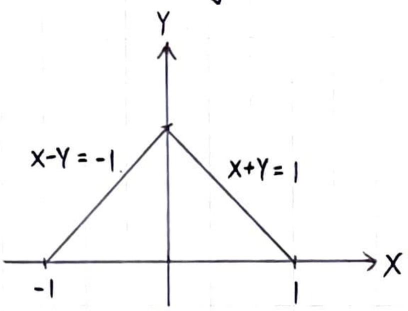
1. Given that joint PDF $f_{XY}(x,y)$ of two continuous random variable X, Y be uniform, shown in the figure

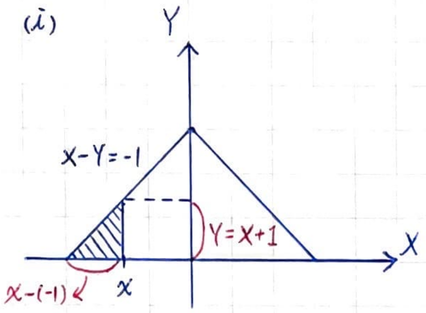


(i) Find PDF $f_X(x)$ of X

(ii) Find PDF $f_Y(y)$ of Y

(iii) Use "derived-distribution" method Find PDF $f_W(w)$ that $W = X+Y$

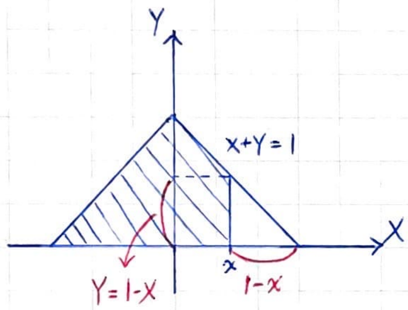
(i)



(a) for $X < 0$

$$F_X(x) = \frac{(x+1)(x-(-1))}{2} = \frac{(x+1)^2}{2} = \frac{x^2+2x+1}{2}$$

$$f_X(x) = x+1 \ \#$$

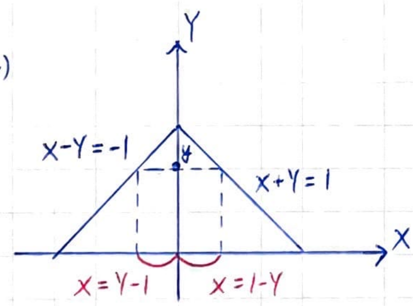


(b) for $X > 0$

$$F_X(x) = 1 - \frac{(1-x)^2}{2} = 1 - \frac{x^2-2x+1}{2} = -\frac{x^2}{2} + x + \frac{1}{2}$$

$$f_X(x) = -x+1 \ \#$$

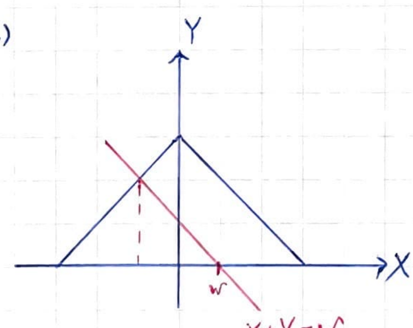
(ii)



$$F_Y(y) = \frac{[2+(1-Y)-(Y-1)] \times Y}{2} = \frac{(4-2Y)Y}{2} = 2Y - Y^2$$

$$f_Y(y) = 2-2y \ \#$$

(iii)



(i)

$$\left.\frac{1}{2}x^2+x\right]_{-1}^{0} + \left.-\frac{1}{2}x^2+x\right]_{0}^{1} = 0-\left(\frac{1}{2}-1\right)+\left(-\frac{1}{2}+1\right)$$
$$= 1$$

(ii)

$$\left.2Y-Y\right]_{-1}^{1} = (2-1)-(-1+1) = 1$$

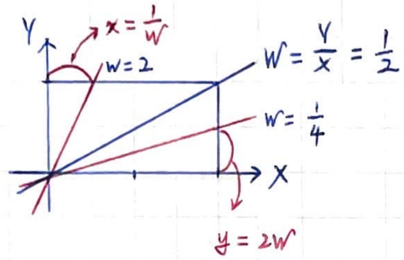
$$F_W(W) = \frac{1}{2} \times \left(\frac{W+1}{\sqrt{2}}\right)^2 = \frac{W^2+2W+1}{4}$$

$$f_W(w) = \frac{1}{2}w + \frac{1}{2} \ \#$$

2021. quiz3

2. Let X and Y be independent continuous random variables that is uniformly on $[0,1]$
What is PDF of random variable $W = Y/2X$

