1. (a) $\Delta f = (f_s - f_p)/F_T = 50/50k = 10^{-3}$ Hz

$$N = \frac{-10\log_{10}(\delta_p \delta_s) - 13}{14.6 \times \Delta f} + 1 = 1850.315 \approx 1851$$

$\frac{1851+1}{2} \times 50 \times 10^3 = 4.63 \times 10^7$ multiplications/sec. (assume linea phase)

(b) Option 1: $D_1 = 10$, $D_2 = 2$

Stage 1: input sampling freq $F_{T1} = F_T = 50 \times 10^3$ Hz

$f_p = 1000$ Hz

$f_s = F_{T1}/10 - f_p = 4000$ Hz

trasition bandwidth $\Delta f = (f_s - f_p)/F_{T1}$

$= 3000/50000 = 0.06$

$\delta_{s1} = \delta_s = 0.001$, $\delta_{p1} = \frac{1}{2}\delta_p = 0.05$

$$N_1 = \frac{-10\log_{10}(\delta_{s1}\delta_{p1}) - 13}{14.6 \times \Delta f} + 1 = 35.2583 \approx 35$$

$\frac{\cancel{35}\,34+1}{2} \times 50 \times 10^3 = 9 \times 10^5$ multiplications/sec.

Stage 2: $F_{T2} = F_{T1}/10 = 5 \times 10^3$ Hz.

$f_p = 1000$ Hz,

$f_s = 1050$ Hz,

$\Delta f = (1050 - 1000)/F_{T2} = 50/5 \times 10^3 = 10^{-2}$

$\delta_{s2} = \delta_s = 0.001$, $\delta_{p2} = \frac{1}{2}\delta_p = 0.05$

$$N_2 = \frac{-10\log(\delta_{p2}\delta_{s2}) - 13}{14.6 \times \Delta f} + 1 = 206.5499 \approx 207$$

$\frac{\cancel{207}\,206+1}{2} \times 5 \times 10^3 = 5.20 \times 10^5$ multiplications/sec

$\therefore$ $9 \times 10^5 + 5.2 \times 10^5 = 14.2 \times 10^5$

Option 2: $D_1 = 2$, $D_2 = 10$, $F_T = 50\,kHz$

Stage 1: $f_p = 1000\,Hz$

$f_s = F_T / D_1 - f_p = 24 \times 10^3\,Hz$

$\Delta f = (f_s - f_p)/F_T = 23/50 = 0.46.$

$\delta_{p1} = \frac{1}{2}\delta_p = 0.05,\quad \delta_{s1} = \delta_s = 0.001$

$$N_1 = \frac{-10\log_{10}(\delta_{p1}\delta_{s1}) - 13}{14.6 \times \Delta f} + 1 = 5.4085 \approx 6$$

$\frac{6+1}{2} \times 50 \times 10^3 = 1.75 \times 10^5$ multiplications/sec

Stage 2: $F_{T2} = F_T/2 = 25\,kHz$

$f_p = 1000\,Hz,$

$f_s = 1050\,Hz.$

$\Delta f = (f_s - f_p)/F_{T2} = 50/25k = 2 \times 10^{-3},$

$\delta_{p2} = \frac{1}{2}\delta_p = 0.05,\quad \delta_{s2} = \delta_s = 0.001$

$$N_2 = \frac{-10\log_{10}(\delta_{p2}\delta_{s2}) - 13}{14.6 \times \Delta f} + 1 = 1028.7499 \approx 1029$$

$\frac{1029+1}{2} \times 25 \times 10^3 = 1.2875 \times 10^7$ multiplications/sec

$\therefore$ $1.2875 \times 10^7 + 1.75 \times 10^5 = 1.305 \times 10^7$ multiplications/sec

(c) Because of the different options of the D_1 and D_2, the resulting filter length and sampling frequencies are very different. In this Question, Option #1 is more efficient than Option #2 since its overall complexity is lower.

