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電磁學（一）
Electromagnetics (I)

7. 電容
Capacitance

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In this lecture, we will introduce a charge-storage device, called capacitor.

- 7.1 Charge storage 電荷儲存
- 7.2 Parallel-plate capacitor 平板電容器
- 7.3 Cylindrical and spherical capacitors 圓柱及球形電容器
- 7.4 Capacitor circuit 電容電路
- 7.5 Review 單元回顧

電容
Capacitance

7.1 電荷儲存
Charge Storage

Observation 1

Faraday's law of electrostatics $\nabla \times \vec{E} = 0$

⇨ $\vec{E} \equiv -\nabla V$ defines electric potential

Gauss Law $\nabla \cdot \vec{D} = \rho \Rightarrow \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon}$

⇨ $-\nabla \cdot \nabla V = \dfrac{\rho}{\varepsilon}$ or $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$

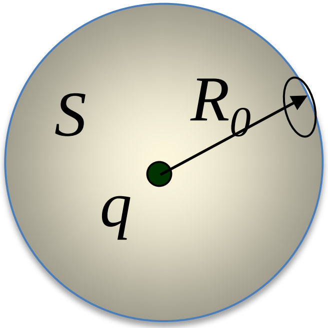
(**Poisson's equation**)

$\boxed{V \propto \rho}$: electric potential V is linearly proportional to charge ρ

Observation 2

● $q' = +1$

$$\vec{E} = E_R \hat{a}_R = \frac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$



$$V(R_0) = -\int_{\infty}^{R_0} \vec{E}\cdot\hat{a}_R\, dR = \frac{q}{4\pi\varepsilon_0 R_0}$$

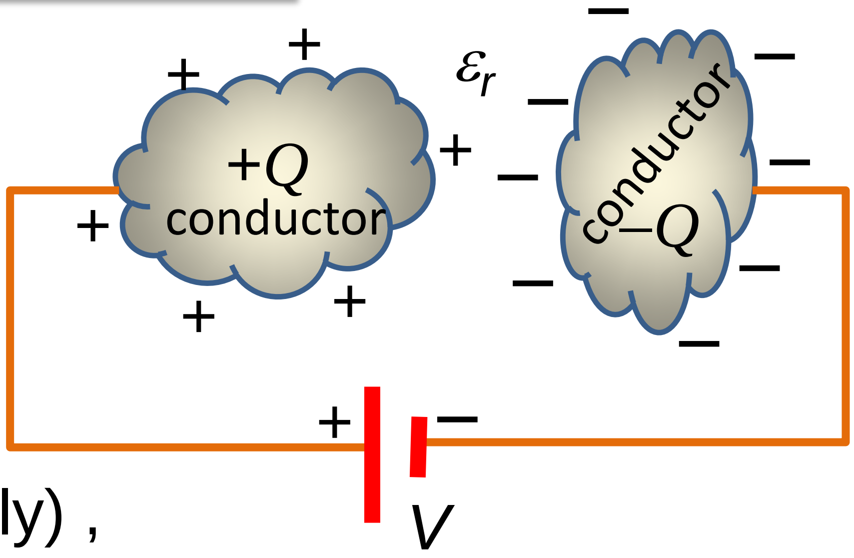
$$\Rightarrow \quad \boxed{V \propto q}$$

electric potential V is linearly proportional to charge q

Capacitance

From previous calculations

$$V \propto Q,$$



define capacitance as

$$C \equiv \frac{Q}{V} \text{ (positive value only) ,}$$

which is the stored charge per unit voltage

(a function of device geometry and relative permittivity ε_r.).

7.1 電荷儲存

Charge Storage

- Electric potential V is linearly proportional to the amount of charges q generating it.
- Connecting a battery (applying a voltage V) to a pair of conductors (electrodes) stores charges into the system.
- Capacitance is the amount of stored charge per unit voltage, depending on the potential difference or voltage between electrodes, nearby material, and geometry of the system.

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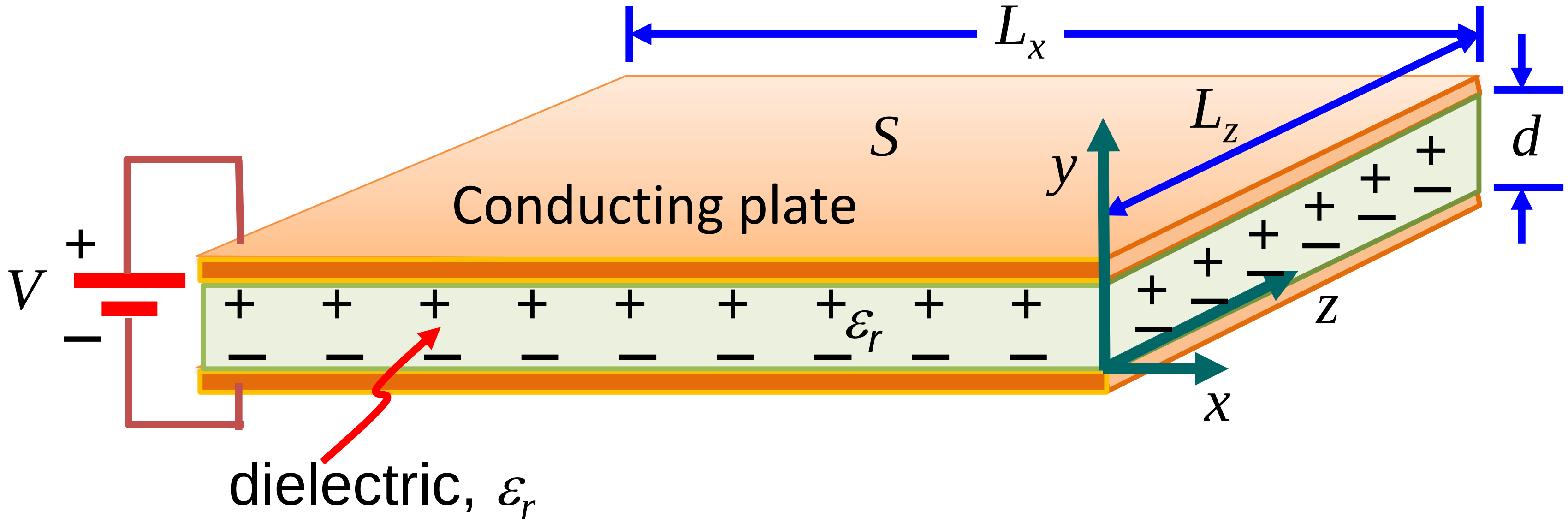
Capacitance
7.2 平板電容器
Parallel-plate Capacitor

