EE6204

Decision Making under Certainty

Example: car purchasing decision problem

- decision alternatives: [1]Accord, [2]Saturn, [3]Cavalier
- criteria: [1]price, [2]MPG, [3]comfort, [4]style

after verbal judgment & numerical rating:

	price	MPG	comfort	style
price	1	3	2	2
MPG	$\frac{1}{3}$	1	$\frac{1}{4}$	$\frac{1}{4}$
comfort	$\frac{1}{2}$	4	1	$\frac{1}{2}$
style	$\frac{1}{2}$	4	2	1
sum	2.333	12	5.25	3.75

①每列相加

normalized

	price	MPG	comfort	style	priority
price	0.429	0.25	0.381	0.533	0.398
MPG	0.143	0.083	0.048	0.067	0.085
comfort	0.214	0.333	0.190	0.133	0.218
style	0.214	0.333	0.381	0.267	0.299

②每个element除以列sum
③每行作平均
④均值×每列，相加
⑤和除以均值

consistency:

① $0.398\begin{bmatrix}1\\1/3\\1/2\\1/2\end{bmatrix} + 0.085\begin{bmatrix}3\\1\\4\\4\end{bmatrix} + 0.218\begin{bmatrix}2\\1/4\\1\\2\end{bmatrix} + 0.299\begin{bmatrix}2\\1/4\\1/2\\1\end{bmatrix}$

$= \begin{bmatrix}1.687\\0.347\\0.907\\1.274\end{bmatrix}$;

② price: $1.687 \div 0.398 = 4.239$
MPG: $0.347 \div 0.085 = 4.082$
comfort: $0.907 \div 0.218 = 4.161$
style: $1.274 \div 0.299 = 4.261$

③ $\lambda_{max} = \frac{1}{4} \times (4.239 + 4.082 + 4.161 + 4.261) = 4.186$ ⑥取平均

④ $CI = \frac{\lambda_{max} - n}{n-1} = \frac{4.186 - 4}{4-1} = 0.062$ (consistency index)

⑤ $CR = \frac{CI}{RI} = \frac{0.062}{0.90} = 0.0689 < 0.1$

n	3	4	5	6	7	8
RI	0.58	0.90	1.12	1.24	1.32	1.41

after pairwise comparison for each alternatives:

• price:

	A	S	C
A	1	1/3	1/4
S	3	1	1/2
C	4	2	1
sum	8	3.33	1.75

①求sum

	A	S	C	priority
A	0.125	0.1	0.143	0.123
S	0.375	0.3	0.286	0.320
C	0.5	0.6	0.572	0.557

②每个element除以sum

• MPG:

	A	S	C
A	1	1/4	1/6
S	4	1	1/3
C	6	3	1
sum	11	4.25	1.5

	A	S	C	priority
A	0.091	0.059	0.111	0.087
S	0.364	0.235	0.222	0.274
C	0.545	0.706	0.667	0.639

• Comfort:

	A	S	C
A	1	2	8
S	1/2	1	6
C	1/8	1/6	1
sum	1.625	3.167	15

	A	S	C	priority
A	0.615	0.632	0.533	0.593
S	0.308	0.316	0.4	0.341
C	0.077	0.052	0.067	0.196

• style:

	A	S	C
A	1	1/3	4
S	3	1	7
C	1/4	1/7	1
sum	4.25	1.476	12

	A	S	C	priority
A	0.235	0.226	0.333	0.265
S	0.706	0.677	0.583	0.655
C	0.059	0.097	0.083	0.080

	price	MPG	comfort	style
Accord	0.123	0.087	0.593	0.265
Saturn	0.320	0.274	0.341	0.655
Cavalier	0.557	0.639	0.196	0.080

overall priority:

$$\begin{bmatrix} 0.123 & 0.087 & 0.593 & 0.265 \\ 0.320 & 0.274 & 0.341 & 0.655 \\ 0.557 & 0.639 & 0.196 & 0.080 \end{bmatrix} \begin{bmatrix} 0.398 \\ 0.085 \\ 0.218 \\ 0.299 \end{bmatrix}$$

$$= \begin{bmatrix} 0.265 \\ 0.421 \\ 0.343 \end{bmatrix} \begin{matrix} \rightarrow \text{Accord} = 0.265 \\ \rightarrow \text{Saturn} = 0.421 \checkmark \\ \rightarrow \text{Cavalier} = 0.343 \end{matrix}$$

with equality & inequality:

$$\min.\ z = -x_1(30-x_1) - x_2(50-2x_2) + 3x_1 + 5x_2 + 10x_3$$

s.t. $x_1 + x_2 - x_3 \le 0$

$x_3 \le 17.25$

$$L = -30x_1 + x_1^2 - 50x_2 + 2x_2^2 + 3x_1 + 5x_2 + 10x_3 + \mu_1(x_1 + x_2 - x_3) + \mu_2(x_3 - 17.25)$$

$\frac{\partial L}{\partial x_1} = -30 + 2x_1 + 3 + \mu_1 = 0.$

$\frac{\partial L}{\partial x_2} = -50 + 4x_2 + 5 + \mu_1 = 0.$

$\frac{\partial L}{\partial x_3} = 10 + \mu_2 - \mu_1 = 0,$

$g_1 = x_1 + x_2 - x_3 \le 0.$

$g_2 = x_3 - 17.25 \le 0,$

$\mu_1 g_1 = \mu_1(x_1 + x_2 - x_3) = 0.$

$\mu_2 g_2 = \mu_2(x_3 - 17.25) = 0.$

$\mu_1, \mu_2 \ge 0.$

$$\min.\ f(x) = (x_1-1)^2 + x_2 - 2$$

s.t. $h(x) = -x_1 + x_2 - 1 = 0$

$g(x) = x_1 + x_2 - 2 \le 0$

$$L = (x_1-1)^2 + x_2 - 2 + \lambda(-x_1 + x_2 - 1) + \mu(x_1 + x_2 - 2)$$

$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - \lambda + \mu = 0.$ ①

$\frac{\partial L}{\partial x_2} = 1 + \lambda + \mu = 0.$ ②

$\frac{\partial L}{\partial \lambda} = -x_1 + x_2 - 1 = 0.$ ③

$g(x) = x_1 + x_2 - 2 \le 0.$ ④

$\mu g(x) = \mu(x_1 + x_2 - 2) = 0.$ ⑤

$\mu \ge 0$ ⑥

$\Rightarrow x_1 = 0.5,\ x_2 = 1.5,\ \lambda = -1,\ \mu = 0$

$$\nabla h^T Y = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -y_1 + y_2 = 0$$

$$Y^T \nabla^2_{xx} L\, Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 > 0$$

convex function:

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2)$$

$\Rightarrow$ H 正定/半正定

$SB_c \ge 0$

$y^T A^* - C_c^{*T} \ge 0$

Lagrange multipliers:

$$\max.\ f = -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 3x_2$$
$$\text{s.t.}\ \ 3x_1 + x_2 = 10$$

Standardized:

$$\min.\ f = 2x_1^2 + x_2^2 - x_1x_2 - 8x_1 - 3x_2$$
$$\text{s.t.}\ \ 3x_1 + x_2 = 10$$

$$L = f + \lambda h$$
$$= 2x_1^2 + x_2^2 - x_1x_2 - 8x_1 - 3x_2 + \lambda(3x_1 + x_2 - 10)$$

$$\frac{\partial L}{\partial x_1} = 4x_1 - x_2 - 8 + 3\lambda = 0 \quad ①$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - x_1 - 3 + \lambda = 0 \quad ②$$

$$\frac{\partial L}{\partial \lambda} = 3x_1 + x_2 - 10 = 0 \quad ③$$

② × 3 − ①, 消去 λ:

$$-7x_1 + 7x_2 - 1 = 0 \quad ④$$

③ and ④, 得到:

$$x_1 = \frac{69}{28}, \quad x_2 = \frac{73}{28}, \quad \lambda = \frac{1}{4}$$

is $[\frac{69}{28}, \frac{73}{28}]$ the minimum point?

