Technical note
Maneuvering model of an idealized ship

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1 Introduction and preliminaries

This technical note documents and investigates an idealized 3 degree of freedom (DOF) maneuvering model developed in matlab. The model is aptly named RV Marley (Research Vessel Marley), after its creator and intended usage in research.

Empirical maneuvering models, based on e.g. Taylor expansions or 2nd order modulus functions, are valid for perturbations about a nominal forward speed. There is no guarantee for the validity in operations with varying speed (e.g. transit+docking operations) or large current velocities (due to the resulting wide range of relative fluid velocities).

RV Marley qualitatively captures the dynamics of ships moving in constant currents. By disregarding all sway-yaw coupling terms other than the speed-dependent Munk moment, the model is sufficiently valid at all speeds, including station-keeping. This makes it suitable as a simulation platform when designing general guidance and navigation control systems.

The ship is assumed to have two symmetry planes; in the fore-aft or port-starboard direction. This may for instance be a double-ended ferry or a canoe. This ensures that the rigid-body and added mass matrices are diagonal (with midship as reference point). The hydrodynamic damping matrix is also assumed diagonal. The above assumptions mean that the only coupling terms present are due to the Coriolis forces (which includes the destabilizing Munk moment). The ship is directionally unstable for surge velocities larger than approximately 4.5 m/s. However, the degree of instability can be tuned by modifying the added mass terms.

1.1 Coordinate system

Define a right-handed inertial coordinate system with z pointing downwards, consistent with the convention used in e.g. [1]. Let $p := [x\ y\ \psi]^\top \in \mathbb{R}^2 \times$

$[-\pi, \pi]$ be the ship pose. With x-axis pointing upwards (ordinate), the y-axis pointing to the right (abscissa), the z-axis points into the paper and the yaw angle ψ is positive in the clockwise direction (relative to the x-axis).

Define a body-fixed coordinate system with origin at midship, which coincides with center of gravity (and center of buoyancy). Let the body-fixed x-axis point in the longitudinal direction towards the bow, y-axis point towards starboard and z-axis downwards. Let $\nu := [u\ v\ r]^\top \in \mathbb{R}^3$ be the 3 DOF velocity vector in body-fixed coordinates.

Remark 1. *For simplicity we use $x-y-z$, without subscripts or superscripts, both for body-fixed and inertial coordinate system. The considered coordinate system should be clear from context.*

2 Kinematics

The kinematic equation relating ship pose with body-fixed motions is given by

$$\dot{p} = R(\psi)\nu, \quad R(\psi) := \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

2.1 Course angle and drift angle

Let $U := \sqrt{u^2 + v^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$ be the ship speed. The course angle satisfies

$$\dot{x} = U\cos(\chi),\ \dot{y} = U\sin(\chi) \implies \tan(\chi) = \frac{\dot{y}}{\dot{x}}. \tag{2}$$

From this we obtain $\chi = \operatorname{atan2}(\dot{y}, \dot{x})$. The drift angle satisfies

$$u = U\cos(\beta),\ v = U\sin(\beta) \implies \tan(\beta) = \frac{v}{u}. \tag{3}$$

From this we obtain $\beta = \operatorname{atan2}(v, u)$. We note that χ and β are only defined for non-zero speed, i.e. $U > 0$. Assuming $u > 0$ we have $\beta = \arcsin\left(\frac{v}{U}\right)$, which is the formulation used in [1].

The relation between course angle, drift angle and ship heading is given by

$$\chi = \psi + \beta \tag{4}$$

2.2 Kinematics of current

We consider a steady (time-invariant) and uniform (in space) current, given by the current speed U_c and flow direction $\beta_c \in [-\pi, \pi]$. The current velocities in inertial reference frame are given by $u_c^n = U_c\cos(\beta_c)$ and $v_c^n = U_c\sin(\beta_c)$. The body-fixed current velocity becomes

$$u_c = U_c\cos(\beta_c - \psi), \quad v_c = U_c\sin(\beta_c - \psi). \tag{5}$$

We define the current velocity vector $\nu_c := [u_c \; v_c \; 0]^\top$, and the relative velocity vector $\nu_r := \nu - \nu_c$. Since $\dot{U}_c = \dot{\beta}_c = 0$ we have that

$$\dot{\nu}_r = \dot{\nu} - \dot{\nu}_c, \quad \dot{\nu}_c = U_c \begin{bmatrix} \sin(\beta_c - \psi) \\ -\cos(\beta_c - \psi) \\ 0 \end{bmatrix} r \tag{6}$$

We also define the relative speed $U_r := \sqrt{u_r^2 + v_r^2}$. The relative drift angle satisfies

$$u_r = U_r \cos(\beta_r), \; v_r = U_r \sin(\beta_r) \implies \tan(\beta_r) = \frac{v_r}{u_r}. \tag{7}$$

from which we obtain $\beta_r = \operatorname{atan2}(v_r, u_r)$.

