Hw 2021 - 6

1. Let $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and line L represented by $y = 3x$
Compute the orthogonal projection w of u on L
and compute the distance d from u to L

$\Rightarrow$ Let $\vec{v}$ be the directional vector of L, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$w = \frac{u \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \frac{7}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 2.1 \end{bmatrix} \ \text{※}$$

$$d = \left| \begin{bmatrix} 4 - 0.7 \\ 1 - 2.1 \end{bmatrix} \right| = \left| \begin{bmatrix} 3.3 \\ -1.1 \end{bmatrix} \right| = 1 - \sqrt{10} \ \text{※}$$

2. Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} \right\}$

(a) Apply Gram-Schmidt process to replace given linearly independent set S by an orthogonal set of nonzero vectors with the same span, and obtain an orthonormal set with the same span as S

(b) Let A be the matrix whose columns are vetors in S
Determine the matrix Q and R in QR factorization of A

(c) Use QR fractorization to solve system $Ax = b$, $b = \begin{bmatrix} 8 \\ 0 \\ 1 \\ 11 \end{bmatrix}$

(a)

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} ; \quad V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} ; \quad V_3 = \begin{bmatrix} +3 \\ 1 \\ 1 \\ 5 \end{bmatrix} - \frac{12}{6} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} - \frac{8}{6} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} \times \frac{1}{3}$$

$$\Rightarrow \text{orthonormal} : \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \ \text{※}$$

(b) $QR = A$

$$Q = \frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{2} & 0 & 3 \\ -\sqrt{2} & 2\sqrt{2} & 1 \\ 0 & \sqrt{2} & -1 \\ 2\sqrt{2} & \sqrt{2} & -1 \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{6} & \frac{4}{3}\sqrt{6} \\ 0 & 0 & \frac{2}{3}\sqrt{6} \end{bmatrix} \ \text{※}$$

(c) $Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^T b$

$$\frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{2} & 0 & 3 \\ -\sqrt{2} & 2\sqrt{2} & 1 \\ 0 & \sqrt{2} & -1 \\ 2\sqrt{2} & \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 5\sqrt{6} \\ 2\sqrt{6} \\ 2\sqrt{3} \end{bmatrix} \ \text{※}$$

3. Let $\{w_1\ w_2\ \dots\ w_n\}$ be orthonormal basis for $\mathbb{R}^n$
Prove that for any vectors u, v in $\mathbb{R}^n$.

(a) $u + v = (u\cdot w_1 + v\cdot w_1)w_1 + (u\cdot w_2 + v\cdot w_2)w_2 + \dots + (u\cdot w_n + v\cdot w_n)w_n$

$$u = \sum_{i=1}^{n}(u\cdot w_i)w_i \quad , \quad v = \sum_{i=1}^{n}(v\cdot w_i)w_i$$

$$u + v = \sum_{i=1}^{n}(u\cdot w_i)w_i + \sum_{i=1}^{n}(v\cdot w_i)w_i = \sum_{i=1}^{n}(u\cdot w_i + v\cdot w_i)\, w_i \quad \#$$

(b) $u\cdot v = (u\cdot w_1)(v\cdot w_1) + (u\cdot w_2)(v\cdot w_2) + \dots + (u\cdot w_n)(v\cdot w_n)$

$$u\cdot v = \sum_{i=1}^{n}(u\cdot w_i)w_i \cdot \sum_{j=1}^{n}(v\cdot w_j)w_j$$

$$= \sum_{i=1}^{n}(u\cdot w_i)(v\cdot w_i)\, w_i^2 = \sum_{i=1}^{n}(u\cdot w_i)(v\cdot w_i) \quad \#$$

(c) $\|u\|^2 = (u_1\cdot w_1)^2 + (u\cdot w_2)^2 + \dots + (u\cdot w_n)^2$

According to (b), let $u = v \Rightarrow u\cdot u = \sum_{i=1}^{n}(u\cdot w_i)^2$ #

4.
Let $u = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, and W be the solution set of $x_1 + 2x_2 - x_3 = 0$

(a) Find the orthogonal projection matrix P_W
(b) Obtain the unique vectors w in W and z in $W^\perp$ such that $u = w + z$
(c) Find the distance from u to W

$$\Rightarrow w = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow P_W = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \#\ (a)$$

(b) $$w = P_W\cdot u = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$z = u - w = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \#$$

(c) $|z| = \sqrt{1 + 4 + 1} = \sqrt{6}$ #

5. An inconsistent system of linear equation $Ax = b$, $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

(a) Obtain the vector z for which $\|Az - b\|$ is minimum

(b) Find the vector z of least norm for which $\|Az - b\|$ is a minimum

(a)

