Q1

Contact me:

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- Assumption: $a_0 = 1,\ a_1 = 8,\ a_{n+1} = 4a_n - 4a_{n-1}$
- To Show: $a_n = (1 + 3\mathrm{n}) * 2^{\mathrm{n}}, \forall n \in \mathbb{N} \cup \{0\}$ ...($\diamondsuit$)

Sol:

- (Basic step) (Note: at least two step) (5 points)
  - $a_0 = (1 + 3 * 0) * 2^0 = 1 * 1 = 1$
  - $a_1 = (1 + 3 * 1) * 2^1 = 4 * 2 = 8$
- (Induction step)
  - Induction hypothesis: (5 points)
    1. "Suppose the statement ($\diamondsuit$) is true for $n \leq k,\ for\ some\ k\ is\ an\ integer$." or
    2. "Suppose the statement ($\diamondsuit$) is true for $n = k\ and\ k - 1,\ for\ some\ k\ is\ an\ integer$."
    - That is, $a_k = (1 + 3k) * 2^k$ and $a_{k-1} = \big(1 + 3(k - 1)\big) * 2^{k-1}$, ...

-
  - Induction step: (5 points)
    - a_{k+1}
    - $= 4a_k - 4a_{k-1}$
    - $= 2^2 * (1 + 3k\ ) * 2^k\ - 2^2 * \big(1 + 3(k-1)\big) * 2^{k-1}$
    - $= (4 + 12\mathrm{K} - 6k + 4) * 2^k$
    - $= 2 * \big(1 + 3(k+1)\big) * 2^k$
    - $= \big(1 + 3(k+1)\big) * 2^{k+1}$
- By M.I., the statement is true for $n \in \mathbb{N} \cup \{0\}$.

Common mistakes

- Prove the equation $a_n = 4a_{n-1} - 4a_{n-2}$ is True
  - This is not our target formula.
- Basic step:
  - Only proof a_0 or a_1 (-2 scores)
    - If so, we can't write $a_0 = (1 + 3 * 0) * 2^0$ or $a_1 = (1 + 3 * 1) * 2^1$, respectively.
    - That may cause some problem when we are proving a_2
  - Only proof a_2 or other (-2 scores)
- Induction hypothesis:
  - Only suppose the statement is true when $n = k$ (-2 scores)
    - If so, a_{k-1} may not equal to $\big(1 + 3(k-1)\big) * 2^{k-1}$

Common mistakes

- Induction step:
  - Write $a_{k+1} = \left(1 + 3(k + 1)\right) * 2^{k+1}$ before you proved it. (-5 scores)
    - If so, you assume that the statement is true for $n = k + 1$

- Errors in proofs are rather difficult to elaborate with text alone, thus feel free to ask me in person if there are any issues.

Q2

李旺陽

2. (20%) Consider all the odd positive integers $1, 3, 5, 7, \ldots, 99$. Show that if we select any 34 distinct numbers from them, we can always find some selected number x divides properly another selected number y.

5% Use Pigeonhole principle

5% Give a way to make holes

10% Above is correct.

10% Give a way to make a legal group and its size is **33**.

10% Prove there is no way to enlarge the group.

Pigeonhole principle

How to make hole?

Ensure there is **at most 1 element** can be select for one hole.

wrong way

1, 3, **9**, **15**, 18...

5, **25**, **35**...

7, 14...

11

13

17

n, 3n, 5n, 7n...

you can select 2 elements from same hole and it is legal

correct way

1, 3, 9, 27, 81

5, 15, 45

7, 21, 63

11, 33, 99

13, 39

n, 3n, 9n, 27n...

2. (20%) Consider all the odd positive integers $1, 3, 5, 7, \ldots, 99$. Show that if we select any 34 distinct numbers from them, we can always find some selected number x divides properly another selected number y.

correct way **33 groups totally**

1, 3, 9, 27, 81

5, 15, 45

7, 21, 63

11, 33, 99

13, 39

17, 51

19, 57

23, 69

25, 75

29, 87

31, 93

35

37

Because there is only 33 holes,
but we need to select 34 elements.

By Pigeonhole principle,
it is impossible to select 34 or more elements to meet the condition.

41	43	47	49	53	55	59	61	65	67
71	73	77	79	83	85	89	91	95	97

Q3

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The Question

- (15%) How many ternary strings of length n we can find, such that each string does not contain the strings 01, 12, and 20 within? Express your answer in terms of n. Show your steps.
- For instance, when n = 4, 1111 is counted, 0211 is counted, but 2122 is not, 1220 is not.

The problem of using the **Inclusion–exclusion principle**

- $all\ -\ |01| - |12| - |20| + |01 \cap 12| + |12 \cap 20| + |20 \cap 01|$
  $-\ |01 \cap 12 \cap 20|$
- It will be very complicated to use this principle to solve this problem
- To calculate the amount of strings contains 01, if you just use

$$\underbrace{(n-2+1)}_{\text{Ways to choose the position to put in string “01”}} \times \underbrace{3^{n-2}}_{\text{Ways to fill-in the n-2 bits left}}$$

to count then the strings that contains multiple 01 will be counted repeatedly , for example 0101 will be counted twice, 010101 will be counted three times.(so as others)

The problem of using the **Inclusion–exclusion principle**

- The strings that contains $01 \cap 12$, not only means the strings that contains 012 , it could be the string that contains 0112 or 1201 or …
- The strings that contains $01 \cap 12 \cap 20$, not definitely means the strings that contains 0120 or 1201 or 2012 , it could be the string that contains 01120 or 011220 or 12201 …
- As n gets bigger, It’s extremely hard to calculate them each by each, and add up them to get the answer.

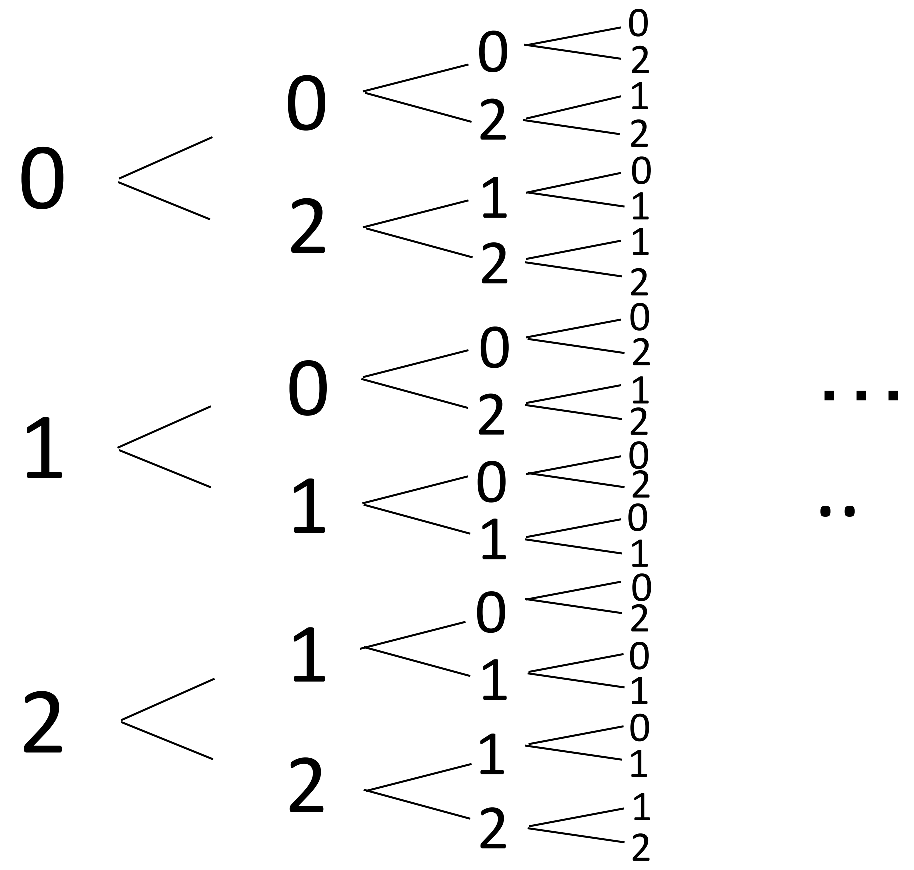
The problem of using the **Inclusion–exclusion principle**

- When n=4,there are :
- $all\ -\ |01| - |12| - |20| + |01 \cap 12| + |12 \cap 20| + |20 \cap 01| - |01 \cap 12 \cap 20|$ (ternary strings match the condition)
- $$= \underbrace{3^4}_{\text{all}} - \overset{\text{The amount of 01 is the same as 12,20}}{3} \times \Big(\underbrace{(4-2+1)\times 3^{4-2} - \underbrace{1}_{\text{0101 will be counted twice}}}_{\text{Strings that contains 01}}\Big) + \overset{\text{The amount of } |01 \cap 12| \text{ is the same as } |12 \cap 20|, |20 \cap 01|}{3} \times \Big(\underbrace{(4-3+1)\times 3^{4-3}}_{\text{Strings that contains 012}} + \underbrace{2}_{\text{String 0112 and 1201}}\Big) - \underbrace{3}_{\text{String 1201,0120, 2012}} = 24$$
- If n becomes higher , the situation will become even more complicated.
- It will be extremely complicated to come up with a formula that is correct for all cases using the Inclusion–exclusion principle.

The “next-bit” method

- The bit next to 0 could only be 0 or 2.(01 is not allowed)
- The bit next to 1 could only be 0 or 1.(12 is not allowed)
- The bit next to 2 could only be 1 or 2.(20 is not allowed)
- The first bit of these ternary strings could be 0, 1, 2.

The "next-bit" method



As the length of ternary string increase 1, the amount of ternary strings that doesn't contain 01,12,20 will double.

By the graph we can know that there are $3 \times 2^{n-1}$ ternary strings of length n that doesn't contain the string 01, 12, 20 within.

$$\underbrace{3}_{\text{The first bit}} \times \underbrace{2 \times 2 \times 2 \times \dots\dots \times 2}_{\text{n-1 bits (the 2-nd bit \textasciitilde{} the n-th bit)}} = 3 \times 2^{n-1}$$

Grading Policy

- **Use the "next-bit" method**
- Use the graph or words to explain that the amount of string (not containg 01,12,20) will double as the length of ternary string increase 1.(10%, 7% if few parts of induction is wrong.)
- Write the correct answer(5%, 3% if few parts of answer is wrong.)
- **Use the Inclusion–exclusion principle or other method**
- The formula you induct is correct for all cases.(15%)
- The formula you induct is correct only when n = specific numbers, you can get "$2 \times$ amount of specific numbers"% of the score.(maximum = 8%)
- For example, If the formula is correct only when n=1,2,4 , you get $2 \times 3 =$ 6%

Q4

陳咨蓉

4. How many ways to arrange the characters **S, U, C, C, E, S, S**, into a 7-letter word, such that the characters C are not next to each other?

For instance, CECUSSS is okay, but SUCCESS is not.

Sol : 1

all possible cases - the characters C are next to each other

: $\frac{7!}{3!2!}$ - $\frac{6!}{3!}$

Part A (all possible cases)

7! : all character arrangement(success)

3! : there are 3 characters ‘s’ repeat.

2! : there are 2 characters ‘c’ repeat.

