1. (10) Find a generating set for the subspace
$$\left\{ \begin{bmatrix} -r+4t \\ r-s+2t \\ 3t \\ r-t \end{bmatrix} \in \mathcal{R}^4 \right\}, \text{ where } r,\ s, \text{ and } t \text{ are scalars}$$

2. (10) Find a basis for (a) (5) the column space and (b) (5) the null space of the matrix
$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & -3 & 5 & 4 \\ 0 & 0 & 3 & -3 \\ 2 & -2 & 1 & 5 \end{bmatrix}$$

3. (10) The reduced row echelon form of a matrix A is given. Determine the dimension of

   (a) (2) *Col A*

   (b) (3) *Null A*

   (c) (2) *Row A*

   (d) (3) *Null* A^T

$$A = \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 2 & -3 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

4. (10) The $\mathcal{B}$-coordinate vector $\mathbf{v}$ is $\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Find $\mathbf{v}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

5. (10) Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be bases for $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. The matrix
$$[[T(\mathbf{b}_1)]_{\mathcal{C}}\ [T(\mathbf{b}_2)]_{\mathcal{C}} \ldots T[(\mathbf{b}_n)]_{\mathcal{C}}]$$
is called the matrix representation of T with respected to $\mathcal{B}$ and $\mathcal{C}$. It is denoted by $[T]_{\mathcal{B}}^{\mathcal{C}}$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

   (a) (5) Prove that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathcal{R}^3$ and $\mathcal{R}^2$, respectively.

   (b) (5) Let $T : \mathcal{R}^3 \to \mathcal{R}^2$ be the linear transformation defined by
$$T = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix}. \text{ Find } [T]_{\mathcal{B}}^{\mathcal{C}}.$$

1 1 -1

6. (10) A matrix and a vector are given. Show that the vector is an eigenvector of the matrix and determine the corresponding eigenvalue.

$$\begin{bmatrix} 6 & 5 & 15 \\ 5 & 6 & 15 \\ -5 & -5 & -14 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

-1·5+15

7. (10) A matrix and its characteristic polynomial are given. Find the eigenvalues of each matrix and determine a basis for each eigenspace

$$\begin{bmatrix} 1 & 6 & -6 & -6 \\ 6 & 7 & -6 & -12 \\ 3 & 3 & -2 & -6 \\ 3 & 9 & -9 & -11 \end{bmatrix}, \ (t+5)(t+2)(t-1)^2$$

8. (10) The trace of a square matrix is the sum of its diagonal entries

   (a) (4) Prove that if A is a diagonalizable matrix, then the trace of A equals the sum of the eigenvalues of A.

   (b) (3) Let A be diagonalizable $n \times n$ matrix with characteristic polynomial $(-1)^n(t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_n)$. Prove that the coefficient of t^{n-1} in this polynomial is $(-1)^{n-1}$ times the trace of A.

   (c) (3) For A as in (b), what is the constant term of the characteristic polynomial of A?

9. (10) Two vectors $\mathbf{u}$ and $\mathbf{v}$ are given. Compute the norms of the vectors and the distance d between them

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

10. (10) Determine whether the given set is orthogonal

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

1
-1

11. (10) Find a basis for the subspace $S^{\perp}$, where $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

12. (10) Find the equation of the least-squares line for the given data $(1, 21), (3, 32), (9, 38), (12, 41), (15, 51)$

13. (10) An $n \times n$ matrix C is said to be positive semidefinite if C is symmetric and $\mathbf{v}^T C \mathbf{v} \geq 0$ for every vector $\mathbf{v} \in \mathcal{R}^n$. Prove that, for any matrix A, the matrices AA^T and A^TA are positive semidefinite.