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電磁學（一）
Electromagnetics (I)

11. 電流與電壓
Electric Current and Electric Voltage

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In this lecture, we will learn about the basic concepts of electric current driven by a voltage.

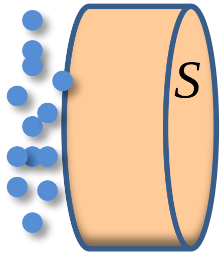
- 11.1 Electric Current 電流
- 11.2 Convection Current 真空電流（對流電流）
- 11.3 Conduction Current 導體電流
- 11.4 Basic Circuit Laws 基本電路定律
- 11.5 Review 單元回顧

電流與電壓
Electric Current and Electric Voltage

11.1 電流
Electric Current

Electric Current – flow of charges

Electric Current = amount charges crossing an area per unit time



$\vec{u}$: velocity of charges

$$I = \frac{dq}{dt} = \int_s \vec{J} \cdot d\vec{s}$$

(SI unit: Ampere ≡ C/sec)

Define **Volume Current Density**,

$$\vec{J} = \rho_v \vec{u}$$

(SI unit: A/m^2)

where ρ_v is the volume charge density.

$$dI = \vec{J} \cdot d\vec{s} \Rightarrow I = \int_S \vec{J} \cdot d\vec{s}$$

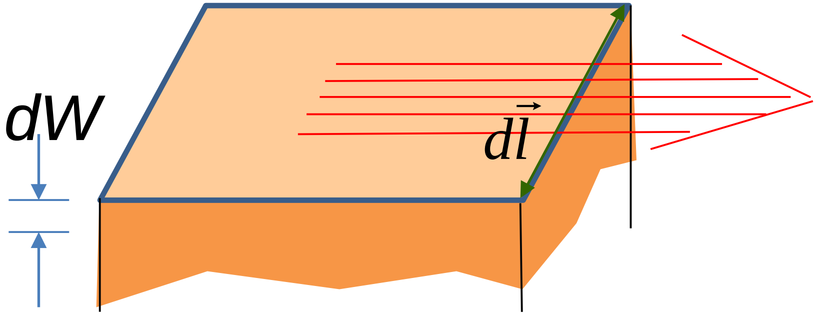
* Dot product: Only the charges flowing along the normal direction of the cross sectional area will move down a wire effectively.

Surface Current Density

In a **perfect** **conductor, the volume charge density $\rho_v \rightarrow \infty$.**
Thus, the volume current density $J = \rho_v u \rightarrow \infty$

There exists a finite **surface current density** in the limit

$$\vec{J}_s = \lim_{dW \rightarrow 0} \vec{J} \times dW$$ (SI unit: A/**m**)

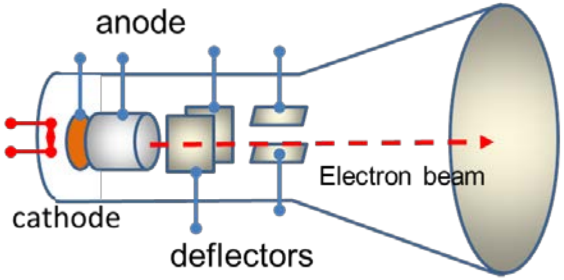


The total current on the surface is the line integration along the transverse direction.

$$I = \int_S \vec{J}_s \cdot d\vec{l}$$

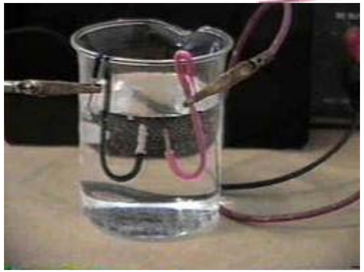
Types of Current

1. **Convection current**: resulting from motion of charged particles in vacuum. The charge density can modify the potential that drives the particles.



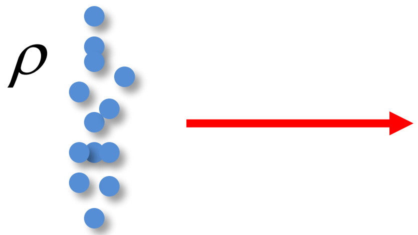
2. **Conduction current**: resulting from motion of electrons and/or holes in a neutral material

3. **Electrolytic current**: resulting from migration of positive and/or negative ions in an aqueous environment.



