Last update: Oct. 5^{th}, 2020

電磁學（一）
Electromagnetics (I)

4. 電磁學的數學工具（三）向量微積分
Mathematic Tools (III) - vector calculus

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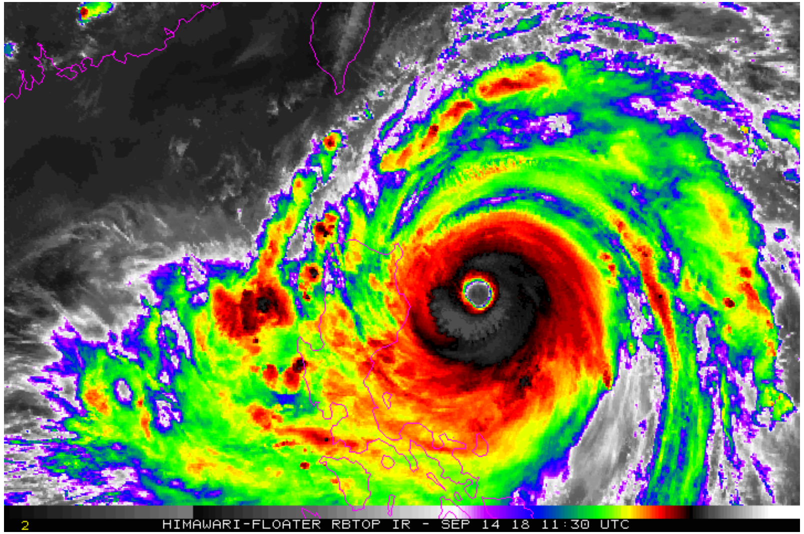
In this lecture, we will learn vector calculus to deal with the flux and circulation of fields in electromagnetics.

- 4.1 Circulation and flux 旋量與通量
- 4.2 Gradient of a scalar 梯度
- 4.3 Divergence of a vector 散度
- 4.4 Curl of a vector 旋度
- 4.5 Combined operators 組合運算子
- 4.6 Review 單元回顧

電磁學的數學工具 (二)- 向量微積分
Mathematic Tools (III) – vector calculus

4.1旋量與通量 Circulation and flux

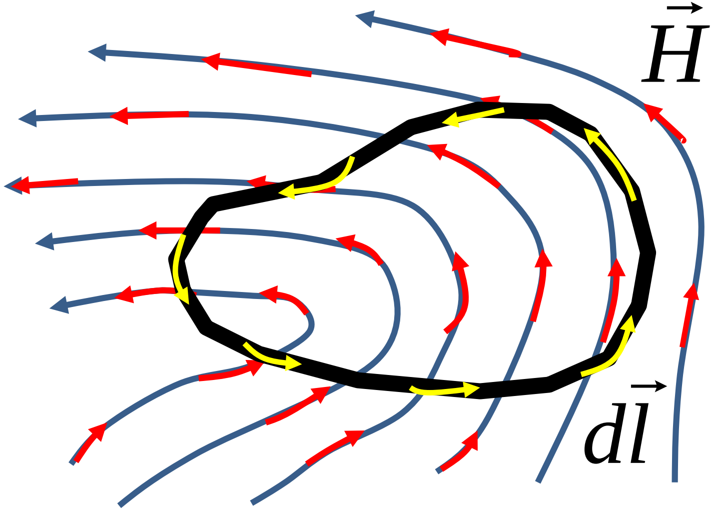
Circulation



資料來源

zh.wikipedia.org/wiki/颱風山竹_(2018年)

line integral of a vector along a close path



Line integral

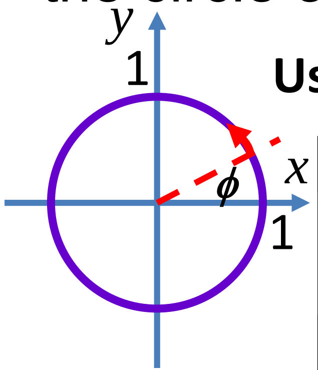
$$\int_L \vec{H}\cdot d\vec{l} = \int_a^b H\cos\theta dl$$

Circulation

$$\oint_L \vec{H}\cdot d\vec{l}$$ (close-path integral)

E.g. Let $\vec{H} = x\hat{a}_x - y\hat{a}_y$ $\Rightarrow$ $\vec{H} = \cos 2\phi\hat{a}_r - \sin 2\phi\hat{a}_\phi$

Calculate the circulation of H along the circle of r = 1, as shown below.



