CONSTRAINED NONLINEAR OPTIMIZATION

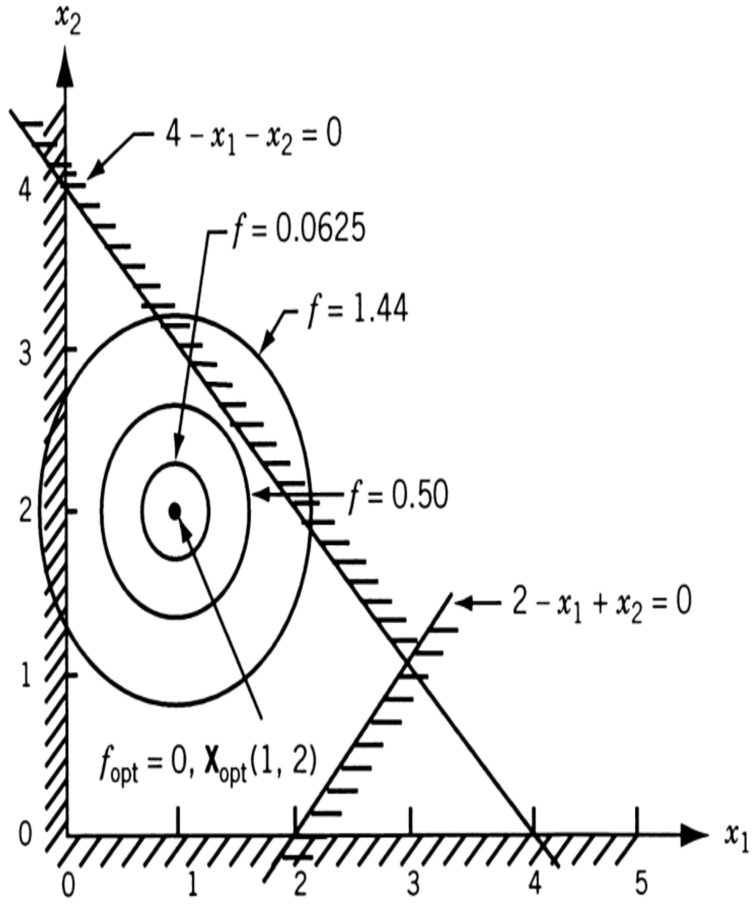
Minimize the nonlinear function ***f(X)*** subject to

$$g_j(\mathbf{X}) \leq 0, \qquad j = 1,2,\ldots,m$$

$$h_k(\mathbf{X}) = 0, \qquad k = 1,2,\ldots,p$$

CHARACTERISTIC OF CONSTRAINED PROBLEM

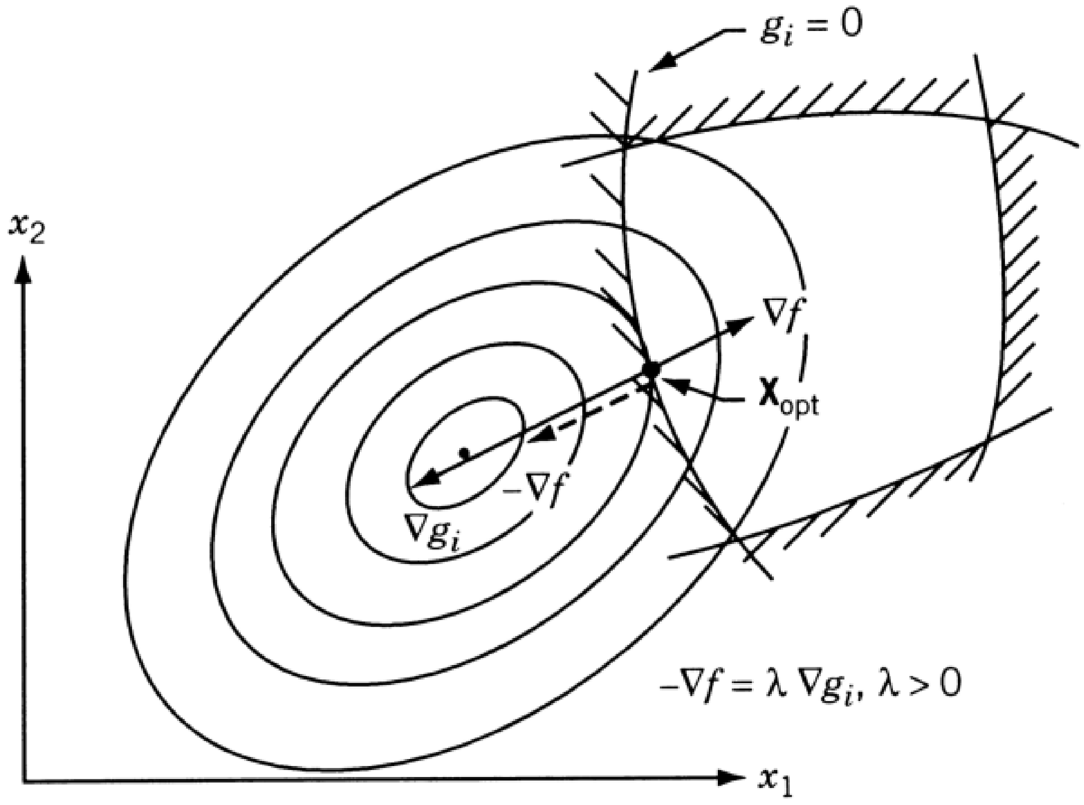
The constraints may have no effect on the optimum point



