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電磁學（一）
Electromagnetics (I)

16. 磁力與磁能
Magnetic Force and Energy

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In this lecture, we will learn about how a moving charge and an inductive system experience a magnetic force.

- **16.1 Magnetic Force on Charges 電荷所受的磁力**
- **16.2 Hall effect 霍爾效應**
- **16.3 Magnetic Energy 磁場能量**
- **16.4 Inductive Magnetic Force 電感系統磁力**
- **16.5 Review 單元回顧**

磁力與磁能
Magnetic Force and Energy

16.1 電荷所受的磁力
Magnetic Force on Charges

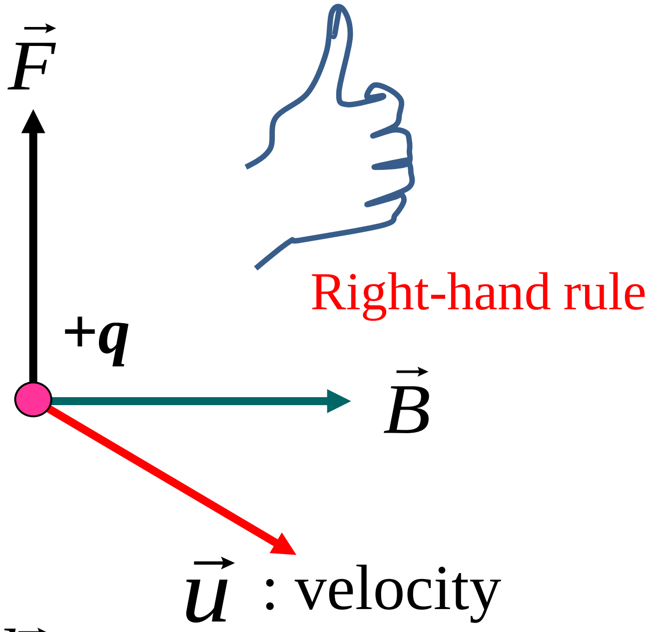
Magnetic Force on a Moving Charge

Recall the Lorentz force (Lecture 1)

$$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$$

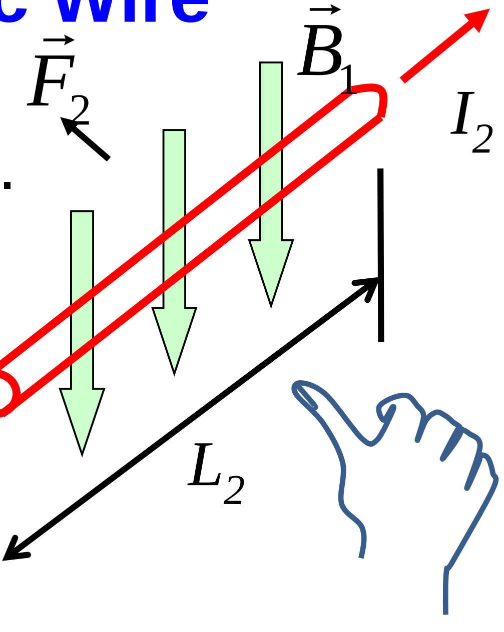
B: magnetic flux density (Tesla or Weber/m^2)

Φ: magnetic flux (Weber) $\Phi = \int_S \vec{B} \cdot d\vec{s}$



Magnetic Force on a DC Electric Wire

Although neutral, a wire carrying a current has moving charges q with an average speed u_2 in it.



The total force on the wire is

$$\vec{F}_2 = N_{total} q\vec{u}_2 \times \vec{B}_1 = \underline{n(a}L_2)\underline{q\vec{u}_2} \times \vec{B}_1$$

n: # of q per volume

The current in the wire is $nq\vec{u}_2 a = \rho\vec{u}a = \vec{J}_2 a = \vec{I}_2$

The force on the wire can be expressed as $\vec{F}_2 = L_2\vec{I}_2 \times \vec{B}_1$

Alternatively, the force per unit length on the wire is $\vec{F}_l = \dfrac{\vec{F}_2}{L_2} = \vec{I}_2 \times \vec{B}_1$

Usually current I is not a vector. We'd better write $\boxed{\vec{F}_2 = I_2\vec{L}_2 \times \vec{B}_1}$

Torque on a Current Loop

Assume a current loop with its pivot axis along y in a magnetic field $\vec{B} = B\hat{a}_x$

Recall *"torque"* - $\vec{T} = \vec{r} \times \vec{F}$

$F_{3,4}$ along y axis produce no torque

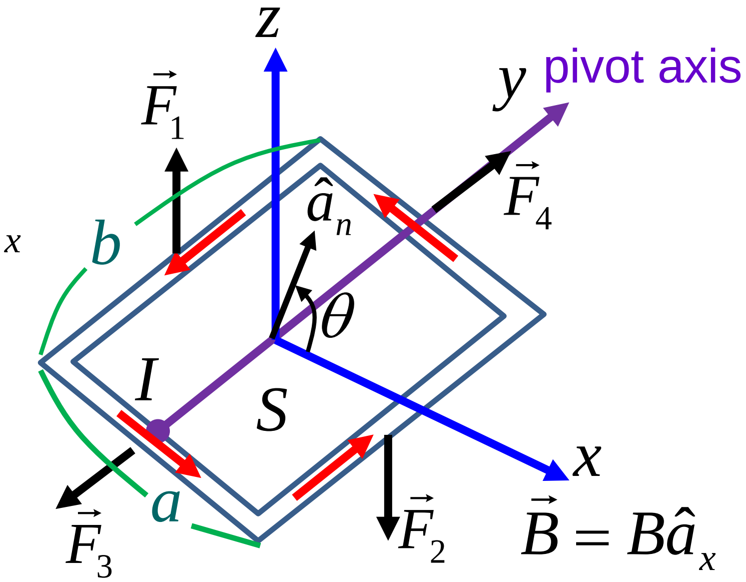
Use $\vec{F} = I\vec{L} \times \vec{B}$ to obtain

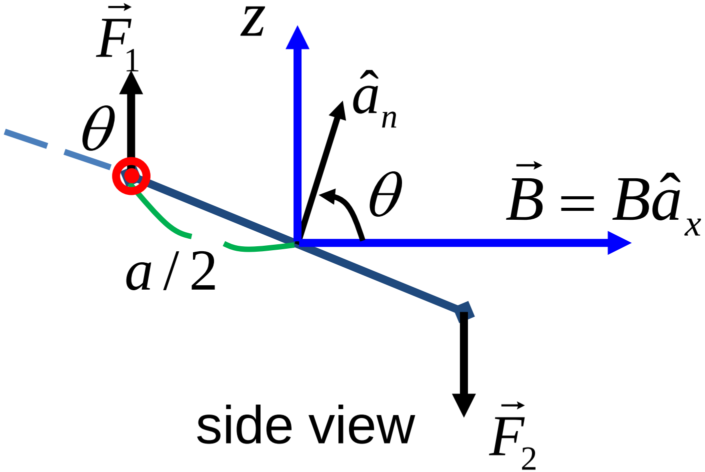
$\vec{F}_1 = IbB\hat{a}_z$ and $\vec{F}_2 = IbB(-\hat{a}_z)$

The total torque on the current loop is given by

$$\vec{T} = 2 \times \frac{a}{2} IbB \sin\theta \hat{a}_y = ISB \sin\theta \hat{a}_y,$$

where $a \times b = S =$ area of the loop.





Magnetic Moment

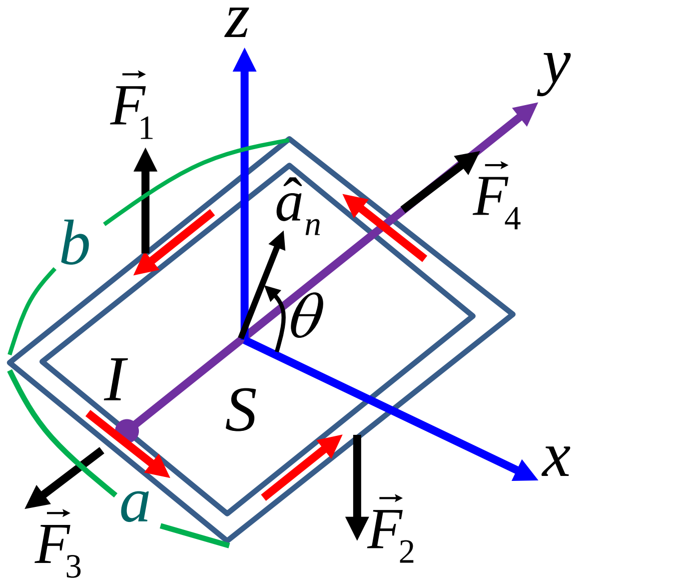
Define the magnetic moment as

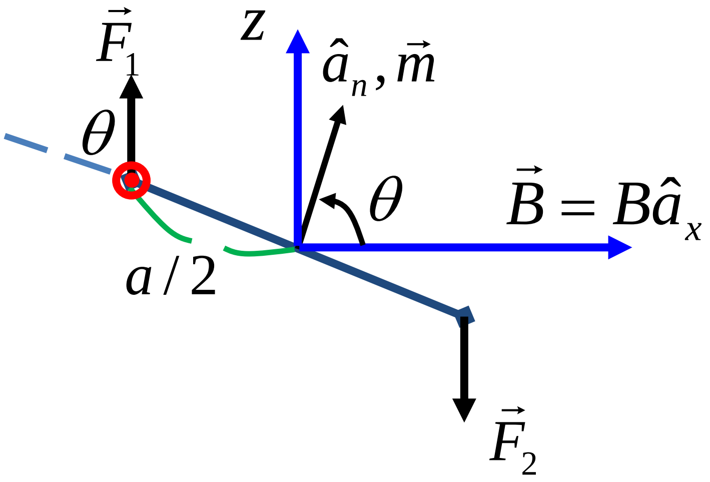
$$\boxed{\vec{m} = I\vec{S}}$$

The direction of S follows the right-hand rule with reference to the direction of I.

