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電磁學（一）
Electromagnetics (I)

9. 邊界值問題

Boundary-value Problems

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In this lecture, we will learn to solve electrostatic problems subject to boundary conditions.

- **9.1 Poisson ＆ Laplace equations**
  Poisson 與 Laplace 方程式
- **9.2 Method of image charge 鏡像電荷方法**
- 9.3 Point image charge 點鏡像電荷
- 9.4 Line image charge 線鏡像電荷
- 9.5 Review 單元回顧

邊界值問題
Boundary-value Problems

9.1 Poisson 與 Laplace 方程式
Poisson & Laplace Equations

Poisson's Equation & Laplace Equation

Recall, the two postulates for electrostatics

$$\left.\begin{aligned} &\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V \\ &\nabla \cdot \vec{D} = \rho \end{aligned}\right\}$$

Poisson's Equation

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

In a charge-free region, $\nabla^2 V = -\cancelto{0}{\frac{\rho}{\varepsilon}} \Rightarrow \nabla^2 V = 0$

Laplace Equation

Laplacian Operator

In Cartesian coordinates, $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

In cylindrical coordinates, $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

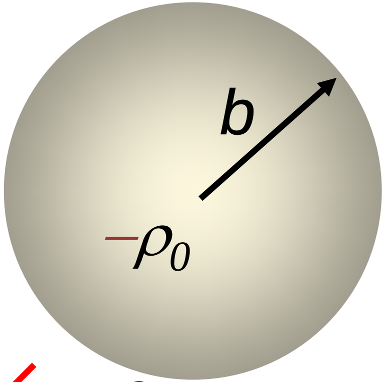
In spherical coordinates,

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Given $\nabla^2 V = -\frac{\rho}{\varepsilon}$ ***and*** $\nabla^2 V = 0$***,***

problems in electrostatics can be solved from known boundary conditions.

E.g. For a ball of radius *b* having a uniform volume charge density of $-\rho_0$, find the electric field intensity inside the ball.



First, identify the boundary conditions

i. At $R = 0$, $E = 0$, ii. At $R \to \infty$, $E \,\&\, V \to 0$

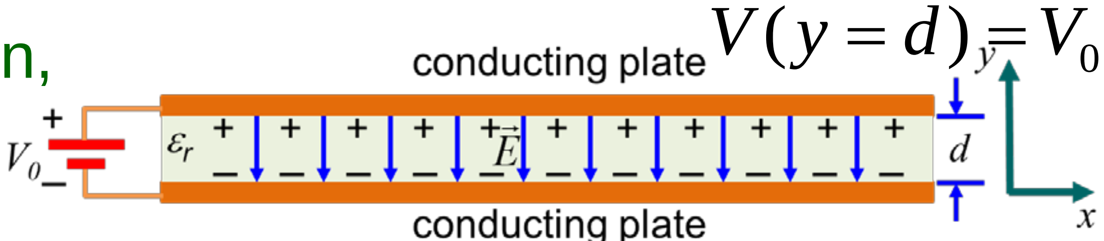
ii. Use the Poisson's equation $\nabla^2 V_i = -\dfrac{\rho}{\varepsilon}$

$$\Rightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V_i}{\partial R}\right) = \frac{\rho_0}{\varepsilon_0} \quad \Rightarrow \frac{dV_i}{dR} = \frac{\rho_0}{3\varepsilon_0}R + \cancelto{0}{\frac{C_1}{R^2}} = \frac{\rho_0}{3\varepsilon_0}R$$

Take $C_1 = 0$ to avoid $E_R \propto \dfrac{dV_i}{dR} \to \infty$ when $R \to 0$.

$$\Rightarrow \vec{E}_i = -\nabla V = -\frac{\rho_0}{3\varepsilon_0} R\hat{a}_R$$ (same solution obtained from Gauss Law. Verify it!)

E.g. Refer to the following plot. Given V_0 and d, find V and E in the parallel-plate capacitor.



1. In a charge-free region, find V from $\nabla^2 V = 0$

$$\nabla^2 V = \frac{\partial^2 V}{\partial y^2} = 0 \Rightarrow V = C_1 y + C_2$$

2. Apply boundary condition (1) $V(y=0)=0 \Rightarrow C_2 = 0$

Apply boundary condition (2) $V(y=d)=V_0 \Rightarrow C_1 = \dfrac{V_0}{d}$

Final solution $\boxed{V = \dfrac{V_0}{d} y}$

3. Find E from $\vec{E} = -\nabla V \Rightarrow \quad \vec{E} = -\nabla V = -\dfrac{V_0}{d}\hat{a}_y$

9.1 Poisson & Laplace 方程式

Poisson & Laplace Equations

- Poisson's equation – $$\nabla^2 V = -\frac{\rho}{\varepsilon}$$
- Laplace equation (in a charge-free region) – $$\nabla^2 V = 0$$
- Electric potential and field are solved from both equations subject to boundary conditions.

