HW 2020-5

1. (a) Prove that if λ is an eigenvalue of matrix A, then λ^2 is an eigenvalue of A^2

If v is an eigenvector of A with corresponding eigenvalue λ, then $Av = \lambda v$

$\Rightarrow A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$

Hence λ^2 is an eigenvalue of A^2

(b) An $n \times n$ matrix A is called nilpotent if, for some positive integer k, $A^k = O$, where O is an $n \times n$ zero matrix.

Prove that 0 is the only eigenvalue of a nilpotent matrix

Suppose that λ is an eigenvalue of a nilpotent matrix A.

Then $Av = \lambda v$, hence $(A - \lambda I_n)v = 0$ for some $v \neq 0$

$\Rightarrow$ we have $A^{k-1}(A - \lambda I_n)v = A^{k-1} 0 = 0$, hence $A^k v = \lambda A^{k-1} v = \lambda^k \cdot v$

$\Rightarrow$ Because $A^k = O$ for some k, $0 = A^k v = \lambda^k v$

$\Rightarrow$ Since $v \neq 0$, we must have $\lambda = 0$

2. Find the eigenvalues of linear operator T and determine a basis for each eigenspace

where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 7x_1 - 10x_2 \\ 5x_1 - 8x_2 \\ -x_1 + x_2 + 2x_3 \end{bmatrix}$

standard matrix $A = \begin{bmatrix} 7 & -10 & 0 \\ 5 & -8 & 0 \\ -1 & 1 & 2 \end{bmatrix}$, $\det(A - \lambda I_3) = \det \begin{bmatrix} 7-\lambda & -10 & 0 \\ 5 & -8-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{bmatrix} = -(\lambda+3)(\lambda-2)^2$

so the eigenvalues of T is -3, 2

① $(A + 3I_3) = \begin{bmatrix} 10 & -10 & 0 \\ 5 & -5 & 0 \\ -1 & 1 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, nullity $= 1$.

② $(A - 2I_3) = \begin{bmatrix} 5 & -10 & 0 \\ 5 & -10 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, nullity $= 2$.

$\Rightarrow$ a basis for the eigenspace corresponding to $\begin{cases} -3 \\ 2 \end{cases}$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$, $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ #

3. Let A and B be $n \times n$ matrices such that $B = P^{-1}AP$
and let λ be the eigenvalue of A and B

(a) Prove that a vector v in $\mathbb{R}^n$ is in the eigenspace of A corresponding to λ if and only if, $P^{-1}v$ is in the eigenspace of B corresponding to λ.

$\Rightarrow$ Suppose v is in the eigenspace of A corresponding to eigenvalue λ
Then $B(P^{-1}v) = P^{-1}AP(P^{-1}v) = P^{-1}Av = P^{-1}(\lambda v) = \lambda(P^{-1}v)$
Hence $P^{-1}v$ is in the eigenspace of B corresponding to eigenvalue λ

$\Rightarrow$ Conversely,
Suppose $P^{-1}v$ is in the eigenspace of B corresponding to eigenvalue λ
Then $Av = PBP^{-1}v = P(\lambda P^{-1}v) = \lambda(PP^{-1}v) = \lambda v$
Hence v is in the eigenspace of A corresponding to eigenvalue λ

(b) If $\{v_1\ v_2\ \dots\ v_k\}$ is a basis for the eigenspace of A corresponding to λ then $\{P^{-1}v_1\ P^{-1}v_2\ \dots\ P^{-1}v_k\}$ is a basis for the eigenspace of B corresponding to λ.

① $\because$ (a) and $\{v_1\ v_2\ \dots\ v_k\}$ is a basis for the eigenspace of A corresponding to λ
$\therefore$ for any v_i, $i = 1, 2, \dots k$, $P^{-1}v_i$ is in the eigenspace of B corresponding to λ

② Suppose that $c_1P^{-1}v_1 + c_2P^{-1}v_2 + \dots + c_kP^{-1}v_k = 0$, for some scalar $c_1 \sim c_k$

$\times P \Rightarrow c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$.

Thus $c_1 = c_2 = \dots = c_k = 0$, because $\{v_1\ v_2\ \cdots\ v_k\}$ is linearly independent.
So, $\{P^{-1}v_1, P^{-1}v_2 \dots P^{-1}v_k\}$ is a linearly independent subset of eigenspace B & eigenvalue λ

③ Let w be any vector in the eigenspace of B corresponding to λ
Since $w = P^{-1}Pw$,
follows (a), that Pw is in the eigenspace of A corresponding to λ
So, there exist scalars $d_1 \sim d_k$ such that $Pw = d_1v_1 + d_2v_2 + \dots + d_kv_k$
Hence, $w = d_1P^{-1}v_1 + d_2P^{-1}v_2 + \dots + d_kP^{-1}v_k$.
Thus, $\{P^{-1}v_1\ P^{-1}v_2\ \dots\ P^{-1}v_k\}$ is a generating set for the eigenspace B & eigenvalue λ.

(c) The eigenspaces of A and B that correspond to the same eigenvalue have same dimension.

by (b),
the eigenspaces of A and B corresponding to λ have bases with equal numbers of vectors
hence, the dimensions of those two subspaces are equal.

4. Find, if possible, an invertible matrix P and diagonal matrix D, such that $A = PDP^{-1}$

where $A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Since A is a upper triangular, its eigenvalue are its diagonal entries $-1, -3, 2$

$\lambda = -1$, $(A + I_3) = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, corresponding eigenspace $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\lambda = -3$, $(A + 3I_3) = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, corresponding eigenspace $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 2$, $(A - 2I_3) = \begin{bmatrix} -3 & 2 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{-1}{5} \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, corresponding eigenspace $\left\{ \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \right\}$

Hence, $P = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ #

5. Find a 3x3 matrix having eigenvalues $-1, 2, 3$ with corresponding eigenvector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

The matrix $A = PDP^{-1}$

where $P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -1 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

(P^{-1})

$$A = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -3 \\ -1 & 2 & 3 \\ -1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -4 \\ 1 & -4 & 4 \\ 2 & 6 & 7 \end{bmatrix} \#$$

$$= \begin{bmatrix} 1 & 6 & -4 \\ 1 & -4 & 4 \\ 2 & -6 & 7 \end{bmatrix} \#$$

6. Let A be a diagonalizable $n\times n$ matrix.
Prove that if the characteristic polynomial of A is $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ then $f(A) = 0$, where

$$f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n \quad \text{(Cayley-Hamilton theorem)}$$

Let λ be an eigenvalue of A, then $f(\lambda) = 0$

Let $A = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$, Then $f(D) = \begin{bmatrix} f(\lambda_1) & 0 & 0 & 0 \\ 0 & f(\lambda_2) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda_n) \end{bmatrix} = 0$

Hence, $f(A) = f(PDP^{-1}) = P f(D) P^{-1} = POP^{-1} = 0$ #

7. A linear operator T on $\mathbb{R}^n$ is given in following
Find, if possible, a basis $\mathcal{B}$ for $\mathbb{R}^n$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Let $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 3x_1 - x_2 + 3x_3 \\ 3x_1 + 2x_3 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$ (standard matrix), $(\lambda+1)^2(2-\lambda)$ ... (characteristic polynomial)

eigenvalue -1, has $\mathcal{B}_1 = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$; eigenvalue 2, has $\mathcal{B}_2 = \left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} \Rightarrow \mathcal{B} = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ #

8. The equation of a plane W through the origin of $\mathbb{R}^3$ is $x+y+z=0$
Determine an explict formula for the reflection T_W of $\mathbb{R}^3$ about W

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (general solution) $\Rightarrow \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for W,

with eigenspace T_W, eigenvalue 1

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the coefficient of the equation $x+y+z=0$, is orthogonal to W

so is an eigenvector of T_W corresponding to eigenvalue -1

Thus $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$, $[T_W]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$A = \mathcal{B}[T_W]_{\mathcal{B}}\mathcal{B}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$T_W\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x_1 - 2x_2 - 2x_3 \\ -2x_1 + x_2 - 2x_3 \\ -2x_1 - 2x_2 + x_3 \end{bmatrix}$$ #