1. (a) $x[n] = 2\cos\left(\frac{\pi q n}{3}\right) = e^{j\left(\frac{\pi q n}{3}\right)} + e^{-j\left(\frac{\pi q n}{3}\right)}$

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(e^{j\left(\frac{\pi q n}{3}\right)} + e^{-j\left(\frac{\pi q n}{3}\right)}\right) e^{-jwn}$$

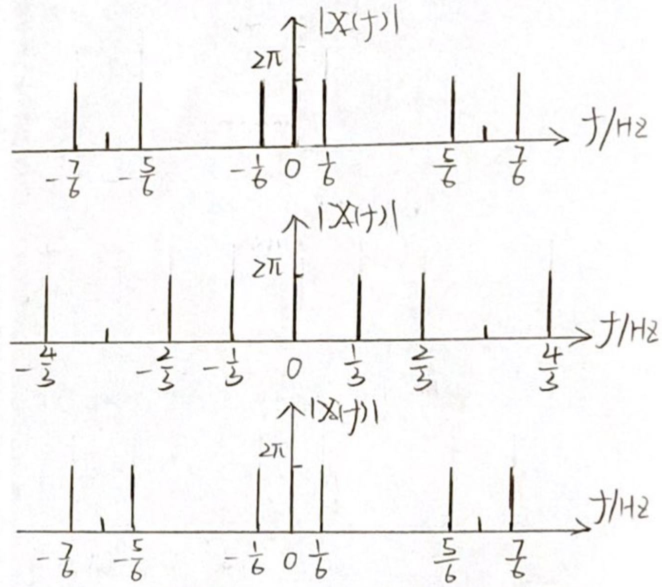
$$= \sum_{n=-\infty}^{\infty} \left(e^{j\left(\frac{\pi q}{3} - w\right)n} + e^{-j\left(\frac{\pi q}{3} + w\right)n}\right)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \left[\delta\left(w - \frac{\pi q}{3} - 2\pi k\right) + \delta\left(w + \frac{\pi q}{3} - 2\pi k\right)\right]$$

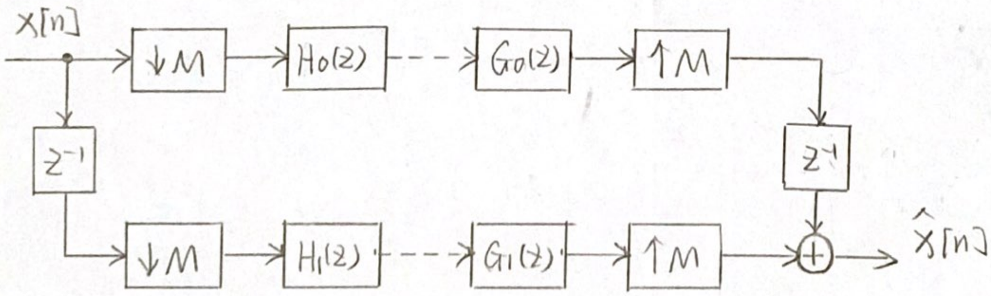
$q = 1$: $\quad f_1 = \frac{w}{2\pi} = \frac{\frac{\pi}{3}}{2\pi} = \frac{1}{6}$,

$q = 2$: $\quad f_2 = \frac{w}{2\pi} = \frac{\frac{2}{3}\pi}{2\pi} = \frac{1}{3}$

$q = 5$: $\quad f_3 = \frac{w}{2\pi} = \frac{\frac{5}{3}\pi}{2\pi} = \frac{5}{6}$



(b)



