Q1. (a)

| | 1 | 2 | 3 | S | $u_i$ | |
|---|---|---|---|---|---|---|
| 1 | 26 1 | 25 10 | 20 40 | 50 | 5 | 5 |
| 2 | 20 70 * | 23 3 | 18 3 | 70 | 0 | 2 |
| 3 | 24 5 | 24 55 | 20 1 | 60 | 4 | 4 |
| D | 75 | 65 | 40 | | | |
| $v_i$ | 20 | 20 | 15 | | | |

| | | | | |
|---|---|---|---|---|
| 1 | 26 | 25 | 10 | 1 |
| 2 | 20 70 | 23 | 70 | 3 |
| 3 | 24 | 24 | 60 | 0 |
| D | 75 | 65 | | |
| | 4 | 1 | | |

| | | | | |
|---|---|---|---|---|
| 1 | 26 | 25 | 10 | 1 |
| 3 | 24 5 | 24 | 60 | 0 |
| D | 5 | 65 | | |
| | 2 | 1 | | |

| | | |
|---|---|---|
| 1 | 25 | 10 |
| 3 | 24 | 55 |
| | 65 | |

(i)

$$z_{min} = 25\times 10 + 20\times 40 + 20\times 70 + 24\times 5 + 24\times 55 = 3890.$$

$c_{ij} - u_i - v_j$ for all non-basic variables, the solution is optimal.

(ii)

$$(26-\Delta)-5-20 \geq 0$$

$$\Rightarrow \Delta \leq 1$$

(b)

| | J1 | J2 | J3 | J4 | J5 |
|---|---|---|---|---|---|
| P1 | 20 | 15 | 16 | 17 | M |
| P2 | M | M | 16 | 20 | 17 |
| P3 | 12 | 15 | 11 | M | 17 |
| P4 | 13 | M | 14 | 20 | 12 |
| P5 | 0 | 0 | 0 | 0 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| P1 | 5 | 0 | 1 | 2 | M |
| P2 | M | M | 0 | 4 | 1 |
| P3 | 1 | 4 | 0 | M | 6 |
| P4 | 1 | M | 2 | 8 | 0 |
| P5 | 0 | 0 | 0 | 0 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| P1 | 4 | 0 | 1 | 0 | M |
| P2 | M | M | 0 | 2 | 1 |
| P3 | 0 | 4 | 0 | M | 6 |
| P4 | 0 | M | 2 | 6 | 0 |
| P5 | 0 | 0 | 0 | 0 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| P1 | 5 | 0* | 2 | 0 | M |
| P2 | M | M | 0* | 1 | 0 |
| P3 | 0* | 3 | 0 | M | 5 |
| P4 | 1 | M | 3 | 6 | 0* |
| P5 | 1 | 0 | 1 | 0* | 0 |

⇒ J1 → P3

J2 → P1

J3 → P2

J4 → P5

J5 → P4

Q2.

(a) length : $x_1$ . width : $x_2$ . height : $\frac{1}{3}x_1$

min. $f(x) = 2\times(x_1x_2 + \frac{1}{3}x_1^2 + \frac{1}{3}x_1x_2)$

$= \frac{8}{3}x_1x_2 + \frac{2}{3}x_1^2$

s.t. $x_1 x_2 \frac{1}{3}x_1 = \frac{1}{3}x_1^2x_2 = 115.2$

$x_1 , x_2 > 0$

(b) min. $f(x) = \frac{2}{3}x_1^2 + \frac{8}{3}x_1x_2$

s.t. $\frac{1}{3}x_1^2x_2 - 115.2 = 0$

先化简

$L = \frac{2}{3}x_1^2 + \frac{8}{3}x_1x_2 + \lambda(\frac{1}{3}x_1^2x_2 - 115.2)$

$\frac{\partial L}{\partial x_1} = \frac{4}{3}x_1 + \frac{8}{3}x_2 + \frac{2}{3}\lambda x_1 = 0 .$

$\frac{\partial L}{\partial x_2} = \frac{8}{3}x_1 + \frac{1}{3}\lambda x_1^2 = 0 .$

$\frac{\partial L}{\partial \lambda} = \frac{1}{3}x_1^2x_2 - 115.2 = 0 .$

$4x_1^2 + 8x_1x_2 + 2\lambda x_1^2 = 0$

$8x_1 + \lambda x_1^2 = 0$

$x_1^2x_2 = 345.6$

$4x_1^2 + 8x_1x_2 - 16x_1 = 0$

$x_1^2 + 2x_1x_2 - 4x_1 = 0$

$x_1^3 + 691.2 - 4x_1^2 = 0$

min. $f(x) = x_1^2 + 4x_1x_2$

s.t. $x_1^2x_2 = 345.6$

$x_1 , x_2 > 0$

$f_{min} = 2\times(x_1x_2 + \frac{1}{3}x_1^2 + \frac{1}{3}x_1x_2)$

$= 156.2912$

$L = x_1^2 + 4x_1x_2 + \lambda(x_1^2x_2 - 345.6)$

$\frac{\partial L}{\partial x_1} = 2x_1 + 4x_2 + 2\lambda x_1x_2 = 0 . \Rightarrow 2x_1 + 4x_2 - 8x_2 = 0 . \ x_1 = 2x_2$

$\frac{\partial L}{\partial x_2} = 4x_1 + \lambda x_1^2 = 0 . \Rightarrow x_1(\lambda x_1 + 4) = 0 . \ \lambda x_1 = -4 . \quad \lambda x_2 = -2$

$\frac{\partial L}{\partial \lambda} = x_1^2x_2 = 345.6 . \Rightarrow (2x_2)^2x_2 = 4x_2^3 = 345.6 , \quad x_2 = 4.42 . \ x_1 = 8.84 .$

$\lambda = -0.452$

$\nabla^T h(x) Y = [2x_1x_2 \quad x_1^2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2x_1x_2y_1 + x_1^2y_2 = 0 . \quad y_1 = -y_2$

$Y^T \nabla^2 L_{xx} Y = [y_1 \quad y_2]\begin{bmatrix} 2+2\lambda x_2 & 4+2\lambda x_1 \\ 4+2\lambda x_1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [y_1 \quad y_2]\begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= [-2y_1 - 4y_2 \quad -4y_1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -2y_1^2 - 8y_1y_2 = -2y_1^2 + 8y$

$= 6y_1^2 > 0$

AY 2017 - 2018

Q3. (a) state space $S = \{B_k : k = 0,1,2,3\}$.

$$P(A_k(t) = k) = \frac{e^{-\lambda t}(\lambda t)}{k!} = \frac{e^{-1}}{k!}.$$

① no demand: $P(A_k(t)=0) = e^{-1}$.

② one demands: $P(A_k(t)=1) = e^{-1}$.

③ two demands: $P(A_k(t)=2) = \frac{1}{2}e^{-1}$.

④ three and more demands: $P(A_k(t) \geq 3) = 1-\frac{5}{2}e^{-1}$

$B_0 = 2$

$P_{00} = \frac{1}{2}e^{-1} + 1 - \frac{5}{2}e^{-1} = 1-2e^{-1}$.

$P_{01} = P(A_k(t)=1) = e^{-1}$.

$P_{02} = P(A_k(t)=0) = e^{-1}$.

$P_{03} = 0$.

$P_{10} = P(A_k(t) \geq 3) = 1-\frac{5}{2}e^{-1}$.

$P_{11} = P(A_k(t)=2) = \frac{1}{2}e^{-1}$.

$P_{12} = P(A_k(t)=1) = e^{-1}$.

$P_{13} = P(A_k(t)=0) = e^{-1}$.

$P_{20} = P(A_k(t)=2) + P(A_k(t) \geq 3) = 1-2e^{-1}$,

$P_{21} = P(A_k(t)=1) = e^{-1}$.

$P_{22} = P(A_k(t)=0) = e^{-1}$.

$P_{23} = 0$

$P_{30} = P(A_k(t) \geq 3) = 1-\frac{5}{2}e^{-1}$.

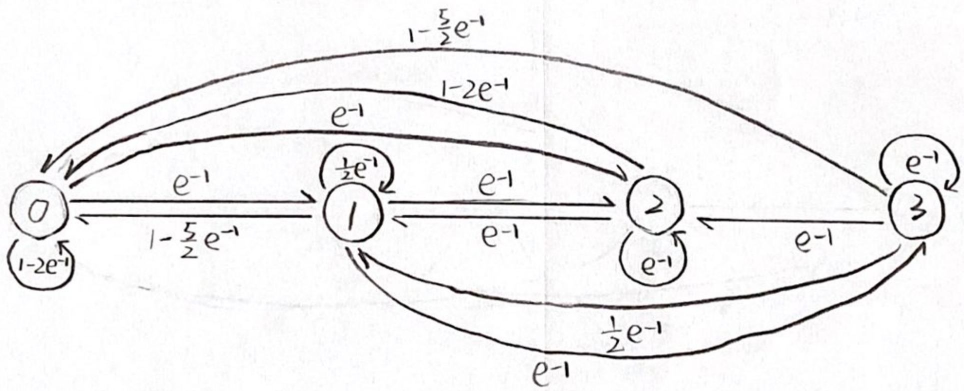
$P_{31} = P(A_k(t)=2) = \frac{1}{2}e^{-1}$.

$P_{32} = P(A_k(t)=1) = e^{-1}$.

$P_{33} = P(A_k(t)=0) = e^{-1}$.

TPM:

$$P = \begin{bmatrix} 1-2e^{-1} & e^{-1} & e^{-1} & 0 \\ 1-\frac{5}{2}e^{-1} & \frac{1}{2}e^{-1} & e^{-1} & e^{-1} \\ 1-2e^{-1} & e^{-1} & e^{-1} & 0 \\ 1-\frac{5}{2}e^{-1} & \frac{1}{2}e^{-1} & e^{-1} & e^{-1} \end{bmatrix}$$



(b) $\{B_k : k = 0, 1, 2, 3\}$.

$P(T_k > n) = P(B_n = k \mid B_{n-1} = k) P(B_{n-1} = k \mid B_{n-2} = k) \cdots = P_{kk}^{n}$.

$P(T_k = n) = \sum_{i=n}^{\infty} P(T_k = i) - \sum_{i=n+1}^{\infty} P(T_k = i)$

$= P(T_k > n-1) - P(T_k > n) = P_{kk}^{n-1} - P_{kk}^{n} = P_{kk}^{n-1}(1 - P_{kk})$.

$T_k$ is a geometric r.v. with success probability $(1 - P_{kk})$.

$E(T_k) = \frac{1}{1 - P_{kk}}$,

$E(T_0) = \frac{1}{1 - 1 + 2e^{-1}} = \frac{1}{2e^{-1}}$, $E(T_1) = \frac{1}{1 - \frac{1}{2}e^{-1}} = \frac{2}{2 - e^{-1}}$

$E(T_2) = \frac{1}{1 - e^{-1}}$, $E(T_3) = \frac{1}{1 - e^{-1}}$

(c) $\pi(n) = [P_0(n) \quad P_1(n) \quad P_2(n) \quad \cdots]$

$= \pi(0) P^n$

$\pi_j = \lim_{n \to \infty} P_j(n) = \lim_{n \to \infty} P_{ij}(n)$

$\Upsilon = \lim_{n \to \infty} \pi(n) = [\pi_0 \quad \pi_1 \quad \cdots \quad \pi_j \quad \cdots]$

$\Upsilon = \Upsilon P$, $\sum_j \pi_j = 1$.

**NANYANG TECHNOLOGICAL UNIVERSITY**

**SEMESTER 2 EXAMINATION 2017-2018**

**EE6204 – SYSTEMS ANALYSIS**

Apr/May 2018 Time Allowed: 3 hours

**INSTRUCTIONS**

1. This paper contains 5 questions and comprises 7 pages.
2. Answer ALL questions.
3. All questions carry equal marks.
4. Appendix A is given on pages 6 and 7.

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1. (a) A food manufacturing company has to produce a certain product to cater to the needs of three supermarkets over a period of time. The company has three factories to supply the product required by the supermarkets. The amount of the product required for each supermarket and the unit cost ($/Kg) in shipping the product from the three factories to the three supermarkets are given in Table 1.

**Table 1: Shipping cost ($) per Kg**

| | | Supermarket | | | Supply (Kg) |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| Factory | 1 | 26 | 25 | 20 | 50 |
| | 2 | 20 | 23 | 18 | 70 |
| | 3 | 24 | 24 | 20 | 60 |
| Demand (Kg) | | 75 | 65 | 40 | |

The company would like to minimize the total shipping cost.

Note: Question No. 1 continues on Page 2

(i) Use Vogel's method to construct an initial basic feasible solution and apply the transportation algorithm to obtain an optimal solution. Show all your calculations.

(ii) If the unit shipping cost of shipping the product from factory 1 to supermarket 1 is reduced by $\$\Delta$ /Kg, determine the range of values of $\Delta$ for which the current basic feasible solution in part 1(a)(i) remains optimal.

