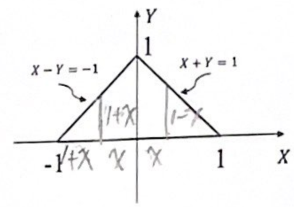
Introduction to Probability

Quiz 4 (Open-book, In-Class Exercise)

December 16, 2022, 9:10 p.m. - 10:100 p.m.



Note: You have to answer the questions with supporting explanations if needed.

1. Given that the joint PDF $f_{X,Y}(x,y)$ of two continuous random variables X and Y is constant (uniform) in the triangle of the figure shown on the right.
   (i) Find the PDF $f_X(x)$ of X. (10%)
   (ii) Find the PDF $f_Y(y)$ of Y. (10%)
   (iii) Use the "derived-distributions" method to find the PDF $f_W(w)$ of the random variable W which is expressed by $W = X + Y$. (10%)
2. Given that X is a continuous random variable with PDF $f_X(x)$ and $Y = aX + b$. Show that the PDF (probability density function) of random variable Y can be expressed as: $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$. (20%)
3. We are told that X, Y, Z are three independent random variables. All of them are uniformly distributed in the interval [0, 1]. (20%)
   (i) Use the "derived-distributions" method to find the PDF $f_W(w)$ of $W = X + 2Y$.
   (ii) Find the PDF of $W = \max(X, Y, Z)$.
4. Suppose that in Taiwan, the number of summer typhoons in a good year is a Poisson random variable with mean 2, whereas the number of typhoons in a bad year is a Poisson random variable with mean 6. If next year will be a good year with probability 0.3 or a bad year with probability 0.7, find the expected value (i.e., mean) and variance of the number of typhoons that will occur in next year. (20%)
   Hint: The number of summer typhoons can be modeled as a random variable. You may use "Law of Iterated Expectations" to calculate
   $\mathbf{E}[X]$ (**Hint:** $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$), and use "Law of Total Variance" to calculate $\mathrm{var}(X)$ (**Hint:** $\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$)
5. Given that X and Y are two independent random variables, each of which has mean equal to 3 and variance 2, respectively. Let $U = 2X + 3Y$ And $V = 3X - 2Y$. Find $\mathrm{var}(U)$ and $\mathrm{cov}(U,V)$. **Hint:** $\mathrm{cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$ (20%)
6. Let X be a discrete random variable with a transform (moment generating function) $M_X(s)$ such that $M_X(s) = a + be^{3s} + ce^{6s}$,
   $\mathbf{E}[X] = 2$ and $\mathrm{var}(X) = \frac{9}{2}$. Find a, b, c and the PMF of X. (20%) Hint: For a discrete random variable X, $M_X(s) = \sum_x e^{sx} p_X(x)$.
7. Given that X is an exponential random variable with its pdf expressed by $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$
   (i) Show that the transform of X (**Hint:** $M_X(s) = \mathbf{E}[e^{sX}] = \int_0^\infty e^{sx} f_X(x)dx$) can be expressed as: $M_X(s) = \frac{\lambda}{\lambda - s}$. (10%)
   (ii) Using $M_X(s)$ to show that $\mathbf{E}[X] = \frac{1}{\lambda}$ and $\mathrm{var}(X) = \frac{1}{\lambda^2}$. (10%)
   (iii) Given that random variable Y can be expressed as $Y = 3X + 2$. Find the transform of Y. (10%)
   (ix) Given that Z is also an exponential random variable with parameter η, and X and Z are independent of each other. Find the transform of random variable $Y = X + Z$. (10%)
8. A biased coin (with probability 0.4 being head) is flipped independently until the first head is obtained. For each flip of the coin, a value of an exponential random variable (with its parameter $\lambda = 2$) is generated. Let the random variable Y be defined as the sum of all the value obtained before the first head. Find the mean, variance and transform of Y. (20%)

Transform of Random variables:

Uniform(a, b)

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b. \qquad M_X(s) = \frac{1}{b-a}\frac{e^{sb}-e^{sa}}{s}.$$

Geometric(p)

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots \qquad M_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

Properties of Sums of a Random Number of Independent Random Variables

Let $X_1, X_2, \ldots$ be random variables with common mean μ and common variance σ^2. Let N be a random variable that takes nonnegative integer values. We assume that all of these random variables are independent, and consider

$$Y = X_1 + \cdots + X_N.$$

Then,

- $\mathbf{E}[Y] = \mu\mathbf{E}[N]$.
- $\text{var}(Y) = \sigma^2\mathbf{E}[N] + \mu^2\text{var}(N)$.
- The transform $M_Y(s)$ is found by starting with the transform $M_X(s)$ and replacing each occurrence of e^s with $M_X(s)$.

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Poisson with Parameter λ:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots,$$

$$\mathbf{E}[X] = \lambda, \qquad \text{var}(X) = \lambda.$$