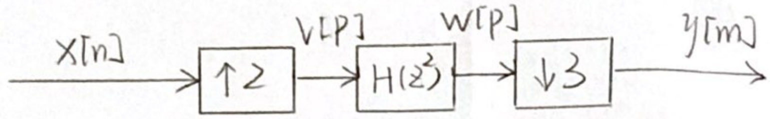
$H_0(z) = P_0(z^2) + z^{-1} P_1(z^2)$

$H_1(z) = P_0(z^2) - z^{-1} P_1(z^2)$

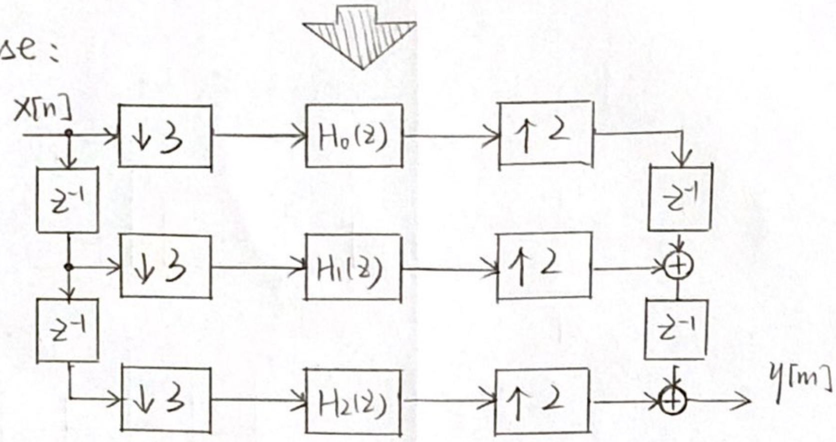
$G_0(z) = P_0(z^2) + z^{-1} P_1(z^2)$

$G_1(z) = -P_0(z^2) + z^{-1} P_1(z^2)$

2. (a) $\frac{I}{D} = \frac{2}{3}$

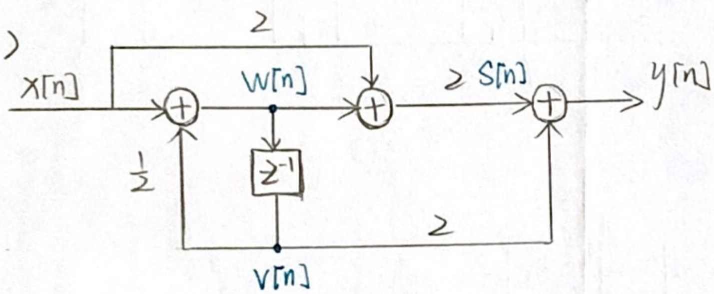


polyphase:



The subfilters are working at the lowest sampling frequency.

(b)



$W(z) = X(z) + \frac{1}{2}V(z)$, $V(z) = z^{-1}W(z)$.

$\Rightarrow W(z) = X(z) + \frac{1}{2}z^{-1}W(z)$, $X(z) = (1 - \frac{1}{2}z^{-1})W(z)$

$S(z) = W(z) + 2X(z)$

$$\begin{aligned} Y(z) = 2V(z) + 2S(z) &= 2z^{-1}W(z) + 2W(z) + 4X(z) \\ &= (2z^{-1} + 2)W(z) + (4 - 2z^{-1})W(z) \\ &= 6W(z) \end{aligned}$$

$$\frac{Y(z)}{X(z)} = \frac{6}{1 - \frac{1}{2}z^{-1}}$$

$\cos w_0 t = \frac{1}{2}(e^{jw_0 t} + e^{-jw_0 t})$, $\sin w_0 t = \frac{1}{2j}(e^{jw_0 t} - e^{-jw_0 t})$

$\Rightarrow F[\cos w_0 t] = \pi[\delta(w+w_0) + \delta(w-w_0)]$

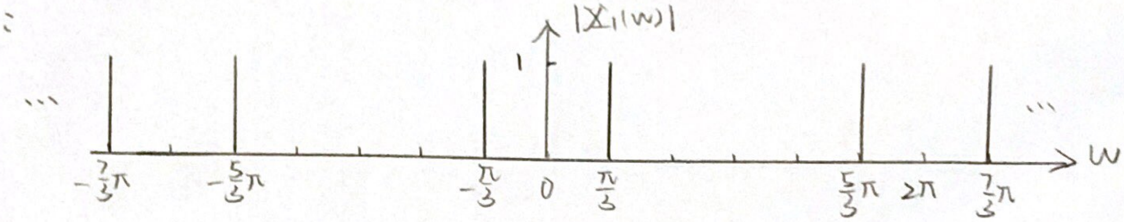
$F[\sin w_0 t] = j\pi[\delta(w+w_0) - \delta(w-w_0)]$

$x[n] = 2\cos(\frac{1}{3}\pi q n) = e^{j(\frac{1}{3}\pi q n)} + e^{-j(\frac{1}{3}\pi q n)}$.

