1. Let X, Y, Z be independent, continuous random variable that are uniform on interval $[0,1]$
What is the PDF of random variable $W = \{2X, 3Y, Z\}$

$\Rightarrow$ Let $X' = 2X$, $Y' = 3Y$

$F_W(w) = P(\max\{X'\ Y'\ Z\} < w) = P(X' < w)P(Y' < w)P(Z < w)$

(i) $0 \leq w \leq 1 \quad \Rightarrow F_W(w) = (\frac{1}{2}w) \times (\frac{1}{3}w) \times w = \frac{1}{6}w^3$,

$\Rightarrow f_W(w) = \frac{1}{2}w^2$

(ii) $1 \leq w \leq 2 \quad \Rightarrow F_W(w) = (\frac{1}{2}w)(\frac{1}{3}w) = \frac{1}{6}w^2$

$\Rightarrow f_W(w) = \frac{1}{3}w$

(iii) $2 \leq w \leq 3 \quad \Rightarrow F_W(w) = \frac{1}{3}w$

$f_W(w) = \frac{1}{3}$

$$\Rightarrow f_W(w) = \begin{cases} \frac{1}{2}w^2, & 0 \leq w \leq 1 \\ \frac{1}{3}w, & 1 \leq w \leq 2 \\ \frac{1}{3}, & 2 \leq w \leq 3 \\ 0, & \text{otherwise} \end{cases}$$ ※

2. If $W = ax + b$, where a, b are scalars ($a \neq 0$)
show that PDF $f_W(w)$ of W can be expressed in terms of PDF $f_X(x)$ of X

by $f_W(w) = \frac{1}{|a|} f_X(\frac{w-b}{a})$

$\Rightarrow F_W(w) = P(ax + b \leq w)$

(i) if $a > 0$

$P(ax+b \leq w) = P(x \leq \frac{w-b}{a}) = F_X(\frac{w-b}{a})$

$f_W(w) = \frac{d F_X(\frac{w-b}{a})}{d(\frac{w-b}{a})} \cdot \frac{d(\frac{w-b}{a})}{dw} = f_X(\frac{w-b}{a}) \times \frac{1}{a} \quad \dots (1)$

(ii) if $a < 0$

$P(ax+b \leq w) = 1 - P(x \leq \frac{w-b}{a}) = 1 - F_X(\frac{w-b}{a})$

$f_W(w) = -\frac{d F_X(\frac{w-b}{a})}{d(\frac{w-b}{a})} \cdot \frac{d(\frac{w-b}{a})}{dw} = -f_X(\frac{w-b}{a}) \times \frac{1}{a} \quad \dots (2)$

$\because$ (1) & (2)

$\therefore f_W(w) = \frac{1}{|a|} f_X(\frac{w-b}{a})$

3. Given a random variable with a transform is $M_X(s) = \frac{1}{3}e^{-3s} + \frac{1}{4} + \frac{1}{4}e^{2s} + \frac{1}{6}e^{3s}$
Calculate $E[X]$ and $var(X)$

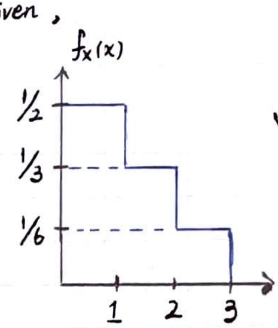
[for discrete random variable X, its transform $M_X(s) = \sum_x e^{sx} p_X(x)$]

$$\Rightarrow E[X] = \frac{d M_X(s)}{ds}\bigg|_{s=0} = -e^{-3s} + \frac{1}{2}e^{2s} + \frac{1}{2}e^{3s}\bigg|_{s=0} = -1 + \frac{1}{2} + \frac{1}{2} = 0 \text{ ※}$$