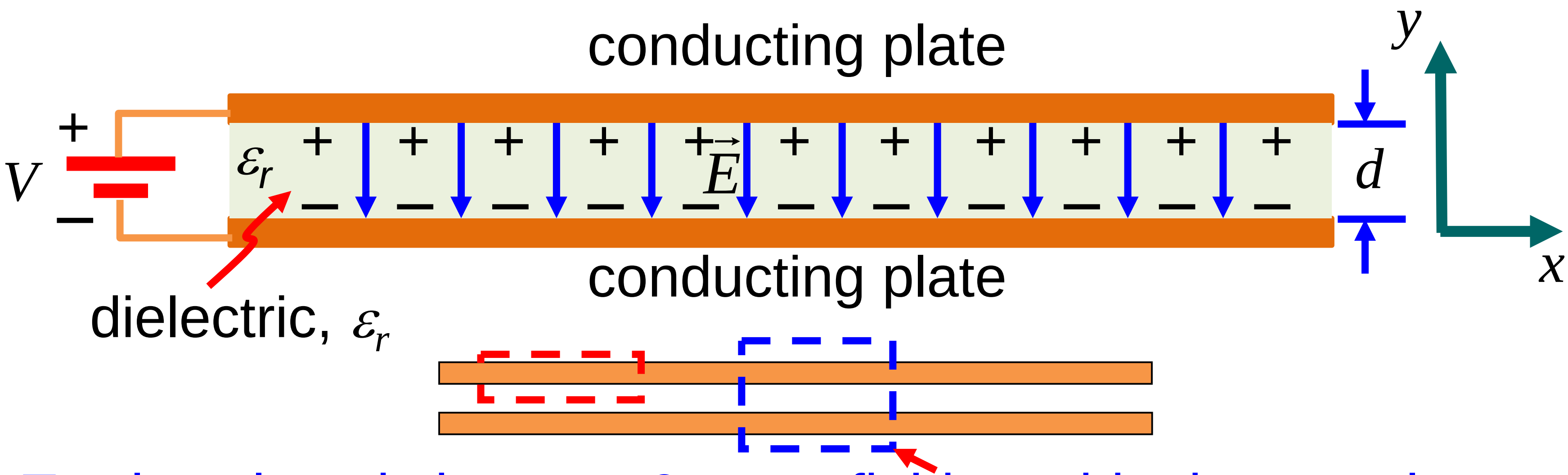
Parallel-plate Capacitance



Assumptions: $L_x, L_z >> d$

$\Rightarrow$ fields are more or less uniform between plates

(fringe fields are ignored during calculation)

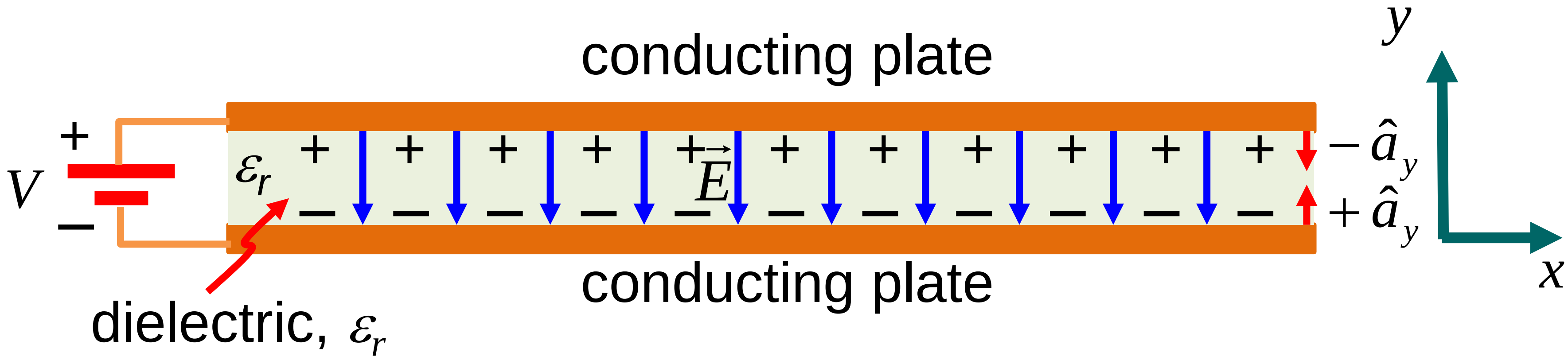


Total enclosed charge = 0 $\Rightarrow$ no field outside the two plates

Apply Gauss law to the surface defined by red dashed line

$$\varepsilon\vec{E}\cdot\vec{S}=Q \;\Rightarrow (-\varepsilon E_y\hat{a}_y)\cdot(-S\hat{a}_y)=Q \;\Rightarrow E_y=\frac{Q}{\varepsilon S}=\frac{\rho_s}{\varepsilon}$$

where S is the total area of the plate, $\varepsilon = \varepsilon_0\varepsilon_r$ is permittivity of the dielectric, and Q is the total charge on S.



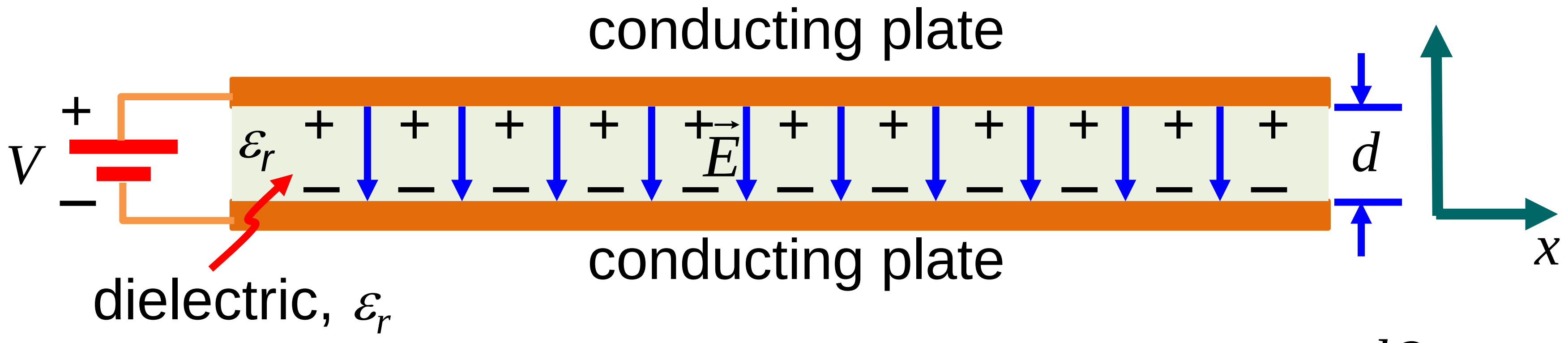
$$\vec{E} = -\frac{\rho_s}{\varepsilon}\hat{a}_y \;\Longrightarrow\; \vec{D} = \varepsilon\vec{E} = -\rho_s\hat{a}_y$$

0 in conductor

Recall the B.C. $\hat{a}_{n2}\cdot(\vec{D}_1 - \cancel{\vec{D}_2}) = \rho_s \;\Longrightarrow\; \hat{a}_{n,conductor}\cdot\vec{D}_{dielectric} = \rho_s$

Upper conducting plate: $-\hat{a}_y\cdot(-\rho_s\hat{a}_y) = \rho_s$

Lower conducting plate: $\hat{a}_y\cdot(-\rho_s\hat{a}_y) = -\rho_s$



The voltage across the two electrodes is $\boxed{V = E_y d} = \dfrac{dQ}{\varepsilon S}$

The capacitance of this parallel plate capacitor is $C \equiv \dfrac{Q}{V} = \dfrac{\varepsilon S}{d}$

A large area S, a high permittivity ε_r, and a small electrode gap d help to store charges under a voltage.

Why does large ε_r give large C?

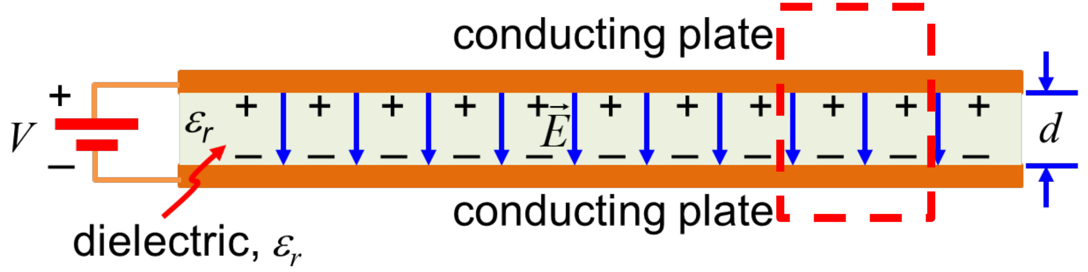
$E = V/d$ is fixed for a given V, but a large ε_r results in a large $D = \varepsilon E$. $\rightarrow$ Q becomes larger due to $D \propto \rho$ in $\nabla \cdot \vec{D} = \rho$.

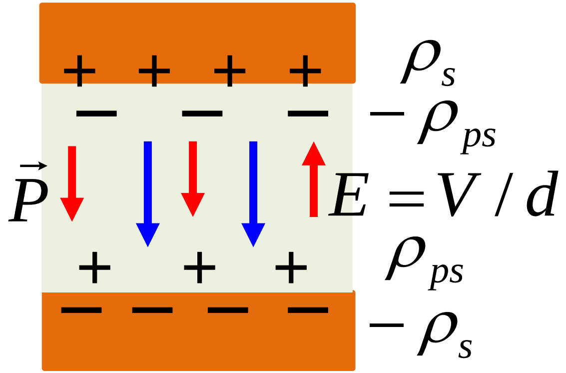
Therefore, $C = \dfrac{Q}{V}$ becomes larger when $\varepsilon_r \uparrow \rightarrow Q\uparrow$ for fixed V.