CHARACTERISTIC OF CONSTRAINED PROBLEM

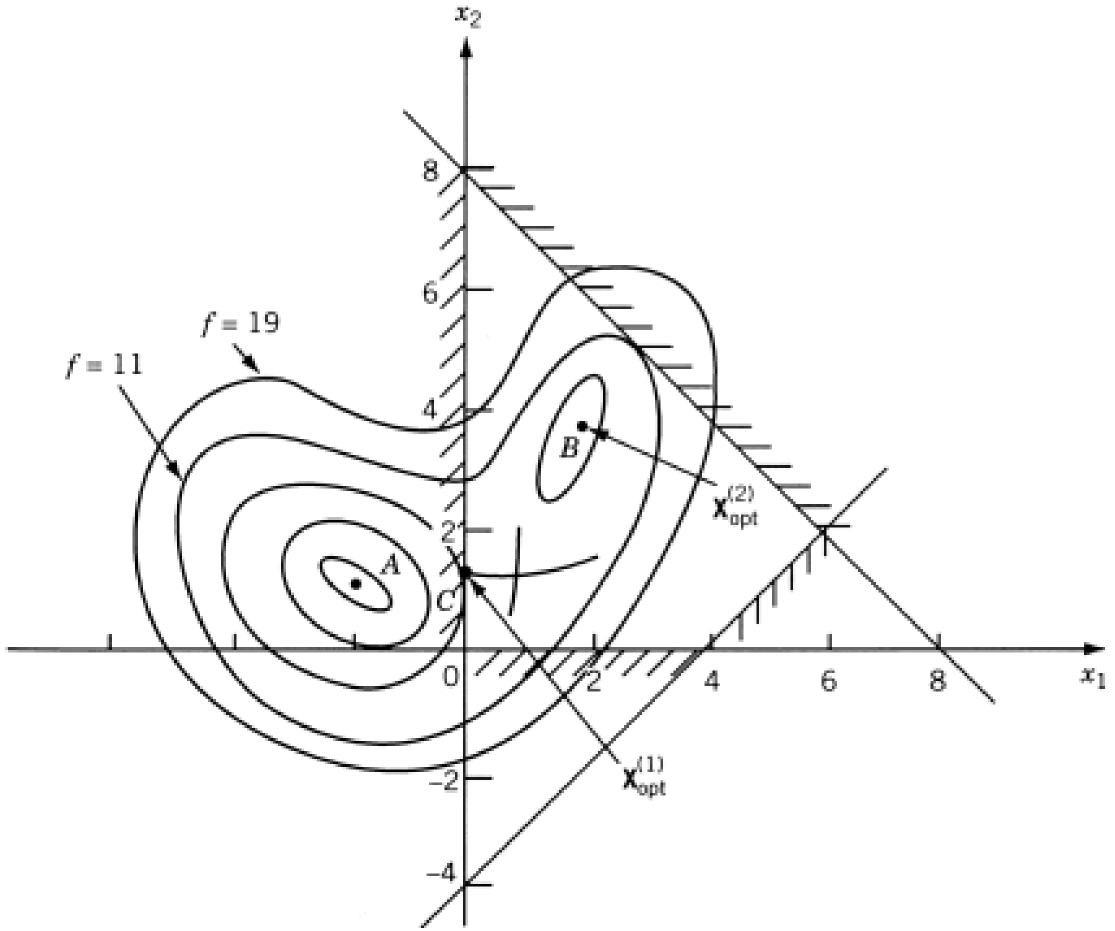
The optimum, unique solution occurs on a constraint boundary.

In this case, the *Kuhn-Tucker necessary conditions* indicate that the negative of the gradient must be expressible as a positive linear combination of the gradients of the active constraints.



