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Step 1 : Express the solution as

$$y = x^r(c_0 + c_1x + c_2x + \cdots \qquad) = \sum_{k=0}^{\infty} c_k x^{k+r}$$

Step 2 : plug in the series to the ODE

$$y' = \sum_{k=0}^{\infty}(k+r)c_k x^{k+r-1}, \quad y'' = \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^{k+r-2}$$

$$x^2(\qquad\qquad) + x(\qquad\qquad) + (x^2-\nu^2)(\qquad\qquad) = 0$$

Step 3 : Find the indicial eq and the roots : r_1, r_2

$$r(r-1) + a_0 r + b_0 = 0$$

$\Rightarrow$

Step 4 : For each r, find the recurrence relation

(Note: After matching the coefficients,

For $r_1 =$ $\qquad$, $C_{2k} = \dfrac{(-1)^k C_0}{2^{2k}k!(1+\nu)(2+\nu)(3+\nu)\cdots(k+\nu)}$, $k = 1, 2, 3, \cdots$

For $r_2 =$ $\qquad$, $C_{2k} = \dfrac{(-1)^k C_0}{2^{2k}k!(1-\nu)(2-\nu)(3-\nu)\cdots(k-\nu)}$, $k = 1, 2, 3, \cdots$

Step 5 : Plug in the coefficients and obtain the general solution

For $r_1 =$ $\qquad \rightarrow y_1 = \sum_{k=0}^{\infty} C_{2k} x^{2k+\nu} = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{k!\Gamma(1+\nu+k)}\left(\dfrac{x}{2}\right)^{2k+\nu}$

For $r_2 =$ $\qquad \rightarrow y_2 = \sum_{k=0}^{\infty} C_{2k} x^{2k-\nu} = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{k!\Gamma(1-\nu+k)}\left(\dfrac{x}{2}\right)^{2k-\nu}$

(About "gamma function $\Gamma(x)$" (in Appendix A):

Def: $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ (Check out the graph of $\Gamma(x)$ in Fig A.1)

properties of $\Gamma(x)$: ① $\Gamma(x+1) = x\Gamma(x)$ ② $\Gamma(n+1) = n!$ when n = positive integer

③ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ ④ $\Gamma(n) \to \infty$ when $n = 0$ or negative integer)

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· However, depending on the value of "ν", there are two possible cases :

1) If ν is

$\Rightarrow$ J_ν and $J_{-\nu}$ are

So, general solution $y =$

2) If ν is

$J_{-\nu}(x) =$, which means J_ν and $J_{-\nu}$ are

We first need to find out the 2nd

Q : How to find the 2nd linearly independent solution if one solution is given ?

In the following, let's do some examples with specific order ν to see how Bessel functions look like :

<u>Bessel's eq of order 0</u> : $x^2y'' + xy' + x^2y = 0$ $(\nu = 0)$

general solution $\Rightarrow$ $y =$

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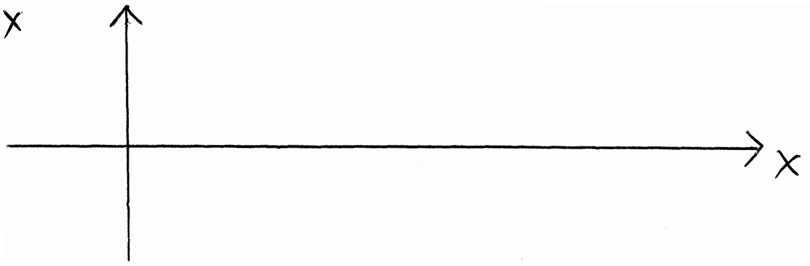
Plot $J_0(x)$ / $Y_0(x)$ v.s. x



Properties of J_0

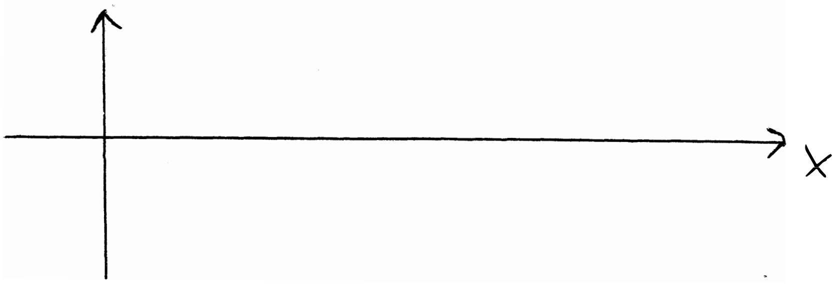
Properties of Y_0

— As $x \to 0$

<u>Bessel's eq of order $\frac{1}{2}$</u> : $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0 \quad (\nu = \frac{1}{2})$

general solution $\Rightarrow$ $y =$

Plot



Properties of

— As $x \to 0$,

— For $x > 0$,

Properties of

— As $x \to 0$

— For $x > 0$

Compared to the order 0 ($\nu = 0$), except a phase shift of

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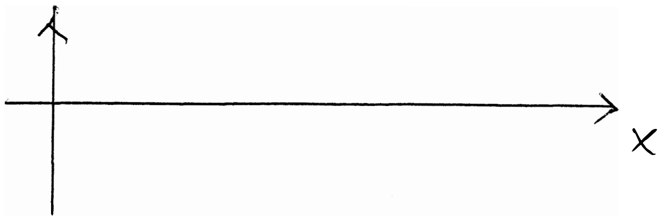
Bessel's eq of order 1 : $x^2y'' + xy' + (x^2-1)y = 0 \quad (\nu = 1)$

general solution $\Rightarrow y =$

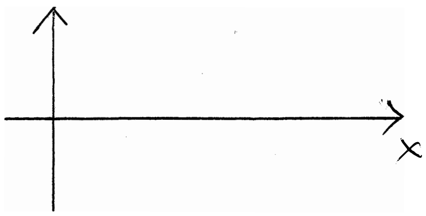
Plot



Properties of
- As $x \to 0$
- As x is large

Properties of
- As $x \to 0$
- As x is large

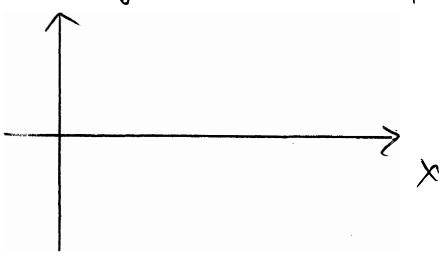
☆ Comparing J_ν (Bessel functions of the first kind) of different orders:



Properties of J_ν

1) $J_{-\nu}(x) =$
2) $J_\nu(-x)$
3) $J_\nu(0) =$
4) When x is large,

☆ Comparing Y_ν (Bessel functions of the second kind) of different orders:



Properties of Y_ν

1) As $x \to 0$
2) When x is large,

A final remark:

From the general solution of Bessel's equation $y =$

Note that $Y_\nu \to$ at the origin $(x=0)$.