1. (a) (i) The solution $y_1(t) = y_p(t) + y_h(t)$

To find $y_p(t)$, we hypothesize $y_p(t) = Ye^{3t}$ for $t>0$

$\Rightarrow 3Ye^{3t} + 2Ye^{3t} = e^{3t} \quad \Rightarrow Y = \frac{1}{5}$

Hence, $y_p(t) = \frac{1}{5}e^{3t}$

To find $y_h(t)$, we hypothesize $y_h(t) = Ae^{st}$

$sAe^{st} + 2Ae^{st} = 0 \Rightarrow Ae^{st}(s+2) = 0$

$s = -2$, A is arbitrary.

$y_1(t) = Ae^{-2t} + \frac{1}{5}e^{3t}$, $t>0$, for any A.

By assume that system is causal,

$y_1(t) = 0$ for $t<0$ since $x_1(t) = e^{3t}u(t)$

$\Rightarrow y_1(0) = \frac{1}{5} + A = 0 \Rightarrow A = -\frac{1}{5}$ Hence, $y_1(t) = \frac{1}{5}(e^{3t} - e^{-2t})u(t)$ #

(ii) Similar to (i) $\Rightarrow y_2(t) = \frac{1}{4}(e^{2t} - e^{-2t})u(t)$ #

(iii) Hypothesize $y_p(t) = K_1e^{3t} + K_2e^{2t}$, $t>0$

$3K_1e^{3t} + 2K_2e^{2t} + 2K_1e^{3t} + 2K_2e^{2t} = \alpha e^{3t} + \beta e^{2t}$ for $t>0$

$\Rightarrow 3K_1 + 2K_1 = \alpha \quad K_1 = \frac{\alpha}{5} \quad 2K_2 + 2K_2 = \beta \quad K_2 = \frac{\beta}{4}$

Hence, $y_p(t) = \frac{\alpha}{5}e^{3t} + \frac{\beta}{4}e^{2t}$

Hypothesize $y_h(t) = Ae^{st}$, Similar to (i) $\Rightarrow s = -2$

$y_3(t) = \frac{\alpha}{5}e^{3t} + \frac{\beta}{4}e^{2t} + Ae^{-2t}$, $t>0$, for any A.

By assume that system is causal,

$y(t) = 0$ for $t<0$ since $x_3(t) = (\alpha e^{3t} + \beta e^{2t})u(t)$

$\Rightarrow y(0) = \frac{\alpha}{5} + \frac{\beta}{4} + A = 0 \Rightarrow A = -\frac{\alpha}{5} - \frac{\beta}{4}$

$y_3(t) = \left[\frac{\alpha}{5}e^{3t} + \frac{\beta}{4}e^{2t} - \left(\frac{\alpha}{5} + \frac{\beta}{4}\right)e^{-2t}\right]u(t)$. From $y_1(t)$ and $y_2(t)$, we can see that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$

#

(iv) From the first-order differential with initial rest,

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t) \qquad y_1(t) = 0 \quad \text{since } x_1(t) = 0, \text{ for } t < t_1.$$

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t) \qquad y_2(t) = 0 \quad \text{Since } x_2(t) = 0, \text{ for } t < t_2.$$

Scaling above two equations,

$$\Rightarrow \alpha \frac{dy_1(t)}{dt} + 2\alpha y_1(t) + \beta \frac{dy_2(t)}{dt} + 2\beta y_2(t) = \alpha x_1(t) + \beta x_2(t),$$

and $y_1(t) + y_2(t) = 0$ for $t < t_1$ & $t < t_2 \Rightarrow t < \min(t_1, t_2)$

It's clear that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when $x_3(t) = \alpha x_1(t) + \beta x_2(t)$,

and $y_3(t) = 0$ for $t < t_3$, where $t_3 = \min(t_1, t_2)$ so $x_3(t) = 0$.

(b) (i) Similar to (a)-(ii) $\Rightarrow y_1(t) = \frac{K}{4}(e^{2t} - e^{-2t})u(t)$

(ii) Hypothesize $y_p(t) = Ye^{2(t-T)}$ for $t > T$

$$2Ye^{2(t-T)} + 2Ye^{2(t-T)} = Ke^{2(t-T)} \Rightarrow Y = \frac{K}{4}$$

Hence, $y_p(t) = \frac{K}{4}e^{2(t-T)}$ for $t > T$

Hypothesize $y_h(t) = Ae^{st}$, Similar to (a)-(ii) $\Rightarrow S = -2$

$$y_2(t) = \frac{K}{4}e^{2(t-T)} + Ae^{-2t}, \quad \text{for } t > T$$

By assume that system is causal,

$$y_2(T) = 0 = \frac{K}{4} + Ae^{-2T} \Rightarrow A = -\frac{K}{4}e^{2T}$$

$$y_2(t) = \left(\frac{K}{4}e^{2(t-T)} - \frac{K}{4}e^{-2(t-T)}\right)u(t-T)$$

$$= \frac{K}{4}\left(e^{2(t-T)} - e^{-2(t-T)}\right)u(t-T)$$

Obviously, $y_2(t) = y_1(t-T)$. #

(iii) From the first-order differential with initial rest,

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t) \quad y_1(t)=0 \text{ since } x_1(t)=0, \text{ for } t<t_0$$

Above system is a causal LTI system,

and the derivative is a time-invariant operation.

Therefore, it has time-invariant property,

$$\Rightarrow \frac{dy_1(t)}{dt} = \frac{dy_1(t-T)}{dt} \text{ Hence, } \frac{dy_1(t-T)}{dt} + 2y_1(t-T) = x_1(t-T),\ y_1(t)=0 \text{ for } t<t_0$$

When $x_2(t) = x_1(t-T)$, it comes to $y_2(t) = y_1(t-T)$, #

and $y_2(t) = 0$ for $t < t_0+T$ since $x_2(t) = 0$ for $t<t_0+T$.

2. $\boxed{2.43(a)}$ prove : $[x(t)*h(t)]*g(t) = x(t)*[h(t)*g(t)]$

left side :

$[x(t)*h(t)]*g(t)$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(a'-\tau)\, g(t-a')\, d\tau\, da'$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(a)\, g(t-a-\tau)\, d\tau\, da$ #

$[\because a' = a-\tau,\ a' = a+\tau,\ da' = da]$

right side :

$x(t)*[h(t)*g(t)]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-a')\, h(\tau)\, g(a'-\tau)\, da'\, d\tau$ $\quad \because a = t-a'$, $a' = t-a$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(a)\, h(\tau)\, g(t-a-\tau)\, da\, d\tau$ $\quad \tau = a$, $a = \tau$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(a)\, g(t-\tau-a)\, d\tau\, da$ #.

$\boxed{2.43(c)}$

$h_1[n] = \sin 8n$

$h_2[n] = a^n u[n],\ |a|<1$

$x[n] = 2\delta[n] - \delta[n-2]$

$y[n] = ?$

$h[n] = h_1[n] * h_2[n]$

$= \sin(8n) * a^n u[n]$

$= a^n \sin(8n)\, u[n]$

let $x_1[n] = \delta[n]$

$\therefore y_1[n] = \delta[n] * a^n \sin(8n)\, u[n]$.

$= a^n \sin(8n)\, u[n]$

$\therefore y[n] = 2x_1[n] - x_1[n-2]$

$= 2a^n \sin(8n)\, u[n] - a^{n-2} \sin[8(n-2)]\, u[n-2]$

$\cancel{= 2\sin[8n] + 2a\sin[8(n-1)] + \sum_{k=2}^{\infty} (2a^k - a^{k-2}) \sin[8(n-k)]}$

#

3. $\boxed{2.49(a)}$ $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, $x[n] = \begin{cases} 0, & h[-n]=0 \\ \frac{h[-n]}{|h[-n]|}, & h[-n]\neq 0 \end{cases}$ bounded input?

when $h[-n] \neq 0$,

$x[n] = \frac{h[-n]}{|h[-n]|}$ $\quad \because -1 \le x[n] \le 1 \Rightarrow$ $\cancel{\sum_{k=-\infty}^{\infty} |x[k]| < \infty}$ $\Rightarrow$ bounded input #

$\Rightarrow |x[n]| \le 1$ for all n #.

$\boxed{2.49(b)}$ $n=0$, output = ?

$y[0] = \sum_{k=-\infty}^{\infty} x[-k]\, h[k]$

$= \sum_{k=-\infty}^{\infty} \frac{h[k]}{|h[k]|} \cdot h[k]$

$= \sum_{k=-\infty}^{\infty} \frac{h^2[k]}{|h[k]|}$

$= \sum_{k=-\infty}^{\infty} |h[k]| = \infty \Rightarrow$ not bounded output.

#.

$\therefore$ the system is not stable,

absolute summability is necessary #.

2.55 (a) Take n=0 into equation

$$y[0]-\frac{1}{2}y[-1]=x[0] \Rightarrow y[n]=1$$

Impulse response $h[n]$

$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

$$\Rightarrow h[n]-\frac{1}{2}h[n-1]=\delta[n]$$

Since initial rest, $h[n]=0 \ \forall n<0$

For $n=0$,

$$h[0]-\frac{1}{2}h[-1]=1 \Rightarrow h[0]=1 \quad \leftarrow \text{auxiliary condition}$$

For $n>0 \Rightarrow h[n]-\frac{1}{2}h[n-1]=0 \leftarrow$ homogeneous equation

$n=1$, $h[1]=\frac{1}{2}h[0]$

$n=2$, $h[2]=\frac{1}{2}h[1]$

$\therefore \ h[n]=(\frac{1}{2})^n h[0] \quad \forall n>0$

$\Rightarrow h[n]=(\frac{1}{2})^n u[n]$ (Initial rest)

(b) Use figure 2.55 and (a)

$$w[n]=(\frac{1}{2})^n u[n]$$

$$\therefore y[n]=w[n]+2w[n-1]$$
$$=(\frac{1}{2})^n u[n]+2(\frac{1}{2})^{n-1}u[n-1]$$

(c)

$$\sum_{m=-\infty}^{\infty} h[n-m]x[m]-\frac{1}{2}\sum_{m=-\infty}^{\infty} h[n-m-1]x[m]$$

$$=\sum_{m=-\infty}^{\infty}(\frac{1}{2})^{n-m}\underbrace{u[n-m]}_{n-m>0 \Rightarrow n>m}x[m]-\frac{1}{2}\sum_{m=-\infty}^{\infty}(\frac{1}{2})^{n-m-1}\underbrace{u[n-m-1]}_{n-m-1>0 \Rightarrow n-1>m}x[m]$$

$$=\sum_{m=-\infty}^{n}(\frac{1}{2})^{n-m}x[m]-\frac{1}{2}\sum_{m=-\infty}^{n-1}(\frac{1}{2})^{n-m-1}x[m]$$

$$=(\frac{1}{2})^{n-n}x[n]$$

$$=x[n]$$

(d) If $a_0 \neq 0$ and $h[n]$ is the impulse response of $y[n]$,

Take $n=0$ $\because$ Initial rest $y[n]=0 \ \forall n<0$

$$a_0 y[0]=x[0]=1$$

$$\Rightarrow y[0]=\frac{1}{a_0}$$

The homogeneous eq.

$$\sum_{k=0}^{N} a_k h[n-k]=0$$

For $n=1$,

$$\sum_{k=0}^{N} a_k h[1-k]=0$$

By the initial rest condition and previous result

$$h[-1]=\cdots=h[1-N]=0, \quad h[0]=\frac{1}{a_0}$$

$\therefore \ a_0 h[1]+a_1 h[0]=0$

$\Rightarrow a_0 h[1]=-\frac{a_1}{a_0}$

$\Rightarrow h[1]=-\frac{a_1}{a_0^2}$

Assume the impulse response of P2.55-4 is $h_1[n]$

Then, by linearity, the impulse response of P2.55-5

$$h[n]=\sum_{k=0}^{M} b_k h_1[n-k]$$

(e)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Rightarrow a_0 y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

$\because$ $x[n] = \delta[n]$

$$\Rightarrow \sum_{k=0}^{M} b_k x[n-k] = b_n \quad \forall 0 \le n \le M$$

and $y[-N] = y[-N+1] = \cdots = y[-1] = 0$

$\therefore$ For $n=0$, $a_0 y[0] = b_0 - 0$

$$\Rightarrow y[0] = \frac{b_0}{a_0}$$

For $n=1$, $a_0 y[1] = b_1 - a_1 y[0]$

$$\Rightarrow y[1] = \frac{1}{a_0}\left(b_1 - \frac{a_1 b_0}{a_0}\right)$$

$\vdots$

For $n=M$, $a_0 y[M] = b_M - \sum_{k=1}^{N} a_k y[n-k]$

Since input are impulse for $0 \le n \le M$,

$$h[0] = y[0],\ h[1] = y[1],\ \cdots h[M] = y[M]$$

with $y[0] = \frac{b_0}{a_0}$, $y[1] = \frac{1}{a_0}\left(b_1 - \frac{a_1 b_0}{a_0}\right)$ .... satisfying the aforementioned equation

For $n > M$,

$$\sum_{k=0}^{N} a_k h[n-k] = 0$$

5.

(a)

$$
\begin{aligned}
E &= \int_a^b |x(t)-\hat{x}_N|^2 dt \\
&= \int_a^b \Big[x(t)-\hat{x}_N(t)\Big]\Big[x^*(t)-\hat{x}_N^*(t)\Big]dt \\
&= \int_a^b \Big[x(t)-\sum_{i=-N}^{N} a_i\phi_i(t)\Big]\Big[x^*(t)-\sum_{i=-N}^{N} a_i^*\phi_i^*(t)\Big]dt \\
&= \int_a^b \Big[x(t)x^*(t)-x(t)\sum_{i=-N}^{N} a_i^*\phi_i^*(t)-x^*(t)\sum_{i=-N}^{N} a_i\phi_i(t)+\sum_{i=-N}^{N} a_i\phi_i(t)\sum_{i=-N}^{N} a_i^*\phi_i^*(t)\Big]dt
\end{aligned}
$$

Since $\{\phi_i(t)\}$ *is an orthonormal set, we know* $\int_a^b \phi_i(t)\phi_i^*(t)dt = \int_a^b |\phi_i(t)|^2 dt = 1$

$Let \quad a_i = b_i + jc_i \quad with \quad b_i, c_i \in \mathbb{R}$

$$\frac{\partial E}{\partial b_i} = -\int_a^b \phi_i^* x(t)dt + 2b_i - \int_a^b \phi_i(t)x^*(t)dt = 0 \qquad \text{—(1)}$$

$$\frac{\partial E}{\partial c_i} = -j\int_a^b \phi_i(t)x^*(t)dt + 2c_i + j\int_a^b \phi_i^*(t)x(t)dt = 0 \quad \text{—(2)}$$

$$\left(\frac{\partial}{\partial b_i}\int_a^b x^*(t)\sum_i (b_i+jc_i)\phi_i(t)dt = \int_a^b x^*(t)\phi_i(t)dt \text{ ,and so on.}\right)$$

from (1),(2) we can solved $b_i + jc_i = \int_a^b x(t)\phi_i^*(t)dt.$

That means $a_i = \int_a^b x(t)\phi_i^*(t)dt.$

(b)

In this case, $\int_a^b |\phi_i(t)|^2 = A_i.$ $Let\ a_i = b_i + jc_i. Then$

$$\frac{\partial E}{\partial b_i} = -\int_a^b \phi_i^*(t)x(t)dt + 2A_ib_i - \int_a^b \phi_i(t)x^*(t)dt = 0 \qquad \text{—(3)}$$

$$\frac{\partial E}{\partial c_i} = -j\int_a^b \phi_i(t)x^*(t)dt + 2A_ic_i + j\int_a^b \phi_i^*(t)x(t)dt = 0 \quad \text{—(4)}$$

From (3), (4), we can know

$$a_i = b_i + jc_i = \frac{1}{A_i}\int_a^b x(t)\phi_i^*(t)dt$$

(c)

Consider $\phi_k(t) = e^{jk\omega_0 t}$*, and compute* $\dfrac{\partial E}{\partial a_k}$.

$$\frac{\partial E}{\partial a_k} = -\int_{T_0} e^{jk\omega_0 t} x^*(t) + a_k^* \int_{T_0} dt = 0$$

$$\Rightarrow a_k^* = \frac{1}{T_0} \int_{T_0} x^*(t) e^{j\omega_0 kt} dt$$

$$\Rightarrow a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 kt} dt$$

6.

(a)

$$z(t) = x(t)y(t) = \sum_n a_n e^{jn\omega_0 t} \sum_k b_k e^{jk\omega_0 t}$$

$$= \sum_n \sum_k a_n b_k e^{j(n+k)\omega_0 t}, \quad let\ k' = k - n$$

$$= \sum_n \sum_{k'-n=-\infty}^{+\infty} a_n b_{k'-n} e^{jk'\omega_0 t}$$

$$= \sum_n \sum_{k'=-\infty}^{+\infty} a_n b_{k'-n} e^{jk'\omega_0 t}$$

$$= \sum_{k'=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a_n b_{k'-n} e^{jk'\omega_0 t} = \sum_{k'=-\infty}^{+\infty} c_{k'} e^{jk'\omega_0 t}$$. Let $k' = k$

$$\Rightarrow c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}$$

(b)

Compute the Fourier series of $e^{-2|t|}$*, where* $T_0 = 2$ *and* $e^{-2|t|} = 0\ for\ |t| > 1$.

$$\frac{1}{2} \int_{-1}^{1} e^{-2|t|} e^{-jk(2\pi/2)t} dt$$

$$= \frac{1}{2} \int_{-1}^{0} e^{2t} e^{-jk\pi t} dt + \frac{1}{2} \int_{0}^{1} e^{-2t} e^{-jk\pi t} dt$$

$$= \frac{1}{4 - 2jk\pi} \left[1 - e^{-2+jk\pi}\right] + \frac{1}{4 + 2jk\pi} \left[1 - e^{-2-jk\pi}\right]$$

$$= \frac{8}{16 + 4(k\pi)^2} - \frac{e^{-2}}{16 + 4(k\pi)^2} \left[(4 + 2jk\pi)e^{jk\pi} + (4 - 2jk\pi)e^{-jk\pi}\right]$$

$$= \frac{1}{4 + (k\pi)^2} \left[2 - 2e^{-2} \cos(k\pi) + e^{-2} k\pi \sin(k\pi)\right] \quad \text{—(5)}$$

Compute the convolution of (5) and Fourier series of $\cos(6\pi t)$

$$\frac{1}{4+(k\pi)^2}\left[2-2e^{-2}\cos(k\pi)+e^{-2}k\pi\sin(k\pi)\right]*\left[\frac{1}{2}\delta(k-6)+\frac{1}{2}\delta(k+6)\right]$$
$$=\frac{1}{8+2((k-6)\pi)^2}\left[2-2e^{-2}\cos((k-6)\pi)+e^{-2}(k-6)\pi\sin((k-6)\pi)\right]$$
$$+\frac{1}{8+2((k+6)\pi)^2}\left[2-2e^{-2}\cos((k+6)\pi)+e^{-2}(k+6)\pi\sin((k+6)\pi)\right]$$

(c)

From the result of (a), we know that if

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t},\quad y(t)=\sum_{k=-\infty}^{+\infty}b_k e^{jk\omega_0 t},\quad z(t)=\sum_{k=-\infty}^{+\infty}c_k e^{jk\omega_0 t},\quad z(t)=x(t)y(t)$$

, then

$$c_k=\sum_{n=-\infty}^{+\infty}a_n b_{k-n}$$

And $b_k=a_{-k}^*$. *(since* $y(t)=x^*(t)$)

$$c_k=\frac{1}{T_0}\int_0^{T_0}|x(t)|^2e^{-j(2\pi/T_0)kt}dt=\sum_{n=-\infty}^{+\infty}a_n b_{k-n}=\sum_{n=-\infty}^{+\infty}a_n a_{n-k}^*$$

Let $k=0$,

$$c_0=\frac{1}{T_0}\int_0^{T_0}|x(t)|^2dt=\sum_{n=-\infty}^{+\infty}a_n a_n^*=\sum_{n=-\infty}^{+\infty}|a_n|^2$$