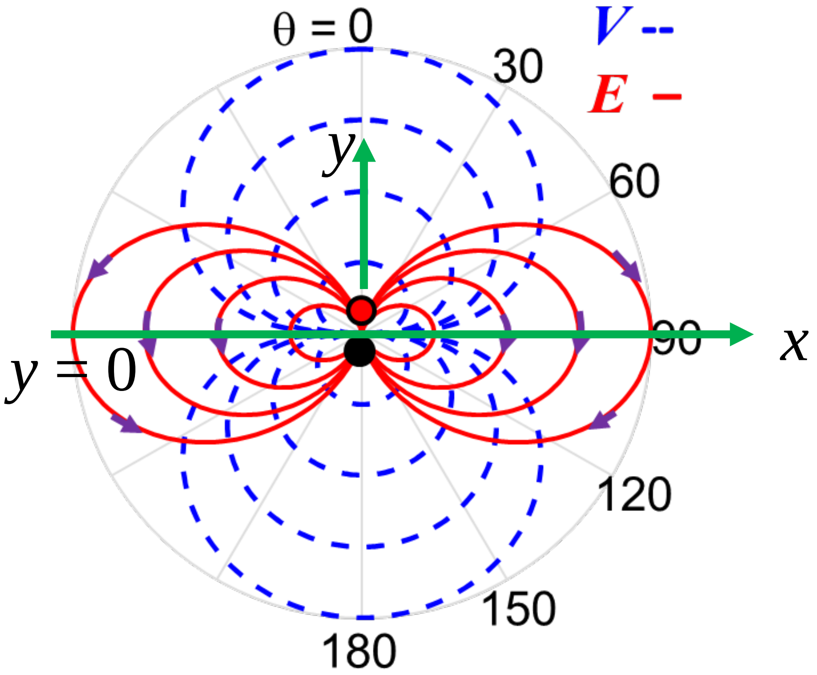
邊界值問題
Boundary-value Problems

9.2 鏡像電荷方法
Method of Image Charge

Method of Image Charges

matching boundary conditions by creating image charges with known solutions to obtain an electrostatic solution

Recall the electric dipole



Boundary Conditions (B.C.'s)

$$V(x, y=0, z) = 0$$

$$V(x, y, z) = V(-x, y, z)$$

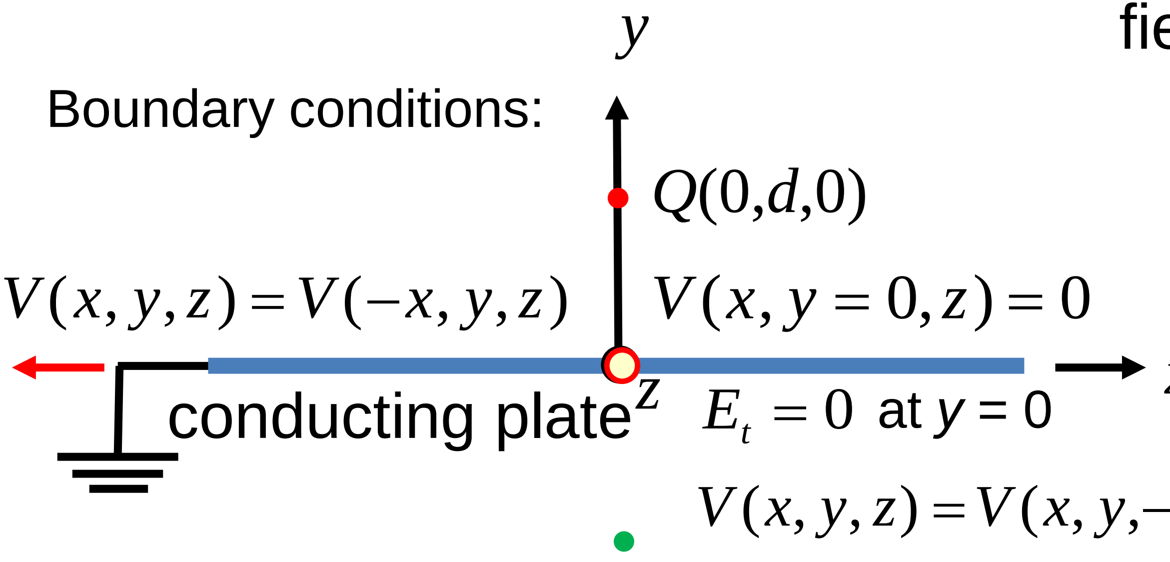
$$V(x, y, z) = V(x, y, -z)$$

$E_t = 0$ (tangential field) at $y = 0$

Consider the example

Replace the problem with the electric dipole for the field above the plate

Boundary conditions:



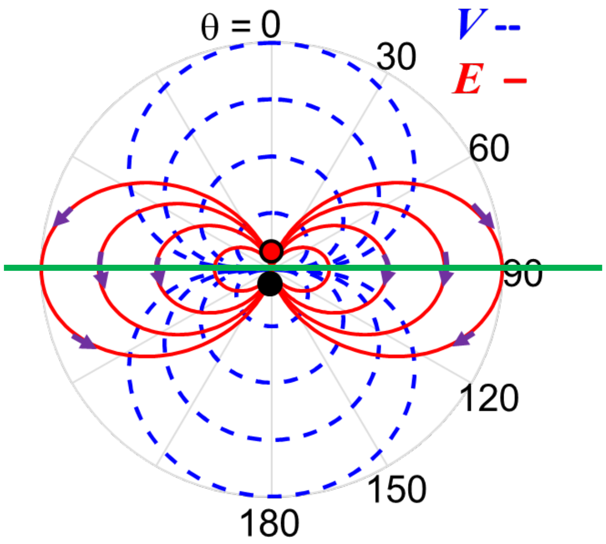
$$V(x,y,z)=V(-x,y,z)$$

$$V(x,y=0,z)=0$$

$E_t=0$ at $y=0$

$$V(x,y,z)=V(x,y,-z)$$

Image charge: $Q_i=-Q(0,-d,0)$



Matching the boundary conditions by creating an imaging charge $-Q$ at $y = -d$

Now, the field quantities above the grounded conduction can be calculated with ease.

