HW1

Q4, 5, 6, 8

林恩德

4. (*) Consider the expression $(p \wedge q) \vee \neg(p \rightarrow q)$. In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.

- Solution 1: draw a truth table

p	q	p∧q	p->q	$(p \wedge q) \vee \neg(p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	F
F	F	F	T	F

This is equal to p

Ans: p

- Solution 2:

$(p \wedge q) \vee \neg(p \rightarrow q)$

$\equiv (p \wedge q) \vee \neg(\neg p \vee q)$ by De Morgan's Laws

$\equiv (p \wedge q) \vee (p \wedge \neg q)$ by *Distributive Laws*

$\equiv p \wedge (q \vee \neg q)$ by Negation Laws

$\equiv p \wedge T$ by Identity Laws

$\equiv p$

5. (*) A compound proposition is *satifiable* if there is a way to assign truth values to each of the propositions, such that the truth value of the compound proposition is true. In other words, a compound proposition is satisfiable if and only if it is not a contradiction. For example, the compound proposition $p \wedge \neg q$ is satisfiable, as setting $p = \texttt{T}$ and $q = \texttt{F}$ will make the $p \wedge \neg q$ true. In contrast, $p \wedge (q \wedge \neg q)$ is not satisfiable.

   Determine whether each of these compound propositions is satisfiable.

   (a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

   (b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$

(a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

We can divide it into three parts: p V ¬q , ¬p V q , ¬p V ¬q

and all of three proposition need to be true ,so if we let p equal to F

and q equal to F, three propositions are true.

-> (a) is satisfiable.

$$(b)\ \ (p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$$

(b) is similar to (a). Divide it into four parts , and four expressions need to be true. However , we can’t avoid this situation : one of four expressions must be false.(No matter we set q F or T , there is a entry is false) You can just draw a truth table to test

-> (b) is not satisfiable.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

6. (*) The following exercises involve the logical operator $\uparrow$ (read as NAND). The proposition $p \uparrow q$ is true when either p, or q, or both, are false.

   (a) Show that $p \uparrow q \equiv \neg(p \wedge q)$.

   (b) Show that $p \uparrow p \equiv \neg p$.

   (c) Express $p \wedge q$ by using only $\uparrow$ operators.

   (d) Express $p \vee q$ by using only $\uparrow$ operators.

(a) truth table

p	**q**	$p \uparrow q$	**¬(p∧q)**
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

(b)

Replace q with p in (a) ,we can get

$p \uparrow p \equiv \neg(p \wedge p)$, by Idempotent Laws , it equals to $\neg p$

(c)

$p \wedge q$

$\equiv (p \wedge q) \vee (p \wedge q)$ by Idempotent Laws

$\equiv \neg(\neg(p \wedge q) \wedge \neg(p \wedge q))$ by De Morgan's Laws

$\equiv (p \uparrow q) \uparrow (p \uparrow q)$

(d)

$$p \vee q$$

$$\equiv \neg(\neg p \wedge \neg q) \qquad \text{by De Morgan's Laws}$$

$$\equiv (p \uparrow p) \uparrow (q \uparrow q)$$

8. (*) What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$ it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

We know that there exists a person y such that S (y , Max) is true. However, we don't know if the person y is Max, thus we can't conclude S(Max , Max).

Moreover , S(Max , Max) can't be true , because Max can't be shorter than himself. This applies to everybody.
Thus, the statement $\exists x S(x, x)$ can't be true.

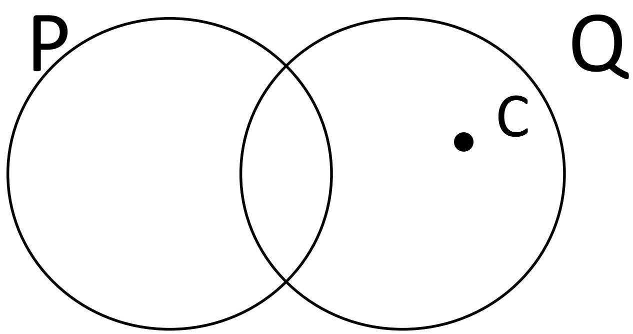
HW1

Q10, 11, 12, 13
陳咨蓉

10. (*) Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

(1)	$\forall x(P(x) \vee Q(x))$	Premise.
(2)	$P(c) \vee Q(c)$	Universal instantiation from (1).
(3)	$P(c)$	Simplification from (2).
(4)	$\forall x P(x)$	Universal generalization from (3).
(5)	$Q(c)$	Simplification from (2).
(6)	$\forall x Q(x)$	Universal generalization from (5).
(7)	$\forall x P(x) \vee \forall Q(x)$	Conjunction from (4) and (6).