Use $x = 1\times\cos\phi, y = 1\times\sin\phi$

$$\begin{bmatrix} H_r \\ H_\phi \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ -y \\ 0 \end{bmatrix}$$

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz$$

$$= \hat{a}_\phi r d\phi$$

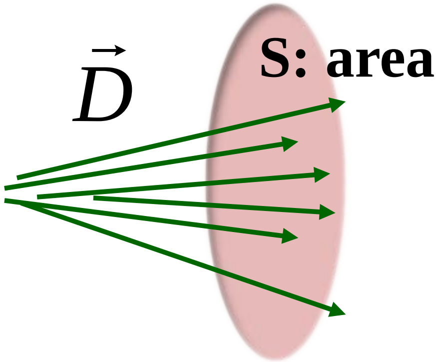
$$\oint_L \vec{H}\cdot d\vec{l} = -\int_0^{2\pi}\sin 2\phi d\phi$$

$$= 0$$

Flux

E.g. Water flux ∝ strength of water source

Electric Flux Φ_e

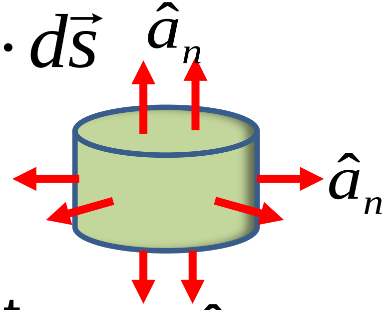


Surface Integral

$$\int_s \vec{A}\cdot d\vec{s} = \int_S A\cos\theta ds = \int_S \vec{A}\cdot\hat{a}_n ds$$

Close-surface Integral

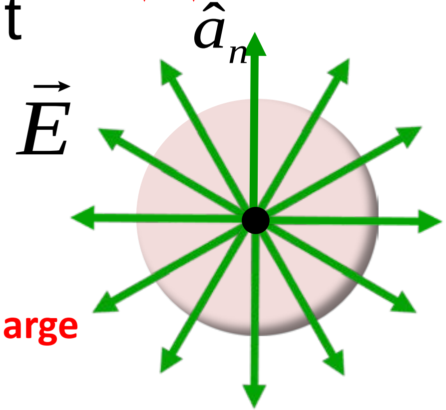
$$\oint_S \vec{A}\cdot d\vec{s}$$



The direction of a surface is always defined outward from a volume.

E.g. Calculate the total electric flux of a point charge of q.

Recall the electric flux density $\vec{D} = \varepsilon\vec{E} = \dfrac{q}{4\pi R^2}\hat{a}_R$



$$\oint_S \vec{A}\cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi}\frac{qR^2\sin\theta}{4\pi R^2}\hat{a}_R\cdot\hat{a}_R d\theta d\phi = q$$

= total charge inside

4.1旋量與通量

Circulation and flux

- Circulation – line integral of a vector field along a close path.
- Flux – surface integral of a vector field projected along the normal direction of the surface.
- The direction of a surface always points outward from a volume.