At $\mathrm{P}(x,y,z)$, the electric potential is

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

where

$$R_\pm = \left[x^2 + (y \mp d)^2 + z^2\right]^{1/2}$$

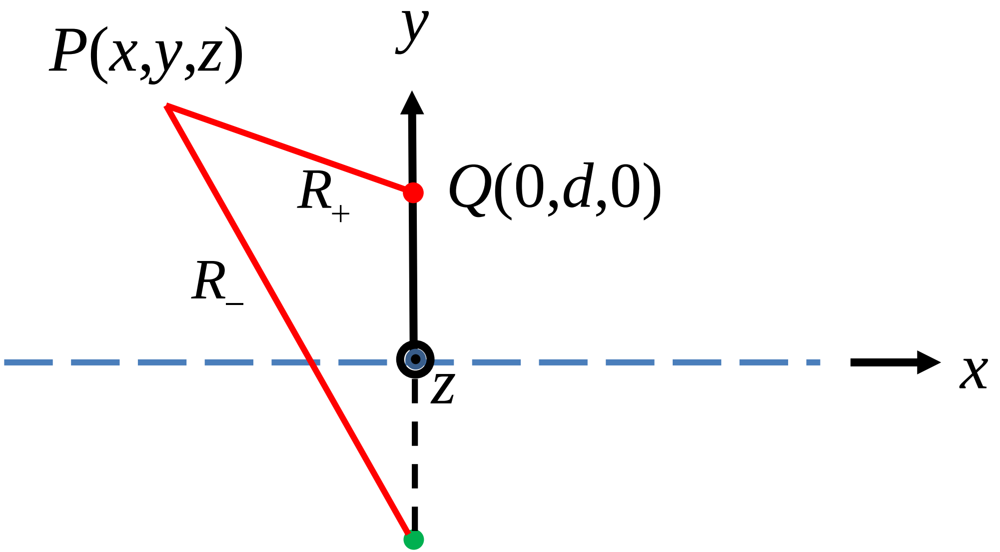


Image charge: $Q_i = -Q(0,-d,0)$

Electric field is $\vec{E} = -\nabla V$

Surface charge on the conducting plate is

$$\hat{a}_{n2} \cdot \vec{D} = \rho_s$$

9.2 鏡像電荷方法

Method of Image Charge

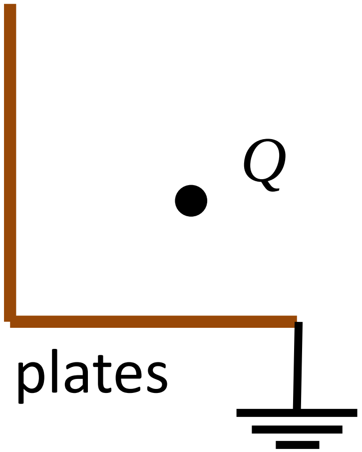
- The solution to an electrostatic problem is **unique** subject to **boundary conditions**.
- Given boundary conditions, one can solve an electrostatic problem by arranging **image charges** in space to satisfy the boundary conditions and solve the problem by using **known solutions** from the image charges.

邊界值問題
Boundary-value Problems

9.3 點鏡像電荷

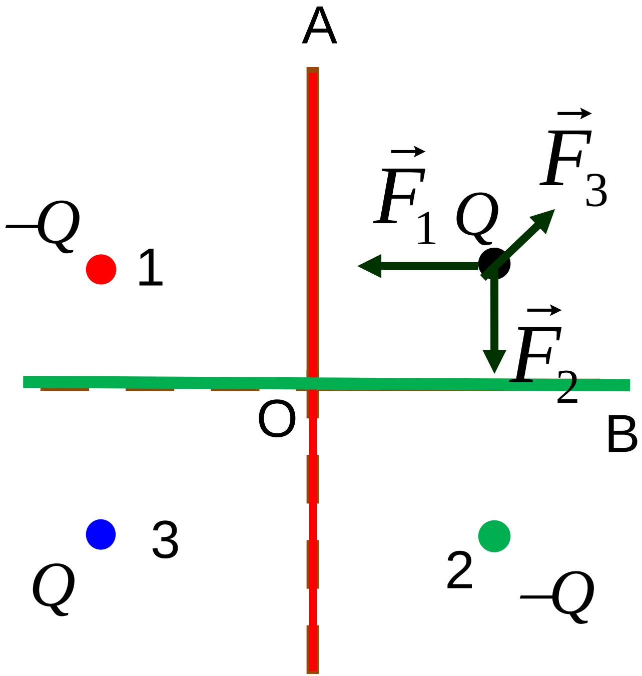
Point Image Charge

E.g. Find the force on Q for the following setup



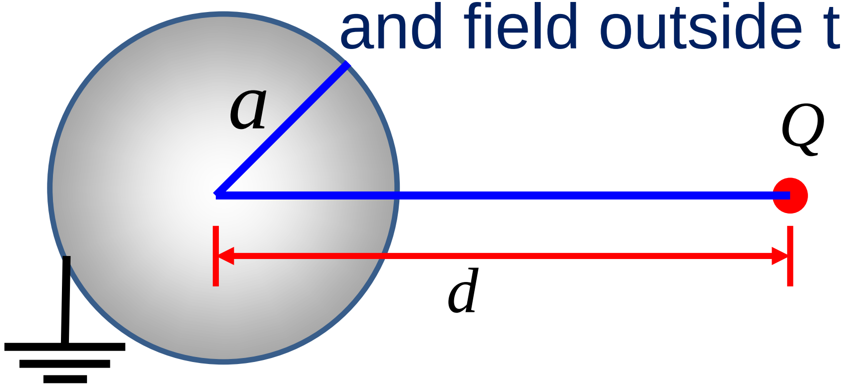
Grounded conducting plates with V = 0

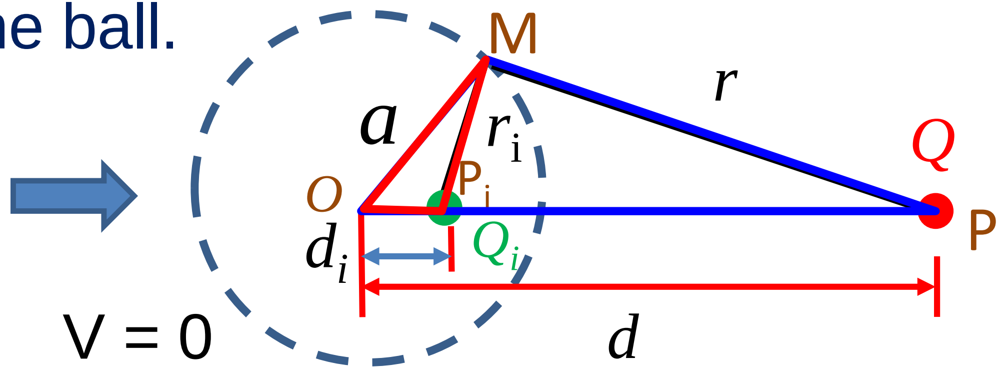
Equivalent problem



The force on Q is therefore the vector sum of F_1, F_2, and F_3

E.g. (1) For a point charge at d from the center of a grounded conducting ball, find the electric potential and field outside the ball.





