HW3

Q1a Q3 Q4

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Problem 1(a) mathematical induction

- $P(n) \coloneqq 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$
- Basic step: for $n = 1$,

  LHS: $1 \times 3 = 3$, RHS: $\frac{1(1+1)(2+7)}{6} = \frac{2\times 9}{6} = 3$
- Inductive step: assume the statement holds for $n = k$, then for $n = k + 1$ we get:

$$
\begin{aligned}
\mathrm{P(k+1)} &= \boxed{1 \times 3 + 2 \times 4 + \cdots + k(k+2)} + (k+1)(k+1+2) \\
&= \frac{k(k+1)(2k+7)}{6} + \boxed{(k+1)(k+3)} \\
&= \frac{(k+1)(2k^2+7k+6k+18)}{6} \\
&= \frac{(k+1)(k+2)(2k+9)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}
\end{aligned}
$$

→By mathematical induction $P(n)$ holds for all integer $n \geq 1$.

Problem 3

- Show that any odd positive integer can be expressed as a product of fractions with each fraction of the form

$$\frac{4q-1}{2q+1}, \text{ where } q \text{ is a positive integer?}$$

- For instance,

$$1 = \frac{3}{3} = \frac{4\times1-1}{2\times1+1}$$

and

$$3 = \frac{15}{9} \times \frac{27}{15} = \frac{4\times4-1}{2\times4+1} \times \frac{4\times7-1}{2\times7+1}$$

Problem 3 – Hint (decompose an odd number)

- $(2q+1) \times \frac{4q-1}{2q+1} = 4q-1$
- $(2q+1) \times \frac{12q+3}{6q+3} = 4q+1$

- $4q-1$ and $4q+1$ with $q \in \mathbb{N}$, is a representation for any odd positive integer.
- For both equations, we will show the fractional parts follow the desired form $\frac{4q-1}{2q+1}$.
- For the first equation it is straightforward, for the second one we get

$$\frac{12q+3}{6q+3} = \frac{4(3q+1)-1}{2(3q+1)+1} = \frac{4k-1}{2k+1}$$

with $k = 3q+1$, which is also a positive integer.

Problem 3 – Prove by Strong Induction

- Since we find one way decompose an odd number ($4q + 1$ or $4q - 1$) into another smaller odd number ($2q + 1$), we are able to prove this by strong induction.

- Basic step: for n=1, we have

$$1 = \frac{3}{3} = \frac{4\times1-1}{2\times1+1}$$

- Inductive step: assume all odd numbers that not greater than k, where k is an odd number, can be expressed in the desired form.
- Then we need to express k in the desired form…

Problem 3 – Prove by Strong Induction

- Inductive step: assume that for all odd numbers $\mathbf{n} < k$, where k is an odd number, can be expressed in the desired form.
- First, if the odd number $k = 4q - 1$ we can express k as

$$k = 4q - 1 = (2q + 1) \times \frac{4q-1}{2q+1}.$$

- Second, if the odd number $k = 4q + 1$ we can express k as

$$k = 4q + 1 = (2q + 1) \times \frac{12q+3}{6q+3}.$$

While $(2q + 1)$ is an odd number that is smaller than k, we now finish the proof that we can express any odd integer k in the desired form.

Discrete Math HW3 Q4

Question: Show that the following expression is always a positive integer, for any $k \geq 1$, by expressing it in terms of k:

$$10(\frac{10^4+324}{4^4+324})(\frac{22^4+324}{16^4+324})...(\frac{(12k-2)^4+324}{(12k-8)^4+324})$$

first, we do prime factorization

$$324 = 2^2 * 3^4$$

then do some factorization

$$\begin{aligned} a^4+324 &= a^4+4*3^4 \\ &= a^4+4b^4\ (let\ b=3) \\ &= (a^2+2b^2)^2-4a^2b^2 \\ &= (a^2+2b^2)^2-(2ab)^2 \\ &= (a^2+2b^2+2ab)(a^2+2b^2-2ab) \\ &= ((a+b)^2+b^2)((a-b)^2+b^2) \\ &= ((a+3)^2+b^2)((a-3)^2+b^2) \end{aligned}$$

finally

$$\begin{aligned} &10\,(\frac{10^4+324}{4^4+324})(\frac{22^4+324}{16^4+324})...(\frac{(12k-2)^4+324}{(12k-8)^4+324}) \\ &= 10\,(\frac{((10-3)^2+b^2)((10+3)^2+b^2)}{((4-3)^2+b^2)((4+3)^2+b^2)})...(\frac{((12k-2-3)^2+b^2)((12k-2+3)^2+b^2)}{((12k-8-3)^2+b^2)((12k-8+3)^2+b^2)}) \\ &= 10\,(\frac{(7^2+b^2)(13^2+b^2)}{(1^2+b^2)(7^2+b^2)})(\frac{(19^2+b^2)(25^2+b^2)}{(13^2+b^2)(19^2+b^2)})... \\ &= 10\,(\frac{13^2+b^2}{1^2+b^2})((\frac{25^2+b^2}{13^2+b^2}))... \\ &= 10\,(\frac{1}{1^2+3^2})(\frac{1}{1})...(\frac{(12k+1)^2+3^2}{1}) \\ &= (12k+1)^2+9 \end{aligned}$$

Q6 Q10 Q14

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Problem 6

Show that it is possible to arrange the numbers $1,2,\ldots,n$ in a row so that the average of any two of these numbers never appears between them.

