Introduction to Probability
Final

December 20, 2022, 3:30 p.m. - 5:30 p.m.

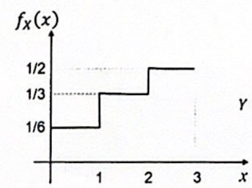
Note: You have to answer the questions with supporting explanations if needed.

1. Given that X is a continuous random variable with a probability density function (PDF) $f_X(x)$ and $Y = aX + b$.

   Show that the PDF of random variable Y can be expressed as: $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$. (10%)

2. We are told that X and Y are two independent random variables, both of which are uniformly distributed in the interval $[0, 1]$.
   (i) Use the "derived-distributions" method to find the PDF $f_W(w)$ of the random variable W which is express by $W = X + Y$. (10%)
   (ii) Use the "derived-distributions" method to find the PDF $f_Z(z)$ of the random variable Z which is express by $Z = 3Y/X$. (10%)
   (iii) Find the probability that $X \le 3Y$ given that $X \ge 2/3$ (namely, $\mathbf{P}(X \le 3Y | X \ge 2/3)$). (5%)

3. Consider a continuous random variable X with the PDF given in the right figure, where we define an auxiliary (discrete) random variable Y.
   (i) Find $\mathbf{E}[X]$ with Law of Iterated Expectations:
   $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$. (10%)
   (ii) Find $\text{var}(X)$ with Law of Total Variance:
   $\text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y])$. (10%)



$$Y = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 2, & \text{if } 1 \le x < 2 \\ 3, & \text{if } 2 \le x < 3 \end{cases}$$

Hint I: for a (continuous) uniform random variable distributed in an interval $[a, b]$, its mean is $(a + b)/2$ and its variance is $(b - a)^2/12$.

4. (i) The covariance of two random variables X and Y is defined by $\text{cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$. Show that $\text{cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$. (5%)
   (ii) Let U be the number of 2's and V be the number of 6's that occur in n rolls of a fair six-sided die. Compute $\text{cov}(U, V)$. (10%)

$\frac{\text{cov}(U.V)}{\sqrt{\text{var}(X)\text{var}(V}}$

5. Let X be a discrete random variable with a moment generating function $M_X(s)$ such that $M_X(s) = a + be^{2s} + ce^{6s}$, $\mathbf{E}[X] = 2$ and $\text{var}(X) = 6$.
   (i) Find the probability mass function (PMF) of X. (10%)
   (ii) Find $\mathbf{E}[(2 + X)^2]$. (5%)
   Hint II: For a discrete random variable X, $M_X(s) = \sum_x e^{sx} P_X(x)$.

$2be^{2s} + 6ce^{6s}$

$4be^{2s} + 36ce^{6s}$

6. Given that X is an exponential random variable with its pdf expressed by $f_X(x) = \lambda e^{-\lambda x}$ $(x \ge 0)$, whose transform can be expressed by $M_X(s) = \frac{\lambda}{\lambda - s}$.
   (i) Using $M_X(s)$ to show that $\mathbf{E}[X] = \frac{1}{\lambda}$ and $\text{var}(X) = \frac{1}{\lambda^2}$. (10%)
   (ii) Given that random variable Y can be expressed as $Y = 3X + 2$. Find the transform of Y. (5%)

$3X+2$

$\lambda e^{-(3X+2)\lambda}$

7. A car driver goes through 10 lights, each of which is found to be red with probability 0.4. The waiting times (in seconds) at each light are modeled as independent uninform random variables uniformly distributed in the interval $[0, 3]$. Let X be the total waiting time at the red lights.
   (i) Find the transform (moment generating function) of X by viewing X as a sum of a random number of random variables. (10%)
   (ii) Find the mean and variance of X. (10%)

$\lambda e^{-3x\lambda} \times e^{-2\lambda}$ $\quad \frac{p^2}{1-p}$

$\frac{0.16}{0.6} = \frac{16}{60} = \frac{4}{15}$

$M_X(s) = E\, e^{\lambda e^{-3x\lambda} \times e^{-2\lambda}}$ $\times P_U P$

$E[e^{\lambda e^{-3x\lambda} \times e^{-2\lambda}}]$