$D_1 = 10, \quad D_2 = 2$:

Stage 1: $F_T = 50kHz, \quad F_1 = \frac{F_T}{D_1} = 5kHz, \quad F_S = F_1 - F_P = 5k - 1k = 4kHz$

$\delta_{p1} = \frac{1}{2}\delta_p = 0.05. \quad \delta_{s1} = \delta_s = 0.001.$

$\Delta f = (F_S - F_P)/F_T = (4k - 1k)/50k = 0.06$

$$N_1 = \frac{-10\log_{10}(0.05 \times 0.001) - 13}{14.6 \times \Delta f} + 1 \approx 35.$$

Stage 2: $F_{T2} = \frac{F_{T1}}{D_1} = 5kHz, \quad F_P = 1k, \quad F_S = 1050.$

$\Delta f = (1050 - 1000)/5k = 0.01$

$$N_2 = \frac{-10\log_{10}(0.05 \times 0.001) - 13}{14.6 \times \Delta f} + 1 = 206.55 \approx 207.$$

$M_1 = \frac{N_1 + 1}{2} \times 50kHz = 900000$

$M_2 = \frac{N_2 + 1}{2} \times 5kHz = 520000$

$\Rightarrow M = 1420000$ multiplications/sec.

$D_1 = 2, \quad D_2 = 10$:

Stage 1: $F_T = 50kHz, \quad F_1 = \frac{F_T}{D_1} = 25kHz, \quad F_S = F_1 - F_P = 24kHz.$

$\Delta f = (24k - 1k)/50k = 0.46,$

$N_1 = 5.4684 \approx 6.$

Stage 2: $F_T = F_1 = 25kHz, \quad F_P = 1050Hz$

$\Delta f = (1050 - 1000)/25k = 2 \times 10^{-3},$

$N_2 = 1028.749 \simeq 1029$

$M_1 = \frac{N_1 + 1}{2} \times 50kHz = 175000$

$M_2 = \frac{N_2 + 1}{2} \times 25kHz = 12875000$

$\Rightarrow M = 13050000$ multiplications/sec.

about 7.26 times of multi/sec saving by using the first option.

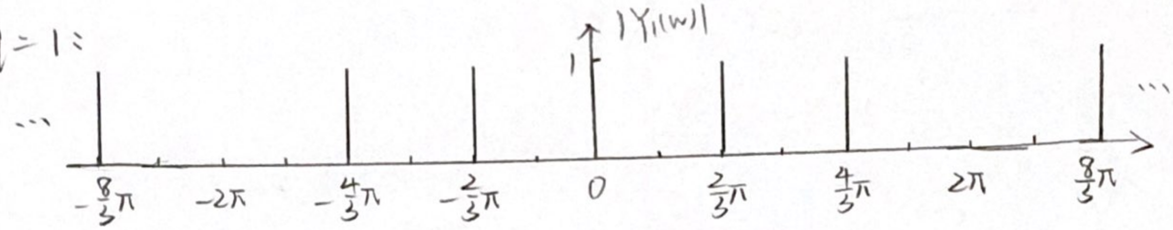
2. (a) $x[n] = 2\cos\left(\frac{2\pi n}{3}\right) = e^{j\left(\frac{2\pi n}{3}\right)} + e^{-j\left(\frac{2\pi n}{3}\right)}$

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(e^{j\left(\frac{2\pi}{3}-w\right)n} + e^{-j\left(\frac{2\pi}{3}+w\right)n} \right)$$

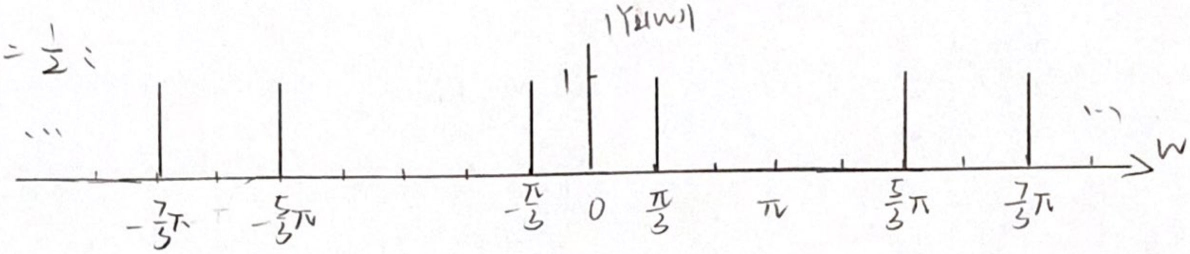
$$= 2\pi \sum_{k=-\infty}^{\infty} \left(\delta\left(w+\frac{2\pi}{3}-2k\pi\right) + \delta\left(w-\frac{2\pi}{3}-2k\pi\right) \right);$$

$$Y(w) = 2\pi \sum_{k=-\infty}^{\infty} \left(\delta\left(w+\frac{2}{3}q\pi-2k\pi\right) + \delta\left(w-\frac{2}{3}q\pi-2k\pi\right) \right);$$