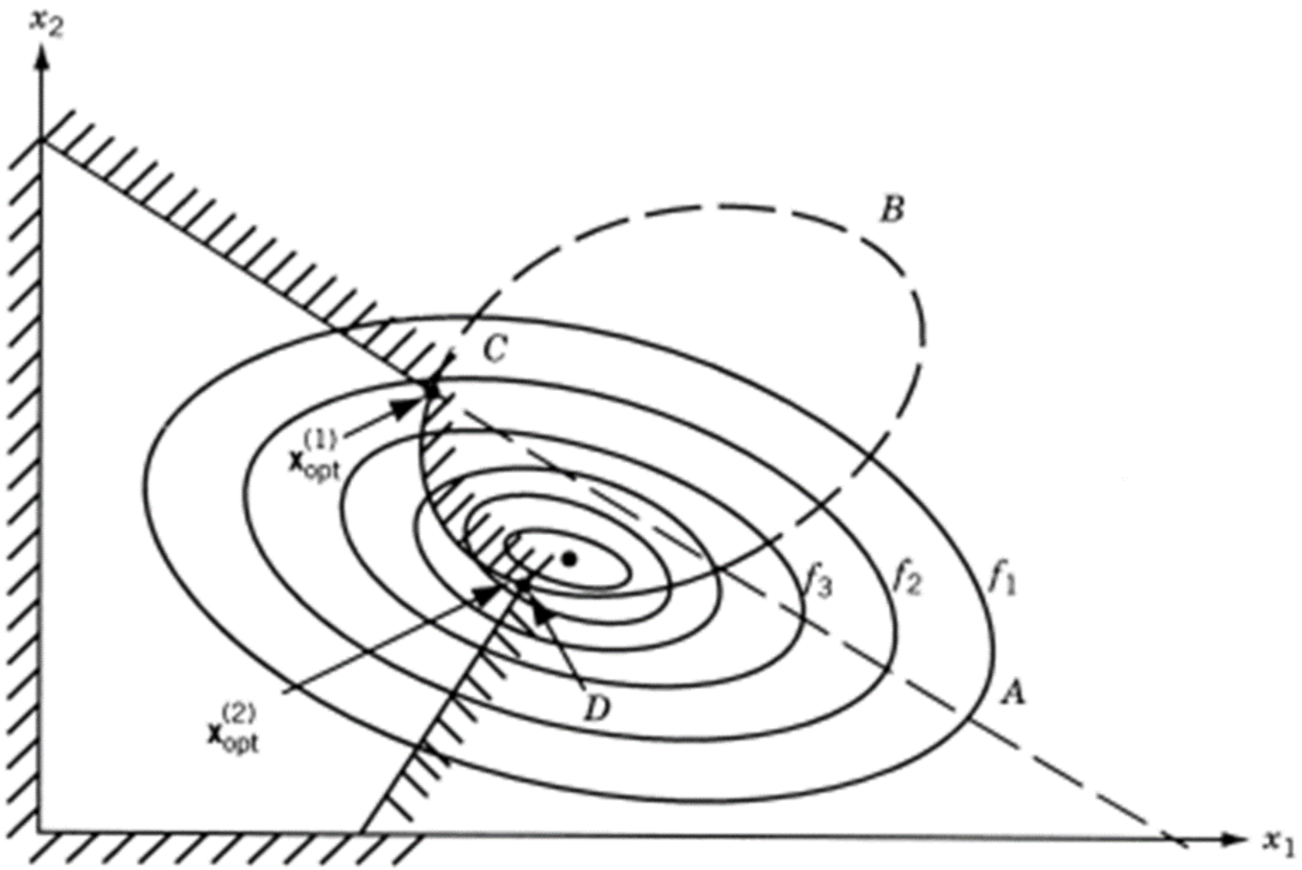
CHARACTERISTIC OF CONSTRAINED PROBLEM

The objective function may have two or more unconstrained local minima



CHARACTERISTIC OF CONSTRAINED PROBLEM

In some cases, even if the objective function has a single unconstrained minimum, the constraints may introduce multiple local minima.



OPTIMIZATION WITH EQUALITY CONSTRAINTS

Minimize the nonlinear n-variable function $f(X)$ subject to

$h_k(X)=0, \quad k=1,2,\ldots,p$

- If $p=n-1$, the problem could be transformed through substitution into a single variable function optimization.

e.g. min $f(X)=\sin x+\cos y$ subject to $2x-y=0$

The problem is equivalent to

- min $\sin x+\cos 2x$
- $y=2x$

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LAGRANGE MULTIPLIER METHOD

Minimize the nonlinear n-variable function $f(X)$ subject to

$h_j(X)=0, \quad j=1,2,\ldots,p\ ,\ p<n$

The **Lagrangian** corresponding to the above equations is given by:

