CS2336 Discrete Mathematics

Homework 3
Tutorial: November 18, 2019
Exam 2: December 02, 2018 (2.5 hours)

Problems marked with * will be explained in the tutorial.

1. Prove each of the following for all integer $n \geq 1$ by mathematical induction.

   (a) (*)
   $$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

   (b)
   $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

2. Use strong induction to prove that $\sqrt{2}$ is irrational.

   Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.

3. (*, Challenging, UKMT MOG 2016) Show that any odd positive integer can be expressed as a product of fractions with each fraction of the form
   $$\frac{4q-1}{2q+1}, \qquad \text{where } q \text{ is a positive integer?}$$
   For instance,
   $$1 = \frac{3}{3} = \frac{4 \times 1 - 1}{2 \times 1 + 1}$$
   and
   $$3 = \frac{15}{9} \times \frac{27}{15} = \frac{4 \times 4 - 1}{2 \times 4 + 1} \times \frac{4 \times 7 - 1}{2 \times 7 + 1}.$$
   Hint: On Page 3; try your best without using the hint.

4. (*, Challenging, Adapted from AIME 1987) Show that the following expression is always a positive integer, for any $k \geq 1$, by expressing it in terms of k:
   $$10 \left(\frac{10^4 + 324}{4^4 + 324} \right) \left(\frac{22^4 + 324}{16^4 + 324} \right) \cdots \left(\frac{(12k-2)^4 + 324}{(12k-8)^4 + 324} \right)$$
   Hint: On Page 3; try your best without using the hint.

5. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$ and so on.

   [*Hint:* For the inductive step, separately consider the case where $k+1$ is even and where it is odd. When it is even, note that $(k+1)/2$ is an integer.]

6. (*) Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them.

   [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

7. There are 50 baskets of apples. Each basket contains at most 24 apples. Show that there are at least 3 baskets containing the same number of apples.

8. Suppose $n + 1$ integers are chosen from 1 to $2n$. Show that there exist two of the chosen numbers which have no common factor.

9. Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$, there are at least two selected integers whose sum is 26.

10. (*) A lecture lasts 50 minutes and 6 students were sleeping for at least 10 minutes during the lecture. Show that two students were sleeping simultaneously at some point during the lecture.

11. (*) Show that in a group of 10 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

12. (**) Show that in a group of 9 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

13. (*) Show that among a group of 100 people, if any two will shake hands at most once, then at least two people will shake hands for the same number of times.

14. (*, Challenging) Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $(b_1, b_2, b_3, b_4, b_5, b_6)$ be two arrangements of the integers $1, 2, 3, 4, 5, 6$. Consider the six pairs of differences $|a_i - b_i|$. Is it possible that all of these differences are not the same?

Hint for Question 3:

$$(2q+1) \times \frac{4q-1}{2q+1} = 4q-1; \qquad\qquad (2q+1) \times \frac{12q+3}{6q+3} = 4q+1.$$

Hint for Question 4:

Sophie Germain Identity: $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$