Hint

(1) Show that it suffices to prove this fact when n is a power of 2.

(2) Then use mathematical induction to prove the result when n is a power of 2

$n = 4$

$(1, 2, 3, 4)$ (x) $(2,4, 1, 3)$ (o)

Proof by Induction

$$n = 4 \Rightarrow (1, 2, 3, 4)$$

$$A_n \quad := (i)_{i=1}^{n},$$

$$n = 4 \Rightarrow (4, 2, 3, 1) \text{ or others}$$

$$P(A_n) := \left(p(i)\right)_{i=1}^{n}, \text{permutation of } A_n$$

$$Q(n) := \forall\, n \in \mathbb{N}, \exists\, P(A_n)\; s.t.\, a_k \neq \frac{a_i + a_j}{2}, 1 \leq i < k < j \leq n$$

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

(ii) Show that $Q(k) \to Q(k-1), \forall\, k \in \mathbb{N}$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

Proof

Basis

$A_{2^0} = A_1$ is a valid $P(A_1) \Rightarrow Q(2^0)$ is true.

Inductive hypothesis

Suppose $Q(2^{k-1})$ is true $\Rightarrow \exists$ valid $P\left(A_{2^{k-1}}\right)$

Show that valid $P(A_{2^k})$ can be generated from valid $P\left(A_{2^{k-1}}\right)$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

Proof

Inductive hypothesis

$P^*(A_n) \coloneqq$ valid $P(A_n)$

$c \cdot P(A_n) \coloneqq \big(c \cdot p(i)\big)_{i=1}^{n}$ $\quad (1,2,3,4) \Rightarrow (c, 2c, 3c, 4c)$

$P(A_n) - d \coloneqq (p(i) - d)_{i=1}^{n}$ $\quad (1,2,3,4) \Rightarrow (1-d, 2-d, 3-d, 4-d)$

$P(A_n)P(A_m) \coloneqq$ concatenation of $P(A_n)$ and $P(A_m)$ $\quad (1,2)(3,4) \Rightarrow (1,2,3,4)$

Claim

$$\exists\, P^*(A_{2^k})\ s.t.\, P^*(A_{2^k}) = \left(2 \cdot P^*(A_{2^{k-1}})\right)\left(2 \cdot P^*(A_{2^{k-1}}) - 1\right)$$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

Claim

$$\exists\, P^*(A_{2^k})\ s.t.\, P^*(A_{2^k}) = \left(2 \cdot P^*(A_{2^{k-1}})\right)\left(2 \cdot P^*(A_{2^{k-1}}) - 1\right)$$

Proof

(1) $\left(2 \cdot P(A_{2^{k-1}})\right)\left(2 \cdot P(A_{2^{k-1}}) - 1\right)$ is $P'(A_{2^k})$ for some permutation function P'

$$\left(2 \cdot P(A_{2^{k-1}})\right)\left(2 \cdot P(A_{2^{k-1}}) - 1\right)$$
$$= \left(\left(2 \cdot p(i)\right)_{i=1}^{2^{k-1}}\right)\left(\left(2 \cdot p(i) - 1\right)_{i=1}^{2^{k-1}}\right)$$
$$= \left(p'(i)\right)_{i=1}^{2^k} = P'(A_{2^k})$$

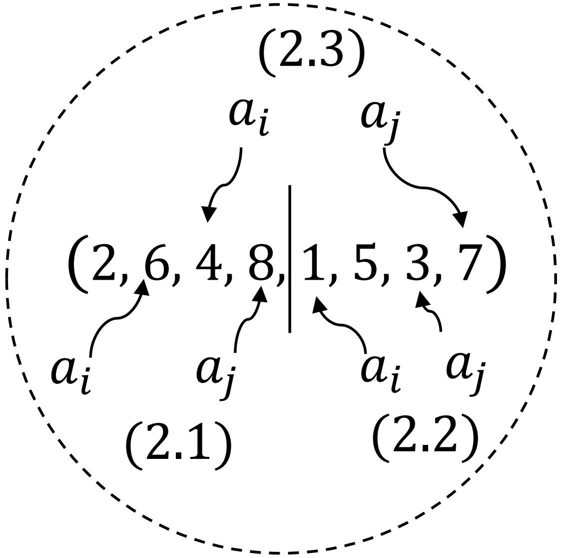
$$P(A_2) = (1, 2)$$
$$2 \cdot P(A_2) = (2, 4)$$
$$2 \cdot P(A_2) - 1 = (1,3)$$
$$\left(2 \cdot P(A_2)\right)\left(2 \cdot P(A_2) - 1\right) = (2,4,1,3)$$
$$= P(A_4)$$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

<u>Claim</u>

$$\exists\, P^*\left(A_{2^k}\right)\, s.t.\, P^*\left(2^k\right) = \left(2 \cdot P^*\left(A_{2^{k-1}}\right)\right)\left(2 \cdot P^*\left(A_{2^{k-1}}\right) - 1\right)$$



<u>Proof</u>

(2) $P'(2^k)$ is a $P^*(2^k)$

Show (2.1) $\forall\, a_i, a_j \in 2 \cdot P^*\left(A_{2^{k-1}}\right), a_k \neq \frac{a_i + a_j}{2}, i < k < j$

