CS2336 Discrete Mathematics

Homework 5
Tutorial: December 30, 2019
Exam 3: January 06, 2020 (2.5 hours)

Problems marked with * will be explained in the tutorial.

1. Determine whether each of these pairs of sets are equal.
   (a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
   (b) $\{\{1\}\}$, $\{1, \{1\}\}$
   (c) $\emptyset$, $\{\emptyset\}$
2. Let $A = \{1, \{\{1\}\}, \{\{2\}\}\}$. Which of the following statements are true?
   (a) $1 \in A$
   (b) $\{1\} \in A$
   (c) $\{1\} \subseteq A$
   (d) $\{\{1\}\} \subseteq A$
   (e) $\{\{2\}\} \subset A$
   (f) $\{\{1\}, \{2\}\} \subset A$
3. (*) Let $A = \{\varnothing, \{\varnothing\}\}$. Determine whether each of the following statements is true or false.
   (a) $\varnothing \in 2^A$
   (b) $\varnothing \subseteq 2^A$
   (c) $\{\varnothing\} \subseteq 2^A$
   (d) $\{\varnothing\} \subseteq A$
   (e) $\{\varnothing\} \in 2^A$
   (f) $\{\varnothing\} \in A$
   (g) $\{\{\varnothing\}\} \subseteq 2^A$
   (h) $\{\{\varnothing\}\} \subseteq A$
   (i) $\{\{\varnothing\}\} \in 2^A$
   (j) $\{\{\varnothing\}\} \in A$
4. A survey of 150 college students reveals that 83 own cars, 97 own bikes, 28 own motorcycles, 53 own a car and a bike, 14 own a car and a motorcycle, 7 own a bike and a motorcycle and 2 own all three. How many students own a bike and nothing else?
5. (*) For two sets A and B, we use $A - B$ to denote $A \cap \overline{B}$. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.
6. (*) For each of the following lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

(a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...

(b) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...

(c) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...

(d) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ...

7. (*) Show that the set $\mathbb{R}$ of real numbers and the set $\mathbb{R}^+$ of positive real numbers have the same cardinality. (That is, give a one-to-one correspondence between the items in the two sets.)

8. (*) After learning the diagonalization technique, Peter has come up with the following proof, showing that the set

$$X = \{\, x \mid x \in (0,1) \;\text{ and }\; x \text{ has } k \text{ decimal places } \;\text{ and }\; k \in \mathbb{N} \,\}$$

is uncountable:

> *We prove this by contradiction. Assume to the contrary that there is a one-to-one correspondence between items in X and items in $\mathbb{N}$. Then, we can list the items in X one by one, say $x_1, x_2, x_3, \ldots$. Now, consider the number x such that its digit in the first decimal place is different from x_1, its digit in the second decimal place is different from x_2, and in general, its digit in the jth decimal palce is different from x_j for all j. Then, x is not listed by the correspondence, and a contradiction occurs as desired.*

However, each number in X is a rational number; for instance, $0.33215 = 33215/100000$. Thus, $X \subseteq \mathbb{Q}$ (where $\mathbb{Q}$ is countable), which implies X must be countable.

So, what's wrong with Peter's proof?

9. (*) Give an example of three sets W, X, Y such that $W \in X$ and $X \in Y$ but $W \notin Y$. (Don't mix up the symbol $\in$ with the symbol $\subseteq$.)

10. (*) Consider all the five-element subsets of $\{1, 2, 3, \ldots, n\}$. It is known that one quarter of these subsets contain the element 7. What is the value of n?

11. (*) Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

    (a) $f(x) = -3x + 4$

    (b) $f(x) = 3x^2 + 4$

    (c) $f(x) = (x+1)/(x+2)$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function where $f(x) = x^2$. For any subset A of $\mathbb{R}$, we use $f(A)$ to denote the set $\{f(x) \mid x \in A\}$. Determine $f(A)$ for the following subsets A taken from the domain $\mathbb{R}$.

    (a) $A = \{2, -3\}$

    (b) $A = (-3, 3)$

    (c) $A = (-3, 2]$

    (d) $A = (-4, -3] \cup [5, 6]$

13. (*) Define $g : \mathbb{Z} \to \mathbb{Z}$, where $g(n) = \lfloor n/2 \rfloor$. Is g a surjection? Is g a bijection?

14. Let S be the set of all strings of English letters. Consider the following relations on S and determine whether

    (a) $R_1 = \{(a, b) \mid a \text{ and } b \text{ have no letters in common}\}$ is reflexive or not.

    (b) $R_2 = \{(a, b) \mid a \text{ and } b \text{ are not of the same length}\}$ is transitive or not.