- (2) $P(c) \lor Q(c)$ Universal instantiation from (1).
- (3) $P(c)$ Simplification from (2).
- →Simplification is "$\land$" not "$\lor$"
- →another view



11. (*) Determine whether these are valid arguments.

(a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

(b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

(a) Counterexample : -1

(b) x^2 ≠ 0, where x is a real number.

which means

(x^2 ≠ 0) ∧ (x is a real number) ≡ (x is a real number) ∧ (x^2 ≠ 0)

→Correct!

12. (*) To describe the various restaurants in the city, we let p denote the statement "The food is good," q denote the statement "The service is good," and r denote the statement "The rating is three-star." Write the following statements in symbolic form.

(a) Either the food is good, or the service is good, or both.

(b) Either the food is good, or the service is good, but not both.

(c) If both the food and services are good, then the rating will be three-star.

(d) It is not true that a three-star rating always means good food and good service.

(a) $p \lor q$

(b) $p \oplus q$

(c) $(p \land q) \rightarrow r$

(d) $\neg[r \rightarrow (p \land q)]$

13. (*) Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

(a) The product of two negative real numbers is positive.

(b) The difference of a real number and itself is zero.

(c) A negative real number does not have a square root that is a real number.

(a) $\forall x \forall y[(x < 0 \wedge y < 0) \rightarrow xy > 0]$

(b) $\forall x(x - x = 0)$

(c) $\forall x[x < 0 \rightarrow \neg \exists y(x = y^2 \wedge y \in R)]$

HW1

Q14, 18, 19
李沛倫

14. (*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

(a) $\forall n \exists m\ (n^2 < m)$

(b) $\exists n \forall m\ (n < m^2)$

(c) $\forall n \exists m\ (n + m = 0)$

(d) $\exists n \forall m\ (nm = m)$

(e) $\exists n \exists m\ (n^2 + m^2 = 5)$

(f) $\exists n \exists m\ (n^2 + m^2 = 6)$

(g) $\exists n \exists m\ (n + m = 4 \land n - m = 1)$

(h) $\exists n \exists m\ (n + m = 4 \land n - m = 2)$

(a) True. Let $m = n^2 + 1$

(b) True. Let n be a negative number. Then $n < 0 \leq m^2 \rightarrow n < m^2$

(c) True. Let $m = -n$

(d) True. Let $n = 1$

(e) True. Let $n = 1$ and $m = 2$

(f) False. There doesn't exist any integer solution.

(g) False. There doesn't exist any integer solution, since $n = 2.5$ and $m = 1.5$

(h) True. Let $n = 3$ and $m = 1$

18. (*) Four cards are displayed on the table as shown in Figure 3. It is known that for each card, both faces are drawn with geometric shapes, such that one is solid while the other is empty. For instance, Card 1 shows a solid circle, which implies its other face will be some empty shape. Similarly, Card 2 shows an empty square, which implies its other face will be some solid shape.

Peter took a look at the other face of each card, and said, "if one face is drawn with a solid circle, then the other face must be drawn with an empty triangle".

(a) You want to double check about Peter's claim. One way is to look at the other face of every card. However, you want to save time. Is it possible to check only some (but not all) of these cards, so that you can be 100% sure that Peter's claim is correct?

(b) What is the minimum number of cards you need to check?

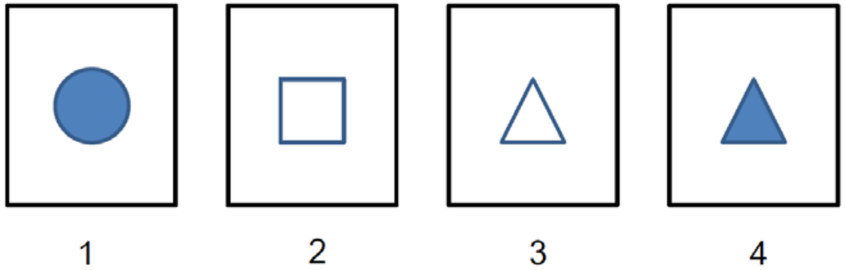


Figure 3: The four cards for Question 18.

For solid, let p = circle. For empty, let q = triangle.

Then we want to check the statement $p \to q \equiv (\neg q) \to (\neg p)$.

Card 1: Since p is true, the other side need to be true (q).

Card 2: Since q is false ($\neg$ q), the other side need to be false ($\neg$ p).

Card 3: Since q is true, we don't have to check the other side.

Card 4: Since p is false, we don't have to check the other side.

(a) It's possible. Just check card 1 and card 2.

(b) 2 cards.

19. (*) Determine the answers of the following questions so that all can be answered correctly.

Q1. Which is the first question where (c) is the correct answer?

(a) Q3
(b) Q4
(c) Q1
(d) Q2

Q2. Which is the first question where (a) is the correct answer?

(a) Q4
(b) Q2
(c) Q3
(d) Q1

Q3. Which is the first question where (d) is the correct answer?

(a) Q1
(b) Q2
(c) Q4
(d) Q3

Q4. Which is the first question where (b) is the correct answer?

(a) Q2
(b) Q4
(c) Q3
(d) Q1

Assume Q1: (a) → Q3: (c) → Q4: (d) → Q1: (b)
It contradicts to the assumption that Q1 is (a).

Assume Q1: (b) → Q4: (c) → Q3: (b)
It contradicts to the assumption that Q1 is the question answer (a).

Assume Q1: (c)

- Assume Q2: (a) → Q4: (a) → Q2: (b) It contradicts to the assumption that Q2 is (a).
- Assume Q2: (b) → Q2: (a) It contradicts to the assumption that Q2 is (b).
- Assume Q2: (c) → Q3: (a) → Q1: (d) It contradicts to the assumption that Q1 is (c).
- Assume Q2: (d) → Q1: (a) It contradicts to the assumption that Q1 is (c).

Assume Q1: (d) → Q2: (c) → Q3: (a) → Q1: (d) and Q4: (b)

Thus, there is only one answer where Q1: (d), Q2: (c), Q3: (a), Q4: (b).

HW2

Q1, 2, 5a, 6
陳弘欣

Q1

Give a direct proof for the following theorem: If n is perfect square, then n + 2 is not a perfect square.

$n = k^2, k \in Z$. It suffices to prove that $n + 2 = (k + a)^2$, where $a \in N$.

$k^2 + 2 = (k + a)^2 = k^2 + 2ka + a^2$ => $2 = 2ka + a^2$.

- $k = 0$: There is no a such that $a^2 = 2$.
- $k >= 1$: $2ka + a^2 > 2$.

Thus, $n+2$ can’ t be a perfect square.

Q2

Use a direct proof to show that any odd integer is the difference of two squares.

Let n be an odd integer.

Let $n = a * b$, where a, b are odd integers. WLOG, suppose $a \geq b$.

Rewrite $a = x + y$, $b = x - y$, where $x = (a + b) / 2$, $y = (a - b) / 2$.

Since a, b are odd, x and y are integers.

So $n = a * b = (x + y) * (x - y) = x^2 - y^2$.

Q5(a)

For all integers m and n, if mn is odd, then m, n are both odd.

Contrapositive: If <u>one of</u> m, n are *even*, then mn is *even*.

- If one of m, n are even, say m, we can write $m = 2k$, where $k \in Z$.
- Thus, $mn = 2k * n$, which is divisible by 2.

Q6(a)

If n is a natural number, then $n^2 + n + 3$ is odd.

- Case 1: n is odd.
  - $n = 2k + 1, k \in \mathbb{N}$. $n^2 + n + 3 = 4k^2 + 4k + 1 + 2k + 1 + 1 = 4k^2 + 6k + 3 = 2(2k^2 + 3k + 1) + 1$.
  - So $n^2 + n + 3$ is odd.
- Case 2: n is even.
  - $n = 2k, k \in \mathbb{N}$. $n^2 + n + 3 = 4k^2 + 2k + 1 = 2(2k^2 + k) + 1$.
  - So $n^2 + n + 3$ is odd.