(17 Marks)

(b) An assignment problem is given in Table 2 with 4 persons ($P_1$, $P_2$, $P_3$, $P_4$) and 5 jobs ($J_1$, $J_2$, $J_3$, $J_4$, $J_5$), where the dash "-" in "$(i, j)$" cell means that the $i^{th}$ person cannot do the $j^{th}$ job. Create a Cost Tableau for this assignment problem. Note: a solution is not required.

**Table 2**

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|---|---|---|---|---|---|
| $P_1$ | 20 | 15 | 16 | 17 | - |
| $P_2$ | - | - | 16 | 20 | 17 |
| $P_3$ | 12 | 15 | 11 | - | 17 |
| $P_4$ | 13 | - | 14 | 20 | 12 |

(3 Marks)

2. A closed rectangular container is to be designed to have volume capacity of 115.2 cm³. One design requirement is that the height of the container must be one third of the length. The aim is to design the container with the least material surface possible.

min. area

(a) Formulate a nonlinear program for the above problem. You may let the length of the container be $x_1$ (cm) and the width be $x_2$ (cm).

(5 Marks)

(b) Apply the method of Lagrange multipliers to the nonlinear program in part 2(a) to determine the dimensions of the container (in cm) as well as the minimal area of material surface. Show all your steps.

(15 Marks)

仓库里2个　店里3个

3. A car dealer sells a brand of sports cars imported from Italy. Because of the high carrying costs, the shop only keeps a maximum of 3 cars in the shop at any time. Import orders can be placed in lots of two cars only and the order can be placed at the end of each month, when the number of cars left in the shop is less than 2. The ordered cars will arrive at the beginning of the following month. For $k = 1, 2, \ldots$, let

$A_k$ be the demand for the car in the $k^{th}$ month,
$B_k$ be the number of cars unsold at the end of the $k^{th}$ month.
left

Assume that $A_k$'s are independent and identical Poisson random variables with parameter $\lambda = 1$, and let $B_0 = 2$ be the initial number of cars in the shop at the $0^{\text{th}}$ month. It is known that $\{B_k\}$ forms a discrete-time Markov chain (DTMC) with transition probability matrix $P = [p_{ij}]$.

(a) Write down the state space $S$ for $\{B_k\}$ and find the transition probability matrix $P$ and draw the transition diagram.

(7 Marks)

(b) Derive an expression of the mean sojourn time for each state of $\{B_k\}$, in terms of the one-step state transition probabilities $p_{ij}$ from state $i$ to state $j$, and calculate the expected sojourn time for each state.

(10 Marks)

(c) Define the steady-state probability $\pi_j$ for state $j$, and write down the equations for the steady-state probabilities $\pi_j$. Solving the equations to obtain the $\pi_j$'s is NOT required.

(3 Marks)

4. (a) A copy of a popular magazine is placed in the reserve section of the library and readers can borrow the magazine to read. If the magazine has been loaned out, then only one reservation can be accepted. Any additional readers that come to make reservation will be turned away and be asked to come back again to make reservation.

Suppose the arrival of borrowers of the magazine follows a Poisson process with rate $\lambda$ and the duration of a loan is exponentially distributed with mean $\frac{1}{\mu}$. Draw the state transition rate diagram for this process. Write down the rate balance equations and find the steady-state probabilities.

Note: DO NOT use the formulae in Appendix A.

(9 Marks)

(b) A canteen has 3 counters selling vegetarian food, Chinese Food and Muslim food, respectively. Customers have to queue up in front of the respective counters for their choice of food. The arrival of customers for vegetarian food, Chinese food and Muslim food follow Poisson processes with rates 1, 3 and 2 per minute, respectively. The service time of each counter is exponentially distributed with a mean of $\frac{1}{4}$ minute. Find, for each counter,

$\mu = 4/min$

$\lambda_1 = 1/min$

$\lambda_2 = 3/min$

$\lambda_3 = 2/min$

(i) the proportion of time it has no customer;
(ii) the mean number of customers in the queue;
(iii) the mean waiting time in the queue.

Note: You may use the formulae in Appendix A.

(8 Marks)

(c) With reference to part 4(b), propose a method to run the canteen so as to reduce the mean number of customers in the queue as well as the mean waiting time in the queue.

