1. The probability density function (PDF) of a continuous random variable X is given

$$f_X(x) = \begin{cases} 1 + ax + bx^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E[x] = 2$, find a, b

$$\int_0^1 f_X(x)\, dx = \int_0^1 (1+ax+bx^2)\, dx = x + \frac{ax^2}{2} + \frac{bx^3}{3}\Big|_0^1 = 1 + \frac{a}{2} + \frac{b}{3} - 0 = 1$$

(normalization property)

$$\Rightarrow \frac{a}{2} + \frac{b}{3} = 0 \quad \dots ①$$

$$E[X] = \int_0^1 x\, f_X(x)\, dx = \int_0^1 (x + ax^2 + bx^3)\, dx$$

$$= \frac{x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}\Big|_0^1 = \frac{1}{2} + \frac{a}{3} + \frac{b}{4} - 0 = 2$$

$$\Rightarrow \frac{a}{3} + \frac{b}{4} = \frac{3}{2} \quad \dots ②$$

$$\begin{cases} \frac{a}{2} + \frac{b}{3} = 0 \\ \frac{a}{3} + \frac{b}{4} = \frac{3}{2} \end{cases} \Rightarrow \begin{cases} 3a + 2b = 0 \\ 4a + 3b = 18 \end{cases} \Rightarrow \begin{cases} 12a + 8b = 0 \\ 12a + 9b = 54 \end{cases} \Rightarrow \begin{array}{l} a = -36 \\ b = 54 \ \# \end{array}$$

2. We are told that X is a normal distribution with mean 20 and variance 100

(i) Find the probability that the value of X is in the interval $[5, 30]$
Given that a few CDF values of a standard normal are given as follows:

$\Phi(0) = 0.5$, $\Phi(1) = 0.6915$, $\Phi(1.5) = 0.9332$, $\Phi(2.0) = 0.9772$, $\Phi(2.5) = 0.9938$, $\Phi(3.0) = 0.9987$

$$\Rightarrow P(5 \leq X \leq 30) = P(x \leq 30) - P(x \leq 5)$$

$$= P\left(\frac{X-20}{10} \leq \frac{30-20}{10}\right) - P\left(\frac{X-20}{10} \leq \frac{5-20}{10}\right) \quad (\because \mu = 20,\ \sigma = \sqrt{100} = 10)$$

$$= P\left(\frac{X-20}{10} \leq 1\right) - P\left(\frac{X-20}{10} \leq -1.5\right)$$

$$= \Phi(1) - \Phi(-1.5) = \Phi(1) - (1 - \Phi(1.5)) = 0.6915 - 1 + 0.9332 = 0.6247 \ \#$$

(ii) Find the mean and variance of W, that has the relation $W = 3X + 2$. Is W normal?

$W = 3X + 2$, $\mu_W = 3 \times 20 + 2 = 62$, $\sigma_W^2 = 900$, $\sigma_W = 30$

W: mean $= 62$, variance $= 900$

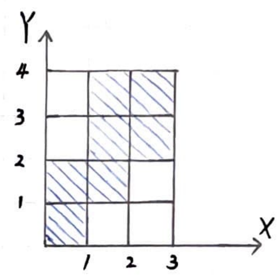
Yes, W is normal. $\because$ X is normal
$\therefore$ The linear transformation of X is also normal.

3. You are allowed to play bowling three times independently, each time your bowling is discrete and take values from the range 111 to 115 ($P = 0.2$). Your final score will be the maximum of the three score and is modeled with X. Calculate the probability mass function (PMF) of X.

$$f_X(111) = (0.2)^3 = 0.008$$
$$f_X(112) = C^3_3(0.2)^3 + C^3_2(0.2)^2(0.2) + C^3_1(0.2)(0.2)^2 = (1+3+3)(0.2)^3 = 0.056$$
$$f_X(113) = C^3_3(0.2)^3 + C^3_2(0.2)^2(0.4) + C^3_1(0.2)(0.4)^2 = 0.152$$
$$f_X(114) = C^3_3(0.2)^3 + C^3_2(0.2)^2(0.6) + C^3_1(0.2)(0.6)^2 = 0.296$$
$$f_X(115) = C^3_3(0.2)^3 + C^3_2(0.2)^2(0.8) + C^3_1(0.2)(0.8)^2 = 0.488$$

$$f_X(x) = \begin{cases} 0.008, & x = 111 \\ 0.056, & x = 112 \\ 0.152, & x = 113 \\ 0.296, & x = 114 \\ 0.488, & x = 115 \end{cases} \quad ※$$

4. We are told that the joint PDF of random variables X, Y is a constant in "shaded" area



(i) Find the PDF $f_Y(Y)$ of Y

$$P_Y(Y) = \begin{cases} \frac{1}{7}, & 0 \le y \le 1 \\ \frac{2}{7}, & 1 \le y \le 1 \\ \frac{2}{7}, & 2 \le y \le 3 \\ \frac{2}{7}, & 3 \le y \le 4 \end{cases}$$

(ii) Find the conditional PDF $f_{Y|X}(y\,|\,2.5)$ of Y

$$f_{Y|X}(y\,|\,2.5) = \frac{1/7}{2/7} = \frac{1}{2} \quad ※$$

(iii) Find the conditional expectation $E[Y|X = 2.5]$ of Y

$$\int_2^4 y\, f_{Y|X}(y|2.5)\,dy = \int_2^4 \frac{1}{2} y\,dy = \frac{1}{2}\cdot\frac{y^2}{2}\Big|_2^4 = \frac{y^2}{4}\Big|_2^4 = \frac{16}{4} - \frac{4}{4} = 3 \quad ※$$

(iv) Given that the event $B = \{1 \le X \le 3 \text{ and } 2 \le Y \le 4\}$ occurs, determine whether X and Y are dependent or independent of each other.

if $f_{XY|B}(x,y) = f_{X|B}(x)\, f_{Y|B}(y)$, then X, Y are independent

$$\Rightarrow \quad 1/4 = 1/2 \times 1/2 \quad ※$$

5. The mediam of a continuous random variable X is a number k that satisfy the cumulative distribution function (CDF) $F_X(k) = 1/2$.
Find the median of the exponetial random variable with a positive parameter λ.
Note that the PDF of an exyonetial random variable is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow f_X(k) = \int_0^k \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_0^k = -e^{-\lambda k} + 1$$

$$= 1 - e^{-\lambda k} = \frac{1}{2}$$

$$\Rightarrow e^{-\lambda k} = \frac{1}{2} \Rightarrow \ln e^{-\lambda k} = \ln 1 - \ln 2 = -\lambda k = -\ln 2$$

$$\Rightarrow \ln 2 = \lambda k \Rightarrow k = \frac{\ln 2}{\lambda} \text{ ※}$$

2020, quiz 3