The magnetic torque of a current loop is in general expressed as

$$\vec{T} = ISB\sin\theta\hat{a}_y \Rightarrow \boxed{\vec{T} = \vec{m}\times\vec{B}}$$





16.1 電荷所受的磁力

Magnetic Force on Charges

- A moving charge q experiences a magnetic force according to $\boxed{\vec{F} = q\vec{u} \times \vec{B}}$
- A wire of length L carrying a flow of charges experiences a magnetic force according to

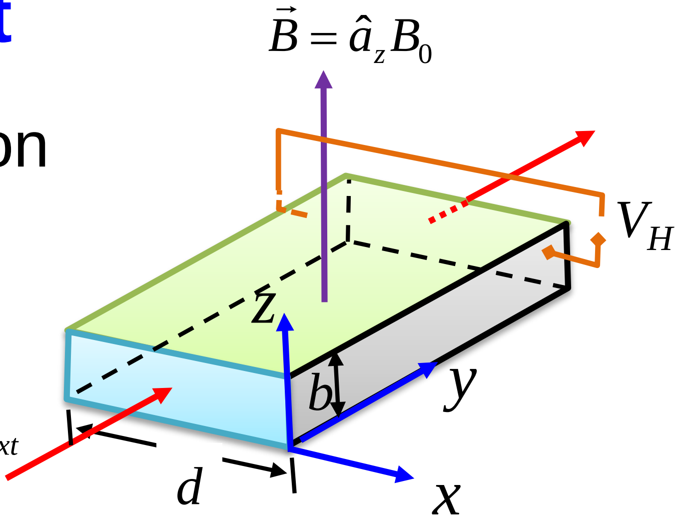
$$\boxed{\vec{F} = I\vec{L} \times \vec{B}}$$

- A current loop with a magnetic moment of m experiences a torque under a magnetic field according to $\boxed{\vec{T} = \vec{m} \times \vec{B}}$

磁力與磁能
Magnetic Force and Energy

16.2 霍爾效應
Hall Effect

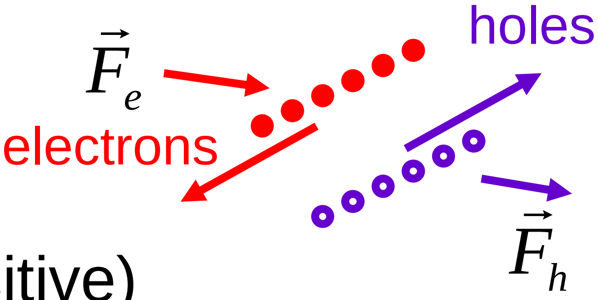
Hall Effect



A transverse voltage, V_H, is induced on a piece of semiconductor carrying a current in a magnetic field.

Assume a field $\vec{B} = \hat{a}_z B_0$ $\quad \vec{J} = \hat{a}_y J_0 \ @ \ V_{ext}$

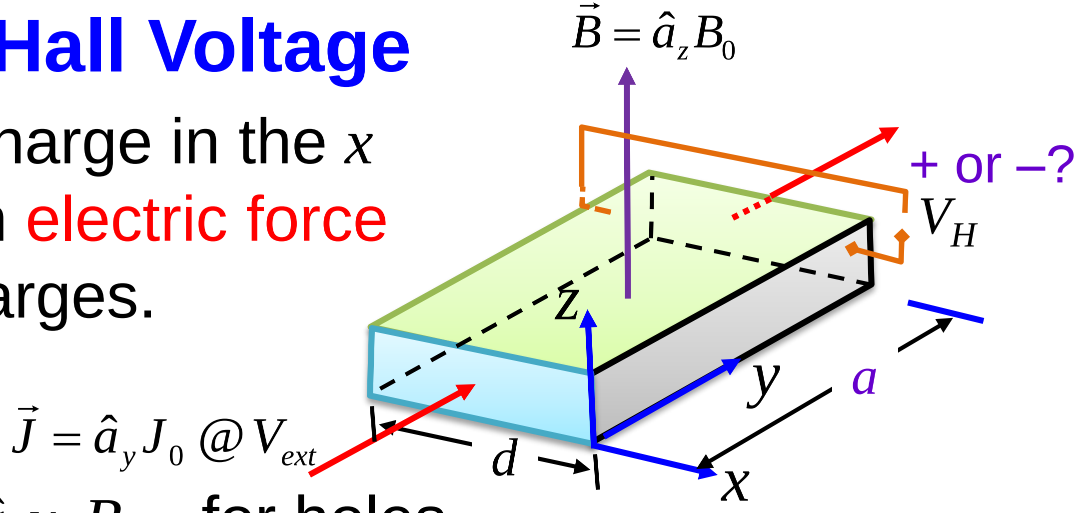
and a current density $\vec{J} = \hat{a}_y J_0$
under an applied voltage V_{ext} along y.



For holes, $\vec{u} = \hat{a}_y u_h$

For electrons, $\vec{u} = -\hat{a}_y u_e$ ($u_{e,h}$ are both positive)

Hall Voltage



The magnetic force on a charge in the x direction is balanced by an electric force due to accumulation of charges.

$$q\vec{E}_H + q\vec{u} \times \vec{B} = 0$$

$$\Rightarrow \vec{E}_{H,h} = -(\hat{a}_y u_h) \times \hat{a}_z B_0 = -\hat{a}_x u_h B_0 \quad \text{for holes}$$

$$\Rightarrow \vec{E}_{H,e} = -(-\hat{a}_y u_e) \times \hat{a}_z B_0 = \hat{a}_x u_e B_0 \quad \text{for electrons}$$

By measuring the polarity of the Hall voltage $V_H = E_H d = u B_0 d$, one can determine the type of the moving charges in the semiconductor material.

