RANDOM PROCESSES

outline:

1. geometric random variable
2. exponential random variable
3. Stochastic process
4. possion process
5. discrete time Markov chain
6. Chapman-Kolmogorov equation
7. continuous time Markov chain
8. Chapman-Kolmogorov equation
9. Kolmogrov differential equation
10. birth-death process

1. geometric random variable X

$$P(X=k) = (1-p)^{k-1}p$$

1st	2nd	...	kth
fail	fail	...	success

e.g.

$$P(X=1) = p$$
$$P(X=2) = (1-p)p$$
$$P(X=3) = (1-p)^2 p$$

$$E(X) = \frac{1}{p}$$

memoryless (Markovian) property

$$P(X=m+n \mid X>m) = P(X=n)$$

Proof: $P(X=m+n \mid X>m)$

$$= \frac{P(X=m+n \cap X>m)}{P(X>m)}$$

$$= \frac{P(X=m+n)}{P(X>m)}$$

$$= \frac{(1-p)^{m+n-1}p}{(1-p)^m}$$

$$= (1-p)^{n-1}p = P(X=n)$$

2. exponential random variable X

$X = \exp(\lambda)$, λ is the rate

cdf: $F_X(x) = P(X \le x)$

$$= \begin{cases} 1-e^{-\lambda x}, & 0 \le x < \infty \\ 0, & x < 0 \end{cases}$$

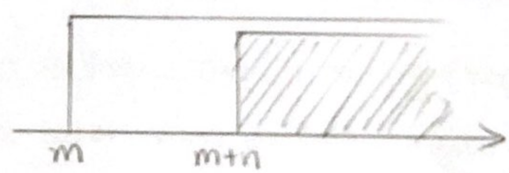
$$P(X>x) = 1 - P(X \le x) = e^{-\lambda x}, \quad x \ge 0.$$

pdf: $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

mean: $E(X) = \frac{1}{\lambda}$

memoryless property

$$P(X>x+y \mid X>x) = P(X>y),$$
$$P(X \le x+y \mid X>x) = P(X \le y),$$
$$x, y \ge 0$$



$\Rightarrow$ $P(X>m+n \mid X>m) = P(X>n)$,
$P(X \le m+n \mid X>m) = P(X \le n)$,
$m, n \ge 1$

3. stochastic process

def: a collection of random variable $\{X(t): t \in T\}$.

T – index set

t – index, of interpreted as time

T is a countable set – discrete time process

T is an interval of $\mathbb{R}$ – continuous time process

state space S – the set of all values $X(t)$ may assum

chain – discrete state space

4. Possion process

def: a collection of Possion r.v. $\{X(t): t>0\}$.

$$P(X(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k=0,1,2,\cdots$$

mean: $E(X(t)) = \lambda t$

Possion process is a continuous time, discrete state space process (continuous time chain).

superposition: the superposition of Possion processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_n$ is also a Possion process (with rate $\sum_{i=1}^{n} \lambda_i$).

decomposition: the original Possion process (with rate λ) is branched out into n streams, with rates $P_1\lambda, P_2\lambda, \cdots, P_n\lambda$, respectively.

5. Discrete-Time Markov Chain

def: DTMC $\{X_n : n \in N\}$ with countable state space S has the following Markov property:

$$P(X_n=j \mid X_{n-1}=i, X_{n-2}=i_2, \cdots X_0=i_n) = P(X_n=j \mid X_{n-1}=i)$$

$\Rightarrow$ given the current state $X_{n-1}=i$. the future $X_n=j$ is independent of the past $(X_{n-2}=i_2, \cdots, X_0=i_n)$.

n-step transition perbability

$$P_{ij}(m, m+n) = P(X_{m+n}=j \mid X_m=i)$$

homogeneous

for all $i, j \in S$ and $m, n \in N$, the transition probability $P_{ij}(m, m+n)$ is independent of m and only depend on n:

$$P_{ij}(m, m+n) = P_{ij}(n)$$

$$P_{ij} \equiv P_{ij}(1) = P(X_n=j \mid X_{n-1}=i)$$

transition probability matrix TPM

$$P = [P_{ij}] = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$0 \le P_{ij} \le 1, \quad i, j \in N$

$\sum_j P_{ij} = 1, \quad i \in N$

sojourn time

def: given a state i of a DTMC $\{X_n \in S : n \in N\}$. the sojourn time T_i of state i is the discrete r.v. that gives the # of time steps DTMC resides in state i before transiting to a different state.