In all possible cases, $n^2 + n + 3$ is odd. So we can conclude that $n^2 + n + 3$ is odd if n is a natural number.

Q6(b)

If a and b are real numbers, $|a - b| = |b - a|$.

- Case 1: $a > b$.
  - $|a - b| = a - b$, $|b - a| = -(b - a) = a - b$.
  - So $|a - b| = |b - a|$.
- Case 2: $a = b$.
  - $|a - b| = 0$, $|b - a| = 0$.
  - So $|a - b| = |b - a|$.
- Case 3: $a < b$.
  - $|a - b| = -(a - b) = b - a$, $|b - a| = b - a$.
  - So $|a - b| = |b - a|$.

In all possible cases, $|a - b| = |b - a|$. So we can conclude that $|a - b| = |b - a|$.

HW2

Q7, 8, 9, 10
薛旻欣

HW2 Q7

- Show that $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$ has an integral root.
- $$x^5 - x^4 + x^3 - x^2 + x - 1 = (x - 1)(x^4 + x^2 + x)$$
- It has an integral root $x = 1$.

HW2 Q8

8. (*, Challenging) Prove that when a white square and a black square are removed from an 8×8 chessboard, you can tile the remaining squares of the checkerboard using dominoes.

   Hint: It is a fun problem! Try it without the hint. See Figure 2 only if you get stuck.

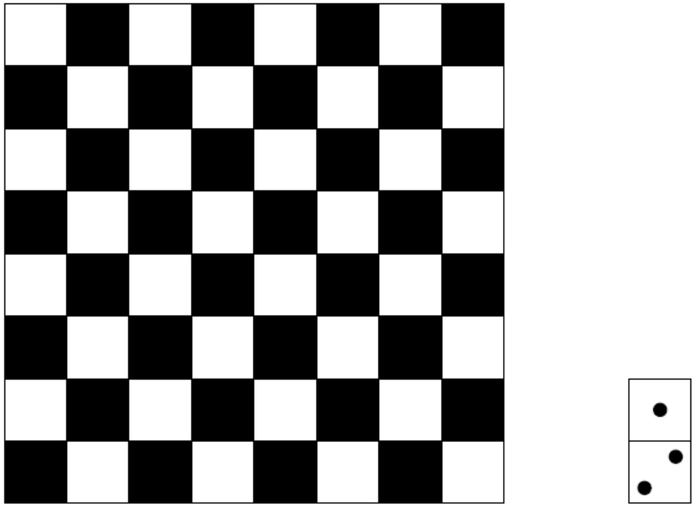


Figure 1: A checkerboard and a domino piece.

HW2 Q8

- It can be reformed to a cycle with length 64.

