Hw 2021-3

1. Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that

$$T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix},\quad T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\0\\1\end{bmatrix},\quad T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)=\begin{bmatrix}5\\4\\3\end{bmatrix}$$

Determine $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ for any $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|ccc}-1&1&1&1&0&0\\1&-1&1&0&1&0\\1&1&-1&0&0&1\end{array}\right]\ \begin{matrix}r_2+r_1\to r_2\\ \\ r_3+r_1\to r_3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}-1&1&1&1&0&0\\0&0&2&1&1&0\\0&2&0&1&0&1\end{array}\right]\ \begin{matrix}-r_1\to r_1\\ \frac{1}{2}r_2\to r_2\\ \frac{1}{2}r_3\to r_3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&-1&-1&-1&0&0\\0&0&1&\frac{1}{2}&\frac{1}{2}&0\\0&1&0&\frac{1}{2}&0&\frac{1}{2}\end{array}\right]$$

$$r_2\leftrightarrow r_3\Rightarrow\left[\begin{array}{ccc|ccc}1&-1&-1&-1&0&0\\0&1&0&\frac{1}{2}&0&\frac{1}{2}\\0&0&1&\frac{1}{2}&\frac{1}{2}&0\end{array}\right]\ r_1+r_2+r_3\to r_1\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{1}{2}&0&\frac{1}{2}\\0&0&1&\frac{1}{2}&\frac{1}{2}&0\end{array}\right]$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=0\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-3\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix}+0\begin{bmatrix}-3\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-3\\0\\1\end{bmatrix}+0\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}1&3&-1\\2&3&1\\2&3&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+3x_2-x_3\\2x_1+3x_2+x_3\\2x_1+3x_2+2x_3\end{bmatrix}\ \#$$

2. Find a generating set for the null space of linear transformation T and determine whether T is one-to-one.

Here $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_3 \\ x_1 + x_2 + 4x_3 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 4 \end{bmatrix} \xRightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & -1 \\ 2 & 0 & 3 \end{bmatrix} \xRightarrow[r_3 - 2r_1 = r_3]{-r_2 + r_1 = r_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 5 \end{bmatrix} \xRightarrow[\frac{1}{5}r_3 \to r_3]{r_1 - r_2 \to r_1,\ r_2 - r_3 \to r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ have 3 pivot column $\Rightarrow$ nullity $= 3 - 3 = 0$ $\therefore$ is one-to-one ※

3. For an invertible linear transformation T defined below, determine a similar definition for its inverse T^{-1}

Here: $T: \mathbb{R}^3 \to \mathbb{R}^3$ is $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + x_2 - x_3 \\ -x_1 - x_2 \\ -5x_1 - 3x_2 + x_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 4 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -5 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xRightarrow{r_1 \leftrightarrow -r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 4 & 1 & -1 & 1 & 0 & 0 \\ -5 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xRightarrow[r_3 + 5r_1 \to r_3]{r_2 - 4r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -3 & -1 & 1 & 4 & 0 \\ 0 & 2 & 1 & 5 & 0 & 1 \end{array}\right]$$

$$\xRightarrow{-r_2 - r_3 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -6 & -4 & -1 \\ 0 & 2 & 1 & 5 & 0 & 1 \end{array}\right] \xRightarrow[r_3 - 2r_2 \to r_3]{r_1 - r_2 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 3 & 1 \\ 0 & 1 & 0 & -6 & -4 & -1 \\ 0 & 0 & 1 & 7 & 4 & 3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 3x_2 + x_3 \\ -6x_1 - 4x_2 - x_3 \\ 7x_1 + 8x_2 + 3x_3 \end{bmatrix} \quad ※$$

4. Suppose that $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear and one-to-one.
Let $\{v_1, v_2, \dots, v_k\}$ be linearly independent subset of $\mathbb{R}^n$
Prove that $\{T(v_1), T(v_2) \dots T(v_k)\}$ is linearly independent subset of $\mathbb{R}^m$.

5. Determine the values of c for which the following matrix is no invertable

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 1 & -1 & c & -1 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 1 & -1 & c & -1 \\ -1 & 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & c+1 & -2 \\ 0 & 1 & -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & c-1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \cancel{\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & c-1 & -1 \\ 0 & 0 & 0 & 2+\frac{1}{c-1} \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & c-1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & (-2c+2-1) \end{bmatrix}$$

$\Rightarrow$ If not invertable $\Rightarrow$ determinent $= 0$ $\Rightarrow$ $-2c+1 = 0$ $\Rightarrow$ $c = \frac{1}{2}$ ※

6. The matrix $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is called Vandermonde matrix.

Show that $\det A = (b-a)(c-a)(c-b)$

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix} \times (b-a) \times (c-a)$$

$$\Rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix} (b-a) \times (c-a)$$

$\Rightarrow \det A = (c-b) \times (b-a) \times (c-a)$ #

7. Find the generating sets for the range and null space of linear transformation T

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 - 5x_3 \\ -x_1 + 2x_2 + 7x_3 \\ 2x_1 - x_2 - 8x_3 \\ 2x_2 + 4x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -5 \\ -1 & 2 & 7 \\ 2 & -1 & -8 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

① Range of T $= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \right\}$

② $\begin{cases} x_1 = 3x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \Rightarrow$ Null space $= \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

8. Find a basis for (a) the column space of $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & -3 & 5 & 4 \\ 0 & 0 & 3 & -3 \\ 2 & -2 & 1 & 5 \end{bmatrix}$
(b) the null space

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & -3 & 5 & 4 \\ 0 & 0 & 3 & -3 \\ 2 & -2 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) column space $= \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \end{bmatrix} \right\}$ #

(b) $\begin{cases} x_1 = x_2 - 3x_4 \\ x_2 = x_2 \\ x_3 = x_4 \\ x_4 = x_4 \end{cases}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Null space $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ #