IMAGE ANALYSIS

1. feature extraction
2. segmentation
3. classification

★ Segmentation by Thresholding

$$g(x,y) = \begin{cases} 1, & \text{if } f(x,y) > T \\ 0, & \text{if } f(x,y) \le T \end{cases}$$

$$T = T[x, y, p(x,y), f(x,y)].$$

global threshold T:

1. select initial T;
2. segment the image using T,
   ⇨ $G_1(>T)$, $G_2(\le T)$;
3. compute average gray-level values μ_1, μ_2;
4. $T_{new} = \frac{1}{2}(\mu_1 + \mu_2)$;
5. repeat 2-4 until $T_{new} < T_0$.

★ Detection of Discontinuities

① point detection:

$$R = \sum_{i=1}^{9} w_i z_i,$$

$$|R| \ge T$$

-1	-1	-1
-1	8	-1
-1	-1	-1

② line detection:

-1	-1	-1
2	2	2
-1	-1	-1

horizontal

-1	-1	2
-1	2	-1
2	-1	-1

+45°

-1	2	-1
-1	2	-1
-1	2	-1

vertical

③ edge detection:

1st-order derivative (gradient)

$$\nabla f(x,y) = \begin{bmatrix} G_x(x,y) \\ G_y(x,y) \end{bmatrix} = \begin{bmatrix} \partial f(x,y)/\partial x \\ \partial f(x,y)/\partial y \end{bmatrix}$$

magnitude:

$$|\nabla f(x,y)| = [G_x^2(x,y) + G_y^2(x,y)]^{\frac{1}{2}}$$

direction:

$$\varphi(x,y) = \tan^{-1}\left(\frac{G_y(x,y)}{G_x(x,y)}\right)$$

⇨ $G_x(x,y) = f(x+1,y) - f(x-1,y)$
$G_y(x,y) = f(x,y+1) - f(x,y-1)$

⇨ $G_x(x,y) = h_x(x,y) * f(x,y)$
$G_y(x,y) = h_y(x,y) * f(x,y)$

0	0	0
1	0	-1
0	0	0

$h_x(x,y)$

0	1	0
0	0	0
0	-1	0

$h_y(x,y)$

Sobel mask:

$$G_x = (z_3 + 2z_6 + z_9) - (z_1 + 2z_4 + z_7)$$
$$G_y = (z_7 + 2z_8 + z_9) - (z_1 + 2z_2 + z_3)$$

-1	0	1
-2	0	2
-1	0	1

G_x

-1	-2	-1
0	0	0
1	2	1

G_y

2nd-order derivative:

$$\nabla^2 f = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)$$
$$= h_L(x,y) * f(x,y)$$

0	-1	0
-1	4	-1
0	-1	0

-1	-1	-1
-1	8	-1
-1	-1	-1

Laplace of Gaussian (LoG):

$$h(x,y) = -e^{-\frac{x^2+y^2}{2\sigma^2}} = -e^{-\frac{r^2}{2\sigma^2}},$$

$$\nabla^2(h*f) = (\nabla^2 h) * f.$$

$$\nabla^2 h = -\left(\frac{r^2-\sigma^2}{\sigma^4}\right)e^{-\frac{r^2}{2\sigma^2}}$$

★ Hough Transform

$y_i = ax_i + b$, $b = y_i - ax_i$

define the output image of the input image as impulse response of the system:

$$h(x,y) \triangleq T\{\delta(x,y)\}$$

for shift invariant system:

$$h(x-i, y-j) = T\{\delta(x-i, y-j)\}$$

* 2-D convolution:

input image $f(x,y)$;

linear shift-invariant LSI system;

output image $g(x,y)$:

$$\begin{aligned} g(x,y) &= T\{f(x,y)\} \\ &= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f(i,j)\,h(x-i,y-j) \\ &\triangleq f(x,y) * h(x,y) \\ &= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} h(i,j)\,f(x-i,y-j) \end{aligned}$$

* impulse response $h(x,y)$:

is also an image, called spatial representation of a filter mask:

$$\begin{aligned} g(x,y) &= f(x,y) * h(x,y) \\ &= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} h(i,j)\,f(x-i,y-j) \end{aligned}$$

