ODE Q3: Online Open book Exam May 26, 2022.

1. (16%) If F(s) is the Laplace transform of f(t) and G(s) is the Laplace transform of g(t).
   (a) (6%) Prove that $e^{-bs}F(s)$ is the transform of f(t − b)H(t − b), where b is real and positive and H(t) is the unit step function, or called Heaviside step function.
   (b) (10%) Let H(s)=F(s)G(s). Prove that h(t) $=\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$
2. (14%) Consider the function f(t) shown in Fig. 1.
   (a) (5%) Express this function f(t) in terms of unit step functions, or called Heaviside step functions, H(t).
   (b) (9%) Find the Laplace Transform of f(t).

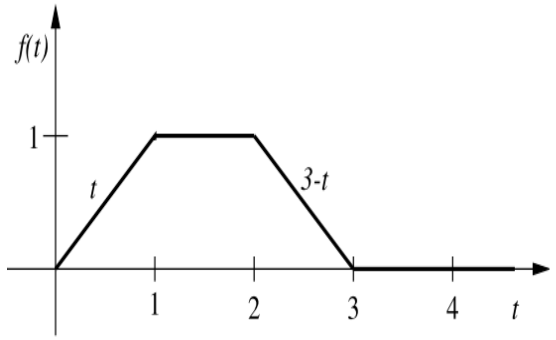

Fig. 1

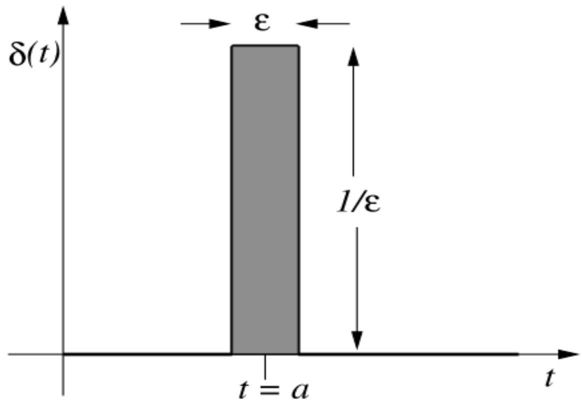

Fig. 2

3. (12%) Consider the Dirac delta function δ (t) shown in Fig. 2 which is a function of ε and a.
   (a) (8%) Find the Laplace Transform of δ (t).
   (b) (4%) Find the Laplace Transform of δ (t) as ε approaches 0.
4. (10%) Find the inverse of the Laplace transform $F(s)=(s + 2)/(s^2 + 6s + 1)$.
5. (24%) Given the nonhomogeneous ordinary differential equation: y''+ 3y'+ 2y = t, for 0 < t < 1, and y''+ 3y'+ 2y = 0, for 1 < t, with y(0) = 0 and y'(0) = 1.
   Find (a) (10%) Y(s) and (b) (14%) y(t) by the Laplace transform.
6. (16%) Let f(t) be a periodic function with the period T. That is, f(t + T) = f(t). Define a function x(t) that equals zero except over the interval (0, T) where it equals f(t), i.e., x(t)=f(t) for t in (0, T); otherwise x(t)=0 for T < t. Let F(s) be the Laplace transform of f(t) and X(s) be the Laplace transform of x(t). Express X(s) in terms of F(s) and T.
7. (8%) Let the Bessel Equation of order one-half is $x^2y''+xy'+(x^2-1/4)y=0$. We assume solutions have the form y(x)= φ (r,x)=$\sum_{n=0}^{\infty} a_n x^{r+n}$. Find the roots of the indicial equation and the recurrence relation of $a_n(r)$.