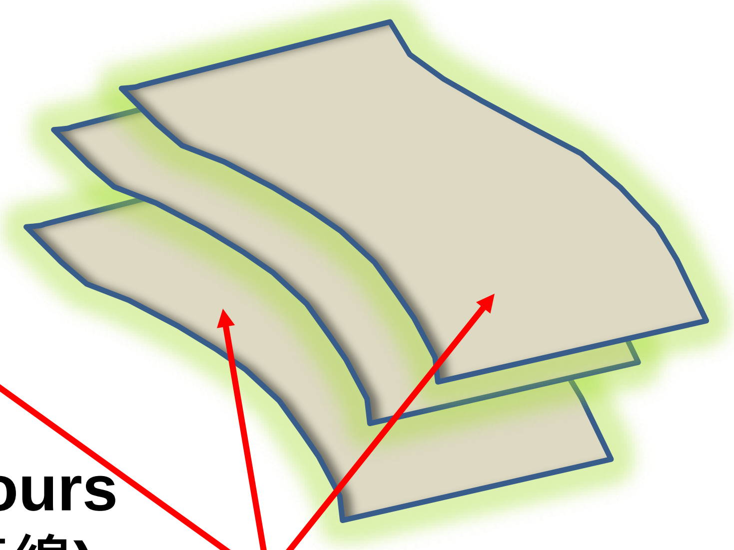
電磁學的數學工具 (三)- 向量微積分
Mathematic Tools (III) – vector calculus

4.2 梯度 Gradient of a scalar

Gradient 梯度

橫山梯田（節錄自新北市觀光旅遊網）
https://tour.ntpc.gov.tw/zh-tw/Food/Detail?wnd_id=60&id=110453

Contours
(等高線)



Equipotential surfaces
(等位面)

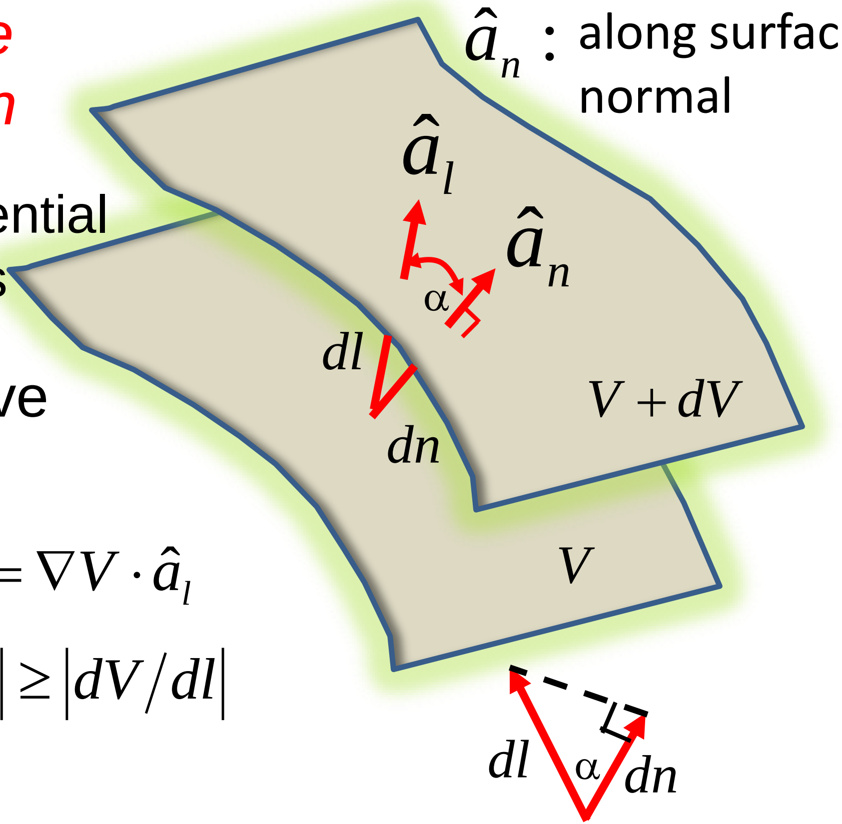
Gradient of a Scalar → a vector

Gradient: *maximum rate of change of a scalar in space, and a direction along the maximum change*

$$\nabla V \equiv \hat{a}_n \frac{dV}{dn}$$

equipotential surfaces

$\hat{a}_n$: along surface normal



- Consider the directional derivative along an arbitrary path l

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{a}_n \cdot \hat{a}_l = \nabla V \cdot \hat{a}_l$$

$$\left|dV/dn\right| \geq \left|dV/dl\right|$$

Useful expression $\frac{dV}{dl} = \nabla V \cdot \hat{a}_l \Rightarrow dV = (\nabla V) \cdot d\vec{l}$

E.g. Work done by a force is the dot product integral of a force along a path. Force is the gradient of work.

