國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 1: sample space

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Why probability

A systematic (mathematical) way of describing our ‘belief’ or the ‘likelihood’ of event occurrences that helps us make decisions

- Especially important in real life:
  - *Randomness, noise, unknown rule*
- Engineering applications
  - Detection (e.g., radar)
  - Communication (e.g., noise effect)
  - Prediction/recognition (e.g., statistical model)
- Other fields: almost everything in life
  - Gambling
  - Store management
  - Economics
  - Risk management
  - etc

Lecture Outline

- Reading: Section 1.1, 1.2
- Probabilistic mathematical model for:
  - Reasoning about uncertainty
  - Developing approaches to inference problems
- Probabilistic models
  - Sample space
  - Probability law
- Axioms of probability
- Simple examples

What is a sample space?

We do an (random) experiment ->

A list (set) of all possible things (outcomes) that may happen during this experiment

e.g., flipping a coin (H & T)

Sample Space Ω

- “List” (set) of possible outcomes

- Requirements:
  - Mutually exclusive
  - Collectively exhaustive

- Art: to be at the “right” granularity
  - Pick (decide) appropriate sample space for your problem
  - Let’s think about ‘*coin-flipping*’ exp. again
  - Real world, real life

Sample space: discrete example

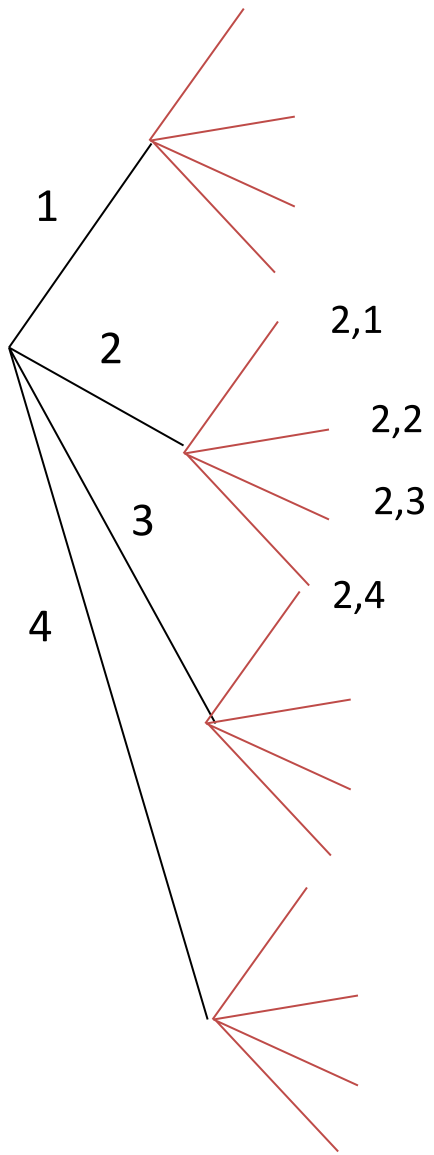
- Two rolls of a tetrahedral die (4-sided)
  - Two rolls as 1 single experiment
- Sample space of this experiment?

Second roll

		1	2	3	4
	1				
First roll	2		{2,2}		
	3		{3,2}		
	4				

– Sample space: 16 elements – a finite set

Tree-diagram of sample space



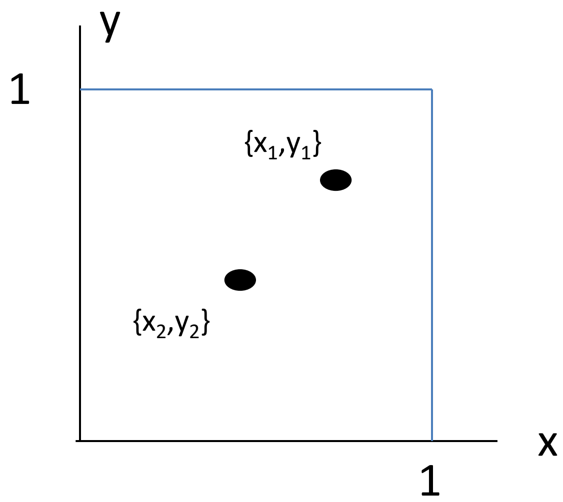
Sequential description

Relation to previous diagram?
Each *outcome* corresponds to each path (leaves) on the tree diagram

Subtly of wordings ('result' not 'outcome')
Outcome = (2, 1)
Experiment consists of 'stages'

