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Method I : Series solutions about an ordinary point (ch6.2 & 6.4)

When $x=0$ is an "ordinary point" of the ODE (That is,

It means
$\Longrightarrow$ ① We can find in the form of a power series

② Each series converges

Example 1 : Solve $y''-xy=0$

✡ observation : $x=0$ is

Step 1 : Express the solution as a power series

Step 2 : plug in the series to the ODE

Step 3 : Match the coefficients to find the recurrence relation

Step 4 : Plug in the coefficients and obtain the general solution

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Example 2: Solve $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ (Ch 6.4)
(Legendre's equation of order n)

☆ Observation: $x=0$ is

Step 1: Express the solution as a power series

Step 2: Plug in the series to the ODE

Step 3: Match the coefficients to find the recurrence relation

Step 4: Plug in the coefficients and obtain the general solution

$$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \right.$$
$$+ c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \right.$$

Remarks: About solutions of Legendre's equation

① For $n=0$, $y=$

$n=1$, $y=$

$n=2$, $y=$

$n=3$, $y=$

⋮ ⋮

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So for each integer n, we obtain an
of Legendre's equation. These polynomials are called

ex: The first few order of

$P_0(x) =$ → the polynomial solution of

$P_1(x) =$ → " "

$P_2(x) =$

$P_3(x) =$

$P_4(x) =$

We can make a plot for Legendre polynomials :

(2) Properties of Legendre polynomials :

Method II : Series solutions about a regular singular point

When $x=0$ is a "regular singular point" of the ODE

(That is,

It means

$\Longrightarrow$ ① We can find

② The convergence of the series can be determined by.

Example 1 : Solve $3xy''+y'-y=0$

* Observation :

Step 1 : Express the solution as

Step 2 : Plug in the series to the ODE

$y' =$

$y'' =$

Step 3 : Match the coefficient from

$3C_0 r(r-1)+$

$\Rightarrow r(3r-2)=0$: This eq is called

$r_1 =$ $\qquad r_2 =$

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Step 4 : For each r,

For $r_1 = \frac{2}{3}$ $\Rightarrow$

For $r_2 = 0$ $\Rightarrow$

Step 5 : Plug in the coefficients and obtain the general solution

$y =$

Remarks :

① In this example,

② The indicial eq is obtained by matching the

There is a general form of indicial eq (can be derived) as

$r(r-1) +$

ex : $2xy'' + (1+x)y' + y = 0$

Example 2 : Solve $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (ch 6.4)

(Bessel's equation of order ν)

★ observation :