Joule's Law

Consider a group of charges pulled by an electric field



$$\vec{F} = q\vec{E}$$

$\vec{u}$: velocity of charges

The differential power in the current is

$$dP = d(\vec{F}\cdot\vec{u}) = (\underbrace{\vec{E}dq}_{\vec{F}=q\vec{E}})\cdot\vec{u} = (\vec{E}\underbrace{\rho dv}_{dq=\rho dv})\cdot\vec{u} = \vec{E}\cdot(\underbrace{\rho\vec{u}}_{\vec{J}=\rho\vec{u}}dv) = \vec{E}\cdot\vec{J}dv$$

$$\Rightarrow \quad P = \int_V \vec{E}\cdot\vec{J}dv$$ (Joule's Law)

where $\vec{E}\cdot\vec{J}$ is a volume power density in Watt/m^3

11.1 電流

Electric Current

- The electric current is defined as the amount of charges crossing a cross sectional area per unit time.
- A convection current flows in vacuum, a conduction current flows in a conductor, and an electrolytic current flows in electrolyte.
- The power carried in a current under an electric field is governed by the Joule's law

$$P = \int_V \vec{E} \cdot \vec{J} dv$$

電流與電壓
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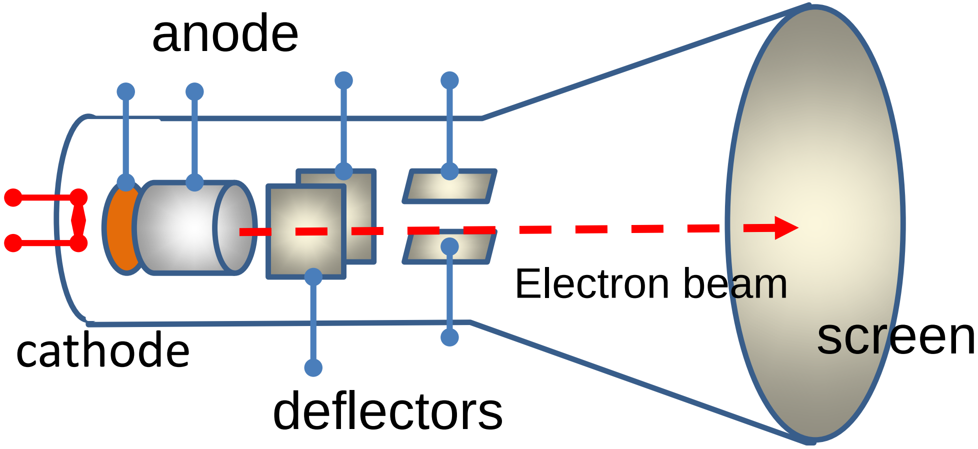
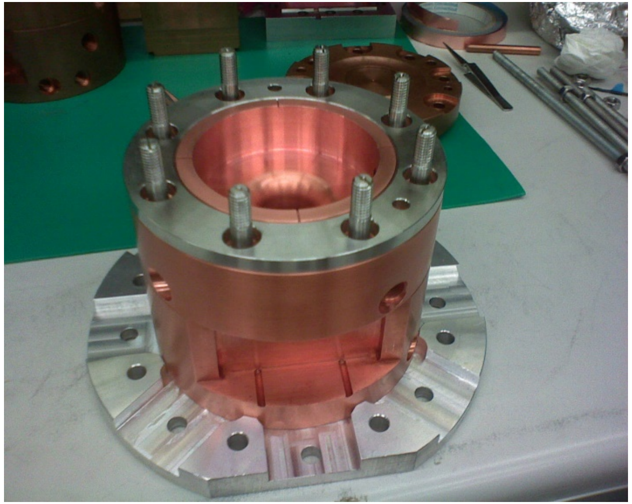
11.2 真空電流（對流電流）
Convection Current

Convection Current

(charged particle moving in vacuum)



cathode Ray Tube (CRT)



cavity of a microwave electron gun

A moving charge experiences the externally applied voltage + the electric potential from adjacent charges.

