Signals and Systems

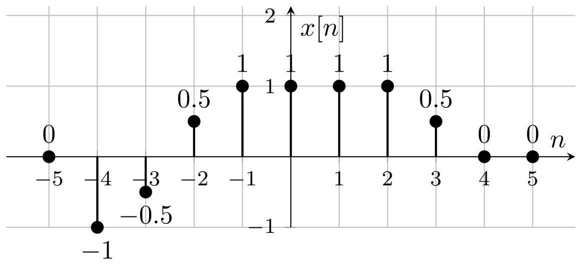
Homework 3 — Due : Mar. 15 2024

Problem 1 (18 pts)**.** A discrete-time signal $x[n]$ is shown in the figure below. Sketch and label carefully each of the following signals:

(a) $x[n]u[3-n]$

(b) $x[n](u[n]+u[-n])$

(c) $x[n-2]\delta[n-2]$



Problem 2 (16 pts)**.** Determine whether or not each of the following continuous-time signal is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = \mathbf{Even}\{\cos(4\pi t)u(t)\}$

(b) $x(t) = \mathbf{Even}\{\sin(4\pi t)u(t)\}$

Problem 3 (30 pts)**.** In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be (1) Memoryless, (2) Time invariant, (3) Linear, (4) Causal, (5) Stable. Determine which of these properties hold and which do not hold for each of the following systems. *Justify your answers with proofs.*

(a) $y(t) = x(t)\cos(7t)$

(b) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-7), & t \geq 0 \end{cases}$

(c) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-7), & x(t) \geq 0 \end{cases}$

(d) $y[n] = \sum_{k=-\infty}^{2n} x[k]$

(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$

(f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

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(a) $y(t) = x(2t)$

(b) $y(t) = \cos(x(t))$

(c) $y[n] = x[2n]$

(d) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{\pi})^{n-k} x[k]$

Problem 5 (24 pts)**.**

(a) (8 pts) Is $y[n] = \mathbf{Re}\left\{e^{\frac{j\pi n}{4}} x[n]\right\}$ additive? Justify your answer. (Do not assume that $x[n]$ is real in this problem.)

(b) (8 pts each) Determine whether each of the following systems is additive and/or homogeneous. Justify your answers by providing a proof for each property if it holds or a counterexample if it does not.

(i) $y(t) = \frac{1}{x(t)} \left[\frac{d}{dt}x(t)\right]^2$

(ii) $y[n] = \begin{cases} \frac{x[n]x[n-2]}{x[n-1]}, & x[n-1] \neq 0 \\ 0, & x[n-1] = 0 \end{cases}$

Problem 1 (18 pts). A discrete-time signal $x[n]$ is shown in the figure below. Sketch and label carefully each of the following signals:

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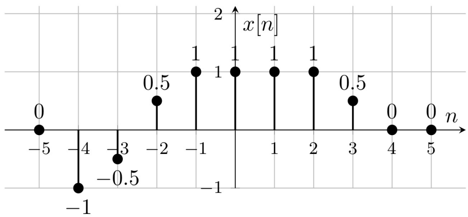
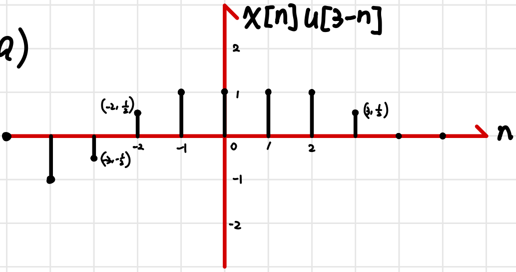
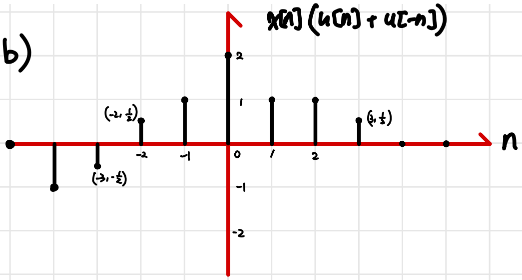


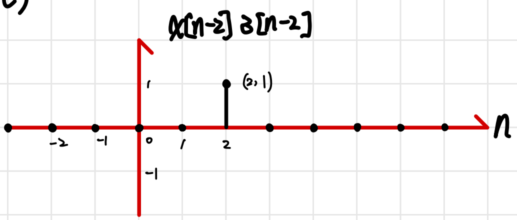
(a)



(b)



(c)



Problem 2 (16 pts). Determine whether or not each of the following continuous-time signal is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = \mathbf{Even}\{\cos(4\pi t)u(t)\}$

(b) $x(t) = \mathbf{Even}\{\sin(4\pi t)u(t)\}$

(a) $x(t) = \frac{1}{2}\left[\cos(4\pi t)u(t) + \cos(-4\pi t)u(-t)\right]$

$= \frac{1}{2}\left[\cos(4\pi t)u(t) + \cos(4\pi t)u(-t)\right]$

$= \frac{1}{2}\cos(4\pi t)$ $\qquad \frac{2\pi}{4\pi} = \frac{1}{2}$

$\Rightarrow x(t)$ is periodic, the fundamental period is $\frac{1}{2}$

(b) $x(t) = \frac{1}{2}\left[\sin(4\pi t)u(t) + \sin(-4\pi t)u(-t)\right]$

$= \frac{1}{2}\left[\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)\right]$

$\Rightarrow x(t)$ is not periodic.

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(a) $y(t) = x(t)\cos(7t)$

(b) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-7), & t \geq 0 \end{cases}$

(c) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-7), & x(t) \geq 0 \end{cases}$

(d) $y[n] = \sum_{k=-\infty}^{2n} x[k]$

(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$

(f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(a) memoryless

let $y_1(t) = x_1(t)\cos(7t)$

let $x_2(t) = x_1(t-n)$

$y_2(t) = x_2(t)\cos(7t)$

$= x_1(t-n)\cos(7t)$

$\neq y_1(t-n)$

$\Rightarrow$ not time invariant

let $y_1(t) = x_1(t)\cos(7t)$

$y_2(t) = x_2(t)\cos(7t)$

let a, b be constants $\neq 0$

$[a x_1(t) + b x_2(t)]\cos(7t)$

$= a x_1(t)\cos(7t) + b x_2(t)\cos(7t)$

$= a y_1(t) + b y_2(t) \Rightarrow$ linear

$\because$ memoryless $\therefore$ Causal

$\because 0 \leq |\cos(7t)| \leq 1, \forall t$

$\therefore |y(t)| \leq |x(t)| \;\forall t$

$\Rightarrow$ If $|x(t)| < \infty$, then $|y(t)| < \infty$

$\Rightarrow$ Stable

(b) not memoryless

causal

$\forall t$

$|x(t)| < \infty \Rightarrow |x(t) + x(t-7)| < \infty$

$\Rightarrow |y(t)| < \infty \Rightarrow$ stable

let $y_1(t) = \begin{cases} 0, & t<0 \\ x_1(t) + x_1(t-7), & t \geq 0 \end{cases}$

$x_2(t) = x_1(t-n)$

$y_2(t) = \begin{cases} 0, & t<0 \\ x_2(t) + x_2(t-7), & t \geq 0 \end{cases}$

$y_2(t) = \begin{cases} 0, & t<0 \\ x_1(t-n) + x_1(t-n-7), & t \geq 0 \end{cases}$

$\neq y_1(t-n) = \begin{cases} 0, & t<n \\ x_1(t-n) + x_1(t-n-7), & t \geq n \end{cases}$

$\Rightarrow$ not time invariant

let $y_1(t) = \begin{cases} 0, & t<0 \\ x_1(t) + x_1(t-7), & t \geq 0 \end{cases}$