(2.2) $\forall\, a_i, a_j \in 2 \cdot P^*\left(A_{2^{k-1}}\right) - 1, a_k \neq \frac{a_i + a_j}{2}, i < k < j$

(2.3) $\forall\, a_i \in 2 \cdot P^*\left(A_{2^{k-1}}\right), \forall\, a_j \in 2 \cdot P^*\left(A_{2^{k-1}}\right) - 1, a_k \neq \frac{a_i + a_j}{2}$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\ k \in \{0\} \cup \mathbb{N}$

<u>Claim</u>

$$\exists\ P^*(A_{2^k})\ s.t.\ P^*(A_{2^k}) = \left(2 \cdot P^*(A_{2^{k-1}})\right)\left(2 \cdot P^*(A_{2^{k-1}}) - 1\right)$$

<u>Proof</u>

$$(2.1)\ \forall\ a_i, a_j \in 2 \cdot P^*(A_{2^{k-1}}), a_k \neq \frac{a_i + a_j}{2}, i < k < j$$

We know that $a'_m = \frac{a_m}{2}, \forall\ a'_m \in P^*(A_{2^{k-1}})$ and $Q(2^{k-1})$ is true

$$\Rightarrow \forall a'_i, a'_j \in P^*(A_{2^{k-1}}), a'_k \neq \frac{a'_i + a'_j}{2}, i < k < j$$

$$\Rightarrow \forall\ \frac{a_i}{2}, \frac{a_j}{2} \in P^*(A_{2^{k-1}}), \frac{a_k}{2} \neq \frac{\frac{a_i}{2} + \frac{a_j}{2}}{2}, i < k < j$$

$$\Rightarrow \forall a_i, a_j \in 2 \cdot P^*(A_{2^{k-1}}), a_k \neq \frac{a_i}{2} + \frac{a_j}{2}, i < k < j$$

$$(2, \underset{a_i}{6}, 4, \underset{a_j}{8} \mid 1, 5, 3, 7)$$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

<u>Claim</u>

$$\exists\, P^*(A_{2^k})\ s.t.\ P^*(A_{2^k}) = \left(2 \cdot P^*(A_{2^{k-1}})\right)\left(2 \cdot P^*(A_{2^{k-1}}) - 1\right)$$

<u>Proof</u>

$$(2.2)\ \forall\, a_i, a_j \in 2 \cdot P^*(A_{2^{k-1}}) - 1, a_k \neq \frac{a_i + a_j}{2}, i < k < j$$

We know that $a'_m = \frac{a_m + 1}{2}, \forall\, a'_m \in P^*(A_{2^{k-1}})$ and $Q(2^{k-1})$ is true

$$\Rightarrow \forall a'_i, a'_j \in P^*(A_{2^{k-1}}), a'_k \neq \frac{a'_i + a'_j}{2}, i < k < j$$

$(2, 6, 4, 8, \mid 1, 5, 3, 7)$

a_i a_j

$$\Rightarrow \forall\, \frac{a_i + 1}{2}, \frac{a_j + 1}{2} \in P^*(A_{2^{k-1}}), \frac{a_k + 1}{2} \neq \frac{\frac{a_i + 1}{2} + \frac{a_j + 1}{2}}{2}, i < k < j$$

$$\Rightarrow \forall a_i, a_j \in 2 \cdot P^*(A_{2^{k-1}}) - 1, a_k \neq \frac{a_i}{2} + \frac{a_j}{2}, i < k < j$$

Proof by Induction

(i) Show that $Q(2^k)$ is true, $\forall\, k \in \{0\} \cup \mathbb{N}$

<u>Claim</u>

$$\exists\, P^*(A_{2^k})\ s.t.\ P^*(A_{2^k}) = \left(2 \cdot P^*(A_{2^{k-1}})\right)\left(2 \cdot P^*(A_{2^{k-1}}) - 1\right)$$

<u>Proof</u>

$$(2.3)\ \forall\, a_i \in 2 \cdot P^*(A_{2^{k-1}}), \forall\, a_j \in 2 \cdot P^*(A_{2^{k-1}}) - 1, a_k \neq \frac{a_i + a_j}{2}$$

a_i is an even number (say $2m$ for some $m \in \mathbb{N}$)

a_j is an odd number (say $2m' + 1$ for some $m' \in \mathbb{N}$)

$$\Rightarrow \frac{a_i + a_j}{2} = \frac{2m + (2m' + 1)}{2} = (m + m') + \frac{1}{2} \notin \mathbb{N}$$

$$\Rightarrow a_k \neq \frac{a_i + a_j}{2}\ (\because a_k \in \mathbb{N})$$

a_i a_j

$(2, 6, 4, 8, \mid 1, 5, 3, 7)$

Proof by Induction

(ii) Show that $Q(k) \rightarrow Q(k-1), \forall\, k \in \mathbb{N}$

$(2,\cancel{4},1,3) \Rightarrow (2,1,3)$

$(2,\cancel{6},4,\cancel{8},1,5,3,\cancel{7}) \Rightarrow (2,4,1,5,3)$

Proof

If $Q(k)$ is true, remove k from valid $P(A_k)$, we get valid $P(A_{k-1})$.

Note

$Q(k)$ is true, $\forall\, k = 2^i,\ i \in \{0\} \cup \mathbb{N}$.