E.g. Convection current across a parallel-plate accelerator. Find J as a function of V_0

Electron kinetic energy = potential energy

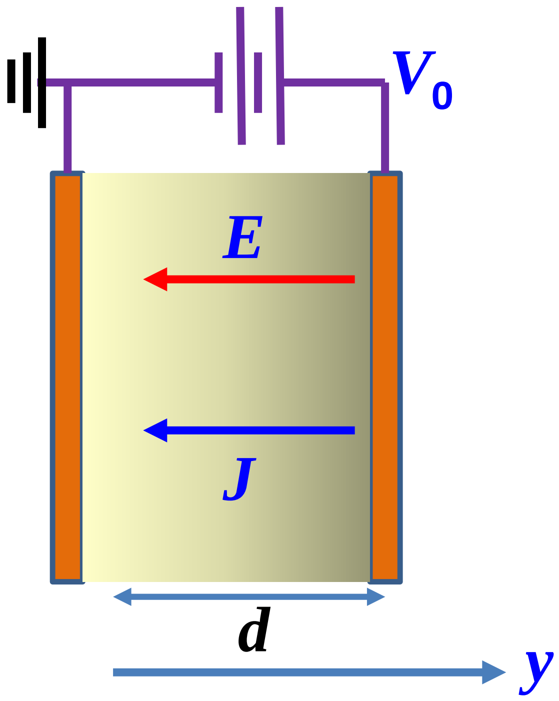
$$\frac{1}{2}mu^2 = eV \Rightarrow u = \sqrt{\frac{2eV}{m}}$$

electron mass   electron speed

At a steady state, the current density is

$$\vec{J} \equiv Nq\vec{u} = \rho(y)\vec{u}(y) = \text{const.}$$

The charge density is $\Rightarrow \rho(y) = -J\sqrt{\dfrac{m}{2eV(y)}}$



From Poisson's equation $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, $\Rightarrow \frac{d^2V}{dy^2} = \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2}$

Use the Math trick:

$\frac{d}{dy}\left(\frac{dV}{dy}\right)^2 = 2\frac{dV}{dy}\left(\frac{d^2V}{dy^2}\right)$ to write $d\left(\frac{dV}{dy}\right)^2 = 2\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2}dV$

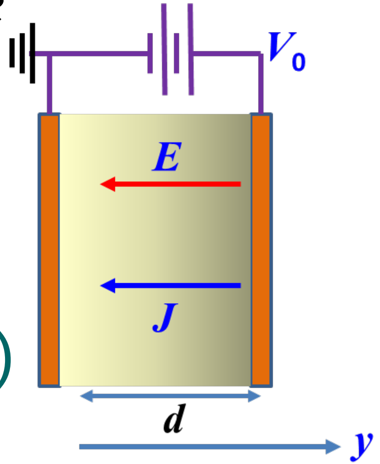
Boundary conditions:

I. at $y = 0$, $V = 0$ and $\frac{dV}{dy} = 0$ II. at $y = d$, $V = V_0$

Space-charge limited emission

$$\boxed{\Rightarrow J = \frac{4\varepsilon_0}{9d^2}\sqrt{\frac{2e}{m}}V_0^{3/2} \propto V^{3/2}}$$

(Child-Langmuir's law)



Ohm's law is not valid for a convection current in vacuum, because charge-in-vacuum is not neutral and can modify the local potential.

11.2 真空電流（對流電流）

Convection Current

- A convection current flows in vacuum.
- The electric potential seen by a charge is modified by adjacent charges.
- At the steady state, the I-V curve of a vacuum diode follows the expression

$$J = \frac{4\varepsilon_0}{9d^2}\sqrt{\frac{2e}{m}}V_0^{3/2} \propto V^{3/2}$$

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11.3 導體電流
Conduction Current

Mobility

From Newton's mechanics, $F = ma$, one would expect acceleration of a charge, *a*, under an electric field or

$$\frac{d\vec{u}}{dt} = \vec{a} \propto \vec{E}$$

But, in fact, in a conductor, $\vec{u} \propto \vec{E}$ is observed.