T_i is a geometric r.v. with success probability $(1-P_{ii})$.

$$P(T_i = n) = P_{ii}^{n-1}(1-P_{ii})$$

mean: $E(T_i) = \frac{1}{1-P_{ii}}$.

$P_{ii} = 0 \Rightarrow E(T_i) = 1$

$P_{ii} = 1 \Rightarrow E(T_i) = \infty$

Example 5.1:

a machine has 2 states: working (state 0) and breakdown (state 1). The state is examined every hour.

the system can be formulated as a homogeneous DTMC $\{X_n \in S : n \in N\}$, where $S = \{0, 1\}$ and the time constants $t_0, t_1, t_2, \cdots$, correspond to 0, 1h, 2h, ⋯, respectively.

let a = probability that machine is failed in a given hour $= P_{01}$.

b = probability that the failed machine gets repaired in a given hour $= P_{10}$.

TPM:

$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

$0 \le a, b \le 1$

case 1: $a = b = 0$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

mean sojourn time:

$$E(T_i) = (1-P_{ii})^{-1} \cdot E(T_0) = \infty = E(T_1).$$

case 2: $a = b = 1$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$E(T_i) = (1-P_{ii})^{-1} \cdot E(T_0) = 1 = E(T_1).$$

case 3: $a, b \in (0,1)$

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$E(T_i) = (1-P_{ii})^{-1} \cdot E(T_0) = a^{-1}.$$

$$E(T_1) = b^{-1}.$$

6. Chapman-Kolmogorov equation

for $m, n \geq 0$, $i, j \in S$:

$$P_{ij}(m+n) = P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j, X_m = k \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j \mid X_m = k, X_0 = i) \cdot P(X_m = k \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j \mid X_m = k) P(X_m = k \mid X_0 = i)$$

$$= \sum_{k \in S} P_{kj}(n) P_{ik}(m)$$

$\Rightarrow$ $P(m+n) = P(m) P(n)$.

n-step transition probabilities:

$P(n) = [P_{ij}(n)]$, $P(0) = I$,

let $m = n-1$, $n = 1$.

$$\underline{P_{ij}(m+n) = \sum_{k \in S} P_{ik}(m) P_{kj}(n)}$$

$\Rightarrow$ $P_{ij}(n) = \sum_{k \in S} P_{ik}(n-1) P_{kj}(1)$

or $P(n) = P(n-1) \cdot P$, $n \geq 1$

$\Rightarrow$ $P(n) = P^n$, $n \geq 1$

State Probabilities

assume $S = \{0, 1, 2, \cdots\}$.

$P_j(n) = P(X_n = j)$, $n = 0, 1, 2, \cdots$
$j = 0, 1, 2, \cdots$,

by total probability theorem:

$$P_j(n) = P(X_n = j)$$

$$= \sum_{i \in S} P(X_n = j, X_0 = i)$$

$$= \sum_{i \in S} P(X_n = j \mid X_0 = i) P(X_0 = i)$$

$$= \sum_{i \in S} P_i(0) P_{ij}(n), \quad j = 0, 1, 2, \cdots$$

let $\pi(n) = [P_0(n) \; P_1(n) \; P_2(n) \cdots]$.

$\pi(n)$ gives the pmf of r.v. X_n.

$\pi(0)$ gives the initial r.v. X_0.

$$\underline{\pi(n) = \pi(0) P(n) = \pi(0) P^n}, \quad n = 0, 1, 2, \cdots$$

thus, knowing $\pi(0)$ and P, we can compute $\pi(n)$.

Steady State analysis

assume the following limiting probabilities exist and are unique

$$\lim_{n\to\infty} P_j(n), \quad j = 0, 1, \cdots$$

$$\lim_{n\to\infty} P_{ij}(n), \quad i, j = 0, 1, \cdots$$

⇒ for any j:

$$y_j = \lim_{n\to\infty} P_j(n) = \lim_{n\to\infty} P_{ij}(n).$$

$$Y = \lim_{n\to\infty} \pi(n) = [y_0 \ y_1 \ y_2 \ \cdots]$$

is called the vector of the steady-state probabilities.

