Chapter 4 - The finite element method for dynamic linear elasticity

In this chapter, the FEM is extended to solve dynamic problems of linear elasticity. Specifically,

1. The principle of virtual work is used to derive a discrete system of equations for the time-varying nodal displacements.
2. Three methods for integrating the equation of motion with respect to time are pre- sented:
   (1) explicit time stepping,
   (2) implicit time stepping, and
   (3) modal analysis.
   The properties of each scheme are illustrated by solving simple problems.
3. As always, example codes are discussed so how the method is actually implemented to explore useful predictions can be investigated.

4.1 Dynamic linear elasticity

1. Governing equations of dynamic linear elasticity
   (1) As before, the governing equations for a standard dynamic linear elasticity problem are summarized first. A generic linear elasticity problem is shown in the following figure. The following informations are given:
   ① The shape of the solid in its unloaded condition R.
   ② The initial stress field in the solid (for simplicity, take this to be zero).
   ③ The mass density ρ and elastic constants for the solid C_{ijkl}.
   ④ The thermal expansion coefficients for the solid and temperature distribution (for simplicity, take this to be zero).
   ⑤ The initial displacement field in the solid $\boldsymbol{u}_0(\boldsymbol{x})$ and the initial velocity field $\boldsymbol{v}_0(\boldsymbol{x})$.
   ⑥ A body force distribution $\boldsymbol{b}(\boldsymbol{x},t)$ acting on the solid (note that, in this section, we will use $\boldsymbol{b}$ to denote force per unit volume rather than force per unit mass, to avoid having to write out the mass density all the time).
   ⑦ Boundary conditions, specifying displacements $\boldsymbol{u}^*(\boldsymbol{x},t)$ on a portion $\partial_1 R$ or tractions $\boldsymbol{t}^*(\boldsymbol{x},t)$ on a portion $\partial_2 R$ of the boundary of R.
   (2) Calculate time-varying displacements, strains, and stresses u_i, ε_{ij}, σ_{ij} satisfying the governing equations of dynamic linear elasticity:
   ① The strain-displacement equation:

$$(4.1)\ \varepsilon_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$$

   ② The elastic stress-strain law:

$$(4.2)\ \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

   ③ The equation of motion for stresses:

$$(4.3)\ \partial\sigma_{ij}/\partial x_i + b_j = \rho\,\partial^2 u_j/\partial t^2$$

   ④ The boundary conditions on displacement and stress:

$$(4.4)\ u_i = u_i^* \ \text{ on } \ \partial_1 R \quad \text{and} \quad \sigma_{ij}n_i = t_j^* \ \text{ on } \ \partial_2 R$$

2. Expressing the governing equations using the principle of virtual work
   (1) Just as for static problems, the principle of virtual work can be used to write the governing equation for the displacement field in a linear elastic solid in an integral form (called the weak form).
   (2) Instead of solving those governing equations listed in the preceding section, the displacements, strains, and stresses u_i, ε_{ij}, and σ_{ij} are calculated as follows:
   ① Find a displacement field $u_i(x_j)$ satisfying the derived virtual work equation:

$$(4.5)\ \int_R \rho\frac{\partial^2 u_i}{\partial t^2}\delta v_i\,dV + \int_R C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\frac{\partial \delta v_i}{\partial x_j}dV - \int_R b_i\delta v_i\,dV - \int_{\partial_2 R} t_i^*\delta v_i\,dA = 0 \qquad u_i = u_i^* \ \text{ on } \ \partial_1 R$$

for all virtual velocity fields δv_i satisfying $\delta v_i = 0$ on $\partial_1 R$.

② Compute the strains from the definition $\varepsilon_{ij} = \dfrac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$.

③ Compute the stresses from the stress-strain law: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$.

(3) Similar to the derivation for statics cases, the stress will automatically satisfy the equation of motion and boundary conditions, so all the field equations and boundary conditions will be satisfied.

