LECTURE NO 3

9. Observability

Observability is a property of a system which is strongly related to the ability to determine the initial state $\mathbf{x}(0)$ on the basis of the input and output data. Consider,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) \qquad ....(9.1)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \qquad ....(9.2)$$

Assume that $y(0), y(1), \cdots y(N-1)$ are given for a certain finite *N*. To simply derivations, assume that $u(k) = 0, k \geq 0$.

$$\begin{cases} y(0) = \mathbf{Cx}(0) \\ y(1) = \mathbf{Cx}(1) = \mathbf{CAx}(0) \\ \vdots \\ y(N-1) = \mathbf{CA}^{N-1}\mathbf{x}(0) \end{cases}$$

$$\Rightarrow \left\{ \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{N-1} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} \right. \qquad .....(9.3)$$

In (9.3), $\mathbf{x}(0)$ can be obtained if and only if $\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{N-1} \end{bmatrix}$ has rank n, and this is true if and only if $N = n$, i.e. the observability matrix $\mathbf{W}_O$ has rank n.

$$\text{rank}\left(\mathbf{W}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \right) = n \qquad ..... (9.4)$$

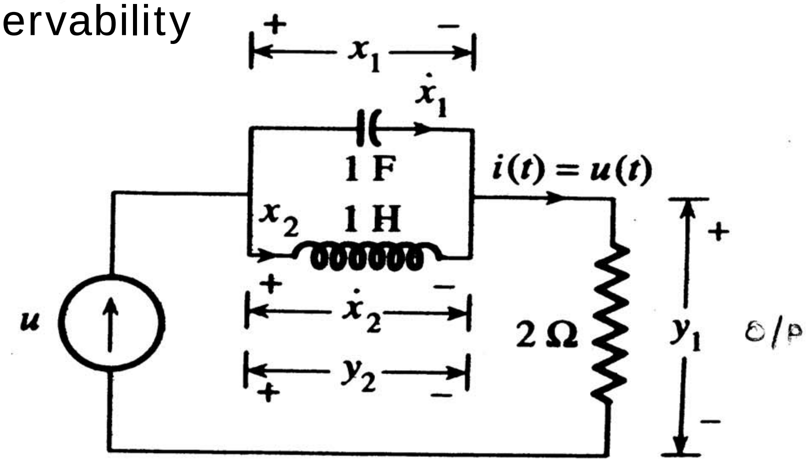
Definition :
The pair $(\mathbf{A}, \mathbf{C})$ is observable if there exists a finite *N* such that the knowledge of the inputs $u(0), u(1), ..., u(N-1)$ and the outputs $y(0), y(1), ..., y(N-1)$ are sufficient to determine the initial state $\mathbf{x}(0)$ of the system.

Theorem : The pair $(\mathbf{A}, \mathbf{C})$ is observable if and only if

$$\text{rank } \mathbf{W}_O = n$$

where *n* is the order of the system.

Example 3.1 Practical example on observability



State variables are :
- voltage $x_1(t)$ across the capacitor
- current $x_2(t)$ through the inductor.

Non-zero $x_1(0)$ and/or $x_2(0)$ will excite a response inside the LC loop. However, $i(t) = u(t)$ and the output $y_1(t) = 2u(t)$ always, no matter what $x_1(0)$ or $x_2(0)$ are. Therefore, there is no way to determine the initial state from $u(t)$ and $y_1(t)$. Thus, the system is not observable.

Note : Controllability and observability depends on what are considered as input and output. If $y_2(t)$ is considered as the output, then the system is observable.

Example 3.2 Consider the following system and investigate its observability.

$$\mathbf{A} = \begin{bmatrix} 1.1 & -0.3 \\ 1 & 0 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$$

$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 0.6 & -0.3 \end{bmatrix}$$

$$\Rightarrow \left|\mathbf{W}_O\right| = 0$$

Not observable.

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Example 3.3 Consider the following system and find the initial state if $u(0) = u(1) = 0$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$$

$$\mathbf{W}_O = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \left|\mathbf{W}_O\right| \neq 0$$

The system is observable and the initial state can be determined.

