1. Given that $X$ is a continuous random variable with PDF $f_X(x)$ & $Y = aX + b$
Show that PDF of $Y$ can be expressed as :

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$\Rightarrow F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$

(i) $a > 0$

$$F_Y(y) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) \Rightarrow f_Y(y) = \frac{d\,F_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(ii) $a < 0$

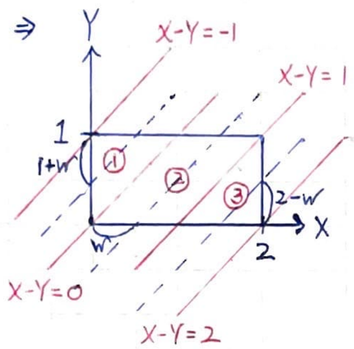
$$F_Y(y) = P\left(X \geq \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right) \Rightarrow f_Y(y) = -\frac{d\,F_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad ※$$

2. We are told $X, Y$ are two independent random variables, $X$ is uniformly distributed in $[0, 2]$
$Y$ is uniformly distributed in $[0, 1]$

(i) use "derived-distributions" to find PDF $f_W(w)$ of $W = X - Y$

$\Rightarrow$




① $-1 \leq w \leq 0$

$$F_W(w) = \frac{(1+w)^2}{2} \times \frac{1}{2} \Rightarrow f_W(w) = \frac{1+w}{2}$$

② $0 \leq w \leq 1$

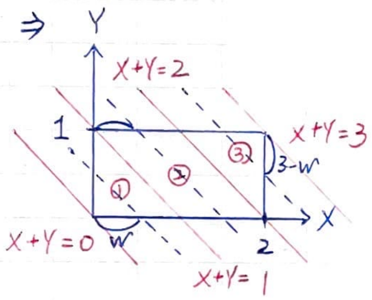
$$F_W(w) = \frac{(1+2w) \times 1}{2} \times \frac{1}{2} \Rightarrow f_W(w) = \frac{1}{2}$$

③ $1 \leq w \leq 2$

$$F_W(w) = \left[2 - \frac{(2-w)^2}{2}\right] \times \frac{1}{2} \Rightarrow f_W(w) = \frac{2-w}{2}$$

(ii) use "derived-distributions" to find PDF $f_W(w)$ of $W = X + Y$

$\Rightarrow$




① $0 \leq w \leq 1$

$$F_W(w) = \frac{w^2}{2} \times \frac{1}{2} \Rightarrow f_W(w) = \frac{w}{2}$$

② $1 \leq w \leq 2$

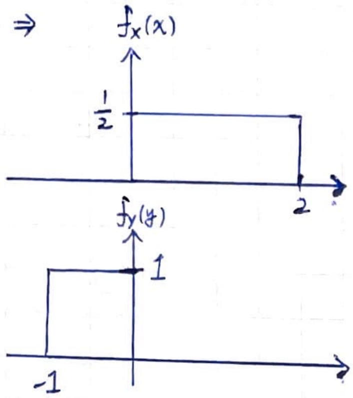
$$F_W(w) = \frac{(2w-1) \times 1}{2} \times \frac{1}{2} \Rightarrow f_W(w) = \frac{1}{2}$$

③ $2 \leq w \leq 3$

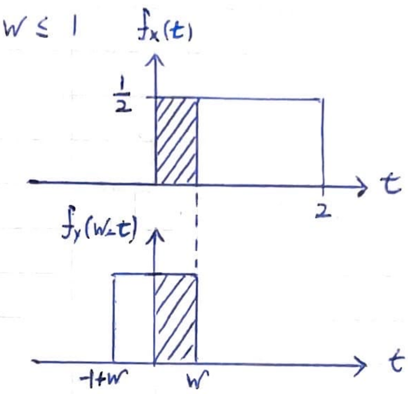
$$F_W(w) = \left[2 - \frac{(3-w)^2}{2}\right] \times \frac{1}{2} \Rightarrow f_W(w) = \frac{3-w}{2}$$