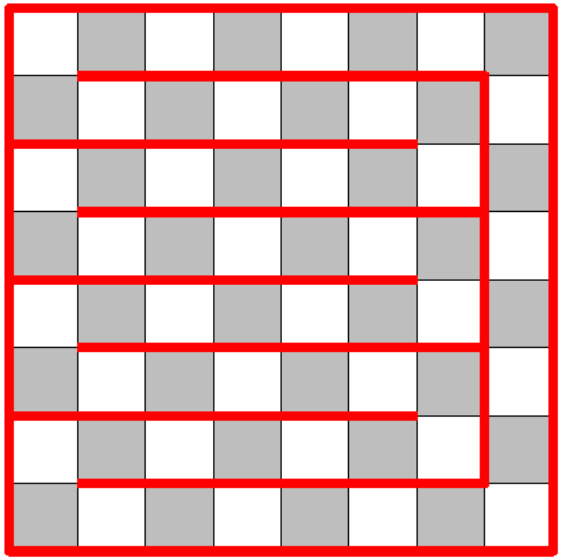
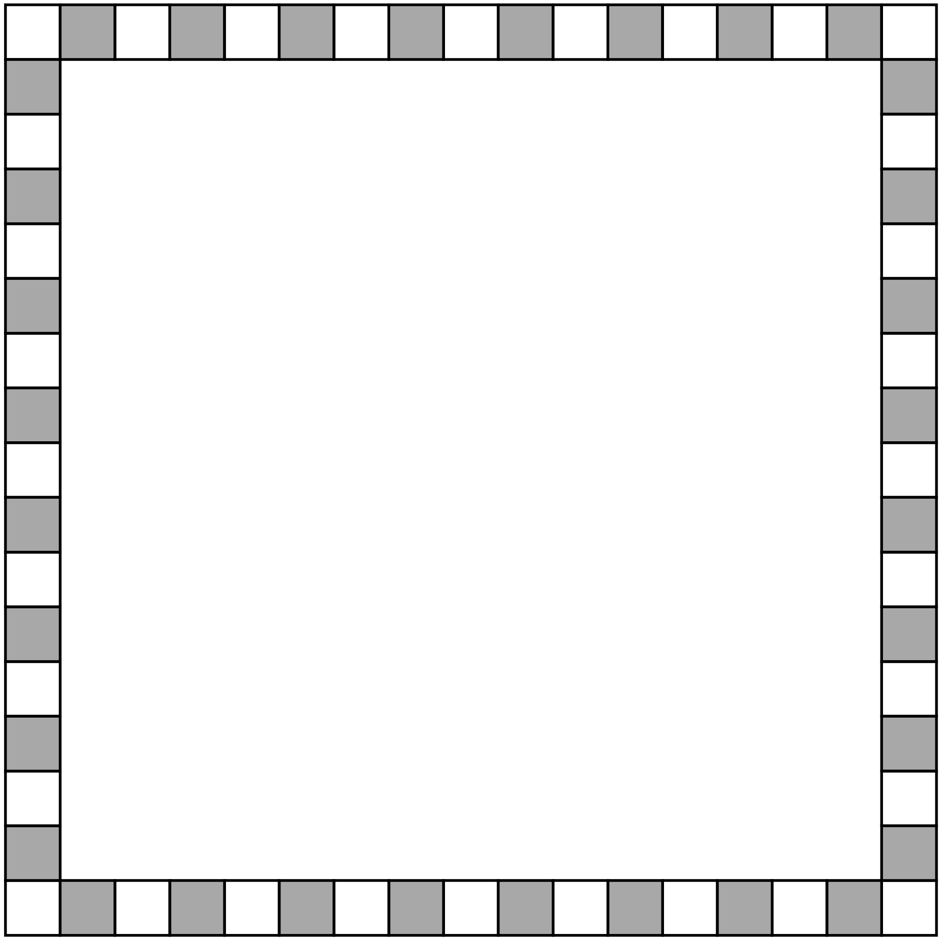
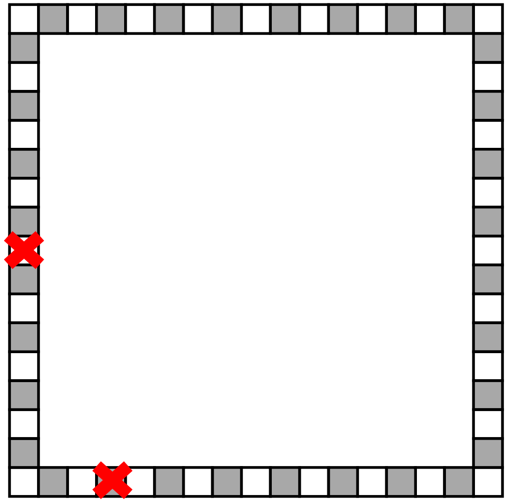


Figure 2: A hint for Question 8.