    (c) $R_3 = \{(a, b) \mid a \text{ is longer than } b\}$ is symmetric or not.

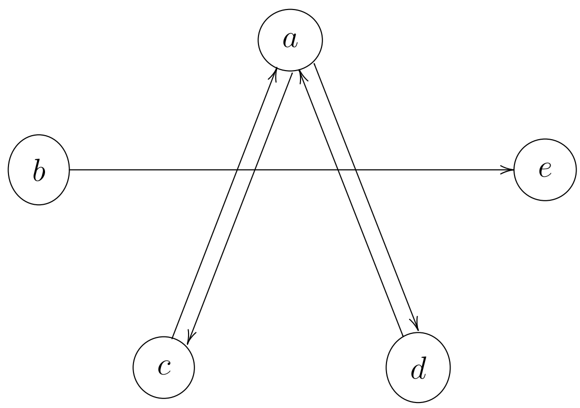
15. (*) What is wrong with the following argument that attempts to show that if R is a relation on a set S that is both symmetric and transitive, then R is also reflexive?

    "Since xRy implies yRx by the symmetric property,
    xRy and yRx imply xRx by the transitive property,
    thus, xRx is true for each x in S, and so R is reflexive."

16. (*)

    (a) Find the reflexive closure of R as represented by the following directed graph.

    (b) Find the transitive closure of R as represented by the following directed graph.



17. Let A be a set of books. Let R be a binary relation on A such that (a, b) is in R if book a costs more and contains fewer pages than book b does. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?

18. Let R be a binary relation on the set of all positive integers such that

$$R = \{(a, b) \mid a - b \text{ is an odd positive integer}\}$$

    Is R reflexive? Symmetric? Antisymmetric? Transitive?

19. Let R be a binary relation on the set of all strings of 0s and 1s such that

$$R = \{(a, b) \mid a \text{ and } b \text{ are strings that have the same number of 0s}\}$$

    Is R reflexive? Symmetric? Antisymmetric? Transitive?

20. (*) Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7, 9\}$. Suppose that P is a relation from A to B defined as follows. In each case, give the elements of P.

(a) $aPb \Leftrightarrow a \equiv b \pmod 4$

(b) $aPb \Leftrightarrow 2$ divides $(a+b)$

(c) $aPb \Leftrightarrow a = b$ or $a - 1 = b$

21. Let R be the relation $\{(a,b) \mid a < b\}$ on the set of integers. What are the reflexive closure and the symmetric closure of R?

22. Find the transitive closure of these relations on $\{1,2,3,4\}$.

    (a) $\{(1,2),(2,1),(2,3),(3,4),(4,1)\}$

    (b) $\{(1,1),(1,4),(2,1),(2,3),(3,1),(3,2),(3,4),(4,2)\}$

23. Construct a relation R such that R is irreflexive but R^2 is reflexive.

24. Construct a relation which is not reflexive, not symmetric, not antisymmetric, and not transitive.

25. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relations the others lack.

    (a) $\{(a,b) \mid a$ and b have the same age$\}$

    (b) $\{(a,b) \mid a$ and b have the same parents$\}$

    (c) $\{(a,b) \mid a$ and b speak a common language$\}$

26. (*) Suppose that R_1 and R_2 are equivalence relation a set S. Is it necessarily true that $R_1 \cap R_2$ is an equivalence relation? How about $R_1 \cup R_2$?

27. Determine whether each of the following collections of sets is a partition for the given set A. If the collection is not a partition, explain why it fails to be.

    (a) $A = \{1,2,3,4,5,6,7,8\}$; $A_1 = \{4,5,6\}, A_2 = \{1,8\}, A_3 = \{2,3,7\}$

    (b) $A = \{a,b,c,d,e,f,g,h\}$; $A_1 = \{d,e\}, A_2 = \{a,c,d\}, A_3 = \{f,h\}, A_4 = \{b,g\}$

28. (*) Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$, and define R on A by $(x_1,y_1)R(x_2,y_2)$ if $x_1+y_1 = x_2+y_2$.

    (a) Verify that R is an equivalence relation on A.

    (b) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$, and $[(1,1)]$.

    (c) Determine the partition induced by R.

29. (*) If $|A| = 30$ and the equivalence relation R on A partitions A into equivalence classes A_1, A_2, and A_3, where $|A_1| = |A_2| = |A_3|$, what is $|R|$?

30. Which of these are posets (partial order sets)?

(a) $(R, =)$ (b) $(R, <)$ (c) $(R, \leq)$ (d) $(R, \neq)$

31. Let $A = \{1, 2, 3, 6, 9, 18\}$, and define R on A by xRy if x divides y. Draw the Hasse diagram for the poset (A, R).