1. Given X be a continuous random variable with PDF $f_X(x)$ and $Y = aX + b$
Show that the PDF of random variable Y can be expressed as $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$$\Rightarrow F_Y(y) = P(Y \le y) = P(aX+b \le y)$$

(i) $a > 0$

$$F_Y(y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) \Rightarrow f_Y(y) = \frac{d\, F_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(ii) $a < 0$

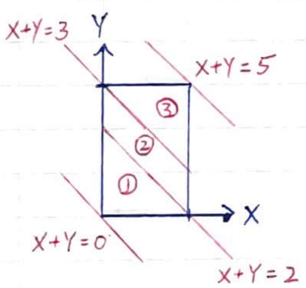
$$F_Y(y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right) \Rightarrow f_Y(y) = -\frac{d\, F_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad ※$$

2. We are told that X, Y, Z are three independent random variables
All of them are uniformly distributed in the interval $[0, 1]$

(i) use "derived-distributed" to find the PDF $f_W(w)$ of $W = 2X + 3Y$
(ii) Find the PDF of $W = \min(X, Y, Z)$

(i) Let $X' = 2X$, $Y' = 3Y$



① $0 \le W \le 2$ $\quad F_W(w) = \frac{1}{6} \times \frac{w^2}{2} \Rightarrow f_W(w) = \frac{w}{6}$ ※

② $2 \le W \le 3$ $\quad F_W(w) = \frac{1}{6} \times [2(w-2)+2] \times 2 \times \frac{1}{2} = \frac{1}{6} \times (2w-2)$

$$\Rightarrow f_W(w) = \frac{1}{3} \quad ※$$

③ $3 \le W \le 5$ $\quad F_W(w) = \frac{1}{6} \times \left[6 - \frac{(5-w)^2}{2}\right] = 1 - \left(\frac{w^2 - 10w + 25}{12}\right)$

$$\Rightarrow f_W(w) = -\frac{w}{6} + \frac{5}{6} \quad ※$$

(ii) $P(w \ge k) = P(X \ge k)P(Y \ge k)P(Z \ge k)$

$$= (1-k)^3$$

$$F_W(w) = 1 - (1-k)^3$$

$$f_W(w) = -3(1-k)^2 \times (-1) = 3(1-k)^2 \quad ※$$

3. Suppose that in Taiwan, the number of summer typhoons in good year is Poisson with mean = 2
bad year is Poisson with mean = 6

there's a probability 0.3 to be good year, 0.7 be bad year.

Find the expected value and variance of the number of typhoon occur

$\Rightarrow$ mean $= 2 \times 0.3 + 6 \times 0.7 = 4.8$ ※

$$\text{variance} = 0.3 \times 2 + (2-4.8)^2 \times 0.3 + 0.7 \times 6 + (6-4.8)^2 \times 0.7$$
$$= 0.6 + 2.352 + 4.2 + 1.008$$
$$= 8.16 \quad ※$$

$[\, var(X+Y) = var(X) + var(Y) \,]$

4. The covariance of any two random variables X, Y is $cov(X,Y) = E[(X-E[X])(Y-E[Y])]$
(i) Show that, for X, Y, Z, $cov(X, Y+Z) = cov(X,Y) + cov(X,Z)$
※ If, $X\,Y\,Z$ are pairwise uncorrelated random variable with mean = 0, variance = 2 for each
(ii) compute the correlations of $2X$ and $3X$
(iii) compute the correlations of $2X-3Y$ and $2W+Y$

(i)

$$cov(X, Y+Z) = E[(X-E[X])(Y+Z-E[Y+Z])] = E[(X-E[X])(Y+Z-E[Y]-E[Z])]$$
$$= E[(X-E[X])(Y-E[Y]) + (X-E[X])(Z-E[Z])]$$
$$= E[(X-E[X])(Y-E[Y])] + E[(X-E[X])(Z-E[Z])] = cov(X,Y) + cov(X,Z)$$

(ii)

$$cov(2X, 3X) = 6\, var(X)$$

$$\rho(2X, 3X) = \frac{cov(2X,3X)}{\sqrt{4var(X)\,9var(X)}} = \frac{6\,var(X)}{6\,var(X)} = 1 \quad ※$$

(iii)

$$\rho(2X-3Y, 2W+Y) = \frac{cov(2X,2W) + cov(2X,Y) + cov(-3Y,W) + cov(-3Y,Y)}{\sqrt{(4var(X) + 9\,var(Y))(4var(W) + var(Y))}}$$

$$= \frac{-3\,var(Y)}{\sqrt{13 \times 2 \times 5 \times 2}} = \frac{-3}{\sqrt{65}} \quad ※$$

5. Let X be a descrete random variable with a transform $M_X(s)$ such that $M_X(s) = a + be^{3s} + ce^{6s}$, $E[X] = 2$, $var(X) = 9/2$