$W = \int dW = \int \vec{F} \cdot d\vec{l}$ or $dW = \vec{F} \cdot d\vec{l}$ or $\vec{F} = \nabla W$

- In the ***x,y, z*** coordinate system, $\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$

4.2 梯度

Gradient of a scalar

- Gradient of a scalar is vector.
- The magnitude of it is the maximum rate of change of the scalar in space.
- The direction of it is along the direction of the maximum change or along the surface normal of the equipotential surface of the scalar field.

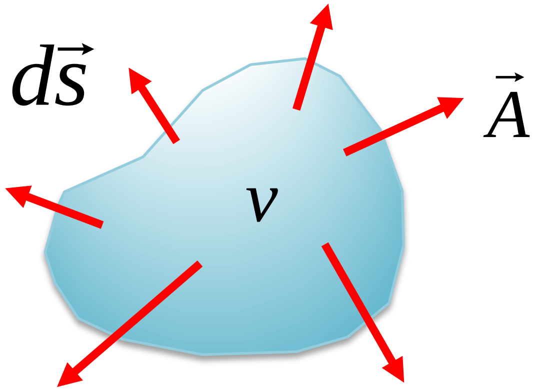
電磁學的數學工具 (三)- 向量微積分
Mathematic Tools (III) – vector calculus

4.3 散度 Divergence of a vector

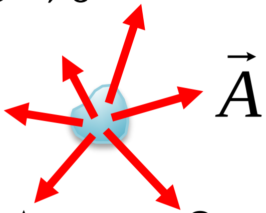
Divergence of a Vector $\rightarrow$ a scalar

Divergence: A scalar equal to the net outward flux of $\vec{A}$ per unit volume at a "point" in space.

$$\oint_S \vec{A} \cdot d\vec{s}$$



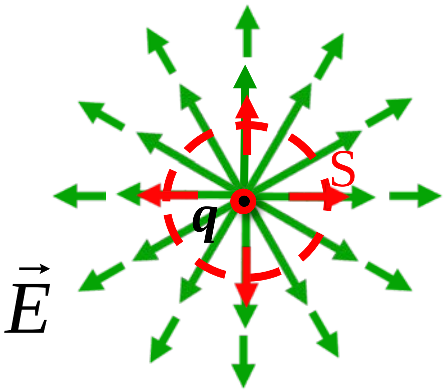
$$\oint_{S \to 0} \vec{A} \cdot d\vec{s}$$



$$\Delta v \rightarrow 0$$

$$\Rightarrow \quad \nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

net outward flux surrounding q

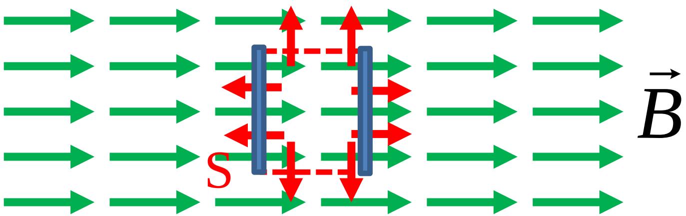


$\hat{a}_n$

$$\oint_S \vec{E}\cdot d\vec{s} \neq 0$$

$$\boxed{\nabla\cdot\vec{E} \neq 0}$$

uniform flux of fields to the right



$$-Bds + Bds$$

$$\oint_S \vec{B}\cdot d\vec{s} = 0$$

$$\boxed{\nabla\cdot\vec{\mathrm{B}} = 0}$$ (solenoidal field)