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$$(9.3) \Rightarrow \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\Rightarrow x_1(0) = 0.2; x_2(0) = 1$$

or

$$y(k) = \mathbf{C}\mathbf{x}(k) \Rightarrow y(0) = \mathbf{C}\mathbf{x}(0)$$

$$\Rightarrow 1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \Rightarrow x_2(0) = 1$$

$$y(1) = \mathbf{C}\mathbf{x}(1) \Rightarrow 1.2 = \mathbf{C}\left[\mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0)\right]$$

$$\Rightarrow 1.2 = x_1(0) + x_2(0) \Rightarrow x_1(0) = 0.2$$

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10.1 Relationship between controllability, observability and transfer function

Theorem :

If the input-output transfer function of a LTI system has pole-zero cancellations, the system will be either uncontrollable, unobservable, or both, depending on how the state variables are defined.

If the input-output transfer function does not have pole-zero cancellation, then the system can always be represented by dynamic equations as a completely controllable and observable system.

Example 3.4 Consider the following plant

$$\frac{Y(z)}{U(z)} = \frac{z+0.2}{(z+0.8)(z+0.2)} = \frac{z+0.2}{z^2+z+0.16}$$

CCF: Controllable but unobservable

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}; \mathbf{B}_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{C}_C = \begin{bmatrix} 0.2 & 1 \end{bmatrix}; d_C = 0$$

$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B}_C & \mathbf{A}_C\mathbf{B}_C \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C}_C \\ \mathbf{C}_C\mathbf{A}_C \end{bmatrix} = \begin{bmatrix} 0.2 & 1 \\ -0.16 & -0.8 \end{bmatrix}$$

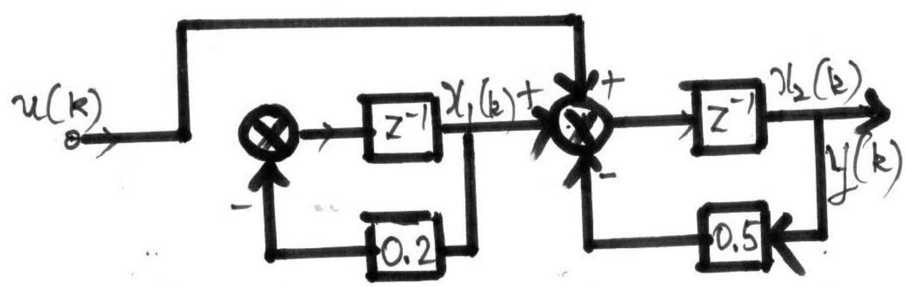
OCF: Observable but uncontrollable

$$\mathbf{A}_O = \begin{bmatrix} 0 & -0.16 \\ 0 & -1 \end{bmatrix}; \mathbf{B}_O = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}; \mathbf{C}_O = \begin{bmatrix} 0 & 1 \end{bmatrix}; d_O = 0$$

$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B}_O & \mathbf{A}_O\mathbf{B}_O \end{bmatrix} = \begin{bmatrix} 0.2 & -0.16 \\ 1 & -0.8 \end{bmatrix}$$

$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C}_O \\ \mathbf{C}_O\mathbf{A}_O \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Example 3.5



The states and output equations are

$$x_1(k+1) = -0.2x_1(k)$$

$$x_2(k+1) = x_1(k) - 0.5x_2(k) + u(k)$$

$$y(k) = x_2(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\mathbf{W}_C = \begin{bmatrix} 0 & 0 \\ 1 & -0.5 \end{bmatrix}$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}\left[z\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} = \frac{z+0.2}{(z+0.2)(z+0.5)}$$

Uncontrollable and there is a pole-zero cancellation.

10.2 Loss of Controllability and/or Observability due to Sampling

When sampling a continuous-time system, the resulting discrete-time system matrices depend on the sampling period *T.*

For a discretised system to be controllable, it is necessary that the initial continuous-time system to be controllable.