Find a, b, c and the PMF of X. $\left[(\text{discrete})\ M_X(s) = \sum_x e^{sx} p_X(x)\right]$

$$E[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0} = 3be^{3s} + 6ce^{6s}\Big|_{s=0} = 3b + 6c = 2$$

$$E[X^2] = \left.\frac{d^2M_X(s)}{ds^2}\right|_{s=0} = \left.\frac{d(3be^{3s} + 6ce^{6s})}{ds}\right|_{s=0} = 9b + 36c$$

$$var(X) = 9b + 36c - 4 = \frac{9}{2}$$

$$\begin{cases} 3b + 6c = 2 \\ 9b + 36c = 17/2 \end{cases} \Rightarrow 9b + 18c = 6 \Rightarrow 18c = 5/2 \Rightarrow c = 5/36$$

$$\Rightarrow b = 7/18 \quad \Rightarrow a = 17/36$$

$$\Rightarrow p_X(x) = \begin{cases} 17/36, & x = 0 \\ 7/18, & x = 3 \\ 5/36, & x = 6 \end{cases} \quad ※$$

7. A biased coin with probability 0.4 being head is flipped independently until first head obtained. For each flip of a coin, a value of an exponential random variable with parameter $\lambda = 2$. Let random variable Y be define as the sum of all value obtained before the first head. Find mean, variance and transform of Y

$$E[N] = \frac{1}{0.4} = \frac{5}{2} \qquad var(N) = \frac{1-0.4}{(0.4)^2} = \frac{60}{16} = \frac{15}{4}$$

$$E[X] = \frac{1}{\lambda} = \frac{1}{2} \qquad var(X) = \frac{1}{\lambda^2} = \frac{1}{4}$$

$$E[Y] = E[N]E[X] = 5/4 \quad ※$$

$$var(Y) = \sigma^2 E[N] + \mu^2 var(N)$$

$$= \frac{1}{4} \times \frac{5}{2} + \left(\frac{1}{2}\right)^2 \times \frac{15}{4} = \frac{25}{16} \quad ※$$

$$M_Y(s) = \frac{p \cdot M_X(s)}{1-(1-p)M_X(s)} = \frac{p \times \frac{2}{2-s}}{1-(1-p)\frac{2}{2-s}} = \frac{\frac{2}{5} \times \frac{2}{2-s}}{1 - \frac{3}{5} \times \frac{2}{2-s}} = \frac{4}{10-5s-6} = \frac{4}{4-5s} \quad ※$$

6. Given X be exponential random variable with PDF $f_X(x) = \lambda e^{-\lambda x}, \ x \geq 0$

(i) Show that the transform of X can be expressed as : $M_X(s) = \frac{\lambda}{\lambda - s}$

$\Rightarrow M_X(s) = E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx = \int_{-\infty}^{\infty} e^{sx} \cdot \lambda e^{-\lambda x}\, dx$

$= \lambda \int_0^{\infty} e^{sx} \times e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{(s-\lambda)x}\, dx$

$= \lambda \left[\frac{1}{s-\lambda} e^{(s-\lambda)x}\right]_0^{\infty} = \lambda \left[-\frac{1}{s-\lambda}\right] = \frac{\lambda}{\lambda - s}$ ※

(ii) Using $M_X(s)$ to show that $E[X] = \frac{1}{\lambda}$, $var(X) = \frac{1}{\lambda^2}$

$\Rightarrow E[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0} = \left.\left(\frac{d\left(\frac{\lambda}{\lambda-s}\right)}{d(\lambda-s)} \cdot \frac{d(\lambda-s)}{ds}\right)\right|_{s=0} = \frac{1}{\lambda}$ ※

$\Rightarrow E[X^2] = \left.\frac{d^2M_X(s)}{ds^2}\right|_{s=0} = \left.\left(\frac{d\left(\frac{\lambda}{\lambda-s}\right)^2}{d(\lambda-s)} \cdot \frac{d(\lambda-s)}{ds^2}\right)\right|_{s=0} = \frac{2}{\lambda^2}$

$\Rightarrow var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ ※

(iii) Given random variable $Y = 3X + 2$.
Find the transform of Y.

$\Rightarrow M_Y(s) = E[e^{(3X+2)s}] = e^{2s} \cdot E[e^{3Xs}] = e^{2s} \times M_X(3s) = \frac{\lambda e^{2s}}{\lambda - 3s}$ ※

(iv) Given Z, an exponential random variable with parameter η, X and Z are independent to each
Find the transform of $Y = X + Z$

$\Rightarrow M_Y(s) = E[e^{s(X+Z)}] = E[e^{Xs}]\, E[e^{Zs}] = M_X(s)\, M_Z(s) = \frac{\lambda}{\lambda - s} \times \frac{\eta}{\eta - s}$ ※

(v) same as (iv), but $Y = 3X + 2Z$

$\Rightarrow M_Y(s) = E[e^{s(3X+2Z)}] = E[e^{3Xs}]\, E[e^{2Zs}] = M_X(3s)\, M_Z(2s) = \frac{\lambda}{\lambda - 3s} \times \frac{\eta}{\eta - 2s}$ ※