Due to grounding,

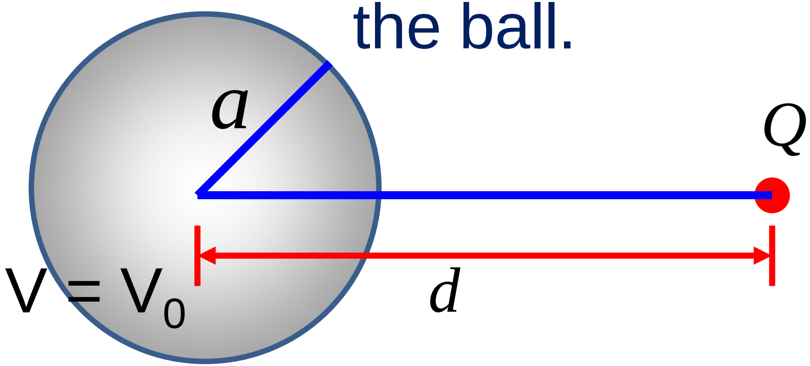
Put an image charge of Q_i at d_i

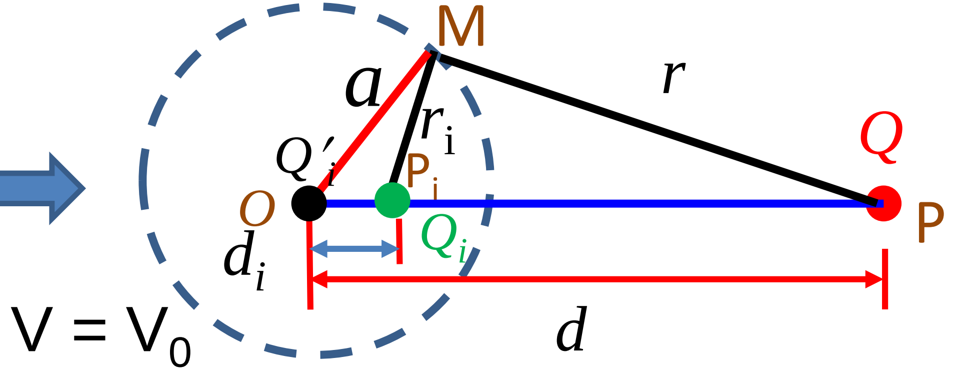
$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r}+\frac{Q_i}{r_i}\right) = 0 \;\Rightarrow\; \frac{r_i}{r} = -\frac{Q_i}{Q} = \text{const.} \;\Rightarrow\; Q_i = -Q\frac{a}{d}$$

Choose d_i such that $\underline{\Delta OPM} \sim \underline{\Delta OMP_i}$ $\Rightarrow \dfrac{r_i}{r} = \dfrac{d_i}{a} = \dfrac{a}{d} = \text{const.} \Rightarrow d_i = \dfrac{a^2}{d}$

The electric potential and field outside the ball can then be solved from Q, Q_i.

E.g. (2) Suppose the conducting ball is maintained at V_0, find the electric potential and electric field outside the ball.





Keep Q_i at d_i and installed another Q'_i at O

$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r}+\frac{Q_i}{r_i}\right)+\frac{1}{4\pi\varepsilon_0}\frac{Q'_i}{a} = 0 + V_0 \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{Q'_i}{a} = V_0$$

$$\text{where } \frac{r_i}{r} = -\frac{Q_i}{Q} = \text{const.} \quad d_i = \frac{a^2}{d}, Q_i = -Q\frac{a}{d} \quad \Rightarrow Q'_i = V_0 \times 4\pi\varepsilon_0 a$$

The electric potential and field outside the ball can be solved from Q, Q_i, and Q'_i.

9.3 點鏡像電荷

Point Image Charge

- The electric potential and field of a point charge are well known.
- If one could generate the boundary conditions of a problem by arranging point image charges properly, the solution to the electrostatic problem can be greatly simplified by using the solutions of point charges.

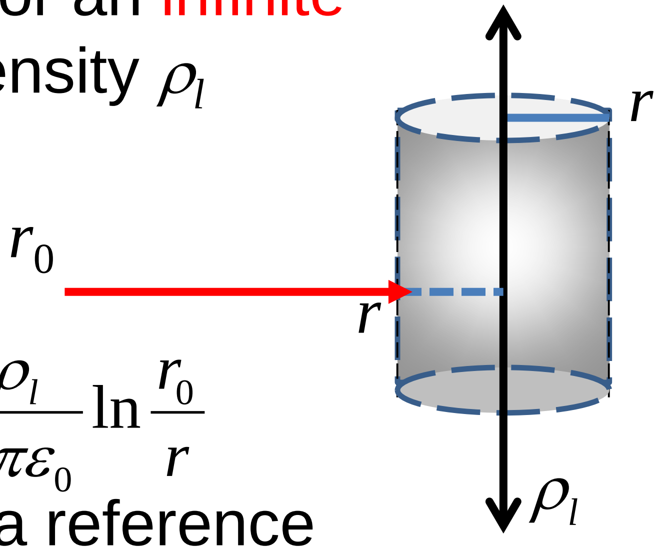
邊界值問題
Boundary-value Problems

9.4 線鏡像電荷
Line Image Charge

Solutions of an Infinite Line Charge

Recall the electric field at r for an infinite line charge with a charge density ρ_l