* Fourier transform:

$$\begin{aligned} F(u) &= \mathcal{F}\{f(x)\} \\ &= \int_{-\infty}^{\infty} f(x)\,e^{-j2\pi ux}\,dx \end{aligned}$$

$$\begin{aligned} f(x) &= \mathcal{F}^{-1}\{F(u)\} \\ &= \int_{-\infty}^{\infty} F(u)\,e^{j2\pi ux}\,du \end{aligned}$$

* 2-D Fourier transform:

$$\begin{aligned} &F(u,v) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{[-j2\pi(ux+vy)]}\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-j2\pi ux}\,e^{-j2\pi vy}\,dx\,dy \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,e^{-j2\pi ux}\,dx\right]e^{-j2\pi vy}\,dy \\ &= \int_{-\infty}^{\infty} F_x(u,y)\,e^{-j2\pi vy}\,dy \\ &= F_x(u)\,F_y(v), \end{aligned}$$

only if $f(x,y) = f_1(x)\,f_2(y)$

Example:

$$\begin{aligned} g(x,y) = {} & f(x,y) + f(x-1,y) + \\ & f(x+1,y) + f(x,y-1) + f(x,y+1). \end{aligned}$$

0	1	0
1	1	1
0	1	0

⇒ impulse response:

$$h(x,y) = \delta(x,y) + \delta(x-1,y) + \delta(x+1,y) + \delta(x,y-1) + \delta(x,y+1)$$

$$F(u,v) = R(u,v) + jI(u,v)$$
$$= |F(u,v)| e^{j\varphi(u,v)},$$

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)},$$

$$\varphi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right],$$

$$R(u,v) = |F(u,v)| \cos\varphi(u,v).$$
$$I(u,v) = |F(u,v)| \sin\varphi(u,v).$$

Fourier transform:

$$F(u) = \mathcal{F}\{f(x)\}$$
$$= \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

$$f(x) = \mathcal{F}^{-1}\{F(u)\}$$
$$= \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

2-D Fourier transform:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy$$
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv$$

* DFT:

$$F(u,v) = \frac{1}{mn} \sum_{x=0}^{m-1}\sum_{y=0}^{n-1} f(x,y) e^{[-j2\pi(\frac{ux}{m} + \frac{vy}{n})]}$$

$$f(x,y) = \frac{1}{mn} \sum_{u=0}^{m-1}\sum_{v=0}^{n-1} F(u,v) e^{[j2\pi(\frac{ux}{m} + \frac{vy}{n})]}$$

* properties of DFT:

1. translation:

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v) e^{[-j2\pi(\frac{x_0u}{m} + \frac{y_0v}{n})]}$$

$$F(u-u_0, v-v_0) \Leftrightarrow f(x,y) e^{[j2\pi(\frac{u_0x}{m} + \frac{v_0y}{n})]}$$

2. rotation:

$x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$.
$u = \omega\cdot\cos\varphi$, $v = \omega\cdot\sin\varphi$.

$$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$$

3. rotation invariant transform:

$$g(u,v) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 e^{-j(2\pi ur^2 + v\theta)} f(r,\theta) dr d\theta$$

4. periodicity:

$$F(u,v) = F(u+m, v) = F(u, v+n)$$
$$= F(u+m, v+n)$$

5. conjugate symmetry for real image:

$$F(u,v) = F^*(-u,-v).$$
$$|F(u,v)| = |F(-u,-v)|$$

6. linearity & scaling:

$$\mathcal{F}\{\alpha f_1(x,y) + \beta f_2(x,y) + \cdots\}$$
$$= \alpha F_1(u,v) + \beta F_2(u,v) + \cdots,$$

$$\mathcal{F}\{f(\alpha x, \beta y)\} = \frac{1}{|\alpha\beta|} F\left(\frac{u}{\alpha}, \frac{v}{\beta}\right)$$

7. convolution theorem:

$$f(x,y) * g(x,y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta) g(x-\alpha, y-\beta) d\alpha d\beta$$

$$f(x,y) * g(x,y) \Leftrightarrow F(u,v)G(u,v)$$
$$f(x,y) g(x,y) \Leftrightarrow F(u,v) * G(u,v)$$

