Q1. (a) (i) x_i : the amount of food F_i.

z : total cost.

$$\min.\quad z = \sum_{i=1}^{m} b_i x_i$$
$$\text{s.t.}\quad \sum_{i=1}^{m} a_{ij} x_i \geq c_j$$

(ii) $\min.\quad z = x_1 + 2x_2 + 0\cdot x_3 + 0\cdot x_4 + M\cdot \bar{x}_5$

s.t. $4x_1 + 3x_2 + x_3 = 12$

$x_1 - 2x_2 - x_4 + \bar{x}_5 = 1$

$x_1, x_2, x_3, x_4, \bar{x}_5 \geq 0$

$X = [x_1 \; x_2 \; x_3 \; x_4 \; \bar{x}_5]^T$.

$C = [1 \; 2 \; 0 \; 0 \; M]^T$.

$A = \begin{bmatrix} 4 & 3 & 1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 1 \end{bmatrix}$.

$B = \begin{bmatrix} 12 \\ 1 \end{bmatrix}$. $X_0 = \begin{bmatrix} x_3 \\ \bar{x}_5 \end{bmatrix}$. $C_0 = \begin{bmatrix} 0 \\ M \end{bmatrix}$.

$C^T - C_0^T A = [1 \; 2 \; 0 \; 0 \; M] -$

$[M \; -2M \; 0 \; -M \; M] = [1-M \;\; 2+2M \;\; 0 \;\; M \;\; 0]$. $-C_0^T B = -M$

	x_1	x_2	x_3	x_4	$\bar{x}_5$	
x_3	4	3	1	0	0	12
$\bar{x}_5$	1*	-2	0	-1	1	1
	1	2	0	0	0	0
	-1	2	0	1	0	-1

x_3	0	11	1	4	-4	8
x_1	1*	-2	0	-1	1	1
	0	4	0	1	-1	-1
	0	0	0	0	1	0

$x_3^* = 8$, $x_1^* = 1$, $z_{min} = 1$.

(b) $z = \frac{1}{2} x^T P x + q^T x + r$

$$= \frac{1}{2} [x_1 \ x_2 \ x_3] \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 22x_1 - 14.5x_2 + 13x_3 + 1$$

$$= \frac{1}{2} [13x_1 + 12x_2 - 2x_3 \quad 12x_1 + 17x_2 + 6x_3 \quad -2x_1 + 6x_2 + 12x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$L = \frac{13}{2} x_1^2 + 6x_1x_2 - x_1x_3 + 6x_1x_2 + \frac{17}{2} x_2^2 + 3x_2x_3 - x_1x_3 + 3x_2x_3 + 6x_3^2 - 22x_1 - 14.5x_2 + 13x_3 + 1 + \mu_1(x_1 - 1) + \mu_2(-x_1 - 1) + \mu_3(x_2 - 1) + \mu_4(-x_2 - 1) + \mu_5(x_3 - 1) + \mu_6(-x_3 - 1).$$

$g_1(x) = x_1 - 1 \le 0,$
$g_2(x) = -x_1 - 1 \le 0,$
$g_3(x) = x_2 - 1 \le 0,$
$g_4(x) = -x_2 - 1 \le 0,$
$g_5(x) = x_3 - 1 \le 0,$
$g_6(x) = -x_3 - 1 \le 0.$

$$\frac{\partial L}{\partial x_1} = 13x_1 + 6x_2 - x_3 + 6x_2 - x_3 - 22 + \mu_1 - \mu_2 = 0,$$

$$\frac{\partial L}{\partial x_2} = 6x_1 + 6x_1 + 17x_2 + 3x_3 + 3x_3 - 14.5 + \mu_3 - \mu_4 = 0$$

$$\frac{\partial L}{\partial x_3} = -x_1 + 3x_2 - x_1 + 3x_2 + 12x_3 + 13 + \mu_5 - \mu_6 = 0.$$

$\mu_1(x_1 - 1) = 0,$
$\mu_2(-x_1 - 1) = 0,$
$\mu_3(x_2 - 1) = 0,$
$\mu_4(-x_2 - 1) = 0,$
$\mu_5(x_3 - 1) = 0,$
$\mu_6(-x_3 - 1) = 0$

$\mu_2 = \mu_4 = 0,$
$\mu_6 = 2$

$13x_1 + 12x_2 - 2x_3 = 22 - \mu_1.$
$12x_1 + 17x_2 + 6x_3 = 14.5 - \mu_3.$
$-2x_1 + 6x_2 + 12x_3 = -13 - \mu_5.$

$\mu_1 = 1, \ x_1 = 1.$
$\mu_3 = 0. \ x_2 = \frac{1}{2}.$
$\mu_5 = 0. \ x_3 = -1$