Controllability (and/or observability) of the discretised system may be lost for certain values of *T.*

Example 3.6 The state space model of a harmonic oscillator with transfer function

$$H(s) = \frac{\omega^2}{s^2+\omega^2}$$

is given as follows. Investigate the controllability and observability of the ZOH equivalent.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \omega \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$\Rightarrow \mathbf{x}(k+1)$$

$$= \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos\omega T \\ \sin\omega T \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\left|\mathbf{W}_C\right| = -\sin\omega T\left(1-\cos\omega T\right); \quad \left|\mathbf{W}_O\right| = \sin\omega T$$

The controllability and observability of the discrete-time system is lost when $\omega T = q\pi$ where q is an integer, although (which can be shown) the continuous-time system is both controllable and observable.

Example 3.7

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \Rightarrow \left|\mathbf{W}_O\right| = T$$

$\mathbf{W}_O$ is non-singular for all $T > 0$.
The system is observable for all $T > 0$.

11 State feedback Design

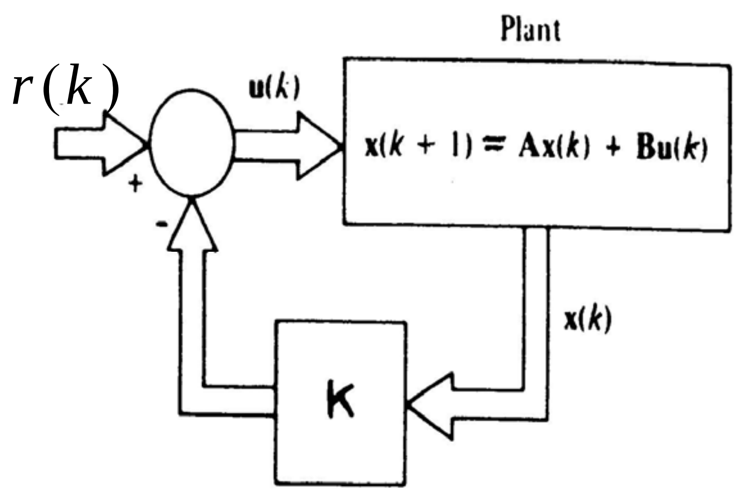


Figure 5. Closed-loop system with a linear state feedback law

The linear state feedback controller shown in Figure 5 can be expressed as follows.

$$u(k) = -\mathbf{K}\mathbf{x}(k)$$

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \qquad \text{...(11.1)}$$

It is called a regulator as $r(k) = 0$.
Substitute (11.1) into the state equation of the plant, the closed-loop system is then given by

$$\mathbf{x}(k+1) = \left[\mathbf{A} - \mathbf{B}\mathbf{K}\right]\mathbf{x}(k) \qquad \text{...(11.2)}$$

Problem Formulation :
Design a state feedback controller, i.e. find $\mathbf{K}$ in (11.1) such that the closed-loop poles in (11.2) are at desirable locations.

The characteristic polynomial of the closed-loop system is

$$\alpha(z) = \det\left[z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right]$$

$$= z^n + \hat{\beta}_1 z^{n-1} + \cdots + \hat{\beta}_{n-1} z + \hat{\beta}_n \qquad \text{...(11.3)}$$

Suppose that the closed-loop poles are desired to be at

$$p_i = p_1, p_2, \cdots, p_n \qquad ...(11.4)$$

Then, the desired closed-loop characteristic polynomial is

$$\alpha_C(z) = (z - p_1)(z - p_2)\cdots(z - p_n)$$

$$= z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n \quad ...(11.5)$$

Equate

(11.3) = (11.5)

And solve for $\mathbf{K}$.

The design method is also known as the pole placement method.

Question : Under what circumstances will there exists a state feedback matrix $\mathbf{K}$ such that the poles of the closed-loop system

$$\mathbf{x}(k+1) = [\mathbf{A} - \mathbf{BK}]\mathbf{x}(k)$$

can be placed arbitrary.

Answer : If the pair $(\mathbf{A}, \mathbf{B})$ is controllable.