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} P(n) = \begin{bmatrix} Y \\ Y \\ \vdots \\ Y \end{bmatrix}.$$

$$\pi(n) = \pi(0)P^n = \pi(0)P^{n-1}P = \pi(n-1)P.$$

so $\lim_{n\to\infty} \pi(n) = \lim_{n\to\infty} \pi(n-1)P$

⇒ $Y = YP$.

with $\sum_j y_j = 1$, $y_j \geq 0$, $j \geq 0$.

therefore, we can compute Y.

Example 6.1

TPM: $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$, $0 \le a, b \le 1$

case 1: $a = b = 0$

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

$P(n) = P^n = I$.

$\pi(n) = \pi(0) P(n) = \pi(0) P^n = \pi(0)$

⇒ if the initial state $X_0 = 0$, then $\pi(0) = [1 \ \ 0] = \pi(n)$, the system will remain forever in state 0;

if $X_0 = 1$, then $\pi(0) = [0 \ \ 1]$, the system will remain forever in state 1.

case 2: $a = b = 1$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$,

$P^3 = P^2 P = P$,

$P^4 = P^3 P = P^2 = I$, ...

⇒ $P^n = \begin{cases} I, & \text{if } n \text{ is even} \\ P, & \text{if } n \text{ is odd} \end{cases}$

Suppose $X_0 = [1 \ \ 0] = \pi(0)$,

$\pi(n) = \pi(0) P^n = \begin{cases} \pi(0) I, & \text{even} \\ \pi(0) P, & \text{odd} \end{cases}$

$= \begin{cases} [1 \ \ 0], & \text{if } n \text{ is even} \\ [0 \ \ 1], & \text{if } n \text{ is odd} \end{cases}$

the system will be in state 0 after even # of steps, and in state 1 after odd # of steps.

case 3: $a, b \in (0, 1)$, $|1-a-b| < 1$

$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$,

$P^n = \begin{bmatrix} \frac{b + a x^n}{a+b} & \frac{a - a x^n}{a+b} \\ \frac{b - b x^n}{a+b} & \frac{a + b x^n}{a+b} \end{bmatrix}$,

where $x = 1-a-b$, $|x| < 1$

$\lim_{n \to \infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

⇒ if the initial state is 0, then $\pi(0) = [1 \ \ 0]$,

$\pi(n) = \pi(0) P^n = \left[\frac{b + a x^n}{a+b} \ \ \frac{a - a x^n}{a+b}\right]$

if the initial state is 1, then $\pi(0) = [0 \ \ 1]$,

$\pi(n) = \pi(0) P^n = \left[\frac{b - b x^n}{a+b} \ \ \frac{a + b x^n}{a+b}\right]$

as $n \to \infty$, both $\pi(n)$ give:

$\pi(\infty) = \lim_{n \to \infty} \pi(n) = \left[\frac{b}{a+b} \ \ \frac{a}{a+b}\right]$

the DTMC settles down to a behavior whereby it visits state 0 $\frac{b}{a+b}$ of time, and state 1 $\frac{a}{a+b}$ of time.

⇒ steady-state probability or limiting probability of state 0 and state 1.

7. CTMC

def: a continuous time discrete state space process $\{X(t): t \geq 0\}$ with state space S is called CTMC if the following Markov property is satisfied:

for all $s \geq 0$, $u \geq 0$, $t > s$, and $i, j, x(u) \in S$:

$$P(X(t)=j \mid X(s)=i, X(u)=x(u) \text{ for } 0 \leq u < s)$$
$$= P(X(t)=j \mid X(s)=i).$$

$P_{ij}(s,t) = P(X(t)=j \mid X(s)=i)$ are called transition probabilities corresponding to states $i, j \in S$ and $t \geq s \geq 0$.

$$P_{ij}(s,s) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}.$$

A CTMC is said to be homogeneous or to have stationary transition probabilities if:

for each $t \geq s \geq 0$, $P_{ij}(s,t)$ does NOT depend on s, and only depends on $t-s$.

$$\Rightarrow P_{ij}(s,t) = P_{ij}(t-s)$$
$$= P(X(u+t-s)=j \mid X(u)=j)$$

sojoum times in states

def: given a state i of a CTMC $\{X(t): t \geq 0\}$, sojourn time T_i of state i is a continuous r.v. that denotes the span of time.

T_i is an exponential r.v.

$P(T_i > s+x \mid T_i > s) = h(x)$,

where $h(x)$ is a function of x only.

$$P(T_i > s+x \mid T_i > s)$$
$$= \frac{P(T_i > s+x, T_i > s)}{P(T_i > s)}$$
$$= \frac{P(T_i > s+x)}{P(T_i > s)} = h(x),$$

letting $s=0$ and $P(T_i > 0) = 1$:

$$\Rightarrow h(x) = P(T_i > x)$$
$$= e^{-\lambda_i x}, \quad x \geq 0$$

if $\lambda_i = 0$: absorbing state;

if $\lambda_i = \infty$: instantaneous state;

if $\lambda_i \in (0, \infty)$: stable state.

8. C-K eqn

def: consider a CTMC $\{X(t): t \geq 0\}$ with state space $\{0, 1, 2, \cdots\}$.

$$P_{ij}(s,t) = P(X(t)=j \mid X(s)=i)$$

can be expressed as matrix form:

$$H(s,t) = [P_{ij}(s,t)].$$

$$H(s,s) = I.$$

for $0 \leq s \leq u \leq t$.

$$
\begin{aligned}
P_{ij}(s,t) &= P(X(t)=j \mid X(s)=i) \\
&= \sum_{k \in S} P(X(t)=j, X(u)=k \mid X(s)=i) \\
&= \sum_{k \in S} P(X(t)=j \mid X(u)=k, X(s)=i) \cdot P(X(u)=k \mid X(s)=i) \\
&= \sum_{k \in S} P(X(t)=j \mid X(u)=k) \cdot P(X(u)=k \mid X(s)=i) \\
&= \sum_{k \in S} P_{ik}(s,u) P_{kj}(u,t)
\end{aligned}
$$

⇒ matrix form:

$$H(s,t) = H(s,u) H(u,t), \quad 0 \leq s \leq u \leq t$$

9. Kolmogorov Differential Eqn

def: let h be an infinitesimal increment in time. $t = t+h$, $u = t$.

$$H(s, t+h) = H(s,t) H(t, t+h).$$

$$H(s,t+h) - H(s,t) = H(s,t) H(t, t+h) - H(s,t)$$

$$H(s,t+h) - H(s,t) = H(s,t) [H(t,t+h) - I]$$

$$
\begin{aligned}
&\lim_{h \to 0} \frac{H(s,t+h) - H(s,t)}{h} \\
&= H(s,t) \lim_{h \to 0} \frac{H(t,t+h) - I}{h} \\
&= H(s,t) Q(t),
\end{aligned}
$$

⇒ partial differential eqn:

$$\frac{\partial H(s,t)}{\partial t} = H(s,t) Q(t),$$

with initial state $H(s,s) = I$, is called the forward Kolmogorov equation.

$Q(t)$ is called the infinitesimal generator of the CTMC, and is also called transition rate matrix.

take $u = s+h$:

⇒ backward Kolmogorov eqn:

$$\frac{\partial H(s,t)}{\partial s} = -Q(s) H(s,t)$$

with initial condition $H(s,s) = I$.

Interpretation of $Q(t)$:

$$Q(t) = \lim_{h\to 0} \frac{H(t, t+h) - I}{h}$$
$$= [q_{ij}(t)].$$

$$q_{ii}(t) = \lim_{h\to 0} \frac{P_{ii}(t, t+h) - 1}{h}$$

$$q_{ij}(t) = \lim_{h\to 0} \frac{P_{ij}(t, t+h)}{h}, \quad i \neq j$$

is the rate at which CTMC moves from state i to state j $(i \neq j)$ in the time interval $(t, t+h)$.

since $\sum_j P_{ij}(s,t) = 1$:

$$q_{ii}(t) + q_{ij}(t) = \sum_j q_{ij}(t) = 0, \quad \text{for all } i$$

⇒ the sum of any row of $Q(t)$ is zero.

