Hw 2021 - 1

1. Determine whether the following system is consistent and if so, find the vector form of its general form

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 = -1 \\ -2x_1 - 6x_2 - x_3 = 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 = 2 \end{cases}$$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ -2 & -6 & -1 & 0 & 5 \\ 1 & 3 & 2 & 3 & 2 \end{array}\right] \xrightarrow[-r_1+r_3 \to r_3]{2r_1+r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow[r_3-r_2 \to r_3]{r_1-r_2 \to r_1} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = -3x_2 + x_4 - 4 \\ x_2 \text{ free} \\ x_3 = -2x_4 + 3 \\ x_4 = \text{free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \#$$

(general form) (vector form)

2. Find the rank and nullity of matrix $\begin{bmatrix} 1 & -2 & -1 & 0 & 3 & -2 \\ 2 & -4 & -2 & -1 & 5 & 9 \\ -1 & 2 & 1 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$

$$\xrightarrow[r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & -1 & -1 & 13 \\ 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow[\substack{r_4-r_3 \to r_4 \\ -r_2 \to r_2}]{r_3+r_2 \to r_3} \begin{bmatrix} 1 & -2 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & 1 & -13 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank $= 3$

$\Rightarrow$ Nullity $= 6 - 3 = 3$ #

3. A company makes three types of fertilizer.
The first type contains 10 % nitrogen and 3 % phosphates by weight
The second type contains 8 % 6 %
The third type contains 6 % 1 %

Can the company mix these types of firtilizers to supply exactly 600 pound contains 7.5 % nintrogen and 5 % phosphates.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600 \\ 10 & 8 & 6 & 4500 \\ 3 & 6 & 1 & 3000 \end{array}\right] \begin{array}{c} \frac{1}{2}r_2 \to r_2 \\ r_2 - 5r_1 \to r_2 \\ r_3 - 3r_1 \to r_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600 \\ 0 & -1 & -2 & -750 \\ 0 & 3 & -2 & 1200 \end{array}\right]$$

$$\begin{array}{c} -r_2 \to r_2 \\ r_3 - 3r_2 \to r_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600 \\ 0 & 1 & 2 & 750 \\ 0 & 0 & -8 & -1050 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600 \\ 0 & 1 & 2 & 750 \\ 0 & 0 & 1 & 131.25 \end{array}\right]$$

$$\begin{array}{c} r_1 - r_3 \to r_1 \\ r_2 - 2r_3 \to r_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 112.5 \\ 0 & 1 & 0 & 487.5 \\ 0 & 0 & 1 & 131.25 \end{array}\right] \; r_1 - r_2 \to r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -18.75 \\ 0 & 1 & 0 & 487.5 \\ 0 & 0 & 1 & 131.25 \end{array}\right]$$

$\Rightarrow$ x 不可為負，故無解 #

4. Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix}$, determine if the equation $Ax = b$ is consistent for every b in R^4

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow Ax = b$ is not consistent for every b in R^4

because there's a zero row in A's reduce row echelon form

A has at most 1 solution

5. Let R be the reduced row echelon form of an $m\times n$ matrix A.
Is the span of the columns of R equal to the span of column of A ?

No, R is the reduced row echelon form of $m\times n$ matrix A,
if any row in A appears to be zero and be removed,
then A will become R^{m-1} instead of R^m

$\Rightarrow$ span $A \neq$ span R #

6. Determine, if possible, a value of r for which the following set of vector is linearly dependent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 9 \\ r \\ -2 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 9 & 0 \\ -1 & 2 & 1 & r & 0 \\ 1 & 1 & 0 & -2 & 0 \end{array}\right] \xrightarrow[r_4 - r_1 \to r_4]{r_3 + r_1 \to r_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 9 & 0 \\ 0 & 2 & 0 & r-1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{cases} r_3 + 2r_2 \to r_3 \\ r_4 + r_2 \to r_4 \\ -r_2 \to r_2 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -9 & 0 \\ 0 & 0 & -2 & r+1 & 0 \\ 0 & 0 & 2 & 8 & 0 \end{array}\right] \xRightarrow[\frac{1}{2}r_4 \to r_4]{r_3 \leftrightarrow r_4} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & r+1 & 0 \end{array}\right]$$

$$\xrightarrow{r_4 + 2r_3 \to r_4} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & r+9 & 0 \end{array}\right]$$

$\Rightarrow$ when $r = -9$, x_4 is free

then the following set of vectors is linearly dependent #

7. Let A and B be $m \times n$ matrices such that B can be obtained by performing a single elementary row operation on A.

Prove that if the row of A are linearly independent, then the rows of B are also L. I.

$\Rightarrow$ 設 A 的 row 分別為 $a_1\ a_2 \dots a_m$，由 independent 定義

$\Rightarrow x_1 a_1 + x_2 a_2 + \dots + x_m a_m = 0$ 有唯一解 $x_1 = x_2 = \dots = x_m = 0$

設 B 為 A 進行列運算 $k a_i + a_j$，$k \in R$，$1 \leq i, j \leq m$

得 B 的 linear combination $\Rightarrow y_1 a_1 + y_2 a_2 + \dots + y_j (k a_i + a_j) + \dots + y_m a_m = 0$

$\Rightarrow y_1 a_1 + y_2 a_2 + \dots + (y_j k + y_i) a_i + \dots + y_j a_j + \dots + y_m a_m = 0$

又 $a_1, a_2 \dots a_m$ 是 linear independent 上式有唯一解 0

$\therefore$ B 的 row vector 亦為 linear independent，得證 #

8. Let A be an $m \times n$ matrix with reduced row echelon form R.
Determine the reduced row echelon form of each of the following matrices.

(a) $[A \quad 0]$ $\Rightarrow [R \quad 0]$

(b) $[a_1, a_2 \dots a_k]$ for $k < n$, where $a_i = Ae_i$

$\Rightarrow [r_1\ r_2 \dots r_k]$, where $r_i = Re_i$

(c) cA, where c is a nonzero scalar $\Rightarrow R$

(d) $[I_m \quad A]$ $\Rightarrow [I_m \quad A]$

(e) $[A \quad cA]$, where c is any scalar $\Rightarrow [R \quad cR]$