Hw 2020-3

1.

Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that

$$T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix},\quad T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\0\\1\end{bmatrix},\quad T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)=\begin{bmatrix}5\\4\\3\end{bmatrix}$$

Determine $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ for any $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|ccc}-1&1&1&1&0&0\\1&-1&1&0&1&0\\1&1&-1&0&0&1\end{array}\right]\xrightarrow[r_3+r_1\to r_3]{r_2+r_1\to r_2}\left[\begin{array}{ccc|ccc}-1&1&1&1&0&0\\0&0&2&1&1&0\\0&2&0&1&0&1\end{array}\right]$$

$$\xrightarrow{r_1-\frac{1}{2}r_2-\frac{1}{2}r_3\to r_1}\left[\begin{array}{ccc|ccc}-1&0&0&0&-\frac{1}{2}&-\frac{1}{2}\\0&0&2&1&1&0\\0&2&0&1&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{1}{2}&0&\frac{1}{2}\\0&0&1&\frac{1}{2}&\frac{1}{2}&0\end{array}\right]$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=0\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-3\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix}+0\begin{bmatrix}-3\\0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-3\\0\\1\end{bmatrix}+0\begin{bmatrix}5\\4\\3\end{bmatrix}=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}1&3&-1\\2&3&1\\2&3&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+3x_2-x_3\\2x_1+3x_2+x_3\\2x_1+3x_2+2x_3\end{bmatrix}$$ #

2. Define the linear transform $T: R^3 \to R^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

T represents the orthogonal projection of R^3 on the xy-plane

(a) Prove that T is linear.

⇒ First proof that T preserves vector addition :

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}$$

⇒ Next proof that T preserves scalar multiplication. Let c be a real number :

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ cx_2 \\ cx_3 \end{bmatrix} = c\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$$

⇒ Thus we have proven that the function T is a linear transformation #

(b) Find the standard matrix of T.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \#$$

(c) Prove that $T(T(v)) = T(v)$ for every v in R^3

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$T(T(v)) = T\left(\begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix} = T(\vec{v}) \quad \#$$

3. Find a generating set for the null space of linear transformation T and determine whether T is one-to-one.

Here $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + 2x_2 - 3x_3 \\ 2x_1 + 3x_3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{bmatrix} \xrightarrow[-2r_1 + r_3 \to r_3]{-r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[r_3 - r_2 \to r_3]{r_1 - 2r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$, not one-to-one $\because$ nullity $= 1 \neq 0$ #

4. For an invertible linear transformation T defined below, determine a similar definition for its inverse T^{-1}

Here $T: \mathbb{R}^3 \to \mathbb{R}^3$ is $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + 2x_2 - 3x_3 \\ 2x_1 + 3x_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 + r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 - 2r_2 \to r_3]{r_1 + r_2 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right] \xrightarrow[r_2 + r_3 \to r_2]{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 3 & -1 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 3x_2 - x_3 \\ -3x_1 - x_2 + x_3 \\ -4x_1 - 2x_2 + x_3 \end{bmatrix}$$ #

5. Evaluate the determinate of matrix A, where $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} \begin{cases} r_2 - 2r_1 \to r_2 \\ r_3 + 4r_1 \to r_3 \\ r_4 - 3r_1 \to r_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{bmatrix} \begin{cases} r_3 - r_2 \to r_3 \\ r_4 - r_2 \to r_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 9 & -6 \end{bmatrix}$$

$$\xrightarrow{r_4 - 3r_3 \to r_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix} = B \Rightarrow \det(B) = 1 \cdot 1 \cdot 3 \cdot 6 = 18$$ #

6. Determine the value of c for which the following matrix in not invertable :

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & c & 0 \end{bmatrix} \xrightarrow[r_4+r_1\to r_4]{r_3-r_1\to r_3} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 2 & -2 \\ 0 & 1 & c-1 & 1 \end{bmatrix} \xrightarrow[r_4+r_2\to r_4]{r_3-r_2\to r_3} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & c+1 & 0 \end{bmatrix} \xrightarrow{r_3\leftrightarrow r_4} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & c+1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\det(A) = \underset{\uparrow \text{ row interchange}}{(-1)} \times (1\times(-1)\times(c+1)\times(-1)) = -c-1$$

$\Rightarrow$ If A is not invertable $\Rightarrow -c-1=0 \Rightarrow c=-1$ #

7. Find the generating sets for the range and null space of linear transformation T

define as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ x_2+x_3 \\ x_1-x_3 \\ x_1+2x_2+x_3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[r_4-r_1\to r_4]{r_3-r_1\to r_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[r_4-r_2\to r_4]{r_3+r_2\to r_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1-r_2\to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

① Range of T = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \right\}$ #

② $\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$ $\Rightarrow$ Null Space of T = span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ #

8. Find a basis for the (a) column space and (b) null space of matrix $\begin{bmatrix} -1 & 1 & 2 & 2 \\ 2 & 0 & -5 & 3 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) column space : $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -1 \\ -2 \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}} \right\}$

(b) $\begin{cases} x_1 = -4x_4 \\ x_2 = -4x_4 \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases}$ null space : $\left\{ \begin{bmatrix} -4 \\ -4 \\ -1 \\ 1 \end{bmatrix} \right\}$ #

9. Find the basis for the subspace of $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 - 3x_2 + 5x_3 = 0 \right\}$

$\begin{cases} x_1 = 3x_2 - 5x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$

basis for subspace $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 - 3x_2 + 5x_3 = 0 \right\}$

general solution :

$$\begin{bmatrix} 3 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[r_2\leftrightarrow r_3]{r_1\leftrightarrow r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ #