Midterm **Linear Algebra** 2022/10/28

1. (10) Compute the matrix-vector product of $\left(\begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}\right)\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

2. (10) Write the coefficient matrix and the augmented matrix of the given system of equations.

$$\begin{aligned} x_1 - 2x_2 + x_4 + 7x_5 &= 5 \\ x_1 - 2x_2 + 10x_5 &= 3 \\ 2x_1 - 4x_2 + 4x_4 + 8x_5 &= 7 \end{aligned}$$

3. (10) Determine whether the system is consistent, and if so, find its general solution.

$$\begin{aligned} x_1 - x_2 + x_4 &= -4 \\ x_1 - x_2 + 2x_4 + 2x_5 &= -5 \\ 3x_1 - 3x_2 + 2x_4 - 2x_5 &= -11 \end{aligned}$$

4. (10) Determine whether the given vector $\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$ is in the span of the set of vectors $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right\}$.

5. (10) Determine whether the given set of vectors are linear dependent. $\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right\}$.

6. (10) Give an example of matrices A and B such that BA is defined, but AB is not.

7. (10) Determine whether $B = A^{-1}$

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 6 & -4 & 3 & 2 \\ -2 & 2 & -2 & 1 \\ -2 & 1 & -1 & 1 \end{bmatrix}.$$

8. (10) Determine whether the matrix is invertible. If so, find its inverse.

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 3 & -1 & 0 \\ 2 & -2 & -2 & 3 \\ 9 & -5 & -3 & -1 \end{bmatrix}$$

9. (10) Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & -2 \end{bmatrix}$. Compute $T_A(\mathbf{e}_1)$ and $T_A(\mathbf{e}_3)$

10. (10) Find a generating set for the range of the linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^3$ defined by $T(\mathbf{v}) = 4\mathbf{v}$ for all $\mathbf{v} \in \mathcal{R}^3$

11. (10) Prove that, for any 2×2 matrices A and B, $\det AB = (\det A)(\det B)$.

12. (10) Let A be an $n \times n$ matrix and $b_{j,k}$ denote the (k, j)-cofactor of A

    (a) (3) Prove that if P is the matrix obtained from A by replacing column k by $\mathbf{e}_j$, then $\det(P) = b_{k,j}$

    (b) (3) Show that for each j, we have

    $$A \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{n,j} \end{bmatrix} = (\det A) \cdot \mathbf{e}_j.$$

    (c) (2) Deduce that if B is the $n \times n$ matrix whose (i, j)-entry is $b_{i,j}$, then $AB = \det(A)\mathbf{I}_n$

    (d) (2) Show that if $\det(A) \neq 0$, then $A^{-1} = \dfrac{1}{\det(A)}B$.