last update: Dec. 21th , 2020
Changes are purple marked

電磁學（一）
Electromagnetics (I)

14. 磁性物質
Magnetic Material

授課老師：國立清華大學 電機工程學系 黃衍介 教授
Yen-Chieh Huang, National Tsing Hua University, Taiwan

In this lecture, we will introduce the magnetic dipole and generalize it to understand magnetic materials.

- **14.1 Magnetic Dipole 磁耦極**
- **14.2 Magnetization 磁化**
- **14.3 Modification of Ampere's Law 安培定律的修正**
- **14.4 Magnetic Boundary Conditions 磁場邊界條件**
- **14.5 Review 單元回顧**

磁性物質
Magnetic Material

14.1 磁耦極
Magnetic Dipole

Magnetic Dipole

(a basic magnetic element in materials)

Due to symmetry $\vec{A} = A_\varphi \hat{a}_\varphi$

Recall $\vec{A} = \dfrac{\mu_0 I}{4\pi} \oint_{C'} \dfrac{d\vec{l}\,'}{R_1}$

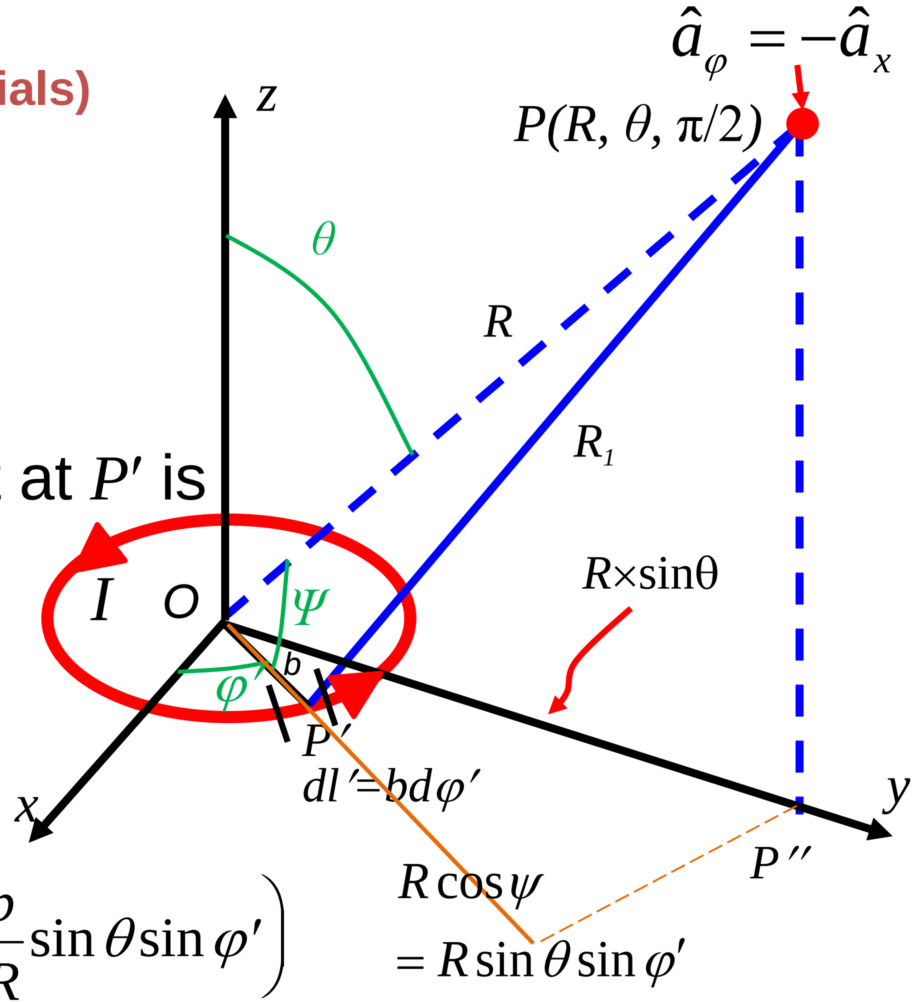
The vector potential due to current at P' is

$$dA_\varphi = dA_{(-\hat{a}_x)} = \frac{\mu_0 I}{4\pi} \times \frac{-b \sin\varphi' d\varphi'}{R_1}$$

where $R_1^2 = R^2 + b^2 - 2bR\cos\psi$

$= R^2 + b^2 - 2bR\sin\theta\sin\varphi'$

In the far zone $R >> b$ ➡ $\dfrac{1}{R_1} \approx \dfrac{1}{R}\left(1 + \dfrac{b}{R}\sin\theta\sin\varphi'\right)$



Magnetic Dipole Moment

Integrate over the whole current loop to obtain

$$A_\varphi = \frac{\mu_0 I \pi b^2}{4\pi R^2}\sin\theta$$

S = area of I

Define the *magnetic dipole moment* $\vec{m} \equiv I\pi b^2 \hat{a}_z = (I\vec{S})$

The vector potential has a general form $\vec{A} = \dfrac{\mu_0 \vec{m}\times\hat{a}_R}{4\pi R^2}$

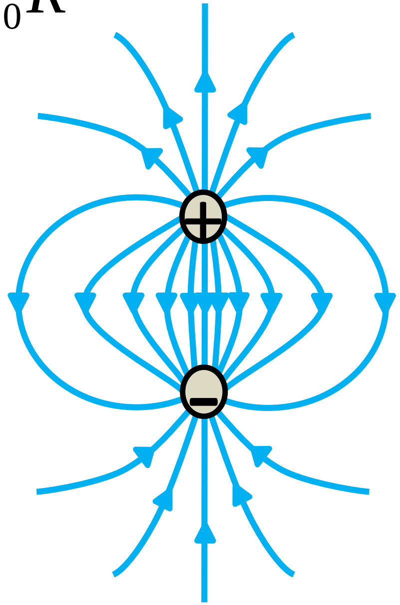
*As a comparison, the electric dipole potential was $V(R) = \dfrac{\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}$

