Exercise 1.2:

P_1: Let $x(n) = Acos(2\pi f_0 n - \phi) + e(n)$, where A, f_0 are constants, and ϕ is uniformly distributed on $[-\pi,\ \pi]$ and statistically independent of the signal $e(n)$. Suppose $e(n)$ is the output of the system $H(z) = \frac{1}{1-\beta z^{-1}}$ excited by a white noise $w(n)$, where $|\beta| < 1$. Compute the autocorrelation function and the power density spectrum $x(n)$.

Solution: Denote $s(n) = Acos(2\pi f_0 n - \phi)$. Since ϕ and $e(n)$ are independent, $s(n)$ and $e(n)$ are uncorrelated. In fact, the joint PDF satisfies $p_{\phi e}(\phi, e) = p_\phi(\phi) p_e(e)$.

$$\begin{aligned}
E[s(n_1)e(n_2)] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(n_1)e(n_2)p_{\phi e}(\phi, e)d\phi de \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(n_1)e(n_2)p_{\phi}(\phi)p_e(e)d\phi de \\
&= \int_{-\infty}^{+\infty} s(n_1)p_{\phi}(\phi)d\phi \int_{-\infty}^{+\infty} e(n_2)p_e(e)de \\
&= E[s(n_1)]E[e(n_2)],
\end{aligned}$$

which means that $s(n)$ and $e(n)$ are uncorrelated.

As known that $e(n)$ and $s(n)$ are all stationary, it then follows from P_2 of **Exercise 1.1** that $x(n)$ is stationary and

$$\gamma_x(m) = \gamma_s(m) + 2m_s m_e + \gamma_e(m).$$

Since $\gamma_s(m) = \frac{A^2}{2} cos(2\pi f_0 m)$ and $m_s = 0$ (In fact, m_e is also zero. Why?), one has

$$\gamma_x(m) = \frac{A^2}{2} cos(2\pi f_0 m) + \gamma_e(m).$$

Noting $e(n)$ is the output of $H(z) = \frac{1}{1-\beta z^{-1}}$ excited with a white noise $w(n)$, that is $e(n)$ is a 1st order AR process, it follows from Theorem 1 that

$$\Gamma_e(z) = H(z)H(z^{-1})\Gamma_w(z) = \frac{1}{1-\beta z^{-1}} \frac{1}{1-\beta z} \sigma_w^2$$

$$= \frac{\sigma_w^2}{1-\beta^2} \frac{1-\beta^2}{(1-\beta z^{-1})(1-\beta z)} \quad \Leftrightarrow \quad \frac{\sigma_w^2}{1-\beta^2}\, \beta^{|m|}$$

See page 35 of the notes.

Therefore,

$$\gamma_x(m) = \frac{A^2}{2} cos(2\pi f_0 m) + \frac{\sigma_w^2}{1-\beta^2}\ \beta^{|m|}$$

and hence the power density function is

$$\begin{aligned}
\Gamma_x(e^{j2\pi f}) &= \frac{A^2}{4}[\delta_c(f+f_0) + \delta_c(f-f_0)] + \Gamma_e(e^{j2\pi f}) \\
&= \frac{A^2}{4}[\delta_c(f+f_0) + \delta_c(f-f_0)] + \frac{1}{1-\beta e^{-j2\pi f}} \frac{1}{1-\beta e^{j2\pi f}} \sigma_w^2 \\
&= \frac{A^2}{4}[\delta_c(f+f_0) + \delta_c(f-f_0)] + \frac{\sigma_w^2}{(1+\beta^2) - 2\beta cos(2\pi f)}.
\end{aligned}$$

Sketch the spectrum.

P_2: Let $x(n) = s(n) + w(n)$, where $s(n)$ is a 2nd order AR (i.e., AR(2)) process and $w(n)$ is a white noise process with variance σ_w^2. Assume that $w(n)$ and $s(n)$ are uncorrelated. Determine the power density spectrum of $x(n)$ and hence show that $x(n)$ can be modelled as ARMA(2,2) process.

Solution: From the fact that $w(n)$ and $s(n)$ are uncorrelated and P_2 of **Exercise 1.1**, one knows $x(n)$ is stationary and

$$\gamma_x(m) = \gamma_s(m) + 2m_s m_w + \gamma_w(m)$$

Since $w(n)$ is white, $m_w = 0$ and $\gamma_w(m) = \sigma_w^2 \delta(m)$. Therefore,

$\gamma_x(m) = \gamma_s(m) + \sigma_w^2 \delta(m)$ and

$$\Gamma_x(z) = \Gamma_s(z) + \sigma_w^2.$$

What about $\Gamma_s(z)$? Noting that $s(n)$ is a 2nd order AR process:

$$s(n) + a_1 s(n-1) + a_2 s(n-2) = v(n)$$

with $v(n)$, a white noise.

Denote $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$. The transfer function between $v(n)$ and $s(n)$ is $1/A(z)$ and hence $\Gamma_s(z) = \frac{1}{A(z)} \frac{1}{A(z^{-1})} \Gamma_v(z)$.

