H.W. 6

109060013 張芯瑜

(1) $\because (A, *)$ is semigroup

$\therefore *$ is an associative operation

$$
\begin{aligned}
\Rightarrow (a*b)*c &= a*b*c \\
&= a*(b*c) \\
&= a*(c*b) \quad (\because b*c = c*b) \\
&= a*c*b \\
&= (a*c)*b \\
&= (c*a)*b \quad (\because a*c = c*a) \\
&= c*a*b \\
&= c*(a*b) \quad \#
\end{aligned}
$$

(2) $\because (A, *)$ is monoid

$\therefore *$ is an associative operation also an identity element exists

$a, b \in A$, $a \neq b$

$\Rightarrow a*a = e$, $b*b = e$

$a*b \overset{?}{=} b*a$

$\Rightarrow b*a*b \overset{?}{=} (b*b)*a = e*a$

$\Rightarrow b*a*b*a \overset{?}{=} e*a*a = e*e = e$

$\Rightarrow (b*a)*(b*a) = e \overset{?}{=} e$

"$(b*a) \in A$"

$\Downarrow$

equal

$\therefore a*b = b*a$

(3) Pick an element $g \in G$ that is not an identity.

Case 1: g is a generator of $(G, \star)$

$\Rightarrow \{g, g^2, \cdots, g^{p^2}\} = G$

Assume $h = g^p \Rightarrow H = \{h, h^2, h^3, \cdots h^p\} \subset G$

$\therefore$ H is a subgroup of G with $|H| = p$

Case 2: g is not a generator of $(G, \star)$

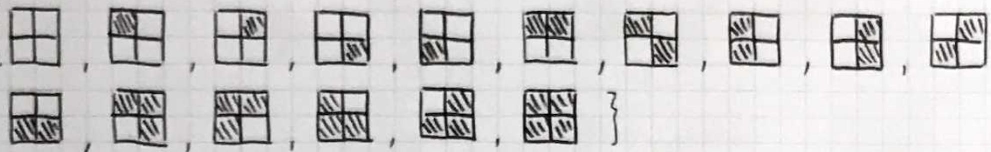
Let G' is a subgroup of G generated by g.

By Lagrange Thm., $|G'|$ must be 1 or p or p^2.

$\left(\begin{array}{l} 1 = \text{only } e, \text{ but } g \neq e \ (\divideontimes) \\ p^2 = \text{than } g \text{ is a generator of } (G, \star) \ (\divideontimes) \end{array} \right.$

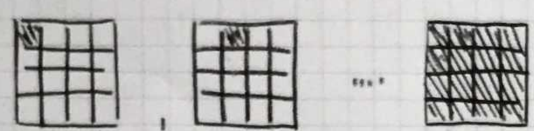
$\therefore |G'| = p$

$\therefore$ exists !!

(4a) $S = \{$  $\}$

$G_S = \{ (I), (90^\circ), (180^\circ), (270^\circ) \}$

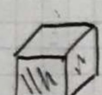
$$\Rightarrow \frac{2^4 + 2 + 2^2 + 2}{4} = \underline{6} \ \#$$

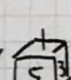
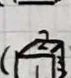
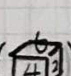
(4b) $S = \{$  $\}$ (num $= 2^{16}$)

$G_S = \{ (I), (90^\circ), (180^\circ), (270^\circ) \}$

$$\Rightarrow \frac{2^{16} + 2^4 + 2^8 + 2^4}{4} = \underline{2^3 + 2^6 + 2^{14}} \ \#$$

$$= \underline{16456} \ \#$$

(5) $S = \{$  .... $\}$ $\Rightarrow$ num $= n^6$.

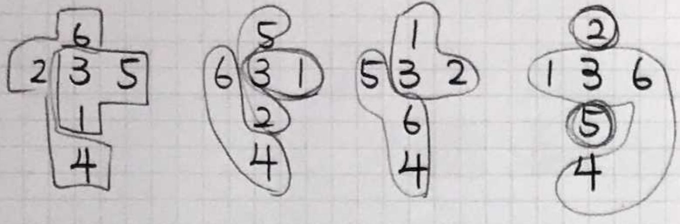
$G_S = \{$
1 on top: (), (90°), (180°), (270°)
2 on top: (), (90°), (180°), (270°)
⋮
6 on top: (), (90°), (180°), (270°) $\}$

对稱!!

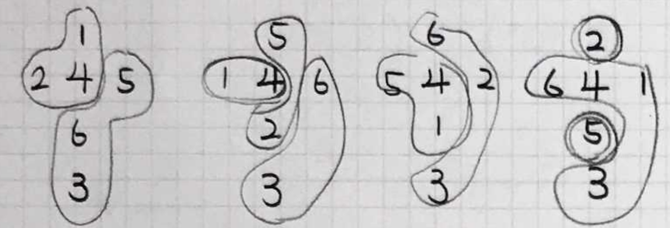
↳ # = 24

$$\Rightarrow \boxed{\overset{I}{\begin{array}{ccc} & 2 & \\ 4 & 1 & 3 \\ & 5 & \\ & 6 & \end{array}}} \quad \begin{array}{ccc} & 3 & \\ 2 & 1 & 5 \\ & 4 & \\ & 6 & \end{array} \quad \begin{array}{ccc} & 5 & \\ 3 & 1 & 4 \\ & 2 & \\ & 6 & \end{array} \quad \begin{array}{ccc} & 4 & \\ 5 & 1 & 2 \\ & 3 & \\ & 6 & \end{array} \quad (n^6, n^3, n^4, n^3)$$

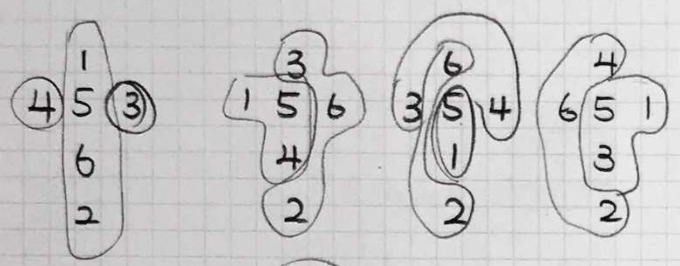
$$\begin{array}{ccc} & 6 & \\ 4 & 2 & 3 \\ & 1 & \\ & 5 & \end{array} \quad \begin{array}{ccc} & 3 & \\ 6 & 2 & 1 \\ & 4 & \\ & 5 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 3 & 2 & 4 \\ & 6 & \\ & 5 & \end{array} \quad \begin{array}{ccc} & 4 & \\ 1 & 2 & 6 \\ & 3 & \\ & 5 & \end{array} \quad (n^3, n^2, n^3, n^2)$$



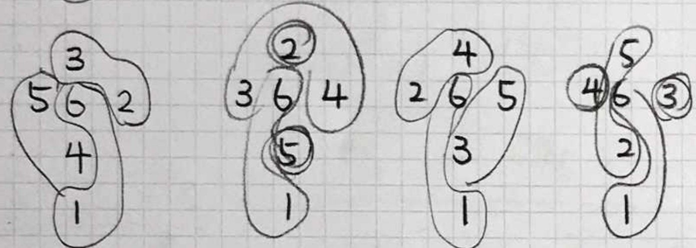
(n^2, n^3, n^2, n^3)



(n^2, n^3, n^2, n^3)



(n^3, n^2, n^3, n^2)



(n^3, n^4, n^3, n^4)

$$\Rightarrow \quad \# = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}$$

#