From $\vec{B} = \nabla\times\vec{A}$, the far-zone dipole field is

$$\vec{B} = \frac{\mu_0 m}{4\pi R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

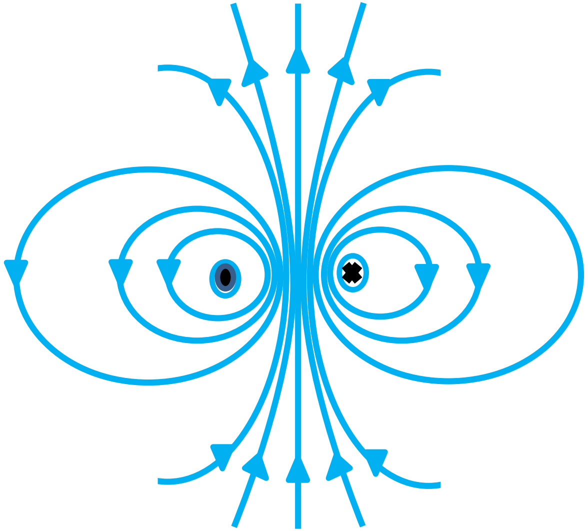
Electric dipole field (note the field starts from positive charge and ends on negative one)

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$



Magnetic dipole field (note the field closes upon itself)

$$\vec{B} = \frac{\mu_0 m}{4\pi R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$



14.1 磁耦極

Magnetic Dipole

- A magnetic dipole is a basic building block of a magnetic material.
- The magnetic moment is defined as the area of a current loop S multiplying the current on the loop I (direction determined from right hand rule). $\vec{m} \equiv I\vec{S}$
- In the far zone, the magnetic-dipole and the electric-dipole fields have the same pattern

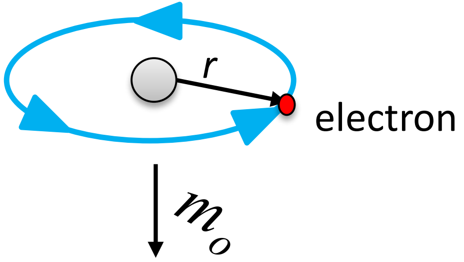
$$\vec{B} = \frac{\mu_0 m}{4\pi R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

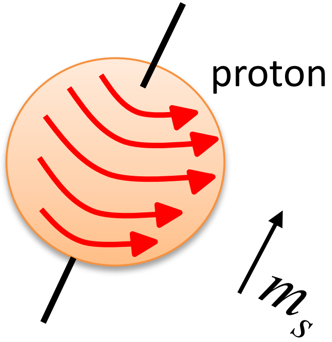
磁性物質
Magnetic Material

14.2 磁化
Magnetization

In a material, an orbiting or spinning charge generates a magnetic dipole moment

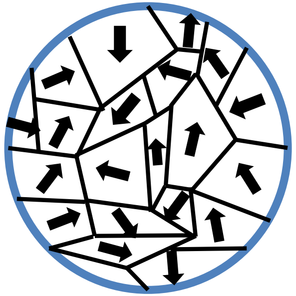


orbiting electron

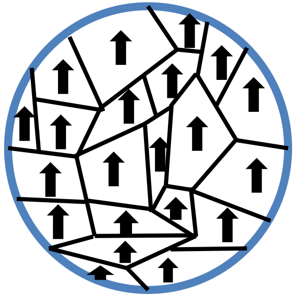


spinning proton

In materials, some magnetic dipoles form a *magnetic domain.*
Those domains could be random or could be aligned.



Random domains



Aligned domains

Magnetization Vector

Define the magnetization vector as the averaged sum of magnetic dipole moments per unit volume in a point volume.

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{m}_k}{\Delta v} \quad \text{(A/m)}$$

From $\vec{A} = \dfrac{\mu_0 \vec{m} \times \hat{a}_R}{4\pi R^2}$, the differential vector potential in a material is

$$d\vec{A} = \frac{\mu_0 \vec{M} \times \hat{a}_R}{4\pi R^2} dv' = \frac{\mu_0 \vec{M} \times \nabla' \left(\dfrac{1}{R} \right)}{4\pi} dv'$$

As a comparison, the polarization charges

$$\nabla' \cdot \vec{P} = -\rho_p, \vec{P} \cdot \hat{a}_n = \rho_{ps}$$

After integration $\vec{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int_{V'} \dfrac{\vec{J}_m}{R} dv' + \dfrac{\mu_0}{4\pi} \oint_{S'} \dfrac{\vec{J}_{ms}}{R} ds'$

where the magnetization volume current density $\vec{J}_m \equiv \nabla' \times \vec{M}$

magnetization surface current density $\vec{J}_{ms} \equiv \vec{M} \times \hat{a}_n$

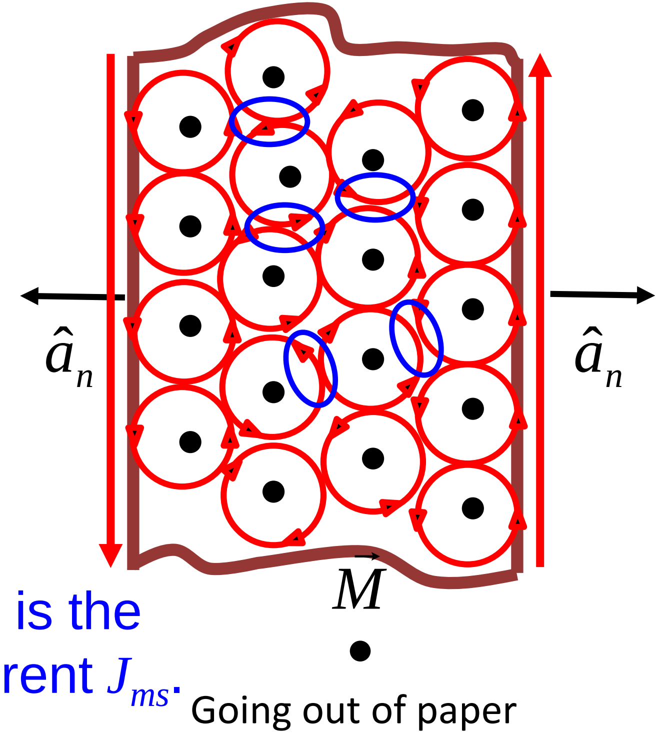
Interface volume currents J_m are all cancelled inside the volume.

