Chapter 6 - Locking effects in FEM and formulation

6.1 Locking phenomena

1. General concepts

(1) Techniques for interpolating the displacement field within 2D and 3D finite elements have been discussed before. In addition, methods for evaluating the volume or area integrals in the principle of virtual work have been investigated. These implementation procedures work well for most application.

(2) However, in some particular situations, the elements with their linear or quadratic shape functions, which interpolate the nodal values at the nodes of those elements, cannot give accurate finite element results for well-defined boundary value problems.

(3) Especially, we focus on *locking effects* in finite elements. Among the element-locking phenomena, we have studied the *shear-locking difficulties* and thus developed corresponding finite element formulations for forming *incompatible elements* to solve the problems.

(4) Finite elements are regarded to "lock" for three reasons in general:

a. The governing equations you are trying to solve are poorly conditioned, which leads to an ill-conditioned system of finite element equations.

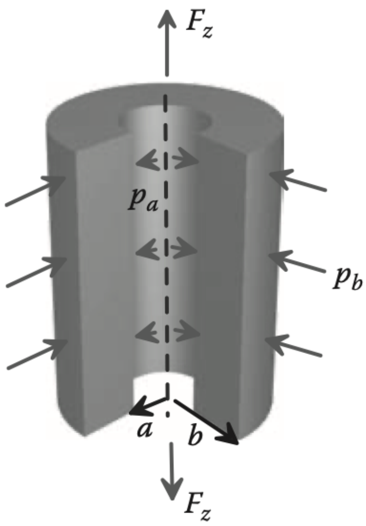
b. The element interpolation functions are unable to approximate accurately the strain distribution in the solid.

c. The displacements and their derivatives in certain element formulations (especially in beam, plate, and shell problems) need to be are interpolated separately. In other words, locking can occur in these elements if the interpolation functions for displacements and their derivatives are not consistent.

(5) In this chapter, we will illustrate more unexpected difficulties related to such elements with unphysically stiff response to deformation including *reduced integration elements* and *hybrid elements*.

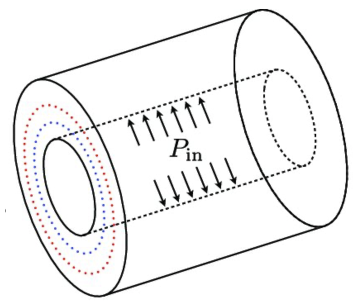
2. Volumetric locking

(1) Consider a long hollow cylinder with internal radius a and external radius b, and the solid is made from a linear elastic material with Young's modulus E and Poisson's ν. The cylinder is loaded by an internal pressure p_a and external pressure p_b, and thus deforms in plane strain. The analytical solution to this problem is given:

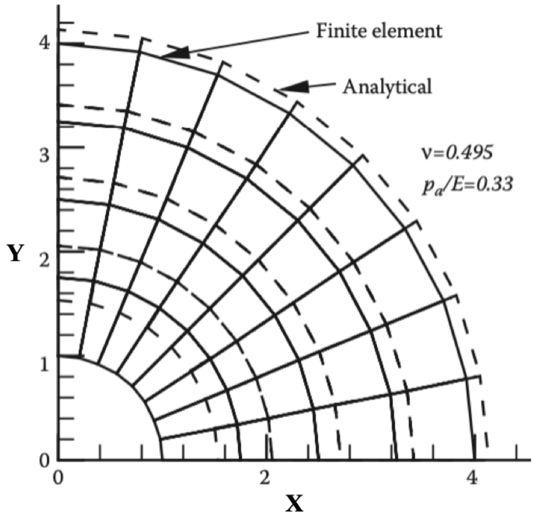


$$\boldsymbol{u}(r)=\frac{(1+\nu)a^2b^2}{E(b^2-a^2)}\left[\frac{(p_a-p_b)}{r}+(1-2\nu)\frac{(p_aa^2-p_bb^2)}{a^2b^2}r\right]\boldsymbol{e}_r \tag{6.1}$$

Volumetric locking can be illustrated using the simple boundary value problem of an internally pressurized thick-walled cylinder shown below ($p_b=0$).