* image sampling:

$$f_d(m,n) = f_c(m\Delta x, n\Delta y) = f_c(x,y)\Big|_{\text{let } x=m\Delta x,\ y=n\Delta y}$$

band-limited:

a 2-D func $f_c(x,y)$ is band-limited if its Fourier transform $F_c(u,v)$ is zero outside a bounded spatial freq support, e.g.,

$F_c(u,v)=0$, for $|u|>U_0$, $|v|>V_0$, where $2U_0$ and $2V_0$ are the bandwidths of $f_c(x,y)$.

2-D sampling func:

$$s(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x-m\Delta x, y-n\Delta y)$$

$$S(u,v) = \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta\left(u-\frac{m}{\Delta x}, v-\frac{n}{\Delta y}\right)$$

sampling:

$$f_d(x,y) = f_c(x,y)\, s(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_c(m\Delta x, n\Delta y)\,\delta(x-m\Delta x, y-n\Delta y)$$

$$F_d(u,v) = F_c(u,v) * S(u,v)$$
$$= \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} F_c(u,v) * \delta\left(u-\frac{m}{\Delta x}, v-\frac{m}{\Delta y}\right)$$
$$= \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} F_c\left(u-\frac{m}{\Delta x}, v-\frac{n}{\Delta y}\right)$$
$$= \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} F_c(u-mf_{xs}, v-nf_{ys}),$$

it is a periodic replication of $F_c(u,v)$ on a rectangular grid with spacing $(1/\Delta x, 1/\Delta y)$.

no overlapping:

$$f_{xs} = \frac{1}{\Delta x} \ge 2U_0$$
$$f_{ys} = \frac{1}{\Delta y} \ge 2V_0$$

$\Rightarrow$ sampling freq is greater than the bandwidth.

$$\Delta x \le \frac{1}{2U_0}, \quad \Delta y \le \frac{1}{2V_0}$$

$\Rightarrow$ sampling interval is smaller than the reciprocal of bandwidth

Then $F_c(u,v)$ can be recovered from $F_d(u,v)$ by using a LPF with freq response:

$$H(u,v) = \begin{cases} \Delta x\Delta y, & (u,v)\in R \\ 0, & \text{otherwise} \end{cases}$$

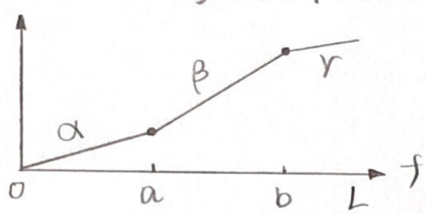
$$F_c(u,v) = F_d(u,v)\,H(u,v)$$
$$f_c(x,y) = f_d(x,y) * h(x,y)$$
$$= h(x,y) * \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_c(m\Delta x, n\Delta y)\,\delta(x-m\Delta x, y-n\Delta y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_c(m\Delta x, n\Delta y)\, h(x,y) * \delta(x-m\Delta x, y-n\Delta y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_c(m\Delta x, n\Delta y)\, h(x-m\Delta x, y-n\Delta y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_d(m,n)\, h(x-m\Delta x, y-n\Delta y)$$

Image Enhancement

* point processing:

- power transformation (gamma correction): $g = cf^{\gamma}$
- log transformation: $g = c\log(1+f)$
- piecewise linear transformation:

$$g = T(f) = \begin{cases} \alpha f, & 0 \le f < a \\ \beta(f-a)+T(a), & a \le f < b \\ \gamma(f-b)+T(b), & b \le f < L \end{cases}$$



- histogram equalization:

objective: obtain a uniform histo-gram

$$c(f) = \sum_{t=0}^{f} P_f(t) = \sum_{t=0}^{f} \frac{n_t}{n}, \quad f = 0, 1, \cdots, L,$$

$$g = T(f) = \text{round}\left[\frac{c(f) - C_{min}}{1 - C_{min}} L\right], \quad c(f) \ge C_{min}$$

where t is a dummy variable of the summation. C_{min} is the smallest positive value of all $c(f)$ obtained. round[·] rounds a real # to an integer. g is uniformly distributed in $[0, L]$.