In a homogeneously magnetized material, there is no spatial variation of magnetization and therefore

$$\boxed{\vec{J}_m \equiv \nabla' \times \vec{M} = 0}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}_m}{R} dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\vec{J}_{ms}}{R} ds'$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\oint_{S'} \frac{\vec{J}_{ms}}{R} ds'$$



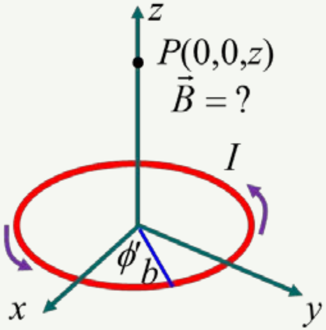
What is left is the surface current J_{ms}.

$\Rightarrow$ A magnet can be modeled with surface magnetization currents

E.g. Find $\vec{B}$ at point $P(0, 0, z)$ with the magnet having $\vec{M} = M_0 \hat{a}_z$, as shown below.

The equivalent magnetization surface current density is $\vec{J}_{ms} \equiv \vec{M} \times \hat{a}_n = M_0 \hat{a}_z \times \hat{a}_r = M_0 \hat{a}_\phi$

Recall (**Lecture 13**)

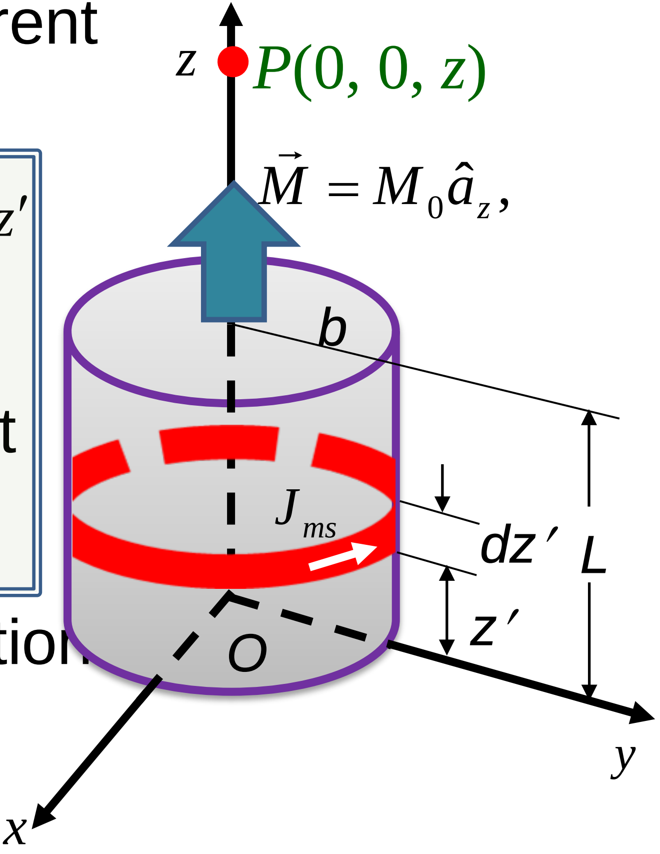


$$\vec{B} = \hat{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$$

at $P(0, 0, z)$ for a single current loop carrying a current I

$I \rightarrow dI = J_{ms} dz'$

$z \rightarrow z - z'$



The total magnetic field is then the integration

$$\vec{B} = \hat{a}_z \int_0^L \frac{\mu_0 b^2 J_{ms} dz'}{2((z - z')^2 + b^2)^{3/2}} = \hat{a}_z \int_0^L \frac{\mu_0 b^2 M_0 dz'}{2((z - z')^2 + b^2)^{3/2}}$$

14.2 磁化

Magnetization

- The magnetization vector is the vector sum of all the magnetic dipole moments in a material.
- Both volume and surface current densities of a magnetic material contribute to the field associated with it, described by

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}_m}{R} dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\vec{J}_{ms}}{R} ds'$$

