Chapter 8 Plane Electromagnetic Waves

From here on, we will only deal with harmonic fields and drop the "hat" in the phasor expression of a field ($\hat{E}, \hat{B}, \hat{D}, \hat{H}, \hat{A}, \hat{V}$ etc.) for convenience. Recall the homogeneous vector Helmholtz's equation for the *E* and *H* fields

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0\,,\ \ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \qquad (8\text{-}1)$$

where *k* is the wave number defined by

$$k = \omega\sqrt{\varepsilon\mu} = 2\pi/\lambda \qquad (8\text{-}2)$$

with ω being the angular frequency and λ being the wavelength of the harmonic field. Create a trial solution of the form for the Helmholtz equation

$$\vec{E}(\vec{R}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{R}} = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z}\,, \qquad (8\text{-}3)$$

where the *wave vector* is defined to be

$$\vec{k} \equiv k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z\,, \qquad (8\text{-}4)$$

and the position vector in the Cartesian coordinate system is understood as

$$\vec{R} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\,. \qquad (8\text{-}5)$$

Substituting (8-3) back into the Helmholtz's equation, the relationship holds

$$k^2 = \omega^2\mu\varepsilon = k_x^2 + k_y^2 + k_z^2 \qquad (8\text{-}6)$$

The *wave vector* shown in (8-4) but satisfying (8-6) can be written as

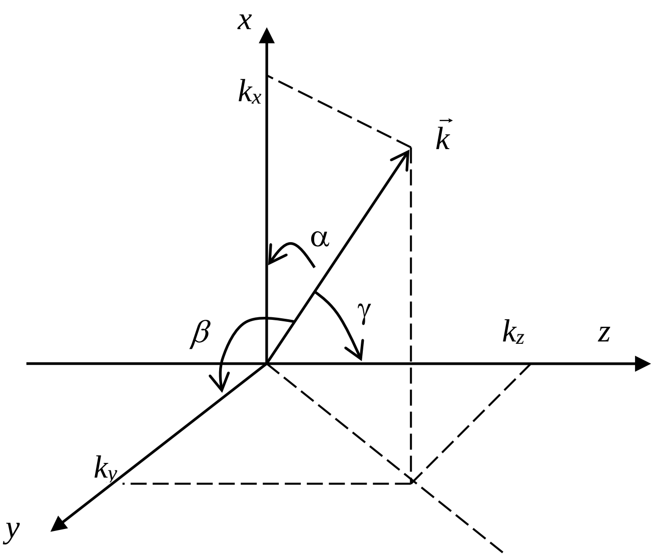
$$\vec{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z = k\hat{a}_n, \tag{8-7}$$

where $\hat{a}_n = \vec{k}/k$ is the unit vector along the wave vector direction .

The field expression now has the concise form

$$\vec{E}(\vec{R}) = \vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{R}} \tag{8-8}$$

There is a very specific meaning associated with this expression. Refer to the following figure.



The x, y, z components of the wave vector can be written as $k_x = |\vec{k}|\cos\alpha$, $k_y = |\vec{k}|\cos\beta$, $k_z = |\vec{k}|\cos\gamma$ where the angles α, β, γ are the angles between the $\vec{k}$ vector and the x, y, z axes, respectively.

The wave vector can be explicitly written as

$$\vec{k} = k\hat{a}_n = k(\cos\alpha\hat{a}_x + \cos\beta\hat{a}_y + \cos\gamma\hat{a}_z)$$

where the unit vector along $\vec{k}$ is given by

$\hat{a}_n = \cos\alpha\hat{a}_x + \cos\beta\hat{a}_y + \cos\gamma\hat{a}_z$. The cosine terms, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are called *direction cosines*, because they define the direction of the wave vector relative to the *x*, *y*, *z* axes.

When we use the expression

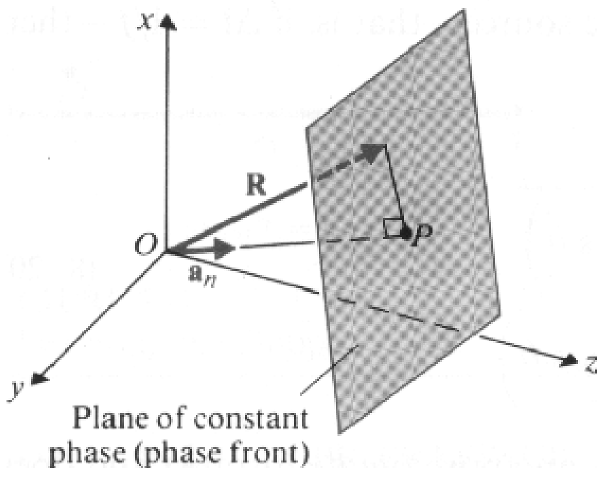
$$\vec{E}(\vec{R}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{R}} = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z},$$

we imply the instantaneous electric field

$$\vec{E}(\vec{R},t) = \mathrm{Re}(\vec{E}_0 e^{-j\vec{k}\cdot\vec{R}} e^{j\omega t}).$$

The phase of the field is $\phi = \omega t - \vec{k}\cdot\vec{R} + \phi_0$, where ϕ_0 is a constant phase associated with E_0. The constant phase surface at a give instance defines a *wavefront* of a wave. Therefore $\hat{a}_n \cdot \vec{R}$ = constant defines the wavefront of the wave expressed by (8-3).

