7. (a) forward difference: $s = \frac{1-z^{-1}}{Tz^{-1}} = \frac{1-z^{-1}}{z^{-1}}$ ($T = 1$ sec)

$$G(z) = \left[\frac{s+1}{s+3}\right]_{s=\frac{1-z^{-1}}{z^{-1}}} = \frac{\frac{1-z^{-1}}{z^{-1}}+1}{\frac{1-z^{-1}}{z^{-1}}+3} = \frac{1}{1+2z^{-1}} = \frac{z}{z+2} = 1+\frac{z}{2}$$

Because $e(kT)$ is a unit-step function: $E(s) = \frac{1}{s}$

$$\Rightarrow C(s) = G(s)\cdot E(s) = \frac{s+1}{s(s+3)} = \frac{1}{3}\frac{1}{s} + \frac{2}{3}\frac{1}{s+3}$$

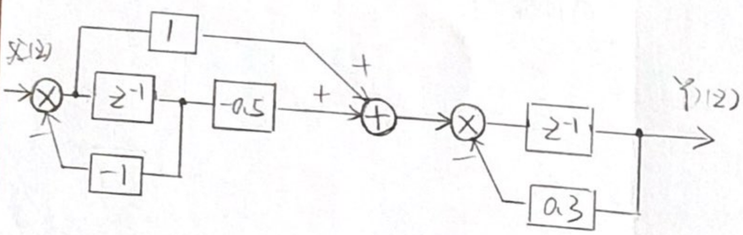
$$\Rightarrow c(kT) = \frac{1}{3}1(k) + \frac{2}{3}e^{-3kT} = \frac{1}{3}1(k) + \frac{2}{3}e^{-3k}, \quad k = 0, 1, 2, \ldots$$

$k=0$: $c(0) = 1$, $k=1$: $c(T) = \frac{1}{3}+\frac{2}{3}e^{-3}$,

$k=2$: $c(2T) = \frac{1}{3}+\frac{2}{3}e^{-6T} = \frac{1}{3}+\frac{2}{3}e^{-6}$.

(b)

$$D(z) = \frac{z-0.5}{z-1} \times \frac{1}{z+0.3} = \frac{1-0.5z^{-1}}{1-z^{-1}} \times \frac{z^{-1}}{1+0.3z^{-1}}$$



(c) backward difference: $s = \frac{1-z^{-1}}{T} = 1-z^{-1}$ ($T = 1$ sec)

$$G(z) = \left[\frac{s+1}{s+3}\right]_{s=1-z^{-1}} = \frac{2-z^{-1}}{4-z^{-1}}$$

has a zero at $\hat{z} = \frac{1}{2}$ and a pole at $\hat{p} = \frac{1}{4}$

$$C(z) = \frac{1}{1+2z^{-1}}\,\frac{1}{1-z^{-1}} = \frac{z^2}{(z+2)(z-1)}$$

$$\frac{C(z)}{z} = \frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}$$

$(A+B)z + (2B-A) = z$, $A = 2B$

$$= \frac{2}{3}\frac{1}{z+2} + \frac{1}{3}\frac{1}{z-1}$$

$B = \frac{1}{3}$, $A = \frac{2}{3}$

$$C(z) = \frac{2}{3}\frac{z}{z+2} + \frac{1}{3}\frac{z}{z-1} = \frac{2}{3}\frac{1}{1+2z^{-1}} + \frac{1}{3}\frac{1}{1-z^{-1}} \Rightarrow c(k) = \frac{2}{3}(-2)^k + \frac{1}{3}(1)^k$$

$c(0) = 1$, $c(1) = -1$, $c(2) = 3$, $c(3) = -5$

2. (a) $G_{ZAS}(z) = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$

$= (1-z^{-1})\mathcal{Z}\left[\frac{2}{s^2(s+6)^2}\right] = (1-z^{-1})$

3. (a) $\dot{x}(t) = Ax(t) + Bu(t)$

$$\Phi(T) = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\frac{\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}\right\} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}$$

$$\Theta(T) = \int_0^T \Phi(\tau) B \, d\tau = \int_0^T \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^T \begin{bmatrix} \sin\tau \\ \cos\tau \end{bmatrix} d\tau = \begin{bmatrix} -\cos\tau \\ \sin\tau \end{bmatrix}_0^T = \begin{bmatrix} -\cos T - 1 \\ \sin T \end{bmatrix}$$

$$x(k+1) = \Phi(T)x(k) + \Theta(T)u(k)$$

$$= \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} x(k) + \begin{bmatrix} -\cos T - 1 \\ \sin T \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 1]\, x(k)$$

(b) $$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \cos T - \sin T & \sin T + \cos T \end{bmatrix}$$

$$|W_o| = \sin T + \cos T - \cos T + \sin T = 2\sin T = 0$$

$\Rightarrow$ $T = k\pi$ (k is an integer)

(c) $$x(k+1) = Ax(k) + Bu(k) = Ax(k) + B(r(k) - y(k))$$

$$y(k) = Cx(k) + du(k) = Cx(k) + dr(k) - dy(k)$$

$$\Rightarrow y(k) = \frac{1}{1+d}(Cx(k) + dr(k)) = \frac{C}{1+d}x(k) + \frac{d}{1+d}r(k)$$

$$u(k) = r(k) - y(k) = \frac{1}{1+d}r(k) - \frac{C}{1+d}x(k)$$

$$\Rightarrow x(k+1) = Ax(k) + \frac{B}{1+d}r(k) - \frac{BC}{1+d}x(k)$$

$$= \left[A - \frac{BC}{1+d}\right]x(k) + \frac{B}{1+d}r(k)$$

$$\therefore A_{cl} = \left[A - \frac{BC}{1+d}\right], \quad B_{cl} = \frac{B}{1+d}, \quad C_{cl} = \frac{C}{1+d}, \quad D_{cl} = \frac{d}{1+d}$$

4. (a) $x(k+1) = Ax(k) + Bu(k) = [A-BK]x(k) + \dots$

$$[A-BK] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.125 \\ 0.25 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 1-0.125k_1 & 1-0.125k_2 \\ -0.25k_1 & 1-0.25k_2 \end{bmatrix}$$

$$|zI - [A-BK]| = \begin{vmatrix} z-1+0.125k_1 & -1+0.125k_2 \\ 0.25k_1 & z-1+0.25k_2 \end{vmatrix}$$

$$= (z-1+\tfrac{1}{8}k_1)(z-1+\tfrac{1}{4}k_2) - \tfrac{1}{4}k_1(\tfrac{1}{8}k_2 - 1)$$

$$= z^2 - z + \tfrac{1}{4}k_2 z - z + 1 - \tfrac{1}{4}k_2 + \tfrac{1}{8}k_1 z - \tfrac{1}{8}k_1 + \cancel{\tfrac{1}{32}k_1k_2} - \cancel{\tfrac{1}{32}k_1k_2} + \tfrac{1}{4}k_1$$

$$= z^2 + (\tfrac{1}{8}k_1 + \tfrac{1}{4}k_2 - 2)z + (\tfrac{1}{8}k_1 - \tfrac{1}{4}k_2 + 1)$$

$$\alpha_c(z) = (z-z_1)(z-z_2) = z^2 - (z_1+z_2)z + z_1z_2 = z^2 - z + 0.25$$

$$\Rightarrow \begin{cases} \tfrac{1}{8}k_1 + \tfrac{1}{4}k_2 - 2 = -1 \\ \tfrac{1}{8}k_1 - \tfrac{1}{4}k_2 + 1 = 0.25 \end{cases} \Rightarrow \tfrac{1}{4}k_1 - 1 = -0.75, \quad \begin{matrix} k_1 = 1 \\ k_2 = 3.5 \end{matrix}$$

$\Rightarrow K = [1 \quad 3.5]$, $y(k) = [1 \quad 1]\, x(k) = x_1(k) + x_2(k) = 1$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.875 & 0.5625 \\ -0.25 & 0.125 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.125k_r \\ 0.25k_r \end{bmatrix} r(k)$$

$$\begin{cases} x_1(k+1) = 0.875x_1(k) + 0.5625x_2(k) + 0.125k_r \\ x_2(k+1) = -0.25x_1(k) + 0.125x_2(k) + 0.25k_r \\ x_1(k) + x_2(k) = 1 \end{cases}$$

$k_r = 0$, $K = [4 \quad 6]$

$$x(k+1) = [A-BK]x(k) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.125 \\ 0.25 \end{bmatrix} [4 \quad 6] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.75 \\ 1 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.25 \\ -1 & -0.5 \end{bmatrix}$$

