Signals and Systems

Homework 10 — Due : May 17 2024

Problem 1 (20 pts)**.** Determine the Fourier transform of the following signals:

(a) $x_1[n] = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{2\pi}{3}n\right)$

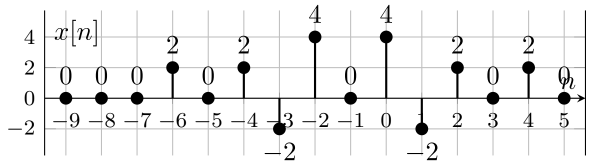
(b) $x_3[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$

Problem 2 (20 pts)**.** Determine the inverse Fourier transform of the following signals:

(a) $X_1(e^{j\omega}) = \cos^2(3\omega) + \sin^2(\omega)$

(b) $X_2(e^{j\omega}) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = \begin{cases} 0, & 0 \le |\omega| < \pi/3 \\ 1, & \pi/3 \le |\omega| < \pi \end{cases}$, and $B(\omega) = -\frac{2}{3}\omega$

Problem 3 (30 pts)**.** Let $X(e^{j\omega})$ be the Fourier transform of the signal $x[n]$.



Perform the following calculations *without* explicitly evaluating $X(e^{j\omega})$.

(a) Find $X(e^{j0})$.

(b) Find $\measuredangle X(e^{j\omega})$.

(c) Find $X(e^{j\pi})$.

(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

(e) Evaluate $\int_{-\pi}^{\pi} |\frac{d}{d\omega}X(e^{j\omega})|^2 d\omega$.

(f) Sketch $y[n]$ such that $\mathfrak{F}\{y[n]\} = \mathbf{Re}\{X(e^{j\omega})\}$.

Problem 4 (20 pts)**.** Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^{\pi} \left[B(\omega)\sin(\omega n) + C(\omega)\cos(\omega n)\right] d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

Problem 5 (10 pts)**.** Consider a signal $y[n]$ with Fourier transform $Y(e^{j\omega})$. Suppose that $y[n] = x_{(4)}[n]$, where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $Y(e^{j\omega}) = Y(e^{j(\omega-\alpha)})$.

Problem 1 (20 pts). Determine the Fourier transform of the following signals:

(a) $x_1[n] = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{2\pi}{3}n\right)$

$$\frac{1}{2\pi}\int_0^{2\pi} 2\pi\delta(\omega-\omega_0)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}e^{j\frac{\pi}{4}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}$$

$$F\left\{\sin\left(\frac{\pi}{4}n\right)\right\} = \sum_{k=-\infty}^{\infty}\left[\frac{\pi}{j}\delta\left(\omega-\frac{\pi}{4}-2\pi k\right) - \frac{\pi}{j}\delta\left(\omega+\frac{\pi}{4}-2\pi k\right)\right]$$

$$\cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n}$$

$$F\left\{\cos\left(\frac{2\pi}{3}n\right)\right\} = \sum_{k=-\infty}^{\infty}\left[\pi\cdot\delta\left(\omega-\frac{2\pi}{3}-2\pi k\right) + \pi\,\delta\left(\omega+\frac{2\pi}{3}-2\pi k\right)\right]$$

$$F\{x_1[n]\} = \sum_{k=-\infty}^{\infty}\left[\frac{\pi}{j}\delta\left(\omega-\frac{\pi}{4}-2\pi k\right) - \frac{\pi}{j}\delta\left(\omega+\frac{\pi}{4}-2\pi k\right) + \pi\,\delta\left(\omega-\frac{2\pi}{3}-2\pi k\right) + \pi\,\delta\left(\omega+\frac{2\pi}{3}-2\pi k\right)\right]$$

(b) $x_3[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$

$$x_3[n] = 2^n\cdot\frac{1}{2j}\cdot\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)u[-n]$$

$$\frac{1}{2j}\sum_{n=-\infty}^{0} 2^n\cdot\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)e^{-j\omega n}$$

$$= \frac{1}{2j}\sum_{n=-\infty}^{0}\left(\left[2\cdot e^{-j(\omega-\frac{\pi}{4})}\right]^n - \left[2\cdot e^{-j(\omega+\frac{\pi}{4})}\right]^n\right)$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{e^{j(\omega-\frac{\pi}{4})}}{2}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{e^{j(\omega+\frac{\pi}{4})}}{2}\right)^n$$

