EE3060 Probability - Quiz 4

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Easy 1.(G3)

The probability density function of Θ is given by

$$f(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } \theta \in [0, 2\pi] \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(XY) = \int_0^{2\pi} \sin\theta \cos\theta \frac{1}{2\pi}\, d\theta = 0, \qquad E(X) = \int_0^{2\pi} \sin\theta \frac{1}{2\pi}\, d\theta = 0,$$

$$E(Y) = \int_0^{2\pi} \cos\theta \frac{1}{2\pi}\, d\theta = 0.$$

Thus $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$

2.(G2)

Mr. Jones has two jobs. Next year, he will get a salary raise of X thousand dollars from one employer and a salary raise of Y thousand dollars from his second. Suppose that X and Y are independent random variables with probability density functions f and g, respectively, where

$$f(x) = \begin{cases} \frac{8x}{15}, & if\ \frac{1}{2} < x < 2 \\ 0, & elsewhere \end{cases}$$

$$g(y) = \begin{cases} \frac{6\sqrt{y}}{13}, & if\ \frac{1}{4} < y < \frac{9}{4} \\ 0, & elsewhere \end{cases}$$

What are the expected value and variance of the total raise that Mr. Jones will get next year?

Ans:

We have that

$$E(X) = \int_{1/2}^{2} \frac{8}{15} x^2\,dx = 1.4, \qquad E(X^2) = \int_{1/2}^{2} \frac{8}{15} x^3\,dx = 2.125,$$

$$E(Y) = \int_{1/4}^{9/4} \frac{6}{13} y^{3/2}\,dy = 1.396, \qquad E(Y^2) = \int_{1/4}^{9/4} \frac{6}{13} y^{5/2}\,dy = 2.252.$$

These give $\mathrm{Var}(X) = 2.125 - 1.4^2 = 0.165$, and $\mathrm{Var}(Y) = 2.252 - 1.396^2 = 0.303$. Hence $E(X+Y) = 1.4 + 1.396 = 2.796$, and by independence of X and Y,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 0.165 + 0.303 = 0.468.$$

Therefore, the expected value and variance of the total raise Mr. Jones will get next year are \$2796 and \$468, respectively.

3.(G3)

Suppose that a random variable X satisfies

$$\mathrm{E}[\mathrm{X}] = 0, E[X^2] = 1, E[X^3] = 0, E[X^4] = 3$$

And let

$$\mathrm{Y} = \mathrm{a} + \mathrm{bX} + cX^2$$

Find the correlation coefficient $\rho(\mathrm{X.Y})$.

Ans:

$$\rho\,(\mathrm{X,Y}) = \frac{\mathrm{cov(x,y)}}{\sigma\,\mathrm{X}\sigma\mathrm{Y}}$$

$$\begin{aligned} \mathrm{cov(X,Y)} &= \mathrm{E[XY]} - \mathrm{E[X]E[Y]} \\ &= \mathrm{E[aX + bX^2 + cX^3]} - \mathrm{E[X]E[Y]} \\ &= \mathrm{aE[X] + bE[X^2] + cE[X^3]} \\ &= \mathrm{b}. \end{aligned}$$

$$\begin{aligned} \mathrm{var(Y)} &= \mathrm{var(a + bX + cX^2)} = \mathrm{E[(a + bX + cX^2)^2]} - (\mathrm{E[a + bX + cX^2]})^2 \\ &= (\mathrm{a^2 + 2ac + b^2 + 3c^2}) - (\mathrm{a^2 + c^2 + 2ac}) \\ &= b^2 + 2c^2 \end{aligned}$$

$$\mathrm{var(X)} = 1,$$

$$\rho\,(\mathrm{X,Y}) = \frac{\mathrm{b}}{\sqrt{\mathrm{b^2 + 2c^2}}}$$

4.(G4)

Given the joint density function of X, Y as below:

$$f_{XY}(x,y) = \begin{cases} \dfrac{x+4y}{a}, 0 \le x \le 4, 0 \le y \le 2 \\ 0, otherwise \end{cases}$$

(a) Find the value of a.

(b) Find the covariance of X & Y, Cov(x,y)

Ans:

a.

