

# The problem

- Given two corresponding point sets:

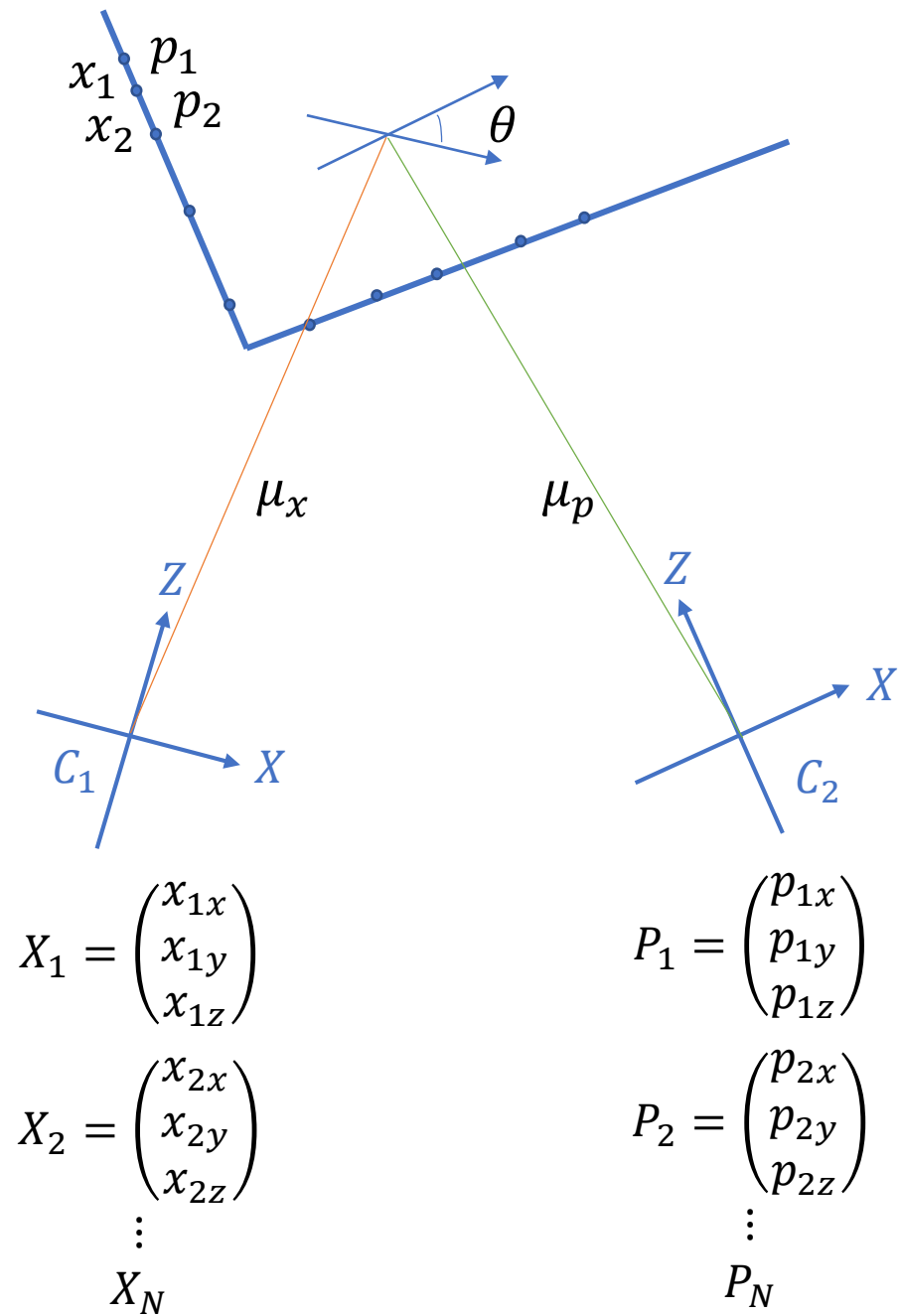
$$X = \{x_1, \dots, x_{N_x}\}$$

$$P = \{p_1, \dots, p_{N_p}\}$$

- Wanted: Translation  $t$  and rotation  $R$  that minimize the sum of the squared error:

$$E(R, t) = \frac{1}{N_p} \sum_{i=1}^{N_p} ||x_i - Rp_i - t||^2$$

where  $x_i$  and  $p_i$  are corresponding points.