$$E[X^2] = \frac{d^2 M_X(s)}{ds^2}\bigg|_{s=0} = \frac{d\left(-e^{-3s} + \frac{1}{2}e^{2s} + \frac{1}{2}e^{3s}\right)}{ds}\bigg|_{s=0} = 3e^{-3s} + e^{2s} + \frac{3}{2}e^{3s}\bigg|_{s=0} = \frac{11}{2}$$

$$var(X) = E[X^2] - E[X]^2 = \frac{11}{2} - 0 = \frac{11}{2} \text{ ※}$$

4. Consider a continuous random variable X with PDF given, where we define an discrete random variable Y.



$$Y = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 2, & \text{if } 1 \le x \le 2 \\ 3, & \text{if } 2 \le x \le 3 \end{cases}$$

(i) Find $E[X]$ with Law of Iterated Expectation
$E[X] = E[E[X|Y]]$

(ii) Find $var(X)$ with Law of Total Variance
$var(X) = E[var(X|Y)] + var(E[X|Y])$

$$\Rightarrow f_Y(1) = \frac{1}{2}, \quad f_Y(2) = \frac{1}{3}, \quad f_Y(3) = \frac{1}{6}$$

$$E[X|Y=1] = \frac{0+1}{2} = \frac{1}{2}, \quad E[X|Y=2] = \frac{1+2}{2} = \frac{3}{2}, \quad E[X|Y=3] = \frac{2+3}{2} = \frac{5}{2}$$

$$\Rightarrow E[E[X|Y]] = \frac{1}{2}\times\frac{1}{2} + \frac{1}{3}\times\frac{3}{2} + \frac{1}{6}\times\frac{5}{2} = \frac{1}{4} + \frac{1}{2} + \frac{5}{12} = \frac{14}{12} = \frac{7}{6} \text{ ※ (i)}$$

$$\Rightarrow var(X|Y=1) = var(X|Y=2) = var(X|Y=3) = \frac{1}{12} \left(\frac{b-a}{12} \text{ in uniform}\right)$$

$$E[var(X|Y)] = \frac{1}{12}\times\frac{1}{2} + \frac{1}{12}\times\frac{1}{3} + \frac{1}{12}\times\frac{1}{6} = \frac{1}{12}$$

$$var(E[X|Y]) = \frac{1}{2}\left(\frac{1}{2}-\frac{7}{6}\right)^2 + \frac{1}{3}\left(\frac{3}{2}-\frac{7}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{2}-\frac{7}{6}\right)^2$$

$$= \frac{1}{2}\times\frac{4}{9} + \frac{1}{3}\times\frac{1}{9} + \frac{1}{6}\times\frac{16}{9} = \frac{2}{9} + \frac{1}{27} + \frac{8}{27} = \frac{5}{9}$$

$$\Rightarrow var(X) = E[var(X|Y)] + var(E[X|Y])$$

$$= \frac{1}{12} + \frac{5}{9} = \frac{23}{36} \text{ ※}$$

5 The covariance of any two random variable X and Y is $\text{cov}(X.Y) = E[(X-E[X])(Y-E[Y])]$

(a) Show that for any three random variables X, Y, Z, $\text{cov}(X, Y+Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$

$\Rightarrow \text{cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$

$= E[XY - YE[X] - XE[Y] + E[X]E[Y]] = E[XY] - E[X]E[Y]$

$\Rightarrow \text{cov}(X, Y+Z) = E[X(Y+Z)] - E[X]E[Y+Z]$

$= E[XY + XZ] - E[X](E[Y]+E[Z])$

$= E[XY] - E[X]E[Y] + E[XZ] - E[X]E[Z]$

$= \text{cov}(X,Y) + \text{cov}(X,Z)$ ※

(b) Show that for any two random variables X, Y independent, then $\text{cov}(X.Y) = 0$

$\Rightarrow \text{cov}(X,Y) = E[XY] - E[X]E[Y]$

$\because X, Y$ independent $\therefore E[XY] = E[X]E[Y] \Rightarrow \text{cov}(X,Y) = 0$ ※