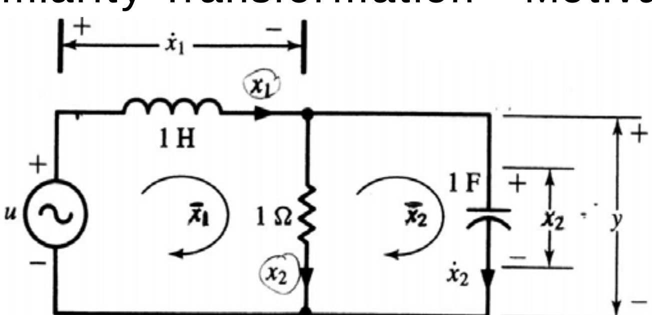
LECTURE NO 2

5 Similarity Transformation - Motivation



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$

The two state space models described the same network and are related by.

$$\begin{cases} x_1(t) = \bar{x}_1(t) \\ x_2(t) = \bar{x}_1(t) - \bar{x}_2(t) \end{cases} \Rightarrow \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Given one state space model of a system, we can derive another state space model through *similarity transformation*. Consider,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \qquad \text{... (5.1)}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + du(k) \qquad \text{... (5.2)}$$

Apply the following linear transformation where $\mathbf{P}$ is a constant non-singular $n \, x \, n$ matrix and $\mathbf{w}(k)$ is the new state vector.

$$\mathbf{x}(k) = \mathbf{P}\mathbf{w}(k) \Rightarrow \mathbf{w}(k) = \mathbf{P}^{-1}\mathbf{x}(k) \quad \text{...(5.3)}$$

Substitute (5.3) into (5.1) and (5.2) :

$$\mathbf{w}(k+1) = \left(\mathbf{P}^{-1}\mathbf{AP}\right)\mathbf{w}(k) + \left(\mathbf{P}^{-1}\mathbf{B}\right)u(k)$$

$$= \mathbf{A}_w\mathbf{w}(k) + \mathbf{B}_w u(k) \qquad \text{... (5.6)}$$

$$y(k) = \left(\mathbf{CP}\right)\mathbf{w}(k) + du(k)$$

$$= \mathbf{C}_w\mathbf{w}(k) + d_w u(k) \qquad \text{...(5.7)}$$

$$\Rightarrow \mathbf{A}_w = \mathbf{P}^{-1}\mathbf{AP}; \quad \mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} \qquad \text{...(5.8)}$$

$$\mathbf{C}_w = \mathbf{CP}; \quad d_w = d \qquad \text{...(5.9)}$$

The characteristic polynomial of the system is unchanged under a similarity transformation :

$$\left|\lambda\mathbf{I} - \mathbf{A}_w\right| = \left|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}\right|$$

$$= \left|\lambda\mathbf{I}\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}\right| = \left|\lambda\mathbf{I} - \mathbf{A}\right|$$

The transfer function of the system is invariant under a similarity transformation (Prove the following yourself) :

$$\mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + d \quad = \mathbf{C}_w[z\mathbf{I} - \mathbf{A}_w]^{-1}\mathbf{B}_w + d_w$$

6 Canonical Forms (Companion Forms)

6.1 Controllable Canonical Form (CCF)

Consider the following system and whose characteristic polynomial is as in (6.3).

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \qquad ...(6.1)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + du(k) \qquad ...(6.2)$$

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 ...(6.3)$$

Suppose that the controllability matrix $\mathbf{W}_C$ is non-singular.

$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} ...(6.4)$$

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Then, there exists a non-singular matrix $\mathbf{P}$ that transforms (6.1) into the CCF :

$$\mathbf{w}(k) = \mathbf{P}^{-1}\mathbf{x}(k) \qquad ....(6.5)$$

$$\mathbf{w}(k+1) = \mathbf{A}_C\mathbf{w}(k) + \mathbf{B}_C u(k) \qquad ....(6.6)$$

$$y(k) = \mathbf{C}_C\mathbf{w}(k) + d_C u(k) \qquad ....(6.7)$$

Let $\widetilde{\mathbf{W}}_C$ be the controllability matrix for the representation (6.6)

$$\tilde{\mathbf{W}}_C = \begin{bmatrix} \mathbf{B}_C & \mathbf{A}_C\mathbf{B}_C & \mathbf{A}_C^2\mathbf{B}_C & \dots & \mathbf{A}_C^{n-1}\mathbf{B}_C \end{bmatrix}$$

$$...(6.8)$$

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \quad ...(6.9)$$

$$\mathbf{B}_C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The transformation matrix $\mathbf{P}$ in (6.5) to obtain the CCF is

$$\mathbf{P} = \mathbf{W}_C \tilde{\mathbf{W}}_C^{-1} \quad ...(6.10)$$

Example 2.1 Find a transformation matrix $\mathbf{P}$ that will transforms the given state space model into the CCF and find this realization.

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

$$\left\| \left[z\mathbf{I} - \mathbf{A} \right] \right\| = \left\| \begin{bmatrix} z+0.5 & 0 \\ 0 & z+0.8 \end{bmatrix} \right\| = z^2 + 1.3z + 0.4$$

$$\Rightarrow \mathbf{A}_C = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}; \quad \mathbf{B}_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{W}}_C = \begin{bmatrix} \mathbf{B}_C & \mathbf{A}_C\mathbf{B}_C \end{bmatrix} \Rightarrow \tilde{\mathbf{W}}_C^{-1} = \begin{bmatrix} 1.3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \Rightarrow \mathbf{P} = \mathbf{W}_C \tilde{\mathbf{W}}_C^{-1} = \begin{bmatrix} 0.8 & 1 \\ 0.5 & 1 \end{bmatrix}$$

$$\text{CCF} \ : \ \mathbf{A}_C = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}; \quad \mathbf{B}_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\mathbf{C}_C = \mathbf{CP} = \begin{bmatrix} -0.2 & -1 \end{bmatrix}; d_C = 0$$

6.2 Observable Canonical Form (OCF)

Let the characteristic polynomial is as given in (6.3) and the observability matrix $\mathbf{W}_O$ is non-singular.

$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad \text{...(6.11)}$$

Then, there exists a non-singular matrix $\mathbf{Q}$ that transforms (6.1) and (6,2) into the OCF.

$$\mathbf{w}(k) = \mathbf{Q}^{-1}\mathbf{x}(k)$$

$$\mathbf{w}(k+1) = \mathbf{A}_O\mathbf{w}(k) + \mathbf{B}_O u(k) \qquad ...(6.12)$$

$$y(k) = \mathbf{C}_O\mathbf{w}(k) + d_O u(k) \qquad ...(6.13)$$

$$\mathbf{A}_O = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

$$\mathbf{C}_O = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad ...(6.14)$$

The transformation matrix $\mathbf{Q}$ to obtain the OCF is

$$\mathbf{Q} = \mathbf{W}_O^{-1}\tilde{\mathbf{W}}_O \qquad .....(6.15)$$

where $\tilde{\mathbf{W}}_{\boldsymbol{O}}$ be the observability matrix for the representation (6.12) and (6.13).

$$\tilde{\mathbf{W}}_O = \begin{bmatrix} \mathbf{C}_O \\ \mathbf{C}_O\mathbf{A}_O \\ \vdots \\ \mathbf{C}_O\mathbf{A}_O^{n-1} \end{bmatrix} \qquad .....(6.16)$$

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6.3 Canonical forms from transfer function

A process with a given transfer function can have the state space model in either the OCF or the CCF form written directly by inspection.

$$G(z)=\frac{b_{n-1}z^{n-1}+\cdots+b_1z+b_0}{z^n+\cdots+a_1z+a_0}$$

$$\mathbf{C}_C=\begin{bmatrix}b_0 & b_1 & \dots & b_{n-2} & b_{n-1}\end{bmatrix}$$

$$\mathbf{B}_O=\begin{bmatrix}b_0 & b_1 & \dots & b_{n-2} & b_{n-1}\end{bmatrix}^T$$

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Example 2.2

$$G(z)=\frac{-2z^3+2z^2-z+2}{z^3+z^2-z-3/4}=\left(\frac{4z^2-3z+1/2}{z^3+z^2-z-3/4}\right)-2$$

$$\text{CCF}:\mathbf{A}_C=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 3/4 & 1 & -1\end{bmatrix};\mathbf{B}_C=\begin{bmatrix}0\\0\\1\end{bmatrix};$$

$$\mathbf{C}_C=\begin{bmatrix}1/2 & -3 & 4\end{bmatrix};d_C=-2$$

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Example 2.3 Consider the system in OCF

$$\mathbf{x}(k+1)=\begin{bmatrix}0 & -a_O\\ 1 & -a_1\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}b_O\\ b_1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}0 & 1\end{bmatrix}\mathbf{x}(k)$$

$$\Rightarrow\left[z\mathbf{I}-\mathbf{A}\right]^{-1}=\begin{bmatrix}z & a_O\\ -1 & z+a_1\end{bmatrix}^{-1}=\frac{\begin{bmatrix}z+a_1 & -a_O\\ 1 & z\end{bmatrix}}{z^2+a_1z+a_O}$$

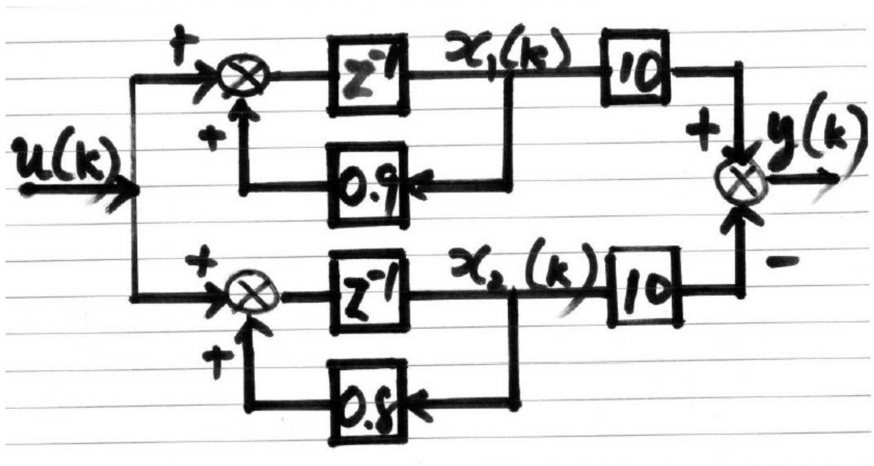
$$\Rightarrow\frac{Y(z)}{U(z)}=\mathbf{C}\left[z\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}=\frac{b_1z+b_O}{z^2+a_1z+a_O}$$

Example 2.4 Consider the given transfer function and its partial fraction expansions,

$$\frac{Y(z)}{U(z)}=\frac{1}{z^2-1.7z+0.72}$$

$$=\left(\frac{1}{z-0.9}\right)\left(\frac{1}{z-0.8}\right)$$

$$=\left(\frac{10}{z-0.9}\right)+\left(\frac{-10}{z-0.8}\right)$$



From the block diagram, the states and output equations are

$$x_1(k+1) = 0.9x_1(k) + u(k)$$

$$x_2(k+1) = 0.8x_2(k) + u(k)$$

$$y(k) = 10x_1(k) - 10x_2(k)$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 10 & -10 \end{bmatrix} \mathbf{x}(k)$$

Note that $\mathbf{A}$ is diagonal.

7 Controller Design

Consider the following plant.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \qquad ...(7.1)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \qquad ...(7.2)$$

Step 1 : Design a state feedback controller (assuming the system states are available for feedback). See Figure 5.