Sample space: infinite example

$$\Omega = \{0 \leq x, y \leq 1\}$$



Darts-playing

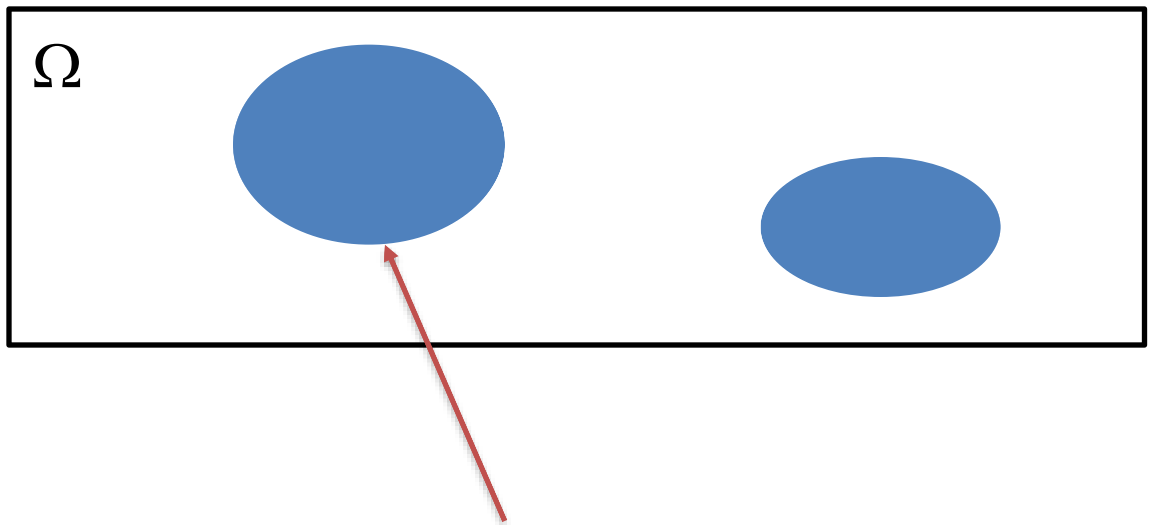
Experiment:
Randomly fall inside the unit square <u>only</u>

Outcomes:
All possible point in unit square are possible outcomes

A typical question raised:
Which outcome is more likely to occur compared to others?

Assign probabilities to individual outcome?
- Not really
- Any single point (real value) with infinite precision
- Zero probability

Instead, assign probabilities to subsets of sample space



Assign a numerical number to represent our belief on how likely this subset would occur

- Subset of sample space : **EVENTS**
- Outcome is a *'point'*, and 'random',
  - If occurred inside the event A , then event A occurred
- Assign probability to events
- How?

Axioms of probability

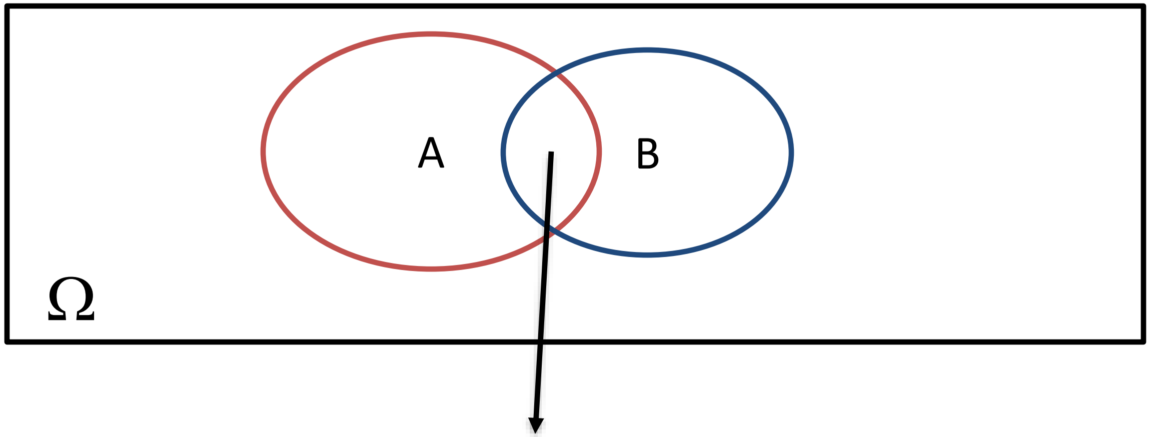
Event: a subset of sample space (collection of elements)