Example 3.8 Consider the servo motor in Example 1.2 where T = 0.1 sec.

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) \quad ...(11.6)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \quad ...(11.7)$$

$x_1(k)$ - position of motor shaft
$x_2(k)$ - shaft velocity

Determine a state feedback gain matrix $\mathbf{K}$ such that the closed-loop poles are at

$0.888 \pm j0.173 = 0.905 \measuredangle \pm 0.193$ radians

These poles give a damping ratio and time constant, respectively, as

$$\zeta = -\frac{\ln\ r}{\sqrt{\left(\ln^2 r + \theta^2\right)}} = 0.46; \tau = -\frac{T}{\ln\ r} = 1 \text{ sec}$$

$$\mathbf{W}_C = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0.00484 & 0.0139 \\ 0.0952 & 0.0862 \end{bmatrix} \Rightarrow \left|\mathbf{W}_C\right| \neq 0$$

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$$\left[z\mathbf{I}-\mathbf{A}+\mathbf{BK}\right]$$

$$=\begin{bmatrix} z-1+0.00484k_1 & -0.0952+0.00484k_2 \\ 0.0952k_1 & z-0.905+0.0952k_2 \end{bmatrix}$$

$$\Rightarrow \alpha(z)=z^2+\left(0.00484k_1+0.0952k_2-1.905\right)z$$
$$+\left(0.00468k_1-0.0952k_2+0.905\right)$$

$$(11.5)\Rightarrow \alpha_C(z)=z^2-1.776z+0.819$$

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$$\alpha(z)\equiv\alpha_C(z)$$

$$\Rightarrow\begin{cases} 0.00484k_1+0.0952k_2-1.905=-1.776 \\ 0.00468k_1-0.0952k_2+0.905=0.819 \end{cases}$$

$$\Rightarrow\ k_1=4.52;\quad k_1=1.12$$

$$u(k)=-\begin{bmatrix}4.52 & 1.12\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$$
$$=-4.52x_1(k)-1.12x_2(k)$$

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11.1 Zeros with/without state feedback

Before state feedback, described by (11.6) and (11.7), the zeros polynomial

$$\left\| \begin{bmatrix} z\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \right\| = \left\| \begin{bmatrix} z-1 & -0.0952 & -0.00484 \\ 0 & z-0.905 & -0.0952 \\ 1 & 0 & 0 \end{bmatrix} \right\|$$

$$= 0$$

$$\Rightarrow z - 0.975 = 0$$

With following control law and the closed-loop system described, respectively, by

$$u(k) = -\begin{bmatrix} 4.52 & 1.12 \end{bmatrix} \mathbf{x}(k) + r(k)$$

$$\mathbf{x}(k+1) = \left[\mathbf{A} - \mathbf{BK}\right]\mathbf{x}(k) + \mathbf{B}r(k)$$

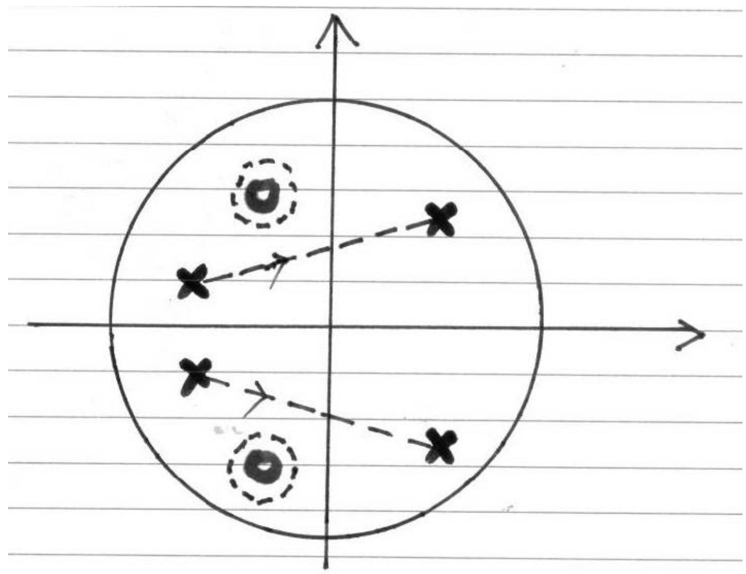
$$y(k) = \mathrm{C}\mathbf{x}(k)$$

$$\left[\mathbf{A} - \mathbf{BK}\right] = \begin{bmatrix} 0.9781 & 0.08978 \\ -0.4303 & 0.7984 \end{bmatrix}$$

The closed-loop zeros polynomial

$$\left\| \begin{bmatrix} z-0.9781 & -0.08978 & -0.00484 \\ 0.4303 & z-0.7984 & -0.0952 \\ 1 & 0 & 0 \end{bmatrix} \right\| = 0$$

$$\Rightarrow z - 0.975 = 0$$



Example 3.9 Consider

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) \qquad ...(11.8)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

A state feedback controller of the form

$u(k) = -[4.52 \quad 19.6694]\mathbf{x}(k) + r(k)$..(11.9)

is to be implemented. *r(k)* is the reference system input. Discuss the controllability and observability of the closed-loop system.

The open-loop plant is controllable. Substitute (11.9) into (11.8), the closed-loop system is shown to be controllable.

$$\mathbf{x}(k+1)=\begin{bmatrix}0.97812 & 0\\ -0.4303 & -0.96753\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0.00484\\ 0.0952\end{bmatrix}r(k)$$

$$\mathbf{W}_{C_{cl}}=\begin{bmatrix}\mathbf{B}_{cl} & \mathbf{A}_{cl}\mathbf{B}_{cl}\end{bmatrix}=\begin{bmatrix}0.00484 & 0.004734\\ -0.0952 & -0.09419\end{bmatrix}$$

$$\Rightarrow\left|\mathbf{W}_{C_{cl}}\right|=-9.07\text{ x }10^{-4}$$

$$\mathbf{W}_{O_{cl}}=\begin{bmatrix}\mathbf{C}_{cl}\\ \mathbf{C}_{cl}\mathbf{A}_{cl}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0.97812 & 0\end{bmatrix};\mathbf{C}_{cl}=\mathbf{C}$$

The closed-loop system is not observable. The closed-loop and open-loop transfer functions are, respectively, given by

$$\frac{Y(z)}{R(z)}=\mathbf{C}_{cl}\left[z\mathbf{I}-\mathbf{A}_{cl}\right]^{-1}\mathbf{B}_{cl}=\frac{0.00484\left(z+0.9676\right)}{\left(z-0.9781\right)\left(z+0.9676\right)}$$

$$\frac{Y(z)}{U(z)}=\mathbf{C}\left[z\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}=\frac{0.00484\left(z+0.9676\right)}{\left(z-1\right)\left(z+0.905\right)}$$

Example 3.10 Consider

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

The discrete-time state space model with sampling period *T* is

$$\mathbf{x}(k+1) = \begin{bmatrix} e^{-T} & 0 \\ 1-e^{-T} & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-e^{-T} \\ T-1+e^{-T} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Design a state feedback controller

$$u(k) = -[k_1 \quad k_2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

such that the closed-loop poles are at $-p_1$ and $-p_2 \Rightarrow \alpha_C(z) = z^2 + (p_1 + p_2)z + p_1 p_2$ and the CL characteristic polynomial is

$$\alpha(z) = z^2 + \left(-e^{-T} + \left(1-e^{-T}\right)k_1 - 1 + \left(T-1+e^{-T}\right)k_2\right)z + \left(\left(-e^{-T} + \left(1-e^{-T}\right)k_1\right)\left(-1 + \left(T-1+e^{-T}\right)k_2\right)\right) - \left(1-e^{-T}\right)k_2\left(e^{-T} - 1 + \left(T-1+e^{-T}\right)k_1\right) \qquad ...(11.10)$$