Part B (the characters C are next to each other)

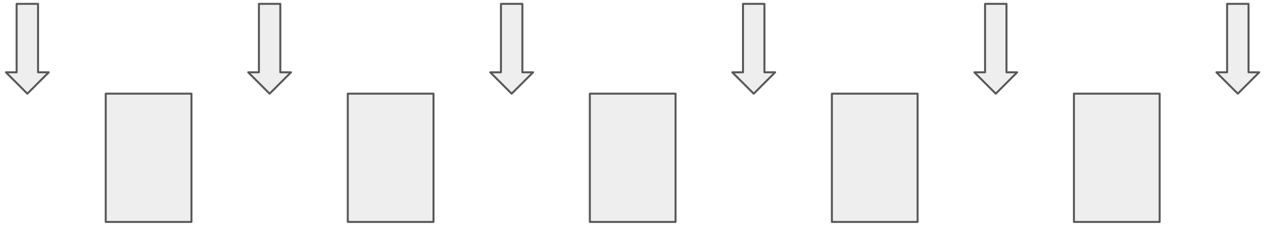
6! : sucess arrangement

3! : there are 3 characters ‘s’ repeat.

Sol : 2

Arrange the remaining characters and then insert 2 characters ‘c’ in different gaps.

: $\frac{5!}{3!} \cdot C_2^6$



Part A

5! : remaining characters arrangement (suess).

3! : there are 3 characters ‘s’ repeat.

Part B

C_2^6 : there 6 gaps between each remaining characters, and we choose 2 different gaps to insert characters ‘c’.

Grading

-2 : there are some mistakes in your answer.(each -2)

e.g. you think there are 2 ‘s’ or 4 ‘s’.(But you need to write down the number of ‘s’ which you think)

-10 : your solution is wrong in some parts. That is, if your solution is sol 1 or sol 2, and you are wrong in part A or B, each -10

-20 : wrong answer without no explaination

Q5

李明哲

5. (15%) Let m and n be positive integers such that $m \leq n$. Prove the following identity using combinatorial arguments:

$$\binom{n+1}{m} \equiv \binom{n}{m} + \binom{n-1}{m-1} + \binom{n-2}{m-2} + \cdots + \binom{n-m}{0},$$

where the notation $\binom{x}{y}$ denotes the number of ways to select y objects out of x objects.

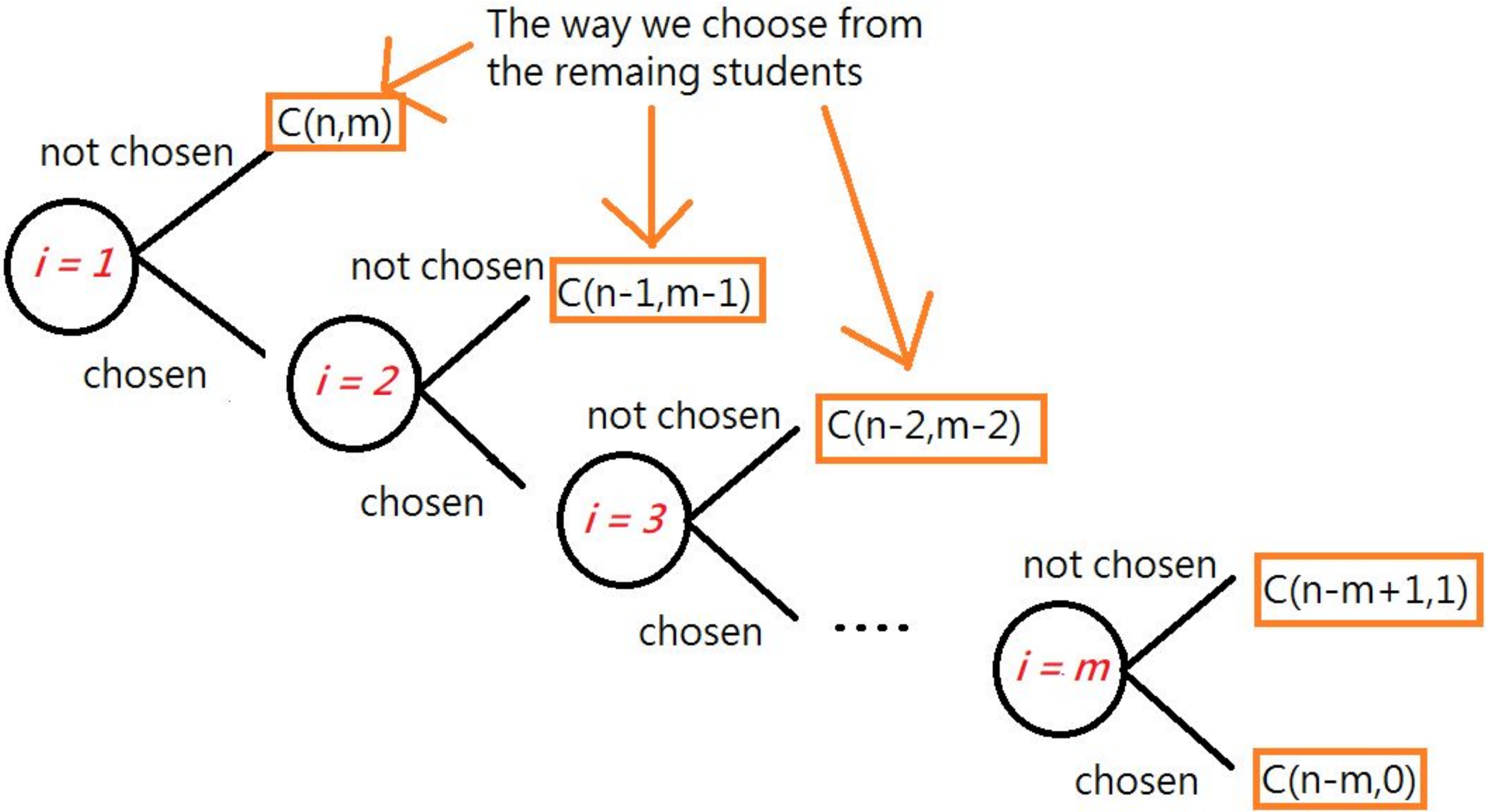
Note: No points will be given if your proof is not a combinatorial proof !

We are going to pick ***m*** students out of a class which has ***n+1*** students. Each student has an unique ID number ***i*** (***i*** is an natural number between 1 to n+1).

See this problem in two different perspectives:

1) Intuitively pick them → C(n+1,m) ways

2) Case by case (see the next page)


The way we choose from
the remaing students
C(n,m)
not chosen
i = 1
chosen
not chosen
C(n-1,m-1)
i = 2
chosen
not chosen
C(n-2,m-2)
i = 3
chosen
....
not chosen
C(n-m+1,1)
i = m
chosen
C(n-m,0)

Another combinatorial proof (a pretty clever method!)

Consider the equation $x_1 + x_2 + \cdots + x_{n-m+1} + x_{n-m+2} = m$ $\quad\cdots\cdots(☆)$

Note: # of nonnegative integer solutions for the equation $x_1 + x_2 + \cdots + x_k = m \Rightarrow \binom{k+m-1}{m}$ ways.

Put $k = n - m + 2$, we know that there are $\binom{(n-m+2)+m-1}{m} = \binom{n+1}{m}$ solutions for (☆).

See (☆) in another way.

Move x_{n-m+2} to the RHS, we get $x_1 + x_2 + \cdots + x_{n-m+1} = m - x_{n-m+2}$

x_{n-m+2} is a nonnegative integer($x_{n-m+2} \geq 0$), so $x_1 + x_2 + \cdots + x_{n-m+1} = m - x_{n-m+2} \leq m$

Thus, # of nonnegative integer solutions of (☆) is equivalent to $x_1 + x_2 + \cdots + x_{n-m+1} \leq m$ …… (◇)

$x_1 + x_2 + \cdots + x_{n-m+1} \leq m$ ...... $(\diamond)$

$$(\diamond) \equiv \left\{ \begin{array}{c} x_1 + x_2 + \cdots + x_{n-m+1} = m \\ x_1 + x_2 + \cdots + x_{n-m+1} = m-1 \\ \vdots \\ x_1 + x_2 + \cdots + x_{n-m+1} = 1 \\ x_1 + x_2 + \cdots + x_{n-m+1} = 0 \end{array} \right\} \equiv \left\{ \begin{array}{c} \binom{n-m+1+m-1}{m} \\ \binom{n-m+1+(m-1)-1}{m-1} \\ \vdots \\ \binom{n-m+1+1-1}{1} \\ \binom{n-m+1+0-1}{0} \end{array} \right\} \equiv \left\{ \begin{array}{c} \binom{n}{m} \\ \binom{n-1}{m-1} \\ \vdots \\ \binom{n-m+1}{1} \\ \binom{n-m}{0} \end{array} \right\}$$

# of nonnegative integer solutions of $(\diamond)$ is $\binom{n}{m} + \binom{n-1}{m-1} + \cdots + \binom{n-m+1}{1} + \binom{n-m}{0}$

Since $(\star) \equiv (\diamond)$, we get $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} + \cdots + \binom{n-m+1}{1} + \binom{n-m}{0}$

Grading

RANK1(15%): Use a convincible combinatorial proof without mistakes.

RANK2(10%-12%): Use a combinatorial proof to prove Pascal’s Identity first, and then use this identity to accomplish your proof. Or use a convincible combinatorial proof with a little mistakes.

RANK3(5%): Use a combinatorial proof, but the logic behind it is not correct.

RANK4(0%): Directly apply Pascal’s Identity without any combinatorial argument to accomplish your proof. Or use a combinatorial proof with lots of mistakes.

Discrete Mathematics

Exam 2 Question 6

資工系碩士班

李峻丞

Question 6

Show that for any collection of 5 positive integers, we can always find three different ways to select integers from them, such that the sums of those selected integers have the same remainder when divided by 15.

For instance, suppose the 5 numbers in the collection are: {3, 5, 16, 17, 22}. In each of the following three ways, the sum has the same remainder (which is 3) when divided by 15:

{3}, {16, 17}, {3, 5, 16, 17, 22}

Question 6

For a given set S with 5 elements, the number of elements in 2^S is 2^5.

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Empty set is a subset of every set.

There are 31 ways to select integers.

Question 6

There are 15 possible remainders when an integer is divided by 15.

Question 6

There are 15 possible remainders when an integer is divided by 15.

By the generalized pigeonhole principle, we can always find at least $\lceil 31/15 \rceil$ different ways to select integers from them, such that the sums of those selected integers have the same remainder when divided by 15.

Question 6

Scoring Guidelines:

3 points are deducted if you miscalculate the number of pigeons(31) or holes(15).

Q7

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Grading Policy:

- 5 points: complete and correct proof
- 3 points: almost complete proof with subtle error
- 0 points: incomplete or wrong proof

Q7

Consider a sequence of $n(n+1)$ distinct numbers.

(5%) Show that we can always find a subsequence of $2n$ numbers, such that within the subsequence, the smallest number and the second smallest number are next to each other, the third smallest and the fourth smallest numbers are next to each other, and in general, the $(2k-1)^{\text{th}}$ smallest and the $2k^{\text{th}}$ smallest numbers are next to each other, for all k with $1 \leq k \leq n$.

For instance, suppose $n = 2$, and we have the following sequence of six numbers:

$$6, \quad 2, \quad 4, \quad 5, \quad 1, \quad 3.$$

Then, there is a subsequence of four numbers:

$$6, \quad 5, \quad 1, \quad 3$$

Q7

- Prove by induction.
- $P(n) =$ "For any sequence of $n(n+1)$ distinct numbers, there exist a subsequence s, $|s| = 2n$, and the $2i - 1^{th}$ smallest and the $2i^{th}$ smallest numbers in s are next to each other, for all i with $1 \le i \le n$."
- $S_n =$ { s | s is subsequence that satisfy $P(n)$}
- We only need to prove, for any sequence of $n(n+1)$ distinct numbers, $S_n \neq \emptyset$.
- $P'(n) =$ "For any sequence of $n(n+1)$ distinct numbers, there exist a subsequence s, $|s| = 2n$, and the $2i - 1^{th}$ smallest and the $2i^{th}$ smallest numbers in s are next to each other and the value of these two numbers between $(i-1)(n+1) + 1^{th}$ and $i(n+1)^{th}$ smallest numbers of sequence, for all i with $1 \le i \le n$."
- $S'_n =$ { s | s is subsequence that satisfy $P'(n)$}
- Since $S'_n \subseteq S_n$, if we can prove $S'_n \neq \emptyset$ then $S_n \neq \emptyset$.