Material Characterization

The Hall effect can measure a few useful material parameters.

Charge density ρ: ρ can be determined from the *Hall coefficient*

$$\frac{E_H}{JB_0} = \frac{\cancel{uB_0}}{\cancel{JB_0}} = \frac{1}{\rho} \qquad (E_H = u_{h,e}B_0 \text{ and } J = \rho u \text{ are used})$$

Charge velocity: $u_{e,h}$ can be deduced from the Hall voltage

$$V_H = E_H d = u_{e,h} B_0 d \Rightarrow u_{e,h} = \frac{V_H}{B_0 d}$$

Charge mobility: Given an applied voltage V_{ext}, the driving electric field is known from $E_{ext} = V_{ext}/a$. One can obtain the charge mobility from

$$\vec{u}_e = -\mu_e \vec{E}_{ext} \quad \text{or} \quad \vec{u}_h = +\mu_h \vec{E}_{ext}$$

Conductivity: σ deduced from known ρ and $\mu \Rightarrow \sigma = \rho\mu_e$

16.2 霍爾效應

Hall Effect

- A current flowing through a semiconductor under a transverse magnetic field induces the Hall voltage across the other transverse direction in the semiconductor.
- The polarity of the Hall voltage can determine the type of the charge carriers in the semiconductor.
- The Hall effect is also useful for characterizing the other material parameters, such as μ, σ, ρ etc.

磁力與磁能
Magnetic Force and Energy

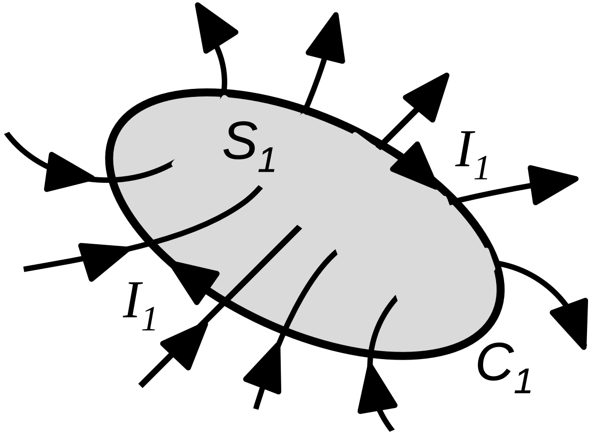
16.3 磁場能量
Magnetic Energy

Magnetic Energy Stored in Inductors

Without considering the sign of the induced voltage, for a *single current loop C_1*,

$$v_1 = \frac{d\phi_{11}}{dt} \quad \text{thus} \quad v_1 = L_{11}\frac{di_1}{dt}$$

$$(\phi = Li)$$

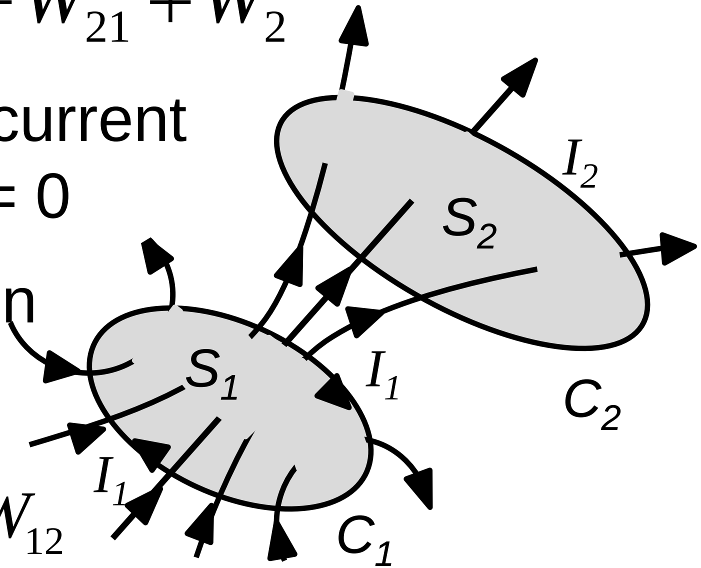


The energy is stored to the current loop when current increases from 0 to I_1, given by

$$W_1 = \int v_1 i_1 dt = \int_0^{I_1} L_{11} i_1 di_1 = \frac{1}{2}L_{11}I_1^2 = \frac{1}{2}\Phi_{11}I_1$$

For two current loops C_1 and C_2, the total stored energy is

$$W_{m,2} = W_1 + W_{12} + W_2 = W_1 + W_{21} + W_2$$



$W_1 = \frac{1}{2} L_{11} I_1 I_1$ is the energy for pumping current I_1 *into* C_1 while keeping $I_2 = 0$

$W_2 = \frac{1}{2} L_{22} I_2 I_2$ is the energy for pumping in current I_2 into C_2

current

$$W_{21} = \int v_{21} I_1 dt = \int_0^{I_2} L_{21} \frac{di_2}{dt} I_1 dt = L_{21} I_1 I_2 = W_{12}$$

loop

is the energy necessary for maintaining I_1 in loop 1 when current i_2 is increased from 0 to I_2 in loop 2.