3 Kinetics

The equation of motion is given by

$$M_{rb}\dot{\nu} + M_a\dot{\nu}_r + C_{rb}(\nu)\nu + C_a\nu + D(\nu_r)\nu_r = F, \tag{8}$$

where $M_{rb}, M_a, C_{rb}, C_a, D \in \mathbb{R}^{3\times 3}$ are the rigid body mass matrix, added mass matrix, rigid body Coriolis matrix, added mass Coriolis matrix and linear damping matrix, respectively. (Note that there are no centripetal forces when M_{rb} and M_a are diagonal.) The vector $F := [F_u \; F_v \; F_r]^\top \in \mathbb{R}^3$ denote all forces that do not depend on body motions. Eq. 8 must be solved using either ν or ν_r as the state. Choosing ν we obtain

$$\dot{\nu} = M^{-1} \left(F - D(\nu_r)\nu_r - C_{rb}(\nu)\nu - C_a(\nu_r)\nu_r + M_a\dot{\nu}_c \right), \tag{9}$$

where $M := M_{rb} + M_a$ is the total mass matrix, and $\dot{\nu}_c$ is given by (6).

3.1 Mass parameters

Reasonable ship parameters are chosen based on engineering judgment. All parameters are given as a function of the ship length L, making scaling possible.

- $L = 100$ m: Length.
- $B = L/10$ m: Breadth
- $D = L/20$ m: Draft
- $m = LBD \times 10^3$ kg: Rigid body mass (assuming fresh water density).
- $l_r = L/4$: Inertia radius in yaw.

This gives $M_{rb} := \operatorname{diag}(m, m, ml_r^2)$. The added mass matrix is given by $M_a := \operatorname{diag}(a_{11}, a_{22}, a_{33})$, where the parameters are chosen as $a_{11} = 0.05m$, $a_{22} = 0.3m$ and $a_{33} = a_{22}l_r^2$.

3.2 Damping parameters

Damping is selected to be linear+quadratic. While the overall damping level may be unrealistically large, the chosen values is believed to yield a reasonable ratio between damping level in the various DOFs. The quadratic damping in surge and sway is chosen as

$$d_{11q} := \frac{1}{2} BDC_{dx} \times 10^3, \quad d_{22q} := \frac{1}{2} LDC_{dy} \times 10^3, \tag{10}$$

where $C_{dx} = 0.5$ and $C_{dy} = 1$ are drag coefficients in surge and sway, respectively, while BD and LD is the projected area. The multiplication 10^3 corresponds to fresh water density. Yaw damping is obtained by integrating the yaw-induced sway force along the hull:

$$d_{33q} := C_{dy} D \int_0^{L/2} x^3 dx \times 10^3 = C_{dy} D \frac{L^4}{64} \times 10^3. \tag{11}$$

Linear damping is added by simply matching the resulting quadratic damping force at selected velocities:

$$d_{11} := d_{11q} u_0, \ d_{22} := d_{22q} v_0, \ d_{33} := d_{33q} r_0, \tag{12}$$

where $u_0 = 6$ m/s (corresponding to Froude number $F_n = 0.19$), $v_0 = 2$ m/s and $r_0 = \pi/90$ rad/s (corresponding to 2 deg/s).

The nonlinear damping matrix is then given by $D(\nu) = D_l + D_{nl}(\nu)$, where $D_l = \mathrm{diag}(d_{11}, d_{22}, d_{33})$ and $D_{nl} = \mathrm{diag}(d_{11q}|u_r|, d_{22q}|v_r|, d_{33q}|r|)$.

3.3 Rigid-body kinetics

The rigid body forces, evaluated in a rotating reference frame about the center of gravity, is given by

$$\begin{bmatrix} m(\dot{u} - vr) \\ m(\dot{v} + ur) \\ ml_r^2 \dot{r} \end{bmatrix} =: M_{rb}\dot{\nu} + C_{rb}(\nu)\nu. \tag{13}$$

Assuming no external forces we have that

$$\dot{u} = vr, \ \dot{v} = -ur \implies \tag{14}$$

$$\dot{U} = \frac{d\sqrt{u^2 + v^2}}{dt} = \frac{u\dot{u} + v\dot{v}}{\sqrt{u^2 + v^2}} = \frac{uvr - vur}{\sqrt{u^2 + v^2}} = 0. \tag{15}$$

This result is expected from conservation of momentum. The Coriolis forces are given by

$$\begin{bmatrix} -mvr \\ mur \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -mv \\ 0 & 0 & mu \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} = \begin{bmatrix} 0 & -mr & 0 \\ mr & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} =: C_{rb}(\nu)\nu. \tag{16}$$

Regardless of how $C_{rb}(\nu)$ is parametrized, we have $\nu^\top C_{rb}(\nu)\nu = 0$, as expected from conservation of energy.