Q7

$P'(n) =$ "For any sequence of $n(n+1)$ distinct numbers, there exist a subsequence s, $|s| = 2n$, and the $2i-1^{th}$ smallest and the $2i^{th}$ smallest numbers are next to each other and the value of these two numbers between $(i-1)(n+1)+1^{th}$ *and* $i(n+1)^{th}$ smallest numbers of sequence, for all i with $1 \le i \le n$."

- Let $SEQ_{n(n+1)}$ be the sequence with $n(n+1)$ distinct numbers.
- Suppose numbers in sequence $SEQ_{n(n+1)}$ are $\{1,2,\dots,n(n+1)\}$.

- Basis case: for $n = 1$, $SEQ_{1(2)}$ only have 2 distinct numbers, pick them all, $S'_1 = \{SEQ_{1(2)}\}$. $P'(1)$ is true.
- Inductive case: suppose $P'(k)$ is true. That is, for $SEQ_{k(k+1)}$, $S'_k \neq \emptyset$.

6	13	1	7	20	11	3	19	9	15	5	12	8	14	18	16	17	4	2	10

Q7

B_i ={x | x is a number in $SEQ_{n(n+1)}$, value of x between $(i-1)(n+1)+1^{th}$ and $i(n+1)^{th}$ smallest numbers of sequence}

- When $n = k + 1$, $SEQ_{(k+1)(k+2)}$ is the sequence of $\{1,2,\ldots,(k+1)(k+2)\}$, partition these numbers into $(k+1)$ buckets $B_1 \sim B_{k+1}$, each bucket contains $(k+2)$ distinct numbers.
- Start from first number of $SEQ_{(k+1)(k+2)}$, check each number x, mark the bucket which x belongs to. By pigeonhole principle, after checking at most $(k+2)$ numbers, we can find the bucket B_r that checked twice.

Frist			Repeat																
6	13	1	7	20	11	3	19	9	15	5	12	8	14	18	16	17	4	2	10
B_2	B_3	B_1	B_2	k+1															

B_1: 1 2 3 4 5

B_2: 6 7 8 9 10

B_3: 11 12 13 14 15

B_4: 16 17 18 19 20

Q7

$$\begin{cases} S_{checked} = \{buckets\ B_i\ checked\ once\} \\ S_{unchecked} = \{buckets\ B_i\ unchecked\} \\ S_{repeat} = \{bucket\ B_r\ checked\ twice\} \end{cases}$$

- After find the repeat checked bucket, $B_1 \sim B_{k+1}$ can be partitioned into 3 set.
- For bucket in S_{repeat}, pick those two checked number into subsequence s and remove whole bucket.
- For bucket in $S_{checked}$, remove the checked number.
- For bucket in $S_{unckecked}$, randomly remove one number.
- Since $|S_{checked}| + |S_{unchecked}| = k$ and B_r contains $(k+2)$ numbers, so it will remove $2(k+1)$ numbers, remain $k(k+1)$ numbers, which is $SEQ_{k(k+1)}$.