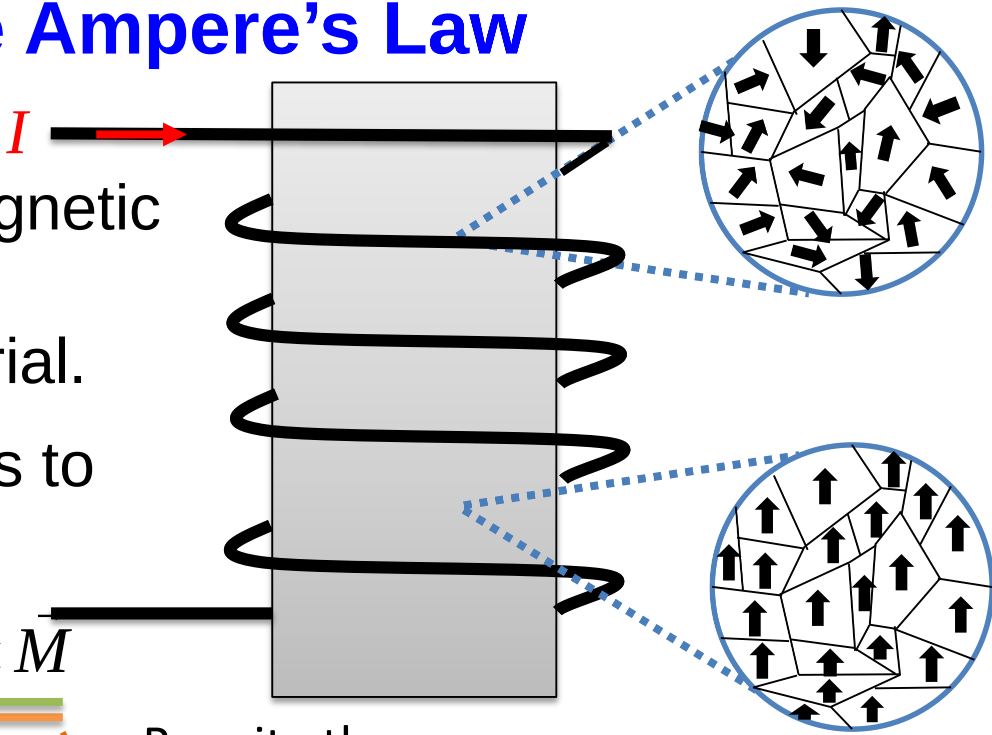
- A homogeneously magnetized material, such as a magnet, can be modeled with surface currents on the magnet.

磁性物質
Magnetic Material

14.3 安培定律的修正
Modification of Ampere’s Law

Modification to the Ampere's Law

In vacuum, $\nabla \times \vec{B} = \mu_0 \vec{J}$



In a magnetic material, the magnetic field is modified due to aligned magnetic domains in the material.

$\Rightarrow$ Magnetization current J_m has to be considered

$$\underline{\nabla \times \vec{B}} = \mu_0 \vec{J} + \mu_0 \underline{\vec{J}_m} = \mu_0 \vec{J} + \underline{\mu_0 \nabla \times \vec{M}}$$

$$\nabla \times \vec{H} = \vec{J}$$

Define magnetic field intensity *H* related to the source current *J*

Rewrite the expression into

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J} \quad \text{or} \quad \vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}$$

Modified Ampere's Circuit Law

Differential form

$$\nabla \times \vec{H} = \vec{J}$$

Integral form

$$\int_{S'} \nabla \times \vec{H} \cdot d\vec{s} = I$$

Don't forget the 1st postulate

$$\nabla \cdot \vec{B} = 0 \qquad \oint_S \vec{B} \cdot d\vec{s} = 0$$

In a simple (linear, isotropic, and nondispersive) material, M is linearly proportional to H

$$\vec{M} = \chi_m \vec{H}$$

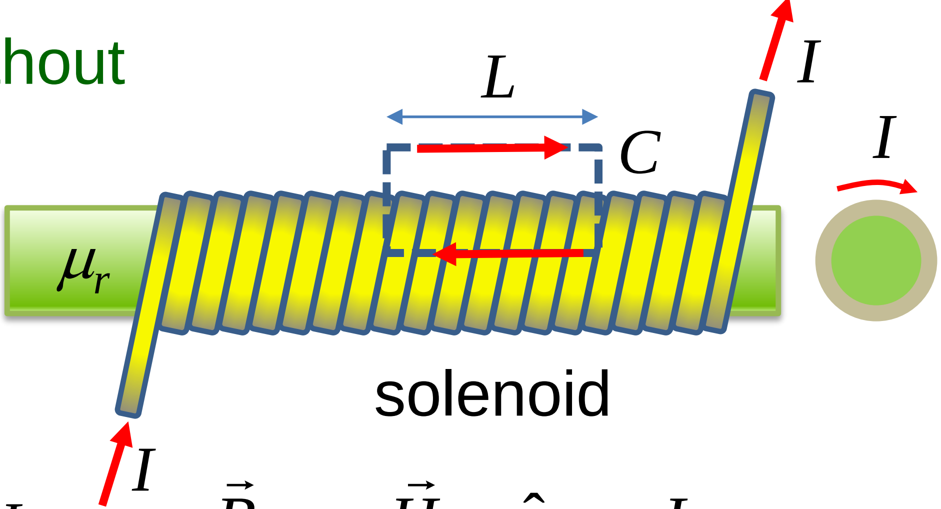
The proportional factor χ_m is called *magnetic susceptibility*.

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \quad \vec{B} = \mu\vec{H} \equiv \mu_0(1+\chi_m)\vec{H} = \mu_0\mu_r\vec{H}$$

The relative permeability is defined as $\mu_r \equiv 1 + \chi_m = \dfrac{\mu}{\mu_0}$

For magnetic materials, $\mu_r \sim 10^6$!

Eg. A long solenoid with and without a ferromagnetic core, find B.



i. Without the ferromagnetic material

From Ampere's law:

$$\oint_C \vec{H} \cdot d\vec{l} = NI \Rightarrow \vec{H} = H_z \hat{a}_z = \hat{a}_z nI \quad \Rightarrow \vec{B} = \mu_0 \vec{H} = \hat{a}_z \mu_0 nI$$

$n = N / L$: # of loops per length

ii. With the ferromagnetic material having μ_r

Again, from Ampere's law $\Rightarrow \vec{H} = H_z \hat{a}_z = \hat{a}_z nI$

$$\Rightarrow \quad \vec{B} = \boxed{\mu_r} \mu_0 \vec{H} = \hat{a}_z \boxed{\mu_r} \mu_0 nI$$

Note that μ_r can be a big number

Magnetic Materials

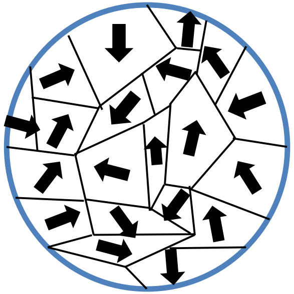
Diamagnetism $\mu_r \leq 1$ or $\chi_m \leq 0$ All materials show diamagnetism

Lenz's law: The induced magnetic dipole moment from electron orbits always opposes the applied external field.