Physically, large $\varepsilon_r = 1 + \chi_e \rightarrow$ large χ_e, but $\vec{P} = \varepsilon_0 \chi_e \vec{E}$.

Recall $\vec{P} \cdot \hat{a}_n = \rho_{ps}$ ➡ large P means large ρ_{ps}

➡ large ρ_{ps} holds a large Q for a fixed V.





7.2 平板電容器

Parallel-plate Capacitor

- Ignoring fringe fields, the capacitance of a parallel-plate capacitor is given by

$$C \equiv \frac{Q}{V} = \frac{\varepsilon S}{d},$$

where S is the area of the electrode plate, d is the separation of the electrodes, and ε is the permittivity of the dielectric between the electrodes.

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7.3 圓柱及球形電容
Cylindrical and Spherical Capacitors

Cylindrical Capacitor (1)

Again, ignore the fringe fields at the edges.

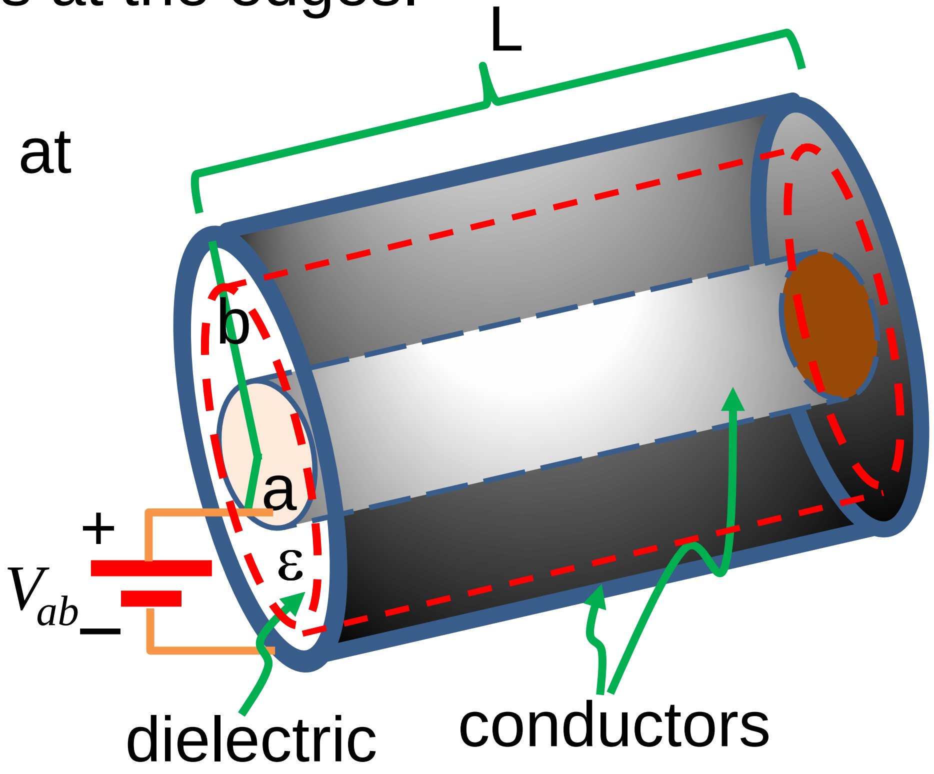
Apply Gauss's law to the cylindrical Gaussian surface at a constant r

$$\varepsilon\oint_S \vec{E}\cdot d\vec{s} = Q$$

$$\Rightarrow \quad \hat{a}_r E_r = \hat{a}_r \frac{Q/\varepsilon}{2\pi rL}$$

The electric potential across the two electrodes is

$$V_{ab} = -\int_{r=b}^{r=a} \vec{E}_r \cdot d\vec{r} = \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$



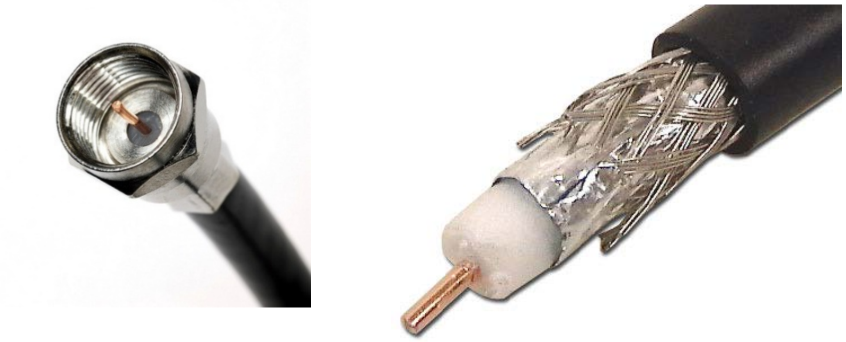
Cylindrical Capacitor (2)