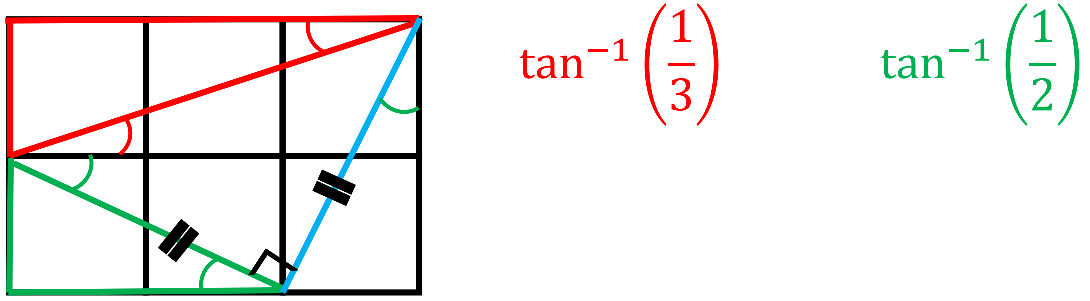
HW2 Q8

- It can be reformed to a cycle with length 64.
- And we can see that if we remove a white and a black square at arbitrary positions, it always breaks the cycle into two lists of squares with length x and $62 - x$. While x and $62 - x$ are always even numbers, we can surely tile them with dominos.



HW2 Q9

- Let α be an angle s.t. $\alpha = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)$ and $0 \le \alpha \le 2\pi$. Show that $\alpha = \frac{\pi}{4}$ without using a calculator.
- The sum formula: $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$.
- Try to solve it by drawing the angles.



HW2 Q10

- Prove or disprove the following:

  If $p_1, p_2, \dots, p_n$ are the n smallest primes, then $k = p_1 p_2 \cdots p_{n+1} + 1$ is prime.

- It's obvious that k can't be divisible by any p_i for $i \leq n + 1$.
- However, given some n, the corresponding k might be divisible by some p_i s.t. k $\neq p_i$ and $i > n + 1$.
- Disprove by existence:

  $$2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$$

HW2 Q10

- We can find it not a rare case if we write a program:

n=1: 2x3 + 1 = 7 -> prime
n=2: 2x3x5 + 1 = 31 -> prime
n=3: 2x3x5x7 + 1 = 211 -> prime
n=4: 2x3x5x7x11 + 1 = 2311 -> prime
n=5: 2x3x5x7x11x13 + 1 = 30031 -> 59 x 509
n=6: 2x3x5x7x11x13x17 + 1 = 510511 -> 19 x 97 x 277
n=7: 2x3x5x7x11x13x17x19 + 1 = 9699691 -> 347 x 27953
n=8: 2x3x5x7x11x13x17x19x23 + 1 = 223092871 -> 317 x 703763
n=9: 2x3x5x7x11x13x17x19x23x29 + 1 = 6469693231 -> 331 x 571 x 34231
n=10: 2x3x5x7x11x13x17x19x23x29x31 + 1 = 200560490131 -> prime
n=11: 2x3x5x7x11x13x17x19x23x29x31x37 + 1 = 7420738134811 -> 181 x 60611 x 676421
n=12: 2x3x5x7x11x13x17x19x23x29x31x37x41 + 1 = 304250263527211 -> 61 x 450451 x 11072701
n=13: 2x3x5x7x11x13x17x19x23x29x31x37x41x43 + 1 = 13082761331670031 -> 167 x 78339888213593
n=14: 2x3x5x7x11x13x17x19x23x29x31x37x41x43x47 + 1 = 614889782588491411 -> 953 x 46727 x 13808181181

HW1

Q16, 17
answers

HW1 Q16

$$\begin{array}{r} S\;E\;N\;D \\ +\;M\;O\;R\;E \\ \hline M\;O\;N\;E\;Y \end{array} \qquad \begin{array}{r} 9\;5\;6\;7 \\ +\;1\;0\;8\;5 \\ \hline 1\;0\;6\;5\;2 \end{array}$$

HW1 Q17

$$
\begin{array}{rl}
 & \,\,\,*\,*\,8\,*\,* \\ \cline{2-2}
y \,\big) & *\,*\,*\,*\,*\,*\,* \\
 & *\,*\,* \\ \cline{2-2}
 & *\,* \\
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 & *\,*\,* \\
 & *\,*\,* \\ \cline{2-2}
 & 0
\end{array}
\qquad\qquad
\begin{array}{rl}
 & 9\,0\,8\,0\,9 \\ \cline{2-2}
12 \,\big) & 1\,0\,8\,9\,7\,0\,8 \\
 & 1\,0\,8 \\ \cline{2-2}
 & 9\,7 \\
 & 9\,6 \\ \cline{2-2}
 & 1\,0\,8 \\
 & 1\,0\,8 \\ \cline{2-2}
 & 0
\end{array}
$$