Midterm 2019

1. Let $\{u_1, u_2 \cdots u_k\}$ be linearly independent set of vectors in $\mathbb{R}^n$, and let v be a vector in $\mathbb{R}^n$ such that $v = c_1u_1 + c_2u_2 + \dots + c_ku_k$ for some scalar $c_1 \sim c_k$ with $c_1 \neq 0$.
Prove that $\{v, u_2 \dots u_k\}$ is linearly independent.

Let $a_1 \sim a_k$ be scalars that $a_1v + a_2u_2 + a_3u_3 + \cdots + a_ku_k = 0$

$$0 = a_1v + a_2u_2 + \cdots + a_ku_k = a_1(c_1u_1 + c_2u_2 + \cdots + c_ku_k) + a_2u_2 + \cdots + a_ku_k$$
$$= a_1c_1u_1 + (a_1c_2 + a_2)u_2 + \cdots + (a_1c_k + a_k)u_k$$

$\because \{u_1\ u_2 \dots u_k\}$ is L.I. $\therefore$ the coefficients of the linear combination is 0

$\Rightarrow a_1c_1 = a_1c_2 + a_2 = \cdots = a_1c_k + a_k = 0$

$\because c_1 \neq 0 \quad \therefore a_1 = 0$ Thus $a_1 = a_2 = \cdots = a_k = 0$

$\Rightarrow \{v, u_2, \cdots, u_k\}$ is linear independent #

2. Let A be an $m \times n$ matrix with reduced row echelon form R
Determine the reduced row echelon form of each of the following matrices

(a) $[A\ \ 0] \Rightarrow [R\ \ 0]$

(b) $[a_1\ a_2 \dots a_k]$ for $k < n$, where $a_i = Ae_i$

$\Rightarrow [r_1\ r_2 \dots r_k]$ for $k < n$, where $r_i = Re_i$

(c) cA, where c is a nonzero scalar $\Rightarrow R$

(d) $[I_m\ \ A] \Rightarrow [I_m\ \ A]$

(e) $[A\ \ cA] \Rightarrow [R\ \ cR]$

3. Suppose that u, v are linearly independent vectors in $\mathbb{R}^3$.
Find the reduced row echelon form of $A = [a_1\ a_2\ a_3\ a_4]$ given that $a_1 = u$, $a_2 = 2u$, $a_3 = u+v$, $a_4 = v$.

$\because u, v$ are L.I. $\therefore$ we can let $u = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$$\Rightarrow A = [a_1\ a_2\ a_3\ a_4] = \begin{bmatrix}1&2&1&0\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

$$r_1 - r_2 \to r_1 \Rightarrow \begin{bmatrix}1&2&0&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$ ※

4. Determine (a) reduced row echelon form R of A
(b) an invertable matrix P such that $PA = R$

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & -2 & 7 & 4 \\ 2 & 1 & 3 & -3 & 1 \end{bmatrix}$$

$\because PA = R \quad \therefore [A \ \ I_n] = [R \ \ P]$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 7 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 3 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_4 - 2r_1 \to r_4]{r_3 - r_1 \to r_3} \left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 5 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -7 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_4 - r_2 \to r_4]{r_3 - r_2 \to r_3} \left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 6 & 3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -6 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{r_4 + r_3 \to r_4} \left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 6 & 3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & -2 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{2}r_4 \to r_4 \\ r_3 - 3r_4 \to r_3 \\ r_1 - r_4 \to r_1 \end{array} \Rightarrow \left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 0 & 5/2 & 1 & -1/2 & -1/2 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 6 & 0 & 7/2 & 2 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & -1 & 1/2 & 1/2 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 2 & 0 & 5/2 & 1 & -1/2 & -1/2 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & -7/4 & -1 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & -1 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[r_2 + r_3 \to r_2]{r_1 - r_3 \to r_1} \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 5 & 0 & 17/4 & 2 & -3/4 & -5/4 \\ 0 & 1 & 0 & -4 & 0 & -7/4 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1 & -3 & 0 & -7/4 & -1 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & -1 & 1/2 & 1/2 \end{array}\right]$$

Ans :

$$R = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \frac{17}{4} & 2 & -\frac{3}{4} & -\frac{5}{4} \\ -\frac{7}{4} & 0 & \frac{1}{4} & \frac{3}{4} \\ -\frac{7}{4} & -1 & \frac{1}{4} & \frac{3}{4} \\ -\frac{3}{2} & -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ #

5. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -2r_1 \to r_1 \\ r_2 + r_1 \to r_2 \\ r_3 + r_1 \to r_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 2 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$$\begin{bmatrix} -3 & 5 & 1 \\ 0 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix} \Rightarrow T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 + x_3 \\ 3x_1 + x_2 + 2x_3 \\ 3x_1 + 2x_2 + 2x_3 \end{bmatrix} \#$$

6. Define the linear transform $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 4x_2 + x_3 + 6x_4 \\ 3x_1 + 7x_2 - x_3 + 11x_4 \\ x_1 + 2x_2 + 2x_4 \\ 2x_1 + 5x_2 - x_3 + 8x_4 \end{bmatrix}$

(a) Find the standard matrix A of T
(b) Show that A is invertable and find its inverse
(c) Use your answer to (b) to find the rule of T^{-1}

(a)

the standard matrix is $\begin{bmatrix} 2 & 4 & 1 & 6 \\ 3 & 7 & -1 & 11 \\ 1 & 2 & 0 & 2 \\ 2 & 5 & -1 & 8 \end{bmatrix} \#$

(b) $[A \ \ I_4] \Rightarrow [R \ \ C]$

$$\left[\begin{array}{cccc|cccc} 2 & 4 & 1 & 6 & 1 & 0 & 0 & 0 \\ 3 & 7 & -1 & 11 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 5 & -1 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 2 & 4 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 2 & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 2 & 4 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 2 & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 2 & 4 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 2 & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 10 & -1 & -12 \\ 0 & 1 & 0 & 0 & 1 & -6 & 2 & 7 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \end{array}\right]$$

$\Rightarrow \because R = I_4$ (rank = 4)

$\therefore A^{-1}$ exist $= \begin{bmatrix} -2 & 10 & -1 & -12 \\ 1 & -6 & 2 & 7 \\ 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix} \#$

(c)

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 10x_2 - x_3 - 12x_4 \\ x_1 - 6x_2 + 2x_3 + 7x_4 \\ x_1 - 2x_2 + 2x_4 \\ x_2 - x_3 - x_4 \end{bmatrix} \#$$

7. (a) Evaluate the determination of the matrix $\begin{bmatrix} 2 & 1 & 5 & 2 \\ 2 & 1 & 8 & 1 \\ 2 & -1 & 5 & 3 \\ 4 & -2 & 10 & 3 \end{bmatrix}$

(a)

$$\begin{bmatrix} 2 & 1 & 5 & 2 \\ 2 & 1 & 8 & 1 \\ 2 & -1 & 5 & 3 \\ 4 & -2 & 10 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 5 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & -2 & 0 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 5 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 5 & 2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$\Rightarrow$ determinent $= 2 \times (-2) \times 3 \times (-3) = 36$ ※

(b) Determine the value(s) of c which matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{bmatrix}$ is not invertable.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c+2 \\ 0 & c & -15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c+2 \\ 0 & 0 & c^2+2c-15 \end{bmatrix}$$

when the last row is a nonzero row, then A is invertable

$\Rightarrow c^2+2c-15 = (c+5)(c-3) = 0 \qquad \Rightarrow c = -5, 3$ ※

☆ 8. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Prove that if W is a subspace of $\mathbb{R}^m$, then $\{u : T(u) \text{ is in } W\}$ is a subspace of $\mathbb{R}^n$

Let $V = \{u : T(u) \text{ is in } W\}$

(a) since $T(0) = 0$ is in W, $0 \in V$

(b) Let u_1 u_2 in V, then $T(u_1)$, $T(u_2)$ are in W, hence $T(u_1) + T(u_2)$ in W
since $T(u_1) + T(u_2) = T(u_1+u_2)$ (T is L.T.) $\Rightarrow (u_1+u_2)$ is in V

(c) For any scalar c, $cT(u_1)$ is in W, so cu_1 is in V ($cT(u_1) = T(cu_1)$)

$\Rightarrow V$ is a subspace of $\mathbb{R}^n$

9. For the linear transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 + x_4 \\ 2x_1 - 5x_2 + x_3 + 3x_4 \\ x_1 - 3x_2 + 2x_4 \end{bmatrix}$

(a) find a basis for the range of T

(b) find a basis for the null space of T

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & -5 & 1 & 3 \\ 1 & -3 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)

basis of the range of T:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \\ -3 \end{bmatrix} \right\}$$ ※

(b)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_4$$

basis of the null space of T

$$\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ ※