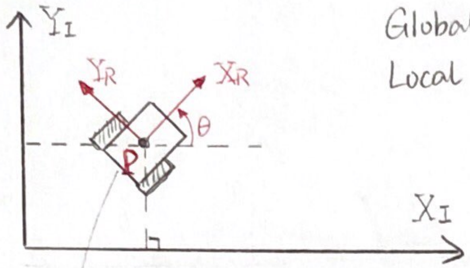
Frames



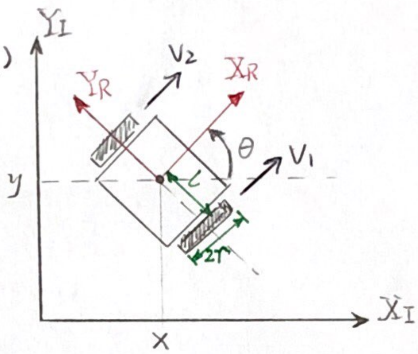
Global frame $\{X_I, Y_I\}$

Local frame $\{X_R, Y_R\}$

点P： $\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$ ， $\dot{\xi}_R = R(\theta)\,\dot{\xi}_I = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$

Forward Kinematics

$\dot{\xi}_R = R(\theta)\,\dot{\xi}_I$ ， $\dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \dot{\varphi}_2)$



① 在 local frame 中定义：

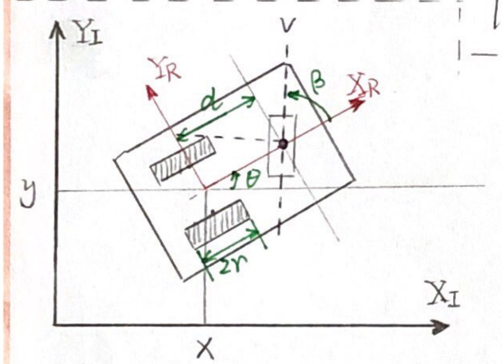
$V_x = \frac{1}{2}(V_1 + V_2)$ ， $V_y = 0$ ， $\dot{\theta} = \frac{V_1}{2l} - \frac{V_2}{2l}$

where $V_1 = \dot{\varphi}_1 r$ ， $V_2 = \dot{\varphi}_2 r$

$\Rightarrow$ $\dot{\xi}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} V_x \\ y_x \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\dot{\varphi}_1 r + \frac{1}{2}\dot{\varphi}_2 r \\ 0 \\ \frac{\dot{\varphi}_1 r}{2l} - \frac{\dot{\varphi}_2 r}{2l} \end{bmatrix}$

② 映射到 global frame：

$\dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R$ $\Rightarrow$ $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\dot{\varphi}_1 r}{2} + \frac{\dot{\varphi}_2 r}{2} \\ 0 \\ \frac{\dot{\varphi}_1 r}{2l} - \frac{\dot{\varphi}_2 r}{2l} \end{bmatrix}$



① 在 local frame 中定义：

$V_x = v \cdot \cos\beta$ ， $V_y = 0$ ， $\dot{\theta} = \frac{v \sin\beta}{d}$ ， $v = \dot{\varphi} r$

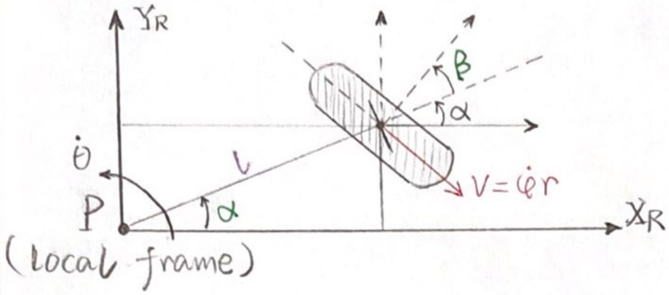
$\dot{\xi}_R = \begin{bmatrix} V_x \\ V_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\varphi} r \cdot \cos\beta \\ 0 \\ \frac{\dot{\varphi} r \cdot \sin\beta}{d} \end{bmatrix}$

② 映射到 global frame：

$\dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R$

Wheel Constraints 7

Fixed Standard Wheel:



rolling constraint:

$$v = \dot{\varphi} r = \dot{x}_r \cdot \sin(\alpha+\beta) - \dot{y}_r \cdot \cos(\alpha+\beta) - \dot{\theta} l \cdot \cos\beta$$

$$= [\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta] \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \end{bmatrix},$$

with $v = \dot{\varphi} r$, $\dot{\xi}_R = R(\theta)\dot{\xi}_I$:

$\Rightarrow$ $\underline{j R(\theta) \dot{\xi}_I - \dot{\varphi} r = 0}$.

where $j = [\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta]$.

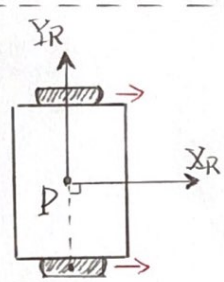
Sliding constraint:

$$\dot{x}_r \cos(\alpha+\beta) + \dot{y}_r \sin(\alpha+\beta) + \dot{\theta} l \sin\beta = 0.$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta] \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \end{bmatrix} = 0.$$

$\Rightarrow$ $\underline{c \dot{\xi}_R = c R(\theta) \dot{\xi}_I = 0}$.

where $c = [\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]$.



由图知: $\alpha = \frac{\pi}{2}$, $\beta = 0$

- rolling constraint:

$$j R(\theta) \dot{\xi}_I - \dot{\varphi} r = 0$$

$$[1 \quad 0 \quad -l] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} - \dot{\varphi} r = 0$$

$$\Rightarrow \dot{x} = \dot{\varphi} r.$$

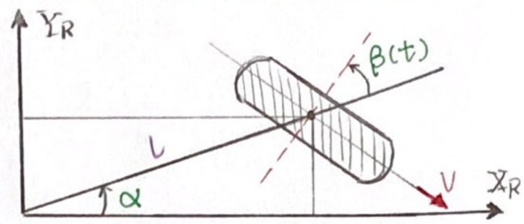
assume $\theta = 0$. $\dot{\theta} = 0$

- sliding constraint:

$$c \dot{\xi}_R = c R(\theta) \dot{\xi}_I = 0.$$

$$[0 \quad 1 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \dot{y} = 0.$$

Steered Standard Wheel:



rolling constraint:

$$j(\beta) R(\theta) \dot{\xi}_I - \dot{\varphi} r = 0$$

where

$$j(\beta) = [\sin(\alpha+\beta(t)) \quad -\cos(\alpha+\beta(t)) \quad -l\cos(\beta(t))]$$

Sliding constraint:

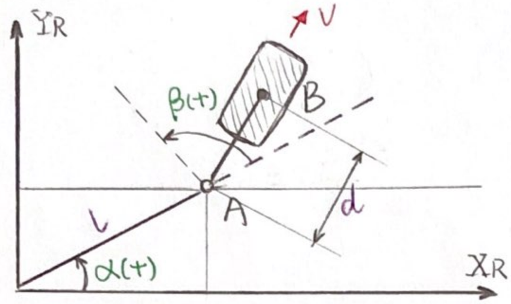
$$c(\beta) R(\theta) \dot{\xi}_I = 0$$

where

$$c(\beta) = [\cos(\alpha+\beta(t)) \quad \sin(\alpha+\beta(t)) \quad l\sin(\beta(t))]$$

Wheel Constraints 2

Castor Wheel:



rolling constraint: (和 steered standard 一致)

$$j(\beta)R(\theta)\dot{\xi}_I - \dot{\varphi}r = 0$$

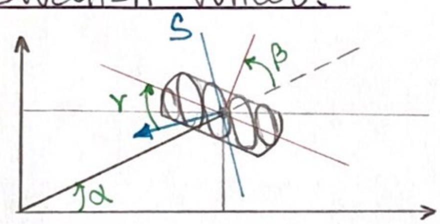
$j(\beta) = [\sin(\alpha+\beta(t)) \quad -\cos(\alpha+\beta(t)) \quad -l\cos(\beta(t))]$

sliding constraint:

$$C(\beta)R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$

$C(\beta) = [\cos(\alpha+\beta(t)) \quad \sin(\alpha+\beta(t)) \quad l\sin(\beta(t))]$

Swedish wheel:



rolling constraint:

$$jR(\theta)\dot{\xi}_I - \dot{\varphi}r\cdot\cos\gamma = 0$$

$j = [\sin(\alpha+\beta+\gamma) \quad -\cos(\alpha+\beta+\gamma) \quad -l\cdot\cos(\beta+\gamma)]$

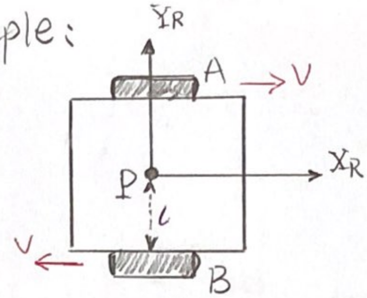
sliding constraint:

$$CR(\theta)\dot{\xi}_I - \dot{\varphi}r\cdot\sin\gamma - \dot{\varphi}_{sw}r_{sw} = 0$$

where r_{sw} is the radius of the small rollers.

$C = [\cos(\alpha+\beta+\gamma) \quad \sin(\alpha+\beta+\gamma) \quad l\sin(\beta+\gamma)]$

Example:



wheel A:

$\alpha = \frac{\pi}{2}$, $\beta = 0$.

wheel B:

$\alpha = -\frac{\pi}{2}$, $\beta = \pi$.

rolling constraint:

$$jR(\theta)\dot{\xi}_I - \dot{\varphi}r = 0$$

$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta]R\theta\dot{\xi}_I - \dot{\varphi}r = 0$

- wheel A:

$[1 \quad 0 \quad -l]R(\theta)\dot{\xi}_I - \dot{\varphi}_{fA}r = 0$.

