Midterm 2020

1. Determine a vector form for the general solution of the following system

$$x_1 - x_2 - 3x_3 + x_4 - x_5 = -2$$
$$-2x_1 + 2x_2 + 6x_3 - 6x_5 = -6$$
$$3x_1 - 2x_2 - 8x_3 + 3x_4 - 5x_5 = -7$$

$$\begin{bmatrix} 1 & -1 & -3 & 1 & -1 & -2 \\ -2 & 2 & 6 & 0 & -6 & -6 \\ 3 & -2 & -8 & 3 & -5 & -7 \end{bmatrix} \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 + 2r_1 \to r_2} \begin{bmatrix} 1 & -1 & -3 & 1 & -1 & -2 \\ 0 & 0 & 0 & 2 & -8 & -10 \\ 0 & 1 & 1 & 0 & -2 & -1 \end{bmatrix}$$

$r_1 + r_3 \to r_1$

$r_2 \leftrightarrow r_3$

$\frac{1}{2}r_2 \to r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & -3 & -3 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -4 & -5 \end{bmatrix} \xrightarrow{r_1 - r_3 \to r_1} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -4 & -5 \end{bmatrix}$$

$$\begin{cases} x_1 = 2x_3 - x_5 + 2 \\ x_2 = -x_3 + 2x_5 - 1 \\ x_3 = \text{free} \\ x_4 = 4x_5 - 5 \\ x_5 = \text{free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ -5 \\ 0 \end{bmatrix} \quad \#$$

2. Determine the value of r for which v is in the span of S

where $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $v = \begin{bmatrix} 2 \\ r \\ -8 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & r \\ 2 & 0 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & r+4 \\ 0 & 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & r+4 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & r+4 \\ 0 & 0 & r+6 \end{bmatrix}$$

$\Rightarrow$ If. consistent, the last row must be zero $\therefore$ $r+6=0$, $r=-6$ #

3. Let $S = \{u_1, u_2 \dots u_k\}$ be a nonempty subset of $\mathbb{R}^n$ and A be an $m \times n$ matrix with rank n.

Prove that if S is linearly independent set, then the set $\{Au_1\ Au_2 \dots Au_k\}$ is also L.I.

Let $c_1Au_1 + c_2Au_2 + \dots + c_kAu_k = 0$, for some $c_1 \sim c_k$

then $A(c_1u_1 + c_2u_2 + \dots + c_ku_k) = 0$

$\because A$ has rank $= n$ $\therefore c_1u_1 + c_2u_2 + \dots + c_ku_k = 0$

$\because S$ is L.I. $\therefore c_1u_1 + c_2u_2 + \dots c_ku_k = 0$ has only solution $c_1 = c_2 = \dots = c_k$

Hence $\{Au_1\ Au_2 \dots Au_k\}$ is linearly independent #

4. Suppose that the reduce row echelon form R and two columns of matrix A given

$$R = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 0 & 0 & 1 & 3 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad a_5 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

(a) Determine the matrix A

(column corresponding property)

① $r_2 + r_1 \to r_2$, $r_3 + r_1 \to r_3$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 1 & -1 & 1 & 1 & 1 & -2 \\ 1 & -1 & 0 & -2 & -3 & 2 \end{bmatrix}$$

② $2r_2 - r_3 \to r_2$, $r_2 \leftrightarrow r_3$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 1 & -1 & 2 & 4 & 5 & 6 \\ 1 & -1 & 1 & 1 & 1 & -2 \end{bmatrix} = A$$

#

(b) Find the rank and nullity of A

$$R = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 0 & 0 & 1 & 3 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

2 pivot column

$\Rightarrow$ Nullity $= 6-2 = 4$ #

5. Find an LU decomposition of matrix $\begin{bmatrix} 3 & 1 & -1 & 1 \\ 6 & 4 & -1 & 4 \\ -3 & -1 & 2 & -1 \\ 3 & 5 & 0 & 3 \end{bmatrix}$

$$\begin{matrix} r_2 - 2r_1 \to r_2 \\ r_3 + r_1 \to r_3 \\ r_4 - r_1 \to r_4 \end{matrix} \Rightarrow \begin{bmatrix} 3 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 2 \end{bmatrix} \quad r_4 - 2r_2 \to r_4 \Rightarrow \begin{bmatrix} 3 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad r_4 + r_3 \to r_4 \Rightarrow \begin{bmatrix} 3 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix} \quad \#$$

6. Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

(a) Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$

$$T(e_1) = \frac{1}{2}\left[T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_2) = \frac{1}{2}\left[T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$T(e_3) = \frac{1}{2}\left[T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)\right] = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Ans: $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ #

(b) Find T^{-1} if invertable

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 3 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -4 \\ 0 & 0 & 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -4 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 3 \end{array}\right]$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 - 4x_3 \\ -x_1 + x_2 + 2x_3 \\ -x_1 + x_2 + 3x_3 \end{bmatrix} \quad \#$$

7. Given matrics $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 \\ -7 & -3 & -2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & -2 & -3 & 8 \\ -3 & 4 & 1 & -1 \\ -2 & 6 & -4 & 18 \end{bmatrix}$

(a) compute the determinant of $A^{-1}B^T$

$$\det A^{-1}B^T = \det A^{-1} \det B^T = \frac{\det B}{\det A} = (-\tfrac{1}{8}) \times (-56) = 7 \quad \#$$

(b) evaluate $\det(A^{-1}C)$ where $C = 2B^T$

$$\det A^{-1}C = \det A^{-1} \det C = \det A^{-1} \det 2B = (-\tfrac{1}{8}) \times (-56 \times 2^4) = 112 \quad \#$$

8. Let V and W be two subspace of $\mathbb{R}^n$.
Use the definition of a subspace to prove that $S = \{ s \in \mathbb{R}^n : s = v + w \text{ for some } v \text{ in } V, w \text{ in } W \}$ is a subspace of $\mathbb{R}^n$.

(1) $\because 0 \in V,\ 0 \in W,\quad 0 + 0 = 0 \in S$

(2) Let $S_1, S_2 \in S$. Then $S_1 = W_1 + V_1$, $S_2 = W_2 + V_2$ for some $V_1, V_2 \in V$, $W_1, W_2 \in W$

Hence, $S_1 + S_2 = (W_1 + V_1) + (W_2 + V_2)$
$= (W_1 + W_2) + (V_1 + V_2)$

$\because$ V and W are closed under vector addition, $V_1 + V_2 \in V$, $W_1 + W_2 \in W$

$\therefore S_1 + S_2 \in S$

(3) For any scalar c, $cS_1 = c(V_1 + W_1) = cV_1 + cW_1$

$\because$ V and W are closed under scalar multiplication, $cV_1 \in V$, $cW_1 \in W$

$\therefore cS_1 \in S$

$\Rightarrow S$ is a subspace of $\mathbb{R}^n$ #

9. A linear transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 - 3x_4 \\ -2x_1 + 3x_2 - 3x_3 + 2x_4 \end{bmatrix}$

(a) Find basis for the range of T

$$A = \begin{bmatrix} 1 & -2 & 1 & -3 \\ -2 & 3 & -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & -3 \\ 0 & -1 & -1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 4 \end{bmatrix} = R \quad \#$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\} \quad \#$$

(b) Find basis for the null space of T if the nullspace is nonzero

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{null space of } T \text{ is } \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \#$$