$\Rightarrow$ Let $X' = 2X$ $\Rightarrow$ PDf $f_x(x) = \frac{1}{2}$



(a) for $0 \leq W \leq \frac{1}{2}$

$$F_w(w) = (2w \times 2 \times \tfrac{1}{2})\, \tfrac{1}{2} = w$$

$$f_w(w) = 1 \quad \#$$

(b) for $\frac{1}{2} \leq w$

$$F_w(w) = (2 - \tfrac{1}{w} \times 1 \times \tfrac{1}{2}) \times \tfrac{1}{2} = 1 - \tfrac{1}{4w}$$

$$f_w(w) = \frac{1}{4w^2} \quad \#$$

3. Given that X is continuous random variable with PDF $f_X(x)$, $Y = aX + b$

Show that $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

4. Given that X and Y are two independent random variable having PMFs

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 1 \\ 1/2, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} 1/3, & \text{if } y = 1.2.3 \\ 0, & \text{otherwise} \end{cases}$$

Find the PMF of $Z = X + Y$ by convolution

$\Rightarrow Z$ is in $[2, 5]$

$$p_Z(2) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$p_Z(3) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$$

$$p_Z(4) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$$

$$p_Z(5) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\Rightarrow p_Z(z) = \begin{cases} 1/6, & z = 2, 5 \\ 1/3, & z = 3, 4 \\ 0, & \text{otherwise} \end{cases} \quad \#$$

5. Given that X and Y are two independent random variables, which mean = 3, variance = 2

Let $U = 2X + 3Y$, $V = 3X - 2Y$

Find $var(U)$, $cov(U, V)$

(i)

$$var(2X + 3Y) = var(2X) + var(3Y)$$

$$= 4\,var(X) + 9\,var(Y) = 13 \times 2 = 26 \ \#$$

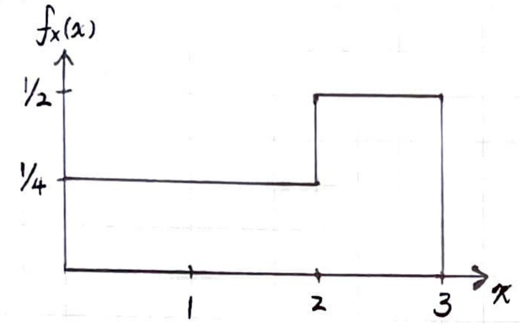
(ii)

$$cov(U, V) = cov(2X + 3Y, 3X - 2Y)$$

$$= cov(2X, 3X) + cov(2X, -2Y) + cov(3Y, 3X) + cov(3Y, -2Y)$$

$$= 6\,var(X) + 0 + 0 + (-6)\,var(Y) = 0 \ \#$$

6. Consider random variable X given PDF below, another random variable Y defined below



$$Y = \begin{cases} 0, & 0 \le x \le 2 \\ 1, & 2 \le x \le 3 \end{cases}$$

Find the mean and variance of "the law of iterated expectations"
"the law of total variance"

(a) law of Iterated expectations : $E[X] = E[\,E[X|Y]\,]$

$\Rightarrow E[X \mid Y=0] = 1$

$E[X \mid Y=1] = \frac{5}{2}$

$\Rightarrow E[X] = 1 \times \frac{1}{2} + \frac{5}{2} \times \frac{1}{2} = \frac{7}{4}$ #

(b) law of total variance : $E[\,var(X|Y)\,] + var(\,E[X|Y]\,)$

$\Rightarrow var(X \mid Y=0) = \frac{(2-0)^2}{12} = \frac{1}{3}$

$var(X \mid Y=1) = \frac{(3-2)^2}{12} = \frac{1}{12}$

$\Rightarrow E[\,var(X|Y)\,] = \frac{1}{3} \times \frac{1}{2} + \frac{1}{12} \times \frac{1}{2} = \frac{5}{24}$

$\Rightarrow var(\,E[X|Y]\,) = (1-\frac{7}{4})^2 \times \frac{1}{2} + (\frac{5}{2}-\frac{7}{4})^2 \times \frac{1}{2} = (\frac{9}{16}+\frac{9}{16})\frac{1}{2} = \frac{9}{16}$

$\Rightarrow var(X) = \frac{5}{24} + \frac{9}{16} = 37/48$ #

7. Given X, Y are two continuous random variables, show that $E[X] = E[\,E[X|Y]\,]$

$$\begin{aligned}
E[\,E[X|Y]\,] &= \int_y \int_x x\, f_{X|Y}(x|y)\, dx\, f_Y(y)\, dy \\
&= \int_x x \int_y \underline{f_{X|Y}(x|y)\, f_Y(y)}\, dy\, dx \\
&= \int_x x \left[\int_y \underline{f_{XY}(x,y)}\, dy\right] dx \\
&= \int_x x \left[f_X(x)\right] dx \\
&= E[X]
\end{aligned}$$
#