Chapter 1 - Introduction to the finite element method in solid mechanics

1.1 Organization

1. Course: 有限元素法 (FINITE ELEMENT METHOD，機械所、機械系應用力學知識領域與奈米科技學程課程 - ME 7112)
2. Instructor: Chien-Kai Wang (王建凱), 617 Engieering Building, Tel.: 02-3366-4515, E-mail: ckwang@ntu.edu.tw
3. Lecture: Tue. 1:20-4:20 pm, 617 Engieering Building
4. Grading policy:
   (1) Assignment (指定作業): 70% - 全學期七次指定作業，指定當天兩週後繳交
   (2) Project (期末報告): 20% - 期末報告，且當天繳交書面報告
   (3) Bonus (平時成績): 10% - 以學期平時表現評分，並依全班整體表現而有調整學期成績總分之可能彈性
5. Suggested Texts:
   (1) J. N. Reddy, An Introduction to the Finite Element Method, 2nd ed., McGraw-Hill, 1993.
   (2) T. J. R. Hughes, The Finite Element Method, Prentice Hall, 1987.
   (3) K. J. Bathe, Finite Element Procedures, Prentice Hall, 1995.
   (4) O. C. Zienkiewicz and R. L. Taylor, The Finite Element Method for Solid and Structural Mechanics, 6th ed., Butterworth-Heinemann, 2005.
6. Office hours: Appointments by e-mail only.
7. Teaching assistant:
   機械所固力組博士班 - 周峮毅, E-mail: r10522543@ntu.edu.tw
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1.2 Description

ME 7112 is a graduate level course on the theory and implementation of the finite element method (FEM) for solving boundary value problems in solid mechanics.
On completing the course of computational methods in solid mechanics, you should be:

1. Familiar with the theoretical foundations of the finite element method;
2. Able to use and extend your own FEM code to solve boundary and initial value problems in mechanics of solids and physics;
3. Able to apply commercial finite element codes.

Experience with MATLAB is necessary, and all students will need to write computer programs for this course study.

1.3 Goal

1. Familiar with the theoretical foundations of the finite element method;
2. Able to use and extend your own FEM code to solve boundary and initial value problems in mechanics of solids and physics;
3. Able to apply commercial finite element codes.

1.4 Schedule

<u>Week 01</u>
Chap. 1 - Introduction to the finite element method in solid mechanics

<u>Week 02</u>
Chap. 2 - The finite element method (FEM) for static linear elasticity
2.1 Derivation and implementation of a basic 2D FE code with triangular constant strain elements

Week 03

2.2 Generalization of finite element procedures for linear elasticity

2.3 Accuracy and convergence

Week 04

Chap. 3 - Advanced element formulations

3.1 Shear locking and incompatible mode elements

Week 05

3.2 Volumetric locking: Reduced integration and Bbar methods

Week 06

3.3 Mixed (hybrid) elements

Week 07

Chap. 4 - The finite element method for dynamic linear elasticity

4.1 Explicit time integration - the Newmark method

4.2 Implicit time integration

Week 08

4.3 Modal analysis and modal time integration

Week 09

Chap. 5 - Finite element method for nonlinear problems

5.1 Small strain hypoelastic materials

Week 10

5.2 Small strain viscoelasticity

Week 11

5.3 Large strain elasticity

Week 12

5.4 Large strain viscoelasticity

Week 13

5.5 Explicity dynamics for nonlinear problems

Week 14

Chap. 6 - Special topic

6.1 Special elements I

Week 15

6.2 Special elements II (Time permitting)

Week 16

Final project presentation

1.5 General comments

1. One of the most important issues engineers and scientists do is to model physical phenomena by virtually describing every event in nature, whether aerospace, biological, chemical, geological, or mechanical, with the aid of the laws of physics or other fields in terms of algebraic, differential, and/or integral equations relating various quantities of interest.
2. Analytical description of physical phenomena and processes are called *mathematical models*. The use of a mathematical algorithm and a computer program to evaluate the mathematical model of a process and thus esitmate its characteristics is calle *numerical modeling*.

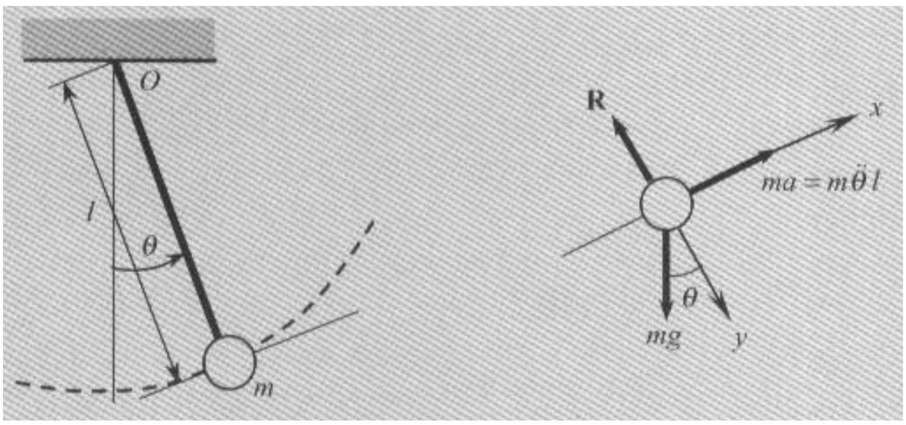
3. Over the last four decades, electronic computers have advanced much so that many mathematical models of pratical engineering problems can be efficiently solved by numerical modeling. It is known as computational mechanics, which is a new and growing body of knowledge.
4. The finite element method (FEM) is one of the numerical modeling methods. It is not an end in itself but rather an aid to design and manufacturing engineering products.
5. Several reasons why an engineering or a scientists should study numerical modeling methods, especially FEM:
   (1) Most pratical problems invol complicated domains, load, and nonlinearities that forbid analytical solution development. Hence, the alternative is to find excellent approximation solutions using adequate numerical methods.
   (2) A numerical method, with the advent of a computer, can be used to investigate the effects of various parameters of the system on its response to gain a better understanding of the process being analyzed.
   (3) People who are quick to use a computer program may need time to interpret the computer-generated results. Even to develop proper input data to the program, engineers and scientitst are required to have a good understanding of the underlying theory of the physics for the problems to be solved.
   (4) Today, generally speaking, the finite element method and its derived generalizations are the most powerful computer-oriented methods in the fields of engineering and applied sciences. Major established industries such as the automobile, aerospace chemical, pharmaceutical, pretroleum, electronics, and communications, as well as nanotechnology and biotechnology. Those emerging technologies rely on FEM to simulate complex phenomenon at different spatial and temporal scales for design and manufacturing of high-technology products.

