EE6203 Computer Control Systems

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LECTURE NO 1

State Space Design Methods and Optimal Control
(15 hrs)

References

i. Digital Control Systems by Benjamin C Kuo, Saunders College Publishing, 1992.

ii. Computer Controlled Systems : Theory and Design by Karl J Astrom and Bjorn Wittenmark, Prentice Hall, 1997.

iii. Digital Control Systems Analysis and Design by Charles L Philips and H Troy Nagle, Prentice Hall, 1995.

iv. Digital Control of Dynamic Systems by Franklin G F, Powell J D and Workman M, Prentice Hall, 1998 (2006).

v. Digital Control Systems : Theory, Hardware and Software by Constantine H Houpis and Gary B Lamont, McGraw Hill, 1992.

1 State variables analysis

Objectives :

Design an algorithm for the computer such that a sequence $u(kT)$ can be generated to control the process and the output $y(t)$ vary according to some pre-specified criteria.

See Figure 1.

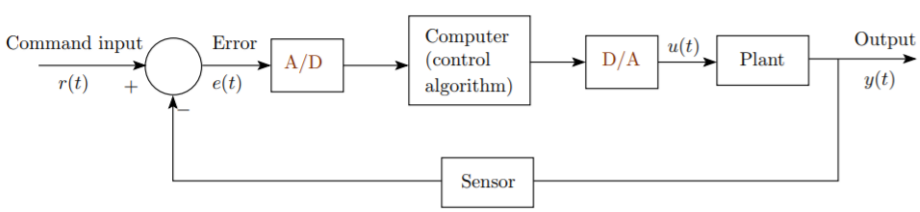


Figure 1. A Digital Control System

Typical specifications :

- Steady-state tracking accuracy :
  $\lim_{t \to \infty} \left(r(t) - y(t) \right)$

- Transient accuracy (dynamic response) :
- Rise time
- Overshoot
- Settling time

- Control effort required :
- Maximum magnitude of $u(kT)$
- Energy of $\sum \left(u^2(kT) \right)$

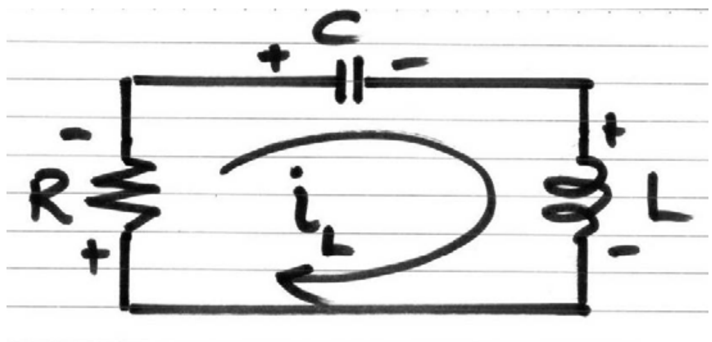
Design approaches :

i. Frequency domain design (classical approach)

ii. State space design (modern approach)

Advantages of state-space approach:
- Convenient for computer applications.
- Allows a unified representation of single-variable and multi-variable systems and various types of sampling schemes.
- Can be applied to nonlinear and time-varying systems.

Motivation : Consider the series RLC circuit as shown. By Kirchoff's Voltage Law,



$$v_L(t) + v_R(t) + v_C(t) = 0$$

$$L\frac{di_L(t)}{dt} + i_L(t)R + \frac{1}{C}\int_o^t i_L(\tau)\,\mathrm{d}\tau + \mathrm{V}_C(0) = 0$$

$$\Rightarrow \frac{d^2 i_L(t)}{dt^2} + \frac{R}{L}\frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = 0$$

The solution gives $i_L(t)$. What about $v_C(t)$?

- Not adequately reflect the behaviour of the circuit.
- Time consuming in finding solutions.

Two important variables : $v_C(t)$ and $i_L(t)$

Knowledge of these two variables enable the computations of other voltages in the circuit.