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$$



The potential at r is

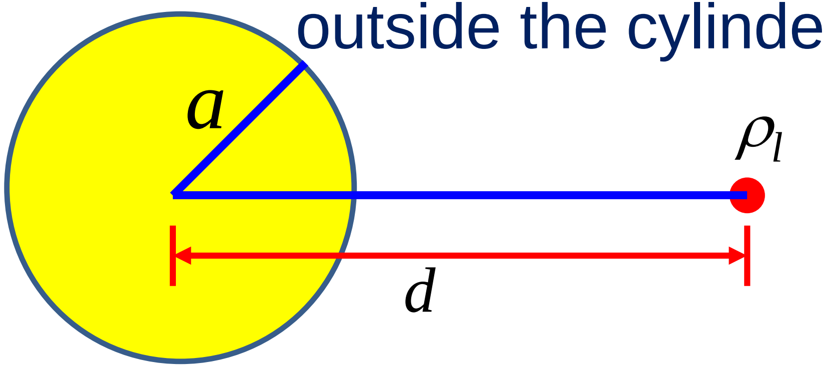
$$V(r) - V(r_0) = -\int_{r_0}^{r} E_r\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

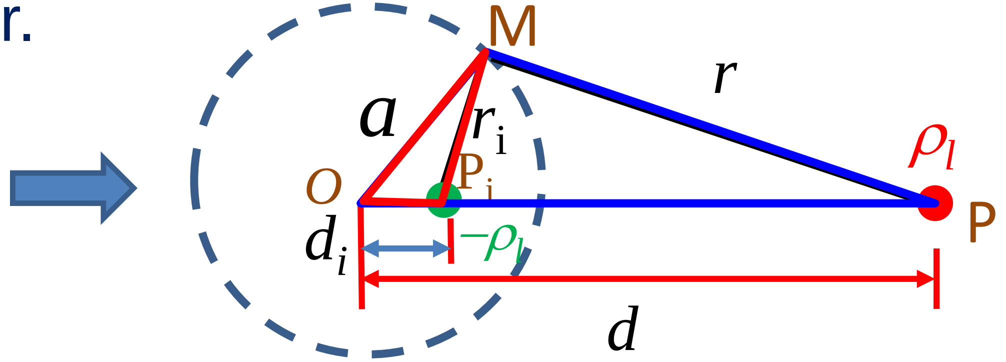
where r_0 is the location at which a reference potential $V(r_0)$ is given.

Notes: 1. an electric potential is the work done on moving a charge between two points – a relative quantity.

2. Since the charge is extended to infinity, we can’t claim $V = 0$ at $r = \infty$

E.g. For a line charge at d from the center of a conducting cylinder, find the electric potential and field outside the cylinder.





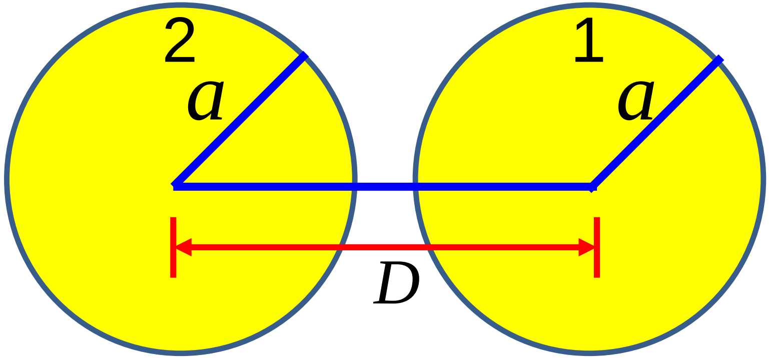
On the conducting surface   Put a line image charge of $-\rho_l$ at d_i

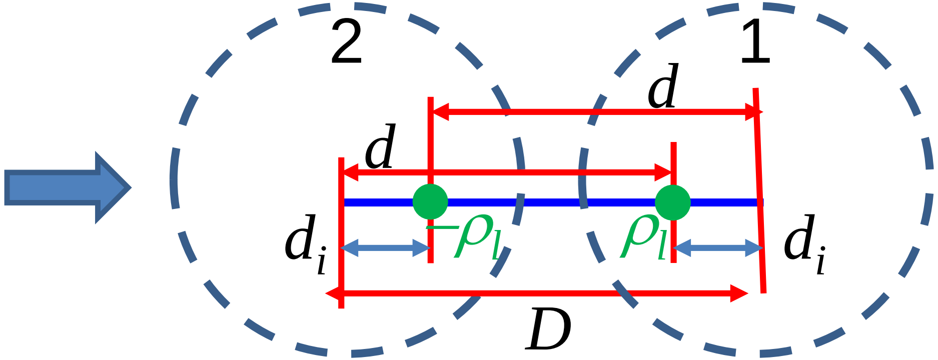
$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r} = \text{constant} \Rightarrow \frac{r_i}{r} = \text{const.}$$

Choose d_i such that $\Delta\text{OPM} \sim \Delta\text{OMP}_\text{i} \Rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{const.} \Rightarrow d_i = \frac{a^2}{d}$

The electric potential and field outside the cylinder can be solved from ρ_l, $-\rho_l$.

E.g. Find the capacitance of the following two-wire transmission line (a and D are known parameters).





