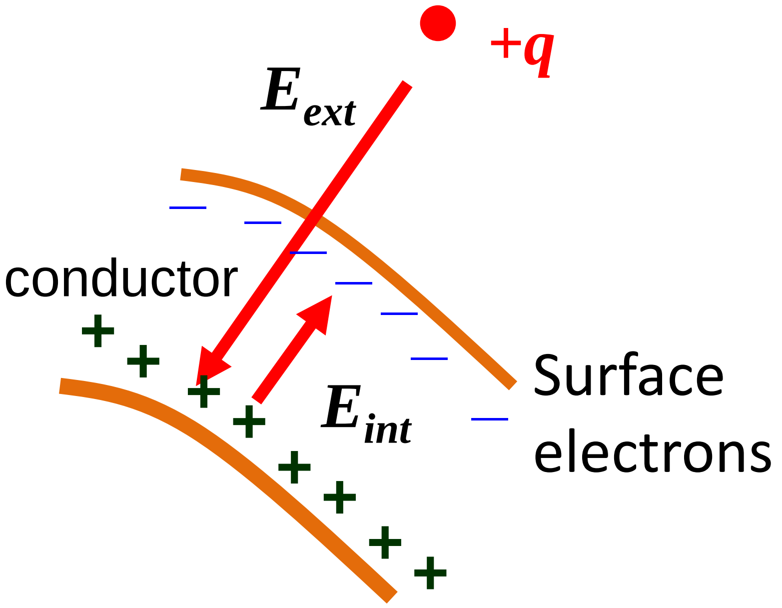


單元回顧

9. In a perfect conductor, both time-varying electric and magnetic fields are zero.

$$E_{net} = E_{ext} - E_{int} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow 0$$



THANK YOU FOR YOUR ATTENTION