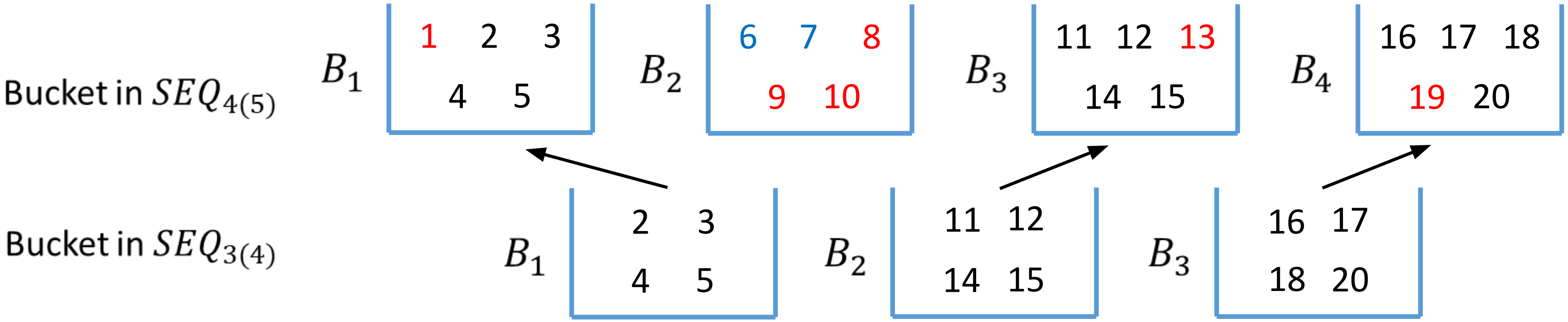
index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$SEQ_{4(5)}$	6	13	1	7	20	11	3	19	9	15	5	12	8	14	18	16	17	4	2	10
$SEQ_{3(4)}$	20	11	3	15	5	12	14	18	16	17	4	2	13	1	19	9	8	10	6	7

$P(k)$ Remove pick

Q7

$P'(n) =$ "For any sequence of $n(n+1)$ distinct numbers, there exist a subsequence s, $|s| = 2n$, and the $2i-1^{th}$ smallest and the $2i^{th}$ smallest numbers are next to each other and the value of these two numbers between $(i-1)(n+1)+1^{th}$ *and* $i(n+1)^{th}$ smallest numbers of sequence, for all i with $1 \le i \le n$."

- By induction hypothesis, we can get subsequence of $2k$ numbers in remaining $k(k+1)$ numbers in $SEQ_{k(k+1)}$. So we have subsequence s with $2k$ (from $P'(k)$) $+$ 2(from B_r) numbers.
- Since B_i in $P'(k)$ for all i with $1 \le i < r$ corresponds to B_i in $P'(k+1)$ for all i with $1 \le i < r$, and B_i in $P'(k)$ for all i with $r \le i \le k$ corresponds to B_i in $P'(k+1)$ for all i with $r+1 \le i \le k+1$, so $(k+1)$ pair of numbers belongs to different bucket in $B_1 \sim B_{k+1}$.



Q7

$P'(n)$ = "For any sequence of $n(n+1)$ distinct numbers, there exist a subsequence s, $|s| = 2n$, and the $2i-1^{th}$ smallest and the $2i^{th}$ smallest numbers are next to each other and the value of these two numbers between $(i-1)(n+1)^{th}$ *and* $i(n+1)^{th}$ smallest numbers of sequence, for all i with $1 \le i \le n$."

- k pair of numbers from $P'(k)$ next to each other, $2r-1^{th}$ smallest and $2r^{th}$ smallest numbers also next to each other.
- So $s \in S'_{k+1}$ of $SEQ_{(k+1)(k+2)}$, $P'(k+1)$ is true.

index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$SEQ_{3(4)}$	20	11	3	15	5	12	14	18	16	17	4	2	13	1	19	9	8	10	6	7
						$P(k)$										Remove			pick	

Bucket in $SEQ_{3(4)}$

B_1		B_3		B_4	
2	3	11	12	16	17
4	5	14	15	18	20

$SEQ_{4(5)}$	6	13	1	7	20	11	3	19	9	15	5	12	8	14	18	16	17	4	2	10
order	3^{rd}			4^{th}		5^{th}				6^{th}					8^{th}	7^{th}		2^{nd}	1^{st}	