$$\nabla h^T(x)\, Y = [3 \ \ 1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3y_1 + y_2 = 0$$

$$\nabla_{xx}^2 L = \begin{bmatrix} \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 \\ \partial^2 L/\partial x_2 \partial x_1 & \partial^2 L/\partial x_2^2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Y^T \nabla_{xx}^2 L\, Y = [y_1 \ \ y_2]\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= [4y_1 - y_2 \ \ -y_1 + 2y_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= 4y_1^2 - y_1y_2 - y_1y_2 + 2y_2^2$$
$$= 4y_1^2 - 2y_1y_2 + 2y_2^2$$
$$= 4y_1^2 - 2y_1(-3y_1) + 2(-3y_1)^2$$
$$= 4y_1^2 + 6y_1^2 + 18y_1^2$$
$$= 28y_1^2 > 0$$

$\Rightarrow$ x is a minimum point.

Fibonacci search:

$f(x) = x(5\pi - x)$, $x \in [0, 20]$, $\varepsilon = 1$

0 1 2 3 4 5 6 7

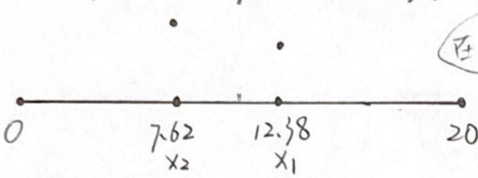
$F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$.

$b - a = 20 - 0 = 20$. $F_7 = 21 > 20$.

$\varepsilon' = \dfrac{b-a}{F_N} = \dfrac{20}{21} = 0.9524$.

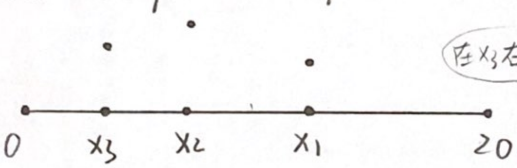
$F_{N-1} \cdot \varepsilon' = F_6 \times 0.9524 = 12.38$.

$x_1 = a + 12.38 = 12.38$.
$x_2 = 20 - 12.38 = 7.62$.
$f(x_1) = 12.38(5\pi - 12.38) = 41.20$.
$f(x_2) = 7.62(5\pi - 7.62) = 61.63$.



在x1右边
减

$F_{N-1} \cdot \varepsilon' = F_5 \times 0.9524 = 7.619$.
$x_3 = 12.38 - 7.619 = 4.761$
$f(x_3) = 4.761(5\pi - 4.761) = 52.12$.



在x3右边
加

$F_{N-1} \cdot \varepsilon' = F_4 \times 0.9524 = 5 \times 0.9524 = 4.762$.
$x_4 = x_3 + 4.762 = 9.523$
$f(x_4) = 58.9$

在x4右边
减

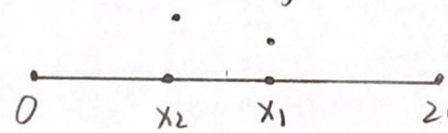
$F_{N-1} \cdot \varepsilon' = F_3 \times 0.9524 = 3 \times 0.9524 = 2.857$.
$x_5 = 9.523 - 2.857 = 6.666$.
$f(x_5) = 60.27$

在x5右边
加

Golden-section search:

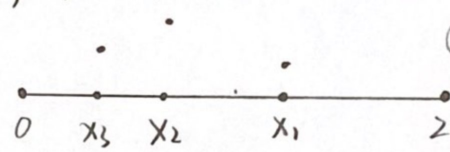
$f(x) = x(5\pi - x)$, $x \in [0, 20]$, $\varepsilon = 1$

