1. Given that X and Y are two discrete random variables.

(i) Show that: $var(X) = E[X^2] - E[X]^2$

(ii) If $E[X] = 3$; $var(X) = 5$, Find $E[(X^2+2)]$

(iii) Show that $E[XY] = E[X]E[Y]$, if X and Y are independent of each other.

(iv) Show that any two events of the forms $P(X \in A)$, $P(Y \in B)$ are independent if X, Y are independent of each other.

(i)

$$var(X) = \sum_x (x - E[X])^2 P_X(x)$$

$$= \sum_x (x^2 - 2xE[X] + E[X]^2) P_X(x)$$

$$= \left[\sum_x x^2 P_X(x)\right] - 2E[X]\left[\sum_x x P_X(x)\right] + E[X]^2 \sum_x P_X(x)$$

$$= E[X^2] - 2E[X]E[X] + E[X]^2 = E[X^2] - E[X]^2 \text{ ※}$$

(ii)

$$E[X^2+2] = E[X^2] + 2$$

$$var(X) = E[X^2] - E[X]^2 \Rightarrow 5 = E[X^2] - 9 \Rightarrow E[X^2] = 14$$

$$\Rightarrow E[X^2+2] = 14 + 2 = 16 \text{ ※}$$

(iv) Let $U = g(x) = P(X \in A)$, $V = h(y) = P(Y \in B)$

$$\Rightarrow P_{UV}(u,v) = \sum_{\{(x,y) \mid g(x)=u,\ h(y)=v\}} P_{X,Y}(x,y)$$

$$= \sum_{\{x \mid g(x)=u\}} P_X(x) \cdot \sum_{\{y \mid h(y)=v\}} P_Y(y)$$

$$= P_U(u) \cdot P_V(v) \text{ ※}$$

(iii) if X, Y independent,

then:

$$P_{X|Y}(x|y) = P_X(x)$$

又 $$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$\Rightarrow P_X(x) = \frac{P_{XY}(x,y)}{P_Y(y)} \Rightarrow P_X(x)P_Y(y) = P_{XY}(x,y) \text{ ※}$$

2021. Mid2

2. Given that the time duration (in minutes) of your week meetup with Prof. Berlin can be modeled by an exponential random variable with parameter $\lambda = 1/5$

(i) Find the probability that a meetup last less than 3 minutes.

(ii) Suppose that a meetup has already lasted for 3 minutes. Find the probability that the meetup will lasted less than 3 more minutes

(iii) Find a number k that satisfy the CDF of X, that $F_X(k) = 1/3$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $F_X(3) = \int_0^3 \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^3 = 1 - e^{-\frac{3}{5}}$ ※

(ii) $F_X(6) = \int_0^6 \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^6 = 1 - e^{-\frac{6}{5}}$

所求 $= F_X(6) - F_X(3) = e^{-3\lambda} - e^{-6\lambda}$

$= e^{-\frac{3}{5}}(1 - e^{2})$ ※

(iii) $F_X(k) = 1 - e^{-\frac{k}{5}} = \frac{1}{3}$

※ Relation between Geomatric & Exponential

$$F_{geo}(n) = \sum_{k=1}^{n} (1-p)^{k-1} p = p\left[\frac{1-(1-p)^n}{1-(1-p)}\right] = 1 - (1-p)^n$$

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda x} = \left[-e^{-\lambda x}\right]_0^x = 1 - e^{-\lambda x}$$

$$\Rightarrow e^{-\lambda x} = (1-p)^n \quad \Rightarrow x = n \cdot \left(\frac{-1}{\lambda} \cdot \ln(1-p)\right) \quad \Rightarrow x = n \cdot \delta$$

$$\Rightarrow F_{exp}(\delta n) = 1 - e^{-\lambda \delta n} = 1 - (1-p)^n = F_{geo}(n)$$

$\Rightarrow$ Let $p = \frac{1}{3}$, $n = 1$ $\quad \left(1 - \left(1 - \frac{1}{3}\right)^1 = \frac{1}{3}\right)$

$\Rightarrow \delta n = 1 \times \left(-\frac{1}{\lambda}\right) \cdot \ln\left(\frac{2}{3}\right) = -\frac{1}{5}(\ln 2 - \ln 3)$ ※

3. Let $X = \max\{X_1, 2X_2, 3X_3, 4X_4\}$, $X_1 \sim X_4$ are independent discrete random variables which may take value $-1, 0, 1$ with probability of $1/3$ for each value.

Find the PMF of X.

$\Rightarrow F_X(-1) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{81}$ $\qquad F_X(2) = 1 \times 1 \times \frac{2}{3} \times \frac{2}{3} = \frac{36}{81}$

$F_X(0) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{16}{81}$ $\qquad F_X(3) = 1 \times 1 \times 1 \times \frac{2}{3} = \frac{54}{81}$

$F_X(1) = 1 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{24}{81}$ $\qquad F_X(4) = 1 \times 1 \times 1 \times 1 = 1$

$$\Rightarrow f_X(x) = \begin{cases} 1/81, & X = -1 \\ 15/81, & X = 0 \\ 8/81, & X = 1, \\ 12/81, & X = 2 \\ 18/81, & X = 3 \\ 27/81, & x = 4 \\ 0, & \text{otherwise} \end{cases} \quad ※$$

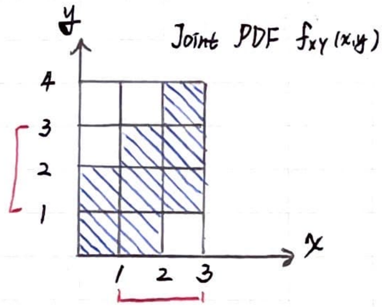
$(1 + 15 + 8 + 12 + 18 + 27 = 81)$

4. We are told that the joint PDF of two random variables X, Y (continuous) is uniform in shaded area,

(i) Find the PDF of $f_X(x)$ of X

(ii) Find the conditional PDF $f_{Y|X}(y|x = 2.5)$

(iii) Given event $A = \{1 \le X \le 3, 1 \le Y \le 3\}$ determine whether X, Y are dependent or not.



$$\text{(i)}\ f_X(x) = \begin{cases} 1/4, & 0 \le x \le 1 \\ 3/8, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{(ii)}\ f_{Y|X}(y|2.5) = \frac{3}{4}$$

(iii) if $f_{XY|B}(x,y) = f_{X|B}(x)\, f_{Y|B}(y)$ then X, Y are independent

$1/4 = \frac{1}{2} \times \frac{1}{2}$ $\quad (1 \le x \le 2, 1 \le y \le 2)$

$\qquad \shortparallel \qquad (1 \le x \le 2, 2 \le y \le 3)$

$1/4 = \frac{1}{2} \times \frac{1}{2}$ $\quad (2 \le x \le 3, 1 \le y \le 2)$ $\quad \Rightarrow$ independent ※

$\qquad \shortparallel \qquad (2 \le x \le 3, 2 \le y \le 3)$

2021. Mid 2

5. We are told that X is normal distribution with mean 2, variance 9

(i) Find the probability $P(-1 \le X \le 8)$

(ii) If $Y = 2X+1$, Find the probability $P(Y \ge 11)$

$\phi(0) = 0.5$
$\phi(0.5) = 0.6915$
$\phi(1.0) = 0.8413$
$\phi(1.5) = 0.9332$
$\phi(2.0) = 0.9772$
$\phi(2.5) = 0.9938$

(i) $$P(-1 \le X \le 8) = P(x \le 8) - P(x \le -1)$$
$$= P\left(\frac{x-2}{3} \le \frac{8-2}{3}\right) - P\left(\frac{x-2}{3} \le \frac{-1-2}{3}\right)$$
$$= P\left(\frac{x-2}{3} \le 2\right) - P\left(\frac{x-2}{3} \le -1\right)$$
$$= \phi(2) - \phi(-1)$$
$$= \phi(2) - [1-\phi(1)] = 0.9772 - 0.3085 = 0.6688 \ \#$$

(ii) $W = 2X+1$, $\mu_W = 2\times 2 + 1 = 5$ $\quad \sigma_W^2 = 36$ $\quad \sigma_W = 6$

$$\Rightarrow P(Y \ge 11) = 1 - P(Y \le 11)$$
$$= 1 - P\left(\frac{Y-5}{6} \le \frac{11-5}{6}\right)$$
$$= 1 - P\left(\frac{Y-5}{6} \le 1\right) = 1 - \phi(1) = 0.1587 \ \#$$

6. Dr. Minsky watch TV for M hours every day,
where M is a discrete random variable that has it's value equally likely to be 1, 2, 3
When watching TV, the number of TV stations N,
that he will browse is random and depends on how long he watch

$$\Rightarrow P_{N|M}(n|m) = \frac{1}{m}, \quad \text{for } n = 1 \dots m$$

(i) Find joint PMF $P_{N,M}(n,m)$ of M, N

(ii) Find marginal PMF of N

(i)
$$P_{N|M}(n|m) = \frac{P_{N,M}(n,m)}{P_M(m)}$$

$P_{N|M}(n|m)$

$M \backslash N$	1	2	3
3	1/3	1/3	1/3
2	1/2	1/2	0
1	1	0	0

$\Rightarrow$

$M \backslash N$	1	2	3
3	1/9	1/9	1/9
2	1/6	1/6	0
1	1/3	0	0

\#

(ii) marginal PMF of N

$$\Rightarrow P_N(n) = \begin{cases} \frac{11}{18}, & n=1 \\ \frac{5}{18}, & n=2 \\ \frac{2}{18}, & n=3 \\ 0, & \text{otherwise} \end{cases} \ \#$$

7. Given that X, Y are two continuous random variables with $Y = e^{x}$

and PDF of X: $f_X(x) = \begin{cases} 1/3, & -1 \le x \le 0 \\ 2/3, & 0 \le x \le 1 \end{cases}$

(i) Find $E[Y]$

(ii) Find $var(Y)$

(i) $E[X] = \int_{-1}^{0} x \cdot \frac{1}{3}\, dx + \int_{0}^{1} x \cdot \frac{2}{3}\, dx$

$= \left[\frac{x^2}{6} \right]_{-1}^{0} + \left[\frac{x^2}{3} \right]_{0}^{1} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ ※

(ii) $E[X^2] = \int_{-1}^{0} x^2 \cdot \frac{1}{3}\, dx + \int_{0}^{1} x^2 \cdot \frac{2}{3}\, dx$

$= \left[\frac{x^3}{9} \right]_{-1}^{0} + \left[\frac{2x^3}{9} \right]_{0}^{1} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$ ※

$var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ ※

2021, mid2