Exam2 Q1
林勺甫

Mathematical Induction

- [Definition] a_n with recurrence relation

$$a_n = a_{n-1} + 2 \times a_{n-2}, \text{ where } a_0 = 8,\ a_1 = 1$$

- [Formula] For any integer $n \geq 0$, a_n would obey the following formula

$$a_n = 3 \times 2^n + 5 \times -1^n$$

- You don't need to prove [Definition], all you need is to prove [Formula] !!!!

Grading Policy

- Base step (3 pts)
  - You should give a explicit validation on the [Formula]
  - `[Formula]` is correct when n = 0 and n = 1

$$3 \times 2^0 + 5 \times -1^0 = 8$$

$$3 \times 2^1 + 5 \times -1^1 = 1$$

Grading Policy

- Inductive assumption (3 pts)
  - You need to make a clear inductive assumption, one of below is accepted
    - Strong induction

      If $a_k = 3 \times 2^k + 5 \times -1^k$ for any $1 \leq k \leq n$,

      then $a_{n+1} = 3 \times 2^{n+1} + 5 \times -1^{n+1}$
    - Induction with 2 term in the sequence

      If $a_k = 3 \times 2^k + 5 \times -1^k$ for $k = n$ and $k = n - 1$,

      then $a_{n+1} = 3 \times 2^{n+1} + 5 \times -1^{n+1}$
  - Induction with 1 term in the sequence is incorrect

Grading Policy

- Inductive step (9 pts)
  - If you could show your inductive assumption clearly with introducing the definition, you would get the points

By strong induction, $a_k = 3 \times 2^k + 5 \times -1^k$ for $k = n$ and $k = n - 1$

By [definition], $a_{n+1} = a_n + a_{n-1}$, so $a_{n+1} = 3 \times 2^n + 5 \times -1^n + 2\times(3 \times 2^{n-1} + 5 \times -1^{n-1})$,

$a_{n+1} = 3 \times 2^n + 5 \times -1^n + 2 \times (3 \times 2^{n-1} + 5 \times -1^{n-1})$

$a_{n+1} = 3 \times (2^n + 2 \times 2^{n-1}) + 5 \times (-1^n + 2 \times -1^{n-1})$

$a_{n+1} = 3 \times (2^{n+1}) + 5 \times -1^n \times (1 + 2 \times -1)$

$a_{n+1} = 3 \times 2^{n+1} + 5 \times -1^{n+1}$

Q2

鄧晉杰

2. (15%)

Let n be a positive integer, and consider an array with 2 rows and $2n$ columns. Each entry in the array is either 0 or 1. It is known that for each row, exactly n entries are 0 and exactly n entries are 1.

For a particular column, if both entries are 0, we call it a 0-column; else, if both entries are 1 we call it a 1-column.

Show that the number of 0-columns is the same as the number of 1-columns.

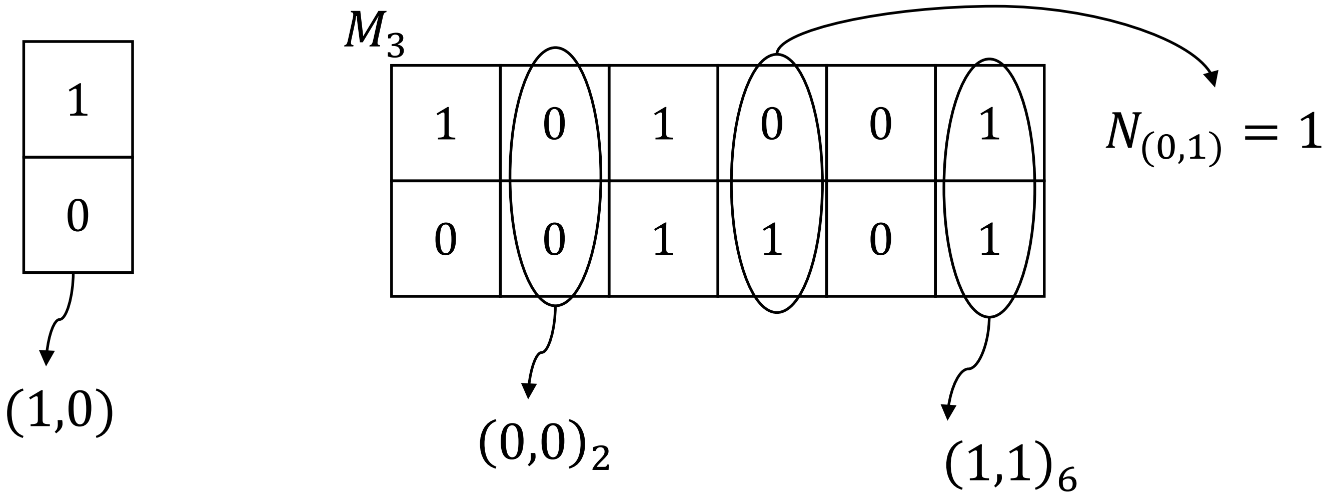
For instance, suppose $n = 3$. Suppose the array looks like the following:

1	0	1	0	0	1
0	0	1	1	0	1

Each row contains exactly n 0s and exactly n 1s. Also, we see that there are two 0-columns (the 2^{nd} one and the 5^{th} one), and there are two 1-columns (the 3^{rd} one and the 6^{th} one).

Notation

c_i $\coloneqq$ value in column j (fixed row)
(c_1, c_2) $\coloneqq$ column representation
$(c_1, c_2)_r$ $\coloneqq$ column representation of row r
$N_{(c_1,c_2)}$ $\coloneqq$ number of (c_1, c_2)
M_n $\coloneqq$ a $2 \times 2n$ matrix whose number of 0s $=$ number of 1s $= n$ in both row



Proof by Induction

Problem Statement

$$P(n) \coloneqq \forall M_n, n \in \mathbb{N}, N_{(0,0)} = N_{(1,1)}$$

Basis $(P(1)$ is true$)$

0	1
0	1

$$N_{(0,0)} = N_{(1,1)} = 1$$

0	1
1	0

$$N_{(0,0)} = N_{(1,1)} = 0$$

1	0
0	1

$$N_{(0,0)} = N_{(1,1)} = 0$$

1	0
1	0

$$N_{(0,0)} = N_{(1,1)} = 1$$

Inductive hypothesis $(P(k)$ is true$)$

Inductive Step $\left(P(k) \to P(k+1)\right)$

Claim $\forall M_{k+1}$,
(1) If $(c_1, c_2)_1 = (0,0), \exists (c_1, c_2)_r = (1,1), 1 < r \leq 2(k+1)$
(2) If $(c_1, c_2)_1 = (0,1), \exists (c_1, c_2)_r = (1,0), 1 < r \leq 2(k+1)$
(3) If $(c_1, c_2)_1 = (1,0), \exists (c_1, c_2)_r = (0,1), 1 < r \leq 2(k+1)$
(4) If $(c_1, c_2)_1 = (1,1), \exists (c_1, c_2)_r = (0,0), 1 < r \leq 2(k+1)$

If $(c_1, c_2)_1$ and $(c_1, c_2)_r$ are removed from M_{k+1}, M_k is obtained.

$\because$ $P(k)$ is true, and $N_{(0,0)} = N_{(1,1)}$ holds after re-insertion of $(c_1, c_2)_1$ and $(c_1, c_2)_r$
$\therefore$ $P(k+1)$ is true

Therefore, $P(n)$ is true, $n \in \mathbb{N}$.

Claim

$$(1)\ \forall M_{k+1}, \text{if } (c_1, c_2)_1 = (0,0), \exists (c_1, c_2)_r = (1,1), 1 < r \leq 2(k+1)$$

Proof

Assume the contrary,
if $(c_1, c_2)_1 = (0,0)$, $(c_1, c_2)_r = (0,0), (0,1), \text{or } (1,0), 1 < r \leq 2(k+1)$.

$$\begin{aligned} \Rightarrow 2 \cdot \left(N_{(0,0)} - 1\right) + N_{(0,1)} + N_{(1,0)} &= 2 \cdot (k+1) - 2 \ (\#\text{ of 0s without } (0,0)_1) \\ N_{(1,0)} + N_{(0,1)} &= 2 \cdot (k+1) \qquad (\#\text{ of 1s without } (0,0)_1) \end{aligned}$$

$$\Rightarrow N_{(0,0)} = 0\ (\rightarrow\leftarrow)$$

Claim (4) can be proved by symmetry.

Claim

$$(2)\ \forall M_{k+1}, \text{if } (c_1, c_2)_1 = (0,1), \exists (c_1, c_2)_r = (1,0), 1 < r \leq 2n$$

Proof

Assume the contrary,
If $(c_1, c_2)_1 = (0,1)$, $(c_1, c_2)_r = (0,0), (0{,}1),$ or $(1{,}1), 1 < r \leq 2n.$

$$\begin{aligned} \Rightarrow N_{(1,1)} &= k+1 && (\#\text{ of 1s in row 1 without } (0{,}1)_1) \\ \left(N_{(0,1)} - 1\right) + N_{(1,1)} &= (k+1) - 1 && (\#\text{ of 1s in row 2 without } (0{,}1)_1) \end{aligned}$$

$$\Rightarrow N_{(0,1)} = 0\ (\rightarrow\leftarrow)$$

Claim (3) can be proved by symmetry.

Short proof

$$\text{total \# of } 0s = N_{(0,1)} + N_{(1,0)} + 2 \cdot N_{(0,0)} = 2 \cdot n$$
$$\text{total \# of } 1s = N_{(1,0)} + N_{(0,1)} + 2 \cdot N_{(1,1)} = 2 \cdot n$$

$$\Rightarrow 2 \cdot \left(N_{(0,0)} - N_{(1,1)}\right) = 0$$

$$\Rightarrow N_{(0,0)} = N_{(1,1)}$$

$$\begin{cases} N_{(0,1)} + N_{(0,0)} = n \\ N_{(0,1)} + N_{(1,1)} = n \end{cases}$$

$$\Rightarrow N_{(0,0)} = N_{(1,1)}$$

Q3

陳咨蓉

3. (15%) How many binary strings of length 6 we can find, such that each string does not contain three or more contiguous 1s?

For instance,

011011 is counted, but 011110 is not.

Discuss with :

1. Calculate directly
   (1) List them all
   (2) Discuss with the strings contain how many 0s or 1s
2. All – those are not included in the request
   (1) List those are not included in the request
   (2) Discuss with the strings contain how many 0s or 1s

Sol:
1.(2) Discuss with the strings contain how many 1s

contain how many 1s : how many strings

(1)0 : 1

$6! / 6!$

(2)1 : 6

$6! / 5!$

(3)2 : 15

$6! / (4! * 2!)$

(4)3 : 16

$6! / (3! * 3!) - 4! / 3!$

(5)4 : 6

3(two 11) + 3(one 11, two 1s)

Total : 1 + 6 +15 + 16 + 6 = 44

Grading Policy

- List out all possible cases
  - -1 point for each missing binary string
- Divide into 5 cases and discuss each of them
  - +3 points for each correct cases
  - If you provided >5 cases, -3 points for each excessive case

Q4

林毓淇

(a)

(15%) How many positive integral solutions are there for $x + y + z = 99$?

NOTE:

Positive integral (not include 0)

<sol>

Balls and bars

96 balls and 2 bars

<ans>

C(96+2, 2)=C(98, 2)=4753

(b)

(10%) How many positive integral solutions are there for x + y + z = 99, if x, y, z are restricted to be all odd integers?

Key:

Design balls and bars

<sol>

Original boxes: x, y, z

Changed boxes: x', y', z' (x=2x'+1)

$$x + y + z = 99$$
$$\rightarrow (2x' + 1) + (2y' + 1) + (2z' + 1) = 99$$
$$\rightarrow x' + y' + z' = 48$$

(b)

(10%) How many positive integral solutions are there for x + y + z = 99, if x, y, z are restricted to be all odd integers?

<sol> (continue)
$x' + y' + z' = 48$

Balls and bars

48 balls and 2 bars

<ans>

C(48+2, 2)=C(50, 2)=1225

Score:

1. (a) answer includes 0 (10 points)
2. (a)(b) concept right, combinatorial equation wrong (half)

Common mistakes:

1. Positive integer (not include 0) (-5 points)
2. Three bars (0 points)

NOTE:

Some use counting method, it’s ok if you show how you count.

But process is tedious……

Q5

陳弘欣

5. (15%) Use a combinatorial argument to show that for any positive integers r, k with $r > k$:

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}.$$

Note: No marks will be given if your proof is not a combinatorial proof.

Wikipedia

In mathematics, the term **combinatorial proof** is often used to mean either of two types of mathematical proof:

- A proof by **double counting**. A combinatorial identity is proven by **counting the number of elements of some carefully chosen set in two different ways** to obtain the different expressions in the identity. Since those expressions count the same objects, they must be equal to each other and thus the identity is established.
- A bijective proof. Two sets are shown to have the same number of members by exhibiting a bijection, i.e. a one-to-one correspondence, between them.

From :
https://en.wikipedia.org/wiki/Combinatorial_proof

Sol: Double Counting

Objective: select 1 team leader and k team members from r students.

1. Select a team leader first, then select k team members from the remaining r-1 students. $\boxed{r\binom{r-1}{k}}$
2. Select k team members first, then select 1 team leader from the remaining r-k students. $\boxed{(r-k)\binom{r}{k}}$

Grading policy

1. 0 points for non-combinatorial proof.
   (e.g. direct proof, algebra transposition)
2. 15 points for correct answer
3. 10 points for saying “select k+1 team members from r students” (or something like this)
   - You must explicitly say that there are two kinds of people

Discrete Mathematics

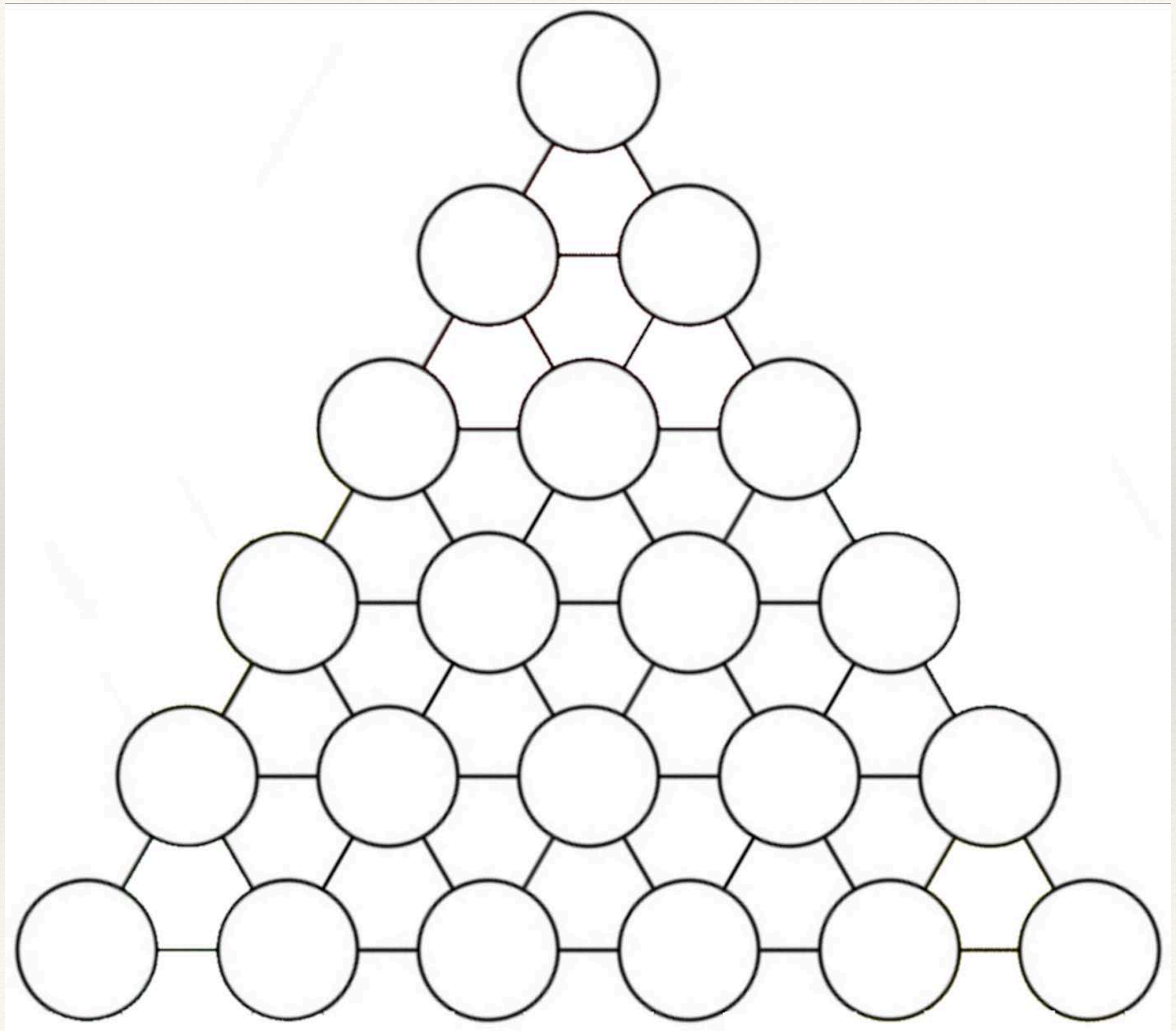
Exam 2 Question 6

李峻丞

Question 6

Consider the triangular grid as shown in Figure 1, which contains 21 nodes. If two nodes u and v in the grid is connected directly by an edge, we say u and v are adjacent.

Question 6

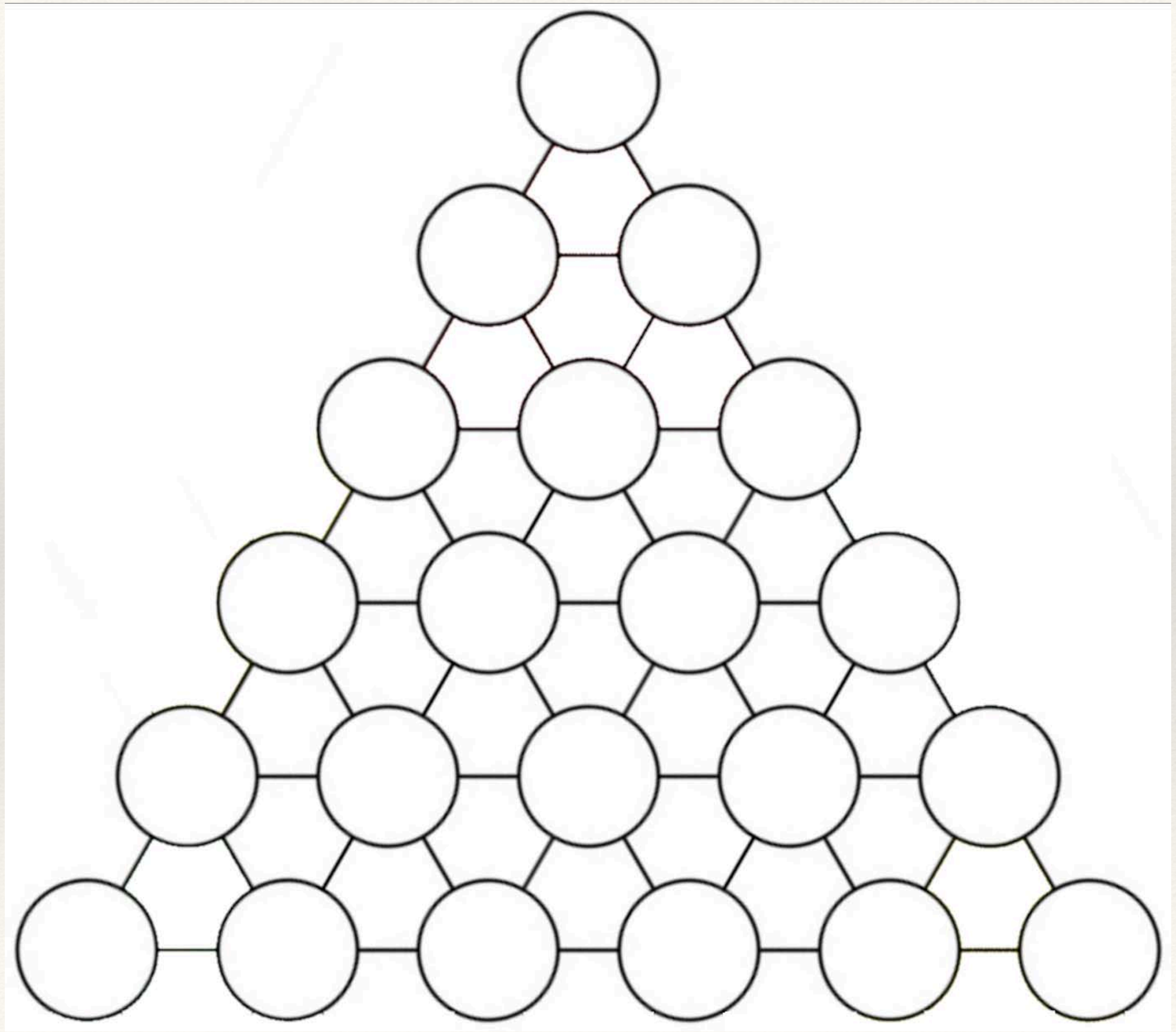


Question 6

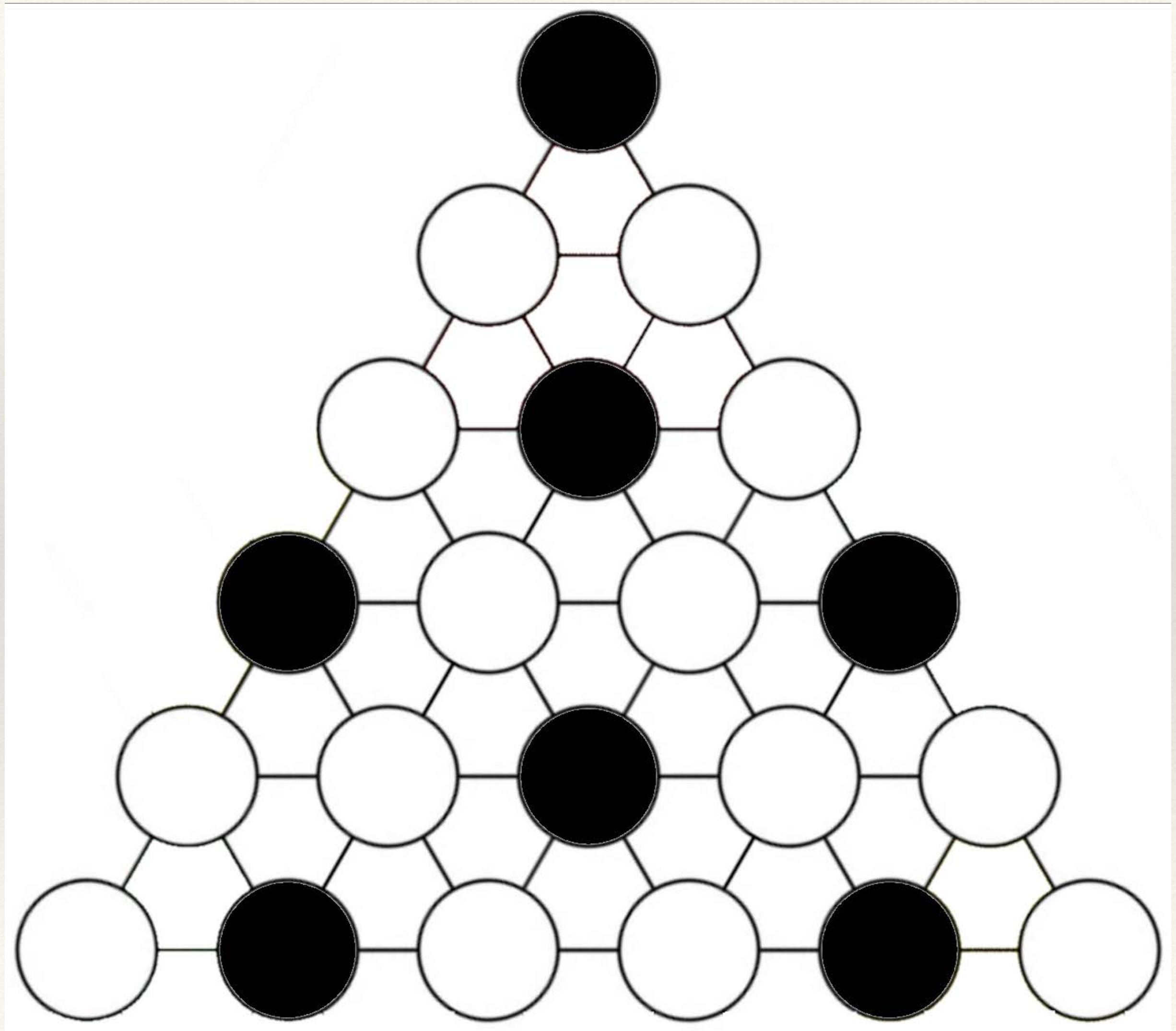
(a) (15%) Show that if we color any 9 of these nodes as black, we can always find two black nodes that are adjacent.

(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent.

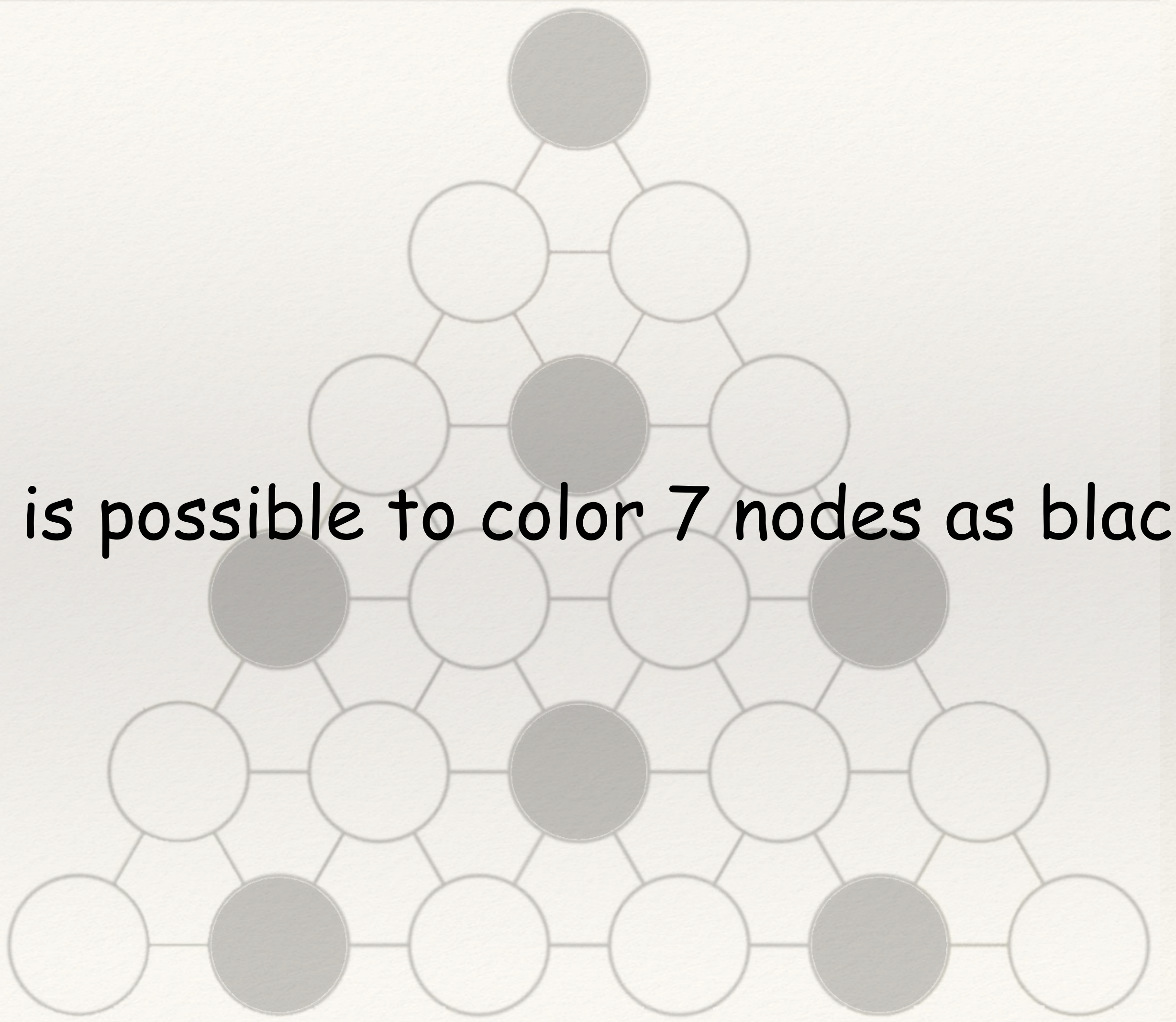
Question 6



Question 6

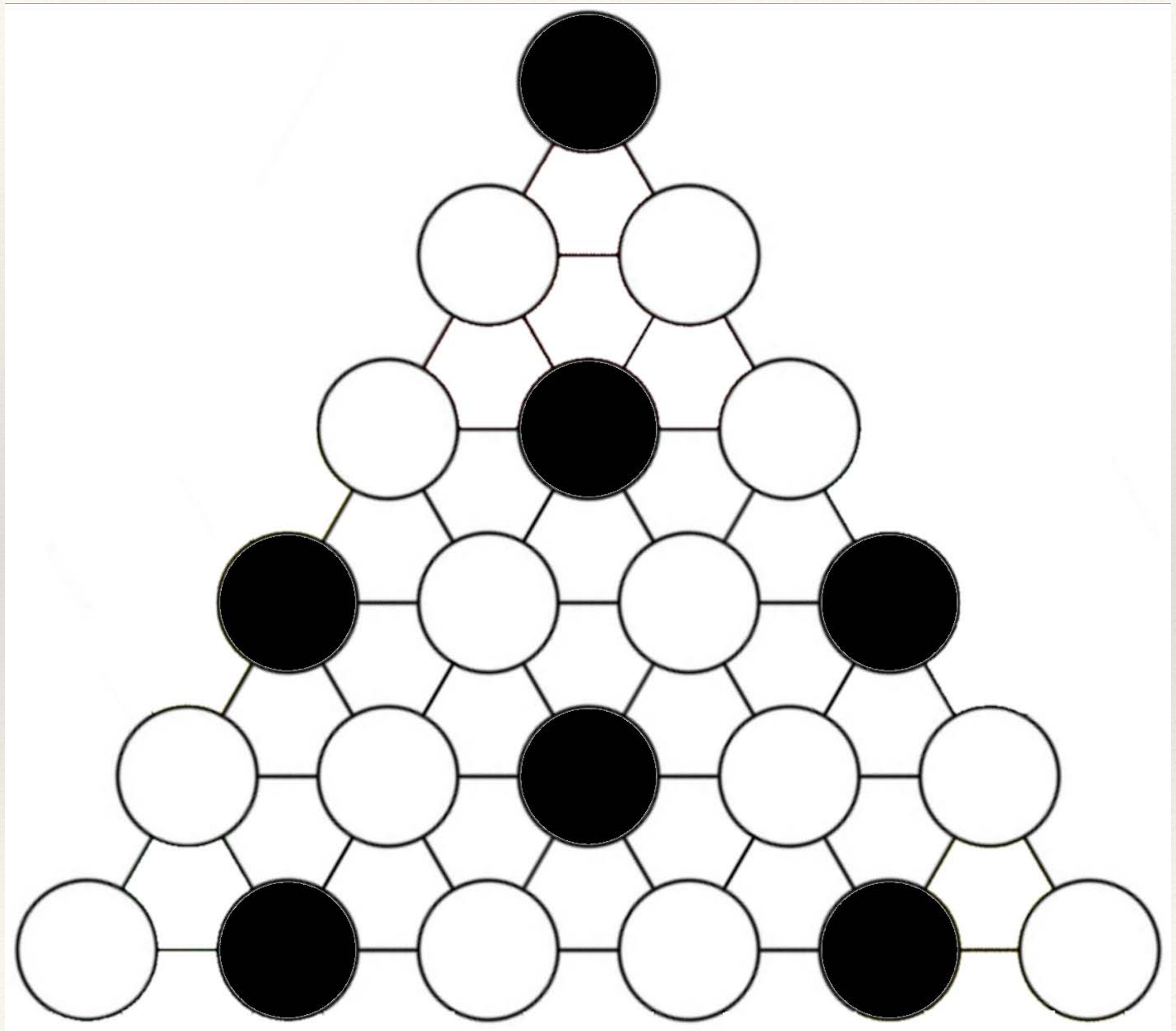


Question 6

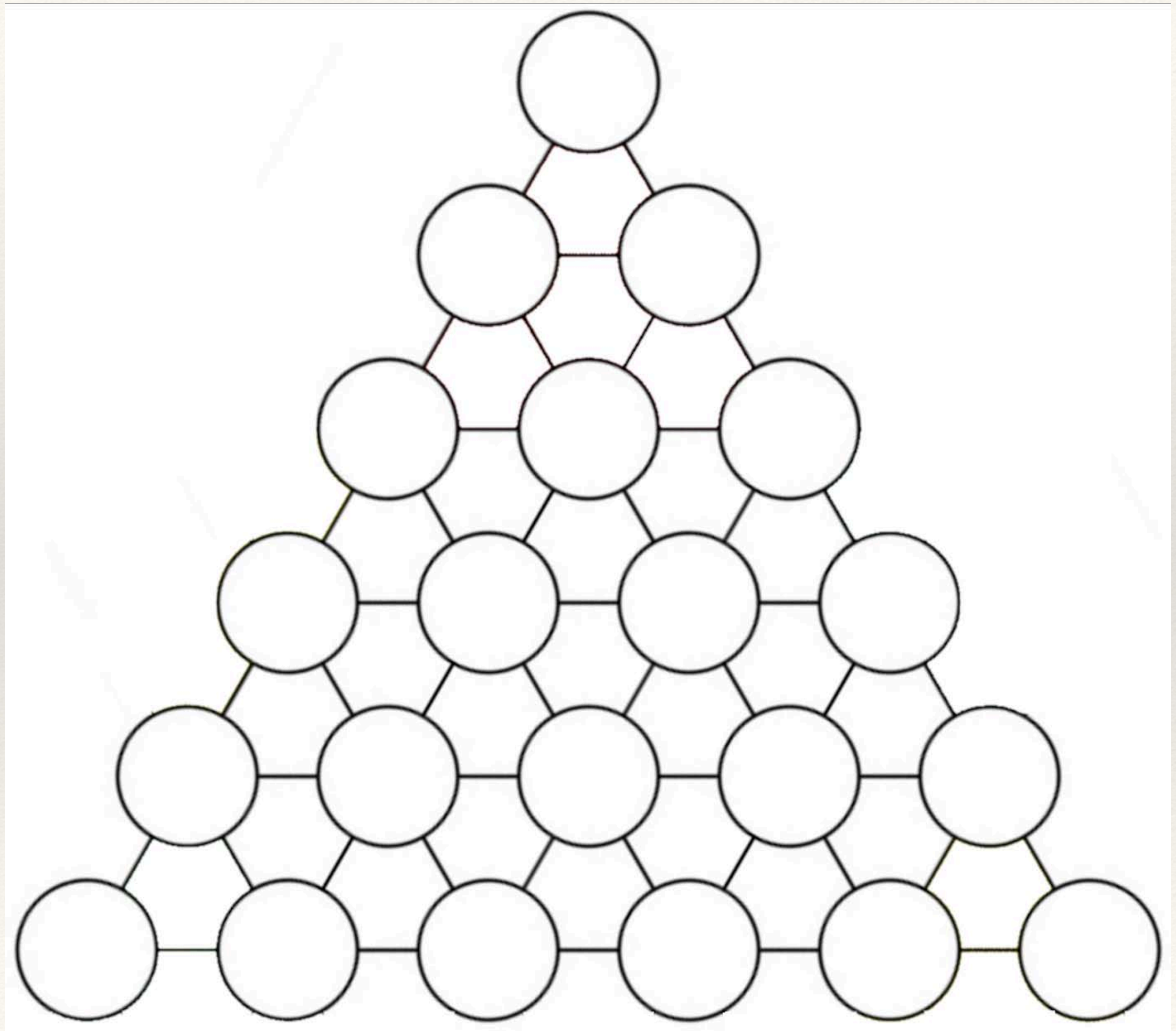


It is possible to color 7 nodes as black.

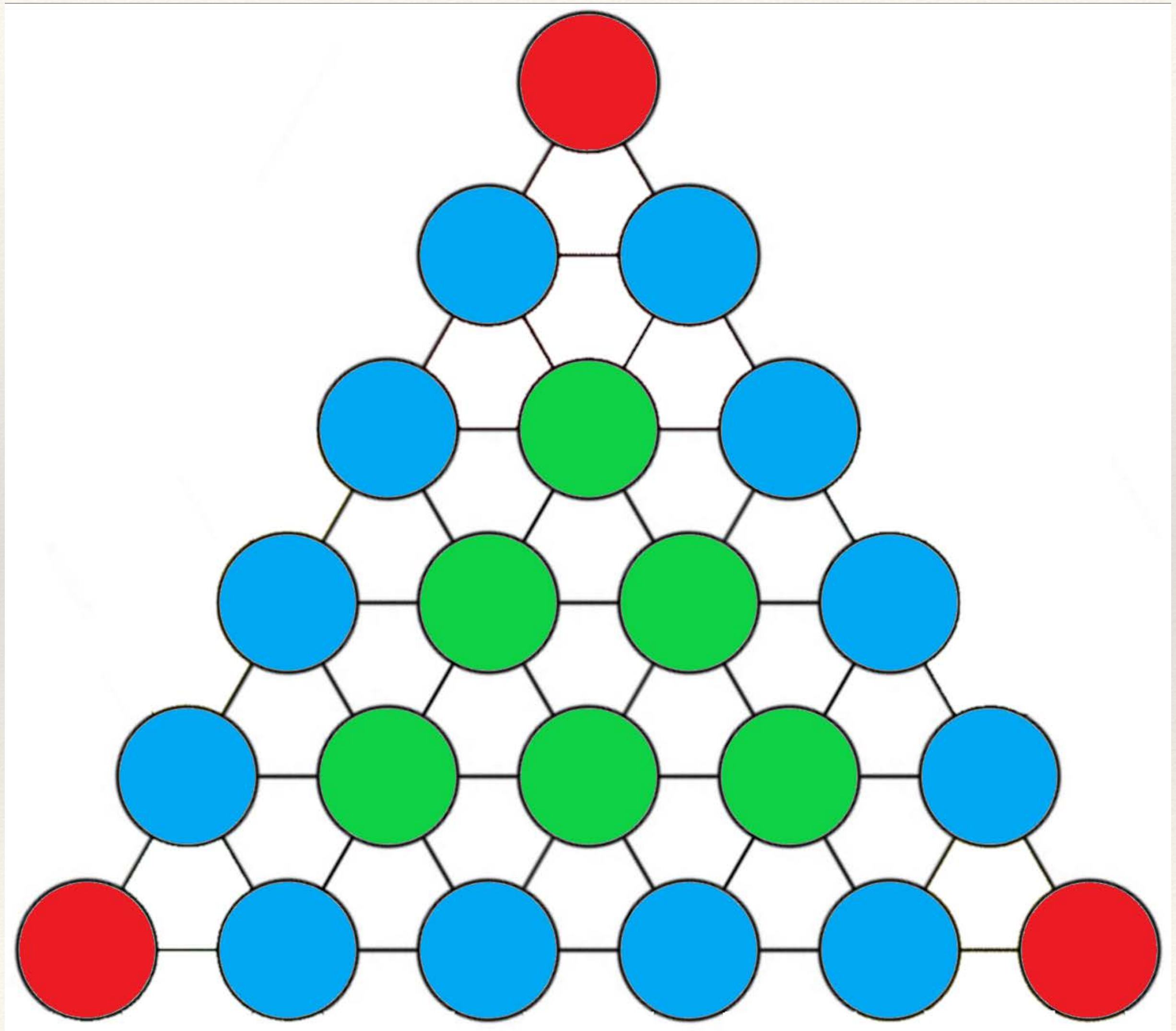
Question 6



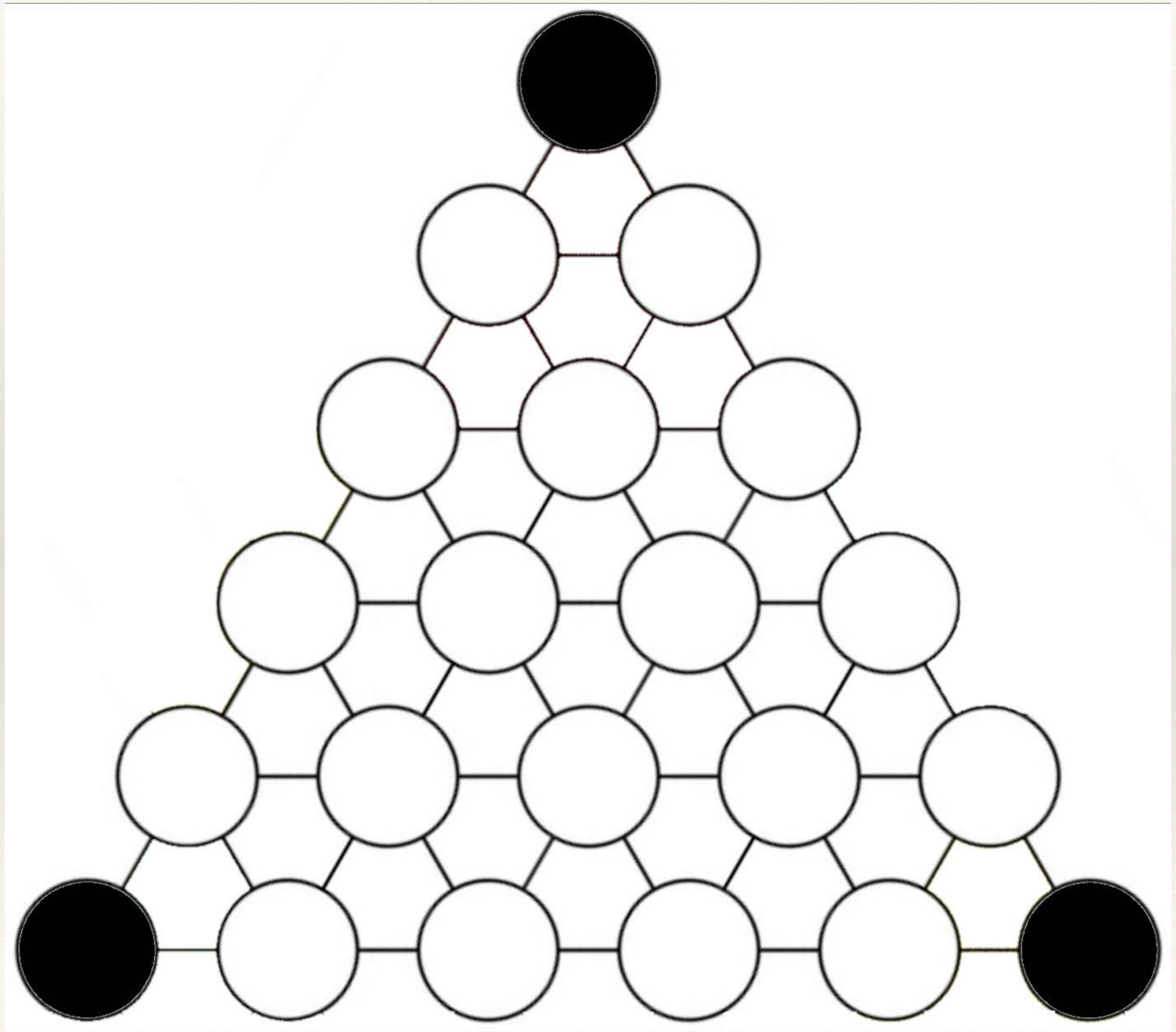
Question 6



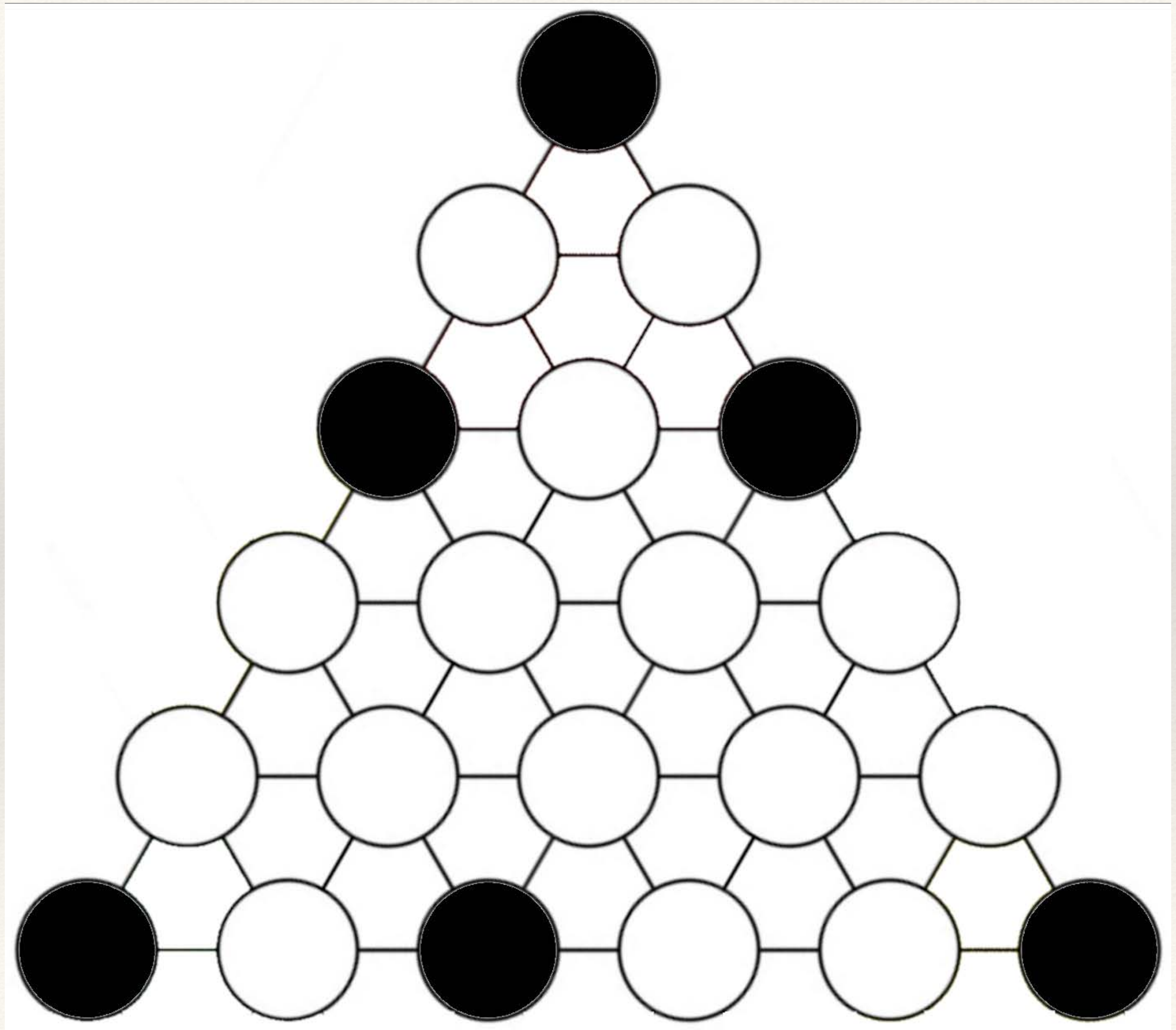
Question 6



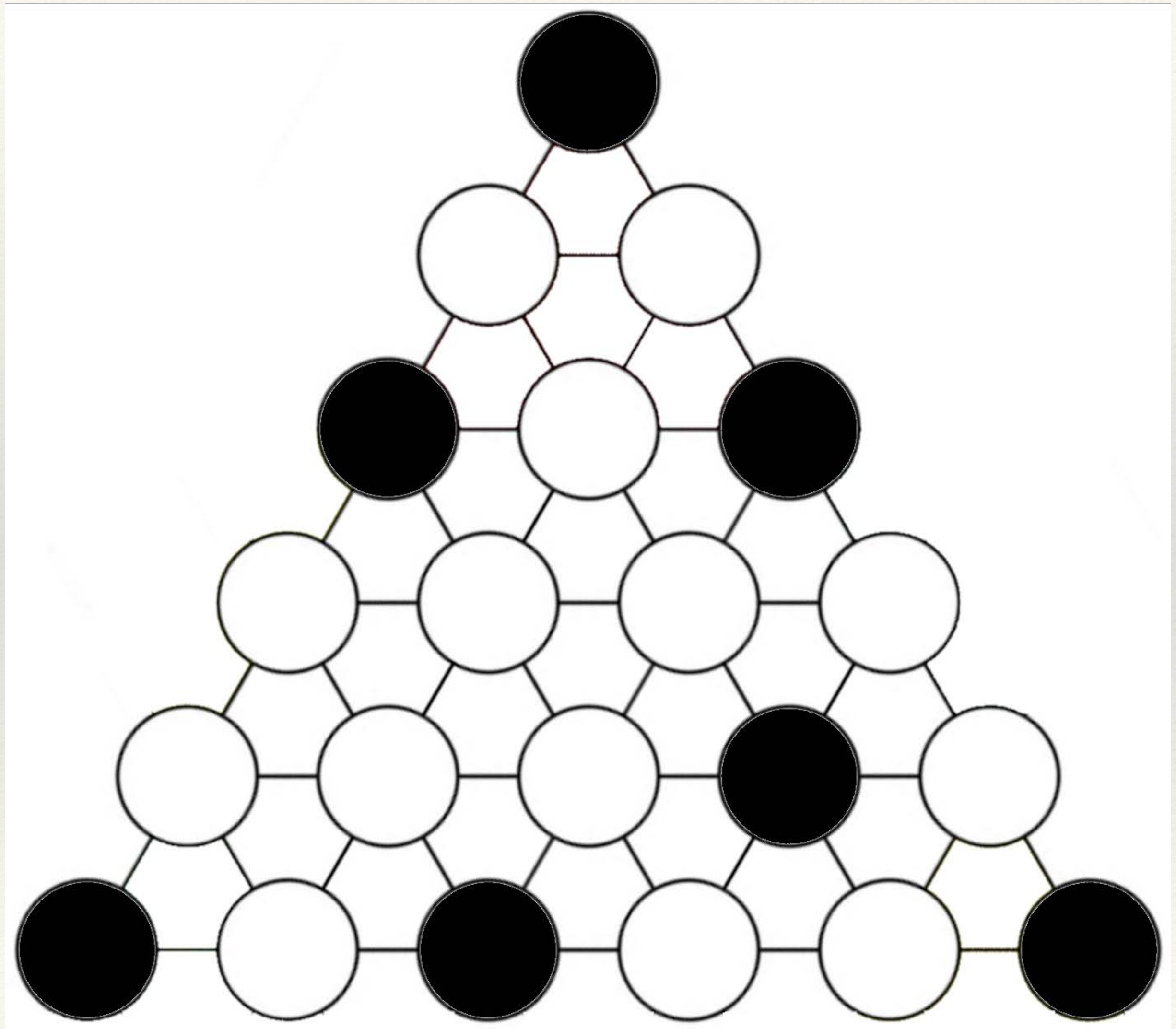
Question 6



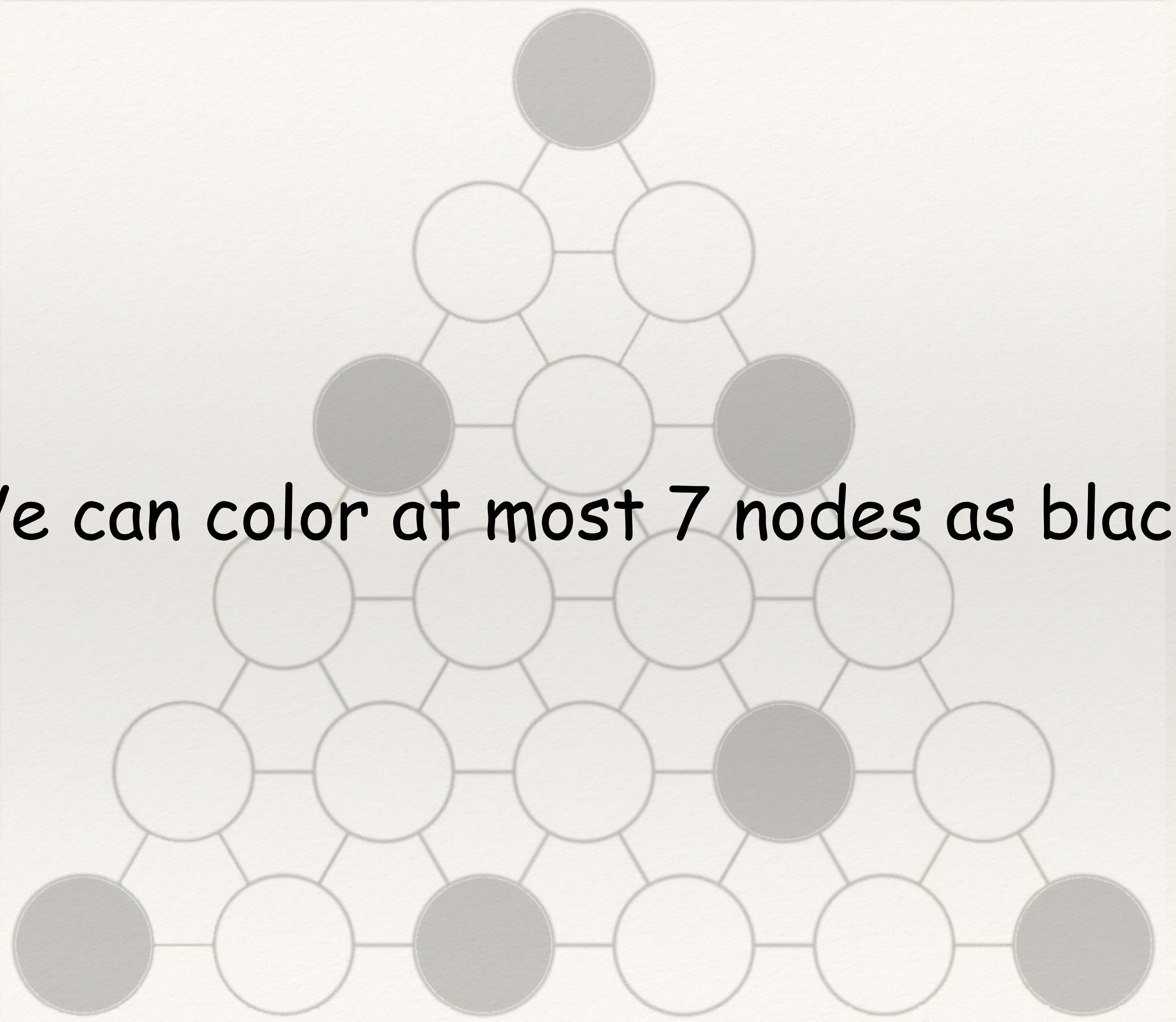
Question 6



Question 6

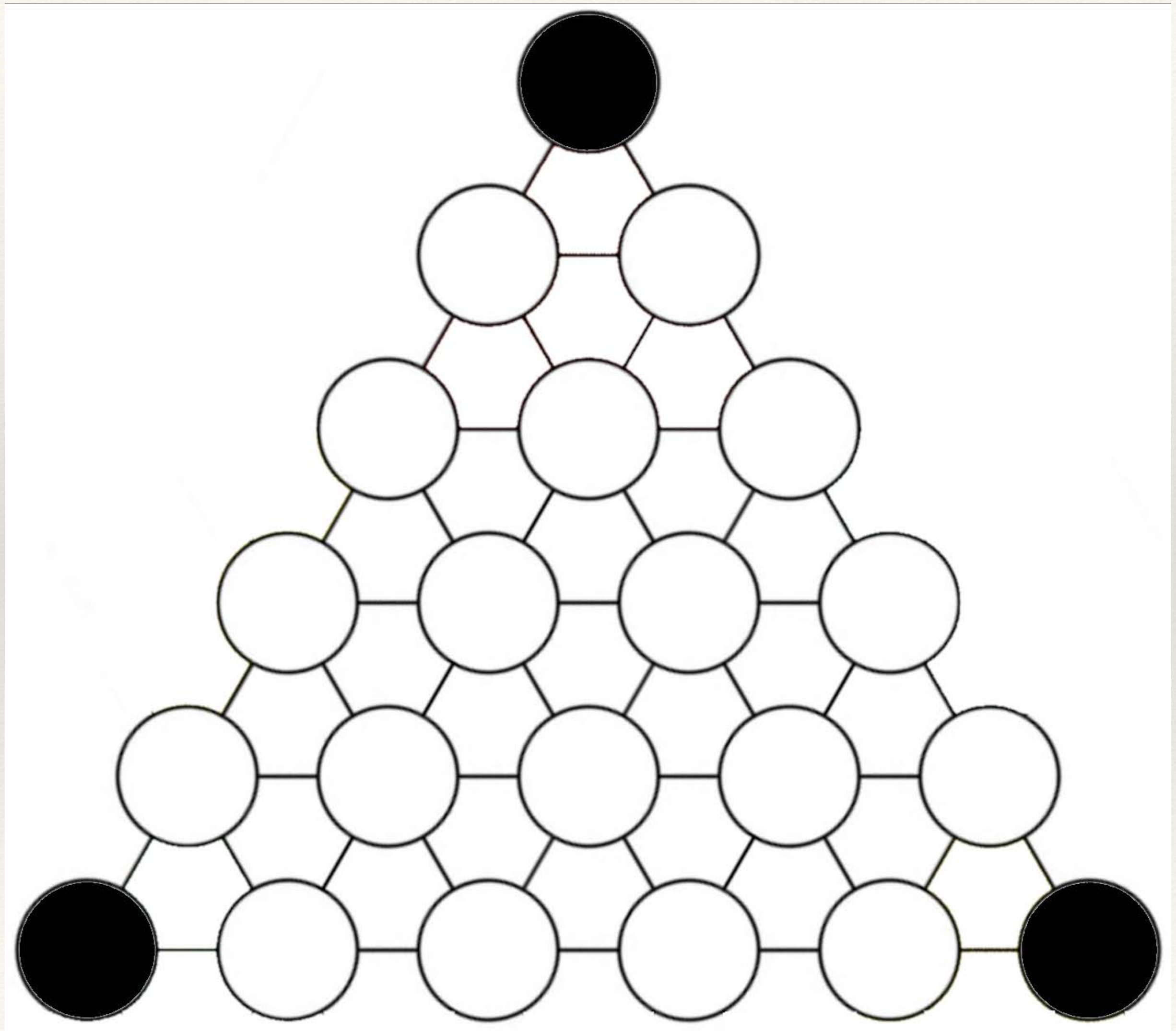


Question 6



We can color at most 7 nodes as black.

Question 6

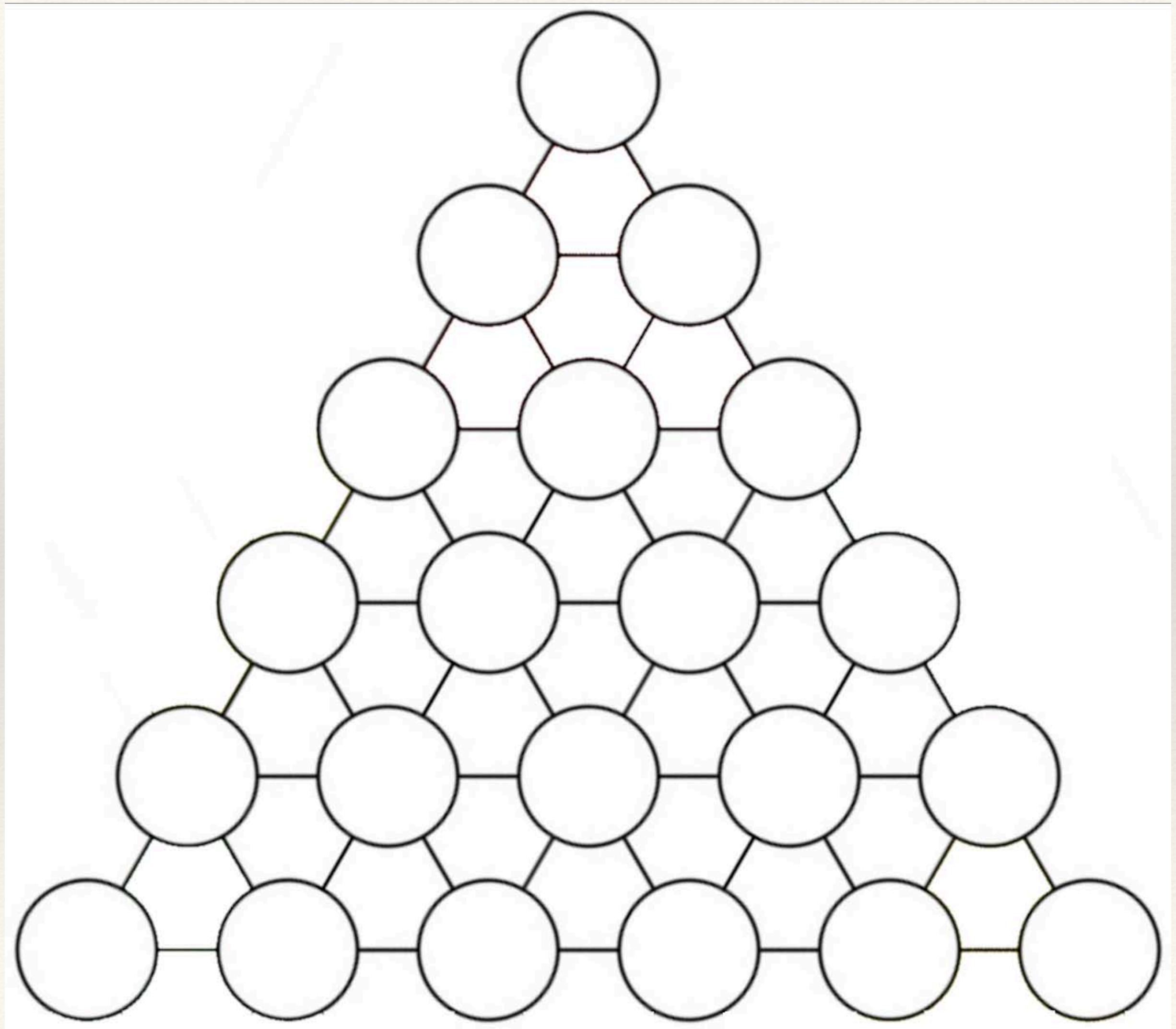