In a conductor, collisions make an electron move at an average speed under an electric field

$$\boxed{\vec{u} = -\mu_e \vec{E},} \qquad (\mathrm{m/s})$$

where $\mu_e\ (\mathrm{m^2/V \cdot s})$ is the *mobility* of an electron.

E.g. $\mu_{e,Al} = 1.4 \times 10^{-4},\ \mu_{e,Cu} = 3.2 \times 10^{-3},\ \mu_{e,Ag} = 5.2 \times 10^{-3}\, m^2/V \cdot s$

Conductivity

The volume current density becomes linearly proportional to the electric field.

$$\vec{J} = \rho\vec{u} = -\rho\mu_e\vec{E} = \sigma\vec{E}$$

where the proportional factor σ is called conductivity in Siemens/m and $1/\sigma$ is called resistivity.

In a semiconductor, there are two types of charges, electron (–) and hole (+). The total conductivity becomes

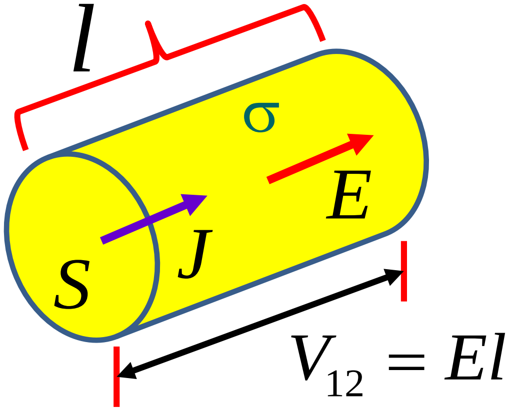
$$\boxed{\sigma = -\underbrace{\rho_e\mu_e}_{\text{Electron conductivity}} + \underbrace{\rho_h\mu_h}_{\text{hole conductivity}}}$$

(conductivity of a semiconductor)

Ohm's Law

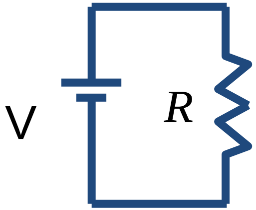
(for conduction materials with collisions or $J = \sigma E$)

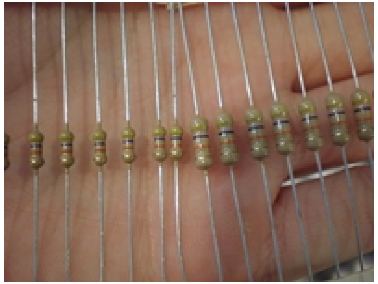
$$\left.\begin{aligned} I &= JS = \sigma ES \\ V_{12} &= El \end{aligned}\right\} \quad \frac{V_{12}}{I} = \frac{l}{\sigma S} = R$$



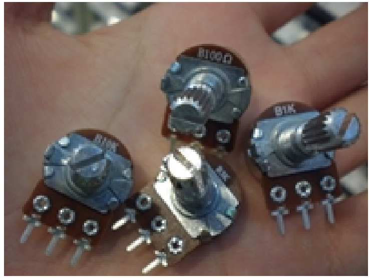
or $\boxed{V = IR}$ (Ohm's Law)

R is resistance in ohm, Ω; $G = 1/R$ is conductance in mho





一般電阻



可變電阻

光敏電阻

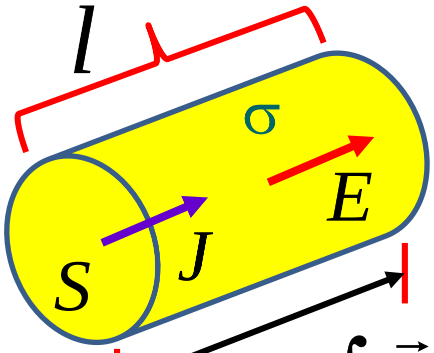
熱敏電阻

Ohmic Loss

Recall the Joule's law $P = \int_V \vec{E} \cdot \vec{J} dv$

In a circuit, $dv = dl ds$

$$P = \int_V \vec{E} \cdot \vec{J} dv = \underbrace{\int_L \vec{E} \cdot d\vec{l}}_{V} \underbrace{\int_S \vec{J} \cdot d\vec{s}}_{I} = V \times I$$



