1. (i) State the three probability axioms

(a) Nonnegativity : $\forall x$, $P(x) \geq 0$

(b) Additivity : $\forall A, B$ disjoint, $P(A \cup B) = P(A) + P(B)$

(c) Normalization : $P(\Omega) = 1$

(ii) Given that $P(A|D) > P(B|D)$ and $P(A|D^c) > P(B|D^c)$ where A, B, D are three events and D^c is the complement of D. Show that $P(A) > P(B)$

$\Rightarrow$ According to Total Probability Theorem

$$\begin{cases} P(A) = P(A|D) + P(A|D^c) \\ P(B) = P(B|D) + P(B|D^c) \end{cases} \text{ and } \begin{cases} P(A|D) > P(B|D) \\ P(A|D^c) > P(B|D^c) \end{cases}$$

$$\Rightarrow P(A) > P(B) \ \#$$

(iii) Show that $P(A|B) = P(A|B \cap D)P(D|B) + P(A|B \cap D^c)P(D^c|B)$

$$\Rightarrow P(A|B \cap D)P(D|B) = \frac{P(A \cap B \cap D)}{P(B \cap D)} \times \frac{P(D \cap B)}{P(B)} = \frac{P(A \cap B \cap D)}{P(B)}$$

$$P(A|B \cap D^c)P(D^c|B) = \frac{P(A \cap B \cap D^c)}{P(B \cap D^c)} \times \frac{P(D^c \cap B)}{P(B)} = \frac{P(A \cap B \cap D^c)}{P(B)}$$

$$\Rightarrow P(A|B \cap D)P(D|B) + P(A|B \cap D^c)P(D^c|B) = \frac{P(A \cap B \cap D) + P(A \cap B \cap D^c)}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)} = P(A|B) \ \#$$

(iv) Let X and Y be two random variables. Show that $E[X+Y] = E[X] + E[Y]$.

$\Rightarrow$ Let $g(x,y) = x + y$,

$$E[X+Y] = E[g(x,y)] = \sum_x \sum_y g(x,y) P_{xy}(x,y)$$

$$= \sum_x \sum_y (x+y) P_{xy}(x,y) = \sum_x X \sum_y P_{xy}(x,y) + \sum_y Y \sum_x P_{xy}(x,y)$$

$$= \sum_x X P_x(X) + \sum_y Y P_y(Y)$$

$$= E[X] + E[Y] \ \#$$

(1) Given the mean and varience of a descrete random X are defined by

$$E[X] = \sum_x X \cdot P_X(x) \quad \text{and} \quad var(X) = E[(X - E(X))^2]$$

respectively, where $P_X(x)$ is the probability mass function (PMF) of X

Show that $E[X^2] = var(X) + (E[X])^2$

$\Rightarrow$ $var(x) = E[(X - E[x])^2] = \sum_x (x - E[x])^2 P_x(x) = \sum_x (x^2 - 2x\,E[x] + E[x]^2)\,P_x(x)$

$= \sum_x x^2 P_x(x) + (-2)\sum_x x\,P_x(x)\,E[x] + \sum_x P_x(x)\,(E[x]^2)$

$= E[x^2] - 2\,E[x]^2 + E[x]^2$

$= E[x^2] - E[x]^2$ # $\therefore\ E[x^2] = var(x) + E[x]^2$

2. A batch of one 50 light bulbs are to be inspected by testing three random selected items.
If one of the three is defective, the batch is rejected.
What is the probability that the batch is accepted if it contains 10 defective bulbs.

$$\Rightarrow \frac{C^{40}_3}{C^{50}_3} = \frac{\frac{40\cdot39\cdot38}{3\cdot2\cdot1}}{\frac{50\cdot49\cdot48}{3\cdot2\cdot1}} = \frac{40\cdot39\cdot38}{50\cdot49\cdot48} = \frac{4}{5}\times\frac{1}{49}\times\frac{13\times19}{8} = \frac{247}{490}$$ #

3. Prof. Berlin likes to submit papers for publication in autumn seasons.
He tends to submit 0, 1 or 2 papers in autumn with probability of $1/6$, $1/2$, $1/3$
each paper will be accepted with the probability of $2/3$, independently.
What is the probability that Prof. Berlin would have at least 1 paper accepted this year.