$\int_0^2 \int_0^4 \frac{x+4y}{a} dxdy = 1$, $\int_0^2 \int_0^4 \frac{x+4y}{a} dxdy = \frac{48}{a}$, $\mathrm{a} = 48$

b.

$$f_X(x) = \int_0^2 \frac{x+4y}{48} dy = \frac{x+4}{24}$$

$$f_Y(y) = \int_0^4 \frac{x+4y}{48} dx = \frac{2y+1}{6}$$

$$\mathrm{E[X]} = \int_0^4 x * \frac{x+4}{24} dx = \frac{20}{9}$$

$$\mathrm{E[Y]} = \int_0^2 y * \frac{2y+1}{6} dy = \frac{11}{9}$$

$$\mathrm{E[XY]} = \int_0^2 \int_0^4 xy * \frac{x+4y}{48} dxdy = \frac{8}{3}$$

$$\mathrm{Cov(X,Y)} = \mathrm{E[XY]} - \mathrm{E[X]E[Y]} = \frac{8}{3} - \left(\frac{20}{9} * \frac{11}{9}\right) = -\frac{4}{81}$$

5.(G5)

Consider four random variables. W,X,Y,Z with

$$\mathrm{E[W]} = \mathrm{E[X]} = \mathrm{E[Y]} = \mathrm{E[Z]} = 0$$
$$\mathrm{var(W)} = \mathrm{var(X)} = \mathrm{var(Y)} = \mathrm{var(Z)} = 1$$

and assume that W,X,Y,Z are pairwise uncorrelated. Find the correlation coefficients $\rho(\mathrm{R}, \mathrm{S})$ and $\rho(\mathrm{R}, \mathrm{T})$, where $\mathrm{R} = \mathrm{W} + \mathrm{X}$, $\mathrm{S} = \mathrm{X} + \mathrm{Y}$, and $\mathrm{T} = \mathrm{Y} + \mathrm{Z}$.

Ans:

$$\text{cov}(R,S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R,S) = \frac{\text{cov}(R,S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R,T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R,T) = 0.$$

Hard 1.(G2)

Let X be a random variable that takes nonnegative integer values, and is associated with a transform of the form $M_x(s)$=k$\frac{(9+6e^s+1e^{2s})}{3-2e^s}$, Find E[X|X $\neq$ 1,2]

Ans:

Since

$$\text{Mx}(0) = \text{E}[1] = 1\text{, so } k = 1/16$$

Also, since X takes nonnegative integer values:

$$M(s) = p_x(0)\, e^0 + p_x(1)\, e^s + p_x(2)\, e^{2s} \ldots\ldots$$

Base on long division, we can know that

$$M(s) = \frac{1}{16}(3 + 4e^s + 3e^{2s} + 2e^{3s} \ldots\ldots..)$$

Thus,

$$p_x(0)=3/16 \; p_x(1)=4/16 \; p_x(2)=3/16 \ldots$$

Base on expectation theorem:

$$E[X]=E[X|X=1]P(X=1)+E[X|X=2]P(X=2)+ E[X|X \neq 1,2]P(X \neq 1,2)$$

Since E[X]=$\frac{d}{ds}$ Mx(s)$|_{s=0}$ = $\frac{5}{2}$ and P(X $\neq$ 1,2)= 1-P(X=1or 2) = 1- $\frac{7}{16}$

Therefore,

$$\frac{5}{2} = 1*\frac{4}{16} + 2*\frac{3}{16} + E[X|X \neq 1,2](1 - \frac{7}{16})$$

$$E[X|X \neq 1,2] = \frac{10}{3}$$

2.(G2)

A vendor machine supplies **‘n’** different types of beverages, and it is visited by a number **Y** of customers in a certain period of time, we know that X is a nonnegative integer random variable with known transform $\boldsymbol{M_Y(s) = E[e^{sY}]}$. Each customer will buy one single beverage, with all types of beverage being equally likely, independent of the number or the types other customers buy. Please derive the formula in terms of $M_Y(.)$ for the **expected value** of **number of different beverage** ordered.