$$= \frac{1}{2j}\lim_{n\to\infty}\frac{1-[2^{-1}e^{j(\omega-\frac{\pi}{4})}]^n}{1-2^{-1}e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{2j}\sum_{n=0}^{\infty}\frac{1-[2^{-1}\cdot e^{j(\omega+\frac{\pi}{4})}]^n}{1-2^{-1}\cdot e^{j(\omega+\frac{\pi}{4})}}$$

$$= \frac{1}{2j}\cdot\frac{2}{2-e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{2j}\cdot\frac{2}{2-e^{j(\omega+\frac{\pi}{4})}}$$

$$= \frac{1}{j}\left(\frac{1}{2-e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{2-e^{j(\omega+\frac{\pi}{4})}}\right)$$

Problem 2 (20 pts). Determine the inverse Fourier transform of the following signals:

(a) $X_1(e^{j\omega}) = \cos^2(3\omega) + \sin^2(\omega)$

$$\sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$\cos^2(3\omega) = \frac{1}{4}e^{j6\omega} + \frac{1}{4}e^{-j6\omega} + \frac{1}{2}$$

$$F^{-1}\{\cos^2(3\omega)\} = \frac{1}{4}\delta[n+6] + \frac{1}{4}\delta[n-6] + \frac{1}{2}\delta[n]$$

$$\sin^2(\omega) = \left(-\frac{1}{4}\right)e^{j2\omega} + \left(-\frac{1}{4}\right)e^{-j2\omega} + \frac{1}{2}$$

$$F^{-1}\{\sin^2(\omega)\} = \left(-\frac{1}{4}\right)\delta[n+2] + \left(-\frac{1}{4}\right)\delta[n-2] + \frac{1}{2}\delta[n]$$

$$F^{-1}\{X_1(e^{j\omega})\} = \delta[n] - \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+6] + \frac{1}{4}\delta[n-6]$$

(b) $X_2(e^{j\omega}) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = \begin{cases} 0, & 0 \le |\omega| < \pi/3 \\ 1, & \pi/3 \le |\omega| < \pi \end{cases}$, and $B(\omega) = -\frac{2}{3}\omega$

$$F^{-1}\{X_2(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega) e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n}\, d\omega$$

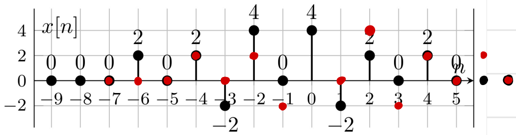
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{-\frac{\pi}{3}} e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n}\, d\omega + \int_{\frac{\pi}{3}}^{\pi} e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n}\, d\omega\right)$$

$$= \frac{1}{2\pi}\left(\left.\frac{e^{j(n-\frac{2}{3})\omega}}{j(n-\frac{2}{3})}\right|_{-\pi}^{-\frac{\pi}{3}} + \left.\frac{e^{j(n-\frac{2}{3})\omega}}{j(n-\frac{2}{3})}\right|_{\frac{\pi}{3}}^{\pi}\right)$$

$$= \frac{1}{\pi(n-\frac{2}{3})} \cdot \left[\sin\left(\left(n-\frac{2}{3}\right)\pi\right) - \sin\left(\left(n-\frac{2}{3}\right)\frac{\pi}{3}\right)\right]$$

$$= \text{sinc}\left(n\pi - \frac{2\pi}{3}\right) - 3\,\text{sinc}\left(\frac{\pi}{3}n - \frac{2\pi}{9}\right)$$

Problem 3 (30 pts). Let $X(e^{j\omega})$ be the Fourier transform of the signal $x[n]$.



$144+64+36+64+4+16+64$

Perform the following calculations *without* explicitly evaluating $X(e^{j\omega})$.

(a) Find $X(e^{j0})$.

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \sum_{n=-\infty}^{\infty} x[n] = 12$$

(b) Find $\measuredangle X(e^{j\omega})$.

$x[n-1]$ is real and even. $\mathcal{F}\{x[n-1]\} = e^{-jw} X(e^{jw})$ is also real and even.

$$\measuredangle\, e^{-jw} X(e^{jw}) = \angle(w_x - w) = \angle 0 \;,\; w_x = \measuredangle X(e^{jw}) = w$$

(c) Find $X(e^{j\pi})$.