Take the ratio of Q and V to obtain the **capacitance** for a cylindrical capacitor

$$C \equiv \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

For a transmission line, what we care is the capacitance per unit length

$$C_l \equiv \frac{\rho_l}{V} = \frac{2\pi\varepsilon}{\ln(b/a)}$$



Coaxial-cable transmission line

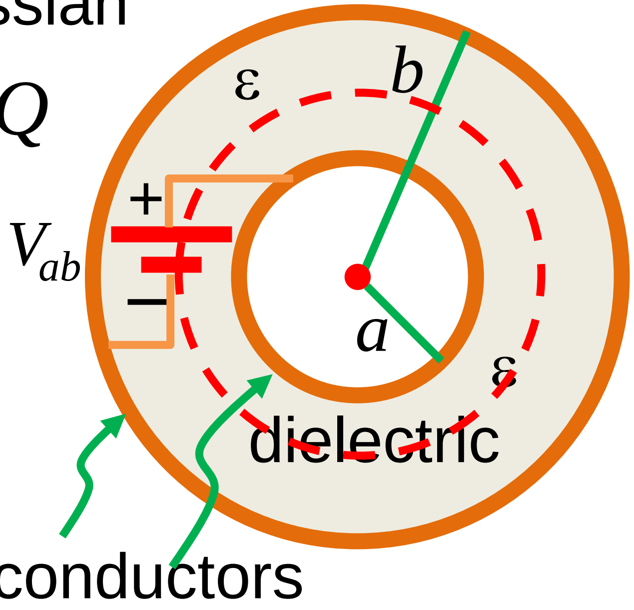
Spherical Capacitor

Again, apply the Gauss law to the Gaussian surface at a constant $R \Rightarrow \varepsilon \oint_S \vec{E} \cdot d\vec{s} = Q$

to obtain $E_R = \dfrac{Q}{4\pi\varepsilon R^2}$

Calculate the electric potential between $R = a$ and $R = b$

$$V_{ab} = -\int_{R=b}^{R=a} \vec{E}_R \cdot d\vec{R} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$



Take the ratio of charge to voltage to obtain the capacitance

$$C \equiv \frac{Q}{V} = \frac{4\pi\varepsilon}{(1/a - 1/b)}$$

7.3 圓柱及球形電容器

Cylindrical and spherical capacitors

- A cylindrical capacitor consists of concentric electrodes filled with dielectric in between.
- The capacitance per unit length for a coaxial cable is given by

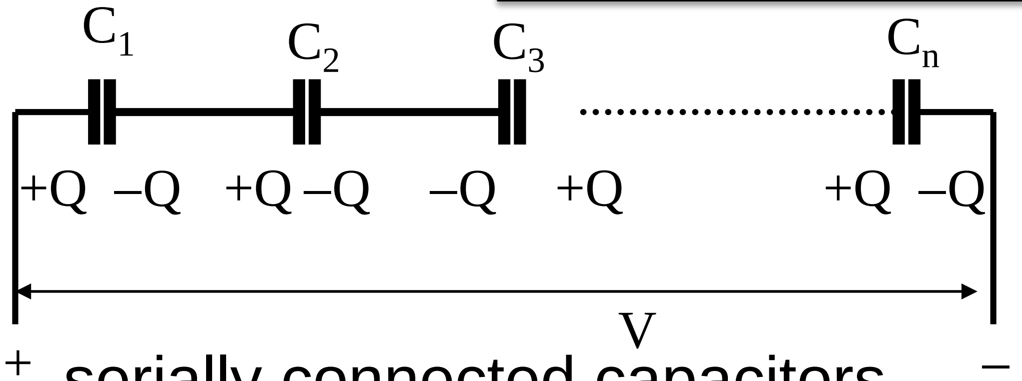
$$C_l = \frac{2\pi\varepsilon}{\ln(b/a)}$$

- A spherical capacitor consists of two concentric spherical electrodes filled with dielectric in between.

電容
Capacitance

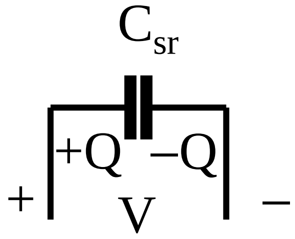
7.4 電容電路
Capacitor Circuit

Serial Capacitors



serially connected capacitors

$\equiv$



equivalence circuit

Circuit expression

$$V = \frac{Q}{C_{sr}} = \sum_i V_i = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \dots + \frac{Q}{C_n} \quad \Rightarrow \frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \dots + \frac{1}{C_n}$$

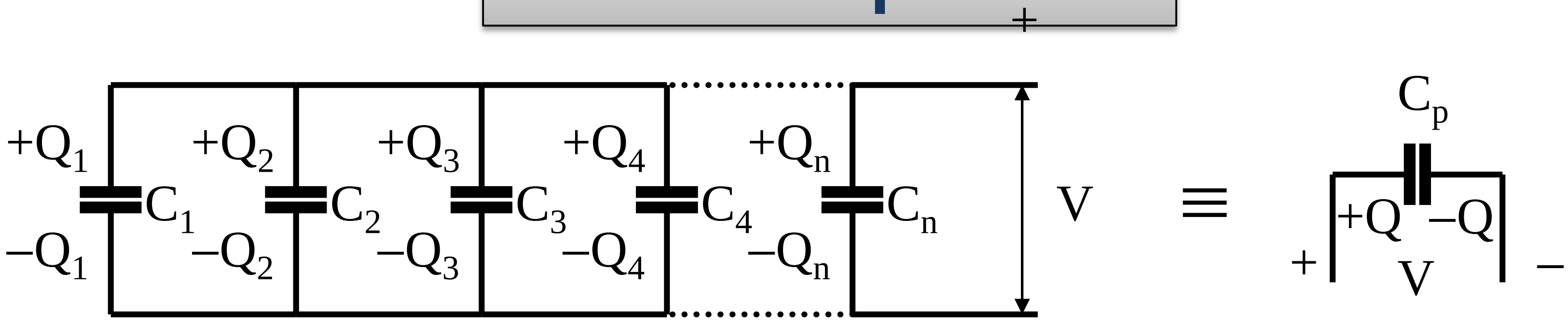
Equivalent capacitance – inverse of the inverse sum of C_i

$$C_{sr} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \dots + \frac{1}{C_n} \right)^{-1}$$