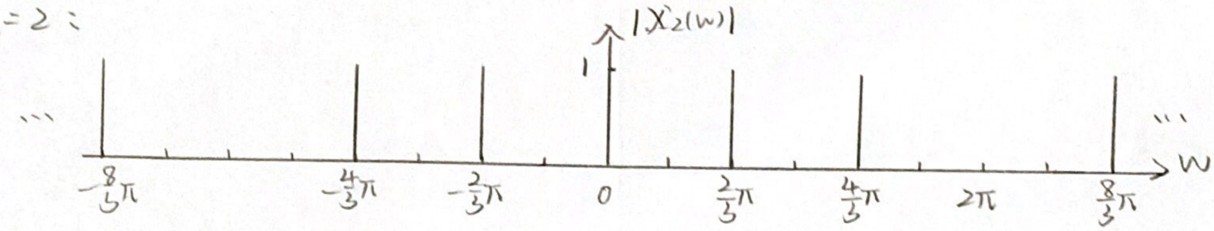
$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(e^{j(\frac{1}{3}\pi q - w)n} + e^{-j(\frac{1}{3}\pi q + w)n} \right)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} [\delta(w + \frac{1}{3}\pi q - 2k\pi) + \delta(w - \frac{1}{3}\pi q - 2k\pi)]$$

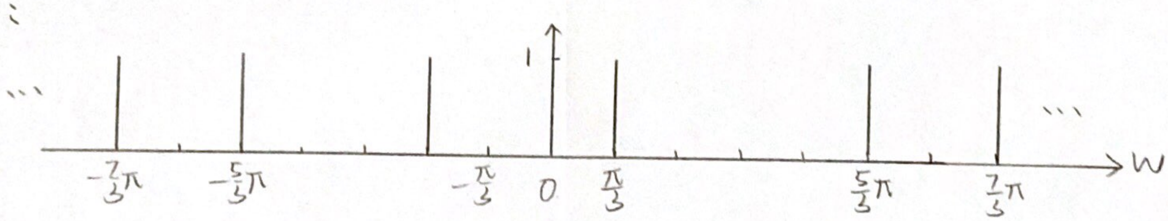
$q=1$:



$q=2$:



$q=5$:



$F[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t) e^{-jwt} dt = e^{-jwt}\Big|_0 = 1$.

$F^{-1}[1(t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 1 \cdot e^{jwt} dw = \delta(t)$.

$\Rightarrow \int_{-\infty}^{+\infty} e^{jwt} dw = 2\pi \delta(t)$

3. (a) parameters used to specify LPF:

① passband freq $\omega_p = 2\pi \frac{F_p}{F_T}$,

② stopband freq $\omega_s = 2\pi \frac{F_s}{F_T}$,

③ passband ripple $\delta_p = 0.5(1 - 10^{-\frac{\alpha_p}{20}})$,

④ stopband ripple $\delta_s = 10^{-\frac{\alpha_s}{20}}$.

⑤ filter order $N = \dfrac{-10\log_{10}(\delta_p\delta_s) - 13}{14.6 \times (\omega_s - \omega_p)/2\pi}$.

$\Rightarrow$ $\delta_p\delta_s \downarrow \rightarrow N\uparrow$; N is inversly proportional to the size of ripples.

$(\omega_s - \omega_p)\downarrow \rightarrow N\uparrow$; N is inversly proportional to the transition bandwidth.

$F_T \downarrow \rightarrow N\downarrow$. N is proportional to the sampling freq.

(b)

$$\longrightarrow \boxed{LPF} \longrightarrow \boxed{\downarrow 3} \dashrightarrow \boxed{LPF} \longrightarrow \boxed{\downarrow 16} \longrightarrow$$

$\delta_p = 0.01$, $\delta_s = 0.001$, $F_T = 48 \times 10^3\,Hz$

Stage 1: $D_1 = 3$, $\delta_{p1} = \frac{1}{2}\delta_p = 0.005$, $\delta_{s1} = \delta_s = 0.001$.

passband freq: $0 \sim 450Hz$.

stopband freq: $450 \sim (48K/3) - 450 = 15550\,Hz$,

$\Delta F_1 = (15550 - 450)/48\times10^3 = 0.31458$

$$N_1 = \frac{-10\log_{10}(\delta_{p1}\delta_{s1}) - 13}{14.6 \times \Delta F_1} + 1 = 9.7 \approx 10$$

Stage 2: $D_2 = 16$, $\delta_{p2} = \frac{1}{2}\delta_p = 0.005$, $\delta_{s2} = \delta_s = 0.001$

passband freq: $0 \sim 450Hz$,

stopband freq: $450 \sim 500\,Hz$.

input sampling freq: $F_{T2} = F_T/3 = 16\times10^3 Hz$, $F_s = \frac{F_T}{2D} = \frac{16\times10^3}{32} = 500Hz$,

ΔF: $(500 - 450)/16\times10^3 = 0.003125$,

$$N_2 = \frac{-10\log_{10}(\delta_{s2}\delta_{p2}) - 13}{14.6 \times \Delta F_2} + 1 = 877.9 \approx 878$$

$N_1 = 10$, $N_2 = 878$

$\frac{(N+1)}{2} \times F_i$ multiplications/sec

$N \times F_i$ additions/sec

$\Rightarrow$ stage 1: $\frac{10+1}{2} \times 48 \times 10^3 = 264 \times 10^3$ multiplications/sec

$10 \times 48 \times 10^3 = 48 \times 10^4$ additions/sec ;

stage 2: $\frac{878+1}{2} \times 16 \times 10^3 = 7032 \times 10^3 = 7.032 \times 10^6$ multiplications/sec

$878 \times 16 \times 10^3 = 1.4048 \times 10^7$ additions/sec.

4. $x[n] = s[n] + w[n]$, $s[n] - 0.4s[n-1] = v[n]$, $\sigma_v^2 = 0.36$

(a) $\Gamma_{dx}(z) = \Gamma_{ss}(z) = \sigma_v^2 H(z) H(z^{-1})$, $H(z) = \dfrac{1}{1-0.4z^{-1}}$

$$= \frac{0.36}{(1-0.4z^{-1})(1-0.4z)} = \frac{0.36}{0.84} \cdot \frac{1}{1-0.4z^{-1}}$$

$$\Gamma_{xx}(z) = \Gamma_{dx}(z) + \Gamma_{ww}(z) = \Gamma_{dx}(z) + 1$$

$$= \frac{(1-0.4z^{-1})(1-0.4z) + 0.36}{(1-0.4z^{-1})(1-0.4z)} = \frac{1.52 - 0.4(z^{-1}+z)}{(1-0.4z^{-1})(1-0.4z)}.$$

$r_{dx}[k] = \frac{3}{7} \times (0.4)^{|k|}$, $r_{xx}[k] = r_{dx}[k] + r_{ww}[k]$

$$= \frac{3}{7}(0.4)^{|k|} + \delta[k]$$

(b)
$$\begin{bmatrix} r_{xx}[0] & r_{xx}[-1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} r_{ss}[0] \\ r_{ss}[1] \end{bmatrix}$$

$$\begin{bmatrix} \frac{10}{7} & \frac{6}{35} \\ \frac{6}{35} & \frac{10}{7} \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{6}{35} \end{bmatrix}$$