3.4 Added mass forces

The hydrodynamic inertia loads are given by

$$\begin{bmatrix} a_{11}\dot{u}_r - a_{22}v_r r \\ a_{22}\dot{v}_r + a_{11}u_r r \\ a_{33}\dot{r} + (a_{22} - a_{11})u_r v_r \end{bmatrix} =: M_a\dot{\nu}_r + C_a(\nu_r)\nu_r \tag{17}$$

Separating out the Coriolis terms we obtain

$$\begin{bmatrix} -a_{22}v_r r \\ a_{11}u_r r \\ (a_{22} - a_{11})u_r v_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_{22}v_r \\ 0 & 0 & a_{11}u_r \\ a_{22}v_r & -a_{11}u_r & 0 \end{bmatrix} \nu_r =: C_a(\nu_r)\nu_r \tag{18}$$

As for rigid body Coriolis forces, we have $\nu_r^\top C_a(\nu_r)\nu_r = 0$, meaning that kinetic energy is preserved. However, since the "apparent mass" is different for surge and sway, $a_{11} \neq a_{22}$, the added mass Coriolis forces can transfer kinetic energy between translation and rotation. The term $(a_{22} - a_{11})u_r v_r$ is the Munk moment. This may also be expressed as

$$\begin{aligned} (a_{22} - a_{11})u_r v_r &= (a_{22} - a_{11})U_r^2 \cos(\beta_r)\sin(\beta_r) \\ &= \frac{1}{2}(a_{22} - a_{11})U_r^2 \sin(2\beta_r). \end{aligned} \tag{19}$$

We recognize this expression from [2, Eq. (6.27)]. The Munk moment is destabilizing in the sway-yaw subsystem. To realize this, consider the induced sideslip velocity during a turn, and assume no current. This will cause an "outwards drift". For a clockwise turn this means $r > 0$ and $v < 0$. Since r and v have opposite sign during a steady-state turn, the sway term $a_{11}u_r r$ and yaw term $(a_{22} - a_{11})u_r v_r$ have opposite sign (note that $a_{22} \gg a_{11}$ for any conventional ship design).

4 Motion characteristics

4.1 Directional stability

To investigate directional stability we consider small perturbations about a forward velocity $u = u_0 > 0$, and $U_c = 0$. Since linear damping dominates for small perturbations the coupled sway-yaw subsystem becomes

$$\begin{bmatrix} m_{22} & 0 \\ 0 & m_{33} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} d_{22} & m_{11}u_0 \\ (a_{22} - a_{11})u_0 & d_{33} \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{20}$$

Written in state-space form we obtain

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = - \begin{bmatrix} \frac{d_{22}}{m_{22}} & \frac{m_{11}u_0}{m_{22}} \\ \frac{(a_{22}-a_{11})u_0}{m_{33}} & \frac{d_{33}}{m_{33}} \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} \tag{21}$$

The eigenvalues of the dynamics matrix is given by

$$\lambda = -\frac{1}{2}\left(\frac{d_{22}}{m_{22}} + \frac{d_{33}}{m_{33}}\right) \pm \frac{1}{2}\left(\left(\frac{d_{22}}{m_{22}} - \frac{d_{33}}{m_{33}}\right)^2 + 4\frac{m_{11}(a_{22}-a_{11})u_0^2}{m_{33}m_{22}}\right)^{1/2} \tag{22}$$

This will always produce two real eigenvalues, one of which may be positive. If both eigenvalues are negative the ship is directionally stable. If one eigenvalue is positive the ship is directionally unstable (a saddle point from the eigenvalue analysis). Hence the ship is directionally stable if

$$d_{22}d_{33} > m_{11}(a_{22}-a_{11})u_0^2. \tag{23}$$

This shows that the ship will become directionally unstable for sufficiently large velocity. With the parameters chosen in previous chapter the ship becomes unstable for

$$u_0 > \sqrt{\frac{d_{22}d_{33}}{m_{11}(a_{22}-a_{11})}} = 4.56. \tag{24}$$

4.2 Pivot point

For control purposes the pivot point is often used as reference point. The sway velocity along the longitudinal axis of the ship is given by

$$v_x(x) = v + rx, \tag{25}$$

The pivot point x_p is defined as the point on the ship where $v_x(x) = 0$, i.e.

$$v + rx_p = 0 \implies x_p := -\frac{v}{r},\ r \neq 0. \tag{26}$$

In steady-state turning we have $x_p > 0$, i.e. the pivot point is in front of COG. However, the pivot point is not a fixed point, but depends on both u and r. For low yaw velocities the linear damping dominates. The pivot point at steady-state turn may then be estimated by

$$v = -\frac{m_{11}ur}{d_{22}} \implies x_p = \frac{m_{11}u_0}{d_{22}}. \tag{27}$$

At larger yaw velocities the yaw-induced sway velocity is lower than predicted by (27), and thus the pivot point moves closer to COG. We also define the fluid-relative pivot point

$$x_{pr} := -\frac{v_r}{r}. \tag{28}$$

References

[1] T. I. Fossen, *Handbook of marine craft hydrodynamics and motion control.* John Wiley & Sons, 2011.

[2] O. Faltinsen, *Sea loads on ships and offshore structures.* Cambridge university press, 1993, vol. 1.