$y_2(t) = \begin{cases} 0, & t<0 \\ x_2(t) + x_2(t-7), & t \geq 0 \end{cases}$

let a, b be constants $\neq 0$

$\begin{cases} 0, & t<0 \\ [a x_1(t) + b x_2(t)] + [a x_1(t-7) + b x_2(t-7)], & t \geq 0 \end{cases}$

$= a y_1(t) + b y_2(t)$

$\Rightarrow$ linear

(c) not memoryless

let $y_1(t) = \begin{cases} 0, & x_1(t) < 0 \\ x_1(t) + x_1(t-7), & x_1(t) \geq 0 \end{cases}$

$x_2(t) = x_1(t-n)$

$y_2(t) = \begin{cases} 0, & x_2(t) < 0 \\ x_2(t) + x_2(t-7), & x_2(t) \geq 0 \end{cases}$

$y_2(t) = \begin{cases} 0, & x_1(t-n) < 0 \\ x_1(t-n) + x_1(t-n-7), & x_1(t-n) \geq 0 \end{cases}$

$= y_1(t-n)$

$\Rightarrow$ time invariant

let $y_1(t) = \begin{cases} 0, & x_1(t) < 0 \\ x_1(t) + x_1(t-7), & x_1(t) \geq 0 \end{cases}$

$y_2(t) = \begin{cases} 0, & x_2(t) < 0 \\ x_2(t) + x_2(t-7), & x_2(t) \geq 0 \end{cases}$

$y(t) = \begin{cases} 0, & x_1(t) + x_2(t) < 0 \\ [x_1(t) + x_2(t)] + [x_1(t-7) + x_2(t-7)], & x_1(t) + x_2(t) \geq 0 \end{cases}$

let $\begin{cases} x_1(t_1) = 0 \\ x_2(t_1) < 0 \end{cases}$, $y_1(t_1) = x(t_1) + x(t_1 - 7) \neq 0$

$y(t_1) = 0 \neq y_1(t_1) + y_2(t_1) = y_1(t_1)$

Counter case

$\Rightarrow$ not linear

causal

let $|x(t)| \overset{\forall t}{<} \infty \Rightarrow |x(t) + x(t-7)| < \infty$

$\Rightarrow |y(t)| < \infty \Rightarrow$ stable

(d) not memoryless

if $n>0$, then $2n>n$ $\Rightarrow$ not causal

let $y_1[n]=\sum_{k=-\infty}^{2n} x_1[k]$

$x_2[k]=x_1[k-h]$

$y_2[n]=\sum_{k=-\infty}^{2n} x_2[k]$

$y_2[n]=\sum_{k=-\infty}^{2n} x_1[k-h]$

let $r=k-h$ $\quad = \sum_{r=-\infty}^{2n-h} x_1[r]$

$\neq y_1[n-h]=\sum_{r=-\infty}^{2n-2h} x_1[r]$

$\Rightarrow$ not time invariant

let $y_1[n]=\sum_{k=-\infty}^{2n} x_1[k]$

$y_2[n]=\sum_{k=-\infty}^{2n} x_2[k]$

let a, b be constants $\neq 0$

$\sum_{k=-\infty}^{2n}(a x_1[k]+b x_2[k])$

$=\sum_{k=-\infty}^{2n} a x_1[k]+\sum_{k=-\infty}^{2n} b x_2[k]$

$=a\sum_{k=-\infty}^{2n} x_1[k]+b\sum_{k=-\infty}^{2n} x_2[k]$

$=a\, y_1[n]+b\, y_2[n]$

$\Rightarrow$ linear

consider $x_1[k]=1$ (bounded)

$\sum_{k=-\infty}^{2n} x_1[k]=\infty$ $\Rightarrow$ not stable

(e) not memoryless

let $y_1[n] = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n = 0 \\ x_1[n+1], & n \leq -1 \end{cases}$

