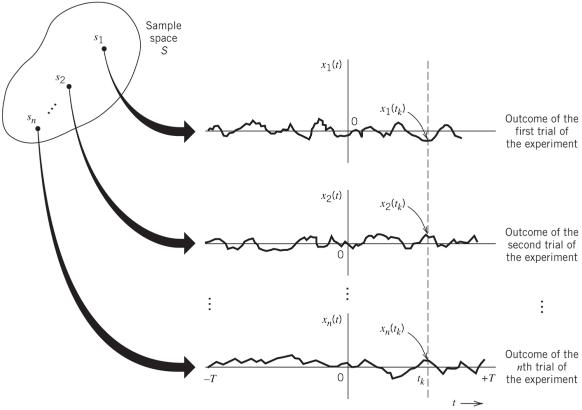
Random Processes

Chi-chao Chao

Department of Electrical Engineering
National Tsing Hua University
ccc@ee.nthu.edu.tw

Introduction

- A random process (or called stochastic process) $X(t)$ is a rule for assigning to every outcome ξ in the sample space Ω a function $x(\xi; t)$.



- Each $x_j(t)$, for $j = 1, 2, \ldots, n$, is called a sample function (or realization) of the random process $X(t)$.

- Given any time instant t_1, $X(t_1)$ is a random variable.
- Given k time instants $t_1, t_2, \ldots, t_k$, consider the k random variables $X(t_1), X(t_2), \ldots, X(t_k)$. The joint cumulative distribution function

$$F_{\boldsymbol{X}(t)}(\boldsymbol{x}) = P(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_k) \leq x_k)$$

and the joint probability density function

$$f_{\boldsymbol{X}(t)}(\boldsymbol{x}) = \frac{\partial^k}{\partial x_1 \partial x_2 \cdots \partial x_k} F_{\boldsymbol{X}(t)}(\boldsymbol{x})$$

where $\boldsymbol{X}(t) = (X(t_1), X(t_2), \ldots, X(t_k))$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$.

Stationarity

Definition

A random process $X(t)$ is strictly stationary if

$$f_{\mathbf{X}(t)}(\mathbf{x}) = f_{\mathbf{X}(t+\tau)}(\mathbf{x}), \quad \textit{for all } \tau$$

i.e., $X(t_1), X(t_2), \ldots, X(t_k)$ have the same distribution as $X(t_1+\tau), X(t_2+\tau), \ldots, X(t_k+\tau)$.

- If $X(t)$ is strictly stationary, then the mean function

$$\mathrm{E}[X(t)] = \mu_X, \quad \text{for all } t$$

and the autocorrelation function

$$R_X(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)] = R_X(t_1 - t_2).$$

Definition

A random process $X(t)$ is wide-sense stationary (or called weakly stationary) if

1. $E[X(t)] = \mu_X$.
2. $E[X(t+\tau)X(t)] = R_X(\tau)$.

- A random process is wide-sense stationary if it is strictly stationary, but not vice versa.
- For a wide-sense stationary process $X(t)$, its autocorrelation function $R_X(\tau)$ satisfies the following properties.

Property

$$R_X(\tau) = R_X(-\tau).$$

Proof. $R_X(\tau) = \mathrm{E}[X(t+\tau)X(t)] = \mathrm{E}[X(t)X(t-\tau)] = \mathrm{E}[X(t-\tau)X(t)]$
$= R_X(-\tau)$. ∎

Property

$$R_X(0) = \mathrm{E}[X^2(t)], \text{ which is the average power.}$$

Property

$$|R_X(\tau)| \le R_X(0).$$

Proof. We have

$$\mathrm{E}\left[(X(t+\tau) \pm X(t))^2\right] \ge 0$$

which gives

$$\mathrm{E}\left[X^2(t+\tau)\right] \pm 2\mathrm{E}\left[X(t+\tau)X(t)\right] + \mathrm{E}\left[X^2(t)\right] \ge 0.$$

We then obtain

$$2R_X(0) \pm 2R_X(\tau) \ge 0$$

and hence

$$R_X(0) \ge \mp R_X(\tau).$$

Example

- Let $X(t) = A\cos(2\pi f_c t + \Theta)$, where Θ is uniformly distributed in $(-\pi, \pi)$, i.e., its probability density function

$$f_\Theta(\theta) = \begin{cases} 1/2\pi, & -\pi < \theta < \pi \\ 0, & \text{elsewhere.} \end{cases}$$

- The autocorrelation function of $X(t)$ is given by

$$\begin{aligned} R_X(\tau) &= \mathsf{E}[X(t+\tau)X(t)] \\ &= \mathsf{E}[A\cos(2\pi f_c(t+\tau)+\Theta)A\cos(2\pi f_c t+\Theta)] \\ &= \frac{A^2}{2}\mathsf{E}[\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] + \frac{A^2}{2}\mathsf{E}[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{2}\int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\, d\theta + \frac{A^2}{2}\cos(2\pi f_c \tau) \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau). \end{aligned}$$

Cross-Correlation Functions

- Consider two random processes $X(t)$ and $Y(t)$.
- The cross-correlation functions

$$R_{XY}(t,u) = \mathrm{E}[X(t)Y(u)]$$

$$R_{YX}(t,u) = \mathrm{E}[Y(t)X(u)].$$

- The correlation matrix

$$\boldsymbol{R}(t,u) = \begin{bmatrix} R_X(t,u) & R_{XY}(t,u) \\ R_{YX}(t,u) & R_Y(t,u) \end{bmatrix}.$$

- If $X(t)$ and $Y(t)$ are each wide-sense stationary and jointly wide-sense stationary, then the correlation matrix

$$\boldsymbol{R}(\tau) = \begin{bmatrix} R_X(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_Y(\tau) \end{bmatrix}$$

where $\tau = t - u$ and $R_{XY}(\tau) = R_{YX}(-\tau)$.

Ergodicity

- Consider the ensemble averages

$$\mu_X = \mathrm{E}[X(t)]$$

$$R_X(\tau) = \mathrm{E}[X(t+\tau)X(t)]$$

and the time averages

$$\mu_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

$$R_x(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)\,dt$$

where $x(t)$ is a sample function of a wide-sense stationary process $X(t)$.

Definition

The process $X(t)$ is ergodic in the mean if

1. $\lim_{T\to\infty} \mu_x(T) = \mu_X$.
2. $\lim_{T\to\infty} Var[\mu_x(T)] = 0$.

Definition

The process $X(t)$ is ergodic in the autocorrelation function if

1. $\lim_{T\to\infty} R_x(\tau, T) = R_X(\tau)$.
2. $\lim_{T\to\infty} Var[R_x(\tau, T)] = 0$.

Power Spectral Density

- For a deterministic signal $g(t)$ with finite power, let

$$g_T(t) = \begin{cases} g(t), & -T \leq t \leq T \\ 0, & \text{elsewhere} \end{cases}$$

and the Fourier transform of $g_T(t)$ is $G_T(f)$:

$$F[g_T(t)] = G_T(f).$$

- The power spectral density of $g(t)$

$$S_g(f) = \lim_{T\to\infty} \frac{1}{2T}|G_T(f)|^2.$$

- For a wide-sense stationary process $X(t)$, the power spectral density

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T}\mathrm{E}\left[|X_T(f)|^2\right]$$

where $X_T(f)$ is the Fourier transform of a truncated version of a sample function of the random process.

Einstein-Wiener-Khintchine Relation

The power spectral density $S_X(f)$ is the Fourier transform of the autocorrelation function $R_X(\tau)$: $F[R_X(\tau)] = S_X(f)$, i.e.,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} \, d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} \, df.$$

Property

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau.$$

Proof. This property follows directly from

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau}\, d\tau$$

by putting $f = 0$. ∎

Property

The average power $\mathrm{E}[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df.$

Proof. This property follows directly from

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau}\, df$$

by putting $\tau = 0$ and noting that $R_X(0) = \mathrm{E}[X^2(t)]$. ∎

Property

$$S_X(f) \geq 0, \quad \text{for all } f.$$

Property

$$S_X(-f) = S_X(f).$$

Proof.

$$\begin{aligned} S_X(-f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{j2\pi f\tau}\, d\tau = \int_{-\infty}^{\infty} R_X(-\tau) e^{-j2\pi f\tau}\, d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau}\, d\tau = S_X(f). \end{aligned}$$

∎

Example

- Consider $X(t) = A\cos(2\pi f_c t + \Theta)$, where Θ is uniformly distributed in $(-\pi, \pi)$.
- The autocorrelation function

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau).$$

- Then the power spectral density

$$S_X(f) = F[R_X(\tau)] = \frac{A^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right].$$

- The average power

$$\mathrm{E}[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df = \frac{A^2}{2}.$$

Cross-Spectral Densities

- Consider two wide-sense stationary processes $X(t)$ and $Y(t)$ which are also jointly wide-sense stationary.
- The cross-spectral densities

$$S_{XY}(f) = F[R_{XY}(\tau)]$$

$$S_{YX}(f) = F[R_{YX}(\tau)].$$

- Since $R_{XY}(\tau) = R_{YX}(-\tau)$,

$$S_{XY}(f) = S_{YX}(-f) = S^*_{YX}(f)$$

where $*$ denotes conjugation.

Example

- Let $Z(t) = X(t) + Y(t)$.
- Then the autocorrelation function

$$\begin{aligned} R_Z(\tau) &= \mathrm{E}[Z(t+\tau)Z(t)] \\ &= \mathrm{E}[(X(t+\tau) + Y(t+\tau))(X(t) + Y(t))] \\ &= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau). \end{aligned}$$

- Therefore, the power spectral density

$$S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f).$$

Cyclostationarity

Definition

A random process $X(t)$ is wide-sense cyclostationary *if $E[X(t)]$ and $R_X(t+\tau, t) = E[X(t+\tau)X(t)]$ are periodic functions of t with period T, i.e.,*

$$E[X(t)] = E[X(t+T)]$$

$$R_X(t+\tau, t) = R_X(t+T+\tau, t+T).$$

- Consider the time-average autocorrelation function

$$\bar{R}_X(\tau) = \frac{1}{T}\int_0^T R_X(t+\tau, t)\, dt.$$

- Then the average power spectral density $\bar{S}_X(f)$ is the Fourier transform of $\bar{R}_X(\tau)$:

$$\bar{S}_X(f) = F[\bar{R}_X(\tau)] = \int_{-\infty}^{\infty} \bar{R}_X(\tau) e^{-j2\pi f\tau}\, d\tau.$$

- Consider a linear time-invariant (LTI) system with impulse response $h(t)$ and transfer function $H(f)$.
- The input to the system is a wide-sense stationary process $X(t)$ with mean μ_X, autocorrelation function $R_X(\tau)$, and power spectral density $S_X(f)$.
- The output is $Y(t)$.

- The cross-correlation function between $X(t)$ and $Y(t)$

$$
\begin{aligned}
& R_{XY}(t+\tau, t) = \mathrm{E}[X(t+\tau)Y(t)] \\
= \;& \mathrm{E}\left[X(t+\tau)\int_{-\infty}^{\infty} X(t-u)h(u)\,du\right] \\
= \;& \int_{-\infty}^{\infty} \mathrm{E}[X(t+\tau)X(t-u)]h(u)\,du = \int_{-\infty}^{\infty} R_X(\tau+u)h(u)\,du \\
= \;& \int_{-\infty}^{\infty} R_X(\tau-v)h(-v)\,dv = R_X(\tau)\star h(-\tau) \\
= \;& R_{XY}(\tau)
\end{aligned}
$$

where $\star$ denotes convolution.