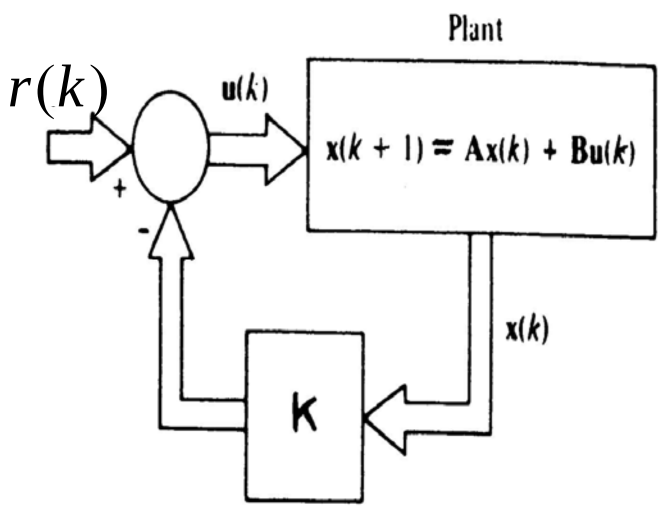


Figure 5. Closed-loop system with a linear state feedback law

Step 2 : Design an observer to estimate the system states from available measurements (if the system states are not available). See Figure 6.

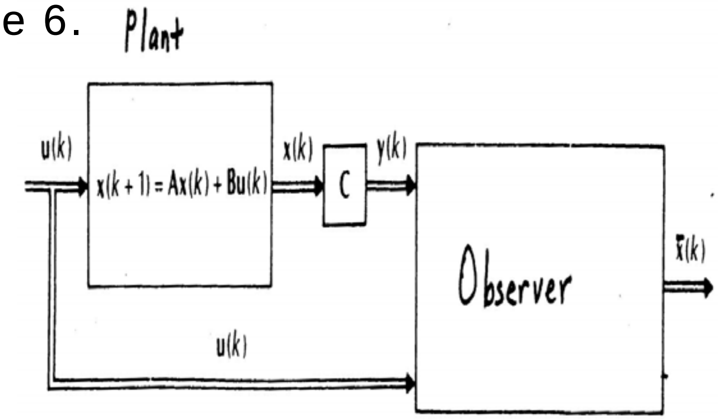


Figure 6. Simplified presentation of the system and the observer

The two steps are independent – Separation Principle. Combine the two steps with the system states replaced by their estimates. See Figure 7.

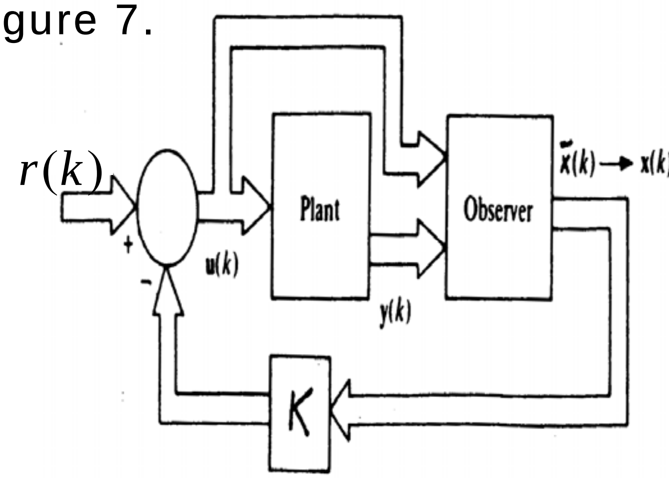


Figure 7. A feedback control system using measurement

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8 Controllability

Consider the following plant.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \qquad ....(8.1)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \qquad ....(8.2)$$

Assume the initial state $\mathbf{x}(0)$ is given and from (4.3)

$$\mathbf{x}(N) = \mathbf{A}^N\mathbf{x}(0) + \sum_{i=0}^{N-1}\left(\mathbf{A}^{(N-i-1)}\mathbf{B}\mathbf{u}(i)\right) \quad ....(8.3)$$

$$= \mathbf{A}^N\mathbf{x}(0) + \mathbf{A}^{(N-1)}\mathbf{B}\mathbf{u}(0) + \cdots + \mathbf{B}\mathbf{u}(N-1)$$

$$= \mathbf{A}^N\mathbf{x}(0) + \mathbf{W}_C\mathbf{U}$$

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$$\mathbf{W}_C = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{N-1}\mathbf{B}\end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix}u(N-1) & \cdots & u(0)\end{bmatrix}^T$$

$$\Rightarrow \mathbf{W}_C\mathbf{U} = \mathbf{x}(N) - \mathbf{A}^N\mathbf{x}(0)$$

$\mathbf{W}_C$ is the controllability matrix.