1.6 Basic examples of mathematical models

1. A mathematical model can broadly defined as a set of equations that expresses the essential features of a physical systems in terms of variables evolving the problem.
2. The mathemtial models of physical phenomena are often based on fundamental scientific laws of physics such as the principle of conservation of mass, conservation of linear momentum, and conservation of energy.
3. Three basic examples drawn from dynamics, heat transfer, and solid mechanics to illustrate how mathematical models of physical problems are formulated in the following contents.

1.7 A dynamics problem

A simple pendulum consists of a bob of mass m (kg) attached to one end of a rod of length ℓ (m) and the other end is pivoted to a fixed point O, as shown in the figure. In order to derive the governing equation of the problem, we must make certain reasonable assumptions concerning the system consistent with the goal of the analysis. If the goal is to study the simplest linear motion of the pendulum, we assume that the bob and the rod are rigid (i.e., not deformable) and the rod is massless (i.e., compared to that of the bob). In addition, we assume that there is no friction at the pivot point O and the resistence offered by the surrounding medium to pendulum is also negligible.



Sol.
Under these assumptions, the equation governing the motion of the system can be formulated using the *principle of conservation of linear momentum* (i.e., the Newton's second law), which states, in the present case, that the vector sum of externally applied forces on a mechanical system is equal to the time rate of change of the linear momentum (i.e., mass times velocity) of the system:

$$\text{(1.1)}\quad \boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = m\boldsymbol{a}$$

where $\boldsymbol{F}$ is the vector sum of all forces acting on the system, m is the mass of the system, $\boldsymbol{v}$ is the velocity vector, and $\boldsymbol{a}$ is the acceleration vector of the system. To write the equation governing the angular motion, we set up a coordinate system, as shown in the figure. Applying the Newton's second law to the x-direction in terms of the weight $m\,g$ of the bob, we obtain:

(1.2) $F_x = m\dfrac{dv_x}{dt}$

where $F_x = -mgsin\theta$, $v_x = \ell d\theta/dt$, θ is the angular displacement (rad), v_x is the component of velocity (m/s) along the x coordinate, and t denotes times (s). Thus, the equation for angular motion becomes:

(1.3) $-mg\,\sin\theta = m\ell\dfrac{d^2\theta}{dt^2}$ or

(1.4) $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{\ell}\sin\theta = 0$

Eq. (1.4) is nonlinear on account of the term $\sin\theta$. For small angular motions, which is consistent with the goal of the problem to be solved, $\sin\theta$ is approximated as θ. Thus, the angular motion is described by the following linear differential equation:

(1.5) $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{\ell}\theta = 0$

Eqs. (1.3) and (1.4) represent mathematical models of nonlinear and linear motions, respectively, of a rigid pendulum. Their solution requires knowledge of conditions at time t=0 on θ and its time derivative $\dot{\theta}$ (i.e., angular velocity). These condition are known as the initial conditions. Thus, the linear problem involves solving the differential equation (1.5) subjected to the initial conditions:

(1.6) $\theta(0) = \theta_0$, $\dfrac{d\theta}{dt}(0) = v_0$

The problem described Eqs. (1.5) and (1.6) is called an *initial-value problem* (IVP). The linear problem described Eqs. (1.5) and (1.6) can be solved analytically. The general analytical solution of the linear system $\ddot{\theta} + \lambda^2\theta = 0$ is written as:

(1.7) $\theta(t) = A\sin(\lambda t) + A, \cos(\lambda t)$

where $\lambda = \sqrt{g/\ell}$ and A and B are constants to be determined using the initial conditions in Eq. (1.6). We obstain:

(1.8) $A = \dfrac{v_0}{\lambda}$, $B = \theta_0$

and the solution to the linear problem is:

(1.9) $\theta(t) = \dfrac{v_0}{\lambda}\sin(\lambda t) + \theta_0\cos(\lambda t)$

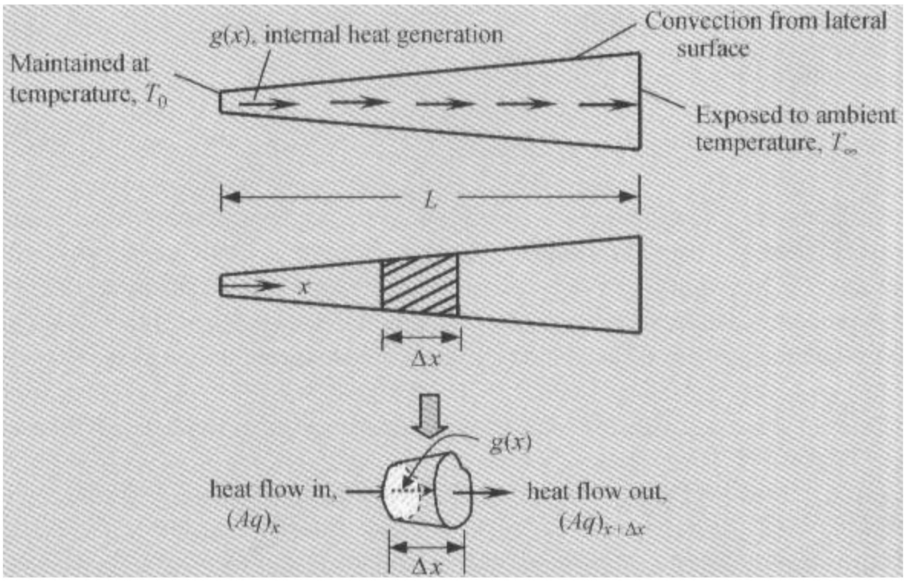
For the case of zero initial velocity and nonzero initial position θ_0, we have:

(1.10) $\theta(t) = \theta_0\cos(\lambda t)$

which represents a simple harmonic motion. If we were to solve the nonlinear equation (Eq. 1.4) subjected to the conditions (Eq. 1.6), we may consider using a proper numerical method because it is not possible to solve such equation exactly for large value of θ.