Therefore, $W_{m,2} = \frac{1}{2} \sum_{k=1}^{2} \sum_{j=1}^{2} L_{jk} I_j I_k$

For N inductor loops, the stored energy is

$$W_{m,N} = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} L_{jk} I_j I_k = \frac{1}{2}\sum_{k=1}^{N} \Phi_k I_k$$

where $\Phi_k = \sum_{j=1}^{N} L_{jk} I_j$ is the total magnetic flux going through loop *k*.

Magnetic Energy in Space

In a distributed inductive system, the calculation of the stored energy becomes an integration

$$W_m = \frac{1}{2}\sum_{j=1}^{N} \Phi_j I_j = \frac{1}{2}\sum_{j=1}^{N} \vec{J}_j \cdot \Delta\vec{s}_j \oint_{C_j} \vec{A}\cdot d\vec{l}_j \qquad W_m \to \frac{1}{2}\int_V \vec{A}\cdot\vec{J}\,dv$$

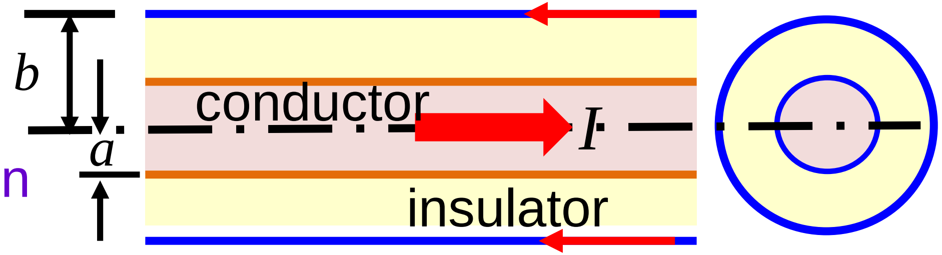
By using $\nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H}) = \vec{H}\cdot\vec{B} - \vec{A}\cdot\vec{J},$

$$\int_V \nabla\cdot(\vec{A}\times\vec{H})dv = \oint \cancelto{1/R}{\vec{A}}\times\cancelto{1/R^2}{\vec{H}}\cdot d\vec{s} \xrightarrow[R\to\infty]{} 0 \quad \Rightarrow \quad W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}dv = \frac{\mu}{2}\int_V H^2 dv = \frac{1}{2\mu}\int_V B^2 dv$$

Eg. Find the internal inductance of the following coaxial cable.

L_i: arises from the magnetic linkage in the current-flow region.

L_{ext}: arises from the magnetic linkage in between the current-flow regions.



The magnetic field intensity in the core is given by

$$H_i 2\pi r = I\frac{r^2}{a^2} \Rightarrow \quad H_i = I\frac{r}{2\pi a^2}$$ along the φ direction.

The stored energy associated with the internal inductance is given by

$$W_{m,i} = \frac{\mu}{2}\int_V H_i^{\,2} dv = \frac{\mu_0}{2}\int_V H_i^{\,2} l 2\pi r dr = \frac{\mu_0 l I^2}{16\pi}$$

From the expression, $W_{m,i} = \dfrac{L_i I^2}{2} = \dfrac{\mu_0 l I^2}{16\pi}$

one can calculate the *internal inductance per unit length* $\dfrac{L_i}{l} = \dfrac{\mu_0}{8\pi}$

16.3 磁場能量

Magnetic Energy

- The magnetic energy stored in an inductor carrying a current I is

$$W = \frac{1}{2}LI^2 = \frac{\Lambda^2}{2L}$$

- The magnetic energy stored in magnetic fields is

$$W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}dv = \frac{\mu}{2}\int_V H^2 dv = \frac{1}{2\mu}\int_V B^2 dv$$

where the magnetic energy density (energy per unit volume) is $w_m = \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{1}{2}\mu H^2 = \frac{1}{2\mu}B^2$

磁力與磁能
Magnetic Force and Energy

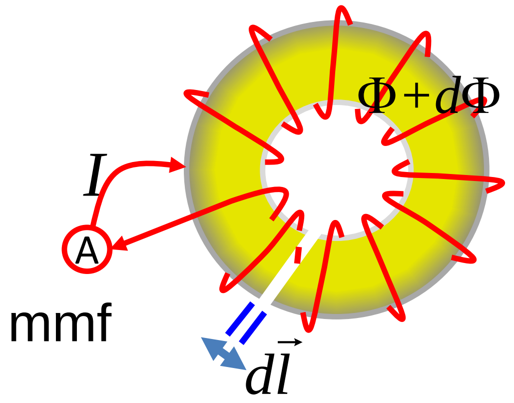
16.4 電感系統磁力
Inductive Magnetic Force

Force (F) and Work (W)

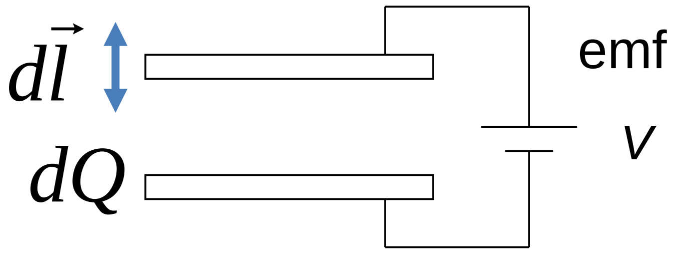
Differential work $dW = \vec{F} \cdot d\vec{l}$ ⇒ $dW = \nabla W \cdot d\vec{l}$ ⇒ $\vec{F} = \nabla W$