(2) The figure below shows the analytical solution with a finite element solution with standard four-noded plane strain quadrilateral elements. Results are shown for the values of Poisson's ratio $\nu = 0.495$. The dashed lines show the analytical solution, and the solid line shows the FEA solution.



a. The solution does not agree very well. The finite element solution significantly underestimates the displacements as Poisson's ratio is increased toward 0.5 (the material is nearly incompressible). In this limit, the finite element displacements almost are equal to zero. This is known as "volumetric locking."

b. The error in the finite element solution occurs because the finite element interpolation functions are unable to properly approximate a *volume preserving* strain field. In the incompressible limit, a nonzero volumetric strain at any of the integration points gives rise to a very large contribution to the virtual power. The interpolation functions can make the volumetric strain vanish at some, but not all, the integration points in the element.

c. In general, volumetric locking is a much more serious problem than shear locking, because it can not be avoided by refining the mesh. In addition, all the standard *fully integrated* finite elements will lock in the incompressible limit, and some elements show very poor performance even for Poisson's ratios as small as 0.45.

d. Fortunately, most materials have Poisson's ratios around 0.3 or less, so the standard elements can be used for most linear and nonlinear elasticity problems. However, to model incompressible materials (e.g., rubbers), the elements must be redesigned to avoid locking.

6.2 B-bar Method

1. General concepts

(1) Like selective reduced integration, the "B-bar method" works by treating the volumetric and deviatoric parts of the stiffness matrix separately.

(2) Instead of separating the volume integral into two parts; however, the B-bar method modifies the definition of the strain in the element. Its main advantage is that the concept can easily be generalized to *finite strain* problems.

(3) The procedure will be illustrated by applying the usual virtual work principle to small strain linear elasticity problems.

$$\int_R \sigma_{ij}[\varepsilon_{ij}(u_k)]\,\delta\varepsilon_{ij}\,dV - \int_R b_i\delta v_i\,dV - \int_{\partial_2 R} t_i^*\delta v_i\,dA = 0$$

by the virtual work principle. See Chapter 2.

(4) The B-bar method is implemetned as follows:

a. We introduce a new variable for characterizing the volumetric (average) strain in elements by defining:

$$\omega \equiv \frac{1}{3V_e}\int_{V_e}\varepsilon_{kk}\,dV = \frac{1}{3V_e}\int_{V_e}\frac{\partial u_k}{\partial x_k}dV = \frac{1}{3V_e}\int_{V_e}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}dV\,u_k^b \equiv B_{bk}^{\mathrm{v}}\,u_k^b \tag{6.2}$$

where

$$B_{bk}^{\mathrm{v}} = \frac{1}{3V_e}\int_{V_e}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}dV \tag{6.3}$$

The volumetric strain ω is evaluated by taking integral over the volume of the element.

(5) The strain variation in each element is replaced by the approximation:

$$\overline{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{kk}}{3}\delta_{ij} + \omega\,\delta_{ij} = \varepsilon_{ij} + \left(\omega - \frac{\varepsilon_{kk}}{3}\right)\delta_{ij} \tag{6.4}$$

2. Volume integral in the virtual work principle

(1) Consequently, the virtual strain in each element is written by:

$$\delta\overline{\varepsilon}_{ij} = \delta\varepsilon_{ij} + \left(\delta\omega - \frac{\delta\varepsilon_{kk}}{3}\right)\delta_{ij} \tag{6.5}$$

This means that the volumetric strain in the element is everywhere equal to its mean value evaluation ω.

(2) The virtual work principle is then written in terms of $\overline{\varepsilon}_{ij}$ and $\delta\overline{\varepsilon}_{ij}$ as:

$$\int_R \sigma_{ij}[\overline{\varepsilon}_{ij}(u_k)]\,\delta\overline{\varepsilon}_{ij}\,dV - \int_R b_i\delta v_i\,dV - \int_{\partial_2 R} t_i^*\delta v_i\,dA = 0 \tag{6.6}$$

For linear elastic materials,

$$\int_R C_{ijk\ell}\,\overline{\varepsilon}_{k\ell}\,\delta\overline{\varepsilon}_{ij}\,dV - \int_R b_i\delta v_i\,dV - \int_{\partial_2 R} t_i^*\delta v_i\,dA = 0 \tag{6.7}$$