Potential on the 1st and 2nd cylinders are

$$V_{2,1} = \frac{\pm \rho_l}{2\pi\varepsilon_0} \ln \frac{a}{d}$$

Put two line image charges $\pm\rho_l$ to create the B.C. with $d_i = \dfrac{a^2}{d}$

However d is an unknown satisfying

$$d = D - d_i = D - \frac{a^2}{d}$$

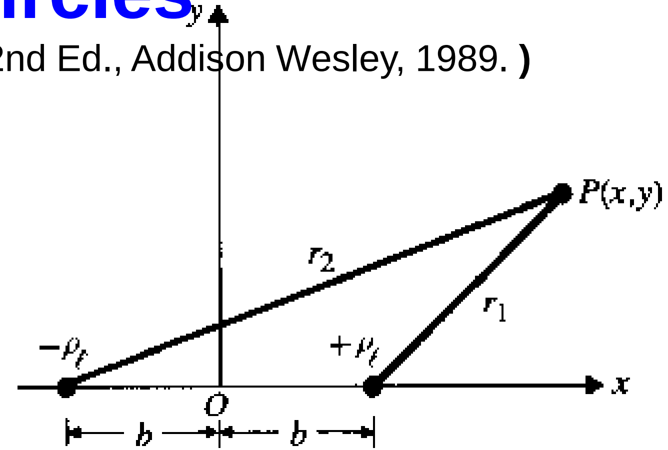
$$\Rightarrow d = \frac{1}{2}\left(D + \sqrt{D^2 - 4a^2}\right)$$

The capacitance/length is then

$$C_l = \rho_l / (V_1 - V_2) = \frac{\pi\varepsilon_0}{\ln(d/a)}$$

Equipotential circles

(David K. Cheng, Field and Wave Electromagnetics 2nd Ed., Addison Wesley, 1989.)

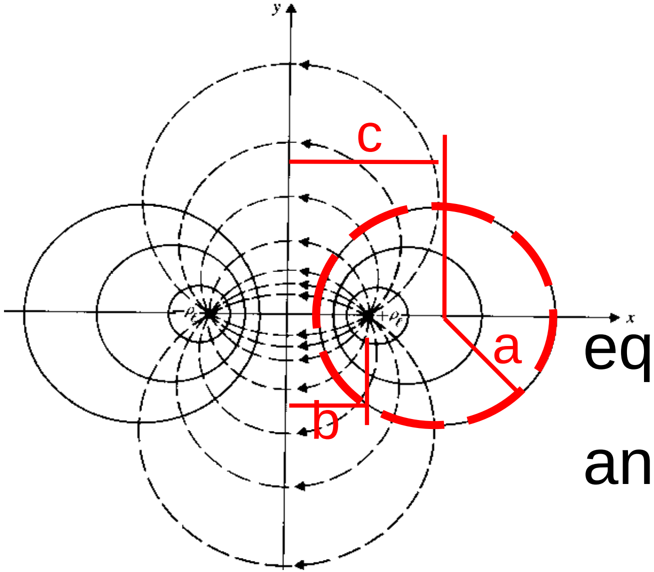


The potential at point *P* is $V_P = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{r_2}{r_1}$

Set V_p = constant to obtain

$$\frac{r_2}{r_1} = k \quad \Rightarrow \quad \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k$$

$$\Rightarrow \quad \Bigg(x - \underbrace{\frac{k^2+1}{k^2-1}b}_{x_0}\Bigg)^2 + y^2 = \left(\frac{2k}{k^2-1}b\right)^2$$



equipotential circles with radii of $a \equiv \left|\dfrac{2kb}{k^2-1}\right|$

and centers at $c \equiv x_0 = \dfrac{k^2+1}{k^2-1}b$, where $\boxed{c^2 = a^2 + b^2}$

Re-do the problem….

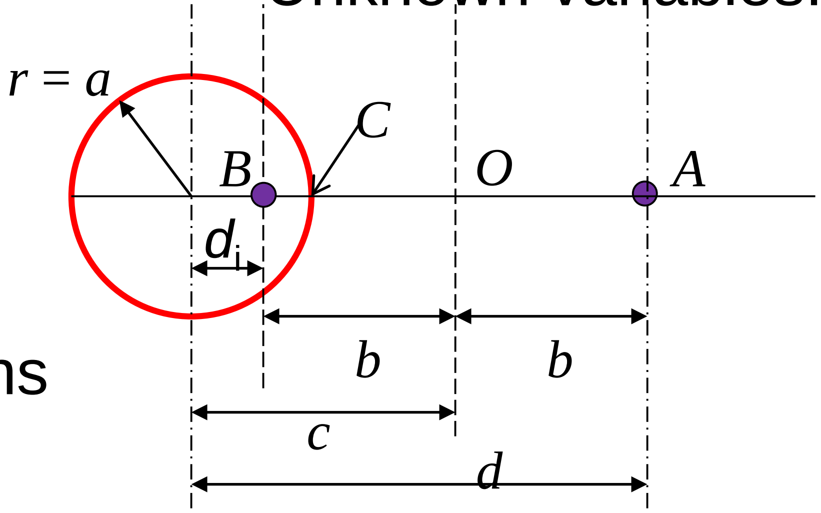
Known variables: d, a

Unknown variables: b, c, d_i

$$c - b = d_i$$
$$c + b = d$$

Use $c^2 = a^2 + b^2$ or

$$(c-b)(c+b) = a^2$$



Immediately, one obtains

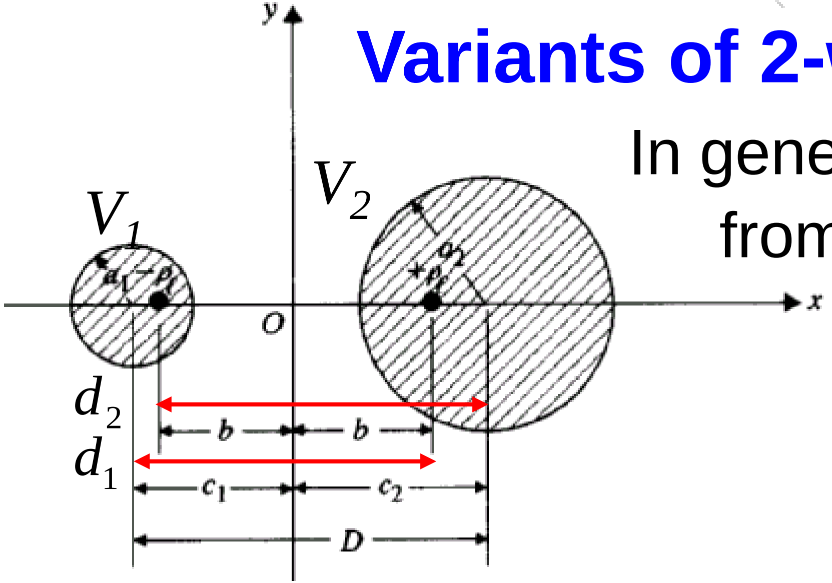
$$d_i = a^2 / d$$

The potential on the conducting cylinder is given by $V_P = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}$