The expression $\hat{a}_n \cdot \vec{R} = \text{constant}$ or $\cos\alpha\cdot x + \cos\beta\cdot y + \cos\gamma\cdot z = \text{constant}$ is a plane in space with its surface normal along $\hat{a}_n$. Therefore Eq. (8-3) manifests itself as a plane wave.



In a charge-free region and in a homogeneous medium

$$\nabla\cdot\vec{E}=0 \;\Rightarrow\; \nabla\cdot(\vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{R}})=0 \tag{8-9}$$

Apply the vector identity $\nabla\cdot f\vec{G}=\nabla f\cdot\vec{G}+f\nabla\cdot\vec{G}$ to (8-9) and obtain

$$\begin{aligned}&\vec{E}_0\cdot\nabla e^{-jk\hat{a}_n\cdot\vec{R}}\\ &=\vec{E}_0\cdot[-j(k_x\hat{a}_x+k_y\hat{a}_y+k_z\hat{a}_z)e^{-jk\hat{a}_n\cdot\vec{R}}]\\ &=-jke^{-jk\hat{a}_n\cdot\vec{R}}(\vec{E}_0\cdot\hat{a}_n)=0\end{aligned} \tag{8-10}$$

The scalar product $\vec{E}_0\cdot\hat{a}_n=0$ indicates that the electric field intensity $\vec{E}_0$ is perpendicular to $\hat{a}_n$ or $\vec{E}_0$ is transverse to the propagation direction of the wavefront.

Also, for a harmonic field,

$$\nabla\times\vec{E}=-j\omega\mu\vec{H} \tag{8-11}$$

Substitute Eq. (8-8) into (8-11) and use the vector identity

$$\nabla\times(f\vec{G})=\nabla f\times\vec{G}+f\nabla\times\vec{G}$$

Since $\vec{E}_0$ is just a constant vector, $\nabla\times\vec{E}_0=0$. The left hand side of (8-11) becomes

$$\begin{aligned}&\nabla\times\vec{E}=\nabla\times(\vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{R}})=\nabla(e^{-jk\hat{a}_n\cdot\vec{R}})\times\vec{E}_0\\ &=-jke^{-jk\hat{a}_n\cdot\vec{R}}\hat{a}_n\times\vec{E}_0\end{aligned} \tag{8-12}$$

Use (8-12) to equate LHS = RHS for (8-11) and obtain the expression

$$\vec{H}(\vec{R}) = \frac{1}{\eta}\hat{a}_n \times \vec{E}(R), \tag{8-13}$$

where the ratio of the electric field intensity to the magnetic field intensity

$$\eta \equiv \sqrt{\mu/\varepsilon} = 377\sqrt{\mu_r/\varepsilon_r}\ \Omega \tag{8-14}$$

is called the *intrinsic wave impedance*, where $\eta_0 \equiv \sqrt{\mu_0/\varepsilon_0} = 377\Omega$ is the intrinsic impedance of an electromagnetic (EM) wave in vacuum. Very often, an EM wave propagates in a nonmagnetic material ($\mu_r = 1$) and the wave impedance $\eta = \eta_0\sqrt{\varepsilon_r}\ \Omega = \eta_0 / n$, where *n* is called the *refractive index.*

Eq. (8-13) further defines the relative directions among $\hat{a}_n, \vec{E}, \vec{H}$ or specifically $\vec{H}$ is perpendicular to both $\hat{a}_n, \vec{E}$.

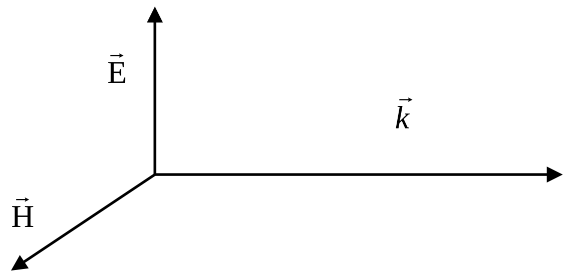
For a harmonic field in a current free region, the Ampere's law gives $\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$, which also gives $\vec{E}(\vec{R}) = -\eta\hat{a}_n \times \vec{H}(\vec{R})$ (8-15)

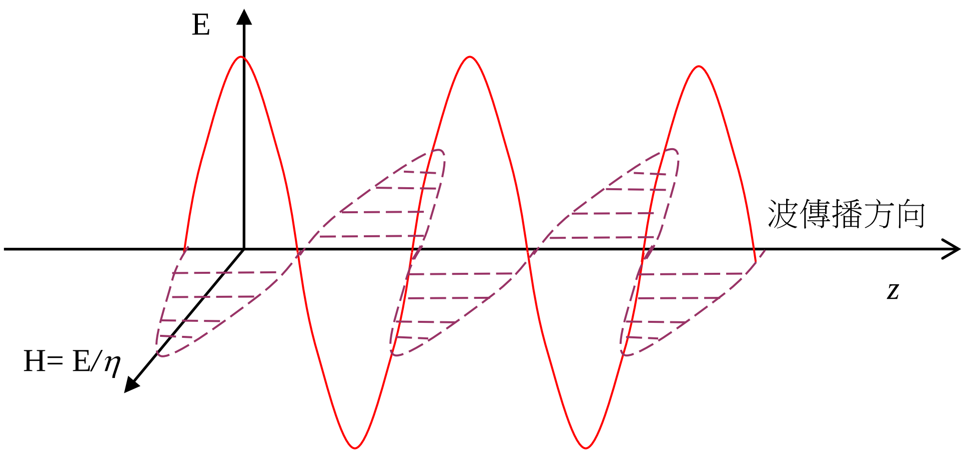
From the three vector relations obtained above,

$$\vec{E}_0 \cdot \hat{a}_n = 0,\ \vec{H}(\vec{R}) = \hat{a}_n \times \vec{E}(R)/\eta,\ \vec{E}(\vec{R}) = -\eta\hat{a}_n \times \vec{H}(\vec{R}),$$

one can conclude $\vec{H} \perp \hat{a}_n$, $\vec{E} \perp \hat{a}_n$, $\vec{H} \perp \vec{E}$. Note that both the electric and magnetic field components are transverse to the wavefront propagation direction. The relative directions among $\vec{E}$, $\vec{H}$, and $\hat{a}_n$ follow a so-called right-hand rule. If one uses one's right hand

and lets the four fingers point toward the $\vec{E}$ direction and curl towards the $\vec{H}$ direction, the thumb will point to the wavefront propagation direction $\hat{a}_n$. This is a general characteristic of a *transverse electromagnetic wave* (TEM wave).





A Plane Wave in Vacuum

For a TEM wave, one can arbitrarily define the direction of its electric field intensity along one of the three directions in an orthogonal coordinate system. Assume that a TEM wave in vacuum is *polarized* in the *x* direction, given by

$$\vec{E} = E_x \hat{a}_x \tag{8-16}$$

The electric field satisfies Helmholtz's equation

$$\nabla^2 \vec{E} + k_0^2 \vec{E} = 0,$$

or in the Cartesian coordinate system

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) E_x = 0 \tag{8-17}$$

where $k_0 = \omega\sqrt{\mu_0 \varepsilon_0}$ is the wave number in vacuum with ω being the angular frequency of the plane wave. Note that the relative permittivity ε_r is 1 in vacuum. Assume the electric field does not have any variation in the *x* and *y* direction or $\frac{\partial^2 E_x}{\partial x^2} = 0$ and $\frac{\partial^2 E_x}{\partial y^2} = 0$. Helmholtz's equation is reduced to

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0 \tag{8-18}$$

with the solution

$$E_x(z) = E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z} \tag{8-19}$$

The first term of (8-19) gives the instantaneous field expression,

$$E_x^+(z,t) = \mathrm{Re}(E_0^+ e^{-jk_0 z} \cdot e^{j\omega t}) = \left|E_0^+\right| \cos(\omega t - k_0 z + \varphi_+) \tag{8-20}$$

where φ_+ is the phase of E_0^+. We look into the propagation of the wavefront by setting the phase of (8-20) to be a constant

$$\omega t - k_0 z + \varphi_+ = \text{constant} \tag{8-21}$$

Take time derivative on the constant-phase expression (8-21) to obtain the propagation velocity of the wavefront or the *phase velocity*, given by

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c \tag{8-22}$$

where $c = 3 \times 10^8$ m/s is the speed of an electromagnetic wave in vacuum.

Therefore the first term of (8-19) indicates a plane wave propagating along the +z direction with a phase velocity of *c*. Likewise, the second term of (8-19) gives the instantaneous field expression,

$$E_x^-(z,t) = \mathrm{Re}(E_0^- e^{jk_0 z} \cdot e^{j\omega t}) = \left|E_0^-\right| \cos(\omega t + k_0 z + \varphi_-) \tag{8-23}$$

Where φ_- is the phase of E_0^-. Take time derivative on $\omega t + k_0 z + \varphi_- = \text{constant}$ and obtain the *phase velocity* of the wave (8-23), given by

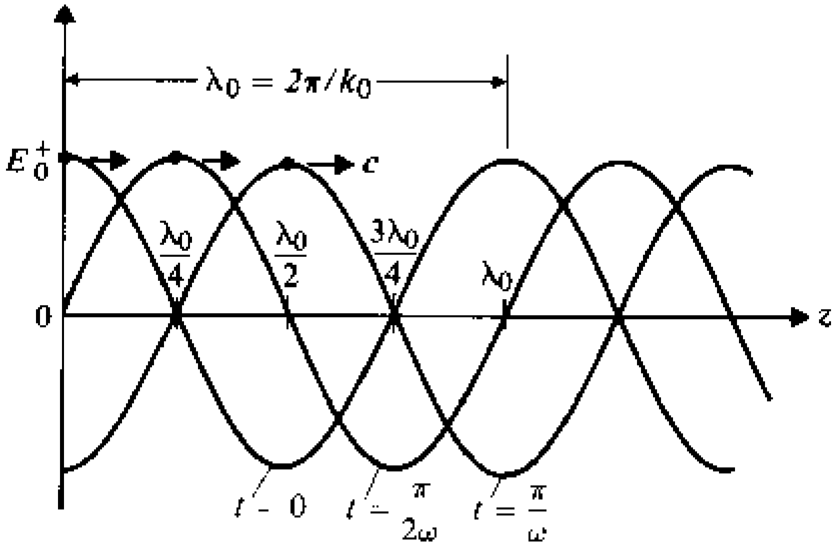
$$u_p = \frac{dz}{dt} = \frac{-\omega}{k_0} = -c \tag{8-24}$$

Therefore the second term of (8-19) indicates a plane wave propagating along the –z direction with a phase velocity of *c*.

The wave number can be expanded as

$$k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0}$$

where *f* is the frequency of the plane waves and $\lambda_0 = c/f$ is the wavelength of an EM wave in vacuum.



Since the relationship between the electric and magnetic field intensities is given by $\vec{H}(\vec{R}) = \hat{a}_n \times \vec{E}(R)/\eta$, the magnetic field intensity of a plane wave can be explicitly solved from a known electric field. For example, the phasor expression of the magnetic field intensity of the +z propagating wave is

$$\vec{H}^+ = \frac{1}{\eta_0} E_0^+ e^{-jkz} \hat{a}_z \times \hat{a}_x = \frac{1}{\eta_0} E_0^+ e^{-jkz} \hat{a}_y \tag{8-25}$$

and the corresponding instantaneous expression is

$$\vec{H}^+(z,t) = \frac{\left|E_0^+\right|}{\eta_0} \cos(\omega t - kz + \varphi_+) \cdot \hat{a}_y \tag{8-26}$$

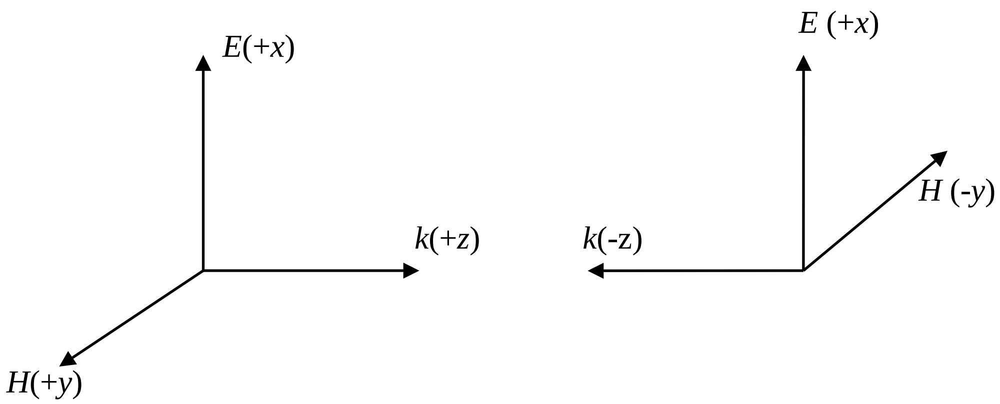
Similarly, the phasor expression of the magnetic field intensity of the –z propagating wave is

$$\vec{H}^- = \frac{-1}{\eta_0} E_0^- e^{jkz} \hat{a}_y , \tag{8-27}$$

and the corresponding instantaneous expression is

$$\vec{H}^-(z,t) = -\frac{|E_0^-|}{\eta_0}\cos(\omega t + kz + \varphi_-)\cdot \hat{a}_y \qquad (8\text{-}28)$$

Note that both the +*z* and –*z* propagating waves follow the right-hand rule of a transverse electromagnetic wave.



Polarization of an Electromagnetic Wave

Linear Polarization

The polarization direction of an electromagnetic wave is referred to the direction of the electric field. For example, a *z*-propagating TEM wave with its electric field along the *x* direction, governed by

$$\vec{E}(z,t) = E_{10}\cos(\omega t - kz + \varphi)\hat{a}_x\,,$$

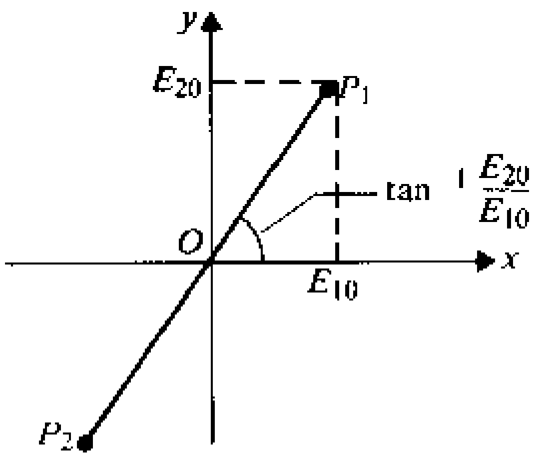
is called a *linearly polarized wave* in the *x* direction. In general, the electric field of a linearly polarized electromagnetic wave propagating in *z* may have both *x* and *y* components in the *xy* plane, given by

$$\vec{E}(z,t) = E_{10}\cos(\omega t - kz + \varphi_0)\hat{a}_x + E_{20}\cos(\omega t - kz + \varphi_0)\hat{a}_y \quad .$$

For this case, the two field components are in the same phase and thus the angle of the net field vector

$$\alpha = \tan^{-1}(E_{20} / E_{10})$$

with respect to the *x* axis is not a function of time (*t*) or location (*z*). In other words, the net electric-field or the polarization direction does not change during wave propagation, although the amplitude of the field indeed varies with time.



