Hw 2021-2

1. A square matrix A is called upper triangular if the (i,j)-entry of A is zero, whenever $i > j$.

Prove that if A and B are both $n \times n$ upper triangular matrices, then AB is also an upper triangular matrix.

$\Rightarrow AB = \sum_{k=1}^{n} a_{ik} b_{kj}$

when $i > j$, $\sum_{k=1}^{n} a_{ik} b_{kj} = 0$ $\Rightarrow$ when A 的 $i > k$
B 的 $k > j$, entry $= 0$

$\Rightarrow AB$ 為 upper triangular

2. The trace of an $n \times n$ matrix A, written trace(A), is defined to be the sum

$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Prove that if A is an $m \times n$ matrix and B is an $n \times m$ matrix, then trace(AB) = trace(BA)

$\Rightarrow$ trace$(AB) = (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn}$

$$\begin{aligned} = \; & a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1k}b_{k1} \\ + \; & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2k}b_{k2} \\ + \; & \cdots \\ + \; & \cdots \qquad\qquad\qquad\quad + a_{nk}b_{kn} \end{aligned} = \text{trace}(BA)$$

3. Suppose that the reduced row echelon form R and two column of matrix A given

$$R = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 0 & 0 & 1 & 3 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad a_5 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

(a) determine the matrix A (b) find rank and nullity

(a)

$r_2 + r_1 \to r_2$
$r_3 + r_1 \to r_2$ $\Rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 1 & -1 & 1 & 1 & 1 & -2 \\ 1 & -1 & 0 & -2 & -3 & 2 \end{bmatrix}$ $r_3 - 2r_1 \to r_3 \Rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 1 & -1 & 1 & 1 & 1 & -2 \\ -1 & 1 & -2 & -4 & -5 & 6 \end{bmatrix}$

$-r_3 \to r_3$
$r_2 \leftrightarrow r_3$ $\Rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & -3 & 2 \\ 1 & -1 & 2 & 4 & 5 & -6 \\ 1 & -1 & 1 & 1 & 1 & -2 \end{bmatrix} = A$ #

(b)

Rank = 2

Nullity = 6 - 2 = 4 #

4. Suppose that u and v are linearly independent vectors in R^3.

Find the reduced row echelon form of $A = [a_1 \ a_2 \ a_3 \ a_4]$

given that $a_1 = u$, $a_2 = 2u$, $a_3 = u+v$, $a_4 = v$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 1 & 4 \\ 4 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 1 & -1 & 2 & 4 & 5 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix}$. Find $A^{-1}B$

$$\left[\begin{array}{ccc|ccccc} 3 & 2 & 4 & 1 & -1 & 0 & -2 & -3 \\ 4 & 1 & 4 & 1 & -1 & 2 & 4 & 5 \\ 4 & 2 & 5 & 1 & -1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccccc} 1 & 0 & 1 & 0 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 & 0 & -1 & -3 & -4 \\ 3 & 2 & 4 & 1 & -1 & 0 & -2 & -3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccccc} 1 & 0 & 1 & 0 & 0 & 1 & 3 & -4 \\ 0 & 1 & 1 & 0 & 0 & -1 & -3 & -4 \\ 0 & 1 & 0 & 1 & -1 & -2 & -8 & -11 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 1 & 0 & -1 & -1 & -2 & -8 & -11 \\ 0 & 0 & 1 & -1 & -1 & 1 & 5 & 17 \end{array}\right]$$

6. Let A and B be $n \times n$ matrices.
We said that A is similar to B if $B = P^{-1}AP$ for some invertable matrix P.

Let A, B, C be $n \times n$ matrices. Prove the following statements:

(a) A is similar to A

Let $P = I_n \Rightarrow P^{-1}AP = AI_n = A$

$\therefore A$ is similar to A #

(b) If A is similar to B, then B is similar to A

$B = P^{-1}AP$, Let $Q = P^{-1}$

$Q^{-1}BQ = (P^{-1})^{-1}[P^{-1}AP]P^{-1} = A$ $\therefore B$ is similar to A

(c) If A is similar to B, and B is similar to C, then A is similar to C

Let $B = P^{-1}AP$, $C = Q^{-1}BQ$ Let $K = PQ$

$$K^{-1}AK = (PQ)^{-1}A(PQ)$$

$$= Q^{-1}P^{-1}APQ = Q^{-1}BQ = C$$ $\therefore A$ is similar to C

7. Assume that A, C, D are $n \times n$ matrices, O is $n \times n$ zero matrix, A, D are invertable

Verify $\begin{bmatrix} C & A \\ D & O \end{bmatrix}^{-1} = \begin{bmatrix} O & D^{-1} \\ A^{-1} & -A^{-1}CD^{-1} \end{bmatrix}$

$$\begin{bmatrix} C & A \\ D & O \end{bmatrix}\begin{bmatrix} O & D^{-1} \\ A^{-1} & -A^{-1}CD^{-1} \end{bmatrix} = \begin{bmatrix} I_n & CD^{-1} - I_nCD^{-1} \\ O & I_n \end{bmatrix}$$

$$= \begin{bmatrix} I_n & O \\ O & I_n \end{bmatrix} = I_n$$ ※

8. Let $A = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 4 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 2 \\ -3 & -2 & 0 & -3 & -5 \end{bmatrix}$

(a) Find a permutation matrix P such that PA has an LU decomposition

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & -2 & -3 & -2 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 4 & 6 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & -2 & -3 & -2 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & -4 & -1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \#$$