(recall $dV = (\nabla V) \cdot d\vec{l}$ from **Lecture 4**)

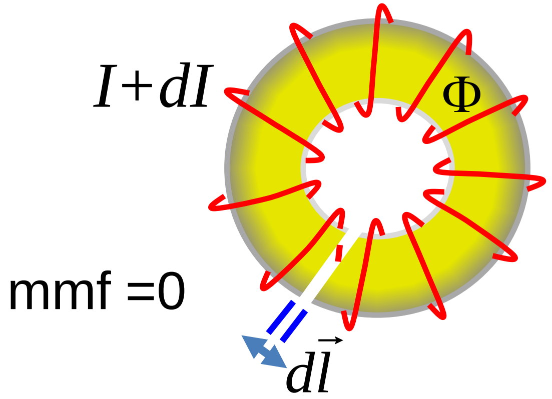
1. **Fixed-current system**: a system connected to a current source (forcing a displacement in the system causes a change to the magnetic flux $d\phi$)



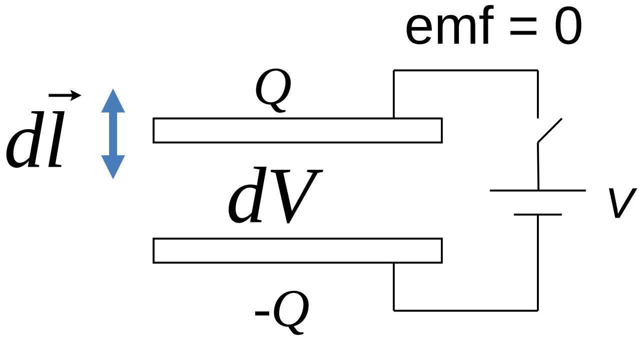
Analogy: **Fixed-voltage system**: a system connected to batteries (forcing a displacement causes a flow of charges)



2. **Fixed-flux system**: an isolated system (forcing a displacement in the system causes a change in the current dI)



Analogy: **Fixed-charge system**: an isolated system (forcing a displacement causes a change in voltages)

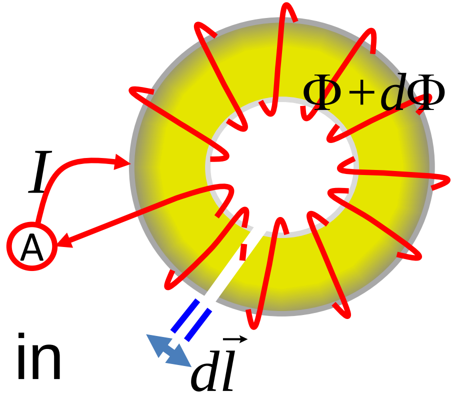


Thought Experiment I: System with fixed currents (maintained by current sources)

Energy conservation requires

$$dW_s = dW + dW_m\Big|_{I=const.}$$

mmf



energy supplied by the sources

mechanical work done to the system

change in internal energy

$$dW_s = \sum_k I_k \left(V_k = \frac{d\Phi_k}{dt}\right) dt \qquad dW = \vec{F}_{I=const} \cdot d\vec{l} \qquad dW_m\Big|_{I=const} = \frac{1}{2}\sum_k I_k d\Phi_k$$

$$= \sum_k I_k d\Phi_k$$

x 1/2

$$\left.\begin{aligned} &\vec{F}_{I=const} \cdot d\vec{l} = dW_m\Big|_{I=const} \\ &\text{But } dW_m\Big|_{I=const} = \left(\nabla W_m\Big|_{I=const}\right) \cdot d\vec{l} \end{aligned}\right\} \Rightarrow \boxed{\vec{F}_{I=const} = \nabla W_m\Big|_{I=const}}$$

Thought Experiment II: System with fixed fluxes (isolated system)

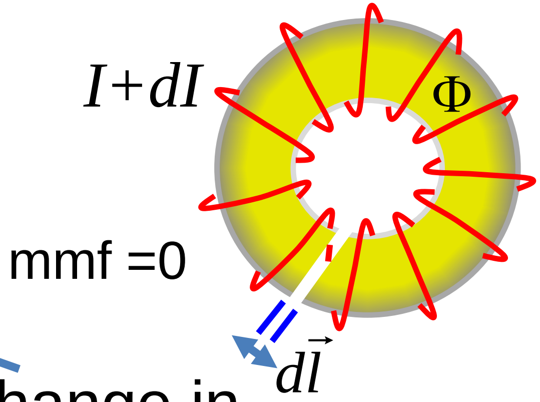
Energy conservation requires

$$\overset{0}{\cancel{dW_s}} = dW + dW_m\big|_{\Phi=const.}$$

No sources

mechanical work done to the system

change in internal energy



$$dW = \vec{F}_{\Phi=const} \cdot d\vec{l}$$

$$dW_m\big|_{\Phi=const} = \frac{1}{2}\sum_k \Phi_k dI_k$$

$$\left.\begin{aligned} -dW_m\big|_{\Phi=const} &= \vec{F}_{\Phi=const} \cdot d\vec{l} \\ \text{But } dW_m &= (\nabla W_m) \cdot d\vec{l} \end{aligned}\right\} \Rightarrow \vec{F}_{\Phi=const} = -\nabla W_m\big|_{\Phi=const}$$