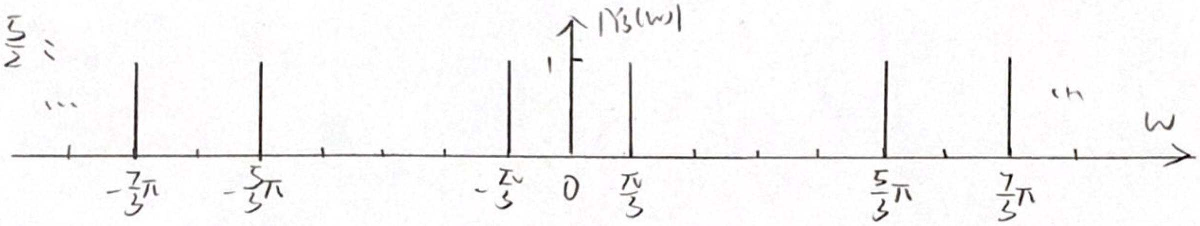
$q=1$:



$q=\frac{1}{2}$:



$q=\frac{5}{2}$:



(b) $W_1(z) = \nu_2 X(z) + z^{-1} W_3(z) = \nu_2 X(z) + z^{-1}\nu_1 X(z) + k_1 z^{-1} Y(z)$

$W_2(z) = z^{-1} W_1(z) + k_2 z^{-1} W_3(z) = Y(z)$

$W_3(z) = \nu_1 X(z) + k_1 Y(z)$

$$Y(z) = z^{-1} W_1(z) + k_2 z^{-1} \left(\nu_1 X(z) + k_1 Y(z) \right)$$
$$= z^{-1} W_1(z) + k_2 \nu_1 z^{-1} X(z) + k_1 k_2 z^{-1} Y(z)$$

$$(1 - k_1 k_2 z^{-1}) Y(z) = k_2 \nu_1 z^{-1} X(z) + \nu_2 z^{-1} X(z) + \nu_1 z^{-2} X(z) + k_1 z^{-2} Y(z)$$

$$(1 - k_1 k_2 z^{-1} - k_1 z^{-2}) Y(z) = \left((k_2 \nu_1 + \nu_2) z^{-1} + \nu_1 z^{-2} \right) X(z)$$

$\Rightarrow$ transfer function: $\frac{Y(z)}{X(z)} = \frac{(k_2\nu_1 + \nu_2) z^{-1} + \nu_1 z^{-2}}{1 - k_1 k_2 z^{-1} - k_1 z^{-2}}$ is canonic

2. (a) $x[n] = 2\cos(\frac{2\pi n}{3}) = e^{j\frac{2}{3}\pi n} + e^{-j\frac{2}{3}\pi n}$

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} (e^{j\frac{2}{3}\pi n} + e^{-j\frac{2}{3}\pi n}) e^{-jwn}$$

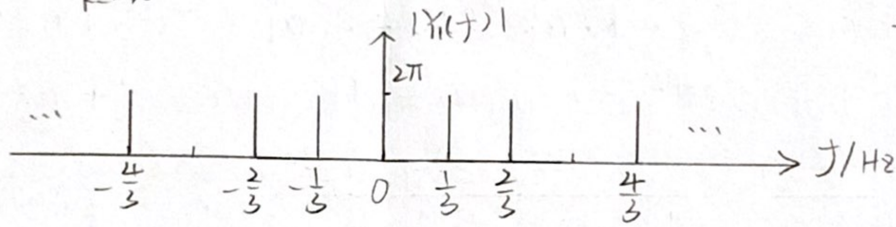
$$= \sum_{n=-\infty}^{\infty} (e^{j(\frac{2}{3}\pi - w)n} + e^{-j(\frac{2}{3}\pi + w)n})$$

$$= 2\pi \sum_{k=-\infty}^{\infty} (\delta(w + \frac{2}{3}\pi - 2\pi k) + (w - \frac{2\pi}{3} - 2\pi k))$$