probability theory:

$g = T(f)$ is single-valued, monotonically increasing in $[0, 1]$, $f = T^{-1}(g)$, the transformed gray level pdf:

$$P_g(g) = P_f(f)\frac{df}{dg} \quad \text{(equal probability)}$$

Consider cdf of f:

$$g = T(f) = \int_0^f P_f(t)\,dt,$$

$$\frac{dg}{df} = P_f(f), \quad P_g(g) = P_f(f)\frac{df}{dg} = P_f(f)\frac{1}{P_f(f)} = 1$$

⇒ the transformed gray value has a uniform distribution.

the histogram equalization

$$c(f) = \sum_{t=0}^{f} P_f(t) = \sum_{t=0}^{f} \frac{n_t}{n}$$

is the discrete version of

$$g = T(f) = \int_0^f P_f(t)\,dt$$

- image smoothing:

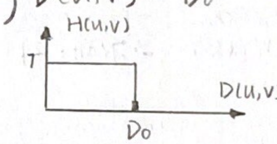
$$g(x,y) = f(x,y) * h(x,y)$$

$$G(u,v) = F(u,v)H(u,v)$$

ideal LPF:

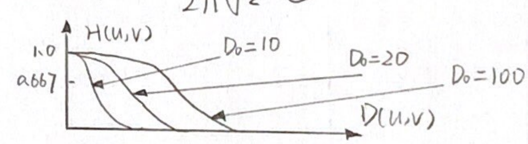
$$H(u,v) = \begin{cases} 1, & \text{if } D(u,v) \le D_0 \\ 0, & \text{if } D(u,v) > D_0 \end{cases},$$

$$D(u,v) = \sqrt{u^2+v^2}$$



Gaussian LPF:

$$G(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2D_0}}$$

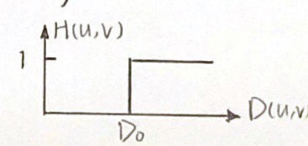


- image sharpening:

ideal HPF:

$$H(u,v) = \begin{cases} 0, & \text{if } D(u,v) \le D_0 \\ 1, & \text{if } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2+v^2}$$



Gaussian HPF:

$$G(u,v) = 1 - e^{-\frac{u^2+v^2}{2D_0}}$$

---

high-boost filtering:

$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y), \quad A > 1$$

$$f_{hb}(x,y) = (A-1)f(x,y) + f(x,y) - f_{lp}(x,y)$$
$$= (A-1)f(x,y) + f_{hp}(x,y)$$

---

* nonlinear processing:

problems of linear filters:

$$\hat{f}(x,y) = f(x,y) * h(x,y)$$
$$= \sum_{i=-a}^{a}\sum_{j=-b}^{b} h(i,j) f(x-i, y-j)$$
$$= \sum_{(s,t)\in S_{xy}} w(s,t) f(s,t)$$

image blurring ⇨ sharpness details are lost

---

order statistic filters:

⇨ median filter:

$$\hat{f}(x,y) = \underset{(s,t)\in S_{xy}}{\text{median}}\{f(s,t)\}$$

10	20	20
20	15	20
25	20	100

⇨ (10, 15, 20, 20, 20, 20, 20, 25, 100)

15 is replaced by 20

★ median filter does not blur the edge. (preserve the edge)

other order statistic filters:

① max filter:

$$\hat{f}(x,y) = \max_{(s,t)\in S_{xy}}\{f(s,t)\}$$

② min filter:

$$\hat{f}(x,y) = \min_{(s,t)\in S_{xy}}\{f(s,t)\}$$

③ mid-point filter:

$$\hat{f}(x,y) = \frac{1}{2}\left[\max_{(s,t)\in S_{xy}}\{f(s,t)\} + \min_{(s,t)\in S_{xy}}\{f(s,t)\}\right]$$