Question 6

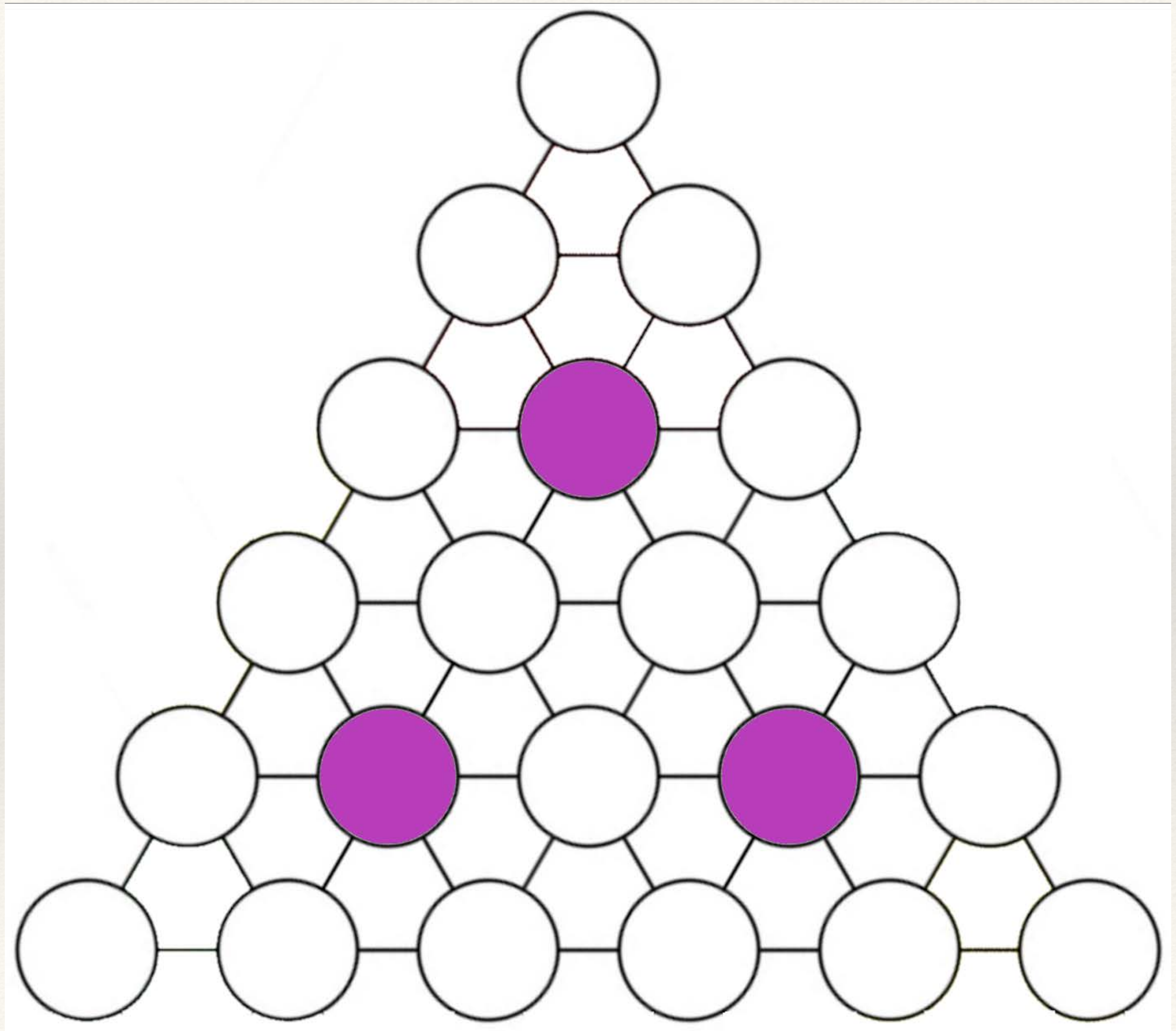
(a) (15%) Show that if we color any 9 of these nodes as black, we can always find two black nodes that are adjacent.

(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent.

Question 6



Question 6



Question 6

claim:

We cannot color more than 1 of these three nodes as black, if we want to color 8 nodes on the triangular grid and make sure that any two black nodes are not adjacent.