Considering Eqs. (6.4) and (6.5):

a. The integration term including ε_{ij} and $\delta\varepsilon_{ij}$:

$$\begin{aligned}\int_R C_{ijk\ell}\,\varepsilon_{k\ell}\,\delta\varepsilon_{ij}\,dV &= \int_R C_{ijk\ell}\,\frac{\partial u_k}{\partial x_\ell}\frac{\partial\delta v_i}{\partial x_j}dV \\ &= \left(\int_R C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}dV\right)u_k^b\,\delta v_i^a\end{aligned} \tag{6.8}$$

b. The integration term including ε_{ij} and $(\delta\omega - \delta\varepsilon_{kk}/3)\delta_{ij}$:

$$\begin{aligned}\int_R C_{ijk\ell}\,\varepsilon_{k\ell}\left(\delta\omega - \frac{\delta\varepsilon_{qq}}{3}\right)\delta_{ij}\,dV &= \int_R C_{ijk\ell}\,\frac{\partial u_k}{\partial x_\ell}\left(\delta\omega - \frac{1}{3}\frac{\partial\delta v_q}{\partial x_q}\right)\delta_{ij}\,dV \\ &= \int_R C_{ijk\ell}\,\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}u_k^b\left(B^{\mathrm{v}}_{aq}\,\delta v_q^a - \frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_q}\delta v_q^a\right)\delta_{ij}\,dV \\ &= \int_R C_{jjk\ell}\,\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}u_k^b\left(B^{\mathrm{v}}_{aq}\,\delta v_q^a - \frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_q}\delta v_q^a\right)dV \\ &= \left[\int_R C_{jjk\ell}\,\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}\left(B^{\mathrm{v}}_{ai} - \frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\right)dV\right]u_k^b\,\delta v_i^a\end{aligned} \tag{6.9}$$

c. The integration term including $(\omega - \varepsilon_{kk}/3)\delta_{ij}$ and $\delta\varepsilon_{ij}$:

$$\begin{aligned}\int_R C_{ijk\ell}\left(\omega - \frac{\varepsilon_{qq}}{3}\right)\delta_{k\ell}\,\delta\varepsilon_{ij}\,dV &= \int_R C_{ijk\ell}\left(\omega - \frac{1}{3}\frac{\partial u_q}{\partial x_q}\right)\delta_{k\ell}\,\frac{\partial\delta v_i}{\partial x_j}dV \\ &= \int_R C_{ijk\ell}\left(B^{\mathrm{v}}_{bq}u_q^b - \frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_q}u_q^b\right)\delta_{k\ell}\,\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\delta v_i^a\,dV \\ &= \int_R C_{ij\ell\ell}\left(B^{\mathrm{v}}_{bq}u_q^b - \frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_q}u_q^b\right)\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\delta v_i^a\,dV \\ &= \left[\int_R C_{ij\ell\ell}\left(B^{\mathrm{v}}_{bk} - \frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}\right)\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}dV\right]u_k^b\,\delta v_i^a\end{aligned} \tag{6.10}$$

d. The integration term including $(\omega - \varepsilon_{kk}/3)\delta_{ij}$ and $(\delta\omega - \delta\varepsilon_{kk}/3)\delta_{ij}$:

$$\int_R C_{ijk\ell}\left(\omega-\frac{\varepsilon_{qq}}{3}\right)\delta_{k\ell}\left(\delta\omega-\frac{\delta\varepsilon_{pp}}{3}\right)\delta_{ij}\,dV=\int_R C_{ijk\ell}\left(\omega-\frac{1}{3}\frac{\partial u_q}{\partial x_q}\right)\delta_{k\ell}\left(\delta\omega-\frac{1}{3}\frac{\partial\delta v_p}{\partial x_p}\right)\delta_{ij}\,dV$$
$$=\int_R C_{ijk\ell}\left(B^{\mathrm{v}}_{bq}u^b_q-\frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_q}u^b_q\right)\delta_{k\ell}\left(B^{\mathrm{v}}_{ap}\,\delta v^a_p-\frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_p}\delta v^a_p\right)\delta_{ij}\,dV$$
$$=\int_R C_{jj\ell\ell}\left(B^{\mathrm{v}}_{bq}u^b_q-\frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_q}u^b_q\right)\left(B^{\mathrm{v}}_{ap}\,\delta v^a_p-\frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_p}\delta v^a_p\right)dV$$
$$=\left[\int_R C_{jj\ell\ell}\left(B^{\mathrm{v}}_{bk}-\frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}\right)\left(B^{\mathrm{v}}_{ai}-\frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\right)dV\right]u^b_k\,\delta v^a_i \tag{6.11}$$