(iii) use "convolution formula" to find PDF of $W = X + Y$ $\left( f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx \right)$

$\Rightarrow$

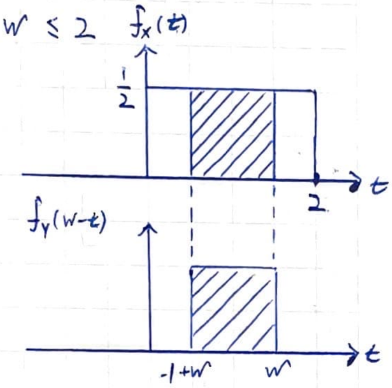



① $0 \le w \le 1$



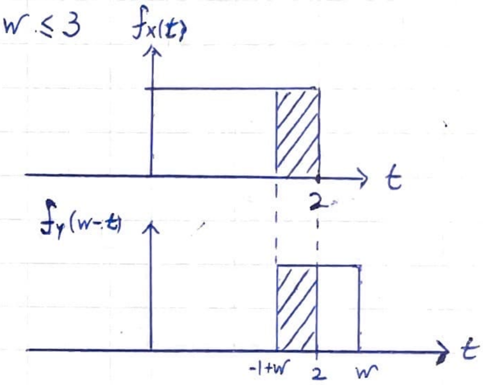

$$f_W(w) = w \times \frac{1}{2} \times 1 = \frac{w}{2}$$

② $1 \le w \le 2$




$$f_W(w) = \frac{1}{2} \times 1 = \frac{1}{2}$$

③ $2 \le w \le 3$




$$f_W(w) = (3-w) \times \frac{1}{2} \times 1 = \frac{3-w}{2}$$ ※

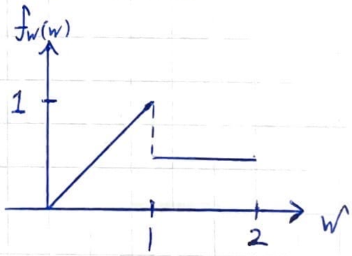
(iv) Find PDF of $W = \max(X, Y)$

$\Rightarrow F_W(w) = P(X \le w, Y \le w) = P(X \le w)\, P(Y \le w)$

① $0 \le w \le 1$

$$F_W(w) = \frac{1}{2} w \times w = \frac{w^2}{2}$$

$$f_W(w) = w$$




② $1 \le w \le 2$

$$F_W(w) = \frac{1}{2} w \times 1 = \frac{w}{2}$$

$$f_W(w) = \frac{1}{2}$$

$$\Rightarrow f_W(w) = \begin{cases} w, & 0 \le w \le 1 \\ \frac{1}{2}, & 1 \le w \le 2 \end{cases}$$ ※

2019. Quiz 4

3. Consider an exponential random variable X and Y are independent random variables with PDF below

$f_X(x) = 2e^{-2x},\ x \geq 0 \quad , \quad f_Y(y) = 3e^{-3y},\ y \geq 0$

(i) Find the transform of $W = 3X + 2Y + 2$

$$f_X(x) = \lambda e^{-\lambda x} \Rightarrow M_X(s) = \frac{\lambda}{\lambda - s} \quad (s < \lambda)$$

$$\therefore M_W(s) = \frac{2}{2-3s} \times \frac{3}{3-2s} \times e^{2s} = \frac{6e^{2s}}{(2-3s)(3-2s)} \quad ※$$

(ii) Find the transform and mean of V, whose PDF $f_V(v) = \frac{1}{3} f_X(x) + \frac{2}{3} f_Y(y)$

$$M_V(s) = \frac{1}{3} \times \frac{2}{2-3s} + \frac{2}{3} \times \frac{3}{3-2s}$$

$$E[V] = \left.\frac{dM_V(s)}{ds}\right|_{s=0} = \frac{2}{3} \times \frac{1}{(2-s)^2} + 2 \times \left.\frac{1}{(3-s)^2}\right|_{s=0} = \frac{2}{3} \times \frac{1}{4} + 2 \times \frac{1}{9} = \frac{7}{18} \text{ ※}$$