Choose point C for calculation $\rightarrow$ $r_1 = \overline{AC} = d - a$

and $r_2 = \overline{BC} = a - d_i = (d-a)a/d$ $\Rightarrow V_P = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{a}{d}$

Variants of 2-wire Transmission Line

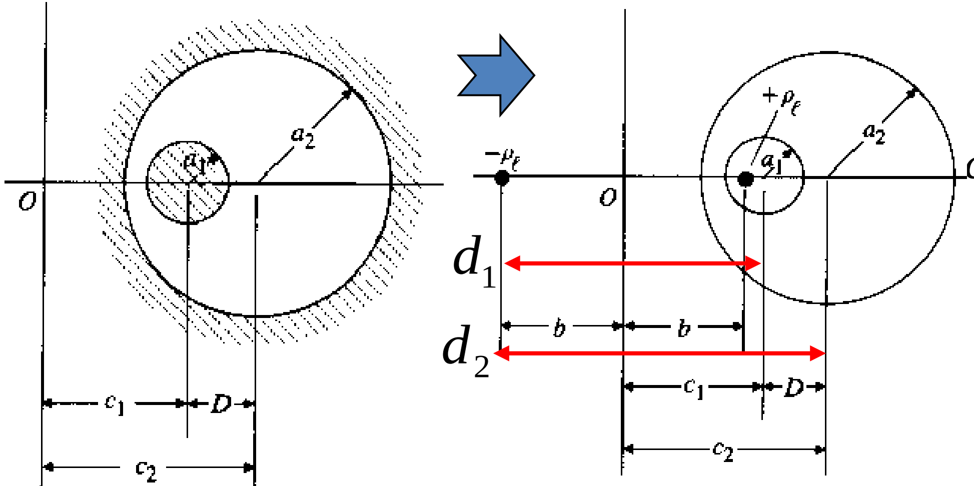


In general a_1, a_2, D are known. Find b, c_1, c_2

from $b^2 = c_1^2 - a_1^2,\ b^2 = c_2^2 - a_2^2,\ c_1 + c_2 = D$

$$V_1 = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{d_1} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{b+c_1}$$

$$V_2 = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{d_2} = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{b+c_2}$$



Again, a_1, a_2, D are known

$$b^2 = c_1^2 - a_1^2 \quad b^2 = c_2^2 - a_2^2 \quad c_2 - c_1 = D$$

$$c_1 = \frac{1}{2D}\left(D^2 + a_1^2 - a_2^2\right) \quad c_2 = \frac{1}{2D}\left(D^2 + a_2^2 - a_1^2\right)$$

$$V_1 = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{d_1} = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_1}{b+c_1}$$

$$V_2 = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{d_2} = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\frac{a_2}{b+c_2}$$

9.4 線鏡像電荷

Line Image Charge

- The electric field and relative potential of an infinite line charge can be derived easily.
- Problems involving two long wires of certain radii (transmission line) can often be replaced with properly arranged line image charges.
- Subject to the same boundary conditions, the problems are solved from the solutions of line charges.

邊界值問題
Boundary-value Problems

9.5 單元回顧 Review

單元回顧

1. The Poisson's equation is one governing the electric potential in a region with charges, given by

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

2. In a charge-free region, the Poisson's equation reduces to the so-called Laplace equation, given by

$$\nabla^2 V = 0$$

單元回顧

3.1 The solution of the electric potential, governed by the Poisson's equation and Laplace equation, are unique for a set of boundary conditions.

3.2 As a result, one can cleverly and properly arrange image charges to satisfy the prescribed boundary conditions of an electrostatic problem.

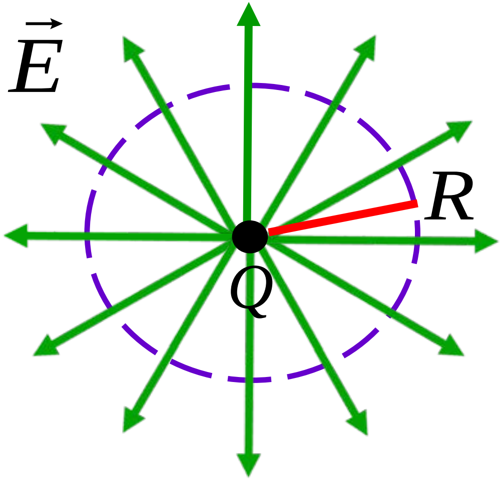
3.3 The electrostatic problem is then solved from the known solutions of the images charges.

單元回顧

4. The simplest image charge is a point charge of Q with the solutions

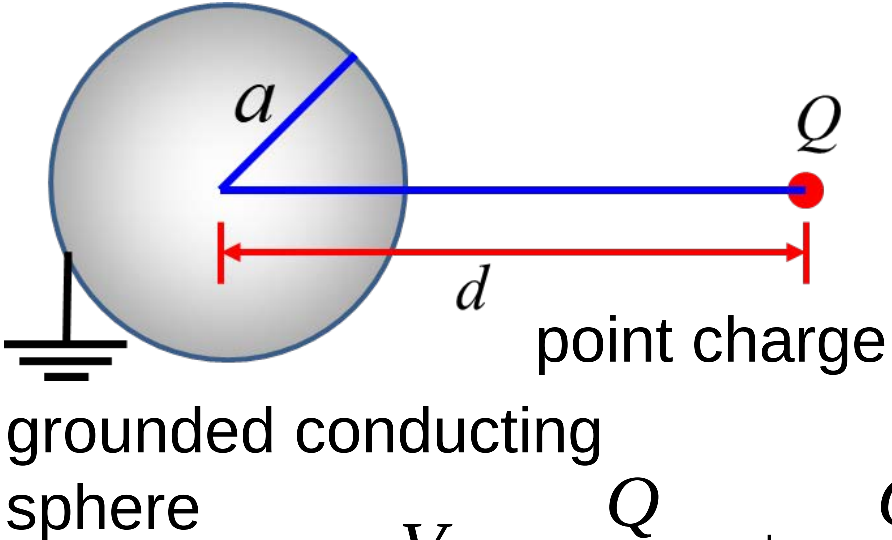
$$\vec{E} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R$$

