1. (i) Let X be a discrete random variable with PMF $p_X(x)$, and its expectation is defined by $E[X] = \Sigma_x \, x\, p(x)$.

Given $g(X)$ be a function of X. Show that the expectation of $g(X)$ can be expressed by

$$E[g(x)] = \Sigma_x \, g(x) \cdot P_X(x)$$

$$\Rightarrow \quad E[g(x)] = E[Y] = \Sigma_y \, y \cdot P_Y(y)$$

$$= \sum_y y \sum_{\{x | g(x)=y\}} P_X(X) = \sum_y \sum_{\{x | g(x)=y\}} g(x) P_X(X)$$

$$= \sum_x g(x) P_X(x) \quad \#$$

(ii) Let X and Y be two independent discrete random variables

Show that $E[XY] = E[X]\,E[Y]$

$$\Rightarrow \quad E[XY] = \sum_x \sum_y x \cdot y \cdot P_{xy}(x, y)$$

$$= \sum_x \sum_y x \cdot y \; P_x(X)\, P_y(Y) = \left[\sum_x x\, P_x(x)\right]\left[\sum_y y\, P_y(y)\right] = E[X]\,E[Y] \quad \#$$

(iii) If $E[X] = 2$ and $var(X) = 4$, find $E[(X+2)^2]$ and $var(3X+5)$

$$\Rightarrow \quad E[(X+2)^2] = E[X^2 + 4X + 4] = E[X^2] + 4 \cdot E[X] + 4$$

$$= \left[var(X) + E[X]^2\right] + 4 \cdot E[X] + 4$$

$$= [4 + 4] + 4 \times 2 + 4 = 20 \quad \#$$

$$\Rightarrow \quad var(3X+5) = 9 \times var(X) = 36$$

2020. quiz 2

2. An experiment consists of 5 independent tosses of a fair coin.
Let random variable X model the number of heads obtained.
Given that a random variable Y is defined by $Y = X \bmod 2$,

Find $E[Y]$ and $var(Y)$, given that $var(Y)$ is defined by $E[(Y-E[Y])^2]$

$$P_X(x) = \begin{cases} C^5_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, & x \le 5 \\ 0 \end{cases}$$

$$P_Y(y) = \begin{cases} C^5_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 + C^5_3 \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + C^5_5 \left(\frac{1}{2}\right)^5 = \frac{5}{32} + \frac{10}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}, & y=1 \\ C^5_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + C^5_4 \left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + C^5_0 \left(\frac{1}{2}\right)^5 = \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}, & y=0 \end{cases}$$

$\Rightarrow E[Y] = 1 \times \frac{1}{2} + 0 \times \frac{1}{2} = \frac{1}{2}$ #

$\Rightarrow var(Y) = E[Y^2] - E[Y]^2 = \left(1^2 \times \frac{1}{2} + 0 \times \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ #

3. Our probability class has 86 students and each student has probability of $4/5$ to pass. (independently)
If the number of students passing in the class is modeled as binomial random variable X,
What is $E[X]$ and $var(X)$

$\because$ Binomial $\therefore E[X] = np$, $var(X) = np(1-p)$

$\Rightarrow p = \frac{4}{5}$, $n = 86$

$\Rightarrow E[X] = \frac{4}{5} \times 86 = 68.8$ #

$\Rightarrow var(X) = 86 \times \frac{4}{5} \times \frac{1}{5} = 13.76$ #

4. Let X and Y be two discrete random variables whose joint PMF is given. A new random variable Z is defined by $Z = X^2 + Y$.

(i) Calculate $E[X \mid X \geq 2 \cap Y \geq 3]$

(ii) Determine whether X and Y are dependent or independent

(iii) Calculate $E[Z]$

joint PMF $P_{xy}(x,y)$

Y \ X	1	2	3
4	2/20	1/20	2/20
3	1/20	2/20	1/20
2	1/20	1/20	2/20
1	2/20	2/20	3/20

(i)

$$E[X \mid X \geq 2 \cap Y \geq 3] = 2 \times \frac{3}{6} + 3 \times \frac{3}{6} = \frac{5}{2} \text{ ※}$$

(ii) If X, Y are independent, $P_{xy}(x,y) = P_x(x)\,P_y(y)$

$\Rightarrow (x,y) = (2,2)$ $\quad P_{xy}(2,2) = 1/20 \neq P_x(2)\,P_y(2) = \frac{6}{20} \times \frac{4}{20} = \frac{3}{50}$

$\Rightarrow X, Y$ are dependent

(iii)

$$P_Z(z) \begin{cases} 2/20 & , z=2 \\ 1/20 & , z=3 \\ 1/20 & , z=4 \\ 2/20 + 2/20 & , z=5 \\ 1/20 & , z=6 \\ 2/20 & , z=7 \\ 1/20 & , z=8 \\ 3/20 & , z=10 \\ 2/20 & , z=11 \\ 1/20 & , z=12 \\ 2/20 & , z=13 \end{cases}$$

$$E[Z] = \frac{4}{20} + \frac{3}{20} + \frac{4}{20} + \frac{20}{20} + \frac{6}{20} + \frac{14}{20} + \frac{8}{20} + \frac{30}{20} + \frac{22}{20} + \frac{12}{20} + \frac{26}{20}$$

$$= \frac{149}{20} \text{ ※}$$

5. We roll two fair 4-sided dice. Each 16 possible outcome is assumed to be equally likely. Let X, Y denote, respectively, the largest and smallest value obtained from each roll

(i) Compute the conditional PMF of Y given X = 1, 2, 3, 4

$$P_X(x) \begin{cases} \frac{1}{16}, & x=1 \\ \frac{3}{16}, & x=2 \\ \frac{5}{16}, & x=3 \\ \frac{7}{16}, & x=4 \end{cases} \Rightarrow P_X(x) = \frac{(2x-1)}{16}, \quad P_Y(y) \begin{cases} \frac{7}{16}, & y=1 \\ \frac{5}{16}, & y=2 \\ \frac{3}{16}, & y=3 \\ \frac{1}{16}, & y=4 \end{cases} \Rightarrow P_Y(y) = \frac{9-2y}{16}$$

$$P_{Y|X}(y|x) \begin{cases} \frac{1}{2x-1}, & y=x \\ \frac{2}{2x-1}, & y<x \end{cases}$$

(ii) Are X, Y independent of each other?

$\forall y$, $P_Y(y) \neq P_{Y|X}(y|x)$, where $y<x$, so x, y are dependent.