If $\mathbf{W}_C$ has rank n, then it is possible to find n equations from which the control signals can be found such that the initial state is transferred to the desired final state $\mathbf{x}(N)$.

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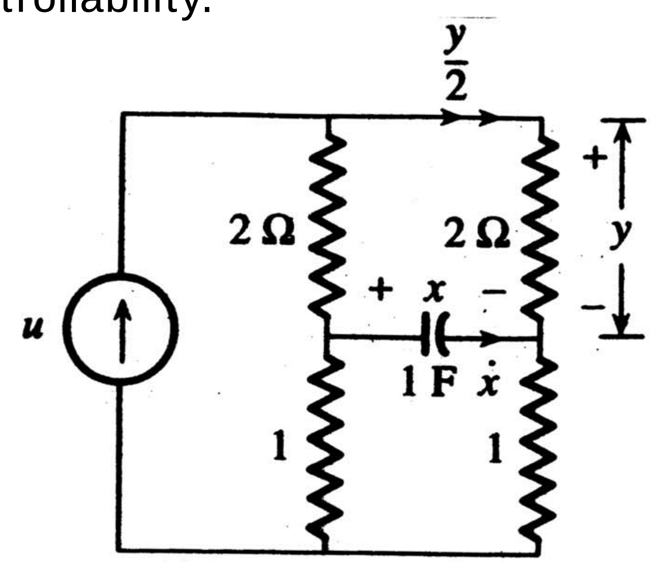
Definition : The pair $(\mathbf{A}, \mathbf{B})$ is controllable if it is possible to find a control sequence $\{u(0), u(1), u(2), \dots, u(N-1)\}$ which allows the system to reach an arbitrary final state $\mathbf{x}(N)$ from any initial state $\mathbf{x}(0)$.

Theorem : The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if

$$\text{rank } \mathbf{W}_C = n$$

where n is the order of the system.

Example 2.5 A practical example on controllability.



The voltage across the capacitor is the only state variable $x(t)$.

If $x(0) = 0$, the voltage across the capacitor is always zero for all $u(t)$. Cannot transfer $x(0) = 0$ to any non-zero $x(t)$.

Thus, the state equation describing the network is uncontrollable.

Example 2.6 Investigate the controllability of the following system.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{W}_C = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow |\mathbf{W}_C| = 0$$

The system is uncontrollable.

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Example 2.7 Investigate the controllability of the following system. Find a control sequence, if it exists, to drive the system to $\mathbf{x}(2) = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$ from the origin.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{W}_C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \left|\mathbf{W}_C\right| \neq 0$$

The system is controllable.

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$$\begin{bmatrix} 1 \\ 1.2 \end{bmatrix} = \mathbf{x}(2) = \mathbf{B}u(1) + \mathbf{AB}u(0)$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(0)$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} = \begin{bmatrix} u(0) \\ u(1) + u(0) \end{bmatrix}$$

$$\Rightarrow u(0) = 1; \quad u(1) = 0.2$$

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Example 2.8 Consider the following system. Find a control sequence, if it exists, to drive the system from $\mathbf{x}(0) = [0 \quad -1 \quad 3]^T$ to $\mathbf{x}(3) = [6 \quad -8 \quad 2]^T$.