$y[n] = x[qn] = 2\cos(\frac{2\pi q n}{3})$, $Y(w) = 2\pi \sum_{k=-\infty}^{\infty} (\delta(w + \frac{2}{3}q\pi - 2\pi k) + \delta(w - \frac{2}{3}q\pi - 2\pi k))$.

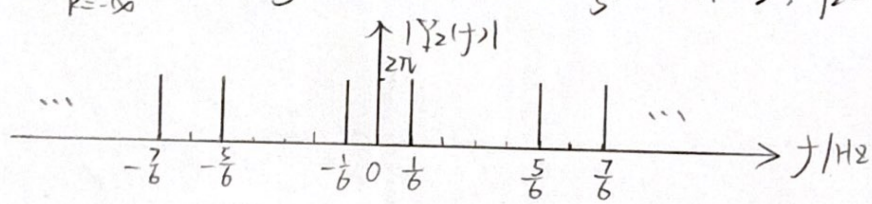
$q = 1$:

$$Y_1(w) = 2\pi \sum_{k=-\infty}^{\infty} (\delta(w - \frac{2}{3}\pi - 2\pi k) + \delta(w + \frac{2}{3}\pi - 2\pi k)), \quad f_1 = \frac{w}{2\pi} = \frac{1}{3}$$



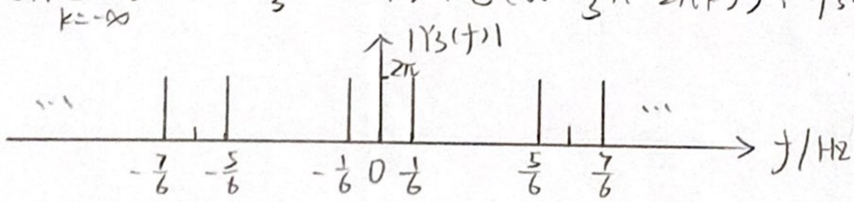
$q = \frac{1}{2}$:

$$Y_2(w) = 2\pi \sum_{k=-\infty}^{\infty} (\delta(w + \frac{1}{3}\pi - 2\pi k) + \delta(w - \frac{1}{3}\pi - 2\pi k)), \quad f_2 = \frac{w}{2\pi} = \frac{1}{6}$$



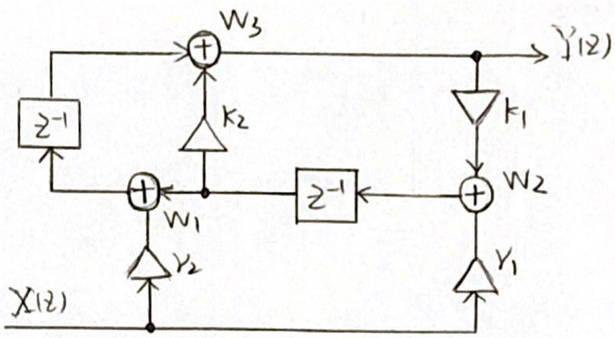
$q = \frac{5}{2}$:

$$Y_3(w) = 2\pi \sum_{k=-\infty}^{\infty} (\delta(w + \frac{5}{3}\pi - 2\pi k) + \delta(w - \frac{5}{3}\pi - 2\pi k)), \quad f_3 = \frac{w}{2\pi} = \frac{5}{6}$$



DTFT has symmetric and periodic properties.

