Hw 2020 - 1

1. Determine whether the following system is consistent and if so, find the vector form of its general solution.

$$\begin{cases} x_1 - x_3 - 2x_4 - 8x_5 = -3 \\ -2x_1 + x_3 + 2x_4 + 9x_5 = 5 \\ 3x_1 - 2x_3 - 3x_4 - 15x_5 = -9 \end{cases}$$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -8 & -3 \\ -2 & 0 & 1 & 2 & 9 & 5 \\ 3 & 0 & -2 & -3 & -15 & -9 \end{array}\right] \xrightarrow[-3r_1+r_3 \to r_3]{2r_1+r_2 \to r_2} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -8 & -3 \\ 0 & 0 & -1 & -2 & -7 & -1 \\ 0 & 0 & 1 & 3 & 9 & 0 \end{array}\right]$$

$$\xrightarrow[-r_2 \to r_2]{r_2+r_3 \to r_3} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -8 & -3 \\ 0 & 0 & +1 & +2 & +7 & +1 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{r_1+r_2 \to r_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 7 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{r_2-2r_3 \to r_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = x_5 - 2 \\ x_2 \text{ free} \\ x_3 = -3x_5 + 3 \\ x_4 = -2x_5 - 1 \\ x_5 \text{ free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -3 \\ -2 \\ 0 \end{bmatrix} x_5 \quad \#$$

(general solution) (vector form)

2. Find the rank and nullity of matrix $\begin{bmatrix} 1 & -2 & 0 & -3 & 1 \\ 2 & -4 & -1 & -8 & 8 \\ -1 & 2 & 1 & 5 & -7 \\ 0 & 0 & 1 & 2 & -6 \end{bmatrix}$

$$\xrightarrow[r_3+r_1 \to r_3]{r_2-2r_1 \to r_2} \begin{bmatrix} 1 & -2 & 0 & -3 & 1 \\ 0 & 0 & -1 & -2 & 6 \\ 0 & 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & 2 & -6 \end{bmatrix} \xrightarrow[r_2+r_4 \to r_4]{r_2+r_3 \to r_3} \begin{bmatrix} 1 & -2 & 0 & -3 & 1 \\ 0 & 0 & 1 & 2 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank = 2

$\Rightarrow$ Nullity = 5 - 2 = 3

3. Let u and v be any vectors in R^n. Prove that the span of $\{u, v\}$ and $\{u+v, u-v\}$ are equal.

Define the span of $\{u, v\} = a_1\vec{u} + a_2\vec{v}$, where a_1, a_2 are scalars
Define the span of $\{u+v, u-v\} = b_1(\vec{u}+\vec{v}) + b_2(\vec{u}-\vec{v})$, where b_1, b_2 are scalars

$\Rightarrow b_1(\vec{u}+\vec{v}) + b_2(\vec{u}-\vec{v}) = (b_1+b_2)\vec{u} + (b_1-b_2)\vec{v}$, (b_1+b_2), (b_1-b_2) are also scalars

which shows that the span of $\{u, v\}$ and $\{u+v, u-v\}$ are equal.

4. Let A be an $m\times n$ matrix with reduced row echelon form R.
Determine the reduce row echelon form of each of the following matrices.

(a) $[A \ \ 0]$ $\Rightarrow$ $[R \ \ 0]$

(b) $[a_1 \ a_2 \ \cdots \ a_k]$ for $k<n$, where $a_i = Ae_i$

$\Rightarrow [r_1 \ r_2 \ \cdots \ r_k]$ for $k<n$, where $r_i = Re_i$

(c) cA, where c is a nonzero scalar $\Rightarrow R$

(d) $[I_m \ \ A]$ $\Rightarrow$ $[I_m \ \ A]$

(e) $[A \ \ cA]$, where c is any scalar $\Rightarrow$ $[R \ \ cR]$

5. Given $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \end{bmatrix}$, determine whether the equation $Ax = b$ is consistent for every b in R^4

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \end{bmatrix} \xrightarrow[-3r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -2 & 2 \end{bmatrix} \xrightarrow{-4r_2+r_3 \to r_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_3 \to r_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[r_3+r_2 \to r_2]{-2r_3+r_1 \to r_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ Rank $A = 3 = m$ $\Rightarrow$ Ans. Yes #

6. The input-output matrix for an economy producing transportation, food and oil follows

$$\begin{array}{cccc} T & F & O & \\ \end{array}$$
$$\begin{bmatrix} .2 & .2 & .3 \\ .4 & .3 & .1 \\ .2 & .25 & .3 \end{bmatrix} \begin{array}{c} T \\ F \\ O \end{array}$$

(a) What is the net production corresponding to a gross production of \$40 million of transportation, \$30 million of food, \$35 million of oil?

