EE203001 SES #3

Overview of key ideas

vectors / matrices / subspaces

Vectors

Linear comb. of vectors

$$x_1 \underline{u} + x_2 \underline{v} + x_3 \underline{w} = \underline{b}$$

Ex

$$\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

collection of all multiples of $\underline{u}$ forms a line via origin

"

collections of all lin. comb. of $\underline{u}$ & $\underline{v}$ forms a plane

all lin. comb. forms a subspace

Matrices

coeff. matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

the product

$$A\underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \underline{u} + x_2 \underline{v} + x_3 \underline{w} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

difference matrix

($x_1 \underline{u} + x_2 \underline{v} + x_3 \underline{w}$: multiplying numbers by vectors

diff. views $\Updownarrow$

$A\underline{x}$: matrix multiplying numbers x_1, x_2, x_3

or A acts on vector $\underline{x}$, the result is

$\underline{b}$: a comb. of col.s of A)

For any input vector $\underline{x}$, the output of "multiplication" by A is some vector $\underline{b}$

Ex:

$$A \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

A deeper question: For what $\underline{x}$, does

$A\underline{x} = \underline{b}$?

$$A\underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Equivalently

$$x_1 = b_1$$
$$x_2 - x_1 = b_2 \Rightarrow x_2 = x_1 + b_2 = b_1 + b_2$$
$$x_3 - x_2 = b_3 \Rightarrow x_3 = x_2 + b_3 = b_1 + b_2 + b_3$$

Vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$

(lin. comb. with scalars b_1, b_2, b_3)

$$\Rightarrow \quad \underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ or } \underline{x} = A^{-1}\underline{b}$$

(S: sum matrix) A^{-1} (inverse) $\underline{b}$

(A^{-1} exists if A is invertible)

($A^{-1}A = I$)

($A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$)

(A : transform $\underline{x} \to \underline{b}$

A^{-1} : inverse transform $\underline{b} \to \underline{x}$)

(If $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

(Sum matrix is the inverse of diff. matrix)

Another example (same $\underline{u}$, $\underline{v}$, diff. $\underline{w}$)

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = [\underline{u} \ \ \underline{v} \ \ \underline{w}^*]$$

$$\Rightarrow C\underline{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

(circular)

(or cyclic diff. matrix)

(Recall: $A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$

$\because A\underline{x} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x} = \underline{0}$)

But here, $C\underline{x} = \underline{0}$ has infinitely many sol.

for any vector $\underline{x}$ with $x_1 = x_2 = x_3 \Rightarrow C\underline{x} = \underline{0}$

(or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix} \Rightarrow C\underline{x} = \underline{0}$)

(inverse does NOT exist since cannot find

C^{-1}. s.t. $C^{-1}C\underline{x} = C^{-1}(\underline{0}) = \underline{x}$)

Note that the corr. system of eqns

in $C\underline{x} = \underline{b}$ is

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \end{aligned}$$

Adding 3 eqns together,

$$0 = b_1 + b_2 + b_3$$

(Sol. only exists when $b_1 + b_2 + b_3 = 0$)

(Ex: $\underline{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ $\Rightarrow$ no sol.

$\Rightarrow$ no comb. of $\underline{u}$, $\underline{v}$, $\underline{w}^*$ produce $\underline{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$\Rightarrow$ the comb. Don't fill entire 3D space

or all lin. comb. of $\underline{u}$, $\underline{v}$, $\underline{w}^*$ lie on the plane $b_1 + b_2 + b_3 = 0$

