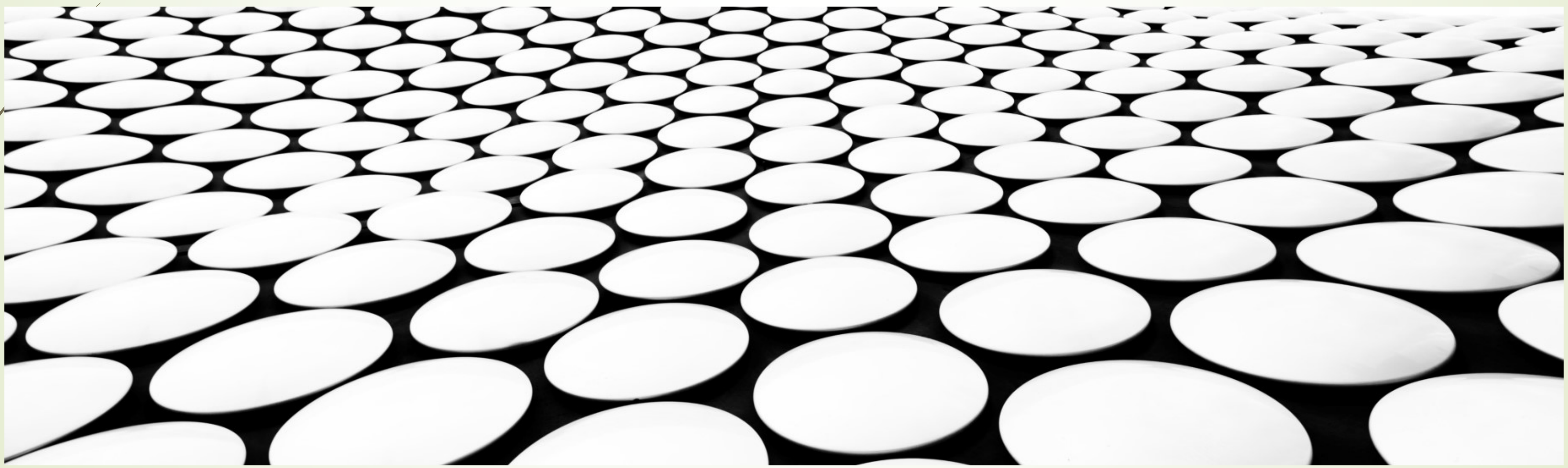
# Linear programming

Simplex algorithm



# definition

- Linear programming (LP), also called linear optimization, is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships.
- Linear function

$$f(x_1, \ldots, x_k) = b + a_1 x_1 + \cdots + a_k x_k,$$

# General LP model

$$\text{minimize / maximize} \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

*subject to*

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\le, \ge, or =) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\le, \ge, or =) b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\le, \ge, or =) b_m$$

$$x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0$$

# Standard Form

- Maximize $c^T x$
- Subject to:
  - $Ax \leq b$
  - $x \geq 0$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

; A is a m.$n$ matrix

# Standard Form

- To apply the simplex algorithm, we should have:
  - The objective function is either maximized or minimized.
  - All the constraints are equalities.
  - All the variables are non-negative.
- To convert a minimization problem to a maximization problem, we can multiply the objective function by -1.
- To convert an inequality constraint to an equality constraint, we can introduce a slack variable (for ≤ constraints) or a surplus variable (for ≥ constraints) with zero cost.
- To convert a variable that is unrestricted in sign to a non-negative variable, we can replace it with the difference of two non-negative variables.

# Standard Form

$$
\begin{array}{ll}
\text{minimize} & 3x_1 - x_2 \\
\text{subject to} & -x_1 + 3x_2 \leq 3 \\
& -2x_1 - x_2 \geq -2 \\
& 2x_1 + x_2 \leq 8 \\
& 4x_1 - x_2 \geq 4 \\
& x_1 \geq 0, x_2 \text{ unrestricted}
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & -3x_1 + x_2 \\
\text{subject to} & -x_1 + 3x_2 + s_1 = 3 \\
& -2x_1 - x_2 - s_3 = -2 \\
& 2x_1 + x_2 + s_2 = 8 \\
& 4x_1 - x_2 - s_4 = 4 \\
& x_1, s_1, s_2, s_3, s_4 \geq 0, x_2 \text{ unrestricted}
\end{array}
$$

# Standard Form

$$\begin{aligned}
&\text{maximize} && -3x_1 + x_2 \\
&\text{subject to} && -x_1 + 3x_2 + s_1 = 3 \\
&&& -2x_1 - x_2 - s_3 = -2 \\
&&& 2x_1 + x_2 + s_2 = 8 \\
&&& 4x_1 - x_2 - s_4 = 4 \\
&&& x_1, s_1, s_2, s_3, s_4 \geq 0, x_2 \text{ unrestricted}
\end{aligned}$$

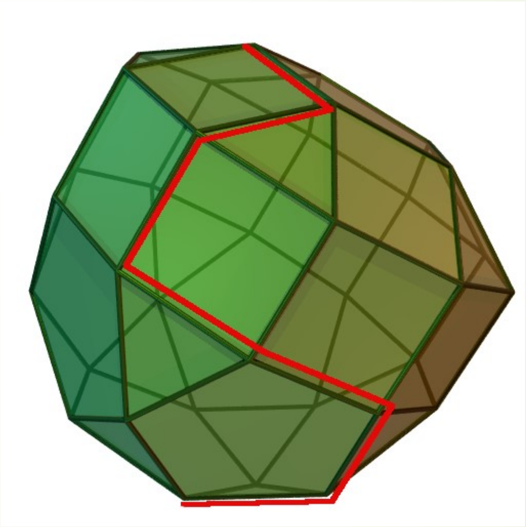
We replace $x_2$ with $x_2' - x_2''$, where $x_2'$ and $x_2''$ are non-negative variables:

$$\begin{aligned}
&\text{maximize} && -3x_1 + x_2' - x_2'' \\
&\text{subject to} && -x_1 + 3x_2' - 3x_2'' + s_1 = 3 \\
&&& -2x_1 - x_2' + x_2'' - s_3 = -2 \\
&&& 2x_1 + x_2' - x_2'' + s_2 = 8 \\
&&& 4x_1 - x_2' + x_2'' - s_4 = 4 \\
&&& x_1, x_2', x_2'', s_1, s_2, s_3, s_4 \geq 0
\end{aligned}$$

# Formulating linear programming problems

## SIMPLEX ALGORITHM

The simplex method involves defining the system of linear inequalities as before, then ‘moving’ along each line, vising vertices trying to improve the objective function on each trip.

# Simplex algorithm

1. Draw the tableaux. (We need a basic variable column on the left, one column for each variable, including the slack, and a value column. We need one row for each constraint and the bottom row for the objective function.
2. Create the initial tableau. (Enter the coefficient of the variable in the appropriate column and row)
3. Look along the objective row for the maximum value entry: this indicates the pivot column.
4. Calculate the $\theta$ values, for each of the constraint rows, where
   $\theta$ = (the term in the value column) ÷ (the term in the pivot column)
5. Select the row with the smallest, positive $\theta$ value to become the pivot row
6. The element in the pivot row and pivot column is the pivot
7. Divide the row found in step 5 by the pivot and change the basic variable at the start of the row to the variable at the top of the pivot column. This is now the pivot row.
8. Use the pivot row to eliminate the pivot's variable from the other rows. This means that the pivot column now contains one 1 and zeros.
9. Repeat steps 3-8 until there are no more strictly positive numbers in the objective row.
10. The tableau is now optimal and the non-zero values can be read off using the basic variable column and value column.