HW 2020-4

1. For the matrix $A = \begin{bmatrix} -1 & 2 & 1 & -1 & -2 \\ 2 & -4 & 1 & 5 & 7 \\ 2 & -4 & -3 & 1 & 3 \end{bmatrix}$, determine the dimension of

(a) Col $A$ (b) Null $A$ (c) Row $A$ (d) Null $A^T$

$$A = \begin{bmatrix} -1 & 2 & 1 & -1 & -2 \\ 2 & -4 & 1 & 5 & 7 \\ 2 & -4 & -3 & 1 & 3 \end{bmatrix} \xrightarrow[r_3+2r_1 \to r_3]{r_2+2r_1 \to r_2} \begin{bmatrix} -1 & 2 & 1 & -1 & -2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow[\frac{1}{3}r_3 \to r_3]{r_3+\frac{1}{3}r_2 \to r_2} \begin{bmatrix} -1 & 2 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Col $A$ = 2 Null $A = 5-2 = 3$

Row $A$ = 2 Null $A^T = 3-2 = 1$ #

2. For the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$, determine

(a) the dimension of the range of $T$
(b) the dimension of the null space of $T$
(c) whether $T$ is one-to-one or on-to

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[r_3+r_1 \to r_3]{r_2+r_1 \to r_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1+r_2 \to r_1} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[r_2-2r_3 \to r_2]{-r_1+3r_3 \to r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) dim Range $T$ = 3 #

(b) dim Null Space = $3-3 = 0$ #

(c)
① $T$ is one-to-one.
($\because$ the nullity of the function is zero)
$T$ is on-to
($\because$ the number of row is equal to the rank)

3. Find the unique representation of $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as linear combination of $b_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ -1 & 2 & 0 & b \\ 1 & 1 & 2 & c \end{array}\right] \xrightarrow[r_3-r_1 \to r_3]{r_2+r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 1 & a+b \\ 0 & 2 & 1 & -a+c \end{array}\right] \xrightarrow[r_3-2r_2 \to r_1]{r_1+r_2 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2a+b \\ 0 & 1 & 1 & a+b \\ 0 & 0 & -1 & -3a-2b+c \end{array}\right]$$

$$\begin{array}{l} r_1+2r_3 \to r_1 \\ r_2+r_3 \to r_2 \\ -r_3 \to r_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4a-3b+2c \\ 0 & 1 & 0 & -2a-b+c \\ 0 & 0 & 1 & 3a+2b-c \end{array}\right]$$

$\Rightarrow [\vec{u}]_B = (-4a-3b+2c)\vec{b_1} + (-2a-b+c)\vec{b_2} + (3a+2b-c)\vec{b_3}$ #

4 Let $\mathcal{B} = \{ b_1 , b_2 , b_3 \}$, where $b_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

(a) show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 + 2r_1 \to r_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-r_3 + r_2 \to r_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[r_2 + r_1 \to r_1]{r_2 - 2r_3 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \mathcal{B}$ is the basis for $\mathbb{R}^3$

(b) Determine the matrix $A = \left[ [e_1]_{\mathcal{B}} \ [e_2]_{\mathcal{B}} \ [e_3]_{\mathcal{B}} \right]$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right] \qquad A = \begin{bmatrix} -3 & -1 & 2 \\ -4 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix} \#$$

(C) What is the relationship between $A$ and $\mathcal{B}$

$$(A\mathcal{B} = I_3) == (A = \mathcal{B}^{-1})$$

5. Let $A = \{u_1, u_2, \dots, u_n\}$ be a basis for $\mathbb{R}^n$ and $c_1, c_2, \dots, c_n$ be nonzero scalars. Let $B = \{c_1u_1, c_2u_2, \dots, c_nu_n\}$, which is also a basis for $\mathbb{R}^n$.

If $v$ is a vector in $\mathbb{R}^n$ and $[v]_A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, compute $[v]_B$

$$\Rightarrow \vec{v} = a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} = A\,[v]_A$$

$$= B\,[v]_B = [c_1u_1 \ \ c_2u_2 \ \cdots \ c_nu_n]\,[v]_B$$

Let $[v]_B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

$$\Rightarrow B\,[v]_B = b_1c_1\vec{u_1} + b_2c_2\vec{u_2} + \cdots + b_nc_n\vec{u_n}$$

$$\Rightarrow a_i = c_i b_i \quad \Rightarrow \quad b_i = \frac{a_i}{c_i} \qquad \Rightarrow [v]_B = \begin{bmatrix} \frac{a_1}{c_1} \\ \frac{a_2}{c_2} \\ \vdots \\ \frac{a_n}{c_n} \end{bmatrix}$$

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6. Determine $[T]_B$ for linear operator $T$ and basis $B$,

where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + 4x_3 \\ 3x_1 \\ -3x_2 + 2x_3 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$

$$\Rightarrow T(\vec{b_1}) = T\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 3 \\ 8 \end{bmatrix}, \quad T(\vec{b_2}) = T\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}, \quad T(\vec{b_3}) = T\left(\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 23 \\ 3 \\ 21 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & -5 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & -1 & -5 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1 \to r_3]{r_2 + 2r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-r_3 \to r_3]{r_3 + r_2 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right] \xrightarrow[r_2 + 3r_3 \to r_2]{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

$$\Rightarrow B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\Rightarrow [T(\vec{b_1})]_B = B^{-1}\,T(\vec{b_1}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 29 \\ 39 \\ -20 \end{bmatrix}$$

$$\Rightarrow [T(\vec{b_2})]_B = B^{-1}\,T(\vec{b_2}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 21 \\ -11 \end{bmatrix} \qquad \Rightarrow [T]_B = \begin{bmatrix} 29 & 17 & 70 \\ 39 & 21 & 92 \\ -20 & -11 & -47 \end{bmatrix}$$

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$$\Rightarrow [T(\vec{b_3})]_B = B^{-1}\,T(\vec{b_3}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 23 \\ 3 \\ 21 \end{bmatrix} = \begin{bmatrix} 70 \\ 92 \\ -47 \end{bmatrix}$$

7. Given a basis $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$, $T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\end{bmatrix}$,

$$T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = 4\begin{bmatrix}1\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = -\begin{bmatrix}1\\1\\-1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\end{bmatrix}$$

determine

(a) $[T]_{\mathcal{B}}$ (b) the standard matrix of $T$ (c) an explicit formula for $T(x)$

(a)

$$[T]_{\mathcal{B}} = \begin{bmatrix}0&4&-1\\1&0&3\\2&-1&2\end{bmatrix}, \quad B = \begin{bmatrix}1&0&1\\1&1&2\\-1&1&3\end{bmatrix} \qquad B^{-1} = \begin{bmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{4}{3}&-\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\1&1&2&0&1&0\\-1&1&3&0&0&1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&1&-1&1&0\\0&1&4&1&0&1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&1&-1&1&0\\0&0&3&2&-1&1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}\\0&1&0&-\frac{5}{3}&\frac{4}{3}&-\frac{1}{3}\\0&0&1&\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\end{array}\right]$$

(b)

$$A = B[T]_{\mathcal{B}}B^{-1} = \begin{bmatrix}1&0&1\\1&1&2\\-1&1&3\end{bmatrix}\begin{bmatrix}0&4&-1\\1&0&3\\2&-1&2\end{bmatrix}\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{4}{3}&-\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\end{bmatrix} = \begin{bmatrix}2&3&1\\5&2&6\\7&-7&10\end{bmatrix}\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{4}{3}&-\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{11}{3}&\frac{13}{3}&-\frac{4}{3}\\\frac{7}{3}&\frac{7}{3}&-\frac{1}{3}\\\frac{62}{3}&-\frac{31}{3}&\frac{10}{3}\end{bmatrix} \#$$

(c)

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-\frac{11}{3}x_1 + \frac{13}{3}x_2 - \frac{4}{3}x_3\\ \frac{7}{3}x_1 + \frac{7}{3}x_2 - \frac{1}{3}x_3\\ \frac{62}{3}x_1 - \frac{31}{3}x_2 + \frac{10}{3}x_3\end{bmatrix} \#$$

8. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$, $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$ be basis for $\mathbb{R}^n$, $\mathbb{R}^m$, respectively.

Let $B$ and $C$ be the matrices whose columns are the vectors in $\mathcal{B}$, $\mathcal{C}$.

Prove

(a) If $A$ is the standard matrix of $T$, then $[T]_{\mathcal{B}}^{\mathcal{C}} = C^{-1}AB$

$$\begin{aligned}\Rightarrow [T]_{\mathcal{B}}^{\mathcal{C}} &= \left[\,[T(\vec{b}_1)]_{\mathcal{C}}\ \ [T(\vec{b}_2)]_{\mathcal{C}}\ \cdots\ [T(b_n)]_{\mathcal{C}}\,\right]\\ &= \left[\,C^{-1}T(\vec{b}_1)\ \ C^{-1}T(\vec{b}_2)\ \cdots\ C^{-1}T(\vec{b}_n)\,\right] = C^{-1}\left[\,T(\vec{b}_1)\ \ T(\vec{b}_2)\ \cdots\ T(\vec{b}_n)\,\right]\\ &= C^{-1}\left[\,A\vec{b}_1\ \ A\vec{b}_2\ \cdots\ A\vec{b}_n\,\right]\\ &= C^{-1}A\left[\,\vec{b}_1\ \ \vec{b}_2\ \cdots\ \vec{b}_n\,\right] = C^{-1}AB \ \#\end{aligned}$$

(b) $[T(v)]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}}$ for any vector $v$ in $\mathbb{R}^n$

$$\begin{aligned}\Rightarrow [T(v)]_{\mathcal{C}} &= C^{-1}T(v) = C^{-1}A\vec{v} = C^{-1}AI_n\vec{v} = C^{-1}A(BB^{-1}\vec{v})\\ &= (C^{-1}AB)(B^{-1}\vec{v}) = [T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}} \ \#\end{aligned}$$

(c) Let $U: \mathbb{R}^m \to \mathbb{R}^p$ be linear, and let $\mathcal{D}$ be a basis for $\mathbb{R}^p$.
Then $[UT]_{\mathcal{B}}^{\mathcal{D}} = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}}$

$\Rightarrow$ Let $A$, $E$ be standard matrices of $U$ and $T$, respectively
Notice that $UT$ is defined for the given linear transformations of $U$ and $T$.

$$\begin{aligned}[UT]_{\mathcal{B}}^{\mathcal{D}} &\underset{(a)}{=} D^{-1}(AE)B = D^{-1}AIEB = D^{-1}ACC^{-1}EB\\ &= (D^{-1}AC)(C^{-1}EB) = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{C}} \ \#\end{aligned}$$