EE205003 Session 12

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Independence, basis, and dimension

Recall: (1)

Suppose $A_{m\times n}$ with $m < n$

Then there are nonzero sol. for $A\mathbf{x} = \mathbf{0}$

(more unknowns than eqn.s)

Reason: A has at least one free var.

$$R = \begin{bmatrix} I & F \end{bmatrix} \text{ or } R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(We will come back to this later)

$\boxed{\text{Def}}$ The vectors $\mathbf{v}_1,\ \mathbf{v}_2, \cdots, \mathbf{v}_n$ are lin. indep. if no combination (except the zero comb.) gives zero vector

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \neq \mathbf{0}$$

(except $x_1 = x_2 = \cdots = x_n = 0$)

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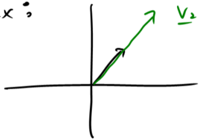
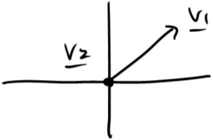
Ex:

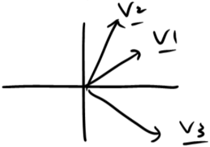


$\underline{v}_2 = 2\underline{v}_1$ (dependent)

$2\underline{v}_1 - \underline{v}_2 = \underline{0}$



$0\underline{v}_1 + \underline{v}_2 = \underline{0}$ (dependent)



$\underline{v}_1, \underline{v}_2$ are indep.

Q: How about $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

Back to (1)

$$A = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 3 \\ -1 \end{matrix}} \end{bmatrix} \\ \begin{matrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$\Rightarrow$ Whether $A\mathbf{x} = 0$ has nonzero sol. is the same as whether $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linear independent

Repeat : When $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are col. of A

They are indep. if $\mathbf{N}(A) = \mathbf{0}$

(rank $= n$, no free var.s)

They are dependent if $A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x}$

(rank $< n$, Yes ! free var.s)

If $m < n \Rightarrow$ At least $n - m$ free var.s

$\Rightarrow$ col.s of A are linear dependent

$\Rightarrow \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ has to be dependent!

(7 dim space, 10 vectors $\Rightarrow m = 7,\ n = 10$
$\Rightarrow$ linear dependent $\because m < n$)

$\boxed{\text{Fact}}$ Any set of n vectors in R^m must be linear dependent if $m < n$

Spanning a space

$\boxed{\text{Def}}$ Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_l$ span a space if the space consists of all comb. of these vectors
(Ex: col.s of A spans $\mathbf{C}(A)$)

$\boxed{\text{Fact}}$ If $\mathbf{v}_1, \cdots, \mathbf{v}_l$ span a space S then S is the smallest space that contains these vectors

Column space

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{C}(A) = R^2$

Ex: $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix}$, $\mathbf{C}(A) = R^2$

(columns may be dependent)

$\boxed{\text{Def}}$ The row space of a matrix is the subspace of R^n spanned by the rows

$\Rightarrow$ row space of $A = \mathbf{C}(A^T)$

$\Rightarrow$ it's the col. space of A^T

Ex:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \quad \Rightarrow \mathbf{C}(A) = \text{plane in } R^3 \text{ spanned by two vectors}$$

$\updownarrow$ same dim but different spaces

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix} \Rightarrow \mathbf{C}(A^T) = R^2$$

(Rows in R^n spanning the row space
Col.s in R^m spanning the col. space)

Basis & dim.

$\boxed{\text{Def}}$ A basis for a space is a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_d$ with two properties:

1. They are independent
2. They span the space

(Tell us everything we need to know about the space)

Ex: space is R^3

one basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (standard basis)

Test independence:

Method 1:

$$x_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = x_2 = x_3 = 0 \Rightarrow$ independent

Method 2:

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{array}{c}\text{no free var. } \mathbf{N}(A) = \{\mathbf{0}\}\\ \text{or } \ (A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0})\end{array} \quad \Rightarrow \text{independent}$$

Q: Is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ a basis?

$$A = \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\2 & 5 & 8\end{bmatrix}$$

Do elimination:
$\Rightarrow$ first two rows are the same
$\Rightarrow$ only two pivot, one free var.
$\Rightarrow$ dependent
$\Rightarrow$ NOT a basis

In general, n vectors in R^n form a basis if they are col.s of n invertible matrix.

Q: Is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ a basis?

Yes! For a plane S spanned by these vectors in R^3

Q: How many basis do we have for R^3?

Infinitely many!

$\boxed{\text{Fact}}$ Every basis for the space has the same number of basis vectors

(This number is the dimension of the space)

More on basis

$\boxed{\text{Fact}}$ There is only one & only one way to write $\mathbf{v}$ as a comb. of basis

Reasons:

$$\begin{array}{rl} \text{Let } \mathbf{v} &= a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \\ -)\ \mathbf{v} &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \\ \hline \mathbf{0} &= (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n \end{array}$$

Since $\mathbf{v}_i$'s are linear independent

$\Rightarrow a_1 - b_1 = 0, \ \cdots, \ a_n - b_n = 0$

$\Rightarrow a_1 = b_1, \ \cdots, \ a_n = b_n$

$\boxed{\text{Fact}}$ The pivot col.s of A are a basis for $\mathbf{C}(A)$, The pivot rows of A are a basis for $\mathbf{C}(A^T)$, So are the pivot rows of R (not true for col.s)

Ex: $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \to R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{array}{l} \text{basis for col. space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ not } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \text{basis for row space: both } \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

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Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

col.3 = col.1 + col.2, col.4 = col.1

col.1 & 2 are independent

$\Rightarrow$ basis for $\mathbf{C}(A)$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$\boxed{\text{Fact}}$ For any matrix A

rank$(A) = \#$ of pivot col.s $A =$ dimension of $\mathbf{C}(A)$

(Matrix has a rank, not a dimension,
subspace has a dimension, not a rank)

Another basis for $\mathbf{C}(A)$:

col.1 & col.3, col.2 & col.3, $\cdots$
(infinitely many basis but dim $= 2$)

Q: How about $\mathbf{N}(A)$?

$$\text{Special sol.s} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ dimension $= 2$

$\boxed{\text{Fact}}$ For any matrix A

dimension of $\mathbf{N}(A) = \#$ of free var.s $= n - r$

(dimension of $\mathbf{N}(A) = 4 - 2 = 2$)