EE306002: Probability **Spring 2022**

Department of Electrical Engineering

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Homework #3 – Solution
Coverage: chapter 5

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Problem 5.1.24 Vincent is a patient with the life threatening blood cancer leukemia, and he is in need of a bone marrow transplant. He asks n people whether or not they are willing to donate bone marrow to him if they are a close bone marrow match for him. Suppose that each person's response, independently of others, is positive with probability p_1 and negative with probability $1 - p_1$. Furthermore, suppose that, independently of the others, the probability is p_2 that a person tested is a close match. Find the probability that Vincent finds at least one close bone marrow match among these n people.
(*Hint*: For $0 \leq i \leq n$, let A_i be the event that i of the n individuals respond positively to Vincent's request. Let X be the number of close bone marrow matches among those who will be tested. To find $P(X \geq 1) = 1 - P(X = 0)$, calculate $P(X = 0)$ by using the Law of Total Probability.)

Solution:
By the Law of Total Probability, we have

$$\begin{aligned}
P(X = 0) &= \sum_{i=0}^{n} P(X = 0 \mid A_i)P(A_i) \\
&= \sum_{i=0}^{n} (1 - p_2)^i \binom{n}{i} p_1^i (1 - p_1)^{n-i} \\
&= \sum_{i=0}^{n} \binom{n}{i} [p_1(1 - p_2)]^i (1 - p_1)^{n-i} \\
&= \sum_{i=0}^{n} \binom{n}{i} [p_1 - p_1 p_2]^i (1 - p_1)^{n-i} \\
&= (p_1 - p_1 p_2 + 1 - p_1)^n = (1 - p_1 p_2)^n.
\end{aligned}$$

The next to last equality follows by Binomial Expansion. Then,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - p_1 p_2)^n.$$

Problem 5.3.14 For certain software, independently of other users, the probability is 0.07 that a user encounters a fault. What are the chances that the 30-th user is the 5-th person encountering a fault?

Solution:
Let X be the number of users until the 5-th person encounters a fault. Thus X is negative binomial distribution given by

$$P(X = n) = \binom{n-1}{i-1} p^5 (1 - p)^{30-5} = \binom{30-1}{5-1} (0.07)^5 (1 - 0.07)^{30-5} = 0.0065.$$

Problem Ch5-Review 2. The time between the arrival of two consecutive customers at a post office is 3 minutes, on average. Assuming that customers arrive in accordance with a Poisson process, find the probability that tomorrow during the lunch hour (between noon and 12:30 p.m.) fewer than seven customers arrive.

Solution:
Let $N(t)$ be the number of customers arriving at the post office at or prior to t is a Poisson process with λ =1/3. Then,

$$P(N(t) \leq n) = \sum_{i=0}^{n} P(N(t) = i) = \sum_{i=0}^{6} P(N(30) = i) = \sum_{i=0}^{6} \frac{e^{-(1/3)\cdot 30}[(1/3)\cdot 30]^i}{i!}.$$

Problem Ch5-Review 20. Passengers are making reservations for a particular flight on a small commuter plane 24 hours a day at a Poisson rate of 3 reservations per 8 hours. If 24 seats are available for the flight, what is the probability that by the end of the second day all the plane seats are reserved?

Solution:
Let X be the number of requests for reservations at the end of the second day. Assume that X is a Poisson distribution with parameter $\lambda = 18$. The probability can be written as

$$P(X \geq n) = 1 - \sum_{i=0}^{n-1} P(X = i) = 1 - \sum_{i=0}^{23} P(X = i) = 1 - \sum_{i=0}^{23} \frac{e^{-18}(18)^i}{i!}.$$

Problem Ch5-Review 26. Suppose that n babies were born at a county hospital last week. Also suppose that the probability of a baby having blonde hair is p. If k of these n babies are blondes, what is the probability that the i-th baby born is blonde? (Note: Please show the detail of your proof.)

Solution:
Let A be the event that k of the n babies are bloneds, and let B be the event that the i-th baby born is blonde. We have

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{\binom{n-1}{k-1} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)