Exercise 2:

P_1: Let $x(n) = cos(\frac{\pi}{4}n - \phi) + e(n) \triangleq s(n) + e(n)$, where ϕ is uniformly distributed on $[-\pi,\ \pi]$ and statistically independent of the signal $e(n)$. Suppose $e(n)$ is the output of the system $H(z) = \frac{1}{1-0.75z^{-1}}$ excited by a white noise $w(n)$ of variance $\sigma_w^2 = 0.01$.

- Compute the signal-to-noise ratio (SNR) of $x(n)$, defined as $SNR_x = \frac{E[s^2(n)]}{E[e^2(n)]}$.
- If $x(n)$ is modelled as a 2nd order AR process, what is the estimate of the true frequency?
- If $x(n)$ is modelled as a sinusoid buried in a white noise (i.e., ARMA(2,2)), estimate the frequency.

Repeat the three questions for $\sigma_w^2 = 0.05$.

Solution: From P_1 of *Exercise 1.2*, one has

$$\gamma_x(m) = \gamma_s(m) + \gamma_e(m) = 0.5cos(\frac{\pi}{4}m) + \frac{\sigma_w^2}{1-0.75^2}0.75^{|m|}.$$

- Compute SNR_x:

$$SNR_x = \frac{E[s^2(n)]}{E[e^2(n)]} = \frac{\gamma_s(0)}{\gamma_e(0)} = \frac{0.5}{0.01/(1-0.75^2)} = 21.8750 \quad (13.3995\, dB)$$

- If $x(n)$ is modelled/considered as a 2nd AR process, i.e.,

$$x(n) + a_1 x(n-1) + a_2(n-2) = v(n)$$

where a_1 and a_2 are constant, and $v(n)$, a white noise sequence.

Approach 1: The frequency is detected from the most significant peak of the 2nd AR power density spectrum:

$$P_x^{RA}(e^{j2\pi f}) = \frac{\sigma_v^2}{|1 + a_1 e^{-j2\pi f} + a_2 e^{-j2\pi 2f}|^2}$$

So, the key is to find out a_1 and a_2 with $\gamma_x(m)$ given:

$$\begin{bmatrix} \gamma_x(0) \; \gamma_x(1) \\ \gamma_x(1) \; \gamma_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} \gamma_x(1) \\ \gamma_x(2) \end{bmatrix}$$

that is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \frac{1}{\gamma_x^2(0) - \gamma_x^2(1)} \begin{bmatrix} \gamma_x(0) & -\gamma_x(1) \\ -\gamma_x(1) & \gamma_x(0) \end{bmatrix} \begin{bmatrix} \gamma_x(1) \\ \gamma_x(2) \end{bmatrix}.$$

With $\gamma_x(m)$ obtained above, one has $\gamma_x(0) = 0.5229, \quad \gamma_x(1) = 0.3707, \quad \gamma_x(2) = 0.0129$ and hence $a_1 = -1.3905, \quad a_2 = 0.9612$

Figure 3 depicts the power density spectrum normalized by σ_v^2 when modelled as a 2nd order AR process.

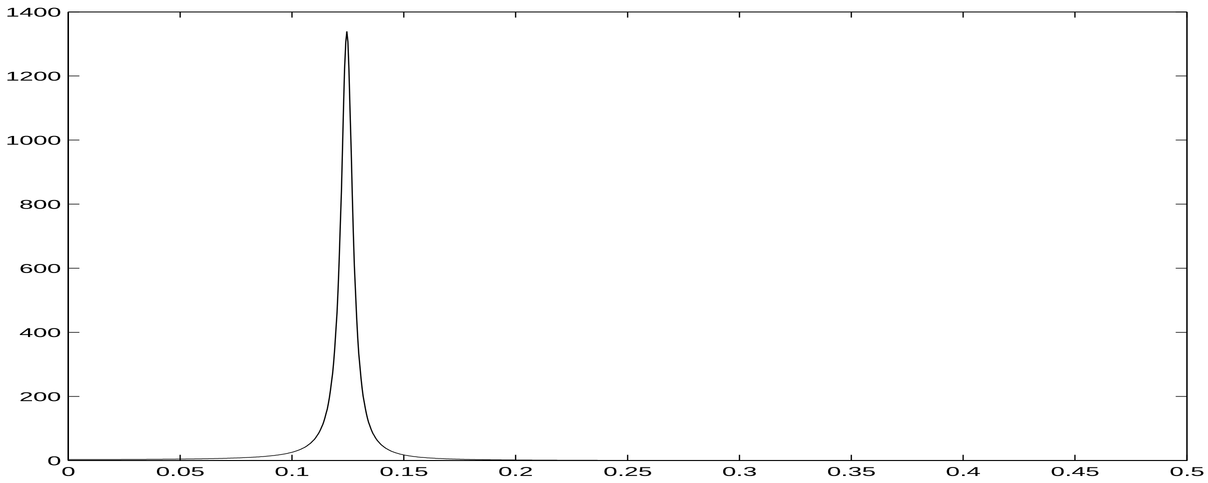

Figure 3: Normalized power density spectrum when modelled as a 2nd order AR process.

