EECS 205003 Session 16

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Orthogonality of the Four subspaces Looking ahead(part I)(part II)

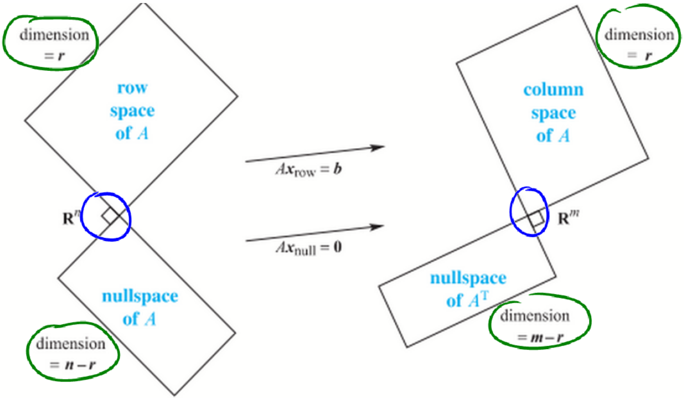
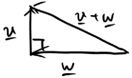


Figure 24: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

Orthogonal vectors



Two vectors are orthogonal (= perpendicular)

if $\mathbf{u^T w} = 0$ **or** $\|\mathbf{u}+\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2$

$$((\mathbf{u}+\mathbf{w})^{\mathbf{T}}(\mathbf{u}+\mathbf{w}) = \mathbf{u^T u} + \mathbf{w^T w} + \mathbf{w^T u} + \mathbf{u^T w}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 \Rightarrow \mathbf{w^T u} = \mathbf{u^T w} = 0)$$

Note: All vectors are orthogonal to zero vector

Orthogonal subspaces

$\boxed{Def}$ **Subspace S is orthogonal to subspace T if every vector in S is orthogonal to every vectors in T**
$(\mathbf{u}^{\mathbf{T}}\mathbf{w} = 0 \ \ \forall \mathbf{u} \in S, \ \forall \mathbf{w} \in T)$

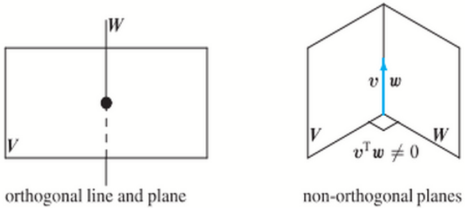


Figure 23: Orthogonality is impossible when dim V +dim W > dimension of whole space.

(If $S \cap T$ contains any vector (except 0)
$\Rightarrow S \cap T$ cannot be orthogonal)
(0 is orthogonal to itself $0^{\mathbf{T}}0 = 0$)

Nullspace is orthogonal to row space

$N(A)$ **&** $C(A^T)$ **are orthogonal subspace of** R^n

Why? $\forall\ \mathbf{x} \in N(A)$, $A\mathbf{x} = \mathbf{0}$

$$A\mathbf{x} = \begin{bmatrix} row\ 1 \\ row\ 2 \\ \vdots \\ row\ m \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \leftarrow row\ 1 \cdot \mathbf{x} = 0 \\ \leftarrow row\ 2 \cdot \mathbf{x} = 0 \\ \vdots \\ \leftarrow row\ m \cdot \mathbf{x} = 0 \end{matrix}$$

$\Rightarrow$ $\mathbf{x}$ **is orthogonal to every row of** A **so it's also orthogonal to all combination of rows of** A $\Rightarrow$ $N(A) \perp C(A^T)$

Left Nullspace is orthogonal to column space

$N(A) \perp C(A^T)$: **both orthogonal subspaces of** R^m

Reason: $\forall \mathbf{y} \in N(A^T), A^T\mathbf{y} = \mathbf{0}$

$$A^T\mathbf{y} = \begin{bmatrix} col\ 1^T \\ col\ 2^T \\ \vdots \\ col\ n^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

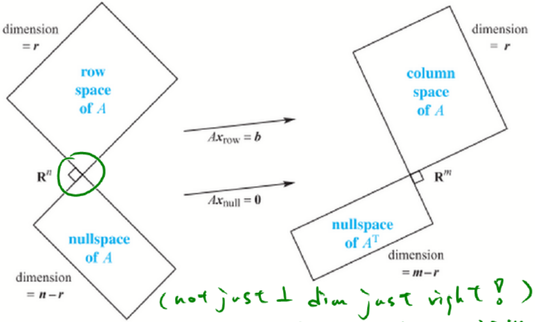
$\Rightarrow$ $\mathbf{y}$ **is orthogonal to every column of** A