This effect is usually weak, $|\chi_m| \approx 10^{-8} \sim 10^{-5}$

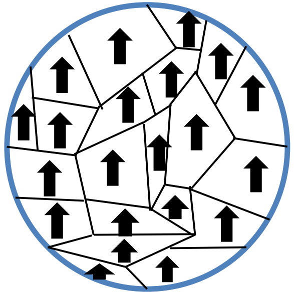
and is often obscured in materials with intrinsic magnetism.

Paramagnetism $\mu_r \geq 1$ or $\chi_m \geq 0$



Dipole moments in electrons, atoms, or molecules tend to partially align with an applied external field.

Ferromagnetism $\mu_r >> 1$ or $\chi_m >> 0$



A small external magnetic field intensity (from, say, a current loop) aligns magnetic domains in such a material, resulting in a large magnetic flux density *B*.

14.3 安培定律的修正

Modification of Ampere's Law

- The modified Ampere's law relates the magnetic field intensity H to a source current density J or current I, given by $\nabla \times \vec{H} = \vec{J}$ and $\int_{S'} \nabla \times \vec{H} \cdot d\vec{s} = I$
- The magnetic flux density B is now expressed as

$$\vec{B} = \mu \vec{H} \equiv \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$$

where μ_0 is the vacuum permeability, μ_r is the relative permeability, and χ_m is the magnetic susceptibility.

磁性物質
Magnetic Material

14.4 磁場邊界條件
Magnetic Boundary Conditions

Electrostatics vs. Magnetostatics

The equations in electrostatics and magnetostatics are *dual equations*. *Systematic substitutions of the following table of symbols transform an equation in one system into another.*

electrostatics	magnetostatics
E	***B***
D	***H***
ε	$1/\mu$
P	***-M***
ρ	***J***
V	***A***
$\bullet$	$\times$
$\times$	$\bullet$

E.g.

$$\nabla \times \vec{E} = 0 \leftrightarrow \nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = \rho \leftrightarrow \nabla \times \vec{H} = \vec{J}$$

$$D = \varepsilon E \leftrightarrow H = \frac{1}{\mu} B$$

etc.

$$\vec{E} = -\nabla V \leftrightarrow \vec{B} = \nabla \times \vec{A}$$

(not totally valid as dual equations)

Boundary Conditions – Normal Components

I. From $\nabla \cdot \vec{B} = 0 \Rightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$

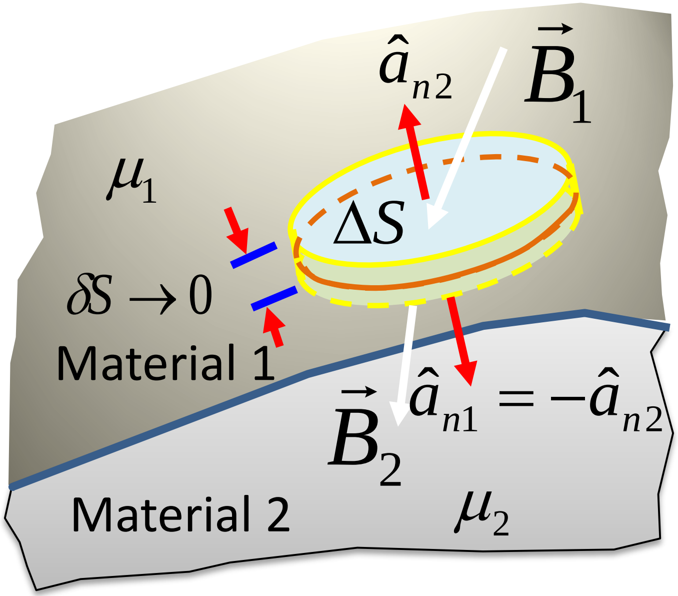
*Recall

$$\nabla \cdot \vec{D} = 0$$

$$D_{1n} = D_{2n}$$

$$\Rightarrow \quad B_{n1}\Delta S = B_{n2}\Delta S$$

$$\Rightarrow \quad B_{n1} = B_{n2} \quad \text{or} \quad \mu_1 H_{n1} = \mu_2 H_{n2}$$



The normal component of a magnetic flux density B_n is continuous across a boundary.

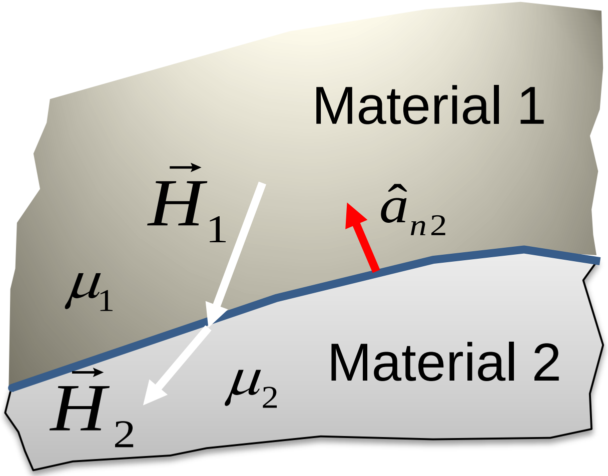
Boundary Conditions – Tangential Components

