1. Consider a random variable W with probability density function (PDF) as follows

$$f_W(w) = \begin{cases} a(1-w^2), & \text{if } -1 \le w \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) What is the cumulative distribution function (CDF) of W

$$\Rightarrow \int_{-1}^{1} a(1-w^2)\,dw = a\left(w - \frac{w^3}{3}\right)\Big|_{-1}^{1} = 1$$

$$\Rightarrow a\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right] = a\left(\frac{4}{3}\right) = 1 \quad \Rightarrow a = \frac{3}{4}$$

$$\Rightarrow \int f_W(w)\,dw = \frac{3}{4}\left(w - \frac{w^3}{3}\right) + C = F_W(w)$$

$$\Rightarrow F_W(w) = 1 \quad \Rightarrow \frac{3}{4}\left(1-\frac{1}{3}\right) + C = 1 \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\therefore F_W(w) = \frac{3}{4}\left(w - \frac{w^3}{3}\right) + \frac{1}{2} = -\frac{1}{4}w^3 + \frac{3}{4}w + \frac{1}{2} \quad \#$$

(ii) Given that $A = \{\frac{1}{4} \le W \le 1\}$. Calculate the conditional probability $P(W \ge \frac{1}{2} \mid A)$

$$\Rightarrow P(W \ge \tfrac{1}{2} \mid A) = \frac{P(W \ge \frac{1}{2} \cap A)}{P(A)} = \frac{1 - F_W(\frac{1}{2})}{1 - F_W(\frac{1}{4})} = \frac{1 - \frac{27}{32}}{1 - \frac{175}{256}} = \frac{40}{81} \quad \#$$

(iii) Let $U = W^2$. Find the expectation $E[U]$ and variance $var(U)$

$$E[U] = E[W^2] = \int_{-\infty}^{\infty} (w^2 \times f_W(w))\,dw = \frac{3}{4}\int_{-1}^{1} (w^2 - w^4)\,dw$$

$$= \frac{3}{4}\left[\frac{w^3}{3} - \frac{w^5}{5}\right]_{-1}^{1} = \frac{3}{4}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}\right) = \frac{1}{5}$$

$$E[U^2] = E[W^4] = \int_{-\infty}^{\infty} (w^4 \times f_W(w))\,dw = \frac{3}{4}\int_{-1}^{1} (w^4 - w^6)\,dw$$

$$= \frac{3}{4}\left[\frac{w^5}{5} - \frac{w^7}{7}\right]_{-1}^{1} = \frac{3}{4}\left(\frac{1}{5} - \frac{1}{7} + \frac{1}{5} - \frac{1}{7}\right) = \frac{3}{35}$$

$$var(U) = E[U^2] - E[U]^2 = \frac{3}{35} - \frac{1}{25} = \frac{8}{175} \quad \#$$

2 The annual rainfall (in inches) in Taipei area is normally distributed with mean 60 and variance 100.

Find the probability that starting from this year (2020), it will at least take over 3 years that consecutively have a rainfall every year over 40 inches.

Given a few CDF values of standard normal:

$\phi(0) = 0.5$, $\phi(1) = 0.8413$, $\phi(1.5) = 0.9332$, $\phi(2.0) = 0.9772$, $\phi(2.5) = 0.9938$, $\phi(3.0) = 0.9987$

$\Rightarrow Y = \dfrac{X-60}{10}$

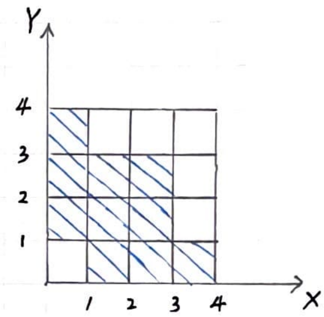
$$P(X \geq 40) = 1 - P(X < 40) = 1 - P\left(\frac{X-60}{10} < \frac{40-60}{10}\right) = 1 - P(Y < -2)$$

$$= 1 - [1 - \phi(2)] = 0.9772 .$$

所求：$(0.9772)^3$ #

3. We are told that the joint PDF of random variable X , Y is constant (uniformly distributed) in the "shaded" area of the figure shown on right.