$x_2[n] = x_1[n-k]$, k is an integer

$y_2[n] = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n = 0 \\ x_2[n+1], & n \leq -1 \end{cases}$

$y_2[n] = \begin{cases} x_1[n-k], & n \geq 1 \\ 0, & n = 0 \\ x_1[n-k+1], & n \leq -1 \end{cases}$

$\neq y_1[n-k]$

$= \begin{cases} x_1[n-k], & n-k \geq 1 \\ 0, & n-k = 0 \\ x_1[n-k+1], & n-k \leq -1 \end{cases}$

$\Rightarrow$ not time invariant

let $y_1 = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n = 0 \\ x_1[n+1], & n \leq -1 \end{cases}$

$y_2 = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n = 0 \\ x_2[n+1], & n \leq -1 \end{cases}$

let a, b be constants

for $n \geq 1$, $a x_1[n] + b x_2[n] = a y_1[n] + b y_2[n]$

for $n = 0$, $a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n]$

$= a \cdot 0 + b \cdot 0 = 0$

for $n \leq -1$, $a x_1[n+1] + b x_2[n+1]$

$= a y_1[n] + b y_2[n+1]$

$\Rightarrow$ linear

not causal

let $|x[n]| < \infty \ \forall n$

$\Rightarrow |y[n]| < \infty \ \forall n$

$\Rightarrow$ stable

(f) memoryless

let $y_1[n] = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n = 0 \\ x_1[n], & n \leq -1 \end{cases}$

$x_2[n] = x_1[n-k]$, k is an integer. $\neq 0$

$y_2[n] = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n = 0 \\ x_2[n], & n \leq -1 \end{cases}$

$y_2[n] = \begin{cases} x_1[n-k], & n \geq 1 \\ 0, & n = 0 \\ x_1[n-k], & n \leq -1 \end{cases}$

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$= \begin{cases} x_1[n-k], & n-k \geq 1 \\ 0, & n-k = 0 \\ x_1[n-k], & n-k \leq -1 \end{cases}$

$\Rightarrow$ not time invariant

$\because$ memoryless $\therefore$ causal

let $y_1[n] = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n = 0 \\ x_1[n], & n \leq -1 \end{cases}$

$y_2[n] = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n = 0 \\ x_2[n], & n \leq -1 \end{cases}$

let a, b be constants $\neq 0$

for $n \geq 1$ and $n \leq -1$

$a x_1[n] + b x_2[n] = a y_1[n] + b y_2[n]$

for $n = 0$

$a x_1[n] + b x_2[n] \longrightarrow a y_1[n] + b y_2[n]$

$= a \cdot 0 + b \cdot 0 = 0$

$\Rightarrow$ linear

let $|x[n]| < \infty \ \forall n$

the $|y[n]| < \infty \ \forall n$

$\Rightarrow$ stable

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(a) $y(t) = x(2t)$

(b) $y(t) = \cos(x(t))$

(c) $y[n] = x[2n]$

(d) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{\pi})^{n-k} x[k]$

$(\frac{1}{\pi})^n * x[n]$

(a) invertible, $w(t) = y(\frac{t}{2}) = x(t)$

(b) not invertible, $y(t) = \cos(t) = \cos(-t)$

(c) not invertible, $x_1[n] = 1 \rightarrow y_1[n]$, $x_2[n] = \begin{cases} 1, & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \rightarrow y_2[n]$

$y_1[n] = y_2[n] = 1 \ \forall n$

(d) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{\pi})^{n-k} x[k]$

$= x[n] + \frac{1}{\pi} x[n-1] + (\frac{1}{\pi})^2 x[n-2] + \cdots$

$y[n-1] = \sum_{k=-\infty}^{n-1} (\frac{1}{\pi})^{n-1-k} x[k]$

$= x[n-1] + \frac{1}{\pi} x[n-2] + (\frac{1}{\pi})^2 x[n-3] + \cdots$

$w[n] = y[n] - \frac{1}{\pi} y[n-1] = x[n]$

invertible

Problem 5 (24 pts).