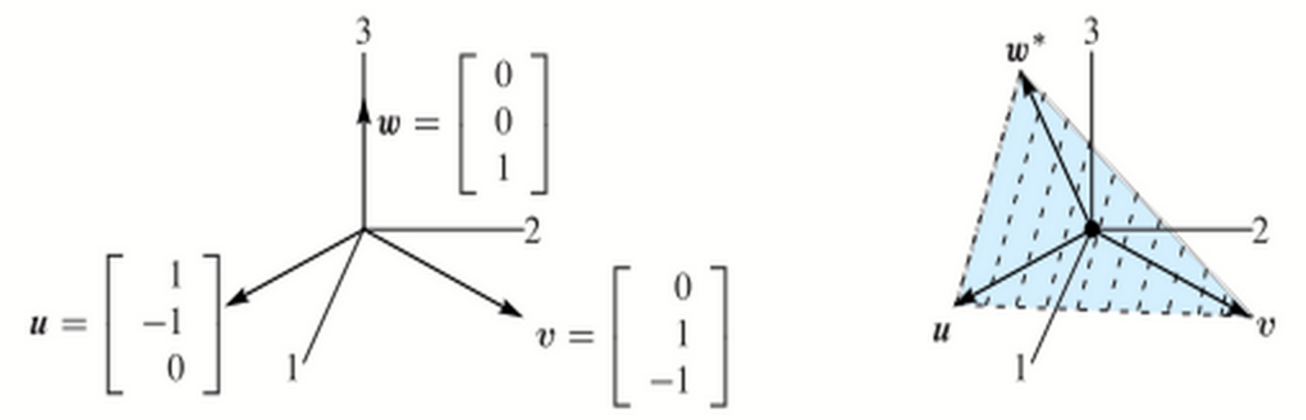

Figure 10: Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane.

$\underline{u}$ & $\underline{v}$ already forms a plane

- lin. indep. if $\underline{w}$ is not in the plane
- lin. depend. if $\underline{w}^*$ is in the plane

(Note that $\underline{u} + \underline{v} + \underline{w}^* = \underline{0}$ (lin. comb. !)

$\Rightarrow \underline{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\underline{u} - \underline{v}$)

Fact

$\underline{u}, \underline{v}, \underline{w}$ are lin. indep. $\Leftrightarrow$ no comb. except

$0\underline{u} + 0\underline{v} + 0\underline{w} = \underline{0}$ gives $\underline{b} = 0$

(lin. indep. col.s: $A\underline{x} = \underline{0}$ has only one sol.

& A is invertible)

$\underline{u}, \underline{v}, \underline{w}$ are lin. depend. $\Leftrightarrow$ Other comb.

gives $\underline{b} = 0$

(lin. depend. col.s: $A\underline{x} = \underline{0}$ has many sol.

& A is singular) ($\exists$ nonzero $\underline{x}$ s.t.

$x_1\underline{u} + x_2\underline{v} + x_3\underline{w} = 0$

$\Rightarrow \underline{w} = \frac{x_1}{-x_3}\underline{u} + \frac{x_2}{-x_3}\underline{v} \Rightarrow$ cannot solve $A\underline{x} = \underline{b}$ $\forall \underline{b}$

$\Rightarrow A$ singular)

Subspaces

Recall: col.s of C are depend.

(col.s of C lie in the same plane)

(Many vectors in $\mathbb{R}^3$ do not lie in that plane)

For $\underline{b}$ not in that plane,

$$C\underline{x} = \underline{b} \text{ has no sol.}$$

Lin. comb. of col.s of C form a subspace of $\mathbb{R}^3$

Recall: col.s of A are indep.

$\Rightarrow$ All comb. of col.s of A forms the entire space $\Rightarrow A\underline{x} = \underline{b}$ has a sol. for every $\underline{b}$

$\Rightarrow$ col.s of A $(\underline{u}, \underline{v}, \underline{w})$ forms a

basis for $\mathbb{R}^3$

<u>More generally</u>

A basis for $\mathbb{R}^n$ is a collection of

n lin. indep. vectors in $\mathbb{R}^n$

or

A comb. of n vectors whose comb. cover the entire $\mathbb{R}^3$

or

A matrix has these n vectors as col. vectors is invertible

<u>Vector space</u>

A collection of vectors closed under lin. comb.

<u>Subspace</u>

A vector space inside another vector space

Ex:
- the origin
- a line through the origin
- a plane through " "
- all of $\mathbb{R}^3$