(3 Marks)

5. A fund manager has 100 million dollars of funds to invest in stocks or bonds. He has been instructed to invest the entire amount for 1 year in either stocks or bonds (but not both) and then to reinvest the entire fund (including the returns/losses) in either stocks or bonds (but not both) for 1 year more. The objective is to maximize the expected monetary value of the fund at the end of the second year.

The annual rates of return on these investments depend on the economic environment, as shown in the following table:

| Economic Environment | Rate of Return | |
|---|---|---|
| | Stock | Bonds |
| Growth | 20% | 5% |
| Recession | -10% | 15% |

The probabilities of growth and recession for the first year are 0.7 and 0.3, respectively. If growth occurs in the first year, these probabilities remain the same for the second year. However, if a recession occurs in the first year, these probabilities change to 0.3 and 0.7, respectively, for the second year.

(a) Construct the decision tree for this problem.

(14 Marks)

(b) Quantitatively analyze the decision tree to identify the optimal policy.

(6 Marks)

**Appendix A**

M/M/1 Queue with Arrival Rate $\lambda$ and Service Rate $\mu$:

$$\rho = \frac{\lambda}{\mu}$$

$$\pi_0 = 1 - \rho$$

$$\pi_k = \rho^k (1-\rho), \qquad k \geq 1$$

$$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

$$Q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$W = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$$

$$D = W - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

M/M/1/N Queue with Arrival Rate $\lambda$ and Service Rate $\mu$:

$$\rho = \frac{\lambda}{\mu}$$

$$\pi_0 = \left( \sum_{k=0}^{N} \rho^k \right)^{-1} = \frac{1-\rho}{1-\rho^{N+1}}$$

$$\pi_k = \rho^k \pi_0 = \frac{\rho^k (1-\rho)}{1-\rho^{N+1}}, \qquad 0 \leq k \leq N$$

$$L = \frac{\rho\left[1-\rho^N - N\rho^N(1-\rho)\right]}{(1-\rho)(1-\rho^{N+1})}$$

$$Q = \frac{\rho^2\left[1-\rho^N - N\rho^{N-1}(1-\rho)\right]}{(1-\rho)(1-\rho^{N+1})}$$

$$W = \frac{1-\rho^N - N\rho^N(1-\rho)}{\mu(1-\rho)(1-\rho^{N+1})}$$

$$D = \frac{\rho\left[1-\rho^N - N\rho^{N-1}(1-\rho)\right]}{\mu(1-\rho)(1-\rho^{N+1})}$$

M/M/m Queue with Arrival Rate $\lambda$ and Service Rate $\mu$:

$$\rho = \frac{\lambda}{m\mu}$$

$$\pi_0 = \left[ \frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} \right]^{-1}$$

$$\pi_k = \pi_0 \begin{cases} \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m-1 \\ \dfrac{m^m \rho^k}{m!}, & k \ge m \end{cases}$$

$$L = \frac{\rho(m\rho)^m \pi_0}{m!(1-\rho)^2} + \frac{\lambda}{\mu}$$

$$Q = \sum_{k=m}^{\infty} (k-m)\pi_k = \frac{\rho(m\rho)^m \pi_0}{m!(1-\rho)^2}$$

$$W = \frac{L}{\lambda} = \frac{\rho(m\rho)^m \pi_0}{m!\lambda(1-\rho)^2} + \frac{1}{\mu}$$

$$D = W - \frac{1}{\mu} = \frac{\rho(m\rho)^m \pi_0}{m!\lambda(1-\rho)^2}$$

$M^b$/M/1 Queue with Arrival Rate $\lambda$ and Service Rate $\mu$:

$$\rho = \frac{b\lambda}{\mu}$$

$$\pi_0 = 1-\rho$$

$$\pi_k = \begin{cases} \left(\dfrac{\lambda+\mu}{\mu}\right)^{k-1} \dfrac{\lambda}{\mu}\pi_0, & 1 \le k \le b \\ \dfrac{\lambda+\mu}{\mu}\pi_{k-1} - \dfrac{\lambda}{\mu}\pi_{k-b-1}, & k \ge b+1 \end{cases}$$

$$L = \frac{\rho(1+b)}{2(1-\rho)}$$

$$Q = L - \rho = \frac{\rho(b-1+2\rho)}{2(1-\rho)}$$

$$W = \frac{L}{\lambda b} = \frac{1+b}{2\mu(1-\rho)}$$

$$D = W - \frac{1}{\mu} = \frac{b+2\rho-1}{2\mu(1-\rho)}$$

**END OF PAPER**