1.8 A heat transfer problem

Here we wish to derive the governing equation (i.e., develop the mathematical model) of steady-state heat transfer through a cylindrical bar of nonuniform cross section. The bar is subjected to a known temperature T_0 (^{o}C) at the left end and exposed, both on the surface and at the right end, to a medium (such as cooling fluid or air) at temperature T_∞ We assume that temperature is uniform at any section of the bar, $T = T(x)$. Due to the differenc between the temperature of the bar and the surrounding medium, there is convective heat transfer across the surface of the body and at the right end. The principle of conservation of energy (i.e., the second law of thermodynamics) can be used to derive the governing equations of the problem. The principle of conservation of energy requires that the rate of change (increase) of internal energy is equal to the sum of heat gained by conduction, convection, and internal heat generation (radiation not included). When the process reaches a steady state, the time rate of internal energy is zero.



Sol. Consider a volume element of Δx and having an area of cross section $A(x)$ (m^2) normal to the x axis shown in the figure. If q denotes the heat flux (heat flow per unit area, W/m^2), then $[Aq]_x$ is the net heat flow into the volume element at x, $[Aq]_{x+\Delta x}$ is the net heat flow out of the volume element at $x+\Delta x$, and $\beta P \Delta x(T_\infty - T)$ is the heat flow through the surface of the rod in the body. Here β denotes the film (that is formed between the body and the medium around) conductance ($W/(m^2\text{-}^oC)$), T_∞ is the temperatrue of the surrounding medium, and P is the perimeter (m). We also assume that there is a heat source within the rod generating energy at a rate of g (W/m^3). In practice, such energy source can be due to nuclear fission or chemical reactions taking place within the rod, or due to the passage of electric current through the medium. Then the energy balance gives:

$$(1.11)\ [Aq]_x - [Aq]_{x+\Delta x} + \beta P \Delta x(T_\infty - T) + gA\Delta x = 0$$

or, dividing throughout by Δx,

$$(1.12)\ -\frac{[Aq]_{x+\Delta x} - [Aq]_x}{\Delta x} + \beta P(T_\infty - T) + Ag = 0$$

and taking the limit $\Delta x \to 0$, we obtain:

$$(1.13)\ -\frac{d}{dx}(Aq) + \beta P(T_\infty - T) + Ag = 0$$

We can relate the heat flux q (W/m^2) to the temperature gradient. Such a relation is provided by the *Fourier law*:

$$(1.14)\ q(x) = -k\frac{dT}{dx}$$

where k denotes the thermal conductivity ($W/(m\text{-}^oC)$) of the material. The minus sign on the right side of the equality in the above equation indicates that heat flows from high temperature to low temperature. Eq. (1.14) is known as a *constitutive ralation* because it contains a meterial property. Now using the Fourier law, we arrive at the heat *conduction equation*:

$$(1.15)\ \frac{d}{dx}\left(kA\frac{dT}{dx}\right) + \beta P(T_\infty - T) + Ag = 0$$

which can be also written as:

$$(1.16)\ -\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + \beta P(T - T_\infty) = Ag$$

Eq. (1.16) is a linear, nonhomegeneous, second-order differential equation, which can be solved with known conditions on the temperature T or heat Aq (not both) at a boundary point (e.g., ends of the bar). The known end conditions for the present case can be expressed as:

$$(1.17)\ T(0) = T_0,\ \left[kA\frac{dT}{dx} + \beta A(T - T_\infty)\right]_{x=L} = 0$$

Those conditions are called *boundary conditions* because they represent condition at the boundary points of the bar. The first condition in Eq. (1.17) is obvious. The second condition represents the balance of heat due to conduction $[kAdT/dx]$ and convection $[\beta A(T - T_\infty)]$. The problem described by Eqs. (1.16) and (1.17) is called a *boundary-value problem*. This completes the mathematical model development of the problem. First, let:

$$(1.18)\ \theta \equiv T - T_\infty,\ a \equiv kA\ (W\,m/^oC),\ c \equiv \beta P\ (W/(m^oC),\ f \equiv Ag\ (W/m)$$

so that Eqs. (1.16) and (1.17) can be simplied as:

$$(1.19)\ -\frac{d}{dx}\left(a\frac{d\theta}{dx}\right) + c\theta = f$$

$$(1.20)\ \theta(0) = T_0 - T_\infty,\ \left[a\frac{d\theta}{dx} + \beta A\theta\right]_{x=L} = 0$$

Now if a and c are constant (e.g., homogeneous bar of constant cross section) and $g = 0$ (no internal heat generation), Eqs. (1.19) and (1.20) become:

$$(1.21)\ -\frac{d^2\theta}{dx^2} + \frac{c}{a}\theta = 0 \text{ where } 0 < x < L$$

$$(1.22)\ \theta(0) = T_0 - T_\infty \equiv \theta_0,\ \left[\frac{d\theta}{dx} + \frac{\beta}{k}\theta\right]_{x=L} = 0$$

The general solution of (1.21): $\theta" - m^2\theta = 0$ is:

$$(1.23)\ \theta(x) = C_1\cosh(mx) + C_2\sinh(mx),\ m \equiv \sqrt{\frac{c}{a}} = \sqrt{\frac{\beta P}{kA}}$$

where C_1 and C_2 are constants that can be determined using the boundary conditions (1.22). Using $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$, we have:

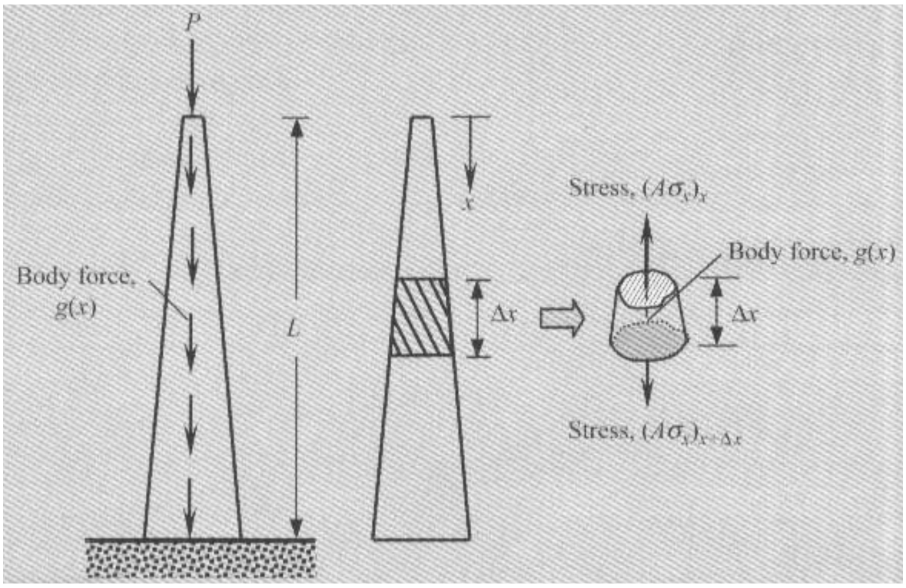
$$(1.24)\ C_1 = \theta_0,\ C_2 = -\theta_0\left[\frac{\sinh(mL) + (\beta/mk)\cosh(mL)}{\cosh(mL) + (\beta/mk)\sinh(mL)}\right]$$

Hence, the solution of Eqs. (1.21) and (1.22) becomes:

$$(1.25)\ \theta(x) = \theta(0)\left[\frac{\cosh(m(L-x)) + (\beta/mk)\sinh(m(L-x))}{\cosh(mL) + (\beta/mk)\sinh(mL)}\right]$$

1.9 A solid mechanics problem

The last example is concerned with the mathematical formulation of the axial deformation of a bar of variable cross section. The term *bar* is used in solid mechanics to mean an element that carries only axial loads (tensile as well as compressive). The governing equations of this simplified problem can be obtained using the Newton's second law and uniaxial stress-strain (constitutive) relation.



The figure shows an element of length Δx with axial forces acting at both ends of the element, where σ_x denotes stress (i.e., force per unit area; N/m^2) in the x-direction, which is taken positive downward, and $g(x)$ denotes the body force measured per unit volume (N/m^3). Hence, $[A\sigma_x]_x$ is the net tensile force on the volume element at x and $[A\sigma_x]_{x+\Delta x}$ is the net tensile force at $x + \Delta x$. Then setting the sum of the forces to zero (i.e., applying the Newton's second law in the x-direction) yields:

(1.26) $-[A\sigma_x]_x + [A\sigma_x]_{x+\Delta x} + gA\Delta X = 0$

where $g(x)$ is the body force (i.e., weight of the body). Dividing throughout by Δx and taking the limit $\Delta x \to 0$, we obtain:

(1.27) $$\frac{d}{dx}(A\sigma_x) + Ag(x) = 0$$

where represents the equilibrium of forces in the x-direction. The stress σ_x can be related to the axial displacement using *Hooke's law*:

(1.28) $$\sigma_x = E\varepsilon_x,\ \varepsilon_x = \frac{du}{dx}$$

where E is Young's modulus (N/m^2), u denotes the axial displacement (m), and ε_x is the axial strain. Again, Eq. (1.28) is a constitutive equation. Note that a system can have several constitutive ralations, each depending on the phenomenon being studied. In the previous example, the study of heat transfer in a bar with nonuniform temperature required us to employ the Fourier's law to relate temperature gradient to heat flow. Now using Eq. (1.28) in Eq. (1.27), we arrive at the equilibrium equation in terms of the displacement:

(1.29) $$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + gA = 0, \quad 0 < x < L$$

This second-order equation can be solved using known boundary conditions at $x = 0$ and $x = L$. The boundary condition of a bar involve specifying either the displacement u or the force $A\sigma_x$ at a boundary point. The known boundary condition of the problem are (see the figure):

(1.30) $$u(L) = 0 \quad \left[EA\frac{du}{dx}\right]_{x=0} = -P$$

where P is the load carried by the bar. The second boundary condition in Eq. (1.30) represents the force equilibrium at $x = L$. Eqs. (1.29) and (1.30) may not admit an analytical solution when $a = a(x)$, requiring us to seek approximate solution using numerical method. For the simple case in which $a \equiv EA$ and $f \equiv Ag$ are constant (i.e., homogeneous and uniform cross-sectional bar), the analytical solution of Eq. (1.29) subjected to the boundary conditions in Eq. (1.30) may be found. Thus, the general solution of Eq. (1.29) in the above case is:

(1.31) $$u(x) = \frac{1}{a}\left(-\frac{f}{2}x^2 + C_1x + C_2\right)$$

Use of boundary conditions give $C_1 = -P$ and $C_2 = fL^2/2 - PL$, and the solution becomes:

$$(1.32)\quad u(x) = \frac{1}{E}\left[\frac{g}{2}(L^2 - x^2) + \frac{P}{A}(L - x)\right]$$

1.10 Numerical solutions

1. While the derivation of the governing equations for most problems is not unduly difficult, their solution by exact methods of analysis is often hard due to intrinsic geometric and material complexities. In such cases, numerical methods of analysis provide alternative means of finding solutions.
2. Numerical methods typically transform differential equations governing a continuum to a set of algebraic equations of a discrete model of that to be solved using computer programming.
3. There exist a number of numerical methods, many of which are developed to solve differential equations. In the *finite difference* approximation of a differential equation, the derivatives are numerically replaced by difference quotients (or the function expanded in a Taylor series).
4. Those numerical difference approximation involves the values of the solution at discrete mesh points of the continuum domain. Hence, the resulting algebraic equations are solved for the values of the solution at the mesh points after imposing the boundary conditions.
5. The above ideas are illustrated with the help of the following two examples of numerial solutions using the finite difference method, one for an initial-value problem (IVP) and the other for a boundary-value problem (BVP).

1.11 An initial-value problem

Here we consider the numerical solution of Eq. (1.5) governing a simple pendulum. To introduce the finite difference method, we consider the first-order differential equation:

$$(1.33)\quad \frac{du}{dt} = f(t, u)$$

where f is a known function. Eq. (1.33) must be solved for $t > 0$ subject to the initial condition $u(0) = u_0$. We approximate the derivative at t_i by:

$$(1.34)\quad \left(\frac{du}{dt}\right)\Bigg|_{t=t_i} \approx \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i}$$

We replaced the derivative at $t = t_i$ by its definition except that we did not take the limit $\Delta t = t_{i+1} - t_i \to 0$; that is why it is an approximation. For increasingly small values of Δt, it is expected that the approximation has decreasingly small error. Substituting Eq. (1.34) into Eq. (1.33) at $t = t_i$, we obtain:

$$(1.35)\quad u_{i+1} = u_i + \Delta t\, f(u_i, t_i), \quad u_i = u(t_i), \quad \Delta t = t_{i+1} - t_i$$