(b)



$$W_1 = r_2 X(z) + z^{-1} W_2 , \quad \Rightarrow \quad W_1 = r_2 X(z) + z^{-1} r_1 X(z) + z^{-1} k_1 Y(z)$$

$$W_2 = r_1 X(z) + k_1 Y(z) , \quad = (r_2 + z^{-1} r_1) X(z) + z^{-1} k_1 Y(z)$$

$$Y(z) = k_2 z^{-1} W_2 + z^{-1} W_1$$

$$= k_2 r_1 z^{-1} X(z) + k_1 k_2 z^{-1} Y(z) + z^{-1} W_1$$

$$(1 - k_1 k_2 z^{-1}) Y(z) - k_2 r_1 z^{-1} X(z) = z^{-1} W_1 = (r_2 z^{-1} + r_1 z^{-2}) X(z) + k_1 z^{-2} Y(z)$$

$$(1 - k_1 k_2 z^{-1} - k_1 z^{-2}) Y(z) = \left[(k_2 r_1 + r_2) z^{-1} + r_1 z^{-2} \right] X(z)$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{(k_2 r_1 + r_2) z^{-1} + r_1 z^{-2}}{1 - k_1 k_2 z^{-1} - k_1 z^{-2}} , \quad \text{is canonic.}$$

3. $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$

(a) condition:

$$X_0(w) = \frac{1}{2}\left[X\left(\frac{w}{2}\right)H_0\left(\frac{w}{2}\right) + X\left(\frac{w-2\pi}{2}\right)H_0\left(\frac{w-2\pi}{2}\right)\right] \quad \text{analysis}$$

$$X_1(w) = \frac{1}{2}\left[X\left(\frac{w}{2}\right)H_1\left(\frac{w}{2}\right) + X\left(\frac{w-2\pi}{2}\right)H_1\left(\frac{w-2\pi}{2}\right)\right]$$

$$\hat{X}(w) = X_0(2w)G_0(w) + X_1(2w)G_1(w) \quad \text{synthesis}$$

$$= \frac{1}{2}\left[X(w)H_0(w) + X(w-\pi)H_0(w-\pi)\right]G_0(w)$$
$$+ \frac{1}{2}\left[X(w)H_1(w) + X(w-\pi)H_1(w-\pi)\right]G_1(w)$$

$$= \frac{1}{2}\left[G_0(w)H_0(w) + G_1(w)H_1(w)\right]X(w)$$
$$+ \frac{1}{2}\left[G_0(w)H_0(w-\pi) + G_1(w)H_1(w-\pi)\right]X(w-\pi)$$

$\Rightarrow$ select $G_0(w) = H_1(w-\pi)$, $G_0(z) = H_1(-z)$

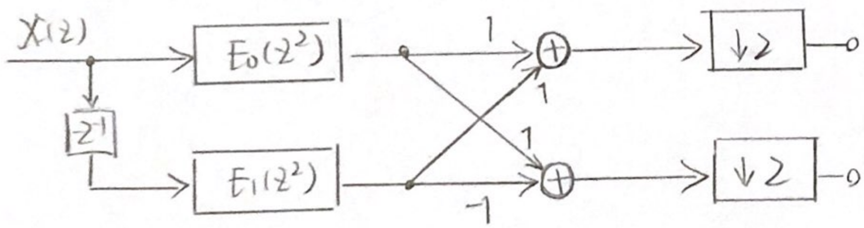
$G_1(w) = -H_0(w-\pi)$, $G_1(z) = -H_0(-z)$

$H_0(w)$, $H_1(w) = H_0(w-\pi)$, $H_1(z) = H_0(-z)$

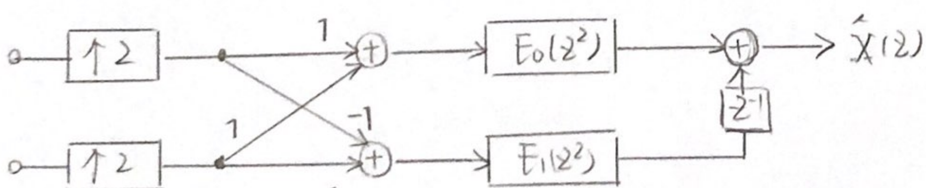
(b) $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$, $H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2)$.

$G_0(z) = H_1(-z) = H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$.