- wheel B:

$[1 \quad 0 \quad l]R(\theta)\dot{\xi}_I - \dot{\varphi}_{fB}r = 0$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & -l \\ 1 & 0 & l \end{bmatrix} R(\theta)\dot{\xi}_I - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \dot{\varphi}_{fA} \\ \dot{\varphi}_{fB} \end{bmatrix} = 0.$$

Sliding constraint:

$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi}_I = 0$.

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} R\theta\dot{\xi}_I = 0$$

合并: $\begin{bmatrix} J_1 \\ C_1 \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix} = 0$.

where $\dot{\varphi} = \begin{bmatrix} \dot{\varphi}_{fA} \\ \dot{\varphi}_{fB} \end{bmatrix}$.

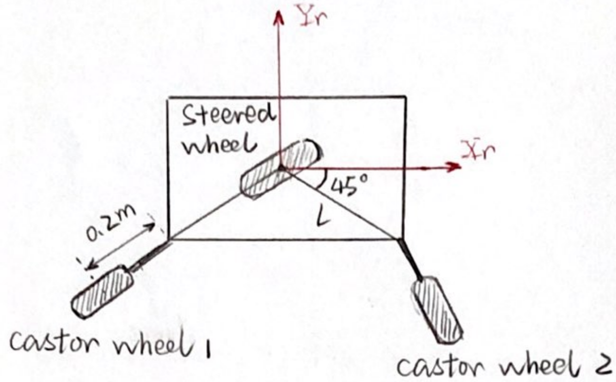
$$\begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} r\dot{\varphi}_{fA} \\ r\dot{\varphi}_{fB} \\ 0 \end{bmatrix}.$$

$$\dot{\xi}_I = R^T(\theta)\begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} r\dot{\varphi}_{fA} \\ r\dot{\varphi}_{fB} \\ 0 \end{bmatrix}.$$

General: $J_1(\beta_s + \beta_c)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0$

$C(\beta_s, \beta_c)R(\theta)\dot{\xi}_I + D\dot{\beta} = 0$

Exercise - Mobile robot



$r = 0.1m$

$L = 0.25m$

$\dot{\varphi}_s$, $\dot{\varphi}_{c1}$, $\dot{\varphi}_{c2}$

For the steered wheel:

$\alpha = 0$, $\beta = \beta_1(t)$, $l = 0$, $r = 0.1$

- rolling constraint:

$$j(\beta) R(\theta) \dot{\xi}_I - 0.1\dot{\varphi}_s = 0$$

$$j(\beta) = [\sin\beta_1(t) \quad -\cos\beta_1(t) \quad 0].$$

$$\Rightarrow \quad 0.1\dot{\varphi}_s = \dot{x}_r \sin\beta_1(t) - \dot{y}_r \cos\beta_1(t).$$

- sliding constraint:

$$c(\beta) R(\theta) \dot{\xi}_I = 0$$

$$c(\beta) = [\cos\beta_1(t) \quad \sin\beta_1(t) \quad 0].$$

$$\Rightarrow \quad \dot{x}_r \cos\beta_1(t) + \dot{y}_r \sin\beta_1(t) = 0.$$

For castor wheel 1:

$\alpha = -135°$, $\beta = \beta_2(t)$, $l = 0.25$,
$r = 0.1$, $d = 0.2$, $\dot{\varphi} = \dot{\varphi}_{c1}$

- rolling constraint:

$$j(\beta) R(\theta) \dot{\xi}_I - 0.1\dot{\varphi}_{c1} = 0.$$

$$j(\beta) = [\sin(-135° + \beta_2(t)) \quad -\cos(-135° + \beta_2(t)) \quad -0.25\cos(\beta_2(t))].$$

$$\Rightarrow \quad 0.1\dot{\varphi}_{c1} = \dot{x}_r \sin(-135° + \beta_2(t)) - \dot{y}_r \cos(-135° + \beta_2(t)) - 0.25\dot{\theta}\cos(\beta_2(t)).$$

- sliding constraint:

$$c(\beta) R(\theta) \dot{\xi}_I + 0.2\dot{\beta}_2(t) = 0.$$

$$c(\beta) = [\cos(-135° + \beta_2(t)) \quad \sin(-135° + \beta_2(t)) \quad 0.25\sin(\beta_2(t))].$$

$$\Rightarrow \quad \dot{x}_r \cos(-135° + \beta_2(t)) + \dot{y}_r \sin(-135° + \beta_2(t)) + 0.25\dot{\theta}\sin(\beta_2(t)) + 0.2\dot{\beta}_2(t) = 0.$$

For castor wheel 2:

$\alpha = -45°$, $\beta = \beta_3(t)$, $l = 0.25$, $r = 0.1$, $d = 0.2$, $\dot{\varphi} = \dot{\varphi}_{c2}$

the calculation is similar as above.