Elliptical Polarization

In a more general situation, the *x*, *y* components of the electric field can have a phase difference of φ. For instance, the electric field intensity is expressed by

$$\vec{E}(z,t) = E_x\hat{a}_x + E_y\hat{a}_y$$
$$= E_{x0}\cos(\omega t - kz + \phi_0)\hat{a}_x + E_{y0}\cos(\omega t - kz + \phi_0 + \varphi)\hat{a}_y \cdot$$

With some algebra (do it as an exercise), it can be shown that the two field components satisfy the conic equation

$$(\frac{E_x}{E_{x0}})^2 - 2\frac{E_y}{E_{y0}}\frac{E_x}{E_{x0}}\cos\varphi + (\frac{E_y}{E_{y0}})^2 = \sin^2\varphi \qquad (8\text{-}29)$$

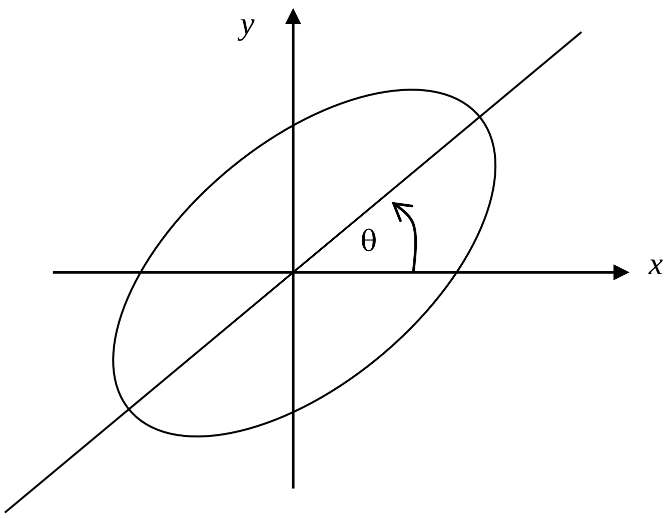
The conic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents an ellipse, if $B^2 - 4AC < 0$. Therefore Eq. (8-29) or the tip of the electric field vector (E_x, E_y) indeed traces out an ellipse in the *xy* plane. The major axis of the ellipse forms an angle θ with respect to the *x* axis, where

$$\tan 2\theta = \frac{B}{A-C} = \frac{2E_{x0}E_{y0}\cos\varphi}{E_{x0}^2 - E_{y0}^2}$$

The kind of polarization is called elliptical polarization.



i. For the case of $\varphi = \pm\pi/2$

$$\vec{E}(z,t) = E_{x0}\cos(\omega t - kz + \phi_0)\hat{a}_x \mp E_{y0}\sin(\omega t - kz + \phi_0)\hat{a}_y \qquad (8\text{-}30)$$

(or $\vec{E}(z) = e^{j\phi_0}(E_{x0}e^{-jkz}\hat{a}_x \pm jE_{y0}e^{-jkz}\hat{a}_y)$ in phasor notation, recall $e^{\pm j\pi/2} = \pm j$)

The conic equation is reduced to an up-right ellipse governed by the expression

$$\left(\frac{E_x}{E_{x0}}\right)^2 + \left(\frac{E_y}{E_{y0}}\right)^2 = 1$$

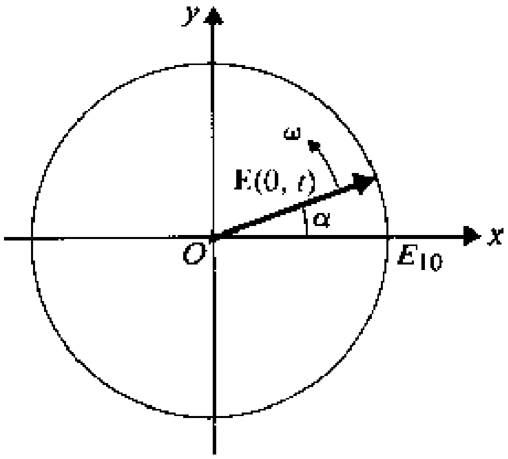
where the major and minor axes of the ellipse are along the *x* and *y* directions. The angle of the electric field vector with respect to the *x* axis is given by

$$\alpha = \tan^{-1}[\mp(E_{y0}/E_{x0})\tan(\omega t - kz)],$$

which is apparently a function of time and *z*.

ii. For the case of $E_{x0} = E_{y0} = E_0$

The conic equation is reduced to an equation of circle of radius $|E_0|$, given by $\frac{E_x^2}{E_0^2} + \frac{E_y^2}{E_0^2} = 1$



The angle between the electric field vector and the *x* axis is

$$\alpha = \tan^{-1}(\frac{E_y}{E_x}) = \mp(\omega t - kz),$$

meaning that the electric field vector rotates about the *xy* plane at a angular frequency of *ω*. This wave is called a circularly polarized wave. For $\alpha = +(\omega t - kz)$, the tip of the net electric field rotates in the counterclockwise direction when the wave is viewed in such a way that the wave propagates towards the observer. This wave is called a right-hand circular polarized wave. On the other hand, if $\alpha = -(\omega t - kz)$ the wave is called a left-hand circularly polarized wave.