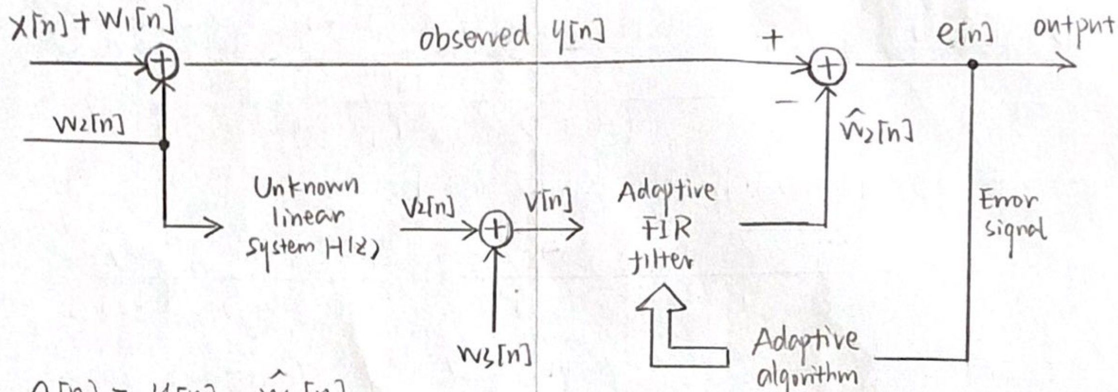
$$\begin{cases} \frac{10}{7} h[0] + \frac{6}{35} h[1] = \frac{3}{7} \\ \frac{6}{35} h[0] + \frac{10}{7} h[1] = \frac{6}{35} \end{cases} \Rightarrow \begin{cases} h[0] + \frac{42}{350} h[1] = \frac{3}{10} \\ h[0] + \frac{350}{42} h[1] = 1 \end{cases}$$

$$\Rightarrow h[1] = \frac{15}{176} \approx 0.0852 , \quad h[0] = \frac{51}{176} \approx 0.2898$$

$$\min \mathcal{E}_2^h = r_{ss}[0] - \begin{bmatrix} r_{ss}[0] & r_{ss}[1] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix}$$

$$= \frac{3}{7} - \left(\frac{3}{7} \times \frac{51}{176} + \frac{6}{35} \times \frac{15}{176}\right) \approx 0.2898.$$

5. adaptive noise-canceling system



(a) $e[n] = y[n] - \hat{w}_2[n]$

$= y[n] - \sum_{k=0}^{P} h[k]\, v[n-k]$, normal equation:

$\sum_{k=0}^{M-1} h[k]\, r_{vv}[l-k] = r_{vy}[l]$, $l = 0, 1, \cdots, M$

$v[n] = v_2[n] + w_3[n]$

$v_2[n] = \sum^{\infty}$

4. (a) $\Gamma_{ss}(z) = \sigma_v^2 H(z^{-1}) H(z)$

$$= \frac{0.36}{(1-0.4z^{-1})(1-0.4z)} = \frac{0.36}{0.84} \frac{1}{1-0.4z^{-1}} = \frac{3}{7} \frac{1}{1-0.4z^{-1}},$$

$$\gamma_{ss}[m] = \frac{3}{7}(0.4)^{|m|}. \qquad \gamma_{xx}[m] = \gamma_{ss}[m] + \gamma_{ww}[m]$$

$$= \frac{3}{7}(0.4)^{|m|} + \delta[m].$$

$$\Gamma_{xx}(z) = \Gamma_{ss}(z) + \Gamma_{ww}(z) = \frac{0.36}{(1-0.4z^{-1})(1-0.4)} + 1.$$

(b) $$\begin{bmatrix} \gamma_{xx}[0] & \gamma_{xx}[1] \\ \gamma_{xx}[-1] & \gamma_{xx}[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} \gamma_{ss}[0] \\ \gamma_{ss}[1] \end{bmatrix}.$$

$$\begin{bmatrix} \frac{10}{7} & \frac{6}{35} \\ \frac{6}{35} & \frac{10}{7} \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{6}{35} \end{bmatrix}.$$