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$$\alpha(z) \equiv \alpha_C(z)$$

$$\begin{cases} \left(-e^{-T} + \left(1-e^{-T}\right)k_1 - 1 + \left(T-1+e^{-T}\right)k_2\right) = p_1 + p_2 \\ \left(\left(-e^{-T} + \left(1-e^{-T}\right)k_1\right)\left(-1+\left(T-1+e^{-T}\right)k_2\right)\right) - \\ \left(1-e^{-T}\right)k_2\left(e^{-T} - 1 + \left(T-1+e^{-T}\right)k_1\right) = p_1 p_2 \end{cases} \quad ...(11.11)$$

Solving the above for k_1 and k_2 is tedious especially if the system order is > 2. But if the state space model is in CCF form $(\mathbf{A}_C, \mathbf{B}_C)$, finding $\mathbf{K}$ is simpler. Then,

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$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$\left|z\mathbf{I} - \mathbf{A}_C\right| = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

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$$\mathbf{A}_C - \mathbf{B}_C\mathbf{K} =$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \cdots & -(a_{n-1} + k_n) \end{bmatrix}$$

$$\Rightarrow \alpha(z) = \left| z\mathbf{I} - \mathbf{A}_C + \mathbf{B}_C\mathbf{K} \right|$$

$$= z^n + (a_{n-1} + k_n) z^{n-1} + \cdots + (a_1 + k_2) z$$

$$+ (a_0 + k_1) \qquad ...(11.12)$$

$$\alpha_C(z) = (z - p_1)(z - p_2)\cdots(z - p_n)$$

$$= z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n \quad ...(11.13)$$

$$(11.12) \equiv (11.13)$$

$$\Rightarrow \quad k_{i+1} = \beta_{n-i} - a_i, \quad i = 0, 1, \cdots, n-1 \quad ...(11.14)$$

12 Ackermann's Formula

In general, a plant need not be represented by a state-space model in CCF.

A more practical procedure for calculating $\mathbf{K}$ is by use of Ackermann's formula which was derived by Ackermann in 1972.

The proof is omitted.

$$\alpha_C(z) = z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n$$

Let the desired characteristic polynomial be as given above.

Ackermann's formula for the controller :

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{W}_C^{-1} \alpha_C(\mathbf{A}) \quad ....(12.1)$$

$$\alpha_C(\mathbf{A}) = \mathbf{A}^n + \beta_1 \mathbf{A}^{n-1} + \cdots + \beta_{n-1}\mathbf{A} + \beta_n \mathbf{I}$$

Example 3.11 Consider the controller design problem in Example 3.8

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\alpha_C(z) = z^2 - 1.776z + 0.819$$

$$\Rightarrow \alpha_C(\mathbf{A}) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}^2 - 1.776 \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} + 0.819\mathbf{I}_2$$

$$\mathbf{W}_C^{-1} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix}^{-1} = \begin{bmatrix} -95.13 & 15.34 \\ 105.1 & -5.342 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{W}_C^{-1} \alpha_C(\mathbf{A}) = \begin{bmatrix} 4.52 & 1.12 \end{bmatrix}$$

Same answer as in Example 3.8

13 Deadbeat Control

$$\mathbf{x}(k+1) = [\mathbf{A} - \mathbf{BK}]\mathbf{x}(k)$$

The solution $\mathbf{x}(k)$ of the above closed-loop system is

$$\mathbf{x}(k) = [\mathbf{A} - \mathbf{BK}]^k \mathbf{x}(0)$$

If the eigenvalues are such that

$$\left|\lambda_i [\mathbf{A} - \mathbf{BK}]\right| < 1; \qquad i = 1, \cdots, n$$

Then, the closed-loop system is stable, i.e.

$$\mathbf{x}(k) \to 0 \quad \text{as} \quad k \to \infty$$

If

$\lambda_i[\mathbf{A} - \mathbf{BK}] = 0$ for all $i = 1, \cdots, n$ ...(13.1)

then the closed-loop system is stable and becomes 0 for $k \leq n$,

i.e. the closed-loop system will arrive and remain at rest in at most *n* sampling periods, after an impulse disturbance in the process state.

This type of control is called *deadbeat control.*

Deadbeat control is a phenomenon of discrete-time systems. Not in continuous-time systems.

