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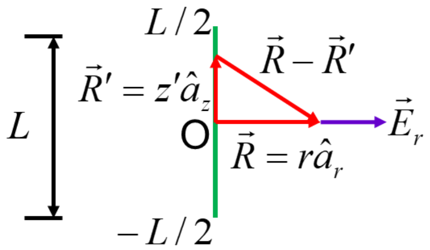
Nov. 9th, 2020

EE214000 Electromagnetics, Fall, 2020
Midterm Exam #1, 10:10 am ~ 12:00 pm, Monday, Nov. 9th, 2020
Homework # 3, due in class, Monday, Nov. 16th, 2020

Problem 1 (15%)

Refer to the following figure. In Lecture 5, we derived the electric potential a distance r from the mid-point of a line with length L carrying a line charge of ρ_l, given by $V(z=0,r)=\frac{\rho_l}{4\pi\varepsilon_0}\ln(z'+\sqrt{r^2+z'^2})\Big|_{-L/2}^{L/2}$. (1) Show that in the limit $L >> r$, the electric potential approximates $V(z=0,r)\sim\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{L}{r}$. (5%) (2) Explain why you are getting a potential that is unphysical when $L\to\infty$. (5%) (3) Use the expression in (1) to calculate the electric field intensity at $(z=0,r)$ when $L >> r$. (5%)



Ans: (1) $V(z=0,r)=\frac{\rho_l}{4\pi\varepsilon_0}\ln\frac{L/2+L/2\sqrt{1+(2r/L)^2}}{-L/2+L/2\sqrt{1+(2r/L)^2}}$. In the limit of L >> r,

$\sqrt{1+(2r/L)^2}\sim 1+\frac{1}{2}(2r/L)^2$. $V(z=0,r)\sim\frac{\rho_l}{4\pi\varepsilon_0}\ln\frac{1+(r/L)^2}{(r/L)^2}\sim\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{L}{r}$

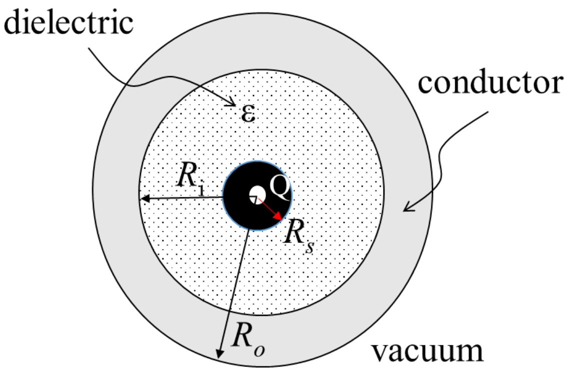
(2) Apparently, the voltage diverges for $L\to\infty$ $V(z=0,r)=\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{L}{r}\xrightarrow[L\to\infty]{}\infty$.

When $L\to\infty$, the total charge is also $\to\infty$, which adds up to an infinite potential at $(z=0,r)$ and is unphysical.

(3) $\vec{E}=-\nabla V\Rightarrow\vec{E}=\frac{-\rho_l}{2\pi\varepsilon_0}\frac{\partial}{\partial r}\ln(\frac{L}{r})=\frac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$. So, for a long enough line charge L >> r, $\vec{E}=\frac{\rho_l}{2\pi\varepsilon_0 r}\hat{a}_r$ obtained from $L\to\infty$ is a good approximation for the electrical field intensity close to the mid-point of the line.

Problem 2 (30%)

A charged ball electrode of radius R_s is embedded into the center of a dielectric ball with an outer radius of R_i. The permittivity of the dielectric between $R_s < R < R_i$ is ε. A total charge Q is uniformly distributed on the surface of the ball electrode. There is a spherical shell of conductor with $R_i < R < R_o$ covering the dielectric, as shown below.