$\Rightarrow$ P(one paper and accepted) + P(two paper and at least one accepted)

$\Rightarrow \left(\frac{1}{2}\times\frac{2}{3}\right) + \frac{1}{3}\left[1 - \frac{1}{9}\right] = \frac{1}{3} + \frac{1}{3}\times\frac{8}{9} = \frac{17}{27}$ #

4. Let X and Y be two discrete random variables whose joint PMF is given below

A new random variable Z is defined by $Z = X + Y$
while event B is defined by $B = \{3 \leq X \leq 4 \text{ and } 1 \leq Y \leq 2\}$

Y \ X	1	2	3	4
3	1/15	2/15	1/15	0
2	0	5/15	1/15	1/15
1	2/15	0	1/15	1/15

(i) What are the marginal PMFs of X and Y

$$\Rightarrow P_x(X) = \begin{cases} 3/15 = 1/5, & x=1 \\ 7/15, & x=2 \\ 3/15 = 1/5, & x=3 \\ 2/15, & x=4 \end{cases} \qquad P_y(Y) = \begin{cases} 4/15, & y=1 \\ 7/15, & y=2 \\ 4/15, & y=3 \end{cases}$$

#

(ii) Calculate $E[z]$

$$\Rightarrow P_z(z) = \begin{cases} 2/15, & z=2 \\ 0, & z=3 \\ 7/15, & z=4 \\ 4/15, & z=5 \\ 2/15, & z=6 \\ 0, & z=7 \end{cases}$$

$$E[z] = 2 \times \frac{2}{15} + 4 \times \frac{7}{15} + 5 \times \frac{4}{15} + 6 \times \frac{2}{15}$$

$$= \frac{4}{15} + \frac{28}{15} + \frac{20}{15} + \frac{12}{15} = \frac{64}{15} \#$$

(iii) Given the event B occurs, determine whether X and Y are dependent or independent

Known that B occurs, if X, Y are independent $\Rightarrow P_{x,y|B}(x,y) = P_{x|B}(x)\, P_{y|B}(y)$

$$(x,y) \begin{cases} (3,1) & \Rightarrow P_{xy|B}(3,1) = 1/4, \quad P_{x|B}(x)\, P_{y|B}(y) = \frac{2}{4} \times \frac{2}{4} = 1/4 \\ (3,2) & \Rightarrow P_{xy|B}(3,2) = 1/4, \quad P_{x|B}(x)\, P_{y|B}(y) = \frac{2}{4} \times \frac{2}{4} = 1/4 \\ (4,1) & \Rightarrow P_{xy|B}(4,1) = 1/4, \quad P_{x|B}(x)\, P_{y|B}(y) = \frac{2}{4} \times \frac{2}{4} = 1/4 \\ (4,2) & \Rightarrow P_{xy|B}(4,2) = 1/4, \quad P_{x|B}(x)\, P_{y|B}(y) = \frac{2}{4} \times \frac{2}{4} = 1/4 \end{cases}$$

$\Rightarrow$ X, Y are independent in the condition of B #

(iv) Calculate $var(Z|B)$

$$\Rightarrow var(Z|B) = E[Z^2|B] - (E[Z|B])^2$$

$$= \left(16 \times \frac{1}{4} + 25 \times \left(\frac{1}{4} + \frac{1}{4}\right) + 36 \times \frac{1}{4}\right) - \left(4 \times \frac{1}{4} + 5 \times \left(\frac{1}{4} + \frac{1}{4}\right) + 6 \times \frac{1}{4}\right)^2$$

$$= \left(4 + \frac{25}{2} + 9\right) - \left(1 + \frac{5}{2} + \frac{3}{2}\right)^2$$

$$= 13 + \frac{25}{2} - 25 = 12.5 - 12 = 1/2 \#$$

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5. The expected number of typographical errors on a page of Allen's academic paper is 5. What is the probability that the next page of the paper you're going to read

contains (i) no typographical error.
(ii) 3 typographical error.

$\Rightarrow E[X] = \lambda = 5$

(i) $P_X(0) = e^{-5} \times \frac{5^0}{0!} = e^{-5}$ #

(ii) $P_X(3) = e^{-3} \times \frac{5^3}{6} = \frac{125}{6} e^{-5}$ #

✱ PMF is defined by $P_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$

✱ the expectation of Poisson random variable is $E[X] = \lambda$