To achieve deadbeat control, $\mathbf{K}$ must be such that (13.1) holds.
If the system is controllable, then the deadbeat controller is,

$$\alpha_C(z) = z^n = \det\left[z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right]$$

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{W}_C^{-1} \alpha_C(\mathbf{A})$$

$$\alpha_C(\mathbf{A}) = \mathbf{A}^n$$

To show deadbeat control when the (13.1) holds :

Consider a controllable system and there is a similarity transformation matrix that transforms the system into the CCF form.

$$\mathbf{w}(k) = \mathbf{P}^{-1}\mathbf{x}(k) \Rightarrow \mathbf{w}(k+1) = \mathbf{A}_C\mathbf{w}(k) + \mathbf{B}_C u(k)$$

$$u(k) = -\mathbf{K}\mathbf{x}(k) = -\mathbf{K}\mathbf{P}\mathbf{w}(k)$$

$$\Rightarrow \mathbf{w}(k+1) = \left[\mathbf{A}_C - \mathbf{B}_C\mathbf{K}\mathbf{P}\right]\mathbf{w}(k)$$

$$\alpha_C(z) = z^n;$$

$$(11.14) \Rightarrow \mathbf{K}\mathbf{P} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

$$= \begin{bmatrix} -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

$$[\mathbf{A}_C - \mathbf{B}_C\mathbf{KP}] =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -a_0 & \cdots & -a_{n-1} \end{bmatrix}$$

$$\Rightarrow [\mathbf{A}_C - \mathbf{B}_C\mathbf{KP}] = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

which is a nilpotent matrix.
A matrix $\mathbf{M}$ is called nilpotent if there is a positive n such that $\mathbf{M}^n = 0$.
In this case, $[\mathbf{A}_C - \mathbf{B}_C\mathbf{KP}]^n = 0$. Thus,

$$\mathbf{x}(n) = \left[\mathbf{A} - \mathbf{BK}\right]^n \mathbf{x}(0)$$
$$= \left[\mathbf{PA}_C\mathbf{P}^{-1} - \mathbf{PB}_C\mathbf{K}\right]^n \mathbf{x}(0)$$
$$= \left[\mathbf{P}\left(\mathbf{A}_C - \mathbf{B}_C\mathbf{KP}\right)\mathbf{P}^{-1}\right]^n \mathbf{x}(0)$$
$$= \mathbf{0}$$

Example 3.12 Consider a double integrator

$$Y(s) = \frac{1}{s^2}U(s) \;\; \Rightarrow \frac{d^2y(t)}{dt^2} = u(t)$$

with state variables as defined.

$$x_1(t) = y(t); \quad x_2(t) = \frac{dy(t)}{dt}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathbf{x}(k+1)=\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}T^2/2\\ T\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(k)$$

The equivalent ZOH discrete-time model can be obtained as given above.
Determine the controller $\mathbf{K}$ for deadbeat control.
Find $u(0)$ and $u(1)$ for various T if $\mathbf{x}(0)=[1 \quad 1]^T$.

$$\mathbf{W}_C^{-1}=\begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix}^{-1}=\begin{bmatrix}-T^2/2 & 3/2T\\ 1/T^2 & -1/2T\end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix}0 & 1\end{bmatrix}\mathbf{W}_C^{-1}\mathbf{A}^2=\begin{bmatrix}1/T^2 & 3/2T\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\mathbf{A}-\mathbf{BK}\end{bmatrix}=\begin{bmatrix}1/2 & T/4\\ -1/T & -1/2\end{bmatrix}$$

Note that $[\mathbf{A}-\mathbf{BK}]^q=0$ for some $q>1$
$\Rightarrow \mathbf{x}(k)=0$ for $k>1$.