Plane Waves in Lossy Media

There is ohmic loss in a conducting material. The induced current density $\vec{J}$ in a conducting material is proportional to the electric field $\vec{E}$ or $\vec{J} = \sigma\vec{E}$, where σ is the conductivity of the conducting material. Recall the Ampere's law in the Maxwell's equations

$$\nabla \times \vec{H} = j\omega\varepsilon'\vec{E} + \vec{J} = j\omega\varepsilon\vec{E} + \sigma\vec{E} = j\omega\varepsilon_c\vec{E} \ ,$$

where $\varepsilon_c \equiv \varepsilon' - j\varepsilon''$ a complex number with $\sigma = \omega\varepsilon''$ in units of (S/m). The ohmic loss contributes to the imaginary part of the complex permittivity. The complex permittivity results in a complex wave number given by

$$k = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')}$$

Note that the loss of an electromagnetic wave in a material is not always due to conductivity, but could be due to atomic or molecular absorption. For the later, the permittivity is also of a complex form.

For convenience, define the γ coefficient

$$\gamma = jk = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\,\varepsilon''/\varepsilon'\right)^{1/2} .$$

For a plan wave propagating in the z direction, the field component is expressed by

$$E, H \propto e^{-\gamma z} = E_0 e^{-\alpha z - j\beta z} .$$

Therefore, the real part of γ (=α) represents attenuation of the wave along the propagation direction. By defining a constant wavefront in $\omega t - \beta z = \text{constant}$, one concludes that the phase velocity of this attenuated wave is $u_p = \omega / \beta$. The imaginary part of γ (= β) is called the propagation constant or the phase constant, and the real part of γ (= α) is called the (field) attenuation constant.

Low Loss Dielectric

In the low loss limit, $\left|\varepsilon''/\varepsilon'\right| << 1$, one can apply Taylor's expansion to the expression $\left(1 - j\,\varepsilon''/\varepsilon'\right)^{1/2}$ and obain

$$jk = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right)$$

Substituting jk into the plane wave solution $E = E_0 e^{-jkz} = E_0 e^{-\alpha z - j\beta z}$, one can conclude that the wave propagates with an *attenuation constant* of

$$\alpha \equiv \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \quad \text{(NP/m)}$$

and a *phase constant* of

$$\beta \equiv \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right].$$

The *wave impedance*, defined to be the ratio of the electric field intensity to the magnetic field intensity, is also a complex number, given by

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\frac{1}{\sqrt{1 - j\,\varepsilon''/\varepsilon'}} \approx \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\,\varepsilon''/2\varepsilon'\right)$$

for a low-loss material. Accord to $H = E/\eta_c$, it can be concluded that *E* and *H* in a lossy medium are not in phase.

The *phase velocity* is also modified by the material loss, given by

$$u_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

in the low-loss limit.

Good Conductor

In a conductor, $\varepsilon'' = \sigma/\omega = \sigma/2\pi f$

Recall the γ coefficient defined by

$$\gamma = jk = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\varepsilon''/\varepsilon'\right)^{1/2}$$

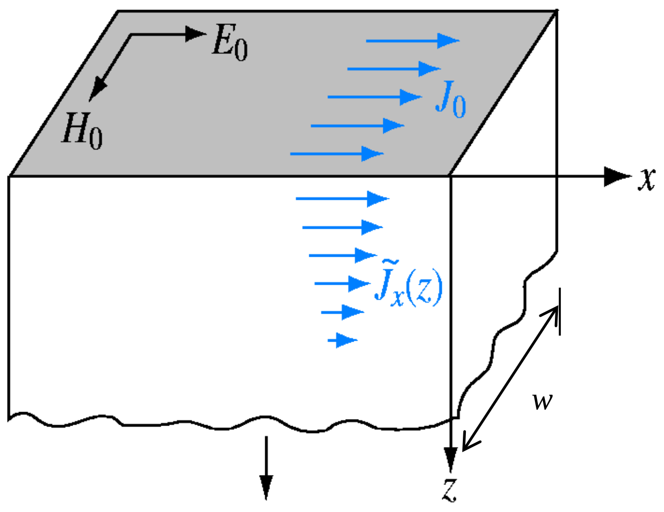
and thus

$$\alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\sigma/2\pi f\varepsilon'\right)^{1/2}$$

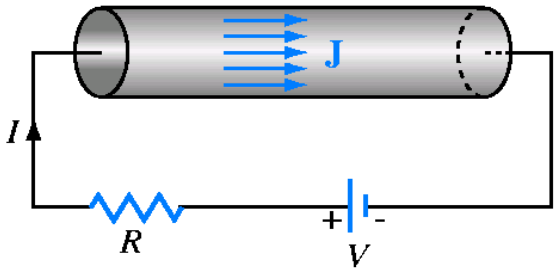
For a good conductor, the conductivity could be big and $\sigma/2\pi f\varepsilon' >> 1$ is possible. The γ coefficient has the approximate expression

$$\alpha + j\beta \approx j\omega\sqrt{\mu\varepsilon'}\left(-j\sigma/2\pi f\varepsilon'\right)^{1/2} = (1+j)\sqrt{\pi f\mu\sigma}$$