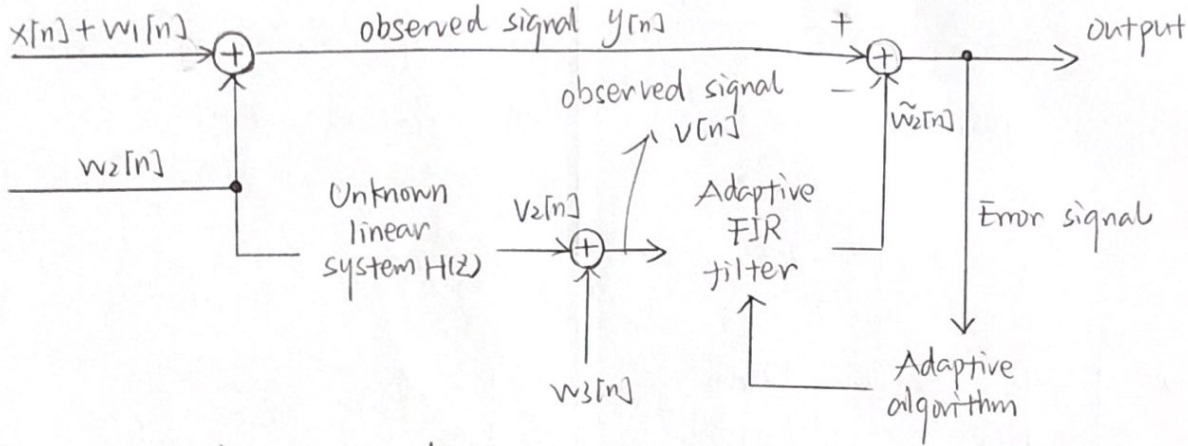
$$\begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} 0.7102 & -0.0852 \\ -0.0852 & 0.7102 \end{bmatrix} \begin{bmatrix} \frac{3}{7} \\ \frac{6}{35} \end{bmatrix}$$

$$= \begin{bmatrix} 0.2898 \\ 0.0852 \end{bmatrix}$$

$$\varepsilon_m^h = \gamma_{ss}[0] - [\gamma_{ss}[0] \quad \gamma_{ss}[1]] \begin{bmatrix} h[0] \\ h[1] \end{bmatrix}$$

$$= \frac{3}{7} - [\tfrac{3}{7} \quad \tfrac{6}{35}] \begin{bmatrix} 0.2898 \\ 0.0852 \end{bmatrix} = 0.2898.$$

5. adaptive noise canceling system:



$x[n]$ is a stationary random process.

$w_1[n]$, $w_2[n]$, $w_3[n]$ are white noise sequence with $\sigma_{w_1}^2 = 0.04$, $\sigma_{w_2}^2 = 0.16$, $\sigma_{w_3}^2 = 1$. $v[n] = v_2[n] + w_3[n]$, $y[n] = x[n] + w_1[n] + w_2[n]$;

$e[n] = y[n] - \tilde{w}_2[n] = y[n] - \sum_{k=0}^{M-1} h[k]\, v[n-k]$; $\quad v_2[n] = \sum_{l=0}^{\infty} h[l]\, w_2[n-l]$;

normal equation: $\gamma_{vy}[l] = \sum_{k=0}^{M-1} h[k]\, \gamma_{vv}[n-l]$.

$$\gamma_{vv}[k] = \frac{1}{N}\sum_{n=0}^{N-1} v[n]\, v[n-k]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{l=0}^{\infty} h[l]\, w_2[n-l] + w_3[n]\Big)\Big(\sum_{p=0}^{\infty} h[p]\, w_2[n-k-p] + w_3[n-k]\Big)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{l=0}^{\infty}\sum_{p=0}^{\infty} h[p]\, h[l]\, w_2[n-k-p]\, w_2[n-l] + w_3[n]\sum_{p=0}^{\infty} h[p]\, w_2[n-k-p] +$$
$$w_3[n-k]\sum_{l=0}^{\infty} h[l]\, w_2[n-l] + w_3[n]\, w_3[n-k]\Big)$$

$$E[\gamma_{vv}[k]] = \gamma_{w_3w_3}[k] + \sum_{l=0}^{\infty} h[l]\, \gamma_{w_2w_3}[k-l] + \sum_{p=0}^{\infty} h[p]\, \gamma_{w_3w_2}[k+p]$$
$$+ \sum_{l=0}^{\infty}\sum_{p=0}^{\infty} h[p]\, h[l]\, \gamma_{w_2w_2}[k+p-l].$$

$$\gamma_{vy}[k] = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\, v[n-k] = \frac{1}{N}\sum_{n=0}^{N-1}(x[n] + w_1[n] + w_2[n])\Big(\sum_{l=0}^{\infty} h[l]\, w_2[n-k-l] + w_3[n-l]\Big)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{\infty}(x[n] + w_1[n] + w_2[n])\, h[l]\, w_2[n-k-l] + \frac{1}{N}\sum_{n=0}^{N-1}(x[n] + w_1[n] + w_2[n])\cdot w_3[n-l],$$

$$E[\gamma_{vy}[k]] = \sum_{l=0}^{\infty} h[l]\big(\gamma_{xw_2}[k+l] + \gamma_{w_1w_2}[k+l] + \gamma_{w_2w_2}[k+l]\big) + \gamma_{xw_3}[l] + \gamma_{w_1w_3}[l]$$
$$+ \gamma_{w_2w_3}[l].$$

