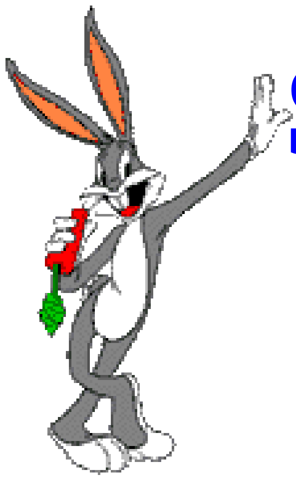
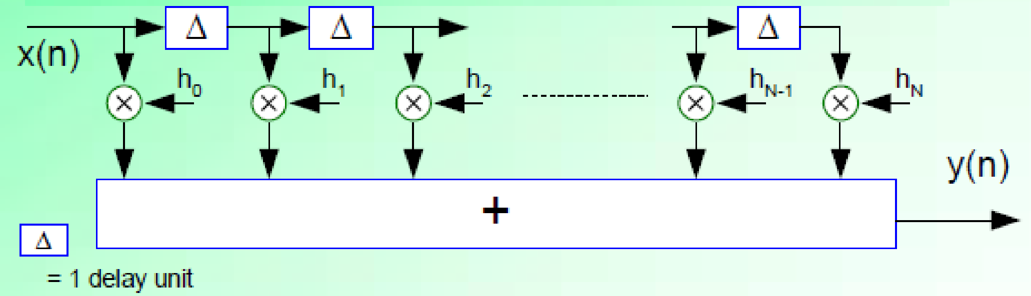
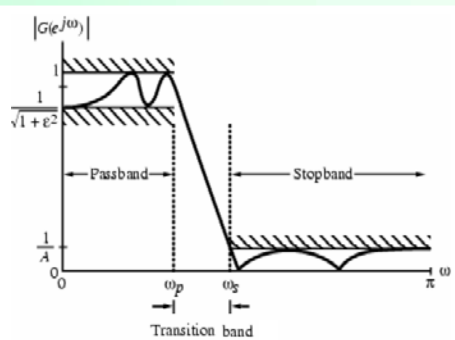


Specifications of Filters

$$y(n) = \sum_{m=0}^{M} h_m x(n-m) = h(n) * x(n)$$





Digital Filter Design

Objective: Determining a **realizable** transfer function $G(z)$ **approximating** a given frequency response specification.

- Generally classified into categories of finite impulse response (FIR) and infinite impulse response (IIR).
- $G(z)$ of IIR filter should be a *stable* and *real rational* function of two polynomials of z.
- $G(z)$ of FIR filter is a polynomial and is always stable, why?
- Digital filter design is the process of deriving the transfer function $G(z)$ to **approximate** the intended specifications.
- Either the *magnitude* and/or the *phase* (*delay*) response of the digital filter are specified for most applications
- The problem of interest is the development of a ***realizable approximation*** to a given magnitude response specification
- We mainly discuss the **magnitude approximation** of **FIR** filter

Digital Filter Design – Filter Specifications

- The magnitude response specifications of a digital filter in the passband and the stopband are given with some acceptable tolerances
- In addition, a *transitional* band is specified between the passband and stopband
- For example, the magnitude response $G(e^{j\omega})$ of a digital *lowpass* filter may be given as indicated in the Figure
- Similarly, other types of filters also have the transitional band.

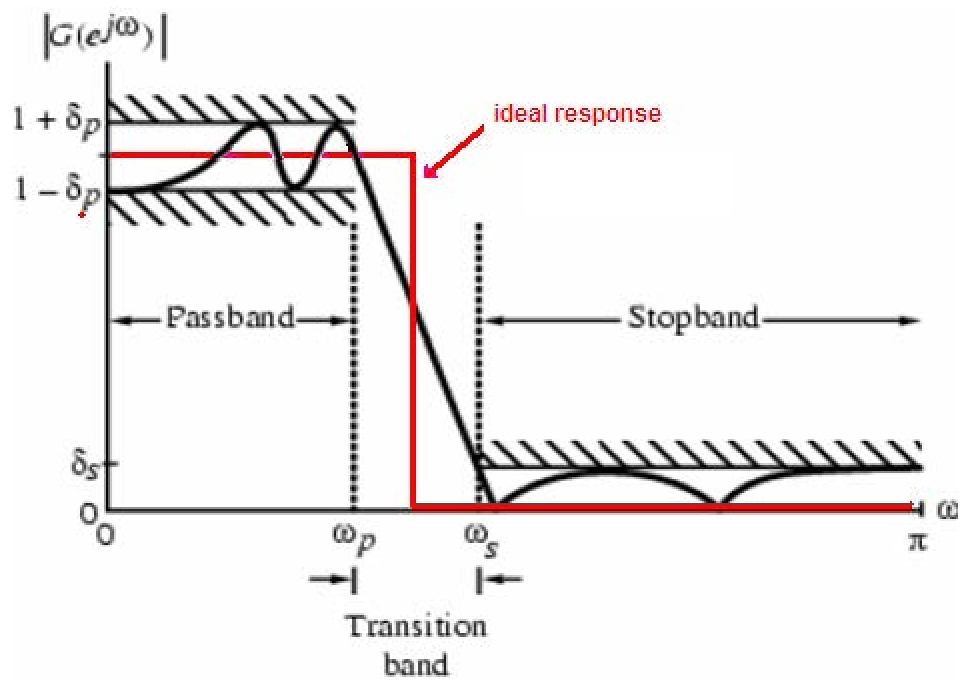


Figure 31 Filter specification parameters

Digital Filter Design – Filter Specifications

- The passband is defined by $0 \le \omega \le \omega_p$, $|G(e^{j\omega})| \approx 1$ with an error $\pm\,\delta_p$, i.e.,

$$1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- The stopband is defined by $\omega_s \le \omega \le \pi$, $|G(e^{j\omega})| \approx 0$ with an error δ_s, that is

$$|G(e^{j\omega})| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

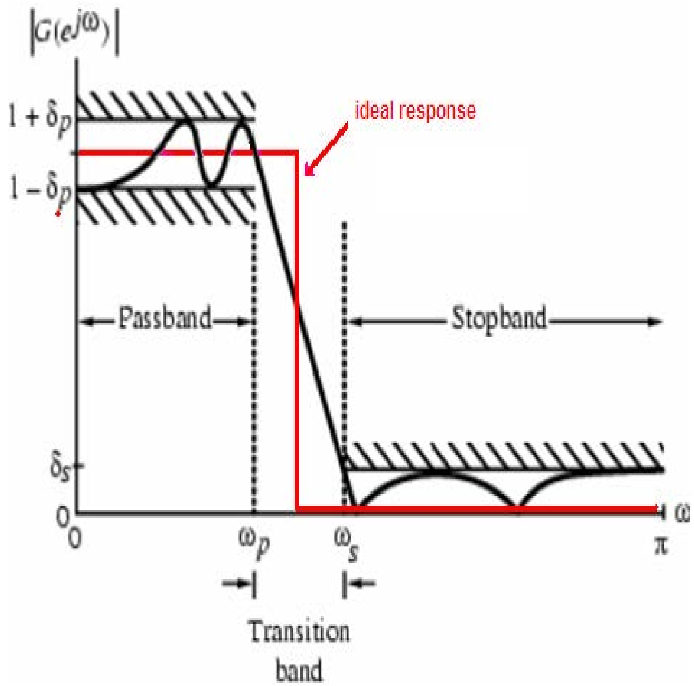
where

ω_p - passband edge frequency;

ω_s - stopband edge frequency,

δ_p - peak ripple in the passband;

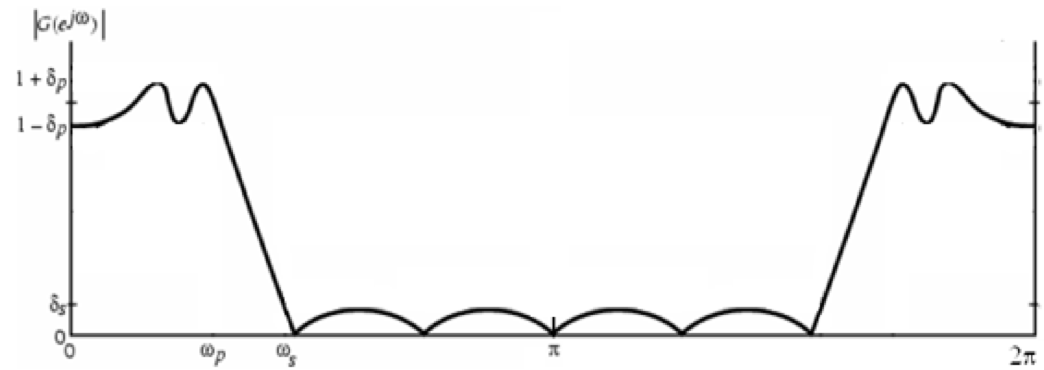
δ_s - peak ripple in the stopband



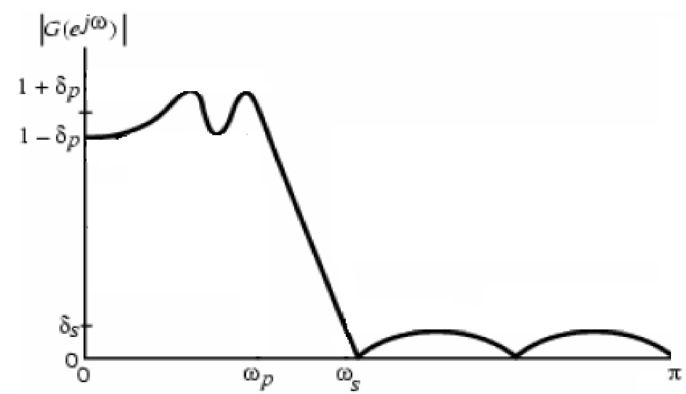
Digital Filter Design – Filter Specifications

- $G(e^{j\omega})$ is a *periodic* function of ω, and $|G(e^{j\omega})|$ is the spectrum magnitude of a *real*-coefficient filter and an *even function* of ω.
- As a result, filter specifications are only for the frequency range $0 \leq \omega \leq \pi$

- **Example**



(a) A complete view of filter magnitude response



(b) Simplified view of filter magnitude response if coefficients are real

Figure 32 Representation of filter magnitude response

Digital Filter Design – Filter Specifications

- Specifications are often given in terms of *loss function* $A(\omega) = -20\log|G(e^{j\omega})|$ in dB
- Peak passband ripple $\alpha_p(\omega) = -20\log_{10}|1 - 2\delta_p|$ dB
- The minimum stopband attenuation is $\alpha_s(\omega) = -20\log_{10}\delta_s$ dB
- Magnitude specifications may be alternately given in a *normalized* form as indicated

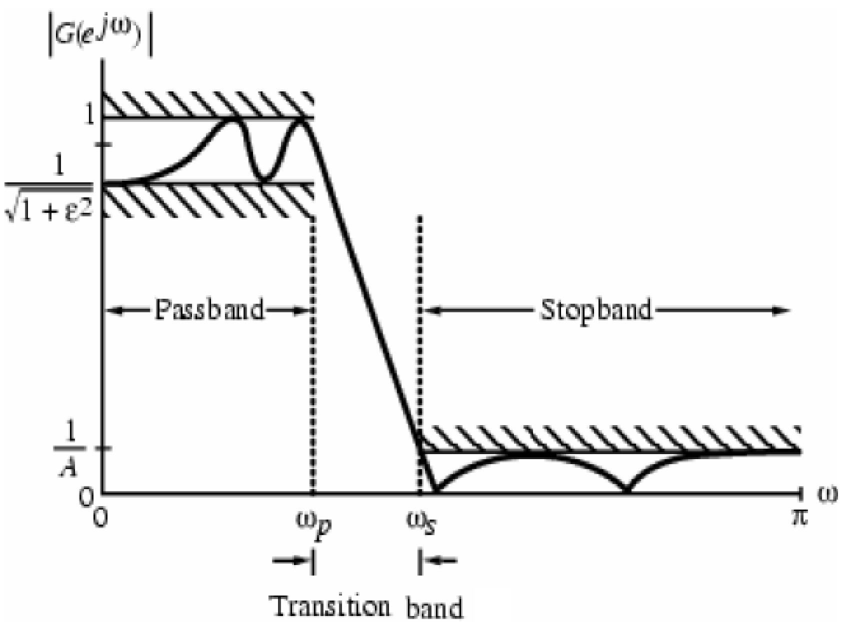
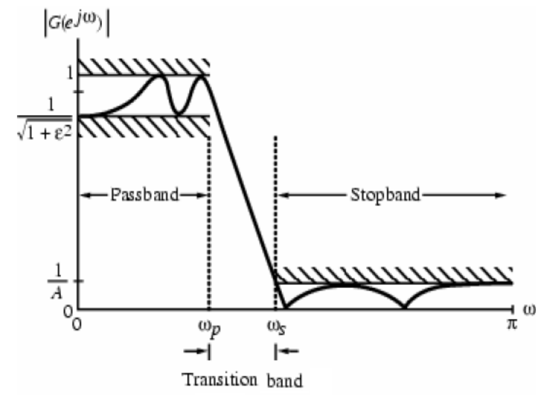


Figure 33 Normalized filter specification

Digital Filter Design – Filter Specifications

- Here, the maximum value of the magnitude in the passband is assigned to be unity (or 0 dB)
- $1/\sqrt{1+\varepsilon^2}$ is the *maximum passband deviation* given by the *minimum* value of the magnitude in the passband. ε^2 is a value larger than 0.
- $1/A$ is the *maximum stopband magnitude*
- For the normalized specification, maximum value of the *gain function* or the minimum value of the *loss function* is 0 dB
- *Maximum passband* attenuation

$$\alpha_{\max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right)\,\text{dB}$$



- For $\delta_p << 1$, it can be shown that

$$\alpha_{\max} \cong -20\log_{10}(1-2\delta_p)\,\text{dB}$$

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz

Digital Filter Design – Filter Specifications

- For digital filter design, *normalized* band edge frequencies need to be computed from specifications in *Hz* using

$$\omega_p = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

where T is the sample interval.

- **Example** : Let F_s = 7 *k*Hz, F_p = 3 *k*Hz and F_T = 25 *k*Hz. Then

$$\omega_s = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$

$$\omega_p = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

Digital Filter Design – FIR Filters

- For FIR digital filter design, the transfer function is a polynomial of z^{-1} with real coefficients:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- To minimize computational complexity, order N of $H(z)$ must be as small as possible
- If a *linear phase* is desired, the filter coefficients must satisfy the constraint:

$$h[n] = \pm\, h[N - n],\ n = 0, 1, \ldots, N/2\text{-}1 \text{ or } (N+1)/2\text{-}1.$$

- Advantages of FIR filter :
  - Can be designed with exact linear phase,
  - Filter structure is always stable with quantized coefficients
- Disadvantages of FIR filter :
  - Order of an FIR filter is considerably higher than the order of an equivalent IIR filter meeting the same *amplitude specifications*, and FIR filter has thus *higher* computational complexity (why?)

How Are the Filter Coefficients Calculated to Meet the Specifications?

Windowed Fourier Series Approach

Digital Filter Design – Basic Approach

- FIR filter design is based on a direct approximation of the specified *magnitude response*, with the added requirement of *linear phase.*
- The design of an FIR filter of order N may be accomplished by finding either the length-(N+1) impulse response samples $\{h[n]\}$ or the (N+1) samples of its frequency response $H(e^{j\omega})$
- Three commonly used approaches to FIR filter design
  - Windowed Fourier series approach
  - Frequency sampling approach
  - Computer-based optimization methods
- We next describe the design approach based on windowed Fourier series approach.

Design of FIR Digital Filters – Basic Approach

- FIR *filter order* can be estimated by the Kaiser's formula as follows:

$$N \cong \frac{-10\log_{10}(\delta_p\delta_s)-13}{14.6(\omega_s-\omega_p)/2\pi} \qquad \delta_p = 0.5(1-10^{-\alpha_p/20}) \qquad \delta_s = 10^{-(\alpha_s/20)}$$

  – **Example:** Estimate the order of a linear-phase lowpass FIR filter with the specifications: passband edge F_p = 1.8 kHz, stopband edge F_s = 2 kHz, peak passband ripple α_p = 0.1 dB, minimum stopband attenuation α_s = 35 dB, and sampling rate F_T = 12 kHz.

- We get δ_p = 0.005723 and δ_s = 0.01778279. Substituting these values and band edge frequencies in the above equation, we get

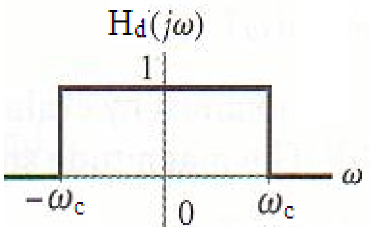
$$N \cong \frac{-10\log_{10}(0.00010177)-13}{14.6(2000-1800)/12,000} = 110.78 \cong 111$$

- Other estimation method for the filter order such as Hermann's formula also exists.

Design of FIR Digital Filters – LSE Approach

- Let $H_d(e^{j\omega})$ be the desired frequency response, and is a periodic function of ω with a period 2π, expressed as a Fourier series

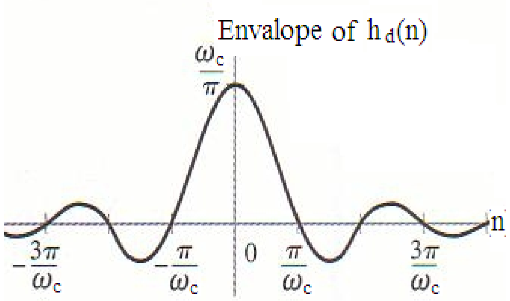
$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}$$



where

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega,$$

$$-\infty < n < \infty$$



- In this case, $\{h_d[n]\}$ is of *infinite* length and *non-causal*
- **Objective:** Find a *finite-duration* $\{h_t[n]\}$ of length N+1 whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$ *in some sense.*

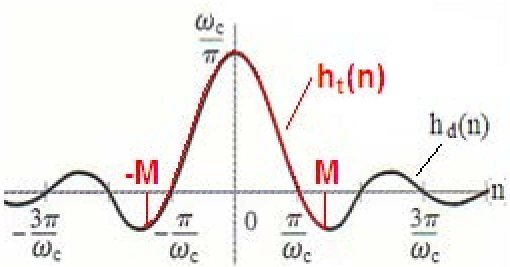
Design of FIR Digital Filters – LSE Approach

- One commonly used approximation criterion is the *integral-squared error*

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_t(e^{j\omega}) - H_d(e^{j\omega})|^2 \, d\omega$$

where

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n]e^{-j\omega n}$$



- Using Parseval's relation we can write

$$\Phi = \sum_{-\infty}^{\infty} |h_t[n] - h_d[n]|^2$$

$$= \sum_{-M}^{M} |h_t[n] - h_d[n]|^2 + \sum_{-\infty}^{-(M+1)} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

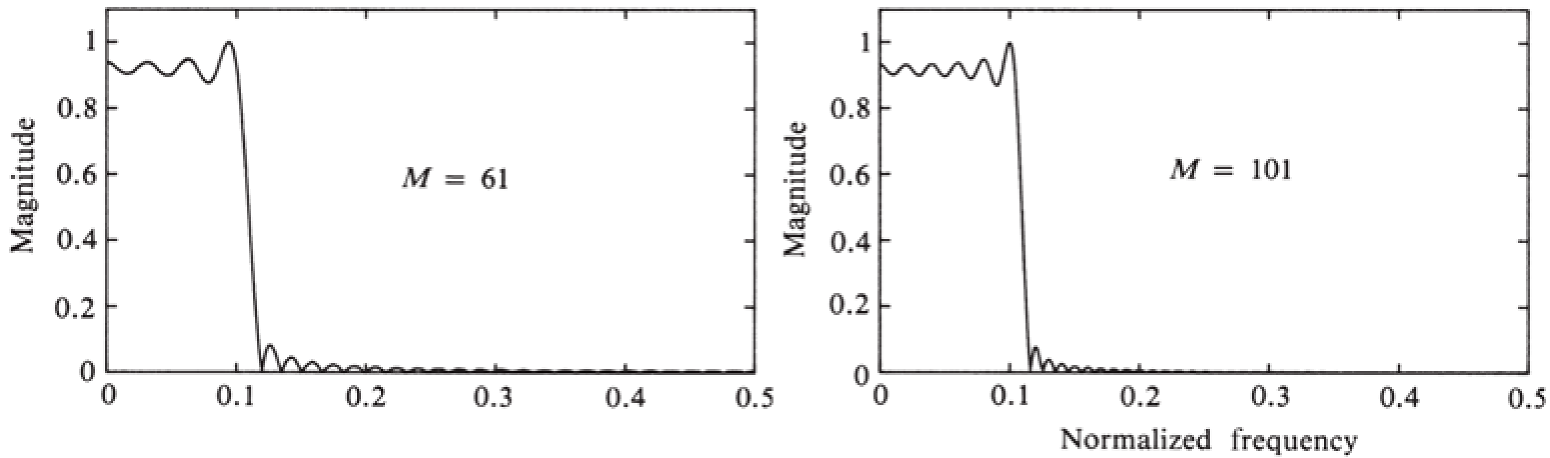
- It follows from above that Φ is minimum when

$$h_t[n] = h_d[n] \quad \text{for } -M \le n \le M$$

- Best finite-length approximation to ideal infinite-length impulse response in the *mean-square sense* is obtained by *truncation*

Design of FIR Digital Filters – Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Lowpass filter designed with a rectangular window: (a) $M = 61$ and (b) $M = 101$.

- The number of ripples in both passband and stopband increases as the length of the lowpass filter is increased, with a corresponding decrease in the ripple widths

Design of FIR Digital Filters – Gibbs Phenomenon

- Height of the largest ripples remain the same, i.e., being independent of filter length
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types (such as high pass, bandpass and bandstop) of ideal filters
- Gibbs phenomenon can be explained by treating the truncation operation as a windowing operation with $w[n]$:

$$h_t[n] = h_d[n] \cdot w[n]$$

- In the frequency domain, we have the convolution

$$H_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\varphi}) W(e^{j(\omega-\varphi)}) d\varphi$$

where $H_t(e^{j\omega})$ and $W(e^{j\omega})$ are the DTFTs of $h_t[n]$ and $w[n]$, respectively

Design of FIR Digital Filters – Gibbs Phenomenon

- Thus, $H_t(e^{j\omega})$ is obtained from a continuous convolution of $H_t(e^{j\omega})$ and $W(e^{j\omega})$

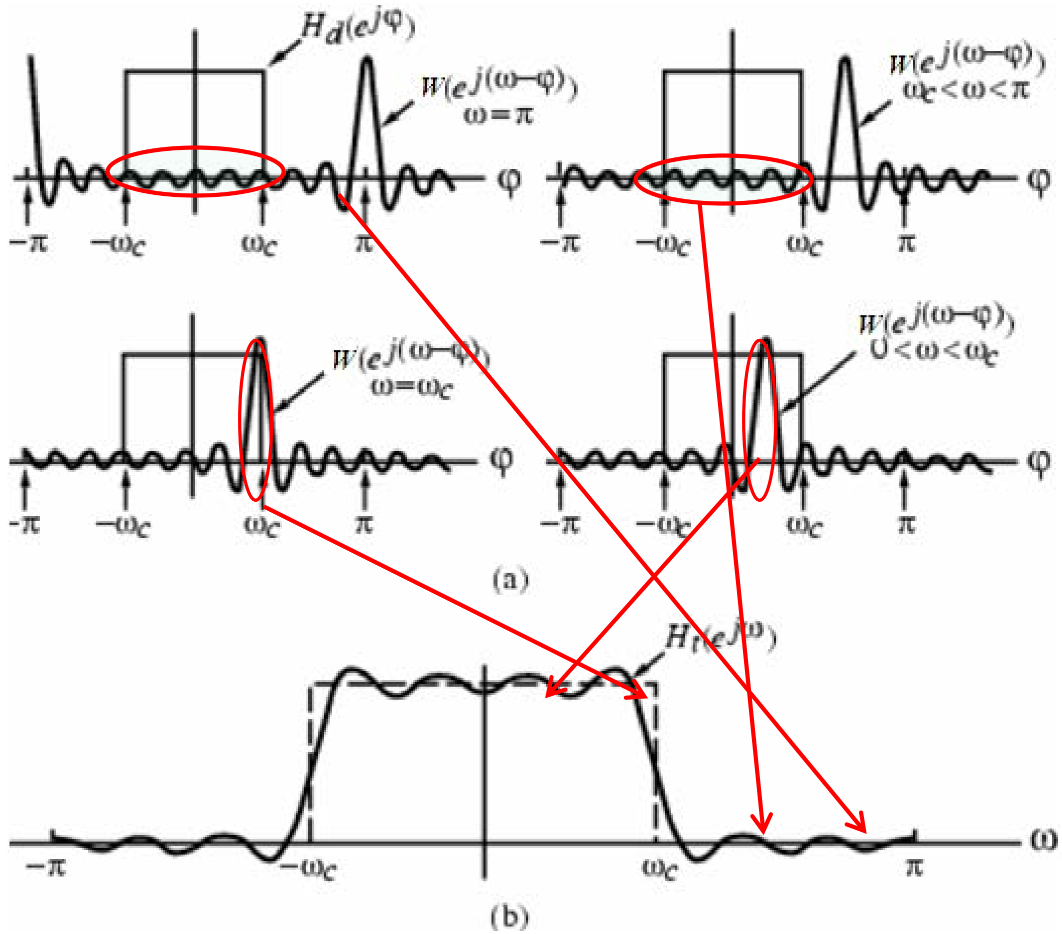


Figure 35 Window based method for generating filter coefficients

Design of FIR Digital Filters – Gibbs Phenomenon

- If $W(e^{j\omega})$ is a very narrow pulse centered $\omega = 0$ (ideally a delta function) compared to variations in $H_d(e^{j\omega})$, then $H_t(e^{j\omega})$ will approximate $H_d(e^{j\omega})$ very closely, which means length N=2M + 1 of $w[n]$ should be very large.
- On the other hand, length 2M + 1 of $h_t[n]$ should be *as small as possible* to reduce computational complexity

- Gibbs phenomenon can be reduced either:
  – using a window that tapers smoothly to zero at each end, or
  – providing a smooth transition from passband to stopband in the magnitude specifications

Design of FIR Digital Filters – Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a *wider transition* at the discontinuity

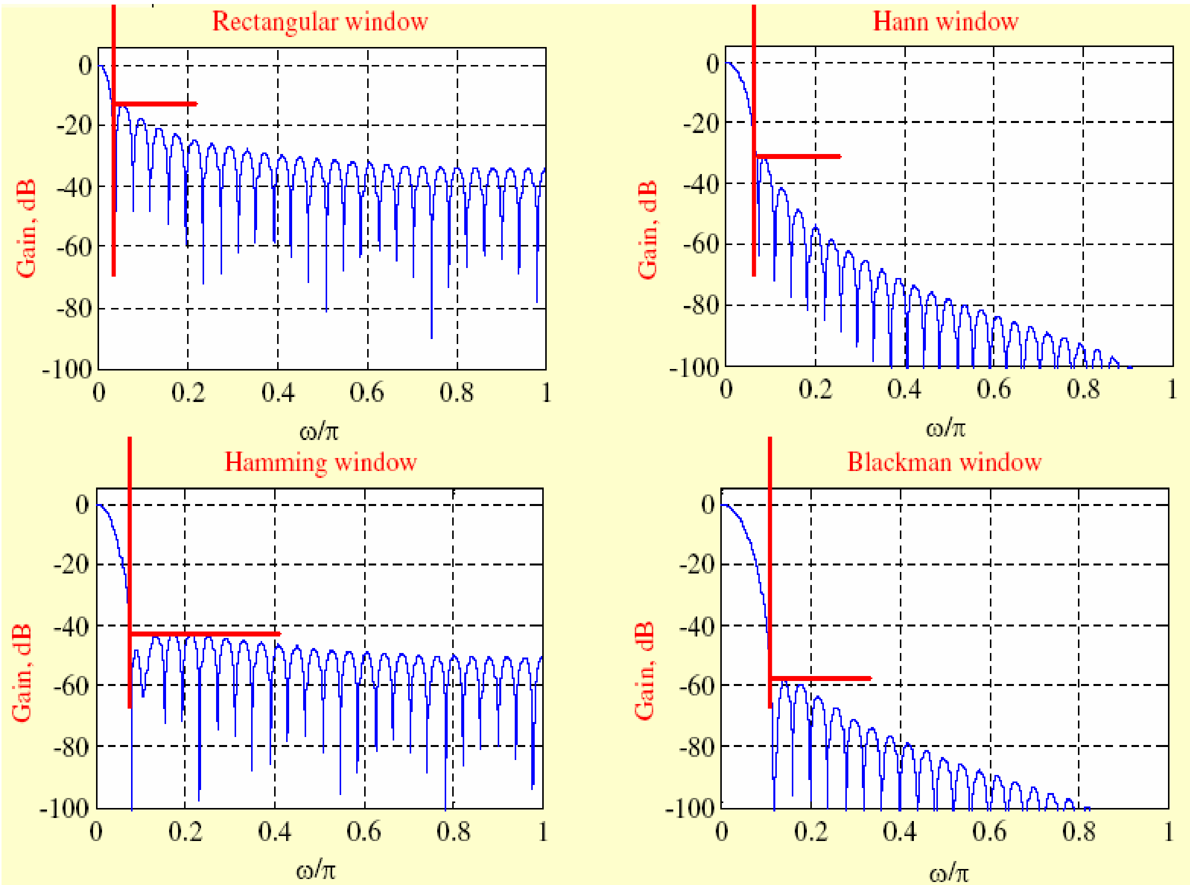
- Illustration of windows

Assuming the window length N = 2M+1, we have

- Hann $$w[n]=0.5+0.5\cos(\frac{2\pi n}{2M+1}), \qquad -M\le n\le M$$
- Hamming: $$w[n]=0.54+0.46\cos(\frac{2\pi n}{2M+1}), \qquad -M\le n\le M$$
- Blackman $$w[n]=0.42+0.5\cos(\frac{2\pi n}{2M+1})+0.08\cos(\frac{4\pi n}{2M+1}),$$

Design of FIR Digital Filters – Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for M = 25 are shown below:

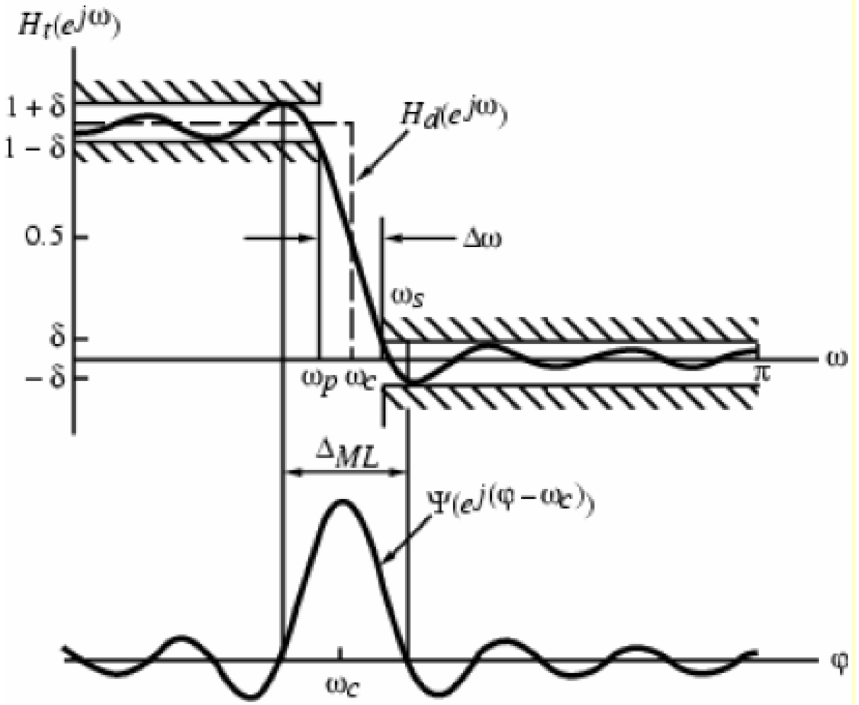


- As the side lobes reduced, the width of the main lobe increases

Design of FIR Digital Filters – Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at $\omega = 0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are
  - –Main lobe width
  - –Relative sidelobe level

Design of FIR Digital Filters – Fixed Window Functions

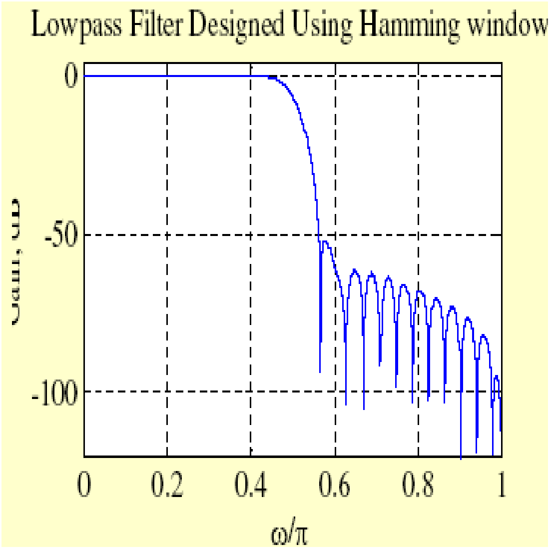


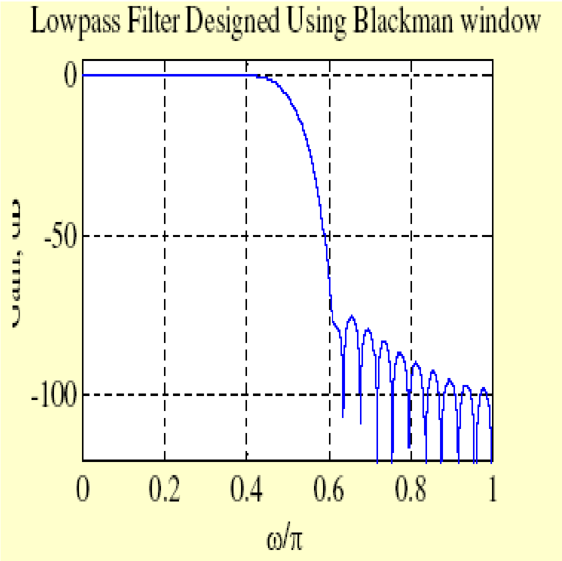
- Observe $$H_t(e^{j(\omega_c+\Delta\omega)})+H_t(e^{j(\omega_c-\Delta\omega)})\cong 1$$
- Thus $$H_t(e^{j(\omega_c)})\cong 0.5$$
- Ripples in passband and stopband are about the same.

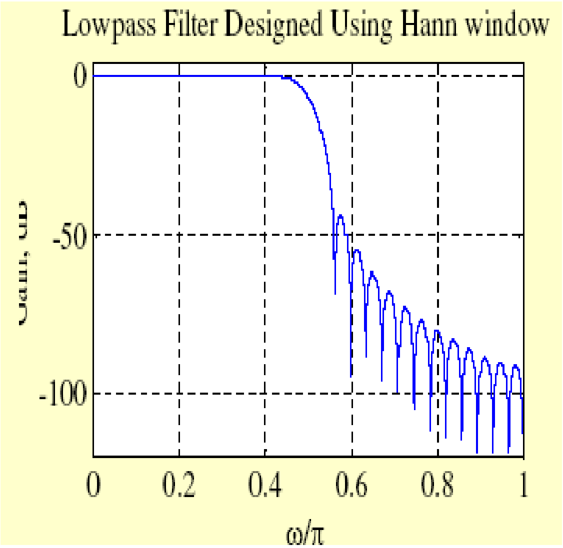
Design of FIR Digital Filters – Design Examples

Design steps:

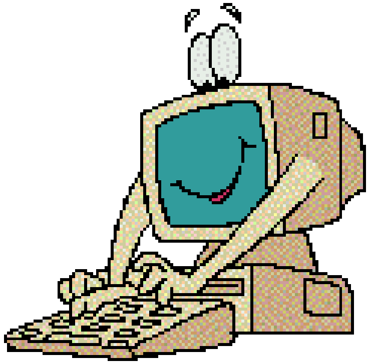
1. Set $\omega_c = (\omega_p + \omega_s)/2$
2. Choose window based on specified α_s
3. Estimating M using $\Delta\omega \approx \dfrac{c}{M}$ where c is a constant value for a given window
4. Lowpass filter of length 51 and $\omega_c = \pi/2$ (See the Figures).