(1) Find E, D, P, V in the three regions, $R_o < R < \infty$, $R_i < R < R_o$, $R_s < R < R_i$). (8%)

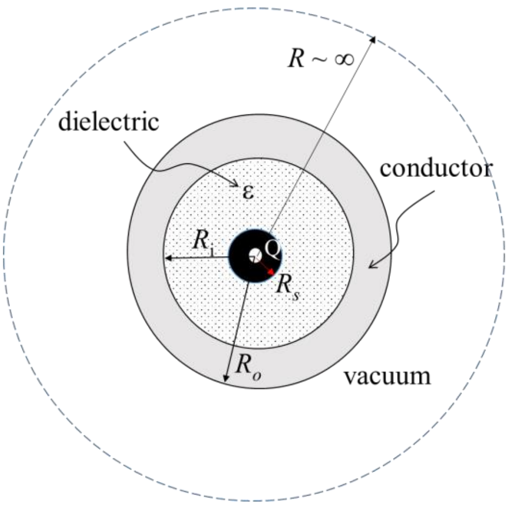
(2) Sketch E, D, P, V versus R in the 3 regions. (8%)

(3) What are the surface charge densities on the conductor at R_i and R_o? Specify the signs of the charges. (4%)

(4) What is the surface polarization charge density on the dielectric at R_i? Specify the sign of the charge in your answer. (2%)

(5) Refer to the following plot. Imagine that another spherical electrode is placed at $R \sim \infty$. Calculate the capacitance of this "capacitor" by first deriving the voltage V between $R = R_s$ and $R = \infty$ and then use $C = Q/V$ to obtain C. (3%)

(6) Calculate the capacitances in $R_o < R < \infty$ and $R_s < R < R_i$ to prove that the answer in "(5)" can be calculated from two serial capacitors connected through the conductor between $R_i < R < R_o$. (5%)



Ans:

(1) **In the vacuum region $R_o < R < \infty$**, use the Gauss law to obtain

$\vec{D} = \hat{a}_R \dfrac{Q}{4\pi R^2}$, $\vec{E} = \hat{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2}$. In a vacuum region, P = 0. The work to bring a unit positive charge from infinity to R is $V = \dfrac{Q}{4\pi\varepsilon_0 R}$.

In the conductor shell region, $R_i < R < R_o$, $D = 0$, $E = 0$, and $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = 0$

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A neutral conductor is an equipotential object. Therefore $V = \dfrac{Q}{4\pi\varepsilon_0 R_o}$ remains a constant equal to that at the outer surface of the conductor.

In the dielectric region, $R_s < R < R_i$, again apply the Gauss law to obtain

$$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}, \quad \vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon R^2}.$$

The polarization density vector is $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \hat{a}_R \dfrac{Q}{4\pi R^2}(1 - \dfrac{\varepsilon_0}{\varepsilon})$.

The electric potential is obtained by adding the work needed to bring the charge into the dielectric from R_i to R, given by $V = \dfrac{Q}{4\pi\varepsilon_0 R_o} + \dfrac{Q}{4\pi\varepsilon}(\dfrac{1}{R} - \dfrac{1}{R_i})$.

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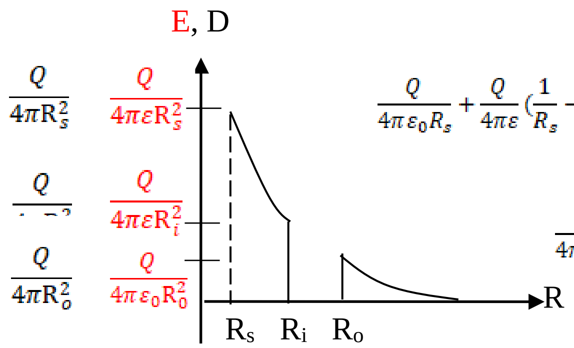
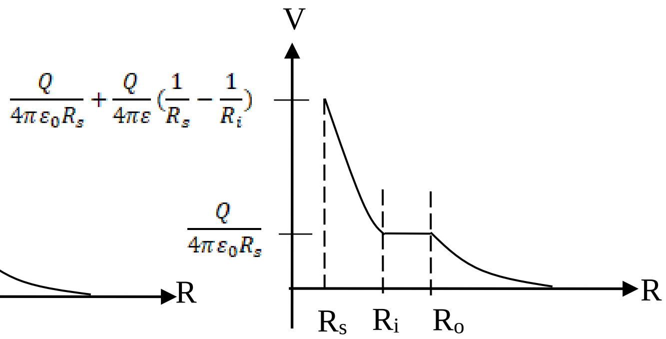
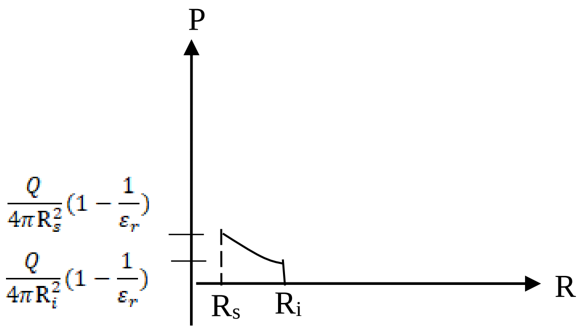
(2)







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(3) At R_i, $\rho_s = -\hat{a}_R \cdot \vec{D} = \dfrac{-Q}{4\pi R_i^2}$. At R_o, $\rho_s = \hat{a}_R \cdot \vec{D} = \dfrac{Q}{4\pi R_o^2}$.

(4) The surface polarization charge at R_i is $\rho_s = +\hat{a}_R \cdot \vec{P} = \dfrac{Q}{4\pi R_i^2}(1-\dfrac{\varepsilon_0}{\varepsilon})$

(5) The voltage between the two electrodes is $V = \dfrac{Q}{4\pi\varepsilon_0 R_o} + \dfrac{Q}{4\pi\varepsilon}(\dfrac{1}{R_s} - \dfrac{1}{R_i})$. The capacitance is $C = \left(\dfrac{V}{Q}\right)^{-1} = \left(\dfrac{1}{4\pi\varepsilon_0 R_o} + \dfrac{1}{4\pi\varepsilon}(\dfrac{1}{R_s} - \dfrac{1}{R_i})\right)^{-1}$

(6) For the capacitor in the vacuum region $R_o < R < \infty$, $V = \dfrac{Q}{4\pi\varepsilon_0 R_o}$ and thus

$$\frac{1}{C_{R_0-\infty}} = \frac{V}{Q} = \frac{1}{4\pi\varepsilon_0 R_o}.$$

For the capacitor in the dielectric region, the voltage between the electrodes at $R = R_s$ and R_i is $V = \dfrac{Q}{4\pi\varepsilon}(\dfrac{1}{R_s} - \dfrac{1}{R_i})$ and thus $\dfrac{1}{C_{R_s-R_i}} = \dfrac{V}{Q} = \dfrac{1}{4\pi\varepsilon}(\dfrac{1}{R_s} - \dfrac{1}{R_i})$.

For the two serially connected capacitors, the total capacitance is

$C = (\dfrac{1}{C_{R_s-R_i}} + \dfrac{1}{C_{R_o-\infty}})^{-1} = [\dfrac{1}{4\pi\varepsilon_0 R_o} + \dfrac{1}{4\pi\varepsilon}(\dfrac{1}{R_s} - \dfrac{1}{R_i})]^{-1}$, which is the same as the answer in (5)

Problem 3 (25%) A *large-area* parallel-plate capacitor is charged up to store an amount of charge Q and then disconnected from the power supply, as shown in (a) below. The parallel-plate capacitor has an area A and an electrode separation d in vacuum. If one inserts a perfect dielectric of area A, thickness $d/2$, and relative permittivity $\varepsilon_r = 2$ into the capacitor, as shown in (b) with an arbitrary y_0,