$\Gamma_M h_M = \gamma_y$, $\quad \Gamma_M[n+1] = \Gamma_M[n] + \frac{1}{2}\Delta[n]\, S[n] = \Gamma_M[n] - \frac{1}{2}\Delta[n]\, g[n]$

$$g[n] = \frac{d\mathcal{E}_M}{d h_M} = 2(\Gamma_M h_M - \gamma_y)$$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2017-2018

EE6401 – ADVANCED DIGITAL SIGNAL PROCESSING

November/December 2017 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 5 questions and comprises 3 pages.
2. Answer all 5 questions.
3. All questions carry equal marks.
4. This is a closed-book examination.

$X(w) = 2\pi \sum_{k=-\infty}^{\infty} (\delta(w + \frac{q}{3}\pi - 2k\pi) + \delta(w - \frac{q}{3}\pi - 2k\pi))$

---

1. (a) Find the discrete-time Fourier transform (DTFT) of the signal

$$x[n] = 2\cos(\frac{\pi q n}{3})$$

for q = 1, 2 and 5. Plot these DTFTs and comment on your observations.

(8 Marks)

(b) For a two-channel alias-free quadrature filter bank, the filters $H_0(z)$, $H_1(z)$, $G_0(z)$ and $G_1(z)$ in the analysis and synthesis sections can be implemented with a two-branch polyphase filter structure. For example, $H_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$, where $P_0(z)$ and $P_1(z)$ are the expressions of the branch filters in the polyphase implementation. Find a computationally efficient polyphase implementation of the two-channel alias-free quadrature filter bank.

(12 Marks)

2. (a) A sampling rate converter is needed to change its input sampling frequency by 2/3 times. To minimize the computational costs, *derive* a polyphase structure in its *most efficient* form, and *explain* why the computational complexity is reduced by the polyphase implementation compared to the implementation directly using a finite impulse response (FIR) filter.

(14 Marks)

(b) Determine the z-transform of the impulse response of the system shown in Figure 1 below.

(6 Marks)

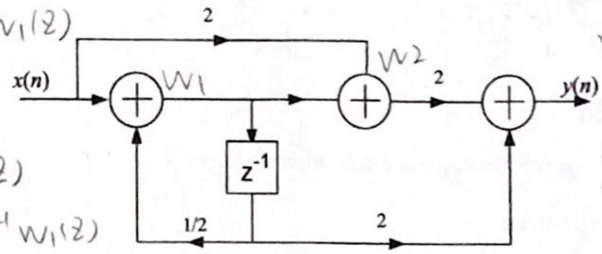
$W_1(z) = X(z) + \frac{1}{2}z^{-1}W_1(z)$

$X(z) = (1-\frac{1}{2}z^{-1})W_1(z)$

$W_2(z) = W_1(z) + 2X(z)$

$Y(z) = 2W_2(z) + 2z^{-1}W_1(z)$

$= 4X(z) + (2+2z^{-1})W_1(z)$



$w_1[n] = x[n] + \frac{1}{2}w_1[n-1]$

$w_2[n] = w_1[n] + 2x[n]$

$y[n] = 2w_2[n] + 2w_1[n-1]$

Figure 1

3. (a) List the parameters that are used to specify a low-pass digital filter and generate the filter coefficients. Briefly discuss how these parameters affect the filter order.