Dominant term is the small C in

$$\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \dots + \frac{1}{C_n}$$

Parallel Capacitors



parallel connected capacitors equivalent circuit

Circuit expression

$$C_pV = Q = Q_1 + Q_2 + Q_3 + \ldots + Q_n$$
$$= C_1V + C_2V + C_3V \ldots + C_nV$$

sum of individual capacitances

$$\Rightarrow C_p = C_1 + C_2 + C_3 \ldots + C_n$$

Dominant term is the large *C* in $C_1 + C_2 + C_3 \ldots + C_n$

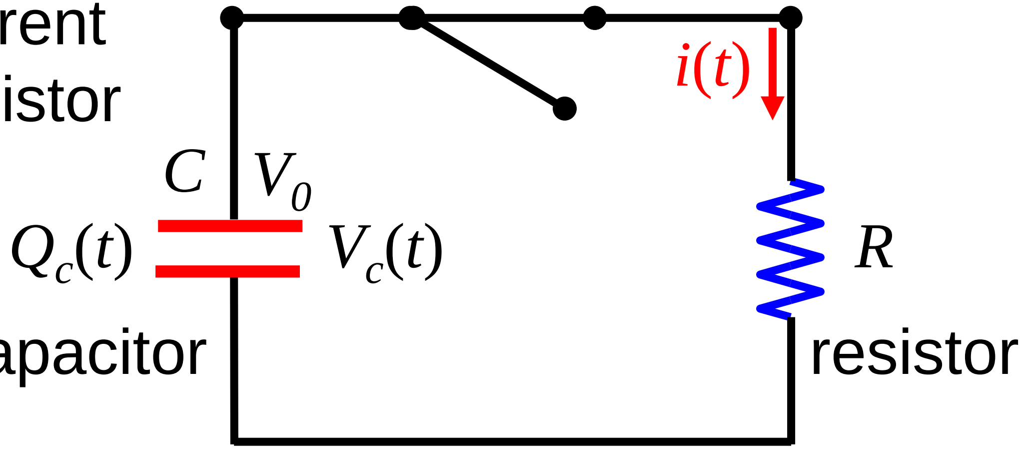
The capacitance becomes larger as the areas for storing Q are added up.

RC Discharging Circuit

capacitor discharging current
= current entering the resistor

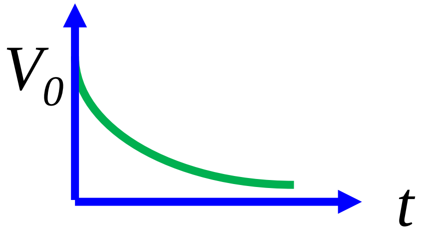
$$\frac{dQ_c}{dt} + \frac{V_c(t)}{R} = 0$$

Recall $C = \dfrac{Q_c}{V_c}$



$$\Rightarrow \quad C\frac{dV_c}{dt} + \frac{V_c(t)}{R} = 0 \quad \Rightarrow \quad V_c(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

Capacitor voltage drops exponentially
with a time constant $\tau = RC$



7.4 電容電路

Capacitor circuit

- The inverse of the equivalent capacitance of serial capacitors is the inverse sum of all the individual capacitances.

$$C_{sr} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \dots + \frac{1}{C_n} \right)^{-1}$$

- The equivalent capacitance of parallel capacitors is the sum of all the individual capacitances.

