double exp(double x);// return e^x double log(double x);// ln(x) double log10(double x);// $\log_{10}(x)$ double pow(double x, double y);// x^y double sqrt(double x)

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2}$

$\frac{d}{dx}(e^u) = e^u u'$

$\frac{d}{dx}(uv) = uv'+vu'$

$\frac{d}{dx}(\ln u) = \frac{u'}{u}$

$\frac{d}{dx}(a^u) = (\ln a)a^u u'$

$\frac{d}{dx}(\log_a u) = \frac{u'}{(\ln a)u}$

$\frac{d}{dx}(\sin u) = (\cos u)u'$

$\frac{d}{dx}(\cos u) = -(\sin u)u'$

Nested multiplication (Horner's method)

$$
\begin{aligned}
P(x) &= 2x^4+3x^3-3x^2+5x-1\\
&= -1+5x-3x^2+3x^3+2x^4\\
&= -1+x(5-3x+3x^2+2x^3)\\
&= -1+x(5+x(-3+3x+2x^2))\\
&= -1+x(5+x(-3+x(3+2x)))
\end{aligned}
$$

- Number of multiplications? 4
- Number of additions? 4

11→8

Fixed point iteration: Approach

Given a function $f(x)$, select a function $g(x)$ such that

$f(x) = 0 \rightarrow g(x) = x$.

Then
- x_0 = initial guess
- $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, \ldots$

Until x_{i+1} satisfies some termination criterion

Convergence
- $S = |g'(r)| < 1$

Number of Roots

In contrast to scalar linear equations

$$mx - n = 0 \Rightarrow x = \frac{n}{m},$$

nonlinear equations have an undetermined number of zeros.

Theorem: Let f be twice continuously differentiable and $f(r) = 0$. If $f'(r) \neq 0$, then Newton's method is quadratically convergent to r, starting with x_0 close to r.

Proof

$$f(r) = f(x_i) + (r-x_i)f'(x_i) + \frac{(r-x_i)^2}{2}f''(x_i)$$

$$0 = f(x_i) + (r-x_i)f'(x_i) + \frac{(r-x_i)^2}{2}f''(x_i)$$

$$-\frac{f(x_i)}{f'(x_i)} = r - x_i + \frac{(r-x_i)^2}{2}\frac{f''(x_i)}{f'(x_i)}$$

$$\underbrace{\left(x_i - \frac{f(x_i)}{f'(x_i)}\right)}_{x_{i+1}} - r = \frac{(r-x_i)^2}{2}\frac{f''(x_i)}{f'(x_i)}$$

$$e_{i+1} = e_i^2\frac{f''(x_i)}{2f'(x_i)}$$

$$\lim_{i\to\infty}\frac{e_{i+1}}{e_i^2} = \left|\frac{f''(r)}{2f'(r)}\right|$$

$$\lim_{i\to\infty}\frac{e_{i+1}}{e_i^2} = M \text{ (a constant).}$$

Bisection

- 猜數字
- Method for finding a root of a scalar equation $f(x) = 0$ in an interval $[a, b]$
- Assumption: $f(a)f(b) < 0$
- Since f is continuous there must be a zero $x^* \in [a, b]$

1. Compute midpoint m of the interval and check $f(m)$
2. Depending on the sign of $f(m)$, we can decide if $x^* \in [a, m]$ or $x^* \in [m, b]$
   - Of course, if $f(m) = 0$ then we are done.

Bisection vs. Fixed-point iteration

Which one is faster?

Depending on $S = |g'(r)|$ is smaller or larger than ½.

Bisection: How accurate and how fast?

The interval length after n bisection steps is: $\frac{b-a}{2^n}$

$$\text{Solution error} = |x_n - x| < \frac{b-a}{2^{n+1}}$$

x_n: the midpoint of the n-th interval

If we want the error to satisfy $|x_n - x| \le \varepsilon$, it suffices to have $(b-a)/2^n \le \varepsilon$, so that

$$\boxed{n > \log_2\left(\frac{b-a}{\varepsilon}\right)}$$

Newton's iteration

$f'(x_0) = f(x_0) / (x_0 - x_1)$

$x_1 = x_0 - f(x_0) / f'(x_0)$

$x_{i+1} = x_i - f(x_i) / f'(x_i),\ i = 0, 1, 2, \ldots$

Newton's method is a fixed point iteration with iteration function

$g(x) = x - f(x) / f'(x)$

Residual x_a = your approximate

$\underline{b} - A\underline{x}_a$

Backward error

$\|\underline{b} - A\underline{x}_a\|_\infty$

Forward error

$\|\underline{x} - \underline{x}_a\|_\infty$

$$\text{error magnification factor} = \frac{\text{relative forward error}}{\text{relative backward error}} = \text{cond}(A) = \|A\| \cdot \|A^{-1}\|$$

Secant method

Replaces the tangent line (the function's derivative) with the secant line.

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Two starting guesses are needed to begin the Secant method.

The secant method is **superlinearly convergent**, meaning that it lies between linearly and quadratically convergent methods.

$$\underbrace{O(n^3)}_{\text{elimination}} + \underbrace{O(n^2)}_{\text{subsitution}} = \underbrace{O(n^3)}_{\text{Gaussian elimination}}$$

The computational cost is dominated by the elimination step!

LU factorization: complexity

Need to solve a number of different problems with the **same A** and **different $\underline{b}$**

The **matrix (absolute row sum) norm**

ex: $A = \begin{bmatrix} 1 & 1 \\ 1.0001 & -1 \end{bmatrix}$

$\|A\|_\infty = 2.0001$

Definition. The $n \times n$ matrix $A = (a_{ij})$ is **strictly diagonally dominant** if, for each $1 \le i \le n$, $|a_{ii}| > \sum_{j \ne i}|a_{ij}|$.

The **infinity norm**, or the $\underline{x} = [x_1, \ldots, x_n]^T$ is

$$\|\underline{x}\|_\infty = \max |x_i|,$$

Multipliers in Gaussian elimination should be kept as small as possible to avoid swamping. From large differences in relative sizes. ex: $\begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Partial pivoting

- Forces the absolute value of multipliers to be no larger than 1

Why using iterative methods?

Can be faster if the input matrix is large
- One step of an iterative method requires only a fraction of the floating operations of a full LU factorization.

A good approximation to the solution is already known.

The input matrix is *sparse*.

Jacobi vs. Gauss-Seidel

$\underline{x}_{k+1} = D^{-1}(\underline{b} - (L+U)\underline{x}_k)$ | $\underline{x}_{k+1} = D^{-1}(\underline{b} - U\underline{x}_k - L\underline{x}_{k+1})$

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_k \\ v_k \end{bmatrix}\right) = \begin{bmatrix} (5-v_k)/3 \\ (5-u_k)/2 \end{bmatrix}$$

Gauss-Seidel

$$u_{k+1} = \frac{4 - v_k + w_k}{3}$$

SOR $w > 1$

$$u_{k+1} = (1-\omega)u_k + \omega\frac{4 - v_k + w_k}{3}$$