HW 2020-2

1\. Let $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 1 & -1 & 3 \\ -1 & -1 & 4 & 3 \\ -5 & 3 & -4 & 7 \end{bmatrix}$. Find a nonzero matrix 4×2 B with rank 2 such that $AB = 0$

Let $B = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \\ b_{14} & b_{24} \end{bmatrix}$ $\quad AB = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 1 & -1 & 3 \\ -1 & -1 & 4 & 3 \\ -5 & 3 & -4 & 7 \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \\ b_{14} & b_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 1 & -1 & 3 \\ -1 & -1 & 4 & 3 \\ -5 & 3 & -4 & 7 \end{bmatrix} \xrightarrow[\substack{r_1+r_3\to r_3 \\ 5r_1+r_4\to r_4}]{2r_1+r_2\to r_2} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & -2 & 6 & 2 \end{bmatrix} \xrightarrow[-2r_2+r_4\to r_4]{\substack{-r_2+r_1\to r_1 \\ -2r_2+r_3\to r_3}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(general solution)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_3 + 2b_4 \\ 3b_3 + b_4 \\ b_3 \\ b_4 \end{bmatrix} = b_3 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \#$$

2\. The trace of an $m\times n$ matrix A, written trace(A), is defined to be the sum

trace$(A) = a_{11} + a_{22} + \cdots + a_{nn}$

Prove that if A is an $m\times n$ matrix and B is an $n\times m$ matrix, than trace(AB) = trace(BA)

$\Rightarrow$ The trace of AB is the sum of the diagonal entries of AB, that is

$$[a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}] + [a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}] + \cdots + [a_{m1}b_{1m} + \cdots + a_{mn}b_{nm}]$$

The trace of BA is

$$[b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1m}a_{m1}] + [b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2m}a_{m2}] + \cdots + [b_{n1}a_{1n} + \cdots + b_{nm}a_{mn}]$$

$\Rightarrow$ We can notice that trace(AB) = trace(BA) #

3. For a given matrix B, find columns b_3 and b_4 as a linear combination of the pivot columns of B

where
$$B = \begin{bmatrix} 1 & 0 & 1 & -3 & -1 & 4 \\ 2 & -1 & 3 & -8 & -1 & 9 \\ -1 & 1 & -2 & 5 & 1 & -6 \\ 0 & 1 & -1 & 2 & 1 & -3 \end{bmatrix}.$$

$$\xrightarrow[r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1 & -3 & -1 & 4 \\ 0 & -1 & 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 2 & 1 & -3 \end{bmatrix} \xrightarrow[r_2 + r_4 \to r_4]{\substack{-r_2 \to r_2 \\ r_2 + r_3 \to r_3}} \begin{bmatrix} 1 & 0 & 1 & -3 & -1 & 4 \\ 0 & 1 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow[r_4 - 2r_1 \to r_4]{\substack{r_1 + r_3 \to r_1 \\ r_2 + r_3 \to r_2}} \begin{bmatrix} 1 & 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & -1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} b_3 &= b_1 - b_2 + 0 b_5 . \\ b_4 &= -3b_1 + 2b_2 + 0 b_5 \end{aligned} \quad \#$$

4. Suppose that u, v are linearly independent vectors in $\mathbb{R}^3$
Find the reduced row echelon form of $A = [a_1\ a_2\ a_3\ a_4]$
given $a_1 = u$, $a_2 = 2u$, $a_3 = u + v$, $a_4 = v$

Let $a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $a_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $a_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $a_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \#$$

6. Let A and B be $n \times n$ matrices. We say that A is similar to B if $B = P^{-1}AP$ for invertable P
Let A, B, C be $n \times n$ matrices. Prove the following statements.

(a) A is similar to A
A is similar to B when $B = P^{-1}AP$
Let $P = I_n$, then $B = I_n^{-1} A I_n = A$ $\Rightarrow$ A is similar to A #

(b) If A is similar to B, then B is similar to A
$B = P^{-1}AP$ $\Rightarrow$ Let $Q = P^{-1}$
then $Q^{-1}BQ = (P^{-1})^{-1} B P^{-1} = PBP^{-1} = P(P^{-1}AP)P^{-1} = A$
$\Rightarrow$ B is similar to A #

(c) If A is similar to B and B is similar to C, then A is similar to C

Let $B = P^{-1}AP$, $C = Q^{-1}BQ$

$\Rightarrow C = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ)$

$\Rightarrow$ A is similar to C #

5. Let $A = \begin{bmatrix} -2 & 3 & 7 \\ -1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 & -1 \\ 1 & 2 & -2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$, Find $A^{-1}B$