Question 6

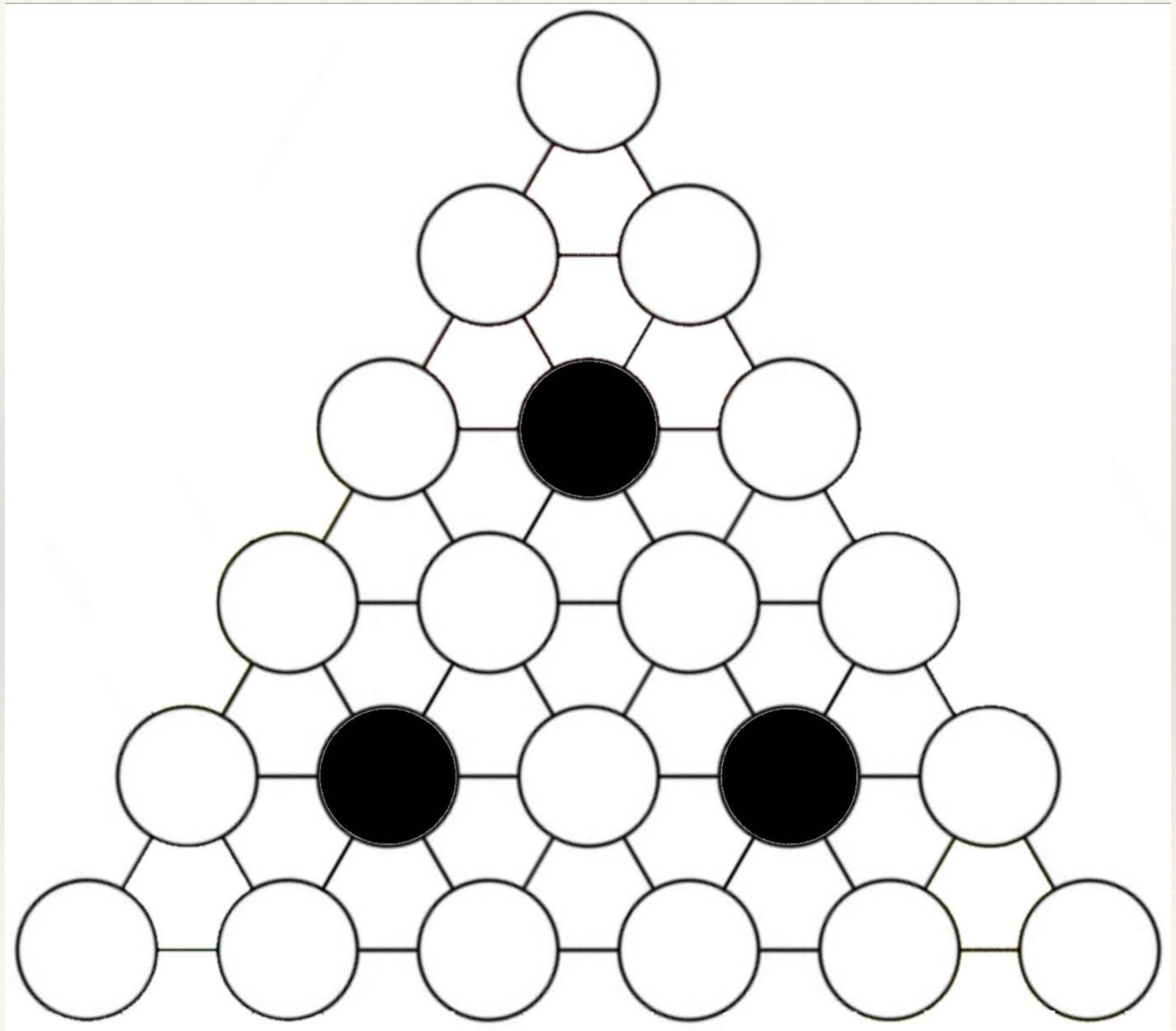
claim:

We cannot color more than 1 of these three nodes as black, if we want to color 8 nodes on the triangular grid and make sure that any two black nodes are not adjacent.

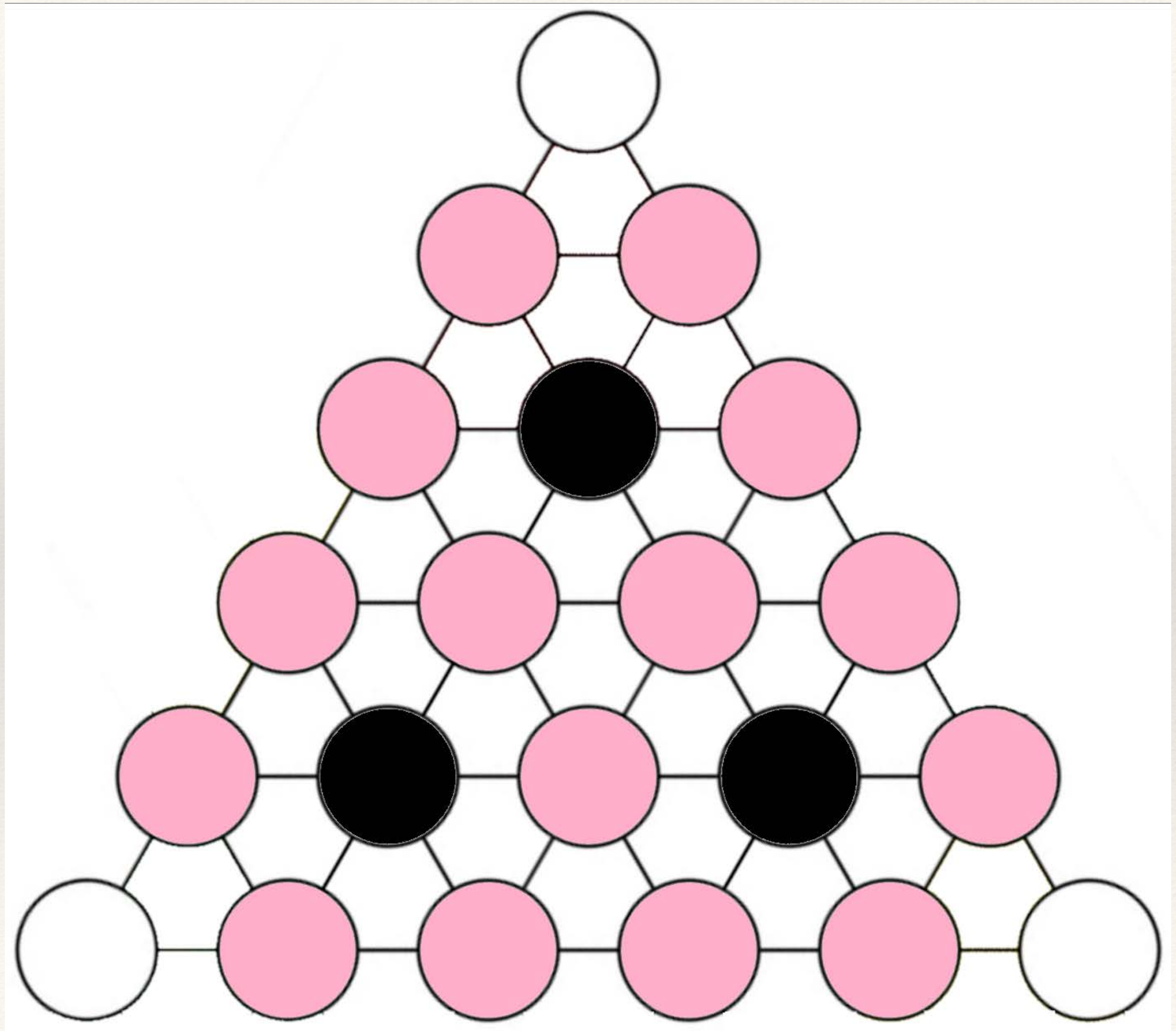
Case 1: We color 2 of these three nodes.

Case 2: We color all the 3 nodes.

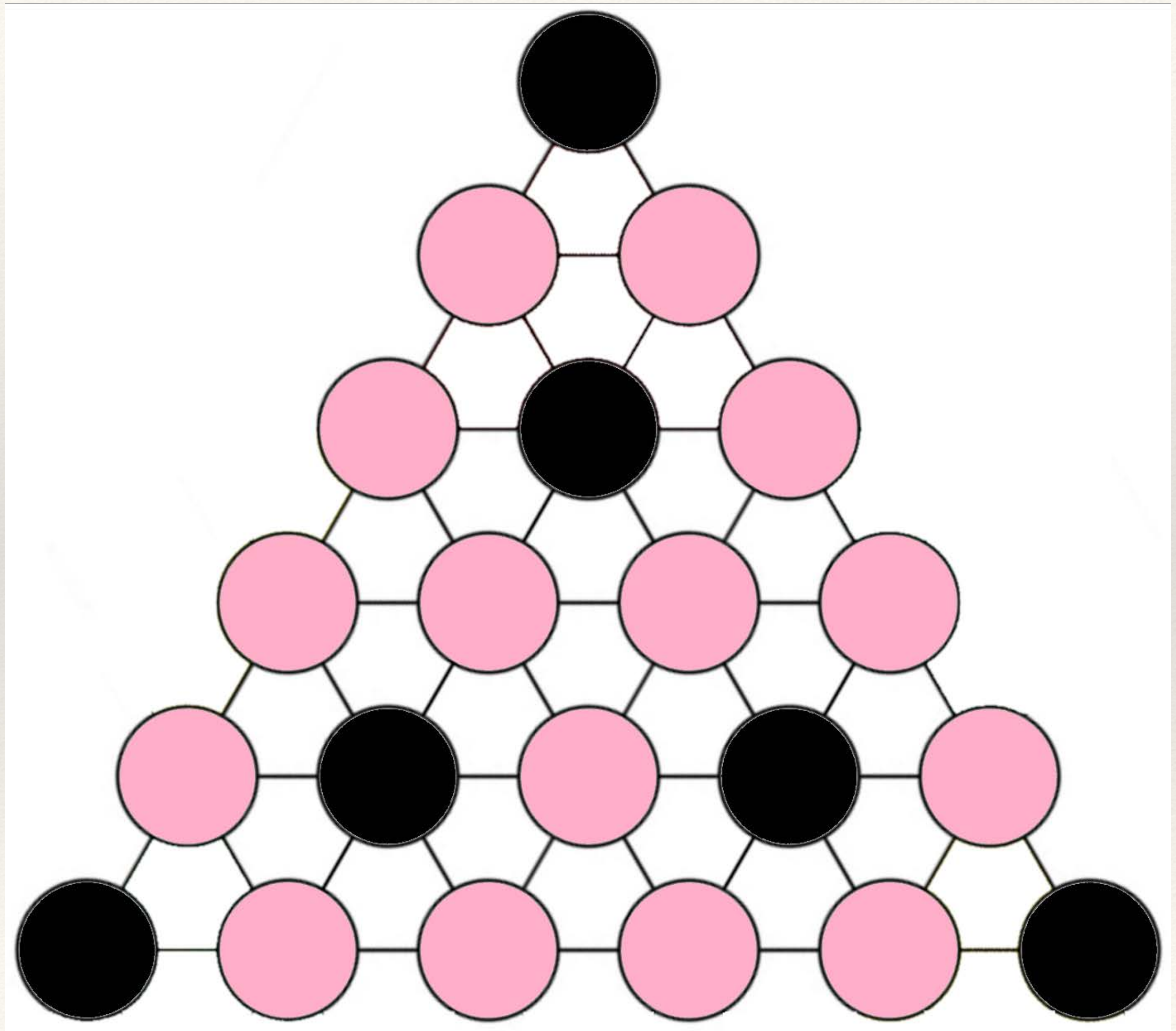
Question 6



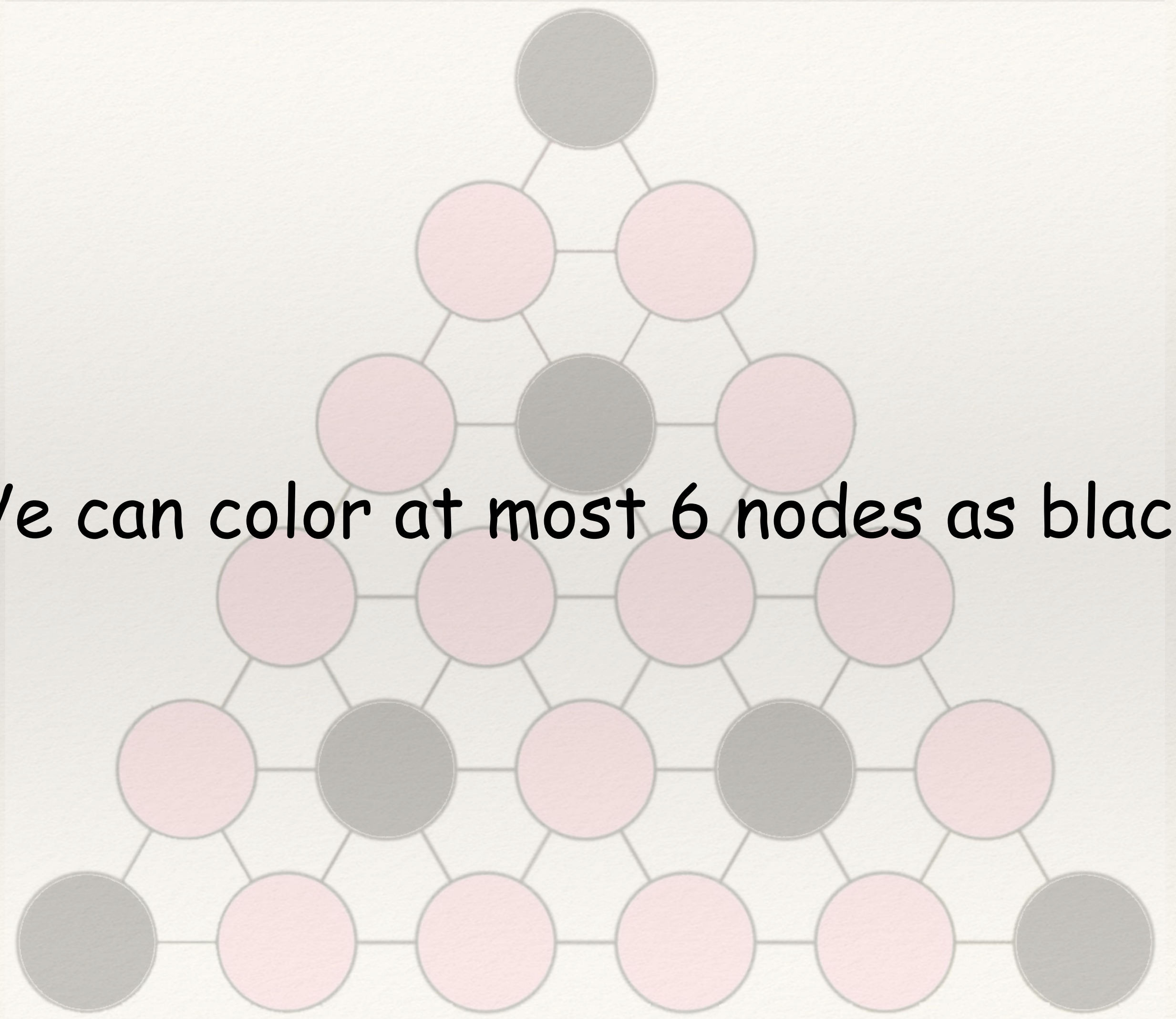
Question 6



Question 6



Question 6



We can color at most 6 nodes as black.

Question 6

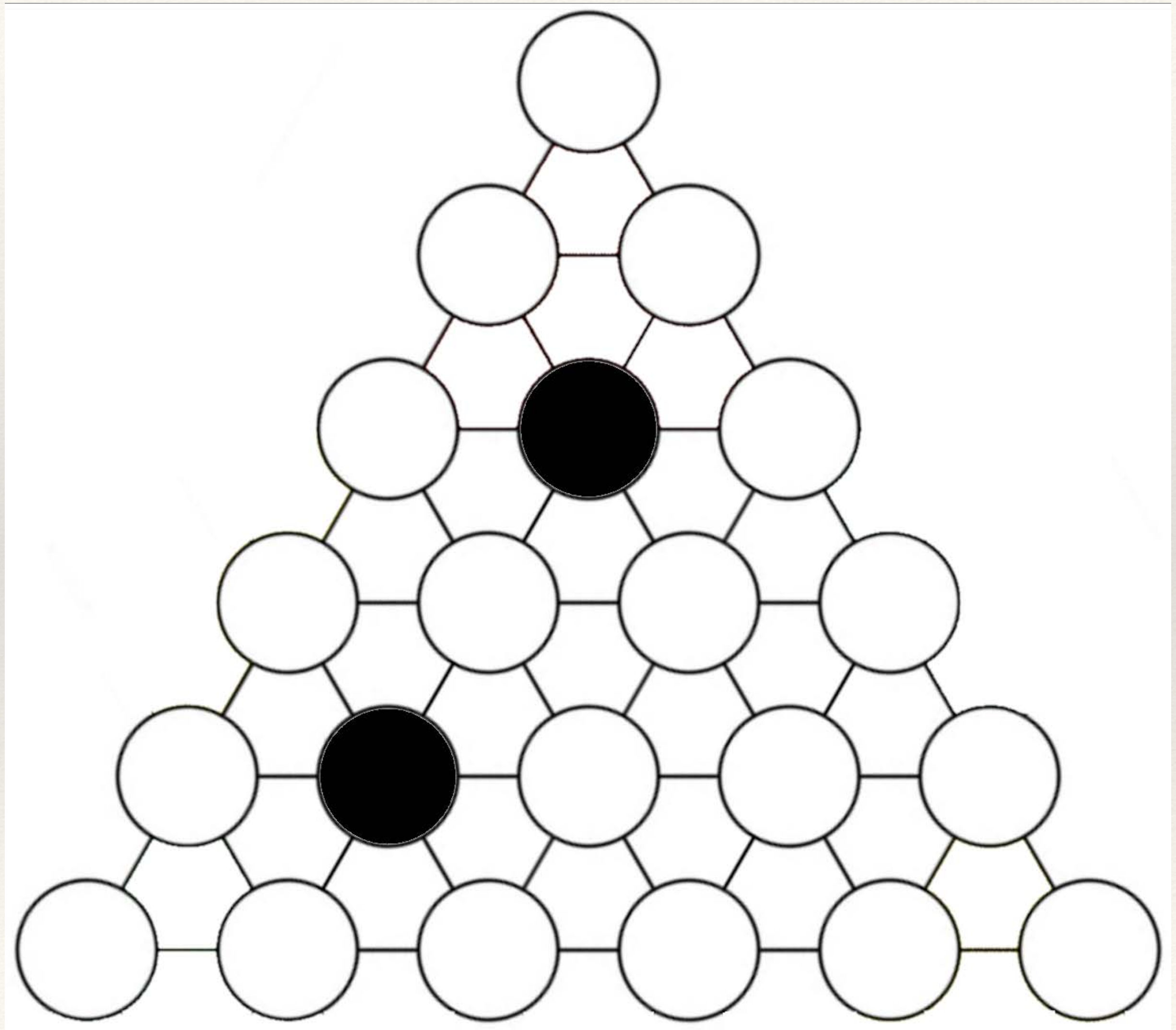
claim:

We cannot color more than 1 of these three nodes as black, if we want to color 8 nodes on the triangular grid and make sure that any two black nodes are not adjacent.