Ans:
Let's first define a random variable X:

$$X_i \begin{cases} 1, \ if\ type\ i\ beverage\ is\ ordered\ by\ customers\ at\ least\ once \\ 0, otherwise \end{cases}$$

Then X = X1+X2+X3…..+Xn, X means number of different drinks ordered.
Using Law of Iterated Expectaiton:

$$\mathrm{E[X]} = \mathrm{E}\big[\mathrm{E[X|Y]}\big] = \mathrm{E}\big[\mathrm{E[X1 + X2 \dots. +Xn|Y]}\big] = \mathrm{nE[E[Xi|Y]]}$$

Now, we know the beverage not be ordered by probability p = (n-1)/n
Then, the expected probability of it being ordered at least once is:

$$\mathrm{E}[X_1|\mathrm{Y = y}] = 1 - (\frac{n-1}{n})^y$$

So, we can know $\mathrm{E}[X_1|\mathrm{Y}] = 1 - (\frac{n-1}{n})^Y$
Lastly, $\mathrm{E[X]} = nE[1 - p^Y] = \mathrm{n - nE}[p^Y] = \mathrm{n - nE}[e^{Ylogp}] = \mathrm{n - n}M_Y(logp)$

3.(G5)
Let X be a random variable that takes nonnegative integer values, and its transform is

$$M_x(s) = \frac{13 - e^s - 6e^{2s}}{15 - 5e^s - 9e^{2s} + 3e^{3s}}$$

Find $p_x(x), E[X]\ and\ var(X)$.

Ans:

$$\mathrm{M(s)} = \frac{13-e^s-6e^{2s}}{15-5e^s-9e^{2s}+3e^{3s}} = \frac{2}{3-e^s} + \frac{1}{5-3e^{2s}} = \frac{\frac{2}{3}}{1-\frac{1}{3}e^s} + \frac{\frac{1}{5}}{1-\frac{3}{5}e^{2s}}$$

$$= \frac{2}{3}\left[1 + \frac{1}{3}e^s + \frac{1}{3}^2 e^{2s} + \frac{1}{3}^3 e^{3s} + \cdots\right] + \frac{1}{5}[1 + \frac{3}{5}e^{2s} + \frac{3}{5}^2 e^{4s} + \frac{3}{5}^4 e^{6s} + \cdots]$$

$$\mathrm{M(s)} = \sum_{x=0}^{\infty} e^{sx} p_X(x) = p_X(0)e^{s0} + p_X(1)e^{s1} + p_X(2)e^{s2} + p_X(3)e^{s3} + \cdots$$

$$p_X(x) = \begin{cases} \frac{2}{3} \times \frac{1}{3}^x + \frac{1}{5} \times \frac{3}{5}^{\frac{x}{2}}, & \mathrm{x} \in \mathrm{even} \\ \frac{2}{3} \times \frac{1}{3}^x, & x \in odd \end{cases}$$

$$\mathrm{E[X]} = \frac{dM(s)}{ds}|_{s=0} = \frac{2e^s}{(3-e^s)^2}|_{s=0} + \frac{6e^{2s}}{(5-3e^{2s})^2}|_{s=0} = 2$$

$$\mathrm{E[X^2]} = \frac{d^2M(s)}{ds^2}|_{s=0}$$

$$= \frac{(3-e^s)^2 \times 2e^s - 2e^s \times (6-2e^s)(-e^s)}{(3-e^s)^4}|_{s=0}$$

$$+ \frac{(5-3e^{2s})^2 \times 12e^{2s} - 6e^{2s} \times (10-6e^{2s}) \times (-6e^{2s})}{(5-3e^{2s})^4}|_{s=0} = 13$$

$$\mathrm{var(X)} = \mathrm{E}[X^2] - (\mathrm{E[X]})^2 = 13 - 2^2 = 9$$

4.(G3)

Suppose we have an integer valued random variable X having the moment generating function shown below

$$M_X(s) = \frac{e^s}{3 - e^{3s} - e^{4s}}$$

Find the

a) $E[X]$

b) P{X = 5}

Hint : Use Maclaurin's series of $\frac{1}{1-x}$

Ans:

a)

$$M_X(s) = \frac{e^s}{3 - e^{3s} - e^{4s}}$$

$$\therefore (3 - e^{3s} - e^{4s})M_X(s) = e^s$$

Differentiate both sides

$$(-3e^{3s} - 4e^{4s})M_X(s) + (3 - e^{3s} - e^{4s})M_X'(s) = e^s$$

Plug in $s = 0$

$$E[X] = M_X'(0) = 8$$

b)

$$\frac{e^s}{3-e^{3s}-e^{4s}} = \frac{1}{3}\left(\frac{e^s}{1-\frac{1}{3}(e^{3s}+e^{4s})}\right)$$

$$= \frac{1}{3}e^s\left(1+\frac{1}{3}(e^{3s}+e^{4s})+\frac{1}{9}(e^{3s}+e^{4s})^2+\cdots\right)$$

$$= \frac{1}{3}e^s+\frac{1}{9}e^{4s}+\frac{1}{9}e^{5s}+\cdots$$

$$\therefore \mathrm{P}\{X=5\} = \frac{1}{9}$$

5.(G1)