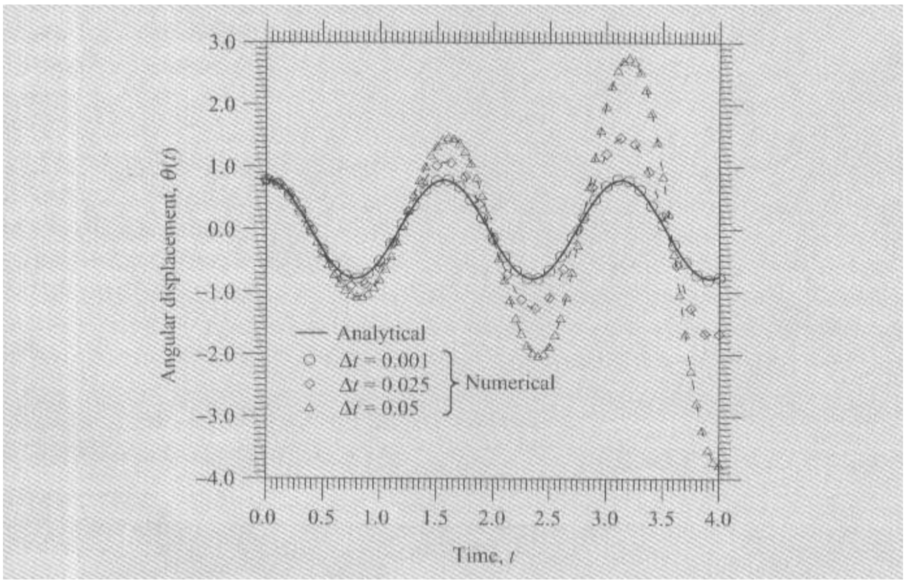
Note that Eq. (1.35) can be solved, starting from the known value u_0 of $u(t)$ at $t = 0$, for $u_1 = u(t_1) = u(\Delta t)$. This process can be repeated to determine the value of u at time $t = \Delta t,\ 2\Delta t, \ldots,\ n\Delta t$. This is known as *Euler's explicit scheme* (or *the first-order Runge-Kutta method*), also known as the *forward difference scheme*. Now we are able to convert the ordinary differential equation (1.33) to an algebraic equation (1.35) that needs to be evaluated at different times to construct the time history of $u(t)$. We further apply the Euler's explicit scheme to the second-order equation (1.5) subjected to its initial conditions. To apply the procedure described above to the equation at hand, we rewrite Eq. (1.5) as a pair of coupled first-order equations (i.e., one cannot be solved without the other):

$$(1.36)\quad \frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2\theta$$

where $\lambda^2 = g/\ell$. Applying the scheme of Eq. (1.35) to the equations above, we obtain:

$$(1.37)\quad \theta_{i+1} = \theta_i + \Delta t\, v_i \quad v_{i+1} = v_i - \Delta t\, \lambda^2\, \theta_i$$

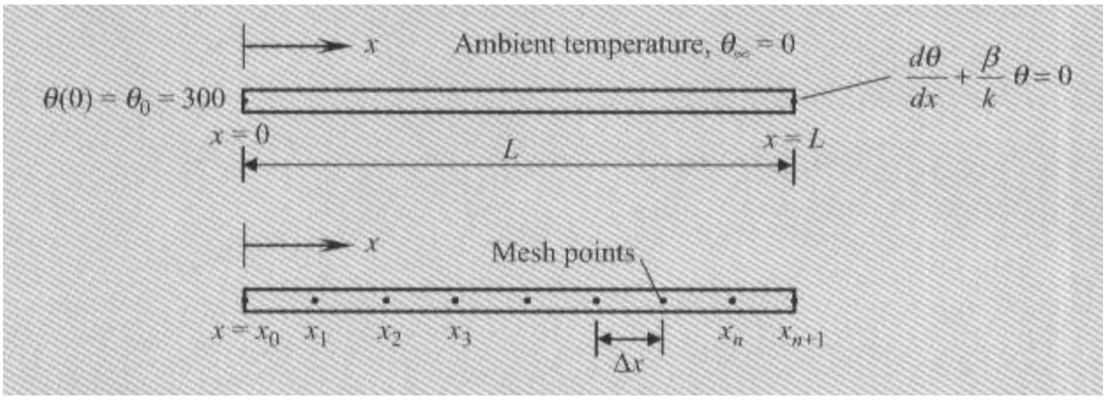
The expression for θ_{i+1} and v_{i+1} in Eq. (1.37) are repeatedly computed using the known solution (θ_i, v_i) from the previous time step. At time $t = 0$, we apply the known initial values (θ_0, v_0). Thus, one needs to use computer language to implement the logic of repeatedly computing θ_{i+1} and v_{i+1} with the help of Eq. (1.37). As an example, the numerial solutions of equation (1.37) for three different time steps, $\Delta t = 0.05,\ 0.025$ and 0.001, along with the exact linear solution (1.10) (with $\ell = 2.0$, $g = 32.2$, $\theta_0 = \pi/4$, and $v_0 = 0$) are presented in the following figure. The accuracy of the numerical solutions depends on the size of the time step; the solution is more accurate with smaller time step. For large time steps, the solution even diverges from the true solution.



1.12 A boundary-value problem

Here we consider the boundary-value problem (BVP) of the heat transfer problem discussed above. The finite difference approximation of a BVP differs from that of an initial value problem. First, we divide the domain $(0, L)$ into a finite set of N intervals of equal length Δx, as shown in the figure. The ends of each interval is called a *mesh point* or *node*. Thus, there are $N+1$ mesh points in the domain. Then we approximate the second-derivative directly using the *centered-difference scheme* where the error is of order $O(\Delta x)^2$:

$$\text{(1.38)} \quad \left(\frac{d^2\theta}{dx^2}\right)_{x=x_i} \approx \left(\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta x^2}\right)$$



Using the above approximation in Eq. (1.21), we obtain:

$$\text{(1.39)} \quad -(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + (m\Delta x)^2\theta_i = 0 \text{ or } -\theta_{i-1} + (2 + (m\Delta x)^2)\theta_i - \theta_{i+1} = 0$$

Eq. (1.39) is valid for any mesh point $x = x_i$, $i = 1, 2, ..., N$, and it contains values of θ both at $x = x_{i-1}$ and $x = x_{i+1}$. Note that Eq. (1.39) is not used at mesh point $x = x_0 = 0$ because the temperature is known there (Eq. 1.20). However, use of Eq. (1.39) at mesh point $x = x_N = L$ requires the knowledge of the fictitious value θ_{N+1}. Hence, the forward finite difference approximation of the second boundary condition in Eq. (1.20) at mesh point $x = L$ gives:

$$\text{(1.40)} \quad \frac{\theta_{N+1} - \theta_N}{\Delta x} + \frac{\beta}{k}\theta_N = 0 \text{ or } \theta_{N+1} = \left(1 - \frac{\beta \Delta x}{k}\right)\theta_N$$