Rule of Probability: Probability is assigned to events

Axioms:

- Non-negativity: $P(A) \geq 0$
- Normalization: $P(\Omega) = 1$
- Additivity

Collectively exhaustive

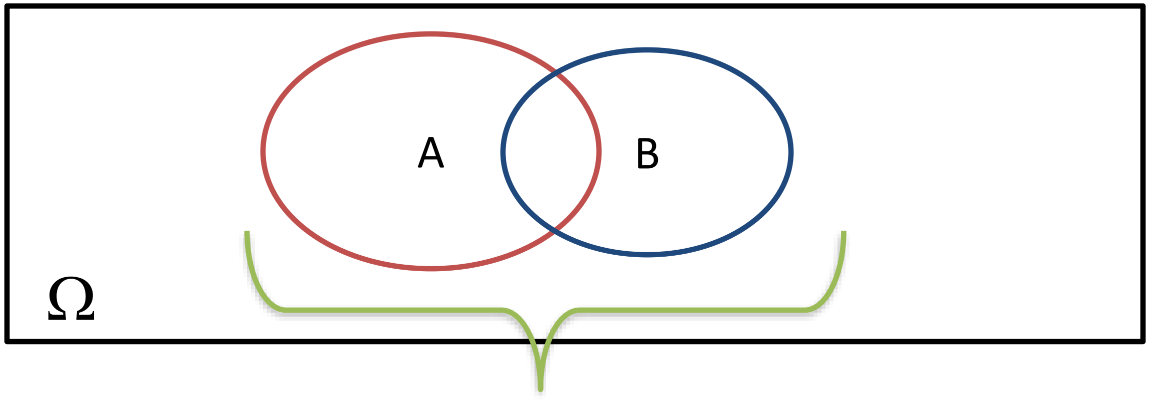


The intersection of A and B

$A \cap B$

An outcome falls in this regions:

Event A **AND** Event B has occurred



The union of A and B

$A \cup B$

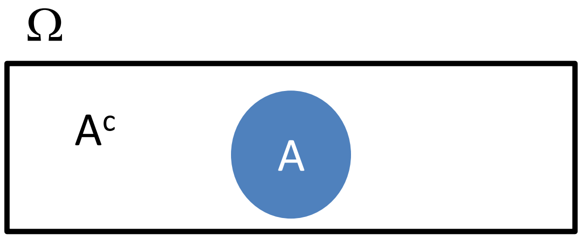
An outcome falls in this regions:

Event A **OR** Event B has occurred

- Axiom 3
  - Additivity
  - If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
  - Imagining probability behaves like mass

Interesting point

No axiom on $P(A) \leq 1$?



Let's try something:

Axiom 2: $P(\Omega) = 1$

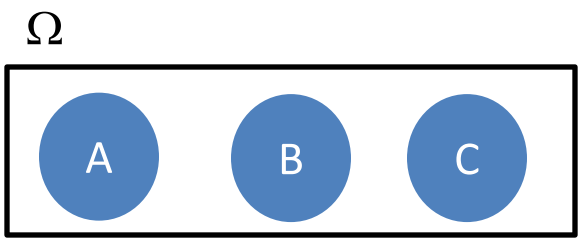
Ω Consists of A and A^c (A complement)

Axiom 3: $P(\Omega) = P(A) + P(A^c) = 1$
Axiom 1: $P(A) \geq 0$

Now we get,

$$P(A) \leq 1$$

Trivial example



$$P(A \cup B \cup C)\ ?$$

Simply, iteratively use axiom 2

$$A' = \{A \cup B\}$$

$$P(A \cup B \cup C) = P(A' \cup C)$$

$$\dots$$

$$= P(A) + P(B) + P(C)$$

Generalize this to ***n*** **disjoint sets**

Special case of this axiom for finite set sample space

- Form 1-element set
  - (note! probability assigned to set)
  - 1-element set(event) is a single outcome

Single element set

$$P(\{s_1, s_2, s_3, \dots, s_k\}) = P(\{s_1\}) + \cdots + P(\{s_k\})$$
$$= P(s_1) + \cdots + P(s_k)$$

Individual outcome

Weird sets exist in a given sample space?
Yes, e.g., the square sample space
Non-visualizable, impossible to assign probability to it – very subtle mathematical point

Probability law: assign probability to each outcome examples with finite sample space

