1. Given the random variables X and Y whose joint PMF $p_{xy}(x,y)$ is given below. and a new random variable Z is defined by $Z = X + Y^2$

Y \ X	1	2	3	4
3	2/12	0	2/12	1/12
2	1/12	1/12	1/12	0
1	0	1/12	1/12	2/12

(i) Calculate $E[Z]$ & $var(Z)$

$E[X] = \frac{3}{12}\times 1 + \frac{2}{12}\times 2 + \frac{4}{12}\times 3 + \frac{3}{12}\times 4 = \frac{31}{12}$

$E[Y^2] = \frac{4}{12}\times 1 + \frac{3}{12}\times 4 + \frac{5}{12}\times 9 = \frac{61}{12}$

$E[Z] = E[X] + E[Y^2] = \frac{92}{12} = \frac{23}{3}$

$E[X^2] = \frac{3}{12}\times 1 + \frac{2}{12}\times 4 + \frac{4}{12}\times 9 + \frac{3}{12}\times 16 = \frac{95}{12}$

$E[Y^4] = \frac{4}{12}\times 1 + \frac{3}{12}\times 16 + \frac{5}{12}\times 81 = \frac{457}{12}$

$E[Z^2] = E[X^2] + 2E[X]E[Y^2] + E[Y^4] = \frac{95}{12} + 2\times\frac{31}{12}\times\frac{61}{12} + \frac{457}{12} = \frac{5203}{72}$

$var(Z) = E[Z^2] - E[Z]^2 = \frac{5203}{72} - \frac{529}{9} = \frac{971}{72}$ #

(ii) Find the conditional PMFs of $P_{y|x}(y|3)$, $P_{x|y}(x|2)$

$$P_{y|x}(Y|3) = \begin{cases} \frac{1}{4}, & (x,y) = (3,2) \text{ or } (3,3) \\ \frac{1}{2}, & (x,y) = (3,1) \end{cases} \qquad P_{x|y}(X|2) = \begin{cases} \frac{1}{3}, & (x,y) = (1,2) \text{ or } (2,2) \text{ or } (3,2) \\ 0, & (x,y) = (4,2), \text{ otherwise} \end{cases}$$

(iii) Calculate $E[X]$ by way of $E[X] = \sum_y P_y(y)\,[X|Y=y]$

$E[X] = \sum_y P_y(y)\, E[x|Y=y]$

$= P_y(1)\, E[x|Y=1] + P_y(2)\, E[x|Y=2] + P_y(3)\, E[x|Y=3]$

$= \frac{4}{12}\times\frac{13}{4} + \frac{3}{12}\times 2 + \frac{5}{12}\times\frac{12}{5} = \frac{31}{12}$ #

$E[x|Y=1] = 1\times 0 + 2\times\frac{1}{12} + 3\times\frac{1}{12} + 4\times\frac{2}{12} = \frac{13}{4}$

$E[x|Y=2] = 1\times\frac{1}{12} + 2\times\frac{1}{12} + 3\times\frac{1}{12} + 4\times 0 = 2$

$E[x|Y=3] = 1\times\frac{2}{12} + 2\times 0 + 3\times\frac{2}{12} + 4\times\frac{1}{12} = \frac{12}{5}$

(iv) Suppose that A is an event expressed by $A = [1 \le X \le 2 \text{ and } 2 \le Y \le 3]$ determine where X and Y are independent of each other given A.

$\Rightarrow$ If independent $P_{xy}(1,2) = P_x(1)\,P_y(2) \Rightarrow \frac{1}{4} = \frac{1}{2}\times\frac{1}{2}$

$P_{xy}(1,3) = P_x(1)\,P_y(3) \Rightarrow \frac{1}{4} = \frac{1}{2}\times\frac{1}{2}$

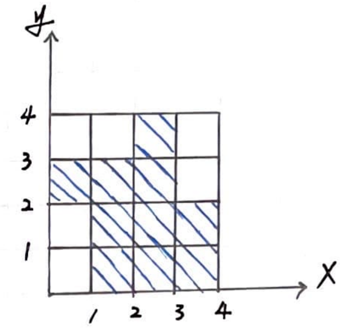
$P_{xy}(2,2) = P_x(2)\,P_y(2) \Rightarrow \frac{1}{4} = \frac{1}{2}\times\frac{1}{2}$

$P_{xy}(2,3) = P_x(2)\,P_y(3) \Rightarrow \frac{1}{4} = \frac{1}{2}\times\frac{1}{2}$ $\Rightarrow$ it's independent #

$(P_x(1) = \frac{1}{2},\ P_x(2) = \frac{1}{2},\ P_y(2) = \frac{1}{2},\ P_y(3) = \frac{1}{2})$

2. We are told that the joint PDF $f_{x,y}(x,y)$ of continuous random variables X and Y is constant (uniform) in the "shaded" area of the figure shown below.

