Hw 2019-5

1. (a) Prove that if λ is an eigenvalue of an invertible matrix A then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}

Let λ be an eigenvalue of an invertable matrix A

* Suppose that 0 is an eigenvalue of A
  Then $Av = 0v = 0$ for some nonzero eigenvector v, eigenvalue $\lambda = 0$
  Hence, A is not invertable

  $\Rightarrow$ By contradiction, $\lambda \neq 0$

* $Av = \lambda v$ for some eigenvector $v \neq 0$, eigenvalue $\lambda \neq 0$
  Thus, $v = A^{-1}(\lambda v) = \lambda(A^{-1}v)$
  Hence, $A^{-1}v = (1/\lambda)v$, so $1/\lambda$ is an eigenvalue of A^{-1}

(b) Let v_1 and v_2 be eigenvalues of a linear transformation T on $\mathbb{R}^n$, and let λ_1 and λ_2, corresponding eigenvalues.

Prove that if $\lambda_1 \neq \lambda_2$, then $\{v_1, v_2\}$ is linearly independent.

Suppose that $c_1v_1 + c_2v_2 = 0$ for some scalars c_1, c_2

Then $0 = T(0) = T(c_1v_1 + c_2v_2) = c_1\lambda_1v_1 + c_2\lambda_2v_2 = \lambda_1(-c_2v_2) + \lambda_2(c_2v_2) = (\lambda_2 - \lambda_1)(c_2v_2)$

Since $\lambda_2 \neq \lambda_1$ and $v_2 \neq 0$, we have $c_2 = 0$, thus $c_1 = 0$,

so $\{v_1\ v_2\}$ is linearly independent.

2. Find the eigenvalues of linear operator T and determine a basis for each eigenspace,

where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -4x_1 + 6x_2 \\ 2x_2 \\ -5x_1 + 5x_2 + x_3 \end{bmatrix}$

standard matrix $A = \begin{bmatrix} -4 & 6 & 0 \\ 0 & 2 & 0 \\ -5 & 5 & 1 \end{bmatrix}$

The characteristic polynomial of A is

$$\det(A - \lambda I_n) = \det \begin{bmatrix} -4-\lambda & 6 & 0 \\ 0 & 2-\lambda & 0 \\ -5 & 5 & 1-\lambda \end{bmatrix} = (-4-\lambda)(2-\lambda)(1-\lambda)$$

① $$A - 1 \cdot I_3 = \begin{bmatrix} -5 & 6 & 0 \\ 0 & 1 & 0 \\ -5 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (rref)}$$

② $$A - 2 \cdot I_3 = \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \\ -5 & 5 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ (rref)}$$

③ $$A + 4 I_3 = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ -5 & 5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (rref)}$$

$\Rightarrow$ when $\lambda = 1$, the corresponding eigenspace is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = -4$, the corresponding eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 2$, the corresponding eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ #

3. Given a matrix $A = \begin{bmatrix} 3 & 2 & -2 \\ -8 & 0 & -5 \\ -8 & -2 & -3 \end{bmatrix}$ and its characteristic polynomial $-(t+5)(t-2)(t-3)$

find, if possible, an invertable matrix P and its diagonal matrix D

such that $A = PDP^{-1}$

(i) $A+5I_3 = \begin{bmatrix} 8 & 2 & -2 \\ -8 & 5 & -5 \\ -8 & -2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 8 & 2 & -2 \\ -8 & 5 & -5 \\ -8 & -2 & 2 \end{bmatrix} \begin{matrix} r_2+r_1\to r_2 \\ r_3+r_1\to r_3 \end{matrix} \Rightarrow \begin{bmatrix} 8 & 2 & -2 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{7}r_2\to r_2 \Rightarrow \begin{bmatrix} 8 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1-2r_2\to r_1 \\ \tfrac{1}{8}r_1\to r_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(ii) $A-2I_3 = \begin{bmatrix} 1 & 2 & -2 \\ -8 & -2 & -5 \\ -8 & -2 & -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & -2 \\ -8 & -2 & -5 \\ -8 & -2 & -5 \end{bmatrix} \begin{matrix} r_2+8r_1\to r_2 \\ r_3+8r_1\to r_3 \\ r_3-r_2\to r_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 14 & -21 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{7}r_2\to r_2 \Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} r_1-r_2\to r_1 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}x_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

(iii) $A-3I_3 = \begin{bmatrix} 5 & 2 & -2 \\ -8 & 2 & -5 \\ -8 & -2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 5 & 2 & -2 \\ -8 & 2 & -5 \\ -8 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 40 & 16 & -16 \\ -40 & 10 & -25 \\ -40 & -10 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 2 & -2 \\ 0 & 26 & -41 \\ 0 & 6 & -21 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 2 & -2 \\ 0 & 26 & -41 \\ 0 & 2 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & 2 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$\Rightarrow$ Ans : $P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ※

5. Let A be a diagonalizable $n \times n$ matrix.
Prove that if the characteristic polynomial of A is $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$
then $f(A) = O$, where $f(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I_n$

(Cayley - Hamilton theorem)

$\Rightarrow$ Let λ be an eigenvalue of A, then $f(\lambda) = 0$.

Let $A = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$

Then $f(D) = \begin{bmatrix} f(\lambda_1) & 0 & 0 & 0 \\ 0 & f(\lambda_2) & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & f(\lambda_n) \end{bmatrix} = O$, Hence $f(A) = f(PDP^{-1}) = P\, f(D)\, P^{-1}$

$= POP^{-1} = O$ ※

6. Given a linear operator T and its characteristic polynomial $f(t)$,
determine all the values of the scalar c for which T on R^3 is not diagonalizable,

where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 \\ -x_1 - 3x_2 - x_3 \\ -8x_1 + x_2 - 5x_3 \end{bmatrix}$, $f(t) = -(t-c)(t+4)^2$

$\Rightarrow A = \begin{bmatrix} c & 0 & 0 \\ -1 & -3 & -1 \\ -8 & 1 & -5 \end{bmatrix}$, $(A - 4I_3) = \begin{bmatrix} c+4 & 0 & 0 \\ -1 & 1 & -1 \\ -8 & 1 & -1 \end{bmatrix}$

$\Rightarrow$ last two rows of $(A-4I_3)$ is linearly independent, so the rank of $A+4I_3$ is at least 2
Hence, the dimension of the eigenspace of T corresponding to -4 is 1.

$\Rightarrow$ Since this dimension does not equal the multiplicity of the eigenvalue -4

$\Rightarrow$ T is not diagonalizable for any scalar c.

7. Let $\{u, v, w\}$ be a basis for $\mathbb{R}^3$
and let T be the linearly operator on $\mathbb{R}^3$ defined by $T(au + bv + cw) = au + bv$
for all scalar a, b, c.

(a) Find the eigenvalues of T and determine a basis for each eigenspace.

$\Rightarrow$ Let $B = \{u, v, w\}$, we can know $T(u) = u$, $T(v) = v$, $T(w) = 0$

Hence, u, v are eigenvectors of T corresponding to eigenvalue $= 1$
w is eigenvectors of T corresponding to eigenvalue $= 0$

① $\{u, v\}$ is a basis for the eigenspace of T corresponding to eigenvalue 1
② $\{w\}$ is a basis for the eigenspace of T corresponding to eigenvalue 0

(b) Is T diagonalizable? Justify the answer

$\Rightarrow$ by (a), there is a basis B for $\mathbb{R}^3$ consisting of eigenvectors of T.

Thus, T is diagonalizable.

8. Let T be the linear operator on $\mathbb{R}^n$ and B be a basis for $\mathbb{R}^n$
Such that $[T]_B$ is a diagonal matrix.

Prove that B must consist of eigenvectors of T.

$\Rightarrow$ Let $[T]_B$ be a diagonal matrix D, $B = \{b_1\ b_2 \ldots b_n\}$.

Since the j^{th} column of $[T]_B$ is $[T(b_j)]_B$

we must have $T(b_j) = d_{jj} \cdot b_j$, $b_j \neq 0$ because B is a basis,

and hence each b_j must be an eigenvector of T #