Define a vector,

$$\mathbf{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} \Rightarrow \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{v}_C(t) \\ \dot{i}_L(t) \end{bmatrix}$$

$$i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt} = C\dot{v}_C(t)$$

$$\Rightarrow \dot{v}_C(t) = \frac{1}{C} i_L(t)$$

$$\dot{i}_L(t) = \frac{1}{L} v_L(t) = \frac{1}{L}\left(-v_C(t) - v_R(t)\right)$$

$$\Rightarrow \dot{i}_L(t) = -\frac{1}{L} v_C(t) - \frac{R}{L} i_L(t)$$

$$\Rightarrow \begin{bmatrix} \dot{v}_C(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$$

This is a first order matrix differential equation of the RLC circuit.

1.1 Review of continuous-time state space representations

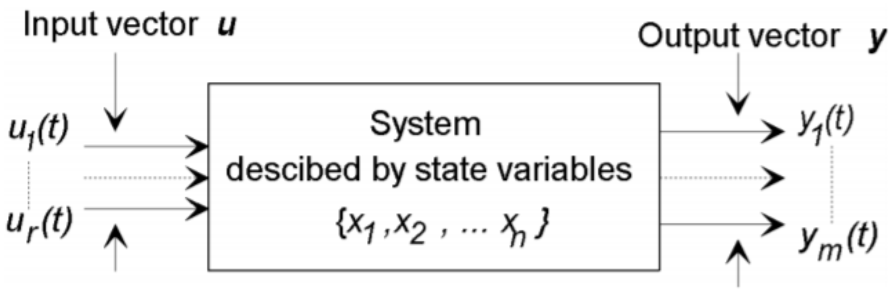


Figure 2. A LTI MIMO continuous-time system with *n* state variables, *r* inputs and *m* outputs

- The state space model of a system is a mathematical description of the system in terms of a minimum set of internal variables $x_1(t), x_2(t), \ldots, x_n(t)$, together with the knowledge of those variables at an initial time t_0 and the system inputs $u_i(t), j = 1,2,\ldots,r$ over $[t_0, \infty)$ are sufficient to predict the future system states and outputs for $t \geq t_0$.
- A continuous-time system can be modelled using these states by a set of first-order differential equations, called *state equations*.

- Together with the *output equation*, a LTI continuous-time system can be represented in state space as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \ldots\ldots. \quad (1.1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \ldots\ldots. \quad (1.2)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}; \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

Example 1.1 State space description of a dynamical system

Consider the Newton's Law $M\ddot{x}(t) = F$ where *M* is the mass, *F* is force and $x(t)$ is the displacement. Define,

$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = \dot{x}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = F/M \end{cases}$$

If the output is $y(t) = x_1(t)$, then a state space model of the system is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} F$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

1.2 Transfer function from the given state space model

Take Laplace transform of (1.1) and (1.2) with $\mathbf{x}(0) = 0$,

$$s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s)$$

$$\Rightarrow \mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{BU}(s) \qquad ...(1.3)$$

$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s) \qquad ...(1.4)$$

$$\Rightarrow \mathbf{Y}(s) = \left(\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B} + \mathbf{D}\right)\mathbf{U}(s)$$

The transfer function

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \left(\mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}+\mathbf{D}\right) \qquad \text{..... (1.5)}$$

$$= \left(\frac{\mathbf{C}\ \mathrm{adj}\left[s\mathbf{I}-\mathbf{A}\right]\ \mathbf{B}}{\det\left[s\mathbf{I}-\mathbf{A}\right]}\right)+\mathbf{D} \qquad \text{..... (1.6)}$$

The adjoint of a matrix is the transpose of the co-factors matrix.

The denominator of the transfer function is $\det[s\mathbf{I}-\mathbf{A}]$.

Hence, the poles of the system are identical to the eigenvalues of the matrix $\mathbf{A}$.

For single-input, single-output systems,
r = 1 and p = 1.

Example 1.2 Obtain a state space model of the servomotor system shown in Figure 3.