(3) Hence, the element stiffness matrix in the B-bar method is:

$$k^e_{aibk}=\int_R C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}dV+\int_R C_{jjk\ell}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}\left(B^{\mathrm{v}}_{ai}-\frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\right)dV+$$
$$\int_R C_{ij\ell\ell}\left(B^{\mathrm{v}}_{bk}-\frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}\right)\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}dV+\int_R C_{jj\ell\ell}\left(B^{\mathrm{v}}_{bk}-\frac{1}{3}\frac{\partial N^b(\boldsymbol{x})}{\partial x_k}\right)\left(B^{\mathrm{v}}_{ai}-\frac{1}{3}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\right)dV \tag{6.12}$$

3. MATLAB code demonstration

We need to modify the *ElemStif* function for the B-bar method:

```
function kel = ElemStif(iel,ndime,nelnd,coor,conn,mate,wglob)
  kel = zeros(ndime*nelnd,ndime*nelnd);
  coorie = zeros(ndime,nelnd);
  wie = zeros(ndime,nelnd);
  xii = zeros(ndime,1);
  dxdxi = zeros(ndime,ndime);
  dNdx = zeros(nelnd,ndime);
  M = numIntegPt(ndime,nelnd);
  xi = IntegPt(ndime,nelnd,M);
  w = integWt(ndime,nelnd,M);
  epsi = zeros(ndime,ndime);
  Bv = zeros(nelnd,ndime);
  for a = 1:nelnd
    for i = 1:ndime
      coorie(i,a) = coor(i,conn(a,iel));
      wie(i,a) = wglob(ndime*(conn(a,iel)-1)+i);
    end
  end
  evol= 0;
  for im = 1:M
    for i = 1:ndime
      xii(i) = xi(i,im);
    end
    dNdxi = ShpFuncDeri(nelnd,ndime,xii);
    dxdxi = 0;
    for i = 1:ndime
      for j = 1:ndime
        for a = 1:nelnd
          dxdxi(i,j) = dxdxi(i,j)+coorie(i,a)*dNdxi(a,j);
        end
      end
    end
    dxidx = inv(dxdxi);
```

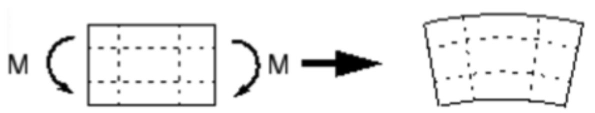
```
  jcb = det(dxdxi);
  dNdx = 0.;
  for a = 1:nelnd
    for i = 1:ndime
      for j = 1:ndime
        dNdx(a,i) = dNdx(a,i)+dNdxi(a,j)*dxidx(j,i);
      end
    end
  end
  for a = 1:nelnd
    for i = 1:ndime
      Bv(a,i) = Bv(a,i)+dNdx(a,i)*w(im)*jcb;
    end
  end
  evol = evol+w(im)*jcb;
end
omega = 0;
for a = 1:nelnd
  for i = 1:ndime
    Bv(a,i) = Bv(a,i)/(ndime*evol);
    omega = omega+Bv(a,i)*wie(i,a);
  end
end
for im = 1:M
  for i = 1:ndime
    xii(i) = xi(i,im);
  end
  dNdxi = ShpFuncDeri(nelnd,ndime,xii);
  dxdxi(:) = 0;
  for i = 1:ndime
    for j = 1:ndime
      for a = 1:nelnd
        dxdxi(i,j) = dxdxi(i,j)+coorie(i,a)*dNdxi(a,j);
      end
    end
  end
  dxidx = inv(dxdxi);
  jcb = det(dxdxi);
  dNdx(:) = 0;
  for a = 1:nelnd
    for i = 1:ndime
      for j = 1:ndime
        dNdx(a,i) = dNdx(a,i)+dNdxi(a,j)*dxidx(j,i);
      end
    end
  end
  ekk = 0;
  epsi(:) = 0;
  for i = 1:ndime
    for j = 1:ndime;
      for a = 1:nelnd
        epsi(i,j) = epsi(i,j)+0.5*(wie(i,a)*dNdx(a,j)+wie(j,a)*dNdx(a,i));
      end
    end
    ekk = ekk+epsi(i,i);
  end
```

```
    epsi(1,1) = epsi(1,1)-ekk/ndime+omega;
    epsi(2,2) = epsi(2,2)-ekk/ndime+omega;
    if(ndime == 3)
      epsi(3,3) = epsi(3,3)-ekk/ndime+omega;
    end
    cmat = MatStif(ndime,mate);
    for a = 1:nelnd
      for i = 1:ndime
        for b = 1:nelnd
          for k = 1:ndime
            ir = ndime*(a-1)+i;
            ic = ndime*(b-1)+k;
            for j = 1:ndime
              for l = 1:ndime
                kel(ir,ic)=kel(ir,ic)+cmat(i,j,k,l)*dNdx(b,l)*dNdx(a,j)*w(im)*jcb;
                kel(ir,ic)=kel(ir,ic)+cmat(j,j,k,l)*dNdx(b,l)*(Bv(a,i)-
dNdx(a,i)/ndime)*w(im)*jcb;
                kel(ir,ic)=kel(ir,ic)+cmat(i,j,l,l)*dNdx(a,j)*(Bv(b,k)-
dNdx(b,k)/ndime)*w(im)*jcb;
                kel(ir,ic)=kel(ir,ic)+...
                  cmat(j,j,l,l)*(Bv(b,k)-dNdx(b,k)/ndime)*(Bv(a,i)-dNdx(a,i)/ndime)*w(im)*jcb;
              end
            end
          end
        end
      end
    end
  end
end
```

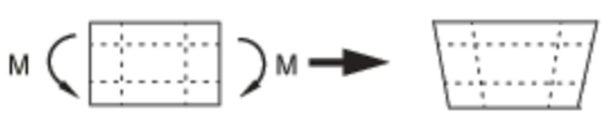
6.3 Reduced integration

1. General concepts

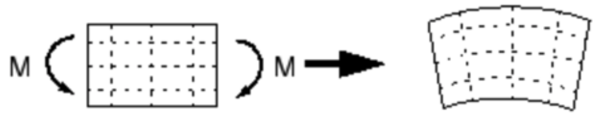
(1) "Reduced integration" is the simplest way to avoid locking. The basic idea is simple: because the fully integrated elements cannot make the strain field volume preserving at all the integration points, it is tempting to reduce the number of integration points so that the constraint can be met.



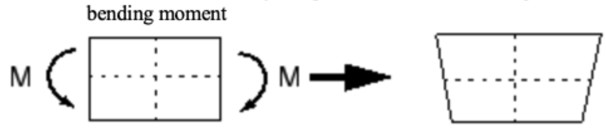
Deformation of material subjected to bending moment



Deformation of a fully integrated, linear element subjected to bending moment



Deformation of a fully integrated and quadratic element subjected to bending moment



Deformation of a linear element with reduced integration subjected to bending moment

(2) Reduced integration usually means that the element stiffness is integrated using an integration scheme that is one order lower than the standard scheme.

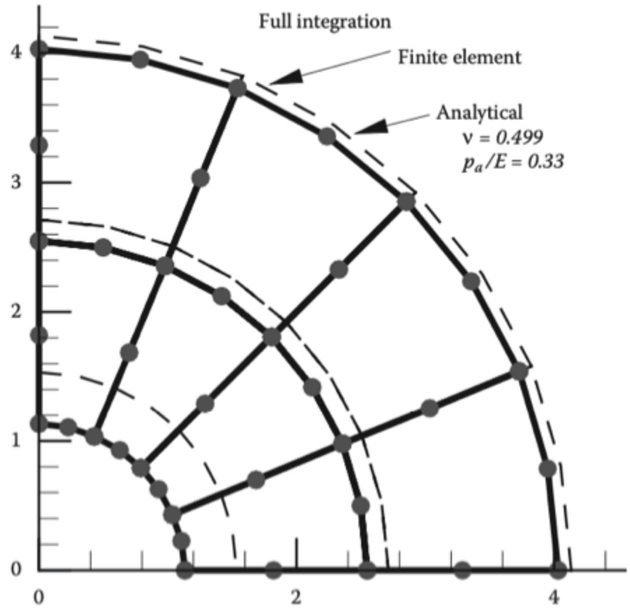
a. The coordinates of the integration points for various element types are identical to the ones discussed before.

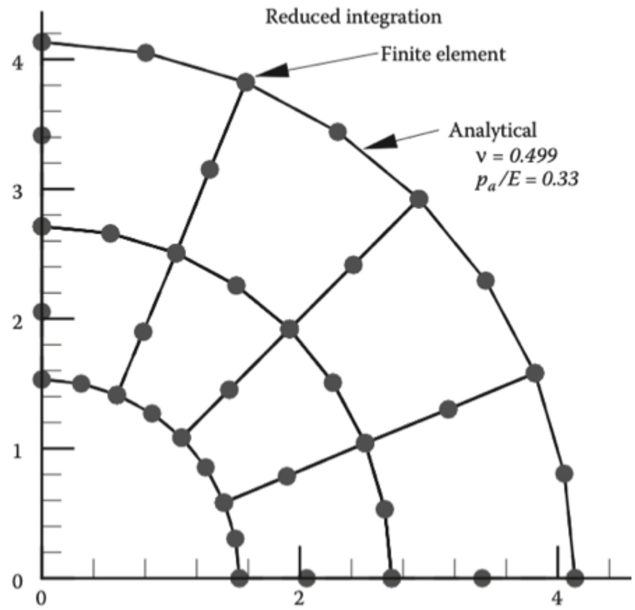
b. The number of reduced integration points for various element types is listed in the tables below.

c. It is important to notice that the integration order cannot be reduced for the *linear triangular and tetrahedral elements*. These elements should not be used to model such nearly incompressible materials. You are suggested to just use very few linear triangular and tetrahedral elements in the regions where the solid cannot be meshed using other elements.

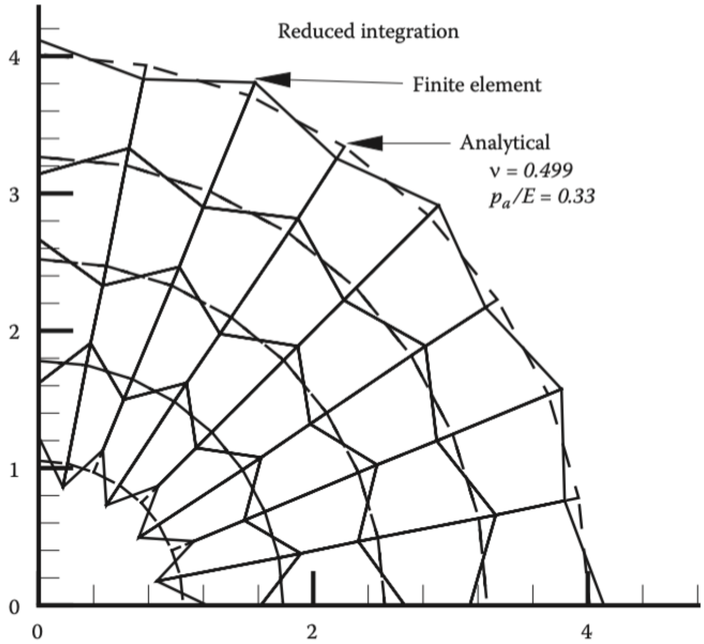
Quadratic triangle (6 nodes): 3 points	Quadratic tetrahedron (10 nodes): 4 points
Linear quadrilateral (4 nodes): 1 point	Linear brick (8 nodes): 1 point
Quadratic quadrilateral (8 nodes): 4 points	Quadratic brick (20 nodes): 8 points

(3) Remarkably, reduced integration completely resolves locking in some elements (especially, the quadratic quadrilateral and brick) and even improves the accuracy of the element. As an example, the figure below shows the solution to the pressurized cylinder problem, using both full and reduced integration for eight-noded quadrilaterals. With reduced integration, the analytical and finite element results are indistinguishable.





(4) However, reduced integration does not work in four-noded quadrilateral elements or eight-noded brick elements. For example, the figure below shows the solution to the pressure vessel problem with linear, four-noded, quadrilateral elements with reduced integration (the displacements have been scaled down). The solution is clearly wrong. The error occurs because the stiffness matrix is nearly singular; the system of equations includes a weakly constrained deformation mode. This phenomenon is known as 'hour-glassing' because of the characteristic shape of the factitious deformation mode.



2. Selectively reduced integration

Selectively reduced integration can be used to cure hourglassing. The procedure will be illustrated clearly by modifying the formulation for static linear elasticity. The method is implemented as follows:

(1) For a Cauchy stress σ_{ij} evaluation, the stress can be divided into its volumetric stress σ^{vol} part and deviatoric stress σ_{ij}^{dev} part:

$$\sigma_{ij} = \left(\sigma_{ij} - \sigma^{\text{vol}}\delta_{ij}\right) + \sigma^{\text{vol}}\delta_{ij} \equiv \sigma_{ij}^{\text{dev}} + \sigma^{\text{vol}}\delta_{ij} \tag{6.13}$$

where

$$\sigma^{\text{vol}} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) = \frac{1}{2}\sigma_{kk} \tag{6.14}$$

in 2D or

$$\sigma^{\text{vol}} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk} \tag{6.15}$$

in 3D

$$\sigma_{ij}^{\text{dev}} = \sigma_{ij} - \sigma^{\text{vol}}\delta_{ij} \tag{6.16}$$

(2) Regarding 3D problems of illustration, the volume integral in the virtual work principle is separated into a deviatoric and volumetric part by writing:

$$\begin{aligned}\int_R \sigma_{ij}\delta\varepsilon_{ij}\,dV &= \int_R \left(\sigma_{ij}^{\text{dev}} + \sigma^{\text{vol}}\delta_{ij}\right)\delta\varepsilon_{ij}\,dV \\ &= \int_R \left(\sigma_{ij}\delta\varepsilon_{ij} - \frac{\sigma_{kk}}{3}\delta\varepsilon_{\ell\ell}\right)dV + \int_R \frac{\sigma_{kk}}{3}\delta\varepsilon_{\ell\ell}\,dV\end{aligned} \tag{6.17}$$

(3) Substituting the linear elastic constitutive equation and the finite element interpolation functions into the virtual work principle, we can find the element stiffness matrix as before:

a. For the integral term containing σ_{ij}

$$\int_R \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_R \sigma_{ij}\frac{1}{2}\left(\frac{\partial\delta v_i}{\partial x_j} + \frac{\partial\delta v_j}{\partial x_i}\right)dV = \int_R \sigma_{ij}\frac{\partial\delta v_i}{\partial x_j}dV \tag{6.18}$$

since $\sigma_{ij} = \sigma_{ji}$.

$$\sigma_{ij} = C_{ijk\ell}\,\varepsilon_{k\ell} = C_{ijk\ell}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\right) = C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell} \tag{6.19}$$

since $C_{ijk\ell} = C_{ij\ell k}$.

Also, through the standard element interpolation approach,

$$\delta v_i = N^a(\boldsymbol{x})\,\delta v_i^a \tag{6.20}$$

$$u_k = N^b(\boldsymbol{x})\,u_k^b \tag{6.21}$$

Hence,

$$\int_R \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_R C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\frac{\partial\delta v_i}{\partial x_j}dV = \left(\int_R C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}dV\right)u_k^b\,\delta v_i^a \tag{6.22}$$

b. For the integral term containing σ_{kk}

$$\int_R \sigma_{pp}\,\delta\varepsilon_{qq}\,dV = \int_R C_{ppk\ell}\,\varepsilon_{k\ell}\frac{\partial\delta v_q}{\partial x_q}dV = \int_R C_{ppk\ell}\frac{\partial u_k}{\partial x_\ell}\frac{\partial\delta v_q}{\partial x_q}dV = \left(\int_R C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_q}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}dV\right)u_k^b\,\delta v_q^a \tag{6.23}$$

since $C_{ijk\ell} = C_{ij\ell k}$ and $\sigma_{ij} = \sigma_{ji}$.

c. Finally, the element stiffness matrix in terms of nodal shape functions can derived as:

$$\int_R \sigma_{ij}\delta\varepsilon_{ij}\,dV - \int_R b_i\delta v_i\,dV - \int_{\partial_2 R} t_i^*\delta v_i\,dA = 0$$

by the virtual work principle. See Chapter 2.

$$\left[\int_R \left(C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} - \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}\right) dV + \int_R \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} dV\right] u_k^b \, \delta v_i^a$$

$$= \left(\int_R b_i N^a(\boldsymbol{x})\, dV + \int_{\partial_2 R} t_i^* N^a(\boldsymbol{x})\, dV\right) \delta v_i^a$$

$$k_{aibk}^e = \int_{V_e} \left(C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} - \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}\right) dV + \int_{V_e} \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} dV \tag{6.24}$$

(4) When selectively reduced integration is applied, the first volume integral:

$$\int_{V_e} \left(C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} - \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell}\right) dV$$

is evaluated using the *full integration* scheme; the second integral:

$$\int_{V_e} \frac{1}{3}C_{ppk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_i}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} dV$$

is evaluated using the scheme of *reduced integration points*.

(5) In several commercial codes, the "fully integrated elements" actually use the scheme of *selective reduced integration*. Please always check the software manuals carefully.