Exercise 1.1

P_1: Let X be a discrete random variable having $S_X = \{x_1, \cdots, x_N\}$, where $x_k < x_{k+1}$, $\forall k$ and $P(X = x_k) = p_k$. Show that (i) $m_x = E[X] = \sum_{k=1}^{N} x_k p_k$. (ii) $var[X] = E[(X - m_x)^2] = \sum_{k=1}^{N}(x_k - m_x)^2 p_k$.

Solution: As mentioned in the lecture notes, X can be considered as a continuous variable with an PDF:

$$p(x) = \sum_{k=1}^{N} p_k \delta_c(x - x_k).$$

According to the definition of $E[X]$, one has

$$m_x = E[X] = \int_{-\infty}^{+\infty} xp(x)dx = \int_{-\infty}^{+\infty} x \sum_{k=1}^{N} p_k \delta_c(x - x_k)dx$$
$$= \sum_{k=1}^{N} p_k \int_{-\infty}^{+\infty} x\delta_c(x - x_k)dx$$

Noting the property:

$$\int_{-\infty}^{+\infty} f(x)\delta_c(x - x_k)dx = f(x_k)$$

for any $f(x)$ and constant x_k, one has $m_x = E[X] = \sum_{k=1}^{N} p_k x_k$.

Similarly,

$$var[X] = E[(X - m_x)^2] = \int_{-\infty}^{+\infty} (x - m_x)^2 p(x) dx$$

$$= \int_{-\infty}^{+\infty} (x - m_x)^2 \sum_{k=1}^{N} p_k \delta_c(x - x_k) dx$$

$$= \sum_{k=1}^{N} p_k \int_{-\infty}^{+\infty} (x - m_x)^2 \delta_c(x - x_k) dx$$

$$= \sum_{k=1}^{N} p_k (x_k - m_x)^2$$

P_2: Let $x(n)$ and $y(n)$ be two (real-valued) wide-sense stationary processes. Show that for any (real) constant c_1, c_2, $s(n) = c_1 x(n) + c_2 y(n)$ is also wide-sense stationary if $x(n)$ and $y(n)$ are uncorrelated and $\gamma_s(m) = c_1^2 \gamma_x(m) + 2c_1 c_2 m_x m_y + c_2^2 \gamma_y(m)$. Do the results hold if $x(n)$ and $y(n)$ are statistically independent?

Solution: Similar to **Example 1.2**. First of all,

$$m_s = E[s(n)] = E[c_1 x(n) + c_2 y(n)] = c_1 E[x(n)] + c_2 E[y(n)]$$

$$= c_1 m_x + c_2 m_y$$

which is constant due to m_x and m_y constant.

Now, look at the aut-correlation function

$$E[s(n)\ \ s^*(n-m)] = E[c_1^2 x(n)x(n-m) + c_1 c_2 x(n)y(n-m)$$
$$+c_2 c_1 y(n)x(n-m) + c_2^2 y(n)y(n-m)] = c_1^2 \gamma_x(m) +$$
$$c_1 c_2 \{E[x(n)y(n-m)] + E[y(n)x(n-m)]\} + c_2^2 \gamma_y(m).$$

Since $x(n)$ and $y(n)$ are uncorrelated and stationary:

$$E[x(n)y(\tilde{n})] = E[x(n)]E[y(\tilde{n})] = m_x m_y,\ \forall n, \tilde{n},$$

$E[x(n)y(n-m)] = m_x m_y,\ E[y(n)x(n-m)] = m_x m_y$ and hence

$$E[s(n)s^*(n-m)] = c_1^2 \gamma_x(m) + 2c_1 c_2 m_x m_y + c_2^2 \gamma_y(m),$$

which is a function of m only. Therefore, $s(n)$ is stationary.

When $x(n)$ and $y(n)$ are statistically independent, they are, as shown on Page 14 of the *lecture notes*, uncorrelated. So, the conclusions above still hold.

P_3: Let $x(n)$ be the output of a real-valued filter $H(z)$ excited by a stationary process $w(m)$, and $y(n)$ be a process such that the cross correlation function between $y(n)$ and $w(n)$ is $\gamma_{yw}(m)$. Show that

- the cross-correlation func. between $y(n)$ and $x(n)$ is $E[y(n)x^*(n-m)] = \gamma_{yx}(m)$, that is *just a function of* m, and
- when all the processes are real valued, $\Gamma_{yx}(z) = H(z^{-1})\Gamma_{yw}(z)$ and $\Gamma_{xy}(z) = H(z)\Gamma_{yw}(z^{-1})$, where $\Gamma_{uv}(z)$ denotes the z-transform of $\gamma_{uv}(m)$.

Solution: Let $H(z) = \sum_{k=-\infty}^{+\infty} h(k) z^{-k}$. Then

$$x(n) = \sum_{k=-\infty}^{+\infty} h(k) w(n-k)$$

Part I:*Cross-correlation function*

$$\begin{aligned}
E[y(n)x^*(n-m)] &= E[y(n) \sum_{k=-\infty}^{+\infty} h^*(k) w^*(n-m-k)] \\
&= E[\sum_{k=-\infty}^{+\infty} h(k) y(n) w^*(n-m-k)] \\
&= \sum_{k=-\infty}^{+\infty} h(k) E[y(n) w^*(n-m-k)]
\end{aligned}$$

Noting that $E[y(n)w^*(n-m)] = \gamma_{yw}(m)$, then $E[y(n)w^*(n-(m+k))] = \gamma_{yw}(m+k)$. Therefore,

$$E[y(n) \quad x^*(n-m)] = \sum_{k=-\infty}^{+\infty} h(k)\gamma_{yw}(m+k) \qquad (k=-l)$$

$$= \sum_{l=-\infty}^{+\infty} h(-l)\gamma_{yw}(m-l) = h(-m) \bigotimes \gamma_{yw}(m) \overset{\triangle}{=} \gamma_{yx}(m)$$

Part II: Since $h(m) \Leftrightarrow H(z)$, $h(-m) \Leftrightarrow H(z^{-1})$. Therefore, it follows from the above convolution (in time domain) that

$$\Gamma_{yx}(z) = H(z^{-1})\Gamma_{yw}(z) \tag{1}$$

Since $\gamma_{yx}(m) = E[y(n)x^*(n-m)] = E[y(n)x(n-m)] = E[x(n-m)y(n-m+m)] = E[x(\tilde{n})y(\tilde{n}+m)] = \gamma_{xy}(-m)$ hence $\gamma_{xy}(m) = \gamma_{yx}(-m)$, one has

$$\Gamma_{xy}(z) = \Gamma_{yx}(z^{-1}) = H(z)\Gamma_{yw}(z^{-1}) \qquad (2)$$

Remarks:

- When $y(n) = x(n)$, (1) $\Rightarrow \Gamma_x(z) = H(z^{-1})\Gamma_{xw}(z)$
- When $y(n) = w(n)$, (2) $\Rightarrow \Gamma_{xw}(z) = H(z)\Gamma_w(z)$.
- So, noting $\Gamma_w(z) = \Gamma_w(z^{-1})$, one has $\Gamma_x(z) = H(z^{-1})H(z)\Gamma_w(z)$, which is Theorem 1 (see Page 29 of the notes).