(i) Find the PDF $f_X(x)$ of X and $f_Y(Y)$ of Y



$$f_X(x) = \begin{cases} \frac{3}{10}, & 0 \leq x \leq 3 \\ \frac{1}{10}, & 3 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(Y) = \begin{cases} \frac{3}{10}, & 0 \leq x \leq 3 \\ \frac{1}{10}, & 3 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

(ii) Given that the event $B = \{0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}$ occurs, determine whether X and Y are dependent or independent.

If independent: $P(0 \leq x \leq 1, 0 \leq y \leq 1) = P(0 \leq x \leq 1) P(0 \leq y \leq 1)$ $\left(0 \neq \frac{1}{3} \times \frac{1}{3}\right)$

$P(0 \leq x \leq 1, 1 \leq y \leq 2) = P(0 \leq x \leq 1) P(1 \leq y \leq 2)$ $\left(\frac{1}{3} \neq \frac{1}{3} \times \frac{2}{3}\right)$

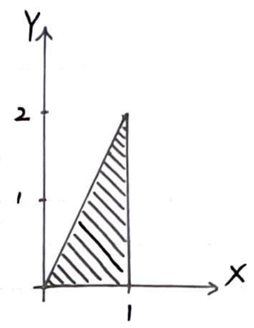
$P(1 \leq x \leq 2, 0 \leq y \leq 1) = P(1 \leq x \leq 2) P(0 \leq y \leq 1)$ $\left(\frac{1}{3} \neq \frac{2}{3} \times \frac{1}{3}\right)$

$P(1 \leq x \leq 2, 1 \leq y \leq 2) = P(1 \leq x \leq 2) P(1 \leq y \leq 2)$ $\left(\frac{1}{3} \neq \frac{2}{3} \times \frac{2}{3}\right)$

$\Rightarrow X, Y$ are dependent #

4. Let X and Y has a joint continuous PDF the "shaded" area of the figure, which is defined by

$$f_{XY}(x,y) = \begin{cases} 2xy, & \text{if } 0 \le y \le 2x < 2, \\ 0, & \text{otherwise} \end{cases}$$



(i) Find the conditional PDF $f_{X|Y}(x|y)$ of X given $Y=y$

$$f_Y(y) = \int_{y/2}^{1} f_{XY}(x,y)\,dx = \int_{y/2}^{1} 2xy\,dx = x^2 y \Big|_{y/2}^{1}$$

$$= y - \frac{y^3}{4}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2xy}{y - y^3/4} = \frac{2x}{1-\frac{y^2}{4}} = \frac{8x}{4-y^2} \quad \#$$

(ii) Find the conditional probability $P(Y \ge \frac{2}{5} \mid X = x)$

$$f_X(x) = \int_0^{2x} 2xy\,dy = y^2 x \Big|_0^{2x} = 4x^3$$

$$f_{Y|X}(y|x) = \frac{2xy}{4x^3} = \frac{y}{2x^2}$$

If $x < \frac{1}{5}$, $P(Y \ge \frac{2}{5} \mid X = x) = 0$

If $x \ge \frac{1}{5}$, $P(Y \ge \frac{2}{5} \mid X = x) = \int_{2/5}^{2x} \frac{y}{2x^2}\,dy = \frac{1}{4}\left[\frac{y^2}{x^2}\right]_{2/5}^{2x} = 1 - \frac{1}{25}x^{-2} \quad \#$

5. Fred goes to the bank to make a withdrawal, and is likely to find 0 (probability $2/3$), 1 (probability $1/3$) customer ahead of him. The service time of the customer ahead, if present, is exponentially distribution with parameter λ. What is the PDF and CDF of Fred's waiting time?

note that the PDF of an exponential random variable is :

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow F_X(x) = P(X \leq x) = P(X \leq x \mid Y=0)\,P(Y=0) + P(X \leq x \mid Y=1)\,P(Y=1)$$

$$= 1 \times \frac{2}{3} + P(X \leq x \mid Y=1) \times \frac{1}{3}$$

CDF:

$$F_X(x) = 1 \times \frac{2}{3} + \frac{1}{3} \cdot \int_0^{1} \lambda e^{-\lambda x}\,dx = \frac{2}{3} + \frac{1}{3}\left(1 - e^{-\lambda x}\right) = 1 - \frac{1}{3} e^{-\lambda x}$$ ※