$|A-BK| = 0$

(b) $S(2)=\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $K(1)=\left([0\ 1]\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+2\right)^{-1}[0\ 1]\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\-2&-1\end{bmatrix}$

$$=[0\ \ 0].$$

$$S(1)=\left(\begin{bmatrix}0&1\\-2&-1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}[0\ 0]\right)^T\begin{bmatrix}0&0\\0&0\end{bmatrix}\left(\begin{bmatrix}0&1\\-2&-1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}[0\ 0]\right)+2\begin{bmatrix}0\\0\end{bmatrix}[0\ 0]+\begin{bmatrix}2&0\\0&2\end{bmatrix}$$

$$=Q=\begin{bmatrix}2&0\\0&2\end{bmatrix},$$

$$K(0)=\left([0\ 1]\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+2\right)^{-1}[0\ 1]\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\-2&-1\end{bmatrix}$$

$$=\frac{1}{4}[-4\ \ -2]=[-1\ \ -\tfrac{1}{2}],$$

$$S(0)=\left(\begin{bmatrix}0&1\\-2&-1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}[-1\ -\tfrac{1}{2}]\right)^T\begin{bmatrix}2&0\\0&2\end{bmatrix}\left(\begin{bmatrix}0&1\\-2&-1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}[-1\ -\tfrac{1}{2}]\right)$$

$$+2\begin{bmatrix}-1\\-\frac{1}{2}\end{bmatrix}[-1\ -\tfrac{1}{2}]+\begin{bmatrix}2&0\\0&2\end{bmatrix}$$

$$=\begin{bmatrix}0&1\\-1&-\frac{1}{2}\end{bmatrix}^T\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\-1&-\frac{1}{2}\end{bmatrix}+\begin{bmatrix}2&1\\1&\frac{1}{2}\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}$$

$$=\begin{bmatrix}0&-1\\1&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\-1&-\frac{1}{2}\end{bmatrix}+\begin{bmatrix}4&1\\1&\frac{5}{2}\end{bmatrix}$$

$$=\begin{bmatrix}0&-2\\2&-1\end{bmatrix}\begin{bmatrix}0&1\\-1&-\frac{1}{2}\end{bmatrix}+\begin{bmatrix}4&1\\1&\frac{5}{2}\end{bmatrix}=\begin{bmatrix}2&1\\1&\frac{5}{2}\end{bmatrix}+\begin{bmatrix}4&1\\1&\frac{5}{2}\end{bmatrix}=\begin{bmatrix}6&2\\2&5\end{bmatrix}$$

$$u^*(0)=-K(0)x(0)=-[-1\ -\tfrac{1}{2}]\begin{bmatrix}1\\1\end{bmatrix}=\frac{3}{2}$$

$$x^*(1)=Ax(0)+Bu^*(0)=\begin{bmatrix}0&1\\-2&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\frac{3}{2}=\begin{bmatrix}1\\-\frac{3}{2}\end{bmatrix}$$

$$u^*(1)=-K(1)x^*(1)=-[0\ \ 0]\begin{bmatrix}1\\-\frac{3}{2}\end{bmatrix}=0$$

5. (a) $H(s) = \dfrac{50}{s(1+0.01s)(1+0.003s)}$. $w_c = 150$ rad/s .

$H_{ZOH}(s) = \dfrac{1-e^{-Ts}}{s} H(s)$, $T = 2ms = 2\times10^{-3}s$. $s = jw_c = j150$

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NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2015-2016

EE6203 – COMPUTER CONTROL SYSTEMS

November/December 2015 Time Allowed: 3 hours

$C(z) = \frac{1}{(1+2z^{-1})(1-z^{-1})}$

INSTRUCTIONS

1. This paper contains 5 questions and comprises 8 pages. $= \frac{A}{1+2z^{-1}} + \frac{B}{1-z^{-1}}$

2. Answer all 5 questions. $(2B-A)z^{-1} + (A+B) = 1$

3. All questions carry equal marks. $2B-A=0$ $B=\frac{1}{3}$

4. The Transform Table is included in Appendix A on pages 6 to 8. $A+B=1$ $A=\frac{2}{3}$

$\Rightarrow C(z) = \frac{2}{3}\frac{1}{1+2z^{-1}} + \frac{1}{3}\frac{1}{1-z^{-1}}$

1. (a) Consider a first order system that has the following transfer function:

$$G(s) = \frac{C(s)}{E(s)} = \frac{s+1}{s+3}.$$

$G(z) = \frac{1}{1+2z^{-1}}$ $C(k) = \frac{2}{3}(-2)^k + \frac{1}{3}(1)^k$

Given a sampling period T of 1 second, discretise the system using the forward difference method.

$C(z) = G(z)E(z) = \frac{1}{(1+2z^{-1})(1-z^{-1})}$

If the input signal $e(kT)$ to the system is a unit-step function, determine the output response sequence $c(kT)$ for $k = 0, 1, 2, 3$.

