1. (i) State the three probability axioms

Nonnegativity , $\forall x$, $P(x) \geq 0$

Addictivity , if A, B disjoint , $P(A) + P(B) = P(A \cup B)$

Normalization , $P(\Omega) = 1$

(ii) Show that $P(A \cap B) \geq P(A) + P(B) - 1$

$\Rightarrow 1 \geq P(A) + P(B) - P(A \cap B)$ (normalization)

$\Rightarrow P(\Omega) \geq P(A \cup B)$ ※

(iii) Show that given a set of disjoint events $A_1, A_2, \ldots, A_n$ ($P(A_n) > 0$) that form a partition of the sample space, the probability of any event B can be express by $P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$

$\Rightarrow P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$

$= \frac{P(B \cap A_1)}{P(A_1)} P(A_1) + \frac{P(B \cap A_2)}{P(A_2)} P(A_2) + \cdots + \frac{P(B|A_n)}{P(A_n)} P(A_n)$

$= P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n) = P(B)[P(A_1) + P(A_2) + \cdots + P(A_n)]$

$\left(\begin{array}{l} \because A_1 \sim A_n \text{ are disjoint and form a partition} \\ \therefore P(A_1) + \cdots + P(A_n) = P(\Omega) = 1 \end{array} \right)$ $= P(B)\,P(\Omega) = P(B)$ ※

(iv) Following (iii), show that given $P(B) > 0$, $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$

$\Rightarrow \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)} = \frac{P(B|A_i)P(A_i)}{P(B)}$ (According to (iii))

$= \frac{P(B \cap A_i)}{P(A_i)} \times \frac{P(A_i)}{P(B)} = \frac{P(B \cap A_i)}{P(B)}$ ... ①

$\Rightarrow P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$ ... ②

$\because$ ① = ② , 故得證 ※

2. A multiple-choice exam gives 4 choices per question.
On 80% of the questions you think you know the answer;
on the other 20%, you just guess at random.
However, when you think you know the answer, you are right only 70% of time.

(i) Find the probability of getting an arbitrary question wrong.

$\Rightarrow P\{\text{I think I know} \cap \text{wrong}\} + P\{\text{guess} \cap \text{wrong}\}$

$= 0.8 \times 0.3 + 0.2 \times 0.75 = 0.24 + 0.15 = 0.39$ ※

(ii) If you get a problem wrong, what's the probability that you think you know the answer.

$\Rightarrow \dfrac{0.8 \times 0.3}{0.39} = \dfrac{0.24}{0.39} = \dfrac{8}{13}$ ※

3. Given that A, B are two mutually exclusive events of an experiment on rolling 4-sided die.

Show that the probability that the event A will occur before event B is $\dfrac{P(A)}{P(A)+P(B)}$.

4. In a class on probability theory, there are 8 freshman boys, 12 freshman girls, and 12 sofomore boys. How many sophomore girls must be present, if sex and grade are to be independent when a student is selected random.

Let sophomore girl be x

$P(\text{girl}) = \dfrac{x+12}{8+12+12+x} = \dfrac{x+12}{x+32}$

$P(\text{sophomore}) = \dfrac{x+12}{8+12+12+x} = \dfrac{x+12}{x+32}$ $\Rightarrow$

$P(\text{girl} \cap \text{sophomore}) = \dfrac{x}{32+x}$

$\because$ independent

$\therefore P(\text{girl} \cap \text{sophomore}) = P(\text{girl})\, P(\text{sophomore})$

$\Rightarrow \dfrac{x}{32+x} = \left(\dfrac{x+12}{32+x}\right)^2$

$\Rightarrow \dfrac{x^2+24x+144}{32+x} = x$

$\Rightarrow x^2+24x+144 = x^2+32x$

$\Rightarrow 144 = 8x \quad \Rightarrow x = 18$ #

5. Let X and Y be discrete random variables and $Y = x^2+1$.

Assume that the PMF of X is $P_X(x) = \begin{cases} k|x|, & \text{if } x = -2, -1, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$ where k is suitable constan

(i) Find the PMF of X

$2k + k + k + 2k + 3k = 1 \quad \Rightarrow k = \frac{1}{9}$ (sum of all event's probability must be 1)

$P_X(x) \begin{cases} \frac{1}{9}|x|, & \text{if } x = -2, -1, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$ $\Rightarrow P_X(x) = \begin{cases} \frac{2}{9}, & x = \pm 2 \\ \frac{1}{9}; & x = \pm 1 \\ \frac{3}{9}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$ #

(ii) Calculate the PMF of Y

$P_Y(Y) = \begin{cases} \frac{4}{9}, & y = 5 \\ \frac{2}{9}, & y = 2 \\ \frac{3}{9}, & y = 10 \\ 0, & \text{otherwise} \end{cases}$ #

6. Given that a random variable X is used to represent the difference between the number of heads and the number of tails obtained when a coin is tossed 3 times. Find the PMF of X (X = head - tail)

$$P_X(x) = \begin{cases} \frac{1}{8}, & x=3 \quad (\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}) \text{ HHH} \\ 0, & x=2 \\ \frac{3}{8}, & x=1 \quad (\frac{1}{8}\times C^3_2 = \frac{3}{8}) \text{ HHT} \\ \frac{3}{8}, & x=-1 \quad (\frac{1}{8}\times C^3_2 = \frac{3}{8}) \text{ HTT} \\ 0, & x=-2 \\ \frac{1}{8}, & x=-3 \quad (\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}) \text{ TTT} \end{cases} \Rightarrow P_X(x) = \begin{cases} \frac{1}{8}, & x=3 \\ \frac{3}{8}, & x=1 \\ \frac{3}{8}, & x=-1 \\ \frac{1}{8}, & x=-3 \\ 0, & \text{otherwise} \end{cases}$$

7. Assume that the average number of car accidents per day in Minsky freeway can be modeled as a Poisson random variable with parameter $\lambda = 3$.

* The PMF of Poisson random variable X with parameter λ is defined by

$$P_X(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2 \ldots$$

(i) No car accidents will occur tomorrow

$$k=0 \Rightarrow e^{-3}\times\frac{1}{1} = e^{-3} \quad ※$$

(ii) At least 3 car accidents will occur tomorrow

$$k=1 \Rightarrow e^{-3}\times\frac{3}{1} = 3e^{-3}$$

$$k=2 \Rightarrow e^{-3}\times\frac{9}{2} = \frac{9}{2}e^{-3}$$

$$\Rightarrow 1 - [e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3}] = 1 - e^{-3}(1+3+\frac{9}{2})$$

$$= 1 - \frac{17}{2}e^{-3} \quad ※$$