Divergence Theorem

$$\int_V \nabla\cdot\vec{A}dv \equiv \oint_S \vec{A}\cdot d\vec{s}$$

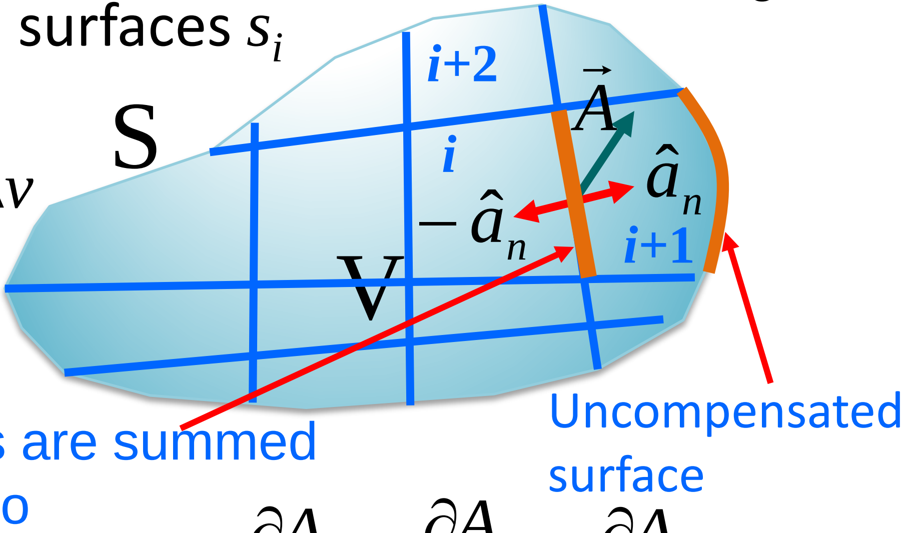
S: surface enclosing V

Partition V into many small v_i surrounded by small surfaces s_i

$$\int_V \nabla\cdot\vec{A}dv$$

$$\sum_i \int_{v_i} \nabla\cdot\vec{A}_i dv \equiv \sum_i \lim_{\Delta v\to 0}\frac{\oint_{S_i}\vec{A}_i\cdot d\vec{s}}{\Delta v}\Delta v$$

$$= \oint_S \vec{A}\cdot d\vec{s}$$



- In x, y, z coordinate system, $\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

4.3 散度

Divergence of a vector

- Divergence of a vector is a scalar.
- It has a value equal to the net outward flux of the vector per unit volume at an infinitesimal point in space.
- Divergence theorem: volume integral of the divergence of a vector field is equal to the total outward flux of the vector field over the enclosed surface.

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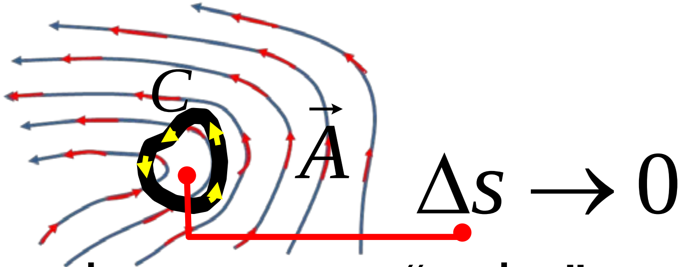
4.4 旋度 Curl of a vector

Curl of a Vector $\rightarrow$ a vector

$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{A} \cdot d\vec{l}}{\Delta s}$$



maximum net circulation of $\vec{A}$ per unit area at a “point” in space.