# Center of Mass

$$\mu_x = \frac{1}{N_x} \sum_{i=1}^{N_x} x_i \quad \text{and} \quad \mu_p = \frac{1}{N_p} \sum_{i=1}^{N_p} p_i$$

are the centers of mass of the two point sets.

## Ideas:

- Subtract the corresponding center of mass from every point in the two point sets before calculating the transformation.
- The resulting point sets are:

$$X' = \{x_i - \mu_x\} = \{x'_i\} \quad \text{and}$$

$$P' = \{p_i - \mu_p\} = \{p'_i\}$$

# Singular Value Decomposition

Let  $W = \sum_{i=1}^{N_p} x_i' p_i'^T$

compute the singular value decomposition (SVD) of  $W$  by:

$$W = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} V^T$$

where  $U, V \in R^{3 \times 3}$  are unitary, and  $\sigma_1 \geq \sigma_2 \geq \sigma_3$  are the singular values of  $W$ .

# SVD

**Theorem** (without proof):

If  $\text{rank}(W)=3$ , the optimal solution of  $E(R, t)$  is unique and is given by:

$$R = UV^T$$

$$t = \mu_x - R\mu_p$$

The minimal value of error function at  $(R, t)$  is :

$$E(R, t) = \sum_{i=1}^{N_p} (||x'_i||^2 + ||y'_i||^2) - 2(\sigma_1 + \sigma_2 + \sigma_3)$$

**Proof:** Assume  $N_x = N_p = N$

$$E(R, t) = \frac{1}{N} \sum_{i=1}^N ||x_i - (Rp_i + t)||^2$$

To find  $t$  that minimize the square error:  $\frac{\partial E}{\partial t} = 0$

Hence 
$$\frac{1}{N} \sum_{i=1}^N 2 (X_i + Rp_i - t) = 0$$

$$\underbrace{\frac{1}{N} \sum_{i=1}^N X_i}_{\mu_x} - \frac{1}{N} \sum_{i=1}^N (Rp_i) - \underbrace{\frac{1}{N} \sum_{i=1}^N t}_t = 0$$

$$\mu_x - R \underbrace{\left( \frac{1}{N} \sum_{i=1}^N p_i \right)}_{\mu_p} - t = 0$$

$$t = \mu_x - R\mu_p$$

$$\begin{aligned}
E(R, t) &= \frac{1}{N} \sum_{i=1}^N ||x_i - Rp_i - t||^2 \\
&= \frac{1}{N} \sum_{i=1}^N ||x_i - Rp_i - (\mu_x - R\mu_p)||^2 \\
&= \frac{1}{N} \sum_{i=1}^N || \underbrace{(x_i - \mu_x)}_{x'_i} - R \underbrace{(p_i - \mu_p)}_{p'_i} ||^2 \\
&= \frac{1}{N} \sum_{i=1}^N ||x'_i - Rp'_i||^2
\end{aligned}$$

$$(ab)^T = b^T a^T$$

$$\begin{aligned} ||x'_i - Rp'_i||^2 &= (x'_i - Rp'_i)^T (x'_i - Rp'_i) \\ &= (x'^T_i - p'^T_i R^T)(x'_i - Rp'_i) \\ &= x'^T_i x'_i - 2p'^T_i R^T x'_i + p'^T_i p'_i \end{aligned}$$

To find  $R$ , we need to minimize the squared error.



$$\operatorname{argmin}_R ||x'_i - Rp'_i||^2 = \operatorname{argmin}_R \left( \underbrace{x_i'^T x'_i}_{\text{independent of } R} - 2p_i'^T R^T x'_i + \underbrace{p_i'^T p'_i}_{\text{independent of } R} \right)$$

$$= \operatorname{argmin}_R (-2p_i'^T R^T x'_i)$$

$$= \operatorname{argmax}_R (p_i'^T R^T x'_i)$$

$$= \operatorname{argmax}_R (x_i'^T Rp'_i)$$

$$\sum (x_i'^T Rp'_i) = \operatorname{trace}(X'^T RP')$$

where  $X' = \{x'_i\}, \quad P' = \{p'_i\}$

$$\operatorname{trace} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} + a_{22} + a_{33}$$

Let

$$X' = \{x'_i\} = \begin{pmatrix} x_1 & x'_1 & x''_1 & \dots \\ x_2 & x'_2 & x''_2 & \dots \\ x_3 & x'_3 & x''_3 & \dots \end{pmatrix}$$

$$P' = \{p'_i\} = \underbrace{\begin{pmatrix} p_1 & p'_1 & p''_1 & \dots \\ p_2 & p'_2 & p''_2 & \dots \\ p_3 & p'_3 & p''_3 & \dots \end{pmatrix}}_N$$

$$\begin{aligned} \sum (x_i'^T R p_i') &= \text{trace} \left[ \begin{pmatrix} x_1 & x_2 & x_3 \\ x'_1 & x'_2 & x'_3 \\ x''_1 & x'_2 & x''_3 \\ \vdots & \vdots & \vdots \end{pmatrix} R \begin{pmatrix} p_1 & p'_1 & p''_1 & \dots \\ p_2 & p'_2 & p''_2 & \dots \\ p_3 & p'_3 & p''_3 & \dots \end{pmatrix} \right] \\ &= \begin{pmatrix} x_1^T R p_1 \\ x_2^T R p_2 \\ \vdots \end{pmatrix} = \text{trace}(X'^T R P') \end{aligned}$$

$$(abc)^T = c^T b^T a^T$$

$$\text{trace}(a \cdot b) = \text{trace}(b \cdot a)$$

$$\text{trace}(a \cdot b) = \text{trace}(a \cdot b)^T$$

$$\begin{aligned} \text{trace} \left( \underbrace{X'^T}_a \underbrace{RP'}_b \right) &= \text{trace} \left[ R \begin{pmatrix} p_1 & p'_1 & p''_1 & \dots \\ p_2 & p'_2 & p''_2 & \dots \\ p_3 & p'_3 & p''_3 & \dots \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ x'_1 & x'_2 & x'_3 \\ x''_1 & x''_2 & x''_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \right] \\ &= \text{trace}(RP'X'^T) \end{aligned}$$

$$\begin{aligned} \text{trace}(X'^T RP') &= \text{trace}(RP'X'^T) \\ &= \text{trace}(RP'X'^T)^T \\ &= \text{trace}(X'P'^T R^T) \end{aligned}$$

Let  $W = X'P'^T \in R^{3 \times 3}$

SVD of  $W$

$$W = U\Sigma V^T$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

singular values  $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$ , and  $R = UV^T$

$$\text{trace}(X'P'^T R^T) = \text{trace}(WR^T)$$

$$= \text{trace} \left( \underbrace{U}_a \underbrace{\Sigma V^T R^T}_b \right)$$

$$= \text{trace} \left( \Sigma \underbrace{V^T R^T U}_I \right)$$

$$= \text{trace}(\Sigma) = \sigma_1 + \sigma_2 + \sigma_3 \quad \underline{\text{maximum!}}$$

$U$  and  $V$  are orthogonal.

$$V^T R^T U = I$$

$$V(V^T R^T U) = V \cdot I$$

$$R^T U = V$$

$$R^T = V U^T$$

$$R = (V U^T)^T = U V^T$$