(i) Find the PDF $f_X(x)$ of X



$$P_x(X) = \begin{cases} \frac{1}{10}, & 0 \leq x \leq 1 \\ \frac{3}{10}, & 1 \leq x \leq 2 \\ \frac{4}{10}, & 2 \leq x \leq 3 \\ \frac{1}{10}, & 3 \leq x \leq 4 \end{cases}$$

#

(ii) Find the PDF $f_y(Y)$ of Y

$$P_y(Y) = \begin{cases} \frac{3}{10}, & 0 \leq y \leq 1 \\ \frac{3}{10}, & 1 \leq y \leq 2 \\ \frac{3}{10}, & 2 \leq y \leq 3 \\ \frac{1}{10}, & 3 \leq y \leq 4 \end{cases}$$

#

(iii) Find the variance of Y

$$E[X] = \frac{1}{10} \times 1 + \frac{3}{10} \times 2 + \frac{4}{10} \times 3 + \frac{2}{10} \times 4 = (1+6+12+8)/10 = \frac{27}{10}$$

$$E[X^2] = \frac{1}{10} \times 1 + \frac{3}{10} \times 4 + \frac{4}{10} \times 3 + \frac{2}{10} \times 16 = (1+12+36+32)/10 = \frac{81}{10}$$

$$Var(x) = E[X^2] - E[X]^2 = \frac{810}{100} - \frac{729}{100} = \frac{81}{100}$$ #

(iv) Are X and Y independent or dependent of each other ?

Ans. dependent #

3. We are told that X is a normal distribution with mean 20 and variance 100

(i) Find the probability that the value of X is in the interval $[10, 50]$ given that $\phi(1) = 0.8413$, $\phi(1.5) = 0.9332$, $\phi(2) = 0.9772$, $\phi(3) = 0.9987$

$$\Rightarrow P(10 \leq x \leq 50) = P(x \leq 50) - P(x \leq 10)$$

$$= P\left(\frac{x-20}{10} \leq \frac{50-20}{10}\right) - P\left(\frac{x-20}{10} \leq \frac{10-20}{10}\right)$$

$$= P\left(\frac{x-20}{10} \leq 3\right) - P\left(\frac{x-20}{10} \leq -1\right)$$

$$= \phi(3) - [1 - \phi(1)]$$

$$= 0.9987 - 1 + 0.8413 =$$

(ii) Find the mean and variance of W, that has the relation $W = 2X + 5$. Is W normal

$\Rightarrow W = 2X + 5$, $\mu_W = 2 \times 20 + 5 = 45$

$\Rightarrow \sigma_W^2 = 100 \times 2^2 = 400 \Rightarrow \sigma_W = 20$

$$\Rightarrow \begin{cases} \text{mean} = 45 \\ \text{variance} = 400 \end{cases} ※$$

Ans: W is normal. $\because X$ is normal

$\therefore$ The linear transformation of X is also normal.

4. Let $X = \min\{X_1, 2X_2, 3X_3\}$, where X_1, X_2, X_3 are three independent discrete random variable. all of which may take values 1, 2, 3 (with the each probability are equal)

Find the PMF of X.

x_1	x_2	x_3
1		
2	2	
3		3
	4	
	6	6
		9

$$P_x(9) = 0$$
$$P_x(6) = 0$$
$$P_x(4) = 0$$
$$P_x(3) = \frac{1}{3} \times \frac{2}{3} \times 1 = \frac{2}{9}$$
$$P_x(2) = \left(\frac{1}{3} \times \frac{2}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{1}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{1}{3} \times 1\right) = \frac{4}{9}$$
$$P_x(1) = 1 \times 1 \times \frac{1}{3} = \frac{1}{3}$$

$$P_x(x) = \begin{cases} \frac{2}{9}, & x = 3 \\ \frac{4}{9}, & x = 2 \\ \frac{1}{3}, & x = 3 \end{cases}$$