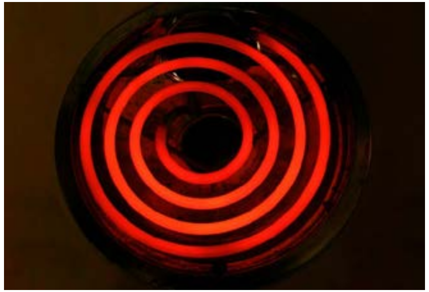
$$V = \int_L \vec{E} \cdot d\vec{l}$$

Use the Ohm's law $V = IR$ to write

$$P = I \times V = I^2 R = \frac{V^2}{R}$$

This is the so-called Ohmic loss in a circuit.

Ohmic loss usually dissipates as heat.



resistive heating (stove)

11.3 導體電流

Conduction Current

- In a conductor (ohmic material), the speed of electrons is linearly proportional to the driving electric field due to collisions of electrons with the material matrix.

$$\vec{u} = -\mu_e \vec{E}$$

- In an ohmic material, the current is linearly proportional to the driving voltage or $V = IR$, where R is the resistance.
- In an ohmic material, collisions induces power loss, described by $P = I^2 R = V^2 / R$

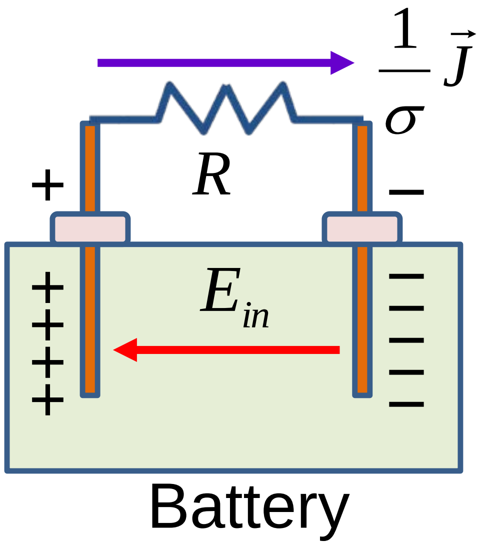
電流與電壓
Electric Current and Electric Voltage

11.4 基本電路定律
Basic Circuit Laws

Electromotive Force

For a conservative force,

$$\nabla \times \vec{E} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{l} = 0 \text{ or } \oint \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = 0$$



A conservative force can't support a current in a circuit loop.

An *electromotive force* from a generator has to be supplied to a circuit with a current.

Add a term of non-conservative force f to the right side

$$\boxed{\nabla \times \vec{E} = \vec{f}} \quad \Rightarrow \oint \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = V_{em} = RI$$