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x[n]\, \{\cos(\pi n) - j\sin(\pi n)\}$$

$$= (2+2+4+2) \times 2 = 20$$

(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

$$\int_{-\pi}^{\pi} X(e^{jw})\,dw = 2\pi \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw}) \cdot e^{jw0}\,dw$$

$$= 2\pi \cdot x[0] = 8\pi$$

(e) Evaluate $\int_{-\pi}^{\pi} |\frac{d}{d\omega}X(e^{j\omega})|^2 d\omega$.

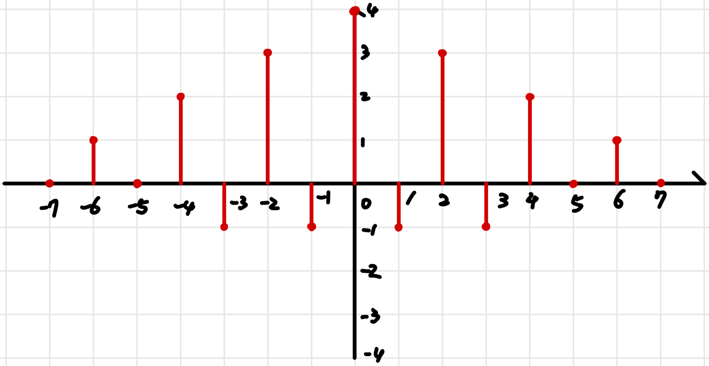
$$\frac{d}{dw} X(e^{jw}) = \frac{d}{dw}\sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \sum_{n=-\infty}^{\infty} (-jn)\, x[n]\, e^{-jwn}$$

$$\mathcal{F}^{-1}\{\tfrac{d}{dw} X(e^{jw})\} = -jn\, x[n]$$

$$\int_{-\pi}^{\pi} \left|\frac{d}{dw} X(e^{jw})\right|^2 dw = 2\pi \sum_{n=-\infty}^{\infty} |-jn\, x[n]|^2 = 2\pi \cdot \sum_{n=-\infty}^{\infty} n^2 x^2[n] = 2\pi \cdot 392 = 784\pi$$

(f) Sketch $y[n]$ such that $\mathfrak{F}\{y[n]\} = \mathbf{Re}\{X(e^{j\omega})\}$.

$$y[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$



Problem 4 (20 pts). Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^\pi \left[B(\omega)\sin(\omega n) + C(\omega)\cos(\omega n)\right] d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\left[\cos(\omega n) + j\sin(\omega n)\right] d\omega$$

$$= \int_{-\pi}^{\pi}\left[\frac{X(e^{j\omega})}{2\pi}\cos(\omega n) + \frac{j}{2\pi}X(e^{j\omega})\sin(\omega n)\right] d\omega$$

$$= \int_0^{\pi}\left[\frac{X(e^{j\omega})}{2\pi}\cos(\omega n) + \frac{jX(e^{j\omega})}{2\pi}\sin(\omega n)\right] d\omega + \int_0^{\pi}\left[\frac{X(e^{-j\omega})}{2\pi}\cos(-\omega n) + \frac{jX(e^{-j\omega})}{2\pi}\sin(-\omega n)\right] d\omega$$

$$= \int_0^{\pi}\frac{1}{2\pi}\left[X(e^{j\omega}) + X(e^{-j\omega})\right]\cos(\omega n)\, d\omega + \int_0^{\pi}\frac{j}{2\pi}\left[X(e^{j\omega}) - X(e^{-j\omega})\right]\sin(\omega n)\, d\omega$$

$$= \int_0^{\pi}\frac{1}{\pi}\, Re\{X(e^{j\omega})\}\cos(\omega n)\, d\omega + \int_0^{\pi}\frac{-1}{\pi}\, Im\{X(e^{j\omega})\}\sin(\omega n)\, d\omega$$

$$B(\omega) = \frac{-1}{\pi}\, Im\{X(e^{j\omega})\}$$

$$C(\omega) = \frac{1}{\pi}\, Re\{X(e^{j\omega})\}$$

Problem 5 (10 pts). Consider a signal $y[n]$ with Fourier transform $Y(e^{j\omega})$. Suppose that $y[n] = x_{(4)}[n]$, where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $Y(e^{j\omega}) = Y(e^{j(\omega-\alpha)})$.

α is a period of $Y(e^{jw})$

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} x_{(4)}[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} x[n] e^{-jw4n} = X(e^{j4w})$$

$$\frac{2\pi}{4} = \frac{\pi}{2} \quad , \alpha = \frac{\pi}{2}$$