Homogeneous CTMC:

def: the transition probabilities $P_{ij}(x, x+t)$ do not depend on x and only depend on t.

$$P_{ij}(t) = P_{ij}(x, x+t).$$

$$H(t) = H(x, x+t) = [P_{ij}(t)].$$

$$q_{ij} = q_{ij}(x).$$

$$Q = Q(x) = [q_{ij}].$$

⇒ $H(x+t) = H(x)\,H(t)$

$$\frac{dH(t)}{dt} = H(t)\,Q, \quad H(0) = I$$

solution: $H(t) = \exp(Qt)$

$$= I + Qt + Q^2\frac{t^2}{2!} + Q^3\frac{t^3}{3!} + \cdots$$

State Probabilities:

assume the state space is $S = \{0, 1, 2, \cdots\}$.

$$P_j(t) = P(X(t) = j)$$

let $\pi(t) = [P_0(t)\ P_1(t)\ P_2(t)\ \cdots]$.

by the total probability thm:

$$P_j(t) = P(X(t) = j)$$
$$= \sum_{i\in S} P(X(t) = j, X(0) = i)$$
$$= \sum_{i\in S} P(X(t) = j \mid X(0) = i)\,P(X(0) = i)$$
$$= \sum_{i\in S} P_i(0)\,P_{ij}(t)$$

⇒ matrix form:

$$\pi(t) = \pi(0)\,H(t)$$
$$= \pi(0)\exp(Qt).$$

$$\frac{d\pi(t)}{dt} = \pi(0)\exp(Qt)\,Q$$
$$= \pi(t)\,Q.$$

Steady State Analysis: $(t \to \infty)$

$$\pi_j = \lim_{t\to\infty} P_j(t), \quad j = 0, 1, \cdots$$

is the long-run proportion of residence time in state j, which satisfies:

1. $\sum_j \pi_j = 1$;
2. unique, independent of the initial state.

$$\pi = [\pi_0\ \pi_1\ \pi_2\ \cdots]$$

is the steady-state probability distribution of the CTMC.

to obtain vector π, set:

$$\frac{d\pi(t)}{dt} = 0 \quad , \quad \pi(t)\, Q = 0 \, ,$$

hence, π can be obtained as the solution of:

$$\begin{cases} \pi Q = 0 \, , \\ \sum_j \pi_j = 1 \, , \\ \pi_j \geq 0 \, , \; j = 0, 1, 2, \ldots \end{cases}$$

$$\pi Q = 0 \Rightarrow q_{jj}\pi_j + \sum_{k \neq j} q_{kj}\pi_k = 0 \qquad = \sum_k q_{jk}\pi_j$$

$$\Rightarrow \pi_j \Big(\sum_{k \neq j} q_{jk} \Big) = \sum_{k \neq j} q_{kj}\pi_k$$

Example 9.1:

states:

0: set up

1: processing

2: failed, being repaired

transition rate:

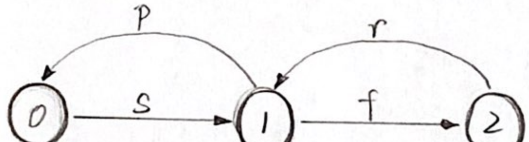
s: setup rate

p: processing rate

f: failure rate

r: repair rate

case 1: the system follows resume policy



$$Q = \begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} -s & s & 0 \\ p & -(p+f) & f \\ 0 & r & -r \end{bmatrix}$$

to find the steady-state probability vector $\pi = [\pi_0 \; \pi_1 \; \pi_2]$:

$\pi Q = 0$,

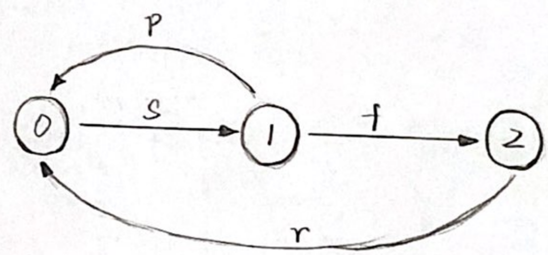
$$\begin{cases} \pi_0 s = \pi_1 p \\ \pi_1 (p+f) = \pi_0 s + \pi_2 r \\ \pi_2 r = \pi_1 f \end{cases} ,$$