$x^* = [1 \ \frac{1}{2} \ -1]^T.$

$$\nabla g_1^T Y = [1 \ 0 \ 0] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 = 0,$$

$\cancel{\nabla g_3^T Y = y_2 = 0}$, $\nabla g_5^T Y = y_3 = 0.$

$$Y^T \nabla^2 L_{xxx} Y = [y_1 \ y_2 \ y_3] \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 13y_1^2 + 12y_1y_2 - 2y_1y_3 + 12y_1y_2 + 17y_2^2 + 6y_2y_3 - 2y_1y_3 + 6y_2y_3 + 12y_3^2$$

$$= 17y_2^2 > 0$$

AY 2015 − 2016

Q4. (a) $\{N_k : 0, 1, 2, 3\}$

0: no book left at the end of the day

1: one book left

2: two books left

3: three books left Possion r.v. $P(X(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

$P(X(t)=0) = e^{-0.5}$.

$P(X(t)=1) = 0.5e^{-0.5}$.

$P(X(t)=2) = \frac{e^{-0.5}(0.5)^2}{2} = 0.125e^{-0.5}$.

$P(X(t)\geq 3) = \frac{e^{-0.5}(0.5)^3}{6} = 1-1.625e^{-0.5}$.

$P_{00} = 1-1.625e^{-0.5}$, $P_{01} = 0.125e^{-0.5}$, $P_{02} = 0.5e^{-0.5}$, $P_{03} = e^{-0.5}$.

$P_{10} =$

Q5. (a) M/M/1/2

$$
\textcircled{0} \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \textcircled{1} \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \textcircled{2}
$$

$$
\begin{cases}
\pi_0 \lambda = \pi_1 \mu \\
\pi_0 \lambda + \pi_2 \mu = (\lambda+\mu)\pi_1 \\
\pi_1 \lambda = \pi_2 \mu \\
\pi_0 + \pi_1 + \pi_2 = 1
\end{cases}
$$

$$\Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0 = \frac{\mu}{\lambda}\pi_2 ,$$

$$\pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0 .$$

$$\pi_0 + \frac{\lambda}{\mu}\pi_0 + \left(\frac{\lambda}{\mu}\right)^2 \pi_0 = 1$$

$$\Rightarrow \pi_0 = \frac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu^2}{\mu^2+\lambda\mu+\lambda^2} .$$

$$\pi_1 = \frac{\lambda\mu}{\mu^2+\lambda\mu+\lambda^2} .$$

$$\pi_2 = \frac{\lambda^2}{\mu^2+\lambda\mu+\lambda^2} .$$

$\cancel{\pi_0 \lambda = \pi_1 \mu}$ $\quad \lambda\pi_0 = \mu\pi_1$

$\cancel{\pi_0 \mu + \pi_2 \lambda} =$ $\quad (\mu+\lambda)\pi_1 = \lambda\pi_0 + \mu\pi_2$

$\mu\pi_2 = \lambda\pi_1$

(b)

utilization: $U = 1-\pi_0 = \frac{\lambda^2+\lambda\mu}{\mu^2+\lambda\mu+\lambda^2}$

production rate:

$$U\mu = \frac{\lambda^2\mu+\lambda\mu^2}{\mu^2+\lambda\mu+\lambda^2}$$

profit: $U\cdot\mu\cdot q - A\mu$

$$P = \frac{\lambda^2\mu+\lambda\mu^2}{\mu^2+\lambda\mu+\lambda^2} q - A\mu$$

$$\frac{\partial P}{\partial \mu} = 0 \Rightarrow \max P$$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2015-2016

EE6204 – SYSTEMS ANALYSIS

April/May 2016 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains FIVE (5) questions and comprises FIVE (5) pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.

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1. (a) (i) Formulate the following problem into a Linear Programming model. There are m different types of food, $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients, $N_1, \ldots, N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j. Let b_i be the price per unit of food, F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

(ii) Use the Simplex Method to solve the following linear Program problem and determine the minimum value z, if applicable. (Show your working and do not use a graphical method.)

Minimize $z = x_1 + 2x_2$

Subject to
$4x_1 + 3x_2 \leq 12$
$x_1 - 2x_2 \geq 1$
$x_1, x_2 \geq 0$

(15 Marks)

Note: Question no. 1 continues on page 2

(b) An optimization problem is given as follows

$$\text{Minimize} \quad z = \frac{1}{2}x^T P x + q^T x + r$$
$$\text{Subject to} \quad -1 \le x_i \le 1, i = 1,2,3$$

$x \le 1$
$x \ge -1$
$-x \le 1$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \ q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \text{ and } r = 1.$$

(i) Formulate the above optimization problem into a Nonlinear Programing problem with Lagrange multipliers and Lagrange function.