$= (4(1-\frac{1}{2}z^{-1}) + 2 + 2z^{-1})W_1(z)$

(5 Marks)

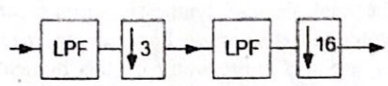
$= 6W_1(z)$

$= \frac{6}{1-\frac{1}{2}z^{-1}}X(z)$

$D=48$

(b) The sampling rate of a signal $x[n]$ is to be reduced, by decimation, from 48 kHz to 1 kHz. The highest frequency of interest after decimation is 450 Hz. Assume that an optimal FIR filter is to be used, with an overall passband ripple, $\delta_p = 0.01$, and stopband ripple, $\delta_s = 0.001$. Design an efficient decimator with two-stage FIR filters shown in Figure 2, and calculate the required computational complexity in terms of the number of additions and multiplications per second.

Stage 1:



$D = \frac{F_T}{2F_{SC}}$

$F_{T1} = 48\text{kHz}$, $F_1 = \frac{48k}{3} = 16\text{kHz}$

Figure 2

(15 Marks)

$F_{SC} = 16k - 450 = 15550\text{ Hz}$,

$\Delta F = (F_{SC} - F_{PC})/F_T = (15550-450)/48k = 0.31458$

$N = \frac{-10\log_{10}(\delta_s\delta_p) - 13}{14.6(\omega_s - \omega_p)/2\pi} + 1$

Stage 2:

$F_{T2} = F_1 = 16\text{kHz}$, $F_2 = \frac{F_T}{D} = 1\text{kHz}$, $F_{SC} = \frac{F_T}{2D} = \frac{16k}{32} = 500\text{Hz}$

$\Delta F = (500-450)/16k$

$\Gamma_{ss}(z) = \sigma_v^2 H(z^{-1}) H(z) = \frac{0.36}{(1-0.4z^{-1})(1-0.4z)}$

$\Gamma_{xx}(z) = \Gamma_{ss}(z) + \Gamma_{ww}(z) = \frac{0.36+1.16-0.4(z^{-1}+z)}{(1-0.4z^{-1})(1-0.4z)} = \frac{1.52-0.4(z^{-1}+z)}{(1-0.4z^{-1})(1-0.4z)}$

4. Consider a signal $x[n] = s[n] + w[n]$ where $\{s[n]\}$ is an $AR(1)$ process that satisfies the difference equation $s[n] = 0.4s[n-1] + v[n]$ where $w[n]$ and $v[n]$ are uncorrelated white noise sequences with variances $\sigma_v^2 = 0.36, \sigma_w^2 = 1$.

   (a) Determine the power density spectrum of $x[n]$.

   (10 Marks)

   (b) Design a Weiner filter of length $M = 2$ to estimate $s[n]$.

   (10 Marks)

5. Consider an adaptive noise-canceling system illustrated in Figure 3 where $x[n]$ is a stationary random process and $w_1[n], w_2[n], w_3[n]$ are white noise sequences with variances $\sigma_{w_1}^2 = 0.04, \sigma_{w_2}^2 = 0.16, \sigma_{w_3}^2 = 1$, respectively.

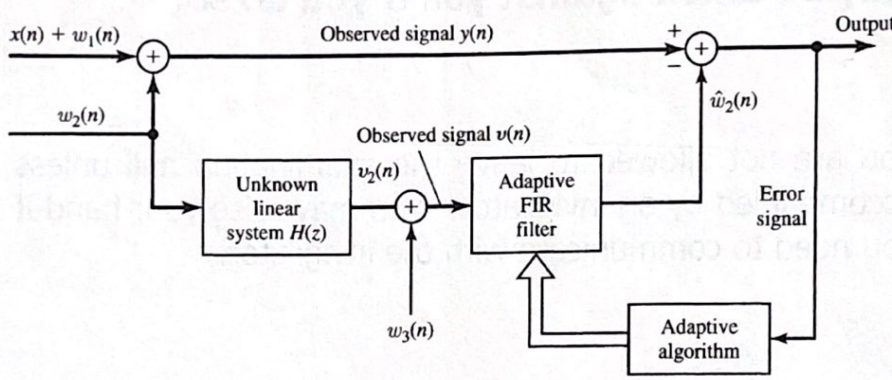


Figure 3

   (a) Determine the equations of the overall system.

   (10 Marks)

   (b) Explain how the Least Mean Square (LMS) Algorithm can be applied in this application.

   (10 Marks)

END OF PAPER