(4) Derivation:

Starting from the original virtual work equation for dynamic problems:

$$\text{(4.6)} \int_R \rho \frac{\partial^2 u_i}{\partial t^2} \delta v_i \, dV + \int_R \sigma_{ij} \delta\varepsilon_{ij} \, dV - \int_R b_i \delta v_i \, dV - \int_{\partial_2 R} t_i^* \delta v_i \, dA = 0 \qquad u_i = u_i^* \text{ on } \partial_1 R.$$

Look at the term including $\sigma_{ij}\delta\varepsilon_{ij}$, and

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i}\right),$$

$$\sigma_{ij}\delta\varepsilon_{ij} = \sigma_{ij}\frac{1}{2}\left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i}\right) = \sigma_{ij}\frac{\partial \delta v_i}{\partial x_j} \qquad \text{(since } \sigma_{ij} = \sigma_{ji}\text{)}$$

$$\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell} = C_{ijk\ell}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\right) = C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell} \qquad \text{(since } C_{ijk\ell} = C_{ij\ell k}\text{)}$$

Hence,

$$\sigma_{ij}\delta\varepsilon_{ij} = C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\frac{\partial \delta v_i}{\partial x_j}$$

Finally, by putting the term including $\sigma_{ij}\delta\varepsilon_{ij}$ of the above equation back to Eq. 4.6, the derived virtual work equation (Eq. 4.5) can be achieved.

3. Finite element equations of motion for linear elastic solids

(1) Following exactly the same setting and procedure that we adopted for static problems, a finite element analysis for dynamics of solids made of linear elastic materials is now introduced. First, the unknown displacements at each nodal point u_i^a ($a = 1 \sim n$), which are now a function of time, are chosen to be calculated at a set of n discrete nodes with the nodal coordinates x_i^a.

(2) Second, as before, the displacement and virtual velocity fields are interpolated between nodal values as:

$$\text{(4.7)}\ u_i(\boldsymbol{x}) = \sum_{a=1}^{N_e} N^a(\boldsymbol{x})\, u_i^a \qquad (N_e\text{: Number of nodes of each element})$$

$$\text{(4.8)}\ \delta v_i(\boldsymbol{x}) = \sum_{a=1}^{N_e} N^a(\boldsymbol{x})\, \delta v_i$$

(3) Substitute into the virtual work equation (Eq. 4.5) to see that:

$$\text{(4.9)} \int_R \rho N^b N^a \frac{\partial^2 u_i^b}{\partial t^2} \delta v_i^a \, dV + \int_R C_{ijk\ell}\frac{\partial N^b}{\partial x_\ell}\frac{\partial N^a}{\partial x_j} u_k^b \delta v_i^a \, dV - \int_R b_i N^a \delta v_i^a \, dV - \int_{\partial_2 R} t_i^* N^a \delta v_i^a \, dA = 0$$
$$u_i = u_i^* \text{ on } \partial_1 R$$

This is once again a matrix-vector system that may be expressed in the form:

$$\text{(4.10)} \left(M_{ab}\,\ddot{u}_i^b + K_{aibk}\,u_k^b - F_i^a\right)\,\delta v_i^a = 0$$

where

$$\text{(4.11)}\ M_{ab} = \int_R \rho N^a(\boldsymbol{x}) N^b(\boldsymbol{x})\, dV$$

$$\text{(4.12)}\ K_{aibk} = \int_R C_{ijk\ell}\frac{\partial N^a(\boldsymbol{x})}{\partial x_j}\frac{\partial N^b(\boldsymbol{x})}{\partial x_\ell} dV$$

$$\text{(4.13)}\ F_i^a = \int_R b_i N^a(\boldsymbol{x})\, dV + \int_{\partial_2 R} t_i^* N^a(\boldsymbol{x})\, dA$$

are the finite element mass matrix, the finite element stiffness matrix, and the force vector respectively.

(4) Finally, because the principal of virtual power holds for all virtual velocity fields such that $\delta v_i^a = 0$ for x_i^a on $\partial_1 R$, it can be concluded that the following equations have to be satisfied:

(4.14) $M_{ab}\dfrac{\partial^2 u_i^b}{\partial t^2} + K_{aibk}\, u_k^b - F_i^a = 0 \qquad \forall(i,a) : x_k^a$ not on $\partial_1 R$

(4.15) $u_i^a = u_i^* \qquad \forall(i,a) : x_k^a$ on $\partial_1 R$

The nodal velocities must also satisfy initial conditions. This is originally a set of coupled second-order linear differential equations that may be integrated with respect to time to determine u_i^a as a function of time. In addition, observe that M_{ab} and K_{aibk} are constant (not time-varying) matrices for linear elastic problems, but F_i^a will in general be a function of time.

(5) To find the solution, these equations have to be integrated with respect to time. For linear problems, there are two choices:

① Brute-force time-stepping methods (e.g., Newmark time integration). This technique can be used for both linear and nonlinear problems.

② Modal methods, which integrate the equations of motion exactly. This method only works for linear problems.