# Questions for Week 1

## Question 1.1

Prove the DTFT of $g_a(n)$

$$G_p(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(\Omega - k\Omega_T) \qquad (1)$$

given in lecture note page 13. You may refer to the text book. Hint: $g_a(n)= g_a(t)*\delta(nT)$

**Solution:**

In the time domain, the samples are obtained by

$$g(n) = g_a(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nTs)$$

where $T_s$ is the sampling period. Based on the convolution theorem, the DTFT of $g(n)$ is the convolution between the Fourier transform of $g_a(t)$ and the Fourier transform of the delta sequence,

$$G_p(\Omega)=G_a(\Omega)*\frac{1}{T_s}\sum \delta(\Omega - \frac{n}{T_s})$$

where the DTFT of the delta sequence is $\frac{1}{T_s}\sum \delta(\Omega - \frac{n}{T_s})$. The result of the convolution is the Aliasing Formula in (1).

## Question 1.2

Figure 1 shows the spectrum of an analogue signal with a bandwidth of 2W. Plot the spectra of the sampled sequence with the following sampling frequencies:

a. $w_s = 3W$; b. $w_s = 2W$; c. $w_s = 1.5W$.

Comments on the recovered signals from the digital sequences obtained from these sampled sequences.

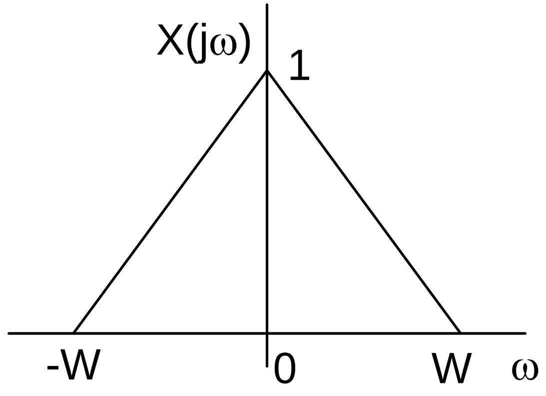
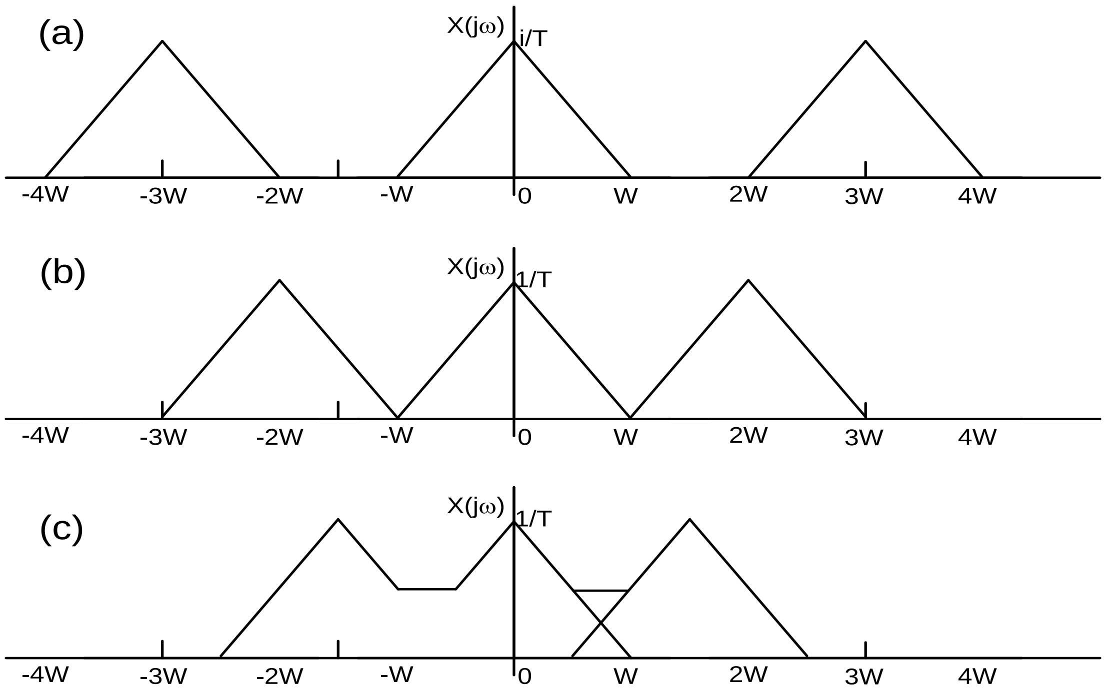



Figure 1

**Solution:**

Based on aliasing formula, the above spectrum is duplicated periodically. The period of the duplication is the sampling frequency. The cases in a, b and c are corresponding to over-sampling, critical sampling and under-sampling. See below and note the consequence of spectrum aliasing in (c).




**Question 1.3**

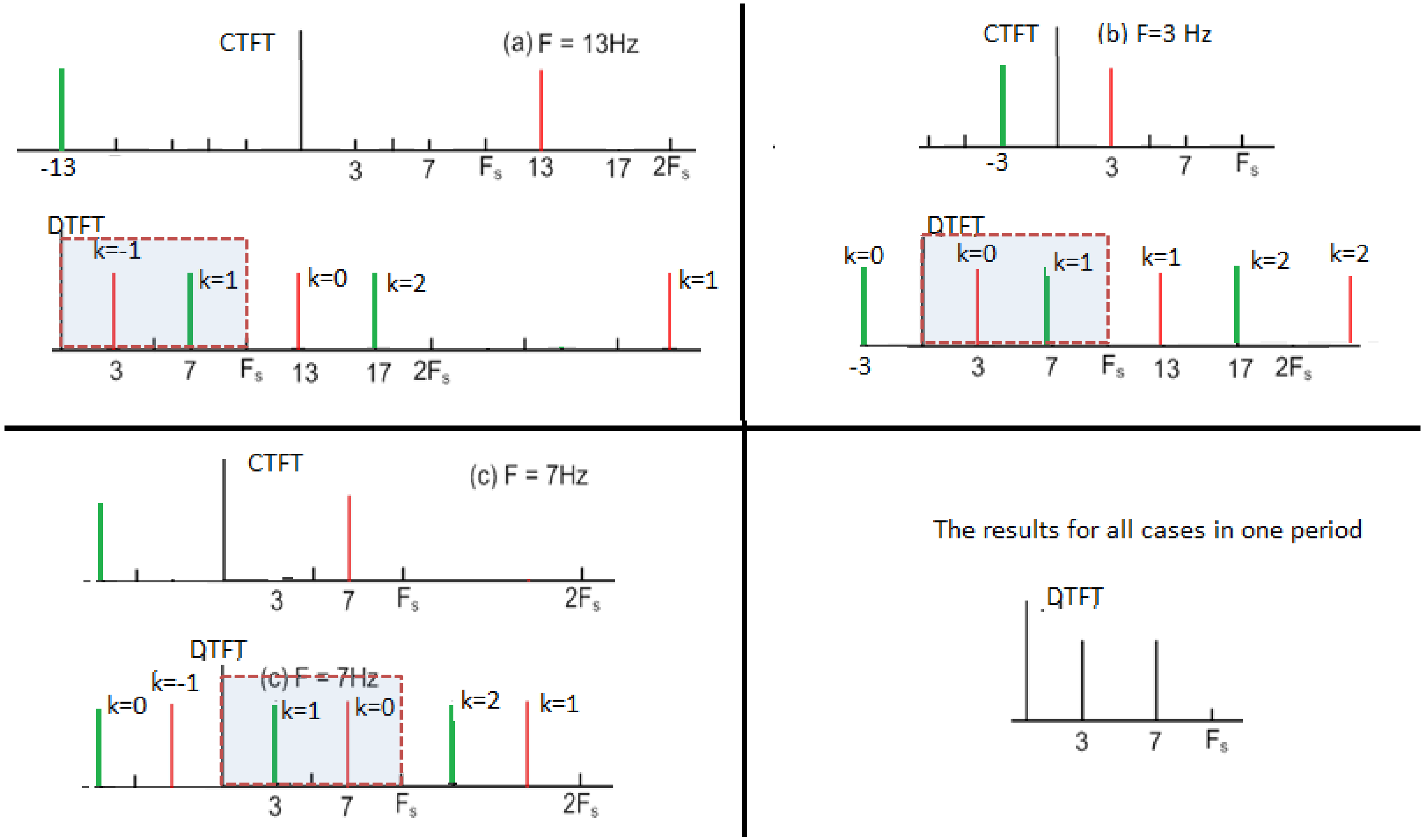
The sampling frequency of a sampler is 10 Hz. Plot the spectra of the sampled sequence from a sinusoidal signal with a combination of frequencies

a. 13 Hz, b. 3 Hz, c. 7 Hz.

Compare your plots and explain your observations. What is the frequency of the signal restored from these sampled sequences?

**Solution:** Using the aliasing formula and the property for real signal to plot the spectrum periodically.

In general, for real signal, **symmetric property** is needed. Also, the **periodic property** is needed in any case.




The spectra from (a), (b) and (c) are the same. The figures above shows the use of Aliasing formula and the final results in one period is shown in the bottom right figure. In fact, the sampling frequency 10 Hz is not high enough for the 7 and 13 Hz waveforms.

In general, the distribution of the frequencies for sampling **a single** sinusoidal component is given by

$$NF_s \pm f \qquad \text{for } N = \pm 0, 1, 2, \ldots$$

For signal reconstruction from the digital samples, we need a lowpass filter with a cup-off frequency of 5 Hz ($0.5F_s$). Therefore the restored signal contains a frequency of 3 Hz only in all these three cases.

Because $F_s$ is not larger or equal to two times of the signal frequencies, (a) and (c) cannot be recovered from the discrete samples.

**Question 1.4**

If Nyquist rate for $x_a(t)$ is $\Omega_s$, what is the Nyquist rate for each of the following signals that are derived from $x_a(t)$?

(a) $y_a(t) = x_a(2t)$, (b) $y_a(t) = x_a^2(t)$, and (c) $y_a(t) = x_a(t)\cos(\omega_0 t)$.

**Hint :** the Nyquist rate for $y_a(t)$ has to be changed if the bandwidth of $y_a(t)$ is not the same as the bandwidth of $x_a(t)$.

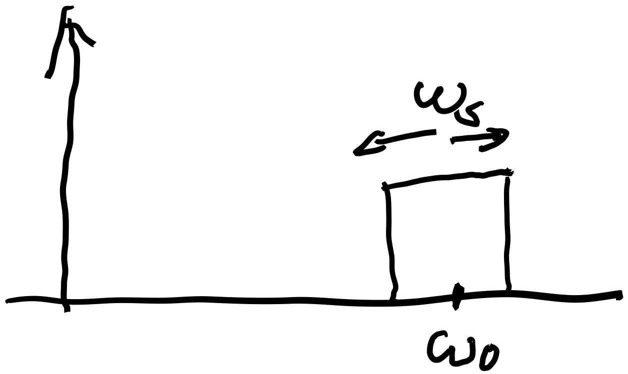
**Solution:**

For all these three cases, we need to know the bandwidth of $y_a(t)$.

(a) For case $y_a(t) = x_a(2t)$, the time scaling means $x_a(t)$ is squeezed by two times in the time domain, the spectrum of $y_a(t)$ should be accordingly expanded by two times (scaling property of the FT). Therefore, the Nyquist rate of $y_a(t)$ should be two times of the Nyquist rate of $x_a(t)$.

(b) Based on the property of DTFT, a multiplication of two signals, $y_a(t) = x_a^2(t) = x_a(t) \cdot x_a(t)$, in the time domain means the convolution of their spectra in the frequency domain (Multiplication property of FT). The output width of a convolution is the sum of the widths of the input signals. Therefore, the bandwidth of $y_a(t)$ is doubled. The Nyquist rate of $y_a(t)$ should be two times of the Nyquist rate of $x_a(t)$.

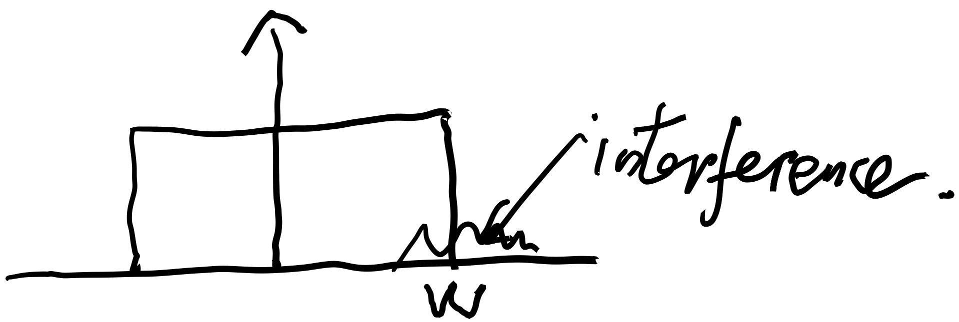
(c) Because the highest frequency of $y_a(t)$ is $W_s/2 + \omega_0$, the Nyquist rate should be $W_s + 2\omega_0$. For bandpass signal, however, it is not necessary to follow the sampling thorem because it has been proved that a proper sampling frequency should be between $2W_s$ and $4W_s$.




**Question 1.5**

Why is antialiasing filter needed? What is the function of reconstruction filter in the DAC? Can you find a practical example in which the antialiasing filter must be used?

**Solution:** Almost all ADCs use anti-aliasing filter to interface with the input signal. The basic function of the filter is to confine the input signal bandwidth so that the selected sampling frequency is valid.




The reconstruction of the filter in the DAC is to select the band of the signal to be reconstructed, or equivalently suppress any unwanted signal that is located outside the signal band. In general, the filter used in DAC also effectively removes any frequency components that is larger than one half of the sampling frequencies. These frequencies mostly come from the quantization word length effects of the sample values.

An example is that in communication, it is likely that the received signal contains various interferences that may be outside the specified bandwidth, and the signal band may be shifted due to the instability of carrier frequency that is used in the demodulation process. The consequences of these phenomena are that the received signal band become wider than specified so that aliasing may occur in the process of sampling.

**Question 1.6**

The DFT deals with complex samples. How many multiplications for real data are used for one multiplication for complex data?

**Solution:**

A complex multiplication can be performed by $(a+jb)(c+jd) = (ac-bd) + j(bc+ad)$.

Therefore, it needs four multiplications and two additions of real data.

Another implementation is using 3 multipliers and 3 adders, $(a+jb)(c+jd) = a(c-d)+d(a-b) + j\,[b(c+d)+d(a-b)]$

**Question 1.7 (optional)**

Read section 8.1.3, page 484 to convince yourself that for a $N$ (being a power of 2) point DFT, the FFT computation requires $(N/2)\log_2 N$ complex multiplications.

**Solution hint:** Take note the number of butterflies in each stage and the number of stages in a radix-2 FFT algorithm. One butterfly uses two complex additions and one complex multiplication.

# Questions for Week 2

## Question 2.1

In the lecture, we have touched a few transforms that are used for spectrum (frequency) analysis of signals. To have a summary,

- Complete the following table by following the example for CTFT.
- Discuss how DFT and STFT are computed
- Discuss issues of frequency resolution and computational complexity of DFT and STFT.

**Solution** (a)

| Transform | Effective for the type of signals | Input of the transform (Analog or Digital) | Output of the transform (Analog or Digital) | Information on applications |
|---|---|---|---|---|
| CTFT | Stationary | Analog | Analog | |
| DTFT | Stationary | Discrete | Analog | Mainly used for theoretical analysis |
| DFT | Stationary | Discrete | Discrete | Widely used for practical applications. The FFT is widely used for calculation of DFT and STFT. The STFT is mainly used for signals whose frequencies change with time. |
| FFT | Stationary | Discrete | Discrete | |
| STFT | Non-stationary Data stream | Discrete | Discrete | |

(b) FFT is used for both DFT and STFT.

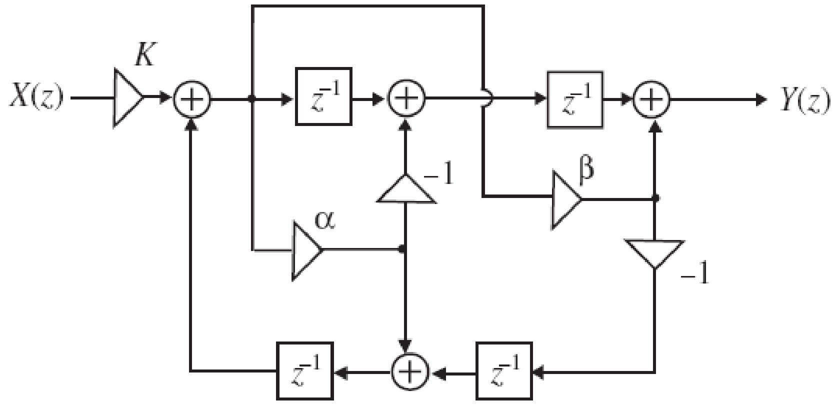
(c) Frequency resolution is the DFT is $F_S/N$, which means that each value of DFT is equivalent to a band of frequencies. Within this band, different frequencies cannot be distinguished.

Computational complexity: Refer to lecture notes. In general, the ratio between the computational complexities needed by the DFT and FFT is in the order of $N/\log_2 N$.

**Question 2.2**

Determine its transfer function, $H(z)=Y(z)/X(z)$, of the digital filter structure in the figure below. Is it canonic?

**Solution:**




From the figure, we have term the outputs of the adder as $W1$, $W2$, and $W3$.

$W1 = KX + z^{-1}W3$; $W2 = (z^{-1} - \alpha)W1$;

$W3 = \alpha W1 - \beta z^{-1}W1 = (\alpha - \beta z^{-1})W1$; $Y = z^{-1}W2 + \beta W1$,

Substituting the third equation in the first one, we have

$W1 = KX + z^{-1}(\alpha - \beta z^{-1})W1$, or $W1 = KX[1 - \alpha z^{-1} + \beta z^{-2}]^{-1}$.

Putting $W1$ and $W2$ into $Y=z^{-1}W2+\beta W1$, we have the 2nd order IIR equation.

$$H(z) = \frac{Y(z)}{X(z)} = K\frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}}$$

The above figure is not a canonic form because the number of delays and the order of the system are not the same.

**Question 2.3**
For the following IIR transfer function

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

(i). Find its canonic form of the block diagram;

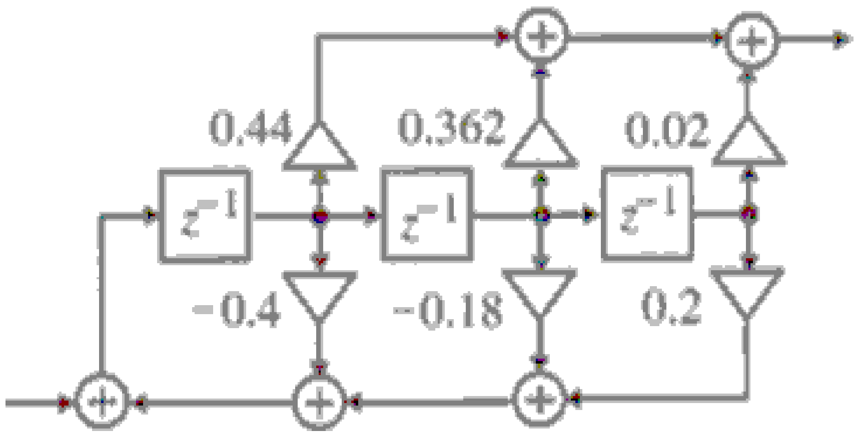
(ii). By factoring the numerator and the denominator of $H(z)$, find its block diagram that is the cascade of a 1st and a 2nd order IIR filters.

**Solution:**

(i). For the equation

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

a standard direct form is



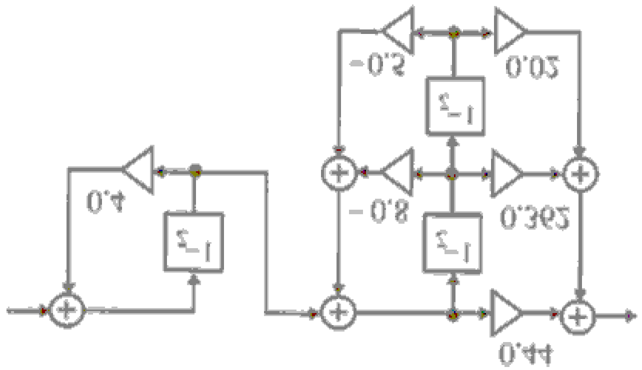
- The block diagram of a general expression should be familiarized.
- If the coefficients on the feedback paths are zero, this figure becomes an FIR system.

(ii). With a long division method, the polynomials of the $H(z)$ can be expressed into

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

$$= \frac{0.44+0.362z^{-1}+0.02z^{-2}}{1+0.8z^{-1}+0.5z^{-2}} \times \frac{z^{-1}}{1-0.4z^{-1}}$$

The corresponding block diagram is




or

$$H(z) = \frac{z^{-1}}{1 + 0.8z^{-1} + 0.5z^{-2}} \times \frac{0.44 + 0.362z^{-1} + 0.02z^{-2}}{1 - 0.4z^{-1}}$$

Please find the block diagram for this equation. Both the above equivalent systems are canonic.

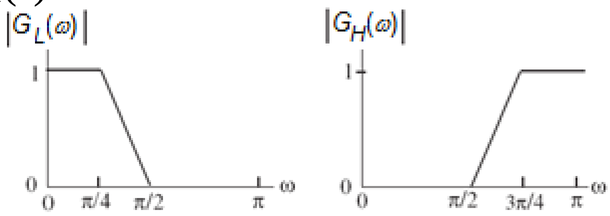
## Question 2.4

The systems are specified as follows:

$$H_0(z)=\frac{Y_0(z)}{X(z)}=G_L(z)G_H(z^2),\quad H_1(z)=\frac{Y_1(z)}{X(z)}=G_H(z)G_H(z^2)$$

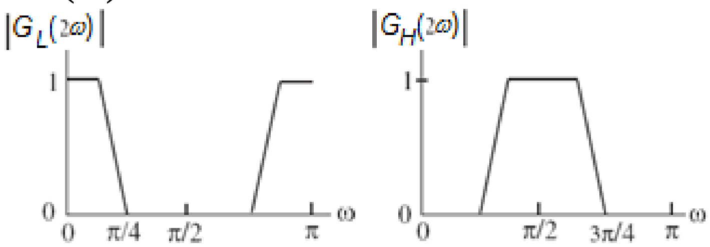
$$H_2(z)=\frac{Y_2(z)}{X(z)}=G_H(z)G_L(z^2),\quad H_3(z)=\frac{Y_3(z)}{X(z)}=G_L(z)G_L(z^2)$$

The magnitude responses of $G_L(z)$ and $G_H(z)$ are



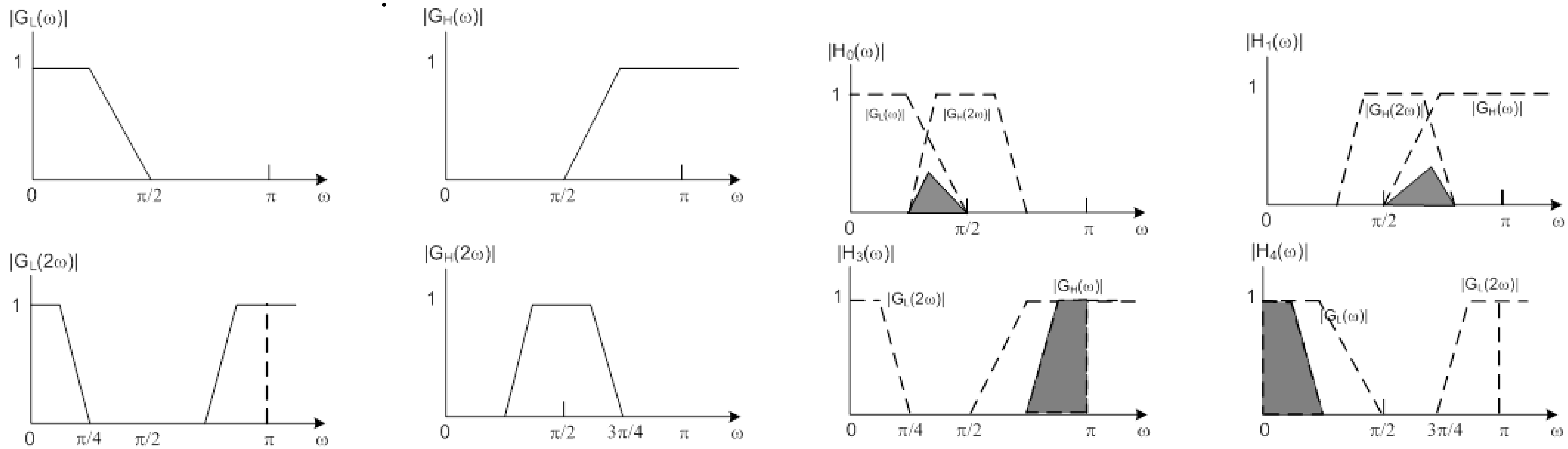

The magnitude responses of $G_L(z^2)$ and $G_H(z^2)$ are

:




Find the magnitude response of the filters $H_i(z)$ for $i$ = 0, 1, 2 and 3.

## Solution




See the above figures. The following steps are

1. Find the spectra for $G_L(z^2)$ and $G_H(z^2)$ as shown in $|G_L(2\omega)|$ and $|G_H(2\omega)|$
2. Find the overlapping areas. The shaded areas are the final frequency responses. Note that the boundary lines of the shaded area should be curves rather than straight lines.

**Question 2.5**

Consider a linear-phase FIR transfer function given by $H(z) = F_1(z)F_2(z)$. Determine the factor $F_2(z)$ of lowest order for each of the following choices for $F_1(z)$:

i. $F_1(z) = 2.1 - 3.5z^{-1} + 4.2z^{-2}$

ii. $F_1(z) = 1.41 + 5.2z^{-1} - 2.2z^{-2} + 3.3z^{-3}$

**Hint:** If $F_1(z)$ is of order $N$ and $F_2(z) = z^{-N}F_1(z^{-1})$, $H(z) = F_1(z)F_2(z)$ is a linear phase transfer function. Can you explain this process how to transform an arbitrary filter into a linear one?

**Solution:**

*i.* $F_1(z) = 2.1(1 - 5z^{-1}/3 + 2z^{-2})$.

$$F_2(z) = z^{-2}F_1(z^{-1}) = 2.1\, z^{-2}(1 - 5z/3 + 2z^2) = 2.1(2 - 5z^{-1}/3 + z^{-2}),$$

resulting in

$$H(z) = F_1(z)F_2(z) = (2.1)^2(2 - 5z^{-1} + (70/9)z^{-2} - 5z^{-3} + 2z^{-4})$$

It is seen that $H(z)$ is a polynomial function of $z^{-1}$ and has a symmetric property. Therefore, it is a linear FIR system.

