1. (i) State the three axioms

(a) Nonnegativity : $\forall x$, $P(x) \geq 0$

(b) Addictivity : $\forall A, B$ disjoint , $P(A \cup B) = P(A) + P(B)$

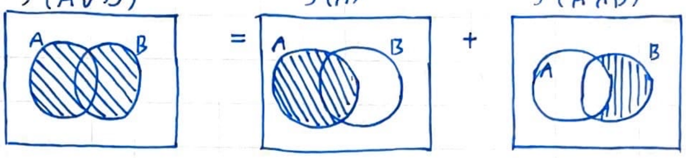
(c) Normalization : $P(\Omega) = 1$

(ii) Use Venn diagram and probability axioms to prove that $P(A \cup B) = P(A) + P(B)P(\bar{A} | B)$

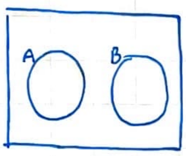
$\Rightarrow P(A) + P(B)P(\bar{A}|B) = P(A) + P(\bar{A} \cap B)$

case 1 :

$P(A \cup B)$ = $P(A)$ + $P(\bar{A} \cap B)$



case 2 : disjoint



$P(A \cup B) = P(A) + P(B)$, $P(\bar{A} \cap B) = P(B)$

2. Given X is a discrete random variable with PMF $p_X(X)$,
it's expectation is defined by $E[X] = \sum_x x\, p_X(x)$.
Let $g(X)$ be a function of X.
Show that the expectation of the random variable $g(X)$ is $E[g(x)] = \sum_x g(X)\, p_X(X)$.

$\Rightarrow$ Let $Y = g(x)$

$$E[Y] = \sum_y Y\, p_Y(Y) = \sum_y Y \sum_{\{x | g(x) = y\}} p_X(x) = \sum_y \sum_{\{x | g(x) = y\}} g(x)\, p_X(x) = \sum_x g(x)\, p_X(x) = E[g(x)]$$

3. Given that a Binomial distribution with parameters (n, p) can be viewed as the sum of n independent and identically distributed Beruoulli variables (with parameter p)

Show that the mean and variance of the Binomial distribution is np and $np(1-p)$.

$$\text{mean} = E[x] = \sum_x x P_x(x) = \sum_{i=1}^{n} P_x(x=i) = n \times p \quad \#$$

$$\text{variance} = var(x) = E[x^2] - E[x]^2 = \sum_x x^2 P_x(x) - \left(\sum_x x P_x(x)\right)^2$$

$$= \frac{n^2p - n^2p^2}{n} = np(1-p) \quad \#$$

4. Three classmates Minsky, Fred and Jelinek will individually and independently pass English oral test with a success probability of 0.8, 0.9, 0.6 respectively

(i) What is the probability that there is at least one of them passing the test

$$\Rightarrow 1 - P(\text{三人皆 fail}) = 1 - (0.2 \times 0.1 \times 0.4) = 1 - \frac{8}{1000} = 1 - \frac{1}{125} = \frac{124}{125} \quad \#$$

(ii) Given that at least one of them pass the test.
What is the probability that only Jelinek pass the test

$$\Rightarrow P(\text{only Jelinek} \mid \text{至少一人 pass}) = \frac{0.2 \times 0.1 \times 0.6}{124/125} = \frac{12/1000}{124/125} = \frac{3/250}{124/125} = \frac{3}{248} \quad \#$$

5. An experiment consists of 4 independent tosses of fair coin
Let random variable X model the number of heads obtained.
Given that a random variable Y is defined by $Y = X \bmod 3$
Find $E[Y]$, $var(Y)$, given that $var(Y)$ is defined by $E[(Y-E[Y])^2]$.

$$p_X(x) \begin{cases} 1/16, & x=0 \\ 4/16, & x=1 \\ 6/16, & x=2 \\ 4/16, & x=3 \\ 1/16, & x=4 \end{cases} \Rightarrow p_Y(Y) = \begin{cases} 5/16, & y=0 \\ 5/16, & y=1 \\ 6/16, & y=2 \end{cases}$$

$$E[Y] = \frac{5}{16}\times 0 + \frac{5}{16}\times 1 + \frac{6}{16}\times 2 = \frac{17}{16}$$

$$E[Y^2] = \frac{5}{16}\times 0 + \frac{5}{16}\times 1 + \frac{6}{16}\times 4 = \frac{29}{16}$$

$$Var(Y) = E[Y^2] - E[Y]^2$$

$$= \frac{29}{16} - \frac{289}{256} = \frac{175}{256} \quad \#$$

$$* \; E[(Y-E[Y])^2] = E[Y^2 - 2Y E[Y] + E[Y]^2]$$

$$= E[Y^2] - 2E[Y]^2 + E[Y]^2$$

$$= E[Y^2] - E[Y]^2$$

6. Given the random variables X and Y whose joint PMF $p_{XY}(x,y)$ is given below.
and a new random variable Z is defined by $Z = X + Y^2$