where V_{em} is called the electromotive force

Kirchhoff's Voltage Law

voltage rises = voltage drops, around a closed-loop circuit

$$\sum_j V_{em_j} = \sum_k R_k I_k$$

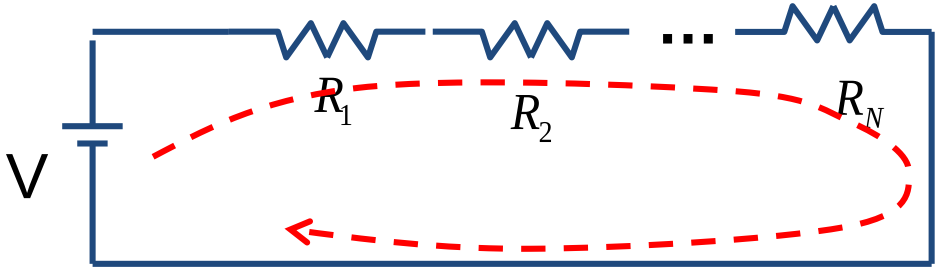
A consequence of energy conservation

Apply the Kirchhoff's voltage law to write

$$V = R_{total}\, I = R_1 I + R_2 I \ldots + R_N I$$

$$\Rightarrow R_{total} = R_1 + R_2 \ldots + R_N$$

Serial resistors

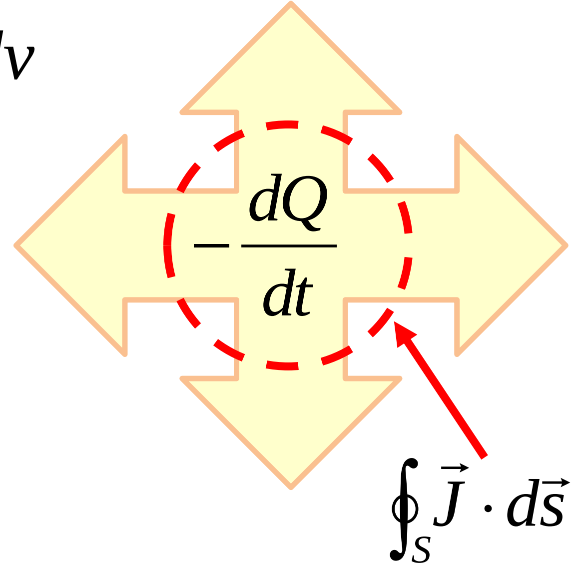


Equation of Continuity

From **charge conservation**

In a close volume, positive (negative) time-rate change of charges = current flowing outward (inward).

$$\oint_S \vec{J} \cdot d\vec{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho dv$$



Apply the divergence theorem to obtain

$$\int_V \nabla \cdot \vec{J} dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

Equation of Continuity $\boxed{\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$

Kirchhoff's Current Law

Charge conservation: In equilibrium, no charge is generated or annihilated at a node of a neutral conductor.

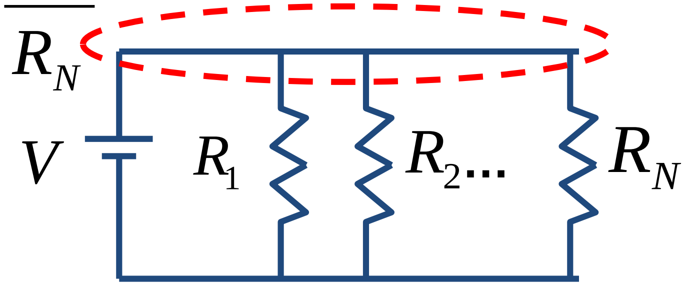
The *algebraic sum of all the currents flowing out of a circuit node is zero*

$$\nabla \cdot \vec{J} + \cancelto{0}{\frac{\partial \rho}{\partial t}} = 0 \Rightarrow \oint_S \vec{J} \cdot d\vec{s} = 0 \;\Rightarrow\; \sum_j I_j = 0$$

Kirchhoff's current law gives the total current

$$\frac{V}{R_{total}} = \frac{V}{R_1} + \frac{V}{R_2} \ldots + \frac{V}{R_N} \Rightarrow \frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} \ldots + \frac{1}{R_N}$$

Parallel resistors



or $G_{total} = G_1 + G_2 \ldots G_N$ (conductance)

11.4 基本電路定律

Basic Circuit Laws

- The electromotive force in a battery or a generator drives a current in a circuit loop.
- Energy conservation requires the voltage rise equal to the voltage drop in a circuit loop (Kirchhoff's voltage law).
- From charge conservation, one derives the equation of continuity.

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

- Charge conservation requires that, in equilibrium, the total current flowing out a circuit node is zero (Kirchhoff's current law).