$\pi_0 + \pi_1 + \pi_2 = 1$,

$$\Rightarrow \pi_0 = \frac{pr}{pr + rs + fs} .$$

$$\pi_1 = \frac{rs}{pr + rs + fs} \, , \quad \pi_2 = \frac{fs}{pr + rs + fs}$$

case 2: the system follows discard policy



$$Q = \begin{bmatrix} -s & s & 0 \\ p & -(p+f) & f \\ r & 0 & -r \end{bmatrix}$$

10、Birth-Death Process

a special case of CTMC.

def: a homogeneous CTMC $\{X(t): t \geq 0\}$ with state space $\{0, 1, 2, \dots\}$ is called a BD process if there exists constants λ_i and μ_i such that the transition rates are given by:

$$q_{i,i+1} = \lambda_i \ , \quad i = 0, 1, 2, \dots$$

$$q_{i,i-1} = \mu_i \ , \quad i = 1, 2, 3, \dots$$

$$q_{ij} = 0 \ , \quad |i-j| > 1$$

in state i, λ_i is birth rate, μ_i is death rate

Steady State Analysis:

rate balance eqn:

$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

$$(\mu_k + \lambda_k)\pi_k = \pi_{k-1}\lambda_{k-1} + \pi_{k+1}\mu_{k+1}$$

QUEUING MODELS

Outline:

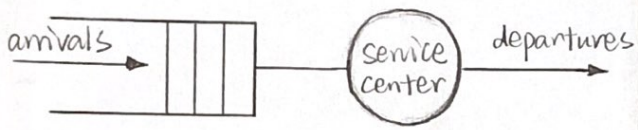
1. Queues
2. Little's result
3. M/M/1
4. M/M/1/N
5. M/M/m
6. M^b/M/1

1. Queues

def: a queue is a system into which customers arrive to receive service.

3 basic elements:

- arrivals
- service mechanism
- queueing discipline



notations:

- A: arrival process
- S: service distribution for the arriving customers
- m: no. of servers
- K: storage space
- L: customer population
- $C_1, C_2, \cdots$: stream of customers
- S_j: service time of C_j
- D_j: waiting time of C_j in queue
- $W_j = D_j + S_j$: waiting time of C_j in system

* in system: either in service or in the queue

2. Little's Result

notations:

- λ: arrival rate
- W: mean waiting time in system
- D: mean waiting time in queue
- L: mean no. of customers in the system
- Q: mean no. of customers in the queue

in the steady state: $(t \to \infty)$

$$L = \lambda W, \quad Q = \lambda D$$

M: Possion process with rate λ
arrival/ service time / # of service

Little's results:
$L = \lambda W$
$Q = \lambda D$

Appendix A

M/M/1 Queue with Arrival Rate λ and Service Rate μ:

$$
\begin{aligned}
\text{utilization: } \rho &= \frac{\lambda}{\mu} = \frac{\text{arrival rate}}{\text{service rate}} \\
\text{steady-state probability: } \pi_0 &= 1-\rho \\
\pi_k &= \rho^k(1-\rho), \qquad k \geq 1 \\
\text{mean \# of customers in system: } L &= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda} \\
\text{mean \# of customers in queue: } Q &= \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)} \\
\text{mean waiting-time in system: } W &= \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda} \\
\text{mean waiting-time in queue: } D &= D = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}
\end{aligned}
$$

M/M/1/N Queue with Arrival Rate λ and Service Rate μ:

production rate $U\mu$

最多同时服务N-1
(<N)

$$
\lambda_k = \begin{cases} \lambda, & 0 \leq k \leq N-1 \\ 0, & k \geq N \end{cases}
$$