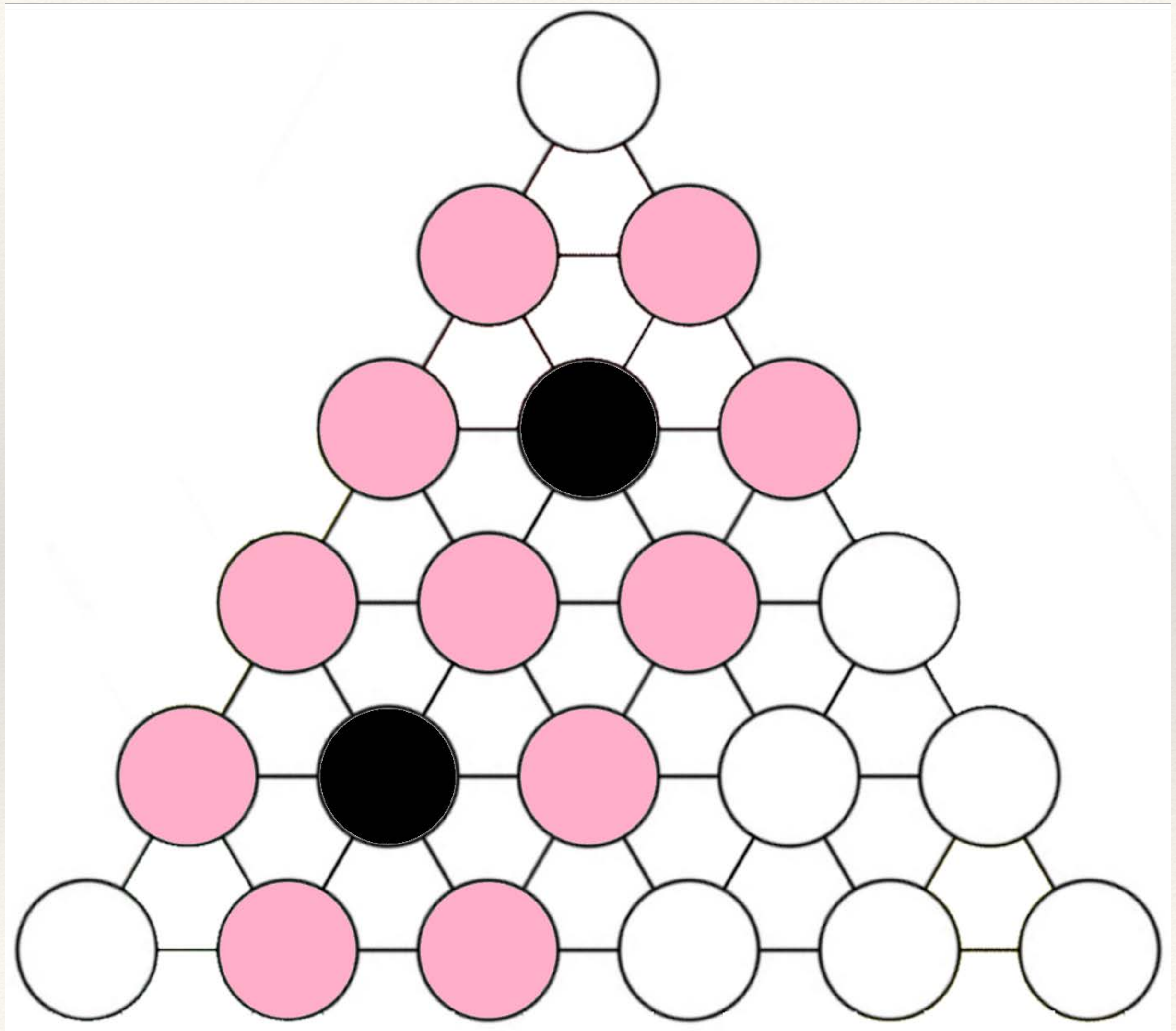
Case 1: We color 2 of these three nodes.

Case 2: We color all the 3 nodes. (proved)

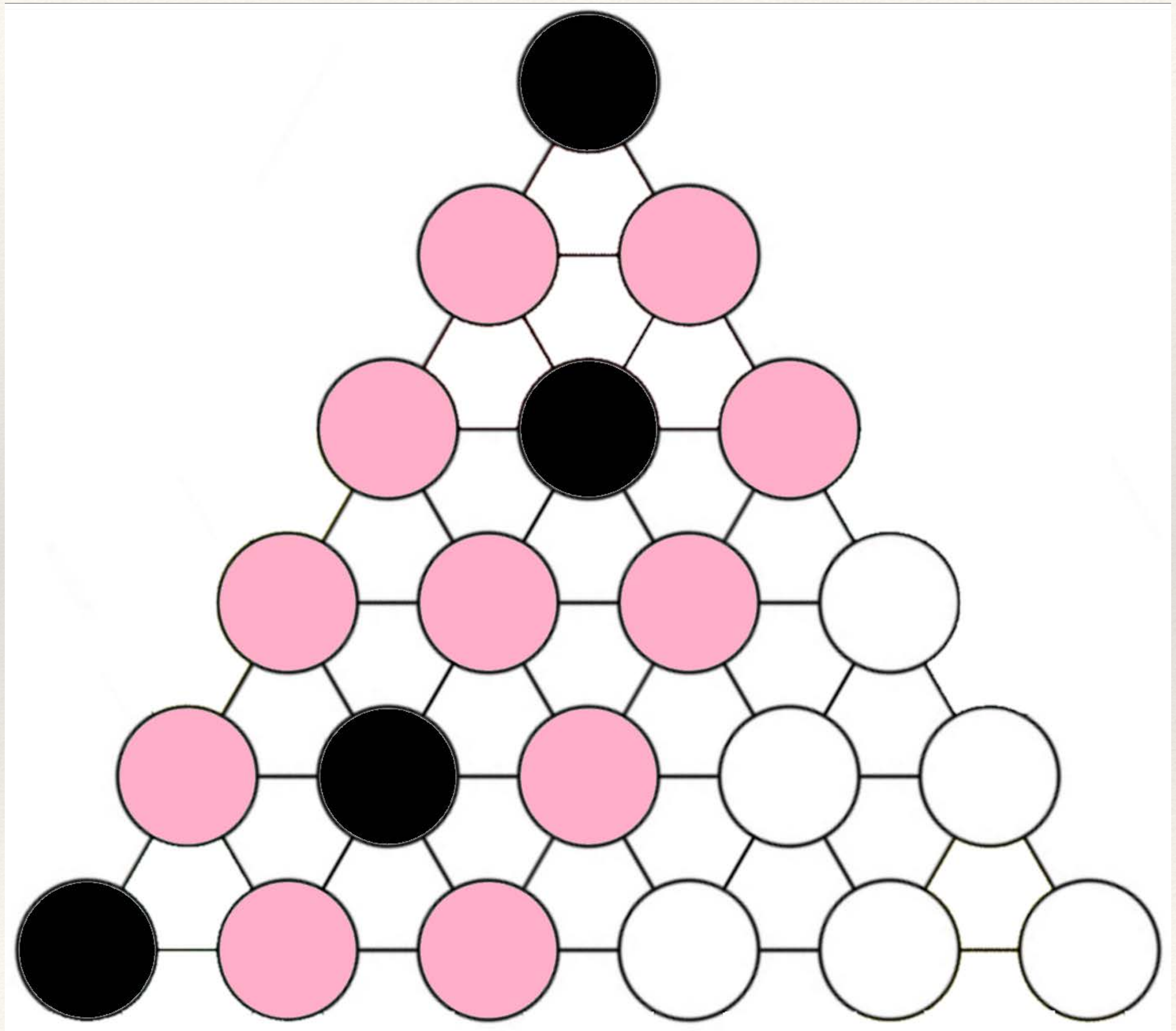
Question 6



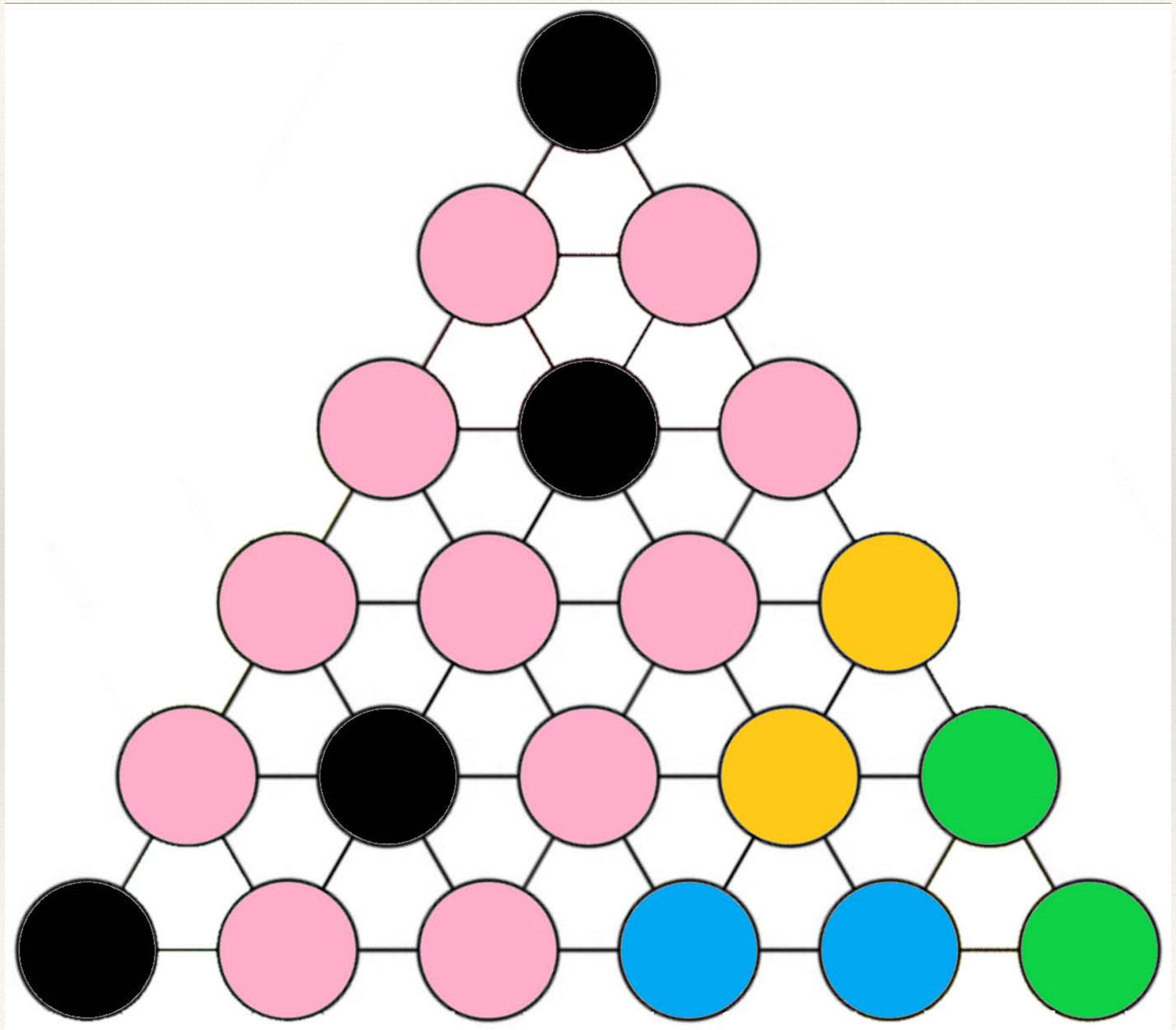
Question 6



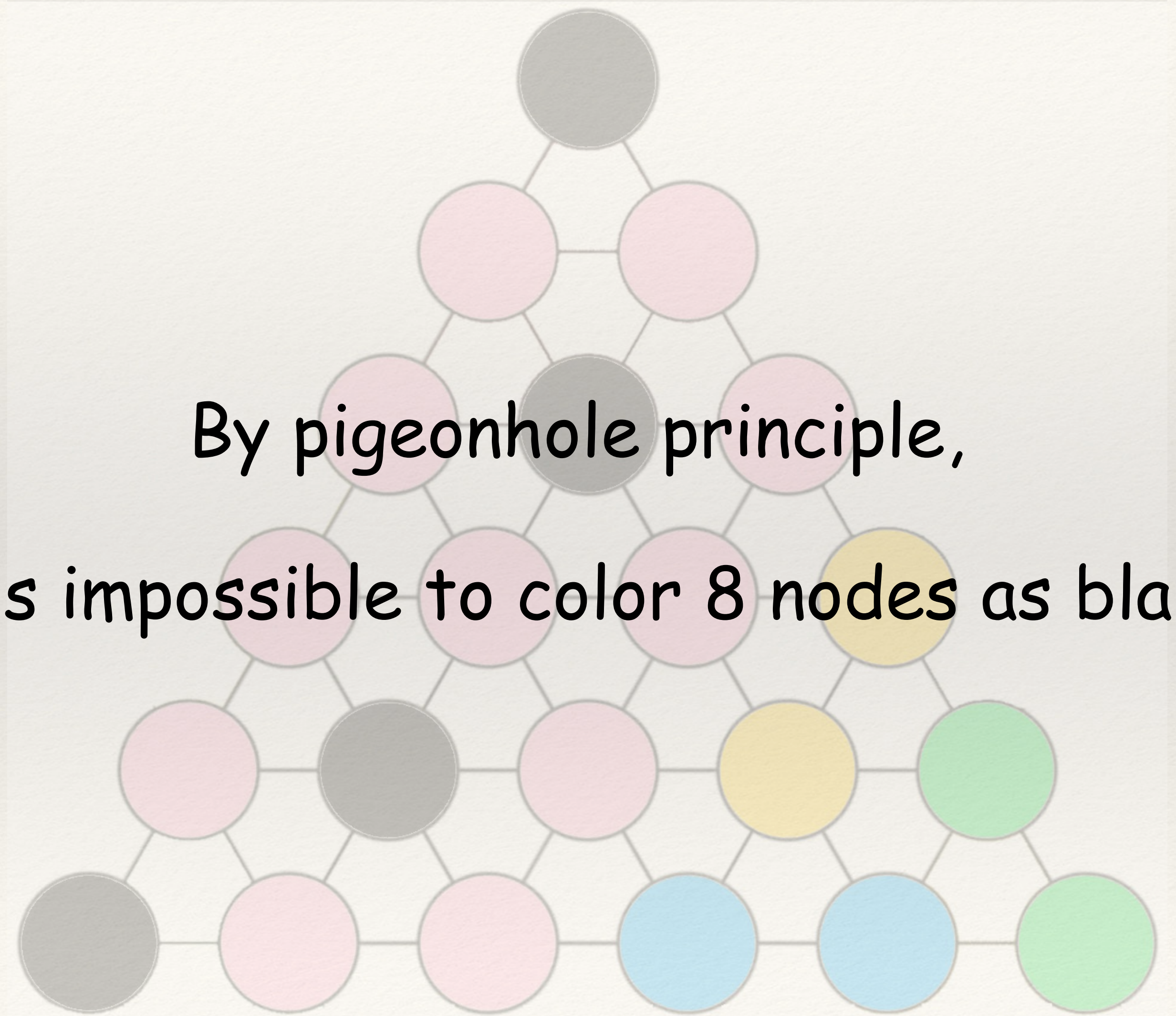
Question 6



Question 6



Question 6



By pigeonhole principle,

it is impossible to color 8 nodes as black.

Question 6

claim:

We cannot color more than 1 of these three nodes as black, if we want to color 8 nodes on the triangular grid and make sure that any two black nodes are not adjacent.