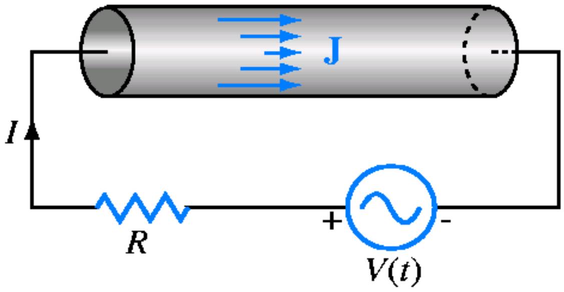
where the ***skin depth*** $\delta = 1/\alpha = 1/\sqrt{\pi f\mu\sigma}$ is the distance where the wave amplitude reduces to $1/e$. The current induced by $\vec{J} = \sigma\vec{E}$ also has a skin depth of $\delta = 1/\alpha = 1/\sqrt{\pi f\mu\sigma}$, as shown below.



Since an electromagnetic wave is attenuated in a conductor with a characteristic length called the skin depth, an AC current tend to flow on the outer surface of a conductor, as shown below.



(a) d-c case



(b) a-c case

The complex intrinsic wave impedance in a good conductor is

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\frac{1}{\sqrt{1-j\varepsilon''/\varepsilon'}} \approx \sqrt{\frac{\mu}{\varepsilon'}}\left(j2\pi f\varepsilon'/\sigma\right)^{1/2}$$

$$= (1+\mathrm{j})\sqrt{\pi f\mu/\sigma} = (1+j)\frac{\alpha}{\sigma} = e^{j\pi/4}\sqrt{2}\,\frac{\alpha}{\sigma}$$

Note that in a good conductor the *H* field lags behind the *E* field by 45°.

When the frequency is sufficiently high in a good conductor (σ is large), the skin depth is very small and the current is purely on the conductor surface, given by $\vec{J}_s = \hat{a}_{n2}\times\vec{H}$ or $J_s = H_y$ for the configuration above. The surface impedance is defined to be

$$Z_s \equiv \frac{E_x}{J_s} = \frac{E_x}{H_y} = \eta_c = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} \equiv R_s + jX_s .$$

The average power-density dissipated on the surface of the conductor is

$$S = \frac{1}{2}\mathrm{Re}(E_x H_y^*) = \frac{1}{2}\left|J_s\right|^2 R_s$$

The total power dissipated in the conductor is

$$P = \frac{1}{2}\left|J_s\right|^2 R_s \cdot A_{x,y} ,$$

where $A_{x,y}$ is the area of the incident plane. The power dissipated on the surface per unit length along *x* is $P_x = \frac{1}{2}\left|J_s\right|^2 R_s \cdot w$ for a given width *w* along *y*. The total current in the width *w* is $I = wJ_s$. Therefore P_x can be rewritten as

$$P_x = \frac{1}{2} I^2 \frac{R_s}{w} .$$

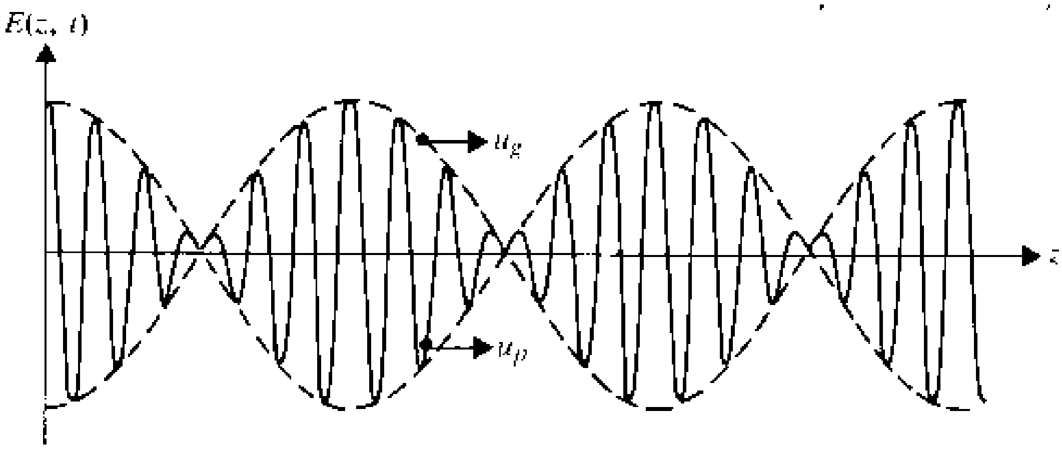
From the definition of phase velocity $u_p = \frac{\omega}{\beta}$, the expression of the phase velocity is given by $u_p \approx \sqrt{\frac{2\omega}{\mu\sigma}}$ for a good conductor. Unlike the phase velocity of a plane wave in vacuum, this phase velocity depends on the frequency of the electromagnetic wave (dispersion) and conductivity of the conductor. So, a good conductor is a dispersive material.