The same can be done for verifying part ii.

**Comments:**

- The multiplication $F_1(z)F_1(z^{-1})$ is to achieve a zero phase system function.
- The multiplication by $z^{-N}$ is to add a linear phase to $F_1(z)F_1(z^{-1})$ so that the $H(z)$ is a casual system.
- In this way, a non-linear phase system of $F_1(z)$ becomes a linear system $H(z)$.

## Question 2.6

The frequency resolution of DFT is defined as $F_T/N$. When the resolution is increased or better, is $F_T/N$ also increased?

**Solution:** No, the value of $F_T/N$ should be degreased so that the resolution is increased.

## Questions for Week 3

### Question 3.1

Discuss the relationship between the estimated filter order $N$ and the parameters of the filter such as ripples in the pass and stop bands, the transition bandwidth and the sampling frequency of the input signal.

$$N \cong \frac{-10\log_{10}(\delta_p \delta_s) - 13}{14.6(\omega_s - \omega_p)/2\pi} = \frac{-10\log_{10}(\delta_p \delta_s) - 13}{14.6(f_{sc} - f_{pc})/F_s}$$

Estimate the number of multiplication and additions per second required for an implementation of the above FIR filter assuming the sampling frequency is given.

**Solution: (a)**

- $N$ is inversely proportional to the size of ripples in the pass and stop bands on a log scale.
- $N$ is inversely proportional to the transitional bandwidth
- $N$ is directly proportional to the sampling frequency

(b) Assuming a linear phase length $N$+1 FIR filter, the number of multiplications required per second is $F_s(N+1)/2$ (assuming the symmetry property is used), and the number of additions required per second is $F_sN$.

(c). Increasing the length of window function reduces the transitional bandwidth and the oscillation frequencies of the ripples in the pass and stop bands. There is no effect on the amplitude of the ripples.

Changing the widow shape may change the amplitude of the ripples.

## Questions for Week 4

### Question 4.1

Figure 1 shows a sampling rate converter with a decimation factor *D*=3, interpolation factor *L*=4, and the spectrum of the input signal $x(n)$. Sketch the signal spectrum of $w(i)$, $v(i)$ and $y(m)$ indicated in the figure.

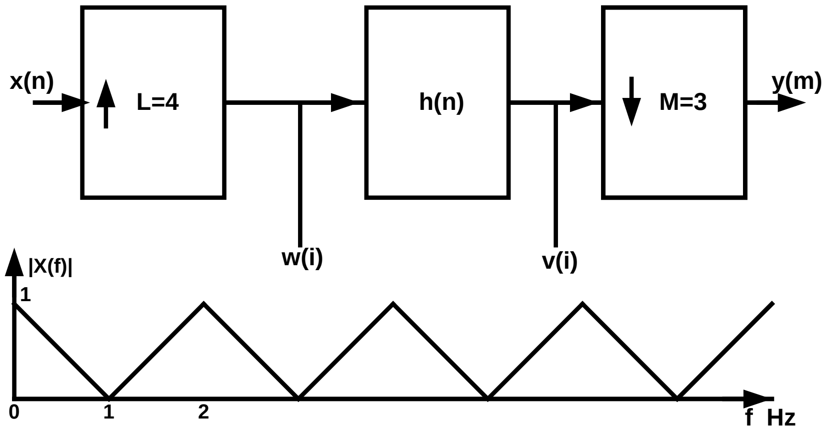
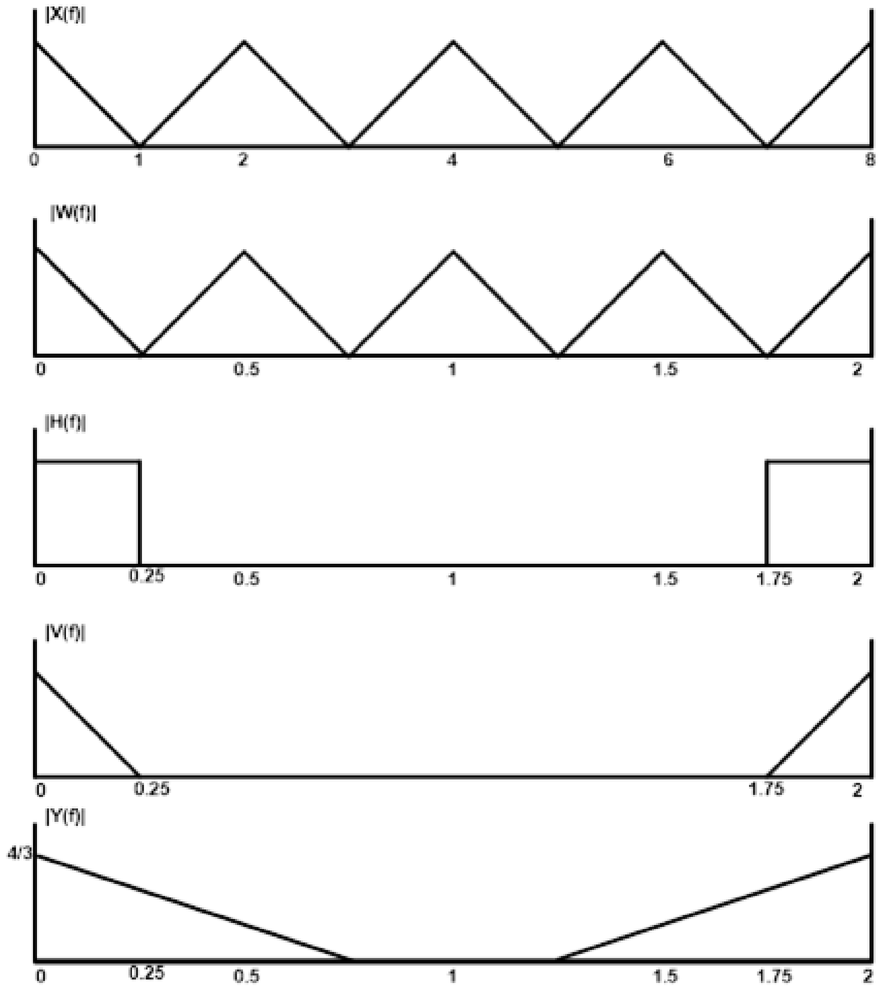



Figure 1

### Solution

From the spectrum of the input, we know the sampling frequency is 2 Hz because its period is 2 Hz.



### Question 4.2

Consider the polyphase realization of a factor of 3 decimator using a length 12 linear phase FIR low pass filter, $h[n]$, with symmetric impulse response.

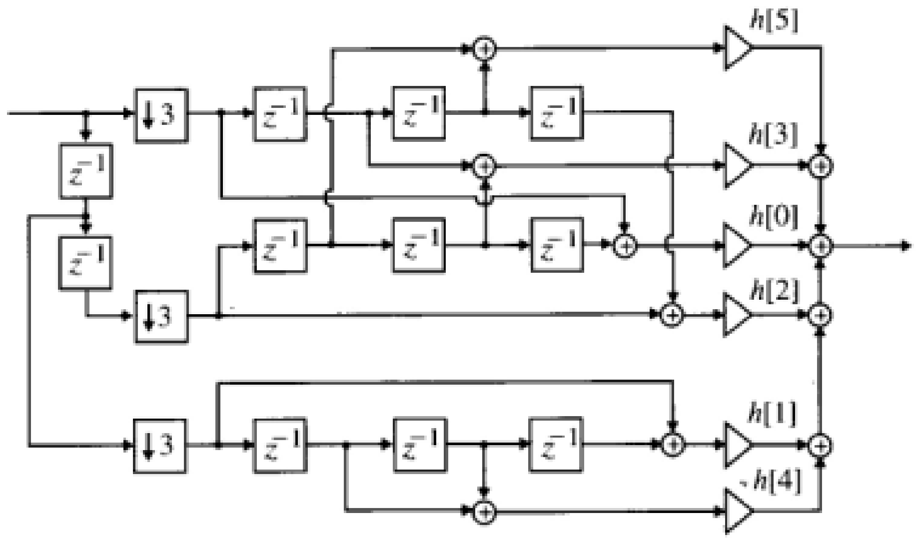
(a) Find the polyphase expression for each branch;
(b) Find the block diagram that is a canonic realization.
(c) Find the number of additions and multiplications per second for the structure shown in Figure 1
(d) Find the number of additions and multiplications per second for the polyphase implementation.

**Solution: (a).**
Because the symmetric property, i.e., $h[i]= h[11-i]$, for $i=0,1,\ldots,5$, the expression is

$$H[z] = E_0[z^3] + E_1[z^3]z^{-1} + E_2[z^3]z^{-2}$$

$$E_0[z] = h[0] + h[3]z^{-1} + h[5]z^{-2} + h[2]z^{-3}$$
$$E_1[z] = h[1] + h[4]z^{-1} + h[4]z^{-2} + h[1]z^{-3}$$
$$E_2[z] = h[2] + h[5]z^{-1} + h[3]z^{-2} + h[0]z^{-3}$$

By making use of the symmetric property of a linear phase filter in the above equation, the canonic structure is in (b).




(b).

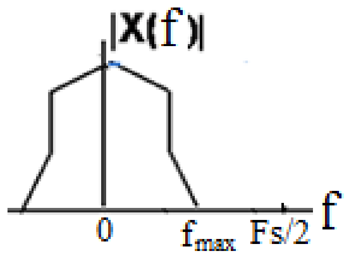
(c). Assume the sampling frequency of.input signal is $F_s$ and the interpolation factor is 1. It is known that the number of filter coefficients is 12 in the question. For linear phase filter, the number of filter coefficients used is 6.

Therefore the number of multiplication needed is 6 x $F_s$.
The number of additions needed is 11 x $F_s$.

(d). With the polyphase structure in (b), the number of multiplications needed per second is
6 x $F_s$/3, and the number of additions per second is 10 x $F_s$/3.

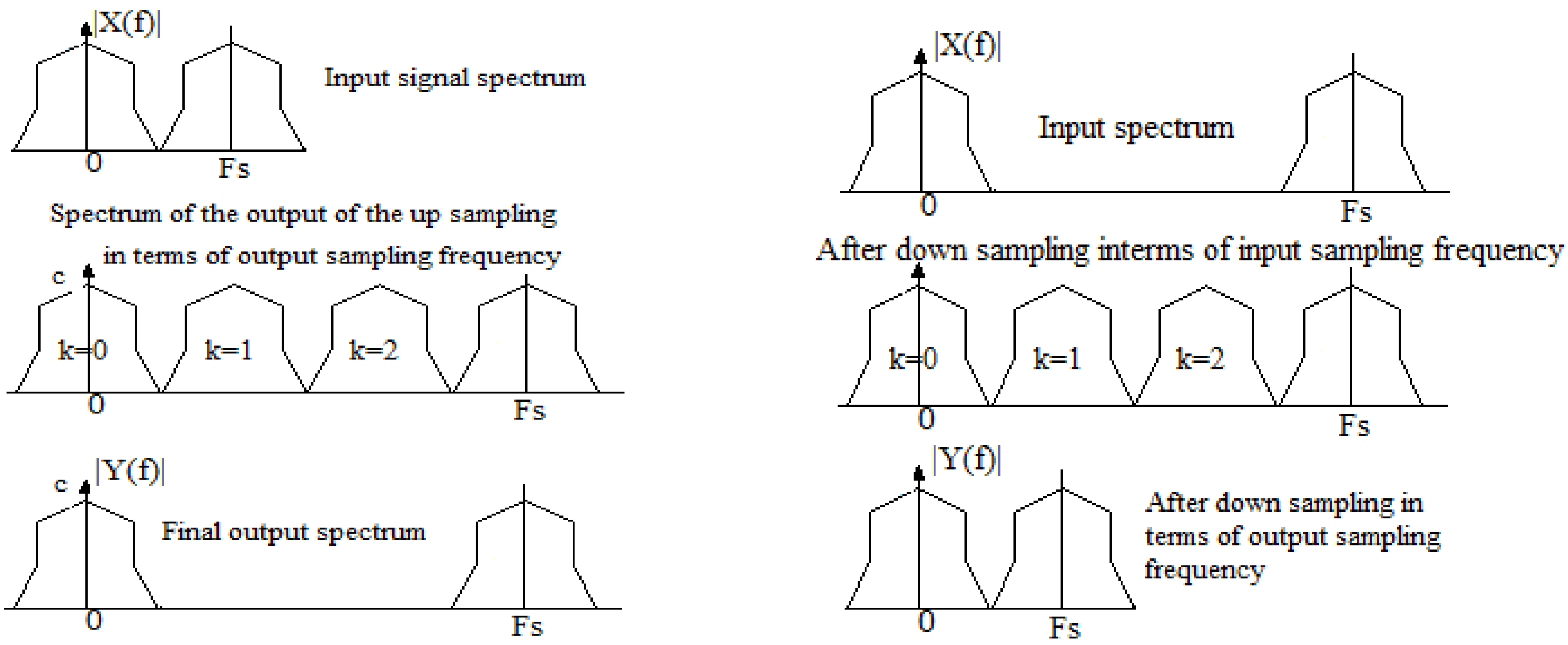
**Question 4.3**

The spectrum of input signal is given in the figure below



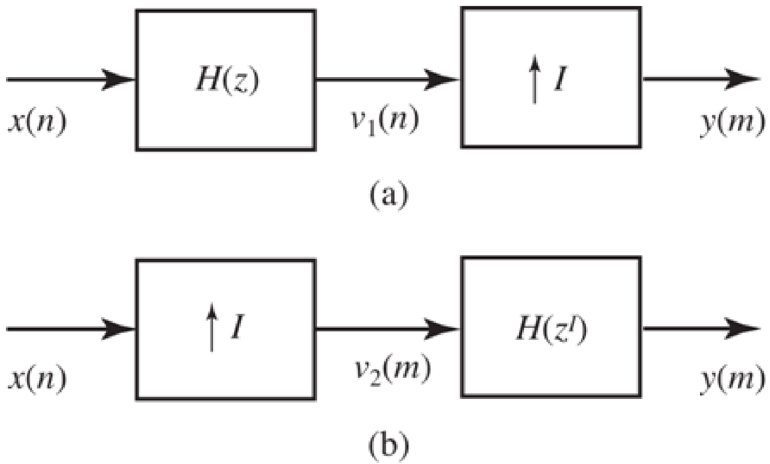

Note that the input spectrum is in terms of frequency in Hz. Plot the output spectra after interpolation and decimation by 3 for the given input signal. The output spectra should be also given in Hz. For decimation, it is assumed that $f_{max}$ is smaller than $F_s/6$, where $F_s$ is the input sampling frequency.

**Solution**



## Question 4.4

Prove the following arrangements are equivalent. You may follow a similar approach used in page 140 of lecture note.



Two equivalent upsampling systems (second noble identity).

**Solution:** An up-sampling operation is expressed by $Y(z)=X(Z^I)$.

Then in (a), $Y(z) = V_1(z^I) = H(z^I)X(z^I)$.

In (b), we have $V_2(z) = X(z^I)$ and $Y(z) = V_2(z)H(z^I) = H(z^I)X(z^I)$.. Therefore, (a) and (b) are equivalent.

## Question 4.5

Solution steps:

1. Simply follow the time domain operations of these samplers to get the outputs.
2. Compare the two outputs.

**Questions for Week 5**

**Question 5.1**

(a). Design a two-stage decimator (D=100) for the following specifications

Passband ripple : $<10^{-3}$ Passband: 0 to 50 Hz

Stopband ripple: $< 10^{-3}$ Stopband: > 55 Hz

Sampling rate 10 kHz

(b). Also calculate the number of multiplications that are needed by the two stage structure, say $D_1$=25 and $D_2$=4 and $D_1$=4 and $D_2$=25.

(c). If each stage is implemented by a polyphase implementation, estimate required number of multiplications per second.

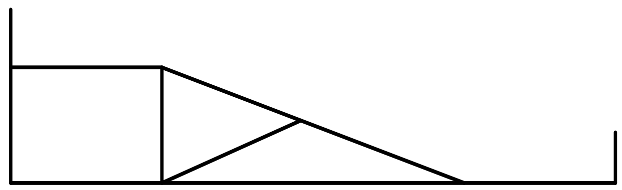
(d). Find the required number of multiplications needed by a direct polyphase structure.

**Solution**

**(a).** Let us assume $D_1 = 25$ and $D_2 = 4$ (other values are also possible) and allow aliasing in the transitional band.

**Specifications for stage 1.**

| | | |
|---|---|---|
| Passband ripple = $10^{-3}/2$ | Stopband ripple = $10^{-3}$ | |
| Input sampling frequency | 10 k Hz | |
| Passband frequency | 50 Hz | |
| Stopband frequency | $10000/D_1 - 50 = 350$ Hz | $\Delta f = (350-50)/10000=0.03$ |




Frequency bands for the first stage

The filter order is

$$N = \frac{-20\log_{10}(10^{-6}/2)^{1/2} - 13}{14.6(\Delta f)} \approx \frac{3.425}{\Delta f} = \frac{3.425}{0.03} \approx 115$$

The filter length for stage 1 is 116

If the transitional band is also protected from aliasing,

Passband frequency 55 Hz

Stopband frequency $10000/D_1 - 55 = 345$ Hz $\Delta f = (345-55)/10000=0.029$

**Specifications for stage 2.**

Passband ripple = $10^{-3}/2$ Stopband ripple = $10^{-3}$

Input sampling frequency 400 Hz

Passband frequency 50 Hz

Stopband frequency 55 Hz

$\Delta f = (55-50)/400=0.0125$

$$N \approx \frac{3.425}{\Delta f} = \frac{3.425}{0.0125} \approx 274$$

The filter length for stage 2 is 275.

**(b).** To calculate the required number of multiplications and additions, we assume the stage filter is a linear phase and has a symmetric property. Therefore, we have

(115+1)/2 ×10 k =580 k multiplications per second

(274+1)/2 ×10 k/25=55 k multiplications per second

for the first and second stages, respectively. The total number of the required multiplications is 635 k multiplications /second.