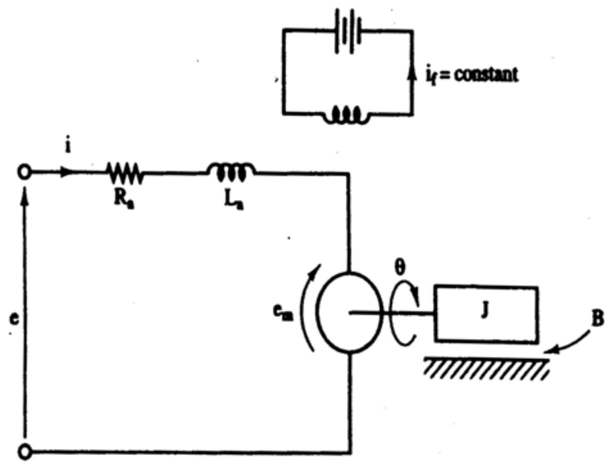


Figure 3. Servomotor system

The motor back emf

$$e_m(t) = K_b\omega(t) = K_b\frac{d\theta(t)}{dt} \qquad ...(1.7)$$

where $\theta(t)$ - motor shaft position

$\omega(t)$ - shaft angular velocity

K_b - motor-dependent constant

If *J* and *B* are the total moment of inertia connected to the shaft and the total viscous friction, respectively, the torque developed by the motor is

$$T(t) = K_T i(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} \qquad ...(1.8)$$

From KVL (neglect L_a)

$$e(t) = i(t)R_a + e_m(t) \qquad \ldots(1.9)$$

Define the following state variables and the output as the shaft position $\theta(t)$,

$$x_1(t) = \theta(t);$$

$$x_2(t) = \frac{d\theta(t)}{dt} = \dot{x}_1(t) \Rightarrow \dot{x}_2(t) = \frac{d^2\theta(t)}{dt^2}$$

$$\Rightarrow \dot{x}_2(t) = -\frac{BR_a + K_T K_b}{JR_a} x_2(t) + \frac{K_T}{JR_a} e(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{BR_a + K_T K_b}{JR_a} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_T}{JR_a} \end{bmatrix} e(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$(1.5) \Rightarrow \frac{\Theta(s)}{E(s)} = \frac{K_T / JR_a}{s\left(s + \left(\dfrac{BR_a + K_T K_b}{JR_a}\right)\right)}$$

1.3 Solution of the state vector $\mathbf{x}(t)$

The Laplace transform of (1.1) gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \qquad \text{... (1.13)}$$

$$\Rightarrow \mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)$$

$$+[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) \qquad \text{...(1.14)}$$

$$\Rightarrow \mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

$$+\int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau \quad \text{...(1.15)}$$

$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \boldsymbol{\gamma}(t), \quad t \geq 0 \qquad \text{... (1.16)}$$

$$\boldsymbol{\gamma}(t) = \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau \qquad \text{.... (1.17)}$$

Alternatively,

$$\boldsymbol{\gamma}(t) = \mathcal{L}^{-1}\left\{\Gamma(s)\right\} \qquad \text{.... (1.18)}$$

$$\Gamma(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) \qquad \text{.... (1.19)}$$

$\mathbf{\Phi}(t)$ is the non-singular state transition matrix and is defined as,

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{[s\mathbf{I} - \mathbf{A}]^{-1}\right\} = e^{\mathbf{A}t} \qquad \text{.... (1.20)}$$

Some properties of $\mathbf{\Phi}(t)$:

1. $\mathbf{\Phi}(0) = \mathbf{I}.$
2. $\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) = \mathbf{\Phi}(t_2 - t_0).$
3. $\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t).$
4. $\mathbf{\Phi}^k(t) = \mathbf{\Phi}(kT), \quad k = 0,1,2,3,...$

Assuming $t = t_0 > 0$, substitute $t = t_0$ into (1.15) and solve for $\mathbf{x}(\mathbf{0})$. The resulting $\mathbf{x}(\mathbf{0})$ is then substituted into (1.15) to yield

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)\ \mathrm{d}\tau, t \geq t_0 \quad .... (1.21)$$

(1.21) is known as the *state transition equation* of the system (1.1).