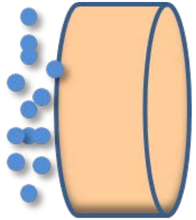
電流與電壓
Electric Current and Electric Voltage

11.5 單元回顧 Review

單元回顧

1. The Volume Current Density is defined as

$$\vec{J} = \rho_v \vec{u} \quad \text{(SI unit: A/m}^2\text{)}$$



$\vec{u}$: velocity of charges

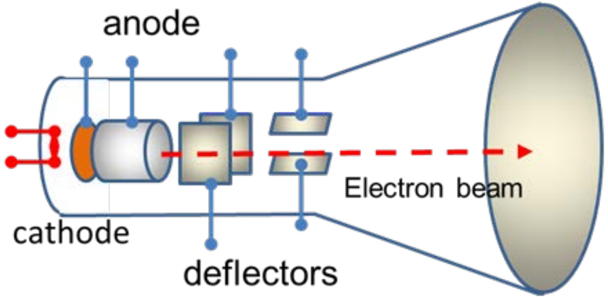
where ρ_v is the volume charge density.

2. An electric current is then the amount of charges crossing an area per unit time:

$$dI = \vec{J} \cdot d\vec{s} \Rightarrow I = \int_S \vec{J} \cdot d\vec{s}$$

單元回顧

3. A convection current is a flow of charges in vacuum. The space charge field modifies the electric potential seen by the charges.



4. A conduction current is a flow of charges in a neutral material. The collisions of the charges with the material matrix makes the charge propagate with a constant speed, giving rise to the Ohm's law: $V = IR$.

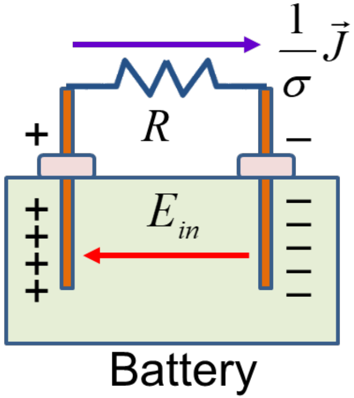
單元回顧

5. Collision of charges in a conducting material results in power loss, called the Ohmic loss, given by

$$\underbrace{P = I \times V}_{\text{Joule's law}} = I^2 R = \frac{V^2}{R}$$

6. The electromotive force of a battery or a generator, V_{em}, drives a current in a circuit loop.

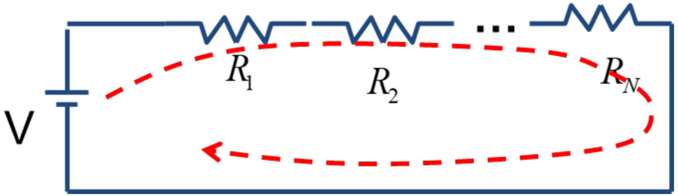
$$\nabla \times \vec{E} = \vec{f} \;\Rightarrow\; \oint \frac{1}{\sigma}\vec{J} \cdot d\vec{l} = V_{em} = RI$$



單元回顧

7. Based on energy conservation, the Kirchhoff's voltage law states

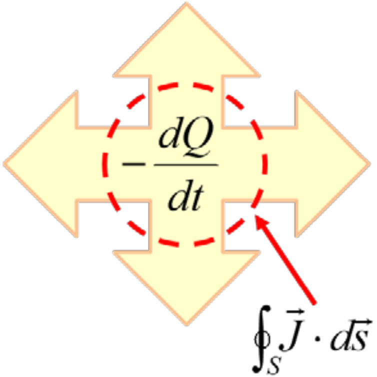
$$\sum_j V_{em_j} = \sum_k R_k I_k$$



voltage rises = voltage drops, around a closed-loop circuit

8. The equation of continuity is a consequence of charge conservation in a close volume, given by

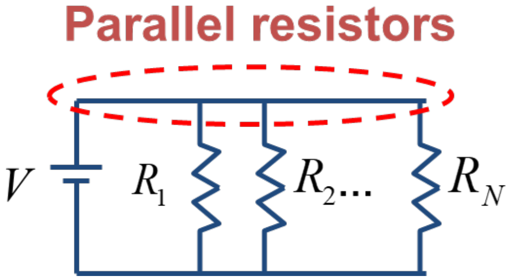
$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$



單元回顧

9. Based on charge conservation in a neutral volume, the Kirchhoff's current law states

$$\sum_j I_j = 0$$



In equilibrium, the *algebraic sum of all the currents flowing out of a circuit node is zero.*

THANK YOU FOR YOUR ATTENTION