Image Restoration

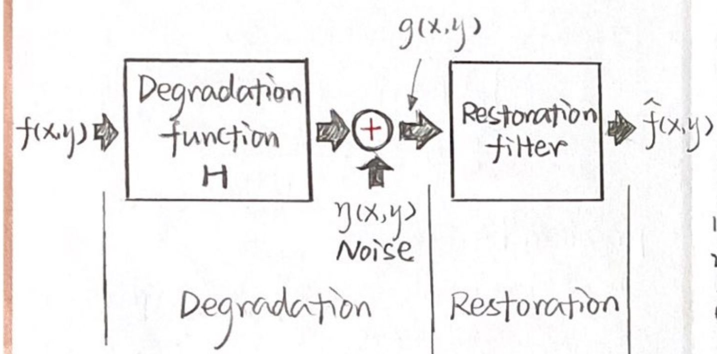
* definition :

Reconstruct or recover a degraded image using priori knowledge of the degration.

It's an objective process and modeling oriented.

common degradation :

- camera noise
- motion blur
- defocus blur
- transmission disturbance



$f(x,y)$: orignal image

H : degradation function

$\eta(x,y)$: additive noise

$g(x,y)$: degraded (observed) image

$\hat{f}(x,y)$: estimated (restored) image

* simple degradation model :

assume :

- LTI/LSI system
- additive uncorrelated noise

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f(i,j)\,h(x-i,y-j) + \eta(x,y)$$

$$\Updownarrow$$

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

assume the impulse response (point spread function) of the degradation process is finite in space :

$$g(x,y) = \sum_{i=-I}^{I}\sum_{j=-J}^{J} h(i,j)\,f(x-i,y-j) + \eta(x,y)$$

* motion blur :

due to relative motion between camera and object.

Example :

an image undergoes planar motion in x- and y- directions with $x_0(t)$ and $y_0(t)$ and T is the duration of the exposure :

$$g(x,y) = \int_0^T f[x - x_0(t),\, y - y_0(t)]\,dt$$

suppose :

$$x_0(t) = \frac{at}{T}, \quad y_0(t) = \frac{bt}{T}$$

$$\Rightarrow H(u,v) = \frac{G(u,v)}{F(u,v)}$$

$$= \frac{T\sin[\pi(ua+vb)]}{\pi(ua+vb)} e^{-j\pi(ua+vb)}$$

the freq response is a directional sinc function.

* rectangular aperture of camera (out-of-focus):

$$h(x,y) = \begin{cases} 1, & -a \le x \le a, -b \le y \le b \\ 0, & \text{otherwise} \end{cases},$$

where a and b depend on the aperture dimensions.

the freq response will be the product of a horizontal and a vertical sinc function.

* atmospheric turbulence:

the freq response is a 2-D Gaussian function with circular contours:

$$H(u,v) = e^{-k(u^2+v^2)^{\frac{5}{6}}},$$

where k is the turbulence parameter.

* noise models:

Typically, noise $\eta(x,y)$ is modeled as:

- zero-mean: $E\{\eta(x,y)\} = 0$
- independent to original image: $E\{\eta(x,y) f(x+i, y+j)\} = 0$
- PDF is Gaussian
- PSD is flat and constant, which means the autocorrelation is an impulse:
  $E\{\eta(x,y)\eta(x+i,y+j)\} = \delta(i,j)$

---

in practice, pseudo-inverse filter:

$$H^{-}(u,v) = \begin{cases} \frac{1}{H(u,v)}, & |H(u,v)| \ge \varepsilon \\ 0, & |H(u,v)| < \varepsilon \end{cases}$$

* inverse filter:

recovers the original image $f(x,y)$ from the observed image $g(x,y)$:

$$G(u,v) \rightarrow \boxed{H^I(u,v)} \rightarrow \hat{F}(u,v)$$

inverse filter: $H^I(u,v) = H^{-1}(u,v)$

---

degraded (observed) image $g(x,y)$:

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

$$\Updownarrow$$

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

$$\begin{aligned} \hat{F}(u,v) &= G(u,v)H^I(u,v) \\ &= [H(u,v)F(u,v) + N(u,v)]H^{-1}(u,v) \\ &= F(u,v) + \frac{N(u,v)}{H(u,v)} \end{aligned}$$

---

problem 1:

$H^I(u,v)$ will NOT exist if $H(u,v)$ has any zero.

problem 2:

inverse filter $H^I(u,v)$ results in noise amplification if $H(u,v)$ is small at certain freq.