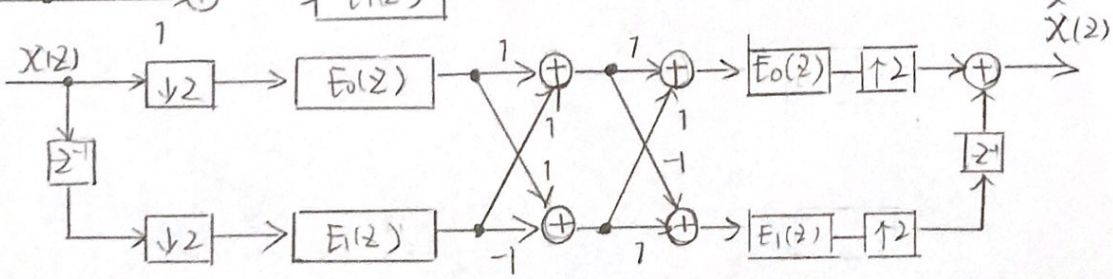
$G_1(z) = -H_0(-z) = -E_0(z^2) + z^{-1}E_1(z^2)$.



analysis



synthesis



(c) $H_0(z) = 1 + z^{-1} = E_0(z^2) + z^{-1}E_1(z^2)$

$\Rightarrow E_0(z^2) = E_0(z) = 1, \quad E_1(z^2) = E_1(z) = 1$

4. (a) $x[n] = \sum_{k=1}^{K} A_k \cos(w_k n + \phi_k) + w[n]$

$A_k, w_k \Rightarrow$ constant, $\phi_k \rightarrow \frac{1}{\pi-(-\pi)} = \frac{1}{2\pi} \Rightarrow f(\phi) = \begin{cases} \frac{1}{\pi-(-\pi)} \\ 0 \end{cases}$

let $Y[n] = \sum_{k=1}^{K} A_k \cos(w_k n + \phi_k)$

$$\gamma_{xx}[m] = \gamma_{yy}[m] + \gamma_{ww}[m]$$
$$= E[\,Y[q]\,Y[q-m]\,] + \sigma_w^2 \delta[m]$$

$$\gamma_{yy}[m] = E[\,Y[q]\,Y[q-m]\,]$$
$$= E\left\{ \sum_{k=1}^{K} A_k \cos(w_k q + \phi_k) \sum_{k=1}^{K} A_k \cos(w_k(q-m) + \phi_k) \right\}$$
$$= \sum_{k=1}^{K} A_k^2\, E\left\{ \sum_{k=1}^{K} \cos(w_k q + \phi_k) \cos(w_k(q-m) + \phi_k) \right\}$$
$$= \sum_{k=1}^{K} A_k^2\, E\left\{ \frac{1}{2} \sum_{k=1}^{K} [\cos(w_k(2q-m) + 2\phi_k) + \cos(w_k m)] \right\}$$
$$= \sum_{k=1}^{K} \frac{A_k^2}{2} \left\{ \underbrace{\sum_{k=1}^{K} E[\cos(w_k(2q-m) + 2\phi_k)]}_{0} + \sum_{k=1}^{K} E[\cos(w_k m)] \right\}$$
$$= \sum_{k=1}^{K} \frac{A_k^2}{2} \cos(w_k m)$$

$$\Rightarrow \gamma_{xx}[m] = \sum_{k=1}^{K} \frac{A_k^2}{2} \cos(w_k m) + \sigma_w^2 \delta[m]$$

(b) $\Gamma_{xx}(w) = \mathrm{DTFT}\{\gamma_{xx}[m]\}$

$$= \sum_{k=1}^{K} \frac{\pi}{2} A_k^2 (\delta(w+w_k) + \delta(w-w_k)) + \sigma_w^2$$

$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2016-2017

EE6401 – ADVANCED DIGITAL SIGNAL PROCESSING

November/December 2016 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 5 questions and comprises 3 pages.
2. Answer all 5 questions.
3. All questions carry equal marks.
4. This is a closed-book examination.

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1. For the filter specification given by

Pass band frequency $f_p = 1000Hz$
Stop band frequency $f_s = 1050Hz$
Pass band ripple $\delta_p = 0.1$
Stop band ripple $\delta_s = 0.001$
Sampling frequency $F = 50kHz$
Decimation factor $D = 20$,

(a)
$\Delta f = (f_s - f_p)/F_T$
$= 10^{-3}$
$N = \frac{-10\log_{10}(10^{-4}) - 13}{14.6 \times 10^{-3}} + 1$
$= 1850.3151 \approx 1850$

determine the multistage implementation based on one-stage and two-stage approaches. Assume an FIR design and use the criteria where aliasing is allowed in the transitional band.