For an inductive system, $L = \frac{\Lambda}{I}, W_m = \frac{1}{2} L I^2 = \frac{\Lambda^2}{2L}$

In practice, with fixed currents

$$\vec{F}_I = \nabla (W_m \big|_I = \frac{1}{2} L I^2) = \frac{I^2}{2} \nabla L$$

With fixed fluxes,

$$\vec{F}_\Phi = -\nabla (W_m \big|_\Phi) = -\nabla (\frac{\Lambda^2}{2L}) = \frac{\Lambda^2}{2L^2} \nabla L = \frac{I^2}{2} \nabla L = \nabla (W_m \big|_I) = \vec{F}_I$$

Magnetostatic Torque

$$(\vec{T}_I)_z = \frac{\partial W_m \big|_I}{\partial \phi}$$ under constant currents

$$(\vec{T}_\Phi)_z = -\frac{\partial W_m \big|_\Phi}{\partial \phi}$$ under constant fluxes

16.4 電感系統磁力

Inductive Magnetic Force

- The relationship between force and work is given by

$$dW = \vec{F} \cdot d\vec{l} \quad \text{or} \quad \vec{F} = \nabla W$$

- In a fixed-current system, the magnetostatic force is

$$\vec{F}_{V=const} = \nabla W_m \Big|_{I=const}$$

- In a fixed-flux system, the magnetostatic force is

$$\vec{F}_{\Phi=const} = -\nabla W_m \Big|_{\Phi=const}$$

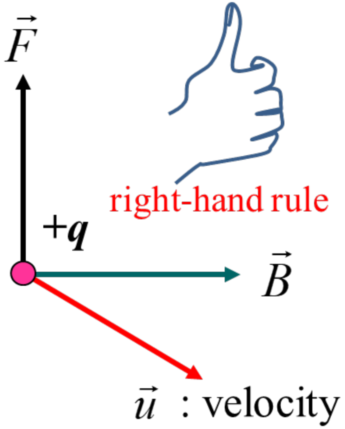
磁力與磁能
Magnetic Force and Energy

16.5 單元回顧 Review

單元回顧

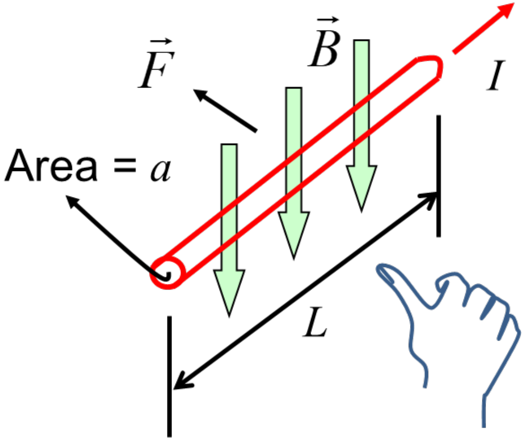
1. A charge q moving with a speed u under a magnetic field B experiences the force

$$\vec{F} = q\vec{u} \times \vec{B}$$



2. A wire of length L carrying a flow of charges (a current I) experiences a magnetic force according to

$$\vec{F} = I\vec{L} \times \vec{B}$$



單元回顧

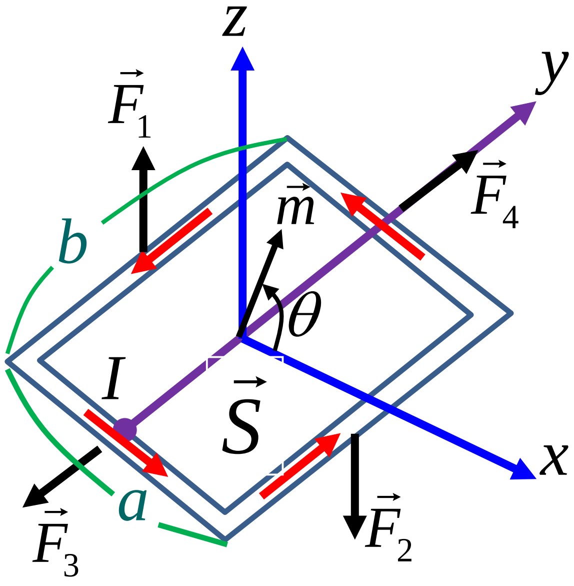
3. A current loop with a magnetic moment of m experiences a torque T under a magnetic field B according to

$$\vec{T} = \vec{m} \times \vec{B},$$

where the magnetic moment is defined as

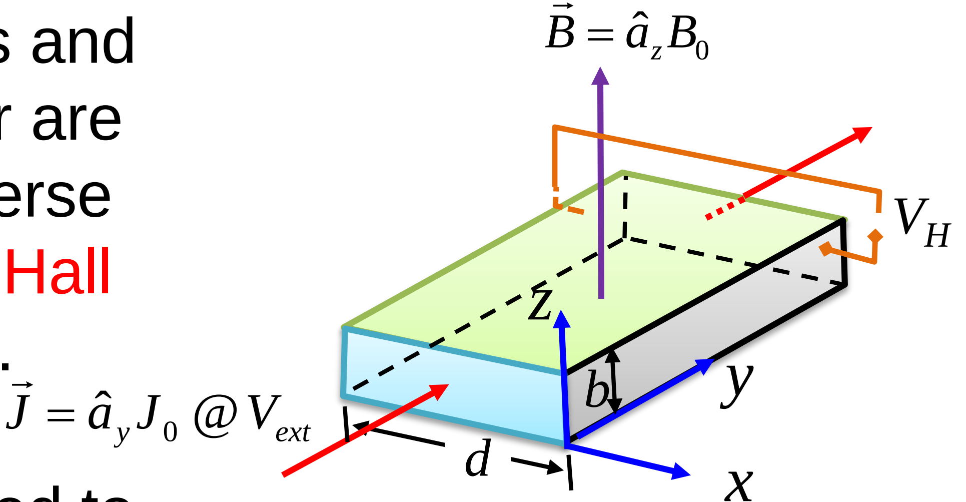
$$\vec{m} = I\vec{S}$$

with S being the area of the current loop.

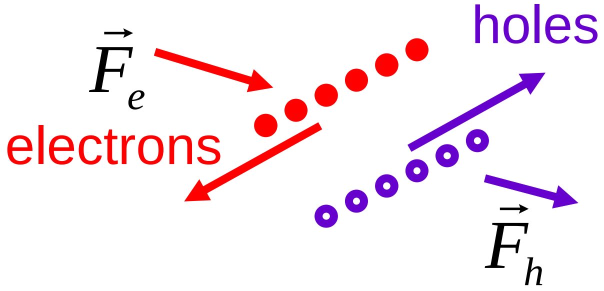


單元回顧

3. **Hall Effect**: Moving holes and electrons in a semiconductor are pushed sideway by a transverse magnetic field, generating a Hall voltage of different polarities.



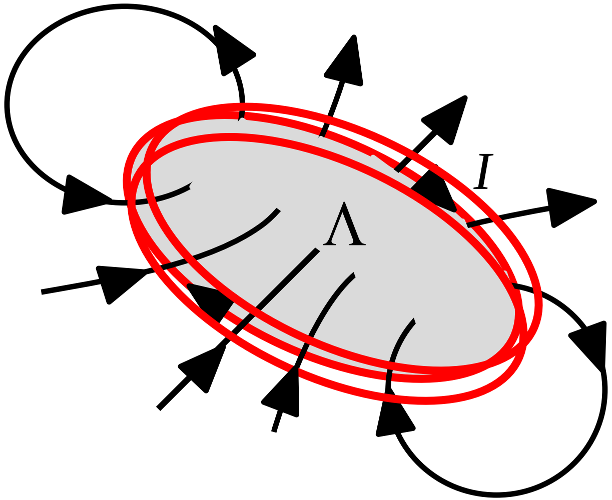
4. The Hall effect can be used to characterize the material parameters of a semiconductor, including the type, mobility, conductivity, charge density of the charge carriers.



單元回顧

5. The magnetic energy stored in an inductor with a inductance of L carrying a current I is

$$W = \frac{1}{2}LI^2 = \frac{\Lambda^2}{2L}$$



6. The magnetic energy density (energy per unit volume) is given by

$$w_m = \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{1}{2}\mu H^2 = \frac{1}{2\mu}B^2$$

單元回顧

7. The magnetic energy associated with magnetic fields is given by the volume integration of the energy density

$$W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}dv = \frac{\mu}{2}\int_V H^2 dv = \frac{1}{2\mu}\int_V B^2 dv$$

8. For an inductor, by using $W_m = \frac{1}{2}LI^2$

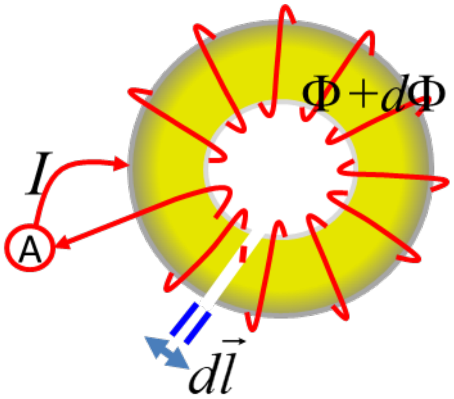
one can calculate the inductance from $L = \frac{2w_m}{I^2}$

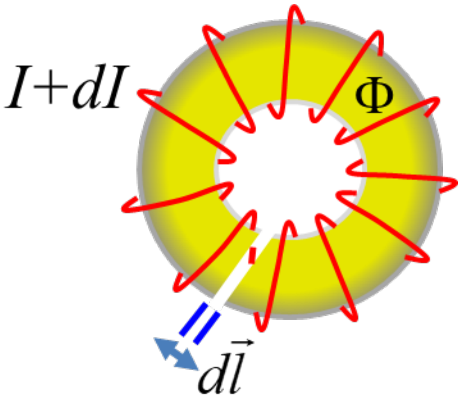
with $W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}dv = \frac{\mu}{2}\int_V H^2 dv = \frac{1}{2\mu}\int_V B^2 dv$

單元回顧

9. In an inductive system, the magnetic force can be calculated from thought experiments with either fixed currents or fixed fluxes, given by

$$\vec{F}_\Phi = -\nabla(W_m|_\Phi) = -\nabla(\frac{\Lambda^2}{2L}) = \frac{\Lambda^2}{2L^2}\nabla L = \frac{I^2}{2}\nabla L = \nabla(W_m|_I) = \vec{F}_I$$





THANK YOU FOR YOUR ATTENTION