- The degradation freq response $H(u,v)$ is a sinc function.
- Thus, $H^I(u,v)$ becomes infinite at some freq points.

generalized inverse filter:

$$H^{-}(u,v) = \begin{cases} \frac{1}{H(u,v)}, & |H(u,v)| \ne 0 \\ 0, & |H(u,v)| = 0 \end{cases}$$

(practically impossible)

* Wiener filter:

problem:

inverse and pseudo-inverse filtering doesn't perform well in the presence of noise.

$$\hat{F}(u,v) = G(u,v) H^{I}(u,v)$$

$$= [F(u,v)H(u,v) + N(u,v)] H^{-1}(u,v)$$

$$= F(u,v) + \frac{N(u,v)}{H(u,v)},$$

if $H(u,v)$ is zero or small at certain freq, then $\frac{N}{H}$ at the output will be large, resulting in noise amplification.

minimum mean square error (MMSE) filter:

$$e^2 = E\{[f(x,y) - \hat{f}(x,y)]^2\}$$

$$= E\{[f(x,y) - h^w(x,y) * g(x,y)]^2\}$$

$$\frac{\partial e^2}{\partial h^w(x,y)} = 0, \text{ with}$$

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

$$G(u,v) = H(u,v) F(u,v) + N(u,v),$$

we have

$$\frac{\partial E\{[f(x,y) - h^w(x,y) * h(x,y) * f(x,y) - h^w(x,y) * \eta(x,y)]^2\}}{\partial h^w(x,y)} = 0$$

assum the noise has zero mean and is uncorrelated with the image:

$$H^w(u,v) = \frac{H^*(u,v) S_f(u,v)}{|H(u,v)|^2 S_f(u,v) + S_\eta(u,v)}$$

$H(u,v)$: degradation function; $H^*(u,v)$: complex conjugate;

$|H(u,v)|^2 = H(u,v) H^*(u,v)$; $S_f(u,v) = |F(u,v)|^2$ is the power spectrum of original image

$S_\eta(u,v) = |N(u,v)|^2$ is the power spectrum of the noise.

The output of Wiener filter: $\hat{F}(u,v) = H^w(u,v) G(u,v)$

Wiener filter:

$$H^w(u,v) = \frac{1}{H(u,v)} \left[\frac{|H(u,v)|^2 S_f(u,v)}{|H(u,v)|^2 S_f(u,v) + S_\eta(u,v)} \right]$$

$$= \frac{1}{H(u,v)} W(u,v)$$

$$W(u,v) = \frac{|H(u,v)|^2 S_f(u,v)}{|H(u,v)|^2 S_f(u,v) + S_\eta(u,v)}$$

Morphological Image Processing

* Set Theory
* Morphological Operations:

dilation:

$$A \oplus B = \{z \mid [(\hat{B})_z \cap A] \neq \phi\}$$

erosion:

$$A \ominus B = \{z \mid (B)_z \subseteq A\}$$

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

opening:

erosion followed by dilation

$$A \circ B = (A \ominus B) \oplus B$$
$$= \cup \{(B)_z \mid (B)_z \subseteq A\}$$

significance:

clears an image of noise whilst retaining the original object size.

flattens the sharp peninsular projections on the object.

closing:

dilation followed by erosion

$$A \bullet B = (A \oplus B) \ominus B$$

significance:

fills holes in a region whilst retaining the original object size

properties:

- $A \circ B$ is a subset of A
- $(A \circ B) \circ B = A \circ B$
- if C is a subset of D, then $C \circ B$ is a subset of $D \circ B$
- A is a subset of $A \bullet B$
- $(A \bullet B) \bullet B = A \bullet B$
- if C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$

* Algorithms & Applications:

boundary extraction:

$$\beta(A) = A - (A \ominus B)$$

region filling:

$$X_k = (X_{k-1} \oplus B) \cap A^c, \quad k = 1, 2, \ldots$$

$$A^f = X_k \cup A$$