超过N，停止服务

$$
\begin{aligned}
\rho &= \frac{\lambda}{\mu} \qquad U = 1-\pi_0 \\
\pi_0 &= \left(\sum_{k=0}^{N} \rho^k\right)^{-1} = \frac{1-\rho}{1-\rho^{N+1}} \\
\pi_k &= \rho^k \pi_0 = \frac{\rho^k(1-\rho)}{1-\rho^{N+1}}, \qquad 0 \leq k \leq N \\
L &= \frac{\rho[1-\rho^N - N\rho^N(1-\rho)]}{(1-\rho)(1-\rho^{N+1})} = \sum_{k=1}^{N} k\pi_k = \pi_1 + 2\pi_2 + 3\pi_3 + \cdots \\
Q &= \frac{\rho^2[1-\rho^N - N\rho^{N-1}(1-\rho)]}{(1-\rho)(1-\rho^{N+1})} \\
W &= \frac{1-\rho^N - N\rho^N(1-\rho)}{\mu(1-\rho)(1-\rho^{N+1})} = \frac{L}{\lambda} \\
D &= \frac{\rho[1-\rho^N - N\rho^{N-1}(1-\rho)]}{\mu(1-\rho)(1-\rho^{N+1})} = \frac{Q}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\text{Utilization } U &= \pi_1 + \pi_2 + \cdots + \pi_N \\
&= 1-\pi_0 \\
&= \frac{\rho(1-\rho^N)}{1-\rho^{N+1}}
\end{aligned}
$$

production rate: $(1-\pi_0)\mu$

$\lambda_k = \lambda, \ k \geq 0$

$$\mu_k = \begin{cases} k\mu, & 0 \leq k \leq m-1 \\ m\mu, & k \geq m \end{cases}$$

M/M/m Queue with Arrival Rate λ and Service Rate μ:

m个 servers，可同时服务m个

average # of busy servers in steady state:

$B = \frac{\lambda}{\mu}$

mean utilization of each server:

$U = \frac{\lambda}{m\mu} = \rho$

$$\rho = \frac{\lambda}{m\mu}$$

$$\pi_0 = \left[\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}\right]^{-1}$$

$$\pi_k = \pi_0 \begin{cases} \dfrac{(m\rho)^k}{k!}, & 0 \leq k \leq m-1 \\ \dfrac{m^m\rho^k}{m!}, & k \geq m \end{cases}$$

$$L = \frac{\rho(m\rho)^m\pi_0}{m!(1-\rho)^2} + \frac{\lambda}{\mu}$$

$$Q = \sum_{k=m}^{\infty}(k-m)\pi_k = \frac{\rho(m\rho)^m\pi_0}{m!(1-\rho)^2}$$

$$W = \frac{L}{\lambda} = \frac{\rho(m\rho)^m\pi_0}{m!\lambda(1-\rho)^2} + \frac{1}{\mu}$$

$$D = W - \frac{1}{\mu} = \frac{\rho(m\rho)^m\pi_0}{m!\lambda(1-\rho)^2}$$

M^b/M/1 Queue with Arrival Rate λ and Service Rate μ:

bulk size：一个batch中有b个 workpieces

rate-balance eqn:

$(\lambda+\mu)\pi_k = \lambda\pi_{k-b} + \mu\pi_{k+1}, \ k \geq b$

$(\lambda+\mu)\pi_k = \mu\pi_{k+1}, \ 1 \leq k \leq b-1$

$\lambda\pi_0 = \mu\pi_1$

utilization: $U = \rho = \frac{b\lambda}{\mu}$

$$\rho = \frac{b\lambda}{\mu}$$

$$\pi_0 = 1-\rho$$

$$\pi_k = \begin{cases} \left(\dfrac{\lambda+\mu}{\mu}\right)^{k-1}\dfrac{\lambda}{\mu}\pi_0 & 1 \leq k \leq b \\ \dfrac{\lambda+\mu}{\mu}\pi_{k-1} - \dfrac{\lambda}{\mu}\pi_{k-b-1} & k \geq b+1 \end{cases}$$

steady-state probabilities

$$L = \frac{\rho(1+b)}{2(1-\rho)}$$ mean # of customers in system

$$Q = L - \rho = \frac{\rho(b-1+2\rho)}{2(1-\rho)}$$ mean # of customers in queue

$$W = \frac{L}{\lambda b} = \frac{1+b}{2\mu(1-\rho)}$$ mean waiting time in system

$$D = W - \frac{1}{\mu} = \frac{b+2\rho-1}{2\mu(1-\rho)}$$ mean waiting time in queue