(ii) Prove that $x^* = [1 \quad 1/2 \quad -1]^T$ is an optimal solution for the optimization problem.

(15 Marks)

2. A hedge fund is planning to make a strategic investment by buying stocks of one company that has a great potential of investment return. They have identified 4 most promising candidates: company A, company B, company C, company D. To decide which company's stocks they should invest in, the hedge fund considers the following criteria: the current market value (M), the current annual revenue (R), the diversity of products (P), and the estimated future market size (S). The pairwise comparison matrices are shown below.

Criterion

	M	R	P	S
M	1	2	0.5	0.25
R	0.5	1	0.25	0.125
P	2	4	1	0.5
S	4	8	2	1

Market Value (M)

	A	B	C	D
A	1	2	1	2
B	0.5	1	0.5	1
C	1	2	1	2
D	0.5	1	0.5	1

Note: Question no. 2 continues on page 3

Annual Revenue (R)

	A	B	C	D
A	1	0.5	1	0.5
B	2	1	2	1
C	1	0.5	1	0.5
D	2	1	2	1

Product Diversity (P)

	A	B	C	D
A	1	1/3	1/6	1/3
B	3	1	0.5	1
C	6	2	1	2
D	3	1	0.5	1

Market Size (S)

	A	B	C	D
A	1	0.5	1/3	0.25
B	2	1	2/3	0.5
C	3	1.5	1	3/4
D	4	2	4/3	1

(a) Compute the priority of each criterion (M, R, P, and S) based on its contribution to the overall goal. Compute the consistency ratio and comment on its acceptability. For the case of 4 criteria, the consistency index of a randomly generated pairwise comparison matrix (RI) is 0.90.

(7 Marks)

(b) Determine the overall priority for each company, and select the best one.

(8 Marks)

3. Jack plans to buy one of the 6 areas for gold mining with the following information:

- Exactly one area contains gold worth \$120M, and all areas have an equal probability.
- The price of each area is \$20M dollars, which is non-refundable.

(a) If Jack is a risk taker, and assume that the utility value of 0 is $U(0)=0$, decide whether Jack should buy one area with reasoning details.

(4 Marks)

(b) A seismologist offers Jack a survey service with a commission fee of \$10M, whose result can indicate whether or not area #1 contains gold (but it does not change area #1's probability of having gold). If no gold is found in area #1, Jack will still have a right to buy another area. Draw a decision tree for this problem and decide whether this survey service is needed if Jack has made his mind to buy one area.

(11 Marks)

4. Joseph recently opened a new computer shop in a shopping mall on Orchard road. The shop is very small in size and the rent is very expensive. As keeping stock in the shop requires space and money, Joseph can only have 3 pieces of an expensive model of ABC notebook in the shop for sale. As business is good due to the good location of the shop, he will order additional notebooks from the supplier at the end of each business day, so that he always has 3 pieces of the notebook for sale on the next business day.

Market research found that the demand for the ABC notebook on each day is an independent and identical Poisson random variable with $\lambda = 0.5$. Then, N_k, the number of ABC notebooks left in the shop at the end of each business day, can be modeled as a Discrete Time Markov Chain (DTMC).

(a) Write down the state space for N_k and obtain the transition probability matrix (TMP).

(10 Marks)

(b) Find the probabilities that Joseph is expected to find that there are 0, 1, 2 and 3 ABC notebooks at the end of each business day, respectively.

(6 Marks)

(c) If Joseph can make a profit of \$300 on each ABC notebook sold after deducting all costs, how much money on average Joseph can expect to make each day? Give comments on the results that you obtain.

(4 Marks)

5. A gymnasium has an exercise machine that is very popular among its members. A chair is placed beside the machine for the next person waiting to use the machine to sit. Due to the limited space in the gymnasium, no one is allowed to wait for using the machine if the chair is occupied. Based on the past records, the arrival of members to use the machine follows a Poison process with a rate of λ, and the time that a member spent when using the machine is exponentially distributed with a mean of $\frac{1}{\mu}$.

(a) Model the above process as a finite birth-death process, and draw the state transition diagram and find the steady-state probabilities.

(8 Marks)

(b) The operating cost of the exercising machine is $\$A\mu$ at a rate of μ, where A is a positive constant. Members need to pay $\$q$ per usage of the machine. Write down an expression for the profit generated by the use of the machine. Derive an equation that μ must satisfy in order to maximize the profit.

(9 Marks)

(c) Suggest a way to show that a solution of the derived equation will indeed maximize the profit.

(3 Marks)

END OF PAPER