Optimization of Filter Design

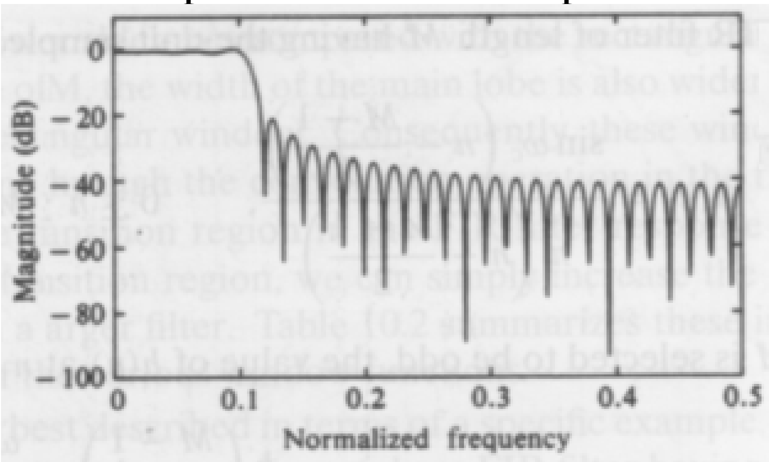


Computer-Aided Design of Digital Filters

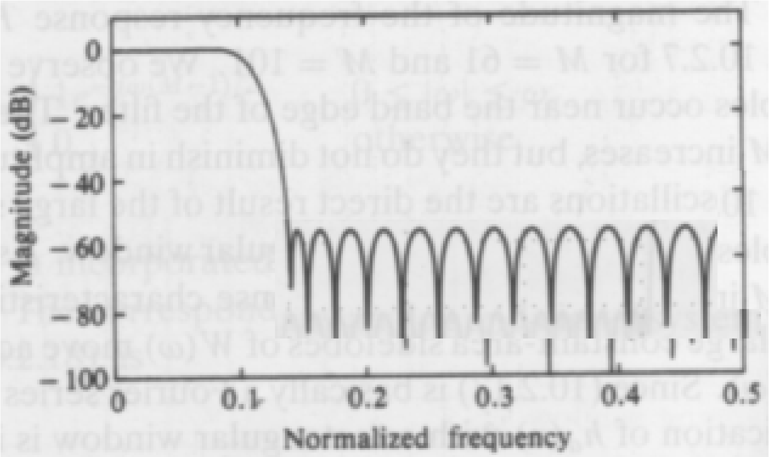
- The FIR filter design techniques discussed so far can be easily implemented with computer programs
- In addition, there are a number of FIR filter design algorithms that rely on some optimization techniques to minimize the errors between the desired frequency response and that of the computer-generated filter
- Let $H(e^{j\omega})$ denote the frequency response of the digital filter $H(z)$ approximating the desired frequency response $D(e^{j\omega})$, given as a piecewise linear function of ω, in some sense
- **Objective -** Determine iteratively the coefficients of $H(z)$ so that the difference between $H(e^{j\omega})$ and $D(e^{j\omega})$ over closed subintervals of $0 < \omega < \pi$ is minimized

Computer-Aided Design of Digital Filters

- One obvious way of optimizing the filter design is to achieve equal ripples so that the side lobe magnitude can be significantly minimized.
- The figures below shows the comparison between the widow-based and the optimized filter response.



Rectangular window



Equal ripple

- Obviously, the optimized one has much lower sidelobe and wider transitional band.

Computer-Aided Design of Digital Filters

- This algorithm is based on iteratively adjusting the coefficients of $\hat{H}(\omega)$ until the peak absolute value of $E(\omega)$ is minimized

- The Parks-McClellan Algorithm or (equi-ripples) widely used for various types of FIR filters.

Example: The LPF specification are

passband edge f_p = 800 Hz, stopband edge f_s = 1000 Hz,

passband ripple R_p = 0.5 dB, stop band attenuation A_s = 40 dB,

sampling frequency F_s = 4000 Hz,

Transitional bandwidth Δf = 200 Hz.

Find the coefficients with the Parks-Mclellan algorithm.

Summary of Review Materials

- Understanding the expressions of filter specifications in different ways and being able to use these expressions
- Understanding the effects of different windows on the characteristics of the designed filters
- Being able to estimate the filter order, or the computational complexity, from a set of given filter specifications.
- Generating filter coefficients is not required.
- Main steps of filter design

- For specifying other type of FIR filters such as highpass and bandpass, the only difference is the passband and stopband frequencies. Other parameters are the same.