EE306002: Probability **Spring 2022**
College of Electrical Engineering
National Tsing Hua University

Homework #2 – Solution
Coverage: chapter 3–4
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Problem 3.1.14. (10 points) Prove that if $P(A) = a$ and $P(B) = b$, then $P(A|B) \geq (a + b - 1)/b$.

Solution:

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(AB)}{b},$$

where

$$P(AB) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = a + b - 1.$$

Hence

$$P(A|B) = \frac{P(AB)}{b} \geq \frac{a + b - 1}{b}.$$

Problem 3.2.14. (10 points) In a series of games, the winning number of the nth game, $n = 1, 2, 3, \ldots$, is a number selected at random from the set of integers $\{1, 2, ..., n + 2\}$. Don bets on 1 in each game and says that he will quit as soon as he wins. What is the probability that he has to play indefinitely?
Hint: Let A_n be the event that Don loses the first n games. To calculate the desired probability, $P(\lim_{n\to\infty} A_n)$, use Theorem 1.8: For any increasing or decreasing sequence of events, $\{E_n\}_{n\geq 1}$,

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n).$$

Solution:
By the multiplication rule,

$$P(A_n) = \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \cdots \times \frac{n+1}{n+2} = \frac{2}{n+2}.$$

Since $A_n \supseteq A_{n+1}$ for $n = 1, 2, 3, \ldots$, by Theorem 1.8,

$$P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \frac{2}{n+2} = 0.$$

Problem 3.3.18. (10 points) A number is selected at random from the set $\{1, 2, 3, \ldots, 20\}$. Then a second number is selected randomly between 1 and the first number selected. What is the probability that the second number is 5?

Solution:
Let A_i be the event that the first number selected is i. Let F be the event that the second number selected is 5. By the law of total probability,

$$P(F) = \sum_{i=1}^{20} P(F \mid A_i)P(A_i) = \sum_{i=5}^{20} \frac{1}{i} \cdot \frac{1}{20} = \frac{1}{20} \sum_{i=5}^{20} \frac{1}{i} \approx 0.075.$$

Problem 3.4.14. (10 points) A certain cancer is found in one person in 5000. If a person does have the disease, in 92% of the cases the diagnostic procedure will show that he or she actually has it. If a person does not have the disease, the diagnostic procedure in one out of 500 cases gives a false positive result. Determine the probability that a person with a positive test result has cancer.

Solution:
Let X be the event that a person has cancer and Y be the event that a person has positive test result. Then the probability that a person with a positive result has cancer is $P(X|Y)$. By Baye's formula,

$$\begin{aligned} P(X|Y) &= \frac{P(XY)}{P(Y)} \\ &= \frac{P(\textit{has cancer and positive result})}{P(\textit{positive result})} \\ &= \frac{P(\textit{has cancer and positive result})}{P(\textit{has cancer and positive result}) + P(\textit{no cancer but positive result})} \\ &= \frac{P(Y|X)P(X)}{P(Y|X)P(X) + P(Y|X^c)P(X^c)} \\ &= \frac{(0.92)(1/5000)}{(0.92)(1/5000) + (1/500)(4999/5000)} \\ &= 0.084. \end{aligned}$$

Problem 3.5.40. (10 points) A fair coin is tossed n times. Show that the events "at least two heads" and "one or two tails" are independent if $n = 3$ but dependent if $n = 4$.

Solution:
Let A be the event that tossed at least two heads and B be the event that tossed with one or two tails.
For $n = 3$, the probabilities of the given events, respectively, are

$$P(A|\ n = 3) = \binom{3}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} + \binom{3}{3} \cdot (\frac{1}{2})^3 = \frac{1}{2},$$

and

$$P(B|\ n = 3) = \binom{3}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^2 + \binom{3}{2} \cdot (\frac{1}{2})^2 \times \frac{1}{2} = \frac{3}{4}.$$

The probability of their joint occurrence is

$$P(AB|\ n=3) = \binom{3}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = P(A|\ n=3) \cdot P(B|\ n=3).$$

So the given events are independent.
For $n = 4$, the probabilities of the given events, respectively, are

$$P(A|\ n=4) = 1 - P(A^c|\ n=4) = 1 - \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 - \binom{4}{0} \cdot (\frac{1}{2})^4 = \frac{11}{16},$$

and

$$P(B|\ n=4) = \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 + \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^2 = \frac{10}{16}.$$

The probability of their joint occurrence is

$$P(AB|\ n=4) = \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 + \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^2 = \frac{10}{16} \neq \frac{11}{16} \cdot \frac{10}{16} = P(A|\ n=4) \cdot P(B|\ n=4).$$

So the given events are *not* independent.