$[A \ \ B] \to [I_n \ \ C]$, $C = A^{-1}B$

$$\left[\begin{array}{ccc|cccc} -2 & 3 & 7 & 2 & 0 & 1 & -1 \\ -1 & 1 & 2 & 1 & 2 & -2 & 1 \\ 1 & 1 & 2 & 3 & 1 & 1 & 3 \end{array}\right] \xrightarrow[r_3+\frac{1}{2}r_1 \to r_3]{r_2-\frac{1}{2}r_1 \to r_2} \left[\begin{array}{ccc|cccc} -2 & 3 & 7 & 2 & 0 & 1 & -1 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 0 & 2 & -\frac{5}{2} & \frac{3}{2} \\ 0 & \frac{5}{2} & \frac{11}{2} & 4 & 1 & \frac{3}{2} & \frac{5}{2} \end{array}\right]$$

$$\xrightarrow[-2r_2 \to r_2]{5r_2+r_3 \to r_3} \left[\begin{array}{ccc|cccc} -2 & 3 & 7 & 2 & 0 & 1 & -1 \\ 0 & 1 & 3 & 0 & -4 & 5 & -3 \\ 0 & 0 & -2 & 4 & 11 & -11 & 10 \end{array}\right] \xrightarrow{-\frac{1}{2}r_3 \to r_3} \left[\begin{array}{ccc|cccc} -2 & 3 & 7 & 2 & 0 & 1 & -1 \\ 0 & 1 & 3 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & -2 & -\frac{11}{2} & \frac{11}{2} & -5 \end{array}\right]$$

$$\xrightarrow[r_1-7r_3 \to r_1]{r_2-3r_3 \to r_2} \left[\begin{array}{ccc|cccc} -2 & 3 & 0 & 16 & \frac{77}{2} & \frac{75}{2} & 34 \\ 0 & 1 & 0 & 6 & \frac{25}{2} & -\frac{23}{2} & 12 \\ 0 & 0 & 1 & -2 & -\frac{11}{2} & \frac{11}{2} & -5 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & 0 & 6 & \frac{25}{2} & -\frac{23}{2} & 12 \\ 0 & 0 & 1 & -2 & -\frac{11}{2} & \frac{11}{2} & -5 \end{array}\right] \Rightarrow A^{-1}B = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 1 \\ 6 & \frac{25}{2} & -\frac{23}{2} & 12 \\ -2 & -\frac{11}{2} & \frac{11}{2} & -5 \end{bmatrix} \ \#$$

7. Assume that B and C are $n \times n$ matrix. Verify the following equation

$$\begin{bmatrix} I_n & B \\ C & I_n \end{bmatrix}^{-1} = \begin{bmatrix} P & -PB \\ -CP & I_n+CPB \end{bmatrix}$$ where $I_n - BC$ is invertable, $P = (I_n - BC)^{-1}$

$$\begin{bmatrix} I_n & B \\ C & I_n \end{bmatrix}\begin{bmatrix} P & -PB \\ -CP & I_n+CPB \end{bmatrix} = \begin{bmatrix} I_nP - BCP & -PB + B + BCPB \\ CP - CP & -CPB + I_n + CPB \end{bmatrix}$$

$$= \begin{bmatrix} (I_n - BC)P & -(I_n - BC)PB + BI_n \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} = I_n$$

$\therefore$ $I_n - BC$ invertable and $P = (I_n - BC)^{-1}$ #

8. Use an LU decomposition to solve the following system of linear equation

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 + 3x_5 = -4 \\ -x_1 + 2x_2 - 2x_4 - 2x_5 = 9 \\ 2x_1 - x_2 + 7x_3 - x_4 + x_5 = -2 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 1 & 3 & -4 \\ -1 & 2 & 0 & -2 & -2 & 9 \\ 2 & -1 & 7 & -1 & 1 & -2 \end{array}\right] \xrightarrow[-2r_1+r_3 \to r_3]{r_1+r_2 \to r_2} \left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 & 1 & 5 \\ 0 & 1 & 3 & -3 & -5 & 6 \end{array}\right]$$

$$\xrightarrow{-r_2+r_3 \to r_3} \left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 & 1 & 5 \\ 0 & 0 & 1 & -2 & -6 & 1 \end{array}\right] \ldots U$$

$$U = \begin{bmatrix} 1 & -1 & 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 & 1 & 5 \\ 0 & 0 & 1 & -2 & 6 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

$$L\vec{y} = \vec{b} \qquad [\,L \;\; \vec{b}\,] = [\,I_n \;\; \vec{y}\,]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 - x_4 - 3x_5 - 4 \\ -2x_3 + x_4 - x_5 + 5 \\ 2x_4 + 6x_5 + 1 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{aligned} x_2 &= -2(2x_4 + 6x_5 + 1) + x_4 - x_5 + 5 \\ &= -3x_4 - 13x_5 + 3 \end{aligned}$$

$$\begin{aligned} x_1 &= (-3x_4 - 13x_5 + 3) - 2(2x_4 + 6x_5 + 1) - x_4 - 3x_5 - 4 \\ &= -8x_4 - 28x_5 - 3 \end{aligned}$$

A: $$\begin{cases} x_1 = -8x_4 - 28x_5 - 3 \\ x_2 = -3x_4 - 13x_5 + 3 \\ x_3 = 2x_4 + 6x_5 + 1 \\ x_4 \text{ free} \\ x_5 \text{ free} \end{cases}$$

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