Example 1.3 Determine the output response of the following system to a unit step-input $u(t) = 1, t \geq 0$ and $\mathbf{x}(0) = [1 \quad -1]^T$.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t)$$

$$\left[s\mathbf{I} - \mathbf{A}\right]^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}}{s(s+2)} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\Rightarrow \mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\right\} = \begin{bmatrix} 1 & 0.5(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

$$(1.18) \Rightarrow \quad \boldsymbol{\gamma}(t) = \begin{bmatrix} 0.5\left(2t + e^{-2t} - 1\right) \\ 1 - e^{-2t} \end{bmatrix}$$

$$(1.16) \Rightarrow \quad \mathbf{x}(t) = \begin{bmatrix} t + e^{-2t} \\ 1 - 2e^{-2t} \end{bmatrix}$$

$$\Rightarrow \quad y(t) = t + e^{-2t}$$

2 Discrete-time state space models with sample and hold

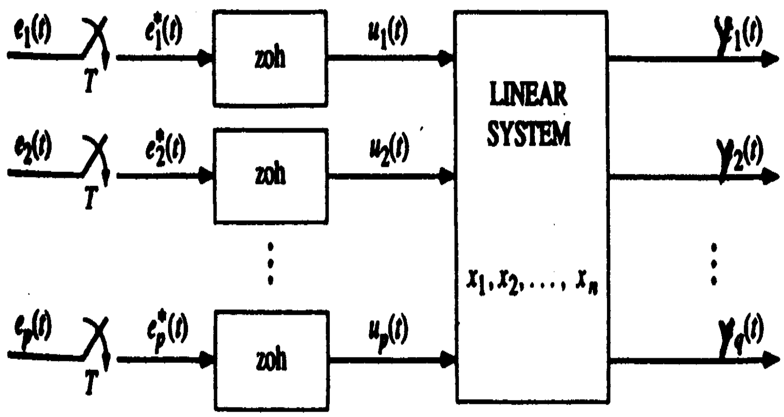


Figure 4. A LTI discrete-time system with ZOH

The aim is to obtain the discrete state equations of the sample-data system (Figure 4) directly from the continuous-time state equations, i.e., to discretise the given continuous-time system.

The outputs of the ZOH are

$$u_i(t) = u_i(kT) \qquad \text{.... (2.1)}$$

$$= \ e_i(kT), \ \ kT \le t < (k+1)T$$

$$k = 0,1,2,..., \ \ i = 0,1,2,..., p.$$

Let the state equation and its solution be respectively, given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)\, \mathrm{d}\tau \qquad ....(2.2)$$

$$\mathbf{u}(\tau) = \mathbf{u}(kT), \quad kT \le \tau < (k+1)T$$

$$\Rightarrow \mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathrm{d}\tau \; \mathbf{u}(kT) \qquad ... (2.3)$$

valid for $kT \le t < (k+1)T$. Set $t_0 = kT$.

$$\Rightarrow \mathbf{x}(t) = \mathbf{\Phi}(t - kT)\mathbf{x}(kT) + \int_{kT}^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathrm{d}\tau \; \mathbf{u}(kT) \qquad ...(2.4)$$

(2.4) describes $\mathbf{x}(t)$ at all times between the sampling instants kT and $(k+1)T$, k = 0, 1, 2, ... Now, let

$$\boldsymbol{\Theta}(t-kT)=\int_{kT}^{t}\boldsymbol{\Phi}(t-\tau)\mathbf{B}\,\mathrm{d}\tau \qquad ...(2.5)$$

$$\Rightarrow \mathbf{x}(t)=\boldsymbol{\Phi}(t-kT)\mathbf{x}(kT)$$
$$+\boldsymbol{\Theta}(t-kT)\ \mathbf{u}(kT) \qquad ...(2.6)$$

The values of $\mathbf{x}(t)$ at successive sampling instants can be derived by setting $t=(k+1)T$ as in the case of numerical iterations.

$$\mathbf{x}((k+1)T)=\boldsymbol{\Phi}(T)\mathbf{x}(kT)+\boldsymbol{\Theta}(T)\ \mathbf{u}(kT) \quad ...(2.7)$$

$$\boldsymbol{\Phi}(t)=\left[\mathcal{L}^{-1}\left\{[s\mathbf{I}-\mathbf{A}]^{-1}\right\}\right]_{t=T} \qquad ...(2.8)$$

$$(2.5)\Rightarrow \boldsymbol{\Theta}(T)=\int_{kT}^{(k+1)T}\boldsymbol{\Phi}\big((k+1)T-\tau\big)\mathbf{B}\,\mathrm{d}\tau$$