Method

step 1 Find the closest power of 2 being greater than k, say 2^c.

step 2 Remove $i,\ k < i \leq 2^c$ from valid $P(A_{2^c})$ to get valid $P(A_k)$.

Problem 10

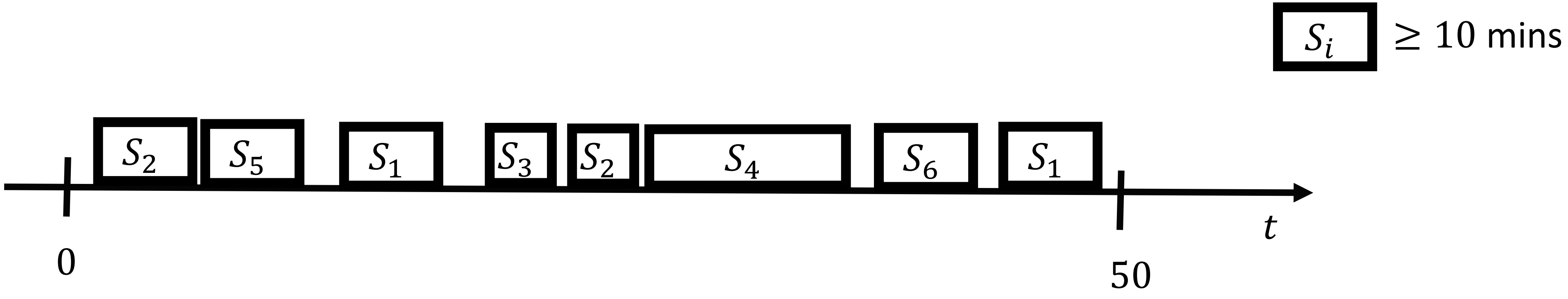
A lecture lasts 50 minutes and 6 students were sleeping for at least 10 minutes during the lecture. Show that two students were sleeping simultaneously at some point during the lecture.

Proof by Contradiction

Assumption

There are no students were sleeping simultaneously at some point during the lecture

$\Rightarrow$ Every student has his (her) mutually exclusive (only for single individual) sleep time from 50 minutes of lecture time.



Proof by Contradiction

Proof

Suppose that there exists an arbitrary small time slot $dt > 0$.

Each of 6 student has his own time function $S_i(t)$
such that $\int_0^{50} S_i(t)\, dt \geq 10$ (mins)

$$\Rightarrow \int_0^{50} [S_1(t) + S_2(t) + S_3(t) + S_4(t) + S_5(t) + S_6(t)]\, dt \geq 60 \text{ (mins)} \; (\rightarrow\leftarrow)$$

Therefore, there are at least two students were sleeping simultaneously at some point during the lecture.

Problem 14

Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $(b_1, b_2, b_3, b_4, b_5, b_6)$ be two arrangements of the integers $1, 2, 3, 4, 5, 6$. Consider the six pairs of differences $|a_i - b_i|$. Is it possible that all of these differences are not the same?

$$a_i\text{'}s = (1, 2, 3, 4, 5, 6)$$

$$b_i\text{'}s = (2\ 4, 6, 5, 3, 1)$$

$$|a_i - b_i|\text{'}s = (1, 2, 3, 1, 2, 3)$$

Proof by Contradiction

Assumption

All 6 differences are distinct $\left(|a_i - b_i| \neq \left|a_j - b_j\right|, \forall\ i \neq j\right)$.

Proof

The only possible values of $|a_i - b_i|$ are $0, 1, 2, 3, 4, 5$.

$$\Rightarrow \sum_{i=1}^{6} |a_i - b_i| = 15 \text{ (by assumption)}$$

, where 15 is an odd number.

Proof by Contradiction

Proof

$$\sum_{i=1}^{6}|a_i - b_i| - \sum_{i=1}^{6}(a_i - b_i) \text{ is an even number}$$

$$(\because |a_i - b_i| - (a_i - b_i) = 0 \text{ or } 2|a_i - b_i|\ )$$

$a_i = 5, b_i = 1$
$\Rightarrow a_i - b_i = 4, |a_i - b_i| = 4$
$\Rightarrow |a_i - b_i| - (a_i - b_i)$
$= 0$

$a_i = 1, b_i = 5$
$\Rightarrow a_i - b_i = -4, |a_i - b_i| = 4$
$\Rightarrow |a_i - b_i| - (a_i - b_i) = 8$
$= 2|a_i - b_i|$

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} b_i$$

($\because$ a_i's and b_i's are permutations of the same tuple)

$$\Rightarrow \sum_{i=1}^{6}(a_i - b_i) = 0$$

$$\Rightarrow \sum_{i=1}^{6}|a_i - b_i| - \sum_{i=1}^{6}(a_i - b_i) = \sum_{i=1}^{6}|a_i - b_i|$$

, where $\sum_{i=1}^{6}|a_i - b_i|$ is an even number (→←)

Therefore, it is impossible that all six differences are not the same.