**(c).** If the polyphase structure is used at each stage, the numbers of multiplications needed by the first and second stages are 1160 k/25 and 110k/4. The total number of the required multiplications is 73.9 k multiplications. The symmetry property of the FIR filter is not used in this case.

**(d).** The number of a direct implementation of polyphase structure is $N/D$, where $N$ is the number of multiplications needed by the proptotype filter and $D$ is the factor of down-sampling.

How is about $D_1$=4 and $D_2$=25? In theory, this is also possible. However, the required computational complexity may be different.

**Question 5.2** (**May ignore this question**)

Figure 1 shows a sampling rate converter with a decimation factor $D$=3, interpolation factor $L$=4,

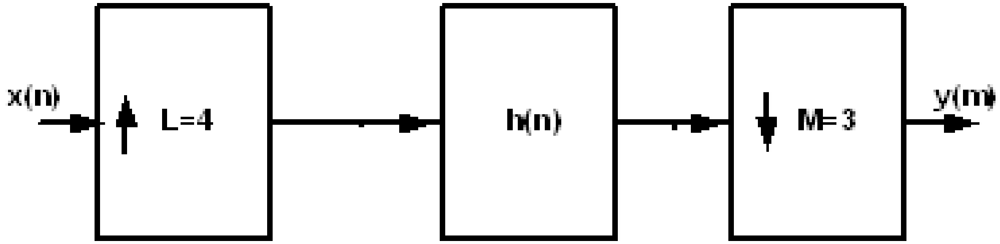
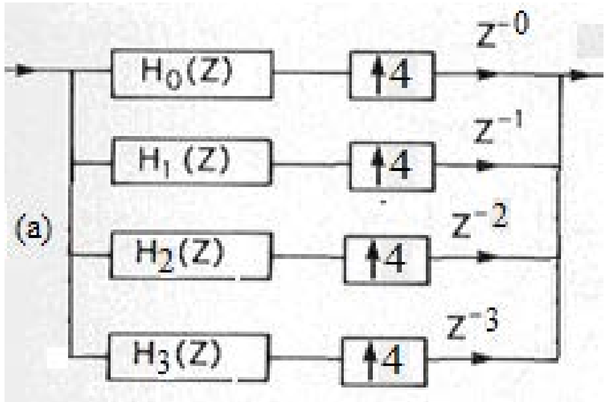



Figure 1

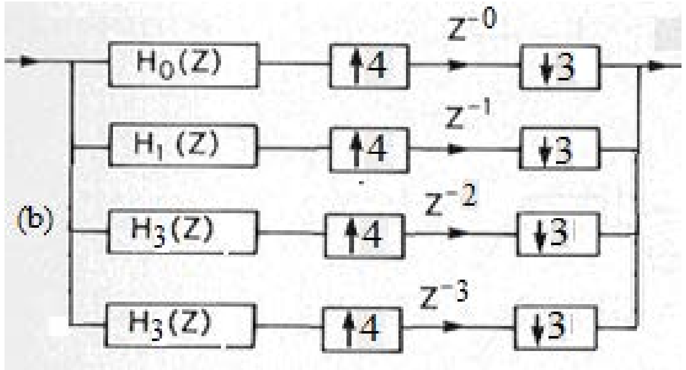
To minimize the computational costs, *derive* a polyphase structure in its *most efficient* form, and *explain* why the computational complexity is reduced by the polyphase implementation compared to the implementation directly using a finite impulse response (FIR) filter.

**Solution:**

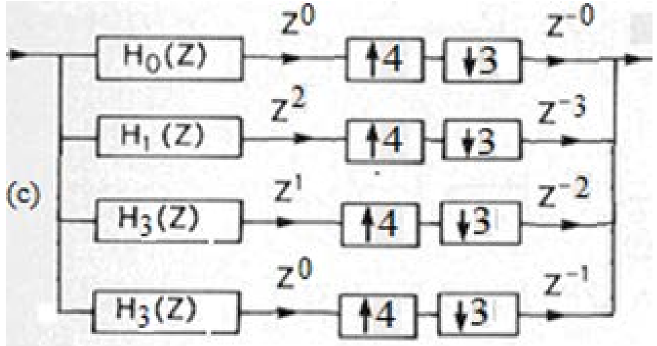
Step 1: Decomposing the interpolation by a factor of 4.



Step 2: Putting the down sampler inside each of the branch in figure (a) to obtain figure (c). Based on the equation $4p_i-3q_i = -i$ for i=0, 1, 2 and 3. The values of $p_i$ and $q_i$ are

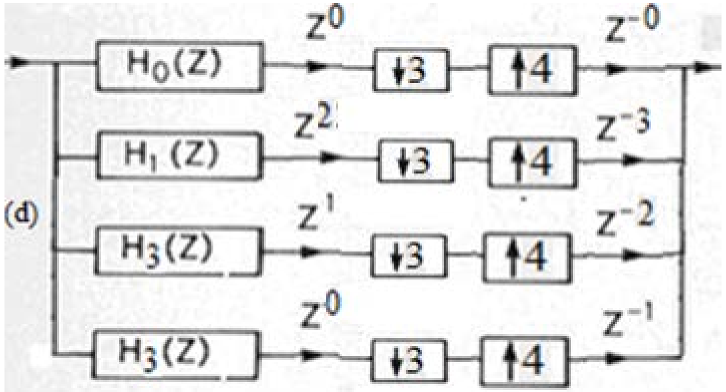


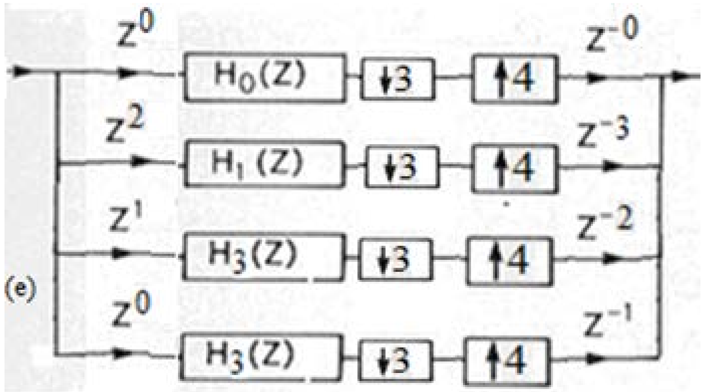

| i | $p_i$ | $q_i$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 2 | 3 |
| 2 | 1 | 2 |
| 3 | 0 | 1 |



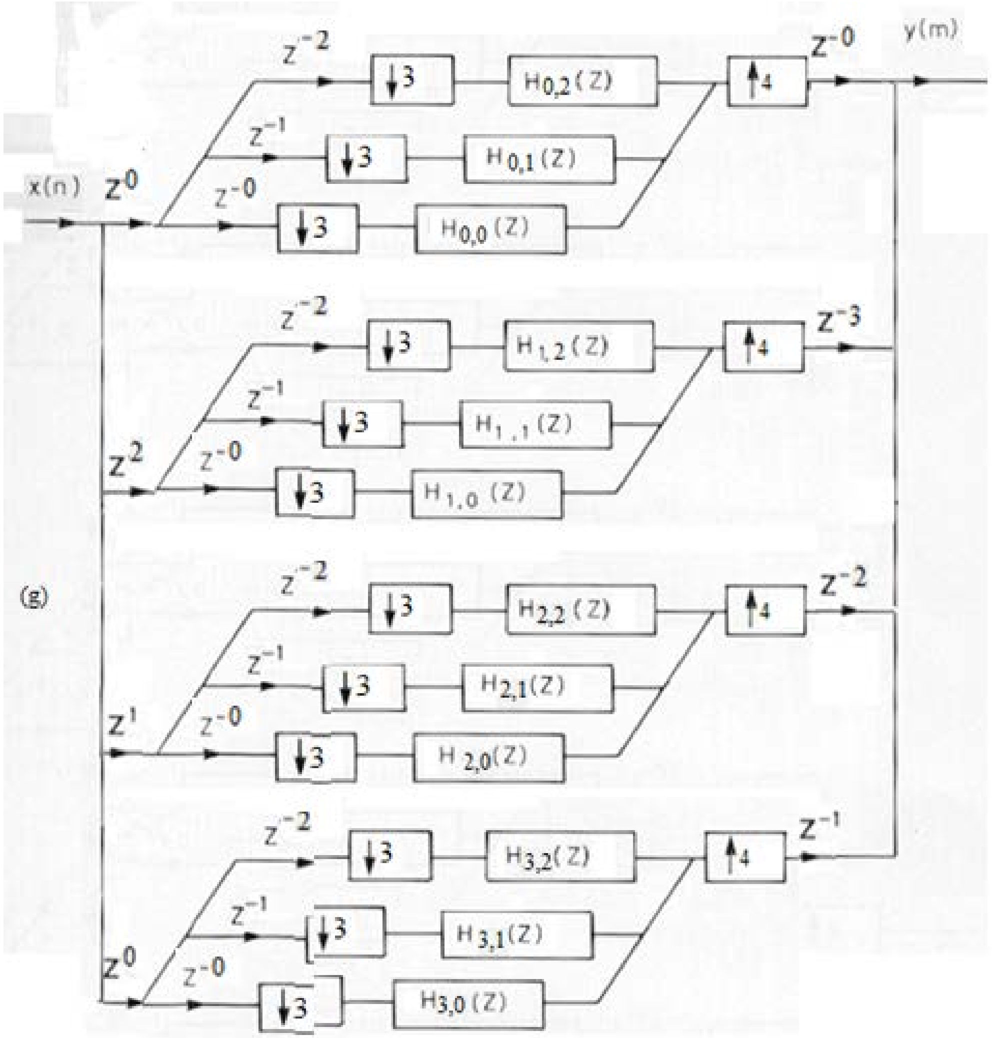

The equivalent delays for those inside the up- and down-samplers in Figure (b) are shown in Figure (c)

Step 3: Because 4 and 3 are mutually prime, the locations of the up-sampler and down sampler can be swapped as shown in figure (d). Note that the delays and the filter in each branch before the decimator is also swapped.



Step 4: Decomposing the decimator (the filter and the down-sampler) in each branch.




You may further rearrange the delays located before the down samplers in each branch. By the above manipulation, the problems of processing zero values data from up-samplers and ignoring the filtered output by the down-samplers are avoided. Also the sub-filters are operated at the places that have the lowest sampling frequencies.

## Question 5.3

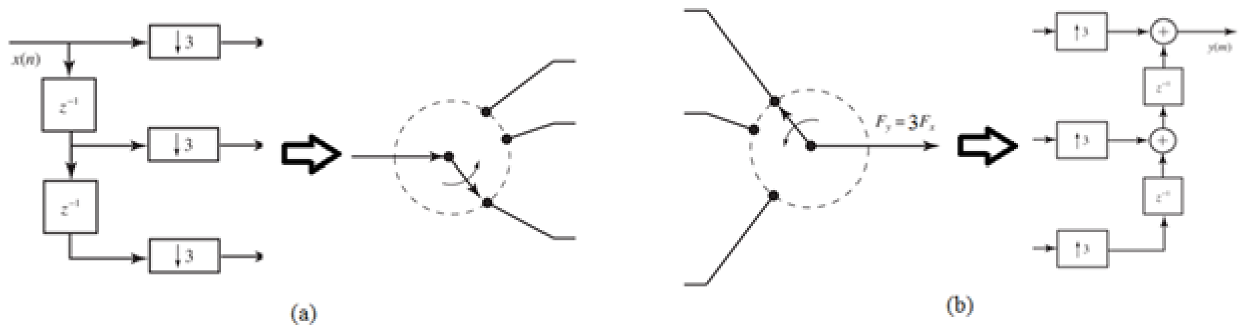



Figure 2

Figure 2 (a) and (b) shows the equivalent implementations for serial-to-parallel and parallel-to serial conversion. Assuming these samplers are synchronously operated, prove their equivalence.

**Solution:**

The first branch outputs Fig (a) are x(3n) for n=0, 1, 2. The sequences arriving at the input of the down-sampler are

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7) x(8) …

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7) x(8) …

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7) x(8) …

in which the data inside the rectangulars are the outputs of the down-samplers. It can be easily shown that the outputs of the commutator are those inside of the rectangulars. Therefore the two implementations perform the operation of 1:3 serial to parallel data conversion.

By using the duality of the system theorem, figure (a) is the inverse operation of figure (b), i.e., using the principle given in lecture note page 62. Therfore, figure (b) are implementations of 3:1 parallel to serial conversion.

## Questions for Week 6
## Question 6.1

The prototype filter in a four channel uniform DFT filter bank is characterized by the transfer function

$$H_0(z) = 1 + z^{-1} + 3z^{-2} + 4z^{-3}$$

(a) Determine the transfer functions of the filter $H_1(z)$, $H_2(z)$, and $H_3(z)$ in the analysis section.

(b) Determine the transfer functions of the filters in the synthesis section.

(c) Sketch the analysis and synthesis sections of the uniform filter bank.

**Solution:**

**(a).** From $H_0(z) = 1 + z^{-1} + 3z^{-2} + 4z^{-3}$, we have $h_0(n) = \{1, 1, 3, 4\}$. Based on $h_k(n) = h_0(n)e^{j2\pi nk/N}$, for $n = 0, 1, 2, 3$, $k$= 1, 2, 3 and $N = 4$, we have

$$h_0(n) = \{1, \quad 1, 3, 4\}$$
$$h_1(n) = \{1, \quad j, -3, -4j\}$$
$$h_2(n)= \{1, -1, 3, -4\}$$
$$h_3(n)= \{1, -j, -3, 4j\}$$

Then

$$H_0(z) = 1 + z^{-1} + 3z^{-2} + 4z^{-3}$$
$$H_1(z) = 1 +jz^{-1} - 3z^{-2} - 4jz^{-3}$$
$$H_2(z) = 1 - z^{-1} + 3z^{-2} - 4z^{-3}$$
$$H_3(z) = 1 -jz^{-1} - 3z^{-2} + 4jz^{-3}$$

**(b).** Let us use the polyphase implementation of the DFT filter bank. Considering the analysis side, we have

$$h_k(n) = h_0(nN-k) \qquad k = 0, 1, 2, 3$$

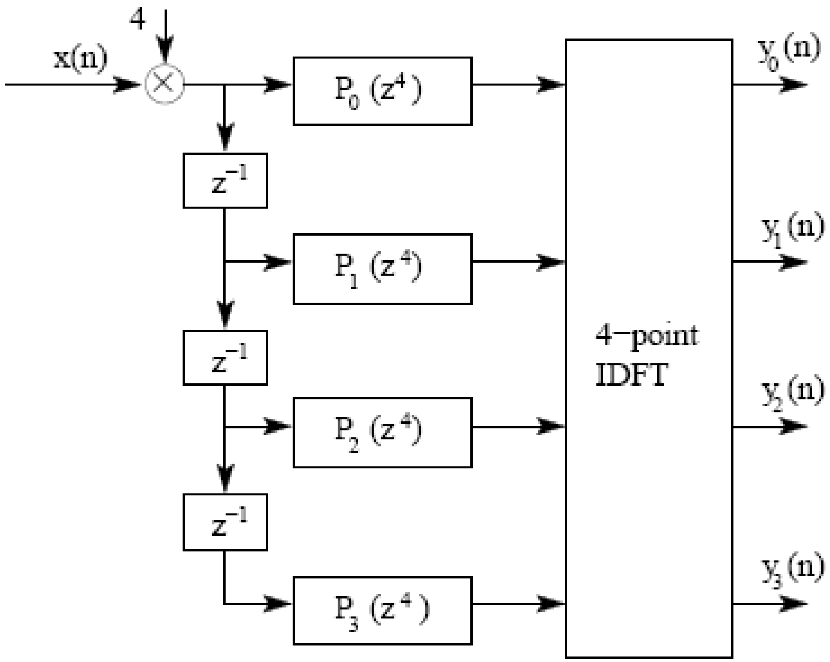
we have its $z$-transform

$$H_k(z) = \sum_{n=0}^{\infty} h_k(n) z^{-n}$$

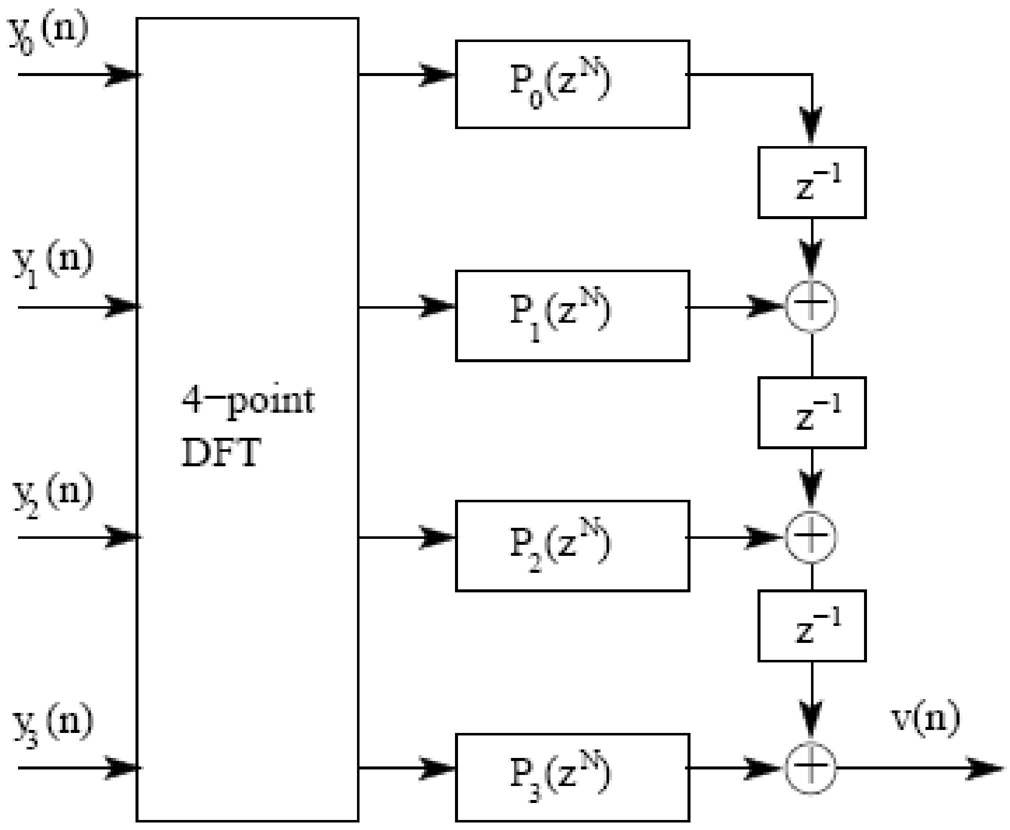
Then

$$\begin{vmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{vmatrix} \begin{vmatrix} 1 \\ z^{-1} \\ 3z^{-2} \\ 4z^{-3} \end{vmatrix} = 4W^{-1} \begin{vmatrix} 1 \\ z^{-1} \\ 3z^{-2} \\ 4z^{-3} \end{vmatrix}$$

where **W** denotes the DFT matrix. If we let $p_0(n) = 1$, $p_1(n) = 1$, $p_2(n) = 3$ and $p_3(n) = 4$, the analysis section is