II. From $\nabla \times \vec{H} = \vec{J} \Rightarrow \oint_C \vec{H} \cdot d\vec{l} = I$

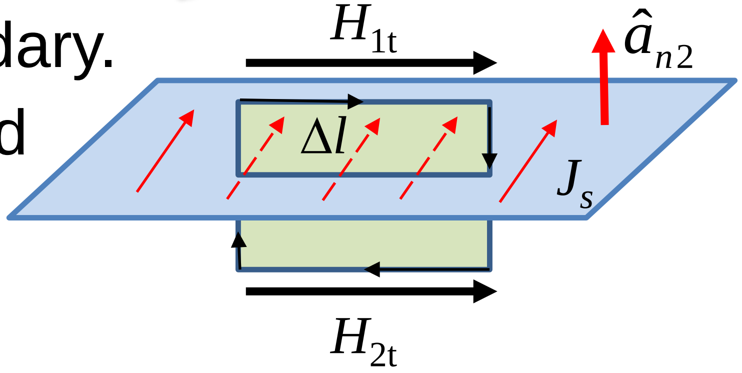
$$\Rightarrow \oint_C \vec{H} \cdot d\vec{l} = H_{1t}\Delta l - H_{2t}\Delta l = \pm J_s \Delta l$$

$$\Rightarrow H_{1t} - H_{2t} = \pm J_s$$

(±? due to uncertainty on directions of H and J)



Tangential components of a magnetic field intensity are discontinuous for a surface current density J_s at a boundary.



The relative direction between H and J_s follows the right-hand rule or

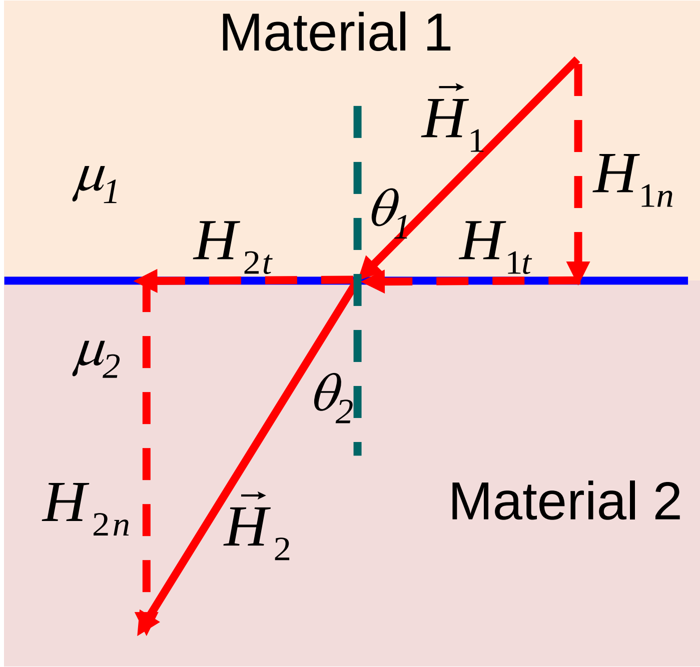
$$\boxed{\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s}$$

For non-conductive materials, $J_s = 0$ and $H_{1t} = H_{2t}$

E.g. Find $\vec{H}_2$ or H_{2t}, H_{2n} in Material 2, assuming no J_s.

Without a surface current, the boundary conditions are

$$H_{1t} = H_{2t} \text{ and } B_{1n} = B_{2n}$$



Tangential component

$$H_{1t} = H_1 \sin\theta_1 \Rightarrow H_{2t} = H_1 \sin\theta_1$$

Normal component

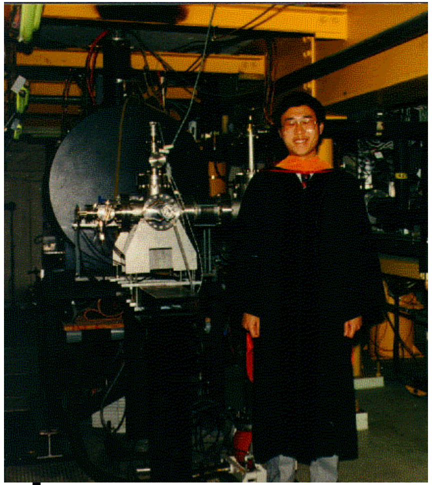
$$H_{1n} = H_1 \cos\theta_1 \Rightarrow B_{2n} = B_{1n} = \mu_1 H_1 \cos\theta_1$$

Thus $$H_{2n} = \frac{B_{2n}}{\mu_2} = \frac{\mu_1}{\mu_2} H_1 \cos\theta_1$$

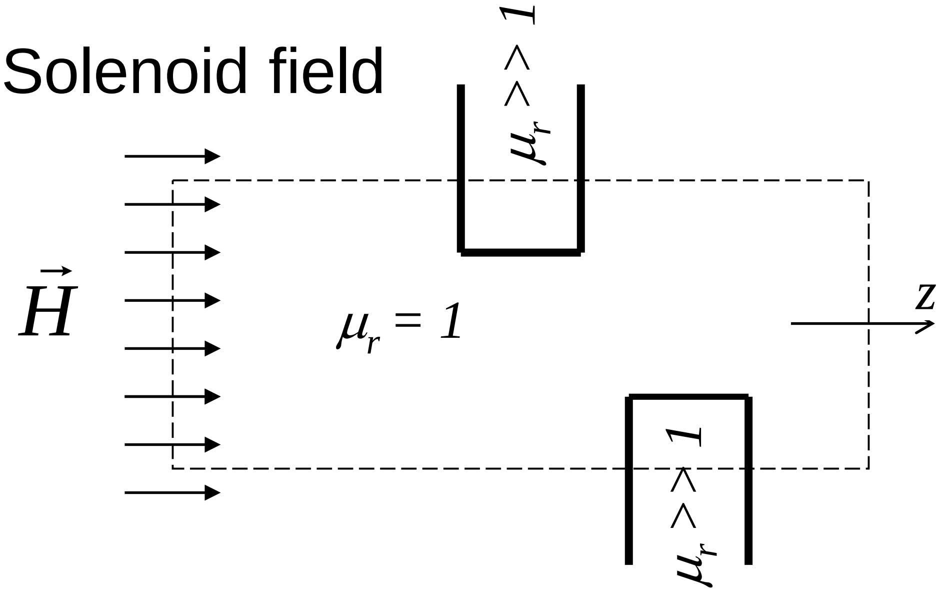
$$H_2 = \sqrt{H_{2t}^2 + H_{2n}^2} \text{ and } \tan\theta_2 = \frac{H_{2t}}{H_{2n}} = \frac{\mu_2}{\mu_1}\tan\theta_1$$

Magnetic flux line enters a high μ_r surface at a right angle

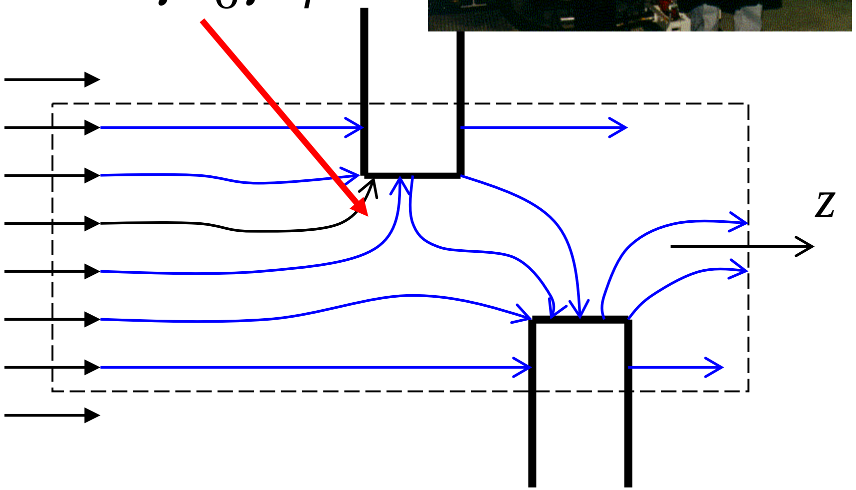
$$\tan\theta_2 = \frac{\mu_2}{\mu_1}\tan\theta_1 \Rightarrow \tan\theta_1 = \frac{\tan\theta_2}{\mu_r} \Rightarrow \theta_1 \xrightarrow[\mu_r\to\infty]{} 0$$



Solenoid field



$$\vec{B} \sim \mu_0\mu_r\vec{H}$$



14.4 磁場邊界條件

Magnetic Boundary Conditions

- Across a boundary, the normal components of the magnetic flux density are continuous, or

$$\boxed{B_{n1} = B_{n2}}$$

- Across a non-conducting boundary, the tangential components of the magnetic field intensity are continuous.
- Across a conducting boundary, the tangential components of the magnetic field intensity are discontinuous, described by

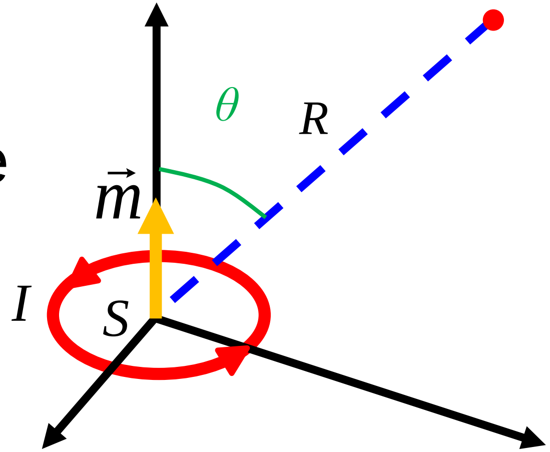
$$\boxed{\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s}$$

磁性物質
Magnetic Material
14.5 單元回顧 Review

單元回顧

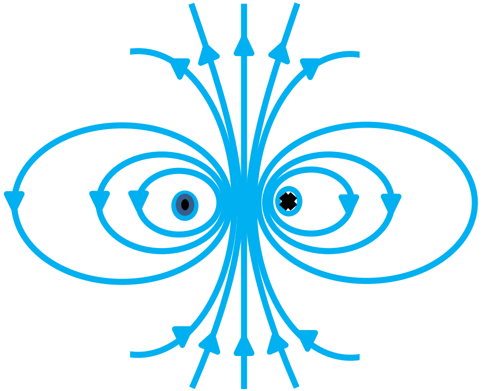
1. A magnetic dipole is the basic building block of a magnetic material and its dipole moment is defined as

$$\vec{m} = I\vec{S}$$



2. In the far zone, the magnetic flux density of a magnetic dipole is described by

$$\vec{B} = \frac{\mu_0 m}{4\pi R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta).$$

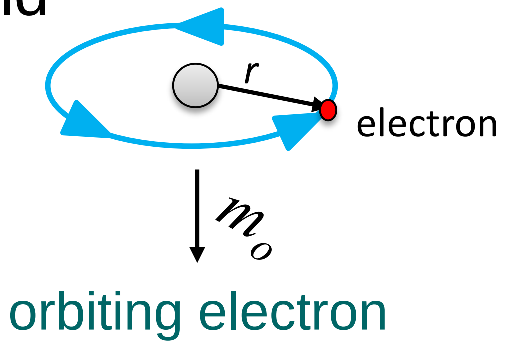


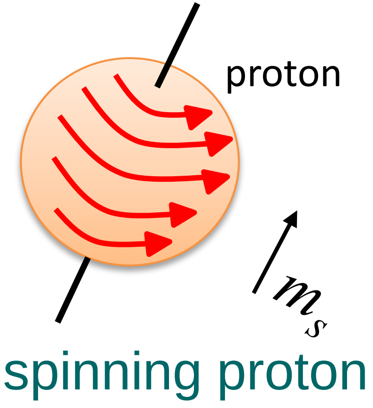
*compare it with the electric dipole field

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$$

單元回顧

3. A material contains orbiting and spinning charges, or magnetic dipoles.



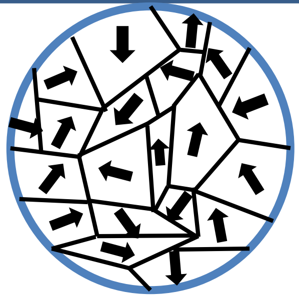


4. The magnetization vector is the vector sum of all the magnetic dipole moments in a material.

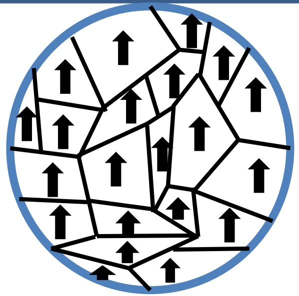
$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{m}_k}{\Delta v} \quad \text{(A/m)}$$

單元回顧

5. Magnetic dipoles can form domains in a material. Aligned domains show strong magnetization.



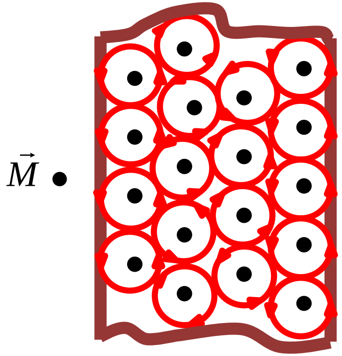
random domains



aligned domains

6. The total vector potential of a magnetic material with magnetization is

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}_m}{R}dv' + \frac{\mu_0}{4\pi}\oint_{S'}\frac{\vec{J}_{ms}}{R}ds'$$



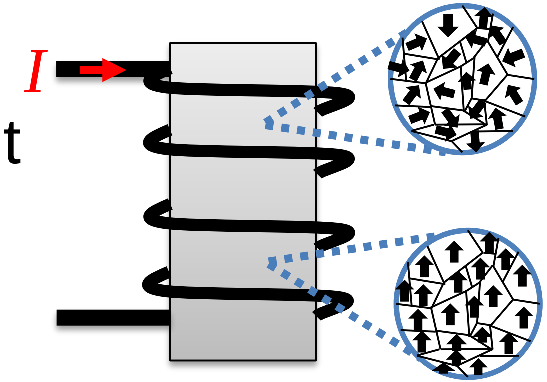
where $\vec{J}_m \equiv \nabla' \times \vec{M}$ is the magnetization volume current density,

and $\vec{J}_{ms} \equiv \vec{M} \times \hat{a}_n$ is the magnetization surface current density.

單元回顧

7. The modified Ampere's law relates the magnetic field intensity H to a source current density J or a source current I, given by

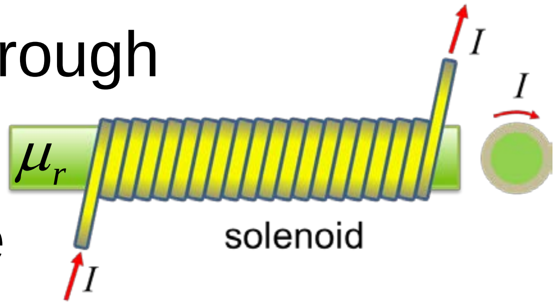
$$\nabla \times \vec{H} = \vec{J} \quad \text{and} \quad \int_{S'} \nabla \times \vec{H} \cdot d\vec{s} = I$$



8. The magnetic flux density B related to H through

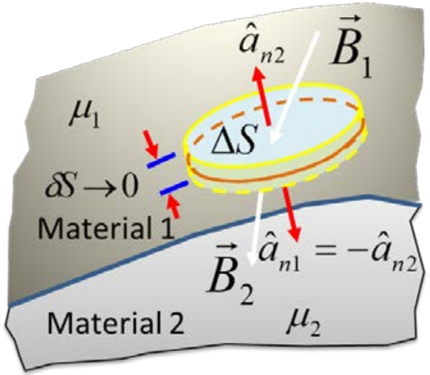
$$\vec{B} = \mu \vec{H} \equiv \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$$

where μ_0 is the vacuum permeability, μ_r is the relative permeability, and χ_m is the magnetic susceptibility.



單元回顧

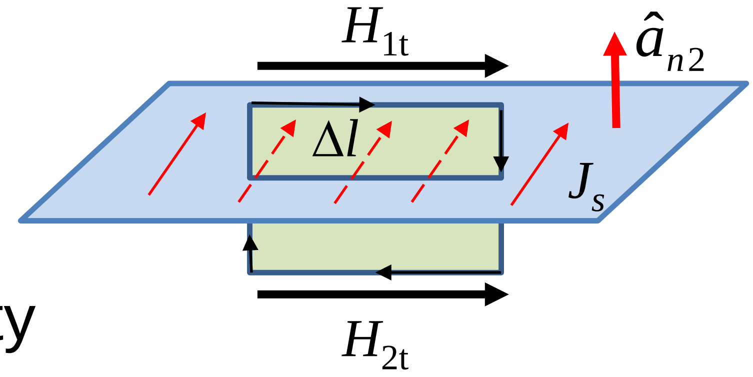
9. Across a boundary, the normal components of the magnetic flux density are continuous, or



$$B_{n1} = B_{n2}$$

10. Across a boundary, the tangential components of the magnetic field intensity are governed by

$$\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s,$$



where J_s is the surface current density existing in a conducting interface.

THANK YOU FOR YOUR ATTENTION