Problem 4.2.16. (10 points) In a small town there are 40 taxis, numbered 1 to 40. Three taxis arrive at random at a station to pick up passengers. What is the probability that the number of at least one of the taxis is less than 5 ?

Solution:
Let X be the minimum of the three numbers.
Then

$$P(X < 5) = 1 - P(X \geq 5) = 1 - \frac{\binom{36}{3}}{\binom{40}{3}} = 0.277.$$

Problem 4.3.8. (10 points) From 18 potential women jurors and 28 potential men jurors, a jury of 12 is chosen at random. Let X be the number of women selected. Find the probability mass function of X.

Solution:
Let p be the probability mass function of X.
Then

$$p(i) = P(X = i) = \frac{\binom{18}{i} \cdot \binom{28}{12-i}}{\binom{46}{12}},\ i = 0, 1, 2, \ldots, 12.$$

Problem 4.4.14. (10 points) The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < -3 \\ 3/8 & -3 \le x < 0 \\ 1/2 & 0 \le x < 3 \\ 3/4 & 3 \le x < 4 \\ 1 & x \ge 4. \end{cases}$$

Calculate $E(X)$, $E(X^2 - 2|X|)$, and $E(X|X|)$.

Solution:
$p(x)$ the probability mass function of X is given by

x	-3	0	3	4
$p(x)$	$3/8$	$1/8$	$1/4$	$1/4$

Hence

$$\begin{aligned} E(X) &= -3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{5}{8}, \\ E(X^2) &= 9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{77}{8}, \\ E(|X|) &= 3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{23}{8}, \\ E(X^2 - 2|X|) &= \frac{77}{8} - 2 \cdot \frac{23}{8} = \frac{31}{8}, \\ E(X|X|) &= -9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{23}{8}. \end{aligned}$$

Problem 4.5.6. (10 points) Let X be a discrete random variable with the set of possible values $\{x_1, x_2, \ldots, x_n\}$; X called a **discrete uniform random variable** if

$$P(X = x_i) = \frac{1}{n},\ 1 \le i \le n.$$

Find $E(X)$ and $Var(X)$ for the special case, where $x_i = i,\ 1 \le i \le n$. Note that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution:

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} (i \cdot \frac{1}{n}) = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}, \\ Var(X) = E(X^2) - [E(X)]^2 &= \sum_{i=1}^{n} (i^2 \cdot \frac{1}{n}) - (\frac{n+1}{2})^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 \\ &= \frac{n+1}{2}(\frac{2n+1}{3} - \frac{n+1}{2}) = \frac{n+1}{2} \cdot \frac{n-1}{6} = \frac{n^2-1}{12}. \end{aligned}$$

Problem 4.6.1 (10 points) Mr. Norton owns two appliance stores. In store 1 the number of TV sets sold by a salesperson is, on average, 13 per week with a standard deviation of 5. In store 2 the number of TV sets sold by a salesperson is, on average, 7 with a standard deviation of 4. Mr. Norton has a position open for a person to sell TV sets. There are two applicants. Mr. Norton asked one of them to work in store 1 and the other in store 2, each for one week. The salesperson in store 1 sold 10 sets, and the salesperson in store 2 sold 6 sets. Based on this information, which person should Mr. Norton hire?

Solution:
Let X_1 be the number of TV sets the salesperson in store 1 sells and X_2 be the number of TV sets the salesperson in store 2 sells. After standardizing X_1 and X_2, we have $X_1^* = (10 - 13)/5 = -0.6$ and $X_2^* = (6 - 7)/4 = -0.25$. Therefore, the number of TV sets the salesperson in store 1 sells is 0.6 standard deviations below the mean, whereas the number of TV sets the salesperson in store 2 sells is 0.25 standard deviations below the mean. So Mr. Norton should hire the salesperson who worked in store 2.

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)