The system will come to a standstill at the second sampling instant.

$$u(0) = -\mathbf{Kx}(0) = -\left(\frac{1}{T^2} + \frac{3}{2T} \right)$$

$$(4.3) \Rightarrow \ \mathbf{x}(k) = \left[\mathbf{A} - \mathbf{BK} \right]^n \mathbf{x}(0)$$

$$\Rightarrow \mathbf{x}(1) = \begin{bmatrix} \frac{1}{2} + \frac{T}{4} \\ -\frac{1}{T} - \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow u(1) = -\mathbf{Kx}(1) = \left(\frac{1}{T^2} + \frac{1}{2T} \right)$$

Table 1. Values of $u(0)$ and $u(1)$ for various T

T	2	1	0.5
$u(0)$	-1	-2.5	-7
$u(1)$	0.5	1.5	5

Figure 8 shows the trajectories of the output $y(kT)$ and input $u(kT)$ for T = 0.5, 1 and 2 secs when $\mathbf{x}(0) = [1 \quad 1]^T$.

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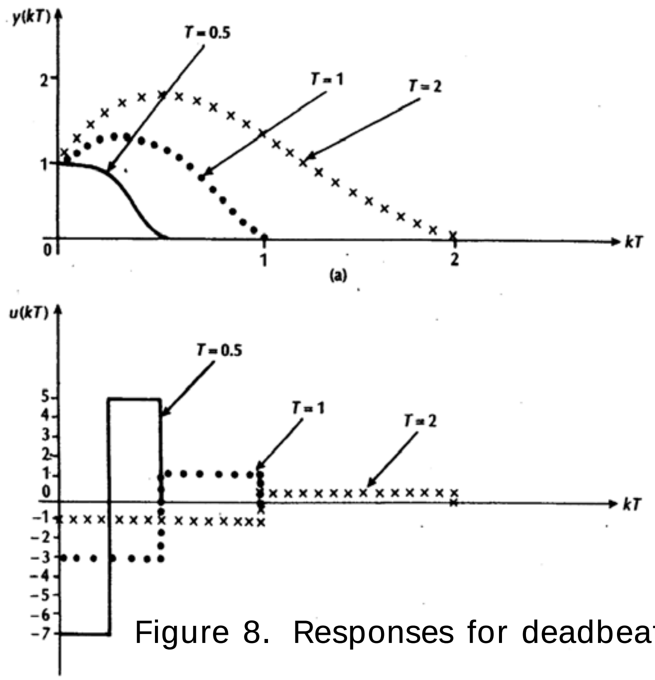

Figure 8. Responses for deadbeat control

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Observations :

The smaller the value of T, the larger the value of $u(kT)$ and vice versa – i.e. the faster we desired the response to settle to zero, the greater will be the control effort. It is thus important to choose the sampling period carefully when using deadbeat control.

e.g. select T to satisfy the Shannon sampling criterion while keeping the magnitude of $u(kT)$ to within acceptable limits.

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Because the error goes to zero in at most n sampling periods, the settling time is at most nT.
The settling time is thus proportional to the sampling period.
The sampling period also influence the magnitude of the control signal, as seen from the example.

Example 3.13 Determine the state feedback gain $\mathbf{K}$ to give a deadbeat system response to any $\mathbf{x}(0)$.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -0.2 & 1.1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\left[z\mathbf{I} - \mathbf{A} + \mathbf{BK} \right] = \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ k_1 + 0.5 & k_2 + 0.2 & z + k_3 - 1.1 \end{bmatrix}$$

$$\alpha(z) = z^3 + \left(k_3 - 1.1\right)z^2 + \left(k_2 + 0.2\right)z + \left(k_1 + 0.5\right)$$

$$\alpha_C(z) = z^3$$

$$\alpha(z) \equiv \alpha_C(z) \Rightarrow \mathbf{K} = \left[-0.5 \quad -0.2 \quad 1.1\right]$$

$$\mathbf{x}(k+1) = \left[\mathbf{A} - \mathbf{BK}\right]\mathbf{x}(k)$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x}(k)$$

The response to any $\mathbf{x}(0) = [a \quad b \quad c]^T$:

$$\begin{bmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ x_3(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} b \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(3) \\ x_2(3) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The response is clearly deadbeat.

$$x(k) = 0, k = 3, 4, 5, \ldots$$