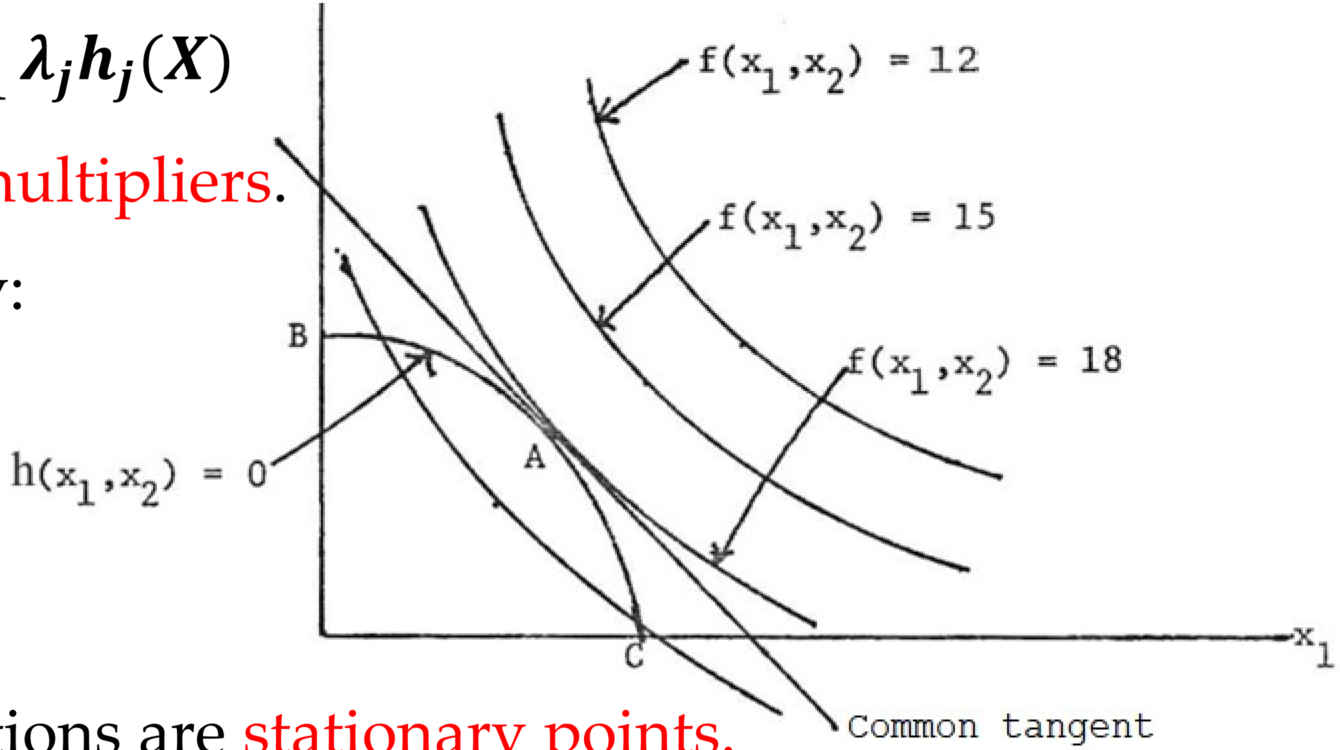
$$L(X,\lambda)=f(X)+\sum_{j=1}^{p}\lambda_j h_j(X)$$

where λ_j, j=1,2,…,p are the Lagrange multipliers.

The necessary conditions of optimality:

$$\frac{\partial L}{\partial x_i}=0$$

$$\frac{\partial L}{\partial \lambda_j}=h_j(X)=0$$



Points obtained by solving these equations are stationary points.

OPTIMIZATION WITH INEQUALITY CONSTRAINTS

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$g_j(X) \leq 0, \quad j=1,2,\ldots,m$

$h_k(X)=0, \quad k=1,2,\ldots,p$

The **Lagrangian** corresponding to the above equations is given by:

$$L(X, \lambda)=f(X)+\sum_{j=1}^{m} \mu_j g_j(X) + \sum_{k=1}^{p} \lambda_k h_k(X) = f(X) + \mu^T g(X) + \lambda^T h(X)$$

OPTIMIZATION WITH INEQUALITY CONSTRAINTS

Karush Kuhn Tucker (KKT) Optimality Conditions

$\nabla f(X) + \mu^T \nabla g(X) + \lambda^T \nabla h(X)) = 0$ *Stationarity condition*

$$\left.\begin{array}{ll} g_j(X) \le 0, & j=1,2,\ldots,m \\ h_k(X) = 0, & k=1,2,\ldots,p \end{array}\right\}$$ *primal feasibility*

$\mu_j \ge 0, \quad j=1,2,\ldots,m$ *Dual feasibility*

$\mu^T g(X) = 0$ *Complementary slackness condition*