The direction of $\nabla \times \vec{A}$ is chosen to be the surface normal direction of the infinitesimal area Δs with which the net circulation is a maximum. (right-hand rule applies)

Stokes' Theorem $\int_S (\nabla \times \vec{A}) \cdot d\vec{s} \equiv \oint_C \vec{A} \cdot d\vec{l}$

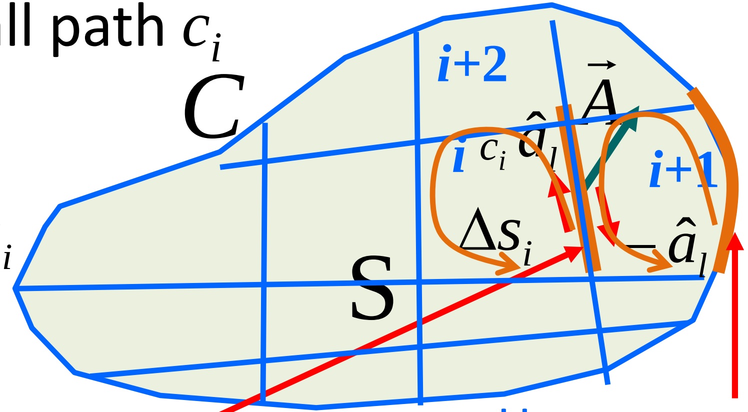
C: path surrounding surface S.

Partition S into many small s_i surrounded by small path c_i

$$\int_S \nabla \times \vec{A} \cdot d\vec{s}$$

$$\sum_i \int_{s_i} \nabla \times \vec{A}_i \cdot d\vec{s} \equiv \sum_i \lim_{\Delta s_i \to 0} \frac{\oint_{c_i} \vec{A}_i \cdot d\vec{l}}{\Delta s_i} \Delta s_i$$

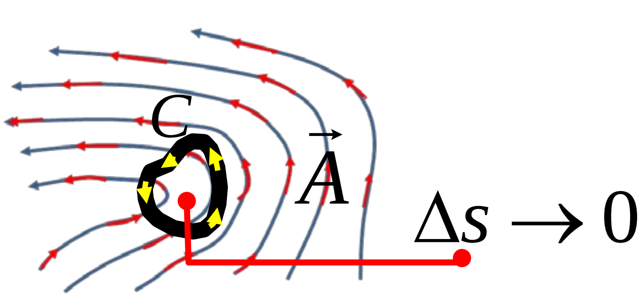
$$= \oint_C \vec{A} \cdot d\vec{l}$$



Line integrals are summed to zero

Uncompensated path

Curl $\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{A} \cdot d\vec{l}}{\Delta s}$



If $\nabla \times \vec{A} = 0$, vector $\vec{A}$ is said to be irrotational .

In ***x*, *y*, *z*** coordinate system,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\hat{a}_x + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\hat{a}_y + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\hat{a}_z$$

4.4 旋度

Curl of a vector

- The curl of a vector is a vector.
- It has a magnitude equal to the maximum circulation of the vector per unit surface at an infinitesimal point in space.
- It has a direction along the surface normal that results in the maximum circulation.
- Stokes theorem: surface integration of the curl of a vector field is equal to the circulation of the vector along the path enclosing the surface.

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4.5 組合運算子
Combined operators

Laplacian Operator ∇^2

Laplacian operator of a scalar field V (scalar Laplacian)

$$\nabla^2 V \equiv \nabla \cdot (\nabla V) \quad \longleftarrow \text{ a scalar}$$

In the *x, y, z* coordinate system, the expression is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian operator of a vector field $\vec{A}$ (vector Laplacian)

$$\nabla^2 \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A} \quad \longleftarrow \text{ a vector}$$

In the *x, y, z* coordinate system, the expression is given by

$$\nabla^2 \vec{A} = \hat{a}_x \nabla^2 A_x + \hat{a}_y \nabla^2 A_y + \hat{a}_z \nabla^2 A_z$$

Null Identities

$$\nabla \times (\nabla V) = 0$$

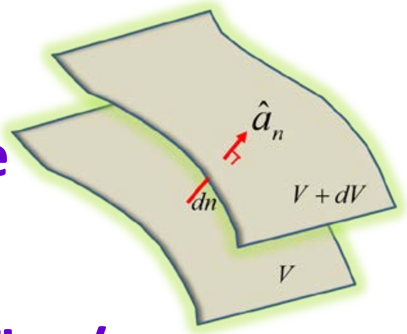
no net circulation along the maximum change of a scalar

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

no net outward flux at the maximum circulation of a vector.

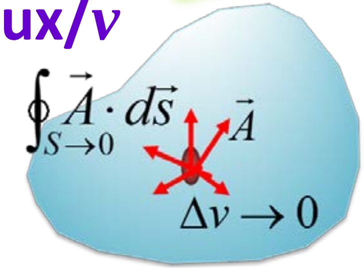
Gradient – max change rate

$$\nabla V \equiv \hat{a}_n \frac{dV}{dn}$$



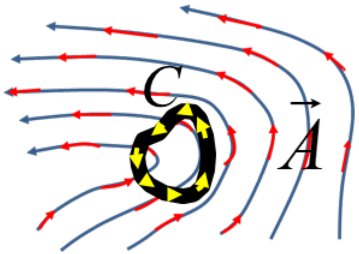
Divergence – net outward flux/*v*

$$\nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$



Curl – max circulation/*s*

$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{A} \cdot d\vec{l}}{\Delta s}$$



Helmholtz's Theorem

"A vector field is determined to within an additive constant, if both its divergence and its curl are specified everywhere." [*]

If $\overbrace{\nabla\cdot\vec{A}=\nabla\cdot\vec{B} \ \& \ \nabla\times\vec{A}=\nabla\times\vec{B}}^{\text{two equations with two variables}} \ \& \ \overbrace{\vec{A}\cdot d\vec{s}=\vec{B}\cdot d\vec{s}}^{\text{boundary conditions}}$

on the surface surrounding the volume in question,

then $\vec{A}=\vec{B}+\underbrace{\text{a constant vector.}}_{\text{not important for spatially and/or temporally varying quantities.}}$

This theorem ensures the uniqueness of a solution in electromagnetics.

*D. K. Cheng, Field and Wave Electromagnetics, 2nd Ed, Addison-Wesley (1989).

4.5 組合運算子

Combined operators

- Scalar Laplacian is a scalar, expressed by $\nabla^2 V$.
- Vector Laplacian is a vector, expressed by $\nabla^2 \vec{A}$.
- Curl of gradient of a scalar is zero $\nabla \times (\nabla V) = 0$
- Divergence of curl of a vector is zero $\nabla \cdot (\nabla \times \vec{A}) = 0$
- The Helmholtz theorem ensures the uniqueness of a field solution.

電磁學的數學工具 (三)- 向量微積分
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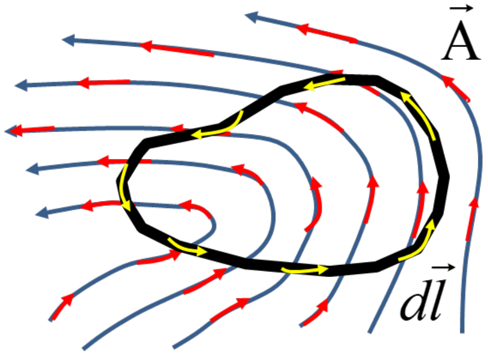
4.6 單元回顧 Review

單元回顧

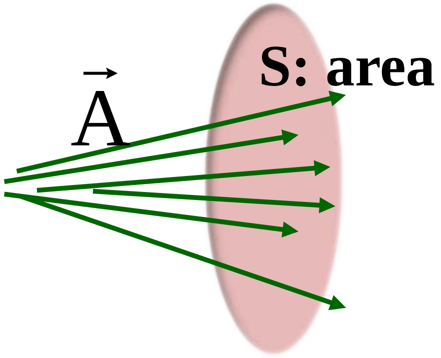
1. Circulation: line integration of a vector projected along a path enclosing a surface.