Y \ X	1	2	3	4
3	2/12	0	2/12	1/12
2	1/12	1/12	1/12	0
1	0	1/12	1/12	2/12

(i) Calculate $E[Z]$ & $var(Z)$

$$E[X] = \frac{3}{12}\times 1 + \frac{2}{12}\times 2 + \frac{4}{12}\times 3 + \frac{3}{12}\times 4 = \frac{31}{12}$$

$$E[Y^2] = \frac{4}{12}\times 1 + \frac{3}{12}\times 4 + \frac{5}{12}\times 9 = \frac{61}{12}$$

$$E[Z] = E[X] + E[Y^2] = 92/12 = 23/3 \quad \#$$

$$var(Z) = E[Z^2] - E[Z]^2$$

$$E[Z^2] = E[X^2] + 2E[X]E[Y^2] + E[Y^4] = \frac{95}{12} + 2\times\frac{31}{12}\times\frac{61}{12} + \frac{457}{12} = \frac{5203}{72}$$

$$var(Z) = \frac{5203}{72} - \frac{529}{9} = \frac{971}{72}$$

(ii) Find the conditional PMFs $p_{Y|X}(y|3)$, $p_{X|Y}(x|2)$

$$p_{Y|X}(Y|3) = \begin{cases} \frac{1}{4}, & (2,3)\ (3,3) \\ \frac{2}{4}, & (1,3) \end{cases}$$

(y, x)

$$p_{X|Y}(x|2) = \begin{cases} \frac{1}{3} & (1,2)\ (2,2)\ (3,2) \\ 0 & (4,2) \end{cases}$$

(x, y)

(iii) Calculate $E[X]$ by way of $E[X] = \sum_y P_Y(y)\,[X \mid Y=y]$

$$E[X] = \sum_y P_Y(y)\, E[X \mid Y=y]$$

$$= P_Y(1)\, E[X \mid y=1] + P_Y(2)\, E[X \mid y=2] + P_Y(3)\, E[X \mid y=3]$$

$$= \frac{4}{12} \times \frac{13}{4} + \frac{3}{12} \times 2 + \frac{5}{12} \times \frac{12}{5} = \frac{31}{12} \#$$

$$E[X \mid y=1] = 1 \times 0 + 2 \times \frac{1}{12} + 3 \times \frac{1}{12} + 4 \times \frac{2}{12} = 13/4$$
$$E[X \mid y=2] = 1 \times \frac{1}{12} + 2 \times \frac{1}{12} + 3 \times \frac{1}{12} + 4 \times 0 = 2$$
$$E[X \mid y=3] = 1 \times \frac{2}{12} + 2 \times 0 + 3 \times \frac{2}{12} + 4 \times \frac{1}{12} = 12/5$$

(iv) Suppose that A is an event expressed by $A = [X=1 \text{ and } 2 \le Y \le 3]$ determine where X and Y are independent of each other given A

$\Rightarrow$ If independent $P_{XY}(1,2) = P_X(1)\, P_Y(2) \Rightarrow \frac{1}{3} = 1 \times \frac{1}{3}$
$P_{XY}(1,3) = P_X(1)\, P_Y(3) \Rightarrow \frac{2}{3} = 1 \times \frac{2}{3}$ $\Rightarrow$ both independent

$P_X(1) = 1$, $P_Y(2) = \frac{1}{3}$, $P_Y(3) = \frac{2}{3}$

7. Jenny and Ted alternate rolling a pair of 6-sided dice, stopping either when Jenny rolls the sum of 5 or Ted rolls the sum of 8. Given that Ted rolls first, find the probability that the final roll is made by Ted

$P(\text{Jenny roll } 5) = 4/36$
$P(\text{Ted roll } 8) = 5/36$

8. Let X be a Passion random variable with parameter λ (PMF is defined by $P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$)
What value λ of that maximizes $P(X=k)$

$$P_X(k) = e^{-\frac{\lambda}{k}} \times \frac{\lambda^k}{k!}$$

$$\Rightarrow \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda}\left[\frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots + \frac{\lambda^k}{k!}\right]$$

$$= e^{-\lambda}$$