- Also

$$
R_{YX}(\tau) = R_{XY}(-\tau) = R_X(-\tau)\star h(\tau) = R_X(\tau)\star h(\tau).
$$

- The autocorrelation function of $Y(t)$

$$
\begin{aligned}
& R_Y(t+\tau, t) = \mathrm{E}[Y(t+\tau)Y(t)] \\
& = \mathrm{E}\left[Y(t+\tau)\int_{-\infty}^{\infty} X(t-u)h(u)\,du\right] \\
= \quad & \int_{-\infty}^{\infty} \mathrm{E}[Y(t+\tau)X(t-u)]h(u)\,du = \int_{-\infty}^{\infty} R_{YX}(\tau+u)h(u)\,du \\
= \quad & \int_{-\infty}^{\infty} R_{YX}(\tau-v)h(-v)\,dv = R_{YX}(\tau)\star h(-\tau) \\
= \quad & R_X(\tau)\star h(\tau)\star h(-\tau) \\
= \quad & R_Y(\tau).
\end{aligned}
$$

- The power spectral density of $Y(t)$

$$
\begin{aligned}
S_Y(f) &= S_X(f)H(f)H^*(f) \\
&= S_X(f)|H(f)|^2.
\end{aligned}
$$

- The mean of $Y(t)$

$$
\begin{aligned}
& \mathrm{E}[Y(t)] \\
= \;& \mathrm{E}\left[\int_{-\infty}^{\infty} X(t-u)h(u)\,du\right] = \int_{-\infty}^{\infty} \mathrm{E}[X(t-u)]h(u)\,du \\
= \;& \int_{-\infty}^{\infty} \mu_X h(u)\,du = \mu_X \int_{-\infty}^{\infty} h(u)\,du \\
= \;& \mu_X H(0) = \mu_Y.
\end{aligned}
$$

- Therefore, $Y(t)$ is also wide-sense stationary.
- The average power of $Y(t)$

$$
\mathrm{E}[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} S_X(f)|H(f)|^2\,df.
$$

Gaussian Process

Definition

A random process $X(t)$ is a Gaussian process if for any $t_1, t_2, \ldots, t_k$, $X(t_1), X(t_2), \ldots, X(t_k)$ are jointly Gaussian random variables.

Property

If a Gaussian process $X(t)$ is wide-sense stationary, then it is also strictly stationary.

Proof. The joint probability density function of $X(t_1), X(t_2), \ldots, X(t_k)$ is completely determined by

$$\mu_X = \mathrm{E}[X(t_i)], \quad \text{for } 1 \leq i \leq k$$

and

$$R_X(t_i - t_j) = \mathrm{E}[X(t_i)X(t_j)], \quad \text{for } 1 \leq i, j \leq k.$$

Property

If a Gaussian process $X(t)$ passes through a linear time-invariant system, then the output $Y(t)$ is also a Gaussian process.

Proof. The output

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} X(u)h(t-u)\,du \\ &= \lim_{\triangle u \to 0} \sum_{k=-\infty}^{\infty} X(k\triangle u)h(t-k\triangle u)\triangle u \end{aligned}$$

where $h(t)$ is the impulse response of the system. The property follows from the fact that linear combinations of jointly Gaussian random variables are still jointly Gaussian. ∎

- This property actually holds for any stable linear filter.

Property

If $X(t)$ is a Gaussian process, then $Y = \int_0^T g(t)X(t)\,dt$, which is called a linear functional of $X(t)$, is a Gaussian random variable.

Proof. Let

$$h(t) = \begin{cases} g(T-t), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

i.e.,

$$h(T-t) = \begin{cases} g(t), & 0 \le t \le T \\ 0, & \text{elsewhere.} \end{cases}$$

Then Y is the output sampled at $t = T$ of $X(t)$ passing through a linear time-invariant system with impulse response $h(t)$. ∎

- This property may be used as the definition for a Gaussian process.