Let $\vec{x} = \begin{bmatrix} 40 \\ 30 \\ 35 \end{bmatrix}$ $C = \begin{bmatrix} .2 & .2 & .3 \\ .4 & .3 & .1 \\ .2 & .25 & .3 \end{bmatrix}$

$$\vec{x} - C\vec{x} = \begin{bmatrix} 40 \\ 30 \\ 35 \end{bmatrix} - \begin{bmatrix} .2 & .2 & .3 \\ .4 & .3 & .1 \\ .2 & .25 & .3 \end{bmatrix}\begin{bmatrix} 40 \\ 30 \\ 35 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 35 \end{bmatrix} - \begin{bmatrix} 24.5 \\ 28.5 \\ 26 \end{bmatrix} = \begin{bmatrix} 15.5 \\ 1.5 \\ 9 \end{bmatrix}$$

$\Rightarrow$ 15.5 million of transportation, 1.5 million of food, 9 million of oil #

(b) What gross production is required to satisfy exactly a demand for \$32 million of transportation, \$48 million of food, \$24 million of oil?

$$\vec{x} - C\vec{x} = (I_n - C)\vec{x} = \vec{d} = \begin{bmatrix} 32 \\ 48 \\ 24 \end{bmatrix}$$

$$I_n - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.4 & 0.3 & 0.1 \\ 0.2 & 0.25 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.2 & -0.3 \\ -0.4 & 0.7 & -0.1 \\ -0.2 & -0.25 & 0.7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.8 & -0.2 & -0.3 & 32 \\ -0.4 & 0.7 & -0.1 & 48 \\ -0.2 & -0.25 & 0.7 & 24 \end{bmatrix} \xrightarrow{\times 100} \begin{bmatrix} 80 & -20 & -30 & 3200 \\ -40 & 70 & -10 & 4800 \\ -20 & -25 & 70 & 2400 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4}r_1 + r_3 \to r_3]{\frac{1}{2}r_1 + r_2 \to r_2} \begin{bmatrix} 80 & -20 & -30 & 3200 \\ 0 & 60 & -25 & 6400 \\ 0 & -30 & 125/2 & 3200 \end{bmatrix} \xrightarrow{r_2 + 2r_3 \to r_3} \begin{bmatrix} 80 & -20 & -30 & 3200 \\ 0 & 60 & -25 & 6400 \\ 0 & 0 & 100 & 12800 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -2 & -3 & 320 \\ 0 & 12 & -5 & 1280 \\ 0 & 0 & 1 & 128 \end{bmatrix} \xrightarrow[r_2 + 5r_3 \to r_2]{r_1 + 3r_3 \to r_1} \begin{bmatrix} 8 & -2 & 0 & 704 \\ 0 & 12 & 0 & 1920 \\ 0 & 0 & 1 & 128 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & 0 & 352 \\ 0 & 1 & 0 & 160 \\ 0 & 0 & 1 & 128 \end{bmatrix} \xrightarrow{r_1 + r_2 \to r_1} \begin{bmatrix} 4 & 0 & 0 & 512 \\ 0 & 1 & 0 & 160 \\ 0 & 0 & 1 & 128 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 128 \\ 0 & 1 & 0 & 160 \\ 0 & 0 & 1 & 128 \end{bmatrix} \quad \Rightarrow \vec{x} = \begin{bmatrix} 128 \\ 160 \\ 128 \end{bmatrix}$$ #

7. Determine, if possible, a value of r for which the following set of vector is linearly independent $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \\ r \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ 3 & 6 & -2 \\ -1 & 1 & r \end{bmatrix} \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 5 \\ 0 & -3 & 1 \\ 0 & 4 & r-1 \end{bmatrix} \xrightarrow[r_1 + r_3 \to r_1]{-\frac{1}{5}r_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & r-3 \end{bmatrix}$$

$$\xrightarrow{r_3 + 3r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & r-3 \end{bmatrix} \xrightarrow[-\frac{1}{2}r_3 \to r_3]{r_2 + \frac{1}{2}r_3 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & r-3 \end{bmatrix}$$

$\Rightarrow$ Rank $= 3 \Rightarrow$ Nullity $= 0$

$\Rightarrow$ This Matrix is L.I.

$\Rightarrow$ impossible #

8. Let $\{u_1, u_2 \cdots, u_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$ and let v be a vector in $\mathbb{R}^n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_ku_k$ for scalars $c_1 \sim c_k$, $c_1 \neq 0$.

Prove that $\{v, u_2, \cdots u_k\}$ is linearly independent.

$\Rightarrow \{u_1, u_2 \cdots u_k\}$ are L.I. $\Rightarrow a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_k\vec{u_k} = 0 \ \forall a_1, a_2 \cdots a_k \in \mathbb{R}$

then $a_1 = a_2 = \cdots = a_k = 0$ is the solution.

Assume $b_1 \sim b_k \in \mathbb{R}$

$$0 = b_1\vec{v} + b_2\vec{u_2} + \cdots + b_k\vec{u_k} = b_1(c_1\vec{u_1} + \cdots + c_k\vec{u_k}) + b_2\vec{u_2} + \cdots + b_k\vec{u_k}$$

$$= (b_1c_1)\vec{u_1} + (b_1c_2 + b_2)\vec{u_2} + \cdots + (b_1c_k + b_k)\vec{u_k}$$

$\because \{u_1, u_2, \cdots u_k\}$ are L.I.

$\therefore b_1c_1 = 0, \ b_1c_2 + b_2 = 0 \ \cdots \ b_1c_k + b_k = 0$

Since $c_1 \neq 0$

$\Rightarrow b_1 = b_2 = \cdots = b_k = 0$, Thus $\{\vec{v}, \vec{u_2}, \cdots, \vec{u_k}\}$ is L.I. #