Application of the formula (1.39) to mesh points at $x_1,\ x_2, ..., \ x_N$ yields:

$$\begin{array}{llll} -\theta_0 & +D\,\theta_1 & -\theta_2 & = & 0 \\ -\theta_1 & +D\,\theta_2 & -\theta_3 & = & 0 \\ -\theta_2 & +D\,\theta_3 & -\theta_4 & = & 0 \\ \cdots & & & & \\ -\theta_{N-1} & +D\,\theta_N & -\theta_{N+1} & = & 0 \end{array} \tag{1.41}$$

where $D = [2 + (m\Delta x)^2]$; Eq. (1.41) consists of N equations in N unknowns, $\theta_1,\ \theta_2, ..., \ \theta_N$. As a specific example, consider a steering rod of diameter $d = 0.02\,m$, length $L = 0.05\,m$, and thermal conductivity $k = 50\,W/(m^oC)$. Suppose that the temperature at the left end is $T_0 = 320\,^oC$, ambient temperature is $T_\infty = 20\,^oC$ and film conductance (or heat transfer coefficient) $\beta = 100\,W/(m^{2o}C)$. For this data set, we have:

$$\frac{\beta}{k} = 2, \quad m^2 = \frac{\beta\,P}{k\,A} = \frac{\beta\,(\pi d)}{k\,(\pi d^2/4)} = 400, \quad \theta_0 = \theta(0) = 300\,^oC \tag{1.42}$$

For a subdivision of four intervals $(N = 4)$, we have $\Delta x = 0.0125\,m$ and $D = 2 + (20 \times 0.0125)^2 = 2.0625$. For this case, there are four equations in four unknowns:

$$\begin{array}{rrrrrr} 2.0625\theta_1 & -\theta_2 & & & = & 300 \\ -\theta_1 & +2.0625\theta_2 & -\theta_3 & & = & 0 \\ & -\theta_2 & +2.0625\theta_3 & -\theta_4 & = & 0 \\ & & -\theta_3 & +1.0875\theta_4 & = & 0 \end{array} \tag{1.43}$$

The above *tridiagonal system* of algebraic equations can be solved using the *Gauss elimination method*, where we need a computer. The solution is given by:

$$[\theta] = [245.81,\ 206.98,\ 181.10,\ 166.52]^{\mathrm{T}} \tag{1.44}$$

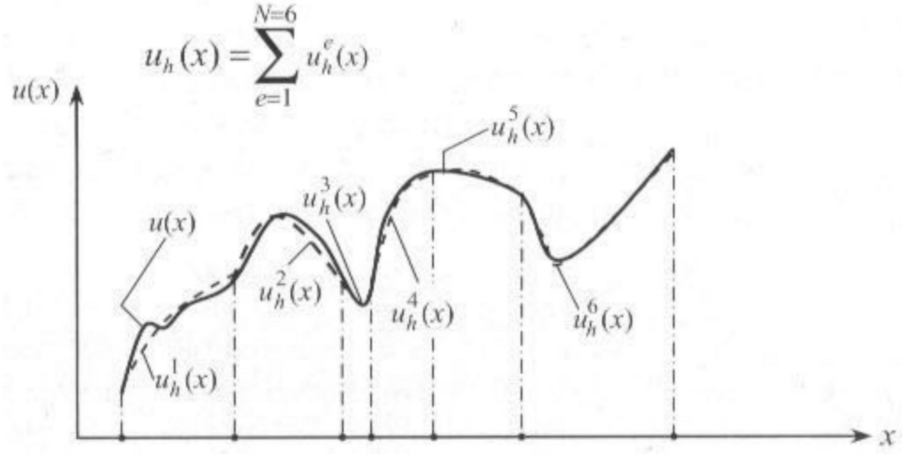
The analytical solution at the same points is:

$$[\theta] = [248.75,\ 213.13,\ 190.90,\ 180.66]^{\mathrm{T}} \tag{1.45}$$

The maximum error is about 7.8 percent. When the number of mesh points is doubled, the maximum error goes down to 4.2 percent, and it is 1 percent when the number of mesh points is increased to 32.

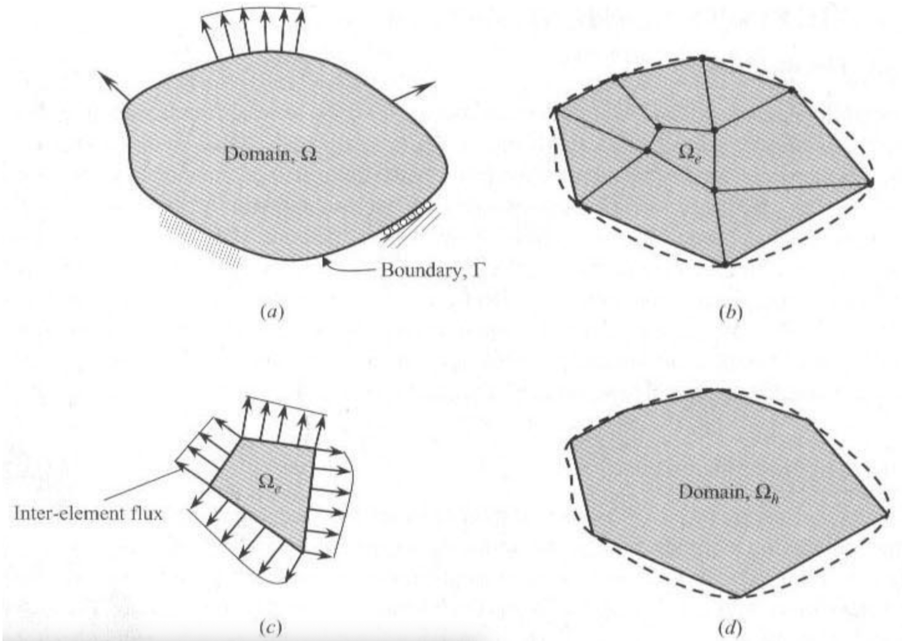
1.13 The finite element method (FEM)

1. Basic idea
   (1) The finite element method is a numerical method like the finite difference method but is more general and powerful in its application to real-world problems that involve complicated physics, geometry, and/or boundary conditions.
   (2) In the finite element method, a given domain is viewed as a *collection of subdomains*, and over each subdomain the governing equation is approximated by any of the *tranditional variational methods*.
   (3) The main reason behind seeking approximate solution on a collection of subdomains is the fact that it is easier to represent a complicated function as a collection of simple polynomials, as can be seen from the figure.
   (4) Each individual segment of the solution should fit with its neighbors in the sense that the function and possibly derivatives up to a chosen order are continuous at the connecting points. These ideas will be addressed more in the course.