Discrete Math HW03

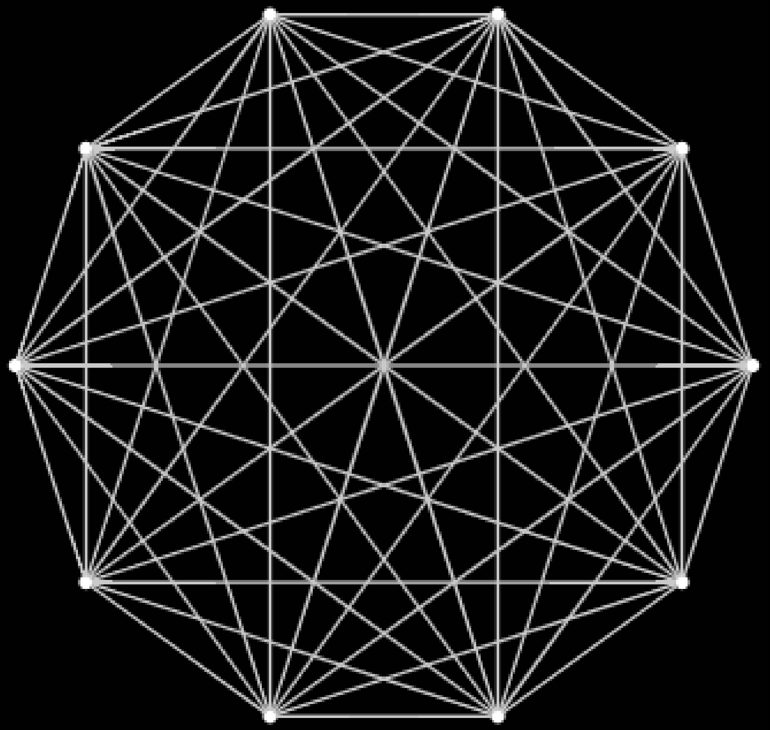
Q11 Q12 Q13

Q11 and Q12

- Show that in a group of 10 people, either there are 3 mutual friends, or 4 mutual enemies, or both.
- Show that in a group of 9 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

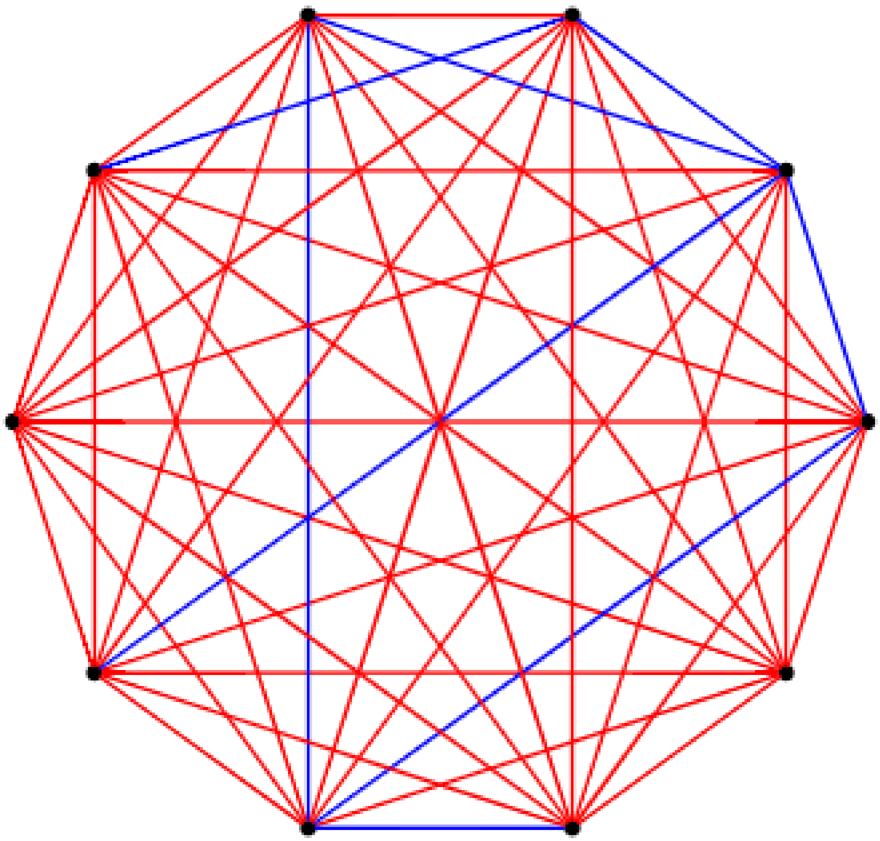
Q11 and Q12

+ Consider a complete graph with 10(or 9) nodes. Each node represents a person, and the edge between them represents their relationship



Q11 and Q12

- Their relationship is either friend(red) or enemy(blue)
- Draw each edge with blue or red.
- We have to prove there must exist at least one in below,
  - a red complete graph with 3 nodes
  - a blue complete graph with 4 nodes