(a) Calculate the number of multiplications per second needed by the one-stage implementation.

(5 Marks)

(b) For the two-stage implementation, the results should be optimized in terms of the required number of multiplications for the options: $D_1 = 10$, $D_2 = 2$, and $D_1 = 2$, $D_2 = 10$. You need to show how the transitional bandwidth at each stage is specified.

(10 Marks)

Note: Question No. 1 continues on page 2

(c) Briefly explain the difference of computational complexity needed by the two options of the two-stage implementations.

(5 Marks)

2. (a) Depict the discrete-time Fourier transform (DTFT) of the signals $x[n]$ and $y[n] = x[qn]$, for $q = 1/2$ and $q = 5/2$, assuming that

$$x[n] = 2\cos(\frac{2\pi n}{3}).$$

Identify which of the two DTFTs are the same and explain why.

(10 Marks)

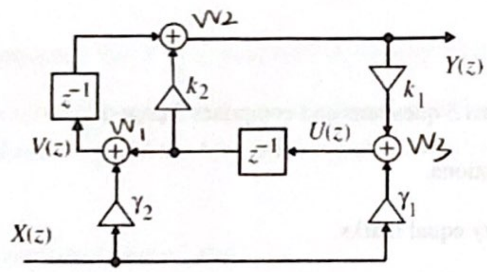

Figure 1

(b) Determine the transfer function of the digital filter structure as shown in Figure 1. Is it canonic?

(10 Marks)

3. Figure 2 below shows a general two channel perfect reconstruction filter bank system.

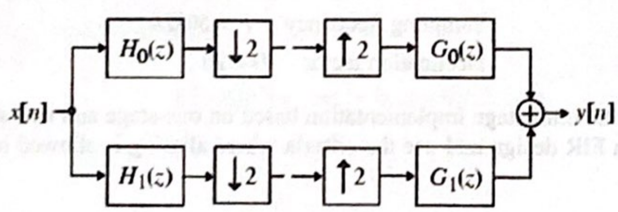

Figure 2

Let us assume that $H_0(z)$ is implemented by a polyphase structure described as $H_0(z) = E_0(z^2)+z^{-1}E_1(z^2)$, where $E_0(z)$ and $E_1(z)$ are sub-systems used in the polyphase structure.

Note: Question No. 3 continues on page 3

$H_0(w) G_0(w) + H_1(w) G_1(w)$, $H_0(z) G_0(z) + H_1(z) G_1(z) = C z^{-l}$

(a) State the condition of perfect reconstruction for a two channel filter bank;

(4 Marks)

(b) Derive the most computational efficient implementation structure in terms of $E_0(z)$ and $E_1(z)$;

(10 Marks)

(c) Based on the result obtained for (b), derive the implementation structure for $H_0(z) = (1+z^{-1})$.

(6 Marks)

4. Consider a random sequence $x[n] = \sum_{k=1}^{K} A_k \cos(\omega_k n + \phi_k) + w[n]$ where $\{A_k\}$ are constant amplitudes, $\{\omega_k\}$ are constant frequencies, $\{\phi_k\}$ are mutually statistically independent and uniformly distributed random phases and the noise sequence $w[n]$ is white with variance σ_w^2.

(a) Determine the autocorrelation of $x[n]$.

(10 Marks)

(b) Determine the power density spectrum of $x[n]$.

(10 Marks)

5. Let $\mathrm{w}[n]$ and $d[n]$ be two stationary random processes. Denote $\hat{\mathrm{d}}[n] = \sum_{l=0}^{q} \mu[l] w[n-l]$ as the linear estimator of $d[n]$. Show that the optimal coefficients $\{\mu[l]\}$ that minimize the estimation error $\mathrm{e}[n]$ are given by

$$\mathrm{E}\left[\mathrm{e[n]} w^*[n-m]\right] = 0, \mathrm{m} = 0,1,2,\ldots,q$$

(20 Marks)

END OF PAPER