$\Rightarrow$ $\mathbf{y}$ **is orthogonal to all combination of columns of** A

$\Rightarrow N(A^T) \perp C(A)$

Orthogonal complements

$\boxed{Def}$ **The orthogonal complement $V^{\perp}$ (V perp) of subspace V contains every vector perpendicular to V**



(not just ⊥ dim just right !)
($r + (n-r) = n$, $r + (m-r) = m$)

Figure 24: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

Fundamental Theorem of Linear Algebra (part II)

(1) $N(A)$ **is orthogonal complement of** $C(A^T)$ **(in** R^n**)**

(2) $N(A^T)$ **is orthogonal complement of** $C(A)$ **(in** R^m**)**

Reason for (1):

$\forall \mathbf{x}$ **orthogonal to rows of** A, $A\mathbf{x} = \mathbf{0}$

$\Rightarrow \mathbf{x} \in N(A) \Rightarrow N(A) = C(A)^{\perp}$

(reverse is also true, i.e., $C(A^T) = N(A)^{\perp}$ **prove by contradiction : if** $\exists \mathrm{v}$ **orthogonal to** $N(A)$ **but not in** $C(A^T)$**, we can add** u **as a new row of matrix:** $A' = \begin{bmatrix} A \\ \mathbf{u}^{\mathbf{T}} \end{bmatrix}$ **without changing** $N(A)$**)**

(If $A\mathbf{x} = \mathbf{0}$**, then** $A'\mathbf{x} = \mathbf{0}$ **since** $\mathbf{u}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$**)**

then $dim\ C(A'^T) = dim\ C(A^T) + 1 = r + 1$**, but**
$dim\ N(A') = dim\ N(A) = n - r \Rightarrow (n - r) + (r + 1) = n + 1 \neq n$
(contradiction!)

(Reason for (2) follows by changing A to A^T)

Ex:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$$

$dim\ C(A^T) = 1 \Rightarrow dim\ N(A) = 3 - 1 = 2$

(basis: $\begin{pmatrix} 1 & 2 & 5 \end{pmatrix}$**)** **(basis: two special solutions)**

By orthogonal complement, $N(A)$ **is the plane**

perpendicular to $\begin{pmatrix} 1 & 2 & 5 \end{pmatrix}$

Row space & Nullspace components

Since $C(A^T)$ & $N(A)$ **are orthogonal complements**

$$(C(A^T) = N(A)^\perp \ \& N(A) = C(A^T)^\perp)$$

every $\mathbf{x} \in R^n$ **can be splitted into**

$\mathbf{x} = \mathbf{x_r} + \mathbf{x_n}$ **(will prove it later)**

$\mathbf{x_r}$: **row space component**

$\mathbf{x_n}$: **nullspace component**

Multiplying by A

Note 1: $A\mathbf{x_n} = \mathbf{0}$ **(nullspace component goes to zero)**

Note 2: $A\mathbf{x_r} = A\mathbf{x} = \mathbf{b}$ **(row space component goes to** $C(A)$**)**

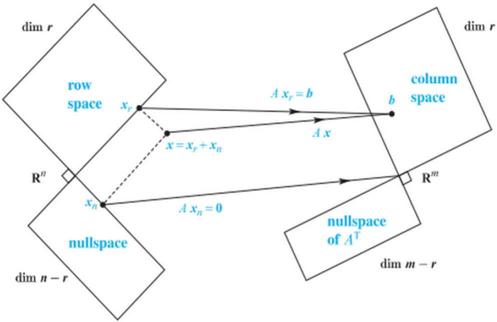


Figure 25: This update of Figure 24 shows the true action of A on $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$. Row space vector $\boldsymbol{x}_r$ to column space, nullspace vector $\boldsymbol{x}_n$ to zero.

Note 3: Every vector $\mathrm{b} \in C(A)$ comes from a unique vector

$$\mathbf{x_r} \in C(A^T)$$

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pf: If $\exists \mathbf{x_r}, \mathbf{x_r'} \in \mathbf{C}(\mathbf{A^T})$ **s.t.**

$$A\mathbf{x_r} = A\mathbf{x_r'} \Rightarrow A\mathbf{x_r} - A\mathbf{x_r'} = \mathbf{0}$$

$$\Rightarrow A(\mathbf{x_r} - \mathbf{x_r'}) = \mathbf{0} \ \Rightarrow \mathbf{x_r} - \mathbf{x_r}^{i} \in N(A)$$

since $\mathbf{x_r}, \mathbf{x_r'} \in C(A^T) \ \Rightarrow \ \mathbf{x_r} - \mathbf{x_r'} \in C(A^T)$

$\Rightarrow \mathbf{x_r} - \mathbf{x_r'} = \mathbf{0}$**, since** $N(A) \perp C(A^T)$

This implies that there is a $r \times r$ **invertible matrix hiding inside** A

(If we throw away two nullspaces)

From $C(A^T) \rightarrow C(A)$, A **is invertible & pseudoinverse will invert it in sec. 7-3**

Ex:

$$A = \begin{bmatrix} \boxed{\begin{matrix} 3 & 0 \\ 0 & 5 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix} \quad \text{invertible}$$

$$B = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \begin{matrix} 2 \\ 2 \end{matrix} & \boxed{\begin{matrix} 3 \\ 4 \end{matrix}} & \begin{matrix} 4 & 5 \\ 5 & 6 \end{matrix} \\ 1 & 2 & 4 & \begin{matrix} 5 & 6 \end{matrix} \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ invertible}$$

Combining Bases from subspaces

Recall: Basis = independent + span the space

But when the count is right, we only need one of them, i.e.,

- Any n independent vector in R^n must span $R^n \Rightarrow$ they are basis
- Any n vectors that span R^n must be independent $\Rightarrow$ they are basis

Equivalent statements: $(A_{n\times n})$

- If n columns of A are independent they span $R^n \Rightarrow A\mathrm{x} = \mathrm{b}$ solvable, $\forall \mathrm{b}$

- if n columns of A span R^n, they are in independent

  $\Rightarrow A\mathrm{x} = \mathrm{b}$ has only one solution

pf: If n columns independent then no free var.s

$\Rightarrow$ sol. x unique $\&$ n pivots

$\Rightarrow$ back sub. solves $A\mathrm{x} = \mathrm{b}$

$\Rightarrow$ solution exists

If n columns span $R^n, A\mathrm{x} = \mathrm{b}$ is solvable $\forall$ b

(solution exists)

$\Rightarrow$ elimination produces no zero rows

$\Rightarrow$ n pivots $\Rightarrow$ no free var.s

$\Rightarrow$ solution unique

Combining bases from $C(A^T)$ & $N(A)$

we have r basis from $C(A^T)$ in R^n

$(n-r)$ basis from $N(A)$ in R^n

combined together

A total of $r+(u-r)=u$ independent vectors in R^n, they span R^n

If $$\underbrace{a_1\mathbf{v_1}+\cdots+a_r\mathbf{v_r}}_{\Downarrow \; \mathbf{x_r}} + \underbrace{a_{r+1}\mathbf{v_{r+1}}+\cdots+a_n\mathbf{v_n}}_{\Downarrow \; \mathbf{x_n}} = \mathbf{0}$$

$$\mathbf{x_r} + \mathbf{x_n} = \mathbf{0}$$

$\Rightarrow \mathbf{x_r} = -\mathbf{x_n} \Rightarrow \mathbf{x_r},\ \mathbf{x_n}$ in both $C(A^T)$ & $N(A)$ but $C(A^T)\perp N(A)$

$\Rightarrow \mathbf{x_r} = \mathbf{x_n} = \mathbf{0}$

since $\mathbf{v_1}\ldots\mathbf{v_r}$ are basis of $C(A^T)$

$\mathbf{v_{r+1}}\ldots\mathbf{v_n}$ are the basis of $N(A)$

$\Rightarrow a_1=a_2=\cdots=a_r=a_{r+1}=\cdots=a_n=0$ n vectors are independent

So for every $\mathbf{x}$ in R^n, we have $\mathbf{x}=\mathbf{x_r}+\mathbf{x_n}$