Q11 and Q12

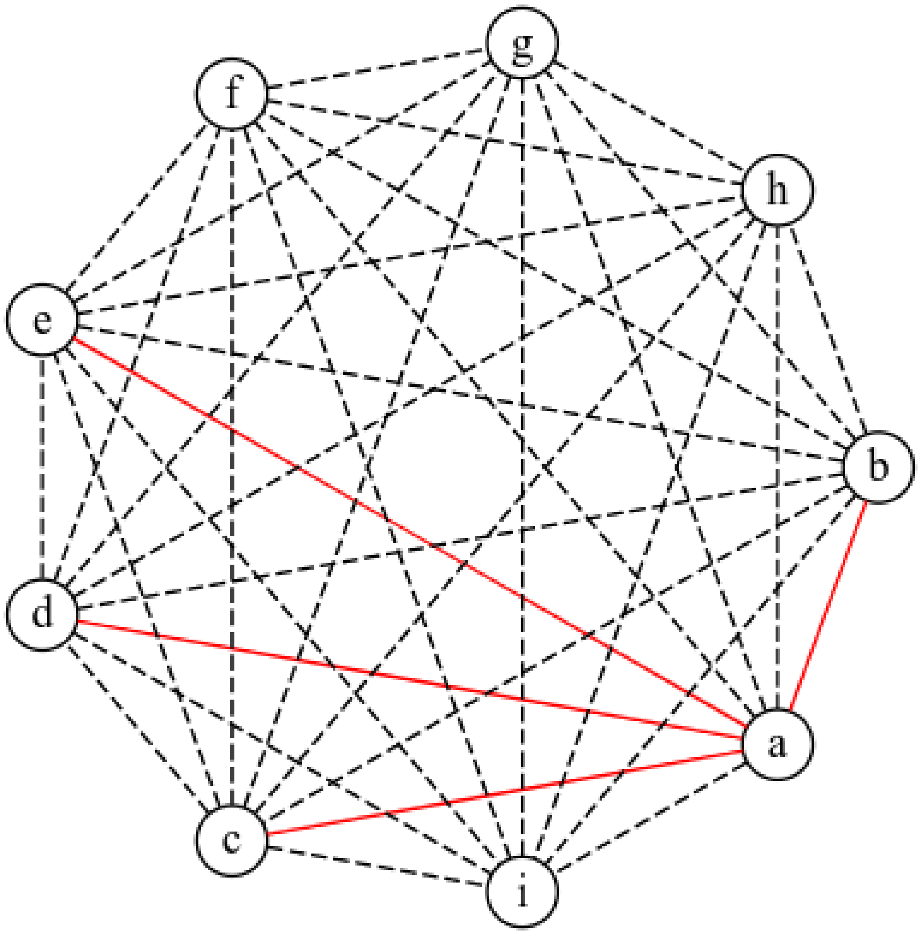
+ Q12 is harder. If we prove Q12, we add a node into any graph described in Q12 we should prove Q11.
+ Let us focus on Q12.

Q12

- For any nodes, it has 8 edges
- For any nodes, >= 4 edges are with the same color.
  - By Pigeonhole Principle.
  - 8 red + 0 blue, 7 red + 1 blue, ...
- Hence, we discuss
  - [Case 1]: it contains >= 4 red edges
  - [Case 2]: it contains >= 6 blue edges
  - [Case 3]: it contains 5 blue edges and 3 red edges

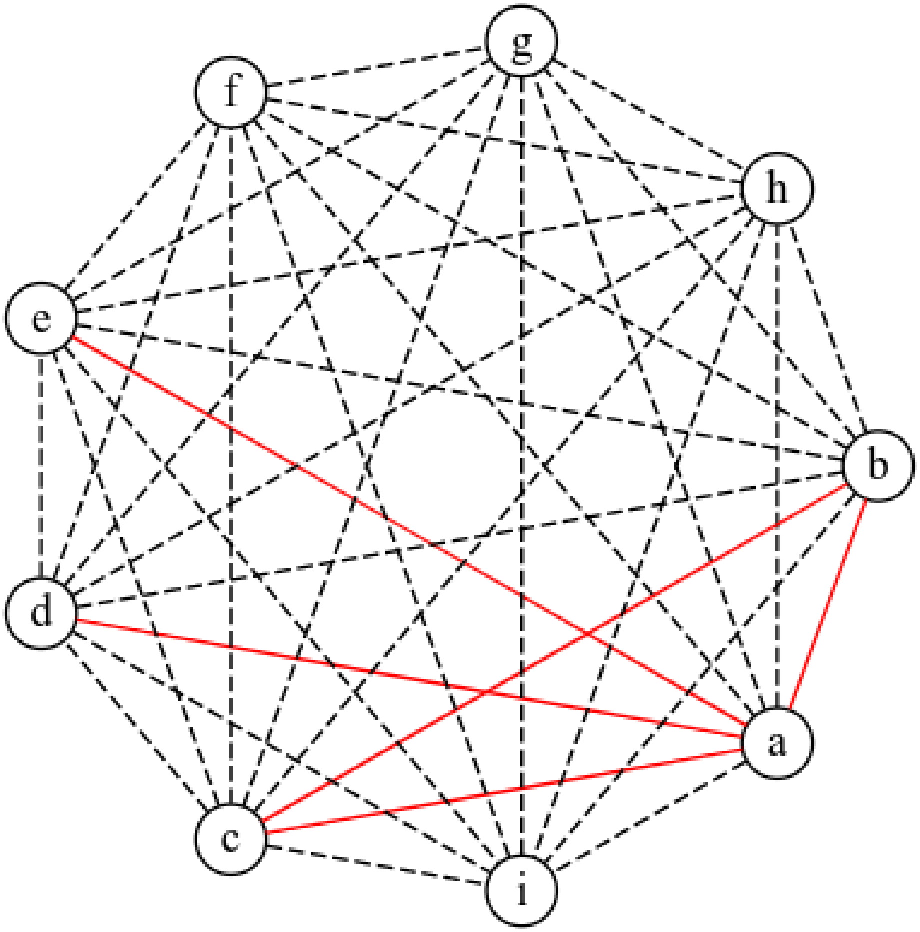
[Case 1] >= 4 red edges

+ The node is labeled ‘a’
+ Assume it is connected to {b,c,d,e}
+ If some pair in {b,c,d,e} is with red edge, there exist a



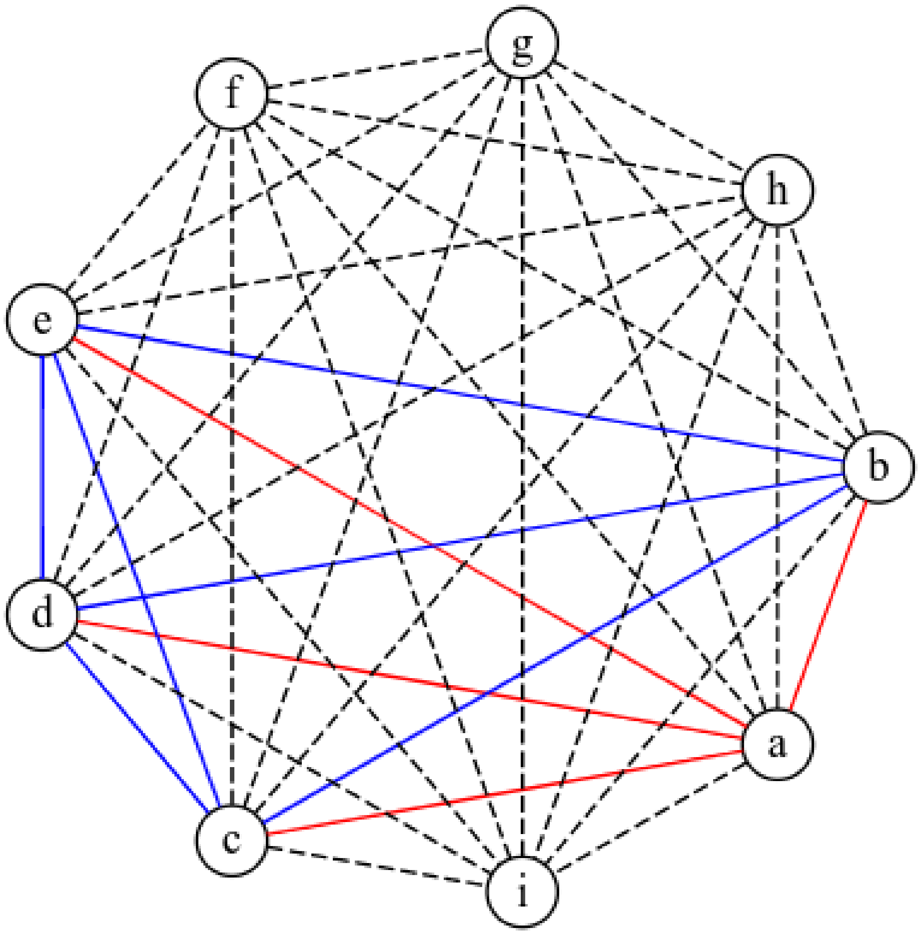
[Case 1] >= 4 red edges

+ If some pair in {b,c,d,e} is with red edge, there exist ‘a red complete graph with 3 nodes’



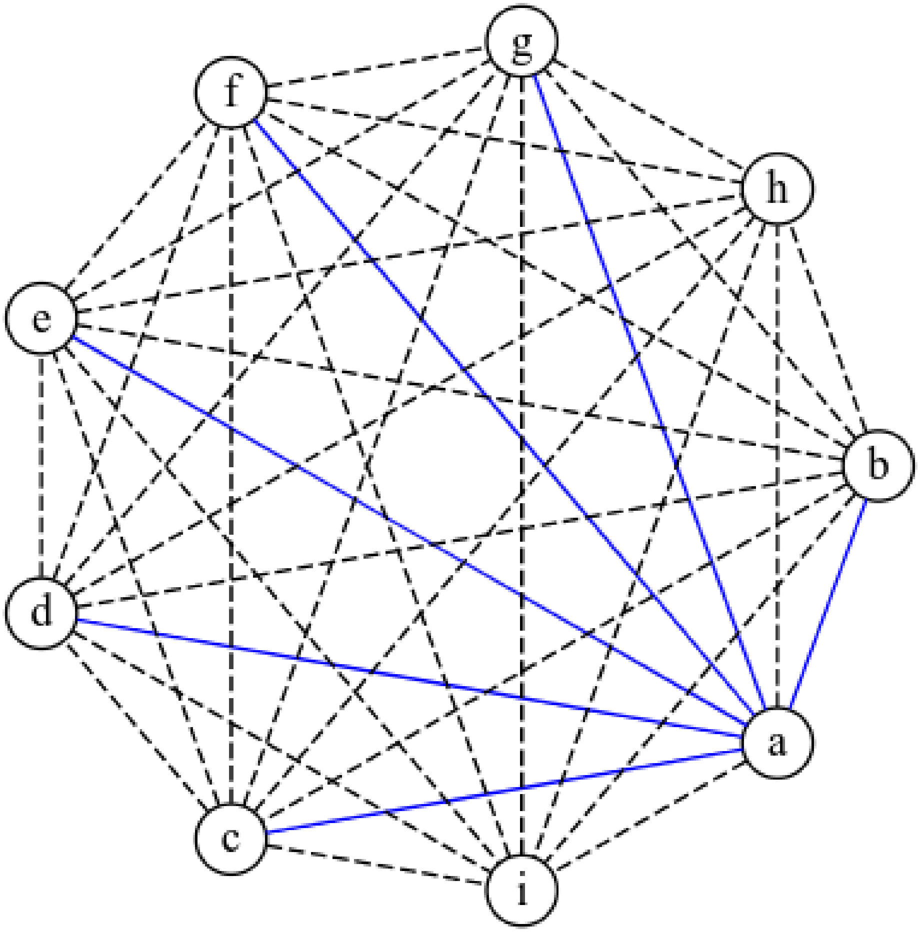
[Case 1] >= 4 red edges

+ If not, any pair in {b,c,d,e} is with blue edge, there exist ‘a blue complete graph with 4 nodes’.



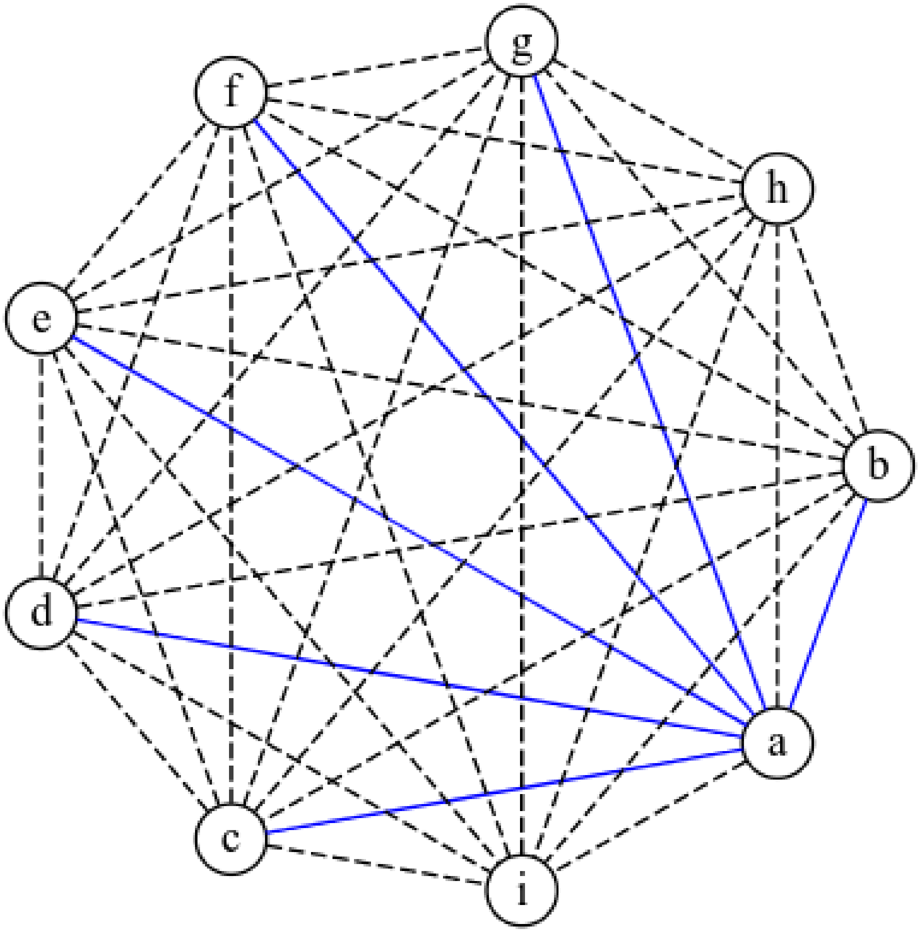
[Case 2] >= 6 blue edges

- We cannot use previous method.
- We could transform the problem into smaller problem but similar to the original one:
- In these 6 nodes, is there contains at least one these
  - a red complete graph with 3 nodes
  - a blue complete graph with 3 nodes
- Proved in lecture.



[Case 2] >= 6 blue edges

- If contains a red complete graph with 3 nodes, then proved
- If contains a blue complete graph with 3 nodes we add ‘a’ and its three blue edge into that graph, we have ‘a blue complete graph with 4 nodes’
- We solve this problem recursively



[Case 3]: 5 blue edges and 3 red edges

+ Every node in this graph must have 5 blue edge and 3 red edge or we could solve this problem with previous methods
+ We have (9*3)/2 = 13.5 edges in this graph
+ This is impossible

Ramsey’s Number

+ If a complete graph with its edge colored with blue and red contains at least one of below
  + a red complete graph with m nodes
  + a blue complete graph with n nodes
+ The smallest size of the possible complete graph is R(m,n) which is known as the Ramsey number.
+ Note that R(m,n) = R(n,m)

Ramsey's Them.

+ $R(m, n) \leq R(m - 1, n) + R(n, m - 1)$
+ You can find the proof on wiki page

Q13

- A person can shake hands with between 0 and 99 people since you cannot shake hands with yourself.
  - That is 100 possibilities and 100 people.
- Number 0 and 99 cannot happen in the same time. If a person has shook with 99 people, then each of them should shake at least 1 time.
  - So there are 99 possibilities and 100 people.
- By Pigeonhole Principle, the proposition is proved.