	1	2	3	4
1				
2				
3				
4				

Assign a **probability law (arbitrarily):** 1/16 for each outcome

- $P((X,Y)\text{is }(1{,}1)\text{ or }(1{,}2))$
  - 1/16 + 1/16
- $P(\{X=1\})$ (reads as: the set of all outcomes such that x is equal to 1)
  - 4/16
- $P(X\ +\ Y\text{ is odd})$
  - Eight of them (8/16)
- $P(\min(X,Y)=2)$
  - Let’s go to the diagram (5/16)

Now you have learned it all: Steps to make probabilistic reasoning

- Setup sample space
- Statement about probability law
- Identify events, and calculate probability for those outcomes of interest

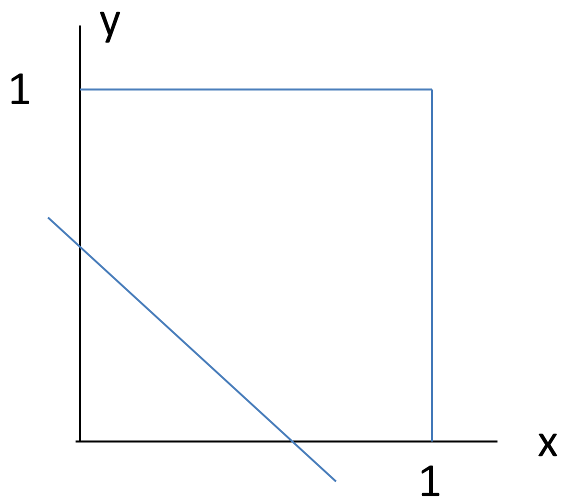
The previous example?
Discrete uniform probability law

- Discrete uniform law
  - Boils down to counting
  - Counting can be extremely complicated really fast

- Discrete uniform law
  - Let all outcomes be equally likely
  - Fair coin, fair dice, well-shuffled cards
  - Then

$$P(A) = \frac{\text{number of elements of A}}{\text{total number of sample points}}$$

Continuous uniform law



Assign a probability law:

- Probability law = area of interest
  - This means:
    two subsets of sample space with equal area, it is equally likely that the outcome with fall into one area vs. the other area

- $P\left(X + Y \leq \frac{1}{2}\right) = 1/8$
- $P\big((X,Y) = (0.5{,}0.3)\big) = 0$

Moral of the story:
- Once probability law in hand, it’s easy
- The hard part really comes from calculus and algebra (not from probability)

Third axiom needs to be strengthened

Think of a new experiment:

- You keep flipping a coin and you wait until you obtain heads for the first time

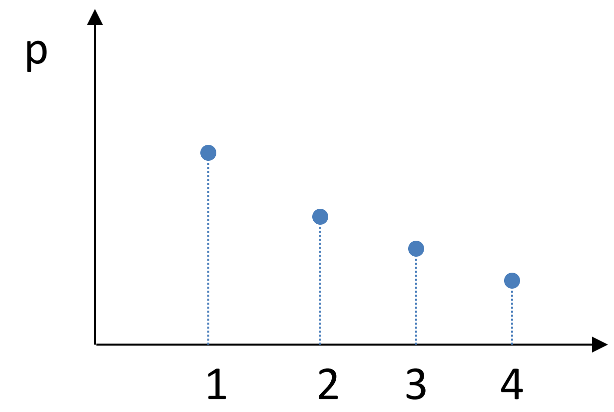
Sample space:

- Outcomes of this experiment is integer (with no upper bounds)
- Set of all possible integers

Let's do this as example

Sample space: {1,2,…}

- Probability law is given:
- $P(n) = 2^{-n},\ n = 1,2,3 \ldots$
- Find P(outcome is even)



Probability of subset that includes just even numbers

- e.g., 2, 4, 6, 8, …, so on
- Add up infinite geometric sequences: 1/3

Can we do it directly from axiom 3?

Axiom 3 applies only for countable finite set of sample space
Here, infinite collections

$P(A\cup B \cup C)$?

Simply, iteratively use axiom 2

$$A' = \{A \cup B\}$$
$$P(A \cup B \cup C) = P(A'\cup C)$$
$$\dots$$
$$= P(A) + P(B) + P(C)$$

Generalize this to ***n* disjoint sets**

Need one more,

Countable additivity axiom

If A1, A2, …are disjoint then:

$$P(A_1 \cup A_2 \cup A_3 \cup, \dots) = P(A_1) + P(A_2) + \cdots$$

Sequence of sets (events), occurred in order, and countable, probability on sequence of sets getting smaller smaller (disjoint) -> you can just add