

Introduction to Probability Midterm II

15:30-17:20, December 6, 2022

Note: You have to answer the questions with supporting explanations if needed.

1. A continuous random variable X has a probability density function (PDF) given below

$$f_X(x) = \begin{cases} a + bx, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

If $E[X] = \frac{10}{9}$, find a and b . (20%)

2. We are told that X is a normal distribution with mean 4 and variance 9. Given that a few CDF values of a standard normal are given as follows:

$$\Phi(0) = 0.5, \Phi(0.5) = 0.6915, \Phi(1.0) = 0.8413, \Phi(1.5) = 0.9332, \Phi(2.0) = 0.9772 \text{ and } \Phi(2.5) = 0.9938.$$

- (i) Find the probability $P(1 \leq X \leq 10)$. (10%)
(ii) If $Y = 2X + 1$, Find the probability $P(Y \leq 18)$. (10%)

3. You are allowed to play a game four times independently, each time your score (in a discrete value) will take values from the range 1 to 5 (that is, 1, 2, 3, 4, or 5), with probability 0.2, uniformly. Your final score will be the maximum (最大值) of the four scores and is modeled with a random variable X . Calculate the probability mass function (PMF) of X . (20%)

4. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 \leq x \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

and Let A be the event $\{X \geq 2\}$. Find $E[X]$, $P(A)$, $f_{X|A}(x)$, and $E[X|A]$. (20%)

5. Allen, a French chef in Taipei, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Allen to cook a duck breast (鴨胸肉) is described by the PDF of a continuous random variable G ; but on a bad day, the time it takes is described by the PDF of a continuous random variable B :

$$f_G(g) = \begin{cases} 2, & \text{if } 1/2 \leq g \leq 1 \\ 0, & \text{otherwise,} \end{cases} \quad f_B(b) = \begin{cases} 1, & \text{if } 1/2 \leq b \leq 3/2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Allen less than $3/4$ hour to cook a duck breast. (20%)

Hint: You can apply Bayes' rule for solving this problem.

6. Given two continuous random variables X and Y , each of which takes values between 0 and 1, with the joint PDF indicated below (the cutoff between probability density 0.8 and 1.6 occurs at $X = 0.5$ and $Y = 0.5$):

- (i) Find the PDF of X . (5%)
(ii) Find the PDF of Y . (5%)
(iii) Find the conditional PDF $f_{X|Y}(x|0.6)$. (5%)
(iv) Are X and Y independent of each other? (5%)