Group Velocity

Consider that an electric field of an EM wave is a superposition of two EM waves having almost the same frequency

$$
\begin{aligned}
E(z,t) &= \cos((\omega + \Delta\omega)t - (\beta + \Delta\beta)z + \phi_1) \\
&\quad + \cos((\omega - \Delta\omega)t - (\beta - \Delta\beta)z + \phi_2) \\
&= 2\cos(\Delta\omega t - \Delta\beta z + \frac{\phi_1 - \phi_2}{2})\cos(\omega t - \beta z + \frac{\phi_1 + \phi_2}{2})
\end{aligned}
$$

where $|\Delta\omega| \ll \omega$ and $|\Delta\beta| \ll |\beta|$. The field amplitude is a fast oscillating component modulated by a slowly varying envelope component, shown in the following plot.



The phase velocity calculated from $u_p = \dfrac{\omega}{\beta}$ is the velocity of the phase of the fast oscillating component.

The expression $u_g = \dfrac{\Delta\omega}{\Delta\beta} \rightarrow \dfrac{d\omega}{d\beta}$ gives the propagation velocity of the envelope or the *group velocity* of a wave. From energy's point of view, what really carries the energy is the wave package defined in the envelope. For most cases, group velocity is the energy propagation velocity of a wave and can only be less than *c*, whereas phase velocity can be larger than *c*.

From the calculation

$$\frac{1}{u_g}=\frac{d\beta}{d\omega}=\frac{d}{d\omega}\left(\frac{\omega}{u_p}\right)=\frac{1}{u_p}-\frac{\omega}{u_p^2}\frac{du_p}{d\omega}$$

one can relate group velocity to phase velocity by the expression

$$u_g=\frac{u_p}{1-\dfrac{\omega}{u_p}\dfrac{du_p}{d\omega}}$$

There are three regimes according to the relative magnitude of u_g and u_p, given by

zero dispersion $\dfrac{du_p}{d\omega}=0 \Rightarrow u_g=u_p$

normal dispersion $\dfrac{du_p}{d\omega}<0 \Rightarrow u_g<u_p$

anomalous dispersion $\dfrac{du_p}{d\omega}>0 \Rightarrow u_g>u_p$

Poynting Vector: a propagation vector of radiation intensity

From the vector identity $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$

and Maxwell's equations, one obtains

$$\nabla\cdot(\vec{E}\times\vec{H})=\vec{H}\cdot(\nabla\times\vec{E})-\vec{E}\cdot(\nabla\times\vec{H})=\vec{H}\cdot(\frac{-\partial\vec{B}}{\partial t})-\vec{E}\cdot(\frac{\partial\vec{D}}{\partial t}+\vec{J})$$

In a simple medium, $\vec{B}=\mu\vec{H}$, $\vec{D}=\varepsilon\vec{E}$, and $\vec{J}=\sigma\vec{E}$, which result in

$$\nabla\cdot(\vec{E}\times\vec{H})=-\frac{\partial(\mu H^2)}{2\partial t}-\frac{\partial(\varepsilon E^2)}{2\partial t}-\sigma E^2$$

Apply the divergence theorem and obtain the following integral form

$$\oint_S(\vec{E}\times\vec{H})\cdot d\vec{s}=\frac{-\partial}{\partial t}\int_V\left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv-\int_V\sigma E^2dv$$

temporal change of stored energy   Ohmic loss

The term on the left hand side, $\oint_S(\vec{E}\times\vec{H})\cdot d\vec{s}$, represents a power flowing out of a closed surface. The first term on the right hand side, $\frac{-\partial}{\partial t}\int_V\left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv$ is the rate of change of the stored energy in the volume; the second term on the right hand side, $-\int_V\sigma E^2dv$, is ohmic loss inside the volume.

Define the ***Poynting Vector*** according to $\vec{P}=\vec{E}\times\vec{H}$. The magnitude of a Poynting vector is the surface power density or intensity of an electromagnetic wave and the direction of a Poynting vector is the direction of power flow of an electromagnetic wave.

Suppose that the electric and magnetic field intensities of a plane wave are expressed by

$$E(z,t) = E_0 \cos(\omega t - kz + \varphi_E)$$ and

$$H(z,t) = H_0 \cos(\omega t - kz + \phi_H)\,.$$

Their corresponding phasor notations are

$$\hat{E}(z) = E_0 e^{-jkz + j\varphi_E}$$ and $$\hat{H}(z) = H_0 e^{-jkz + j\phi_H}.$$

The magnitude of the *instantaneous power density* is given by the product of the two fields,

$$P(z,t) = E_0 H_0 \cos(\omega t - kz + \varphi) \times \cos(\omega t - kz)$$

$$= \frac{1}{2} E_0 H_0 [\cos(2\omega t - 2kz + \varphi_E + \varphi_\mathrm{H}) + \cos(\varphi_E - \phi_\mathrm{H})]$$

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The *average power density* is calculated from

$$P_{av} = \frac{E_0 H_0}{2T} \int_T P(t) \cdot dt = \frac{E_0 H_0}{2} \cos(\varphi_E - \varphi_H) = \mathrm{Re}\left(\hat{E}(z)\hat{H}^*(z)\right)/2$$

where T is a time duration much longer than the period of the wave. Taking into account the direction, the time-averaged power density can be written as

$$\vec{P}_{av} = \mathrm{Re}\left(\vec{E}(z) \times \vec{H}^*(z)\right)/2 = \mathrm{Re}(\vec{s})$$

where $\vec{S} = \left(\hat{\vec{E}}(z) \times \hat{\vec{H}}^*(z)\right)/2$ is called the complex Poynting vector.