$$C_p = C_1 + C_2 + C_3 \dots + C_n$$

- An RC circuit charges/discharges with a time constant equal to

$$\tau = RC$$

電容
Capacitance

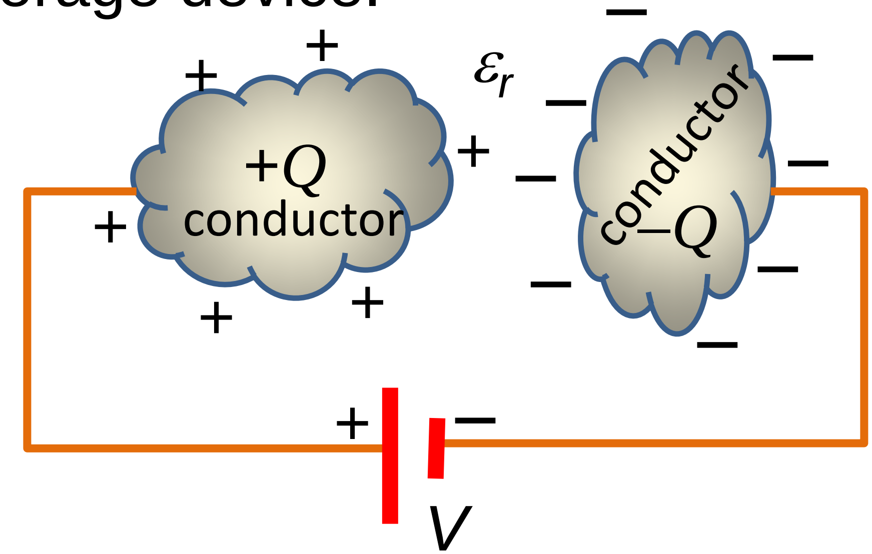
7.5 單元回顧 Review

單元回顧

1. A capacitor is a charge storage device.

2. Capacitance is defined as the amount of charges stored in a capacitor per unit voltage.

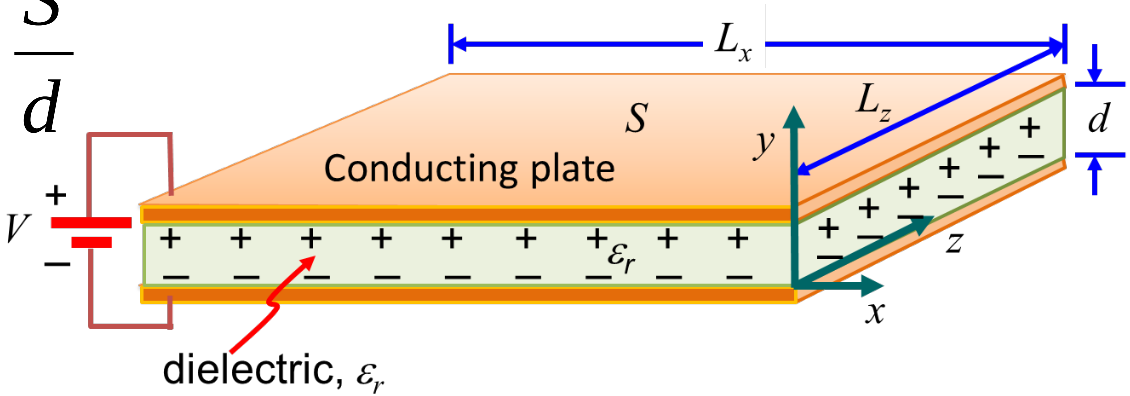
$$C \equiv \frac{Q}{V}$$



單元回顧

3. A parallel-plate capacitor has a capacitance of

$$C = \varepsilon \frac{S}{d} = \varepsilon_0 \varepsilon_r \frac{S}{d}$$



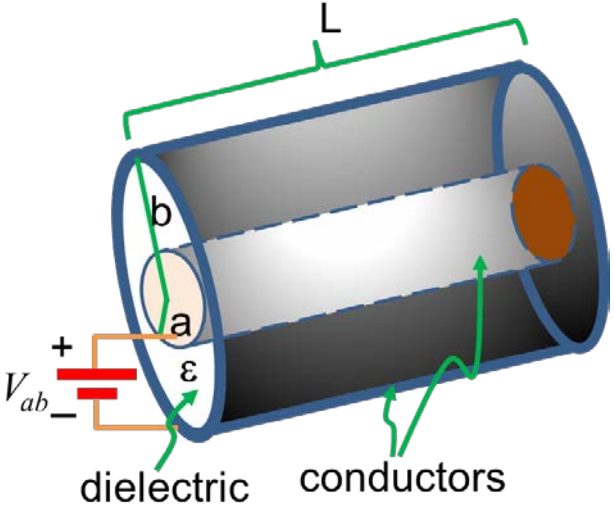
4. In general, a large area S, a high permittivity ε_r, and a small electrode gap d give a high capacitance.

單元回顧

5. Calculation of a cylindrical capacitor leads to a formula for the capacitance per unit length of a coaxial cable

$$C_l \equiv \frac{\rho_l}{V} = \frac{2\pi\varepsilon}{\ln(b/a)}$$

Coaxial-cable transmission line

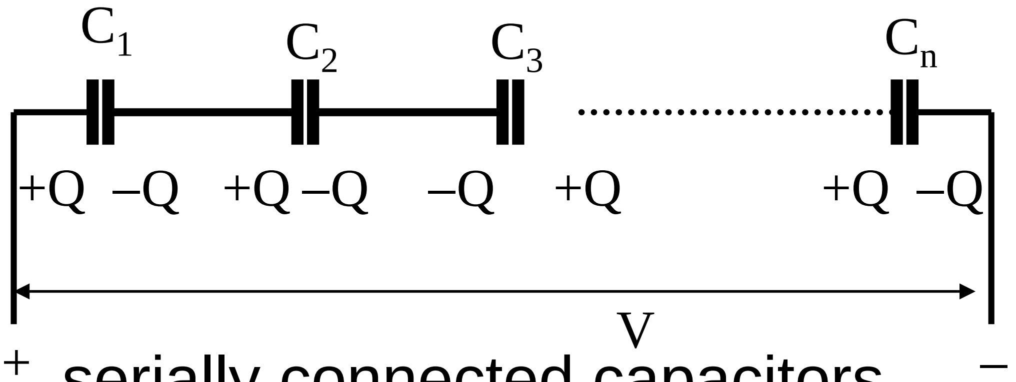


Cylindrical capacitor

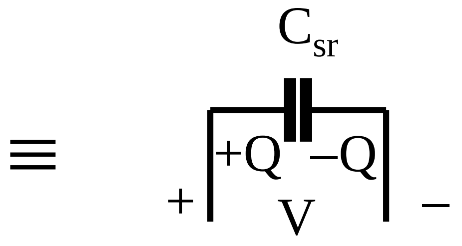
單元回顧

6. The equivalent capacitance of serially connected capacitors is the inverse of the inverse sum of individual capacitances.