Case 1: We color 2 of these three nodes. (proved)

Case 2: We color all the 3 nodes. (proved)

Question 6

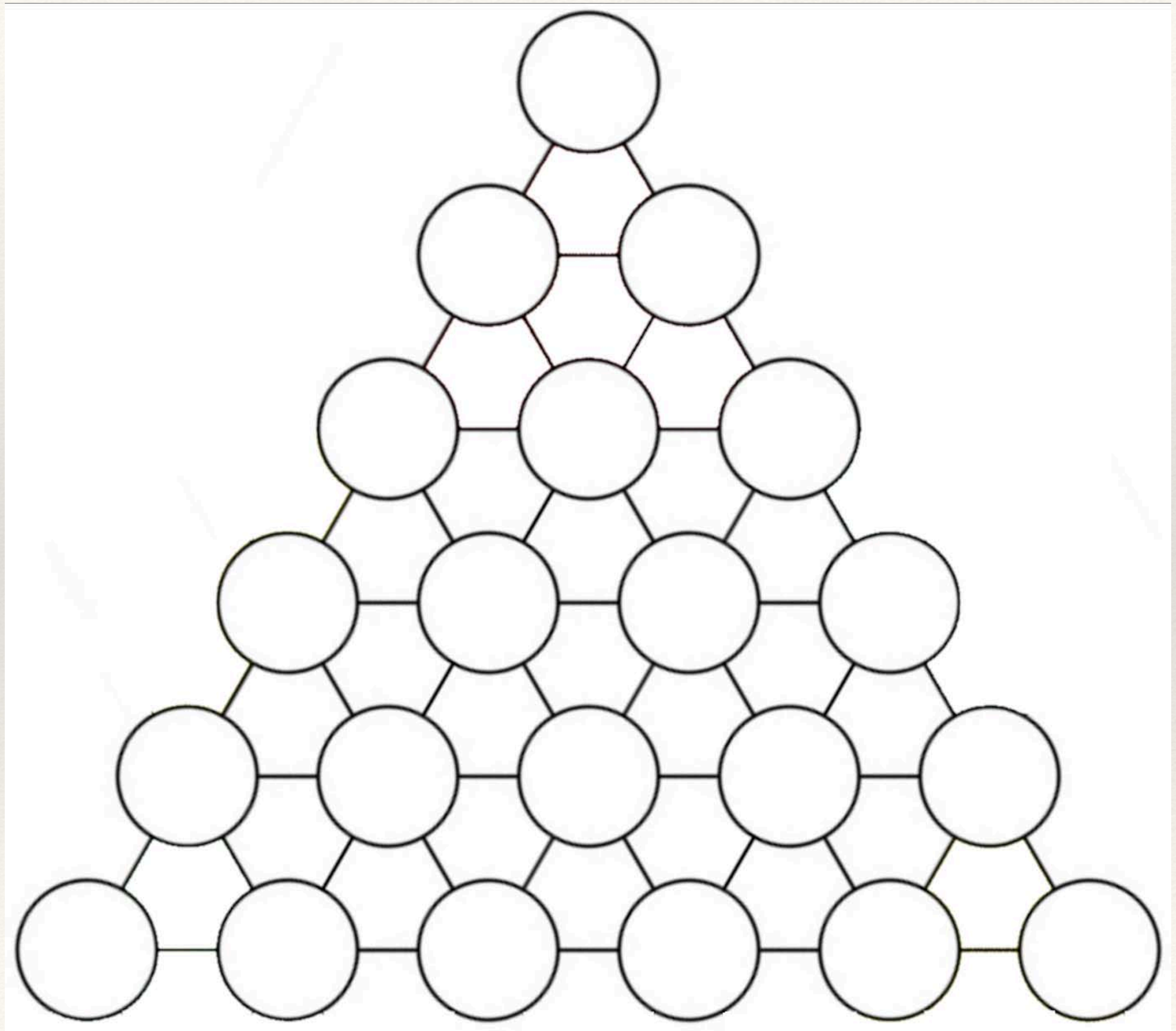
(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent.

Question 6

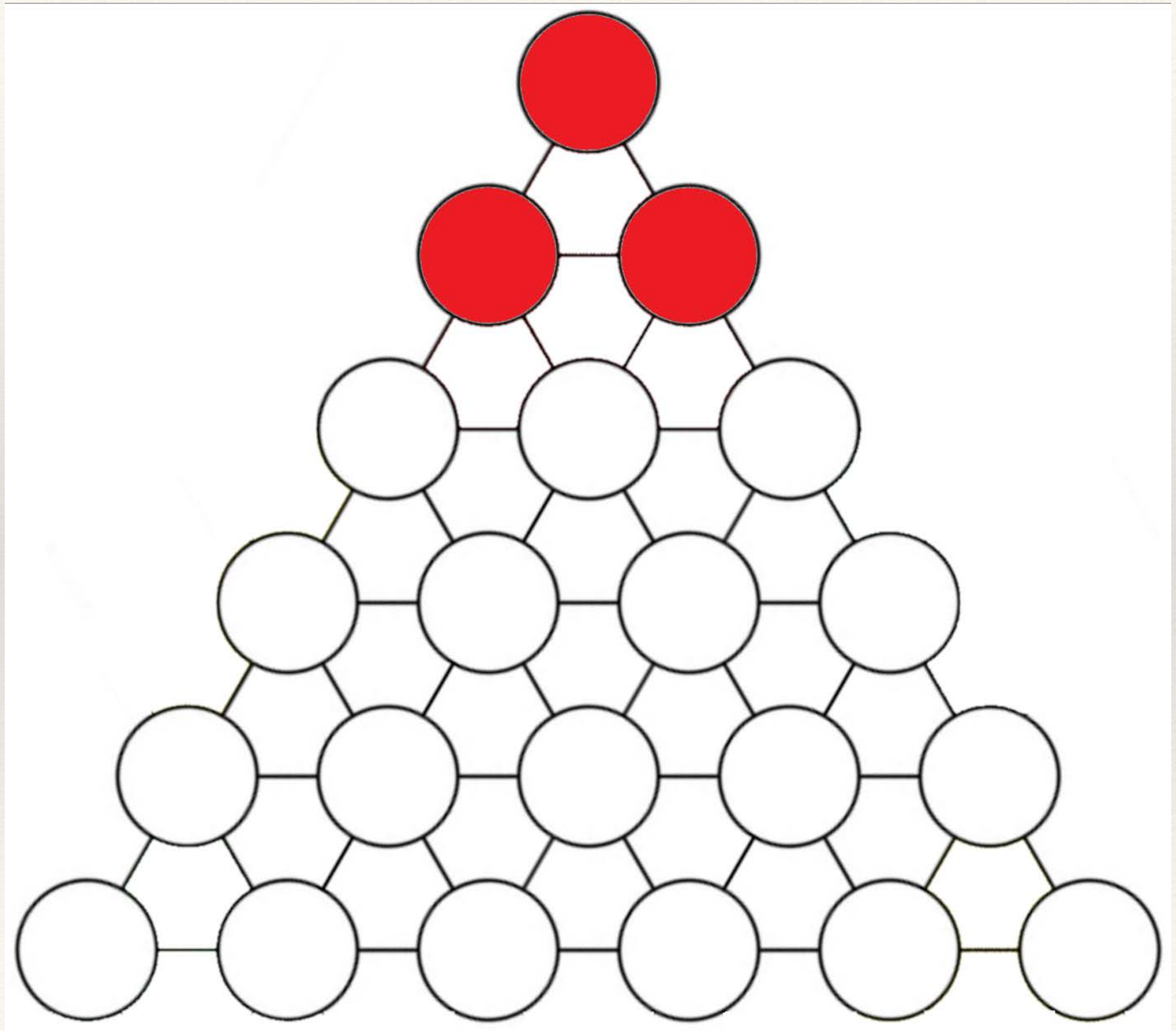
(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent.

(b) (5%, Challenging) If we make sure that any two black nodes are not adjacent, it is impossible to color 8 nodes on the triangular grid.

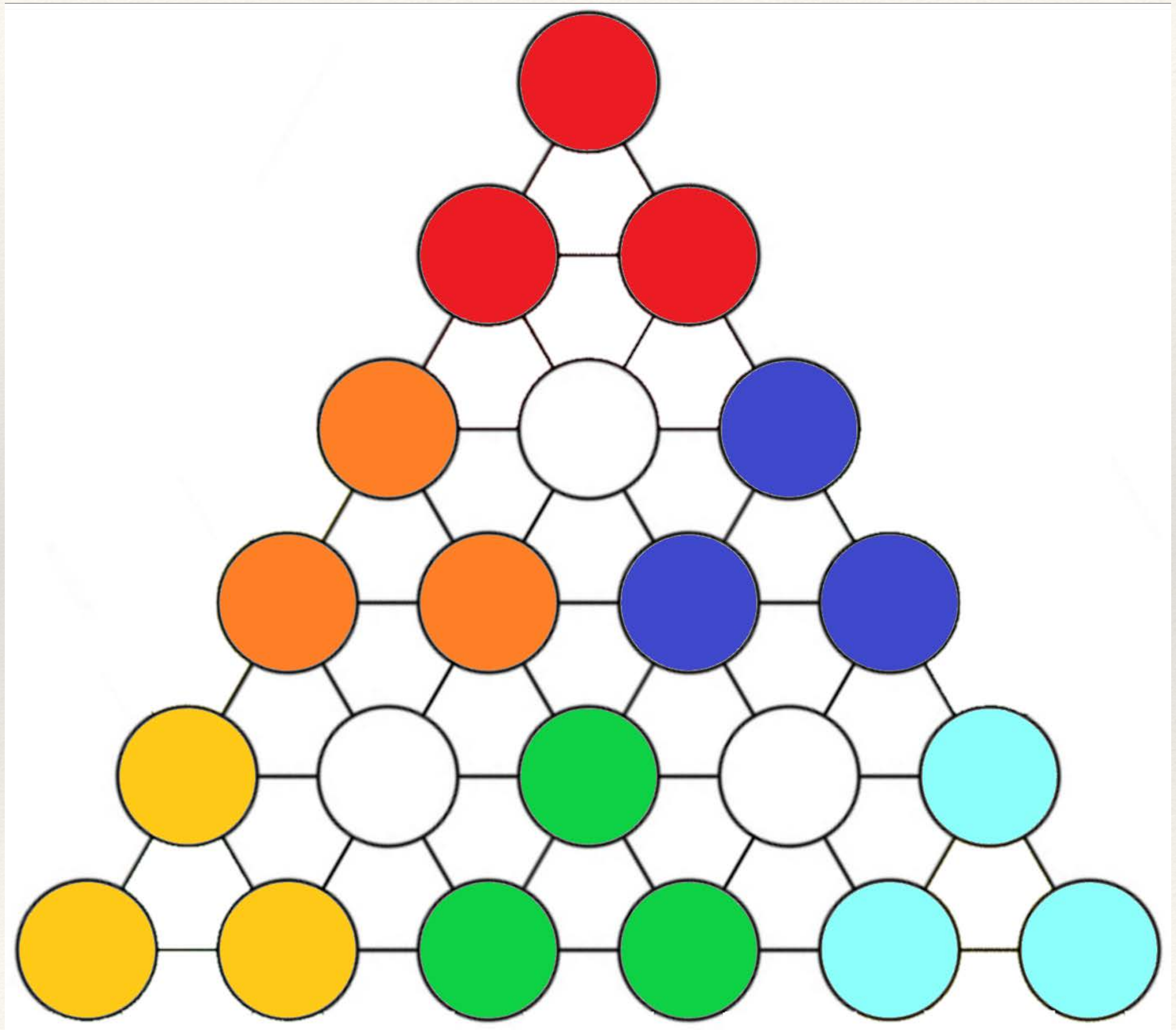
Question 6



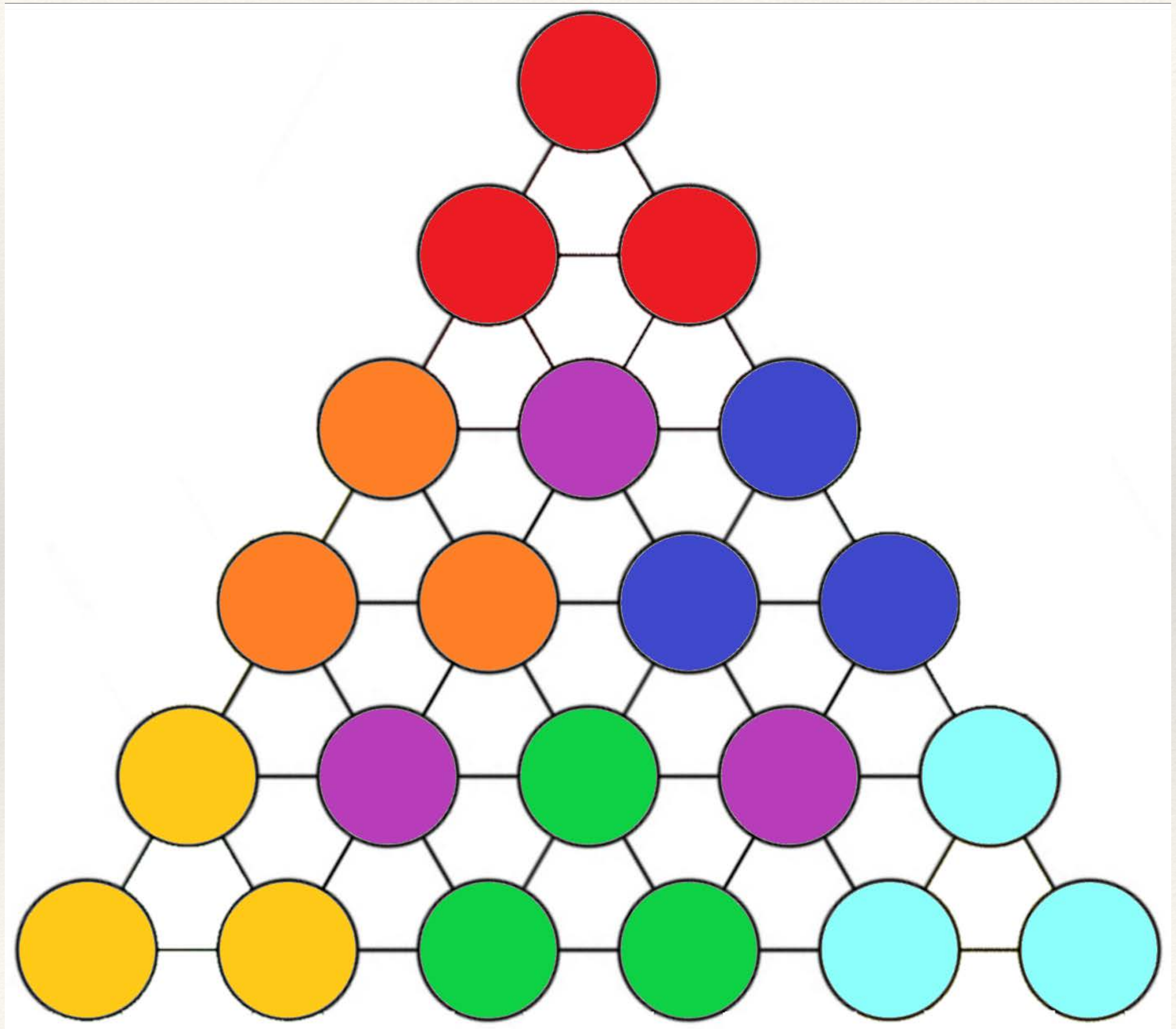
Question 6



Question 6



Question 6

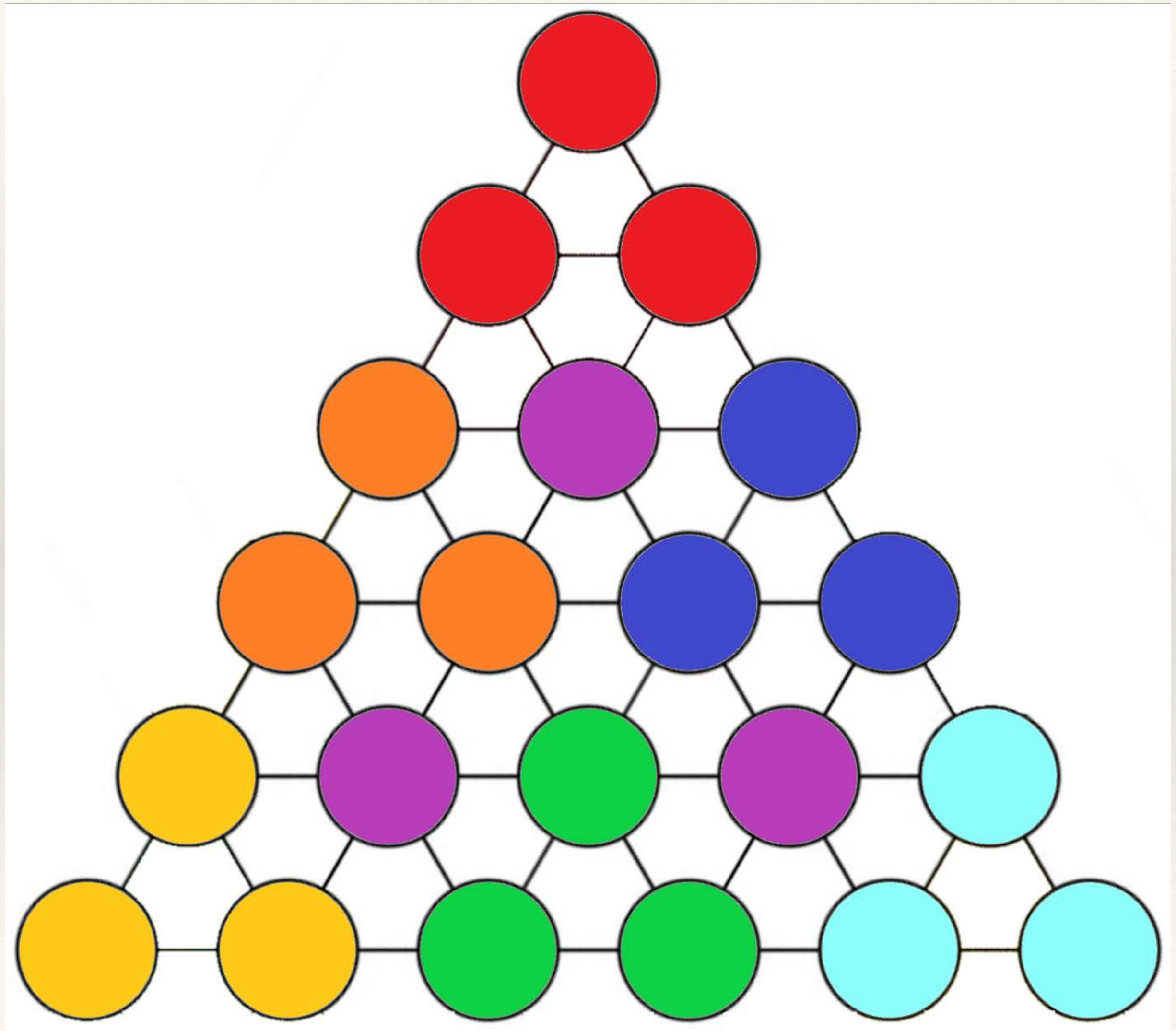


Question 6

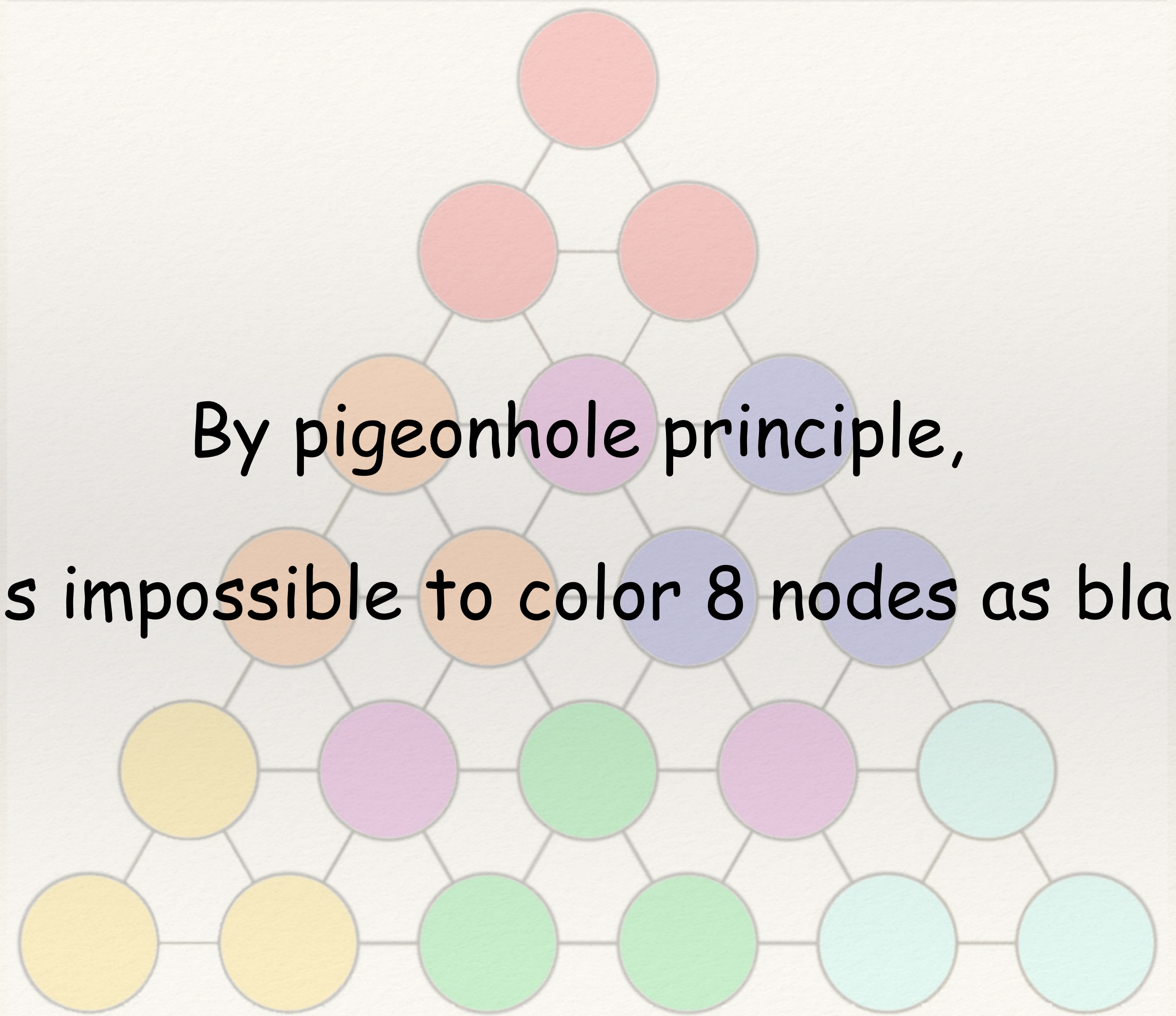
claim:

We cannot color more than 1 of these three nodes as black, if we want to color 8 nodes on the triangular grid and make sure that any two black nodes are not adjacent.

Question 6



Question 6



By pigeonhole principle,

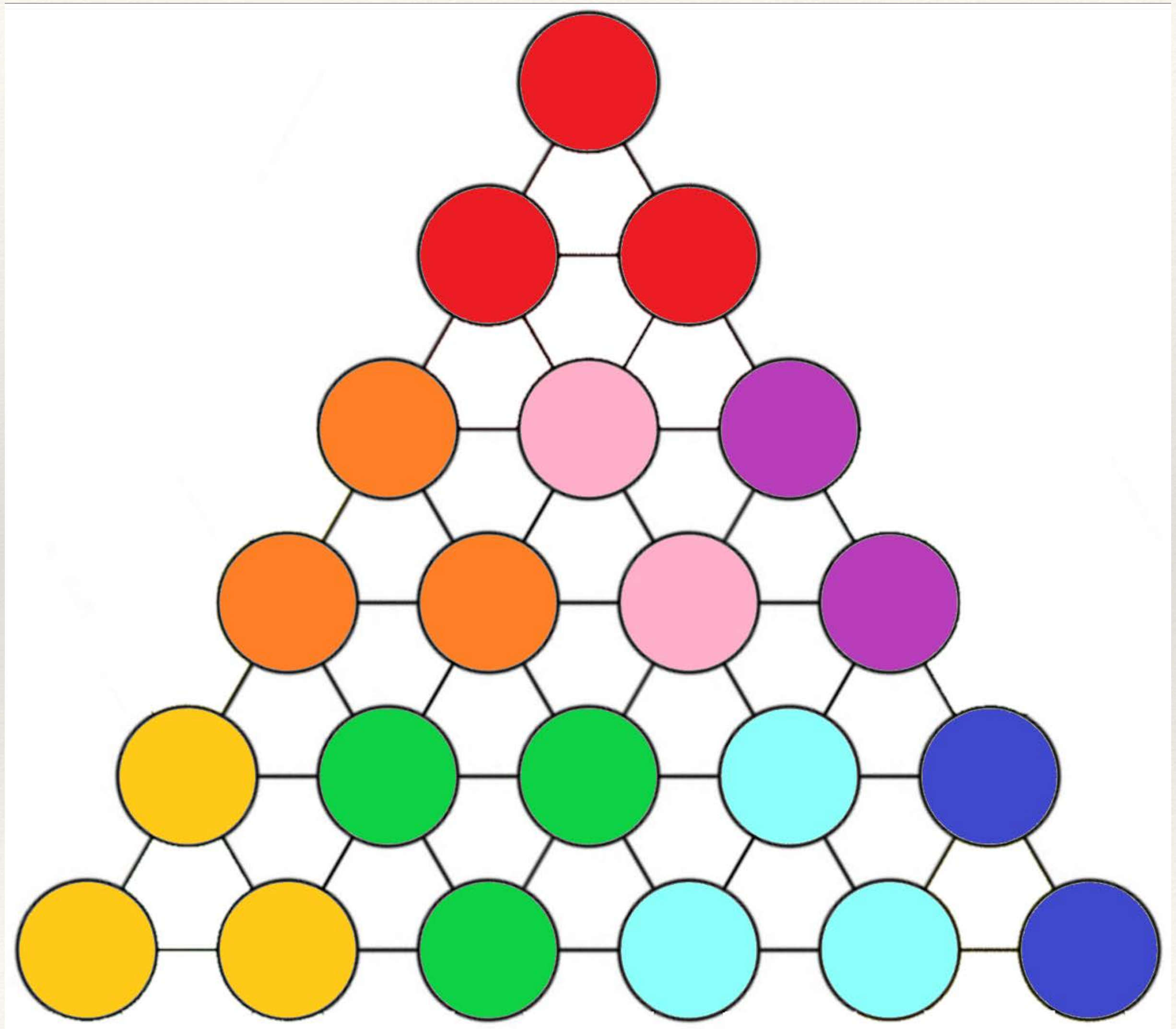
it is impossible to color 8 nodes as black.

Question 6

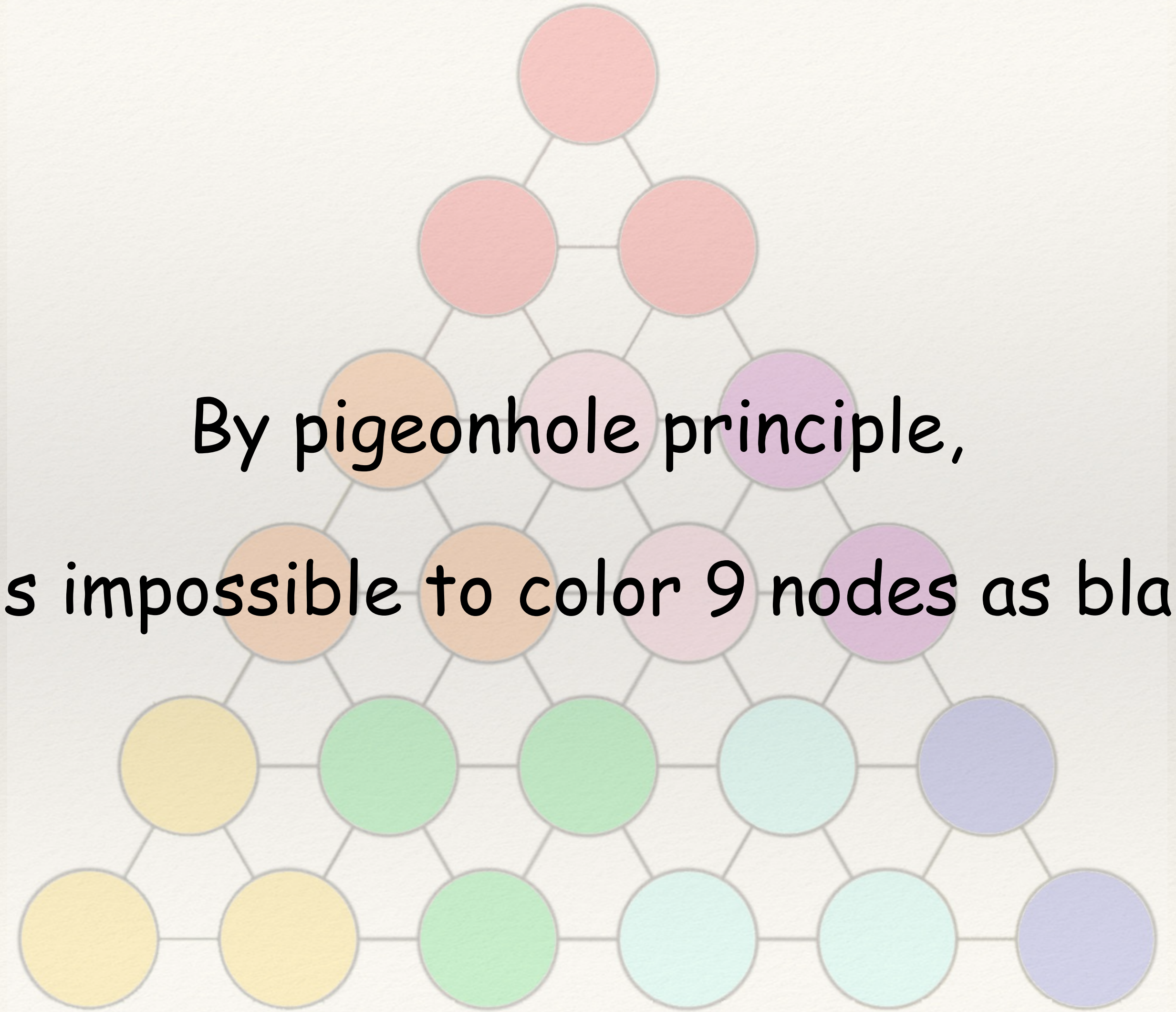
(a) (15%) Show that if we color any 9 of these nodes as black, we can always find two black nodes that are adjacent.

(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent. (proved)

Question 6



Question 6



By pigeonhole principle,

it is impossible to color 9 nodes as black.

Question 6

(a) (15%) Show that if we color any 9 of these nodes as black, we can always find two black nodes that are adjacent. (proved)

(b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent. (proved)

Q7

高紀威

Statistics

- The question is from 19th IMO 1977 Problem 6.
- Link: https://www.imo-official.org/year_statistics.aspx?year=1977

- In IMO(8 points), Mean = 3.054, Max = 8
- In Class(5 points), Mean = 0.000, Max = 0

Question

Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set.

Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then

$$f(n) = n \text{ for each } n.$$

Solution

The first step is show that $f(n)$ is non-decreasing, that is $f(1) < f(2) < f(3) \ldots$. this can be done through induction on n, where S_n means $f(n) < f(m)$ for every $m > n$.

$f(m) > f(f(m-1))$ for $m > 1$, so $f(m)$ is not the smallest number in $\{f(1), f(2), \ldots)\}$. But $f(n)$ has all its values in positive integers, it must have a smallest element. That means $f(1)$ is the unique smallest element, S_1 is true.

Suppose S_n is true, $1 \leq f(1) < f(2) \cdots < f(n)$, so $f(n) \geq n$. $f(m) > f(f(m-1))$ for $m > n+1$, and $f(m-1)$ always larger than n, so $f(m)$ is not the smallest number in $\{f(n+1), f(n+2), \ldots)\}$. Same as S_1, $f(n)$ has all its values in positive integers, it must have a smallest element in $\{f(n+1), f(n+2), \ldots)\}$. That means $f(n+1)$ is the unique smallest element in that set, S_{n+1} is true. by induction, S_n is true for every n.

So now, we have $f(n) > f(m)$, for every $n > m$. Since $f(1) \geq 1$ and $f(n)$ is non-decreasing, that implies $f(n) \geq n$. But if $f(n) \geq n+1$, $f(f(n)) \geq f(n+1)$, that makes contradiction. Hence $f(n) = n$ for each n.

Solution - steps

- First, we want to show $f(1) < f(2) < \ldots. < f(n) < \ldots$ (by induction)
- find the lower and upper bound of $f(n)$
- Finally, $f(n) = n$ for all n

Solution - induction

- Sn means f(n) is unique smallest element in { f(n), f(n+1), ..}
- Proof S1 is true
  - f(m) > f(f(m – 1)) for m > 1
  - f(m) is not smallest, because we can find f(k) is smaller.
  - k = f(m-1)
  - So f(1) is unique smallest element.
  - We only proof f(1) is smallest, not f(1) = 1

Solution - induction

- Suppose Sn is true, that is f(1) < f(2) < … < f(n) < f(n+1), f(n+2), …
- f(1) >= 1 -> f(n) >= n -> f(m) > n for m > n
- Prove Sn+1 is true
  - We want to show f(n+1) is unique smallest in S = { f(n+1), f(n+2), …}
  - f(m) > f(f(m – 1)) for m > n + 1
  - f(m) is not smallest, because we can find f(k) is smaller.
  - k = f(m-1), and k > n
  - So f(n+1) is unique smallest element in S.
- By induction, Sn is true for all n.

Solution - bound

- $f(1) < f(2) < \ldots < f(n) < \ldots$, means $f(n) > f(m)$ for $n > m$
- $f(1)$ is positive, $f(1) >= 1$
- $f(2) >= 2$, $f(3) >= 3$, … $f(n) >= n$

- If $f(n) >= n + 1$, $f(f(n)) >= f(n+1)$, contradiction.
- So $f(n) = n$

Grading Policy

- 5 points for correct answer
- partial score for proving $f(n)$ is non-decreasing or $f(1) = 1$