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} u(k)$$

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{i=0}^{k-1} \left(\mathbf{A}^{(k-i-1)} \mathbf{B}\mathbf{u}(i) \right)$$

$$\Rightarrow \mathbf{x}(3) = \mathbf{A}^3 \mathbf{x}(0) + \mathbf{A}^2 \mathbf{B} u(0) + \mathbf{A}\mathbf{B} u(1) + \mathbf{B} u(2)$$

$$\Rightarrow \begin{bmatrix} 24 \\ -17 \\ -64 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ -2 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} u(1) + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} u(2)$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -9 \\ 1 & -5 & 9 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 24 \\ -17 \\ -64 \end{bmatrix}$$

$$\Rightarrow u(0) = \frac{249}{8}; u(1) = -\frac{7}{4}; u(2) = -\frac{2447}{8}$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0)$$

$$\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) + \mathbf{B}u(1)$$

$$\mathbf{x}(3) = \mathbf{A}\mathbf{x}(2) + \mathbf{B}u(2)$$

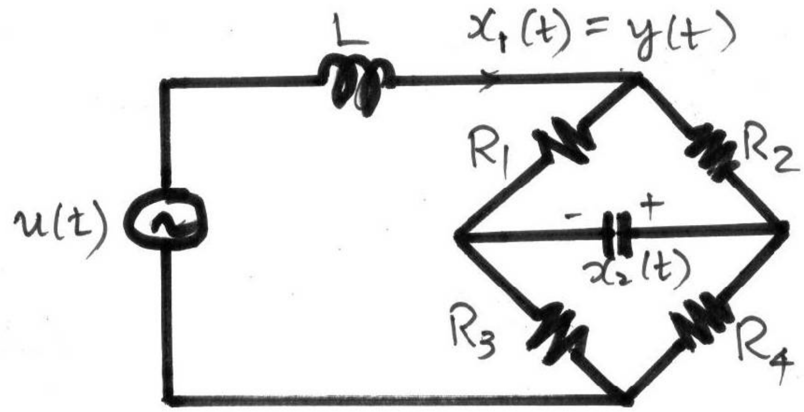
$$\mathbf{x}(1) = \begin{bmatrix} -265/8 \\ 257/8 \\ 8 \end{bmatrix}; \mathbf{x}(2) = \begin{bmatrix} 263/8 \\ -1331/8 \\ 448/8 \end{bmatrix}; \mathbf{x}(3) = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix}$$

If it is desired to move from $\mathbf{x}(5) = [0 \quad -1 \quad 3]^T$ to $\mathbf{x}(8) = [6 \quad -8 \quad 2]^T$,

$$\mathbf{x}(8) = \mathbf{A}^3\mathbf{x}(5) + \mathbf{A}^2\mathbf{B}u(5) + \mathbf{A}\mathbf{B}u(6) + \mathbf{B}u(7)$$

$$\Rightarrow u(5) = \frac{249}{8}; u(6) = -\frac{7}{4}; u(7) = -\frac{2447}{8}$$

Example 2.9 Another practical example on controllability – Bridge Circuit



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \mathbf{B}u(t)$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{L}\left(\frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4}\right) & -\frac{1}{L}\left(\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right) \\ -\frac{1}{C}\left(\frac{R_2}{R_1+R_2} - \frac{R_4}{R_3+R_4}\right) & -\frac{1}{C}\left(\frac{1}{R_1+R_2} + \frac{1}{R_3+R_4}\right) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

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$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{L} & -\dfrac{1}{L^2}\left(\dfrac{R_1 R_2}{R_1 + R_2} + \dfrac{R_3 R_4}{R_3 + R_4} \right) \\ 0 & -\dfrac{1}{LC}\left(\dfrac{R_2}{R_1 + R_2} - \dfrac{R_4}{R_3 + R_4} \right) \end{bmatrix}$$

If $\dfrac{R_4}{R_3+R_4} = \dfrac{R_2}{R_1+R_2} \Rightarrow |\mathbf{W}_C| = 0$

$\Rightarrow$ uncontrollable. This is the condition to balance the resistance bridge.

In this case, the voltage across the capacitor $v_C(t) \big(= x_2(t)\big)$ cannot be varied by any external input $u(t)$.

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