$$\oint_L \vec{A}\cdot d\vec{l}$$



2. Flux: surface integration of a vector projected along the normal direction of a surface.

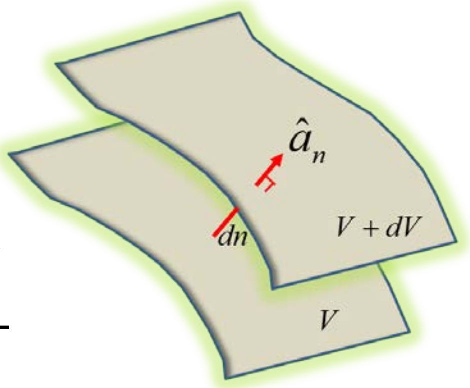
$$\int_S \vec{A}\cdot d\vec{s}$$



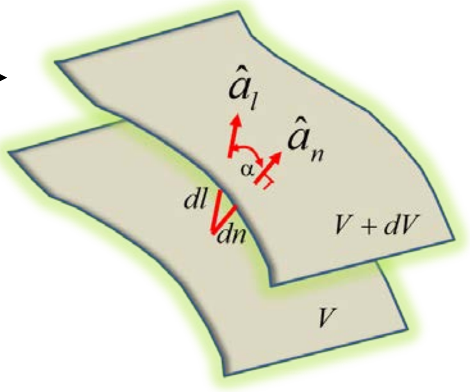
單元回顧

3. Gradient: maximum rate of change of a scalar in space, and a direction along the maximum change.

$$\nabla V \equiv \hat{a}_n \frac{dV}{dn}$$

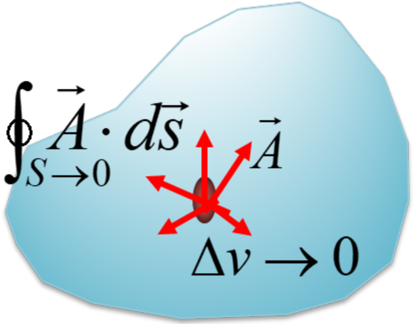


4. Useful expression $dV = (\nabla V) \cdot d\vec{l}$



單元回顧

5. Divergence: A scalar equal to the net outward flux of $\vec{A}$ per unit volume at a "point" in space.



$$\nabla \cdot \vec{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

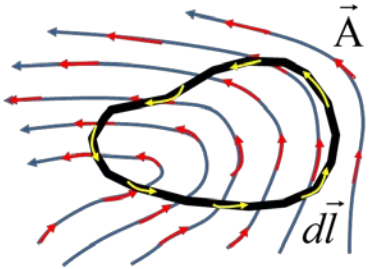
6. Divergence theorem: volume integration of the divergence of a vector is equal to the total outward flux of the vector over the enclosed surface.

$$\int_V \nabla \cdot \vec{A} dv \equiv \oint_S \vec{A} \cdot d\vec{s}$$

單元回顧

7. Curl: maximum net circulation of $\vec{A}$ per unit area at a "point" in space. Its direction follows the right-hand rule, along the normal of the chosen surface.

$$\nabla \times \vec{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{a}_n \oint_C \vec{A} \cdot d\vec{l}}{\Delta s}$$



8. Stokes theorem: $\int_S (\nabla \times \vec{A}) \cdot d\vec{s} \equiv \oint_C \vec{A} \cdot d\vec{l}$ surface integration of the curl of a vector is equal to the circulation of the vector along the path enclosing the surface.

單元回顧

9. Definition of the scalar Laplacian:

$$\nabla^2 V \equiv \nabla \cdot (\nabla V) \longleftarrow \text{ a scalar}$$

10. Definition of the vector Laplacian:

$$\nabla^2 \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A} \longleftarrow \text{ a vector}$$

11. Curl of gradient of a scalar is zero: $\nabla \times (\nabla V) = 0$

12. Divergence of curl of a vector is zero: $\nabla \cdot (\nabla \times \vec{A}) = 0$

13. The Helmholtz theorem ensures the uniqueness of a field solution.

THANK YOU FOR YOUR ATTENTION