The peak occurs at $\hat{f}_0 = 0.1245$, the detected frequency, while the true frequency is $f_0 = 1/8 = 0.1250$.

Approach 2: The peak of the spectrum is strongly related to the position of the roots:

$$z^2 + a_1 z + a_2 = 0 \Rightarrow z^2 - 1.3905z + 0.9612 = 0$$

Solving it, one has $p_{1,2} = 0.9804e^{\pm j0.7825}$ and the estimated frequency $\hat{f}_0$ can be obtained from

$$2\pi \hat{f}_0 = 0.7825, \quad \Rightarrow \hat{f}_0 = 0.1245$$

which is the same as one obtained with Approach 1.

- If $x(n)$ is considered as a sinusoid buried in a white noise (though $e(n)$ is not white), then (see the notes) one can estimate the frequency with the following procedure.

$$det[\lambda I_3 - \begin{bmatrix} \gamma_x(0) & \gamma_x(1) & \gamma_x(2) \\ \gamma_x(1) & \gamma_x(0) & \gamma_x(1) \\ \gamma_x(2) & \gamma_x(1) & \gamma_x(0) \end{bmatrix}] = 0$$

With $\gamma_x(0) = 0.5229, \gamma_x(1) = 0.3707, \gamma_x(2) = 0.0129$, using MATLAB command:

$$[V, \lambda] = eig(\Phi),$$

where λ is diagonal, providing the eigenvalues, and V is the eigenvector matrix such that $\Phi V = V\lambda$, one has $\lambda_1 = 1.0536$, $\lambda_2 = 0.5100$, and $\lambda_3 = 0.0050$, which is the smallest.

One eigenvector to λ_3 is $[1 \quad -1.4317 \quad 1]^{\mathcal{T}} = [1 \quad a_1 \quad a_2]^{\mathcal{T}}$.

The roots of $z^2 + a_1 z + a_2 = 0$ are $z_{1,2} = e^{\pm j0.7730}$ and hence the estimated frequency is

$$2\pi \hat{f}_0 = 0.7730 \quad \Rightarrow \hat{f}_0 = 0.1230$$

Compared with the modelling as 2nd order AR: both approaches yield the same $\hat{f}_0 = 0.1250$, the ARMA(2,2) modelling yields a poorer performance.

When $\sigma_w^2 = 0.05$, $SNR_x = 4.3750 \quad (6.4098\ dB)$ and one can solve the problems using the same procedure. Here are the results:

- *2nd order AR modelling:* $a_1 = -1.3102, \quad a_2 = 0.8323$ and the frequencies detected are

$$\hat{f}_0 = 0.1216 \quad (peak), \quad \hat{f}_0 = 0.1225 \quad (pole)$$

- *ARMA(2,2) modelling:* $a_1 = -1.4893, \quad a_2 = 1$ and the frequency detected is

$$\hat{f}_0 = 0.1163$$

For this example, AR(2) modelling is better than ARMA(2,2). Why ?

Exercise 3:

P_1: Let $x(n)$ be an AR(2) process, i.e.,

$$x(n) + a_1^0(n-1) + a_2^0 x(n-2) = w(n).$$

- Derive the one-step forward linear predictor of order p and the corresponding mean-square prediction error in terms of the auto-correlation function $\gamma_x(m)$ for $p = 1, 2$, respectively.
- With $a_1^0 = -1$, $a_2^0 = 0.6$ and $\sigma_w^2 = 1$, compute $\gamma_x(m)$ for $m = 0, 1, 2, 3, 4, 5$. Then specify the predictors obtained above.