4. If X, Y, Z, W are pairwise uncorrelated random variables, each have mean = 0, variance = 1

compute the correlations of (i) X+Y and 2Y−Z (ii) X−2Y and 3W+Z

(i)

$$\rho(x+y,\ 2y-z)$$

$$= \frac{cov(X+Y,\ 2Y-Z)}{\sqrt{var(X+Y)\, var(2Y-Z)}} = \frac{\overset{0}{cov(X,2Y)} + \overset{0}{cov(X,-Z)} + \overset{2var(Y)=2}{cov(Y,2Y)} + \overset{0}{cov(Y,-Z)}}{\sqrt{[\underset{1}{var(X)} + \underset{1}{var(Y)}][\underset{4}{4\,var(Y)} + \underset{1}{var(Z)}]}}$$

$$= 2/\sqrt{10} = \sqrt{10}/5 \quad ※$$

(ii)

$$\rho(X-2Y,\ 3W-Z)$$

$$= \frac{\overset{=0}{(cov(X-2Y,\ 3W-Z))}}{\sqrt{var(X-2Y)\, var(3W-Z)}} = 0 \text{ ※}$$

5. Type $i$ lightbulbs function for a random amount of time have mean $\mu_i$, standard deviation $\delta_i$, $i = 1, 2$

A lightbulb randomly chosen is type 1 with probability $p$ and type 2 for $1-p$. Let $X$ denote the lifetime of this bulb.

Find (i) $E[X]$ (ii) $var(X)$

(i)

$$E[X] = E[X|Y=1]\,P(Y=1) + E[X|Y=2]\,P(Y=2)$$

$$= p \cdot \mu_1 + (1-p)\,\mu_2$$

(ii)

$$var(X) = E[\,var(X|Y)\,] + var(\,E[X|Y])$$

(a) $$E[\,var(X|Y)\,] = var(X|Y=1)\,P(Y=1) + var(X|Y=2)\,P(Y=2)$$

$$= p \cdot \delta_1^2 + (1-p) \cdot \delta_2^2$$

(b) $$var(\,E[X|Y]) = p(\,E[X|Y=1] - E[X])^2 + (1-p)(\,E[X|Y=2] - E[X])^2$$

$$= p(\,\mu_1 - E[X])^2 + (1-p)(\,\mu_2 - E[X])^2$$

$$= p\mu_1^2 + (1-p)\,\mu_2^2 - [\,p\mu_1 + (1-p)\,\mu_2\,]^2$$

$$\Rightarrow var(X) = p \cdot \delta_1^2 + (1-p) \cdot \delta_2^2 + p\mu_1^2 + (1-p)\,\mu_2^2 - [\,p\mu_1 + (1-p)\,\mu_2\,]^2$$

2019. quiz 4

6. A fair coin is flipped independently until the first head obtained.
For each tail observed before the first head,
a value of a continuous random variable with uniform PDF over interval $[0,2]$
Let random variable $Y$ be defined as the sum of all value obtain before the first head.

Find the mean, variance and transform of $Y$.

$\Rightarrow E[Y] = \mu_i \, E[N] = \frac{2}{2} \times \frac{1}{1/2} = 2$

$\Rightarrow var(Y) = \sigma_i^2 \, E[N] + \mu_i^2 \, var(N) = \frac{4}{12} \times \frac{1}{2} + 1^2 \times 2 = \frac{13}{6}$

$\Rightarrow var(X_i) = (b-a)^2 / 12 = \frac{4}{12}$

$\Rightarrow E[X_i] = \frac{a+b}{2} = \frac{2}{2} = 1$

$$\Rightarrow M_Y(s) = \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{e^2-1}{s}}{1-(1-\frac{1}{2})\frac{1}{2} \times \frac{e^2-1}{s}}$$

※