EE 205003 Session 13

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The Four Fundamental Subspaces

Any $m \times n$ matrix A determines four subspaces (possibly containing only the zero vector)

- Column space, $\mathbf{C}(A)$: (in $\mathbb{R}^m$)
  All comb. of col.s of A
- Null space, $\mathbf{N}(A)$: (in $\mathbb{R}^n$)
  All sol.s of $\mathbf{x}$ of $A\mathbf{x} = \mathbf{0}$
- Row space, $\mathbf{C}(A^\intercal)$: (in $\mathbb{R}^n$)
  All comb.s of row vectors of A
  (same as col. space of $A^\intercal \Rightarrow \mathbf{C}(A^\intercal)$)
- Left null space, $\mathbf{N}(A^\intercal)$: (in $\mathbb{R}^m$)
  Null space of $A^\intercal \Rightarrow$ All sol.s of $\mathbf{y}$ of $A^\intercal\mathbf{y} = \mathbf{0}$
  ($A^\intercal\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{y}^\intercal A = \mathbf{0}^\intercal$ so called left null space)

Ch 3.6 Dimensions of Four Subspaces

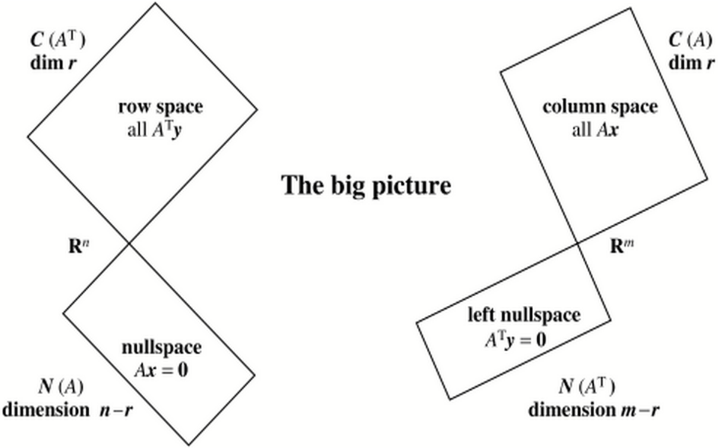

Figure 22: The dimensions of the Four Fundamental Subspaces (for R and for A).

Basis & dimension

Ex :

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\left(\begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)$$

Column space $\mathbf{C}(A)$

- Dimension :

$$\text{dim}(\mathbf{C}(A)) = \text{rank}(A) = \# \text{ of pivot col.s}$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = \text{rank}(R) = 2$$

Column space $\mathbf{C}(A)$ (cont.)

- Basis :

  the r pivot col.s form a basis for $\mathbf{C}(A)$

$$A = \left[\begin{array}{cccc} \boxed{\begin{array}{c|c} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{array}} & \begin{array}{c} 3 \\ 2 \\ 3 \end{array} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \end{array}\right] \to \cdots \to \left[\begin{array}{cccc} \boxed{\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}} & \begin{array}{c} 1 \\ 1 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \end{array}\right] = R$$

($\mathbf{C}(A) \neq \mathbf{C}(R)$, but positions of pivot columns are the same

$\because A\mathbf{x} = \mathbf{0}$ exactly when $R\mathbf{x} = \mathbf{0}$

so $x_1\mathbf{r}_1 + x_2\mathbf{r}_2 = \mathbf{0}$ only when $x_1 = x_2 = 0$

$\Rightarrow x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{0} \cdots \ \cdots \ \cdots$)

(independent pivot columns in $R \Rightarrow$ independent columns in A)

Null space $\mathbf{N}(A)$

- Dimension :

$$\begin{aligned}\text{dim}(\mathbf{N}(A)) &= \#\text{ of free columns of } A\\ &= \#\text{ of free columns of } R\\ &= n - r\end{aligned}$$

$$A = \begin{bmatrix}1 & 2 & 3 & 1\\1 & 1 & 2 & 1\\1 & 2 & 3 & 1\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix} = R$$

$$\begin{array}{cccc} & & \uparrow & \uparrow\end{array} \qquad\qquad \begin{array}{cccc} & & \uparrow & \uparrow\end{array}$$

free col.s free col.s

$\Rightarrow \text{dim}(\mathbf{N}(A)) = 4 - 2 = 2$

Null space $\mathbf{N}(A)$ (cont.)

- Basis :

  special solutions to $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

col.3 $= 1 \cdot$ col.1 $+ 1 \cdot$ col.2 $\quad \left(\begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)$

col.4 $= 1 \cdot$ col.1

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{(basis)}$$

Row space $\mathbf{C}(A^\intercal)$

- Dimension :

$$\begin{aligned}\dim(\mathbf{C}(A^\intercal)) &= \#\text{ of pivot rows}\\ &= \#\text{ of pivot columns} = r\end{aligned}$$

$$A = \begin{bmatrix}1 & 2 & 3 & 1\\ 1 & 1 & 2 & 1\\ 1 & 2 & 3 & 1\end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix}\boxed{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}} & \begin{matrix}1 & 1\\ 1 & 0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \begin{matrix}0 & 0\end{matrix}\end{bmatrix} = R$$

(# of indep. columns = # of indep. rows)

Row space $\mathbf{C}(A^\intercal)$ (cont.)