Therefore,

$$\Gamma_x(z) = \frac{1}{A(z)}\frac{1}{A(z^{-1})}\sigma_v^2 + \sigma_w^2.$$

There power density spectrum of $x(n)$ is

$$\Gamma_x(e^{j2\pi f}) = \frac{1}{A(e^{j2\pi f})}\frac{1}{A(e^{-j2\pi f})}\sigma_v^2 + \sigma_w^2.$$

Noting

$$\begin{aligned}\Gamma_x(z) &= \frac{1}{A(z)A(z^{-1})}\sigma_v^2 + \sigma_w^2 = \frac{\sigma_v^2 + \sigma_w^2 A(z)A(z^{-1})}{A(z)A(z^{-1})} \\ &\stackrel{\triangle}{=} \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}\sigma_u^2,\end{aligned}$$

where $B(z) = 1 + b_1 z^{-1} + b_2 z^{-2}$ can be determined from

$$B(z)B(z^{-1})\sigma_u^2 = \sigma_v^2 + \sigma_w^2 A(z)A(z^{-1})$$

Denote $\rho = \frac{\sigma_w^2}{\sigma_u^2}$. Comparing the coefficients of both sides leads to

$$\begin{cases} b_2 = \rho a_2 & (z^{-2}, \quad z^2) \\ b_1 + b_1 b_2 = \rho(a_1 + a_1 a_2) & (z^{-1}, \quad z^1) \\ 1 + b_1^2 + b_2^2 = \rho(1 + a_1^2 + a_2^2 + \frac{\sigma_v^2}{\sigma_w^2}) & (z^0) \end{cases}$$

Given a_1, a_2, σ_v^2 and σ_w^2 one can solve the above (non-linear) equations for b_1, b_2 and ρ (i.e., σ_u^2).

For example, when $a_1 = 0$, $a_2 = -\frac{1}{4}, \sigma_w^2 = \sigma_v^2 = 1$, one has $b_1(1 + b_2) = 0$, yielding the following

- Case II: $b_2 = -1$.

  Then $\rho = -a_2^{-1} = 4$ and hence $\sigma_u^2 = \sigma_w^2/\rho = 1/4$. Also, with the third equation one has

$$1 + b_1^2 + (-1)^2 = 4 \times [1 + 0^2 + (-1/4)^2 + 1/1]$$

  and hence $b_1^2 = 25/4$, that is $b_1 = \pm\frac{5}{2}$. So, $B(z) = 1 \pm \frac{5}{2} z^{-1} - z^2$ and $\sigma_u^2 = 1/4$.

- Case I: $b_1 = 0$. (Forget about it.)

So, $x(n)$ can be modelled as an ARMA(2,2) process, driven by $u(n)$, the output of the system $A(z)/B(z)$ excited with $x(n)$.

P_3: An ARMA process $x(n)$ has an autocorrelation function whose z-transform is given as

$$\Gamma_x(z) = 9 \times \frac{(z-\frac{1}{3})(z-3)}{(z-0.5)(z-2)}, \ \ ROC: \ 0.5 < |z| < 2$$

- What is the power density spectrum of $x(n)$?
- Determine the innovation process filter $G(z)$ for generating $x(n)$ from a white noise input sequence. Is $G(z)$ unique?
- Determine the minimum-phase $G(z)$ and then give the corresponding difference equation.

Solution:

- Power density spectrum:

$$\Gamma_x(e^{j2\pi f}) = 9 \times \frac{(e^{j2\pi f} - \frac{1}{3})(e^{j2\pi f} - 3)}{(e^{j2\pi f} - 0.5)(e^{j2\pi f} - 2)}$$

$$= 9 \times \frac{e^{j2\pi f} - \frac{1}{3}}{e^{j2\pi f} - 0.5} \frac{e^{-j2\pi f} - \frac{1}{3}}{e^{-j2\pi f} - \frac{1}{2}} \frac{-3e^{j2\pi f}}{-2e^{j2\pi f}}$$

$$= \frac{27}{2} \frac{[1 + (\frac{1}{3})^2] - 2 \times \frac{1}{3}cos(2\pi f)}{[1 + (\frac{1}{2})^2] - 2 \times \frac{1}{2}cos(2\pi f)}$$

$$= \frac{27}{2} \frac{10/9 - \frac{2}{3}cos(2\pi f)}{5/4 - cos(2\pi f)} = \frac{60 - 36cos(2\pi f)}{5 - 4cos(2\pi f)}$$

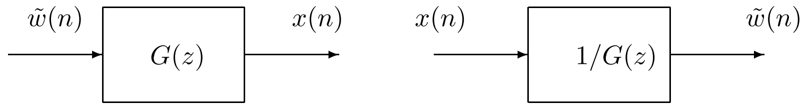

Figure 1: Block diagram of the innovation representation of stationary process $x(n)$.

- If $\tilde{w}(n)$ is a white noise of unit variance (i.e., $\sigma_{\tilde{w}}^2 = 1$), then according to Theorem 1

$$\Gamma_x(z) = G(z)G(z^{-1}) = 9 \times \frac{(z - \frac{1}{3})(z - 3)}{(z - 0.5)(z - 2)}$$

Clearly, one has the following fours choices for such $G(z)$:

$$G_1(z) = 3\sqrt{\frac{3}{2}} \times \frac{z - \frac{1}{3}}{z - 0.5}, \quad G_2(z) = \sqrt{\frac{3}{2}} \times \frac{z - 3}{z - 0.5}$$

$$G_3(z) = 3\sqrt{6} \times \frac{z - \frac{1}{3}}{z - 2}, \qquad G_4(z) = 3\sqrt{\frac{2}{3}} \times \frac{z - 3}{z - 2}$$

- The minimum-phase $G(z)$ is the first one, that is $G_1(z) = 3\sqrt{\frac{3}{2}} \times \frac{z-\frac{1}{3}}{z-0.5}$ since its zero ($z = 1/3$) and pole ($p = 0.5$) are all inside the unit circle, i.e., both are absolutely smaller than one.