Solution:

- *Optimal one-step forward linear predictor:*

$$\hat{x}(n) = -a_1 x(n-1) - a_2 x(n-2) - \cdots - a_p x(n-p)$$

Denote $e(n) = x(n) - \hat{x}(n)$ as the prediction error. The optimal predictor is the solution to $\min_{\{a_k\}} E[|e(n)|^2]$. According Theorem 2, one has

$$E[e(n)x^*(n-m)] = 0,\ 1 \le m \le p$$

which leads to (i) the optimal coefficients

$$E[e(n)x^*(n-m)] = 0 \ \Rightarrow\ E[\{x(n)+\sum_{m=1}^{p} a_m x(n-m)\}x^*(n-m)] = 0$$

that is,

$$\gamma_x(m) + a_1\gamma_x(m-1) + \cdots + a_p\gamma_x(m-p) = 0, \ , \quad 1 \leq m \leq p$$

and (ii) the minimum MSE: $E_p^f = \min_{\{a_k\}} E[|e(n)|^2]$

$$E_p^f = \ E[e(n)e^*(n)] = E[e(n)\{x^*(n) + \sum_{k=m}^{p} a_m^* x^*(n-m)\}]$$

$$= E[e(n)x^*(n)] = E[\{x(n) + \sum_{m=1}^{p} a_m x(n-m)\}x^*(n-m)]$$

$$= \gamma_x(0) + \sum_{m=1}^{p} a_m \gamma_x(-m)$$

1st order ($p = 1$):

$$\gamma_x(1) + a_1\gamma_x(0) = 0 \;\Rightarrow\; a_1 = -\frac{\gamma_x(1)}{\gamma_x(0)}$$

and

$$E_1^f = \gamma_x(0)[1 - |\frac{\gamma_x(1)}{\gamma_x(0)}|^2]$$

2nd order ($p = 2$):

$$\begin{bmatrix} \gamma_x(0) & \gamma_x(-1) \\ \gamma_x(1) & \gamma_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} \gamma_x(1) \\ \gamma_x(2) \end{bmatrix}$$

and $E_2^f = \gamma_x(0) + a_1\gamma_x(-1) + a_2\gamma_x(-2)$.

In fact, $a_1 = a_1^0, \;\; a_2 = a_2^0$. Why ?

• For the 2nd order AR $x(n)$, one has

$$\gamma_x(m) + a_1^0 \gamma_x(m-1) + a_2^0 \gamma_x(m-2) = \sigma_w^2 h(-m)$$

where $\frac{1}{1+a_1^0 z^{-1}+a_2^0 z^{-2}} = \sum_{m=0}^{+\infty} h(m) z^{-m}$. See Eqn. (32) of the notes.

Since $h(0) = 1$ and $h(m) = 0$ for $m < 0$, with $m = 0, 1, 2$

$$\gamma_x(0) + a_1^0 \gamma_x(-1) + a_2^0 \gamma_x(-2) = \sigma_w^2$$

$$\gamma_x(1) + a_1^0 \gamma_x(0) + a_2^0 \gamma_x(-1) = 0$$

$$\gamma_x(2) + a_1^0 \gamma_x(1) + a_2^0 \gamma_x(0) = 0$$

Since for real-valued $x(n)$, $\gamma_x(-m) = \gamma_x(m)$. Therefore,

$$\begin{bmatrix} 1 & a_1^0 & a_2^0 \\ a_1^0 & 1 + a_2^0 & 0 \\ a_2^0 & a_1^0 & a_2^0 \end{bmatrix} \begin{bmatrix} \gamma_x(0) \\ \gamma_x(1) \\ \gamma_x(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sigma_w^2$$

With $a_1^0 = -1$, $a_2^0 = 0.6$ and $\sigma_w^2 = 1$, one has $\gamma_x(0) = 2.5641$, $\gamma_x(1) = 1.6026$, $\gamma_x(2) = 0.0640$ and hence

$$\gamma_x(3) + a_1^0 \gamma_x(2) + a_2^0 \gamma_x(1) = 0 \;\Rightarrow \gamma_x(3) =$$

$$\gamma_x(4) + a_1^0 \gamma_x(3) + a_2^0 \gamma_x(2) = 0 \;\Rightarrow \gamma_x(4) =$$

$$\gamma_x(5) + a_1^0 \gamma_x(4) + a_2^0 \gamma_x(3) = 0 \;\Rightarrow \gamma_x(5) =$$

1st order ($p = 1$):

$$a_1 = -\frac{\gamma_x(1)}{\gamma_x(0)} = \qquad\qquad , E_1^f = \gamma_x(0)[1 - |\frac{\gamma_x(1)}{\gamma_x(0)}|^2]$$

2nd order ($p = 2$):

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} \gamma_x(0) \ \gamma_x(1) \\ \gamma_x(1) \ \gamma_x(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma_x(1) \\ \gamma_x(2) \end{bmatrix} = \begin{bmatrix} \gamma_x(1) \\ \gamma_x(2) \end{bmatrix},$$

which satisfy $a_1 = a_1^0, \quad a_2 = a_2^0$, and

$$E_2^f = \gamma_x(0) + a_1\gamma_x(1) + a_2\gamma_x(2) = 1 = \sigma_w^2.$$