(d)Based on the duality property and the rules of transformation discussed in the lecture notes, the synthesis sections is



**Question 6.2**

The perfect reconstruction condition for a 2 channel QMF is given in terms of $H_0(\omega)$, $H_1(\omega)$,$G_0(\omega)$ and $G_1(\omega)$. Express this condition in terms of $H_0(z)$, $H_1(z)$,$G_0(z)$ and $G_1(z)$.

**Solution:**

To meet the conditions of an aliasing free and perfect reconstruction, we have

$$H_1(z) = H_0(-z)$$
$$G_0(z) = H_1(-z) \qquad\qquad G_1(z) = -H_0(-z)$$

**Question 6.3**

Consider an arbitrary digital filter with transfer function

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

(a) Perform a two-component polyphase decomposition of $H(z)$ by grouping the even-numbered samples $h_0(n)=h(2n)$ and the odd-numbered sample $h_1(n)=h(2n+1)$. Thus show that $H(z)$ can be expressed as

$$H(z) = H_0(z^2) + z^{-1} H_1(z^2)$$

and determine $H_0(z)$ and $H_1(z)$.

(b) Generalize the result in part (a) by showing that $H(z)$ can be decomposed into a $D$-component polyphase filter structure with the transfer function

$$H(z) = \sum_{k=0}^{D-1} z^{-k} H_k(z^D)$$

Determine $H_k(z)$.

(c) For the IIR filter with transfer function

$$H(z) = \frac{1}{1 - az^{-1}}$$

Determine $H_0(z)$ and $H_1(z)$ for the two-component decomposition.

# Solution:

(a)

$$\begin{aligned}
H(z) &= \sum_n h(2n) z^{-2n} + \sum_n h(2n+1) z^{-2n-1} \\
&= \sum_n h(2n)(z^2)^{-n} + z^{-1} \sum_n h(2n+1)(z^2)^{-n} \\
&= H_0(z^2) + z^{-1} H_1(z^2) \\
\text{Therefore } H_0(z) &= \sum_n h(2n) z^{-n} \\
H_1(z) &= \sum_n h(2n+1) z^{-n}
\end{aligned}$$

(b)

$$\begin{aligned} H(z) &= \sum_n h(nD)z^{-nD} + \sum_n h(nD+1)z^{-nD-1} + \dots \\ &\quad + \sum_n h(nD+D-1)z^{-nD-D+1} \\ &= \sum_{k=0}^{D-1} z^{-k} \sum_n h(nD+k)(z^D)^{-n} \end{aligned}$$

$$\text{Therefore } H_k(z) = \sum h(nD+k)z^{-n}$$

(c)

$$\begin{aligned} H(z) &= \frac{1}{1-az^{-1}} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \end{aligned}$$

$$\begin{aligned} H_0(z) &= \sum_{n=0}^{\infty} a^{2n} z^{-n} & \qquad H_1(z) &= \sum_{n=0}^{\infty} a^{2n+1} z^{-n} \\ &= \frac{1}{1-a^2z^{-1}} & &= \frac{a}{1-a^2z^{-1}} \end{aligned}$$

**Question 6.4**

Consider the FIR transfer function

$$H_0(z) = 1-2z^{-1}+4.5z^{-2}+6z^{-3}+z^{-4}+0.5z^{-5}$$

Using $H_0(z)$ as one of the analysis filter, determine the remaining three filters of the corresponding two-channel orthogonal filter bank. Show that the filter bank is aliasing free and satisfies the perfect construction condition.

**Solution:** According to the solution in Question 6.2, we have

$$H_1(z) = H_0(-z)$$

$$G_0(z) = H_1(-z) \qquad G_1(z) = -H_0(-z)$$

The above are the conditions of perfect reconstruction in terms of z-transform of the filter function.

Therefore, these filters are

$$H_1(z) = 1+2z^{-1}+4.5z^{-2} - 6z^{-3}+z^{-4}-0.5z^{-5}$$

$$G_0(z) = H_0(z) \text{ and } G_1(z) = -H_1(-z)$$

To be aliasing free, we need to have

$$H_0(\omega-\pi)G_0(\omega)+H_1(\omega-\pi)G_1(\omega)=0 \qquad (1)$$

Or equivalently

$$H_0(-z)G_0(z)+H_1(-z)G_1(z)=0$$

For the perfect reconstruction condition, we need to have

$$H_0(z)G_0(z)+H_1(z)G_1(z)=Cz^{-l} \qquad (2)$$

where $l$ is an integer. In the time domain, we have $y[n] = Cx[n-1]$ where $y[n]$ is the output of the synthesis section and $x[n]$ is the input of the analysis section.

The verification can be made by putting $H_0(z)$, $H_1(z)$, $G_0(z)$, $G_1(z)$ into equations (1) and (2).

**Question 6.5**

Considering the example in the Figure below. The prototype filter is

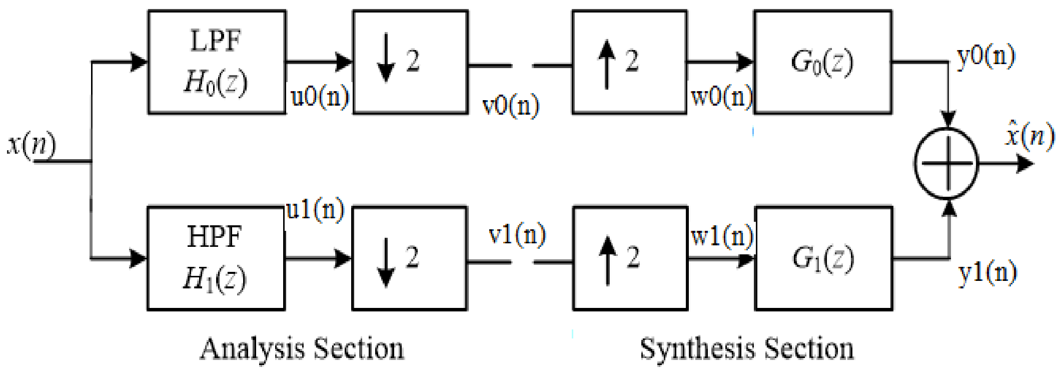
$$H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$$

a) Derive the other filters used in the figure according to the condition of perfect reconstruction.

b) Prove the two-channel QMF achieves perfect reconstruction by deriving the time domain signals, such as

$$u_0[n] = \frac{1}{\sqrt{2}}(x[n]+x[n-1]), \qquad u_1[n] = \frac{1}{\sqrt{2}}(x[n]-x[n-1])$$

$$v_0[n] = ?, v_1[n] = ?, \ldots, y_0[n] = ?, y_1[n] = ? \quad \text{and} \quad \hat{x}[n] = ?$$



**Solution**

(b) After filtering in the analysis section, we have

$$u_0[n] = \frac{1}{\sqrt{2}}(x[n] + x[n-1]), \qquad u_1[n] = \frac{1}{\sqrt{2}}(x[n] - x[n-1])$$

The outputs after the down sampling by a factor of 2 are

$$u_0'[n] = u_0[2n], \qquad u_1'[n] = u_1[2n]$$

The outputs after up sampling by a factor of 2 are

$$u_0''[n] = \begin{cases} u_0'[n/2] & even \quad n \\ 0 & odd \quad n \end{cases}, \qquad u_1''[n] = \begin{cases} u_1'[n/2] & even \quad n \\ 0 & odd \quad n \end{cases}$$

The outputs after the filtering in the synthesis section are

$$u_0'''[n] = \frac{1}{\sqrt{2}}(u_0''[n] + u_0''[n-1]), \qquad u_1'''[n] = \frac{-1}{\sqrt{2}}(u_1''[n] - u_1''[n-1])$$

Then finally we add the above two outputs to get

$$\mathrm{y}[n] = u_0'''[n] + u_1'''[n] = \mathrm{x(n\text{-}1)/2}.$$

Thank you for your support!

Wish you have a happy DSP learning

and