Cards are drawn from an ordinary deck of 52, one at a time, randomly and with replacement. Let X and Y denote the number of draws until the first ace and the first king are drawn, respectively. Find E(X|Y=5).

Ans:

Let $p(x, y)$ be the joint probability mass function of X and Y. Clearly,

$$p_Y(5) = \left(\frac{12}{13}\right)^4\left(\frac{1}{13}\right),$$

and

$$p(x,5) = \begin{cases} \left(\frac{11}{13}\right)^{x-1}\left(\frac{1}{13}\right)\left(\frac{12}{13}\right)^{4-x}\left(\frac{1}{13}\right) & x<5 \\ 0 & x=5 \\ \left(\frac{11}{13}\right)^4\left(\frac{1}{13}\right)\left(\frac{12}{13}\right)^{x-6}\left(\frac{1}{13}\right) & x>5. \end{cases}$$

Using these, we have that

$$E(X \mid Y=5) = \sum_{x=1}^{\infty} x p_{X|Y}(x|5) = \sum_{x=1}^{\infty} x\,\frac{p(x,5)}{p_Y(5)}$$

$$= \sum_{x=1}^{4}\frac{1}{11}x\left(\frac{11}{12}\right)^x + \sum_{x=6}^{\infty} x\left(\frac{11}{12}\right)^4\left(\frac{1}{13}\right)\left(\frac{12}{13}\right)^{x-6}$$

$$= 0.72932 + \left(\frac{11}{12}\right)^4\left(\frac{1}{13}\right)\sum_{y=0}^{\infty}(y+6)\left(\frac{12}{13}\right)^y$$

$$= 0.72932 + \left(\frac{11}{12}\right)^4\left(\frac{1}{13}\right)\left[\sum_{y=0}^{\infty}y\left(\frac{12}{13}\right)^y + 6\sum_{y=0}^{\infty}\left(\frac{12}{13}\right)^y\right]$$

$$= 0.702932 + \left(\frac{11}{12}\right)^4\left(\frac{1}{13}\right)\left[\frac{12/13}{(1/13)^2} + 6\frac{1}{1-(12/13)}\right] = 13.412.$$

Remark: In successive draws of cards from an ordinary deck of 52 cards, one at a time, randomly, and with replacement, the expected value of the number of draws until the first ace is $1/(1/13) = 13$. This exercise shows that knowing the first king occurred on the fifth trial will increase, on the average, the number of trials until the first ace 0.412 draws.

TA Part

1. Please write down the definition of the *moment-generating function* of a random variable *X*.
2. Please write down the **actual formula** of the *moment-generating function* (i.e. The relation with probability density function, $f_x(x)$, to be more specifically).
3. Find the n^{th} moment from the derivation of *moment-generating function*.
4. Explain why is this called *moment-generating function*. (Your explanation should include two parts: i) what is the moments of X, ii) From ***Taylor series*** perspective, how does MGF generate these moments.)

Ans:

1.

$$M_x(s) = E[e^{sX}]$$

2.

$$M_x(s) = \int_{-\infty}^{\infty} e^{sx} f_x(x)dx$$

3.

$$\frac{d^n}{ds^n} M_x(s)|_{s=0} = \int_{-\infty}^{\infty} x^n e^{sx}|_{s=0}\, f_x(x)dx = \int_{-\infty}^{\infty} x^n f_x(x)dx = E[X^n]$$

4.

The n^{th} moment of X is defined as $E[X^n]$.

Since the t^{th} moment of X is the coefficient of $\frac{s^t}{t\,!}$ in the Taylor series of $M_x(s)$. Thus, we can write down

$$M_x(s) = \sum_{t=0}^{\infty} E[X^t] \frac{s^t}{t\,!}$$

and so all the moments of X could be obtained by

$$E[X^t] = \frac{d^k}{ds^k} M_x(s)|_{s=0}$$