Introduction to Probability
Midterm I

15:30-17:20, November 1, 2022

Note: You have to answer the questions with supporting explanations if needed.

1. (a) State and explain the three probability axioms. (5%)
   (b) let $A_1, \cdots, A_n, \cdots, A_N$ be disjoint events that form a partition of the sample space and assume that $\mathbf{P}(A_n) > 0$, for all n. Then, for any event B, show that: $\mathbf{P}(B) = \sum_{i=1}^{N} \mathbf{P}(A_i)\mathbf{P}(B|A_i)$. (5%)

2. Allen has the habit of collecting different types of hats made by a company named **FashionX**. There are n types of hats made by **FashionX**, and each new one Allen collects will belong to type i with probability p_i, $\sum_{i=1}^{n} p_i = 1$. Furthermore, the types of different hats Allen will collect are independent of one another. Suppose that m hats are to be collected by Allen. If H_i is the event that there is at least one type i hat among those collected.
   (1) Find $\mathbf{P}(H_i)$. (10%)
   (2) Find $\mathbf{P}(H_i \cup H_j)$, where $i \neq j$. (5%)
   (3) Find $\mathbf{P}(H_i | H_j)$, where $i \neq j$. (5%)

3. Two random variables X and Y whose joint probability mass function (PMF) is given in the right figure.
   (1) Calculate $\mathbf{E}[X]$ and $\mathbf{E}[Y]$. (10%)
   (2) Determine whether X and Y are independent of each other or not. (10%)
   (3) Given an event $A = \{X \leq 2, Y \geq 2\}$ occurs, determine whether X and Y are independent of each other or not. (10%)

Joint PMF $p_{X,Y}(x,y)$

$y \backslash x$	1	2	3
3	1/4	1/6	1/12
2	1/6	1/9	1/18
1	1/12	1/18	1/36

4. A pair of fair four-sided dice is thrown once. Each die has faces labeled 1, 2, 3, and 4. Discrete random variable X is defined to be the product of the down-face values (四面骰子面朝下那一面的數值).
   (1) Determine probability mass function (PMF) of X. (10%)
   (2) Determine the conditional expectation and variance of X, given that the sum of the down-face values is greater than the product of the down-face values (denoted by event D); that is, $\mathbf{E}[X|D]$ and $\mathrm{var}(X|D)$. (10%)

5. Let X and Y be independent random variables, both of which take values in the set $\{1, 2, 3\}$ with equal probability. Let $V = X + Y$, and $W = X - Y$.
   (1) Find the PMF V. (10%)
   (2) Determine $\mathbf{E}[V]$ and $\mathrm{var}(V)$. (10%)
   (3) Are V and W independent of each other? Explain (no calculations needed). (10%)

2+6+6+10+12

var[V] = E[· E[V]]

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

$\frac{1}{7}+\frac{4}{7}+\frac{6}{7}+\frac{8}{7}$

$\frac{1+8+18+32}{7}$

	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

$\frac{59\times7}{7}$ $\frac{361}{49}$

$\frac{413}{49}$ 413 13+39 52

4 36 18 50 48

40 40 108 48 156

$\frac{156}{9} - \frac{46\times4}{9}$

156 − 144 = 12.