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \cdots + \frac{1}{C_n}$$



serially connected capacitors

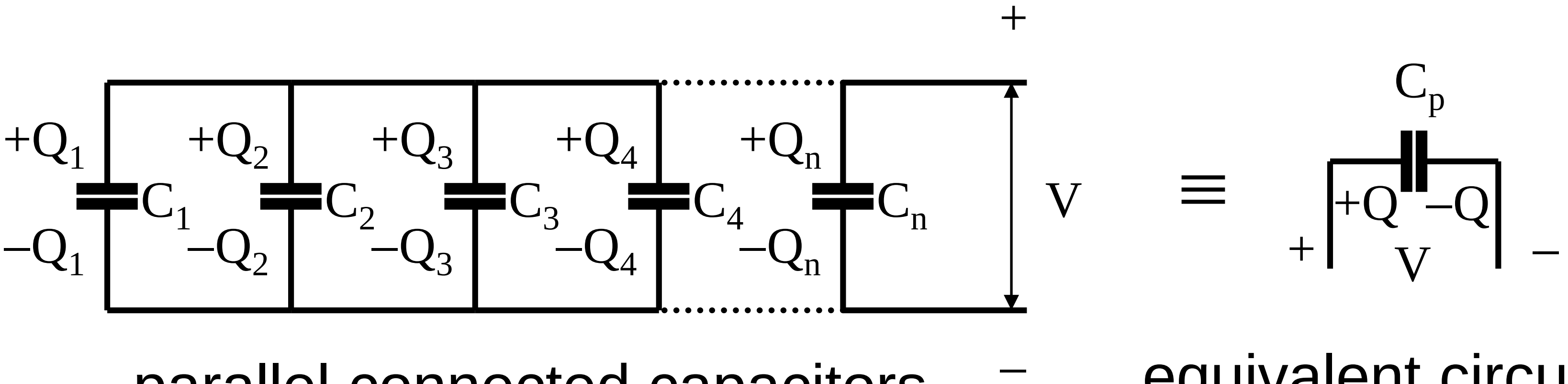


equivalence circuit

單元回顧

7. The equivalent capacitance of parallel connected capacitors is the sum of individual capacitances.

$$C_p = C_1 + C_2 + C_3 \ldots + C_n$$

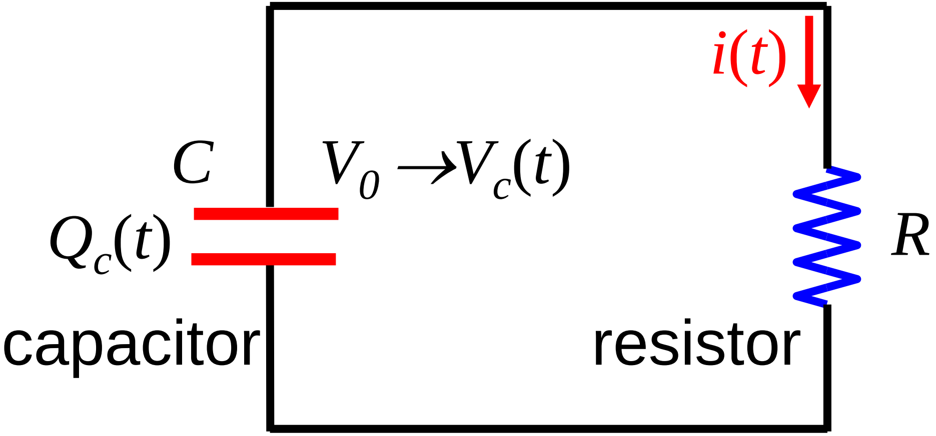


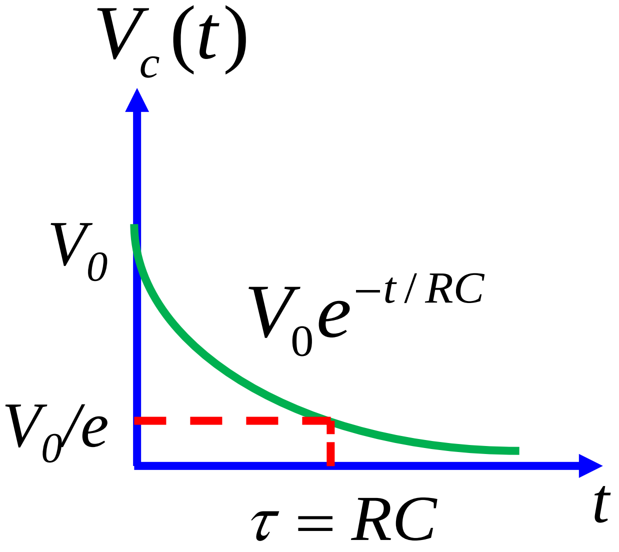
parallel connected capacitors

equivalent circuit

單元回顧

8. The characteristic charging/discharging time of an RC circuit is $\tau = RC$.





THANK YOU FOR YOUR ATTENTION

Review Questions

1. When you design a capacitor, what are the key parameters to increase its capacitance?

Ans: From the formula of a parallel-plate capacitor,

$$C = \frac{\varepsilon S}{d}$$

one could in general increase the capacitance of a capacitor by increasing the electrode areas and the permittivity of the dielectric between the electrodes, and decreasing the separation of the electrodes.

2. If you have a few capacitors in your hands and you want to connect them together to have a high capacitance for your circuit, would you choose serial or parallel connections for your capacitors?

Ans: To solve this problem, one could of course prove $C_p > C_{sr}$ from the following two formulas for serial and parallel capacitors

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots + \frac{1}{C_n}$$

$$C_p = C_1 + C_2 + C_3 \ldots + C_n.$$

However, from the circuit diagrams shown in Sec. 7.4, one can already see that the parallel capacitors store more charges from a increased area. Therefore, to increase the capacitance, parallel connection is the choice.

3. For a high-speed circuit containing R and C, if you would like to have a signal bandwidth > 1 GHz, what is the requirement on the RC time constant of the circuit?

Ans: Consider a sinusoidal signal of 1 GHz in the circuit. Since the charging and discharging time in the circuit has to be less than 1/1 GHz ~ 1 ns to support the 1 GHz signal, the RC time constant of the circuit has to be less than 1 ns.