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(b) (8 pts each) Determine whether each of the following systems is additive and/or homogeneous. Justify your answers by providing a proof for each property if it holds or a counterexample if it does not.

(i) $y(t) = \frac{1}{x(t)}\left[\frac{d}{dt}x(t)\right]^2$

(ii) $y[n] = \begin{cases} \frac{x[n]x[n-2]}{x[n-1]}, & x[n-1] \neq 0 \\ 0, & x[n-1] = 0 \end{cases}$

(a) $e^{j\frac{\pi}{4}n} x[n] = x[n]\left\{\cos\left(\frac{\pi}{4}n\right) + j\sin\left(\frac{\pi}{4}n\right)\right\}$

let $x[n] = A_n(\cos(\theta_n) + j\sin(\theta_n))$, $A_n \in \mathbb{R}$

$y[n] = A_n\cos(\theta_n)\cos\left(\frac{\pi}{4}n\right) - A_n\sin(\theta_n)\sin\left(\frac{\pi}{4}n\right)$

$= A_n\cos\left(\left(\theta_n + \frac{\pi}{4}n\right)\right)$

let $x_1[n] \to y_1[n]$, $x_2[n] \to y_2[n]$

$Re\left\{e^{\frac{j\pi n}{4}}(x_1[n] + x_2[n])\right\} = Re\left\{e^{\frac{j\pi n}{4}}x_1[n] + e^{\frac{j\pi n}{4}}x_2[n]\right\}$

$= A_{1n}\cos\left(\left(\theta_{1n} + \frac{\pi}{4}n\right) + A_{2n}\cos\left(\left(\theta_{2n} + \frac{\pi}{4}n\right)\right)\right.$

$= y_1[n] + y_2[n] \Rightarrow$ additive

(b) (i) let $x_1(t) = 2t$, $y_1(t) = \frac{4}{2t} = \frac{2}{t}$

let $x_2(t) = t^2$, $y_2(t) = \frac{4t^2}{t^2} = 4$

$$\frac{1}{2t+t^2}(2t+2)^2 = \frac{4(t+1)^2}{t(t+2)} \neq y_1(t) + y_2(t) = \frac{2+4t}{t}$$

$\Rightarrow$ not additive

let k be a constants $\neq 0$, $y(t) = \frac{1}{x(t)}\left(\frac{d}{dt}x(t)\right)^2$

$$\frac{1}{kx(t)}\left(\frac{d}{dt}kx(t)\right)^2 = k \cdot \frac{1}{x(t)}\left(\frac{d}{dt}x(t)\right)^2 = ky(t)$$

$\Rightarrow$ homogeneous

(ii) let $x_1[n] \longrightarrow y_1[n]$, $x_2[n] \rightarrow y_2[n]$

let $\begin{cases} x_1[0]=1, & x_1[1]=2, & x_1[2]=3 \\ x_2[0]=1, & x_2[1]=-2, & x_2[2]=4 \end{cases}$

$y_1[2] = \frac{3}{2}$, $y_2[2] = \frac{4}{-2}$

let $y_{12}[2] = \frac{(x_1[2]+x_2[2])(x_1[0]+x_2[0])}{x_1[1]+x_2[1]} = 0 \because x_1[1]+x_2[1]=0$

$y_{12}[2] \neq y_1[2]+y_2[2] \Rightarrow$ not additive

let $x[n] \rightarrow y[n]$, k is a constants $\neq 0$

for $x[n-1]=0$, $kx[n] \rightarrow ky[n] = k \cdot 0 = 0$

for $x[n-1] \neq 0$, $\frac{kx[n]\,kx[n-2]}{kx[n-1]} = k \cdot \frac{x[n]\,x[n-2]}{x[n-1]} = k \cdot y[n]$

$\Rightarrow$ homogeneous