(1) what is the ratio $|V_b / V_a|$, where V_a, V_b are the voltages across the conducting electrodes before and after inserting the perfect dielectric, respectively. (10%)

Ans: It is well known that $C_a = \frac{\varepsilon_0 A}{d}$. With the inserted dielectric slab, consider the capacitance as that equivalent to 3 serially connected capacitors with thickness y_0, $d/2$, and $d/2-y_0$, given by $C_b = \dfrac{1}{\frac{y_0}{\varepsilon_0 A}+\frac{d/2-y_0}{\varepsilon_0 A}+\frac{d/2}{\varepsilon_0\varepsilon_r A}} = \frac{4\varepsilon_0 A}{3d} = \frac{4}{3}C_a$, which is independent of y_0.

Since $V = Q/C$ with a constant Q, the voltage ratio is $|V_b/V_a| = C_a \big/ C_b = \frac{3}{4}$.

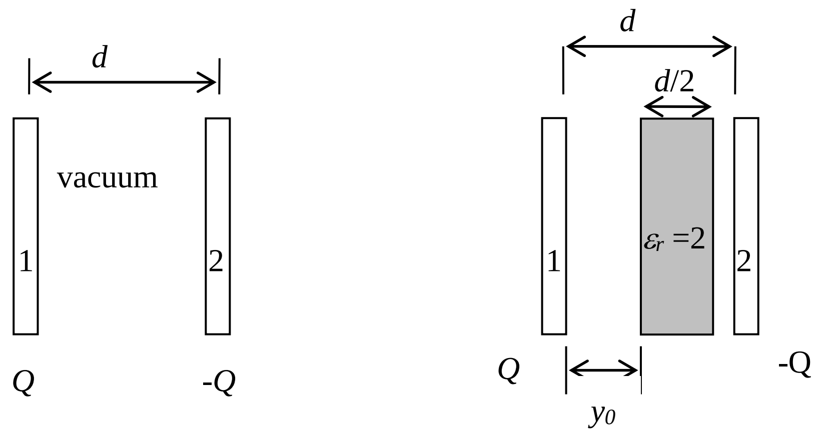
(2) what is the ratio $|F_b/F_a|$, where F_a, F_b are the forces that are needed for a structure to hold the conducting electrodes before and after inserting the perfect dielectric, respectively. (10%)

Ans: $|F| = \frac{dW_e|_Q}{dy} = \frac{1}{2}Q^2\frac{d(1/C)}{dy} \Rightarrow 1/C_a(y) = \frac{y}{\varepsilon_0 A}$, $|F_a| = \frac{1}{2}\frac{Q^2}{dC_a}$;

$1/C_b(y) = \frac{3y}{4\varepsilon_0 A}$, $|F_b| = \frac{1}{2}\frac{Q^2}{dC_b} \Rightarrow |F_b/F_a| = \frac{C_a}{C_b} = \frac{3}{4}$.

(3) what is the polarization charge density induced on the surface of the dielectric facing electrode 1? (5%)

Ans: The electric flux density D in the capacitor is the normal component and is continuous across the boundary. The voltage across the two conducting plates for capacitor (b) is $V_b = \frac{Q}{C_b} = \frac{3Qd}{4\varepsilon_0 A} = \frac{D}{2\varepsilon_0}\frac{d}{2} + \frac{D}{\varepsilon_0}\frac{d}{2}$. $\Rightarrow D = \frac{Q}{A}$. According to $P = D - \varepsilon_0 E = (1-\frac{1}{\varepsilon_r})D = \frac{1}{2}D = \frac{Q}{2A}$. Given the positive charge on electrode 1, the induced polarization charge facing it must be negative. The surface polarization charge density on the dielectric facing electrode 1 is therefore $\vec{P}\cdot\hat{a}_s = \frac{-Q}{2A}$.



