CS2336 Discrete Mathematics

Exam 3
January 06, 2020 (10:10–12:30)

Answer all questions. Total marks = 105. Maximum Score = 105/100. Large portion of marks may be deducted for incomplete proofs.

1. (15%) Let A, B, and C be three sets. It is known that the following are true.

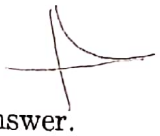
   - $|A| = 12$;
   - $|A - C| = 5$;
   - $|B \cup C| = 27$;
   - $|(A \cup B) - C| = 15$;
   - $|(B - C) - A| = 10$.

   Find $|C|$. (Hint: Draw a Venn diagram to help.)

2. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Consider the function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ with

   $$f(x) = 2/x.$$

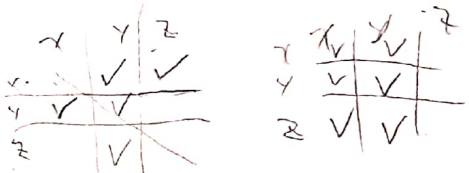
   

   (a) (10%) Determine if f is one-one. Explain your answer.

   (b) (10%) Determine if f is onto. Explain your answer.

3. (15%) Let $S = \{x, y, z\}$. Construct a binary relation R on S such that R is not reflexive, not symmetric, not antisymmetric, **but** transitive.

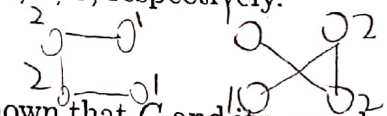
   (Note: You may describe R with a directed graph.)

4. (35%)

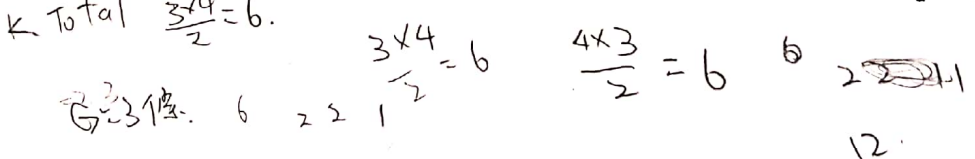
   (a) (15%) Draw two non-isomorphic simple undirected graphs H_1 and H_2, each with 6 vertices, and the degrees of these vertices are 2, 2, 2, 2, 3, 3, respectively.

   (b) (20%) Show that H_1 and H_2 are non-isomorphic.

5. Let G be a simple undirected graph with 4 vertices. It is known that G and its complement are isomorphic.

   (15%) Draw G.

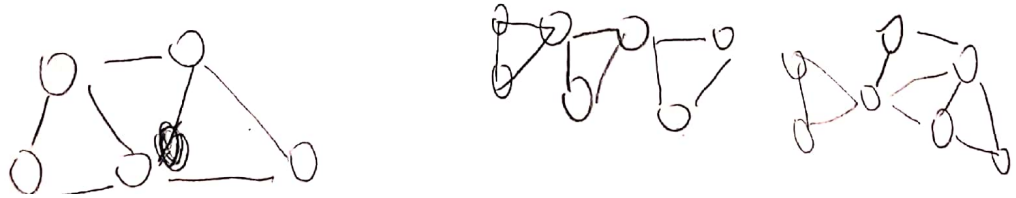
   

6. (Challenging)

   It is known that in a class with $N \geq 3$ students, the following conditions hold:

   - For any two students x and y, they are either friends or enemies, but not both.
   - For any two students x and y, there always exists one, and only one, common friend.

   (5%) Show that N is odd, and there is a student who is a friend with all other students.

CS2336 Discrete Mathematics

Exam 1
October 28, 2019 (10:10–12:30)

Answer all 7 questions. Total marks = 105. Maximum score = 100. For all the proofs, if it is incomplete, large portion of marks may be deducted.

1. Consider the proposition $(p \wedge \neg q) \leftrightarrow (p \vee q)$.

   (15%) Find an equivalent proposition that is as short as possible. Show your steps.

2. Given a collection of premises, say $p, q, r, s, \ldots$, we say a premise p is *redundant* if it is the case that when all premises except p are true would imply p is true. So, p is redundant if

   $$(q \wedge r \wedge s \wedge \cdots) \rightarrow p$$

   is always true.

   Intuitively, a redundant premise will not give us extra knowledge, so we hope to remove it. If after removal, the remaining premises still contain a redundant premise, we will continue to remove it. Our goal is to remove as many redundant premises (so as to keep as few premises in the end) as possible.

   Example: Consider the premises (i) $p \leftrightarrow q$, (ii) $p \rightarrow q$, (iii) $q \rightarrow p$. It is easy to check that $p \leftrightarrow q$ is redundant, since if the remaining two premises are true, then $p \leftrightarrow q$ is true. So, we can just keep premises (ii) and (iii).

   However, we can also say both (ii) and (iii) are redundant, because if we know $p \leftrightarrow q$ is true, then both the remaining premises are true. So, we can just keep premise (i), and remove the other two.

   In this example, we just need to keep one premise in the best case.

   (a) (10%) Consider the premises (i) p, (ii) q, (iii) $p \leftrightarrow q$, (iv) $p \rightarrow q$. What is the minimum number of premises that we need to keep, and which one(s)? Justify your answer.

   (b) (10%) Consider the premises (i) $p \rightarrow q$, (ii) $r \rightarrow (p \rightarrow q)$, (iii) $(p \rightarrow r) \vee ((\neg r) \rightarrow q)$. What is the minimum number of premises that we need to keep, and which one(s)? Justify your answer.

3. (15%) Prove or disprove:

   Let a and b be integers. If $a+b$ is a multiple of 3, then a^3+b^3 is a multiple of 3.

4. (15%) Find three different pairs of integers n and m such that $2^n = 3 + 5^m$.

5. Consider the equation $z^{13} - z^2 - 15015 = 0$.

   (a) (10%) Show that the equation does not have any integral root.

   Hint: Show that for any z that satisfies the above equation, (i) z cannot be an odd number, and (ii) z cannot be an even number.

   (b) (10%) Show that the equation does not have any rational root.

6. (Adapted from a logical puzzle in an online competition)

Raymond is visiting the famous country, *Pureland*, where citizens there are either honest (always tell the truth) or dishonest (always lie). Raymond sees three people, let us identify them as A, B, and C, and chats with them. Suddenly, one of them said "A and B are liars", and then another one of them said "A and C are liars."

(15%) How many liars are there among these three people? Justify your answer.

7. (Extremely Challenging: Estimated time to solve is more than 1 hour)

Two super smart boys, Sam and Peter, are present in a room. Teacher Hans goes to them, one by one, secretly telling each of them something. Now, Teacher Hans says: I have chosen two distinct integers, x and y, such that $1 < x < y$ and $x + y \leq 65$. I have just told Sam the sum $x + y$, and told Peter the product xy.

Next, the boys begin the following interesting conversation:

Peter: I don't know the numbers x and y.
Sam: I already knew that you didn't know.
Peter: Oh, now I know x and y.
Sam: Oh, I also know x and y now.

(5%) What are the numbers x and y? Justify your answer.