$C(kT) = \frac{2}{3}(-2)^k + \frac{1}{3}(1)^k$ (10 Marks)

(b) Given the following digital controller $C(0) = 1$,

$$D(z) = \frac{z - 0.5}{(z-1)(z+0.3)}$$

Show its implementation using the series programming approach.

(7 Marks)

(c) Give your comment on part 1(a) if the backward difference approach is used.

(3 Marks)

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2. (a) Given the following position control system for a robotic arm

$$G(s) = \frac{2}{s(s+6)^2}$$

find its discretized transfer function $G_{ZAS}(z)$ if a zero-order-hold is used. Assume that the desired sampling period is 0.2 s.

(10 Marks)

(b) For a finite settling time of k samples, we have the following deadbeat controller:

$$C(z) = \frac{1}{G_{ZAS}(z)}\left[\frac{z^{-k}}{1-z^{-k}}\right]$$

Determine the deadbeat controller for the position control system in part 2(a) for $k = 2$.

(6 Marks)

(c) Comment on how the time response of the controller in part 2(b) can be improved.

(4 Marks)

3. (a) A continuous-time system has a state-space representation given by

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $x_1(t)$ and $x_2(t)$ are the states, $u(t)$ and $y(t)$ are the input and output variables, respectively. Suppose that the system is sampled with a zero-order hold.

Obtain a discretised state-space model for the system in terms of the sampling period T.

(9 Marks)

(b) Determine the values of the sampling period T that make the discretised system obtained in part 3(a) unobservable.

(4

Marks)

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Note:Question No. 3 continues on Page 3

(c) A discrete-time system has a state-space representation given by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k) + du(k)$$

where $\mathbf{x}(k)$, $u(k)$ and $y(k)$ are the states, input and output variables, respectively. If

$$u(k) = r(k) - y(k)$$

where $r(k)$ is the reference input, obtain an overall closed-loop state-space representation of the following form:

$$\mathbf{x}(k+1) = \mathbf{A}_{cl}\mathbf{x}(k) + \mathbf{B}_{cl}r(k)$$
$$y(k) = \mathbf{C}_{cl}\mathbf{x}(k) + \mathbf{D}_{cl}r(k)$$

State the assumptions that you have made, if any.

(7 Marks)

4. (a) A discrete-time system has a state-space representation given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B}u(k)$$
$$y(k) = \mathbf{C}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.125 \\ 0.25 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, $u(k)$ and $y(k)$ are the states, input and output variables, respectively. A controller of the following form

$$u(k) = -\mathbf{K}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + k_r r(k)$$

is to be implemented, where $r(k)$ is the reference input.

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Note: Question No. 4 continues on Page 4

(i) Design k_r and $\mathbf{K}$ such that the closed-loop poles are at $z_{1,2} = 0.5 \pm j0.5$ and the final value of $y(k)$ is unity for a unit step input $r(k)$.

(9 Marks)

(ii) If $k_r = 0$ and $\mathbf{K} = [4 \quad 6]$, discuss the type of response that the closed-loop system will exhibit when subjected to a non-zero $\mathbf{x}(0)$.

(4 Marks)

(b) A process is described by the following state-space representation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The performance index for the system is given by

$$J = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}(N)\mathbf{x}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left(\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + ru^2(k)\right)$$

and the design equations for $k = N-1, \ldots, 1, 0$ are

$$\mathbf{K}(k) = \left(\mathbf{B}^T\mathbf{S}(k+1)\mathbf{B} + r\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A}$$

$$u^*(k) = -\mathbf{K}(k)\mathbf{x}(k)$$

$$\mathbf{S}(k) = \left(\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right)^T\mathbf{S}(k+1)\left(\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right) + r\mathbf{K}(k)^T\mathbf{K}(k) + \mathbf{Q}$$

If $\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{S}(N) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $r = 2$ and $N = 2$, find the optimal controls $u^*(0)$ and $u^*(1)$ such that J is minimised. Let the initial state $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(7 Marks)

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5. (a) A conveyor belt system uses a DC motor for direct drive for its position control and has the following loop transfer function:

$$H(s) = \frac{50}{s(1+0.01s)(1+0.003s)}$$

What is the system phase margin at $\omega_c = 150$ rad/s if a ZOH is used and the sampling period is 2 ms?

(10 Marks)

(b) Given a digital controller that has the following transfer function

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z}{z+0.9}$$

If the input is a unit impulse at $k = 0$, what is the output of this controller for the time steps $k = 0, 1, 2, \ldots, 6$ assuming that there is a rounding error of one decimal significant digit. Repeat for the case if it is a truncation error of one decimal significant digit.

(8 Marks)

(c) Comment on the results that you obtained in part 5(b).

(2 Marks)