(a) (b)

Problem 4 (30%)

Find the electrostatic energy stored in a sphere of charges with its charge density varying with $\rho = \rho_0 \dfrac{b}{R}$ between the radius distance $0 < R < R_0$, where ρ_0 and b are constants. The sphere of charges is in a vacuum. Present your calculation by using (1) the assembling technique, (10%) (2) the decomposing technique, (10%) (3) field-energy-density technique. (10%) Cross check your results.

Ans:

Method 1 – assembling technique

First calculate the charge in a radius of *R*, then the potential *V* at *R* for a charge ball of radius *R*, and finally the work to assemble the charges layer by layer.

$$Q_R = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho_0 \frac{b}{R} R^2 \sin\theta dR d\theta d\phi = 2\pi b R^2 \rho_0$$

$$V_R = \frac{Q_R}{4\pi\varepsilon_0 R} = \frac{Rb\rho_0}{2\varepsilon_0}$$

$$dW_e = V_R dQ_R$$

$$W_e = \int_0^{R_0} V_R dQ_R = \int_0^{R_0} \frac{2\pi\rho_0^2 b^2 R^2}{\varepsilon_0} dR = \frac{2\pi\rho_0^2 b^2 R_0^{\ 3}}{3\varepsilon_0}$$

Method 2 – decomposing technique

Assume all the charges are fully in place

$$R \geq R_0 \qquad E_R = \frac{Q_{R_0}}{4\pi\varepsilon_0 R^2} = \frac{\rho_0 b R_0^2}{2\varepsilon_0 R^2}$$

$$0 < R \leq R_0 \qquad E_R = \frac{Q_R}{4\pi\varepsilon_0 R^2} = \frac{\rho_0 b}{2\varepsilon_0}$$

$$V = -\int_\infty^R E dR = -\int_\infty^{R_0} \frac{\rho_0 b R_0^2}{2\varepsilon_0 R^2} dR - \int_{R_0}^R \frac{\rho_0 b}{2\varepsilon_0} dR = \frac{\rho_0 b R_0}{\varepsilon_0} - \frac{\rho_0 b R}{2\varepsilon_0}$$

$$W_e = \frac{1}{2} Q \times V = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R_0} \rho_0 \frac{b}{R}\left(\frac{\rho_0 b R_0}{\varepsilon_0} - \frac{\rho_0 b R}{2\varepsilon_0}\right) \times R^2 \sin\theta \times dR d\theta d\phi = \frac{2\pi{\rho_0}^2 b^2}{\varepsilon_0}\left(\frac{R_0^3}{2} - \frac{R_0^3}{6}\right) = \frac{2\pi\rho_0^2 b^2 {R_0}^3}{3\varepsilon_0}$$

Method 3 – field-energy-density technique

$$R \geq R_0 \qquad E_R = \frac{Q_{R_0}}{4\pi\varepsilon_0 R^2} = \frac{\rho_0 b R_0^2}{2\varepsilon_0 R^2}$$

$$0 < R \leq R_0 \qquad E_R = \frac{Q_R}{4\pi\varepsilon_0 R^2} = \frac{\rho_0 b}{2\varepsilon_0}$$

$$W_e = \frac{1}{2}\oint_V \varepsilon_0 E^2 dV = \frac{\varepsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\left(\int_{R_0}^{\infty}\left(\frac{\rho_0 b R_0^2}{2\varepsilon_0 R^2}\right)^2 R^2 dR + \int_0^{R_0}\left(\frac{\rho_0 b}{2\varepsilon_0}\right)^2 R^2 dR\right)\sin\theta d\theta d\phi\right)$$

$$= \frac{\varepsilon_0}{2}\left(\frac{\rho_0 b}{2\varepsilon_0}\right)^2 4\pi\left(\int_{R_0}^{\infty}\frac{R_0^4}{R^2} dR + \int_0^{R_0} R^2 dR\right) = \frac{\pi\rho_0^2 b^2}{2\varepsilon_0}\left(R_0^3 + \frac{R_0^3}{3}\right) = \frac{2\pi\rho_0^2 b^2 {R_0}^3}{3\varepsilon_0}$$