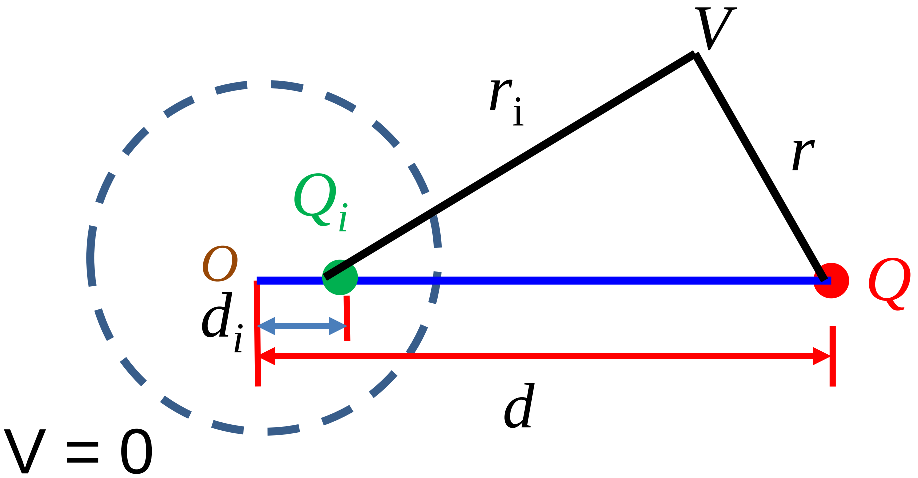
$$V = \frac{Q}{4\pi\varepsilon R}$$



單元回顧

5. With a point image charge Q_i, the left problem reduces to the right problem





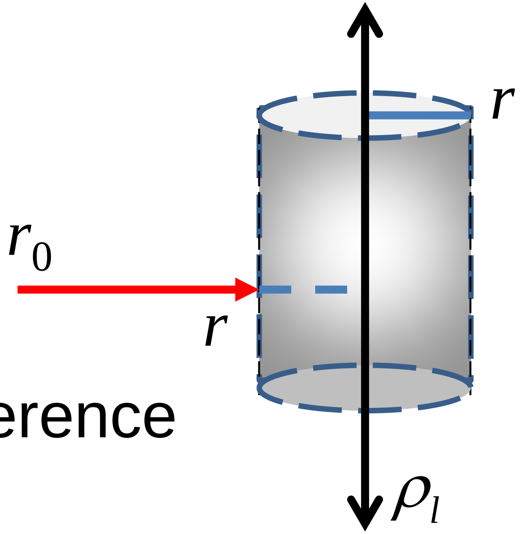
$$V = \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q_i}{4\pi\varepsilon_0 R_i}, \text{ where } Q_i = -Q\frac{a}{d}, \quad d_i = \frac{a^2}{d}$$

單元回顧

6. The solutions of an infinitely long line charge with a charge density of ρ_l can also be derived with ease, given by

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$$

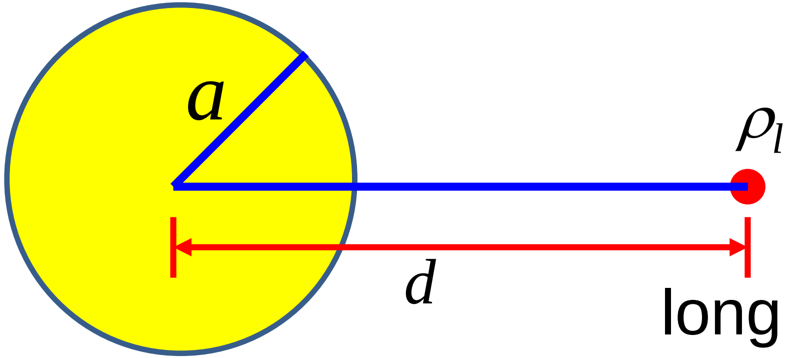
$$V(r) - V(r_0) = -\int_{r_0}^{r} E_r dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$



where r_0 is the location at which a reference potential is given.

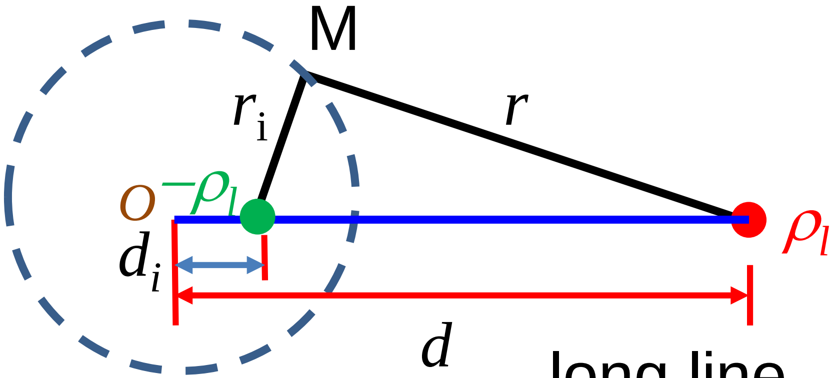
單元回顧

7. With a line image charge $-\rho_l$, the left problem reduces to the right problem



long line charge

long conducting cylinder



long line charge

$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}, \text{ with } d_i = \frac{a^2}{d}$$

THANK YOU FOR YOUR ATTENTION