Set $\eta=(k+1)T-\tau$. Then, d$\eta\ =-d\tau$

$$\boldsymbol{\Theta}(T)=\int_{T}^{0}\boldsymbol{\Phi}(\eta)(-\mathrm{d}\eta)\,\mathbf{B}$$
$$=\int_{0}^{T}\boldsymbol{\Phi}(\eta)\ \mathrm{d}\eta\mathbf{B} \qquad ...(2.9)$$

(2.7) represents a set of first-order difference equations, referred to as the discrete state equations of the discrete-time system of Figure 4.

Similarly, the output equation in (1.2) is discretised by setting $t = kT$,

$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT) \qquad ...(2.10)$$

Note : Conventionally, it is common to drop "*T*" from the above models (2.7) and (2.10) for ease of presentations.

Example 1.4 A servomotor has a continuous-time state space representation as shown below.
Sample the system with a sampling period of *T* = 0.1 sec and obtain a discrete-time state space model of the motor.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$\Rightarrow \left[s\mathbf{I}-\mathbf{A}\right]^{-1} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+1)} \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

$$\Rightarrow \mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\right\} = \begin{bmatrix} 1 & 1-e^{-t} \\ 1 & e^{-t} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{\Phi}(T) = \begin{bmatrix} 1 & 1-e^{-T} \\ 1 & e^{-T} \end{bmatrix} = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}$$

$$\mathbf{\Theta}(T) = \left[\int_0^T \mathbf{\Phi}(\tau)\,\mathrm{d}\,\tau\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) \qquad ...(2.11)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k) \qquad ...(2.12)$$

3 Discrete-time transfer function and state space models.

Consider,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \qquad \text{... (3.1)}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \qquad \text{... (3.2)}$$

Taking z – transform and with $\mathbf{x}(0) = 0$,

$$z\mathbf{X}(z) = \mathbf{A}\mathbf{X}(z) + \mathbf{B}\mathbf{U}(z)$$

$$\Rightarrow \mathbf{X}(z) = [z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(z) \qquad \text{... (3.3)}$$

$$\mathbf{Y}(z) = \mathbf{C}\mathbf{x}(z) + \mathbf{D}\mathbf{U}(z) \qquad \text{... (3.4)}$$

$$= \left(\mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}(z) \qquad \text{...(3.5)}$$

The system transfer function is

$$\frac{\mathbf{Y}(z)}{\mathbf{U}(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \qquad \text{... (3.6)}$$

3.1 Discrete time poles

A pole is a value of z such that (3.1) has a non-trivial solution when the forcing input is zero. From (3.3), this implies that

$$[z\mathbf{I} - \mathbf{A}]\mathbf{X}(z) = 0$$

has a non-trivial solution. This means that

$$\det\,[z\mathbf{I} - \mathbf{A}] = 0$$

i.e. the poles of the transfer function are identical to the eigenvalues of the matrix $\mathbf{A}$.

3.2 Discrete time zeros

A system zero is a value of z_0 such that the system output is zero even with a non-zero state-and-input combination.

That is, if we are able to find a non-trivial solution for $\mathbf{X}(z_0)$ and $\mathbf{U}(z_0)$ such that $\mathbf{Y}(z_0)$ is zero, then z_0 is a zero of the system.

Combining (3.3) and (3.4),

$$\begin{bmatrix} z_0\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{X}(z_0) \\ \mathbf{U}(z_0) \end{bmatrix} = 0$$

which implies that the condition for the existence of non-trivial solutions is that

$$\det \begin{bmatrix} z_0\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = 0$$

Example 1.5 Consider

$$\mathbf{x}(k+1) = \begin{bmatrix} 1.35 & 0.55 \\ -0.45 & 0.35 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(k)$$

$$\left[z\mathbf{I}-\mathbf{A} \right] = \begin{bmatrix} z-1.35 & -0.55 \\ 0.45 & z-0.35 \end{bmatrix}$$

$$\Rightarrow \left[z\mathbf{I}-\mathbf{A} \right]^{-1} = \frac{\begin{bmatrix} z-0.35 & 0.55 \\ -0.45 & z-1.35 \end{bmatrix}}{z^2-1.7z+0.72}$$

$$(3.6) \Rightarrow \frac{Y(z)}{U(z)} = \mathbf{C}\left[z\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B} = \frac{1}{z^2-1.7z+0.72}$$