- Basis :

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \left[\begin{array}{cccc} \boxed{\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}} \\ 0 \quad 0 \quad 0 \quad 0 \end{array} \right] = R$$

$\Rightarrow R = EA$

so rows of R are combinations of rows of A

reversible $\Rightarrow A = E^{-1}R$

this implies rows of A are comb. of rows of R (only pivot rows)

$\Rightarrow \mathbf{C}(A^\intercal) = \mathbf{C}(R^\intercal)$

& first r rows of R form the basis of $\mathbf{C}(A^\intercal)$

Left nullspace $\mathbf{N}(A^\intercal)$

- Dimension :

  matrix $A^\intercal$ has m columns

  From $\dim(\mathbf{C}(A^\intercal)) \Rightarrow \text{rank}(A^\intercal) = r$

  $\Rightarrow \#$ of pivot columns of $A^\intercal = r$

  $\Rightarrow \#$ of free columns of $A^\intercal = m - r$

  $\Rightarrow \dim(\mathbf{N}(A^\intercal)) = m - r$

Left nullspace $\mathbf{N}(A^\mathsf{T})$ (cont.)

- Basis :

Recall : Gauss-Jordan

$$[A_{n\times n}\ I_{n\times n}] \to [I_{n\times n}\ \underset{\substack{\shortparallel \\ E_{n\times n}}}{A^{-1}_{n\times n}}]$$

Similarly,

$$[A_{m\times n}\ I_{m\times m}] \to [R_{m\times n}\ E_{m\times m}]$$

$EA = R$ (This is how we obtain E directly)

$$\underbrace{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \boxed{-1 \quad 0 \quad 1} \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}}_{R}$$

Left nullspace $\mathbf{N}(A^\intercal)$ (cont.)

- Basis :

Recall :

$$A^\intercal \mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{y}^\intercal A = \mathbf{0}^\intercal$$

(so we have $\mathbf{y}^\intercal = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$)

($\because m - r = 3 - 2 = 1$, we only need one basis vector)

In general, the bottom $m - r$ rows of E describes lin. dependencies of rows of A, since the button $m - r$ rows of R are zero

$\Rightarrow$ the bottom $m - r$ rows of E satisfies $\mathbf{y}^\intercal A = \mathbf{0}^\intercal$

$\Rightarrow$ they are basis for $\mathbf{N}(A^\intercal)$

Summary

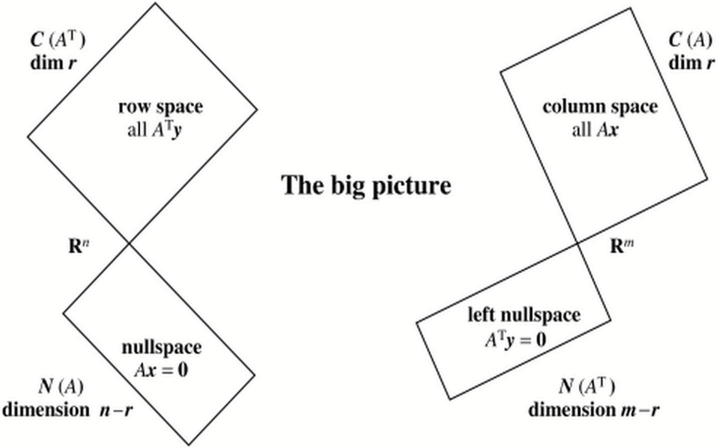

Figure 22: The dimensions of the Four Fundamental Subspaces (for R and for A).

Summary (cont.)

Basis :

$\mathbf{C}(A)$ – r pivot col.s of A ($\neq \mathbf{C}(R)$)

$\mathbf{N}(A)$ – $n-r$ special sol.s are a basis of $\mathbf{N}(A)$ & $\mathbf{N}(R)$ (same space)

$\mathbf{C}(A^\intercal)$ – r pivot rows of R are a basis of $\mathbf{C}(A^\intercal)$ & $\mathbf{C}(R^\intercal)$ (same space)

$\mathbf{N}(A^\intercal)$ – last $m-r$ rows of E are a basis of $\mathbf{N}(A^\intercal)$

Fundamental Theorem of Linear Algebra (part I)

$\mathbf{C}(A)$ & $\mathbf{C}(A^\intercal)$ both have dim. $= r$

dim$(\mathbf{N}(A)) = n - r$, dim$(\mathbf{N}(A^\intercal)) = m - r$