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$P_W = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

$$Az = P_W b = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 3/2 \\ 0 & 1 & 0 & 1 & 3/2 \\ 1 & -1 & 1 & 2 & 1/2 \\ 0 & -1 & 1 & 0 & 1/2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow z = \begin{bmatrix} 0 \\ 3/2 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \#$$

(b)

$$D = \begin{bmatrix} -2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$P_z = \begin{bmatrix} -2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} -2 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & 2 & 2 & -2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3/2 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 4 & 2 & 2 & -2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ -2 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3/2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3/2 \\ 1/2 \end{bmatrix} \quad \#$$

6 Find an orthogonal operator T on $\mathbb{R}^3$ such that $T(v) = w$

where $v = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\\0\end{bmatrix}$, $w = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix}$

⇒ Let A be standard matrix of T

$$v = I_n v = A^T A v = A^T \times \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix} = A^{-1} \times \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix}$$

$$v = A^{-1} \times \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\\0\end{bmatrix}$$

$$\Rightarrow A\begin{bmatrix}3\\1\\0\end{bmatrix} = \sqrt{2}\begin{bmatrix}0\\-2\\1\end{bmatrix} \qquad \Rightarrow A = \begin{bmatrix}0 & 0 & 1\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}$$ #

7. Given a symmetric matrix $A = \begin{bmatrix}0 & 2 & 2\\ 2 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}$

(i) find an orthonormal basis of eigenvector and corresponding eigenvalue

(ii) use (i) to obtain spectral decomposition of A

$$\Rightarrow \det\begin{bmatrix}-t & 2 & 2\\ 2 & -t & 2\\ 2 & 2 & -t\end{bmatrix} = -t(t^2-4) - 2(-2t-4) + 2(4+2t) = -(t-4)(t+2)^2$$

(i) $\lambda = 4$ $\begin{bmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix} \Rightarrow \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

(ii) $\lambda = -2$ $\begin{bmatrix}-2 & 2 & 2\\ 2 & -2 & 2\\ 2 & 2 & -2\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \Rightarrow \left\{\begin{bmatrix}-1\\0\\1\end{bmatrix} \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$

⇒ Gram-schmidt

$$V_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad V_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad V_3 = \begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\1\\-\frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

$$\Rightarrow \text{orthonormal}: \left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\2\\-1\end{bmatrix}\right\}$$ #

$$\Rightarrow A = 4 \times \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix} - 2 \times \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\end{bmatrix} - 2 \times \frac{1}{2}\begin{bmatrix}-\frac{1}{2}\\1\\-\frac{1}{2}\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & 1 & -\frac{1}{2}\end{bmatrix}$$ #

8. Let A be an $n \times n$ symmetric matrix.
Prove that A is positive definite if and only if $\sum_{ij} a_{ij} u_i u_j > 0$ for $u_1 \sim u_n$ not all zero

- An $n \times n$ matrix C is said to be positive definite if C is symmetric and $v^T C v > 0$ for every nonzero vector v in $\mathbb{R}^n$.

- An $n \times n$ matrix C is said to be positive semidefinite if C is symmetric and $v^T C v \geq 0$ for every vector v in $\mathbb{R}^n$

$\Rightarrow$ Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$v^T A v = [v_1 \cdots v_n] \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & a_{ij} & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = [v_1 \cdots v_n] \begin{bmatrix} a_{11} v_1 + \cdots + a_{1n} v_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_{n1} v_n + \cdots + a_{nn} v_n \end{bmatrix}$$

$$\begin{aligned} = & \ a_{11} v_1^2 + a_{12} v_1 v_2 + \cdots + a_{1n} v_1 v_n \\ & + a_{21} v_1 v_2 + a_{22} v_2^2 + \cdots + a_{2n} v_2 v_n \\ & \vdots \\ & + a_{n1} v_n v_1 + a_{n2} v_n v_2 + \cdots + a_{nn} v_n^2 = \sum_{ij} a_{ij} v_i v_j \end{aligned}$$

$\Rightarrow$ If A is positive define $\Rightarrow$ $v^T A v > 0$ $\left(\sum_{ij} a_{ij} v_i v_j > 0\right)$

$\Rightarrow$ If $\sum_{ij} a_{ij} v_i v_j > 0$ $\Rightarrow$ $v^T A v > 0$ $\Rightarrow$ A is positive define #