Hw 2021-4

1. For the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_2 \end{bmatrix},$$ determine

(a) the dimension of the range of T
(b) the dimension of the null space of T
(c) whether T is one-to-one or onto.

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} r_2 + r_1 \to r_2 \\ r_3 + r_1 \to r_3 \\ -r_1 \to r_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \; r_1 - r_2 \to r_2 \Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} r_1 + 3r_3 \to r_1 \\ r_2 - 2r_3 \to r_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ (a) dim Range T = 3

$\Rightarrow$ (b) dim Null T = 3 - 3 = 0

$\Rightarrow$ (c) T is one-to-one ($\because$ the nullity of the function is zero)

T is on-to ($\because$ the number of row is equal to the rank)

2. (a) show that for any vector u in $\mathbb{R}^n$, $u^T u = 0$ if and only if $u = 0$

$\Rightarrow u^T u = u_1^2 + u_2^2 + \dots + u_n^2$ and for square of everything ≥ 0

$\therefore u^T u = 0$, if and only if $u = 0$

(b) Prove for any matrix A, if u is in Row A and v is in Null A, then $u^T v = 0$

$\Rightarrow$ Let $u \in \text{Row} A = \text{Col} A^T$, $v \in \text{Null} A \Rightarrow Av = 0$

There exists w such that $A^T w = u$

Hence $u^T v = (A^T w)^T v = w^T A v = w^T \cdot 0 = 0$

(c) Show that, for any matrix A, if u belongs to both Row A and Null A, then $u = 0$

use (b), and let $v = u$

$\Rightarrow u^T v = u^T u = 0$ $\quad\Rightarrow$ use (a), if $u^T u = 0$

then $u = 0$ ※

3. Let $\mathcal{B} = \{b_1, b_2, b_3\}$, where $b_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $b_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $b_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

(a) Show that $\mathcal{B}$ is a basis for $\mathcal{R}^3$

$\Rightarrow$ Let $B = [b_1 \ b_2 \ b_3]$ $\Rightarrow$ the rref of B is $I_3 = R_B$

Since rank $R_B = 3$, $\mathcal{B}$ is linear independent subset of $\mathcal{R}^3$ containing 3 vectors ※

(b) Determine the matrix $A = [\,[e_1]_\mathcal{B} \ [e_2]_\mathcal{B} \ [e_3]_\mathcal{B}\,]$

$$\Rightarrow A = [B^{-1}e_1 \ B^{-1}e_2 \ B^{-1}e_3] = B^{-1} I_3 = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix} \text{※}$$

(c) What is the relationship between A and $B = [b_1 \ b_2 \ b_3]$

$\Rightarrow A = B^{-1}$ ※

4. Determine $[T]_\mathcal{B}$ for linear operator T and basis $\mathcal{B}$

where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - 2x_3 \\ 2x_1 - x_2 + 3x_3 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$

$$[T]_\mathcal{B} = B^{-1}[\,T(b_1) \ T(b_2) \ T(b_3)\,] = B^{-1} A B$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 5 & 3 \\ -1 & -1 & -2 \\ 4 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 19 & 16 \\ -5 & -8 & -8 \\ 2 & 2 & 3 \end{bmatrix} \text{※}$$

5. Given a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = -\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

determine

(a) $[T]_{\mathcal{B}}$

(b) the standard matrix of T

(c) an explict formula for $T(x)$

(a)
$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 0 & 5 \\ -2 & 4 & 0 \end{bmatrix}$$

(b)
$$A = \mathcal{B}[T]_{\mathcal{B}}\mathcal{B}^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 2 \\ 3 & 0 & 5 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -7 & -1 \\ -8 & -8 & 11 \\ -4 & -9 & 6 \end{bmatrix} \text{※}$$

(c)
$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -7 & -1 \\ -8 & -8 & 11 \\ -4 & -9 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 7x_2 - x_3 \\ -8x_1 - 8x_2 - 11x_3 \\ -4x_1 - 9x_2 + 6x_3 \end{bmatrix} \text{※}$$

7. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation,
$\mathcal{B} = \{b_1\ b_2\ \dots\ b_n\}$ be basis for $\mathbb{R}^n$, $C = \{c_1\ c_2\ \dots\ c_m\}$ be basis for $\mathbb{R}^m$.
Let B and C be the matrices whose columns are vectors in $\mathcal{B}$ and C.

(a) If A is the standard matrix of T, then $[T]_{\mathcal{B}}^{C} = C^{-1}AB$

$$\Rightarrow [T]_{\mathcal{B}}^{C} = [\,[T(b_1)]_C\ [T(b_2)]_C\ \dots\ [T(b_n)]_C\,] = [\,C^{-1}T(b_1)\ C^{-1}T(b_2)\ \dots\ C^{-1}T(b_n)\,]$$

$$= C^{-1}[\,T(b_1)\ T(b_2)\ \dots\ T(b_n)\,] = C^{-1}[\,Ab_1\ Ab_2\ \dots\ Ab_n\,] = C^{-1}A\mathcal{B} \text{※}$$

(b) $[T(v)]_C = [T]_{\mathcal{B}}^{C}[v]_{\mathcal{B}}$ for any vector v in $\mathbb{R}^n$

$$\Rightarrow [T(v)]_C = C^{-1}T(v) = C^{-1}Av = C^{-1}A\mathcal{B}\mathcal{B}^{-1}v = [T]_{\mathcal{B}}^{C}[v]_{\mathcal{B}} \text{※}$$

(c) Let $U: \mathbb{R}^m \to \mathbb{R}^p$ be linear, and let D be a basis for $\mathbb{R}^p$
Then $[UT]_{\mathcal{B}}^{D} = [U]_C^D[T]_{\mathcal{B}}^{C}$

by (a), $[UT]_{\mathcal{B}}^{D} = D^{-1}PA\mathcal{B} = D^{-1}PCC^{-1}A\mathcal{B} = [U]_C^D[T]_{\mathcal{B}}^{C}$ ※

6.

8. (a) Prove that if λ is an eigenvalue of matrix A, then λ^2 is an eigenvalue of A^2

$\Rightarrow$ Let $Av = \lambda v$

$\Rightarrow A^2 v = A(\lambda v) = \lambda (Av) = \lambda^2 v$ ※

(b) Let v_1 and v_2 be eigenvectors of a linear transformation T on $\mathbb{R}^n$
Let λ_1, λ_2 respectively, be the corresponding eigenvalues

Prove that if $\lambda_1 \neq \lambda_2$, then $\{v_1, v_2\}$ is linearly independent

$\Rightarrow$ Let $c_1 v_1 + c_2 v_2 = 0$

$\Rightarrow 0 = T(0) = T(c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$

$= \lambda_1 (-c_2 v_2) + \lambda_2 (c_2 v_2) = (\lambda_2 - \lambda_1)(c_2 v_2)$

$\because \lambda_1 \neq \lambda_2$ and $v_2 \neq 0$ $\therefore c_2 = 0$ and we have $c_1 = 0$

$\Rightarrow$ so $\{v_1, v_2\}$ is linearly independent. ※