2. Features

(1) FEM is endowed with three distinct features that account for its superiority over other competing methods.



(2) First, a geometrically complex domain Ω of the problem to be solved, such as the one in the figure, is represented as a collection of geometrically simple subdomains, call *finite elements*. Each finite element Ω_e is viewed as an independent domain by itself. Here, the word "domain" refers to the geometric region over the governing equations are solved.

(3) Second, over each finite element, algebraic equations among the quantities of interest are developed using the governing equations of the problems.

(4) Third, the relationships from all elements are assembled (i.e., elements are put back into their original positions of the total domain) as indicated in the figure, using certain interelement relationships.

(5) Approximations enter engineering analyses at several stages. The division of the whole domain into finite elements may not be exact (i.e., the assemblage of elements, Ω_h, does not match the original domain Ω), introducing error in the domain being modeled.

(6) Afterwards, when element equations are derived, the dependent unknowns of the problem are typically approximated using the basic idea that any continuous function can be used.

(7) The continuous function used for approximating the unknowns of the problem is represented by a *linear combination of known functions* ϕ_i *and undetermined coefficients* c_i, i.e.,

(1.46) $u \approx u_h = \sum c_i \, \phi_i$

where h denotes typical size of the meshes.

(8) The approximation functions ϕ_i are often taken to be polynomials, and they are derived using concepts from *interpolation theory*. Thus, they are termed *interpolation functions*.

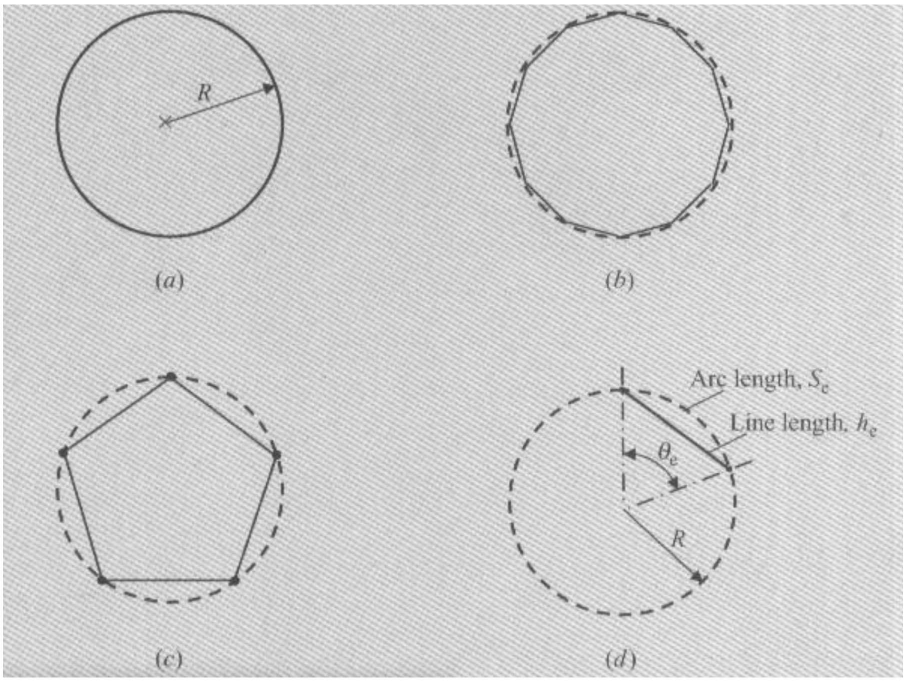
(9) Algebraic relations among the undetermined coefficients c_j are obtained by satisfying the governing equations, in a weighted-integral sense, over each element.

3. Illustration examples

(1) Approximation of the perimeter of a circle

① Finite element discretization

The domain of the circle perimeter is represented as a collection of a finite number n of subdomains, namely, line segments. Each line segment is called an *element*. The collection of elements is called *finite element mesh*. The elements are connected to each other at points called *nodes*.



② Element equations

When all the elements are of the same length, the mesh is said to be *uniform*; otherwise, it is called a *nonuniform* mesh. The *element equation* for determining the element length is:

$$(1.47)\ h_e = 2R\sin(\frac{1}{2}\theta_e)$$

where R is the radius of the circle, and $\theta_e < \pi$ is the angle designed to be subtended by the line segment.

③ Assembly of element equations and solution

The total perimeter of the polygon Ω_h (assembly of element) is equal to the sum of the lengths of individual elements:

$$(1.48)\ P_n = \sum_{e=1}^{n} h_e$$

where P_n represents an approximation to the actual perimeter, P. If the mesh is uniform, $\theta = 2\pi/n$, and we have:

$$(1.49)\ P_n = n\left(2R\sin\frac{\pi}{n}\right)$$

④ Convergence and error estimation

The error in the approximation is equal to the difference between the sector length of the patial circle and the segment length of the element:

$$(1.50)\ E_e = |S_e - h_e|$$

where $S_e = R\theta_e$ is the sector length of the patial circle. Thus, the error estimation for an element in the mesh design is:

$$(1.51)\ E_e = R\left(\frac{2\pi}{n} - 2\sin\frac{\pi}{n}\right)$$

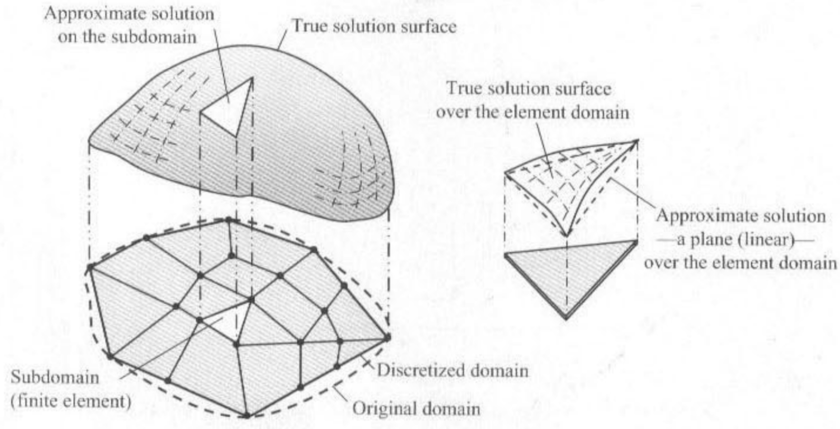
Consequently, the total error (i.e., called *global error*) is:

$$(1.52)\ E_n = n\,E_e = 2\pi R - P_n = P - P_n$$

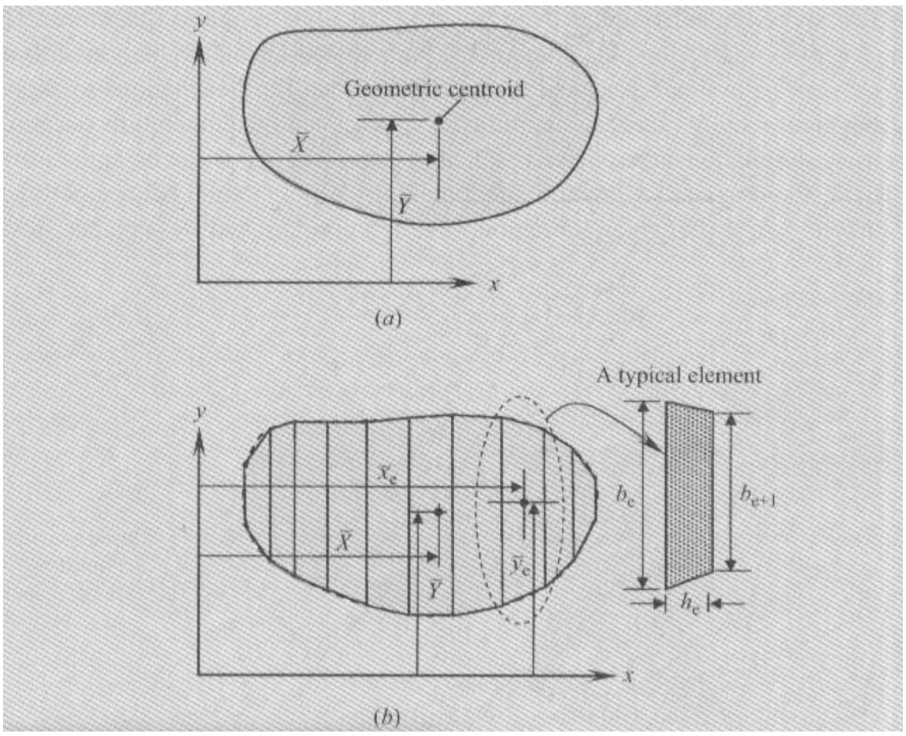
Checking the error E as $n \to 0$ by setting $x = 1/n$:

$$(1.53)\ \lim_{n\to\infty} P_n = \lim_{x\to 0} 2R\frac{\sin\pi x}{x} = \lim_{x\to 0} 2\pi R\frac{\cos\pi x}{1} = 2\pi R$$

Hence, E_n goes to zeros as $n \to \infty$. This is the proof for convergence of discretization.
(2) Approximation of a curved surface by a planar surface
The above example illustrates how the idea of piecewise approximation is used to approximate irregular geometry and calculate required quantities. Here, considering another example, the temperature variation in a two-dimensional domain can be viewed as a curved surface, and it can be approximated over any part of the domain, i.e., subdomain.



In the figure shown above, a curved surface over a triangular subregion may be approximated by a planar surface.
(3) Solution of geometric centroid of an irregular body
As shown in the figure, the center of mass of the whole body is obtained using *the moment principle of Varignon*, which is also a basis for the assembly of element equations:



(1.54) $(m_1 + m_2 + ... + m_n)\bar{X} = m_1\bar{x}_1 + ... + m_n\bar{x}_n$

where $\bar{X}$ is the x-coordinate of the centerof mass of the whole body, m_e is the mass of the e-th part, and $\bar{x}_e$ is the x-coordinate of the center of mass of the e-th part. Similarly, the y and z coordinates of the center of mass of the whole body.Note that analogous relations hold for composite lines, areas, and volumes, where the masses are replaced by lengths, areas, and volumes, respectively. As an example, consider the problem of finding the centroid $(\bar{X}, \bar{Y})$ of the irregular region shown in the figure. The region can be divided into a finite number of trapezoidal strips (i.e., elements). A typical element has width h_e and heights b_e and b_{e+1}, and the area of the e-th strip is given by:

(1.55) $A_e = \frac{1}{2}h_e(b_e + b_{e+1})$

The area A_e is an approximation of the true area of the element because $(b_e + b_{e+1})/2$ is an estimated average height of the element. The coordinates of the centroid of the region are obtained by applying the moment principle:

$$\bar{X} = \frac{\sum_e A_e \bar{x}_e}{\sum_e A_e}, \bar{Y} = \frac{\sum_e A_e \bar{y}_e}{\sum_e A_e} \tag{1.56}$$

where $\bar{x}_e$ and $\bar{y}_e$ are the coordinates of the centroid of the e-th element with respect to the coordinate system used for the whold body. It is should be noted that the accuracy of the approximation will be improved by increasing the number of strips (i.e., decreasing their width h_e).

1.14 Remarks of FEM

1. In summary, in FEM a given domain is divided into subdomains, called *finite elements*, and an approximate solution to the problem is developed over each element. The subdivision of a whole domain into parts has two advantages:
   (1) Allow accurate representation of complex geometry and inclusion of dissimilar material properties.
   (2) Enable easy representation of the total solution by functions defined within each element that capture local effects (e.g., large gradients of the solution).
2. The three fundamental steps of FEM that are illustrated via the examples discussed above:
   (1) Divide the whole domain into parts (both to represent the geometry and solution of the problem).
   (2) Over each part, seek an approximation to the solution as a linear combination of nodal values and approximation functions, and derive the algebraic relations among the nodal values of the solution over each part.
   (3) Assemble the parts and obtain the solution to the whole.
3. Over each finite element, the physical process is approximated by functions of the desired type (polynomials or otherwise), and algebraic equations relating physical quantities at selective points, called nodes, of the element are developed.