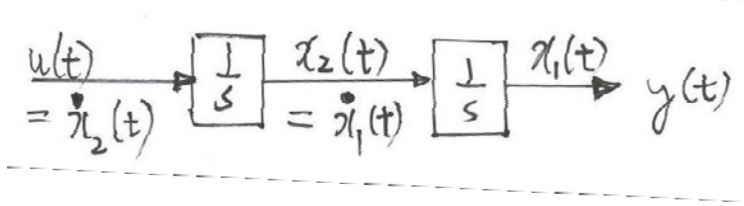
Zeros polynomial :

$$\left\|\begin{bmatrix} z\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\right\| = \left\|\begin{bmatrix} z-1.35 & -0.55 & -0.5 \\ 0.45 & z-0.35 & -0.5 \\ 1 & -1 & 0 \end{bmatrix}\right\| = 1$$

Poles polynomial :

$$\det\left[z\mathbf{I}-\mathbf{A}\right] = z^2 - 1.7z + 0.72$$

Example 1.6 Consider a double integrator with state variables as defined.



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The discrete-time state space model :

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

$$\Rightarrow \mathbf{\Phi}(t) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{\Theta}(T) = \left[\int_0^T \mathbf{\Phi}(\tau)\,\mathrm{d}\tau\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}$$

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)$$

$$\Rightarrow \frac{Y(z)}{U(z)} = \begin{bmatrix} 1 & 0 \end{bmatrix}\left[z\mathbf{I} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right]^{-1}\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}$$

$$= \frac{1}{2}T^2\left(\frac{z+1}{(z-1)^2}\right)$$

Example 1.7 A discrete-time system is represented by the following 4-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, d)$. Find the poles and zeros of the system.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}, d = 0$$

$$\det \begin{bmatrix} z\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & d \end{bmatrix} = \left\| \begin{bmatrix} z+1 & 0 & -1 \\ 0 & z+2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \right\| = z$$

$$\det\,[z\mathbf{I} - \mathbf{A}] = \left\| \begin{bmatrix} z+1 & 0 \\ 0 & z+2 \end{bmatrix} \right\| = (z+1)(z+2)$$

The zero is at z = 0 and the poles at z = -1, -2.

4 The state transition equation
The most straightforward way of solving the state equation is by recursion.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad .... \quad (4.1)$$

$$k = 0,$$

$$\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{u}(0) \quad .... \quad (4.2)$$

$$k = 1,$$

$$\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) + \mathbf{B}\mathbf{u}(1)$$

$$= \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1)$$

Thus, the general solution for (4.1) given $\mathbf{x}(0)$ and $\mathbf{u}(i)$ for $i = 0, 1, \ldots, (k-1)$ is

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{i=0}^{k-1}\left(\mathbf{A}^{(k-i-1)}\mathbf{B}\mathbf{u}(i)\right) \quad ...(4.3)$$

(4.3) is defined as the *state transition equation* of the discrete-time system (4.1).

Example 1.8 Consider

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad \text{.....} \ (4.4)$$

$$y(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}(k) \quad \text{.....} \ (4.5)$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad u(k) = 1; k = 0, 1, 2, \ \ldots$$

$$(4.3) \ \Rightarrow \mathbf{x}(k) = \sum_{i=0}^{k-1} \left(\mathbf{A}^{(k-i-1)} \mathbf{B} \right) \quad \text{....} \ (4.6)$$

$$\mathbf{x}(1) = \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(1) = \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}(1) = 1$$

$$\mathbf{x}(2) = \sum_{i=0}^{1} \left(\mathbf{A}^{(1-i)} \mathbf{B} \right) = \mathbf{AB} + \mathbf{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y(2) = \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}(2) = 1$$

Hence, the states and output can be determined at successive time instants.