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Solving ODEs by "series solutions" (Ch 6)

☆ Condition :

Review : Taylor series

A function has a Taylor series expansion about $x = x_0$

$f(x) =$

ex : $f(x) = e^x$. Use a Taylor seies to expand e^x around $x=0$

Idea : We know a function can be expressed by the Taylor series. So we can guess a solution having a form

That is, solution of the DE is expressed as

Remarks :

① About "convergence" of a power series

— There is a simple "　　　　" to evaluate in what range of x the series converges / diverges.

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$\boxed{\text{ratio test}}$

For a power series : $C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + C_3(x-x_0)^3 + \cdots$

$$\lim_{k\to\infty}\left|\frac{C_{k+1}(x-x_0)^{k+1}}{C_k(x-x_0)^k}\right| = |x-x_0| \lim_{k\to\infty} \qquad = L$$

ex: For a power series $\sum_{k=1}^{\infty}(-1)^{k+1} k(x-2)^k$.

— Define : "radius of convergence" R

② If a function f(x) can be expressed as a power series at $x=x_0$ with a radius of convergence we say this function is " " at point

ex: $f(x) = \frac{1}{1-x}$

For simplicity, in the following, we will focus on expanding the function at

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Let's just use a simple 1st-order ODE to show how the series solutions can be used to solve ODEs.

Example of 1st-order ODE

Ex: For $y' + y = 0$, find a power series solution.

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Series solutions of 2nd-order ODEs

The series solutions are particularly useful to solve some 2nd-order ODEs that are widely used in physics and engineering. Some examples are

These commonly used 2nd-order ODEs are

General procedures to solve 2nd-order ODEs by series solutions:

1) Express the ODE by

2) check

3) Apply method I or method II to solve the ODE.