Efficient Extraction of QBF (Counter)models from Long-Distance Resolution Proofs

(Supplementary Material)

We give an alternative definition of phase function in the following table. Here we simplified the notation by assuming that a literal is associated to its underlying clause, which should be clear from the context.

Table 1. Attributes of each vertex $v \in V_\Pi$ of an LQ-refutation Π of a false QBF Φ.

<table>
<tr><th>Attribute</th><th>Definition</th></tr>
<tr><td>Parent literal</td><td>Literal $l' \in u.clause$ is called a parent literal of $l \in v.clause$ if $var(l') = var(l)$ and $(u, v) \in E_\Pi$. Note that l can only have 0, 1, or 2 parent literals.</td></tr>
<tr><td>Phase function</td><td>The phase function, denoted $l.phase$, of literal $l \in v.clause$ is defined as follows:
– if l is positive, then $l.phase = 1$;
– if l is negative, then $l.phase = 0$;
– if l is merged and l has only one parent literal l', then $l.phase = l'.phase$;
– if l is merged and l has two parent literals $l_1 \in u_1.clause$ and $l_2 \in u_2.clause$ with $(u_1, v) \in E_\Pi$ and $(u_2, v) \in E_\Pi$, then $l.phase = (l_1.phase \wedge \overline{p}) \vee (l_2.phase \wedge p)$, where pivot $p \in X_\exists$, $p \in u_1.clause$ and $\overline{p} \in u_2.clause$.</td></tr>
<tr><td>Effective literal</td><td>The effective literal, denoted $l.elit$, of $l \in v.clause$ is a literal that satisfies $l.elit \;\leftrightarrow\; (x \leftrightarrow l.phase)$, where $x = var(l)$.</td></tr>
<tr><td>Shadow clause</td><td>The shadow clause, denoted $v.shadclause$, of $v \in V_\Pi$ is the clause of effective literals of v:
$$v.shadclause = \bigcup_{l \in v.clause} (l.elit).$$</td></tr>
</table>

The phase function intuitively represents the induced phase of a literal in a clause under a particular assignment to existential variables. The effective literal, on the other hand, represents the induced value of the corresponding literal in its underlying clause. By these definitions, observe that $l.elit \leftrightarrow l$ whenever l is not a merged literal. Effective literals are important in the sense that they unify the semantics of both merged and not-merged literals, and there is no further need to distinguish them in a clause.

As mentioned in Section 2, merged literals do not follow the same semantics as not-merged literals. The following example illustrates the problem of α-implication in the presence of tautological clauses.

Example 1. Consider a simple QBF $\Phi = \exists a \forall x \exists b.(a + x + b)_1(\overline{a} + \overline{x} + b)_2(\overline{b})_3$, and the corresponding LQ-resolution proof:

$$\Pi = \left\{ \begin{array}{l} 1.\ C_4 = \mathit{resolve}(C_1, C_2) \\ 2.\ C_5 = \mathit{resolve}(C_3, C_4) \\ 3.\ C_{\mathit{empty}} = \mathit{reduce}(C_5) \end{array} \right\}$$

Note that $C_5 = \{x^*\}$, and if the semantics of a merged literal x^* is to be treated similarly to an ordinary literal, then $C_5|_\alpha = 1$ for any assignment α. Therefore under no circumstances C_{empty} can be α-implied. On the other hand, the empty clause is soundly deduced following the LQ-resolution proof system [1].

To overcome the above difficulty, the attributes defined in Table 1 provide necessary information to keep track of the origin of a merged literal under some assignment. Consider the merged literal $x^* \in C_5$. By its effective literal $x^*.\mathit{elit} = (\overline{a} \leftrightarrow x)$, under partial assignment $a = 0$ we have $x^*.\mathit{elit} = x$. Moreover, under $a = 0$, C_2 evaluates to 1, and thus proof Π can be simplified to

$$\Pi|_{\{\overline{a}\}} = \left\{ \begin{array}{l} 1.\ C'_4 = \mathit{resolve}(C_1|_{\{\overline{a}\}}, C_3|_{\{\overline{a}\}}) \\ 2.\ C'_{\mathit{empty}} = \mathit{reduce}(C'_4) \end{array} \right\}$$

Observe, that the merged literal $x^* \in C_5$ from the original proof Π now corresponds to the ordinary literal $x \in C'_4$, which is exactly equivalent to $x^*.\mathit{elit}$ under assignment $a = 0$. Similar analysis can be performed for assignment $a = 1$. Therefore intuitively, effective literals represent how the semantics of the merged literals should be interpreted under a given assignment.