$x_1 = a + 20 \times 0.618 = 12.36$,
$x_2 = b - 20 \times 0.618 = 7.64$,
$f(x_1) = 41.38$, $f(x_2) = 61.64$



在x1右边
减

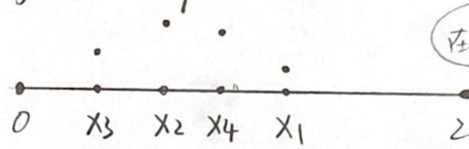
➡ $[0, 12.36]$, $L = 12.36 - 0 = 12.36$
$x_3 = 12.36 - L \times 0.618 = 4.722$,
$f(x_3) = 51.88$.



在x3右边
加

➡ $[4.722, 12.36]$, $L = 12.36 - 4.722 = 7.638$.

$x_4 = 4.722 + L \times 0.618 = 9.442$
$f(x_4) = 59.16$



在x4右边
减

➡ $[4.722, 9.442]$. $L = 4.720$
$x_5 = 9.442 - L \times 0.618 = 6.525$,
$f(x_5) = 59.92$.

在x5右边
加

➡ $[6.525, 9.442]$, $L = 2.917$
$x_6 = 6.525 + L \times 0.618 = 8.328$.
$f(x_6) = 61.46$.

在x6右边
减

➡ $[6.525, 8.328]$, $L = 1.803$.
$x_7 = 8.328 - L \times 0.618 = 7.214$
$f(x_7) = 61.28$

Newton's method:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

find min. of $f(x) = x^2 + \frac{54}{x}$, $x_0 = 2$,

Stop when $|x_{n+1} - x_n| < 10^{-5}$.

$$f'(x) = 2x - \frac{54}{x^2}, \quad f''(x) = 2 + 108\frac{1}{x^3}.$$

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 2 - \frac{4 - 54/4}{2 + 108/8} = 2.613.$$

Multivariable optimization:

$$f(x_1, x_2) = x_1^3 + 2x_1x_2 + x_2^2$$

$$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 + 2x_2 \\ 2x_1 + 2x_2 \end{bmatrix}.$$

$$\nabla^2 f = H = \begin{bmatrix} 6x_1 & 2 \\ 2 & 2 \end{bmatrix}.$$

Stationary point: $\nabla f(x^*) = 0$

$\nabla^2 f(x^*)$ 正定 $\Rightarrow$ minimum

$\nabla^2 f(x^*)$ 负定 $\Rightarrow$ maximum

$$x_{n+1} = x_n - \frac{\nabla f(x_n)}{\nabla^2 f(x_n)}$$

max. $z = -(x_1 - \sqrt{5})^2 - (x_2 - \pi)^2 - 10$.

$x_0 = [6.597, 5.891]^T$, $\varepsilon = 0.05$.

$$\nabla f = \begin{bmatrix} -2(x_1 - \sqrt{5}) \\ -2(x_2 - \pi) \end{bmatrix}, \quad H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

$$H^{-1} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}. \quad \nabla f(x_0) = \begin{bmatrix} -8.722 \\ -5.499 \end{bmatrix}.$$

$$x_1 = x_0 - \frac{\nabla f(x_0)}{H} = \begin{bmatrix} 6.597 \\ 5.891 \end{bmatrix} - \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} -8.722 \\ -5.499 \end{bmatrix}$$

$$= \begin{bmatrix} 2.236 \\ 3.142 \end{bmatrix}. \quad f(x_1) = -10, \quad f(x_1) - f(x_0) > 0.05.$$

$$\nabla f(x_1) = \begin{bmatrix} -0.0001 \\ 0.0008 \end{bmatrix}. \quad x_2 = x_1 - H^{-1}\nabla f(x_1)$$

Q1: decision hierarchy

Select the best job — overall goal

salary | advancement | location | training — criteria

salary	advancement	location	training
job A	job A	job A	job A
job B	job B	job B	job B
job C	job C	job C	job C

decision alternatives

② element 除以 sum
③ 每行作平均

Q2:

	salary	adv	loc	tra
salary	1	1/7	1/2	1/3
adv	7	1	5	3
loc	2	1/5	1	1/2
tra	3	1/3	2	1
① sum	13	1.676	8.5	4.833

salary	adv	loc	tra	supriority
0.077	0.085	0.059	0.069	0.0725
0.538	0.597	0.588	0.621	0.586
0.154	0.119	0.118	0.103	0.1235
0.231	0.199	0.235	0.207	0.218

$$0.0725\begin{bmatrix}1\\7\\2\\3\end{bmatrix}+0.586\begin{bmatrix}1/7\\1\\1/5\\1/3\end{bmatrix}+0.1235\begin{bmatrix}1/2\\5\\1\\2\end{bmatrix}+0.218\begin{bmatrix}1/3\\3\\1/2\\1\end{bmatrix}=\begin{bmatrix}0.291\\2.365\\0.495\\0.878\end{bmatrix}$$

salary: $0.291 \div 0.0725 = 4.014$,
adv: $2.365 \div 0.586 = 4.036$,
loc: $0.495 \div 0.1235 = 4.008$,
tra: $0.878 \div 0.218 = 4.027$.

$$\lambda_{max} = \frac{1}{4}\times(4.014+4.036+4.008+4.027) = 4.021$$

$$CI = \frac{\lambda_{max}-n}{n-1} = \frac{4.021-4}{4-1} = 0.007$$

$$CR = \frac{CI}{RI} = \frac{0.007}{0.90} = 0.0078 < 0.1$$

∴ the consistency is acceptable.

Q3:

$$\begin{bmatrix}0.2213 & 0.2063 & 0.4545 & 0.2748\\0.0934 & 0.7147 & 0.0909 & 0.6572\\0.6853 & 0.0788 & 0.4545 & 0.0682\end{bmatrix}\begin{bmatrix}0.0725 & 0.586 & 0.1235 & 0.218\end{bmatrix}^T$$

$$=\begin{bmatrix}0.2530\\0.5791\\0.1669\end{bmatrix}$$

∴ job B should be selected.

Q4. (a) M/M/1/1 Queue

arrival rate : λ

service rate : μ

$$\textcircled{0} \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \textcircled{1}$$

rate balance eqn : $\pi_0 \lambda = \pi_1 \mu$, $\pi_0 + \pi_1 = 1$.

$\Rightarrow \pi_1 = 1 - \pi_0$, $\pi_0 \lambda = (1-\pi_0)\mu = \mu - \pi_0 \mu$. $(\lambda+\mu)\pi_0 = \mu$,

$\Rightarrow \pi_0 = \dfrac{\mu}{\mu+\lambda}$, $\pi_1 = \dfrac{\lambda}{\mu+\lambda}$

(b) (i) V: $\rho = \dfrac{\lambda_1}{\mu} = \dfrac{1}{4}$. $\pi_0 = 1-\rho = \dfrac{3}{4}$.

C: $\rho = \dfrac{\lambda_2}{\mu} = \dfrac{3}{4}$. $\pi_0 = 1-\rho = \dfrac{1}{4}$.

M: $\rho = \dfrac{\lambda_3}{\mu} = \dfrac{1}{2}$. $\pi_0 = 1-\rho = \dfrac{1}{2}$.

(ii) V: $Q = \dfrac{\rho^2}{1-\rho} = \dfrac{\frac{1}{16}}{\frac{3}{4}} = \dfrac{1}{12}$.

C: $Q = \dfrac{\frac{9}{16}}{\frac{1}{4}} = \dfrac{9}{4}$.

M: $Q = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$.

(iii) V: $D = \dfrac{\lambda}{\mu(\mu-\lambda)} = \dfrac{1}{4\times 3} = \dfrac{1}{12}$

C: $D = \dfrac{3}{4\times 1} = \dfrac{3}{4}$

M: $D = \dfrac{2}{4\times 2} = \dfrac{1}{4}$

(c)