EE306002: Probability **Spring 2022**
Department of Electrical Engineering
National Tsing Hua University

Homework #5
Coverage: Chapters 8 and 9
Due date: May 26, 2022

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Problem 8.1.3 (10 points) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} k(x^2+y^2) & \text{if } (x,y) = (1,1),\ (1,3),\ (2,3), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of the constant k.

(b) Determine the marginal probability mass functions of X and Y.

(c) Find $E(X)$ and $E(Y)$.

Problem 8.1.14 (10 points) Let X be the proportion of customers of an insurance company who bundle their auto and home insurance policies. Let Y be the proportion of customers who insure at least their car with the insurance company. An actuary has discovered that for, $0 \le x \le y \le 1$, the joint distribution function of X and Y is $F(x,y) = x(y^2 + xy - x^2)$. Find the expected value of the proportion of the customers of the insurance company who bundle their auto and home insurance policies.

Problem 8.2.16 (10 points) Let X and Y be independent exponential random variables both with mean 1. Find $E[\max(X,Y)]$.

Problem 8.3.10 (10 points) The random variable Y is selected at random from the interval (0,1); the random variable X is then selected at random from the interval $(Y,1)$. Find the probability density function of X.

Problem 8.3.13 (10 points) The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} ce^{-x} & \text{if } x \ge 0,\ |y| < x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(c) Calculate $E(Y|X=x)$ and $Var(Y|X=x)$.

Problem 9.1.14 (10 points) Let $X_1, X_2, ..., X_n$ be identically distributed, independent, exponential random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that

$$E[\min(X_1, ..., X_n)] < \min\{E(X_1), ..., E(X_n)\}.$$

Problem 9.2.5 (10 points) Let $X_1, X_2, ..., X_n$ be a sequence of nonnegative, identically distributed, and independent random variables. Let F be the distribution function of X_i, $1 \leq i \leq n$. Prove that

$$E[X_{(n)}] = \int_0^\infty (1 - [F(x)]^n)dx.$$

Problem Ch9-Review 8 (10 points) A system consists of n components whose lifetimes form an independent sequence of random variables. Suppose that the system works as long as at least one of its components works. Let $F_1, F_2, ..., F_n$ be the cumulative distribution functions (CDF) of the lifetimes of the components of the system. In terms of $F_1, F_2, ..., F_n$, find the CDF of the lifetime of the system.

Problem 9.2.9 (10 points) Let X_1 and X_2 be two independent random variables $N(0, \sigma^2)$, and $\{X_{(1)}, X_{(2)}\}$ be the ordered statistics of $\{X_1, X_2\}$. Let $f_{12}(x_1, x_2)$ be the joint probability density function of $X_{(1)}$ and $X_{(2)}$. Find $E[X_{(1)}] = \int\int x_1 f_{12}(x_1, x_2)dx_1dx_2$, where the integration is taken over an appropriate region.

Problem Ch9-Review 9 (10 points) A bar of length ℓ is broken into three pieces at two random spots. What is the probability that the length of at least one piece is less than $\ell/20$?

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)